Vector Differentiation and vector operators

Differentiation of a vector function:-

Let 's' be a set of real numbers, Corresponding to each scalar $t \in s$, let there be associated a unique vector $\bar{f}$. Then $\bar{f}$ is said to be a vector (vector valued) function. 's' is called domain of $\bar{f}$.

We write $\bar{f} = \bar{f}(t)$

Let $\bar{i}, \bar{j}, \bar{k}$ be three mutually perpendicular unit vectors in 3D space. We can write $\bar{f} = \bar{f}(t) = f_1(t)\bar{i} + f_2(t)\bar{j} + f_3(t)\bar{k}$

where $f_1(t), f_2(t), f_3(t)$ are real valued functions.

Derivative:- Let $\bar{f}$ be a vector function on an interval I and $a \in I$. Then $\lim_{t \to a} \frac{\bar{f}(t) - \bar{f}(a)}{t - a}$, if exists, is called the derivative of $\bar{f}$ at a and is denoted by $\bar{f}'(a)$ or $\left(\frac{d\bar{f}}{dt}\right)$ at $t = a$.

Properties of Differentiable functions:-

① Derivative of a Constant vector is $\bar{0}$.

If $\bar{a}$ and $\bar{b}$ are differentiable vector point functions, then

② $\frac{d}{dt}(\bar{a} \pm \bar{b}) = \frac{d\bar{a}}{dt} \pm \frac{d\bar{b}}{dt}$

③ $\frac{d}{dt}(\bar{a} \cdot \bar{b}) = \frac{d\bar{a}}{dt} \cdot \bar{b} + \bar{a} \cdot \frac{d\bar{b}}{dt}$

④ $\frac{d}{dt}(\bar{a} \times \bar{b}) = \frac{d\bar{a}}{dt} \times \bar{b} + \bar{a} \times \frac{d\bar{b}}{dt}$

⑤ If $\bar{f}$ is a differentiable vector pt function and φ is a Scalar differenti~~able~~ function, then

$$\frac{d}{dt}(\varphi \bar{f}) = \varphi \cdot \frac{d\bar{f}}{dt} + \frac{d\varphi}{dt} \cdot \bar{f}$$

(6) If $\bar{f} = f_1(t)\bar{i} + f_2(t)\bar{j} + f_3(t)\bar{k}$ where $f_1(t), f_2(t)$ & $f_3(t)$ are Cartesian Components of the vector $\bar{f}$

then $\frac{d\bar{f}}{dt} = \frac{df_1}{dt}\bar{i} + \frac{df_2}{dt}\bar{j} + \frac{df_3}{dt}\bar{k}$.

7) The Necessary and sufficient Condition for $\bar{f}(t)$ to be a Constant vector function is $\frac{d\bar{f}}{dt} = \bar{0}$

Scalar and vector point functions :-

Consider a region in 3D space. To Each point $P(x, y, z)$ Suppose we associate a unique real number (called scalar) say φ. This $\varphi(x, y, z)$ is called a Scalar point function.

Hly if to Each point $P(x, y, z)$ we associate a unique vector $\bar{f}(x, y, z)$. $\bar{f}$ is called vector point function.

Eg:- 1) Let $\varphi(P)$ be the density at any point 'P' of a material body occupying a region R. Then φ is called Scalar point function.

2) Consider a particle moving in space. At Each point 'P' on its path, the article will be having a velocity $\bar{v}$ which is a vector point function.

Tangent vector to a Curve in Space :-

Consider an interval $[a\ b]$

Let $x = x(t)$ $y = y(t)$ $z = z(t)$ be Continuous and derivable for $a \leq t \leq b$.

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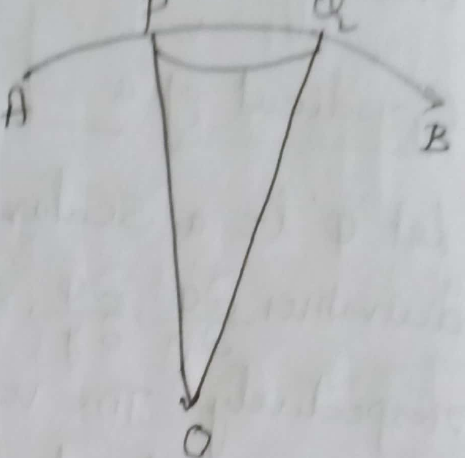
Then the set of all points $(x(t), y(t), z(t))$ is called a curve in space.

Let $A = (x(a), y(a), z(a))$ and $B = (x(b), y(b), z(b))$.

Then A, B are called the end points of the curve. If $A = B$, the curve is said to be a closed curve.



Let 'P' and 'Q' be the neighbouring points on the curve

Let $\overline{OP} = \bar{r}(t)$ $\quad \overline{OQ} = \bar{r}(t+\delta t)$

$= \bar{r} + \delta\bar{r}$

Then $\delta\bar{r} = \overline{OQ} - \overline{OP}$

$= \overline{PQ}$

Then $\frac{\delta\bar{r}}{\delta t}$ is along the vector $\overline{PQ}$. As $Q \to P$, $\overline{PQ}$ and hence $\frac{\overline{PQ}}{\delta t}$ tends to be along the tangent to the curve at P.

Hence $\lim_{\delta t \to 0} \frac{\delta\bar{r}}{\delta t} = \frac{d\bar{r}}{dt}$ will be the tangent vector to the curve at 'P'. (This $\frac{d\bar{r}}{dt}$ may not be a unit vector)

Suppose Arc length $AP = s$. If we take the parameter as the arc length parameter, we can observe $\frac{d\bar{r}}{ds}$ is unit tangent vector at P to the curve.

Vector Differential operator :-

The vector differential operator ∇ (read as del) is defined as $\nabla \equiv \bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}$.

This operator possesses properties analogous to those of ordinary vectors as well as differentiation operator.

Gradient of a Scalar point function :-

Let 'ϕ' be a Scalar point function having the direction derivatives $\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z}$ in the directions of i, j, k respectively. The vector function $i\frac{\partial \phi}{\partial x} + j\frac{\partial \phi}{\partial y} + k\frac{\partial \phi}{\partial z}$ is called the gradient of ϕ. It is written as grad ϕ (or) $\nabla\phi$

$$\boxed{\therefore \text{grad}\,\phi = \nabla\phi = \bar{i}\frac{\partial \phi}{\partial x} + \bar{j}\frac{\partial \phi}{\partial y} + \bar{k}\frac{\partial \phi}{\partial z}}$$

Note :- 1) $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$, $d\bar{r} = \bar{i}\,dx + \bar{j}\,dy + \bar{k}\,dz$

2) If ϕ is a scalar point function, then

$$d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz$$

$$= \left(i\frac{\partial \phi}{\partial x} + j\frac{\partial \phi}{\partial y} + k\frac{\partial \phi}{\partial z}\right)\cdot\left(i\,dx + j\,dy + k\,dz\right)$$

$$= (\nabla\phi)\cdot d\bar{r}$$

Properties :- ① If 'f' and 'g' are two scalar functions then grad $(f \pm g)$ = grad f $\pm$ grad g.

② The Necessary and sufficient Condition for a Scalar point function to be Constant is that $\nabla f = \bar{0}$

③ grad (fg) = f (grad g) + g (grad f)

④ If 'c' is a Constant, grad (cf) = c (grad f)

5) grad $\left(\frac{f}{g}\right) = \frac{g(\text{grad} f) - f(\text{grad} g)}{g^2}$, $g \neq 0$.

6) Let $\bar{r} =$

Directional Derivative:-

Let $\varphi(x,y,z)$ be a scalar point function defined throughout some region of space. Let this function have a value φ at a point P whose position vector referred to the origin O is $\bar{OP} = \bar{r}$.

Let $\varphi + \Delta\varphi$ be the value of the function at neighbouring point Q. If $\bar{OQ} = \bar{r} + \Delta\bar{r}$, then $\bar{PQ} = \Delta\bar{r}$.

Let Δr be the length of $\Delta\bar{r}$

$\frac{\Delta\varphi}{\Delta r}$ gives a measure of the rate at which φ change when we move from P to Q.

The limiting value of $\frac{\Delta\varphi}{\Delta r}$ as $\Delta r \to 0$ is called the derivative of φ in the direction of $\bar{PQ}$ and is denoted by $\frac{d\varphi}{dr}$.

Th:- The directional derivative of a scalar point function φ at a point $P(x,y,z)$ in the direction of a unit vector $\bar{e}$ is equal to $\bar{e} \cdot \text{grad}\,\varphi = \bar{e} \cdot \nabla\varphi$.

Level Surface:- If a surface $\varphi(x,y,z) = c$ be drawn through any point $P(\bar{r})$, such that at each point on it, the function has the same value as at P. Such a surface is called level surface of the function φ through P.

Th:- $\nabla\phi$ at any point is a vector normal to level surface $\phi(x,y,z)=c$ at P and is in increasing direction.

Its magnitude is equal to the greatest rate of increase of ϕ.

Greatest value of directional derivative of ϕ at any point 'P' $= |\text{grad}\phi|$ at that point.

1) If $a = x+y+z$, $b = x^2+y^2+z^2$, $c = xy+yz+zx$ Prove that $[\text{grad}\, a, \text{grad}\, b, \text{grad}\, c] = 0$

Sol:- Given $a = x+y+z$

$$\frac{\partial a}{\partial x} = 1, \frac{\partial a}{\partial y} = 1, \frac{\partial a}{\partial z} = 1$$

$$\therefore \text{grad}\, a = \bar{i}\frac{\partial a}{\partial x} + \bar{j}\frac{\partial a}{\partial y} + \bar{k}\frac{\partial a}{\partial z}$$

$$= \bar{i}+\bar{j}+\bar{k}$$

$$b = x^2+y^2+z^2$$

$$\frac{\partial b}{\partial x} = 2x, \frac{\partial b}{\partial y} = 2y, \frac{\partial b}{\partial z} = 2z$$

$$\text{grad}\, b = i\frac{\partial b}{\partial x} + j\frac{\partial b}{\partial y} + k\frac{\partial b}{\partial z}$$

$$= i(2x) + j(2y) + k(2z)$$

$$c = xy+yz+zx$$

$$\text{grad}\, c = i(y+z) + j(x+z) + k(y+x)$$

$$[\text{grad}\, a\ \text{grad}\, b\ \text{grad}\, c] = \begin{vmatrix} 1 & 1 & 1 \\ 2x & 2y & 2z \\ y+z & x+z & x+y \end{vmatrix} = 0$$

2) Show that $\nabla[f(r)] = \frac{f'(r)}{r}.\bar{r}$ where $\bar{r} = x\bar{i}+y\bar{j}+z\bar{k}$

Sol: Since $\bar{r} = x\bar{i}+y\bar{j}+z\bar{k}$

$r = |\bar{r}| = \sqrt{x^2+y^2+z^2}$

$r^2 = x^2+y^2+z^2$

Differentiating partially w.r.t. 'x'

$2r.\frac{\partial r}{\partial x} = 2x.$

$\boxed{\frac{\partial r}{\partial x} = \frac{x}{r}}$ Lly $\boxed{\frac{\partial r}{\partial y} = \frac{y}{r}}$ $\boxed{\frac{\partial r}{\partial z} = \frac{z}{r}}$

$\therefore \nabla[f(r)] = \left[i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right] f(r)$

$= \sum i.f'(r).\frac{\partial r}{\partial x}.$

$= \sum i.f'(r).\frac{x}{r}.$

$= \frac{f'(r)}{r}\sum x i$

$= \frac{f'(r)}{r}.\bar{r}$

3) Evaluate $\nabla(\log r)$

3) $\nabla[\log r] = \left[i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right] \log r.$

