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ON THE OPTIMAL SEARCH FOR A RANDOMLY MOVING TARGET*

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Abstract. The problem involved in the search for a randomly moving target is considered in the case where the probability density function of the location of the target satisfies an equation of type (1). The searching effort is expressed in terms of a time-dependent search density function, and a necessary condition for the optimality of the search density function is derived. In the case of a stationary target this condition becomes the familiar one of Koopman [4]. Since most applications would involve the solution of a partial differential equation of parabolic type with appropriate initial and boundary conditions, the application of the optimality condition is difficult. Obviously a special numerical technique needs to be introduced, a task which we shall not attempt in the present paper.

1. Motion of the target and the search. We shall assume that the target is moving randomly in a region R of the n-dimensional space R_n, and that the probability density function of the location of the target, denoted here by $u(\mathbf{x}, t)$, is continuous for all $\mathbf{x} \in R$ and $t \in (0, \infty)$. We shall also assume that it satisfies, if no additional information becomes available about the location of the target, the equation

$$\frac{\partial}{\partial t}u(\mathbf{x}, t) = L\{u(\mathbf{x}, t)\}, \tag{1}$$

where $u(\mathbf{x}, 0) = f(\mathbf{x})$ for all $\mathbf{x} \in R, f(\mathbf{x})$ a given probability density function, and where, at all times t, $u(\mathbf{x}, t) \equiv 0$ for $\mathbf{x} \notin R$. We shall make the following assumptions about the operator L:

(a) L is linear and such that $L\{h(t)g(\mathbf{x})\} = h(t)L\{g(\mathbf{x})\}$ for all continuous functions $h(t)$ defined for $t \in (0, \infty)$.

(b) The probability density function $v(\mathbf{x}, t)$ which satisfies (1), the initial condition that $v(\mathbf{x}, 0) = \delta(\mathbf{x})$ and the boundary condition that $v(\mathbf{x}, t) \equiv 0$ for $\mathbf{x} \notin R$ are such that the function $\max_{\mathbf{x}\in R} v(\mathbf{x}, t)$ possesses a Laplace transform with respect to t.

In most applications (1) would be a partial differential equation. If the motion of the target is a diffusion process, one has an equation of this kind. For instance, if the target is moving in a diffuse way along the x-axis at average speed w, the probability density function $u(\mathbf{x}, t)$ of the target satisfies the partial differential equation

$$\frac{\partial}{\partial t}u(\mathbf{x}, t) = a\left(\frac{\partial}{\partial \mathbf{x}}\right)^2 u(\mathbf{x}, t) - w\left(\frac{\partial}{\partial \mathbf{x}}\right)u(\mathbf{x}, t). \tag{2}$$

The operator $L = a(\partial/\partial \mathbf{x})^2 - w(\partial/\partial \mathbf{x})$ obviously has property (a). Also condition (b) is satisfied, because now, as is well known,

$$v(\mathbf{x}, t) = (4\pi at)^{-1/2} \exp\left[-(\mathbf{x} - wt)^2/(4at)\right], \tag{3}$$

so that $\max_{\mathbf{x}} v(\mathbf{x}, t) = (4\pi at)^{-1/2}$ with Laplace transform $(4as)^{-1/2}$.

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The probability density function $u(\mathbf{x}, t)$ discussed above may be called the a priori probability density of the target. With the search going on, additional information about the location of the target is gained and the probability density function of the target no longer satisfies (1). An equation for the a posteriori probability density function of the location of the target at time t, i.e., for the probability density function given that the search during time $(0, t)$ has not been successful, will be introduced in § 2.

The search will be represented by the search density function $\phi(\mathbf{x}, t)$ with the following properties [1], [4]:

(i) $\phi(\mathbf{x}, t) \geqq 0$ for all $\mathbf{x} \in R$ and $t > 0$;

(ii) $\int_R \phi(\mathbf{x}, t)\, dV_x = \Phi = \text{const.}$ for all $t > 0$;

(iii) $\phi(\mathbf{x}, t)\Delta t + o(\Delta t)$ is the conditional probability that the object will be located during time $(t, t + \Delta t)$, given that the object is at point $\mathbf{x}$.

