Project 3

Application of Marked Branching Techniques to XVA Calculation

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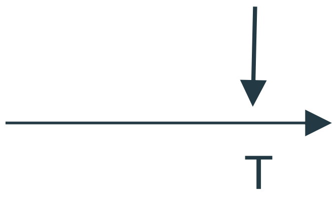
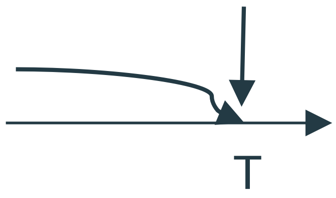
Plan of the Presentation

- Financial and Mathematical Background
- Marked Branching Diffusion
- CVA with Marked Branching Diffusion
- Pricing an American Option with Marked Branching Diffusion

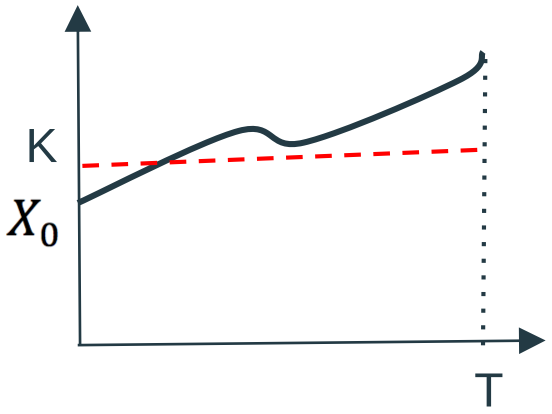
Financial Background

- Option
- Types of the option:
    - European / American





K - strike price
T - time of maturity



Mathematical Background

- Black-Scholes formulas -- closed-form solution of European option pricing

$$C(X_t, t) = N(d_1)X_t - N(d_2)Ke^{-r(T-t)}$$

$$d_1 = \frac{1}{\sigma\sqrt{T-t}}\left[\ln\left(\frac{X_t}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)\right]$$

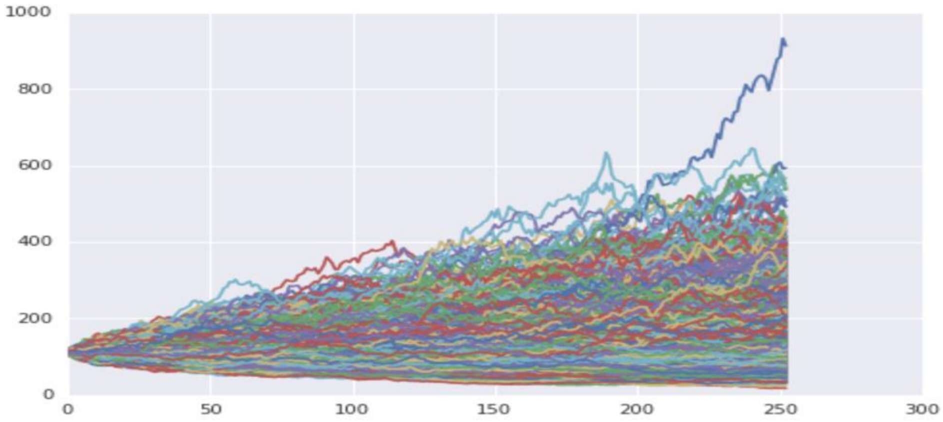
$$d_2 = d_1 - \sigma\sqrt{T-t}$$

Financial Background -- Credit Value Adjustment(CVA)

- Measure of the price of risk
- CVA = (1-R) * Default probability * Exposure

Mathematical Background

- Monte Carlo method -- Simulate the random variable (the stock price) along multiple paths and take the mean



