CENTERS OF PERFECTOID PURITY

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Abstract. In this paper, we introduce a mixed characteristic analog of log canonical centers in characteristic 0 and centers of F-purity in positive characteristic, which we call centers of perfectoid purity. We show that their existence detects (the failure of) normality of the ring. We also show the existence of a special center of perfectoid purity that detects the perfectoid purity of R, analogously to the splitting prime of Aberbach and Enescu, and investigate its behavior under étale morphisms.

1. Introduction

Let R be a commutative Noetherian ring of characteristic $p > 0$. Since the late sixties, the singularities of the ring R have been studied using Frobenius (see [Kun69, HH90, Hoc07, HH89, Smi97, MR85] and many more). These are broadly known as F-singularities and have been linked to singularities of the minimal model program [Sch09, Smi97, Smi00b]. Of particular importance, are the *centers of F-purity* [Sch10], a special type of *compatible ideals* [MR85]. They tell us where the ring fails to be (strongly) F-regular. These are related to the log canonical centers, an important object in the study of singularities of the minimal model program [Amb11]. The aim of this paper is to define an analog object in mixed characteristic.

Although there is no Frobenius in mixed characteristic, we have a good analog of perfection: perfectoidization [Sch12, BS22]. Our strategy is then to express what centers of F-purity are in terms of perfection in positive characteristic before writing an analog definition in mixed characteristic via perfectoidization. This strategy has been used successfully to define analogs of test ideals, F-signature, and F-purity in mixed characteristic [MS18, CLM+22, BMP+24].

Let $F\colon R \longrightarrow R^{1/p}$ be the Frobenius map on R and assume $R^{1/p}$ is a finite R-module and that the natural map $R \longrightarrow R^{1/p}$ is pure. An ideal $\mathfrak{a}$ of R is said to be *uniformly compatible* if for all $e \in \mathbb{N}$, $\phi \in \operatorname{Hom}_R(R^{1/p^e}, R)$, $\phi(\mathfrak{a}^{1/p^e}) \subset \mathfrak{a}$. Let $R_{\mathrm{perf}} \coloneqq \bigcup_e R^{1/p^e}$ and $\mathfrak{a}_{\mathrm{perf}} \coloneqq \bigcup_e \mathfrak{a}^{1/p^e}$. Then, $\mathfrak{a}$ is uniformly compatible if and only if for all $\psi \in \operatorname{Hom}_R(R_{\mathrm{perf}}, R)$, $\psi(\mathfrak{a}_{\mathrm{perf}}) \subset \mathfrak{a}$ (see Corollary 2.20). When $\mathfrak{a}$ is prime, we say that it is a *center of F-purity* of R.

Now, let $(R, \mathfrak{m})$ be a complete Noetherian local ring with perfect residue field k of characteristic $p > 0$. Let $A = W(k)[\![x_1, \dots, x_d]\!]$ be a regular local ring such that R is an A-algebra that is also finite as an A-module (for instance a Noether normalization). Denoting by perfd the perfectoidization of a ring or an ideal (see [BS22, Section 10] or Definition 2.2), we define

$$R^A_\infty \coloneqq \left(R \otimes_A W(k)\left[p^{1/p^\infty}, x_1^{1/p^\infty}, \dots, x_d^{1/p^\infty}\right][\![x_1, \dots, x_d]\!]^{\wedge p}\right)_{\mathrm{perfd}}.$$

Note that if R itself has characteristic p then $R^A_\infty = R_{\mathrm{perf}}$ (Corollary 2.20). Let $\mathfrak{a} \subset R$ be an ideal and fix $\phi \in \operatorname{Hom}_R(R^A_\infty, R)$. We say that $\mathfrak{a}$ is *ϕ-compatible* if

$$\phi\left(\left(\mathfrak{a} R^A_\infty\right)_{\mathrm{perfd}}\right) \subset \mathfrak{a}.$$

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If this holds for every possible choice of $\phi \in \operatorname{Hom}_R(R^A_\infty, R)$, we say that $\mathfrak{a}$ is *uniformly perfectoid compatible*. This does not depend on the choice of A by Corollary 2.30. Moreover, like in positive characteristic, these are closed under sums, intersections, and minimal primes, see Proposition 2.34, Proposition 2.37. This allows us to prove that the following theorem:

Theorem A (Corollary 2.40)**.** *Let $(R, \mathfrak{m})$ be a complete Noetherian local ring with perfect residue field of characteristic $p > 0$. If R is perfectoid pure, there are only finitely many uniformly perfectoid compatible ideals.*

In fact, there are finitely ϕ-compatible ideals for a surjective ϕ (Corollary 2.39). If R is perfectoid pure and $\mathfrak{p}$ is a uniformly perfectoid compatible prime ideal of R, we say that it is a *center of perfectoid purity* of R. Not only are there finitely many of them but we actually have a bound thanks to [ST10], see Remark 2.41. In positive characteristic, log canonical centers are centers of F-purity [Sch10]. The same holds true in mixed characteristic when R is quasi-Gorenstein (that is, when R has a canonical module isomorphic to itself):

Theorem B (Theorem 4.11, Corollary 4.12)**.** *Let $(R, \mathfrak{m})$ be a complete Noetherian normal quasi-Gorenstein local ring with perfect residue field of characteristic $p > 0$. The multiplier ideal $\mathscr{J}$ of R is a uniformly perfectoid compatible ideal of R. Moreover, if $\mathfrak{p} \subset R$ is a log canonical center of R and R is perfectoid pure, then $\mathfrak{p}$ is a center of perfectoid purity of R. In particular, if R has no uniformly perfectoid compatible ideal, then it is klt.*

An important part of the theory in positive characteristic is that compatible ideals detect whether the ring is normal: if a ring of characteristic $p > 0$ has no uniformly compatible ideal, it must be normal. This is also true in mixed characteristic.

Theorem C (Proposition 5.2, Remark 5.3)**.** *Let $(R, \mathfrak{m})$ be a complete Noetherian local ring with perfect residue field of characteristic $p > 0$. Then, the conductor ideal $\mathfrak{c}$ of R is a nonzero uniformly perfectoid compatible ideal of R. In particular, if R has no uniformly perfectoid compatible ideal, then it is normal.*

We also show the existence of a special uniformly perfectoid compatible ideal $\beta(R) \subset R$ that detects perfectoid purity, analog to the splitting prime of Aberbach and Enescu [AE05].

Theorem D (Proposition 6.7)**.** *Let $(R, \mathfrak{m})$ be a complete Noetherian local ring with perfect residue field of characteristic $p > 0$. Then, there is an ideal $\beta(R)$ of R with $\beta(R) \neq R$ if and only R is perfectoid pure. In that case, $\beta(R)$ is a prime and is the largest center of perfectoid purity of R. In particular, $R/\beta(R)$ is normal.*

We then generalize this construction to create an analog of the Cartier core map [Bad21, Bro23, CRF24] in Section 6.3. We also show that this map behaves well under étale morphisms in Section 7.

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2. Definition and first properties

Notation 2.1. For this whole paper, all rings are assumed to be commutative with unity. If $(R, \mathfrak{m})$ is a local ring and M is an R-module, then $\hat{M}$ or $M^\wedge$ is the classical $\mathfrak{m}$-adic completion of M. If $\mathfrak{a}$ is any other ideal of R, $M^{\wedge\mathfrak{a}}$ is the classical $\mathfrak{a}$-adic completion of M unless explicitly stated otherwise. Denoting by p the characteristic of $R/\mathfrak{m}$, if S is an R-algebra, and $\mathfrak{a} \subset S$ is an ideal, then $\mathfrak{a}^-$ is the p-adic closure of $\mathfrak{a}$ in S.

2.1. **Generalities about perfectoid rings and ideals.**

Definition 2.2 ([BS22, Section 10])**.** Let R be a perfectoid ring and $\mathfrak{a} \subset R$ an ideal. We say that $\mathfrak{a}$ is a perfectoid ideal if $R/\mathfrak{a}$ is also perfectoid. If $\mathfrak{b} \subset R$ is any ideal, we define $\mathfrak{b}_{\mathrm{perfd}} := \ker(R \longrightarrow (R/\mathfrak{b})_{\mathrm{perfd}})$ and $\mathfrak{b}$ is perfectoid if and only if $\mathfrak{b} = \mathfrak{b}_{\mathrm{perfd}}$.

We write down a couple of facts about perfectoid ideals. These are well known to experts but will be useful throughout the paper.

Proposition 2.3. *Let $f\colon R \longrightarrow S$ be a morphism between perfectoid rings. Then,* $\ker f$ *is a perfectoid ideal of R.*

Proof. As the kernel of a map of two p-complete rings, $\ker f$ is also p-complete. In particular, $R/\ker f$ is semiperfectoid so it surjects onto its perfectoidization by [BS22, Theorem 7.4]. On the other hand, $R/\ker f \hookrightarrow S$ factors through $(R/\ker f)_{\mathrm{perfd}}$ so $R/\ker f \longrightarrow (R/\ker f)_{\mathrm{perfd}}$ must be injective, hence an isomorphism. □

Proposition 2.4 ([DI24, Proposition 2.8])**.** *Let R be a perfectoid ring and $\mathfrak{a} \subset R$ an ideal containing p. Then, $\mathfrak{a}$ is perfectoid if and only if it is radical.*

Proposition 2.5 ([BS22, Example 7.9], [CLM$^+$22, Lemma 2.3.2])**.** *Let R be a perfectoid ring and $f \in R$ be such that f has a compatible system of p-th power roots, which we denote by $\{f^{1/p^\infty}\}$. Then, $(f)_{\mathrm{perfd}} = (f^{1/p^\infty})^-$.*

Proposition 2.6. *Let R be a perfectoid ring and $\{\mathfrak{a}_i\}_{i\in I}$ be a set of perfectoid ideals. Then, $\cap_{i\in I}\mathfrak{a}_i$ is a perfectoid ideal of R. Moreover, if I is a finite set and $\{\mathfrak{b}_i\}_{i\in I}$ are p-complete ideals of R, then $\cap_i(\mathfrak{b}_i)_{\mathrm{perfd}} = (\cap_i\mathfrak{b}_i)_{\mathrm{perfd}}$.*

Proof. By [BIM19, Example 3.8 (8)], the product

$$\prod_{i\in I} R/\mathfrak{a}_i$$

is perfectoid. Now,

$$\bigcap_{i\in I}\mathfrak{a}_i = \ker\left(R \longrightarrow \prod_{i\in I} R/\mathfrak{a}_i\right)$$

so the result follows from Proposition 2.3. For the second part, since sheafification commutes with finite limits, [BS22, Corollary 8.11] gives

$$\left(\prod_{i\in I} R/\mathfrak{b}_i\right)_{\mathrm{perfd}} = \prod_{i\in I}(R/\mathfrak{b}_i)_{\mathrm{perfd}}.$$

In particular,

$$\bigcap_i(\mathfrak{b}_i)_{\mathrm{perfd}} = \ker\left(R \longrightarrow \prod_{i\in I}(R/\mathfrak{b}_i)_{\mathrm{perfd}}\right) = \ker\left(R \longrightarrow \left(\prod_{i\in I} R/\mathfrak{b}_i\right)_{\mathrm{perfd}}\right) = \left(\bigcap_i \mathfrak{b}_i\right)_{\mathrm{perfd}},$$

which is what we wanted. □

Proposition 2.7. *Let R be a perfectoid ring and $\mathfrak{a}$ and $\mathfrak{b}$ be two perfectoid ideals of R. Then, $\mathfrak{a}+\mathfrak{b}$ is also a perfectoid ideal of R.*