Sol

$= \sum i \frac{\partial}{\partial x}(\log r)$

$= \sum i.\frac{1}{r}.\frac{\partial r}{\partial x}.$

$= \sum i.\frac{1}{r}.\frac{x}{r}.$

$= \frac{1}{r^2}\sum x i = \frac{\bar{r}}{r^2}$

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Evaluate $\nabla(r^n)$

(4) $\nabla(r^n) = \sum i \cdot \frac{\partial}{\partial x}(r^n)$

$= \sum i \cdot n \cdot r^{n-1} \cdot \frac{\partial r}{\partial x}$

$= \sum i \, n r^{n-1} \cdot \frac{x}{r}$

$= \sum i \, n r^{n-2} \cdot x$

$= n r^{n-2} \sum x i$

$= n r^{n-2} (x\bar{i} + y\bar{j} + z\bar{k})$

$= n r^{n-2} (\bar{r})$

Show that $\nabla r = \frac{\bar{r}}{r}$

(5) $\nabla r = \frac{\bar{r}}{r}$

Sol: Let $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

$r = |\bar{r}| = \sqrt{x^2+y^2+z^2}$

$r^2 = x^2+y^2+z^2$

$2r \cdot \frac{\partial r}{\partial x} = 2x$

$\boxed{\frac{\partial r}{\partial x} = \frac{x}{r}}$ Hly $\boxed{\frac{\partial r}{\partial y} = \frac{y}{r}}$ $\boxed{\frac{\partial r}{\partial z} = \frac{z}{r}}$

$\nabla r = i\frac{\partial r}{\partial x} + j\frac{\partial r}{\partial y} + k\frac{\partial r}{\partial z}$

$= i\left(\frac{x}{r}\right) + j\left(\frac{y}{r}\right) + k\left(\frac{z}{r}\right)$

$= \frac{\bar{r}}{r}$

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6) find the directional derivative of $f = xy+yz+zx$ in the direction of vector $\bar{i}+2\bar{j}+2\bar{k}$ at the point $(1,2,0)$.

Sol:- Given $f = xy+yz+zx$

$$\text{grad} f = \bar{i}\frac{\partial f}{\partial x} + \bar{j}\frac{\partial f}{\partial y} + \bar{k}\frac{\partial f}{\partial z}$$

$$= i(y+z) + j(x+z) + k(y+x).$$

If $\bar{e}$ is the unit vector in the direction of the vector $\bar{i}+2\bar{j}+2\bar{k}$, then

$$\bar{e} = \frac{\bar{i}+2\bar{j}+2\bar{k}}{\sqrt{1+4+4}} = \frac{1}{3}\left(\bar{i}+2\bar{j}+2\bar{k}\right)$$

$\therefore$ D.D of f along the given direction $= \bar{e}\cdot\nabla f$

$$= \frac{1}{3}\left(\bar{i}+2\bar{j}+2\bar{k}\right)\cdot\left[\bar{i}(y+z)+j(x+z)+k(y+x)\right]$$

$$= \frac{1}{3}\Big[(y+z)+2(x+z)+2(x+y)\Big]_{(1,2,0)}$$

$$= \frac{10}{3}.$$

7) find the directional derivative of the function $xy^2+yz^2+zx^2$ along the tangent to the curve $x=t,\ y=t^2,\ z=t^3$ at the pt $(1,1,1)$.

Sol:- Here $f = xy^2+yz^2+zx^2$

$$\nabla f = \bar{i}\frac{\partial f}{\partial x} + \bar{j}\frac{\partial f}{\partial y} + \bar{k}\frac{\partial f}{\partial z}$$

$\nabla f = (y^2+2xz)\bar{i} + (z^2+2xy)\bar{j} + (x^2+2yz)\bar{k}$

At $(1,1,1)$ $\nabla f = 3\bar{i}+3\bar{j}+3\bar{k}$

Let $\bar{r}$ be the position vector of any point on the curve

$x=t,\ y=t^2,\ z=t^3$ Then

$\bar{r} = x\bar{i}+y\bar{j}+z\bar{k} = t\bar{i}+t^2\bar{j}+t^3\bar{k}$

$\therefore \frac{d\bar{r}}{dt} = \bar{i}+2t\bar{j}+3t^2\bar{k}$

$\left(\frac{d\bar{r}}{dt}\right)_{(1,1,1)} = \bar{i}+2\bar{j}+3\bar{k}$

we know that $\frac{d\bar{r}}{dt}$ is the vector along the tangent to the curve.

unit vector along the target $\bar{t} = \frac{\bar{i}+2\bar{j}+3\bar{k}}{\sqrt{1+4+9}} = \frac{\bar{i}+2\bar{j}+3\bar{k}}{\sqrt{14}}$

$\therefore$ Directional Derivative along the tangent $= \bar{t}\cdot\nabla f$

$= \frac{1}{\sqrt{14}}(\bar{i}+2\bar{j}+3\bar{k})\cdot(3\bar{i}+3\bar{j}+3\bar{k})$

$= \frac{3+6+9}{\sqrt{14}} = \frac{18}{\sqrt{14}}$

Ex(8):- find the d.d of the function $f=x^2-y^2+2z^2$ at the point $P=(1,2,3)$ in the direction of the line PQ where $Q=(5,0,4)$

Sol The Position vectors of 'P' and 'Q' w.r.to origin are $\bar{OP}=\bar{i}+2\bar{j}+3\bar{k}$, $\bar{OQ}=5\bar{i}+4\bar{k}$

Let $\bar{e}$ be the unit vector in the direction of $\bar{PQ}$

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Then $\bar{e} = \dfrac{4\bar{i}-2\bar{j}+\bar{k}}{\sqrt{21}}$

$$\text{grad } f = \bar{i}\frac{\partial f}{\partial x} + \bar{j}\frac{\partial f}{\partial y} + \bar{k}\frac{\partial f}{\partial z}$$

$$= 2x\bar{i} - 2y\bar{j} + 4z\bar{k}$$

The D.D of f at $P(1,2,3)$ in the direction of $\overline{PQ} = \bar{e}\cdot\nabla f$

$$= \frac{1}{\sqrt{21}}(4\bar{i}-2\bar{j}+\bar{k})\cdot(2x\bar{i}-2y\bar{j}+4z\bar{k})$$

$$= \frac{1}{\sqrt{21}}(8x+4y+4z)(1,2,3)$$

$$= \frac{1}{\sqrt{21}}(28)$$

Eg: 9) find the greatest value of the directional derivative of the function $f = x^2yz^3$ at $(2,1,-1)$

Sol: we have $\text{grad } \bar{f} = \bar{i}\dfrac{\partial f}{\partial x} + \bar{j}\dfrac{\partial f}{\partial y} + \bar{k}\dfrac{\partial f}{\partial z}$

$$= 2xyz^3\bar{i} + x^2z^3\bar{j} + 3x^2yz^2\bar{k}$$

$$(\text{grad } \bar{f})_{(2,1,-1)} = -4\bar{i} - 4\bar{j} + 12\bar{k}$$

$\therefore$ greatest value of the directional derivative of

$$f = |\nabla f| = \sqrt{16+16+144} = 4\sqrt{11}$$

10) find the directional derivative of $xyz^2 + xz$ at $(1,1,1)$ in a direction of the normal to the surface $3xy^2 + y = z$ at $(0,1,1)$

$3xy^2 + y - z = 0$

Sol:- Let $f(x,y,z) = 3xy^2 + y - z = 0$

Let us find the unit normal $\bar{e}$ to this surface at $(0,1,1)$

Then $\frac{\partial f}{\partial x} = 3y^2$; $\frac{\partial f}{\partial y} = 6xy+1$, $\frac{\partial f}{\partial z} = -1$

$\nabla f = i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z}$

$= i(3y^2) + j(6xy+1) + k(-1)$

$= 3y^2\bar{i} + (6xy+1)\bar{j} - \bar{k}$

$(\nabla f)_{(0,1,1)} = 3\bar{i} + \bar{j} - \bar{k} = \bar{n}$

$\bar{e} = \frac{\nabla f}{|\nabla f|} = \frac{3\bar{i}+\bar{j}-\bar{k}}{\sqrt{9+1+1}} = \frac{3\bar{i}+\bar{j}-\bar{k}}{\sqrt{11}}$

Let $g(x,y,z) = xyz^2 + xz$.

Then $\nabla g = \bar{i}\frac{\partial g}{\partial x} + \bar{j}\frac{\partial g}{\partial y} + \bar{k}\frac{\partial g}{\partial z}$

$= \bar{i}(yz^2+z) + \bar{j}(xz^2) + \bar{k}(2xyz + x)$

$(\nabla g)_{(1,1,1)} = 2\bar{i} + \bar{j} + 3\bar{k}$

D.D of the given function in the direction of $\bar{e}$

at $(1,1,1) = \nabla g \cdot \bar{e}$

$= (2\bar{i}+\bar{j}+3\bar{k}) \cdot \left(\frac{3i+j-k}{\sqrt{11}}\right) = \frac{4}{\sqrt{11}}$

11) find the directional derivative of $2xy+z^2$ at $(1,-1,3)$ in the direction of $\bar{i}+2\bar{j}+3\bar{k}$

Sol:- Let $f = 2xy+z^2$

$$\frac{\partial f}{\partial x} = 2y,\ \frac{\partial f}{\partial y} = 2x,\ \frac{\partial f}{\partial z} = 2z$$

$$\therefore \text{grad} f = \sum \bar{i}\frac{\partial f}{\partial x}$$

$$= 2y\bar{i}+2x\bar{j}+2z\bar{k}$$

$$(\text{grad} f)_{(1,-1,3)} = -2\bar{i}+2\bar{j}+6\bar{k}$$

Given vector $\bar{a} = \bar{i}+2\bar{j}+3\bar{k}$

$$|\bar{a}| = \sqrt{1+4+9} = \sqrt{14}.$$

$\therefore$ Directional Derivative of 'f' in the direction $\bar{a}$ is

$$\frac{\bar{a}\cdot\nabla f}{|\bar{a}|} = \frac{(\bar{i}+2\bar{j}+3\bar{k})\cdot(-2\bar{i}+2\bar{j}+6\bar{k})}{\sqrt{14}}$$

$$= \frac{-2+4+18}{\sqrt{14}} = \frac{20}{\sqrt{14}}$$

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12) find the directional derivative of $\varphi = x^2yz + 4xz^2$ in the direction $2\bar{i}-\bar{j}-2\bar{k}$. at $(1,-2,-1)$ in the direction $2\bar{i}-\bar{j}-2\bar{k}$.

