Solving Rational Expectations Models with Partial Information Structure: a Perturbations Approach

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Abstract

This paper presents an algorithm and Matlab codes to solve up to second order of approximation rational expectations models with partial information structure, and provides simple examples to demonstrate how the algorithm works.

Keywords: DSGE models; Timing constraints; Second-order approximation; Matlab code

1 Introduction

Medium-scale DSGE models have become increasingly popular in macroeconomic research. Most models have an implicit assumption that realizations of all shocks are known in the beginning of a period, and that all decisions are made after uncertainty realizes. As a result, all control variables in such models immediately respond to innovations in exogenous variables. However, it seems overly restrictive to expect that all shocks occur simultaneously; moreover, imperfect knowledge and slow acquisition of information may delay and slow down the response of the economy to some shocks compared with the others. There are theories that introduce specific restrictions on the timing of events in dynamic models, such as limited participation models of money, models of labor

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hoarding and models with sticky prices;[1] however, these theories are not widely used in medium-scale DSGE modeling. The goal of this research is to facilitate the use of timing restrictions in rational expectations models by offering an alternative algorithm to obtain approximate dynamics of model variables.

This paper extends the first- and second-order approximation algorithms in Schmitt-Grohé and Uribe (2004) by allowing specific timing structure in an otherwise standard model of rational expectations. The timing structure can possibly consist of multiple information sets, when different shocks occur at different times within a period, and the decisions of economic agents are spread within a period as well. Similar algorithms to solve models with timing structure were offered in Christiano (2002) and King and Watson (2002). Differently from these authors, this paper provides both first- and second-order approximate solutions to the policy functions. The algorithm relies on the solution to the more standard, "full information" version of the model. If the solution to the full information model is available, then to solve the model with timing restrictions, one just needs to group all shocks, control variables, and decision making equations into different information sets, and run the Matlab codes that accompany this paper.

The paper proceeds as follows. Section 2 starts by introducing the standard rational expectation problem with full information structure. Section 3 then develops the algorithm to solve the model with two informational subperiods. In Section 4, the algorithm is extended to allow several subperiods. Section 5 provides the algorithm to obtain the second-order approximate dynamics to the model with timing constraints. Finally, Section 6 concludes.

2 The Problem

I assume that a standard rational expectations model is described by n_F expectational equations in n_F variables, which admit a sequential representation, so that this nonlinear

[1] See, for example, Fuerst (1992), Christiano and eichenbaum (1992), Burnside and Eichenbaum (1996), and Chari, Kehoe, and McGrattan (2000). For other references, see Christiano (2002).

system can be described by only current and future model variables, X, Y, and X', Y':

$$E[f(Y',\, X',\, Y,\, X)] = 0. \tag{1}$$

In this notation, the prime superscript denotes a variable in the next period, and E is the expectations operator conditional on information available by the beginning of a period, Y is the $n_Y \times 1$ vector of control variables, and $n_X \times 1$ vector X contains state variables. The control vector contains all endogenous variables the decision on which is made within the period, while state variable vector X contains endogenous predetermined variables and exogenous shock processes. For convenience, I assume that all model variables have zero means, or presented as deviations from their steady state values.

It is commonly assumed that all shocks have realizations in the beginning of each period, and the optimal choice of all control variables is thus conditioned on all variables in the state vector X. I call this arrangement the full information structure. A well-known fact is that the solution to system (1) assuming full information structure is

$$Y = \bar{G}(X, \sigma), \tag{2}$$

and

$$X' = \bar{H}(X, \sigma) + \sigma e', \tag{3}$$

where the elements of the $n_X \times 1$ vector e' are either zeros or mean-zero i.i.d. stochastic processes. Schmitt-Grohé and Uribe (2004) show that up to the first order of approximation, σ does not affect functions G and H, thus the linearized model dynamics can be presented as follows:

$$Y = \bar{G}_X X, \tag{4}$$

and

$$X' = \bar{H}_X X + \sigma \epsilon', \tag{5}$$

where $\bar{G}_X$ and $\bar{H}_X$ are matrices of first-order derivatives of $\bar{G}(X)$ and $\bar{H}(X)$ respectively, evaluated at a non-stochastic steady state, with sizes $n_Y \times n_X$ and $n_X \times n_X$ respectively.[2]

[2] At this point, I take into account the fact proved in Schmitt-Grohé and Uribe (2004) that the size of

A number of methods exist to solve rational expectations models described by the dynamics system (1) with full information structure (see Blanchard and Kahn (1980), Anderson and Moore (1985), Klein (2000), King and Watson (2002), Christiano (2002), Schmitt-Grohé and Uribe (2004)), and various applications packages are available to implement these methods.[3]

3 Partial Information Structure with Two Subperiods: First-Order Approximation.

In case of the partial information structure, there are timing constraints imposed on decision variables and on realizations of shocks within a period. As a result, not all realizations of stochastic processes occur in the beginning of a period, and decisions on different control variables must be made before or after these shocks realize. Thus, any period can be divided into several subperiods, each of which starts with realizations of some particular shocks, and choices for endogenous variables must be made in different subperiods. First, I focus on a simple case of information structure with two subperiods. Let I^0 and I^1 denote information in the first subperiods 0 and the second subperiod 1 respectively. Note that information contained in I^0 is also available in the subset I^1; thus $I^0 \subset I^1$.

Notice that within-a-period information structure does not change the system of equations $f(Y', X', Y, X)$. To take into account the informational structure in the solution mechanism, it is convenient to partition these vectors as follows:

$$X = [x;\, \theta], \quad \textit{and} \tag{6}$$

$$Y = [y;\, z], \tag{7}$$

uncertainty does not influence the first order approximate policy functions.

[3]See, for example, Dynare package, Uhlig's toolkit, codes of Christiano, and Schmitt-Grohe and Uribe among others.

where $n_x \times 1$ vector x consists of endogenous predetermined variables, and exogenous variables with realization in the beginning of first subperiod, θ contains n_θ exogenous variables with realizations in the second subperiod, y is the $n_y \times 1$ vector of full information control variables. These are endogenous variables the decision for which is made in the beginning of second subperiod, when realizations of all shocks are known. Finally, z is the $n_z \times 1$ vector of partial information control variables. These endogenous variables must be decided on in the first subperiod, when realizations of only a subset of shocks are known. Suppose equations in f are arranged as follows

$$f = [f^0;\ f^1;\ f^\theta].$$

The set of equations in f^0 includes n_z equations determining the choice of partially endogenous variables in z, while f^1 includes n_y equations that determine fully endogenous variables y, and n_x equations determining the dynamics of the state variables in vector x. The set of equations in f^θ describes the evolution of exogenous shocks in θ, which presumably can be represented as $AR(1)$ processes:

$$\theta' = P\theta + \sigma\epsilon'_\theta, \tag{8}$$

where P is the $n_\theta \times n_\theta$ matrix of autoregressive coefficients, and ϵ'_θ is an $n_\theta \times 1$ vector of *iid* shocks with mean 0, and variance Σ.

Denoting $\mathcal{E}$ the expectations operator that takes into account this timing structure, System (1) becomes

$$\mathcal{E}[f(Y',\ X',\ Y,\ X)] = 0. \tag{9}$$

The solution to this system of expectational difference equations can be presented in general form as

$$y = g(x, \theta, \theta_{-1}, \sigma), \tag{10}$$

$$z = j(x, \theta_{-1}, \sigma), \tag{11}$$

and

$$x' = h(x, \theta, \theta_{-1}, \sigma) + \sigma\epsilon'_x, \tag{12}$$

where subscript "-1" denotes previous period value. This system implies that partially endogenous variables in vector z cannot respond to the current realization of shocks in θ because information on θ is not available yet at the time when decision on z is made. Thus, the best option for a constrained decision maker is to rely on the conditional forecast of θ that is determined by θ_{-1} due to the autoregressive structure of the shock process. When choosing fully endogenous variables in y in the second subperiod, z is treated as a state variable. As a result, through their response to z, fully endogenous variables in y will implicitly take into account previous period's realization of the shock θ_{-1}. Finally, endogenous predetermined variables in x will respond to θ_{-1} for the same reason.

Given the stochastic process for θ in Equation (8), the first-order approximations to Equations (10), (11), and (12) are

$$y = g_x x + g_\theta \theta + g_{\theta_{-1}} \theta_{-1}, \tag{13}$$

$$z = j_x x + j_{\theta_{-1}} \theta_{-1}, \tag{14}$$

and

$$x' = h_x x + h_\theta \theta + h_{\theta_{-1}} \theta_{-1} + \sigma\epsilon'_x, \tag{15}$$

respectively, where g_x, g_θ, $g_{\theta_{-1}}$, j_x, $j_{\theta_{-1}}$, h_x, h_θ, and $h_{\theta_{-1}}$ are matrices of coefficients to be determined. [4] These matrices have sizes $n_y \times n_x$, $n_y \times n_\theta$, $n_y \times n_\theta$, $n_z \times n_x$, $n_z \times n_\theta$, $n_x \times n_x$, $n_x \times n_\theta$, and $n_x \times n_\theta$ respectively. The first-order approximate solution can be presented in a more compact way as follows

$$Y = G_X \begin{bmatrix} x \\ \theta \\ \theta_{-1} \end{bmatrix}, \qquad X' = H_X \begin{bmatrix} x \\ \theta \\ \theta_{-1} \end{bmatrix} + \sigma\epsilon', \tag{16}$$

[4] In Appendix, I show that same as in the full information model, the size of uncertainty does not influence the first order approximate policy functions in models with timing restrictions.

where

$$G_X = \left[\begin{array}{c|c|c} G_x & g_\theta & g_{\theta_{-1}} \\ \cline{2-3} & 0_{n_z \times n_\theta} & j_{\theta_{-1}} \end{array}\right] \quad with \quad G_x = \begin{bmatrix} g_x \\ j_x \end{bmatrix},$$

$$H_X = \left[\begin{array}{c|c|c} h_x & h_\theta & h_{\theta_{-1}} \\ \hline 0 & P & 0 \end{array}\right], \quad and \quad \epsilon = \begin{bmatrix} \epsilon_x \\ \epsilon_\theta \end{bmatrix}.$$

I will sometimes refer to another partitioning of matrices G_X and H_X:

$$G_X = [G_x,\, G_\theta,\, G_{\theta_{-1}}], \quad and \quad H_X = [H_x,\, H_\theta,\, H_{\theta_{-1}}],$$

where

$$G_\theta = \begin{bmatrix} g_\theta \\ 0_{n_z \times n_\theta} \end{bmatrix}, \quad G_{\theta_{-1}} = \begin{bmatrix} g_{\theta_{-1}} \\ j_{\theta_{-1}} \end{bmatrix},$$

$$H_x = \begin{bmatrix} h_x \\ 0_{n_\theta \times n_\theta} \end{bmatrix}, \quad H_\theta = \begin{bmatrix} h_\theta \\ P \end{bmatrix}, \quad and \quad H_{\theta_{-1}} = \begin{bmatrix} h_{\theta_{-1}} \\ 0_{n_\theta \times n_\theta} \end{bmatrix}.$$

The following proposition describes how the solution to the model with partial information structure can be obtained from the solution to the full information version of the model, namely from matrices $\bar{G}_X$ and $\bar{H}_X$. For convenience, I partition these matrices as follows:

$$\bar{G}_X = \begin{array}{c} \begin{array}{cc} \overbrace{\phantom{\bar{G}_x}}^{n_x} & \overbrace{\phantom{\bar{g}_\theta}}^{n_\theta} \end{array} \\ \left[\begin{array}{c|c} \bar{G}_x & \bar{g}_\theta \\ \cline{2-2} & \bar{j}_\theta \end{array}\right] \end{array} \begin{array}{l} \\ \}n_y \\ \}n_z \end{array}, \qquad \bar{H}_X = \begin{array}{c} \begin{array}{cc} \overbrace{\phantom{\bar{h}_x}}^{n_x} & \overbrace{\phantom{\bar{h}_\theta}}^{n_\theta} \end{array} \\ \left[\begin{array}{c|c} \bar{h}_x & \bar{h}_\theta \\ \hline 0 & P \end{array}\right] \end{array} \begin{array}{l} \\ \}n_x \\ \}n_\theta \end{array}.$$

Proposition 1 *Suppose $\bar{h}_x$, $\bar{G}_x$, $\bar{g}_\theta$, $\bar{j}_\theta$, $\bar{h}_\theta$, and P are submatrices of $\bar{H}_X$ and $\bar{G}_X$ as defined above. The relationship between solutions to the full-information and partial information versions of the model is the following:*

$$h_x = \bar{h}_x, \qquad G_x = \bar{G}_x,$$

$$g_{\theta_{-1}} + g_\theta P = \bar{g}_\theta P,$$

$$h_{\theta_{-1}} + h_\theta P = \bar{h}_\theta P,$$

and

$$j_{\theta_{-1}} = \bar{j}_\theta P, \tag{17}$$

where $h_{\theta_{-1}}$ and $g_{\theta_{-1}}$ solve the linear system of equations

$$\Delta(f^1)_{[x',y]} \begin{bmatrix} h_{\theta_{-1}} \\ g_{\theta_{-1}} \end{bmatrix} = -f^1_z j_{\theta_{-1}}, \tag{18}$$

in which

$$\Delta(f^1)_{[x',y]} = [f^1_{Y'} G_x + f^1_{x'},\ f^1_y] \tag{19}$$

is the jacobian of the system of equations f^1 with respect to the vector $[x',\ y]$, and $f^1_{Y'}$, $f^1_{x'}$, f^1_y, and f^1_z are derivative matrices of a vector f^1 with respect to vectors Y', x', y and z correspondingly, evaluated at a steady state.