Proof. We have an exact sequence

$$0 \longrightarrow R/\left(\mathfrak{a}\cap\mathfrak{b}\right) \longrightarrow R/\mathfrak{a}\oplus R/\mathfrak{b} \longrightarrow R/\left(\mathfrak{a}+\mathfrak{b}\right) \longrightarrow 0, \tag{2.7.1}$$

which we claim implies $R/\left(\mathfrak{a}+\mathfrak{b}\right)$ is also perfectoid. Indeed,

$$V\left(\mathfrak{a}\right)\cup V\left(\mathfrak{b}\right)\cup V\left(\mathfrak{a}+\mathfrak{b}\right) \longrightarrow \operatorname{Spec}\left(R/\mathfrak{a}\cap\mathfrak{b}\right)$$

is a universal topological epimorphism onto its image so it is an arc cover by [BM21, Proposition 2.6] and [Ryd10, Theorem 2.8]. Taking the sheafification in the arc topology and using [BS22, Corollary 8.11] and [BS17, Theorem 2.9], we see that the above sequence remains exact after perfectoidization *i.e.* the sequence

$$0 \longrightarrow R/\left(\mathfrak{a}\cap\mathfrak{b}\right) \longrightarrow R/\mathfrak{a}\oplus R/\mathfrak{b} \longrightarrow \left(R/\left(\mathfrak{a}+\mathfrak{b}\right)\right)_{\text{perfd}} \longrightarrow 0$$

is exact. In particular,

$$R/\left(\mathfrak{a}+\mathfrak{b}\right) \longrightarrow \left(R/\left(\mathfrak{a}+\mathfrak{b}\right)\right)_{\text{perfd}}$$

must be an isomorphism. □

Corollary 2.8. *Let R be a perfectoid ring and $\mathfrak{a}=(x_1,\dots,x_n)$ a finitely generated ideal. Assume furthermore that each x_i has a compatible system of p-power roots in R. Then, $\mathfrak{a}_{\text{perfd}}=(x_1)_{\text{perfd}}+\dots+(x_n)_{\text{perfd}}=(\sqrt{x_1})^{-}+\dots+(\sqrt{x_n})^{-}$.*

Proof. The first equality is a direct consequence of Proposition 2.7. The second one is Proposition 2.5. □

Proposition 2.9. *Let $R\longrightarrow S$ be a p-completely flat morphism between perfectoid rings. Let $\mathfrak{a}=(h_1,\dots,h_r)$ be a finitely generated ideal of R. Then,*

$$(\mathfrak{a}S)_{\text{perfd}}=\sum_{i=1}^{r}\left((h_iR)_{\text{perfd}}\,S\right)^{-}=\left((\mathfrak{a}R)_{\text{perfd}}\,S\right)^{-}.$$

Proof. By Proposition 2.7, it suffices to show this in the case $\mathfrak{a}$ is principal, generated by $h\in R$. This was proved in [CLM+22, Lemma 2.3.2] but we write it again here for the convenience of the reader. We have

$$\left(S/(hR)_{\text{perfd}}\,S\right)^{\wedge p}=(R/hR)_{\text{perfd}}\,\hat{\otimes}^{L}_{R}S=\left((R/hR)\hat{\otimes}^{L}_{R}S\right)_{\text{perfd}}=(S/hS)_{\text{perfd}},$$

where the first completion is taken in a derived way. Indeed, the first and third equalities follow from the facts that S is p-complete flat over R and that S/hS is derived p-complete, and the second equality follows from [BS22, Proposition 8.13]. This shows that the derived p-adic completion of $S/(hR)_{\text{perfd}}\,S$ is an honest ring and is equal to a classically p-complete ring hence equal to the classical p-adic completion. Then, there are natural quotient maps from S to both side of the equality with kernels $(hS)_{\text{perfd}}$ and $\left((hR)_{\text{perfd}}\,S\right)^{-}$ which must be equal. □

2.2. **Compatible ideals and centers of perfectoid purity.** We are ready to define the main object of study of this paper.

Notation 2.10. Let $(R, \mathfrak{m}, k)$ be a complete Noetherian local ring with residue field of characteristic $p > 0$. Fix a Cohen ring (a complete unramified mixed characteristic DVR with residue field k) $C_k \subset R$. Fix once and for all an inclusion $C_k \longrightarrow W(k^{1/p^\infty})$. There is $A \coloneqq C_k[\![x_1, \ldots, x_d]\!]$, such that R is a module-finite A-algebra and C_k in A maps to C_k in R (e.g. A a Noether–Cohen normalization or $A \twoheadrightarrow R$). Let A_∞ be the p-adic completion of

$$\left(A \hat{\otimes}_{C_k} W\left(k^{1/p^\infty}\right)\right)\left[p^{1/p^\infty}, x_1^{1/p^\infty}, \ldots, x_d^{1/p^\infty}\right],$$

which is perfectoid by [BIM19, Example 3.8 (4)]. Now, let

$$R_\infty^A \coloneqq (R \otimes_A A_\infty)_{\mathrm{perfd}},$$

which is a perfectoid R-algebra by [BS22, Theorem 10.11]. We write $\mathscr{A}$ for the class of all $A = C_k[\![x_1, \ldots, x_d]\!]$ with R a module finite A-algebra such that C_k in A maps to C_k in R.

Remark 2.11. Note that A_∞, and therefore R_∞^A, depend on the choice of a regular system of parameter for A.

Remark 2.12. If $(S, \mathfrak{n})$ is a ramified regular ring with residue field k of characteristic $p > 0$ then $p \in \mathfrak{n}^2$. Let C_k be a Cohen ring for S, $d = \dim S + 1$. The Cohen structure theorem allows us to write $S = A/(p - f)$ for $A \coloneqq C_k[\![x_1, \ldots, x_d]\!]$ and f in $(x_1, \ldots, x_d)^2$. The ring

$$S_\infty = \left(\left(S \hat{\otimes}_{C_k} W\left(k^{1/p^\infty}\right)\right)\left[p^{1/p^\infty}, x_1^{1/p^\infty}, \ldots, x_d^{1/p^\infty}\right] / (p - f) \right)^{\wedge p} = (S \otimes_A A_\infty)^{\wedge p} = S_\infty^A$$

is perfectoid by [Shi16, Proposition 4.9]. Now, if R is a complete local ring that is finite as an S-module, it is also finite as an A-module. Then, if C_k in S maps to C_k in R, by the universal property of perfectoidization,

$$R_\infty^A \cong (R \otimes_S S_\infty)_{\mathrm{perfd}}$$

so we do not lose generality by working with unramified regular local rings.

Definition 2.13 ([BMP+24])**.** Let R, A be as in Notation 2.10. We say that R is perfectoid pure if the natural map $R \longrightarrow R_\infty^A$ is pure. By [BMP+24, Lemma 4.23], this does not depend on the choice of A.

Definition 2.14. In the setting of Notation 2.10, let $\mathfrak{a} \subset R$ be an ideal and ϕ be in $\mathrm{Hom}_R(R_\infty^A, R)$. We call the data of (R, ϕ) a *pair*. Let

$$\mathfrak{a}_\infty^A \coloneqq \left(\mathfrak{a} R_\infty^A\right)_{\mathrm{perfd}}.$$

If $\phi\left(\mathfrak{a}_\infty^A\right) \subset \mathfrak{a}$, we say that $\mathfrak{a}$ is *ϕ-compatible*. If B is any perfectoid R-algebra, we say that $\mathfrak{a}$ is *B-compatible* if for all $\phi \in \mathrm{Hom}_R(B, R)$, $\phi((\mathfrak{a}B)_{\mathrm{perfd}}) \subset \mathfrak{a}$. If $\mathfrak{a}$ is R_∞^A-compatible for all choices of $A \in \mathscr{A}$, we say that $\mathfrak{a}$ is *uniformly perfectoid compatible*.

Remark 2.15. Since R is complete, for any R-module M, a map $R \longrightarrow M$ is pure if and only if it is split by [Fed83, Lemma 1.2]. This is something we often use, especially when dealing with a perfectoid pure R since this gives us the existence of a pair (R, ϕ) with ϕ surjective.

Remark 2.16. In Proposition 2.31, we show that being uniformly perfectoid compatible does not depend on the choices of embeddings $C_k \subset R$ and $C_k \subset W(k^{1/p^\infty})$. Moreover, if k is perfect, there is no choice to be made since an embedding $C_k \subset R$ is equivalent to choosing a p-basis for k and a lift of that p-basis to R (see [Hoc14, Theorem 23]). This justifies that the name uniformly perfectoid compatible does not have a reference to the choices we made in Notation 2.10.

Definition 2.17. Let (R, ϕ) be a pair with ϕ surjective. If $\mathfrak{p} \in \operatorname{Spec} R$ is a ϕ-compatible ideal, we say that $\mathfrak{p}$ is a *center of perfectoid purity of* (R, ϕ). Similarly, if $\mathfrak{p} \in \operatorname{Spec} R$ is a uniformly perfectoid compatible ideal of R then we say that $\mathfrak{p}$ is a *center of perfectoid purity of* R if R is perfectoid pure.

We show that our notion of uniformly perfectoid compatible ideal agrees with the classical notion of a compatible ideal when R is of characteristic $p > 0$, F-finite, and F-pure.

Lemma 2.18. *Let R be a reduced local ring of positive characteristic $p > 0$ and assume that it is F-finite. Let R_{perf} be the perfection of R i.e. $R_{\text{perf}} = \bigcup_e R^{1/p^e}$. Let $f \in R$, Then, the map $R \longrightarrow F^e_* R$, $1 \mapsto F^e_* f$ is split if and only if there is a map $R_{\text{perf}} \longrightarrow R$ sending f^{1/p^e} to 1.*

Proof. The backwards direction is straightforward as one can just restrict said splitting to R^{1/p^e} to get a splitting of $R \longrightarrow R^{1/p^e} \cong F^e_* R$. Suppose, then, that R is e-th Frobenius split along f, say by $\psi_1 \colon R^{1/p^e} \longrightarrow R$ and let $\phi \coloneqq \psi_1 \circ \left(\cdot f^{1/p^e}\right) : R^{1/p^e} \longrightarrow R$. This is a splitting of the inclusion map $R \longrightarrow R^{1/p^e}$ (see e.g. [SS24, Proposition 1.4.6]). We define $\psi \colon R_{\text{perf}} \longrightarrow R$ as follows: for $j > 1$, let

$$\begin{aligned} \psi_j \colon R^{1/p^{ej}} &\longrightarrow R^{1/p^{e(j-1)}} \\ r^{1/p^{ej}} &\mapsto \left(\phi\left(r^{1/p^e}\right)\right)^{1/p^{e(j-1)}} \end{aligned}$$

and

$$\psi \colon r^{1/p^{ei}} \mapsto \psi_1 \circ \psi_2 \circ \cdots \circ \psi_i \left(r^{1/p^{ei}}\right).$$

We need to show that it is well defined *i.e.* that $\psi\left(\left(r^{p^e}\right)^{1/p^{ei}}\right) = \psi\left(r^{1/p^{e(i-1)}}\right)$. This follows from the fact that

$$\psi_i\left(\left(r^{p^e}\right)^{1/p^{ei}}\right) = \phi\left(\left(r^{p^e}\right)^{1/p^e}\right)^{1/p^{e(i-1)}} = \phi(r)^{1/p^{e(i-1)}} = (r\phi(1))^{1/p^{e(i-1)}} = r^{1/p^{e(i-1)}}.$$

We also have that ψ is an R-module homomorphism as it is the composition of R-module homomorphisms and is a splitting since $\psi(f^{1/p^e}) = 1$ by definition so we are done. $\square$

Remark 2.19. Note that the proof implies that as long as R is F-pure, for any $e > 0$, every map in $\operatorname{Hom}_R(R^{1/p^e}, R)$ extends to a map in $\operatorname{Hom}_R(R_{\text{perf}}, R)$.