- Nested Monte Carlo for American option and CVA of American option

Mathematical Background

- Most important PDE

$$\partial_t u + \mathcal{L}u + \beta(t)(F(u) - u) = 0, \qquad u(T, x) = g(x)$$

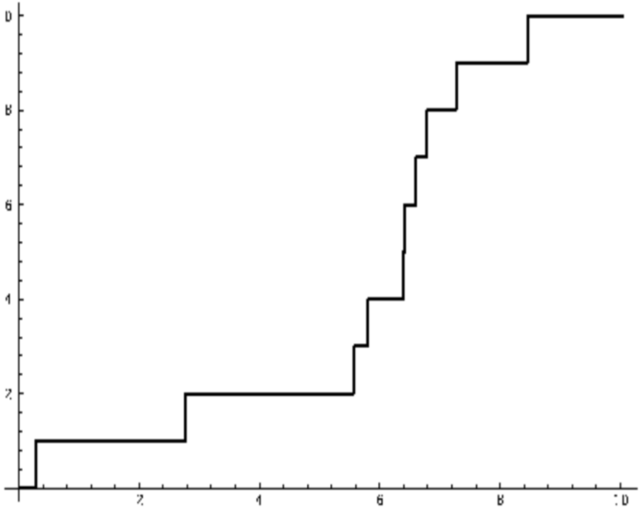
Where

$$\mathcal{L} = \sum_{i=1}^{n} b_i(t,x)\partial_i + \frac{1}{2}\sum_{i,j=1}^{n}\sum_{k=1}^{d}\sigma_{i,k}(t,x)\sigma_{j,k}(t,x)\partial_{ij}$$

and $\beta(t)$ is the intensity of a Poisson jump Process

Mathematical Background

A Poisson Jump Process with intensity = 1



- Poisson Jump Process

$$P(X_t - X_s = k) = \frac{e^{-\beta(t-s)}(\beta(t-s))^k}{k!}$$

Problem Formulation

$$\partial_t u + \mathcal{L}u + \beta(t)(F(u) - u) = 0$$

$$u(T, x) = \psi(x)$$

$$\mathcal{L} = \mu\partial_x + \frac{1}{2}\sigma^2\partial_{xx}$$

$$F(u) = \sum_{i=0}^{M} a_i u^i$$

Feynman – Kac Formula

$$u(t,x) = \mathbb{E}^{\mathbb{Q}}_{t,x}[e^{-\int_t^T \beta(s)ds}\psi(X_T) + \int_t^T e^{-\int_t^r \beta(s)ds}\beta(r)F(u(r,X_r))dr]$$

$$dX_t = \mu dt + \sigma dW_t$$

$$\mathcal{L} = \mu\partial_x + \frac{1}{2}\sigma^2\partial_{xx}$$

Branching Diffusion Process

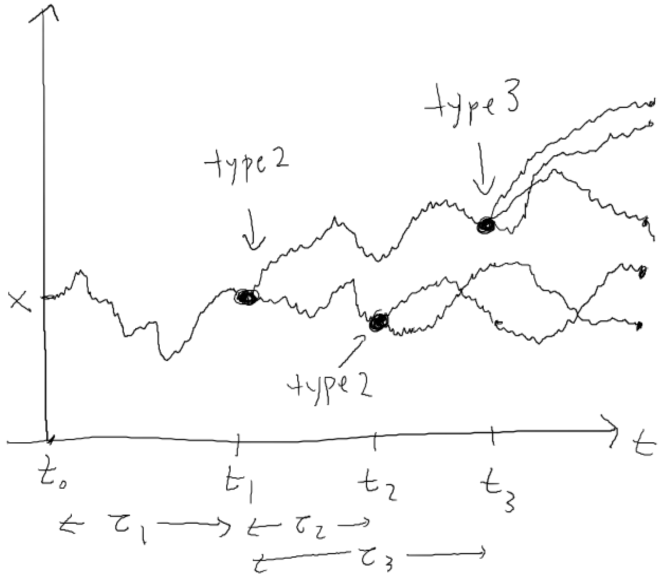
$$dz_t^i = \mu dt + \sigma dW_t^i$$

$$\mathcal{L} = \mu\partial_x + \frac{1}{2}\sigma^2\partial_{xx}$$

$$z^1(t) = x$$

$$\tau_i \sim exp(\beta)$$

$$p_k = \frac{|a_k| * \|\psi\|_\infty^k}{\sum_{i=0}^M |a_i| * \|\psi\|_\infty^i}$$



Useful Notation

$$n(z^{t,x}(T))$$

- Total number of live particles at time T from the branching started by the particle at time t located at position x

$$w_y(z^{t,x}(T))$$

- Total number of y-type branchings until time T from the branching started by the particle at time t located at position x

Marked Branching Diffusion

$$u(t,x) = \mathbb{E}_{t,x}\left[\prod_{i=0}^{n(z_i^{t,x})(T)} \left(\psi(z_i^{t,x}(T))\prod_{j=1}^{M}\left(\frac{a_j}{p_j}\right)^{\omega_j(z_i^{t,x}(T))}\right]$$

Implementation

Goal: compute the expectation

$$\mathbb{E}_{t,x}\left[\prod_{i=1}^{N_T} g(z_T^i) \prod_{k=0}^{\infty} \overline{a_k}^{\Omega_k}\right]$$

to solve a differential equation of the form

$$\partial_t u + \mathcal{L}u + \beta(t)(F(u) - u) = 0, \qquad u(T, x) = g(x)$$

Implementation, cont.

