15 Support Vector Machines

In this chapter and the next we discuss a very useful machine learning tool: the support vector machine paradigm (SVM) for learning linear predictors in high dimensional feature spaces. The high dimensionality of the feature space raises both sample complexity and computational complexity challenges.

The SVM algorithmic paradigm tackles the sample complexity challenge by searching for "large margin" separators. Roughly speaking, a halfspace separates a training set with a large margin if all the examples are not only on the correct side of the separating hyperplane but also far away from it. Restricting the algorithm to output a large margin separator can yield a small sample complexity even if the dimensionality of the feature space is high (and even infinite). We introduce the concept of margin and relate it to the regularized loss minimization paradigm as well as to the convergence rate of the Perceptron algorithm.

In the next chapter we will tackle the computational complexity challenge using the idea of *kernels*.

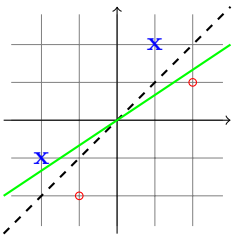
15.1 Margin and Hard-SVM

Let $S = (\mathbf{x}_1, y_1), \ldots, (\mathbf{x}_m, y_m)$ be a training set of examples, where each $\mathbf{x}_i \in \mathbb{R}^d$ and $y_i \in \{\pm 1\}$. We say that this training set is linearly separable, if there exists a halfspace, $(\mathbf{w}, b)$, such that $y_i = \text{sign}(\langle \mathbf{w}, \mathbf{x}_i \rangle + b)$ for all i. Alternatively, this condition can be rewritten as

$$\forall i \in [m], \quad y_i(\langle \mathbf{w}, \mathbf{x}_i \rangle + b) > 0.$$

All halfspaces $(\mathbf{w}, b)$ that satisfy this condition are ERM hypotheses (their 0-1 error is zero, which is the minimum possible error). For any separable training sample, there are many ERM halfspaces. Which one of them should the learner pick?

Consider, for example, the training set described in the picture that follows.



While both the dashed-black and solid-green hyperplanes separate the four examples, our intuition would probably lead us to prefer the black hyperplane over the green one. One way to formalize this intuition is using the concept of *margin*.

The margin of a hyperplane with respect to a training set is defined to be the minimal distance between a point in the training set and the hyperplane. If a hyperplane has a large margin, then it will still separate the training set even if we slightly perturb each instance.

We will see later on that the true error of a halfspace can be bounded in terms of the margin it has over the training sample (the larger the margin, the smaller the error), regardless of the Euclidean dimension in which this halfspace resides.

Hard-SVM is the learning rule in which we return an ERM hyperplane that separates the training set with the largest possible margin. To define Hard-SVM formally, we first express the distance between a point $\mathbf{x}$ to a hyperplane using the parameters defining the halfspace.

CLAIM 15.1 *The distance between a point* $\mathbf{x}$ *and the hyperplane defined by* $(\mathbf{w}, b)$ *where* $\|\mathbf{w}\| = 1$ *is* $|\langle \mathbf{w}, \mathbf{x} \rangle + b|$.

Proof The distance between a point $\mathbf{x}$ and the hyperplane is defined as

$$\min\{\|\mathbf{x} - \mathbf{v}\| : \langle \mathbf{w}, \mathbf{v} \rangle + b = 0\}.$$

Taking $\mathbf{v} = \mathbf{x} - (\langle \mathbf{w}, \mathbf{x} \rangle + b)\mathbf{w}$ we have that

$$\langle \mathbf{w}, \mathbf{v} \rangle + b = \langle \mathbf{w}, \mathbf{x} \rangle - (\langle \mathbf{w}, \mathbf{x} \rangle + b)\|\mathbf{w}\|^2 + b = 0,$$

and

$$\|\mathbf{x} - \mathbf{v}\| = |\langle \mathbf{w}, \mathbf{x} \rangle + b| \, \|\mathbf{w}\| = |\langle \mathbf{w}, \mathbf{x} \rangle + b|.$$

Hence, the distance is at most $|\langle \mathbf{w}, \mathbf{x} \rangle + b|$. Next, take any other point $\mathbf{u}$ on the hyperplane, thus $\langle \mathbf{w}, \mathbf{u} \rangle + b = 0$. We have

$$\begin{aligned}
\|\mathbf{x} - \mathbf{u}\|^2 &= \|\mathbf{x} - \mathbf{v} + \mathbf{v} - \mathbf{u}\|^2 \\
&= \|\mathbf{x} - \mathbf{v}\|^2 + \|\mathbf{v} - \mathbf{u}\|^2 + 2\langle \mathbf{x} - \mathbf{v}, \mathbf{v} - \mathbf{u} \rangle \\
&\geq \|\mathbf{x} - \mathbf{v}\|^2 + 2\langle \mathbf{x} - \mathbf{v}, \mathbf{v} - \mathbf{u} \rangle \\
&= \|\mathbf{x} - \mathbf{v}\|^2 + 2(\langle \mathbf{w}, \mathbf{x} \rangle + b)\langle \mathbf{w}, \mathbf{v} - \mathbf{u} \rangle \\
&= \|\mathbf{x} - \mathbf{v}\|^2,
\end{aligned}$$

where the last equality is because $\langle \mathbf{w}, \mathbf{v} \rangle = \langle \mathbf{w}, \mathbf{u} \rangle = -b$. Hence, the distance

between $\mathbf{x}$ and $\mathbf{u}$ is at least the distance between $\mathbf{x}$ and $\mathbf{v}$, which concludes our proof. □

On the basis of the preceding claim, the closest point in the training set to the separating hyperplane is $\min_{i\in[m]} |\langle \mathbf{w}, \mathbf{x}_i\rangle + b|$. Therefore, the Hard-SVM rule is

$$\operatorname*{argmax}_{(\mathbf{w},b):\|\mathbf{w}\|=1} \min_{i\in[m]} |\langle \mathbf{w}, \mathbf{x}_i\rangle + b| \quad \text{s.t.} \quad \forall i,\ y_i(\langle \mathbf{w}, \mathbf{x}_i\rangle + b) > 0.$$

Whenever there is a solution to the preceding problem (i.e., we are in the separable case), we can write an equivalent problem as follows (see Exercise 1):

$$\operatorname*{argmax}_{(\mathbf{w},b):\|\mathbf{w}\|=1} \min_{i\in[m]} y_i(\langle \mathbf{w}, \mathbf{x}_i\rangle + b). \tag{15.1}$$

Next, we give another equivalent formulation of the Hard-SVM rule as a quadratic optimization problem.[1]

Hard-SVM

input: $(\mathbf{x}_1, y_1), \ldots, (\mathbf{x}_m, y_m)$

solve:

$$(\mathbf{w}_0, b_0) = \operatorname*{argmin}_{(\mathbf{w},b)} \|\mathbf{w}\|^2 \text{ s.t. } \forall i,\ y_i(\langle \mathbf{w}, \mathbf{x}_i\rangle + b) \geq 1 \tag{15.2}$$

output: $\hat{\mathbf{w}} = \frac{\mathbf{w}_0}{\|\mathbf{w}_0\|}, \quad \hat{b} = \frac{b_0}{\|\mathbf{w}_0\|}$

The lemma that follows shows that the output of hard-SVM is indeed the separating hyperplane with the largest margin. Intuitively, hard-SVM searches for $\mathbf{w}$ of minimal norm among all the vectors that separate the data and for which $|\langle \mathbf{w}, \mathbf{x}_i\rangle + b| \geq 1$ for all i. In other words, we enforce the margin to be 1, but now the units in which we measure the margin scale with the norm of $\mathbf{w}$. Therefore, finding the largest margin halfspace boils down to finding $\mathbf{w}$ whose norm is minimal. Formally:

LEMMA 15.2 *The output of Hard-SVM is a solution of Equation (15.1).*

Proof Let $(\mathbf{w}^\star, b^\star)$ be a solution of Equation (15.1) and define the margin achieved by $(\mathbf{w}^\star, b^\star)$ to be $\gamma^\star = \min_{i\in[m]}\ y_i(\langle \mathbf{w}^\star, \mathbf{x}_i\rangle + b^\star)$. Therefore, for all i we have

$$y_i(\langle \mathbf{w}^\star, \mathbf{x}_i\rangle + b^\star) \geq \gamma^\star$$

or equivalently

$$y_i(\langle \tfrac{\mathbf{w}^\star}{\gamma^\star}, \mathbf{x}_i\rangle + \tfrac{b^\star}{\gamma^\star}) \geq 1.$$

Hence, the pair $(\frac{\mathbf{w}^\star}{\gamma^\star}, \frac{b^\star}{\gamma^\star})$ satisfies the conditions of the quadratic optimization

[1] A quadratic optimization problem is an optimization problem in which the objective is a convex quadratic function and the constraints are linear inequalities.

problem given in Equation (15.2). Therefore, $\|\mathbf{w}_0\| \leq \|\frac{\mathbf{w}^\star}{\gamma^\star}\| = \frac{1}{\gamma^\star}$. It follows that for all i,

$$y_i(\langle \hat{\mathbf{w}}, \mathbf{x}_i \rangle + \hat{b}) = \frac{1}{\|\mathbf{w}_0\|} y_i(\langle \mathbf{w}_0, \mathbf{x}_i \rangle + b_0) \geq \frac{1}{\|\mathbf{w}_0\|} \geq \gamma^\star.$$

Since $\|\hat{\mathbf{w}}\| = 1$ we obtain that $(\hat{\mathbf{w}}, \hat{b})$ is an optimal solution of Equation (15.1). □

15.1.1 The Homogenous Case

It is often more convenient to consider homogenous halfspaces, namely, halfspaces that pass through the origin and are thus defined by $\text{sign}(\langle \mathbf{w}, \mathbf{x} \rangle)$, where the bias term b is set to be zero. Hard-SVM for homogenous halfspaces amounts to solving

$$\min_{\mathbf{w}} \|\mathbf{w}\|^2 \quad \text{s.t.} \quad \forall i, \; y_i \langle \mathbf{w}, \mathbf{x}_i \rangle \geq 1. \tag{15.3}$$

As we discussed in Chapter 9, we can reduce the problem of learning nonhomogenous halfspaces to the problem of learning homogenous halfspaces by adding one more feature to each instance of $\mathbf{x}_i$, thus increasing the dimension to $d + 1$.