Corollary 2.20. *Let R be as in Notation 2.10 and assume further that it has characteristic $p > 0$, is F-finite, and F-pure. Then, $R_{\text{perf}} = R^A_\infty$. Let $\mathfrak{a} \subset R$ be an ideal. It is uniformly compatible with the classical definition if and only if is also uniformly perfectoid compatible with our definition. Moreover, our notion of compatible ideal of pairs agrees with the classical notion when working with surjective maps. Indeed, let $\psi_1 \colon R^{1/p^e} \longrightarrow R$ be surjective, say $\psi_1(f^{1/p^e}) = 1$ for some $f \in R$. Let $\psi \colon R_{\text{perf}} \longrightarrow R$ be the corresponding splitting as in Lemma 2.18. Then, $\mathfrak{a}$ is ψ-compatible if and only if it is ψ_1 compatible i.e. $\psi_1(\mathfrak{a}^{1/p^e}) \subset \mathfrak{a}$.*

Proof. The first part follows from the definitions and Lemma 2.18 since $W(k^{1/p^\infty})/p = k^{1/p^\infty}$. If $\psi(\mathfrak{a}_{\mathrm{perf}}) \subset \mathfrak{a}$, then $\psi_1(\mathfrak{a}^{1/p^e}) = \psi(\mathfrak{a}^{1/p^e}) \subset \psi(\mathfrak{a}_{\mathrm{perf}}) \subset \mathfrak{a}$. On the other hand if $\psi_1(\mathfrak{a}^{1/p^e}) \subset \mathfrak{a}$, letting ψ_j and ϕ be as in Lemma 2.18 for $j > 0$, we have

$$\psi_j\left(\mathfrak{a}^{1/p^{ej}}\right) = \left(\phi\left(\mathfrak{a}^{1/p^e}\right)\right)^{1/p^{e(j-1)}} = \left(\psi_1\left(f^{1/p^e}\mathfrak{a}^{1/p^e}\right)\right)^{1/p^{e(j-1)}} \subset \mathfrak{a}^{1/p^{e(j-1)}}.$$

By induction, $\psi\left(\mathfrak{a}^{1/p^{ej}}\right) \subset \mathfrak{a}$ so $\psi(\mathfrak{a}_{\mathrm{perf}}) \subset \mathfrak{a}$. $\square$

Proposition 2.21. *Let (R,ϕ) be a pair and $\mathfrak{a} \subset R$ be an ideal. Then, $\mathfrak{a}$ is ϕ-compatible if and only if ϕ descends to a map $\bar{\phi}\colon (R/\mathfrak{a})^A_\infty \longrightarrow R/\mathfrak{a}$ i.e. if and only if the diagram below commutes.*

(2.21.1)
$$\begin{tikzcd} R^A_\infty \arrow[r, "\phi"] \arrow[d] & R \arrow[d] \\ (R/\mathfrak{a})^A_\infty \arrow[r, "\bar{\phi}"] & R/\mathfrak{a}. \end{tikzcd}$$

and the corresponding result also holds for uniformly perfectoid compatible ideals. Moreover, there is a bijection between the ϕ-compatible ideals of R containing $\mathfrak{a}$ and the $\bar{\phi}$-compatible ideals of $R/\mathfrak{a}$.

Proof. The ideal $\mathfrak{a}$ is ϕ-compatible if and only if the composition map

$$R^A_\infty \xrightarrow{\phi} R \longrightarrow R/\mathfrak{a}$$

factors through $(R/\mathfrak{a})^A_\infty = R^A_\infty/\mathfrak{a}^A_\infty$. The second part of the statement follows from the isomorphism theorems. $\square$

Lemma 2.22. *Let $(R,\mathfrak{m},k)$ be a complete local Noetherian ring of residue characteristic $p>0$ and let B be any R-algebra. Let $\phi \in \mathrm{Hom}_R(B,R)$ be arbitrary. Let $\mathfrak{a} \subset B$ and $\mathfrak{b} \subset R$ be ideals. Then, $\phi(\mathfrak{a}) \subset \mathfrak{b}$ if and only if $\phi(\bar{\mathfrak{a}}) \subset \mathfrak{b}$ where $\bar{\mathfrak{a}}$ is the p-adic closure of $\mathfrak{a}$ in B.*

Proof. We only need show the "only if" direction. We know that $\mathfrak{b}$ is p-adically closed since R is p-adically complete and Noetherian. Then, $\mathfrak{b} = \cap_{n\in\mathbb{N}}(\mathfrak{b} + (p^n))$. Now, let $x \in \bar{\mathfrak{a}} = \cap_{n\in\mathbb{N}}(\mathfrak{a} + (p^n))$. We have

$$\phi(x) \in \bigcap_{n\in\mathbb{N}} \phi(\mathfrak{a}) + \phi((p^n)) \subset \bigcap_{n\in\mathbb{N}} \mathfrak{b} + p^n\phi(B) \subset \bigcap_{n\in\mathbb{N}} (\mathfrak{b} + (p^n)) = \mathfrak{b},$$

as desired. $\square$

Proposition 2.23 (*cf.* [Sch10, Lemma 5.1])**.** *Let (R,ϕ) be a pair and let $\mathfrak{a} \subset R^A_\infty$ and $\mathfrak{b} \subsetneq R$ be ideals. The following are equivalent:*

(a) $\phi(\mathfrak{a}) \subset \mathfrak{b}$.

(b) For any $x \in \mathfrak{a}$, the composition

$$R^A_\infty \xrightarrow{\times x} R^A_\infty \xrightarrow{\phi} R \longrightarrow R/\mathfrak{b}$$

is zero.

(c) For any $x \in \mathfrak{a}$, the composition

$$\mathrm{Hom}_R(R,R) \xrightarrow{\mathrm{Hom}_R(\phi,R)} \mathrm{Hom}_R\left(R^A_\infty, R\right) \xrightarrow{\mathrm{Hom}_R(\times x,R)} \mathrm{Hom}_R\left(R^A_\infty, R\right) \longrightarrow \mathrm{Hom}_R(R,R) \cong R$$

has image in $\mathfrak{b}$.

(d) *For any $x \in \mathfrak{a}$, the composition*

$$E_{R/\mathfrak{b}} \longrightarrow E_R \longrightarrow E_R \otimes_R R_\infty^A \xrightarrow{\mathrm{id} \otimes_R (\times x)} E_R \otimes_R R_\infty^A \xrightarrow{\mathrm{id} \otimes \phi} E_R$$

is zero, where E_R is the injective hull of the residue field k of R over R and $E_{R/\mathfrak{b}}$ is the injective hull of k over $R/\mathfrak{b}$.

and if it holds for $\mathfrak{a} = \mathfrak{b}_\infty^A$ then $\mathfrak{b}$ is ϕ-compatible.

Proof. The equivalence of (a) and (b) readily follows from the definition. The implication (b) implies (c) is direct whereas if we do not have (a) then for some $x \in \mathfrak{a}_\infty^A$, $\phi(x) \notin \mathfrak{b}$ so (c) does not hold. The equivalence between (d) and (c) is a standard application of Matlis duality. □

A similar result holds for uniformly perfectoid compatible ideals.

Proposition 2.24 (*cf.* [Sch10, Lemma 5.1])**.** *Let $(R, \mathfrak{m}, k)$ be a complete Noetherian local ring and let B be an R-algebra. Let $\mathfrak{a} \subset B$ and $\mathfrak{b} \subsetneq R$ be ideals. The following are equivalent:*

(a) *$\mathfrak{a}$ gets sent to $\mathfrak{b}$ under all maps $\phi \in \mathrm{Hom}_R(B, R)$.*

(b) *For any $x \in \mathfrak{a}$ and $\phi \in \mathrm{Hom}_R(B, R)$ the composition*

$$B \xrightarrow{\times x} B \xrightarrow{\phi} R \longrightarrow R/\mathfrak{b}$$

is zero.

(c) *For any $x \in \mathfrak{a}$, the composition*

$$\mathrm{Hom}_R(B, R) \xrightarrow{\mathrm{Hom}_R(\times x, R)} \mathrm{Hom}_R(B, R) \longrightarrow \mathrm{Hom}_R(R, R) \cong R \longrightarrow R/\mathfrak{b}$$

is zero.

(d) *For any $x \in \mathfrak{a}$, the composition*

$$E_{R/\mathfrak{b}} \longrightarrow E_R \longrightarrow E_R \otimes_R B \xrightarrow{\mathrm{id} \otimes_R (\times x)} E_R \otimes_R B$$

is zero, where E_R is the injective hull of k of R over R and $E_{R/\mathfrak{b}}$ is the injective hull of k over $R/\mathfrak{b}$.

If R has residue characteristic $p > 0$ and one (equivalently all) of these hold for B a perfectoid R-algebra, $\mathfrak{a} = (\mathfrak{b}B)_{\mathrm{perfd}}$, we have that $\mathfrak{b}$ is B-compatible. In particular, keeping the notation of Notation 2.10, if this holds for $B = R_\infty^A$ for all possible choices of $A \in \mathscr{A}$ and for $\mathfrak{a} = \mathfrak{b}_\infty^A$, we have that $\mathfrak{b}$ is uniformly perfectoid compatible.

Proof. Same as Proposition 2.23. □

2.3. **New characterizations of uniform perfectoid compatibility.** In this section, we show that uniform perfectoid compatibility can be checked on any choice of $A \in \mathscr{A}$[1] and that it does not depend on the embedding choices that we made in Notation 2.10.

This next lemma is well known to experts but the author does not know a good reference.

Lemma 2.25. *Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring with residue characteristic $p > 0$. If $A \longrightarrow B$ is a p-completely faithfully flat map of R-modules and E is the injective hull of k over R, then $A \otimes_R E \longrightarrow B \otimes_R E$ is injective. In particular, if R is $\mathfrak{m}$-adically complete, $\mathrm{Hom}_R(B, R) \longrightarrow \mathrm{Hom}_R(A, R)$ is surjective. Moreover, if $A = R$, then $A \longrightarrow B$ is pure.*

[1]This proof was suggested by Karl Schwede and the author is very grateful to be able to include it here.