The proof of the proposition is given in Appendix. The proposition suggests that the solution to a model with intratemporal informational restrictions can be obtained from the solution to the full-information version of the model using simple linear transformations. One may notice that it is not important which method is used to obtain the solution to the full information version of the model, as long as the required matrices $\bar{H}_X$ and $\bar{G}_X$ can be recovered with this method. Based on this proposition, the algorithm to solve the rational expectations model with partial information structure is the following:

Step 1. Sort the equilibrium conditions into vectors f^0, f^1, and f^θ, and arrange them into vector f.[5]

Step 2. Arrange variables in X and Y according to formulas (6) and (7).

Step 3. Solve the full-information version of the model to obtain matrices $\bar{G}_X$ and $\bar{H}_X$, and partition them accordingly.

Step 4. Assign $h_x = \bar{h}_x$, $G_x = \bar{G}_x$, and $j_{\theta_{-1}} = \bar{j}_\theta P$.

[5] There must be exactly $n_y + n_x$ equilibrium conditions in f^1 so that matrix $[f^1_{Y'} G_x + f^1_{x'},\ f^1_y]$ is square.

Step 5. Obtain the responses of x' and y to the lag of shocks, θ_{-1} using the formula

$$\begin{bmatrix} h_{\theta_{-1}} \\ g_{\theta_{-1}} \end{bmatrix} = -[f^1_{Y'} G_x + f^1_{x'},\ f^1_y]^{-1} f^1_z j_{\theta_{-1}}.$$

Step 6. Calculate g_θ and h_θ as as follows

$$g_\theta = \bar{g}_\theta - g_{\theta_{-1}} P^{-1}, \quad \textit{and} \quad h_\theta = \bar{h}_\theta - h_{\theta_{-1}} P^{-1}.$$

One may notice that the unique solution to the model with timing restrictions is determined by the assumption that $j_\theta = 0$, which reflects the inability to observe some shocks at the time of decision making. This assumption can be generalized by assuming $j_\theta = J^*$, where J^* is a matrix of size $n_z \times n_\theta$. In this situation, agents cannot perfectly observe realizations of shocks in θ when decisions are made, so they under- or overestimate j_θ in a certain way. Such identifying assumption will not affect the main results of Proposition 1, except for Equation (17) which has to be substituted with

$$j_{\theta_{-1}} + J^* P = \bar{j}_\theta P.$$

3.1 Example 1. Simple New-Keynesian Model: Part 1.

This model is a simple prototype New-Keynesian model.There is no capital, labor supply is inelastic, and monetary policy is motivated by nominal price rigidities. The model equilibrium is described by three equations:

$$c_t = E_t c_{t+1} - \sigma^{-1}(i_t - E_t \pi_{t+1}) + \epsilon_t, \tag{20}$$

$$\pi_t = \beta E_t \pi_{t+1} + \kappa c_t + \upsilon_t, \tag{21}$$

and

$$i_t = \alpha \pi_t, \tag{22}$$

where E_t is a conditional expectations operator, c_t is consumption and output, i_t is the nominal interest rate, π_t is inflation, υ_t is the aggregate supply shock, and ϵ_t is the aggregate demand shock, $\sigma^{-1} > 0$ is the parameter of intertemporal elasticity of substitution, $\beta \in (0, 1)$ is the discount factor, $\kappa > 0$ is the slope of the supply curve[6], and α is the parameter of monetary policy rule. The dynamics for the shocks are described by AR(1) processes

$$\upsilon_{t+1} = \rho_\upsilon \upsilon_t + e_t^\upsilon,$$

and

$$\epsilon_{t+1} = \rho_\epsilon \epsilon_t + e_t^\epsilon,$$

respectively, where ρ_υ and ρ_ϵ are autoregressive coefficients, while e_t^υ and e_t^ϵ are mean zero *i.i.d.* processes. Equation (20) is the Euler equation for consumption, (21) is the Phillips curve, and (22) is the monetary policy rule. I focus on the model with unique equilibrium by imposing $\alpha > 1$.

Suppose realizations of the aggregate demand shock ϵ occur in the middle of a period, however, producers must set prices upon before the realization of this shock. One can substitute Equation (22) into (20) to reduce the system to two equations and two endogenous variables. Then, the state and control vectors are then $X = [\upsilon; \epsilon]$, and $Y = [c; \pi]$, with the implied partitioning being $x = \upsilon$, $\theta = \epsilon$, $y = c$, and $z = \pi$. The set of equilibrium conditions is $\mathcal{E}[f^0; f^1; f^\theta] = 0$, where

$$f^0 = -\pi + \beta\pi' + \kappa c + \upsilon,$$

$$f^1 = \begin{bmatrix} -c + c' - \sigma^{-1}(\alpha\pi - \pi') + \epsilon \\ -\upsilon' + \rho_\upsilon \upsilon + e^\upsilon \end{bmatrix}, \tag{23}$$

and

$$f^\theta = -\epsilon' + \rho_\epsilon \epsilon + e^\epsilon.$$

The first-order approximate solution to the full information version of the model can

[6] In the case of Calvo-Yun type price staggering, $\kappa = \frac{(1-\omega)(1-\beta\omega)}{\omega}$ where ω is the probability of a firm not being able to adjust the price.

be obtained as follows. After substituting the monetary policy rule into Equation (20), the dynamics of the new-Keynesian model is described by the 2-dimensional system of expectational difference equations:

$$\Lambda Y' + \Gamma Y + X = 0, \tag{24}$$

where $\Lambda = \begin{bmatrix} 0 & \beta \\ 1 & \sigma^{-1} \end{bmatrix}$, and $\Gamma = \begin{bmatrix} \kappa & -1 \\ -1 & -\alpha\sigma^{-1} \end{bmatrix}$. Then, G_X can be derived from

$$\Lambda G_X H_X X + \Gamma G_X X + X = 0, \tag{25}$$

where $H_X = \begin{bmatrix} \rho_\upsilon & 0 \\ 0 & \rho_\epsilon \end{bmatrix}$. For simplicity, I further assume $\rho_\upsilon = \rho_\epsilon = \rho$, then System (25) becomes

$$(\Lambda\rho + \Gamma)\bar{G}_X X = -X, \tag{26}$$

and $\bar{G}_X$ can be calculated as

$$\bar{G}_X = -(\Lambda\rho + \Gamma)^{-1} = -\begin{bmatrix} \kappa & \beta\rho - 1 \\ \rho - 1 & (\rho - \alpha)\sigma^{-1} \end{bmatrix}^{-1} = \frac{1}{|D|}\begin{bmatrix} (\alpha - \rho)\sigma^{-1} & \beta\rho - 1 \\ \rho - 1 & -\kappa \end{bmatrix},$$

where $|D| = \kappa\sigma^{-1}(\rho - \alpha) - (\rho - 1)(\beta\rho - 1)$. The first-order dynamics of the shock process is

$$X' = \bar{H}_X X + e', \tag{27}$$

where $e = \begin{bmatrix} e^\upsilon \\ e^\epsilon \end{bmatrix}$ and $\bar{H}_X = \begin{bmatrix} \rho & 0 \\ 0 & \rho \end{bmatrix}$. To solve the model with the timing restrictions, matrices $\bar{G}_X$ and $\bar{H}_X$ should be partitioned as follows:

$$\bar{G}_X = \left[\begin{array}{c:c} \bar{G}_x & \begin{array}{c} \bar{g}_\theta \\ \hdashline \bar{j}_\theta \end{array} \end{array}\right], \quad and \quad \bar{H}_X = \left[\begin{array}{c:c} \bar{h}_x & \bar{h}_\theta \\ \hdashline 0 & P \end{array}\right],$$

where $\bar{G}_x = \frac{1}{|D|}\begin{bmatrix} (\alpha - \rho)\sigma^{-1} \\ \rho - 1 \end{bmatrix}$, $\bar{g}_\theta = \frac{\beta\rho - 1}{|D|}$, and $\bar{j}_\theta = \frac{-\kappa}{|D|}$, $\bar{h}_x = \rho$, $\bar{h}_\theta = 0$, and $P = \rho$. The general form of the first-order approximate solution to the model with timing restrictions is:

$$\begin{bmatrix} c \\ \pi \end{bmatrix} = \left[\begin{array}{c|c|c} G_x & g_\theta & g_{\theta_{-1}} \\ \cline{2-3} & 0 & j_{\theta_{-1}} \end{array}\right] \begin{bmatrix} X \\ \epsilon_{-1} \end{bmatrix},$$

and

$$\begin{bmatrix} \upsilon' \\ \epsilon' \end{bmatrix} = \left[\begin{array}{c|c|c} h_x & h_\theta & h_{\theta_{-1}} \\ \hline 0 & P & 0 \end{array}\right] \begin{bmatrix} X \\ \epsilon_{-1} \end{bmatrix}.$$

According to Proposition 1, G_x, h_x, and $j_{\theta_{-1}}$ are determined as:

$$G_x = \bar{G}_x = \frac{1}{|D|} \begin{bmatrix} (\alpha - \rho)\sigma^{-1} \\ \rho - 1 \end{bmatrix}, \quad h_x = \rho, \quad and \quad j_{\theta_{-1}} = \bar{j}_\theta \rho = \frac{-\kappa\rho}{|D|}.$$

The elements $h_{\theta_{-1}}$ and $g_{\theta_{-1}}$ are determined as a solution to the linear system (18), and f^1 is defined by (23), from which it follows that $f^1_{Y'} = \begin{bmatrix} 1 & \sigma^{-1} \\ 0 & 0 \end{bmatrix}$, $f^1_{x'} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$, $f^1_y = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$, and $f^1_z = \begin{bmatrix} -\alpha\sigma^{-1} \\ 0 \end{bmatrix}$, this solution is

$$h_{\theta_{-1}} = 0, \quad and \quad g_{\theta_{-1}} = \frac{\alpha\sigma^{-1}\kappa\rho}{|D|}.$$

As a result,

$$h_\theta = \bar{h}_\theta = 0, \quad and \quad g_\theta = \bar{g}_\theta - g_{\theta_{-1}}\rho^{-1} = \frac{1}{|D|}(\beta\rho - 1 - \alpha\sigma^{-1}\kappa).$$

Thus, the solution to the model with partial information structure can be expressed in matrix form as

$$\begin{bmatrix} c \\ \pi \end{bmatrix} = \frac{1}{|D|} \begin{bmatrix} (\alpha - \rho)\sigma^{-1} & \beta\rho - 1 - \alpha\sigma^{-1}\kappa & \alpha\sigma^{-1}\kappa\rho \\ \rho - 1 & 0 & -\kappa\rho \end{bmatrix} \begin{bmatrix} \upsilon \\ \epsilon \\ \epsilon_{-1} \end{bmatrix}.$$

Because the state vector consists of only exogenous variables, its dynamics is unaffected by the timing structure of the model, so the evolution of the state vector is given by Equation (27).

4 Partial Information Structure with Multiple Subperiods.

Suppose there are $M+1$ subperiods within each period, and each subperiod is indexed by m, for $m = \overline{0, M}$. The information set of the initial subperiod 0, contains information on endogenous state variables and may be some shocks with realization in the beginning of this subperiod. To maintain analogy with the problem of 2 subperiods, these state variables are grouped into a vector x. In the beginning of each subperiod $m = 1, ...M$, realizations of different shocks θ^m become public information. All such shocks are grouped into an $n_\theta \times 1$ vector θ as follows:

$$\theta = [\theta^1;\, \theta^2;\, \cdots;\, \theta^M],$$

where θ^m are vectors of length $n_\theta^m \geq 1$. The decisions for all endogenous variables are spread within a period. I call z^m endogenous variables, for which decisions must be made in a subperiod m, $m = \overline{0, M-1}$. The $n_z \times 1$ vector of partial information control variables z is constructed as follows:

$$z = [z^{M-1};\, z^{M-2};\, \cdots;\, z^0], \tag{28}$$

where z^m are vectors with length $n_z^m \geq 1$. Decisions on fully endogenous variables y are made in the last subperiod M. The length of the vector y is n_y. Figure 1 provides visual representation of the timing arrangement in the model.

The elements of the vector of equilibrium conditions f can be organized to support the following partitioning

$$f = [f^0;\, f^1;\, \cdots;\, f^M;\, f^\theta],$$

where for $m = \overline{0, M-1}$, each vector f^m contains n_z^m equations determining the choice of

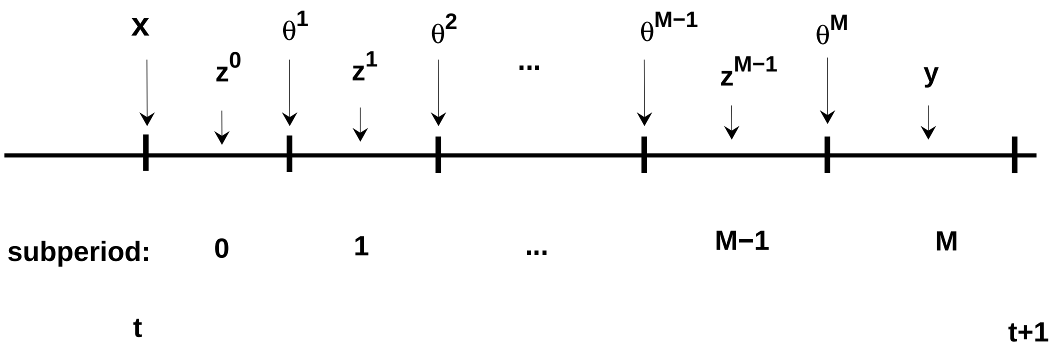

Figure 1: Timing of events in the model with multiple information sets

variables in z^m. Also, f^M contains n_y equations related to the optimal choice of variables in y and n_x equations determining the equilibrium for the state variables in vector x, and f^θ contains n_θ dynamic processes of all shocks with realizations in the middle of the period. I assume that these shocks follow an AR(1) stochastic processes as in Equation (8), where P is a block-diagonal matrix with matrices P_m of sizes $n_\theta^m \times n_\theta^m$ for $m = \overline{1, M}$ along the diagonal.

