DESIGN PROJECT: EXTREMA OF FUNCTIONS

MTE 203 – ADVANCED CALCULUS

Due Date: Monday, November 12, 2018

Project Description:

On topographic maps of mountainous regions, the curves that represent constant elevation (height above sea level) are level curves (usually referred as contours) of complicated functions that represent the terrain. Modeling the variation in temperature at the surface requires the solution of constrained extreme value problems. Extreme value problems with algebraic constraints on the variables can be solved using the method of Lagrange Multipliers which usually results on a system of non-linear equations. The use of mathematical software such as MATLAB® makes it possible to graph complex functions and to also solve non-linear equations with only a few keystrokes which is helpful for the analysis of complex multivariable problems.

In the following project you will use MATLAB® to help you solve a constrained extreme value problem.

Project Submission:

Reports should reflect your individual work!!!

Present your analysis and results in a clear and concise report. The report and calculation process can be typed or handwritten. However, presentation should be neat and professional.

Please, save your report and all associated *.m files as a *.zip file and post it in the dropbox created in LEARN for that purpose. If all your project files are not zipped in a single *.zip file, 5% of your mark will be deducted.

The report should include the following sections:

- **Cover Page**: The cover page should include the name and course number, the date of submission, your name and your ID. Please use the template posted in UW-Learn (MTE203 – Project 1 Cover Page Template.docx)
- **Section 1:** Summary of Analysis
  - Detailed explanation of all the techniques used

- Explanation of all your results
- Required plots clearly including axis names, labels and legends
- Problems encountered during the analysis of the project

- **Section 2:** Summary of Results
    - A summary of the results need to be tabulated appropriately. Use as many tables as you consider relevant and make sure to show all the required answers. See an example of a table that can be used to report all critical points at the end of this project description.
    - If you decide to type your project, you can attach all hand calculations as an appendix. If you do so, when reporting, clearly indicate the section of the appendix you are referring to.
- **Section 3**: Appendices
    - Reproduce your Matlab codes as appendices
    - Attach all hand calculations as appendices with appropriate numbering for referencing

Notes on Appendices:

- Indicate clearly the section to which each appendix belongs.
- Appendices can be typed or handwritten. If handwritten, they should be neat and clear. If we do not understand your hand writing, that specific section will not be graded

Project Details:

Functions representing terrains are usually very complex but can easily provide important information such as the volume of the surface and the degree and orientation of the slopes. They are also a very flexible way to lay the groundwork for other site modeling enterprises. The terrain studied here can be modeled by the following function

$$z = f(x,y) = \ln(x^4+1)\,(4x^4+(2y)^2)\,e^{(-0.5x^2-(y-0.8)^2-3)}\,(sin(2x+0.05y^4)+2\,\cos(0.75y))$$

Where x, y and z are in kilometers and the domain of the function is defined by the open region[1],

$$R: \{-5 < x < 5, -2 < y < 4\}$$

In your analysis, consider that the xy-plane represents the sea level and that the east and north directions are represented by the direction of the positive x and positive y axes respectively (see Figure 1).

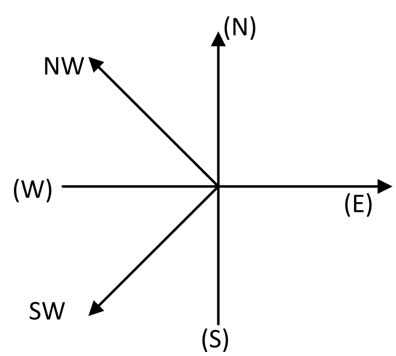

Figure 1. Directions of the Terrain

[1] The region does not include any points on the edge of the rectangle. In other words, we aren't allowing the region to include its boundary and so it's open.

The temperature distribution within the limits of the terrain is a function of the height and it is given by

$$T(x,y,z) = -2 * z^2 - \exp(-0.1 * ((0.1 * x - 2) - (0.05 * y - 3)^2 - (z-1)^2)) + 10 \quad [°C]$$

Where x, y and z are in kilometers and T is in degrees Celsius.

Part I:

In this part of the problem you will be required to plot the terrain and its contour map (level curves at heights above sea level), and find the location of highest and lowest elevation of the terrain.

a. Use MATLAB® to plot the terrain and its contours, showing a reasonable number (25-30) of different elevations.
b. By observation of the contour lines, determine where in the terrain the slope is the steepest. Justify your answer and verify your finding mathematically
c. Calculate the highest and lowest elevation of the terrain using the first and second partial derivative tests. List and classify all the critical points you have found in a table (see template for the table at the end of this document)
   Hint: You might want to check for a non-obvious critical point around the south-west and south-east vicinities of the terrain

Part II:

In this part of the project you are required to calculate the temperature at different locations of the terrain.

a. Calculate the temperatures at the highest and lowest elevation of the terrain
b. Suppose that a hiker is standing at the point $(2.2, 0.5)$ on the terrain.
   i. Calculate the temperature that the hiker feels at this point
   ii. Use MATLAB® to draw a reasonable number of isotherms at the hiker's elevation

c. Suppose that the hiker decides to walk in the north direction (see note 1 below and Figure 1)
   i. Using partial differentiation determine if the hiker would be ascending or descending
   ii. Calculate the rate of change in temperature that the hiker would experiences as she walks in this direction.
d. Suppose now that the hiker decides to walk in the southwest direction (see note 2 below)
   i. Using partial differentiation determine if the hiker would be ascending or descending
   ii. Calculate the rate of change in temperature that the hiker experiences as he walks in this direction.
e. Using MATLAB®, plot again the terrain $f(x, y)$ but this time, use the temperature function as the color map for the terrain. Explain how this plot is useful to the hiker.
f. Note that $T(x, y, z)$ can be written entirely in terms of x and y. On a separate graph, plot $T(x, y)$ simply as a function of x and y. This means that T is shown on the vertical axis. Explain how this plot is useful to the hiker.
g. Using Lagrange Multipliers, determine the location of the lowest temperature on the surface of the island.

Note:

1. To determine the direction of the hiker as he/she ascends/descends in the northwest/southwest direction, use a vector that defines such direction in the xy-plane (i.e. a 2D vector)

2. To calculate the rate of change of temperature that the hiker experiences in the northwest/southwest direction, use a vector that defines such direction in the tangent plane to the surface at the hiker's initial position $(2.2, 0.5)$.

Example of a table listing critical points:

List of critical point(s) of the terrain

Point #	(x,y)	f(x,y)	A	B	C	D = B^2-AC	Type of point[2]
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

Note:

1. A, B, and C are the Hessian parameters and they are given by:

$$A = \frac{\partial^2 f}{\partial x^2}; B = \frac{\partial^2 f}{\partial x \partial y}; C = \frac{\partial^2 f}{\partial y^2}$$

2. List all the points you have found and clearly distinguish which points are acceptable and which are not. Explain your reasoning.

[2] Indicate clearly whether the critical point corresponds to a relative maximum, minimum, saddle or fails the second derivative test