Now

$$\int_0^T \int_R \phi(\mathbf{x}, \eta)\, dV_x\, d\eta = \Phi t,$$

which is a special case of Koopman's effort constraint

$$\int_0^T \int_R \phi(\mathbf{x}, \eta)\, dV_x\, d\eta \leqq \Phi_0,$$

where Φ_0 is a constant (cf. also [1]).

2. The a posteriori probability density function of the target. Let $u(\mathbf{x}, \tau; \phi)$ denote the probability density function of the location of the target, given that the search during time $(0, \tau)$, represented by the search density function $\phi(\mathbf{x}, \tau)$, has not been successful. The probability that the target is not found during time $(0, t)$ now becomes

$$P_\phi = \exp\left[- \int_0^t d\tau \int_R \phi(\mathbf{x}, \tau) u(\mathbf{x}, \tau; \phi)\, dV_x \right]. \tag{4}$$

It can be shown ([3] and [5]) that $u(\mathbf{x}, \tau; 0)$ satisfies the equation

$$\frac{\partial}{\partial \tau} u(\mathbf{x}, \tau; \phi) = L\{u(\mathbf{x}, \tau; \phi)\} + u(\mathbf{x}, \tau; \phi)\left[\int_R \phi(\xi, \tau) u(\xi, \tau; \phi)\, dV_\xi - \phi(\mathbf{x}, \tau) \right], \tag{5}$$

where $u(\mathbf{x}, 0; \phi) = u(\mathbf{x}, 0) = f(\mathbf{x})$ and where $u(\mathbf{x}, \tau; \phi) \equiv 0$ for $\mathbf{x} \notin R$.

Substitution of

$$y(\mathbf{x}, \tau; \phi) = u(\mathbf{x}, \tau; \phi) \exp\left[- \int_0^\tau d\tau_1 \int_R \phi(\xi, \tau_1) u(\xi, \tau_1; \phi)\, dV_\xi \right] \tag{6}$$

brings (5) into the form

$$\frac{\partial}{\partial \tau} y(\mathbf{x}, \tau; \phi) = L\{y(\mathbf{x}, \tau; \phi)\} - \phi(\mathbf{x}, \tau) y(\mathbf{x}, \tau; \phi). \tag{7}$$

The function $y(\mathbf{x}, \tau; \phi)\, dV_x$ is obviously the probability that the target is not detected during time $(0, \tau)$ and the target is within the small volume dV_x which contains point $\mathbf{x}$.

Since $\int_R u(\mathbf{x}, \tau; \phi)\, dV_x = 1$, it follows from (4) and (6) that

$$P_\phi = \int_R y(\mathbf{x}, t; \phi)\, dV_x. \tag{8}$$

Further it follows from (6) that

$$\begin{aligned}\int_R &\phi(\mathbf{x}, \tau) y(\mathbf{x}, \tau; \phi)\, dV_x \\ &= \left(\int_R \phi(\mathbf{x}, \tau) u(\mathbf{x}, \tau; \phi)\, dV_x\right) \exp\left(-\int_0 d\tau_1 \int_R \phi(\boldsymbol{\xi}, \tau_1) u(\boldsymbol{\xi}, \tau_1; \phi)\, dV_\xi\right) \\ &= -\frac{d}{d\tau} \exp\left(-\int_0^\tau d\tau_1 \int_R \phi(\boldsymbol{\xi}, \tau_1) u(\boldsymbol{\xi}, \tau; \phi)\, dV_\xi\right),\end{aligned}$$

which yields

$$\begin{aligned}\int_0^\tau d\tau_1 \int_R \phi(\mathbf{x}, \tau_1) y(\mathbf{x}, \tau_1; \phi)\, dV_x &= 1 - \exp\left(-\int_0^\tau d\tau_1 \int_R \phi(\boldsymbol{\xi}, \tau_1) u(\boldsymbol{\xi}, \tau_1; \phi)\, dV_\xi\right) \\ &= 1 - P_\phi.\end{aligned} \tag{9}$$

