The following shows the systematic application of the Todd-Coxeter algorithm to

$$< A, B | A^4, B^2, ABA^2B >$$

After defining $2 = 1A$ we have the following link table and list of chains. Notice that instead of erasing the bits I know, I have inserted numbers in the chains for the elements corresponding to the links. Notice, too that, near the end of the first chain in the second row, I have been able to insert a 1 since I know the link $1A2$.

	A	B
1	**2**	
2		

$1A2A\ \ A\ \ A1$	$1B\ \ B1$	$1A2B\ \ A\ \ A\ \ B1$
$2A\ \ A\ \ A1A2$	$2B\ \ B2$	$2A\ \ B\ \ A\ \ A\ \ B2$

The definition of the new element is shown in **bold** and the new information in the chain table is shown in red.

Now, defining $3 = 1B$, we get

	A	B
1	**2**	**3**
2		
3		1

$1A2A\ \ A\ \ A1$	$1B3B1$	$1A2B\ \ A\ \ A3B1$
$2A\ \ A\ \ A1A2$	$2B\ \ B2$	$2A\ \ B\ \ A\ \ A\ \ B2$
$3A\ \ A\ \ A\ \ A3$	$3B1B3$	$3A\ \ B\ \ A\ \ A1B3$

Notice that we have discovered the link $3B1$, and that the first and last chains in the middle column have been completely filled in. These completed chains are shown in green.

Next, I choose to define $4 = 2A$, and we get

	A	B
1	**2**	**3**
2	**4**	
3		1
4		

$1A2A4A\ \ A1$	$1B3B1$	$1A2B\ \ A\ \ A3B1$
$2A4A\ \ A1A2$	$2B\ \ B2$	$2A4B\ \ A\ \ A\ \ B2$
$3A\ \ A\ \ A\ \ A3$	$3B1B3$	$3A\ \ B\ \ A\ \ A1B3$
$4A\ \ A1A2A4$	$4B\ \ B4$	$4A\ \ B\ \ A\ \ A\ \ B4$

After this, I choose to define $5 = 3B$ and obtain

	A	B
1	**2**	**3**
2	**4**	**5**
3		1
4		
5		2

$1A2A4A\ \ A1$	$1B3B1$	$1A2B5A\ \ A3B1$
$2A4A\ \ A1A2$	$2B5B2$	$2A4B\ \ A\ \ A5B2$
$3A\ \ A\ \ A\ \ A3$	$3B1B3$	$3A\ \ B\ \ A\ \ A1B3$
$4A\ \ A1A2A4$	$4B\ \ B4$	$4A\ \ B\ \ A\ \ A\ \ B4$
$5A\ \ A\ \ A\ \ A5$	$5B2B5$	$5A\ \ B\ \ A1A2B5$

which completes two more chains in the middle column, and shows us that $5B = 2$.

Now, I choose to define $6 = 3A$, and this gives

	A	B
1	**2**	**3**
2	**4**	**5**
3	**6**	1
4		
5		2
6		

$1A2A4A\ \ A1$	$1B3B1$	$1A2B5A\ \ A3B1$
$2A4A\ \ A1A2$	$2B5B2$	$2A4B\ \ A\ \ A5B2$
$3A6A\ \ A\ \ A3$	$3B1B3$	$3A6B\ \ A\ \ A1B3$
$4A\ \ A1A2A4$	$4B\ \ B4$	$4A\ \ B\ \ A\ \ A\ \ B4$
$5A\ \ A\ \ A\ \ A5$	$5B2B5$	$5A\ \ B\ \ A1A2B5$
$6A\ \ A\ \ A3A6$	$6B\ \ B6$	$6A\ \ B\ \ A\ \ A\ \ B6$

Notice that we have made more progress than you might think towards completing the chains. For example, the missing fragment in the first chain in the left column is $4A\ \ A1$, which is exactly the same as the missing fragment in the second chain and fourth chain in that column. Similarly,$6A\ \ A\ \ A3$ occurs twice. So while there are 14 incomplete chains, there are, in fact, only 11 incomplete chain fragments.

