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AMATH 515 Homework Set 4

Due: Wednesday, March 16th.

(1) Prove the following identity for $\alpha \in \mathbb{R}$:

$$\|\alpha x + (1-\alpha)y\|^2 + \alpha(1-\alpha)\|x-y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2.$$

Answer to 1: Starting with the left-hand side of the equation:

$$\begin{aligned}
&\|\alpha x + (1-\alpha)y\|^2 + \alpha(1-\alpha)\|x-y\|^2 \\
&= \|\alpha x\|^2 + \|y-\alpha y\|^2 + 2(\alpha x)^T(y-\alpha y) + (\alpha-\alpha^2)(\|x\|^2 + \|y\|^2 - 2x^T y) \\
&= \|\alpha x\|^2 + \|y\|^2 + \|\alpha y\|^2 - 2(\alpha x)^T y + 2(\alpha x)^T y - 2(\alpha x)^T(\alpha y) + \\
&\quad \alpha\|x\|^2 + \alpha\|y\|^2 - 2\alpha x^T y - \alpha^2\|x\|^2 - \alpha\|y\|^2 + 2\alpha^2 x^T y \\
&= \|y\|^2 + \|\alpha y\|^2 - 2y^T\alpha y + \alpha\|x\|^2 \\
&= \|y-\alpha y\|^2 + \alpha\|x\|^2 \\
&= (1-\alpha)\|y\|^2 + \alpha\|x\|^2
\end{aligned}$$

As a result, the left-hand side equals the right-hand side of the identity.

(2) An operator T is *nonexpansive* if $\|Tx - Ty\| \leq \|x-y\|$ for all (x,y). For any such nonexpansive operator T, define

$$T_\lambda = (1-\lambda)I + \lambda T.$$

(a) Show that T_λ and T have the same fixed points.

Answer to 2a: If T_λ and T have the same fixed points, knowing that $T = z$.

$$\begin{aligned}
T_\lambda &= (1-\lambda)x + \lambda T(x) \\
x &= (1-\lambda)x + \lambda T(x) \\
&= x - \lambda x + \lambda x \\
&= x
\end{aligned}$$

Since $T = z$ and $T_\lambda = z$, we can conclude that they have the same fixed points.

(b) Use problem 1 to show

$$\|T_\lambda z - \overline{z}\|^2 \leq \|z - \overline{z}\|^2 - \lambda(1-\lambda)\|z - Tz\|^2.$$

where $\overline{z}$ is any fixed point of T, i.e. $T\overline{z} = \overline{z}$.

Answer 2b: Starting with the equation above, we know that $\|T_\lambda z - \overline{z}\|^2 = \|T_\lambda z - T_\lambda \overline{z}\|^2$ and factoring it out below:

$$\begin{aligned} \|T_\lambda z - T_\lambda \overline{z}\|^2 =& \|(1-\lambda)Iz +_z -(1-\lambda)I\overline{z} - \lambda T\overline{z}\|^2 \\ =& \|\lambda(T_z - T_{\overline{z}} + (1-\lambda)(z - \overline{z})\|^2 \end{aligned}$$

From looking at problem 1, we can see that $x = T_z - T_{\overline{z}}$ and $y = z - \overline{z}$ therefore giving us:

$$\|\alpha x + (1-\alpha)y\|^2 + \alpha(1-\alpha)\|x-y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2$$

$$\|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x-y\|^2$$

$$\begin{aligned} \|T_\lambda z - \overline{z}\|^2 =& \|\lambda x + (1-\lambda)y\|^2 \\ =& \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2 \\ =& \|y\|^2 + \lambda(\|x\|^2 - \|y\|^2) - \lambda(1-\lambda)\|x-y\|^2 \\ =& \|z - \overline{z}\|^2 + \lambda(\|T_z - T_{\overline{z}}\|^2 - \|z - \overline{z}\|^2) - \lambda(1-\lambda)\|x-y\|^2 \\ \leq& \|z - \overline{z}\|^2 - \lambda(1-\lambda)\|x-y\|^2 \\ \leq& \|z - \overline{z}\|^2 - \lambda(1-\lambda)\|(T_z - T_{\overline{z}}) - (z - \overline{z})\|^2 \\ \leq& \|z - \overline{z}\|^2 - \lambda(1-\lambda)\|T_z - \overline{z} - z + \overline{z}\|^2 \\ \leq& \|z - \overline{z}\|^2 - \lambda(1-\lambda)\|z - T_z\|^2 \end{aligned}$$

As a result, $\|T_\lambda z - \overline{z}\|^2 \leq \|z - \overline{z}\|^2 - \lambda(1-\lambda)\|z - T_z\|^2$.

(3) An operator T is *firmly nonexpansive* when it satisfies

$$\|Tx - Ty\|^2 + \|(I-T)x - (I-T)y\|^2 \le \|x-y\|^2.$$

(a) Show T is firmly nonexpansive if and only if

$$\langle x-y, Tx-Ty\rangle \ge \|Tx-Ty\|^2.$$

Answer to 3a: Obtaining the equation above when operator T is firmly nonexpanisve:

$$\|Tx - Ty\|^2 + \|(I-T)x - (I-T)y\|^2 \le \|x-y\|^2$$
$$\|Tx - Ty\|^2 + \|x-y\|^2 - 2\langle x-y, (I-T)x-(I-T)y\rangle \le \|x-y\|^2$$