Equations (8) and (9) together show that

$$P_\phi = 1 - \int_0^t d\tau \int_R \phi(\boldsymbol{\xi}, \tau) y(\boldsymbol{\xi}, \tau; \phi)\, dV_\xi = \int_R y(\mathbf{x}, t; \phi)\, dV_x. \tag{10}$$

Equations (6) and (10) now imply that

$$u(\mathbf{x}, \tau; \phi) = y(\mathbf{x}, \tau; \phi)\left[1 - \int_0^\tau d\tau_1 \int_R \phi(\boldsymbol{\xi}, \tau_1) y(\boldsymbol{\xi}, \tau_1; \phi)\, dV_\xi\right]^{-1}. \tag{11}$$

Furthermore,

$$\begin{aligned}&y(\mathbf{x}, 0; \phi) = u(\mathbf{x}, 0; \phi) = f(\mathbf{x}) \quad \text{and} \\ &y(\mathbf{x}, t; \phi) \equiv 0 \quad \text{for all } \mathbf{x} \notin R \quad \text{and} \quad t \in (0, \infty).\end{aligned} \tag{12}$$

The properties of the function $y(\mathbf{x}, \tau; \phi)$, which will be used later, are given in the following lemma.

LEMMA. *Let operator L of equation* (7) *have the properties given in connection with equation* (1) *and let $\phi(\mathbf{x}, \tau)$ be a nonnegative function defined for all $\mathbf{x} \in R$ and $\tau \in (0, t]$ such that $\int_R \phi(\mathbf{x}, \tau)\, dV_x = \Phi$, where Φ is a constant. Then equation* (7) *has a unique solution $y(\mathbf{x}, \tau; \phi)$. Furthermore, let $\varepsilon > 0$ and let the function $\psi(\mathbf{x}, \tau)$ be such that $\phi(\mathbf{x}, \tau) + \varepsilon\psi(\mathbf{x}, \tau) \geqq 0$ for all $\mathbf{x} \in R$ and $\tau \in (0, t]$, where, in addition, $\int_R \psi(\mathbf{x}, \tau)\, dV_x = 0$. Then the function $y(\mathbf{x}, \tau; \phi + \varepsilon\psi)$ may be expanded in an absolutely convergent power series of ε.*

Proof. Equation (7) and conditions (12) may be combined in the integral equation (cf. (1) and the properties of the operator L)

$$\begin{aligned}y(\mathbf{x}, \tau; \phi) = &\int_R v(\mathbf{x} - \boldsymbol{\xi}, \tau) f(\boldsymbol{\xi})\, dV_\xi \\ &- \int_0^\tau d\tau_1 \int_R v(\mathbf{x} - \boldsymbol{\xi}, \tau - \tau_1) \phi(\boldsymbol{\xi}, \tau_1) y(\boldsymbol{\xi}, \tau_1; \phi)\, dV_\xi.\end{aligned} \tag{13}$$

This integral equation may be solved by iteration in the familiar manner:

$$y(\mathbf{x}, \tau; \phi) = F(\mathbf{x}, \tau) + \int_0^\tau d\tau_1 \int_R Q(\mathbf{x}, \tau; \mathbf{s}, \tau_1) F(\mathbf{s}, \tau_1)\, dV_s, \tag{14}$$

where we write more briefly,

$$F(\mathbf{x}, \tau) = \int_R v(\mathbf{x} - \boldsymbol{\xi}, \tau) f(\boldsymbol{\xi})\, dV_\xi, \tag{15}$$

and where

$$Q(\mathbf{x}, \tau; \mathbf{s}, \tau_1) = \sum_{n=1}^{\infty} (-1)^n K_n(\mathbf{x}, \tau; \mathbf{s}, \tau_1). \tag{16}$$