Note, however, that the optimization problem given in Equation (15.2) does not regularize the bias term b, while if we learn a homogenous halfspace in $\mathbb{R}^{d+1}$ using Equation (15.3) then we regularize the bias term (i.e., the $d+1$ component of the weight vector) as well. However, regularizing b usually does not make a significant difference to the sample complexity.

15.1.2 The Sample Complexity of Hard-SVM

Recall that the VC-dimension of halfspaces in $\mathbb{R}^d$ is $d + 1$. It follows that the sample complexity of learning halfspaces grows with the dimensionality of the problem. Furthermore, the fundamental theorem of learning tells us that if the number of examples is significantly smaller than d/ϵ then no algorithm can learn an ϵ-accurate halfspace. This is problematic when d is very large.

To overcome this problem, we will make an additional assumption on the underlying data distribution. In particular, we will define a "separability with margin γ" assumption and will show that if the data is separable with margin γ then the sample complexity is bounded from above by a function of $1/\gamma^2$. It follows that even if the dimensionality is very large (or even infinite), as long as the data adheres to the separability with margin assumption we can still have a small sample complexity. There is no contradiction to the lower bound given in the fundamental theorem of learning because we are now making an additional assumption on the underlying data distribution.

Before we formally define the separability with margin assumption, there is a scaling issue we need to resolve. Suppose that a training set $S = (\mathbf{x}_1, y_1), \ldots, (\mathbf{x}_m, y_m)$ is separable with a margin γ, namely, the maximal objective value of Equation (15.1) is at least γ. Then, for any positive scalar $\alpha > 0$, the training set

$S' = (\alpha \mathbf{x}_1, y_1), \ldots, (\alpha \mathbf{x}_m, y_m)$ is separable with a margin of $\alpha\gamma$. That is, a simple scaling of the data can make it separable with an arbitrarily large margin. It follows that in order to give a meaningful definition of margin we must take into account the scale of the examples as well. One way to formalize this is using the definition that follows.

DEFINITION 15.3 Let $\mathcal{D}$ be a distribution over $\mathbb{R}^d \times \{\pm 1\}$. We say that $\mathcal{D}$ is separable with a (γ, ρ)-margin if there exists $(\mathbf{w}^\star, b^\star)$ such that $\|\mathbf{w}^\star\| = 1$ and such that with probability 1 over the choice of $(\mathbf{x}, y) \sim \mathcal{D}$ we have that $y(\langle \mathbf{w}^\star, \mathbf{x} \rangle + b^\star) \geq \gamma$ and $\|\mathbf{x}\| \leq \rho$. Similarly, we say that $\mathcal{D}$ is separable with a (γ, ρ)-margin using a homogenous halfspace if the preceding holds with a halfspace of the form $(\mathbf{w}^\star, 0)$.

In the advanced part of the book (Chapter 26), we will prove that the sample complexity of Hard-SVM depends on $(\rho/\gamma)^2$ and is independent of the dimension d. In particular, Theorem 26.13 in Section 26.3 states the following:

THEOREM 15.4 *Let $\mathcal{D}$ be a distribution over $\mathbb{R}^d \times \{\pm 1\}$ that satisfies the (γ, ρ)-separability with margin assumption using a homogenous halfspace. Then, with probability of at least $1 - \delta$ over the choice of a training set of size m, the 0-1 error of the output of Hard-SVM is at most*

$$\sqrt{\frac{4\,(\rho/\gamma)^2}{m}} + \sqrt{\frac{2\log(2/\delta)}{m}}.$$

Remark 15.1 (Margin and the Perceptron) In Section 9.1.2 we have described and analyzed the Perceptron algorithm for finding an ERM hypothesis with respect to the class of halfspaces. In particular, in Theorem 9.1 we upper bounded the number of updates the Perceptron might make on a given training set. It can be shown (see Exercise 2) that the upper bound is exactly $(\rho/\gamma)^2$, where ρ is the radius of examples and γ is the margin.

15.2 Soft-SVM and Norm Regularization

The Hard-SVM formulation assumes that the training set is linearly separable, which is a rather strong assumption. Soft-SVM can be viewed as a relaxation of the Hard-SVM rule that can be applied even if the training set is not linearly separable.

The optimization problem in Equation (15.2) enforces the hard constraints $y_i(\langle \mathbf{w}, \mathbf{x}_i \rangle + b) \geq 1$ for all i. A natural relaxation is to allow the constraint to be violated for some of the examples in the training set. This can be modeled by introducing nonnegative slack variables, $\xi_1, \ldots, \xi_m$, and replacing each constraint $y_i(\langle \mathbf{w}, \mathbf{x}_i \rangle + b) \geq 1$ by the constraint $y_i(\langle \mathbf{w}, \mathbf{x}_i \rangle + b) \geq 1 - \xi_i$. That is, ξ_i measures by how much the constraint $y_i(\langle \mathbf{w}, \mathbf{x}_i \rangle + b) \geq 1$ is being violated. Soft-SVM jointly minimizes the norm of $\mathbf{w}$ (corresponding to the margin) and the average of ξ_i (corresponding to the violations of the constraints). The tradeoff between the two

terms is controlled by a parameter λ. This leads to the Soft-SVM optimization problem:

Soft-SVM

input: $(\mathbf{x}_1, y_1), \ldots, (\mathbf{x}_m, y_m)$
parameter: $\lambda > 0$
solve:

$$\min_{\mathbf{w},b,\boldsymbol{\xi}} \left(\lambda \|\mathbf{w}\|^2 + \frac{1}{m} \sum_{i=1}^{m} \xi_i \right) \quad \text{s.t. } \forall i, \quad y_i(\langle \mathbf{w}, \mathbf{x}_i \rangle + b) \geq 1 - \xi_i \quad \text{and} \quad \xi_i \geq 0 \tag{15.4}$$

output: $\mathbf{w}, b$

We can rewrite Equation (15.4) as a regularized loss minimization problem. Recall the definition of the hinge loss:

$$\ell^{\text{hinge}}((\mathbf{w}, b), (\mathbf{x}, y)) = \max\{0, 1 - y(\langle \mathbf{w}, \mathbf{x} \rangle + b)\}.$$

Given $(\mathbf{w}, b)$ and a training set S, the averaged hinge loss on S is denoted by $L_S^{\text{hinge}}((\mathbf{w}, b))$. Now, consider the regularized loss minimization problem:

$$\min_{\mathbf{w},b} \left(\lambda \|\mathbf{w}\|^2 + L_S^{\text{hinge}}((\mathbf{w}, b)) \right). \tag{15.5}$$

CLAIM 15.5 *Equation (15.4) and Equation (15.5) are equivalent.*

Proof Fix some $\mathbf{w}, b$ and consider the minimization over $\boldsymbol{\xi}$ in Equation (15.4). Fix some i. Since ξ_i must be nonnegative, the best assignment to ξ_i would be 0 if $y_i(\langle \mathbf{w}, \mathbf{x}_i \rangle + b) \geq 1$ and would be $1 - y_i(\langle \mathbf{w}, \mathbf{x}_i \rangle + b)$ otherwise. In other words, $\xi_i = \ell^{\text{hinge}}((\mathbf{w}, b), (\mathbf{x}_i, y_i))$ for all i, and the claim follows. □

We therefore see that Soft-SVM falls into the paradigm of regularized loss minimization that we studied in the previous chapter. A Soft-SVM algorithm, that is, a solution for Equation (15.5), has a bias toward low norm separators. The objective function that we aim to minimize in Equation (15.5) penalizes not only for training errors but also for large norm.

It is often more convenient to consider Soft-SVM for learning a homogenous halfspace, where the bias term b is set to be zero, which yields the following optimization problem:

$$\min_{\mathbf{w}} \left(\lambda \|\mathbf{w}\|^2 + L_S^{\text{hinge}}(\mathbf{w}) \right), \tag{15.6}$$

where

$$L_S^{\text{hinge}}(\mathbf{w}) = \frac{1}{m} \sum_{i=1}^{m} \max\{0, 1 - y\langle \mathbf{w}, \mathbf{x}_i \rangle\}.$$

15.2.1 The Sample Complexity of Soft-SVM

We now analyze the sample complexity of Soft-SVM for the case of homogenous halfspaces (namely, the output of Equation (15.6)). In Corollary 13.8 we derived a generalization bound for the regularized loss minimization framework assuming that the loss function is convex and Lipschitz. We have already shown that the hinge loss is convex so it is only left to analyze the Lipschitzness of the hinge loss.