Cancelling out the $\|x-y\|^2$ we get:

$$\|Tx - Ty\|^2 - 2\langle x-y, (I-T)x-(I-T)y\rangle \le 0$$
$$\|Tx - Ty\|^2 \le 2\langle x-y, (I-T)x-(I-T)y\rangle$$
$$\|Tx - Ty\|^2 \le 2\langle x-y, (I-T)(x-y)\rangle$$
$$\|Tx - Ty\|^2 \le 2\langle x-y, x-y\rangle - 2\langle x-y, T(x-y)\rangle$$

where $2\langle x-y, x-y\rangle$ equals:

$$2\langle x-y, x-y\rangle$$
$$\langle x-y, x-y\rangle - \langle x-y, Tx-Ty\rangle$$
$$\langle x-y, Tx\rangle - \langle x-y, Ty\rangle - \langle x-y, Tx\rangle + \langle x-y, Ty\rangle$$
$$2\langle x-y, Ty-Tx\rangle$$
$$-2\langle x-y, Tx-Ty\rangle$$

Substituting it back into the equation, we get:

$$\|Tx - Ty\|^2 \le 2\langle x-y\rangle - 2\langle x-y, T(x-y)\rangle$$
$$\|Tx - Ty\|^2 \le -2\langle x-y, T(x-y)\rangle - 2\langle x-y, T(x-y)\rangle$$
$$\|Tx - Ty\|^2 \le -4\langle x-y, Tx-Ty\rangle$$
$$-4\langle x-y, Tx-Ty\rangle \ge \|Tx-Ty\|^2$$

Comparing $-4\langle x-y, Tx-Ty)\rangle \ge \|Tx-Ty\|^2$ to what we have to show: $\langle x-y, Tx-Ty\rangle \ge \|Tx-Ty\|^2$, we can see that there is a -4 on the left-hand side of the equation. This means that T is firmly nonexpansive because it is always less than 0 when $\langle x-y, Tx-Ty\rangle \ge \|Tx-Ty\|^2$.

(b) Show T is firmly nonexpansive if and only if

$$\langle Tx-Ty, (I-T)x-(I-T)y\rangle \ge 0.$$

Answer to 3b: Taking the nonexpansive equation:

$$\|Tx - Ty\|^2 + \|(I-T)x - (I-T)y\|^2 \le \|x-y\|^2$$
$$\|Tx - Ty\|^2 \le \|x-y\|^2 - \|(I-T)x - (I-T)y\|^2$$
$$\|Tx - Ty\|^2 \le$$
$$\|x-y\|^2 - (\|x-y\|^2 - \langle x-y, Tx-Ty\rangle - \langle Tx-Ty, x-y\rangle + \|Tx-Ty\|^2)$$
$$\|Tx - Ty\|^2 \le \langle x-y, Tx-Ty\rangle + \langle Tx-Ty, x-y\rangle - \|Tx-Ty\|^2$$
$$2\|Tx - Ty\|^2 \le \langle x-y, Tx-Ty\rangle + \langle Tx-Ty, x-y\rangle$$
$$2\|Tx - Ty\|^2 \le 2\langle x-y, Tx-Ty\rangle$$

$$\|Tx - Ty\|^2 \leq \langle x - y, Tx - Ty \rangle$$

This shows that T is nonexpansive when $\langle Tx - Ty, (I - T)x - (I - T)y \rangle \geq 0$ assuming that x, y, Tx and $Ty \in \epsilon$.

(c) Suppose that $S = 2T - I$. Let

$$\mu = \|Tx - Ty\|^2 + \|(I - T)x - (I - T)y\|^2 - \|x - y\|^2$$

and let

$$\nu = \|Sx - Sy\|^2 - \|x - y\|^2.$$

Show that $2\mu = \nu$ (you may find it helpful to use problem (1)). Conclude that T is firmly nonexpansive exactly when S is nonexpansive.

Answer to 3c: Taking v, we can subsitute $S = 2T - I$,

$$\begin{aligned}
\nu &= \|Sx - Sy\|^2 - \|x - y\|^2 \\
&= \|(2T - I)x - (2T - I)y\| - \|x - y\|^2 \\
&= 4\|Tx - Ty\| - 4\langle Tx - Ty, x - y\rangle + \|x - y\|^2 - \|x - y\|^2 \\
&= 2\|Tx - Ty\|^2 + 2(\|Tx - Ty\|^2 - 2\langle Tx - Ty, x - y\rangle + \|x - y\|^2) - 2\|x - y\|^2 \\
&= 2\|Tx - Ty\|^2 + 2\|(Tx - Ty) - (x - y)\|^2 - 2\|x - y\|^2
\end{aligned}$$

We know that $\|(Tx - Ty) - (x - y)\|^2 = \|(I - T)x - (I - T)y\|^2$. Since $\mu = \|Tx - Ty\|^2 + \|(I - T)x - (I - T)y\|^2 - \|x - y\|^2$, we have the following:

$$\begin{aligned}
\nu = 2\|Tx - Ty\|^2 + 2\|(I - T)x - (I - T)y\|^2 - 2\|x - y\|^2 \\
\nu = 2(\|Tx - Ty\|^2 + \|(I - T)x - (I - T)y\|^2 - \|x - y\|^2) \\
\nu = 2(\mu)
\end{aligned}$$

As a result, T is firmly nonexpansive when S is nonexpansive.

Coding Assignment

Please download `515Hw4_Coding.ipynb` and `solvers.py` to complete problem (4).

(4) Implement an interior point method to solve the problem

$$\min_x \frac{1}{2}\|Ax - b\|^2 \quad \text{s.t.} \quad Cx \le d.$$

Let the user input A, b, C, and d. Test your algorithm using a box constrained problem (where you can apply the prox-gradient method).