Proof. We use the same technique as in the proof of [BMP$^+$24, Lemma 4.5]. We know that $A/p^n \longrightarrow B/p^n$ is pure for every n. Let E be the injective hull of $R/\mathfrak{m}$ over R. For every finitely generated submodule N of E, N is p^n-torsion for some n. Therefore, $A \otimes_R N \longrightarrow B \otimes_R N$ can be identified as $A/p^n \otimes_R N \longrightarrow B/p^n \otimes_R N$, which is injective by the purity of $A/p^n \longrightarrow B/p^n$. By taking a direct limit over all such N, we find that $A \otimes_R E \longrightarrow B \otimes_R E$ is injective. The surjectivity of $\operatorname{Hom}_R(A, R) \longrightarrow \operatorname{Hom}_R(B, R)$ follows from Matlis duality. The last statement follows by [HR74, Proposition 6.11]. $\square$

Lemma 2.26. *Let $(R, \mathfrak{m}, k)$ be a complete Noetherian local ring with residue characteristic $p > 0$ and let $\mathfrak{a} \subset R$ be any ideal. Let $B \longrightarrow C$ be a p-completely faithfully flat morphism of perfectoid R-algebras. Then, $\mathfrak{a}$ is B-compatible if and only if it is C-compatible.*

Proof. By Lemma 2.25, we have a surjection $\operatorname{Hom}_R(C, R) \longrightarrow \operatorname{Hom}_R(B, R)$ and by Proposition 2.9, $(\mathfrak{a}C)_{\mathrm{perfd}} = ((\mathfrak{a}B)_{\mathrm{perfd}}\, C)^-$. Suppose that $\mathfrak{a}$ is C-compatible and let $\phi \in \operatorname{Hom}_R(B, R)$. There is $\psi \in \operatorname{Hom}_R(C, R)$ that extends ϕ to C. Then,

$$\phi((\mathfrak{a}B)_{\mathrm{perfd}}) \subset \psi((\mathfrak{a}B)_{\mathrm{perfd}}) \subset \psi((\mathfrak{a}C)_{\mathrm{perfd}}) \subset \mathfrak{a}$$

so $\mathfrak{a}$ is B-compatible. Now, suppose that $\mathfrak{a}$ is B-compatible. By Lemma 2.22, it suffices to show that $(\mathfrak{a}B)_{\mathrm{perfd}}\, C$ gets sent to $\mathfrak{a}$ under any map in $\operatorname{Hom}_R(C, R)$. By Proposition 2.24, it suffices to show that for any $x \in (\mathfrak{a}B)_{\mathrm{perfd}}$, $y \in C$, the following composition is 0:

$$\operatorname{Hom}_R(C, R) \xrightarrow{\operatorname{Hom}_R(\times xy, R)} \operatorname{Hom}_R(C, R) \longrightarrow \operatorname{Hom}_R(R, R) \cong R \longrightarrow R/\mathfrak{a}$$

Since $R \longrightarrow C$ factors through $R \longrightarrow B$, the following diagram commutes:

$$\begin{tikzcd}
\operatorname{Hom}_R(C,R) \arrow[r, "\operatorname{Hom}_R(\times xy,R)"] \arrow[d, "\operatorname{Hom}_R(\times y,R)"] & \operatorname{Hom}_R(C,R) \arrow[r] \arrow[d] & \operatorname{Hom}_R(R,R) \cong R \arrow[r] & R/\mathfrak{a} \\
\operatorname{Hom}_R(B,R) \arrow[r, "\operatorname{Hom}_R(x,R)"] & \operatorname{Hom}_R(B,R) \arrow[ur] & &
\end{tikzcd}$$

but by Proposition 2.24, the composition through the bottom is 0 so we are done. $\square$

Proposition 2.27. *Let $(R, \mathfrak{m}, k)$ be a complete Noetherian local ring with residue characteristic $p > 0$ and $\mathfrak{a} \subset R$ be an ideal. Let B be a perfectoid R-algebra. Then, there is a perfectoid B-algebra C that contains all the p-th power roots of elements of R such that $\mathfrak{a}$ is B-compatible if and only if it is C-compatible. In fact, we can even assume that C is absolutely integrally closed. We can also assume that C is $\mathfrak{m}$-adically complete.*

Proof. The first part follows from Lemma 2.26 and André's flatness lemma [And18a, Théorème 2.5.1], [BS22, Theorem 7.14]. The second part follows from the facts that $C \longrightarrow \hat{C}$ is faithfully flat and that $\hat{C}$ is perfectoid [BIM19, Example 3.8]. $\square$

Lemma 2.28. *Let R and $A \in \mathscr{A}$ be as in Notation 2.10. Let $h_1, \ldots, h_r \in R$ be arbitrary. Let $B \coloneqq A[\![y_1, \ldots, y_r]\!]$ and make R into a B-algebra by sending y_i to h_i. Then, the natural morphism $R_\infty^A \longrightarrow R_\infty^B$ is p-completely faithfully flat.*

Proof. There are natural ring maps

$$R \to R \otimes_A A_\infty \to R \otimes_A B_\infty \to R \otimes_B B_\infty \to R_\infty^B \tag{2.28.1}$$

so by the universal property of perfectoidization, the map $R \longrightarrow R_\infty^B$ factors through R_∞^A. By André's flatness lemma ([And18a, Théorème 2.5.1], [BS22, Theorem 7.14]), there is a perfectoid R^A_∞-algebra, say C, such that $R^A_\infty \longrightarrow C$ is p-completely faithfully flat and C contains a compatible system of p-th power roots for each of the h_is. This gives a map A_∞ to C and then a map $B_\infty \longrightarrow C$ by sending the y_i^{1/p^e}s to the p^e-th power root of the h_is in C in a compatible way. We also have a natural map $R \longrightarrow C$ and therefore maps from all the rings in (2.28.1) to C that commute with each other. This gives us a factorization $R \longrightarrow R^A_\infty \longrightarrow R^B_\infty \longrightarrow C$ so the map $R^A_\infty \longrightarrow R^B_\infty$ is also p-completely faithfully flat. □

Proposition 2.29. *Let R and $A \in \mathscr{A}$ be as in Notation 2.10. Let $\mathfrak{a} \subset R$ be an ideal and let $h_1, \dots, h_r$ be arbitrary elements of R. Let $B \coloneqq A[\![y_1, \dots, y_r]\!]$ and make R into a B-algebra by sending y_i to h_i. Then, $\mathfrak{a}$ is R^A_∞-compatible if and only if it is R^B_∞-compatible.*

Proof. This is direct from Lemma 2.28 and Lemma 2.26. □

Corollary 2.30. *Let R be as in Notation 2.10 and $\mathfrak{a} \subset R$ an ideal. Fix any $A \in \mathscr{A}$. If $\mathfrak{a}$ is compatible with all maps in $\mathrm{Hom}_R(R^A_\infty, R)$, then $\mathfrak{a}$ is uniformly perfectoid compatible. That is, one can test uniform perfectoid compatibility on only one $A \in \mathscr{A}$.*

Proof. Let B be any other ring in $\mathscr{A}$. There is a regular local ring C with

$$C = A[\![x_1, \dots, x_r]\!] = B[\![y_1, \dots, y_s]\!]$$

for some $r,\ s \in \mathbb{N}$ such that the maps $A \longrightarrow R$ and $B \longrightarrow R$ factor through $C \longrightarrow R$. Here, we are using that the Cohen ring C_k for k that we fixed is in A and B and maps to the same C_k in R. By Proposition 2.29, $\mathfrak{a}$ is R^A_∞-compatible if and only if it is R^C_∞-compatible if and only if it is R^B_∞-compatible. □

Proposition 2.31. *Let R be as in Notation 2.10 and let $\mathfrak{a} \subset R$ be any ideal. Then, $\mathfrak{a}$ is uniformly perfectoid compatible if and only if for every perfectoid R-algebra B, $\mathfrak{a}$ is B-compatible. In particular, for any ideal $\mathfrak{a} \subset R$, being uniformly perfectoid compatible does not depend on the choices of embeddings $C_k \hookrightarrow R$ and $C_k \hookrightarrow W(k^{1/p^\infty})$.*

Proof. We only need to show that if $\mathfrak{a}$ is uniformly perfectoid compatible, then for any perfectoid R-algebra B, $\mathfrak{a}$ is B-compatible. We show the contrapositive so let B be any perfectoid R-algebra and suppose that there is $\phi \in \mathrm{Hom}_R(B, R)$ with $\phi((\mathfrak{a}B)_{\mathrm{perfd}}) \not\subset \mathfrak{a}$. By Proposition 2.27, we can assume that B is $\mathfrak{m}$-adically complete and has a compatible system of p-th power roots for all the elements of R. By Corollary 2.8 and Lemma 2.22, by possibly modifying the above ϕ, we can assume that there is $x \in \mathfrak{a}$ with a p^e-th root $y \in B$ such that $\phi(y) \notin \mathfrak{a}$. By the proof of [BMP$^+$24, Lemma 4.23], for any $A \in \mathscr{A}$, there is a map $A_\infty \longrightarrow B$ making B into an A_∞-algebra that agrees with the map $R \longrightarrow B$ when restricted to (the image of) A. Now, make R into an $A[\![z]\!]$-algebra by sending z to x. By sending the p-th power roots of z in $A_\infty[z^{1/p^\infty}]$ to a compatible system of p-th power roots of x in B that has y as its p^e-th root, we extend $A_\infty \longrightarrow B$ to $A[\![z]\!]_\infty \longrightarrow B$. By the universal properties of tensor products and perfectoidization, this gives a ring map $i\colon R^{A[\![z]\!]}_\infty \longrightarrow B$. Let $\psi \coloneqq \phi \circ i\colon R^{A[\![z]\!]}_\infty \longrightarrow R$. Denoting by w the p^e-th root of x in R^A_∞ that goes to y in B, we have $\psi(z) \notin \mathfrak{a}$ so $\mathfrak{a}$ is not uniformly perfectoid compatible. □

2.4. **Intersections, sums, and associated primes.** In this section we show that, just like in positive characteristic, compatible ideals behave well under basic ideal operations.

Proposition 2.32 ([BMP$^+$24, Lemma 4.29])**.** *If R is perfectoid injective, in particular if R is perfectoid pure, then R is reduced and weakly normal.*

Corollary 2.33. *Let (R,ϕ) be a pair and $\mathfrak{a}\subset R$ a ϕ-compatible ideal. If ϕ is surjective, then $\mathfrak{a}$ is radical. In particular, if R is perfectoid pure, then the uniformly perfectoid compatible ideals of R are radical.*

Proof. The surjectivity of ϕ implies that of $\bar{\phi}\colon R^A_\infty/\mathfrak{a}^A_\infty \longrightarrow R/\mathfrak{a}$ as in Proposition 2.21. But $\bar{\phi}$ being surjective implies that $R/\mathfrak{a}$ is perfectoid pure so we are done by Proposition 2.32. The statement about uniformly perfectoid compatible ideals then follows from the fact that R is complete so there is a splitting $\phi\colon R^A_\infty \longrightarrow R$ for some/any $A\in\mathscr{A}$. ☐

Proposition 2.34. *Let (R,ϕ) be a pair and $\{\mathfrak{a}_i\}_{i\in I}$ be ϕ-compatible ideals. Then, $\cap_{i\in I}\mathfrak{a}_i$ and $\sum_{i\in I}\mathfrak{a}_i$ are ϕ-compatible. The corresponding result also holds for uniformly perfectoid compatible ideals.*

Proof. By Proposition 2.6, $\cap_{i\in I}(\mathfrak{a}_i)^A_\infty$ is perfectoid. Then,

$$\phi\left(\left(\bigcap_{i\in I}\mathfrak{a}_i\right)^A_\infty\right)\subset\phi\left(\bigcap_{i\in I}(\mathfrak{a}_i)^A_\infty\right)\subset\bigcap_{i\in I}\phi\left((\mathfrak{a}_i)^A_\infty\right)\subset\bigcap_{i\in I}\mathfrak{a}_i$$

so $\cap_{i\in I}\mathfrak{a}_i$ is ϕ compatible. Since we are in a Noetherian ring, any sum of ideals is finite. In particular, to show $\sum_{i\in I}\mathfrak{a}_i$ is ϕ-compatible, it suffices to show that if $\mathfrak{a}$ and $\mathfrak{b}$ are two ϕ-compatible ideals, then so is $\mathfrak{a}+\mathfrak{b}$. Using Proposition 2.7,

$$\phi\left((\mathfrak{a}+\mathfrak{b})^A_\infty\right)\subset\phi\left(\mathfrak{a}^A_\infty+\mathfrak{b}^A_\infty\right)=\phi\left(\mathfrak{a}^A_\infty\right)+\phi\left(\mathfrak{b}^A_\infty\right)\subset\mathfrak{a}+\mathfrak{b}$$

so $\mathfrak{a}+\mathfrak{b}$ is also compatible. ☐

To show that compatible ideals of pairs are closed under associated primes, we will first need the corresponding statement about uniformly perfectoid compatible ideals.