Sol:- Given $\varphi = x^2yz+4xz^2$

$$\frac{\partial \varphi}{\partial x} = 2xyz+4z^2,\ \frac{\partial \varphi}{\partial y} = x^2z,\ \frac{\partial \varphi}{\partial z} = x^2y+8xz$$

Hence $\nabla\varphi = \sum \bar{i}\frac{\partial \varphi}{\partial x}$

$$= \bar{i}(2xyz+4z^2)+\bar{j}(x^2z)+\bar{k}(x^2y+8xz)$$

$$(\nabla\phi)_{(1,-2,-1)} = \bar{i}(4+4)+\bar{j}(-1)+\bar{k}(-2\cdot 8)$$
$$= 8\bar{i}-\bar{j}-10\bar{k}$$

The unit vector in the direction $2\bar{i}-\bar{j}-2\bar{k}$

$$\bar{a} = \frac{2\bar{i}-\bar{j}-2\bar{k}}{\sqrt{4+1+4}} = \frac{2\bar{i}-\bar{j}-2\bar{k}}{3}$$

$\therefore$ Required directional derivative

$$= (\nabla\phi)\cdot\bar{a}$$
$$= (8\bar{i}-\bar{j}-10\bar{k})\left(\frac{2\bar{i}-\bar{j}-2\bar{k}}{3}\right)$$
$$= \frac{1}{3}(16+1+20) = \frac{37}{3}$$

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13) Find the directional derivative of $f(x,y,z) = xy^2+yz^3$ at the point $(2,-1,1)$ in the direction of the vector $\bar{i}+2\bar{j}+2\bar{k}$

Sol:- Given:- $f(x,y,z) = xy^2+yz^3$

$$\therefore \frac{\partial f}{\partial x} = y^2, \frac{\partial f}{\partial y} = 2xy+z^3, \frac{\partial f}{\partial z} = 3yz^2$$

Hence $\nabla f = \bar{i}\frac{\partial f}{\partial x} + \bar{j}\frac{\partial f}{\partial y} + \bar{k}\frac{\partial f}{\partial z}$

$$= \bar{i}(y^2)+\bar{j}(2xy+z^3)+\bar{k}(3yz^2)$$

$$(\nabla f)_{(2,-1,1)} = \bar{i}+\bar{j}(-4+1)+\bar{k}(-3)$$
$$= \bar{i}-3\bar{j}-3\bar{k}$$

The unit vector in the direction of the vector

$\bar{i}+2\bar{j}+2\bar{k}$ is

$$\bar{e} = \frac{\bar{i}+2\bar{j}+2\bar{k}}{\sqrt{1+4+4}} = \frac{\bar{i}+2\bar{j}+2\bar{k}}{3}$$

∴ Required directional derivative along the given direction

$$= \bar{e}\cdot\nabla f$$

$$= \frac{1}{3}(\bar{i}+2\bar{j}+2\bar{k})\cdot(\bar{i}-3\bar{j}-3\bar{k})$$

$$= \frac{1}{3}(1-6-6) = \frac{-11}{3}.$$

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14) In what direction from the point $(-1,1,2)$ is the directional derivative of $\phi = xy^2z^3$ a maximum. what is the magnitude of this maximum.

Sol:- We have $\phi = xy^2z^3$.

$$\Rightarrow \frac{\partial\phi}{\partial x} = y^2z^3, \quad \frac{\partial\phi}{\partial y} = 2xyz^3, \quad \frac{\partial\phi}{\partial z} = 3xy^2z^2$$

We know that the vector $\nabla\phi$ is along the normal to the surface $\phi(x,y,z)=c$ at a point $P(x,y,z)$

Now $$\nabla\phi = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}$$

$$= y^2z^3\bar{i} + \bar{j}(2xyz^3) + \bar{k}(3xy^2z^2)$$

$$(\nabla\phi)_{(-1,1,2)} = 8\bar{i} - 16\bar{j} - 12\bar{k}$$

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15) find the directional derivative of $\frac{1}{r}$ in the direction of $\bar{r} = x\bar{i}+y\bar{j}+z\bar{k}$ at $(1,1,2)$

Sol:- Given $\bar{r} = x\bar{i}+y\bar{j}+z\bar{k}$

$$r = |\bar{r}| = \sqrt{x^2+y^2+z^2}, \quad r^2 = x^2+y^2+z^2$$

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$$\frac{1}{r} = \frac{1}{\sqrt{x^2+y^2+z^2}}$$

$$\frac{\partial}{\partial x}\left(\frac{1}{r}\right) = \frac{-1}{2}\cdot\frac{1}{(x^2+y^2+z^2)^{3/2}}(2x) = -\frac{x}{(x^2+y^2+z^2)^{3/2}}$$

$$\text{lly } \frac{\partial}{\partial y}\left(\frac{1}{r}\right) = -\frac{y}{(x^2+y^2+z^2)^{3/2}}, \quad \frac{\partial}{\partial z}\left(\frac{1}{r}\right) = -\frac{z}{(x^2+y^2+z^2)^{3/2}}$$

$$\therefore \text{grad}\left(\frac{1}{r}\right) = \bar{i}\frac{\partial}{\partial x}\left(\frac{1}{r}\right) + \bar{j}\cdot\frac{\partial}{\partial y}\left(\frac{1}{r}\right) + \bar{k}\cdot\frac{\partial}{\partial z}\left(\frac{1}{r}\right)$$

$$= \bar{i}\left(\frac{-x}{(x^2+y^2+z^2)^{3/2}}\right) + \bar{j}\left(\frac{-y}{(x^2+y^2+z^2)^{3/2}}\right) + \bar{k}\left(\frac{-z}{(x^2+y^2+z^2)^{3/2}}\right)$$

$$= -\frac{(x\bar{i}+y\bar{j}+z\bar{k})}{(x^2+y^2+z^2)^{3/2}} = -\frac{\bar{r}}{r^3} \rightarrow (1)$$

Hence the directional derivative of $\left(\frac{1}{r}\right)$ in the direction of

$$\bar{r} = \text{grad}\left(\frac{1}{r}\right)\cdot\frac{\bar{r}}{|\bar{r}|}$$

$$= \frac{-\bar{r}}{r^3}\cdot\frac{\bar{r}}{|\bar{r}|}$$

$$= -\frac{(x^2+y^2+z^2)}{r^4} = \frac{1}{(x^2+y^2+z^2)}$$

Hence the directional derivative of $\left(\frac{1}{r}\right)$ in the direction of $\bar{r}$ at $(1,1,2) = -1/6$.

Eg:- 16) find the d.d of $\nabla\cdot\nabla\phi$ at that point $(1,-2,1)$ in the direction of normal to the surface $xy^2z = 3x+z^2$ where $\phi = 2x^3y^2z^4$.

Sol:- Given $\varphi = 2x^3y^2z^4$.

Let $g = \nabla\cdot\nabla\varphi = \nabla^2\varphi = \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2}$

$g = 12xy^2z^4 + 4x^3z^4 + 24x^3y^2z^2$

Given Surfaces is $f(x,y,z) = xy^2z - 3x - z^2$

$\frac{\partial f}{\partial x} = y^2z - 3$, $\frac{\partial f}{\partial y} = 2xyz$, $\frac{\partial f}{\partial z} = xy^2 - 2z$

Normal to the Surface is

$\nabla f = \bar{i}(y^2z - 3) + \bar{j}(2xyz) + \bar{k}(xy^2 - 2z)$

$\bar{n}$ = Normal to the Surface at the point $(1,-2,1)$

$= \bar{i} - 4\bar{j} + 2\bar{k}$. $\bar{e} = \frac{\bar{n}}{|\bar{n}|} = \frac{\bar{i} - 4\bar{j} + 2\bar{k}}{\sqrt{1+16+4}}$

and

$\nabla g = \bar{i}(12y^2z^4 + 12x^2z^4 + 72x^2y^2z^2) + (24xyz^4 + 48x^3yz^2)\bar{j}$
$+ (48xy^2z^3 + 16x^3z^3 + 48x^3y^2z)\bar{k}$

$(\nabla g)_{(1,-2,1)} = 348\bar{i} - 144\bar{j} + 400\bar{k}$

Thus the required d.d is $\bar{e}\cdot\nabla g$

$= (348\bar{i} - 144\bar{j} + 400\bar{k}) \cdot \frac{(\bar{i} - 4\bar{j} + 2\bar{k})}{\sqrt{21}}$

$= \frac{1724}{\sqrt{21}}$

---

17) find the d.d of $\varphi(x,y,z) = x^2yz + 4xz^2$ at the pt $(1,-2,-1)$ in the direction of the normal to the surface $f(x,y,z) = x\log z - y^2$ at $(1,-2,1)$

Sol: Given $\phi(x,y,z) = x^2yz + 4xz^2$ at $(1,-2,-1)$

and $f(x,y,z) = x\log z - y^2$ at $(-1,2,1)$

Now $\nabla\phi = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}$

$= (2xyz + 4z^2)\bar{i} + (x^2z)\bar{j} + (x^2y + 8xz)\bar{k}$

$\therefore (\nabla\phi)_{(1,-2,-1)} = 8\bar{i} - \bar{j} - 10\bar{k} \longrightarrow (1)$

Unit normal to the surface $f(x,y,z) = x\log z - y^2$ is $\frac{\nabla f}{|\nabla f|}$

Now $\nabla f = \bar{i}\frac{\partial f}{\partial x} + \bar{j}\frac{\partial f}{\partial y} + \bar{k}\frac{\partial f}{\partial z}$

$= (\log z)\bar{i} + (-2y)\bar{j} + \frac{x}{z}\bar{k}$

At $(-1,2,1)$, $(\nabla f) = \log 1\,\bar{i} - 2(2)\bar{j} + \frac{-1}{1}\bar{k}$

$= -4\bar{j} - \bar{k}$

$\therefore \frac{\nabla f}{|\nabla f|} = \frac{-4\bar{j}-\bar{k}}{\sqrt{16+1}} = \frac{-4\bar{j}-\bar{k}}{\sqrt{17}}$

Directional Derivative $= \nabla\phi \cdot \frac{\nabla f}{|\nabla f|}$

$= (8\bar{i} - \bar{j} - 10\bar{k}) \cdot \left(\frac{-4\bar{j}-\bar{k}}{\sqrt{17}}\right)$

$= \frac{4+10}{\sqrt{17}} = \frac{14}{\sqrt{17}}$

18) Find a unit normal vector to the surface $x^2y + 2xz = 4$ at the point $(2,-2,3)$.

Sol:- Let the given surface be $f = x^2y + 2xz - 4$.

$\frac{\partial f}{\partial x} = 2xy + 2z$, $\frac{\partial f}{\partial y} = x^2$, $\frac{\partial f}{\partial z} = 2x$.

$\text{grad}\,\bar{f} = \sum \bar{i}\frac{\partial F}{\partial x}$

$= (2xy+2z)\bar{i} + x^2\bar{j} + 2x\bar{k}$

$(\text{grad}\,\bar{f})_{(2,-2,3)} = -2\bar{i} + 4\bar{j} + 4\bar{k}$

$\text{grad}\,\bar{f}$ is the normal vector to the given surface at the given point

Hence the req unit normal vector $= \frac{\nabla f}{|\nabla f|} = \frac{2(-\bar{i}+2\bar{j}+2\bar{k})}{2\sqrt{1+4+4}}$

$= \frac{-\bar{i}+2\bar{j}+2\bar{k}}{3}.$

---

19) Evaluate the angle between the normals to the surface $xy = z^2$ at the points $(4,1,2)$ and $(3,3,-3)$

Sol:- Given surface is $f(x,y,z) = xy - z^2$

Let $\bar{n}_1$ and $\bar{n}_2$ be the normals to this surface at $(4,1,2)$ and $(3,3,-3)$ respectively.

