Jacobian Matrix Part 1

$$H_j = \begin{bmatrix} \frac{\partial h_1}{\partial x_1} & \frac{\partial h_1}{\partial x_2} & \cdots & \frac{\partial h_1}{\partial x_n} \\ \frac{\partial h_2}{\partial x_1} & \frac{\partial h_2}{\partial x_2} & \cdots & \frac{\partial h_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial h_m}{\partial x_1} & \frac{\partial h_m}{\partial x_2} & \cdots & \frac{\partial h_m}{\partial x_n} \end{bmatrix}$$

$$h(x) \approx h(\mu) + \underbrace{\frac{\partial h(\mu)}{\partial x}}_{\text{Jacobian}}(x - \mu)$$

$$H_j = \begin{bmatrix} \frac{\partial h_1}{\partial x_1} & \frac{\partial h_1}{\partial x_2} & \cdots & \frac{\partial h_1}{\partial x_n} \\ \frac{\partial h_2}{\partial x_1} & \frac{\partial h_2}{\partial x_2} & \cdots & \frac{\partial h_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial h_m}{\partial x_1} & \frac{\partial h_m}{\partial x_2} & \cdots & \frac{\partial h_m}{\partial x_n} \end{bmatrix}$$

$$H_j = \begin{bmatrix} \frac{\partial \rho}{\partial p_x} & \frac{\partial \rho}{\partial p_y} & \frac{\partial \rho}{\partial v_x} & \frac{\partial \rho}{\partial v_y} \\ \frac{\partial \varphi}{\partial p_x} & \frac{\partial \varphi}{\partial p_y} & \frac{\partial \varphi}{\partial v_x} & \frac{\partial \varphi}{\partial v_y} \\ \frac{\partial \dot{\rho}}{\partial p_x} & \frac{\partial \dot{\rho}}{\partial p_y} & \frac{\partial \dot{\rho}}{\partial v_x} & \frac{\partial \dot{\rho}}{\partial v_y} \end{bmatrix}$$

$$z = \begin{pmatrix} \rho \\ \varphi \\ \dot{\rho} \end{pmatrix} \begin{matrix} \leftarrow \text{Range} \\ \leftarrow \text{Bearing} \\ \leftarrow \text{Range rate} \end{matrix}$$

$$x = \begin{pmatrix} p_x \\ p_y \\ v_x \\ v_y \end{pmatrix} \begin{matrix} \text{Position} \\ \\ \text{Velocity} \end{matrix}$$

$$H_j = \begin{bmatrix} \frac{\partial h_1}{\partial x_1} & \frac{\partial h_1}{\partial x_2} & \cdots & \frac{\partial h_1}{\partial x_n} \\ \frac{\partial h_2}{\partial x_1} & \frac{\partial h_2}{\partial x_2} & \cdots & \frac{\partial h_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial h_m}{\partial x_1} & \frac{\partial h_m}{\partial x_2} & \cdots & \frac{\partial h_m}{\partial x_n} \end{bmatrix} \quad H_j = \begin{bmatrix} \frac{\partial \rho}{\partial p_x} & \frac{\partial \rho}{\partial p_y} & \frac{\partial \rho}{\partial v_x} & \frac{\partial \rho}{\partial v_y} \\ \frac{\partial \varphi}{\partial p_x} & \frac{\partial \varphi}{\partial p_y} & \frac{\partial \varphi}{\partial v_x} & \frac{\partial \varphi}{\partial v_y} \\ \frac{\partial \dot{\rho}}{\partial p_x} & \frac{\partial \dot{\rho}}{\partial p_y} & \frac{\partial \dot{\rho}}{\partial v_x} & \frac{\partial \dot{\rho}}{\partial v_y} \end{bmatrix}$$

Jacobian

$$H_j = \begin{bmatrix} \frac{p_x}{\sqrt{p_x^2+p_y^2}} & \frac{p_y}{\sqrt{p_x^2+p_y^2}} & 0 & 0 \\ -\frac{p_y}{p_x^2+p_y^2} & \frac{p_x}{p_x^2+p_y^2} & 0 & 0 \\ \frac{p_y(v_x p_y - v_y p_x)}{(p_x^2+p_y^2)^{3/2}} & \frac{p_x(v_y p_x - v_x p_y)}{(p_x^2+p_y^2)^{3/2}} & \frac{p_x}{\sqrt{p_x^2+p_y^2}} & \frac{p_y}{\sqrt{p_x^2+p_y^2}} \end{bmatrix}$$

We're going to calculate, step by step, all the partial derivatives in H_j:

$$H_j = \begin{bmatrix} \frac{\partial \rho}{\partial p_x} & \frac{\partial \rho}{\partial p_y} & \frac{\partial \rho}{\partial v_x} & \frac{\partial \rho}{\partial v_y} \\ \frac{\partial \varphi}{\partial p_x} & \frac{\partial \varphi}{\partial p_y} & \frac{\partial \varphi}{\partial v_x} & \frac{\partial \varphi}{\partial v_y} \\ \frac{\partial \dot{\rho}}{\partial p_x} & \frac{\partial \dot{\rho}}{\partial p_y} & \frac{\partial \dot{\rho}}{\partial v_x} & \frac{\partial \dot{\rho}}{\partial v_y} \end{bmatrix}$$