Here

$$K_{n+1}(\mathbf{x}, \tau; \mathbf{s}, \tau_1) = \int_{\tau_1}^{\tau} d\tau_2 \int_R K(\mathbf{x}, \tau; \mathbf{s}_1, \tau_2) K_n(\mathbf{s}_1, \tau_2; \mathbf{s}, \tau_1)\, dV_{s_1} \tag{17}$$

with

$$K_1(\mathbf{x}, \tau; \mathbf{s}_1, \tau_1) = K(\mathbf{x}, \tau; \mathbf{s}_1, \tau_1) = v(\mathbf{x} - \mathbf{s}_1, \tau - \tau_1)\phi(\mathbf{s}_1, \tau_1). \tag{18}$$

The series (16) converges absolutely for all $\mathbf{x}$, $\mathbf{s} \in R$ and $\tau, \tau_1 \in (0, t)$. Indeed, it follows easily from (17) and (18) that

$$K_n(\mathbf{x}, \tau; \mathbf{s}, \tau_1) \leqq \Phi^{n-1}\phi(\mathbf{s}, \tau_1)[\mu(\tau - \tau_1)]^{*n}, \tag{19}$$

where

$$[\mu(\tau)]^{*2} = \int_0^\tau \mu(\tau - \tau_1)\mu(\tau_1)\, d\tau_1, \tag{20}$$

$$[\mu(\tau)]^{*(n+1)} = \int_0^\tau \mu(\tau - \tau_1)[\mu(\tau_1)]^{*n}\, d\tau_1, \tag{21}$$

and where

$$\mu(\tau) = \max_{\mathbf{x}} v(\mathbf{x}, \tau). \tag{22}$$

Therefore, as is immediate from (16), (19) and (14),

$$Q(\mathbf{x}, \tau; \mathbf{s}, \tau_1) \leqq \phi(\mathbf{s}, \tau_1) \sum_{n=1}^{\infty} \Phi^{n-1}\mu^{*n}(\tau - \tau_1) \tag{23}$$

and

$$y(\mathbf{x}, \tau; \phi) \leqq \mu(\tau) + \sum_{n=1}^{\infty} \Phi^n \mu^{*(n+1)}(\tau). \tag{24}$$

Let $\mathscr{L}\{\cdot\}$ denote the Laplace transformation and let

$$\bar{\mu}(s) = \mathscr{L}\{\mu(\tau)\}. \tag{25}$$

Then

$$\mathscr{L}\left\{\sum_{n=1}^{N} \Phi^n \mu^{*(n+1)}(\tau)\right\} = \Phi\bar{\mu}^2(s)\frac{1-\Phi^N\bar{\mu}^N(s)}{1-\Phi\bar{\mu}(s)}. \tag{26}$$

The series (26) converges, therefore, for all $\Phi < \infty$. The uniqueness of the solution (14) as well as the fact that the function $y(\mathbf{x}, \tau; \phi_0 + \varepsilon\psi)$ may be expanded in an absolutely convergent power series of ε are now immediate. It is clear from (17), (18) and (16) that $Q(\mathbf{x}, \tau; \mathbf{s}, \tau_1)$ satisfies the integral equation

$$\begin{aligned} Q(\mathbf{x}, \tau; \mathbf{s}, \tau_1) &= -v(\mathbf{x}-\mathbf{s}, \tau-\tau_1)\phi(\mathbf{s}, \tau) \\ &\quad - \int_{\tau_1}^{\tau} d\tau_2 \int_R v(\mathbf{x}-\mathbf{s}_1, \tau-\tau_2)\phi(\mathbf{s}_1, \tau_2)Q(\mathbf{s}_1, \tau_2; \mathbf{s}, \tau_1)\, dV_{s_1}. \end{aligned} \tag{27}$$

A special searching problem involving the equations of the present section is treated in [5].