$\frac{\partial f}{\partial x} = y$, $\frac{\partial f}{\partial y} = x$, $\frac{\partial f}{\partial z} = -2z$

$\text{grad}\,f = y\bar{i} + x\bar{j} - 2z\bar{k}$

$\bar{n}_1 = (\text{grad}\,\bar{f})_{(4,1,2)} = \bar{i} + 4\bar{j} - 4\bar{k}$

$\bar{n}_2 = (\text{grad}\,\bar{f})_{(3,3,-3)} = 3\bar{i} + 3\bar{j} - 6\bar{k}$

Let 'θ' be the angle between the two normals

$$\cos\theta = \frac{\bar{n}_1 \cdot \bar{n}_2}{|\bar{n}_1||\bar{n}_2|}$$

$$= \frac{(\bar{i}+4\bar{j}-4\bar{k}) \cdot (3\bar{i}+3\bar{j}+6\bar{k})}{\sqrt{33} \cdot \sqrt{54}}$$

$$= \frac{3+12-24}{\sqrt{33} \cdot \sqrt{54}} = \frac{-9}{\sqrt{33} \cdot \sqrt{54}}.$$

2) Find a unit normal vector to the surface $x^2+y^2+2z^2=26$ at the point $(2,2,3)$

Let the Given Surface be $f(x,y,z) = x^2+y^2+2z^2-26=0$

Then $\frac{\partial f}{\partial x} = 2x$, $\frac{\partial f}{\partial y} = 2y$, $\frac{\partial f}{\partial z} = 4z$

$$\text{grad}\,\bar{f} = \sum i \frac{\partial f}{\partial x}$$

$$= 2x\bar{i} + 2y\bar{j} + 4z\bar{k}$$

Normal vector at $(2,2,3) = (\nabla f)_{(2,2,3)} = 4\bar{i}+4\bar{j}+12\bar{k}$

$$\text{Unit normal vector} = \frac{\nabla f}{|\nabla f|} = \frac{4\bar{i}+4\bar{j}+12\bar{k}}{\sqrt{16+16+144}}$$

$$= \frac{\bar{i}+\bar{j}+3\bar{k}}{\sqrt{11}}$$

4) Find the constants 'a' and 'b' so that the surface $ax^2-byz=(a+2)x$ will be orthogonal to the surface $4x^2y+z^3=4$ at the point $(-1,1,2)$

Sol:- Let the given Surfaces be

$f(x,y,z) = ax^2 - byz - (a+2)x \longrightarrow (1)$

$g(x,y,z) = 4x^2y + z^3 - 4 \longrightarrow (2)$

Given the two Surfaces meet at the point $(1,-1,2)$

Substituting the point in Eq (1), we get

$a + 2b - (a+2) = 0$

$\boxed{b = 1}$

Now, $\frac{\partial f}{\partial x} = 2ax - (a+2)$

$\frac{\partial f}{\partial y} = -bz, \quad \frac{\partial f}{\partial z} = -by.$

$\nabla f = \sum \bar{i} \frac{\partial f}{\partial x} = [2ax - (a+2)]\bar{i} - bz\bar{j} - by\bar{k}$

$(\nabla f)_{(1,-1,2)} = [2a - (a+2)]\bar{i} - 2b\bar{j} + b\bar{k}$

$= (a-2)\bar{i} - 2b\bar{j} + b\bar{k}$

$= (a-2)\bar{i} - 2\bar{j} + \bar{k} = \bar{n}_1$, normal to the Surface (1)

Also $\frac{\partial g}{\partial x} = 8xy, \quad \frac{\partial g}{\partial y} = 4x^2, \quad \frac{\partial g}{\partial z} = 3z^2$

$\nabla g = \sum \bar{i} \frac{\partial g}{\partial x} = 8xy\bar{i} + 4x^2\bar{j} + 3z^2\bar{k}$

$(\nabla g)_{(1,-1,2)} = -8\bar{i} + 4\bar{j} + 12\bar{k} = \bar{n}_2$, normal to the Surface (2).

Given the Surfaces $f(x,y,z) \cdot g(x,y,z)$ are orthogonal at the point $(1,-1,2)$

$[\nabla f][\nabla g] = 0 \Leftrightarrow ((a-2)\bar{i} - 2\bar{j} + \bar{k}) \cdot (-8\bar{i} + 4\bar{j} + 12\bar{k}) = 0$

$\Rightarrow -8a + 16 - 8 + 12 = 0, \quad a = 5/2$

22) Find the angle between the normals to the Surface $x^2 = yz$ at the points $(1,1,1)$ and $(2,4,1)$

Sol:- Given surface is $f(x,y,z) = x^2 - yz$

Let $\bar{n}_1$ and $\bar{n}_2$ be the normals to this surface at the points $(1,1,1)$ and $(2,4,1)$ respectively.

$\frac{\partial f}{\partial x} = 2x, \quad \frac{\partial f}{\partial y} = -z, \quad \frac{\partial f}{\partial z} = -y$

$$\text{grad}\,\bar{f} = \bar{i}\frac{\partial f}{\partial x} + \bar{j}\frac{\partial f}{\partial y} + \bar{k}\frac{\partial f}{\partial z}$$

$$= \bar{i}(2x) + \bar{j}(-z) + \bar{k}(-y)$$

$$\bar{n}_1 = (\text{grad}\,\bar{f})_{(1,1,1)} = 2\bar{i} - \bar{j} - \bar{k}$$

$$\bar{n}_2 = (\text{grad}\,\bar{f})_{(2,4,1)} = 4\bar{i} - \bar{j} - 4\bar{k}$$

Let 'θ' be the angle between the two surfaces. Then

$$\cos\theta = \frac{\bar{n}_1 \cdot \bar{n}_2}{|\bar{n}_1|\,|\bar{n}_2|}$$

$$= \frac{(4\bar{i} - 2\bar{j} + 4\bar{k}) \cdot (4\bar{i} - 2\bar{j} - \bar{k})}{\sqrt{36} \cdot \sqrt{21}}$$

$$= \frac{16 + \cancel{4} - \cancel{4}}{6\sqrt{21}} = \frac{16}{6\sqrt{21}} = \frac{8}{3\sqrt{21}}$$

$$\theta = \cos^{-1}\left(\frac{8}{3\sqrt{21}}\right)$$

Divergence of a vector:- Let $\bar{f}$ be any continuously differentiable vector point function. Then $\bar{i}\frac{\partial f}{\partial x} + \bar{j}\frac{\partial f}{\partial y} + \bar{k}\frac{\partial f}{\partial z}$ is called the divergence of $\bar{f}$ and is written as $\text{div}\,\bar{f}$.

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i.e $div\bar{f} = \bar{i}\frac{\partial f}{\partial x} + \bar{j}\frac{\partial \bar{f}}{\partial y} + \bar{k}\frac{\partial \bar{f}}{\partial z}$

$= \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)\cdot\bar{f}$

$\boxed{div\bar{f} = \nabla\cdot\bar{f}}$ This is a Scalar point function.

Note :- $\nabla\cdot\bar{f} \neq \bar{f}\cdot\nabla$.

Th:- If the vector function $\bar{f} = f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$, then

$div\bar{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$.

Note :- If $\bar{f}$ is a Constant vector, then $div\bar{f} = 0$

Solenoidal vector :-

A vector function $\bar{f}$ is Said to be Solenoidal if

$div\bar{f} = 0$

This Equation is called the Equation of Continuity a Conservation of mass.

① If $\bar{f} = xy^2\bar{i} + 2x^2yz\bar{j} - 3yz^2\bar{k}$ find $div\bar{f}$ at $(1,-1,1)$

Sol Given $\bar{f} = xy^2\bar{i} + 2x^2yz\bar{j} - 3yz^2\bar{k}$

$div\bar{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$

$= y^2 + 2x^2z - 6yz$

$(div\bar{f})_{(1,-1,1)} = 1 + 2 + 6 = 9$.

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② find $div\bar{f}$ when $\bar{f} = grad\,(x^3+y^3+z^3-3xyz)$.

Sol:- Let $\varphi = x^3+y^3+z^3-3xyz$.

Then $\frac{\partial\varphi}{\partial x} = 3x^2-3yz$, $\frac{\partial\varphi}{\partial y} = 3y^2-3xz$, $\frac{\partial\varphi}{\partial z} = 3z^2-3xy$.

$$\text{grad}\,\varphi = \bar{i}\frac{\partial\varphi}{\partial x} + \bar{j}\frac{\partial\varphi}{\partial y} + \bar{k}\frac{\partial\varphi}{\partial z}$$

$$= 3\left[(x^2-yz)\bar{i} + (y^2-zx)\bar{j} + (z^2-xy)\bar{k}\right]$$

Hence $\text{div}\,\bar{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$

$$= \frac{\partial}{\partial x}\left[3(x^2-yz)\right] + \frac{\partial}{\partial y}\left[3(y^2-zx)\right] + \frac{\partial}{\partial z}\left[3(z^2-xy)\right]$$

$$= 3(2x) + 3(2y) + 3(2z)$$

$$= 6(x+y+z)$$

3) If $\bar{f} = (x+3y)\bar{i} + (y-2z)\bar{j} + (x+pz)\bar{k}$ is solenoidal, find p

Sol:- Let $\bar{f} = (x+3y)\bar{i} + (y-2z)\bar{j} + (x+pz)\bar{k}$

$\text{div}\,\bar{f} = 1+1+p$

Since $\bar{f}$ is solenoidal, we have $\text{div}\,\bar{f} = 0$.

$\Rightarrow p+2 = 0$.

$\Rightarrow p = -2$.

4) find $\text{div}\,\bar{f}$ where $\bar{f} = r^n\bar{r}$. find 'n' if it is solenoidal

(or) P.T $r^n\bar{r}$ is solenoidal if $n=-3$

where $\bar{r} = x\bar{i}+y\bar{j}+z\bar{k}$ and $r = |\bar{r}|$

Sol:- Given $\bar{f} = r^n\bar{r}$ where $\bar{r} = x\bar{i}+y\bar{j}+z\bar{k}$

we have $r^2 = x^2+y^2+z^2$

$\boxed{\frac{\partial r}{\partial x} = \frac{x}{r}}$ Ily we have $\boxed{\frac{\partial r}{\partial y} = \frac{y}{r}}$ $\boxed{\frac{\partial r}{\partial z} = \frac{z}{r}}$

$\therefore \bar{f} = r^n(x\bar{i}+y\bar{j}+z\bar{k})$.

$\therefore \text{div}\,\bar{f} = \frac{\partial}{\partial x}(r^n x) + \frac{\partial}{\partial y}(r^n y) + \frac{\partial}{\partial z}(r^n z)$

$= n r^{n-1} \frac{\partial r}{\partial x}(x) + r^n + n r^{n-1} \frac{\partial r}{\partial y}(y) + r^n + n r^{n-1} \frac{\partial r}{\partial z} \cdot z + r^n$

$= n \cdot r^{n-1} \left[\frac{x^2}{r} + \frac{y^2}{r} + \frac{z^2}{r} \right] + 3 r^n.$

$= n \cdot r^{n-1} \left[\frac{r^2}{r} \right] + 3 r^n.$

$= n \cdot r^n + 3 \cdot r^n.$

$= (n+3) r^n$

Let $\bar{f} = r^n \bar{r}$ be solenoidal, Then $\operatorname{div} \bar{f} = 0$.

$(n+3) \cdot r^n = 0 \Rightarrow n = -3.$

5) Evaluate $\nabla \cdot \left(\frac{\bar{r}}{r^3} \right)$ where $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ and $r = |\bar{r}|$.

