System Analysis of the Lorenz Attractor

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1 Introduction

The Lorenz Attractor is a non-linear system of Differential Equations. It is defined as a Chaotic system, which means that its behaviour over time highly depends on the given initial condition. This system is also known as the "butterfly effect" due to the famous quote stating that even the flap of the wings of a butterfly has the power to drastically alter the course of the world by causing a hurricane. The representation of the system in a three-dimensional graph also resembles a butterfly

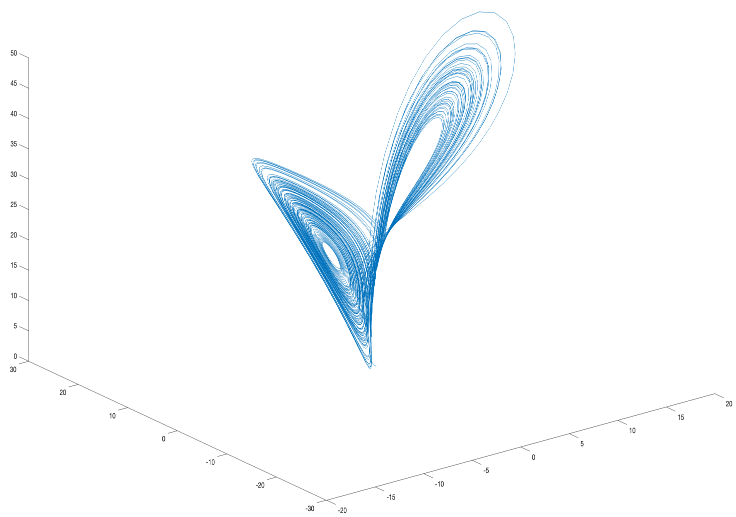

Figure 1: 3-D plot of the "Butterfly Effect".

The system consists of three differential equations. It was developed by Edward Lorenz, together with Ellen Fetter and Margaret Hamilton while coming up with a simplified system for atmospheric convention. The equations represent the warming up from the bottom and cooling from the top of a 2-dimensional fluid:

$$\frac{dx}{dt} = \sigma(y - x)$$
$$\frac{dy}{dt} = x(\rho - z) - y$$
$$\frac{dz}{dt} = xy - \beta z$$

Where x is proportional to the rate of convection, y to the horizontal temperature variation, and z to the vertical temperature variation.

One could write the system in a form such that it resembles the structure used in class. In that case, we would have the following system:

$$\dot{x}_1 = \sigma(x_2 - x_1)$$
$$\dot{x}_2 = x_1(\rho - x_3) - x_2$$
$$\dot{x}_3 = x_1 x_2 - \beta x_3$$

In this case, we will use the following values for the constants: $\sigma = 10$, $\rho = 28$, and $\beta = 8/3$. Therefore, we obtain the following system:

$$\dot{x}_1 = 10(x_2 - x_1)$$
$$\dot{x}_2 = x_1(28 - x_3) - x_2$$
$$\dot{x}_3 = x_1 x_2 - \frac{8}{3} x_3$$

When this system is applied to the Real-World, the constants represent the following:

- σ represents the Prandtl number, which is related to the ratio of kinematic viscosity to thermal diffusivity.
- ρ is the Rayleigh number, which is a dimensionless parameter that is related to the temperature difference across the system.
- β represents the aspect ratio of the system.

1.1 Applications

Applications of the Lorenz Attractor extend beyond atmospheric convection and meteorology, which is the first purpose it was developed for. It has also been examined as a classic example of chaotic behaviour in dynamic systems in a number of scientific fields, including engineering, mathematics, and physics. Studies of chaos, bifurcations, and other nonlinear phenomena usually use the Lorenz equations as a test-bed.

2 Equilibrium Points

Equilibrium points refer to the states at which the derivative of each variable becomes zero. Thus, the system is at rest and there is no change in the state variables. We can find the equilibrium points by:

$$\dot{x}_1 = 10(x_2 - x_1) = 0 \tag{1}$$

$$\dot{x}_2 = x_1(28 - x_3) - x_2 = 0 \tag{2}$$

$$\dot{x}_3 = x_1 x_2 - \frac{8}{3}x_3 = 0 \tag{3}$$

One can easily see that the point (0,0,0) satisfies the equations. To find the other two solutions we can set $x_1 \neq 0$, thus, from (1) we get:

$$x_1 = x_2$$

Which one can then substitute in (2) and (3) to obtain the following:

$$\begin{aligned} 28x_1 - x_1x_3 - x_1 &= 0 \\ x_1(27 - x_3) &= 0 \end{aligned}$$