3. Otimization of search. Let us now consider the following more general searching problem: Find a continuous function $\phi(\mathbf{x}, \tau) \geqq 0$, such that $\int_R \phi(\mathbf{x}, \tau) dV_x = \Phi$, defined for $\mathbf{x} \in R$ (cf. § 1) and for $\tau \in (0, t]$, such that

$$\int_R y(\mathbf{x}, t; \phi)\, dV_x = \min, \tag{28}$$

while

$$\frac{\partial}{\partial \tau} y(\mathbf{x}, \tau; \phi) = L\{y(\mathbf{x}, \tau; \phi)\} - \phi(\mathbf{x}, \tau) y(\mathbf{x}, \tau; \phi), \tag{29}$$

where $y(\mathbf{x}, \tau; \phi) \geqq 0$, $y(\mathbf{x}, 0; \phi) = f(\mathbf{x})$ and $y(\mathbf{x}, \tau; \phi) \equiv 0$ for $\mathbf{x} \notin R$. If the time available for the search is t, we obtain the following theorem concerning the optimal search density function $\phi_0(\mathbf{x}, \tau)$ defined for all $\mathbf{x} \in R$ and for all $\tau \in (0, t]$.

THEOREM. *Let $Q_0(\mathbf{x}, t; \mathbf{s}, \tau)$ be the solution of the integral equation*

$$\begin{aligned} Q_0(\mathbf{x}, t; \mathbf{s}, \tau) &= v(\mathbf{x}-\mathbf{s}, t-\tau)\phi_0(\mathbf{s}, \tau) \\ &\quad - \int_{\tau}^{t} d\tau_1 \int dV_{s_1} v(\mathbf{x}-\mathbf{s}_1, t-\tau_1)\phi_0(\mathbf{s}_1, \tau_1)Q_0(\mathbf{s}_1, \tau_1; \mathbf{s}, \tau), \end{aligned} \tag{30}$$

where $\phi_0(\mathbf{x}, \tau)$ is optimal and where the probability density function $v(\mathbf{x}, \tau)$ is the solution of $(\partial/\partial\tau)v(\mathbf{x}, \tau) = L\{v(\mathbf{x}, \tau)\}$ such that $v(\mathbf{x}, 0) = \delta(\mathbf{x})$. Let A be the set of points $(\mathbf{x}, \tau)$, $\mathbf{x} \in R$ and $\tau \in (0, t]$, such that $\phi_0(\mathbf{x}, \tau) > 0$, and let $\bar{A}$ denote the set of points $(\mathbf{x}, \tau)$, where $\phi_0(\mathbf{x}, \tau) \equiv 0$. Then

$$y(\mathbf{x}_1, \tau_1; \phi)\left[1 + \int_R dV_{\xi_1}\int_{\tau_1}^{t} d\tau \int_R dV_{\xi_2} Q_0(\xi_1, t; \xi_2, \tau) v(\xi_2 - \mathbf{x}_1, \tau - \tau_1)\right] \geqq \lambda, \tag{31}$$

where equality and inequality hold for $(\mathbf{x}_1, \tau) \in A$ and $(\mathbf{x}_1, \tau) \in \bar{A}$, respectively. Here λ is a constant which is to be chosen such that

$$\int_R \phi_0(\xi, \tau)\, dV_\xi = \Phi$$

and

$$y(\mathbf{x}_1, t_1; \phi) = F(\mathbf{x}_1, t_1) + \int_0^{t_1} d\tau \int_R dV_s Q_0(\mathbf{x}_1, t_1; \mathbf{s}, \tau) F(\mathbf{s}, \tau) \tag{32}$$

with

$$F(\mathbf{x}_1, t_1) = \int_R v(\mathbf{x}_1 - \boldsymbol{\xi}, t_1) f(\boldsymbol{\xi}) \, dV_\xi. \tag{33}$$