CLAIM 15.6 *Let $f(\mathbf{w}) = \max\{0, 1 - y\langle \mathbf{w}, \mathbf{x}\rangle\}$. Then, f is $\|\mathbf{x}\|$-Lipschitz.*

Proof It is easy to verify that any subgradient of f at $\mathbf{w}$ is of the form $\alpha\mathbf{x}$ where $|\alpha| \leq 1$. The claim now follows from Lemma 14.7. □

Corollary 13.8 therefore yields the following:

COROLLARY 15.7 *Let $\mathcal{D}$ be a distribution over $\mathcal{X} \times \{0, 1\}$, where $\mathcal{X} = \{\mathbf{x} : \|\mathbf{x}\| \leq \rho\}$. Consider running Soft-SVM (Equation (15.6)) on a training set $S \sim \mathcal{D}^m$ and let $A(S)$ be the solution of Soft-SVM. Then, for every $\mathbf{u}$,*

$$\mathop{\mathbb{E}}_{S\sim\mathcal{D}^m}[L^{\text{hinge}}_{\mathcal{D}}(A(S))] \;\leq\; L^{\text{hinge}}_{\mathcal{D}}(\mathbf{u}) + \lambda\|\mathbf{u}\|^2 + \frac{2\rho^2}{\lambda m}.$$

Furthermore, since the hinge loss upper bounds the $0-1$ loss we also have

$$\mathop{\mathbb{E}}_{S\sim\mathcal{D}^m}[L^{0-1}_{\mathcal{D}}(A(S))] \;\leq\; L^{\text{hinge}}_{\mathcal{D}}(\mathbf{u}) + \lambda\|\mathbf{u}\|^2 + \frac{2\rho^2}{\lambda m}.$$

Last, for every $B > 0$, if we set $\lambda = \sqrt{\frac{2\rho^2}{B^2 m}}$ then

$$\mathop{\mathbb{E}}_{S\sim\mathcal{D}^m}[L^{0-1}_{\mathcal{D}}(A(S))] \;\leq\; \mathop{\mathbb{E}}_{S\sim\mathcal{D}^m}[L^{\text{hinge}}_{\mathcal{D}}(A(S))] \;\leq\; \min_{\mathbf{w}:\|\mathbf{w}\|\leq B} L^{\text{hinge}}_{\mathcal{D}}(\mathbf{w}) + \sqrt{\frac{8\rho^2 B^2}{m}}.$$

We therefore see that we can control the sample complexity of learning a halfspace as a function of the norm of that halfspace, independently of the Euclidean dimension of the space over which the halfspace is defined. This becomes highly significant when we learn via embeddings into high dimensional feature spaces, as we will consider in the next chapter.

Remark 15.2 The condition that $\mathcal{X}$ will contain vectors with a bounded norm follows from the requirement that the loss function will be Lipschitz. This is not just a technicality. As we discussed before, separation with large margin is meaningless without imposing a restriction on the scale of the instances. Indeed, without a constraint on the scale, we can always enlarge the margin by multiplying all instances by a large scalar.

15.2.2 Margin and Norm-Based Bounds versus Dimension

The bounds we have derived for Hard-SVM and Soft-SVM do not depend on the dimension of the instance space. Instead, the bounds depend on the norm of the

examples, ρ, the norm of the halfspace B (or equivalently the margin parameter γ) and, in the nonseparable case, the bounds also depend on the minimum hinge loss of all halfspaces of norm $\leq B$. In contrast, the VC-dimension of the class of homogenous halfspaces is d, which implies that the error of an ERM hypothesis decreases as $\sqrt{d/m}$ does. We now give an example in which $\rho^2 B^2 \ll d$; hence the bound given in Corollary 15.7 is much better than the VC bound.

Consider the problem of learning to classify a short text document according to its topic, say, whether the document is about sports or not. We first need to represent documents as vectors. One simple yet effective way is to use a *bag-of-words* representation. That is, we define a dictionary of words and set the dimension d to be the number of words in the dictionary. Given a document, we represent it as a vector $\mathbf{x} \in \{0,1\}^d$, where $x_i = 1$ if the i'th word in the dictionary appears in the document and $x_i = 0$ otherwise. Therefore, for this problem, the value of ρ^2 will be the maximal number of distinct words in a given document.

A halfspace for this problem assigns weights to words. It is natural to assume that by assigning positive and negative weights to a few dozen words we will be able to determine whether a given document is about sports or not with reasonable accuracy. Therefore, for this problem, the value of B^2 can be set to be less than 100. Overall, it is reasonable to say that the value of $B^2\rho^2$ is smaller than 10,000.

On the other hand, a typical size of a dictionary is much larger than 10,000. For example, there are more than 100,000 distinct words in English. We have therefore shown a problem in which there can be an order of magnitude difference between learning a halfspace with the SVM rule and learning a halfspace using the vanilla ERM rule.

Of course, it is possible to construct problems in which the SVM bound will be worse than the VC bound. When we use SVM, we in fact introduce another form of inductive bias – we prefer large margin halfspaces. While this inductive bias can significantly decrease our estimation error, it can also enlarge the approximation error.

15.2.3 The Ramp Loss*

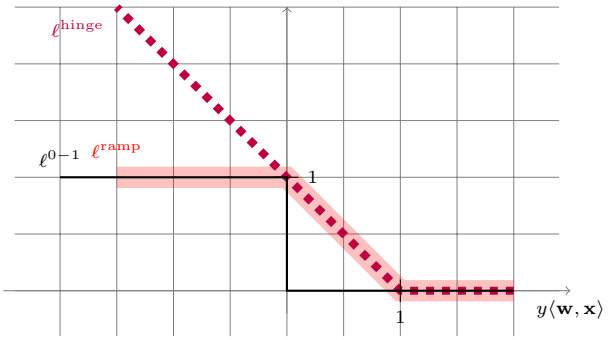
The margin-based bounds we have derived in Corollary 15.7 rely on the fact that we minimize the hinge loss. As we have shown in the previous subsection, the term $\sqrt{\rho^2 B^2/m}$ can be much smaller than the corresponding term in the VC bound, $\sqrt{d/m}$. However, the approximation error in Corollary 15.7 is measured with respect to the hinge loss while the approximation error in VC bounds is measured with respect to the $0-1$ loss. Since the hinge loss upper bounds the $0-1$ loss, the approximation error with respect to the $0-1$ loss will never exceed that of the hinge loss.

It is not possible to derive bounds that involve the estimation error term $\sqrt{\rho^2 B^2/m}$ for the $0-1$ loss. This follows from the fact that the $0-1$ loss is scale

insensitive, and therefore there is no meaning to the norm of $\mathbf{w}$ or its margin when we measure error with the 0−1 loss. However, it is possible to define a loss function that on one hand it is scale sensitive and thus enjoys the estimation error $\sqrt{\rho^2 B^2/m}$ while on the other hand it is more similar to the 0−1 loss. One option is the *ramp loss*, defined as

$$\ell^{\text{ramp}}(\mathbf{w}, (\mathbf{x}, y)) = \min\{1, \ell^{\text{hinge}}(\mathbf{w}, (\mathbf{x}, y))\} = \min\{1\,,\, \max\{0, 1 - y\langle \mathbf{w}, \mathbf{x}\rangle\}\}.$$

The ramp loss penalizes mistakes in the same way as the 0−1 loss and does not penalize examples that are separated with margin. The difference between the ramp loss and the 0−1 loss is only with respect to examples that are correctly classified but not with a significant margin. Generalization bounds for the ramp loss are given in the advanced part of this book (see Appendix 26.3).



The reason SVM relies on the hinge loss and not on the ramp loss is that the hinge loss is convex and, therefore, from the *computational* point of view, minimizing the hinge loss can be performed efficiently. In contrast, the problem of minimizing the ramp loss is computationally intractable.

15.3 Optimality Conditions and "Support Vectors"*

The name "Support Vector Machine" stems from the fact that the solution of hard-SVM, $\mathbf{w}_0$, is supported by (i.e., is in the linear span of) the examples that are exactly at distance $1/\|\mathbf{w}_0\|$ from the separating hyperplane. These vectors are therefore called *support vectors*. To see this, we rely on **Fritz John optimality conditions**.

THEOREM 15.8 *Let $\mathbf{w}_0$ be as defined in Equation (15.3) and let $I = \{i : |\langle \mathbf{w}_0, \mathbf{x}_i\rangle| = 1\}$. Then, there exist coefficients $\alpha_1, \ldots, \alpha_m$ such that*

$$\mathbf{w}_0 = \sum_{i \in I} \alpha_i \mathbf{x}_i.$$

The examples $\{\mathbf{x}_i : i \in I\}$ are called *support vectors*.

The proof of this theorem follows by applying the following lemma to Equation (15.3).

LEMMA 15.9 (Fritz John) *Suppose that*

$$\mathbf{w}^\star \in \operatorname*{argmin}_{\mathbf{w}} f(\mathbf{w}) \quad s.t. \;\; \forall i \in [m],\; g_i(\mathbf{w}) \leq 0,$$

where $f, g_1, \ldots, g_m$ *are differentiable. Then, there exists* $\boldsymbol{\alpha} \in \mathbb{R}^m$ *such that* $\nabla f(\mathbf{w}^\star) + \sum_{i \in I} \alpha_i \nabla g_i(\mathbf{w}^\star) = \mathbf{0}$*, where* $I = \{i : g_i(\mathbf{w}^\star) = 0\}$*.*

15.4 Duality*

Historically, many of the properties of SVM have been obtained by considering the *dual* of Equation (15.3). Our presentation of SVM does not rely on duality. For completeness, we present in the following how to derive the dual of Equation (15.3).