And since we set $x_1 \neq 0$ then $x_3 = 27$.
From (3) we get:

$$\begin{aligned} {x_1}^2 &= \frac{8}{3}x_3 \\ x_1 = x_2 &= \pm\sqrt{\frac{8}{3}27} \end{aligned}$$

Therefore, all in all, we get the following equilibrium points:

$$\begin{aligned} x_{eq_1} &= (0,0,0) \\ x_{eq_1} &= (\sqrt{\frac{8}{3}27}, \sqrt{\frac{8}{3}27}, 27) \\ x_{eq_1} &= (-\sqrt{\frac{8}{3}27}, -\sqrt{\frac{8}{3}27}, 27) \end{aligned}$$

3 Linearization

In the last step the three equilibrium points were found. To linearize the system, one may first try to use the Indirect Method, namely Taylor's expansion. It represents an infinite sum of expressions formulated based on the derivatives of a function at a specific point. The formulas are defined the following way:

$$\begin{aligned}
\dot{x_1}(x_1,x_2,x_3) &= \dot{x_1}(x_{1eq},x_{2eq},x_{3eq}) + (x_1 - x_{1eq})\frac{\partial f}{\partial x_1} + (x_2 - x_{2eq})\frac{\partial g}{\partial x_2} + (x_3 - x_{3eq})\frac{\partial h}{\partial x_3} \\
\dot{x_2}(x_1,x_2,x_3) &= \dot{x_2}(x_{1eq},x_{2eq},x_{3eq}) + (x_1 - x_{1eq})\frac{\partial f}{\partial x_1} + (x_2 - x_{2eq})\frac{\partial g}{\partial x_2} + (x_3 - x_{3eq})\frac{\partial h}{\partial x_3} \\
\dot{x_3}(x_1,x_2,x_3) &= \dot{x_3}(x_{1eq},x_{2eq},x_{3eq}) + (x_1 - x_{1eq})\frac{\partial f}{\partial x_1} + (x_2 - x_{2eq})\frac{\partial g}{\partial x_2} + (x_3 - x_{3eq})\frac{\partial h}{\partial x_3}
\end{aligned}$$

Beforehand, to find the equilibrium points, we set the derivatives to zero, therefore: $\dot{x_1}(x_1, x_2, x_3) = \dot{x_2}(x_1, x_2, x_3) = \dot{x_3}(x_1, x_2, x_3) = 0$. Then, we can express the formulas in matrix form (Jacobian) for each equilibrium point.

$$\begin{pmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \frac{\partial f}{\partial x_3} \\ \frac{\partial g}{\partial x_1} & \frac{\partial g}{\partial x_2} & \frac{\partial g}{\partial x_3} \\ \frac{\partial h}{\partial x_1} & \frac{\partial h}{\partial x_2} & \frac{\partial h}{\partial x_3} \end{pmatrix}$$

We can then construct the three matrices for each equilibrium point, since these were found in the previous section:

$$Jx_1 = \begin{pmatrix} -10 & 10 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -\frac{8}{3} \end{pmatrix}$$
$$Jx_2 = \begin{pmatrix} -10 & 10 & 0 \\ 1 & -1 & \sqrt{72} \\ \sqrt{72} & \sqrt{72} & -\frac{8}{3} \end{pmatrix}$$
$$Jx_3 = \begin{pmatrix} -10 & 10 & 0 \\ 1 & -1 & -\sqrt{72} \\ -\sqrt{72} & -\sqrt{72} & -\frac{8}{3} \end{pmatrix}$$

Using the Jacobian matrix, one can express the system the following way:

$$\dot{x} = Jx(x - x_{eq})$$

Where Jx is the Jacobian matrix of the system evaluated at an equilibrium point. From this system one may obtain a solution of the form:

$$x = x_{eq} + ve^{\lambda t}$$

where λ are the eigenvalues, and can be calculated the following way:

$$det(J - \lambda I) = 0$$

We call the third-degree polynomial obtained in this case the characteristic equation. This equation will have three roots, which can be real, complex or repeated (by the Fundamental Theorem of Algebra). For each λ_i there exists and v_i called eigenvector. The general solution of the system may also be calculated the following way, using the aforementioned eigenvectors and eigenvalues:

$$x = x_{eq} + c_1 v_1 e^{\lambda_1 t} + c_2 v_2 e^{\lambda_2 t} + c_3 v_3 e^{\lambda_3 t}$$