Proof. We shall apply the method of proof used by Koopman (of [4]). Let $\phi_0(\mathbf{x}, \tau)$ be the optimal search density function and let us define the set $S(\mathbf{x}, t; \delta, h)$ as follows: $(\boldsymbol{\xi}, \tau) \in S(\mathbf{x}, t; \delta, h)$ if $\|\boldsymbol{\xi} - \mathbf{x}\| < \delta$ and $|\tau - t| < h$; let points $(\mathbf{x}_1, t_1)$ and $(\mathbf{x}_2, t_1)$, $t_1 \in (0, t]$, be such that $S(\mathbf{x}_1, t_1; \delta, h) \cap S(\mathbf{x}_2, t_1; \delta, h) = 0$ and such that $\phi_0(\mathbf{x}, \tau) > 0$ for $(\mathbf{x}, \tau) \in S(\mathbf{x}_i, t_1; \delta, h)$, $i = 1, 2$. Furthermore, let us define function $\psi(\mathbf{x}, \tau)$ as follows:

(i) $\psi(\mathbf{x}, \tau)$ is continuous in $\mathbf{x}$ and τ;

(ii) $\psi(\mathbf{x}, \tau) \equiv 0$ for $(\mathbf{x}, \tau) \notin S(\mathbf{x}_1, t_1; \delta, h) \cup S(\mathbf{x}_2, t_1; \delta, h)$;

(iii) $\psi(\mathbf{x}, \tau) > 0$ for $(\mathbf{x}, \tau) \in S(\mathbf{x}_1, t_1; \delta, h)$ and $\psi(\mathbf{x}, \tau) < 0$ for $(\mathbf{x}, \tau) \in S(\mathbf{x}_2, t_1; \delta, h)$ so that $\int_{s_1(\tau)} \psi(\mathbf{x}, \tau) \, dV_x = -\int_{s_2(\tau)} \psi(\mathbf{x}, \tau) \, dV_x$.

Here we write more briefly $s_i(\tau) = \{\mathbf{x} : (\mathbf{x}, \tau) \in S(\mathbf{x}_i, t_1; \delta, h)\}$. Then, for any constant ε and for all $\tau \in (0, t]$, $\int_R [\phi_0(\mathbf{x}, \tau) + \varepsilon\psi(\mathbf{x}, \tau)] \, dV_x = \int_R \phi_0(\mathbf{x}, \tau) \, dV_x = \Phi$ and, for a sufficiently small $\varepsilon > 0$, $\phi_0(\mathbf{x}, \tau) + \varepsilon\psi(\mathbf{x}, \tau) \geqq 0$ for all $\mathbf{x} \in R$ and $\tau \in (0, t]$. The density $y(\mathbf{x}, t; \phi_0 + \varepsilon\psi)$ is now defined and we obviously have that

$$\int_R y(\mathbf{x}, t; \phi_0) \, dV_x \leqq \int_R y(\mathbf{x}, t; \phi_0 + \varepsilon\psi) \, dV_x.$$

Due to the lemma of § 2, $y(\mathbf{x}, t; \phi_0 + \varepsilon\psi)$ possesses an absolutely convergent power series expansion in ε:

$$y(\mathbf{x}, t; \phi_0 + \varepsilon\psi) = y_0(\mathbf{x}, t; \phi_0) + \varepsilon y_1(\mathbf{x}, t; \phi_0) + \cdots.$$