We start by rewriting the problem in an equivalent form as follows. Consider the function

$$g(\mathbf{w}) \;=\; \max_{\boldsymbol{\alpha} \in \mathbb{R}^m : \boldsymbol{\alpha} \geq \mathbf{0}} \sum_{i=1}^{m} \alpha_i (1 - y_i \langle \mathbf{w}, \mathbf{x}_i \rangle) \;=\; \begin{cases} 0 & \text{if } \forall i, y_i \langle \mathbf{w}, \mathbf{x}_i \rangle \geq 1 \\ \infty & \text{otherwise} \end{cases} .$$

We can therefore rewrite Equation (15.3) as

$$\min_{\mathbf{w}} \left(\|\mathbf{w}\|^2 + g(\mathbf{w}) \right). \tag{15.7}$$

Rearranging the preceding we obtain that Equation (15.3) can be rewritten as the problem

$$\min_{\mathbf{w}} \max_{\boldsymbol{\alpha} \in \mathbb{R}^m : \boldsymbol{\alpha} \geq \mathbf{0}} \left(\frac{1}{2} \|\mathbf{w}\|^2 + \sum_{i=1}^{m} \alpha_i (1 - y_i \langle \mathbf{w}, \mathbf{x}_i \rangle) \right). \tag{15.8}$$

Now suppose that we flip the order of min and max in the above equation. This can only decrease the objective value (see Exercise 4), and we have

$$\begin{aligned} &\min_{\mathbf{w}} \max_{\boldsymbol{\alpha} \in \mathbb{R}^m : \boldsymbol{\alpha} \geq \mathbf{0}} \left(\frac{1}{2} \|\mathbf{w}\|^2 + \sum_{i=1}^{m} \alpha_i (1 - y_i \langle \mathbf{w}, \mathbf{x}_i \rangle) \right) \\ &\geq \max_{\boldsymbol{\alpha} \in \mathbb{R}^m : \boldsymbol{\alpha} \geq \mathbf{0}} \min_{\mathbf{w}} \left(\frac{1}{2} \|\mathbf{w}\|^2 + \sum_{i=1}^{m} \alpha_i (1 - y_i \langle \mathbf{w}, \mathbf{x}_i \rangle) \right). \end{aligned}$$

The preceding inequality is called *weak duality*. It turns out that in our case, *strong duality* also holds; namely, the inequality holds with equality. Therefore, the *dual* problem is

$$\max_{\boldsymbol{\alpha} \in \mathbb{R}^m : \boldsymbol{\alpha} \geq \mathbf{0}} \min_{\mathbf{w}} \left(\frac{1}{2} \|\mathbf{w}\|^2 + \sum_{i=1}^{m} \alpha_i (1 - y_i \langle \mathbf{w}, \mathbf{x}_i \rangle) \right). \tag{15.9}$$

We can simplify the dual problem by noting that once $\boldsymbol{\alpha}$ is fixed, the optimization

problem with respect to $\mathbf{w}$ is unconstrained and the objective is differentiable; thus, at the optimum, the gradient equals zero:

$$\mathbf{w} - \sum_{i=1}^{m} \alpha_i y_i \mathbf{x}_i \;=\; 0 \quad \Rightarrow \quad \mathbf{w} = \sum_{i=1}^{m} \alpha_i y_i \mathbf{x}_i .$$

This shows us that the solution must be in the linear span of the examples, a fact we will use later to derive SVM with kernels. Plugging the preceding into Equation (15.9) we obtain that the dual problem can be rewritten as

$$\max_{\boldsymbol{\alpha} \in \mathbb{R}^m : \boldsymbol{\alpha} \geq \mathbf{0}} \left(\frac{1}{2} \left\| \sum_{i=1}^{m} \alpha_i y_i \mathbf{x}_i \right\|^2 + \sum_{i=1}^{m} \alpha_i \left(1 - y_i \left\langle \sum_j \alpha_j y_j \mathbf{x}_j , \mathbf{x}_i \right\rangle \right) \right) . \tag{15.10}$$

Rearranging yields the dual problem

$$\max_{\boldsymbol{\alpha} \in \mathbb{R}^m : \boldsymbol{\alpha} \geq \mathbf{0}} \left(\sum_{i=1}^{m} \alpha_i - \frac{1}{2} \sum_{i=1}^{m} \sum_{j=1}^{m} \alpha_i \alpha_j y_i y_j \langle \mathbf{x}_j , \mathbf{x}_i \rangle \right) . \tag{15.11}$$

Note that the dual problem only involves inner products between instances and does not require direct access to specific elements within an instance. This property is important when implementing SVM with kernels, as we will discuss in the next chapter.

15.5 Implementing Soft-SVM Using SGD

In this section we describe a very simple algorithm for solving the optimization problem of Soft-SVM, namely,

$$\min_{\mathbf{w}} \left(\frac{\lambda}{2} \|\mathbf{w}\|^2 + \frac{1}{m} \sum_{i=1}^{m} \max\{0, 1 - y \langle \mathbf{w}, \mathbf{x}_i \rangle\} \right) . \tag{15.12}$$

We rely on the SGD framework for solving regularized loss minimization problems, as described in Section 14.5.3.

Recall that, on the basis of Equation (14.15), we can rewrite the update rule of SGD as

$$\mathbf{w}^{(t+1)} = -\frac{1}{\lambda t} \sum_{j=1}^{t} \mathbf{v}_j ,$$

where $\mathbf{v}_j$ is a subgradient of the loss function at $\mathbf{w}^{(j)}$ on the random example chosen at iteration j. For the hinge loss, given an example $(\mathbf{x}, y)$, we can choose $\mathbf{v}_j$ to be $\mathbf{0}$ if $y \langle \mathbf{w}^{(j)}, \mathbf{x} \rangle \geq 1$ and $\mathbf{v}_j = -y\,\mathbf{x}$ otherwise (see Example 14.2). Denoting $\boldsymbol{\theta}^{(t)} = -\sum_{j<t} \mathbf{v}_j$ we obtain the following procedure.

SGD for Solving Soft-SVM

goal: Solve Equation (15.12)
parameter: T
initialize: $\boldsymbol{\theta}^{(1)} = \mathbf{0}$
for $t = 1, \ldots, T$
  Let $\mathbf{w}^{(t)} = \frac{1}{\lambda t}\boldsymbol{\theta}^{(t)}$
  Choose i uniformly at random from $[m]$
  If $(y_i \langle \mathbf{w}^{(t)}, \mathbf{x}_i \rangle < 1)$
    Set $\boldsymbol{\theta}^{(t+1)} = \boldsymbol{\theta}^{(t)} + y_i \mathbf{x}_i$
  Else
    Set $\boldsymbol{\theta}^{(t+1)} = \boldsymbol{\theta}^{(t)}$
output: $\bar{\mathbf{w}} = \frac{1}{T}\sum_{t=1}^{T} \mathbf{w}^{(t)}$

15.6 Summary

SVM is an algorithm for learning halfspaces with a certain type of prior knowledge, namely, preference for large margin. Hard-SVM seeks the halfspace that separates the data perfectly with the largest margin, whereas soft-SVM does not assume separability of the data and allows the constraints to be violated to some extent. The sample complexity for both types of SVM is different from the sample complexity of straightforward halfspace learning, as it does not depend on the dimension of the domain but rather on parameters such as the maximal norms of $\mathbf{x}$ and $\mathbf{w}$.

The importance of dimension-independent sample complexity will be realized in the next chapter, where we will discuss the embedding of the given domain into some high dimensional feature space as means for enriching our hypothesis class. Such a procedure raises computational and sample complexity problems. The latter is solved by using SVM, whereas the former can be solved by using SVM with kernels, as we will see in the next chapter.

15.7 Bibliographic Remarks

SVMs have been introduced in (Cortes & Vapnik 1995, Boser, Guyon & Vapnik 1992). There are many good books on the theoretical and practical aspects of SVMs. For example, (Vapnik 1995, Cristianini & Shawe-Taylor 2000, Schölkopf & Smola 2002, Hsu, Chang & Lin 2003, Steinwart & Christmann 2008). Using SGD for solving soft-SVM has been proposed in Shalev-Shwartz et al. (2007).

15.8 Exercises

1. Show that the hard-SVM rule, namely,

$$\operatorname*{argmax}_{(\mathbf{w},b):\|\mathbf{w}\|=1} \min_{i\in[m]} |\langle \mathbf{w}, \mathbf{x}_i\rangle + b| \quad \text{s.t.} \quad \forall i,\ y_i(\langle \mathbf{w}, \mathbf{x}_i\rangle + b) > 0,$$

   is equivalent to the following formulation:

$$\operatorname*{argmax}_{(\mathbf{w},b):\|\mathbf{w}\|=1} \min_{i\in[m]} y_i(\langle \mathbf{w}, \mathbf{x}_i\rangle + b). \tag{15.13}$$

   Hint: Define $\mathcal{G} = \{(\mathbf{w}, b) : \forall i, y_i(\langle \mathbf{w}, \mathbf{x}_i\rangle + b) > 0\}$.
   1. Show that

$$\operatorname*{argmax}_{(\mathbf{w},b):\|\mathbf{w}\|=1} \min_{i\in[m]} y_i(\langle \mathbf{w}, \mathbf{x}_i\rangle + b) \in \mathcal{G}$$

   2. Show that $\forall (\mathbf{w}, b) \in \mathcal{G}$,

$$\min_{i\in[m]} y_i(\langle \mathbf{w}, \mathbf{x}_i\rangle + b) = \min_{i\in[m]} |\langle \mathbf{w}, \mathbf{x}_i\rangle + b|$$

2. **Margin and the Perceptron** Consider a training set that is linearly separable with a margin γ and such that all the instances are within a ball of radius ρ. Prove that the maximal number of updates the Batch Perceptron algorithm given in Section 9.1.2 will make when running on this training set is $(\rho/\gamma)^2$.
3. **Hard versus soft SVM:** Prove or refute the following claim:
   There exists $\lambda > 0$ such that for every sample S of $m > 1$ examples, which is separable by the class of homogenous halfspaces, the hard-SVM and the soft-SVM (with parameter λ) learning rules return exactly the same weight vector.
4. **Weak duality:** Prove that for any function f of two vector variables $\mathbf{x} \in \mathcal{X}, \mathbf{y} \in \mathcal{Y}$, it holds that

$$\min_{\mathbf{x}\in\mathcal{X}} \max_{\mathbf{y}\in\mathcal{Y}} f(\mathbf{x}, \mathbf{y}) \geq \max_{\mathbf{y}\in\mathcal{Y}} \min_{\mathbf{x}\in\mathcal{X}} f(\mathbf{x}, \mathbf{y}).$$

16 Kernel Methods

In the previous chapter we described the SVM paradigm for learning halfspaces in high dimensional feature spaces. This enables us to enrich the expressive power of halfspaces by first mapping the data into a high dimensional feature space, and then learning a linear predictor in that space. This is similar to the AdaBoost algorithm, which learns a composition of a halfspace over base hypotheses. While this approach greatly extends the expressiveness of halfspace predictors, it raises both sample complexity and computational complexity challenges. In the previous chapter we tackled the sample complexity issue using the concept of margin. In this chapter we tackle the computational complexity challenge using the method of *kernels*.