Sol. We have $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

$r = |\bar{r}| = \sqrt{x^2+y^2+z^2}$

$r^2 = x^2+y^2+z^2$

$\boxed{\frac{\partial r}{\partial x} = \frac{x}{r}}, \boxed{\frac{\partial r}{\partial y} = \frac{y}{r}}, \boxed{\frac{\partial r}{\partial z} = \frac{z}{r}}$

$\frac{\bar{r}}{r^3} = \bar{r} \cdot r^{-3} = r^{-3} x \bar{i} + r^{-3} y \bar{j} + r^{-3} z \bar{k} = f_1 \bar{i} + f_2 \bar{j} + f_3 \bar{k}$

Hence $\nabla \cdot \left(\frac{\bar{r}}{r^3} \right) = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$

We have $f_1 = r^{-3} x.$

$\Rightarrow \frac{\partial f_1}{\partial x} = r^{-3}(1) + x \cdot (-3) \cdot r^{-4} \frac{\partial r}{\partial x}$

$= r^{-3} - 3x r^{-4} \cdot \frac{x}{r} = r^{-3} - 3x^2 r^{-5}$

$\therefore \nabla \cdot \left(\frac{\bar{r}}{r^3} \right) = \sum \frac{\partial f_1}{\partial x} = \sum \left(r^{-3} - 3x^2 r^{-5} \right)$

$= 3 r^{-3} - 3 r^{-5} \sum x^2$

$= 3 r^{-3} - 3 r^{-5} (r^2) = 0.$

Eg:- find div $\bar{r}$ where $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

Sol:- we have $\bar{r} = x\hat{i} + y\bar{j} + z\hat{k} = f_1\hat{i} + f_2\hat{j} + f_3\hat{k}$

$$\operatorname{div} \bar{r} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

$$= 1 + 1 + 1 = 3.$$

Curl of a vector:- Let $\bar{f}$ be any Continuously differentiable vector point function. Then the vector function defined by $\bar{i} \times \frac{\partial \bar{f}}{\partial x} + \bar{j} \times \frac{\partial \bar{f}}{\partial y} + \bar{k} \times \frac{\partial \bar{f}}{\partial z}$ is called Curl of $\bar{f}$ and is denoted by Curl $\bar{f}$ or $(\nabla \times \bar{f})$.

Th:- If $\bar{f}$ is a differentiable vector point function given by $\bar{f} = f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$ then

$$\operatorname{Curl} \bar{f} = \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right)\hat{i} + \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right)\bar{j} + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)\bar{k}$$

Note:- 1) $\operatorname{Curl} \bar{f} = \nabla \times \bar{f} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$

2) If $\bar{f}$ is a Constant vector then Curl $\bar{f} = \bar{0}$.

3) $\operatorname{Curl}(\bar{a} \pm \bar{b}) = \operatorname{Curl} \bar{a} \pm \operatorname{Curl} \bar{b}$

Irrotational motion:- Any motion in which the Curl of the velocity vector is a null vector is said to be irrotational.

i.e Curl $\bar{v} = 0$ is said to be irrotational.

Irrotational vector:- A vector $\bar{f}$ is said to be irrotational if Curl $\bar{f} = \bar{0}$

If $\bar{f}$ is irrotational, there will always exist a scalar function $\phi(x,y,z)$ such that $\bar{f} = \text{grad}\,\phi$.
This ϕ is called scalar potential of $\bar{f}$.
It is easy to prove that if $\bar{f} = \text{grad}\,\phi$, then $\text{curl}\,\bar{f} = 0$.
Hence $\nabla \times \bar{f} = 0 \Leftrightarrow$ there exist a scalar function ϕ such that $\bar{f} = \nabla\phi$.

Ex:- ① If $\bar{f} = xy^2\bar{i} + 2x^2yz\bar{j} - 3yz^2\bar{k}$ find curl $\bar{f}$ at the point $(1,-1,1)$

Sol:- Let $\bar{f} = xy^2\bar{i} + 2x^2yz\bar{j} - 3yz^2\bar{k}$

$$\text{curl}\,\bar{f} = \nabla \times \bar{f} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy^2 & 2x^2yz & -3yz^2 \end{vmatrix}$$

$$= \bar{i}\left[\frac{\partial}{\partial y}(-3yz^2) - \frac{\partial}{\partial z}(2x^2yz)\right] - \bar{j}\left[\frac{\partial}{\partial x}(-3yz^2) - \frac{\partial}{\partial z}(xy^2)\right] + \bar{k}\left[\frac{\partial}{\partial x}(2x^2yz) - \frac{\partial}{\partial y}(xy^2)\right]$$

$$= \bar{i}\left[-3z^2 - 2x^2y\right] - \bar{j}\left[0-0\right] + \bar{k}\left[4xyz - 2xy\right]$$

$$= -(3z^2 + 2x^2y)\bar{i} + (4xyz - 2xy)\bar{k}$$

$$(\text{curl}\,\bar{f})_{(1,-1,1)} = -\bar{i} - 2\bar{k}$$

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② find curl $\bar{f}$ where $\bar{f} = \text{grad}\,(x^3+y^3+z^3-3xyz)$.

Sol:- Let $\phi = x^3+y^3+z^3-3xyz$

$$\bar{f} = \text{grad}\,\phi = \sum \bar{i}\frac{\partial \phi}{\partial x} = \bar{i}(3x^2-3yz) + \bar{j}(3y^2-3xz) + \bar{k}(3z^2-3xy)$$

$$\text{curl}\,\bar{f} = \text{curl}(\text{grad}\,\phi) = \nabla \times \nabla\phi = 3\begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x^2-yz) & (y^2-zx) & (z^2-xy) \end{vmatrix} = \bar{0}$$

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Note:- we can prove in general that $\text{Curl}(\text{grad}\,\phi) = \bar{0}$ i.e $\text{grad}\,\phi$ is always irrotational.

(3) If $\bar{f} = (x+y+1)\bar{i} + \bar{j} - (x+y)\bar{k}$ then show that $\bar{f} \cdot \text{Curl}\,\bar{f} = 0$

Sol: Given: $\bar{f} = (x+y+1)\bar{i} + \bar{j} - (x+y)\bar{k}$

$$\text{Curl}\,\bar{f} = \nabla \times \bar{f} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x+y+1) & 1 & -(x+y) \end{vmatrix}$$

$$= i(-1-0) - j(-1-0) + k(0-1)$$

$$= -i + j - \bar{k}$$

$$\therefore \bar{f} \cdot \text{Curl}\,\bar{f} = \left[(x+y+1)i + \bar{j} - (x+y)\bar{k}\right] \cdot \left(-\bar{i} + \bar{j} - \bar{k}\right)$$

$$\Rightarrow -x-y-1+x+y = 0$$

$$= 0.$$

(4) Show that $\text{Curl}(r^n \bar{r}) = \bar{0}$

Sol:- Let $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

$$r = |\bar{r}| = \sqrt{x^2+y^2+z^2}$$

$$r^2 = x^2+y^2+z^2$$

Differentiating partially w.r.to x

$$\boxed{\frac{\partial r}{\partial x} = \frac{x}{r}} \quad \boxed{\frac{\partial r}{\partial y} = \frac{y}{r}} \quad \boxed{\frac{\partial r}{\partial z} = \frac{z}{r}}$$

we have $r^n \bar{r} = r^n(x\bar{i} + y\bar{j} + z\bar{k})$

$$= (r^n x)\bar{i} + (r^n y)\bar{j} + (r^n z)\bar{k}$$

$= \bar{i}\left\{\frac{\partial}{\partial y}(r^n z) - \frac{\partial}{\partial z}(r^n y)\right\} - \bar{j}\left\{\frac{\partial}{\partial x}(r^n z) - \frac{\partial}{\partial z}(r^n x)\right\} +$

$\bar{k}\left\{\frac{\partial}{\partial x}(r^n y) - \frac{\partial}{\partial y}(r^n x)\right\}.$

$= \sum \bar{i}\left\{ z \cdot n r^{n-1}\frac{\partial r}{\partial y} - y n r^{n-1} \cdot \frac{\partial r}{\partial z}\right\}$

$= n r^{n-1} \sum \bar{i}\left\{ z\left(\frac{y}{r}\right) - y\left(\frac{z}{r}\right)\right\}.$

$= n r^{n-2}\left[(zy - yz)\bar{i} + (xz - zx)\bar{j} + (xy - yz)\bar{k}\right]$

$= n r^{n-2}\left[0\bar{i} + 0\bar{j} + 0\bar{k}\right]$

$= \bar{0}$

Hence $r^n \bar{r}$ is irrotational.

---

(3) Prove that Curl $\bar{r} = \bar{0}$.

$\nabla \times \bar{r} = \begin{vmatrix} i & j & k \\ & & \\ & & \end{vmatrix}$

Let $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

Curl $\bar{r} = \sum \bar{i} \times \frac{\partial}{\partial x}(\bar{r})$

$= \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix}$

$= \bar{i}\left[\frac{\partial z}{\partial y} - \frac{\partial y}{\partial z}\right] - \bar{j}\left[\frac{\partial z}{\partial x} - \frac{\partial x}{\partial z}\right] + \bar{k}\left[\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y}\right]$

$= \sum(\bar{i} \times \bar{i}) \quad = (\bar{i}\times\bar{i}) + (\bar{j}\times\bar{j}) + (\bar{k}\times\bar{k})$

$= \bar{0} + \bar{0} + \bar{0}$

$= \bar{0}$

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(6) If $\bar{a}$ is a Constant vector, Prove that

$$\text{Curl}\left(\frac{\bar{a}\times\bar{r}}{r^3}\right) = -\frac{\bar{a}}{r^3} + \frac{3\cdot\bar{r}}{r^5}(\bar{a}\cdot\bar{r})$$

Sol:- We have $\bar{r} = x\bar{i}+y\bar{j}+z\bar{k}$

$\therefore \frac{\partial\bar{r}}{\partial x} = \bar{i}, \frac{\partial\bar{r}}{\partial y} = \bar{j}, \frac{\partial r}{\partial z} = \bar{k}$ ✓

If $r = |\bar{r}|$ then $r^2 = x^2+y^2+z^2$

we have $\frac{\partial r}{\partial x} = \frac{x}{r}, \frac{\partial r}{\partial y} = \frac{y}{r}, \frac{\partial r}{\partial z} = \frac{z}{r}$,

$$\text{Curl}\left(\frac{\bar{a}\times\bar{r}}{r^3}\right) = \sum i\times\frac{\partial}{\partial x}\left(\frac{\bar{a}\times\bar{r}}{r^3}\right)$$

Now

$$\frac{\partial}{\partial x}\left(\frac{\bar{a}\times\bar{r}}{r^3}\right) = \bar{a}\times\frac{\partial}{\partial x}\left(\frac{\bar{r}}{r^3}\right)$$

$$= \bar{a}\times\left[\frac{1}{r^3}\cdot\frac{\partial\bar{r}}{\partial x} - \frac{3}{r^4}\frac{\partial r}{\partial x}\cdot\bar{r}\right]$$

$$= \bar{a}\times\left[\frac{1}{r^3}\bar{i} - \frac{3}{r^4}\left(\frac{x}{r}\right)\cdot\bar{r}\right]$$

$$= \frac{(\bar{a}\times i)}{r^3} - \frac{3x\cdot(\bar{a}\times\bar{r})}{r^5}$$

$$i\times\frac{\partial}{\partial x}\left(\frac{\bar{a}\times\bar{r}}{r^3}\right) = i\times\left[\frac{(\bar{a}\times i)}{r^3} - \frac{3x}{r^5}(\bar{a}\times\bar{r})\right]$$

$$= \frac{i\times(\bar{a}\times i)}{r^3} - \frac{3x}{r^5}\, i\times(\bar{a}\times\bar{r})$$

$$= \frac{(i\cdot i)\bar{a}-(i\cdot\bar{a})i}{r^3} - \frac{3x}{r^5}\left[(i\cdot\bar{r})\bar{a}-(i\cdot\bar{a})\bar{r}\right]$$

Let $\bar{a} = a_1\bar{i} + a_2\bar{j} + a_3\bar{k}$. Then

$i \cdot \bar{a} = a_1$

$\therefore i \times \frac{\partial}{\partial x}\left(\frac{\bar{a}\times\bar{r}}{r^3}\right) = \frac{(\bar{a} - a_1 i)}{r^3} - \frac{3x}{r^5}(x\bar{a} - a_1\bar{r})$.

