

Name:

AMATH 515

Homework Set 4

Due: Wednesday, March 16th.

- (1) Prove the following identity for $\alpha \in \mathbb{R}$:

$$\|\alpha x + (1 - \alpha)y\|^2 + \alpha(1 - \alpha)\|x - y\|^2 = \alpha\|x\|^2 + (1 - \alpha)\|y\|^2.$$

- (2) An operator T is *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all (x, y) . For any such nonexpansive operator T , define

$$T_\lambda = (1 - \lambda)I + \lambda T.$$

- (a) Show that T_λ and T have the same fixed points.
(b) Use problem 1 to show

$$\|T_\lambda z - \bar{z}\|^2 \leq \|z - \bar{z}\|^2 - \lambda(1 - \lambda)\|z - Tz\|^2.$$

where \bar{z} is any fixed point of T , i.e. $T\bar{z} = \bar{z}$.

- (3) An operator T is *firmly nonexpansive* when it satisfies

$$\|Tx - Ty\|^2 + \|(I - T)x - (I - T)y\|^2 \leq \|x - y\|^2.$$

- (a) Show T is firmly nonexpansive if and only if

$$\langle x - y, Tx - Ty \rangle \geq \|Tx - Ty\|^2.$$

- (b) Show T is firmly nonexpansive if and only if

$$\langle Tx - Ty, (I - T)x - (I - T)y \rangle \geq 0.$$

- (c) Suppose that $S = 2T - I$. Let

$$\mu = \|Tx - Ty\|^2 + \|(I - T)x - (I - T)y\|^2 - \|x - y\|^2$$

and let

$$\nu = \|Sx - Sy\|^2 - \|x - y\|^2.$$

Show that $2\mu = \nu$ (you may find it helpful to use problem (1)). Conclude that T is firmly nonexpansive exactly when S is nonexpansive.

Coding Assignment

Please download `515Hw4.Coding.ipynb` and `solvers.py` to complete problem (4).

- (4) Implement an interior point method to solve the problem

$$\min_x \frac{1}{2} \|Ax - b\|^2 \quad \text{s.t.} \quad Cx \leq d.$$

Let the user input A , b , C , and d . Test your algorithm using a box constrained problem (where you can apply the prox-gradient method).