We start the chapter by describing the idea of embedding the data into a high dimensional feature space. We then introduce the idea of kernels. A kernel is a type of a similarity measure between instances. The special property of kernel similarities is that they can be viewed as inner products in some Hilbert space (or Euclidean space of some high dimension) to which the instance space is virtually embedded. We introduce the "kernel trick" that enables computationally efficient implementation of learning, without explicitly handling the high dimensional representation of the domain instances. Kernel based learning algorithms, and in particular kernel-SVM, are very useful and popular machine learning tools. Their success may be attributed both to being flexible for accommodating domain specific prior knowledge and to having a well developed set of efficient implementation algorithms.

16.1 Embeddings into Feature Spaces

The expressive power of halfspaces is rather restricted – for example, the following training set is not separable by a halfspace.

Let the domain be the real line; consider the domain points $\{-10, -9, -8, \ldots, 0, 1, \ldots, 9, 10\}$ where the labels are $+1$ for all x such that $|x| > 2$ and -1 otherwise.

To make the class of halfspaces more expressive, we can first map the original instance space into another space (possibly of a higher dimension) and then learn a halfspace in that space. For example, consider the example mentioned previously. Instead of learning a halfspace in the original representation let us

first define a mapping $\psi : \mathbb{R} \to \mathbb{R}^2$ as follows:

$$\psi(x) = (x, x^2).$$

We use the term *feature space* to denote the range of ψ. After applying ψ the data can be easily explained using the halfspace $h(x) = \text{sign}(\langle \mathbf{w}, \psi(x) \rangle - b)$, where $\mathbf{w} = (0, 1)$ and $b = 5$.

The basic paradigm is as follows:

1. Given some domain set $\mathcal{X}$ and a learning task, choose a mapping $\psi : \mathcal{X} \to \mathcal{F}$, for some *feature space* $\mathcal{F}$, that will usually be $\mathbb{R}^n$ for some n (however, the range of such a mapping can be any *Hilbert space*, including such spaces of infinite dimension, as we will show later).
2. Given a sequence of labeled examples, $S = (\mathbf{x}_1, y_1), \ldots, (\mathbf{x}_m, y_m)$, create the image sequence $\hat{S} = (\psi(\mathbf{x}_1), y_1), \ldots, (\psi(\mathbf{x}_m), y_m)$.
3. Train a linear predictor h over $\hat{S}$.
4. Predict the label of a test point, $\mathbf{x}$, to be $h(\psi(\mathbf{x}))$.

Note that, for every probability distribution $\mathcal{D}$ over $\mathcal{X} \times \mathcal{Y}$, we can readily define its image probability distribution $\mathcal{D}^\psi$ over $\mathcal{F} \times \mathcal{Y}$ by setting, for every subset $A \subseteq \mathcal{F} \times \mathcal{Y}$, $\mathcal{D}^\psi(A) = \mathcal{D}(\psi^{-1}(A))$.[1] It follows that for every predictor h over the feature space, $L_{\mathcal{D}^\psi}(h) = L_{\mathcal{D}}(h \circ \psi)$, where $h \circ \psi$ is the composition of h onto ψ.

The success of this learning paradigm depends on choosing a good ψ for a given learning task: that is, a ψ that will make the image of the data distribution (close to being) linearly separable in the feature space, thus making the resulting algorithm a good learner for a given task. Picking such an embedding requires prior knowledge about that task. However, often some generic mappings that enable us to enrich the class of halfspaces and extend its expressiveness are used. One notable example is polynomial mappings, which are a generalization of the ψ we have seen in the previous example.

Recall that the prediction of a standard halfspace classifier on an instance $\mathbf{x}$ is based on the linear mapping $\mathbf{x} \mapsto \langle \mathbf{w}, \mathbf{x} \rangle$. We can generalize linear mappings to a polynomial mapping, $\mathbf{x} \mapsto p(\mathbf{x})$, where p is a multivariate polynomial of degree k. For simplicity, consider first the case in which $\mathbf{x}$ is 1 dimensional. In that case, $p(x) = \sum_{j=0}^{k} w_j x^j$, where $\mathbf{w} \in \mathbb{R}^{k+1}$ is the vector of coefficients of the polynomial we need to learn. We can rewrite $p(x) = \langle \mathbf{w}, \psi(x) \rangle$ where $\psi : \mathbb{R} \to \mathbb{R}^{k+1}$ is the mapping $x \mapsto (1,\ x,\ x^2,\ x^3, \ldots, x^k)$. It follows that learning a k degree polynomial over $\mathbb{R}$ can be done by learning a linear mapping in the $(k+1)$ dimensional feature space.

More generally, a degree k multivariate polynomial from $\mathbb{R}^n$ to $\mathbb{R}$ can be written as

$$p(\mathbf{x}) = \sum_{J \in [n]^r : r \leq k} w_J \prod_{i=1}^{r} x_{J_i}. \tag{16.1}$$

[1] This is defined for every A such that $\psi^{-1}(A)$ is measurable with respect to $\mathcal{D}$.

As before, we can rewrite $p(\mathbf{x}) = \langle \mathbf{w}, \psi(\mathbf{x}) \rangle$ where now $\psi : \mathbb{R}^n \to \mathbb{R}^d$ is such that for every $J \in [n]^r$, $r \leq k$, the coordinate of $\psi(\mathbf{x})$ associated with J is the monomial $\prod_{i=1}^{r} x_{J_i}$.

Naturally, polynomial-based classifiers yield much richer hypothesis classes than halfspaces. We have seen at the beginning of this chapter an example in which the training set, in its original domain ($\mathcal{X} = \mathbb{R}$), cannot be separable by a halfspace, but after the embedding $x \mapsto (x,\ x^2)$ it is perfectly separable. So, while the classifier is always linear in the feature space, it can have highly nonlinear behavior on the original space from which instances were sampled.

In general, we can choose any feature mapping ψ that maps the original instances into some *Hilbert space*.[2] The Euclidean space $\mathbb{R}^d$ is a Hilbert space for any finite d. But there are also infinite dimensional Hilbert spaces (as we shall see later on in this chapter).

The bottom line of this discussion is that we can enrich the class of halfspaces by first applying a nonlinear mapping, ψ, that maps the instance space into some feature space, and then learning a halfspace in that feature space. However, if the range of ψ is a high dimensional space we face two problems. First, the VC-dimension of halfspaces in $\mathbb{R}^n$ is $n + 1$, and therefore, if the range of ψ is very large, we need many more samples in order to learn a halfspace in the range of ψ. Second, from the computational point of view, performing calculations in the high dimensional space might be too costly. In fact, even the representation of the vector $\mathbf{w}$ in the feature space can be unrealistic. The first issue can be tackled using the paradigm of large margin (or low norm predictors), as we already discussed in the previous chapter in the context of the SVM algorithm. In the following section we address the computational issue.

16.2 The Kernel Trick

We have seen that embedding the input space into some high dimensional feature space makes halfspace learning more expressive. However, the computational complexity of such learning may still pose a serious hurdle – computing linear separators over very high dimensional data may be computationally expensive. The common solution to this concern is kernel based learning. The term "kernels" is used in this context to describe inner products in the feature space. Given an embedding ψ of some domain space $\mathcal{X}$ into some Hilbert space, we define the kernel function $K(\mathbf{x}, \mathbf{x}') = \langle \psi(\mathbf{x}), \psi(\mathbf{x}') \rangle$. One can think of K as specifying similarity between instances and of the embedding ψ as mapping the domain set

[2] A Hilbert space is a vector space with an inner product, which is also complete. A space is complete if all Cauchy sequences in the space converge.
In our case, the norm $\|\mathbf{w}\|$ is defined by the inner product $\sqrt{\langle \mathbf{w}, \mathbf{w} \rangle}$. The reason we require the range of ψ to be in a Hilbert space is that projections in a Hilbert space are well defined. In particular, if M is a linear subspace of a Hilbert space, then every $\mathbf{x}$ in the Hilbert space can be written as a sum $\mathbf{x} = \mathbf{u} + \mathbf{v}$ where $\mathbf{u} \in M$ and $\langle \mathbf{v}, \mathbf{w} \rangle = 0$ for all $\mathbf{w} \in M$. We use this fact in the proof of the representer theorem given in the next section.