Proposition 2.35. *Let R be as in Notation 2.10. Let $\mathfrak{a}\subset R$ be a uniformly perfectoid compatible ideal. Then, the minimal primes $\mathfrak{q}_1,\dots,\mathfrak{q}_r$ are also uniformly perfectoid compatible ideals.*

Proof. Without loss of generality, it suffices to show that $\mathfrak{q}_1 =: \mathfrak{q}$ is uniformly perfectoid compatible. Let $\{h_1,\dots,h_s\}$ be a set of generators for $\mathfrak{q}$. Let $A\in\mathscr{A}$ be such that R^A_∞ has a compatible system of p-th power roots for the h_is which we denote by $\{z_{j,e}\}$ where $z_{j,e}^{p^e}=h_j$ By Corollary 2.30, it suffices to show that $\mathfrak{q}$ is compatible with all $\phi\in\operatorname{Hom}_R(R^A_\infty,R)$. Fix w in $(\cap_{i>1}\mathfrak{q}_i)\setminus\mathfrak{q}$ and $\phi\in\operatorname{Hom}_R(R^A_\infty,R)$. We have

$$w\phi\left(z^e_jR^A_\infty\right)=\phi\left(wz_{j,e}R^A_\infty\right)\subset\phi\left(\mathfrak{a}^A_\infty\right)\subset\mathfrak{a}\subset\mathfrak{q}.$$

The first containment follows from noting that $(wz_{j,e})^{p^e}=w^{p^e}h_j\in\mathfrak{a}$ and the fact that perfectoid are radical. Since $\mathfrak{q}$ is prime and $w\notin\mathfrak{q}$, we must have that $\phi\left(z_{j,e}R^A_\infty\right)\subset\mathfrak{q}$. Now,

$$\mathfrak{q}^A_\infty=\left(\sum_{j,e}\left(z_{j,e}R^A_\infty\right)\right)^-$$

so we are done by Lemma 2.22. ☐

Corollary 2.36. *Let R be as in Notation 2.10. Then, the minimal primes of R are uniformly perfectoid compatible.*

Proposition 2.37. *Let (R,ϕ) be a pair and $\mathfrak{a} \subset R$ a compatible ideal. Then, the minimal primes $\mathfrak{q}_1,\ldots,\mathfrak{q}_r$ of $\mathfrak{a}$ are also ϕ-compatible.*

Proof. By Proposition 2.21, ϕ descends to a map $\bar{\phi}\colon (R/\mathfrak{a})^A_\infty \longrightarrow R/\mathfrak{a}$. By Corollary 2.36, the associated primes of $R/\mathfrak{a}$ are uniformly perfectoid compatible hence $\bar{\phi}$-compatible. But these are the images of the $\mathfrak{q}_i$ in $R/\mathfrak{a}$ so by Proposition 2.21, the $\mathfrak{q}_i$s are ϕ-compatible. □

We now recall a classical result of Enescu and Hochster.

Proposition 2.38. [EH08, Corollary 3.2] *A family of radical ideals in an excellent local ring closed under sum, intersection, and primary decomposition is finite.*

Corollary 2.39. *Let (R,ϕ) be a pair. If ϕ is surjective, then there are finitely many ϕ-compatible ideals.*

Proof. This is immediate from Proposition 2.38,Proposition 2.37, and Proposition 2.34. □

Corollary 2.40. *Let R be as in Notation 2.10 and assume further that it is perfectoid pure. Then, there are only finitely many uniformly perfectoid compatible ideals.*

Proof. This follows from Corollary 2.39. □

Remark 2.41. In fact, if n is the embedding dimension of R and R is perfectoid pure, [ST10, Theorem 4.2], says that there are at most $\binom{n}{d}$ centers of perfectoid purity of R of height d. Similarly for the centers of perfectoid purity of (R,ϕ) for a surjective ϕ.

Before we move on to the next section, we give two easy examples of uniformly perfectoid compatible ideals.

Proposition 2.42. *Let $\mathfrak{a} \subset R$ be any ideal of R. Then, $\operatorname{Ann}_R(\mathfrak{a})$ is uniformly perfectoid compatible.*

Proof. Let $A \in \mathscr{A}$ be such that R^A_∞ has a compatible system of p-th power roots for generators of $\operatorname{Ann}_R(\mathfrak{a})$. Let h be one of these generators and let x_e be a p^e-th root of h in R^A_∞. Let $\phi \in \operatorname{Hom}_R\left(R^A_\infty, R\right)$ be arbitrary. Then,

$$\phi\left(x_e R^A_\infty\right)\mathfrak{a} = \phi\left(x_e \mathfrak{a} R^A_\infty\right)$$

but $(x_e\mathfrak{a})^{p^e} = 0$ and R^A_∞ is perfectoid hence reduced so $x_e\mathfrak{a} = 0$. This implies $\phi(x_e R^A_\infty)\mathfrak{a} = 0$ so $\phi(x_e R^A_\infty) \in \operatorname{Ann}_R(\mathfrak{a})$, as desired. □

Proposition 2.43. *Let R be as in Notation 2.10. Then, $H^0_\mathfrak{m}(R)$ is a uniformly perfectoid compatible ideal.*

Proof. Let $h_1,\ldots,h_r$ be generators of $H^0_\mathfrak{m}(R)$ and let $A \in \mathscr{A}$ have a compatible system of p-th power roots for the h_is. We denote these roots by $x_{e,i}$ where $x_{e,i}^{p^e} = h_i$. Let $\phi \in \operatorname{Hom}_R\left(R^A_\infty, R\right)$ be arbitrary. We have

$$\phi\left(x_{e,i} R^A_\infty\right)\mathfrak{m} = \phi\left(x_{e,i} R^A_\infty \mathfrak{m}\right).$$

Now, $(x_{e,i}\mathfrak{m})^n = 0$ for some large enough n but R^A_∞ is perfectoid hence reduced so $x_{e,i}\mathfrak{m} = 0$. Then, $\phi\left(x_{e,i} R^A_\infty\right)\mathfrak{m} = 0$ so $\phi\left(x_{e,i} R^A_\infty\right) \in H^0_\mathfrak{m}(R)$ so $H^0_\mathfrak{m}(R)$ is compatible. □

Remark 2.44. We expect that for each $i = 0,\ldots,\dim R - 1$, $\operatorname{Ann}_R(H^i_\mathfrak{m}(R))$ will be a uniformly perfectoid compatible ideal. This would imply that a ring with no uniformly perfectoid compatible ideal has to be Cohen–Macaulay.

3. Perfectoid purity along elements

In this section, we study a variant of perfectoid purity and link it to compatible ideals.

Definition 3.1 (*cf.* [Ram91], [Smi00a])**.** Let R be a Noetherian ring admitting a perfectoid algebra B, and let x be an element of B. If the R-module map $R \longrightarrow B$ sending 1 to x is pure, we say that R is *perfectoid pure along x*.

We now state a well known fact about pure maps in our specific situation since we will repeatedly use it.

Lemma 3.2. *Let R be a Noetherian ring with p in its Jacobson radical, B a perfectoid R-algebra, and C be a perfectoid B-algebra. Let $b \in B$ and $c \in C$ be such that the R-module map $R \longrightarrow C$ sending 1 to c factors through the R-module map $R \longrightarrow B$ sending 1 to b. If R is perfectoid pure along c, then it is perfectoid pure along b.*

Proposition 3.3 (*cf.* [BMP+24, Lemma 4.5])**.** *Let R be a Noetherian ring with p in its Jacobson radical, $r \in R$, and B a perfectoid R algebra. Suppose that for some $e \in \mathbb{N}_{>0}$, r has an p^e-th root in B, which we denote by x. If R is perfectoid pure along x, then there is a perfectoid B-algebra B' that contains all the p-th power roots of r such that R is perfectoid pure along $x_{B'}$ where $x_{B'}$ is the image of x in B'. In fact, we can even assume that B' is integrally closed. Moreover, if B already contains some other p-th power root y of r that is compatible with x, we can choose B' with a system of p-th power roots that is compatible with y.*

Proof. The proof of [BMP+24, Lemma 4.5] works *mutadis mutandis* here. ☐

Proposition 3.4 (*cf.* [BMP+24, Lemma 4.8])**.** *Let $(R, \mathfrak{m})$ be a Noetherian local ring of residue characteristic $p > 0$, $r \in R$, $n \in \mathbb{N}_{>}0$. Then, there is a perfectoid R-algebra B that contains an n-th root x of r such that R is perfectoid pure along x if and only if there is a perfectoid $\hat{R}$ algebra B' that contains an n-th root $x_{B'}$ of r such that R is perfectoid pure along $x_{B'}$.*

Proof. Again, the proof is essentially the same as in [BMP+24, Lemma 4.8]. Suppose that there is a perfectoid R-algebra B that contains an n-th root x of r with R perfectoid pure along x. Then, by [ČS24, Proposition 2.1.11 (e)], $\hat{B}$ is a perfectoid $\hat{R}$-algebra and the completion of the pure map $R \longrightarrow B$, $1 \mapsto x$ is pure: $E \coloneqq E(\hat{R}/\mathfrak{m}) = E(R/\mathfrak{m}) \longrightarrow E \otimes B \cong E \otimes \hat{B}$ is injective. On the other hand, suppose that there is a perfectoid $\hat{R}$-algebra B' containing an n-th root $x_{B'}$ of r with $\hat{R}$ perfectoid pure along $x_{B'}$. Since the map $R \longrightarrow \hat{R}$ is faithfully flat, it is pure, so the composition map $R \longrightarrow B'$, $1 \mapsto x_{B'}$ is pure. ☐

Proposition 3.5 (*cf.* [BMP+24, Lemma 4.23])**.** *Let R be as in Notation 2.10. Let $h \in R$ be any element, $e \in \mathbb{N}_{>0}$, and suppose that there is a perfectoid R-algebra B such that h has a p^e-th root, say x, in B such that R is perfectoid pure along x. Then, there is a choice of $A \in \mathscr{A}$ such that R_∞^A has a p^e-th root of x, say y, and R is perfectoid pure along y.*

Proof. By Proposition 3.3 and Proposition 3.4, we can assume that B is algebraically closed and $\mathfrak{m}$-adically complete. By the proof of Proposition 2.31, there is an $A \in \mathscr{A}$ such that R_∞^A has a p^e-th root y of h and $R \longrightarrow B$ factors through $i\colon R_\infty^A \longrightarrow B$ with $i(y) = z$. The result then follows from Lemma 3.2. ☐

Proposition 3.6. *Let $(R, \mathfrak{m})$ be as in Notation 2.10. If $\mathfrak{m}$ is a uniformly perfectoid compatible ideal of R, then for all $h \in \mathfrak{m}$ and $n \in \mathbb{N}_{>0}$, there are no perfectoid R-algebra B with an n-th root x of h such that R is perfectoid pure along x.*

Proof. Let B be any perfectoid R-algebra. The ideal $\mathfrak{m}$ contains p so $(\mathfrak{m}B)_{\mathrm{perfd}} = \sqrt{\mathfrak{m}B}$ by Proposition 2.4. If $\mathfrak{m}$ is uniformly perfectoid compatible, then by Proposition 2.31 for any $\phi \in \mathrm{Hom}_R(B, R)$,

$$\phi\left(\sqrt{\mathfrak{m}B}\right) \subset \mathfrak{m}$$

so for no root x of an element of $\mathfrak{m}$, the map $R \longrightarrow B$ sending 1 to x is split. Since R is complete, this means that it is never pure by [Fed83, Lemma 1.2]. □

Conversely, we have the following result when p is in the ideal.