With this timing structure, the first order approximation to the policy functions take the following matrix form:

$$\begin{bmatrix} y \\ z^{M-1} \\ z^{M-2} \\ \cdots \\ z^0 \end{bmatrix} = \left[\begin{array}{c|c|c|c|c|c|c|c|c} & & & \cdots & g_\theta^M & & & \cdots & g_{\theta_{-1}}^M \\ G_x & g_\theta^1 & g_\theta^2 & & & g_{\theta_{-1}}^1 & g_{\theta_{-1}}^2 & & \\ & & & & 0_z^M & & & & j_{\theta_{-1}}^M \\ & & 0_z^2 & \cdots & & & j_{\theta_{-1}}^2 & \cdots & \\ & 0_z^1 & & & & j_{\theta_{-1}}^1 & & & \end{array}\right] \begin{bmatrix} x \\ \theta \\ \theta_{-1} \end{bmatrix},$$

and

$$\begin{bmatrix} x' \\ \theta' \end{bmatrix} = \left[\begin{array}{c|cccc|cccc} h_x & h_{\theta^1} & h_{\theta^2} & \cdots & h_{\theta^M} & h_{\theta_{-1}^1} & h_{\theta_{-1}^2} & \cdots & h_{\theta_{-1}^M} \\ \hline 0_{n_\theta \times n_x} & & P & & & & 0_{n_\theta \times n_\theta} & & \end{array}\right] \begin{bmatrix} x \\ \theta \\ \theta_{-1} \end{bmatrix} + \epsilon,$$

$$
\bar{G}_X = \begin{array}{c}
\begin{array}{ccccc} \overbrace{\quad}^{n_x} & \overbrace{\quad}^{n_\theta^1} & \overbrace{\quad}^{n_\theta^2} & & \overbrace{\quad}^{n_\theta^M} \end{array} \\
\left[\begin{array}{c:c:c:c:c}
 & & & \cdots & \bar{g}_\theta^M \\ \hdashline
 & & \bar{g}_\theta^2 & & \\
\bar{G}_x & \bar{g}_\theta^1 & & & \bar{j}_\theta^M \\
 & & \bar{j}_\theta^2 & \cdots & \\
 & \bar{j}_\theta^1 & & &
\end{array}\right]
\end{array}
\begin{array}{l} \}n_y \\ \}n_z^M \\ \}n_z^{M-1} \\ \cdots \\ \}n_z^1 \end{array} ,
$$

$$
\bar{H}_X = \begin{array}{c}
\begin{array}{ccccc} \overbrace{\quad}^{n_x} & \overbrace{\quad}^{n_\theta^1} & \overbrace{\quad}^{n_\theta^2} & & \overbrace{\quad}^{n_\theta^M} \end{array} \\
\left[\begin{array}{c:c:c:c:c}
\bar{h}_x & \bar{h}_{\theta^1} & \bar{h}_{\theta^2} & \cdots & \bar{h}_{\theta^M} \\ \hdashline
 & P_1 & 0_{n_\theta^1 \times n_\theta^2} & & 0_{n_\theta^1 \times n_\theta^M} \\
0_{n_\theta \times n_x} & 0_{n_\theta^2 \times n_\theta^1} & P_2 & & 0_{n_\theta^2 \times n_\theta^M} \\
 & \cdots & \cdots & \cdots & \cdots \\
 & 0_{n_\theta^M \times n_\theta^1} & 0_{n_\theta^M \times n_\theta^2} & & P_M
\end{array}\right]
\end{array}
\begin{array}{l} \}n_x \\ \}n_\theta^1 \\ \}n_\theta^2 \\ \cdots \\ \}n_\theta^M \end{array} ,
$$

Table 1: Partitioning of full information linear solution

where G_x, and h_x are matrices with sizes $n_y + n_z \times n_x$, and $n_x \times n_x$ respectively. For $m = \overline{1, M}$, g_θ^m and $g_{\theta_{-1}}^m$ are matrices with sizes $(n_y + n_z - \sum_{j=1}^m n_z^j) \times n_\theta^m$, $j_{\theta_{-1}}^m$ are matrices with sizes $\sum_{j=1}^m n_z^j \times n_\theta^m$, and 0_z^m is a zero matrices of the same size as $j_{\theta_{-1}}^m$.

Such linear form of policy functions suggests that control variables with the decision before a shock in θ are unresponsive to the shock within a period, since the information set at the time the decision is made does not include current realization of the shock. Because of that, it is optimal for such control variables to respond to the best forecast of the shock, which is a function of θ_{-1}. The optimal choice of such control variables is included in the information set for control variables with decisions later in a period, and thus the decisions of all control variables will be determined implicitly by θ_{-1}.

Before formulating the algorithm to solve the problem, it is convenient to partition the full information model matrices $\bar{G}_X$ and $\bar{H}_X$ as shown in Table 1,
where $\bar{G}_x$, and $\bar{h}_x$ are matrices with sizes $n_y + n_z \times n_x$, and $n_x \times n_x$ respectively. For $m = \overline{1, M}$, $\bar{g}_\theta^m$ are matrices with sizes $(n_y + n_z - \sum_{j=1}^m n_z^j) \times n_\theta^m$, $\bar{j}_\theta^m$ are matrices with sizes $\sum_{j=1}^m n_z^j \times n_\theta^m$, and $\bar{h}_\theta^m$ are matrices with sizes $n_x \times n_\theta^m$.

The algorithm presented below helps to find the elements of these matrices G_X and H_X.

Algorithm.

Step 1. Set $G_x = \bar{G}_x$, and $h_x = \bar{h}_x$.

Start with vectors of full and partial controls y and z as defined by Equations (7) and (28). For $m = M, M-1, \cdots, 0$, repeat steps (2) through (5).

Step 2. Set $j^m_{\theta_{-1}} = \bar{j}^m_\theta P_m$.

Step 3. Then, $g^m_{\theta_{-1}}$ and $h_{\theta^m_{-1}}$ are determined as a solution to the linear system

$$\Delta(f^{m:M})_{[x',y]} \begin{bmatrix} h_{\theta^m_{-1}} \\ g^m_{\theta_{-1}} \end{bmatrix} = -f^{m:M}_z j^m_{\theta_{-1}},$$

where

$$\Delta(f^{m:M})_{[x',y]} = [f^{m:M}_{Y'} G_x + f^{m:M}_{x'},\ f^{m:M}_y],$$

and $f^{m:M} = [f^m; f^{m+1}; ...; f^M]$.

Step 4. Calculate g^m_θ and h_{θ^m} using the formulas

$$g^m_\theta = \bar{g}^m_\theta - g^m_{\theta_{-1}} P^{-1}_m, \quad \textit{and} \quad h_{\theta^m} = \bar{h}_{\theta^m} - h_{\theta^m_{-1}} P^{-1}_m.$$

Step 5. Update he composition of full and partial control variables, y and z, as follows

$$y = [y;\ z^m], \quad \textit{and} \quad z = [z^{m-1};\ z^{m-2};\ \cdots;\ z^1;\ z^0].$$

4.1 Example 1, Part 2

In Example 1, suppose prices must be set before all shocks realize, while consumption is to be decided upon after the aggregate supply shock υ but prior to the aggregate demand shock ϵ. In this situation, neither consumption nor inflation will respond contemporaneously to the aggregate demand shock, and output is the only variable to respond to the aggregate supply. In this setting, there are three subperiods: 0, 1, and 2. The state and full control vectors x and y are empty, $\theta = [\theta_1;\ \theta_2]$, where $\theta_1 = \upsilon$, $\theta_2 = \epsilon$, and $z = [z^1; z^0]$, with $z^0 = \pi$, and $z^1 = c$. The equilibrium conditions should be structured as follows:

$$\mathcal{E}[f^0; f^1; f^2;\ f^\theta] = 0,$$

$$f^0 = -\pi + \beta E\pi' + \kappa c + \upsilon,$$

$$f^1 = -c + c' - \sigma^{-1}(\alpha\pi - \pi') + \epsilon, \tag{29}$$

f^2 is empty, and

$$f^\theta = \begin{bmatrix} -\upsilon' + \rho_\upsilon \upsilon + e^\upsilon \\ -\epsilon' + \rho_\epsilon \epsilon + e^\epsilon \end{bmatrix}.$$

The dynamics of endogenous variables is

$$\begin{bmatrix} c \\ \pi \end{bmatrix} = \left[\begin{array}{c:c:c:c} g_\theta^1 & & g_{\theta^1_{-1}} & \\ \hdashline 0 & 0 & j^1_{\theta_{-1}} & j^2_{\theta_{-2}} \end{array}\right] \begin{bmatrix} \upsilon \\ \epsilon \\ \upsilon_{-1} \\ \epsilon_{-1} \end{bmatrix}.$$

With the implied timing structure, the required partitioning of $\bar{G}_X$ and $\bar{H}_X$ is

$$\bar{G}_X = \left[\begin{array}{c:c} \bar{g}_\theta & \\ \hdashline \bar{j}^1_\theta & \bar{j}^2_\theta \end{array}\right], \quad \textit{and} \quad \bar{H}_x = \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix},$$

where $\bar{g}^1_\theta = \frac{(\alpha-\rho)\sigma^{-1}}{|D|}$, $\bar{j}^1_\theta = \frac{\rho-1}{|D|}$, and $\bar{j}^2_\theta = \frac{1}{|D|}\begin{bmatrix} \beta\rho - 1 \\ -\kappa \end{bmatrix}$, $P_1 = \rho$, and $P_2 = \rho$. Following the proposed solution algorithm, at the first step determine $j^2_{\theta_{-1}} = \bar{j}^2_\theta P_2 = \frac{\rho}{|D|}\begin{bmatrix} \beta\rho - 1 \\ -\kappa \end{bmatrix}$. At the second step, redefine $y = c$, and $z = \pi$, and obtain $j^1_{\theta_{-1}} = \bar{j}^1_\theta P_1 = \frac{(\rho-1)\rho}{|D|}$. Next,

$f_y^{1:2} = -1$, and $f_z^{1:2} = -\alpha\sigma^{-1}$, thus

$$g_{\theta_{-1}}^1 = -\frac{f_z^{1:2}}{f_y^{1:2}} j_{\theta_{-1}}^1 = \frac{-\alpha\sigma^{-1}(\rho-1)\rho}{|D|},$$

and

$$g_\theta = \bar{g}_\theta^1 + \frac{\alpha\sigma^{-1}(\rho-1)}{|D|} = \frac{\sigma^{-1}(\alpha-1)\rho}{|D|}.$$

As a result, the dynamics of the endogenous variables in matrix form can be written as

$$\begin{bmatrix} c \\ \pi \end{bmatrix} = \frac{1}{|D|} \begin{bmatrix} \sigma^{-1}(\alpha-1)\rho & 0 & -\alpha\sigma^{-1}(\rho-1)\rho & (\beta\rho-1)\rho \\ 0 & 0 & (\rho-1)\rho & -\kappa\rho \end{bmatrix} \begin{bmatrix} \upsilon \\ \epsilon \\ \upsilon_{-1} \\ \epsilon_{-1} \end{bmatrix}.$$

5 Second-Order Approximation

To derive the second-order approximation to the model dynamics, I rely on the notion of n-mode multiplication as defined by De Lathauwer (2000). According to this definition, for any N-dimensional tensor $\mathcal{A}$ with a size $I_1 \times I_2, \cdots, I_N$, and a matrix U with a size $J_n \times I_n$, the result of n-mode multiplication,

$$\mathcal{B} = \mathcal{A} \times_n U,$$

is an N-dimensional tensor with a size $I_1 \times I_2, \cdots I_{n-1}, J_n, I_{n+1}, \cdots, I_N$, each element of which is calculated according to the formula:

$$\mathcal{B}(i_1, i_2, \cdots, i_{n-1}, j_n, i_{n+1}, \cdots, i_N) = \sum_{i_n=1}^{I_n} \mathcal{A}(i_1, i_2, \cdots, i_{n-1}, i_n, i_{n+1}, \cdots, i_N) U(j_n, i_n).$$

For future reference, it is useful to state the product rule of matrix differentiation using n-mode multiplication:

$$\Delta(AB)_x = A_x \times_2 B^T + B_x \times_1 A,$$

where the operator $\Delta(\cdot)_x$ denotes the total derivative with respect to the argument vector x with the length n_x, and A_x and B_x are three-dimensional tensors of first order derivatives of A and B, with sizes $n_a \times n \times n_x$ and $n \times n_b \times n_x$ correspondingly. Also, for two matrix functions A and B with sizes $n_a \times n$ and $n \times n_b$ respectively, $B \times_1 A \equiv AB$, and $A \times_2 B^T = AB$. The chain rule to calculate matrix derivative using n-mode multiplication is

$$\Delta(A(c(x)))_x = A_c \times_3 c_x,$$

where matrix A has the size of $n_a \times n_b$, c is a vector of length n_c, vector x has the length of n_x, A_x and c_x are derivatives of A and c with respect to x, with sizes $n_a \times n_b \times n_x$ and $n_c \times n_x$ respectively.