The next step I choose to make is to define $7 = 4A$ which yields some dramatic information. First I obtain

	A	B
1	**2**	**3**
2	**4**	**5**
3	**6**	1
4	**7**	
5		2
6		4
7	1	

$1A2A4A7A1$	$1B3B1$	$1A2B5A\ \ A3B1$
$2A4A7A1A2$	$2B5B2$	$2A4B\ \ A\ \ A5B2$
$3A6A\ \ A\ \ A3$	$3B1B3$	$3A6B4A7A1B3$
$4A7A1A2A4$	$4B\ \ B4$	$4A\ \ B\ \ A\ \ A\ \ B4$
$5A\ \ A\ \ A\ \ A5$	$5B2B5$	$5A\ \ B\ \ A1A2B5$
$6A\ \ A\ \ A3A6$	$6B\ \ B6$	$6A\ \ B\ \ A\ \ A\ \ B6$
$7A1A2A4A7$	$7B\ \ B7$	$7A\ \ B\ \ A\ \ A\ \ B7$

where the third chain in the third column has shown me that $6B = 4$. Continuing to fill in the chains using the known information we discover

	A	B
1	**2**	**3**
2	**4**	**5**
3	**6**	1
4	**7**	6
5		2
6		4
7	1	

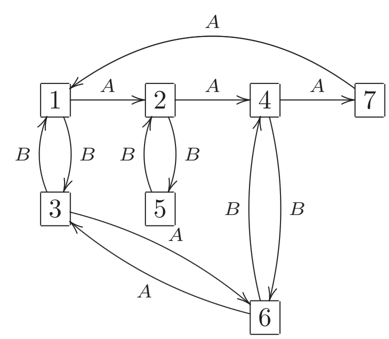
$1A2A4A7A1$	$1B3B1$	$1A2B5A\ \ A3B1$
$2A4A7A1A2$	$2B5B2$	$2A4B\ \ A\ \ A5B2$
$3A6A\ \ A\ \ A3$	$3B1B3$	$3A6B4A7A1B3$
$4A7A1A2A4$	$4B\ \ B4$	$4A\ \ B\ \ A\ \ A\ \ B4$
$5A\ \ A\ \ A\ \ A5$	$5B2B5$	$5A\ \ B\ \ A1A2B5$
$6A\ \ A\ \ A3A6$	$6B4B6$	$6A\ \ B\ \ A\ \ A\ \ B6$
$7A1A2A4A7$	$7B\ \ B7$	$7A\ \ B\ \ A\ \ A\ \ B7$

where the information $4B = 6$ has come from the second last chain in the middle column. Progessing still further, we get

	A	B
1	**2**	**3**
2	**4**	**5**
3	**6**	1
4	**7**	6
5		2
6	3	4
7	1	

$1A2A4A7A1$	$1B3B1$	$1A2B5A\ \ A3B1$
$2A4A7A1A2$	$2B5B2$	$2A4B\ \ A\ \ A5B2$
$3A6A\ \ A\ \ A3$	$3B1B3$	$3A6B4A7A1B3$
$4A7A1A2A4$	$4B6B4$	$4A7B\ \ A3A6B4$
$5A\ \ A\ \ A\ \ A5$	$5B2B5$	$5A\ \ B7A1A2B5$
$6A\ \ A\ \ A3A6$	$6B4B6$	$6A3B1A2A4B6$
$7A1A2A4A7$	$7B\ \ B7$	$7A\ \ B\ \ A\ \ A\ \ B7$

where we have now discovered that $6A = 3$ from the second last chain in the last column. By the way, at this stage, the link table is showing that the graph of the group looks like



But now something interesting happens. A bit more work shows

	A	B
1	**2**	**3**
2	**4**	**5**
3	**6** $=(5)$	1
4	**7**	6
5		2
6	3	4
7	1	

$1A2A4A7A1$	$1B3B1$	$1A2B5A\ \ A3B1$
$2A4A7A1A2$	$2B5B2$	$2A4B6A3A5B2$
$3A6A3A6A3$	$3B1B3$	$3A6B4A7A1B3$
$4A7A1A2A4$	$4B6B4$	$4A7B\ \ A3A6B4$
$5A\ \ A\ \ A\ \ A5$	$5B2B5$	$5A\ \ B7A1A2B5$
$6A\ \ A\ \ A3A6$	$6B4B6$	$6A3B1A2A4B6$
$7A1A2A4A7$	$7B\ \ B7$	$7A\ \ B\ \ A\ \ A\ \ B7$

where the second link in the last column has told us that $3A = 5$. Earlier, we had given $3A$ the name 6, but we have now discovered that 5 and 6 are both names for the same element. This has further consequences. For instance, it means that $5A = 6A = 3$, but it also means that $2 = 5B = 6B = 4$, and so 2 and 4 are the same element.