4 Stability Analysis

For constants c_1, c_2, c_3, one can analyse the stability of the equilibrium points depending on whether the eigenvalues are negative, positive, zero, or complex. Thus, one should first calculate said eigenvalues for each equilibrium point:

Equilibrium point $(0, 0, 0)$

$$P_1(\lambda) = \begin{vmatrix} -10-\lambda & 10 & 0 \\ 1 & -1-\lambda & 0 \\ 0 & 0 & -\frac{8}{3}-\lambda \end{vmatrix} = (-10-\lambda)(-1-\lambda)(-\frac{8}{3}-\lambda) - 10(-\frac{8}{3}-\lambda)$$

From which we obtain the following eigenvalues:

$$\lambda_1 = -11$$
$$\lambda_2 = -\frac{8}{3}$$
$$\lambda_3 = 0$$

Equilibrium point $(\sqrt{\frac{8}{3}27}, \sqrt{\frac{8}{3}27}, 27)$

$$P_1(\lambda) = \begin{vmatrix} -10 & 10 & 0 \\ 1 & -1 & \sqrt{72} \\ \sqrt{72} & \sqrt{72} & -\frac{8}{3} \end{vmatrix} = (-10-\lambda)(-1-\lambda)(-\frac{8}{3}-\lambda) + 10 \times 72 - [10(-\frac{8}{3}-\lambda) + 72(10-\lambda)]$$

From which we obtain the following eigenvalues:

$$\lambda_1 = -11.3 + 5.7i$$
$$\lambda_2 = -11.3 - 5.7i$$
$$\lambda_3 = 8.97$$

Equilibrium point $(-\sqrt{\frac{8}{3}27}, -\sqrt{\frac{8}{3}27}, 27)$

$$P_1(\lambda) = \begin{vmatrix} -10 & 10 & 0 \\ 1 & -1 & \sqrt{72} \\ \sqrt{72} & \sqrt{72} & -\frac{8}{3} \end{vmatrix} = (-10-\lambda)(-1-\lambda)(-\frac{8}{3}-\lambda) + 10 \times 72 - [10(-\frac{8}{3}-\lambda) + 72(10-\lambda)]$$

From which we obtain the same eigenvalues as for the previous equilibrium point:

$$\lambda_1 = -11.3 + 5.7i$$
$$\lambda_2 = -11.3 - 5.7i$$
$$\lambda_3 = 8.97$$

Once all this information has been obtained, one can analyse their stability. It is known that if the Real part of the eigenvalue is negative it will be a stable equilibrium point since the solution will converge.

For $P_1(\lambda)$:

- λ_1 and λ_2 are both Real and negative, therefore they are stable.
- $\lambda_3 = 0$ so it requires further evaluation.

For $P_2(\lambda)$ and $P_3(\lambda)$:

- Re(λ_1) and Re(λ_2) are both negative, therefore they are stable.
- $Re(\lambda_3) > 0$ so it is unstable.

It is important to note that for an equilibrium to be stable, ALL eigenvalues need to have a negative real part. Therefore, as seen for the second and third equilibrium points, since one of the eigenvalues is 8.97, and therefore, positive, the whole equilibrium point is unstable.

Since for P_1 $\lambda_3 = 0$ we need to check the stability through a different method, we can try to do so using the Direct Lyapunov Method.

For that, we need a Candidate Lyapunov Function that satisfies:

$$V(x_{1eq}, x_{2eq}, x_{3eq}) = 0$$
$$V(x_1, x_2, x_3) > 0 \dot{V}(x_1, x_2, x_3) < 0$$

First, we can use a general form to try to find such function:

$$\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \gamma \end{bmatrix} \times \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$
$$V(x_1, x_2, x_3) = x_1^2 + \alpha x_2^2 + \gamma x_3^2$$