$\sum i \times \frac{\partial}{\partial x}\left(\frac{\bar{a}\times\bar{r}}{r^3}\right) = \sum \frac{\bar{a} - a_1 i}{r^3} - \frac{3}{r^5}\sum(x^2\bar{a} - a_1 x\bar{r})$

$= \frac{3\bar{a} - \bar{a}}{r^3} - \frac{3\bar{a}}{r^5}(r^2) + \frac{3\bar{r}}{r^5}(a_1x + a_2y + a_3z)$.

$= \frac{2\bar{a}}{r^3} - \frac{3\bar{a}}{r^3} + \frac{3\bar{r}}{r^5}(\bar{r}\cdot\bar{a})$.

$= \frac{-\bar{a}}{r^3} + \frac{3\bar{r}}{r^5}(\bar{r}\cdot\bar{a})$.

---

(7) Show that the vector $(x^2 - yz)\bar{i} + (y^2 - zx)\bar{j} + (z^2 - xy)\bar{k}$ is irrotational and find scalar potential.

Sol:- Let $\bar{f} = (x^2 - yz)i + (y^2 - zx)\bar{j} + (z^2 - xy)\bar{k}$

Then Curl $\bar{f} = \nabla \times f$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x^2 - yz) & (y^2 - zx) & (z^2 - xy) \end{vmatrix}$$

$= \sum i(-x + x) = \bar{0}$.

$\therefore f$ is irrotational. Then $\exists\ \phi$ such that $\bar{f} = \nabla\phi$.

$\Rightarrow \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z} = (x^2 - yz)\bar{i} + (y^2 - zx)\bar{j} + (z^2 - xy)\bar{k}$

Comparing Components, we get

$$\frac{\partial\phi}{\partial x} = x^2 - yz \Rightarrow \phi = \int (x^2 - yz)\,dx = \frac{x^3}{3} - xyz + f_1(y, \ldots) \rightarrow (1)$$

$$\frac{\partial\phi}{\partial y} = y^2 - zx \Rightarrow \phi = \int (y^2 - zx)\,dy = y^3/3 - xyz + f_2(x, \ldots) \rightarrow (2)$$

$$\frac{\partial\phi}{\partial z} = z^2 - xy \Rightarrow \phi = \int (z^2 - xy)\,dz = \frac{z^3}{3} - xyz + f_3(x, y) \rightarrow (3)$$

from (1) (2) + (3), $\phi = \dfrac{x^3 + y^3 + z^3}{3} - xyz$.

$\therefore \phi = \frac{1}{3}(x^3 + y^3 + z^3) - xyz +$ Const which is req^d Scalar potential.

---

(8) find Constants a, b and c if the vector $\bar{f} = (2x + 3y + az)\bar{i} + (bx + 2y + 3z)\bar{j} + (2x + cy + 3z)\bar{k}$ is irrotational.

Sol- Given $\bar{f} = (2x + 3y + az)\bar{i} + (bx + 2y + 3z)\bar{j} + (2x + cy + 3z)\bar{k}$

$$\text{Curl } \bar{f} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x+3y+az & bx+2y+3z & 2x+cy+3z \end{vmatrix}$$

$$= \bar{i}(c - 3) - \bar{j}(2 - a) + \bar{k}(b - 3)$$

If the vector is irrotational then Curl $\bar{f} = \bar{0}$

$\Rightarrow c - 3 = 0,\quad 2 - a = 0,\quad b - 3 = 0$

$c = 3 \qquad a = 2 \qquad b = 3$

---

(9) If $f(r)$ is differentiable show that Curl$(\bar{r} f(r)) = \bar{0}$ where $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

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→① →② →③

Sol :- Given $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

$r = |\bar{r}| = \sqrt{x^2+y^2+z^2}$

$r^2 = x^2+y^2+z^2$

Differentiating Partially w.r.to x, y, z we have.

$\boxed{\frac{\partial r}{\partial x} = \frac{x}{r}}$ $\boxed{\frac{\partial r}{\partial y} = \frac{y}{r}}$ $\boxed{\frac{\partial r}{\partial z} = \frac{z}{r}}$

$curl(\bar{r} \cdot f(r)) = Curl(f(r) \cdot x\bar{i} + y\bar{j} + z\bar{k})$

$Curl(x \cdot f(r)\bar{i} + y \cdot f(r)\bar{j} + z \cdot f(r)\bar{k})$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x f(r) & y f(r) & z f(r) \end{vmatrix}$$

$$= \sum i \left[\frac{\partial}{\partial y}(z \cdot f(r)) - \frac{\partial}{\partial z}[y \cdot f(r)]\right]$$

$$= \sum i \left[z \cdot f'(r)\frac{\partial r}{\partial y} - y \cdot f'(r) \cdot \frac{\partial r}{\partial z}\right]$$

$$= \sum i \left[z \cdot f'(r) \cdot \frac{y}{r} - y f'(r) \frac{z}{r}\right]$$

$$= \bar{0}$$

---

⑩ find Constants a, b, c So that the vector $\bar{A} = (x+2y+az)\bar{i} + (bx-3y-z)\bar{j} + (4x+cy+2z)\bar{k}$ is irrotational. Also find φ such that $\bar{A} = \nabla\varphi$.

Sol: Given $\bar{A} = (x+2y+az)\bar{i} + (bx-3y-z)\bar{J} + (4x+cy+2z)\bar{k}$

Vector $\bar{A}$ is irrotational $\Rightarrow$ Curl $\bar{A} = \bar{0}$

$$\Rightarrow \begin{vmatrix} i & J & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+2y+az & bx-3y-z & 4x+cy+2z \end{vmatrix} = \bar{0}$$

$$\Rightarrow \bar{i}(c+1) + \bar{J}(a-4) + \bar{k}(b-2) = \bar{0}$$

$$\Rightarrow \bar{i}(c+1) + \bar{J}(a-4) + \bar{k}(b-2) = 0\bar{i} + 0\bar{J} + 0\bar{k}$$

$$c = -1, \quad a = 4, \quad b = 2$$

Now $\bar{A} = (x+2y+4z)\bar{i} + (2x-3y-z)\bar{J} + (4x-y+2z)\bar{k}$

we have $\bar{A} = \nabla\varphi$.

$$\Rightarrow (x+2y+4z)\bar{i} + (2x-3y-z)\bar{J} + (4x-y+2z)\bar{k} = \bar{i}\frac{\partial\varphi}{\partial x} + \bar{J}\frac{\partial\varphi}{\partial y} + \bar{k}\frac{\partial\varphi}{\partial z}$$

Comparing both sides, we have

$$\frac{\partial\varphi}{\partial x} = (x+2y+4z) \Rightarrow \varphi = \frac{x^2}{2} + 2xy + 4xz + f_1(y,z)$$

$$\frac{\partial\varphi}{\partial y} = (2x-3y-z) \Rightarrow \varphi = 2xy - \frac{3y^2}{2} - zy + f_2(z,x)$$

$$\frac{\partial\varphi}{\partial z} = (4x-y+2z) \Rightarrow \varphi = 4xz - yz + z^2 + f_3(x,y)$$

$$\text{Hence } \varphi = \frac{x^2}{2} - \frac{3y^2}{2} + z^2 + 2xy + 4zx - yz + c$$

---

(11) find whether the function $\bar{F} = (x^2-y^3)\bar{i} + (y^2-3x)\bar{J} + (z^2-xy)\bar{k}$ is irrotational and hence find scalar potential function corresponding to it

Sol:-
$$\nabla \times \bar{f} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2-y^3 & y^2-3x & z^2-xy \end{vmatrix}$$

$$= \bar{i}\left[\frac{\partial}{\partial y}(z^2-xy) - \frac{\partial}{\partial z}(y^2-3x)\right] - \bar{j}\left[\frac{\partial}{\partial x}(z^2-xy) - \frac{\partial}{\partial z}(x^2-y^3)\right]$$
$$+ \bar{k}\left[\frac{\partial}{\partial x}(y^2-3x) - \frac{\partial}{\partial y}(x^2-y^3)\right]$$

$$= \bar{i}\left[(0-x)-0\right] - \bar{j}\left[(-y)-0\right] + \bar{k}\left[(-3)-(0-3y^2)\right]$$

$$= -x\bar{i} + y\bar{j} + \bar{k}(3y^2-3)$$

Since $\nabla \times \bar{f} \neq 0$, $\bar{f}$ is not irrotational.

① Prove that $\text{div}(\text{grad}\, r^m) = m(m+1)r^{m-2}$ (or)

$\nabla^2(r^m) = m(m+1)r^{m-2}$ (or) $\nabla^2(r^n) = n(n+1)r^{n-2}$

Sol:- Let $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

$$r = |\bar{r}| = \sqrt{x^2+y^2+z^2}$$

$$r^2 = x^2+y^2+z^2$$

Differentiating partially w.r.to x, y, z, we get

$$2r\frac{\partial r}{\partial x} = 2x$$

$\boxed{\frac{\partial r}{\partial x} = \frac{x}{r}}$ lly $\boxed{\frac{\partial r}{\partial y} = \frac{y}{r}}$, $\boxed{\frac{\partial r}{\partial z} = \frac{z}{r}}$

Now $\text{grad}(r^m) = \sum i \frac{\partial}{\partial x}(r^m)$

$$= \sum i \cdot m r^{m-1} \cdot \frac{\partial r}{\partial x} = \sum i\, m r^{m-1} \cdot \frac{x}{r}$$

$= \sum i\, m r^{m-2} x$

$\therefore \operatorname{div}(\operatorname{grad} r^m) = \sum \frac{\partial}{\partial x}\left[m r^{m-2} x\right]$

$= m \cdot \sum\left[(m-2) r^{m-3} \frac{\partial r}{\partial x} \cdot x + r^{m-2}\right]$

$= m \cdot \sum\left[(m-2) r^{m-3} \cdot \frac{x}{r} \cdot x + r^{m-2}\right]$

$= m \sum\left[(m-2) r^{m-4} x^2 + r^{m-2}\right]$

$= m\left[(m-2) r^{m-4} \sum x^2 + \sum r^{m-2}\right]$

$= m\left[(m-2) r^{m-4}(r^2) + 3 r^{m-2}\right]$

$= m\left[(m-2) r^{m-2} + 3 r^{m-2}\right]$

$= m \cdot r^{m-2}\left[m-2+3\right]$

$= m(m+1) r^{m-2}$

Note:- <u>If $m = -1$, then $\nabla^2(r^m) = 0$.</u>

) Show that $\nabla^2\left[f(r)\right] = \frac{d^2 f}{dr^2} + \frac{2}{r} \cdot \frac{df}{dr}$

$= f''(r) + \frac{2}{r} \cdot f'(r)$ where $r = |\bar{r}|$.

f:- $\operatorname{grad}\left[f(r)\right] = \nabla\left[f(r)\right]$

$= \sum i \frac{\partial}{\partial x}\left[f(r)\right]$

$= \sum i \cdot f'(r) \cdot \frac{\partial r}{\partial x}$

$$= \sum i \cdot f'(r) \cdot \frac{x}{r}.$$

$$\therefore div[grad f(r)] = \nabla^2[f(r)]$$

$$= \nabla \cdot [\nabla f(r)]$$

$$= \sum \frac{\partial}{\partial x}\left[f'(r) \cdot \frac{x}{r}\right]$$

$$= \sum \frac{r \cdot \frac{\partial}{\partial x}[f'(r) x] - f'(r) \cdot x \frac{\partial}{\partial x}(r)}{r^2}$$

$$= \sum \frac{r \cdot \left(f''(r) \frac{\partial r}{\partial x} \cdot x + f'(r)\right) - f'(r) x \cdot \left(\frac{x}{r}\right)}{r^2}.$$

$$= \sum \frac{r \cdot f''(r) \cdot \frac{x}{\cancel{r}} \cdot x + r f'(r) - x^2 \frac{f'(r)}{r}}{r^2}$$

$$= \frac{f''(r)}{r^2} \sum x^2 + \frac{1}{r} \sum f'(r) - \frac{1}{r^3} f'(r) \sum x^2$$

$$= \frac{f''(r)}{r^2} (r^2) + \frac{3}{r} f'(r) - \frac{1}{r^3} f'(r) \cdot r^2$$

$$= f''(r) + \frac{2}{r} \cdot f'(r).$$

<u>NoTe</u>:- from the above, we can have

① $\nabla^2\left(\frac{1}{r}\right) = 0$, $\nabla^2(\log r) = \frac{1}{r^2}$

$\nabla^2(r^n) = n(n+1) r^{n-2}$.

