Name:

AMATH 515 Homework Set 1

Due: Monday Jan 23rd, by midnight.

(1) Let $g : \mathbb{R}^m \to \mathbb{R}$ is a twice differentiable function, $A \in \mathbb{R}^{m\times n}$ any matrix, and h is the composition $g(Ax)$, then we have two simple generalizations of the chain rule that combine linear algebra with calculus:

$$\nabla h(x) = A^T \nabla g(Ax)$$

and

$$\nabla^2 h(x) = A^T \nabla^2 g(Ax) A.$$

(a) Show what happens when you apply the above chain rules to the special case

$$h(x) = g(a^T x)$$

where a is a vector.

(b) Compute the gradient and hessian of the regularized logistic regression objective:

$$\left(\sum_{i=1}^{n} \log(1 + \exp(a_i^T x)) - b^T Ax \right) + \lambda \|x\|^2$$

where a_i denote the rows of A.

(c) Compute the gradient and hessian of the regularized poisson regression objective:

$$\left(\sum_{i=1}^{n} \exp(a_i^T x) - b^T Ax \right) + \lambda \|x\|^2$$

where a_i denote the rows of A.

(d) Compute the gradient and hessian of the regularized 'concordant' regression objective

$$\|Ax - b\|_2 + \lambda \|x\|_2.$$

Give conditions on a point x that ensure the gradient and Hessian exist at x.

(2) Show that each of the following functions is convex.

(a) Indicator function to a convex set: $\delta_C(x) = \begin{cases} 0 & \text{if} \quad x \in C \\ \infty & \text{if} \quad x \notin C. \end{cases}$

(b) Support function to any set:

$$\sigma_C(x) = \sup_{c \in C} c^T x.$$

(c) Any norm (see Chapter 1 for definition of a norm).

(3) Convexity and composition rules. Suppose that f and g are $\mathcal{C}^2$ functions from $\mathbb{R}$ to $\mathbb{R}$, with $h = f \circ g$ their composition, defined by $h(x) = f(g(x))$.
   (a) If f and g are convex, show it is possible for h to be nonconvex (give an example). Give additional conditions that ensure the composition is convex.
   (b) If f is convex and g is concave, what additional hypothesis that guarantees h is convex?
   (c) Show that if $f : \mathbb{R}^m \to \mathbb{R}$ is convex and $g : \mathbb{R}^n \to \mathbb{R}^m$ affine, then h is convex.
   (d) Show that the following functions are convex:
      (i) Logistic regression objective: $\sum_{i=1}^n \log(1 + \exp(a_i^T x)) - b^T Ax$
      (ii) Poisson regression objective: $\sum_{i=1}^n \exp(a_i^T x) - b^T Ax$.

(4) A function f is *strictly convex* if

$$f(\lambda x + (1-\lambda)y) < \lambda f(x) + (1-\lambda) f(y), \quad \lambda \in (0,1).$$

   (a) Give an example of a strictly convex function that does not have a minimizer.
   (b) Show that a sum of a strictly convex function and a convex function is strictly convex.
   (c) Characterize all solutions to the problem

$$\min_x \frac{1}{2}\|Ax - b\|^2$$

(5) A function f is β-smooth when its gradient is β-Lipschitz continuous.
   (a) Find a global bound for β of the least-squares objective $\frac{1}{2}\|Ax - b\|^2$.
   (b) Find a global bound for β of the regularized logistic objective

$$\sum_{i=1}^n \log(1 + \exp(\langle a_i, x\rangle)) + \frac{\lambda}{2}\|x\|^2.$$

   (c) Do the gradients for Poisson regression admit a global Lipschitz constant?

(6) Please complete the coding homework (starting with the notebook uploaded to Canvas).