And now we want to find $\dot{V}(x_1, x_2, x_3)$

$$\begin{aligned} \dot{V}(x_1, x_2, x_3) &= \frac{\partial V}{\partial x_1} \times \frac{\partial x_1}{\partial t} + \frac{\partial V}{\partial x_2} \times \frac{\partial x_2}{\partial t} + \frac{\partial V}{\partial x_3} \times \frac{\partial x_3}{\partial t} \\ &= \begin{pmatrix} \frac{\partial V}{\partial x_1} & \frac{\partial V}{\partial x_2} & \frac{\partial V}{\partial x_3} \end{pmatrix} \times \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} \\ &= \begin{pmatrix} 2x_1 & 2\alpha x_2 & 2\gamma x_3 \end{pmatrix} \times \begin{pmatrix} 10(x_2 - x_1) \\ x_1(28 - x_3) - x_2 \\ x_1 x_2 - \frac{8}{3} x_3 \end{pmatrix} \\ &= 20x_1(x_2 - x_1) + 2\alpha x_1 x_2(28 - x_3) - 2\alpha x_2^2 + 2\gamma x_1 x_2 x_3 - 2\frac{8}{3}\gamma x_3^2 \end{aligned}$$

We want $\dot{V}(x_1, x_2, x_3) < 0$, $\forall x_1, x_2, x_3 \in R$, thus we evaluate the abovementioned equation:

$$20x_1x_2 - 20x_1^2 + 56\alpha x_1x_2 - 2\alpha x_1x_2x_3 - 2\alpha x_2^2 + 2\gamma x_1x_2x_3 - 2\frac{8}{3}\gamma x_3^2$$

If we consider $\alpha, \gamma > 0$ then some of the terms will always have negative values, namely $-20x_1^2$, $-2\alpha x_2^2$ and $-2\frac{8}{3}\gamma x_3^2$. Therefore we can consider only the leftover terms:

$$20x_1x_2 + 56\alpha x_1x_2 - 2\alpha x_1x_2x_3 + 2\gamma x_1x_2x_3$$

Therefore we have:

$$2\alpha x_1x_2x_3 = 2\gamma x_1x_2x_3$$

And:

$$-20x_1x_2 = 56\alpha x_1x_2 \tag{4}$$

However, from (4) we should get a negative value of α, which contradicts the first assumption that said α should be positive.

From this approach, we have failed to find a Lyapunov Function. However that is not enough to say that this equilibrium point is unstable.

What we can do now is take a look at the representation of the system that was seen during the Introduction section. More specifically, inspect what happens around the origin so that we can analyse its stability. In the following Figure, the system was initialized at a point quite close to the origin, $(0.1, 0.1, 0.1)$. The system then has a similar behaviour to the one it had with $(1, 1, 1)$ as initial condition. If the system would be stable at the origin, we would expect the solution to converge towards a value. However, that does not occur, and therefore we can conclude $(0, 0, 0)$ is an unstable equilibrium point.

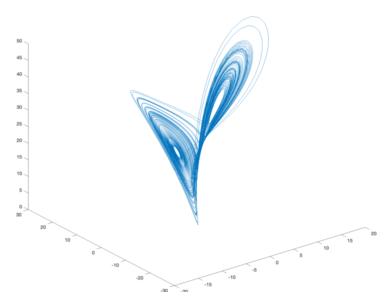

Figure 2: Lorenz Attractor when the initial condition is $(0.1, 0.1, 0.1)$

5 Simulation

Additionally, it may be interesting to take a look at the simulation obtained from Simulink. This gives us an insight of how the systems behaves through time. To do so, one first needs to map the system. A possible representation of the Lorenz Attractor can be seen in Figure 3.

