1 Introduction

- Thank you to the organizers for the opportunity to speak
- don't hesitate to interrupt to ask questions
- explain what Delta conjecture is and discuss some recent results obtained with Alessandro Iraci
- this sentence contains three elements

2 Symmetric functions

- e_n is the sum of all the products of n distinct variables
- e_2 is an example of a symmetric function of homogeneous degree 2, indeed all its monomials are of degree two
- it is easy to see that any symmetric function may be written as a finite sum of homogeneous symmetric functions
- in other words the space $\Lambda_{\mathbb{K}}$ is graded by homogeneous degree
- the dimension of the space $\Lambda_{\mathbb{K}}^{(n)}$ of symmetric functions of homogeneous degree n is exactly the number of partitions of n
- a partition of n is a decreasing vector of positive integers that sum to n
- the Ferrers diagram of a partition has as many boxes in the i-th row from the bottom as the i-th entry of the partition
- We discuss here some famous basis of $\Lambda_{\mathbb{K}}^{(n)}$ that will be relevant to this talk.
- the monomial associated to a SSYT encodes its filling, that is the exponent of x_i equals the number of i's in the filling.
- If we take $\mathbb{K} = \mathbb{Q}(q,t)$ to be the smallest field containing to parameters q and t then the so-called *Macdonald polynomials* are a remarkable basis of $\Lambda_{\mathbb{K}}$.
- Not only do they generalise other important families of symmetric functions (indeed for suitable choices of q and t we recover Schur functions, Hall-Littlewood polynomials and others), they seem to also have connections to Hilbert Schemes, Affine Hecke algebras and statistical physics.

3 Interesting symmetric functions

- why some symmetric functions are more interesting then others has to do with their connection to the representation theory of the symmetric group
- this connection is given by the Frobenius map which is an isomorphism between class functions of the n-th symmetric group and homogeneous symmetric functions of degree n
- it gives a correspondence between
- irreducible representations (Specht modules) and schur functions
- Since, by Maschke's theorem, any finite dimensional representation is decomposable into irreducibles, this gives a correspondence between any representation of the symmetric group and positive integer combinations of schur functions.
- If we now consider a bi-graded representation of the symmetric group and define a bi-graded version of the Frobenius characteristic map to keep track of the gradation as follows, *then* bi-graded representations of the symmetric group correspond exactly to symmetric functions that can be written as a combination of schur functions with coefficients in the polynomial ring in the variables q and t and positive integer coefficients
- this is what we refer to as Schur positivity
- let us discuss some examples of interesting, that is schur positive symmetric functions
- since their introduction in the 80's, a version of Macdonald polynomials have been conjectured to be schur positive
- in the 90's Garsia and Haiman proved Macdonald positivity by showing that they are the Frobenius image of their Garsia-Haiman modules. They first reduced their proof to showing that the dimension of their modules equals $n!$, which Haiman proved in 2001.
- In the process of proving Macdonald positivity, Garsia and Haiman introduced the space of diagonal coinvariants, a natural polynomial ring construction.
- Its Frobenius characteristic is ∇e_n, where ∇ is an diagonal operator on Macdonald polynomials. (This was conjectured by Garsia Bergeron and proved by Haiman)
- Zabrocki introduced the space of *super*diagonal coinvariants, and conjectured its Frobenius characteristic to be $\Delta'_{e_{n-k-1}} e_n$.
- This is the symmetric function of the *Delta conjecture*.
- Δ and Δ' are also diagonal operators on Macdonald polynomials that generalise ∇ in a sense.

4 Combinatorics of lattice paths

- a square path of size n starts at $(0,0)$, ends at (n,n), employs only unit north and east steps and ends with an east steps. Here we show a path of size 8.
- A Dyck path is a square path that stays above the line $x = y$.
- Since all Dyck paths are also square paths, I will give all the definitions on the larger set of square paths.
- the 0's are qualitatively different from the other labels.
- As for SSYT, the monomial of the path encodes the filling/labelling of it.
- in this definition, we set $x_0 \mapsto 1$, so it's as if the 0's are not present, hence *partially labelled*.
- A *contractible valley* is a vertical step preceded by another vertical step, which, if we were to delete it, would still yield a valid Dyck paths. In other words a contractible valley may not be preceded directly by a smaller label on the same diagonal.