The second order approximation to policy functions (2) and (3) of the full-information version of the model can be written in terms of n-mode multiplication as follows:[7]

$$Y = \bar{G}_X X + \frac{1}{2}\bar{G}_{X,X} \times_2 X^T \times_3 X^T + \frac{1}{2}\bar{G}_{\sigma,\sigma}\sigma^2, \tag{30}$$

and

$$X' = \bar{H}_X X + \frac{1}{2}\bar{H}_{X,X} \times_2 X^T \times_3 X^T + \frac{1}{2}\bar{H}_{\sigma,\sigma}\sigma^2 + \sigma\epsilon', \tag{31}$$

where $\bar{G}_{X,X}$ and $\bar{H}_{X,X}$ are three-dimensional tensors of sizes $n_Y \times n_X \times n_X$ and $n_X \times n_X \times n_X$ respectively, such that for $U = \bar{G}$, $\bar{H}$ and for each i, j and k, $U_{X,X}(i,j,k) = \frac{\partial}{\partial X_k}\left(\frac{\partial \bar{U}^i}{\partial X_j}\right)$ evaluated at a steady state, and $\bar{H}_{\sigma,\sigma}$ and $\bar{G}_{\sigma,\sigma}$ are vectors of lengths n_X and n_Y correspondingly.[8] In these formulas, matrices $\bar{G}_X$ and $\bar{H}_X$ result from the first-order approximate solution, while $G_{X,X}$ and $\bar{H}_{X,X}$ can be calculated using the algorithm outlined in Schmitt-Grohé and Uribe (2004).

The second-order approximate solution in the same model with timing constraints can

[7]In these formulas, I acknowledge the fact proved in Schmitt-Grohé and Uribe (2004) that $\bar{G}_{x,\sigma}$ and $\bar{G}_{\sigma,x}$ are zero valued arrays.

[8]In matrix form, $U_{X,Y} = \frac{\partial}{\partial Y}\left(\frac{\partial U}{\partial X}\right)$.

be stated in general form as

$$y = [g_x,\, g_\theta,\, g_{\theta_{-1}}] \begin{bmatrix} x \\ \theta \\ \theta_{-1} \end{bmatrix} +$$

$$\frac{1}{2} \begin{bmatrix} g_{x,x} & g_{x,\theta} & g_{x,\theta_{-1}} \\ g_{\theta,x} & g_{\theta,\theta} & g_{\theta,\theta_{-1}} \\ g_{\theta_{-1},x} & g_{\theta_{-1},\theta} & g_{\theta_{-1},\theta_{-1}} \end{bmatrix} \times_2 \begin{bmatrix} x \\ \theta \\ \theta_{-1} \end{bmatrix}^T \times_3 \begin{bmatrix} x \\ \theta \\ \theta_{-1} \end{bmatrix}^T + \frac{1}{2} g_{\sigma,\sigma}\sigma^2,$$

$$z = [j_x,\, j_{\theta_{-1}}] \begin{bmatrix} x \\ \theta_{-1} \end{bmatrix} + \frac{1}{2} \begin{bmatrix} j_{x,x} & j_{x,\theta_{-1}} \\ j_{\theta_{-1},x} & j_{\theta_{-1},\theta_{-1}} \end{bmatrix} \times_2 \begin{bmatrix} x \\ \theta_{-1} \end{bmatrix}^T \times_3 \begin{bmatrix} x \\ \theta_{-1} \end{bmatrix}^T + \frac{1}{2} j_{\sigma,\sigma}\sigma^2,$$

and

$$x' = [h_x,\, h_\theta,\, h_{\theta_{-1}}] \begin{bmatrix} x \\ \theta \\ \theta_{-1} \end{bmatrix} +$$

$$\frac{1}{2} \begin{bmatrix} h_{x,x} & h_{x,\theta} & h_{x,\theta_{-1}} \\ h_{\theta,x} & h_{\theta,\theta} & h_{\theta,\theta_{-1}} \\ h_{\theta_{-1},x} & h_{\theta_{-1},\theta} & h_{\theta_{-1},\theta_{-1}} \end{bmatrix} \times_2 \begin{bmatrix} x \\ \theta \\ \theta_{-1} \end{bmatrix}^T \times_3 \begin{bmatrix} x \\ \theta \\ \theta_{-1} \end{bmatrix}^T + \frac{1}{2} h_{\sigma,\sigma}\sigma^2 + \sigma\eta_x\epsilon'_x,$$

where the unknowns to be evaluated are the elements of arrays $g_{m,k}$ with sizes $n_y \times n_m \times n_k$ for $m = x,\, \theta,\, \theta_{-1}$, $j_{m,k}$ with sizes $n_z \times n_m \times n_k$ for $m = x,\, \theta_{-1}$, $h_{m,k}$ with sizes $n_x \times n_m \times n_k$ for $m = x,\, \theta,\, \theta_{-1}$, as well as the vectors $h_{\sigma,\sigma}$, $g_{\sigma,\sigma}$, and $j_{\sigma,\sigma}$. If second order derivatives of the policy functions are continuous in a neighborhood of approximation point, then arrays $p_{m,k}$ and $p_{k,m}$ are symmetric along the third mode, which implies that for each i, j, l, $p_{m,k}(i, j, l) = p_{k,m}(i, l, j)$, for $p = g,\, j,\, h$, $m = x, \theta, \theta_{-1}$, and $k = x,\, \theta,\, \theta_{-1}$.

The following proposition provides an algorithm to evaluate policy functions up to second order of approximation in a model with timing restrictions by relying on the solution to the full information version of the model in the form (30) and (31). In this proposition, I use partitioning of three dimensional tensors $\bar{G}_{X,X}$ and $\bar{H}_{X,X}$ by analogy with the first-order problem in Section 3:

$$\bar{G}_{XX}(:,:,1:n_x) = \left[\begin{array}{c:c} \bar{G}_{x,x} & \begin{array}{c}\bar{g}_{\theta,x} \\ \hdashline \bar{j}_{\theta,x}\end{array} \end{array}\right], \qquad \bar{G}_{XX}(:,:,n_x+1:n_X) = \left[\begin{array}{c:c} \bar{g}_{x,\theta} & \bar{g}_{\theta,\theta} \\ \hdashline \bar{j}_{x,\theta} & \bar{j}_{\theta,\theta} \end{array}\right],$$

$$\bar{H}_{XX}(:,:,1:n_x) = \left[\begin{array}{c:c} \bar{h}_{x,x} & \bar{h}_{\theta,x} \\ \hdashline \multicolumn{2}{c}{0_{n_\theta \times n_X \times n_x}} \end{array}\right], \qquad \bar{H}_{XX}(:,:,n_x+1:n_X) = \left[\begin{array}{c:c} \bar{h}_{x,\theta} & \bar{h}_{\theta,\theta} \\ \hdashline \multicolumn{2}{c}{0_{n_\theta \times n_X \times n_\theta}} \end{array}\right],$$

and

$$\bar{G}_{\sigma,\sigma} = \begin{bmatrix} \bar{g}_{\sigma,\sigma} \\ \bar{j}_{\sigma,\sigma} \end{bmatrix}, \quad \textit{and} \quad \bar{H}_{\sigma,\sigma} = \begin{bmatrix} \bar{h}_{\sigma,\sigma} \\ 0_{n_\theta \times 1} \end{bmatrix},$$

Proposition 2 *The algorithm to obtain the second order approximation to the partial information model dynamics from the full-information model dynamics is the following.*

Step 1. *Set* $g_{\sigma,\sigma} = \bar{g}_{\sigma,\sigma}$*,* $j_{\sigma,\sigma} = \bar{j}_{\sigma,\sigma}$*,* $h_{\sigma,\sigma} = \bar{h}_{\sigma,\sigma}$*,* $g_{x,x} = \bar{g}_{x,x}$*,* $j_{x,x} = \bar{j}_{x,x}$*,* $h_{x,x} = \bar{h}_{x,x}$*,* $j_{x,\theta_{-1}} \equiv \bar{j}_{x,\theta} \times_3 P^T$*, and*

$$j_{\theta_{-1},\theta_{-1}} = \bar{j}_{\theta,\theta} \times_2 P^T \times_3 P^T.$$

Step 2. *To get* $n_x n_\theta^2$ *elements of* $h_{\theta_{-1},\theta}$ *and* $n_y n_\theta^2$ *elements of* $g_{\theta_{-1},\theta}$*, find*

$$\begin{bmatrix} h_{\theta_{-1},\theta} \\ g_{\theta_{-1},\theta} \end{bmatrix} = \tag{32}$$

$$-[\Delta(f^1_{[Y',x']})_\theta \times_2 \begin{bmatrix} G_x \\ I_{n_x} \end{bmatrix}^T + G_{x,X} \times_1 f^1_{Y'} \times_3 H^T_\theta,\ \Delta(f^1_Y)_\theta] \times_2 \begin{bmatrix} h_{\theta_{-1}} \\ G_{\theta_{-1}} \end{bmatrix}^T \times_1 \Delta(f^1)^{-1}_{[x,y]}.$$

In this formula,

$$\Delta(f^1_{[Y',x']})_\theta = [\Delta(f^1_{Y'})_\theta,\ \Delta(f^1_{x'})_\theta],$$

and for $q = Y',\ x',\ Y$*,*

$$\Delta(f^1_q)_\theta = f^1_{q,[Y',X',\theta]} \times_3 (\begin{bmatrix} G_X \\ I_{n_X+n_\theta} \end{bmatrix} \begin{bmatrix} H_\theta \\ I_{n_\theta} \end{bmatrix})^T + f^1_{q,y} \times_3 g^T_\theta.$$

and $\Delta(f^1)_{[x,y]}$ *is determined by Equation (19).*

Step 3. *Obtain* $h_{\theta_{-1},\theta_{-1}}$ *and* $g_{\theta_{-1},\theta_{-1}}$ *as follows*

$$\begin{bmatrix} h_{\theta_{-1},\theta-1} \\ g_{\theta_{-1},\theta_{-1}} \end{bmatrix} = -j_{\theta_{-1},\theta_{-1}} \times_1 f_z^1 - \tag{33}$$

$$[\Delta(f^1_{[Y',x']})_{\theta_{-1}} \times_2 \begin{bmatrix} G_x \\ I_{n_x} \end{bmatrix}^T + G_{x,x} \times_1 f^1_{Y'} \times_3 h^T_{\theta_{-1}},\ \Delta(f^1_Y)_{\theta_{-1}}] \times_2 \begin{bmatrix} h_{\theta_{-1}} \\ G_{\theta_{-1}} \end{bmatrix}^T \times_1 \Delta(f^1)^{-1}_{[x,y]},$$

where for $q = Y', x', Y$,

$$\Delta(f^1_q)_{\theta_{-1}} = f^1_{q,[Y',x']} \times_3 (\begin{bmatrix} G_x \\ I_{nx} \end{bmatrix} h_{\theta_{-1}})^T + f^1_{q,Y} \times_3 G^T_{\theta_{-1}}.$$

Step 4. *calculate* $g_{\theta,\theta_{-1}}$ *and* $h_{\theta,\theta_{-1}}$ *by transposing the second and third dimensions of* $g_{\theta_{-1},\theta}$ *and* $h_{\theta_{-1},\theta}$, *and obtain* $g_{\theta,\theta}$ *and* $h_{\theta,\theta}$ *from*

$$g_{\theta,\theta} = \bar{g}_{\theta,\theta} - (g_{\theta,\theta_{-1}} \times_3 (P^T)^{-1} + g_{\theta_{-1},\theta} \times_2 (P^T)^{-1} + g_{\theta_{-1},\theta_{-1}} \times_2 (P^T)^{-1} \times_3 (P^T)^{-1}),$$

$$h_{\theta,\theta} = \bar{h}_{\theta,\theta} - (h_{\theta,\theta_{-1}} \times_3 (P^T)^{-1} + h_{\theta_{-1},\theta} \times_2 (P^T)^{-1} + h_{\theta_{-1},\theta_{-1}} \times_2 (P^T)^{-1} \times_3 (P^T)^{-1}),$$

Step 5. *Obtain* $g_{x,\theta_{-1}}$ *and* $h_{x,\theta_{-1}}$ *from*

$$\begin{bmatrix} h_{x,\theta_{-1}} \\ g_{x,\theta_{-1}} \end{bmatrix} = -j_{x,\theta_{-1}} \times_1 f_z^1 - \tag{34}$$

$$[\Delta(f^1_{[Y',x']})_{\theta_{-1}} \times_2 \begin{bmatrix} G_x \\ I_{n_x} \end{bmatrix}^T + G_{x,x} \times_1 f^1_{Y'} \times_3 h^T_{\theta_{-1}},\ \Delta(f^1_{[Y,x]})_{\theta_{-1}}] \times_2 \begin{bmatrix} h_x \\ G_x \\ I_{n_x} \end{bmatrix}^T \times_1 \Delta(f^1)^{-1}_{[x,y]}.$$

Step 6. *Set* $g_{x,\theta}$ *and* $h_{x,\theta}$ *to*

$$g_{x,\theta} \equiv \bar{g}_{x,\theta} - g_{x,\theta_{-1}} \times_3 (P^T)^{-1},$$

and

$$h_{x,\theta} \equiv \bar{h}_{x,\theta} - h_{x,\theta_{-1}} \times_3 (P^T)^{-1}.$$

The proof of the proposition is given in Appendix. The solution is derived from two sets of fundamental restrictions imposed by the timing constraints. One set of restrictions prevents variables in vector z from responding to current realizations of shocks in θ. The other restriction states that the choices of full information endogenous variables must take into account old information in θ_{-1}, even though the new realization of θ is already available. This is so because the optimal choice for y depends on the partial information variables in z, the choice of which was made based on previous period realizations of θ. In Appendix, I show how the algorithm can be extended to allow multiple subperiods.