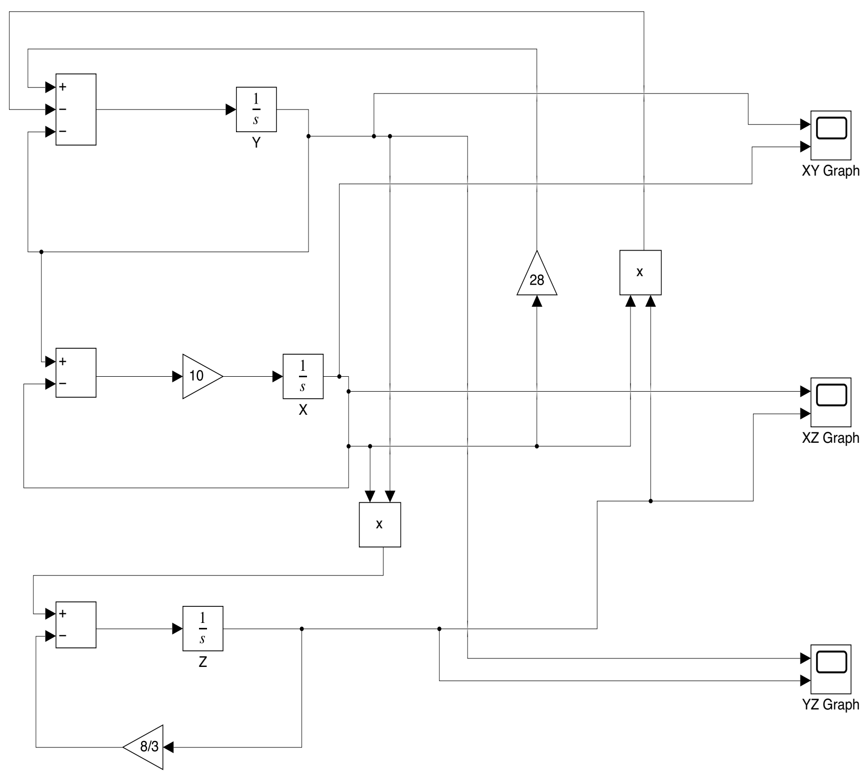

Figure 3: Representation of the system in Simulink.

This map will give us an insight of how the systems behaves (XY, YZ, and XZ), through the *scope* block.

In the following graphs each axis is represented by a different colour: X is dark blue, Y is orange and Z is light blue.

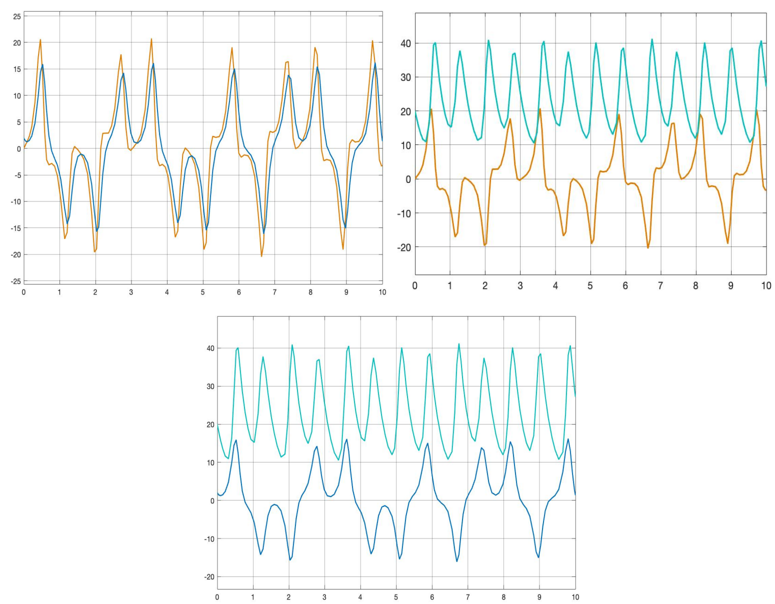

Figure 4: Behaviour of the system, XY, YZ and XZ graphs.

6 Stabilizing

In the previous section, we concluded that the Lorenz Attractor is not stable. Throughout this section we will find a way to stabilize it. A method usually used is adding an input to the system. Currently, the system has a structure that would resemble the following:

$$\begin{pmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_3} \end{pmatrix} = A \times \begin{pmatrix} x_1 \\ x_2 \\ x_{23} \end{pmatrix}$$

Therefore it is missing an input. If we would add it, it would look the following:

$$\dot{x} = Ax + Bu$$

By adding the term Bu we can try to stabilize the system. We can do so by using a function in MATLAB called **place**. Which *moves* the eigenvalues and *places* them to the wanted positions. Beforehand, it was mentioned that if the system has negative eigenvalues then it will be stable, therefore the chosen eigenvalues need to be negative if one wants the system to be stable. Furthermore, since the Lorenz Attractor exhibits chaotic behaviour, it is better to choose poles with large magnitudes, which will force the controller to react faster. After choosing the negative values, the function will return three values, which will be the input Bu that we need to use in our case to obtained the chosen eigenvalues.

Since we have added a new term to the system, the Simulink model has to be changed accordingly. Which is represented in Figure 5. Furthermore, in this case, we do not use ode45 but instead ode23s, which is a stiff solver.