Proposition 3.7. *Let R be as in Notation 2.10, and let $\mathfrak{p} \in \operatorname{Spec} R$ containing p. Suppose that for any $h \in \mathfrak{p}$ and any choice of $A \in \mathscr{A}$ with R^A_∞ containing a p-th power root x of h, the map $R_\mathfrak{p} \longrightarrow R^A_{\infty,\mathfrak{p}}$ sending 1 to x is not pure, then $\mathfrak{p}$ is uniformly perfectoid compatible.*

Proof. Suppose not and let $A \in \mathscr{A}$, $\phi \in \mathrm{Hom}_R(R^A_\infty, R)$, $y \in \mathfrak{p}^A_\infty$ be such that $\phi(y) \notin \mathfrak{p}$. By Proposition 2.4, $y^{p^e} \in \mathfrak{p}$ for some large enough e so the map $R_\mathfrak{p} \longrightarrow R^A_{\infty,\mathfrak{p}}$ sending $1 \longrightarrow y$ is pure, a contradiction. □

Remark 3.8. One can rephrase the assumption of Proposition 3.7 by saying that for all such A, h, and x, denoting by z the image of x in $(R^A_{\infty,\mathfrak{p}})^{\wedge p}$, $R_\mathfrak{p}$ is perfectoid pure along z.

Corollary 3.9. *Let R be as in Notation 2.10, $\mathfrak{p} \in \operatorname{Spec} R$ with $p \in \mathfrak{p}$. If $\mathfrak{p}\hat{R}_\mathfrak{p}$ is a uniformly perfectoid compatible ideal of $\hat{R}_\mathfrak{p}$ then $\mathfrak{p}$ is a uniformly perfectoid compatible ideal of R.*

Proof. This follows from Proposition 3.6, Proposition 3.4, [BIM19, Example 3.8(7)], and Proposition 3.7. □

Characterizing uniform perfectoid compatibility in terms of non-purity holds more generally.

Proposition 3.10. *Let R be as in Notation 2.10, $\mathfrak{p} \in \operatorname{Spec} R$ be an ideal with generators $h_1, \ldots, h_r$. Let B be a perfectoid R-algebra that has a compatible system of p-th power roots for $h_1, \ldots, h_r$. We denote these roots by $\{x_{i,e}\}$ where $x_{i,e}^{p^e} = h_i$. If for any $e \gg 0,\ i = 1, \ldots, r$, the map*

$$\begin{aligned} R_\mathfrak{p} &\longrightarrow B_\mathfrak{p} \\ 1 &\mapsto x_{i,e} \end{aligned} \tag{3.10.1}$$

is not pure, then $\mathfrak{p}$ is B-compatible. In particular, if $B = R^A_\infty$ for some $A \in \mathscr{A}$, then $\mathfrak{p}$ is uniformly perfectoid compatible.

Proof. By Corollary 2.8, we know that $(\mathfrak{p}B)_{\mathrm{perfd}} = (\sum_{i=1}^r \sum_e (x_{i,e}))^-$. Suppose that $\mathfrak{p}$ is not compatible with $\phi \in \mathrm{Hom}_R(B, R)$. Using Lemma 2.22, this means that

$$\phi\left(\sum_{i=1}^r \sum_e (x_{i,e})\right) \not\subset \mathfrak{p}.$$

In particular, there must be i and e with $\phi((x_{i,e})) \not\subset \mathfrak{p}$ so there is $y \in B$ with $\phi(yx_{i,e}) \notin \mathfrak{p}$. Then, $\phi_\mathfrak{p}$ is a splitting of the map $R_\mathfrak{p} \longrightarrow B_\mathfrak{p}$ sending $1 \longrightarrow yx_{i,e}$. This implies that the map $R_\mathfrak{p} \longrightarrow B_\mathfrak{p}$ sending $1 \longrightarrow x_{i,e}$ is pure, a contradiction. This shows that $\mathfrak{p}$ is compatible with all maps in $\mathrm{Hom}_R(B, R)$. If $B = R^A_\infty$ for some $A \in \mathscr{A}$, it must be uniformly perfectoid compatible by Corollary 2.30. $\square$

4. Connection with log canonical centers

In this section, we prove that the log canonical centers of a perfectoid pure ring are centers of perfectoid purity. Many of the statements and proofs in this section are adaptations of the ones in [BMP+24].

Definition 4.1. Let X be a normal Noetherian integral scheme with a dualizing complex $\omega^\bullet_X$ and a canonical divisor K_X. Let Δ be an effective divisor on X with coefficients ≤ 1. We say that (X, Δ) is log canonical if $K_X + \Delta$ is $\mathbb{Q}$-Cartier and for every proper birational map $\pi\colon Y \longrightarrow X$ with Y normal, we have that the coefficients of $K_Y - \pi^*(K_X + \Delta)$ are ≥ -1. Equivalently, for every proper birational map $\pi\colon Y \longrightarrow X$ with Y normal and reduced exceptional divisor E, we need that

$$\pi_*(\mathscr{O}_Y(\lceil K_Y - \pi^*(K_X + \Delta) + E\rceil) = \mathscr{O}_X.$$

See for instance [KM98, Corollary 2.31]. If R is quasi-Gorenstein (*i.e.* $K_X \sim 0$), then it suffices to show $\pi_*(\omega_Y(E - \lfloor \pi^*(\Delta)\rfloor)) \cong \omega_X$.

Definition 4.2. Let $X = \operatorname{Spec} R$ be a normal Noetherian integral scheme with a dualizing complex $\omega^\bullet_X$ and a canonical divisor K_X. A log canonical center $Z \subset X$ is a closed subscheme of X such that X is log canonical at the generic point of Z and for any $f \in \mathscr{I}_Z$, the ideal of R defining Z, and any $1 \gg \varepsilon > 0$, the pair $(X, \varepsilon \operatorname{div}(f))$ is not log canonical. We will abuse notation and say $\mathscr{I}_Z$ is a log canonical center of R.

Lemma 4.3 (*cf.* [BMP+24, Lemma 5.3])**.** *Let R, A be as in Notation 2.10. Let $h \in R$ be arbitrary and assume that for some y in a regular system of parameter of A, $y \mapsto h$. In particular, we can assume that R^A_∞ contains a compatible system of p-th power roots for h, which we denote by $\{z_e\}$ where $z_e^{p^e} = h$ for $e \in \mathbb{N}_{>0}$. Fix an arbitrary $e \in \mathbb{N}_{>0}$ and let $S := R[x]/x^{p^e} - h$. Let $\pi\colon Y \longrightarrow X := \operatorname{Spec} S$ be a birational map. Let Z be the subset of X outside of which π is an isomorphism and set $E := \pi^{-1}(Z)_{\mathrm{red}}$. Let $\Delta := \operatorname{div} x$ on Y and $C^\bullet$ be the following pullback in the (unbounded) derived ∞-category of S-modules*

$$\begin{tikzcd} C^\bullet \arrow[r] \arrow[d] & \mathbf{R}\Gamma(Y, \mathscr{O}_Y) \arrow[d] \\ \mathbf{R}\Gamma(Z, \mathscr{O}_Z) \arrow[r] & \mathbf{R}\Gamma(E, \mathscr{O}_E). \end{tikzcd}$$

Then, the map $R \longrightarrow R^A_\infty$, $1 \mapsto z_e$ factors as

$$R \longrightarrow S \longrightarrow C^\bullet \longrightarrow R^A_\infty$$

where the first map is multiplication by x and the second is the natural map $S \longrightarrow C^\bullet$.

Proof. We need to show that $C^\bullet$ maps to R^A_∞. The (ring) morphism $R \longrightarrow R^A_\infty$ extends to a morphism $S \longrightarrow R^A_\infty$ by sending $x \longrightarrow z_e$. By [BMP+24, Lemma 5.3], there is a map $C^\bullet$ to S_{perfd}. By the universal property of perfectoidization, there is a map $S_{\mathrm{perfd}} \longrightarrow R^A_\infty$ and the result follows. $\square$

Lemma 4.4 (*cf.* [BMP$^+$24, Proposition 5.15])**.** *Let $(R, \mathfrak{m})$ be a Noetherian normal quasi-Gorenstein domain with a dualizing complex $\omega_R^\bullet$. Let $h \in R$ be arbitrary and let $S := R[x]/x^{p^e} - h$ for some $e > 0$. Note that S is S_2 and quasi-Gorenstein. Let $f \colon \operatorname{Spec} S \longrightarrow \operatorname{Spec} R$ be the induced map. Suppose that for any birational $\mu \colon Y \longrightarrow \operatorname{Spec} S$ with*

(a) μ is an isomorphism outside a set $V(J) \subset \operatorname{Spec} S$ of codimension ≥ 2
(b) Y is $G1$ and $S2$
(c) If $F = \mu^{-1}(V(J))_{\mathrm{red}}$, we have that F has pure codimension 1 and that Y is regular at each generic point of F (that is, F can be viewed as a divisor),

we have that the composition

$$f_*\mu_*(x \cdot \mathscr{H}\mathrm{om}_Y(\mathscr{I}_F, \omega_Y)) \longrightarrow f_*(x \cdot \omega_S) \longrightarrow \omega_R \tag{4.4.1}$$

induced by the map $R \longrightarrow S$, $1 \mapsto x$ surjects. Then, if $\lfloor 1/p^e \operatorname{div} h \rfloor$ is 0 on R, the pair $(R, 1/p^e \operatorname{div} h)$ is log canonical.

Proof. This is essentially the same proof as [BMP$^+$24, Proposition 5.15]. Let $\pi \colon X \longrightarrow \operatorname{Spec} R$ be a blowup with X normal and which is an isomorphism over $U = X \setminus \pi(D)$ for some (exceptional) divisor D on X. Let $I \subset R$ be an ideal whose blowup produces $\pi \colon X \longrightarrow \operatorname{Spec} R$. Let $Y_0 \longrightarrow \operatorname{Spec} S$ denote the blowup of IS and note we have a finite map $Y_0 \longrightarrow X$. Let $V \subset Y_0$ be the inverse image of U and note that it is quasi-Gorenstein since it is an open subset of $\operatorname{Spec} S$. Let $i \colon V \longrightarrow Y_0$ be the inclusion. Let $\mathscr{C}$ be the integral closure of $\mathscr{O}_{Y_0}$ in $i_*\mathscr{O}_U$ *i.e.* $\mathscr{C} = \mathscr{O}_{Y_0}^N \cap i_*\mathscr{O}_V$ where the intersection is taken in the fraction field of Y_0. Let $Y := \mathbf{Spec}_{Y_0}(\mathscr{C})$. Then, Y is $G1$ and $S2$ and has a finite map to Y_0 since our base is excellent. Therefore, there is a finite map $g \colon Y \longrightarrow X$ and the induced map $Y \longrightarrow \operatorname{Spec} S$ is an isomorphism over V. Let E and F denote the reduced exceptional sets of the maps $X \longrightarrow \operatorname{Spec} R$ and $Y \longrightarrow \operatorname{Spec} S$, respectively. By [BMP$^+$24, Theorem 5.14, Proposition 5.15], F is a divisor in the sense of [Kol13]. We have the following commutative diagram

$$\begin{tikzcd}
F \arrow[r, two heads, "h"] \arrow[d, hook] & E \arrow[d, hook] \\
Y \arrow[r, two heads, "g"] \arrow[d, "\mu"] & X \arrow[d, "\pi"] \\
\operatorname{Spec} S \arrow[r, two heads, "f"] & \operatorname{Spec} R
\end{tikzcd}$$

where all the horizontal maps are finite, by construction. This induces the following diagram of canonical modules

$$\begin{tikzcd}
0 \arrow[r] & g_*\omega_Y \arrow[r] \arrow[d] & g_*\omega_Y(F) \arrow[r] \arrow[d] & h_*\omega_F \arrow[d] \\
0 \arrow[r] & \mathscr{O}_X(K_X) \arrow[r] & \mathscr{O}_X(K_X + E) \arrow[r] & \omega_E
\end{tikzcd}$$

where the notation $\mathscr{H}\mathrm{om}_Y(\mathscr{I}_F, \omega_Y) = \omega_Y(F) = \mathscr{O}_Y(K_Y + F)$ is reasonable as Y is G_1 and S_2.