5.1 Example 2: Real Business Cycle Model with Labor Decision

This is a simple model that follows Christiano (2002). The social planner maximizes the expected life-time utility $E_0 \sum_{t=0}^{\infty} U(C_t, N_t)$ subject to the resource constraints

$$C_t + K_{t+1} - (1-\delta)K_t = e^{a_t} K_t^{\alpha} N_t^{1-\alpha}, \tag{35}$$

where $0 < \beta < 1$ is the discount factor, C_t is consumption, K_t is the capital stock, N_t is labor, $0 \leq \delta \leq 1$ is the depreciation rate of capital, and a_t is an exogenous process of technology, given by

$$a_{t+1} = \rho^a a_t + \epsilon^a_{t+1},$$

where $e_t \sim i.i.d.(0, \sigma^2)$. For simplicity, I assume that the intratemporal utility is separable in consumption and labor, and is defined as

$$U(C_t, N_t) = log(C_t) - \frac{N_t^2}{2}.$$

Given the stochastic process for a_t and the initial capital level K_0, the equilibrium of the model is described by the sequences $\{C_t, N_t, K_{t+1}\}_{t=0}^{\infty}$ that satisfy the system of equilibrium conditions:

$$\begin{array}{ll} (C): & C_t = e^{a_t} K_t^{\alpha}, N_t^{1-\alpha} - K_{t+1} + (1-\delta) K_t \\ (N): & N_t = \frac{1}{C_t} e^{a_t} (1-\alpha)(K_t/N_t)^{\alpha}, \\ (K'): & \frac{1}{C_t} = \beta E_t [\frac{1}{C_{t+1}} (e^{a_{t+1}} \alpha (K_{t+1}/N_{t+1})^{\alpha-1} + (1-\delta))] \end{array} . \tag{36}$$

Suppose, as in Christiano (2002), that the realization of the technology shock ϵ^a_{t+1} occurs in the second-half of a period. If future capital decision must be made before the realization of the shock, but labor and consumption can be chosen after the shock realizes, then $y = [C; N]$, and $z = (K)'$, $x = K$, and $\theta = a$, where for all variables, current period subscript t is omitted, and prime is used to replace future period subscript $t+1$. Since capital enters both the state and control vectors, the following restriction must be added to the vector of equilibrium conditions:

$$x' = z. \tag{37}$$

The expectational nonlinear system is then $\mathcal{E} f(Y' X', Y, X) = 0$, where $f = [f^0; f^1; f^\theta]$, in which

$$f^0 = -\frac{1}{y(1)} + \beta \frac{1}{y'(1)} [e^{\theta'} f_K(z, y'(2)) + 1 - \delta],$$

$$f^1 = \begin{bmatrix} -y(1) + e^{\theta} f(x, y(2)) - z + (1-\delta) x \\ -y(2) + \frac{1}{y(1)} e^{\theta} f_N(x, y(2)) \\ -x' + z \end{bmatrix},$$

and

$$f^\theta = -\theta' + \rho^a \theta + (\epsilon^a)'.$$

For calibration of the model summarized in Table 2, the jacobians of vector function f^1 with respect to Y', x', z, and y that are required to derive the partial information solution from the model with full information structure are, respectively,

$$f^1_{Y'} = \left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right], \quad f^1_{x'} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}, \quad f^1_z = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \quad and \quad f^1_y = \begin{bmatrix} -1 & 1.8165 \\ -0.5013 & -1.300 \\ 0 & 0 \end{bmatrix}.$$

Moreover, note for a future reference that

$$(\Delta(f^1)_{[x,y]})^{-1} = [f^1_{[Y',x']} \begin{bmatrix} G_x \\ I_{n_x} \end{bmatrix}, f^1_y]^{-1} = \begin{bmatrix} 0 & 0 & -1 \\ -0.5880 & -0.8217 & 0 \\ 0.2268 & -0.4523 & 0 \end{bmatrix}. \tag{38}$$

5.1.1 First Order Approximation

$\bar{G}_X$ and $\bar{H}_X$ are the resulting matrices of coefficients in the full information version of the model are:

$$\bar{G}_X = \left[\begin{array}{c|c} 0.0430 & 0.5423 \\ -0.0070 & 0.5249 \\ \hline 0.9517 & 2.8877 \end{array}\right], \quad and \quad \bar{H}_X = \left[\begin{array}{c|c} 0.9517 & 2.8877 \\ \hline 0 & 0.9 \end{array}\right]. \tag{39}$$

The solution of the model with partial information structure should be constructed as follows. First, set $g_x = \bar{g}_x$, $j_x = \bar{j}_x$, and $h_x = \bar{h}_x$, which means

$$g_x = \begin{bmatrix} 0.0430 \\ -0.0070 \end{bmatrix}, \quad h_x = 0.9517, \quad and \qquad j_x = 0.9517.$$

Also, $j_{\theta_{-1}} = \rho^a \bar{j}_\theta = 0.9 \cdot 2.8877 = 2.5989$ Then, calculate

$$\begin{bmatrix} h_{\theta_{-1}} \\ g_{\theta-1} \end{bmatrix} = -\Delta(f^1)^{-1}_{[x,y]} f^1_z j_{\theta_{-1}} =$$

$$
-\begin{bmatrix} 0 & 0 & -1 \\ -0.5880 & -0.8217 & 0 \\ 0.2268 & -0.4523 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} 2.5989 = \begin{bmatrix} 2.5989 \\ -1.5283 \\ 0.5899 \end{bmatrix},
$$

which means that $h_{\theta_{-1}} = 2.5989$ and $g_{\theta_{-1}} = \begin{bmatrix} -1.5283 \\ 0.5899 \end{bmatrix}$. Then,

$$
\begin{bmatrix} h_\theta \\ g_\theta \end{bmatrix} = \begin{bmatrix} \bar{h}_\theta \\ \bar{g}_\theta \end{bmatrix} - \frac{1}{\rho^a} \begin{bmatrix} h_{\theta_{-1}} \\ g_{\theta_{-1}} \end{bmatrix} = \begin{bmatrix} 2.8877 \\ 0.5423 \\ 0.5249 \end{bmatrix} - \frac{1}{0.9} \begin{bmatrix} 2.5989 \\ -1.5283 \\ 0.5899 \end{bmatrix} = \begin{bmatrix} 0 \\ 2.2405 \\ -0.1299 \end{bmatrix},
$$

which implies $h_\theta = 0$ and $g_\theta = \begin{bmatrix} 2.2405 \\ -0.1299 \end{bmatrix}$.

To summarize, the first-order approximate solution to the problem is described by the system (16), where

$$
G_X = \begin{bmatrix} 0.0430 & 2.2405 & -1.5283 \\ -0.007 & -0.1299 & 0.5894 \\ 0.9517 & 0 & 2.59894 \end{bmatrix}, \quad \textit{and} \quad H_X = \begin{bmatrix} 0.9517 & 0 & 2.5989 \\ 0 & 0.9 & 0 \end{bmatrix}. \tag{40}
$$

5.1.2 Second-Order Approximation

The solution of the full-information version of the model obtained using the second-order approximation algorithms of Schmitt-Grohé and Uribe (2004) is given by Formulas (39) and

$$
\bar{H}_{XX}(:,:,1) = \begin{bmatrix} -0.0001 & 0.0306 \\ 0 & 0 \end{bmatrix}, \qquad \bar{H}_{XX}(:,:,2) = \begin{bmatrix} 0.0306 & 4.1431 \\ 0 & 0 \end{bmatrix},
$$

$$
\bar{G}_{XX}(:,:,1) = \begin{bmatrix} -0.0007 & 0.0031 \\ 0.0003 & -0.0003 \\ -0.0001 & 0.0306 \end{bmatrix}, \qquad \bar{G}_{XX}(:,:,2) = \begin{bmatrix} 0.0031 & 0.4210 \\ -0.0003 & 0.1860 \\ 0.0306 & 4.1431 \end{bmatrix},
$$

and assuming the standard deviation of shock ϵ^a is 0.1 and setting $\sigma = 1$,

$$\bar{H}_{\sigma,\sigma} = \begin{bmatrix} -0.0135 \\ 0 \end{bmatrix}, \quad and \quad \bar{G}_{\sigma,\sigma} = \begin{bmatrix} 0.0080 \\ -0.0031 \\ -0.0135 \end{bmatrix}. \tag{41}$$

Following the strategy outlined in this paper, steps 1 through 6 are presented below:

Step 1. Set $j_{\sigma,\sigma} = -0.0135$, $g_{\sigma,\sigma} = \begin{bmatrix} 0.0080 \\ -0.0031 \end{bmatrix}$, $h_{\sigma,\sigma} = -0.0135$, $g_{x,x} = \begin{bmatrix} -0.0007 \\ 0.0003 \end{bmatrix}$, $j_{x,x} = -0.0001$, $h_{x,x} = -0.0001$, $j_{x,\theta_{-1}} \equiv \bar{j}_{x,\theta}\rho^a = 0.0306 \cdot 0.9 = 0.0275$, and $j_{\theta_{-1},\theta_{-1}} = \bar{j}_{\theta,\theta}(\rho^a)^2 = 4.1431 \cdot 0.81 = 3.3559$.

Step 2.

Since $\Delta(f^1_{Y'})_\theta = 0_{3\times3}$, $\Delta(f^1_{x'})_\theta = 0_{3\times1}$,

$$\Delta(f^1_{[Y',x']})_\theta \times_2 \begin{bmatrix} G_x \\ I_{n_x} \end{bmatrix}^T + G_{x,X} \times_1 f^1_{Y'} \times_3 H^T_\theta = 0_{3\times4},$$

and Equation (32) can be simplified as follows

$$\begin{bmatrix} h_{\theta_{-1},\theta} \\ g_{\theta_{-1},\theta} \end{bmatrix} = -\Delta(f^1)^{-1}_{[x,y]}\Delta(f^1_Y)_\theta G^T_{\theta_{-1}},$$

where

$$\Delta(f^1_Y)_\theta = f^1_{Y,[Y',X',\theta]} \times_3 (\begin{bmatrix} G_X \\ I_{n_X+n_\theta} \end{bmatrix} \begin{bmatrix} H_\theta \\ I_{n_\theta} \end{bmatrix})^T + f^1_{Y,y} \times_3 g_\theta = \begin{bmatrix} 0 & 1.8907 & 0 \\ 0.6583 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus,

$$\begin{bmatrix} h_{\theta_{-1},\theta} \\ g_{\theta_{-1},\theta} \end{bmatrix} =$$

$$
-\begin{bmatrix} 0 & 0 & -1 \\ -0.5880 & -0.8217 & 0 \\ 0.2268 & -0.4523 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1.8907 & 0 \\ 0.6583 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -1.5283 \\ 0.5894 \\ 2.5989 \end{bmatrix} = \begin{bmatrix} 0 \\ -01714 \\ -0.7078 \end{bmatrix}.
$$

Thus, $h_{\theta_{-1},\theta} = 0$, and $g_{\theta_{-1},\theta} = \begin{bmatrix} -0.1714 \\ -0.7078 \end{bmatrix}$.

Step 3. Since $\Delta(f^1_{[Y',x']})_{\theta_{-1}}$ and $f^1_{Y'}$ are zero-valued matrices, Equation (59) becomes

$$
\begin{bmatrix} h_{\theta_{-1},\theta-1} \\ g_{\theta_{-1},\theta_{-1}} \end{bmatrix} = -f^1_z j_{\theta_{-1},\theta_{-1}} - \Delta(f^1)^{-1}_{[x,y]} \Delta(f^1_Y)_{\theta_{-1}} G_{\theta_{-1}},
$$

where

$$
\Delta(f^1_Y)_{\theta_{-1}} = f^1_{Y,[Y',x']} \times_3 (\begin{bmatrix} G_x \\ I_{nx} \end{bmatrix} h_{\theta_{-1}})^T + f^1_{Y,Y} \times_3 G^T_{\theta_{-1}} = \begin{bmatrix} 0 & -0.3366 & 0 \\ -0.7121 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.
$$

Then,

$$
\begin{bmatrix} h_{\theta_{-1},\theta-1} \\ g_{\theta_{-1},\theta_{-1}} \end{bmatrix} = -3.3559 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} -
$$

$$
\begin{bmatrix} 0 & 0 & -1 \\ -0.5880 & -0.8217 & 0 \\ 0.2268 & -0.4523 & 0 \end{bmatrix} \begin{bmatrix} 0 & -0.3366 & 0 \\ -0.7121 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -1.5283 \\ 0.5894 \\ 2.5989 \end{bmatrix} = \begin{bmatrix} 3.3559 \\ 0.7777 \\ -2.8187 \end{bmatrix},
$$

so that $h_{\theta_{-1},\theta_{-1}} = 3.3559$, and $g_{\theta_{-1},\theta_{-1}} = \begin{bmatrix} 0.7777 \\ -2.8187 \end{bmatrix}$.