Consequently,

$$0 \leqq \varepsilon \int_R y_1(\mathbf{x}, t; \phi_0) \, dV + \cdots.$$

Since $\varepsilon > 0$, it is necessary that

$$\int_R y_1(\mathbf{x}, t; \phi_0) \, dV_x \geqq 0. \tag{34}$$

Now we obtain from (13) the following equation for $y_1(\mathbf{x}, t; \phi_0)$:

$$\begin{aligned} y_1(\mathbf{x}, t; \phi_0) = \int_0^t d\tau \int_R v(\mathbf{x} - \boldsymbol{\xi}, t - \tau)[\psi(\boldsymbol{\xi}, \tau) y_0(\boldsymbol{\xi}, \tau; \phi_0) \\ + \phi_0(\boldsymbol{\xi}, \tau) y_1(\boldsymbol{\xi}, \tau; \phi_0)] \, dV_\xi. \end{aligned} \tag{35}$$

Since

$$\begin{aligned} &\int_0^t d\tau \int_R v(\mathbf{x} - \boldsymbol{\xi}, t - \tau)\psi(\boldsymbol{\xi}, \tau) y_0(\boldsymbol{\xi}, \tau; \phi_0) \, dV_\xi \\ &= \sigma[v(\mathbf{x} - \mathbf{x}_{10}, t - t_{10}) y_0(\mathbf{x}_{10}, t_{10}, \phi_0) - v(\mathbf{x} - \mathbf{x}_{20}, t - t_{20}) y_0(\mathbf{x}_{20}, t_{20}, \phi_0)], \end{aligned} \tag{36}$$

where we write more briefly

$$\sigma = \int_0^t d\tau \int_{s_1(\tau)} \psi(\xi, \tau)\, dV_\xi \tag{37}$$

and where $(\mathbf{x}_{i0}, t_{i0}) \in S(\mathbf{x}_i, t_1; \delta, h)$, $i = 1, 2$. It follows from (35) that (cf. (13) and (14))

$$\begin{aligned} y_1(x, t; \phi_0) = {} & \sigma y_0(\mathbf{x}_{10}, t_{10}; \phi_0)\Big[v(\mathbf{x} - \mathbf{x}_{10}, t - t_{10}) \\ & + \int_{t_{10}}^t d\tau \int_R Q_0(\mathbf{x}, t; \mathbf{s}, \tau) v(\mathbf{s} - \mathbf{x}_{10}, \tau - t_{10})\, dV_s \Big] \\ & - \sigma y_0(\mathbf{x}_{20}, t_{20}; \phi_0)\Big[v(\mathbf{x} - \mathbf{x}_{20}, t - t_{20}) \\ & + \int_{t_{20}}^t d\tau \int_R Q_0(\mathbf{x}, t; \mathbf{s}, \tau) v(\mathbf{s} - \mathbf{x}_{20}, \tau - t_{20})\, dV_s \Big], \end{aligned} \tag{38}$$

where $Q_0(\mathbf{x}, t; \mathbf{s}, \tau)$ belongs to the optimal $\phi_0(\mathbf{x}, t)$. From (34) and (35) it follows that

$$\begin{aligned} & y_0(\mathbf{x}_{10}, t_{10}; \phi_0)\Big[1 + \int_R dV_x \int_{t_{10}}^t d\tau \int Q_0(\mathbf{x}, t; \mathbf{s}, \tau) v(\mathbf{s} - \mathbf{x}_{10}, \tau - t_{10})\, dV_s \Big] \\ & \geqq y_0(\mathbf{x}_{20}, t_{20}; \phi_0)\Big[1 + \int_R dV_x \int_{t_{20}}^t d\tau \int Q_0(\mathbf{x}, t; \mathbf{s}, \tau) v(\mathbf{s} - \mathbf{x}_{20}, \tau - t_{20})\, dV_s \Big], \end{aligned} \tag{39}$$

where use is made of the fact that $\int_R dV_x v(\mathbf{x} - \mathbf{x}_{i0}, t - t_{i0}) = 1$, $i = 1, 2$. Since points $(\mathbf{x}_1, t_1)$ and $(\mathbf{x}_2, t_1)$ may be interchanged, we obtain from (39), in the limit as $\delta \to 0$ and $h \to 0$, that for $(\mathbf{x}_1, t_1) \in A$,

