Assignment 1 Questions

(1) **Probability and Bayes:**

Liam is planning to play a one-person game at the royal casino. The rules of the game are very simple. At the start of the i^{th} turn (0 indexed), Liam chooses to bet $\$s_i$ amount of money - the casino bets the same amount. A coin toss happens and if the result is heads, Liam gets all the money and nothing if the result is a tails. The probability of the coin turning a head is p and of it turning a tail is $1-p$. It is known that the coin is biased in favour of the casino.

(a) Consider the case when Liam has a total of $\$Y$. He choses to play the game with $s_i = 1\,\forall i$ till he has earned $\$X$ or lost everything.

(i) What is the probability that Liam wins $\$X$ before he loses everything? Answer in terms of X, Y and the Casino's odds ratio $\beta = (1-p)/p$.

(ii) Performing appropriate substitutions in the above expression, what is the probability that Liam loses everything before winning $\$X$?

(iii) Is there any chance that Liam goes on playing forever?

(iv) Let $X = 1$, what is the expected gain when Liam plays this game? Show that this gain is always bounded above by $(Y+1)/\beta - Y$.

(b) Consider the case when Liam has $\$2^Y - 1$. He chooses to play the game with $s_i = 2^i\,\forall i$ till he has earned $\$1$ or lost everything.

(i) What is the probability that Liam wins $\$1$ before he loses everything? Answer in terms of Y and p.

(ii) What is the expected gain when Liam plays this game? Is this positive or negative?

(2) **Probability and Bayes:**

You have X rupees and are betting on a fair coin flip. You can bet any percentage of the amount you have. If you win, you gain 1.4 times your bet (and your bet back of course), but if you lose, you lose your bet. What is the optimal bet size to maximize long-run expected earnings? Explain your approach.

(3) **Linear Algebra**

(a) There are no $n \times n$ square matrices X, Y with the property that $XY - YX = I_{n\times n}$. Prove this statement or provide a counterexample.

(b) Assume A_n is a sequence of $n \times n$ upper-triangular random matrices for which each entry is either 4 or 5 and 4 is chosen with probability $p = \frac{1}{5}$.

(i) What can you say about $\lim_{n\longrightarrow\infty} \frac{tr(A_n)}{n}$?

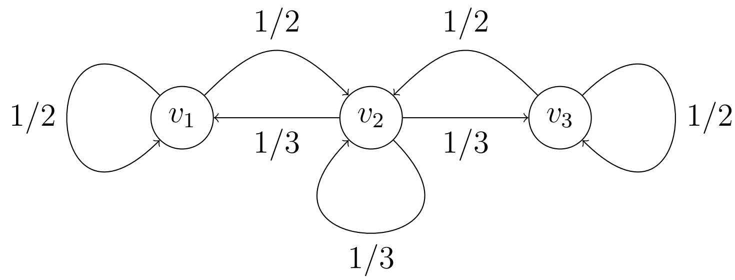
(ii) What can you say about $\lim_{n\longrightarrow\infty} \frac{log(det(A_n))}{n}$?

(4) **Probability and Bayes:**

Consider a set of nodes V and a function $T : V \times V \rightarrow \mathbb{R}$. A particle, when placed at a node $v \in V$ transits to node w (which can be v itself) with a probability of $T(v, w)$.

(a) What conditions must T satisfy to model this situation?

(b) Such a $\langle V, T \rangle$ pair can be represented as a weighted directed graph where edge from v to w has a weight of $T(v, w)$ and only edges with non-zero weights get drawn. Write the transition function for the following graph :



(c) The probabilistic position of a particle at a certain time $p^{(t)}$ on the above graph can be represented as a 3D vector (probability for the 3 points). If the particle starts at v_i, $p^{(0)} = e_i$ (the unit vector in dimension i). Show that irrespective of i, $p^{(t)}$ converges to a fixed value as t grows. Find this fixed value.

(d) Draw a 3-node graph such that $p^{(t)}$ for no starting point converges as t grows.

(5) **About Loss Functions:**

Let y be generated, conditional on $x \in \mathbb{R}$, by the following process:

$$y = ax + \epsilon \ where\ \epsilon \sim \mathcal{N}\left(0, \sigma^2\right),$$

.

Consider you are given n training instances (x_i, y_i)

(a) Write the correct loss function equation

(b) What is the maximum likelihood estimate of a in terms of x_i 's and y_i 's?

(6) **Regression and Loss Functions:**

Scientific studies have shown that there is a correlation between the amount of rainfall and the tree cover at a location. Suppose that rainfall is denoted as r(in mm) and the amount of the ground with tree cover is denoted as t (on a scale from 0 to 100). (r_i, t_i) pairs are recorded for 52 places. In the data collected, r has a sample mean of 1292.6 and a sample standard deviation of 201.0, while t has a sample mean of 57.1 and a sample standard deviation of 10.9. The sample correlation between x and y is 0.722.

(a) In simple linear regression on this data t = a + br+ ϵ, what assumptions do we make about ϵ and how do they affect the least-squares loss function that we use ?

(b) Find the values of parameters (a, b) for regression.

(c) Suppose a new observation $(r_{53}, t_{53})$is added to the data set. Which of the following would change the parameters the most?

- r = 901, t = 46
- r = 1200, t = 40
- r = 1220, t = 61
- r = 2400, t = 80
- r = 2000, t = 30

(d) Suppose we are now given that the parameters (a,b) comes from distributions $(\mathcal{N}(30, 6), \mathcal{N}(0, 2))$ and the absolute value of the error ϵ is follows an exponential distribution with parameter $\lambda = 2$. Construct a custom loss function for regression using this new information.