Rolling without slipping

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Let us consider the problem of a ball rolling down a slope (defined by a known function) without slipping. To solve it, we are going to use the Lagrange formalism (i.e. the Lagrange equations). The target of this text is to find the differential equation of the system considering the following as known information:

- R: Radius of the ball.
- m: Mass of the ball.
- g: Local gravity acceleration.
- $f(x)$: Function that defines the slope. Note that all of its derivatives –up to the third one– must be analitically known.

This system would initially have three degrees of freedom (two for translation and one for rotation) since the ball is restricted to move in the XY plane. However, two holonomic constrains are provided by the *rolling without slipping* condition; one concerning the angle and another concerning the y-coordinate. Therefore, only one degree of freedom is left and, thus, we just need to use one generalized coordinate. So let us choose the x coordinate of the contact point between the ball and the ground as the generalized coordinate x.

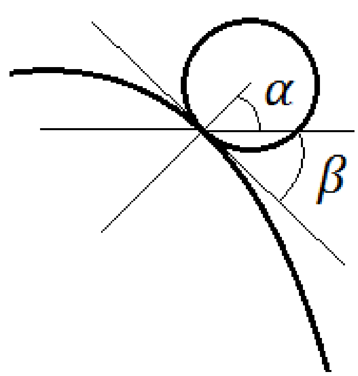

Figure 1: Angles definitions, note that in this case $\beta < 0$

The next step consists of writing the expression of the center of mass position and the rotated angle with respect to our coordinate x, for they appear in the expression of the Lagrangian. Note that since the ball is always in contact with the ground, the center of mass is always displaced a distance R from the curve in the direction that is normal to it. Recalling that the first derivative $f'(x)$ returns the slope $\frac{\Delta y}{\Delta x}$, we can write the angles α and β in figure 1 as:

$$\beta = arctan\left(f'(x)\right)$$

$$\alpha = \frac{\pi}{2} + \beta$$

Then, the center of mass position becomes:

$$X_{CM} = x + Rcos(\alpha)$$

$$Y_{CM} = f(x) + Rsin(\alpha)$$

Concerning the rotated angle, we are going to describe it using ϕ, the rotated angle with respect to the vertical. This angle gets a contribution from the truly rotated angle θ and another from the angle of the slope β. The former is the rotated angle with respect to the line that is normal to the curve and can be computed considering that the "gone by" longitude and the "rotated" ball perimeter are equal. The latter is the same β mentioned before (which is negative if the slope is so, such as in figure 2).

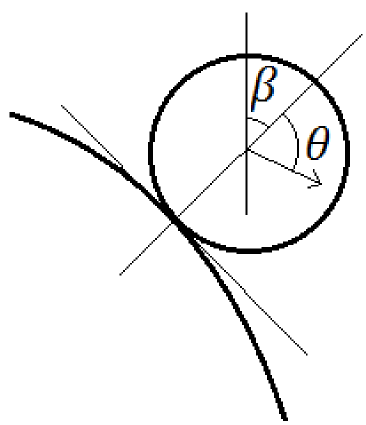

Figure 2: Rotation angles.

So

$$\phi = \theta - \beta$$

$$\theta = \frac{longitude}{R} = \frac{1}{R}\int_{x_o}^{x} \sqrt{1 + \left(f'(x)\right)^2}dx$$

Where x_o is the initial value for x. Now it is time to compute the derivatives of what we have just defined to build our Lagrangian as $L = T - V$, being T the kinetic energy and V the potential energy. The coordinates of the center of mass can be differentiated using the chain rule.

$$\dot{X}_{CM} = X'_{CM}\dot{x} = (1 - Rsin(\alpha)\alpha')\,\dot{x}$$
$$\dot{Y}_{CM} = Y'_{CM}\dot{x} = (f'(x) + Rcos(\alpha)\alpha')\,\dot{x}$$

Where

$$\alpha' = \beta' = \frac{1}{1+(f'(x))^2}f''(x)$$

To take the derivative of ϕ, we recall the fundamental theorem of calculus; which states that

$$\frac{d}{dx}\int_{x_o}^{x} g(x')dx' = g(x)$$

and we use, once again, the chain rule.