$$y(\mathbf{x}_1, t_1; \phi_0)\Big[1 + \int_R dV_\xi \int_{t_1}^t d\tau \int_R dV_s Q_0(\xi, t; \mathbf{s}, \tau) v(\mathbf{s} - \mathbf{x}_1, \tau - t_1) \Big] = \lambda,$$

where λ is a constant. Now let $(\mathbf{x}_1, t_1) \in \bar{A}$ but $(\mathbf{x}_2, t_1) \in A$. It then follows from (39), in the limit as $\delta \to 0$ and $h \to 0$, that for $(\mathbf{x}_1, t_1) \in \bar{A}$,

$$y(\mathbf{x}_1, t_1; \phi_0)\Big[1 + \int_R dV_\xi \int_{t_1}^t d\tau \int_R dV_s Q_0(\xi, t; \mathbf{s}, \tau) v(\mathbf{s} - \mathbf{x}_1, \tau - t_1) \Big] \geqq \lambda.$$

This completes the proof.

It is obvious that the application of the theorem for finding an optimal $\phi_0(\mathbf{x}, t)$ is difficult even in simple cases such as that of equation (2). A special numerical algorithm needs to be derived, a problem which we shall not discuss in the present paper. We shall only demonstrate that the optimal distribution of search of Koopman [4] for a stationary target results from the theorem. Now $F(\mathbf{x}, t) = f(\mathbf{x})$ for all $t \in (0, \infty)$. Therefore, $v(\mathbf{x}, t) = \delta(\mathbf{x})$. Equation (30) becomes

$$Q_0(\mathbf{x}, t; \mathbf{s}, \tau) = -\delta(\mathbf{x} - \mathbf{s})\phi_0(\mathbf{s}, \tau) - \int_\tau^t d\tau_1 \phi_0(\mathbf{x}, \tau_1) Q_0(\mathbf{x}, \tau_1; \mathbf{s}, \tau), \tag{40}$$

which is easily solved to yield

$$Q(\mathbf{x}, t; \mathbf{s}, \tau) = -\delta(\mathbf{x} - \mathbf{s})\phi_0(\mathbf{s}, \tau) \exp\left(-\int_\tau^t \phi_0(\mathbf{x}, \tau_1)\, d\tau_1\right). \tag{41}$$

Consequently,

$$y_0(\mathbf{x}_1, \tau_1; \phi) = f(\mathbf{x}_1) \exp\left(-\int_0^{\tau_1} \phi_0(\mathbf{x}, \tau)\, d\tau\right) \tag{42}$$

and

$$\int_R dV_\xi \int_{\tau_1}^t d\tau \int_R dV_s Q(\boldsymbol{\xi}, t; \mathbf{s}, \tau) v(\mathbf{s} - \mathbf{x}_1, \tau - \tau_1) = -1 + \exp\left(-\int_{\tau_1}^t \phi_0(\mathbf{x}_1, \tau)\, d\tau\right). \tag{43}$$

We now obtain from expression (31),

$$f(\mathbf{x}_1) \exp\left(-\int_0^t \phi_0(\mathbf{x}_1, \tau)\, d\tau\right) = \lambda \quad \text{for } \mathbf{x}_1 \text{ such that } \int_0^t \phi_0(\mathbf{x}_1, \tau)\, d\tau > 0 \tag{44}$$

and

$$f(\mathbf{x}_1) \exp\left(-\int_0^\tau \phi_0(\mathbf{x}_1, \tau)\, d\tau\right) \geqq \lambda \quad \text{for } \mathbf{x}_1 \text{ such that } \int_0^t \phi_0(\mathbf{x}_1, \tau)\, d\tau = 0. \tag{45}$$

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