So all of H_j's elements are calculated as follows:

$$\frac{\partial \rho}{\partial p_x} = \frac{\partial}{\partial p_x}(\sqrt{p_x^2 + p_y^2}) = \frac{2p_x}{2\sqrt{p_x^2+p_y^2}} = \frac{p_x}{\sqrt{p_x^2+p_y^2}}$$

$$\frac{\partial \rho}{\partial p_y} = \frac{\partial}{\partial p_y}(\sqrt{p_x^2 + p_y^2}) = \frac{2p_y}{2\sqrt{p_x^2+p_y^2}} = \frac{p_y}{\sqrt{p_x^2+p_y^2}}$$

$$\frac{\partial \rho}{\partial v_x} = \frac{\partial}{\partial v_x}(\sqrt{p_x^2 + p_y^2}) = 0$$

$$\frac{\partial \rho}{\partial v_y} = \frac{\partial}{\partial v_y}(\sqrt{p_x^2 + p_y^2}) = 0$$

$$\frac{\partial \varphi}{\partial p_x} = \frac{\partial}{\partial p_x} arctan(p_y/p_x) = \frac{1}{(\frac{p_y}{p_x})^2+1}(-\frac{p_y}{p_x^2}) = -\frac{p_y}{p_x^2+p_y^2}$$

$$\frac{\partial \varphi}{\partial p_y} = \frac{\partial}{\partial p_y} arctan(p_y/p_x) = \frac{1}{(\frac{p_y}{p_x})^2+1}(\frac{1}{p_x}) = \frac{p_x^2}{p_x^2+p_y^2}\frac{1}{p_x} = \frac{p_x}{p_x^2+p_y^2}$$

$$\frac{\partial \varphi}{\partial v_x} = \frac{\partial}{\partial v_x} arctan(p_y/p_x) = 0$$

$$\frac{\partial \varphi}{\partial v_y} = \frac{\partial}{\partial v_y} arctan(p_y/p_x) = 0$$

$$\frac{\partial \dot{\rho}}{\partial p_x} = \frac{\partial}{\partial p_x}\left(\frac{p_x v_x + p_y v_y}{\sqrt{p_x^2+p_y^2}}\right)$$

In order to calculate the derivative of this function we use the quotient rule.

Given a function z that is quotient of two other functions, f and g:

$z = \frac{f}{g}$

its derivative with respect to \$x\$ is defined as: $\frac{\partial z}{\partial x} = \frac{\frac{\partial f}{\partial x}g - \frac{\partial g}{\partial x}f}{g^2}$

in our case: $f = p_x v_x + p_y v_y$

$g = \sqrt{p_x^2 + p_y^2}$

Their derivatives are: $\frac{\partial f}{\partial p_x} = \frac{\partial}{\partial p_x}(p_x v_x + p_y v_y) = v_x$

$\frac{\partial g}{\partial p_x} = \frac{\partial}{\partial p_x}\left(\sqrt{p_x^2 + p_y^2}\right) = \frac{p_x}{\sqrt{p_x^2+p_y^2}}$

Putting everything together into the derivative quotient rule we have:

$$\frac{\partial \dot{\rho}}{\partial p_x} = \frac{v_x\sqrt{p_x^2+p_y^2} - \frac{p_x}{\sqrt{p_x^2+p_y^2}}(p_x v_x + p_y v_y)}{p_x^2+p_y^2} = \frac{p_y(v_x p_y - v_y p_x)}{(p_x^2+p_y^2)^{3/2}}$$

Similarly $\frac{\partial \dot{\rho}}{\partial p_y} = \frac{\partial}{\partial p_y}\left(\frac{p_x v_x + p_y v_y}{\sqrt{p_x^2+p_y^2}}\right) = \frac{p_x(v_y p_x - v_x p_y)}{(p_x^2+p_y^2)^{3/2}}$

$$\frac{\partial \dot{\rho}}{\partial v_x} = \frac{\partial}{\partial v_x}\left(\frac{p_x v_x + p_y v_y}{\sqrt{p_x^2+p_y^2}}\right) = \frac{p_x}{\sqrt{p_x^2+p_y^2}}$$

$$\frac{\partial \dot{\rho}}{\partial v_y} = \frac{\partial}{\partial v_y}\left(\frac{p_x v_x + p_y v_y}{\sqrt{p_x^2+p_y^2}}\right) = \frac{p_y}{\sqrt{p_x^2+p_y^2}}$$

So now, after calculating all the partial derivatives, our resulted Jacobian, H_j is:

$$H_j = \begin{bmatrix} \frac{p_x}{\sqrt{p_x^2+p_y^2}} & \frac{p_y}{\sqrt{p_x^2+p_y^2}} & 0 & 0 \\ -\frac{p_y}{p_x^2+p_y^2} & \frac{p_x}{p_x^2+p_y^2} & 0 & 0 \\ \frac{p_y(v_x p_y - v_y p_x)}{(p_x^2+p_y^2)^{3/2}} & \frac{p_x(v_y p_x - v_x p_y)}{(p_x^2+p_y^2)^{3/2}} & \frac{p_x}{\sqrt{p_x^2+p_y^2}} & \frac{p_y}{\sqrt{p_x^2+p_y^2}} \end{bmatrix}$$