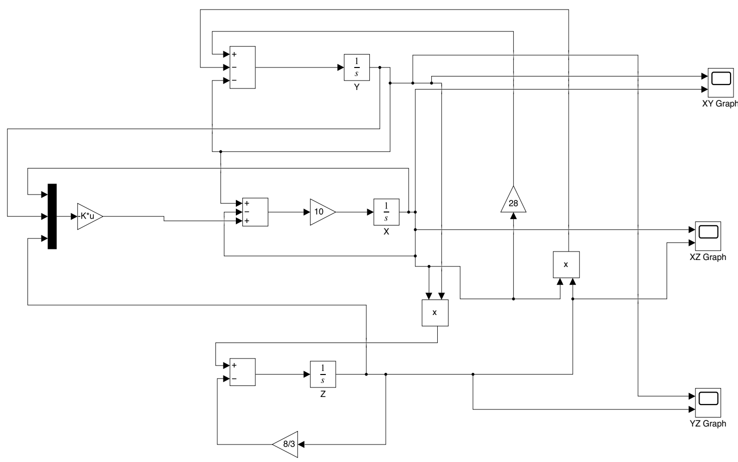

Figure 5: Representation of the stabilized system in Simulink

It can be seen that a Gain of matrix multiplication is being used $(K * u)$. Simulink will obtain the necessary values from the MATLAB function described beforehand. An example of the code would be the following:

```
sigma = 10;
beta = 8/3;
rho = 28;
A = [-sigma sigma 0; rho -1 0; 0 0 -beta];
eig(A)
B = [1 ; 1; 1]
K = place(A,B,[-10,-12,-13])
```

Where first, the constants are defined. Then Matrix A, which is the Jacobian matrix at an equilibrium point, in this case the origin. And then we can see how the function **place** is used, and the chosen eigenvalues are -10, -12, and -13. With this code, we obtain the following values of K:

$$K = \begin{pmatrix} 10.8403886776513 \\ 12.9132234896494 \\ -2.42027883396710 \end{pmatrix}$$

With this last step we have everything that is necessary to plot the stabilized system. Since now, the Simulink file can used the previously obtained values of K as the input. Then, we obtain the following behaviour of the system, where each axis is represented by a different colour: X is dark blue, Y is orange and Z is light blue.

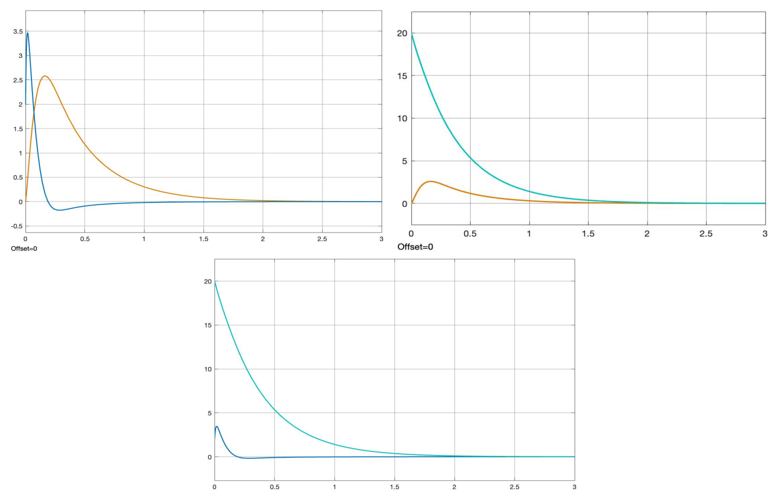

Figure 6: Behaviour of the stabilized system, XY, YZ and XZ graphs (Stop Time 3.0).

In all three graphs one can see that the values converge to 0, which is a clear indication that the new representation of the system is indeed stable.

Therefore, our approach to stabilize the system works. The eigenvalues are $\lambda = -10, -12, -13$ respectively, and the input are the values K.

7 Findings

This report focused on the system analysis of the Lorenz Attractor. Firstly, the equilibrium points of the system were found. This was done by forcing the derivative of each variable to become 0. Once all the equilibrium points are found, one can linearize the model so that the stability analysis can be conducted. The linearization was done using Taylor's expansion, which lead to the Jacobian matrices at each equilibrium point. Looking at the three-dimensional model presented in the introduction section, the results obtained were quite expected. The reasoning for this is different for the equilibrium point placed at the origin and for the other two. Early on, we could have guessed the origin was an equilibrium point since, as it was discussed in the Stabilizing section, there is no input. For the other two equilibrium points, we can see how the *butterfly* shape of the graph has two *holes* in what would be the wings. Which correspond to the other two equilibrium points. Again, just looking at the graph, one can see they have the same *height* since the z values are equal.