Step 4. Set

$$
g_{\theta,\theta} = \bar{g}_{\theta,\theta} - g_{\theta,\theta_{-1}}/\rho^a - g_{\theta_{-1},\theta}/\rho^a - g_{\theta_{-1},\theta_{-1}}/(\rho^a)^2 =
$$

$$
\begin{bmatrix} 0.4210 \\ 0.1860 \end{bmatrix} - 2 \begin{bmatrix} -0.1714 \\ -0.7078 \end{bmatrix} /0.9 - \begin{bmatrix} 0.7777 \\ -2.8187 \end{bmatrix} /0.81 = \begin{bmatrix} -0.1581 \\ 5.2387 \end{bmatrix},
$$

$$h_{\theta,\theta} = \bar{h}_{\theta,\theta} - h_{\theta,\theta_{-1}}/\rho^a - h_{\theta_{-1},\theta}/\rho^a - h_{\theta_{-1},\theta_{-1}}/(\rho^a)^2 = 4.1431 - 3.3559/0.81 = 0.$$

Step 5. Again, since $\Delta(f^1_{[Y',x']})_{\theta_{-1}}$ and $f^1_{Y'}$ are zero-valued matrices, expression (34) can be simplified as follows:

$$\begin{bmatrix} h_{x,\theta_{-1}} \\ g_{x,\theta_{-1}} \end{bmatrix} = -f^1_z j_{x,\theta_{-1}} - \Delta(f^1)^{-1}_{[x,y]} \Delta(f^1_{[Y,x]})_{\theta_{-1}} \begin{bmatrix} G_x \\ I_{n_x} \end{bmatrix}.$$

Given that

$$\Delta(f^1_x)_{\theta_{-1}} = f^1_{x,[Y',x']} \times_3 \left(\begin{bmatrix} G_x \\ I_{nx} \end{bmatrix} h_{\theta_{-1}} \right)^T + f^1_{x,Y} \times_3 G^T_{\theta_{-1}} = \begin{bmatrix} 0.0140 \\ 0.0077 \\ 0 \end{bmatrix},$$

then

$$\Delta(f^1_{[Y,x]})_{\theta_{-1}} = \begin{bmatrix} 0 & -0.3366 & 0 & 0.0140 \\ -0.7121 & 0 & 0 & 0.0077 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

and

$$\begin{bmatrix} h_{x,\theta_{-1}} \\ g_{x,\theta_{-1}} \end{bmatrix} = -0.0275 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} -$$

$$\begin{bmatrix} 0 & 0 & -1 \\ -0.5880 & -0.8217 & 0 \\ 0.2268 & -0.4523 & 0 \end{bmatrix} \begin{bmatrix} 0 & -0.3366 & 0 & 0.0140 \\ -0.7121 & 0 & 0 & 0.0077 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0.0430 \\ -0.0007 \\ 0.9517 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.0275 \\ -0.0092 \\ -0.0416 \end{bmatrix},$$

so that $h_{x,\theta_{-1}} = 0.0275$, and $g_{x,\theta_{-1}} = \begin{bmatrix} -0.0092 \\ -0.0416 \end{bmatrix}$.

Step 6. Set

$$g_{x,\theta} \equiv \bar{g}_{x,\theta} - g_{x,\theta_{-1}}/\rho^a = \begin{bmatrix} 0.0031 \\ -0.0003 \end{bmatrix} - \begin{bmatrix} -0.0092 \\ -0.0416 \end{bmatrix} /0.9 = \begin{bmatrix} 0.0134 \\ 0.0459 \end{bmatrix},$$

and

$$h_{x,\theta} \equiv \bar{h}_{x,\theta} - h_{x,\theta_{-1}}/\rho^a = 0.0306 - 0.0275/0.9 = 0.$$

To summarize,the second-order approximate solution to the model is described by the dynamic system

$$Y = G_X \begin{bmatrix} x \\ \theta \\ \theta_{-1} \end{bmatrix} + \frac{1}{2} G_{X,X} \times_2 \begin{bmatrix} x \\ \theta \\ \theta_{-1} \end{bmatrix}^T \times_3 \begin{bmatrix} x \\ \theta \\ \theta_{-1} \end{bmatrix}^T + \frac{1}{2} G_{\sigma,\sigma}\sigma^2,$$

and

$$X' = H_X \begin{bmatrix} x \\ \theta \\ \theta_{-1} \end{bmatrix} + \frac{1}{2} H_{X,X} \times_2 \begin{bmatrix} x \\ \theta \\ \theta_{-1} \end{bmatrix}^T \times_3 \begin{bmatrix} x \\ \theta \\ \theta_{-1} \end{bmatrix}^T + \frac{1}{2} H_{\sigma,\sigma}\sigma^2 + \sigma\epsilon',$$

where G_X, and H_X are given by Formulas (40),

$$h_{XX}(:,:,1) = \begin{bmatrix} -0.0001 & 0 & 0.0275 \\ 0 & 0 & 0 \end{bmatrix}, \quad h_{XX}(:,:,2) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$h_{XX}(:,:,3) = \begin{bmatrix} 0.0275 & 0 & 3.3559 \\ 0 & 0 & 0 \end{bmatrix},$$

$$g_{XX}(:,:,1) = \begin{bmatrix} -0.0007 & 0.0134 & -0.0092 \\ 0.0003 & 0.0459 & -0.0416 \\ -0.0001 & 0 & 0.0275 \end{bmatrix}, \quad g_{XX}(:,:,2) = \begin{bmatrix} 0.0134 & -0.1581 & -0.1714 \\ 0.0459 & 5.2387 & -0.7078 \\ 0 & 0 & 0 \end{bmatrix},$$

$$g_{XX}(:,:,3) = \begin{bmatrix} -0.0092 & -0.1714 & 0.7777 \\ -0.0416 & -0.7078 & -2.8186 \\ 0.0275 & 0 & 3.3559 \end{bmatrix},$$

$$H_{\sigma,\sigma} = \begin{bmatrix} -0.0135 \\ 0 \end{bmatrix} \quad and \quad G_{\sigma,\sigma} = \begin{bmatrix} 0.0080 \\ -0.0031 \\ -0.0135 \end{bmatrix}.$$

6 Conclusion

This paper describes the algorithm and provides Matlab codes to solve up to the second order of approximation rational expectations models with partial information structure using perturbations approach. Two-subperiod setting is considered first, and then the algorithm is extended to multiple subperiods. This method can be used to study limited participation models of money, models of labor hoarding, models with sticky prices, and other models where timing restrictions are important.

7 Appendix

Table 2: Example 2: Calibration

Variable/parameter	Value
δ	0.025
α	0.3
β	0.9926
ρ^a	0.9
a	1
H	0.95
K	22.92
C	1.90

7.1 Proof of Proposition 1.

Consider the model with partial information structure described by System (9). Denote $\tilde{f}(\epsilon, x, \theta, \theta_{-1}, \sigma) \equiv f(G(H(x, \theta, \theta_{-1}, \sigma)+\sigma\epsilon', P\theta+\sigma\epsilon'_\theta, \theta, \sigma), H(x, \theta, \theta_{-1}, \sigma)+\sigma\epsilon', G(x, \theta, \theta_{-1}, \sigma), [x; \theta])$. Then, System (9) can be presented as

$$\mathcal{E}\tilde{f}(\epsilon, x, \theta, \theta_{-1}, \sigma) = 0. \tag{42}$$

Denote $F = \mathcal{E}\tilde{f}(\epsilon, x, \theta, \theta_{-1}, \sigma)$, and let $F = [F^0, F^1]$. Then, $F^0 \equiv F^0(x, \theta_{-1}, \sigma)$, and $F^1 \equiv F^1(x, \theta, \theta_{-1}, \sigma)$, so that the first-order approximate solution is pinned down by the following set of conditions

$$\Delta(F)_x = \mathcal{E}\tilde{f}_x = 0,$$

$$\Delta(F)_\sigma = \mathcal{E}\tilde{f}_\sigma = 0,$$

$$\Delta(F^0)_{\theta_{-1}} = \mathcal{E}[\tilde{f}^0_\theta P] + \mathcal{E}\tilde{f}^0_{\theta_{-1}} = 0,$$

$$\Delta(F^1)_{\theta_{-1}} = \mathcal{E}\tilde{f}^1_{\theta_{-1}} = 0, \quad \textit{and}$$

$$\Delta(F^1)_\theta = \mathcal{E}\tilde{f}^1_\theta = 0.$$

Notice that if $\Delta(F^1)_{\theta_{-1}} = 0$, then condition $\Delta(F^1)_\theta P + \Delta(F^1)_{\theta_{-1}} = 0$ implies $\Delta(F^1)_\theta = 0$. Therefore, the above system is equivalent to the following set of equations:

$$\mathcal{E}\tilde{f}_x = 0,$$

$$\mathcal{E}\tilde{f}_\sigma = 0,$$

$$\mathcal{E}\tilde{[}f_\theta P] + \mathcal{E}\tilde{f}_{\theta_{-1}} = 0, \quad and$$

$$\mathcal{E}\tilde{f}^1_{\theta_{-1}} = 0.$$

This implies that

$$\mathcal{E}\tilde{f}_x = f_{Y'}G_x h_x + f_{x'}h_x + f_Y G_x + f_x = 0, \tag{43}$$

$$\mathcal{E}\tilde{f}_\sigma = \mathcal{E}[f_{Y'}[G_X(H_\sigma + \eta\epsilon') + G_\sigma] + f_{X'}(H_\sigma + \eta\epsilon') + f_Y G_\sigma] = 0,$$

or

$$\mathcal{E}\tilde{f}_\sigma = f_{Y'}[G_X H_\sigma + G_\sigma] + f_{X'}H_\sigma + f_Y G_\sigma = 0, \tag{44}$$

$$\mathcal{E}[\tilde{f}_\theta P] + \mathcal{E}\tilde{f}_{\theta_{-1}} = f_{Y'}(G_X(h_\theta P + h_{\theta_{-1}}) + G_\theta P + G_{\theta_{-1}}) + \tag{45}$$

$$f_{x'}(h_\theta P + h_{\theta_{-1}}) + f_Y(G_\theta P + G_{\theta_{-1}}) + f_{\theta'}P^2 + f_\theta P = 0,$$

and

$$\mathcal{E}\tilde{f}^1_{\theta_{-1}} = \mathcal{E}[f^1_{Y'}G_x h_{\theta_{-1}} + f^1_{x'}h_{\theta_{-1}} + f^1_Y G_{\theta_{-1}}] = 0. \tag{46}$$

Now consider the full-information version of the same model, described by System (1). Denote $\bar{f}(\epsilon, x, \theta, \sigma) \equiv f(\bar{G}(\bar{H}(x,\theta,\sigma) + \sigma\epsilon', P\theta + \sigma\epsilon'_\theta, \sigma), \bar{H}(x,\theta,\sigma) + \sigma\epsilon', \bar{G}(x,\theta,\sigma), [x;\theta])$. Then, System (1) can be presented as

$$E\bar{f}(\epsilon, x, \theta, \theta_{-1}, \sigma) = 0. \tag{47}$$

Denote $F = \mathcal{E}\tilde{f}(\epsilon, x, \theta, \theta_{-1}, \sigma)$, then $F \equiv F(x, \theta, \sigma)$. The set of equations that pins down the elements of $\bar{H}_X$ and $\bar{G}_X$ for the full information version of the model is

$$\Delta(\bar{F})_x = E\bar{f}_x = 0,$$

$$\Delta(\bar{F})_\sigma = E\bar{f}_\sigma = 0, \quad and$$

$$\Delta(\bar{F})_\theta = E\bar{f}_\theta = 0,$$

which results in the system

$$E\bar{f}_x = f_{Y'}\bar{G}_x\bar{h}_x + f_{x'}\bar{h}_x + f_Y\bar{G}_x + f_x = 0, \tag{48}$$

$$E\bar{f}_\sigma = E[f_{Y'}[\bar{G}_X(\bar{H}_\sigma + \eta\epsilon'] + \bar{G}_\sigma) + f_{X'}(\bar{H}_\sigma + \eta\epsilon') + f_Y\bar{G}_\sigma] = 0,$$

or

$$E\bar{f}_\sigma = f_{Y'}[\bar{G}_X\bar{H}_\sigma + \bar{G}_\sigma] + f_{X'}\bar{H}_\sigma + f_Y\bar{G}_\sigma = 0, \quad and \tag{49}$$

$$E\bar{f}_\theta = \tilde{f}_{Y'}(\bar{G}_x\bar{h}_\theta + \bar{G}_\theta P) + f_{x'}\bar{h}_\theta + f_Y\bar{G}_\theta + f_{\theta'}P + f_\theta = 0. \tag{50}$$

Notice that Equations (43) - (44) and (48) - (49) are equivalent, which implies $H_\sigma = \bar{H}_\sigma$ and $G_\sigma = \bar{G}_\sigma$, $h_x = \bar{h}_x$ and $G_x = \bar{G}_x$. Also, notice that system (50) is linear in $\bar{h}_\theta$ and $\bar{G}_\theta$, an its linear transformation $E\bar{f}_\theta P = 0$ coincides with System (45), if the latter is considered a function of $h_\theta + h_{\theta_{-1}}P^{-1}$ and $G_\theta + G_{\theta_{-1}}P^{-1}$, from which it naturally follows that

$$\bar{h}_\theta = h_\theta + h_{\theta_{-1}}P^{-1}, \quad and \tag{51}$$

$$\bar{G}_\theta = G_\theta + G_{\theta_{-1}}P^{-1}. \tag{52}$$

Since $G_\theta = [g_\theta; 0]$, and $G_{\theta_{-1}} = [g_{\theta_{-1}}; j_{\theta_{-1}}]$, it means the last $n_z \times n_\theta$ equations in (52) pin down the elements of $j_{\theta_{-1}}$ as follows

$$j_{\theta_{-1}} = \bar{j}_\theta P. \tag{53}$$

The system (46) consists of $(nx + ny) \times n_\theta$ equations, so given $j_{\theta_{-1}}$, it pins down all elements of $g_{\theta_{-1}}$ and $h_{\theta_{-1}}$, which can be calculated from

$$[f_{Y'} G_x + f_{x'}, f_y] \begin{bmatrix} h_{\theta_{-1}} \\ g_{\theta_{-1}} \end{bmatrix} = -f_z j_{\theta_{-1}}.$$

Once $g_{\theta_{-1}}$ and $h_{\theta_{-1}}$ are determined, g_θ and h_θ can be recovered from the first $n_x + ny \times n_\theta$ equations of (51) and (52).

One may notice that in the model with timing constraints, Equations (51) and (52) alone cannot uniquely identify all elements in h_θ, $h_{\theta_{-1}}$, G_θ, and $G_{\theta_{-1}}$. The unique solution is pinned down by Equation (53), which is derived from the identifying assumption $j_\theta = 0$. Another unique solution can result under a more general restriction $j_\theta = J^*$, where J^* is a $n_z \times n_\theta$ matrix of parameters, not necessarily zero. Such identifying restriction could reflect an environment with imperfect information acquisition, when just a portion of information about θ is revealed at the time decisions about variables in z are made.