③ Shao that (i) $(\bar{a}\cdot\nabla)\varphi = \bar{a}\cdot\nabla\varphi$ (ii) $(\bar{a}\cdot\nabla)\bar{r} = \bar{a}$

Sol:- Let $\bar{a} = a_1\bar{i} + a_2\bar{j} + a_3\bar{k}$

$$\bar{a}\cdot\nabla = (a_1\bar{i} + a_2\bar{j} + a_3\bar{k})\cdot\left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)$$

$$= a_1\frac{\partial}{\partial x} + a_2\frac{\partial}{\partial y} + a_3\frac{\partial}{\partial z}$$

$$\therefore (\bar{a}\cdot\nabla)\varphi = a_1\frac{\partial\varphi}{\partial x} + a_2\frac{\partial\varphi}{\partial y} + a_3\frac{\partial\varphi}{\partial z}$$

Hence $(\bar{a}\cdot\nabla)\varphi = \bar{a}\cdot(\nabla\varphi)$

ii) Let $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

$$\therefore \frac{\partial\bar{r}}{\partial x} = \bar{i},\quad \frac{\partial\bar{r}}{\partial y} = \bar{j},\quad \frac{\partial\bar{r}}{\partial z} = \bar{k}$$

Hence $(\bar{a}\cdot\nabla)\bar{r} = \sum a_1\frac{\partial}{\partial x}(\bar{r})$

$$= \sum a_1\frac{\partial\bar{r}}{\partial x}$$

$$= a_1\bar{i} + a_2\bar{j} + a_3\bar{k}$$

$$= \bar{a}$$

---

④ Prove that $(\bar{f}\times\nabla)\cdot\bar{r} = 0$

Sol:- $(\bar{f}\times\nabla)\cdot\bar{r} = \left\{f\times\sum i\frac{\partial}{\partial x}\right\}\cdot\bar{r}$

$$= f\times\left\{\sum i\cdot\frac{\partial\bar{r}}{\partial x}\right\}$$

$$= \sum(f\times i)\cdot\frac{\partial\bar{r}}{\partial x}\}.$$

$$= \sum(f\times i)\cdot i = 0$$

(5) Prove that $(f \times \nabla) \times r = -2f$

Sol:- $(f\times\nabla) = (f\times i)\frac{\partial}{\partial x} + (f\times j)\frac{\partial}{\partial y} + (f\times k)\frac{\partial}{\partial z}$

$(f\times\nabla)\times r = (f\times i)\times\frac{\partial r}{\partial x} + (f\times j)\times\frac{\partial r}{\partial y} + (f\times k)\times\frac{\partial r}{\partial z}$

$(a\times b)\times c = (a\cdot b)c - (a\cdot c)b$

$= \sum (f\times i)\times i$

$= \sum \left((f\cdot i)i - f\right)$

$= (f\cdot i)i + (f\cdot j)j + (f\cdot k)k - 3f.$

$= f - 3f = -2f.$

---

(6) find $\operatorname{div}\bar{f}$ where $\bar{f} = \operatorname{grad}(x^3+y^3+z^3-3xyz)$

Sol:- Let $\phi = x^3+y^3+z^3-3xyz$. Then

$\bar{f} = \operatorname{grad}\phi$

$= \sum i\frac{\partial\phi}{\partial x}$

$= \cancel{\sum}\, i(3x^2-3yz) + j(3y^2-3zx) + k(3z^2-3xy)$

$= f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$

$\operatorname{div}\bar{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}.$

$= 6(x+y+z)$

---

(7) If $f = (x^2+y^2+z^2)^{-n}$ then find div grad f and determine 'n' if div grad $f = 0$

Sol:- Let $f = (x^2+y^2+z^2)^{-n}$, $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

$r^2 = x^2+y^2+z^2$.

$\therefore r = |\bar{r}| = \sqrt{x^2+y^2+z^2}$,

$\Rightarrow f(r) = (r^2)^{-n} = r^{-2n}$

$\therefore f'(r) = -2n \cdot r^{-2n-1}$

$f''(r) = (-2n)(-2n-1) r^{-2n-2}$

$= 2n(2n+1) r^{-2n-2}$

we have

$\text{div grad} f = \nabla^2 f(r)$

$= f''(r) + \frac{2}{r} \cdot f'(r)$

$= 2n(2n+1) r^{-2n-2} - 4n \cdot r^{-2n-2}$

$= r^{-2n-2}\left[4n^2 + 2n - 4n\right]$

$= r^{-2n-2}\left[4n^2 - 2n\right]$

If div grad $f(r)$ is zero we have $n = 0$ (or) $n = 1/2$

Vector Identities

1) If 'A' is a differentiable vector function and ϕ is differentiable scalar function, then prove that

$\text{div}(\phi A) = (\text{grad}\,\phi) \cdot A + \phi \, \text{div} A$

(or) $\nabla \cdot (\phi A) = (\nabla \phi) \cdot A + \phi (\nabla \cdot A)$

Pf:- $\text{div}(\phi A) = \nabla \cdot (\phi A)$

$= \sum i \cdot \frac{\partial}{\partial x}(\phi A)$

$= \sum i \left(\frac{\partial \phi}{\partial x} A + \phi \frac{\partial A}{\partial x}\right)$

$= \sum \left(i \frac{\partial \phi}{\partial x}\right) \cdot A + \sum \left(i \cdot \frac{\partial A}{\partial x}\right) \phi$

$= (\nabla \phi) \cdot A + (\nabla \cdot A) \phi$

2) Prove that Curl$(\phi A) = (\text{grad}\,\phi) \times A + \phi$ Curl A

(or) $\nabla \times (\phi A) = (\nabla \phi) \times A + \phi \cdot (\nabla \times A)$.

Sol:- Curl$(\phi A) = \nabla \times (\phi A)$

$$= \sum i \times \frac{\partial}{\partial x}(\phi A)$$

$$= \sum i \times \left(\frac{\partial \phi}{\partial x} A + \phi \frac{\partial A}{\partial x}\right).$$

$$= \sum\left(i \frac{\partial \phi}{\partial x}\right) \times A + \sum\left(i \times \frac{\partial A}{\partial x}\right)\phi$$

$$= \nabla \phi \times A + (\nabla \times A)\phi$$

3) Prove that

grad$(A \cdot B) = (B \cdot \nabla)A + (A \cdot \nabla)\bar{B} + \bar{B} \times \text{Curl}\,\bar{A} + \bar{A} \times \text{curl}\,\bar{B}$

Sol:- Now

$A \times \text{Curl}\,B = A \times (\nabla \times B)$

$$A \times \sum i \times \frac{\partial B}{\partial x} = \sum A \times \left(i \times \frac{\partial B}{\partial x}\right)$$

$$= \sum \left\{\left(A \cdot \frac{\partial B}{\partial x}\right) i - (A \cdot i)\frac{\partial B}{\partial x}\right\}.$$

$$= \sum i \left(A \cdot \frac{\partial B}{\partial x}\right) - \left(A \cdot \sum i \cdot \frac{\partial}{\partial x}\right) B.$$

$$= \sum i \left(A \cdot \frac{\partial B}{\partial x}\right) - (A \cdot \nabla)\bar{B}$$

$$A \times \text{Curl}\,B + (A \cdot \nabla) B = \sum i \left(A \cdot \frac{\partial B}{\partial x}\right)$$

lly $B \times \text{Curl}\,A = \sum i \left(B \cdot \frac{\partial A}{\partial x}\right) - (B \cdot \nabla)\bar{A}$

$$B \times \text{Curl} A + (B \cdot \nabla) A = \sum i \left(B \cdot \frac{\partial A}{\partial x}\right) \rightarrow (2)$$

Eq (1) + Eq (2)

$\therefore A \times \text{Curl} B + (A \cdot \nabla) B + B \times \text{Curl} A + (B \cdot \nabla) A$

$$= \sum i \left(A \cdot \frac{\partial B}{\partial x}\right) + \sum i \left(B \cdot \frac{\partial A}{\partial x}\right).$$

$$= \sum i \left(A \cdot \frac{\partial B}{\partial x} + \frac{\partial A}{\partial x} \cdot B\right).$$

$$= \sum i \cdot \frac{\partial}{\partial x} (\bar{A} \cdot \bar{B}).$$

$$= \nabla (\bar{A} \cdot \bar{B}) = \text{grad} (\bar{A} \cdot \bar{B})$$

0

4) Prove that

$$\text{div} (\bar{A} \times \bar{B}) = B \cdot \text{Curl} A - A \cdot \text{Curl} \bar{B} \text{ (or)}$$

$$\nabla \cdot (\bar{A} \times \bar{B}) = B \cdot (\nabla \times A) - A \cdot (\nabla \times \bar{B})$$

Sol:-

$$\text{div} (\bar{A} \times \bar{B}) = \nabla \cdot (\bar{A} \times \bar{B})$$

$$= \sum i \cdot \frac{\partial}{\partial x} (\bar{A} \times \bar{B})$$

$$= \sum i \cdot \left(\frac{\partial A}{\partial x} \times \bar{B} + \bar{A} \times \frac{\partial B}{\partial x}\right)$$

$$= \sum i \cdot \left(\frac{\partial A}{\partial x} \times B\right) + \sum i \cdot \left(\bar{A} \times \frac{\partial B}{\partial x}\right)$$

$$= \sum \left(i \times \frac{\partial A}{\partial x}\right) \cdot B - \sum i \cdot \left(\frac{\partial B}{\partial x} \times A\right).$$

$$= \sum \left(i \times \frac{\partial A}{\partial x}\right) \cdot B - \left(\sum \left(i \times \frac{\partial B}{\partial x}\right)\right) \cdot A$$

$$= (\nabla \times A) \cdot B - (\nabla \times B) \cdot A$$

$$= \bar{B} \cdot \text{Curl} \bar{A} - \bar{A} \cdot \text{Curl} \bar{B}$$

Note:- If $\bar{A}$ and $\bar{B}$ are irrotational then $\bar{A}\times\bar{B}$ is solenoidal.