- Used Python classes to simulate the particle diffusion
- Choose probabilities to minimize the variance of the random variable
- Run multiple simulations to compute the expectation

$$\mathbb{E}_{t,x}\left[\prod_{i=1}^{N_T} g(z_T^i) \prod_{k=0}^{\infty} \overline{a_k}^{\Omega_k}\right]$$

Numerical Experiments

We replicated a numerical example from Henry-Labordere's paper with the parameters

$$u(T, x) = \mathbb{I}_{x>1}, \qquad F(u) = \frac{1}{3}(u^3 - u^2 - u^4)$$

Results

Monte Carlo Trials (powers of 2)	Published Result	Published Standard Dev	Our Result	Our Standard Dev
12	.2114	.0078	.2111	.0071
14	.2156	.0038	.2145	.0036
16	.2162	.0019	.2135	.0018
18	.2131	.0010	.2142	.0009

MBD for CVA

Suppose r = 0

R - recovery rate

λ_C - intensity of Poisson Jump process

$$\partial_t u + \mathcal{L}u - (1-R)\lambda_C(u^+ - u) = 0, \qquad u(T,x) = (X_s - K)^+$$

Implementation for CVA Pricing

- Necessity of $(X_s - K)^+$ being bounded
- Necessity of $||(X_s - K)^+||^\infty$ being equal to 1
- Polynomial approximation of positive part

Results and Problems

- We edited our implementation described above to price CVA using the differential equation above
- Problem for larger default probabilities
- Attempted workarounds: rescaling, high precision mathematics library

Sample Results

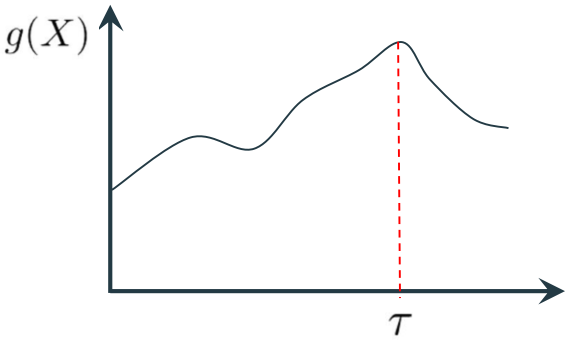
Default Probability	Analytic CVA Pricer	MBD Result
0	.22270258921	.22560051870746
.00001	.222692861783	.226228499639864
.0001	.22262627991	.220218509027449
.001	.221552768189	.233720340286112
.01	.211410346282	.225062397763253
.1	.150147510736	.480409331220962

MBD for American option

$g(x)$ - exercise payoff

Price of the option:

$$\boxed{u(t,x) = \sup_{\tau \in [t,T]} \mathbb{E}^{\mathbb{Q}}_{t,x}[g(x)]}$$



From definition to MBD

$$u(t,x) = \sup_{\tau\in[t,T]} \mathbb{E}^{\mathbb{Q}}_{t,x}[g(x)]$$

$$\Updownarrow$$

$$\max(\partial_t u(t,x) + \mathcal{L}u(t,x),\ \ g(x) - u(t,x)) = 0$$

$$\Updownarrow$$

$$\boxed{\partial_t u(t,x) + \mathcal{L}u(t,x)} + \mathcal{L}g\mathbb{I}_{g(x)=u(t,x)} = 0, \qquad u(T,x) = g(x)$$

Problem

We can solve: $$\partial_t u + \mathcal{L}u + \beta(F(u) - u) = 0; \qquad F(u) = \sum_{k=0}^{M} a_k u^k, \quad u(T,x) = g(x)$$

We need to solve: $$\partial_t u(t,x) + \mathcal{L}u(t,x) + \mathcal{L}g\mathbb{I}_{g(x)=u(t,x)} = 0, \qquad u(T,x) = g(x)$$

The main problem: $$\mathbb{I}_{g(x)=u(t,x)} \backsim F(u)$$

Results:

- Financial background
- Option pricing
- Valuation adjustment (CVA)

Results:

- Marked Branching Diffusion
  - CVA
  - American option pricing