7.2 Proof of Proposition 2

The similar analysis of the second-order approximation problem will lead to the following set of cross-model restrictions:

$$G_{x,x} \equiv \bar{G}_{x,x}, \qquad h_{x,x} \equiv \bar{h}_{x,x}, \quad G_{\sigma,\sigma} = \bar{G}_{\sigma,\sigma}, \quad h_{\sigma,\sigma} = \bar{h}_{\sigma,\sigma},$$

$$G_{x,\sigma} = \bar{G}_{x,\sigma} = 0_{n_Y \times 1}, \quad h_{x,\sigma} = \bar{h}_{x,\sigma} = 0_{n_x \times 1},$$

$$G_{\theta,\theta} \equiv \bar{G}_{\theta,\theta} + G_{\theta,\theta_{-1}} \times_3 (P^T)^{-1} + G_{\theta_{-1},\theta} \times_2 (P^T)^{-1} + G_{\theta_{-1},\theta_{-1}} \times_2 (P^T)^{-1} \times_3 (P^T)^{-1},$$

$$h_{\theta,\theta} \equiv \bar{h}_{\theta,\theta} + h_{\theta,\theta_{-1}} \times_3 (P^T)^{-1} + h_{\theta_{-1},\theta} \times_2 (P^T)^{-1} + h_{\theta_{-1},\theta_{-1}} \times_2 (P^T)^{-1} \times_3 (P^T)^{-1},$$

and

$$G_{x,\theta} \equiv \bar{G}_{x,\theta} - G_{x,\theta_{-1}} \times_3 (P^T)^{-1},$$

$$h_{x,\theta} \equiv \bar{h}_{x,\theta} - h_{x,\theta_{-1}} \times_3 (P^T)^{-1},$$

As I show below, $j_{x,\theta_{-1}}$ and $j_{\theta_{-1},\theta_{-1}}$ are pinned down by the fact that $j_{\theta,\theta}=0$, $j_{\theta,\theta_{-1}}=0$, and $j_{\theta_{-1},\theta}=0$, and the elements of $g_{x,\theta_{-1}}$, $h_{x,\theta_{-1}}$, $g_{\theta,\theta_{-1}}$, $h_{\theta,\theta_{-1}}$, $g_{\theta_{-1},\theta_{-1}}$, and $h_{\theta_{-1},\theta_{-1}}$ can be determined by the solution to the full-information version of the model, and the following set of restrictions:

$$\tilde{f}^1_{\theta_{-1},\theta_{-1}}=0,\quad \tilde{f}^1_{\theta_{-1},\theta}=0,\quad \tilde{f}^1_{x,\theta_{-1}}=0.$$

I show that the first set of these restrictions will help to find the elements of $g_{\theta_{-1},\theta_{-1}}$ and $h_{\theta_{-1},\theta_{-1}}$, the second set will determine $g_{\theta_{-1},\theta}$ and $h_{\theta_{-1},\theta}$, and the third set will provide $g_{x,\theta_{-1}}$ and $h_{x,\theta_{-1}}$. Due to the continuity of second order derivatives of the policy functions, any element $(i,\,j,\,k)$ of arrays $a_{b,c}$ is determined as the $(i,\,k,\,j)$ element of matrices $a_{c,b}$ for $a=\{g,h\}$, and $c,b=\{x,\theta,\theta_{-1}\}$. Also notice that because of the smoothness of the equilibrium system, conditions $\tilde{f}^1_{\theta_{-1},\theta}=0$ and $\tilde{f}^1_{x,\theta_{-1}}=0$ imply $\tilde{f}^1_{\theta,\theta_{-1}}=0$ and $\tilde{f}^1_{\theta_{-1},x}=0$ correspondingly.

Equation (46) can be presented in the matrix form as functions of states as follows:

$$\mathcal{E}\tilde{f}^1_{\theta_{-1}}=[[f^1_{Y'}f^1_{x'}]\begin{bmatrix} g_x(x',\theta',\theta)\\ j_x(x',\theta)\\ I_{n_x}\end{bmatrix},\ f^1_y,\ f^1_z]\begin{bmatrix} h_{\theta_{-1}}(x,\theta,\theta_{-1})\\ g_{\theta_{-1}}(x,\theta,\theta_{-1})\\ j_{\theta_{-1}}(x,\theta_{-1})\end{bmatrix}=0,$$

where f^1_q are also assumed to be functions of states, for $q=Y',x',y$, and z. Then

$$\mathcal{E}\tilde{f}^1_{\theta_{-1},\theta_{-1}}=\Delta([f^1_{[Y',x']}\begin{bmatrix} g_x(x',\theta',\theta)\\ j_x(x',\theta)\\ I_{n_x}\end{bmatrix},\ f^1_Y])_{\theta_{-1}}\times_2\begin{bmatrix} h_{\theta_{-1}}\\ g_{\theta_{-1}}\\ j_{\theta_{-1}}\end{bmatrix}^T+\tag{54}$$

$$\Delta(\begin{bmatrix} h_{\theta_{-1}}(x,\theta,\theta_{-1})\\ g_{\theta_{-1}}(x,\theta,\theta_{-1})\\ j_{\theta_{-1}}(x,\theta_{-1})\end{bmatrix})_{\theta_{-1}}\times_1[f^1_{[Y',x']}\begin{bmatrix} g_x\\ j_x\\ I_{n_x\times n_x}\end{bmatrix},\ f^1_y,\ f^1_z]=0,$$

where

$$\Delta([f^1_{[Y',x']}\begin{bmatrix} g_x(x',\theta',\theta)\\ j_x(x',\theta)\\ I_{n_x}\end{bmatrix},\ f^1_Y])_{\theta_{-1}}=\tag{55}$$

$$[\Delta(f^1_{[Y',x']})_{\theta_{-1}} \times_2 \begin{bmatrix} g_x \\ j_x \\ I_{n_x \times n_x} \end{bmatrix}^T + \Delta(\begin{bmatrix} g_x(x',\theta',\theta) \\ j_x(x',\theta) \\ I_{n_x \times n_x} \end{bmatrix})_{\theta_{-1}} \times_1 f^1_{[Y',x']},\, \Delta(f^1_Y)_{\theta_{-1}}],$$

in which

$$\Delta(\begin{bmatrix} g_x(x',\theta',\theta) \\ j_x(x',\theta) \\ I_{n_x \times n_x} \end{bmatrix})_{\theta_{-1}} = \begin{bmatrix} g_{x,x} \times_3 h^T_{\theta_{-1}} \\ j_{x,x} \times_3 h^T_{\theta_{-1}} \\ 0_{n_x \times n_x \times n_\theta} \end{bmatrix}, \tag{56}$$

and

$$\Delta(\begin{bmatrix} h_{\theta_{-1}}(x,\theta,\theta_{-1}) \\ g_{\theta_{-1}}(x,\theta,\theta_{-1}) \\ j_{\theta_{-1}}(x,\theta_{-1}) \end{bmatrix})_{\theta_{-1}} = \begin{bmatrix} h_{\theta_{-1},\theta-1} \\ g_{\theta_{-1},\theta_{-1}} \\ j_{\theta_{-1},\theta_{-1}} \end{bmatrix}, \tag{57}$$

where

$$\Delta(f^1_{[Y',x']})_{\theta_{-1}} = [\Delta(f^1_{Y'})_{\theta_{-1}},\, \Delta(f^1_{y'})_{\theta_{-1}},\, \Delta(f^1_{x'})_{\theta_{-1}}],$$

and for $q = Y',\, x',\, Y$,

$$\Delta(f^1_q)_{\theta_{-1}} = f^1_{q,y'} \times_3 (g_x h_{\theta_{-1}})^T + f^1_{q,z'} \times_3 (j_x h_{\theta_{-1}})^T + f^1_{q,x'} \times_3 h^T_{\theta_{-1}} + f^1_{q,y} \times_3 g^T_{\theta_{-1}} + f^1_{q,z} \times_3 j^T_{\theta_{-1}},$$

which can be written in matrix form as

$$\Delta(f^1_q)_{\theta_{-1}} = f^1_{q,Y'} \times_3 (\begin{bmatrix} G_x \\ I_{n_x} \end{bmatrix} h_{\theta_{-1}})^T + f^1_{q,Y} \times_3 G^T_{\theta_{-1}}.$$

Substituting formulas (55)-(57) into Equation (54), one can obtain the system of $(n_y + n_x)n_\theta^2$ equations to solve for $h_{\theta_{-1},\theta_{-1}}$ and $g_{\theta_{-1},\theta_{-1}}$:

$$[\Delta(f^1_{[Y',x']})_{\theta_{-1}} \times_2 \begin{bmatrix} g_x \\ j_x \\ I_{n_x \times n_x} \end{bmatrix} + \begin{bmatrix} g_{x,x} \times_3 h^T_{\theta_{-1}} \\ j_{x,x} \times_3 h^T_{\theta_{-1}} \\ 0_{n_x \times n_x \times n_\theta} \end{bmatrix} \times_1 f^1_{[Y',x']},\, \Delta(f^1_Y)_{\theta_{-1}}] \times_2 \begin{bmatrix} h_{\theta_{-1}} \\ g_{\theta_{-1}} \\ j_{\theta_{-1}} \end{bmatrix}^T +$$

$$\begin{bmatrix} h_{\theta_{-1},\theta-1} \\ g_{\theta_{-1},\theta_{-1}} \\ j_{\theta_{-1},\theta_{-1}} \end{bmatrix} \times_1 [f^1_{[Y',x']} \begin{bmatrix} g_x \\ j_x \\ I_{n_x} \end{bmatrix}, \, f^1_y, \, f^1_z] = 0,$$

which implies

$$\begin{bmatrix} h_{\theta_{-1},\theta-1} \\ g_{\theta_{-1},\theta_{-1}} \end{bmatrix} \times_1 [f^1_{[Y',x']} \begin{bmatrix} G_x \\ I_{n_x \times n_x} \end{bmatrix}, \, f^1_y] = -j_{\theta_{-1},\theta_{-1}} \times_1 f^1_z -$$

$$[\Delta(f^1_{[Y',x']})_{\theta_{-1}} \times_2 \begin{bmatrix} G_x \\ I_{n_x} \end{bmatrix}^T + G_{x,x} \times_1 f^1_{Y'} \times_3 h^T_{\theta_{-1}}, \, \Delta(f^1_Y)_{\theta_{-1}}] \times_2 \begin{bmatrix} h_{\theta_{-1}} \\ G_{\theta_{-1}} \end{bmatrix}^T.$$

Next,

$$\mathcal{E}\tilde{f}^1_{\theta_{-1},\theta} = \Delta([f^1_{[Y',x']} \begin{bmatrix} g_x(x',\theta',\theta) \\ j_x(x',\theta) \\ I_{n_x} \end{bmatrix}, \, f^1_y, \, f^1_z])_\theta \times_2 \begin{bmatrix} h_{\theta_{-1}} \\ g_{\theta_{-1}} \\ j_{\theta_{-1}} \end{bmatrix}^T + \tag{58}$$

$$\Delta(\begin{bmatrix} h_{\theta_{-1}}(x,\theta,\theta_{-1}) \\ g_{\theta_{-1}}(x,\theta,\theta_{-1}) \\ j_{\theta_{-1}}(x,\theta_{-1}) \end{bmatrix})_\theta \times_1 [f^1_{[Y',x']} \begin{bmatrix} g_x \\ j_x \\ I_{n_x} \end{bmatrix}, \, f^1_y, \, f^1_z] = 0,$$

where

$$\Delta([f^1_{[Y',x']} \begin{bmatrix} g_x(x',\theta',\theta) \\ j_x(x',\theta) \\ I_{n_x \times n_x} \end{bmatrix}, \, f^1_Y])_\theta =$$

$$[\Delta(f^1_{[Y',x']})_\theta \times_2 \begin{bmatrix} g_x \\ j_x \\ I_{n_x} \end{bmatrix}^T + \Delta(\begin{bmatrix} g_x(x',\theta',\theta) \\ j_x(x',\theta) \\ I_{n_x} \end{bmatrix})_\theta \times_1 f^1_{[Y',' x']}, \, \Delta(f^1_Y)_\theta],$$

in which

$$\Delta(\begin{bmatrix} g_x(x',\theta',\theta) \\ j_x(x',\theta) \\ I_{n_x \times n_x} \end{bmatrix})_\theta = \begin{bmatrix} g_{x,x} \times_3 h^T_\theta + g_{x,\theta} \times_3 P^T + g_{x,\theta_{-1}} \\ j_{x,x} \times_3 h^T_\theta + j_{x,\theta_{-1}} \\ 0_{n_x \times n_x \times n_\theta} \end{bmatrix} = \begin{bmatrix} G_{x,X} \\ 0_{n_x \times n_x \times n_x + n_\theta} \end{bmatrix} \times_3 H^T_\theta,$$

$$\Delta(\begin{bmatrix} h_{\theta_{-1}}(x,\theta,\theta_{-1}) \\ g_{\theta_{-1}}(x,\theta,\theta_{-1}) \\ j_{\theta_{-1}}(x,\theta_{-1}) \end{bmatrix})_\theta = \begin{bmatrix} h_{\theta_{-1},\theta} \\ g_{\theta_{-1},\theta} \\ 0_{n_z\times n_\theta\times n_\theta} \end{bmatrix},$$

and

$$\Delta(f^1_{[Y',x']})_\theta = [\Delta(f^1_{Y'})_\theta,\, \Delta(f^1_{x'})_\theta],$$

where for $q = Y'\, x'$, Y,

$$\Delta(f^1_q)_\theta = f^1_{q,y'}\times_3(g_x h_\theta + g_\theta P + g_{\theta_{-1}})^T + f^1_{q,z'}\times_3(j_x h_\theta + j_{\theta_{-1}})^T + f^1_{q,y}\times_3 g_\theta^T + f^1_{q,x'}\times_3 h_\theta^T + f^1_{q,\theta'}\times_3 P^T + f^1_{q,\theta},$$

which is in matrix form

$$\Delta(f^1_q)_\theta = f^1_{q,[Y',X',\theta]} \times_3 (\begin{bmatrix} G_X \\ I_{n_X+n_\theta} \end{bmatrix} \begin{bmatrix} H_\theta \\ I_{n_\theta} \end{bmatrix})^T + f^1_{q,y} \times_3 g_\theta^T.$$

