Name:

AMATH 515 Homework Set 0

Brief Solution Set.

The goal of this homework is to make sure you are comfortable with all prerequisites for this class, to set-up Python and Jupyter Notebook, and to try submitting your work to Gradescope Autograder. The theoretical portion of the homework will be graded based on completeness, and is intended as a primer on calculus and linear algebra.

1. Theory

(1) Submit your write-up to Gradescope. Look for the assignment "Homework 0 – theory".

(2) **Calculus primer**. For a function $f : \mathbb{R}^n \to \mathbb{R}$, we define the *gradient* to be the vector of partial derivatives:

$$\nabla f(x) = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{bmatrix}$$

and the *Hessian* to be the matrix of second partial derivatives:

$$\nabla^2 f(x) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \vdots & & \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_n} \end{bmatrix}$$

Compute the gradients and hessians of the following functions, with $x \in \mathbb{R}^4$ in all three examples.

(a) $f(x) = \sin(x_1 + x_2 + x_3 + x_4)$

We can write $f(x) = \sin(\mathbf{1}^T x)$, where $\mathbf{1}$ is a vector of ones in $\mathbb{R}^4$, and so

$$\nabla f(x) = \cos(\mathbf{1}^T x)\mathbf{1}, \quad \nabla^2 f(x) = -\sin(\mathbf{1}^T x)\mathbf{1}\mathbf{1}^T$$

(b) $f(x) = \|x\|^2 = x_1^2 + x_2^2 + x_3^2 + x_4^2$

$$\nabla f(x) = 2x, \quad \nabla^2 f(x) = 2I.$$

(c) $f(x) = \ln(x_1 x_2 x_3 x_4)$.

Much easier to rewrite as sum of logs, and so we have

$$\nabla f(x) = \begin{bmatrix} x_1^{-1} \\ x_2^{-1} \\ x_3^{-1} \\ x_4^{-1} \end{bmatrix}, \quad \nabla^2 f(x) = -\begin{bmatrix} x_1^{-2} & 0 & 0 & 0 \\ 0 & x_2^{-2} & 0 & 0 \\ 0 & 0 & x_3^{-2} & 0 \\ 0 & 0 & 0 & x_4^{-2} \end{bmatrix}$$

(3) **Linear algebra primer.**

(a) What are the eigenvalues of the following matrix:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ \pi & 2 & 0 & 0 \\ 64 & -15 & 3 & 0 \\ 321 & 0 & 0 & 5 \end{bmatrix}$$

For any diagonal matrix, the eigenvalues appear on the diagonal.

(b) Write down bases for the range and nullspace of the following matrix, written as the outer product of two vectors:

$$A = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$$

The range is the span of $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ while the nullspace is the plane orthogonal to $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Many explicit bases are possible (each must contain two elements); see one example below.

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right\}$$

(c) Let A be a 10×5 matrix, and b a vector in $\mathbb{R}^{10}$. The notation A^T denotes the *transpose* of A, where the columns of A are rows of A^T.

- What is the size of $A^T A$? What is the size of $A^T b$?

  The sizes are 5×5 and 5×1 respectively.

- How many solutions might there be to the system $Ax = b$?

  General linear systems may have one, zero, or infinitely many solutions. There are no solutions if the system is inconsistent, at least one solution if it is consistent, and infinitely many solutions when A has a nontrivial nullspace. Any of these may happen for a 10×5 system.

- How many solutions might there be to the system $A^T Ax = A^T b$?

  A fundamental theorem of linear algebra tells us that the range of $A^T A$ is always equal to the range of A^T. Thus the system above is always consistent, eliminating the 'no solutions' case. Thus there are either 1 solution (when $A^T A$ is invertible) or infinitely many solutions (when $A^T A$ has a nontrivial nullspace).

- Suppose the columns of A are linearly independent. How many solutions might there be to the system $Ax = b$? To the system $A^T Ax = A^T b$?

  Linearly independent columns means that the nullspace of A only has the element 0. This means the infinite solutions case is eliminated. Thus $Ax = b$ either has one solution (if it is consistent) or no solutions if it is not. The system $A^T Ax = A^T b$ has exactly one solution, since the nullspace of A is the same as the nullspace of $A^T A$.