$\mathcal{X}$ into a space where these similarities are realized as inner products. It turns out that many learning algorithms for halfspaces can be carried out just on the basis of the values of the kernel function over pairs of domain points. The main advantage of such algorithms is that they implement linear separators in high dimensional feature spaces without having to specify points in that space or expressing the embedding ψ explicitly. The remainder of this section is devoted to constructing such algorithms.

In the previous chapter we saw that regularizing the norm of $\mathbf{w}$ yields a small sample complexity even if the dimensionality of the feature space is high. Interestingly, as we show later, regularizing the norm of $\mathbf{w}$ is also helpful in overcoming the computational problem. To do so, first note that all versions of the SVM optimization problem we have derived in the previous chapter are instances of the following general problem:

$$\min_{\mathbf{w}} \left(f\left(\langle \mathbf{w}, \psi(\mathbf{x}_1) \rangle, \ldots, \langle \mathbf{w}, \psi(\mathbf{x}_m) \rangle \right) + R(\|\mathbf{w}\|) \right), \tag{16.2}$$

where $f : \mathbb{R}^m \to \mathbb{R}$ is an arbitrary function and $R : \mathbb{R}_+ \to \mathbb{R}$ is a monotonically nondecreasing function. For example, Soft-SVM for homogenous halfspaces (Equation (15.6)) can be derived from Equation (16.2) by letting $R(a) = \lambda a^2$ and $f(a_1, \ldots, a_m) = \frac{1}{m} \sum_i \max\{0, 1 - y_i a_i\}$. Similarly, Hard-SVM for nonhomogenous halfspaces (Equation (15.2)) can be derived from Equation (16.2) by letting $R(a) = a^2$ and letting $f(a_1, \ldots, a_m)$ be 0 if there exists b such that $y_i(a_i + b) \geq 1$ for all i, and $f(a_1, \ldots, a_m) = \infty$ otherwise.

The following theorem shows that there exists an optimal solution of Equation (16.2) that lies in the span of $\{\psi(\mathbf{x}_1), \ldots, \psi(\mathbf{x}_m)\}$.

THEOREM 16.1 (Representer Theorem) *Assume that ψ is a mapping from $\mathcal{X}$ to a Hilbert space. Then, there exists a vector $\boldsymbol{\alpha} \in \mathbb{R}^m$ such that $\mathbf{w} = \sum_{i=1}^m \alpha_i \psi(\mathbf{x}_i)$ is an optimal solution of Equation (16.2).*

Proof Let $\mathbf{w}^\star$ be an optimal solution of Equation (16.2). Because $\mathbf{w}^\star$ is an element of a Hilbert space, we can rewrite $\mathbf{w}^\star$ as

$$\mathbf{w}^\star = \sum_{i=1}^m \alpha_i \psi(\mathbf{x}_i) + \mathbf{u},$$

where $\langle \mathbf{u}, \psi(\mathbf{x}_i) \rangle = 0$ for all i. Set $\mathbf{w} = \mathbf{w}^\star - \mathbf{u}$. Clearly, $\|\mathbf{w}^\star\|^2 = \|\mathbf{w}\|^2 + \|\mathbf{u}\|^2$, thus $\|\mathbf{w}\| \leq \|\mathbf{w}^\star\|$. Since R is nondecreasing we obtain that $R(\|\mathbf{w}\|) \leq R(\|\mathbf{w}^\star\|)$. Additionally, for all i we have that

$$\langle \mathbf{w}, \psi(\mathbf{x}_i) \rangle = \langle \mathbf{w}^\star - \mathbf{u}, \psi(\mathbf{x}_i) \rangle = \langle \mathbf{w}^\star, \psi(\mathbf{x}_i) \rangle,$$

hence

$$f\left(\langle \mathbf{w}, \psi(\mathbf{x}_1) \rangle, \ldots, \langle \mathbf{w}, \psi(\mathbf{x}_m) \rangle \right) = f\left(\langle \mathbf{w}^\star, \psi(\mathbf{x}_1) \rangle, \ldots, \langle \mathbf{w}^\star, \psi(\mathbf{x}_m) \rangle \right).$$

We have shown that the objective of Equation (16.2) at $\mathbf{w}$ cannot be larger than the objective at $\mathbf{w}^\star$ and therefore $\mathbf{w}$ is also an optimal solution. Since $\mathbf{w} = \sum_{i=1}^m \alpha_i \psi(\mathbf{x}_i)$ we conclude our proof. □

On the basis of the representer theorem we can optimize Equation (16.2) with respect to the coefficients $\boldsymbol{\alpha}$ instead of the coefficients $\mathbf{w}$ as follows. Writing $\mathbf{w} = \sum_{j=1}^{m} \alpha_j \psi(\mathbf{x}_j)$ we have that for all i

$$\langle \mathbf{w}, \psi(\mathbf{x}_i) \rangle = \left\langle \sum_j \alpha_j \psi(\mathbf{x}_j), \psi(\mathbf{x}_i) \right\rangle = \sum_{j=1}^{m} \alpha_j \langle \psi(\mathbf{x}_j), \psi(\mathbf{x}_i) \rangle.$$

Similarly,

$$\|\mathbf{w}\|^2 = \left\langle \sum_j \alpha_j \psi(\mathbf{x}_j), \sum_j \alpha_j \psi(\mathbf{x}_j) \right\rangle = \sum_{i,j=1}^{m} \alpha_i \alpha_j \langle \psi(\mathbf{x}_i), \psi(\mathbf{x}_j) \rangle.$$

Let $K(\mathbf{x}, \mathbf{x}') = \langle \psi(\mathbf{x}), \psi(\mathbf{x}') \rangle$ be a function that implements the kernel function with respect to the embedding ψ. Instead of solving Equation (16.2) we can solve the equivalent problem

$$\min_{\boldsymbol{\alpha} \in \mathbb{R}^m} f\left(\sum_{j=1}^{m} \alpha_j K(\mathbf{x}_j, \mathbf{x}_1), \ldots, \sum_{j=1}^{m} \alpha_j K(\mathbf{x}_j, \mathbf{x}_m) \right) + R\left(\sqrt{\sum_{i,j=1}^{m} \alpha_i \alpha_j K(\mathbf{x}_j, \mathbf{x}_i)} \right). \tag{16.3}$$

To solve the optimization problem given in Equation (16.3), we do not need any direct access to elements in the feature space. The only thing we should know is how to calculate inner products in the feature space, or equivalently, to calculate the kernel function. In fact, to solve Equation (16.3) we solely need to know the value of the $m \times m$ matrix G s.t. $G_{i,j} = K(\mathbf{x}_i, \mathbf{x}_j)$, which is often called the *Gram* matrix.

In particular, specifying the preceding to the Soft-SVM problem given in Equation (15.6), we can rewrite the problem as

$$\min_{\boldsymbol{\alpha} \in \mathbb{R}^m} \left(\lambda \boldsymbol{\alpha}^T G \boldsymbol{\alpha} + \frac{1}{m} \sum_{i=1}^{m} \max\left\{0, 1 - y_i (G\boldsymbol{\alpha})_i\right\} \right), \tag{16.4}$$

where $(G\boldsymbol{\alpha})_i$ is the i'th element of the vector obtained by multiplying the Gram matrix G by the vector $\boldsymbol{\alpha}$. Note that Equation (16.4) can be written as quadratic programming and hence can be solved efficiently. In the next section we describe an even simpler algorithm for solving Soft-SVM with kernels.

Once we learn the coefficients $\boldsymbol{\alpha}$ we can calculate the prediction on a new instance by

$$\langle \mathbf{w}, \psi(\mathbf{x}) \rangle = \sum_{j=1}^{m} \alpha_j \langle \psi(\mathbf{x}_j), \psi(\mathbf{x}) \rangle = \sum_{j=1}^{m} \alpha_j K(\mathbf{x}_j, \mathbf{x}).$$

The advantage of working with kernels rather than directly optimizing $\mathbf{w}$ in the feature space is that in some situations the dimension of the feature space

is extremely large while implementing the kernel function is very simple. A few examples are given in the following.

Example 16.1 (Polynomial Kernels) The k degree polynomial kernel is defined to be

$$K(\mathbf{x}, \mathbf{x}') = (1 + \langle \mathbf{x}, \mathbf{x}' \rangle)^k.$$

Now we will show that this is indeed a kernel function. That is, we will show that there exists a mapping ψ from the original space to some higher dimensional space for which $K(\mathbf{x}, \mathbf{x}') = \langle \psi(\mathbf{x}), \psi(\mathbf{x}') \rangle$. For simplicity, denote $x_0 = x'_0 = 1$. Then, we have

$$\begin{aligned}
K(\mathbf{x}, \mathbf{x}') = (1 + \langle \mathbf{x}, \mathbf{x}' \rangle)^k &= (1 + \langle \mathbf{x}, \mathbf{x}' \rangle) \cdot \cdots \cdot (1 + \langle \mathbf{x}, \mathbf{x}' \rangle) \\
&= \left(\sum_{j=0}^{n} x_j x'_j \right) \cdot \cdots \cdot \left(\sum_{j=0}^{n} x_j x'_j \right) \\
&= \sum_{J \in \{0,1,\ldots,n\}^k} \prod_{i=1}^{k} x_{J_i} x'_{J_i} \\
&= \sum_{J \in \{0,1,\ldots,n\}^k} \prod_{i=1}^{k} x_{J_i} \prod_{i=1}^{k} x'_{J_i}.
\end{aligned}$$

Now, if we define $\psi : \mathbb{R}^n \to \mathbb{R}^{(n+1)^k}$ such that for $J \in \{0,\ 1, \ldots, n\}^k$ there is an element of $\psi(\mathbf{x})$ that equals $\prod_{i=1}^{k} x_{J_i}$, we obtain that

$$K(\mathbf{x}, \mathbf{x}') = \langle \psi(\mathbf{x}), \psi(\mathbf{x}') \rangle.$$

Since ψ contains all the monomials up to degree k, a halfspace over the range of ψ corresponds to a polynomial predictor of degree k over the original space. Hence, learning a halfspace with a k degree polynomial kernel enables us to learn polynomial predictors of degree k over the original space.