Substituting everything into (58), one will get

$$[\Delta(f^1_{[Y',x']})_\theta \times_2 \begin{bmatrix} g_x \\ j_x \\ I_{n_x} \end{bmatrix}^T + \begin{bmatrix} G_{x,X} \\ 0_{n_x\times n_x\times n_x+n_\theta} \end{bmatrix} \times_3 H_\theta^T \times_1 f^1_{[Y',x']},\, \Delta(f^1_Y)_\theta] \times_2 \begin{bmatrix} h_{\theta_{-1}} \\ g_{\theta_{-1}} \\ j_{\theta_{-1}} \end{bmatrix}^T +$$

$$\begin{bmatrix} h_{\theta_{-1},\theta} \\ g_{\theta_{-1},\theta} \\ 0_{n_z\times n_\theta\times n_\theta} \end{bmatrix} \times_1 [f^1_{[Y',x']} \begin{bmatrix} g_x \\ j_x \\ I_{n_x} \end{bmatrix},\, f^1_y,\, f^1_z] = 0,$$

or

$$\begin{bmatrix} h_{\theta_{-1},\theta} \\ g_{\theta_{-1},\theta} \end{bmatrix} \times_1 [f^1_{Y'} \begin{bmatrix} G_x \\ I_{n_x} \end{bmatrix},\, f^1_y] =$$

$$-[\Delta(f^1_{[Y',x']})_\theta \times_2 \begin{bmatrix} G_x \\ I_{n_x} \end{bmatrix}^T + G_X \times_3 H_\theta^T \times_1 f^1_{Y'},\, \Delta(f^1_Y)_\theta] \times_2 \begin{bmatrix} h_{\theta_{-1}} \\ G_{\theta_{-1}} \end{bmatrix}^T.$$

Since all elements except $g_{\theta_{-1},\theta}$ and $h_{\theta_{-1},\theta}$ are known, the system of $(n_y+n_x)n_\theta^2$ equations $\tilde{f}^1_{\theta_{-1},\theta} = 0$ allows to obtain all $n_y n_\theta^2$ elements of the array $g_{\theta_{-1},\theta}$ and $n_x n_\theta^2$ elements of the array $h_{\theta_{-1},\theta}$.

Finally, denoting $f^1_{[z,x]} = [f^1_z,\ f^1_x]$,

$$\tilde{f}^1_x = [f^1_{[Y',x']} \begin{bmatrix} g_x(x',\theta',\theta) \\ j_x(x',\theta) \\ I_{n_x} \end{bmatrix},\ f^1_y] \begin{bmatrix} h_x(x,\theta,\theta_{-1}) \\ g_x(x,\theta,\theta_{-1}) \end{bmatrix} + f^1_{[zx]} \begin{bmatrix} j_x(x,\theta_{-1}) \\ I_{n_x} \end{bmatrix},$$

which can be used to derive

$$\Delta(\tilde{f}^1_x)_{\theta_{-1}} = \Delta([f^1_{[Y',x']} \begin{bmatrix} g_x(x,\theta',\theta) \\ j_x(x,\theta) \\ I_{n_x\times n_x} \end{bmatrix},\ f^1_y])_{\theta_{-1}} \times_2 \begin{bmatrix} h_x \\ g_x \end{bmatrix}^T +$$

$$\Delta(\begin{bmatrix} h_x(x,\theta,\theta_{-1}) \\ g_x(x,\theta,\theta_{-1}) \end{bmatrix})_{\theta_{-1}} \times_1 [f^1_{[Y',x']} \begin{bmatrix} g_x \\ j_x \\ I_{n_x} \end{bmatrix},\ f^1_y] +$$

$$\Delta(f^1_{[z,x]})_{\theta_{-1}} \times_2 \begin{bmatrix} j_x \\ I_{n_x} \end{bmatrix}^T + \Delta(\begin{bmatrix} j_x(x,\theta_{-1}) \\ I_{n_x\times n_x} \end{bmatrix})_{\theta_{-1}} \times_1 f^1_{[z,x]} = 0,$$

where

$$\Delta([f^1_{[Y',x']} \begin{bmatrix} g_x(x,\theta',\theta) \\ j_x(x,\theta) \\ I_{n_x\times n_x} \end{bmatrix},\ f^1_y])_{\theta_{-1}} =$$

$$[\Delta(f^1_{[Y',x']})_{\theta_{-1}} \times_2 \begin{bmatrix} g_x \\ j_x \\ I_{n_x} \end{bmatrix}^T + \Delta(\begin{bmatrix} g_x(x,\theta',\theta) \\ j_x(x,\theta) \\ I_{n_x\times n_x} \end{bmatrix})_{\theta_{-1}} \times_1 f^1_{[Y',x']},\ \Delta(f^1_y)_{\theta_{-1}}],$$

$$\Delta\left(\begin{bmatrix} g_x(x',\theta',\theta) \\ j_x(x',\theta) \\ I_{n_x} \end{bmatrix}\right)_{\theta_{-1}} = \begin{bmatrix} g_{x,x} \times_3 h^T_{\theta_{-1}} \\ j_{x,x} \times_3 h^T_{\theta_{-1}} \\ 0_{n_x\times n_x\times n_\theta} \end{bmatrix}, \quad and \quad \Delta\left(\begin{bmatrix} h_x(x,\theta,\theta_{-1}) \\ g_x(x,\theta,\theta_{-1}) \end{bmatrix}\right)_{\theta_{-1}} = \begin{bmatrix} h_{x,\theta_{-1}} \\ g_{x,\theta_{-1}} \end{bmatrix}.$$

Thus, $g_{x,\theta_{-1}}$, and $h_{x,\theta_{-1}}$ can be obtained as a solution to the system

$$[\Delta(f^1_{[Y',x']})_{\theta_{-1}} \times_2 \begin{bmatrix} g_x \\ j_x \\ I_{n_x} \end{bmatrix}^T + \begin{bmatrix} g_{x,x} \\ j_{x,x} \\ 0_{n_x} \end{bmatrix} \times_3 h^T_{\theta_{-1}} \times_1 f^1_{[Y',x']},\ \Delta(f^1_y)_{\theta_{-1}}] \times_2 \begin{bmatrix} h_x \\ g_x \end{bmatrix}^T +$$

$$\begin{bmatrix} h_{x,\theta_{-1}} \\ g_{x,\theta_{-1}} \end{bmatrix} \times_1 [f^1_{[Y'x']} \begin{bmatrix} g_x \\ j_x \\ I_{n_x} \end{bmatrix}, f^1_y] + \Delta(f^1_{[z,x]})_{\theta_{-1}} \times_2 \begin{bmatrix} j_x \\ I_{n_x} \end{bmatrix}^T + \begin{bmatrix} j_{x,\theta_{-1}} \\ 0_{n_x \times n_x \times n_\theta} \end{bmatrix} \times_1 f^1_{[z,x]} = 0,$$

which can be presented more compactly as

$$\begin{bmatrix} h_{x,\theta_{-1}} \\ g_{x,\theta_{-1}} \end{bmatrix} \times_1 [f^1_{[Y',x']} \begin{bmatrix} G_x \\ I_{n_x} \end{bmatrix}, f^1_y] =$$

$$-[\Delta(f^1_{[Y',x']})_{\theta_{-1}} \times_2 \begin{bmatrix} G_x \\ I_{n_x} \end{bmatrix}^T + G_{x,x} \times_3 h^T_{\theta_{-1}} \times_1 f^1_{Y'},\ \Delta(f^1_{[Y,x]})_{\theta_{-1}}] \times_2 \begin{bmatrix} h_x \\ G_x \\ I_{n_x} \end{bmatrix}^T - j_{x,\theta_{-1}} \times_1 f^1_z.$$

7.3 Second Order Approximation in Case of M-subperiods.

To address the multiple subperiod problem with the second order approximation, one should extend the algorithm in Section 4 with the following steps:

Step 4.1. $j_{x,\theta^m_{-1}} \equiv \bar{j}_{x,\theta^m} \times_3 P^T_m$, and

$$j_{\theta^m_{-1},\theta^m_{-1}} = \bar{j}_{\theta^m,\theta^m} \times_2 P^T_m \times_3 P^T_m.$$

Step 4.2.

$$\begin{bmatrix} h_{\theta^m_{-1},\theta^m} \\ g_{\theta^m_{-1},\theta^m} \end{bmatrix} =$$

$$-[\Delta(f^{m:M}_{[Y',x']})_{\theta^m} \times_2 \begin{bmatrix} G_x \\ I_{n_x} \end{bmatrix}^T + G_{x,X} \times_3 H^T_{\theta^m} \times_1 f^{m:M}_{Y'},\ \Delta(f^{m:M}_Y)_{\theta^m}] \times_2 \begin{bmatrix} h_{\theta^m_{-1}} \\ G_{\theta^m_{-1}} \end{bmatrix}^T \times_1 \Delta(f^{m:M})^{-1}_{[x,y]}.$$

In this formula,

$$\Delta(f^{m:M}_{[Y',x']})_{\theta^m} = [\Delta(f^{m:M}_{Y'})_{\theta^m},\, \Delta(f^{m:M}_{x'})_{\theta^m}],$$

and for $q = Y',\, x',\, Y$,

$$\Delta(f^1_q)_{\theta^m} = f^{m:M}_{q,[Y',X',\theta^m]} \times_3 (\begin{bmatrix} G_X \\ I_{n_X+n_{\theta^m}} \end{bmatrix} \begin{bmatrix} H_{\theta^m} \\ I_{n_{\theta^m}} \end{bmatrix})^T + f^{m:M}_{q,y} \times_3 g^T_{\theta^m}.$$

Step 4.3.

$$\begin{bmatrix} h_{\theta^m_{-1},\theta-1^m} \\ g_{\theta^m_{-1},\theta^m_{-1}} \end{bmatrix} = -j_{\theta^m_{-1},\theta^m_{-1}} \times_1 f^{m:M}_{z^m} - \tag{59}$$

$$[\Delta(f^{m:M}_{[Y',x']})_{\theta^m_{-1}} \times_2 \begin{bmatrix} G_x \\ I_{n_x} \end{bmatrix}^T + G_{x,x} \times_1 f^{m:M}_{Y'} \times_3 h^T_{\theta^m_{-1}},\, \Delta(f^{m:M}_{Y})_{\theta^m_{-1}}]$$

$$\times_2 \begin{bmatrix} h_{\theta^m_{-1}} \\ G_{\theta^m_{-1}} \end{bmatrix}^T \times_1 \Delta(f^{m:M})^{-1}_{[x,y]},$$

where for $q = Y', x', Y$,

$$\Delta(f^1_q)_{\theta^m_{-1}} = f^{m:M}_{q,[Y',x']} \times_3 (\begin{bmatrix} G_x \\ I_{nx} \end{bmatrix} h_{\theta^m_{-1}})^T + f^{m:M}_{q,Y} \times_3 G^T_{\theta^m_{-1}}.$$

Step 4.4.

$$g_{\theta^m,\theta^m} = \bar{g}_{\theta^m,\theta^m} - (g_{\theta^m,\theta^m_{-1}} \times_3 (P^T_m)^{-1} + g_{\theta^m_{-1},\theta^m} \times_2 (P^T_m)^{-1} + g_{\theta^m_{-1},\theta^m_{-1}} \times_2 (P^T_m)^{-1} \times_3 (P^T_m)^{-1}),$$

$$h_{\theta^m,\theta^m} = \bar{h}_{\theta^m,\theta^m} - (h_{\theta^m,\theta^m_{-1}} \times_3 (P^T_m)^{-1} + h_{\theta^m_{-1},\theta^m} \times_2 (P^T_m)^{-1} + h_{\theta^m_{-1},\theta^m_{-1}} \times_2 (P^T_m)^{-1} \times_3 (P^T_m)^{-1}),$$

Step 4.5.

$$\begin{bmatrix} h_{x,\theta^m_{-1}} \\ g_{x,\theta^m_{-1}} \end{bmatrix} = -j_{x,\theta^m_{-1}} \times_1 f^{m:M}_{z^m} -$$

$$[\Delta(f^{m:M}_{[Y',x']})_{\theta^m_{-1}} \times_2 \begin{bmatrix} G_x \\ I_{n_x} \end{bmatrix}^T + G_{x,x} \times_3 h^T_{\theta^m_{-1}} \times_1 f^{m:M}_{Y'},\ \Delta(f^{m:M}_{[Y,x]})_{\theta^m_{-1}}] \times_2 \begin{bmatrix} h_x \\ G_x \\ I_{n_x} \end{bmatrix}^T \times_1 \Delta(f^{m:M})^{-1}_{[x,y]}.$$

Step 4.6.

$$g_{x,\theta^m} \equiv \bar{g}_{x,\theta^m} - g_{x,\theta^m_{-1}} \times_3 (P^T_m)^{-1},$$

$$h_{x,\theta^m} \equiv \bar{h}_{x,\theta^m} - h_{x,\theta^m_{-1}} \times_3 (P^T_m)^{-1}.$$

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