Claim 4.5. *The image of $g_*(x \cdot \omega_Y(F)) \longrightarrow \mathscr{O}_X(K_X + E)$ is contained in the sheaf $\mathscr{O}_X(K_X + E - \lfloor 1/p^e \operatorname{div} h \rfloor)$.*

Proof. Since all sheaves are $S2$, we can check this in codimension 1. On V, we can reduce to the affine case and there is nothing to show. At the generic points of F, Y is normal and the result follows from our choice of rounding. $\square$

By pushing forward to R, we get a map

$$\pi_* g_*(x \cdot \omega_Y(F)) \longrightarrow \pi_*(\mathscr{O}_X(K_X + E - \lfloor 1/p^e \operatorname{div} h \rfloor)) \longrightarrow R \cong \omega_R.$$

But this can also be factored as

$$f_* \mu_*(x \cdot \mathscr{H}\mathrm{om}_Y(\mathscr{I}_F, \omega_Y)) \longrightarrow f_*(x \cdot \omega_S) \longrightarrow \omega_R,$$

which we assumed to be surjective. Then, $\pi_*(\mathscr{O}_X(K_X + E - \lfloor 1/p^e \operatorname{div} h \rfloor)) \longrightarrow \omega_R$ is surjective and since X was arbitrary, R is log canonical. $\square$

Remark 4.6. Note the proof actually shows that it suffices to check (4.4.1) for all birational maps $Y \longrightarrow \operatorname{Spec} S$ built from a blowup $\pi\colon X \longrightarrow \operatorname{Spec} R$ with X normal as in the proof.

Lemma 4.7 (*cf.* [BMP+24, Claim 5.5]). *Let R, h, e, and S be as in Lemma 4.3. Let $X \longrightarrow \operatorname{Spec} R$ be a blowup with X normal. Let $\omega_R^\bullet$ be a dualizing complex for R and let $\mathbf{D}$ denote Grothendieck duality $\operatorname{RHom}_R(-, \omega_R^\bullet)$. Constructing Y as in Lemma 4.4 and $C^\bullet$ from the birational map $\mu\colon Y \longrightarrow \operatorname{Spec} S$, as in Lemma 4.3, we have*

$$H^{-d}(\mathbf{D}(C^\bullet)) = \Gamma(Y, \omega_Y(F))$$

for F the reduced exceptional divisor of μ. The corresponding map $\Gamma(Y, \omega_Y(F)) \longrightarrow \omega_R$ factors through $\Gamma(Y, x \cdot \omega_Y(F))$. Here, we interpret $\omega_Y(F)$ as in Lemma 4.4.

Proof. This follows from [BMP+24, Claim 5.5] and the choice of map $R \longrightarrow C^\bullet$. $\square$

Before our next lemma, we need a result about pure maps in the derived category, which we state for the convenience of the reader.

Lemma 4.8 ([BMP+24, Proposition 2.11]). *Let R be a Noetherian ring, $\mathfrak{a} \subset R$ an ideal, and let $f\colon M \longrightarrow N$ be a pure map in $D(R)$ in the sense of* [BMP+24, Section 2.1]. *Then, $H^i \mathbf{R}\Gamma_{\mathfrak{a}} M \longrightarrow H^i \mathbf{R}\Gamma_{\mathfrak{a}} N$ is injective for all i.*

Proposition 4.9 (*cf.* [BMP+24, Proposition 5.4], [KSS10]). *Let R, A, h, e be as in Lemma 4.3. Let $\mathfrak{p} \in \operatorname{Spec} R$ be a prime that contains p. If R is quasi-Gorenstein and e is such that $\lfloor 1/p^e \operatorname{div} h \rfloor$ is 0 on R, the map $R_{\mathfrak{p}} \longrightarrow R^A_{\infty,\mathfrak{p}}$ sending 1 to h^{1/p^e} is pure, then $(R_{\mathfrak{p}}, 1/p^e \operatorname{div} h)$ is log canonical.*

Proof. By taking the $\mathfrak{p}$-adic completion of $R_{\mathfrak{p}}$ and $R^A_{\infty,\mathfrak{p}}$ and using Proposition 3.4, we can reduce to the case where $\mathfrak{p}$ is the maximal ideal. Let $C^\bullet$ be as in Lemma 4.7. The map $R \longrightarrow C^\bullet$ is pure by Lemma 4.3 so $H^d_{\mathfrak{m}}(R) \longrightarrow H^d_{\mathfrak{m}}(\mathbf{R}\Gamma(X, C^\bullet))$ is injective by Lemma 4.8. Then, the dual map $f_*\mu_*(x \cdot \omega_Y(F)) \longrightarrow \omega_X$, surjects for f and μ as in Lemma 4.4 and the result follows from Lemma 4.4. $\square$

Proposition 4.10 (*cf.* [BMP+24, Proposition 5.4], [KSS10]). *Let R, A, h, e, S, and x be as in Lemma 4.3. Let $\mathfrak{p} \in \operatorname{Spec} R$ be a prime that does not contain p and assume that $h \in \mathfrak{p}$. If R is quasi-Gorenstein, $e \in \mathbb{N}_{>0}$ is such that $\lfloor 1/p^e \operatorname{div} h \rfloor$ is 0 on R, and the map $R_{\mathfrak{p}} \longrightarrow R^A_{\infty,\mathfrak{p}}$ sending 1 to h^{1/p^e} is pure, then the pair $(R_{\mathfrak{p}}, 1/p^e \operatorname{div} h)$ is log canonical.*

Proof. Let $\pi_{\mathfrak{p}}\colon X_{\mathfrak{p}} \longrightarrow \operatorname{Spec} R_{\mathfrak{p}}$ be a blowup, say of the ideal $I \subset R_{\mathfrak{p}}$ and let $\pi\colon X \longrightarrow \operatorname{Spec} R$ be the blowup of $\operatorname{Spec} R$ at the ideal $I \cap R$. Assume that both X and $X_{\mathfrak{p}}$ are normal. Keeping the notation of Lemma 4.4 applied to π and construct S, $C^\bullet$ as in Lemma 4.3 and Y, F with maps $f\colon \operatorname{Spec} S \longrightarrow \operatorname{Spec} R$ and $\mu\colon Y \longrightarrow \operatorname{Spec} S$ as in Lemma 4.4. Let $C^\bullet_{\mathfrak{p}} \coloneqq C^\bullet \otimes_R R_{\mathfrak{p}}$. Note that $C^\bullet_{\mathfrak{p}}$ is the following pullback in the (unbounded) derived ∞-category of $S_{\mathfrak{p}}$-modules

$$\begin{tikzcd}
C^\bullet_{\mathfrak{p}} \arrow[r] \arrow[d] & \mathbf{R}\Gamma(Y_{\mathfrak{p}}, \mathscr{O}_{Y_{\mathfrak{p}}}) \arrow[d] \\
\mathbf{R}\Gamma(Z_{\mathfrak{p}}, \mathscr{O}_{Z_{\mathfrak{p}}}) \arrow[r] & \mathbf{R}\Gamma(F_{\mathfrak{p}}, \mathscr{O}_{F_{\mathfrak{p}}}).
\end{tikzcd}$$

where $X_{\mathfrak{p}}$, $Y_{\mathfrak{p}}$ and $F_{\mathfrak{p}}$ are the base changes of X, Y and F, respectively, from $\operatorname{Spec} R$ to $\operatorname{Spec} R_{\mathfrak{p}}$. Let $\mu_{\mathfrak{p}}\colon Y_{\mathfrak{p}} \longrightarrow \operatorname{Spec} S_{\mathfrak{p}}$ and $f_{\mathfrak{p}}\colon \operatorname{Spec} S_{\mathfrak{p}} \longrightarrow \operatorname{Spec} R_{\mathfrak{p}}$ be the corresponding maps. Since the proof of Lemma 4.7 (*cf.* [BMP+24, Claim 5.5]) did not make use of the p-completeness of R, letting $\omega^\bullet_{R_{\mathfrak{p}}}$ be a normalized dualizing complex for $R_{\mathfrak{p}}$ and $\mathbf{D}$ denote Grothendieck duality $\operatorname{RHom}_R(-, \omega^\bullet_{R_{\mathfrak{p}}})$, we have

$$H^{-r}(\mathbf{D}(C^\bullet)) = \Gamma(Y_{\mathfrak{p}}, \omega_{Y_{\mathfrak{p}}}(F_{\mathfrak{p}}))$$

for $r = \dim R_{\mathfrak{p}}$. Since the multiplication-by-x map $R_{\mathfrak{p}}$ to $C_{\mathfrak{p}}$ is pure, by Lemma 4.8, the corresponding map $H^r_{\mathfrak{p}R_{\mathfrak{p}}}(R_{\mathfrak{p}}) \longrightarrow H^r_{\mathfrak{p}R_{\mathfrak{p}}}(C^\bullet_{\mathfrak{p}})$ injects and its dual surjects. Then, the map

$$(f_{\mathfrak{p}})_*(\mu_{\mathfrak{p}})_*(x \cdot \omega_{Y_{\mathfrak{p}}}(F_{\mathfrak{p}})) \longrightarrow \omega_{R_{\mathfrak{p}}}$$

surjects and by Lemma 4.4 and Remark 4.6, we are done. $\square$

Theorem 4.11 (*cf.* [Sch10, Theorem 6.7])**.** *Let R be as in Notation 2.10 and assume further that it is quasi-Gorenstein and normal. Let $\mathfrak{p} \in \operatorname{Spec} R$ be a log canonical center of R. Then, $\mathfrak{p}$ is uniformly perfectoid compatible. In particular, if R has no uniformly perfectoid compatible ideal, it must be klt.*

Proof. By Proposition 3.10, it is enough to show that for $A \in \mathscr{A}$ with a subset of a system of parameters mapping to generators $\{h_1, \dots, h_r\}$ of $\mathfrak{m}$, the maps $R \longrightarrow R^A_\infty$, $1 \mapsto h_i^{1/p^e}$ are not pure for $e \gg 0$. Since none of the pairs $(R_{\mathfrak{p}}, \operatorname{div}(h^{1/p^e}))$ are log canonical, the maps $R_{\mathfrak{p}} \longrightarrow R^A_{\infty,\mathfrak{p}}$, $1 \mapsto h^{1/p^e}$ are not pure by Proposition 4.9 and Proposition 4.10 so we are done. $\square$

Corollary 4.12. *Let R be as in Notation 2.10 and assume further that it is quasi-Gorenstein and normal. Then, the multiplier ideal $\mathscr{J} \subset R$ is uniformly perfectoid compatible.*

Proof. This follows from the fact that $\mathscr{J}$ is the intersection of all the log canonical centers of R, Theorem 4.11, and Proposition 2.34. $\square$

Remark 4.13. Based on the positive characteristic result [Sch10], we expect this to hold even when R is not quasi-Gorenstein. A first step could be to generalize this to the $\mathbb{Q}$-Gorenstein case with index not divisible by p as in [BMP+24, Corollary 5.11].

5. Normality

In this section, we show that the conductor ideal is uniformly perfectoid compatible, implying that the presence of compatible ideals detects (the failure of) normality. We use this to deduce various properties of perfectoid pure rings.