5 Combinatorial formulas

5.1 Shuffle theorem

- A combinatorial formula for ∇e_n in terms of labelled Dyck paths.
- Recall that ∇e_n is the Frobenius image of the diagonal coinvariants.
- This combinatorial formula was conjectured by Haglund et al. in 2005
- proved in 2018 by Carlson and Mellit

5.2 Delta conjecture

- The Delta conjecture was made by Haglund Remmel and Wilson in 2015
- it is an interpretation of $\Delta'_{e_{n-k-1}} e_n$ in terms of Dyck paths of size n with k decorations
- We present here the *valley version* of the Delta conjecture, which is still an open problem.
- The also exists a version where the decorations are on *rises* instead of valleys. This version was proved very recently by D'Adderio and Mellit.
- This proof made use of the novel Theta operators, introduced by Iraci, D'Adderio and myself.

- We can rewrite the Delta conjecture symmetric function using Theta: we get $\Theta_k \nabla e_{n-k}$
- Comparing with the Shuffle theorem symmetric function, it seems that applying Θ_k has the effect of adding k decorated steps to the combinatorics.

5.3 Generalised Delta conjecture

- The generalised Delta conjecture was made by the same people in the same paper.
- It essentially asserts that applying Δ_{h_m} in the symmetric function side of the Delta conjecture, corresponds to adding m steps labelled 0 on the combinatorial side.

5.4 Square theorem

- Backing up in time, in 2007 Loehr and Warrington proposed a formula for ∇p_n, up to a sign, in terms of labelled *square* paths.
- It was proved by Sergel in 2018 to be a consequence of the Shuffle theorem.

5.5 Generalised Delta Square theorem

- It was natural to look for a decorated, partially labelled equivalent of the generalised Delta conjecture on the square side.
- The obvious interpretations did not work, but with the introduction of Θ and fiddling a bit with the combinatorial set, Iraci and I made the Generalised Delta square conjecture.

6 Two implications

- These two implications put together makes the generalised Delta square conjecture conditional only on the Delta conjecture.
- For the remainder of the talk, I will talk a bit about the proof of these implications.
- I will mainly sketch the combinatorial ideas behind them and omit all the symmetric function manipulations.

6.1 Delta $\Rightarrow$ generalised Delta

- Start out with the set of paths for the Delta conjecture: i.e. labelled decorated Dyck paths.
- We describe an algorithm to go from this set to the set of *partially* labelled Dyck paths. In other words, we add 0 labels.
- The algorithm will allow us to keep track of the changes in statistics and labelling.
- Combined with a fancy identity reflecting the same behavior on the symmetric function side, this will yield a proof a the implication.

6.2 Delta $\Rightarrow$ Delta square

- We follow the same general approach as Sergel used to prove that the square theorem follows from the shuffle theorem.
- build from scratch the set of square paths with a fixed set of labels in the diagonals
- during this process we keep track of the dinv (notice that for fixed labels in diagonals the area and monomial are constant)
- this allows us to get a factorization of the q, t, x enumeration of the set, sometimes called a *schedule formula*
- this formula, combined with some classical symmetric function identities, allows us to *shift* all the labels one diagonal up
- thus we are able to go from square paths to Dyck paths.
- So the last thing I will show you all today is how to construct the set of paths with these labels in the diagonals.
- We start from the empty path and then add the biggest label, 3 into the diagonal containing the line $x = y$, which we will call the 0-diagonal.
- Then we add the smaller labels, two 2's into the same diagonal.
- there are three ways to do this, each creating a different amount of dinv
- the dinv contributions will generally be counted by q-binomials, (for those who are familiar with them)