Note that here the complexity of implementing K is $O(n)$ while the dimension of the feature space is on the order of n^k.

Example 16.2 (Gaussian Kernel) Let the original instance space be $\mathbb{R}$ and consider the mapping ψ where for each nonnegative integer $n \geq 0$ there exists an element $\psi(x)_n$ that equals $\frac{1}{\sqrt{n!}} e^{-\frac{x^2}{2}} x^n$. Then,

$$\begin{aligned}
\langle \psi(x),\ \psi(x') \rangle &= \sum_{n=0}^{\infty} \left(\frac{1}{\sqrt{n!}} e^{-\frac{x^2}{2}} x^n \right) \left(\frac{1}{\sqrt{n!}} e^{-\frac{(x')^2}{2}} (x')^n \right) \\
&= e^{-\frac{x^2 + (x')^2}{2}} \sum_{n=0}^{\infty} \left(\frac{(xx')^n}{n!} \right) \\
&= e^{-\frac{\|x - x'\|^2}{2}}.
\end{aligned}$$

Here the feature space is of infinite dimension while evaluating the kernel is very

simple. More generally, given a scalar $\sigma > 0$, the Gaussian kernel is defined to be

$$K(\mathbf{x}, \mathbf{x}') = e^{-\frac{\|\mathbf{x}-\mathbf{x}'\|^2}{2\sigma}}.$$

Intuitively, the Gaussian kernel sets the inner product in the feature space between $\mathbf{x}, \mathbf{x}'$ to be close to zero if the instances are far away from each other (in the original domain) and close to 1 if they are close. σ is a parameter that controls the scale determining what we mean by "close." It is easy to verify that K implements an inner product in a space in which for any n and any monomial of order k there exists an element of $\psi(\mathbf{x})$ that equals $\frac{1}{\sqrt{n!}} e^{-\frac{\|\mathbf{x}\|^2}{2}} \prod_{i=1}^{n} x_{J_i}$. Hence, we can learn any polynomial predictor over the original space by using a Gaussian kernel.

Recall that the VC-dimension of the class of all polynomial predictors is infinite (see Exercise 12). There is no contradiction, because the sample complexity required to learn with Gaussian kernels depends on the margin in the feature space, which will be large if we are lucky, but can in general be arbitrarily small.

The Gaussian kernel is also called the RBF kernel, for "Radial Basis Functions."

16.2.1 Kernels as a Way to Express Prior Knowledge

As we discussed previously, a feature mapping, ψ, may be viewed as expanding the class of linear classifiers to a richer class (corresponding to linear classifiers over the feature space). However, as discussed in the book so far, the suitability of any hypothesis class to a given learning task depends on the nature of that task. One can therefore think of an embedding ψ as a way to express and utilize prior knowledge about the problem at hand. For example, if we believe that positive examples can be distinguished by some ellipse, we can define ψ to be all the monomials up to order 2, or use a degree 2 polynomial kernel.

As a more realistic example, consider the task of learning to find a sequence of characters ("signature") in a file that indicates whether it contains a virus or not. Formally, let $\mathcal{X}_d$ be the set of all strings of length at most d over some alphabet set Σ. The hypothesis class that one wishes to learn is $\mathcal{H} = \{h_v : v \in \mathcal{X}_d\}$, where, for a string $x \in \mathcal{X}_d$, $h_v(x)$ is 1 iff v is a substring of x (and $h_v(x) = -1$ otherwise). Let us show how using an appropriate embedding this class can be realized by linear classifiers over the resulting feature space. Consider a mapping ψ to a space $\mathbb{R}^s$ where $s = |\mathcal{X}_d|$, so that each coordinate of $\psi(x)$ corresponds to some string v and indicates whether v is a substring of x (that is, for every $x \in \mathcal{X}_d$, $\psi(x)$ is a vector in $\{0,1\}^{|\mathcal{X}_d|}$). Note that the dimension of this feature space is exponential in d. It is not hard to see that every member of the class $\mathcal{H}$ can be realized by composing a linear classifier over $\psi(x)$, and, moreover, by such a halfspace whose norm is 1 and that attains a margin of 1 (see Exercise 1). Furthermore, for every $x \in \mathcal{X}$, $\|\psi(x)\| = O(d)$. So, overall, it is learnable using SVM with a sample

complexity that is polynomial in d. However, the dimension of the feature space is exponential in d so a direct implementation of SVM over the feature space is problematic. Luckily, it is easy to calculate the inner product in the feature space (i.e., the kernel function) without explicitly mapping instances into the feature space. Indeed, $K(x, x')$ is simply the number of common substrings of x and x', which can be easily calculated in time polynomial in d.

This example also demonstrates how feature mapping enables us to use halfspaces for nonvectorial domains.

16.2.2 Characterizing Kernel Functions*

As we have discussed in the previous section, we can think of the specification of the kernel matrix as a way to express prior knowledge. Consider a given similarity function of the form $K : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$. Is it a valid kernel function? That is, does it represent an inner product between $\psi(\mathbf{x})$ and $\psi(\mathbf{x}')$ for some feature mapping ψ? The following lemma gives a sufficient and necessary condition.

LEMMA 16.2 *A symmetric function $K : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ implements an inner product in some Hilbert space if and only if it is positive semidefinite; namely, for all $\mathbf{x}_1, \ldots, \mathbf{x}_m$, the Gram matrix, $G_{i,j} = K(\mathbf{x}_i, \mathbf{x}_j)$, is a positive semidefinite matrix.*

Proof It is trivial to see that if K implements an inner product in some Hilbert space then the Gram matrix is positive semidefinite. For the other direction, define the space of functions over $\mathcal{X}$ as $\mathbb{R}^{\mathcal{X}} = \{f : \mathcal{X} \to \mathbb{R}\}$. For each $\mathbf{x} \in \mathcal{X}$ let $\psi(\mathbf{x})$ be the function $\mathbf{x} \mapsto K(\cdot, \mathbf{x})$. Define a vector space by taking all linear combinations of elements of the form $K(\cdot, \mathbf{x})$. Define an inner product on this vector space to be

$$\left\langle \sum_i \alpha_i K(\cdot, \mathbf{x}_i), \sum_j \beta_j K(\cdot, \mathbf{x}'_j) \right\rangle = \sum_{i,j} \alpha_i \beta_j K(\mathbf{x}_i, \mathbf{x}'_j).$$

This is a valid inner product since it is symmetric (because K is symmetric), it is linear (immediate), and it is positive definite (it is easy to see that $K(\mathbf{x}, \mathbf{x}) \geq 0$ with equality only for $\psi(\mathbf{x})$ being the zero function). Clearly,

$$\langle \psi(\mathbf{x}), \psi(\mathbf{x}') \rangle = \langle K(\cdot, \mathbf{x}), K(\cdot, \mathbf{x}') \rangle = K(\mathbf{x}, \mathbf{x}'),$$

which concludes our proof. □

16.3 Implementing Soft-SVM with Kernels

Next, we turn to solving Soft-SVM with kernels. While we could have designed an algorithm for solving Equation (16.4), there is an even simpler approach that

directly tackles the Soft-SVM optimization problem in the feature space,

$$\min_{\mathbf{w}} \left(\frac{\lambda}{2}\|\mathbf{w}\|^2 + \frac{1}{m}\sum_{i=1}^{m} \max\{0, 1 - y\langle \mathbf{w}, \psi(\mathbf{x}_i)\rangle\} \right), \tag{16.5}$$

while only using kernel evaluations. The basic observation is that the vector $\mathbf{w}^{(t)}$ maintained by the SGD procedure we have described in Section 15.5 is always in the linear span of $\{\psi(\mathbf{x}_1), \ldots, \psi(\mathbf{x}_m)\}$. Therefore, rather than maintaining $\mathbf{w}^{(t)}$ we can maintain the corresponding coefficients $\boldsymbol{\alpha}$.

Formally, let K be the kernel function, namely, for all $\mathbf{x}, \mathbf{x}'$, $K(\mathbf{x}, \mathbf{x}') = \langle \psi(\mathbf{x}), \psi(\mathbf{x}')\rangle$. We shall maintain two vectors in $\mathbb{R}^m$, corresponding to two vectors $\boldsymbol{\theta}^{(t)}$ and $\mathbf{w}^{(t)}$ defined in the SGD procedure of Section 15.5. That is, $\boldsymbol{\beta}^{(t)}$ will be a vector such that

$$\boldsymbol{\theta}^{(t)} = \sum_{j=1}^{m} \beta_j^{(t)} \psi(\mathbf{x}_j) \tag{16.6}$$

and $\boldsymbol{\alpha}^{(t)}$ be such that

$$\mathbf{w}^{(t)} = \sum_{j=1}^{m} \alpha_j^{(t)} \psi(\mathbf{x}_j). \tag{16.7}$$

The vectors $\boldsymbol{\beta}$ and $\boldsymbol{\alpha}$ are updated according to the following procedure.