$$\dot{\phi} = \left(\frac{1}{R}\sqrt{1+(f'(x))^2} - \beta'\right)\dot{x}$$

The kinetic energy combines two energies, one due to the translation movement and another due to the rotational one. The expression for it is then:

$$T = \frac{1}{2}m\left((\dot{X}_{CM})^2 + (\dot{Y}_{CM})^2\right) + \frac{1}{2}I(\dot{\phi})^2 =$$

$$= \frac{1}{2}m\left((X'_{CM})^2 + (Y'_{CM})^2\right)\dot{x}^2 + \frac{1}{2}I\left(\frac{1}{R}\sqrt{1+(f'(x))^2} - \beta'\right)^2\dot{x}^2$$

Where I is the moment of inertia, for a sphere: $I = \frac{2}{5}mR^2$.
The potential energy only gets one contribution, that of the gravity force.

$$V = mgY_{CM}$$

So, finally, we can build the Lagrangian of the system:

$$L = T - V = \frac{1}{2}m\left((X'_{CM})^2 + (Y'_{CM})^2\right)\dot{x}^2 + \frac{1}{5}mR^2\left(\frac{1}{R}\sqrt{1+(f'(x))^2} - \beta'\right)^2\dot{x}^2 - mgY_{CM}$$

Now it is time for us to write the Lagrange equation, which is

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0$$

So

$$\frac{\partial L}{\partial \dot{x}} = \dot{x}\left(m\left((X'_{CM})^2 + (Y'_{CM})^2\right) + \frac{2}{5}mR^2\left(\frac{1}{R}\sqrt{1+(f'(x))^2} - \beta'\right)^2\right)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = \ddot{x}\left(m\left((X'_{CM})^2 + (Y'_{CM})^2\right) + \frac{2}{5}mR^2\left(\frac{1}{R}\sqrt{1-(f'(x))^2} - \beta'\right)^2\right) +$$

$$+\dot{x}^2\left(2m\left(X'_{CM}X''_{CM} + Y'_{CM}Y''_{CM}\right) + \frac{4}{5}mR^2\left(\frac{1}{R}\sqrt{1+(f'(x))^2} - \beta'\right)\left(\frac{f'(x)f''(x)}{R\sqrt{1+(f'(x))^2}} - \beta''\right)\right)$$

$$\frac{\partial L}{\partial x} = \dot{x}^2 \left(m\left(X'_{CM}X''_{CM} + Y'_{CM}Y''_{CM}\right) + \frac{2}{5}mR^2 \left(\frac{1}{R}\sqrt{1+\left(f'(x)\right)^2} - \beta'\right)\left(\frac{f'(x)f''(x)}{R\sqrt{1+\left(f'(x)\right)^2}} - \beta''\right)\right) - mgY'_{CM}$$

Where

$$X''_{CM} = -R\left(cos(\alpha)(\alpha')^2 + sin(\alpha)\alpha''\right)$$

$$Y''_{CM} = f''(x) + R\left(-sin(\alpha)(\alpha`)^2 + cos(\alpha)\alpha''\right)$$

$$\beta'' = \alpha'' = -\frac{2f'(x)\left(f''(x)\right)^2}{\left(1+\left(f'(x)\right)^2\right)^2} + \frac{f'''(x)}{1+\left(f'(x)\right)^2}$$

Finally, combining the prepared ingredients, we get the desired differential equation.

$$\ddot{x} = \frac{-\dot{x}^2 \left(m\left(X'_{CM}X''_{CM} + Y'_{CM}Y''_{CM}\right) + \frac{2}{5}mR^2 \left(\frac{1}{R}\sqrt{1+\left(f'(x)\right)^2} - \beta'\right)\left(\frac{f'(x)f''(x)}{R\sqrt{1+\left(f'(x)\right)^2}} - \beta''\right)\right) - mgY'_{CM}}{m\left((X'_{CM})^2 + (Y'_{CM})^2\right) + \frac{2}{5}mR^2\left(\frac{1}{R}\sqrt{1-\left(f'(x)\right)^2} - \beta'\right)^2}$$

In this particular case, the ground-function that is going to be used is the Gaussian function. So the function and its derivatives are:

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\frac{-(x-\mu)^2}{2\sigma^2}$$

$$f'(x) = -\frac{x-\mu}{\sigma^2}f(x)$$

$$f''(x) = -\left(1 - \frac{(x-\mu)^2}{\sigma^2}\right)\frac{f(x)}{\sigma^2}$$

$$f'''(x) = -\left(\frac{-2(x-\mu)}{\sigma^2} - \left(1 - \frac{(x-\mu)^2}{\sigma^2}\right)\frac{(x-\mu)}{\sigma^2}\right)\frac{f(x)}{\sigma^2}$$