To do the stability analysis one has to look at the eigenvalues from said matrices. It is known that if the eigenvalues have negative Real parts then they will be stable. However if that is not the case further analysis needs to be done. For the second and third equilibrium points two eigenvalues had negative real parts, but the other one had a positive real part. Therefore these equilibrium points are unstable. We can again see if the results obtained make sense with the three-dimensional graph of the system. If the equilibrium points are unstable then that means that the point is repellent. Therefore, if we initialize the system close to the equilibrium point, it will not converge to the equilibrium, but instead the trajectory will migrate away from it. This can be seen in the following figure (Figure 7), where the system was initialized close to both equilibrium points.

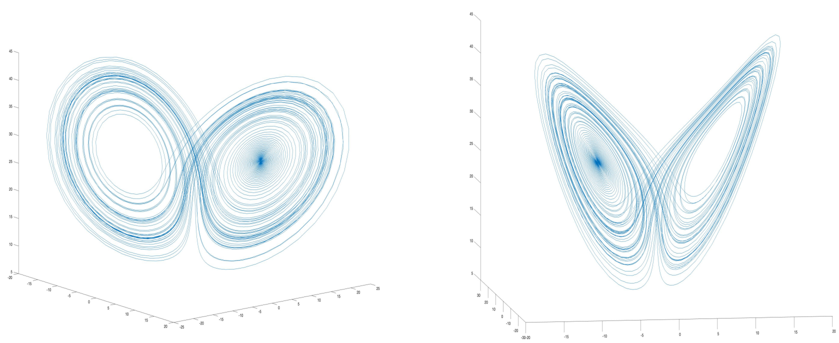

Figure 7: 3-D Graphs when the system is initialised close to the equilibrium points $(\pm\sqrt{72}, \pm\sqrt{72}, 27)$

The equilibrium point at the origin needed further analysis, since one of the eigenvalues was equal to zero. In this case, we tried the Lyapunov method, trying to find a Lyapunov function. However, this could not be done. The Lyapunov method states that if one is able to find a Lyapunov function then the equilibrium point is stable. Nevertheless, if one is not able to find it, it does not automatically mean that the equilibrium point is unstable. Thus, from this method we still did not have a clear conclusion. To be able to figure out the behaviour at the origin, we looked at the three-dimensional graph of the system when it was initialized at a point close to the origin. And then it was seen that the system exhibited a similar behaviour as the one seen in the Introduction section (Figure 1). Therefore we concluded that the origin was an unstable equilibrium point, since the system did not converge. Additionally, when the system is analysed with a different initial condition, the repellent behaviour of the origin can still be seen. For example, in the following figure (Figure 8), we can see how when values get close to the origin (where the drawn lines intercept) they are repelled and therefore forced to migrate away.

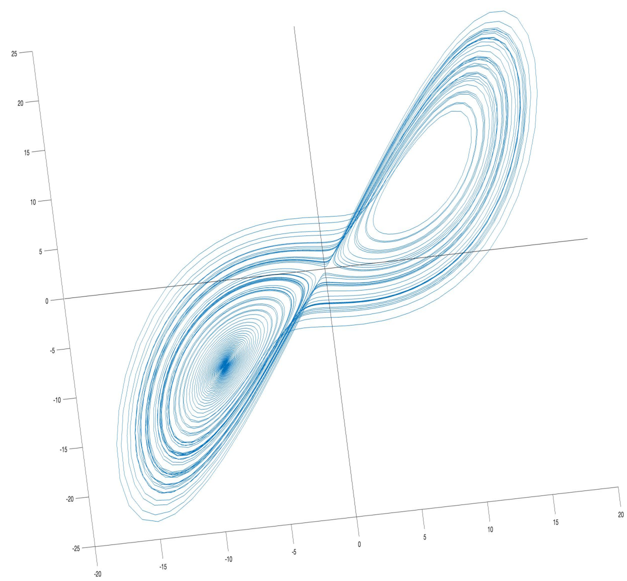

Figure 8: 3-D Graph of the system where the repellent behaviour of the origin can be observed

The next step was to model the system in Simulink to obtain the graphs XY, YZ, and XZ of the system. In this case, we saw that the system did not converge (Figure 4). This result was expected since before it was determined that the equilibrium points were not stable. And if they are not stable, then they will not have a converging behaviour. Additionally, all the graphs have an aperiodic oscillating behaviour. This was to be expected since non-linear chaotic dynamical systems exhibit non-periodic behaviour. In chaotic behaviour, the type that often exhibit periodic behaviour are limit cycles. This refer to a closed trajectory or orbit to which a dynamic system converges over time. Which is not the case for the Lorenz Attractor. The non-converging behaviour can also be observed if the Stop Time is increased. Until now, we have been working with 10.000 as Stop Time. If we increase this, lets say to 100000 we get the following result:

In Figure 9, not much of the behaviour of the system can be observed, however it does become clear that it does not converge.