SGD for Solving Soft-SVM with Kernels

Goal: Solve Equation (16.5)
parameter: T
Initialize: $\boldsymbol{\beta}^{(1)} = \mathbf{0}$
for $t = 1, \ldots, T$
  Let $\boldsymbol{\alpha}^{(t)} = \frac{1}{\lambda t}\boldsymbol{\beta}^{(t)}$
  Choose i uniformly at random from $[m]$
  For all $j \neq i$ set $\beta_j^{(t+1)} = \beta_j^{(t)}$
  If $(y_i \sum_{j=1}^{m} \alpha_j^{(t)} K(\mathbf{x}_j, \mathbf{x}_i) < 1)$
    Set $\beta_i^{(t+1)} = \beta_i^{(t)} + y_i$
  Else
    Set $\beta_i^{(t+1)} = \beta_i^{(t)}$
Output: $\bar{\mathbf{w}} = \sum_{j=1}^{m} \bar{\alpha}_j \psi(\mathbf{x}_j)$ where $\bar{\boldsymbol{\alpha}} = \frac{1}{T}\sum_{t=1}^{T} \boldsymbol{\alpha}^{(t)}$

The following lemma shows that the preceding implementation is equivalent to running the SGD procedure described in Section 15.5 on the feature space.

LEMMA 16.3 *Let $\hat{\mathbf{w}}$ be the output of the SGD procedure described in Section 15.5, when applied on the feature space, and let $\bar{\mathbf{w}} = \sum_{j=1}^{m} \bar{\alpha}_j \psi(\mathbf{x}_j)$ be the output of applying SGD with kernels. Then $\bar{\mathbf{w}} = \hat{\mathbf{w}}$.*

Proof We will show that for every t Equation (16.6) holds, where $\boldsymbol{\theta}^{(t)}$ is the result of running the SGD procedure described in Section 15.5 in the feature

space. By the definition of $\boldsymbol{\alpha}^{(t)} = \frac{1}{\lambda t}\boldsymbol{\beta}^{(t)}$ and $\mathbf{w}^{(t)} = \frac{1}{\lambda t}\boldsymbol{\theta}^{(t)}$, this claim implies that Equation (16.7) also holds, and the proof of our lemma will follow. To prove that Equation (16.6) holds we use a simple inductive argument. For $t = 1$ the claim trivially holds. Assume it holds for $t \geq 1$. Then,

$$y_i \left\langle \mathbf{w}^{(t)}, \psi(\mathbf{x}_i) \right\rangle = y_i \left\langle \sum_j \alpha_j^{(t)} \psi(\mathbf{x}_j), \psi(\mathbf{x}_i) \right\rangle = y_i \sum_{j=1}^{m} \alpha_j^{(t)} K(\mathbf{x}_j, \mathbf{x}_i).$$

Hence, the condition in the two algorithms is equivalent and if we update $\boldsymbol{\theta}$ we have

$$\boldsymbol{\theta}^{(t+1)} = \boldsymbol{\theta}^{(t)} + y_i \psi(\mathbf{x}_i) = \sum_{j=1}^{m} \beta_j^{(t)} \psi(\mathbf{x}_j) + y_i \psi(\mathbf{x}_i) = \sum_{j=1}^{m} \beta_j^{(t+1)} \psi(\mathbf{x}_j),$$

which concludes our proof. □

16.4 Summary

Mappings from the given domain to some higher dimensional space, on which a halfspace predictor is used, can be highly powerful. We benefit from a rich and complex hypothesis class, yet need to solve the problems of high sample and computational complexities. In Chapter 10, we discussed the AdaBoost algorithm, which faces these challenges by using a weak learner: Even though we're in a very high dimensional space, we have an "oracle" that bestows on us a single good coordinate to work with on each iteration. In this chapter we introduced a different approach, the kernel trick. The idea is that in order to find a halfspace predictor in the high dimensional space, we do not need to know the representation of instances in that space, but rather the values of inner products between the mapped instances. Calculating inner products between instances in the high dimensional space without using their representation in that space is done using kernel functions. We have also shown how the SGD algorithm can be implemented using kernels.

The ideas of feature mapping and the kernel trick allow us to use the framework of halfspaces and linear predictors for nonvectorial data. We demonstrated how kernels can be used to learn predictors over the domain of strings.

We presented the applicability of the kernel trick in SVM. However, the kernel trick can be applied in many other algorithms. A few examples are given as exercises.

This chapter ends the series of chapters on linear predictors and convex problems. The next two chapters deal with completely different types of hypothesis classes.

16.5 Bibliographic Remarks

In the context of SVM, the kernel-trick has been introduced in Boser et al. (1992). See also Aizerman, Braverman & Rozonoer (1964). The observation that the kernel-trick can be applied whenever an algorithm only relies on inner products was first stated by Schölkopf, Smola & Müller (1998). The proof of the representer theorem is given in (Schölkopf, Herbrich, Smola & Williamson 2000, Schölkopf, Herbrich & Smola 2001). The conditions stated in Lemma 16.2 are simplification of conditions due to Mercer. Many useful kernel functions have been introduced in the literature for various applications. We refer the reader to Schölkopf & Smola (2002).

16.6 Exercises

1. Consider the task of finding a sequence of characters in a file, as described in Section 16.2.1. Show that every member of the class $\mathcal{H}$ can be realized by composing a linear classifier over $\psi(x)$, whose norm is 1 and that attains a margin of 1.
2. **Kernelized Perceptron:** Show how to run the Perceptron algorithm while only accessing the instances via the kernel function. *Hint:* The derivation is similar to the derivation of implementing SGD with kernels.
3. **Kernel Ridge Regression:** The ridge regression problem, with a feature mapping ψ, is the problem of finding a vector $\mathbf{w}$ that minimizes the function

$$f(\mathbf{w}) = \lambda \|\mathbf{w}\|^2 + \frac{1}{2m}\sum_{i=1}^{m}(\langle \mathbf{w}, \psi(\mathbf{x}_i)\rangle - y_i)^2, \tag{16.8}$$

   and then returning the predictor

$$h(\mathbf{x}) = \langle \mathbf{w}, \mathbf{x}\rangle.$$

   Show how to implement the ridge regression algorithm with kernels.

   Hint: The representer theorem tells us that there exists a vector $\boldsymbol{\alpha} \in \mathbb{R}^m$ such that $\sum_{i=1}^m \alpha_i \psi(\mathbf{x}_i)$ is a minimizer of Equation (16.8).

   1. Let G be the Gram matrix with regard to S and K. That is, $G_{ij} = K(\mathbf{x}_i, \mathbf{x}_j)$. Define $g : \mathbb{R}^m \to \mathbb{R}$ by

$$g(\boldsymbol{\alpha}) = \lambda \cdot \boldsymbol{\alpha}^T G \boldsymbol{\alpha} + \frac{1}{2m}\sum_{i=1}^{m}(\langle \boldsymbol{\alpha}, G_{\cdot,i}\rangle - y_i)^2, \tag{16.9}$$

      where $G_{\cdot,i}$ is the i'th column of G. Show that if $\boldsymbol{\alpha}^*$ minimizes Equation (16.9) then $\mathbf{w}^* = \sum_{i=1}^m \alpha_i^* \psi(\mathbf{x}_i)$ is a minimizer of f.
   2. Find a closed form expression for $\boldsymbol{\alpha}^*$.
4. Let N be any positive integer. For every $x, x' \in \{1, \ldots, N\}$ define

$$K(x, x') = \min\{x, x'\}.$$

Prove that K is a valid kernel; namely, find a mapping $\psi : \{1, \ldots, N\} \to H$ where H is some Hilbert space, such that

$$\forall x, x' \in \{1, \ldots, N\},\ K(x, x') = \langle \psi(x), \psi(x') \rangle.$$

5. A supermarket manager would like to learn which of his customers have babies on the basis of their shopping carts. Specifically, he sampled i.i.d. customers, where for customer i, let $x_i \subset \{1, \ldots, d\}$ denote the subset of items the customer bought, and let $y_i \in \{\pm 1\}$ be the label indicating whether this customer has a baby. As prior knowledge, the manager knows that there are k items such that the label is determined to be 1 iff the customer bought at least one of these k items. Of course, the identity of these k items is not known (otherwise, there was nothing to learn). In addition, according to the store regulation, each customer can buy at most s items. Help the manager to design a learning algorithm such that both its time complexity and its sample complexity are polynomial in s, k, and $1/\epsilon$.
6. Let $\mathcal{X}$ be an instance set and let ψ be a feature mapping of $\mathcal{X}$ into some Hilbert feature space V. Let $K : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ be a kernel function that implements inner products in the feature space V.

   Consider the binary classification algorithm that predicts the label of an unseen instance according to the class with the closest average. Formally, given a training sequence $S = (\mathbf{x}_1, y_1), \ldots, (\mathbf{x}_m, y_m)$, for every $y \in \{\pm 1\}$ we define

$$c_y = \frac{1}{m_y} \sum_{i: y_i = y} \psi(\mathbf{x}_i).$$

   where $m_y = |\{i : y_i = y\}|$. We assume that m_+ and m_- are nonzero. Then, the algorithm outputs the following decision rule:

$$h(\mathbf{x}) = \begin{cases} 1 & \|\psi(\mathbf{x}) - c_+\| \leq \|\psi(\mathbf{x}) - c_-\| \\ 0 & \text{otherwise.} \end{cases}$$

   1. Let $\mathbf{w} = c_+ - c_-$ and let $b = \frac{1}{2}(\|c_-\|^2 - \|c_+\|^2)$. Show that

$$h(\mathbf{x}) = \text{sign}(\langle \mathbf{w}, \psi(\mathbf{x}) \rangle + b).$$

   2. Show how to express $h(\mathbf{x})$ on the basis of the kernel function, and without accessing individual entries of $\psi(\mathbf{x})$ or $\mathbf{w}$.