$\bar{A}\times\bar{B}$ are irrotational. $\bar{A},\bar{B}$ are irrotational

$\Rightarrow \nabla\times\bar{A}=0,\ \nabla\times\bar{B}=\bar{0},$

$$\nabla(\bar{A}\times\bar{B}) = \bar{B}\cdot(\nabla\times\bar{A}) - \bar{A}\cdot(\nabla\cdot\bar{B})$$
$$= 0$$

$\therefore \bar{A}\times\bar{B}$ is solenoidal.

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5) Prove that

$$\text{Curl}(\bar{A}\times\bar{B}) = \bar{A}\,\text{DIV}\,\bar{B} - \bar{B}\,\text{div}\,\bar{A} + (\bar{B}\cdot\nabla)A - (\bar{A}\cdot\nabla)\bar{B}$$

Sol:- $\text{Curl}(\bar{A}\times\bar{B}) = \nabla\times(\bar{A}\times\bar{B})$

$$= \sum i\times\frac{\partial}{\partial x}(\bar{A}\times\bar{B})$$

$$= \sum i\times\left(\frac{\partial A}{\partial x}\times\bar{B} + \bar{A}\times\frac{\partial B}{\partial x}\right)$$

$$= \sum i\times\left(\frac{\partial\bar{A}}{\partial x}\times B\right) + \sum i\times\left(\bar{A}\times\frac{\partial B}{\partial x}\right)$$

$$= \sum\left\{(i\cdot\bar{B})\frac{\partial A}{\partial x} - \left(i\cdot\frac{\partial A}{\partial x}\right)\bar{B}\right\} + \sum\left\{\left(i\cdot\frac{\partial B}{\partial x}\right)\bar{A} - (i\cdot\bar{A})\frac{\partial B}{\partial x}\right\}$$

$$= \sum(B\cdot i)\frac{\partial A}{\partial x} - \sum\left(i\cdot\frac{\partial A}{\partial x}\right)B + \sum\left(i\cdot\frac{\partial B}{\partial x}\right)\bar{A} - \sum(A\cdot i)\frac{\partial B}{\partial x}$$

$$= \left(B\cdot\sum i\frac{\partial}{\partial x}\right)A - \left(\sum i\cdot\frac{\partial A}{\partial x}\right)B + \left(\sum i\cdot\frac{\partial B}{\partial x}\right)A - \left(A\sum i\frac{\partial}{\partial x}\right)\bar{B}$$

$$= (\bar{B}\cdot\nabla)\bar{A} - (\nabla\cdot\bar{A})\bar{B} + (\nabla\cdot\bar{B})\bar{A} - (\bar{A}\cdot\nabla)\bar{B}$$

$\therefore \text{Curl}(\bar{A}\times\bar{B}) = A\,\text{div}\,\bar{B} - \bar{B}\,\text{div}\,A + (B\cdot\nabla)A - (\bar{A}\cdot\nabla)\bar{B}$

6) Prove that Curl grad $\phi = \bar{0}$

Sol:- $\text{grad}\,\phi = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}$

$$\text{Curl}(\text{grad}\,\phi) = \nabla\times(\text{grad}\,\phi)$$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} & \frac{\partial\phi}{\partial z} \end{vmatrix}$$

$$= i\left(\frac{\partial^2\phi}{\partial y\partial z} - \frac{\partial^2\phi}{\partial z\partial y}\right) - j\left(\frac{\partial^2\phi}{\partial x\partial z} - \frac{\partial^2\phi}{\partial z\partial x}\right) + \bar{k}\left(\frac{\partial^2\phi}{\partial x\partial y} - \frac{\partial^2\phi}{\partial y\partial x}\right)$$

$$= \bar{0}$$

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7) Prove that div Curl $\bar{f} = 0$

Sol Let $\bar{f} = f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$

$$\text{Curl}\,\bar{f} = \nabla\times\bar{f}$$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

$$\nabla\times\bar{f} = i\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right) - j\left(\frac{\partial f_3}{\partial x} - \frac{\partial f_1}{\partial z}\right) + \bar{k}\left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)$$

$$\therefore \text{div}(\text{Curl}\,\bar{f}) = \nabla\cdot(\nabla\times\bar{f})$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right) - \frac{\partial}{\partial y}\left(\frac{\partial f_3}{\partial x} - \frac{\partial f_1}{\partial z}\right) + \frac{\partial}{\partial z}\left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)$$

$$= \frac{\partial^2 f_3}{\partial x \partial y} - \frac{\partial^2 f_2}{\partial x \partial z} - \frac{\partial^2 f_3}{\partial y \partial x} + \frac{\partial^2 f_1}{\partial y \partial z} + \frac{\partial^2 f_2}{\partial x \partial z} - \frac{\partial^2 f_1}{\partial z \partial y}$$

$$= 0$$

$\therefore$ div(curl f) $= 0$, Curl f is always solenoidal.

8) Prove that

$$\nabla \times (\nabla \times A) = \nabla(\nabla \cdot A) - \nabla^2 A$$

Sol:-

$$\nabla \times (\nabla \times A) = \sum i \times \frac{\partial}{\partial x}(\nabla \times A).$$

Now

$$i \times \frac{\partial}{\partial x}(\nabla \times A) = i \times \frac{\partial}{\partial x}\left(i \times \frac{\partial A}{\partial x} + j \times \frac{\partial A}{\partial y} + k \times \frac{\partial A}{\partial z}\right)$$

$$= i \times \left(i \times \frac{\partial^2 A}{\partial x^2} + j \times \frac{\partial^2 A}{\partial x \partial y} + k \times \frac{\partial^2 A}{\partial x \partial z}\right)$$

$$= \cancel{i \frac{\partial}{\partial x}\left(i \cdot \frac{\partial A}{\partial x}\right) + j \frac{\partial}{\partial y}\left(i \frac{\partial A}{\partial x}\right) + k \frac{\partial}{\partial z}\Big(i}$$

$$= \left(i \cdot \frac{\partial^2 A}{\partial x^2}\right) i - \frac{\partial^2 A}{\partial x^2} + \left(i \cdot \frac{\partial^2 A}{\partial x \partial y}\right) j + \left(i \cdot \frac{\partial^2 A}{\partial x \partial z}\right) k$$

$$= i \frac{\partial}{\partial x}\left(i \frac{\partial A}{\partial x}\right) + j \frac{\partial}{\partial y}\left(i \frac{\partial A}{\partial x}\right) + k \frac{\partial}{\partial z}\left(i \frac{\partial A}{\partial x}\right) - \frac{\partial^2 A}{\partial x^2}$$

$$= i \frac{\partial A}{\partial x}\left[i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z}\right] - \frac{\partial^2 A}{\partial x^2}$$

$$\therefore \sum i \times \frac{\partial}{\partial x}(\nabla \times A) = \nabla \sum i \cdot \frac{\partial A}{\partial x} - \sum \frac{\partial^2 A}{\partial x^2}$$

$$= \nabla(\nabla \cdot A) - \left(\frac{\partial^2 A}{\partial x^2} + \frac{\partial^2 A}{\partial y^2} + \frac{\partial^2 A}{\partial z^2}\right)$$

$$\boxed{\nabla \times (\nabla \times A) = \nabla(\nabla \cdot A) - \nabla^2 A}$$

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(1) If 'f' and 'g' are two scalar point functions, Prove that $div(f\nabla g) = f\nabla^2 g + \nabla f \cdot \nabla g$

Sol: $\nabla g = \bar{i}\frac{\partial g}{\partial x} + \bar{j}\frac{\partial g}{\partial y} + \bar{k}\frac{\partial g}{\partial z}$

$f\nabla g = \bar{i}f\frac{\partial g}{\partial x} + \bar{j}f\frac{\partial g}{\partial y} + \bar{k}f\frac{\partial g}{\partial z}$

$\therefore \nabla\cdot(f\nabla g) = \frac{\partial}{\partial x}\left(f\frac{\partial g}{\partial x}\right) + \frac{\partial}{\partial y}\left(f\frac{\partial g}{\partial y}\right) + \frac{\partial}{\partial z}\left(f\frac{\partial g}{\partial z}\right)$

$= f\left(\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} + \frac{\partial^2 g}{\partial z^2}\right) + \frac{\partial f}{\partial x}\cdot\frac{\partial g}{\partial x} + \frac{\partial f}{\partial y}\cdot\frac{\partial g}{\partial y} + \frac{\partial f}{\partial z}\cdot\frac{\partial g}{\partial z}$

$= f\nabla^2 g + \left(i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z}\right)\cdot\left(i\frac{\partial g}{\partial x} + j\frac{\partial g}{\partial y} + k\frac{\partial g}{\partial z}\right)$

$= f\nabla^2 g + (\nabla f)(\nabla g)$

2) Prove that $div\{(r\times a)\times \bar{b}\} = -2(\bar{a}\cdot\bar{b})$ where $\bar{a}$ and $\bar{b}$ are Constant vectors.

Sol $div\{(\bar{r}\times\bar{a})\times\bar{b}\} = div\{(r\cdot\bar{b})\bar{a} - (\bar{a}\cdot\bar{b})\bar{r}\}$

$= div\{(\bar{r}\cdot\bar{b})\bar{a}\} - div(\bar{a}\cdot\bar{b})r$

from theorem (1)

$= (\bar{r}\cdot\bar{b})div\bar{a} + \bar{a}\cdot grad(\bar{r}\cdot\bar{b}) - \{(\bar{a}\cdot\bar{b})div\bar{r} + \bar{r}\cdot grad(\bar{a}\cdot\bar{b})\}$

$= (\bar{r}\cdot\bar{b})div\bar{a} + \bar{a}\cdot grad(\bar{r}\cdot\bar{b}) - (\bar{a}\cdot\bar{b})3 + 0$

$= 0 + \bar{a}\cdot grad(r\cdot\bar{b}) - (\bar{a}\cdot\bar{b})3$

$= a\sum i\frac{\partial}{\partial x}(\bar{r}\cdot\bar{b}) - 3(\bar{a}\cdot\bar{b})$

$= a\cdot\sum i\left(\frac{\partial\bar{r}}{\partial x}\cdot\bar{b}\right) - 3(\bar{a}\cdot\bar{b})$

$= a\sum i(i\cdot\bar{b}) - 3(a\cdot b)$

$= (a\cdot\bar{b}) - 3(\bar{a}\cdot\bar{b}) = -2(\bar{a}\cdot\bar{b})$

(3) Prove that $\text{Curl}((r \times a) \times b) = b \times a$ where a and b are constant vectors

Sol:- $\text{Curl}((r \times a) \times b) = \text{Curl}\left((r \cdot b)a - (a \cdot b)r\right)$.

$= \text{Curl}(r \cdot b)a - \text{Curl}(\bar{a} \cdot \bar{b})\bar{r}$

$= (r \cdot b)\,\text{curl}\, a + \text{grad}(r \cdot b) \times \bar{a} - (\bar{a} \cdot \bar{b})\,\text{curl}\,\bar{r}$

$= 0 + \nabla(\bar{r} \cdot \bar{b}) \times \bar{a} - 0$

$= \bar{b} \times \bar{a}$

(4) Prove that $\text{Curl}(f\, \text{grad}\,\varphi) = (\text{grad}\,\bar{f}) \times (\text{grad}\,\varphi)$

Sol $\text{Curl}(\bar{f}\, \text{grad}\,\varphi) = \nabla \times (f \nabla \varphi)$

$= f \cdot \text{Curl}(\text{grad}\,\varphi) + (\text{grad}\, f) \times (\text{grad}\,\varphi)$

$= (\text{grad}\, f) \times (\text{grad}\,\varphi) \quad (\because \text{Curl}(\text{grad}\,\varphi) = 0)$

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