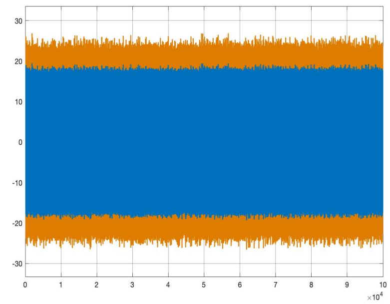

Figure 9: Behaviour of the system when the Stop Time is increased.

The following step consisted on stabilizing the system and then analysing the behaviour of this new system. This was done by adding an input to the system, which altered it. However, one needs to make sure that this input does indeed stabilize the system. As mentioned beforehand, a system is stable when the Real part of the eigenvalues is negative. Therefore, one can find the fit values for the input using MATLAB. Then this input needs to be added to the Simulink model so that the new graphs can be obtained. These are seen in Figure 6, where it can be seen that in all graphs (XY, YZ, and XZ) the values converge to 0. It is a clear indication that the new system is indeed stable. It might be interesting to note that the eigenvalues chosen had a large magnitude. The reason to do so is because this way the controller is forced to react faster. If we would not choose large magnitudes we would get the following (choosing $(-1, -2, -3)$):

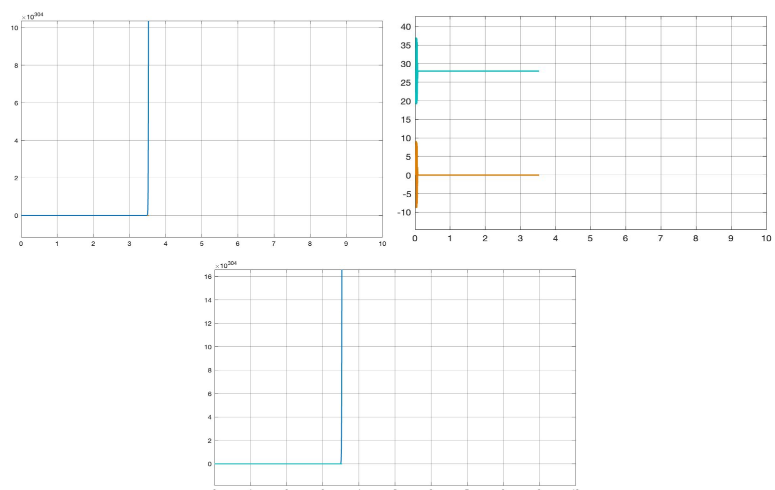

Figure 10: Behaviour of the stabilized system when the eigenvalues do not have a large magnitude, XY, YZ and XZ graphs.

In Figure 10 it may seem that the behaviour of the system is not as smooth, which is due to the fact that the significant changes on the system happen in a shorter period of time. If we zoom in we can see how it still is smooth:

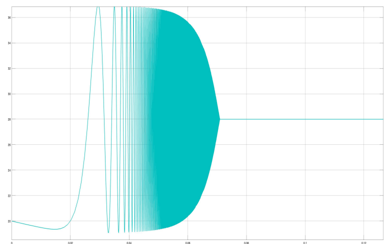

Figure 11: Zoomed in picture of the YZ graph in Figure 10.

Here we can still see how the values converge, but beforehand it oscillates. The system however, is still stable, since all the graphs converge.

All this time we have been talking about the stability/instability of the system without really getting a deeper insight on what does that mean for the system. The study of stability tells us how the system will behave in response to perturbations, disturbances, or changes in initial conditions. As it was mentioned before, we started out with a chaotic system which is highly sensitive to initial conditions, and unpredictable. At the end, we obtained a stable system. Therefore in the case it experiences some kind of disturbance or changes, the system will still return to the equilibrium state over time. These systems are also predictable, robust and less prone to unexpected behaviour.