	A	B
1	**2**	**3**
2	**4**	**5**
3	**6** $=(5)$	1
4	**7**	5
5	3	2 $=(4)$
7	1	

$1A2A2A7A1$	$1B3B1$	$1A2B5A\ \ A3B1$
$2A2A7A1A2$	$2B5B2$	$2A2B5A3A5B2$
$3A5A3A5A3$	$3B1B3$	$3A5B2A7A1B3$
$2A7A1A2A2$	$2B5B2$	$2A7B\ \ A3A5B2$
$5A\ \ A\ \ A\ \ A5$	$5B2B5$	$5A\ \ B7A1A2B5$
$5A\ \ A\ \ A3A5$	$5B2B5$	$5A3B1A2A2B5$
$7A1A2A2A7$	$7B\ \ B7$	$7A\ \ B\ \ A\ \ A\ \ B7$

The identification of elements continues. We see below that $2A = 2 = 7$, from which we rapidly deduce that $2 = 7 = 1$ and hence that that $3 = 1A = 2A = 5$.

	A	B
1	**2**	**3**
2	2	**5**
3	5	1
2	7	5
5	3	2
7	1	

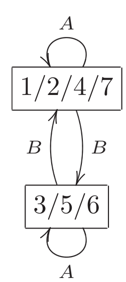
	A	B
1	1	**3**
1	1	3
3	3	1
1	1	3
3	3	
1	1	

so our link table and list of chains finally ends up being completed as

	A	B
1	1	**3**
3	3	1

$1A1A1A1A1$	$1B3B1$	$1A1B3A3A3B1$
$3A3A3A3A3$	$3B1B3$	$3A3B1A1A1B3$

so our group is the cyclic group of order 2. Note that our earlier graph wasn't entirely wrong. It is just that we later discovered that 1, 2, 4 and 7 are all the same element, and that 3, 5 and 6 are all the same element. Collapsing these onto one another, the graph becomes



Now that you have read this, I suggest that you practise the algorithm on this same presentation but using the choices $2 = 1A$, $3 = 2B$, $4 = 3A$, $5 = 4A$ and $6 = 4B$. This gets to the answer via a slightly quicker route. Just before you define $6 = 4B$ you should have 7 uncompleted chain fragments, which is considerably better than the 9 uncompleted chain fragments just before 6 was defined in my working. In the last step, when you follow my suggestion for practice, you will find that $2 = 6 = 1$ and that $4 = 5 = 3$.

Of course, as many of you noticed in class, we could have discovered all of this collapsing algebraically. It turns out that our relations mean that

$$\begin{aligned} A &= B^{-1}A^{-2}B^{-1} \\ &= B^{-1}A^2B \qquad \text{because } B^2 = A^4 = e \\ &= (B^{-1}AB)(B^{-1}AB) \\ &= A^2A^2 \qquad \text{because } B^{-1}AB = A^2 \\ &= A^4 \\ &= e \end{aligned}$$

and so the group is generated by B, which has order 2.

It is important to know two things about the Todd-Coxeter algorithm. First, it is not guaranteed to terminate. For instance, if you apply it to the group

$$< A, B | A^2, B^2 >$$

it will never terminate. In this case, it is because these relations can be satisfied by an infinite group. For instance, if $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} \cos 1 & \sin 1 \\ \sin 1 & -\cos 1 \end{pmatrix}$, it is easy to see that $A^2 = B^2 = I$, so the subgroup of the invertible matrices generated by these two matrices satisfies these relations. However, it is easy to check that $AB = \begin{pmatrix} \cos 1 & \sin 1 \\ -\sin 1 & \cos 1 \end{pmatrix}$, and then to prove by induction that $(AB)^n = \begin{pmatrix} \cos n & \sin n \\ -\sin n & \cos n \end{pmatrix}$. This means that $(AB)^n = I$ only if $n = 0$ (because 0 is the only integer multiple of π as π is not a rational number), and so the group contains an infinite subgroup generated by AB. Notice that this means that the Todd-Coxter algorithm cannot terminate because the elements AB, $(AB)^2$, $(AB)^3$, $\ldots$ really are all different.

But it is worse. It can be proved that there is no method for choosing the new links in the Todd-Coxeter algortihm which will **always** cause the algorithm to terminate when the presentation is the presentation of a finite group. The (quite complicated) proof of this works by supposing that such a method did exist, and constructing from the supposed method itself a presentation of a finite group for which following the method will produce an infinite number of links.