Lemma 5.1. *Let R and A be as in Notation 2.10 and R^N be the normalization of R. Let $\mathfrak{c}$ be the conductor ideal of R i.e. the largest ideal of R that is also an ideal of R^N. Then, $\mathfrak{c}_\infty^A$ is an ideal both of R_∞^A and $(R^N \otimes_A A_\infty)_{\mathrm{perfd}} =: R_\infty^{N,A}$.*

Proof. We first show that $R \otimes_A A_\infty, \mathfrak{c} \otimes_A A_\infty$, and $R^N \otimes_A A_\infty$ are all derived p-complete A_∞-modules. Note that R, $\mathfrak{c}$, and R^N are all finitely presented A-modules since R is excellent hence N-1 and Noetherian. This implies that $R \otimes_A A_\infty, R^N \otimes_A A_\infty$, and $\mathfrak{c} \otimes_A A_\infty$ are all finitely presented A_∞-modules. Since a finitely presented module over a derived p-complete ring is derived p-complete (see, e.g., [Ked24, Corollary 6.3.2]), we are done. Now, by [BS22, Corollary 8.12], we have a pullback diagram

$$\begin{tikzcd} R_\infty^A \arrow[r] \arrow[d] & R_\infty^{N,A} \arrow[d] \\ ((R/\mathfrak{c}) \otimes_A A_\infty)_{\mathrm{perfd}} \arrow[r] & \big((R^N/\mathfrak{c}) \otimes_A A_\infty\big)_{\mathrm{perfd}} \end{tikzcd}$$

which we claim implies that $\mathfrak{c}_\infty^A$ is both an ideal of R_∞^A and $R_\infty^{N,A}$. Indeed, the diagram can be rewritten as

$$\begin{tikzcd} R_\infty^A \arrow[r] \arrow[d] & R_\infty^{N,A} \arrow[d] \\ R_\infty^A/\mathfrak{c}_\infty^A \arrow[r] & R_\infty^{N,A}/\big(\mathfrak{c}R_\infty^{N,A}\big)_{\mathrm{perfd}}, \end{tikzcd}$$

so the image of $\mathfrak{c}_\infty^A$ in $R_\infty^{N,A}$ is exactly the kernel of the quotient $R_\infty^{N,A} \longrightarrow R_\infty^{N,A}/\big(\mathfrak{c}R_\infty^{N,A}\big)_{\mathrm{perfd}}$ hence is an ideal of $R_\infty^{N,A}$. □

Proposition 5.2 (*cf.* [BK05, Exercise 1.2.4(E)])**.** *Let (R, ϕ) be a pair and $\mathfrak{c} \subset R$ be the conductor ideal. Then, for any map $\phi\colon R_\infty^A \longrightarrow R$, $\mathfrak{c}$ is ϕ-compatible. In particular, if there are no non-trivial ϕ-compatible ideals, R is normal.*

Proof. By Lemma 5.1, if $s \in R^N$ then $s\mathfrak{c}_\infty^A \subset \mathfrak{c}_\infty^A$ where we abuse notation and think about all these elements as part of R_∞^N. Then, if W is the set of nonzero divisors of R and

$$\phi_W \coloneqq \phi \otimes_R \mathrm{id}_{W^{-1}R}\colon R_\infty^A \otimes_R W^{-1}R \longrightarrow R \otimes_R W^{-1}R,$$

seeing R as a subset of $W^{-1}R \cong R \otimes_R W^{-1}R$ via $r \longrightarrow r \otimes_R 1$ gives

$$s\phi(\mathfrak{c}_\infty^A) = s\phi_W(\mathfrak{c}_\infty^A) = \phi_W(s\mathfrak{c}_\infty^A) \subset \phi_W(\mathfrak{c}_\infty^A) = \phi(\mathfrak{c}_\infty^A)$$

so $\phi(\mathfrak{c}_\infty^A) \subset \mathfrak{c}$ hence $\mathfrak{c}$ is compatible. Now, since $\mathfrak{c} = \mathrm{Ann}_R(R^N/R)$, it is generically K hence nonzero. In particular, if R has no nontrivial compatible ideals, $\mathfrak{c} = R$ so R is normal. □

Remark 5.3. Note that this is not dependent on the choice of ϕ (nor on the choice of $A \in \mathscr{A}$) so the conductor is uniformly perfectoid compatible. In particular, if R has no uniformly perfectoid compatible ideals then it is normal.

Corollary 5.4 (*cf.* [Sch10, Proposition 7.11], [BK05, Exercise 1.2.E(4)])**.** *Let $\phi\colon R_\infty^A \longrightarrow R$ be an R-linear map and R^N be the normalization of R. Then, ϕ has a unique extension $\phi^N\colon R_\infty^{N,A} \longrightarrow R^N$. In particular, if (R, ϕ) is perfectoid split, so is (R^N, ϕ^N).*

Proof. Let $a \in \mathfrak{c}$ be a nonzerodivisor and K be the total ring of fraction of R. We define $\phi^N \colon R^{N,A}_\infty \longrightarrow K$ as $\phi^N(x) := \frac{\phi(xa)}{a}$. We first show that this is an R^N-linear map. Let W be the set of nonzero divisors of R, $\phi_W := \phi \otimes_R \mathrm{id}_{W^{-1}R}$, and $s \in R^N$. Then,

$$s\phi^N(x) = s\frac{\phi(xa)}{a} = s\frac{\phi_W(xa)}{a} = \frac{\phi_W(xas)}{a} = \frac{\phi(xsa)}{a} = \phi^N(sx).$$

Moreover, for $x, y \in R^{N,A}_\infty$,

$$\phi^N(x+y) = \frac{\phi(xa+ya)}{a} = \frac{\phi(xa)+\phi(ya)}{a} = \frac{\phi(xa)}{a} + \frac{\phi(ya)}{a} = \phi^N(x) + \phi^N(y).$$

We want to show that the image of ϕ^N lands in R^N. For any $c \in \mathfrak{c}$,

$$c\phi^N(x) = \phi^N(cx) \in \mathfrak{c} \subset R.$$

More generally,

$$c\left(\phi^N(x)\right)^n = c\phi^N(x)\left(\phi^N(x)\right)^{n-1} \subset \mathfrak{c}\left(\phi^N(x)\right)^{n-1}$$

which implies $\mathfrak{c}\left(\phi^N(x)\right)^n \subset \mathfrak{c}\left(\phi^N(x)\right)^{n-1}$, and so by induction $\mathfrak{c}\left(\phi^N(x)\right)^n \subset \mathfrak{c} \subset R$. Therefore, $\phi^N(x)$ is integral over R by [HS06, Proposition 2.4.8].

It remains to show the uniqueness of such map. Suppose ϕ_1^N and ϕ_2^N are both extensions of ϕ to $R^{N,A}_\infty$. Then, for any $c \in \mathfrak{c}$ a nonzerodivisor,

$$c\phi_1^N(x) = \phi_1^N(xc) = \phi(xc) = \phi_2^N(xc) = c\phi_2^N(x)$$

and since c is a nonzerodivisor, we must have $\phi_1^N(x) = \phi_2^N(x)$. In particular, the definition of ϕ^N does not depend on the chosen $a \in \mathfrak{c}$. $\square$

Remark 5.5. It is well known that

$$\mathfrak{c} = \mathrm{Im}\left(\mathrm{Hom}_R\left(R^N, R\right) \xrightarrow{\phi \mapsto \phi(1)} R\right).$$

This is a specific case of a trace ideal, which are known to be compatible in positive characteristic (see for instance [PS23, Lemma 2.2]). It is a natural question to ask if this is also true in mixed characteristic. That is, if $R \longrightarrow S$ is a finite extension of complete local Noetherian rings, is

$$\mathrm{Im}\left(\mathrm{Hom}_R(S, R) \xrightarrow{\phi \mapsto \phi(1)} R\right)$$

a uniformly perfectoid compatible ideal of R? Unfortunately, it is not true in this generality. Indeed, let $R := \mathbb{Z}_p[\![x, y, z]\!]/x^2 + y^3 + z^3$ with $p \equiv 1 \pmod 3$. This is a perfectoid pure ring by [BMP+24, Example 7.3] since going modulo p is the (x, y, z)-adic completion of the cone over an ordinary elliptic curve. Then, all the uniformly perfectoid compatible ideals have to be radical by Corollary 2.33. Let $S = R$ with map $R \longrightarrow S$ induced by multiplication by p^n on the elliptic curve. By [BMP+23a, Example 4.14], the image of the trace map is the ideal (p^n, x, y, z) which is not radical and therefore not compatible.

We can also use the compatibility of the conductor ideal to show that perfectoid pure rings are (WN1). When R has char $p > 0$ and is F-split, this was shown by Schwede and Zhang in [SZ13]. The mixed characteristic proof follows their strategy.

Definition 5.6 ([CM81])**.** Let $(R, \mathfrak{m})$ be a reduced local weakly normal ring. We say that $(R, \mathfrak{m})$ is (WN1) if the normalization morphism $R \longrightarrow R^N$ is unramified in codimension 1. That is, for every prime ideal $\mathfrak{q}$ of height 1 in R^N, and $\mathfrak{p} = \mathfrak{q} \cap R$, $\mathfrak{p}S_\mathfrak{q} = \mathfrak{q}S_\mathfrak{q}$ and $R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p} \subset S_\mathfrak{q}/\mathfrak{q}S_\mathfrak{q}$ is separable.

Proposition 5.7 (*cf.* [SZ13, Theorem 7.3])**.** *Let (R, ϕ) be a pair and suppose that ϕ is surjective. Then, R is* (WN1)*. In particular, if R is a perfectoid pure complete local Noetherian ring, then it is* (WN1)*.*

Proof. By Proposition 2.32, we only need to show that $R \longrightarrow R^N$ is unramified in codimension 1. Localization commutes with normalization so we may assume without loss of generality that $(R, \mathfrak{m})$ is local of dimension 1 and so R^N is semilocal of dimension 1. Note that R (and therefore R^N) may now have characteristic 0. Let $(S, \mathfrak{n})$ be the localization of R^N at one of its maximal ideals (which has to lie over $\mathfrak{m}$). Since R is perfectoid pure, the conductor is radical in R and R^N by Corollary 2.33,Proposition 5.2, and Corollary 5.4. In particular, $\mathfrak{c} = \mathfrak{m}$ and $\mathfrak{c}S = \mathfrak{m}S$ is also radical and therefore must be equal to $\mathfrak{n}$. It remains to show that $R/\mathfrak{m} \longrightarrow S/\mathfrak{n}$ is separable. If $R/\mathfrak{m}$ has characteristic 0, there is nothing to show so assume it has characteristic $p > 0$ *i.e.* $p \in \mathfrak{m} = \mathfrak{c}$. We have a commutative diagram

$$\begin{tikzcd} R_\infty^{N,A}/\mathfrak{c}_\infty^A \arrow[r] & R^N/\mathfrak{c} \\ R_\infty^A/\mathfrak{c}_\infty^A \arrow[r] \arrow[u] & R/\mathfrak{c} \arrow[u] \end{tikzcd}$$

and since $p \in \mathfrak{c}$, this can be rewritten as

$$\begin{tikzcd} (R^N/\mathfrak{c})_{\mathrm{perf}} \arrow[r] & R^N/\mathfrak{c} \\ (R/\mathfrak{c})_{\mathrm{perf}} \arrow[r] \arrow[u] & R/\mathfrak{c} \arrow[u] \end{tikzcd}$$

Notice that $(R/\mathfrak{c})^{1/p} \subset (R^N/\mathfrak{c})^{1/p}$ so restricting the maps from the bottom left to $(R/\mathfrak{c})^{1/p}$ and localizing $(R^N/\mathfrak{c})^{1/p} \longrightarrow R^N/\mathfrak{c}$ at $\mathfrak{n} \cap R^N$ gives the following diagram

$$\begin{tikzcd} (S/\mathfrak{n})^{1/p} \arrow[r] & S/\mathfrak{n} \\ (R^N/\mathfrak{c})^{1/p} \arrow[r] \arrow[u] & R^N/\mathfrak{c} \arrow[u] \\ (R/\mathfrak{c})^{1/p} \arrow[r] \arrow[u] & R/\mathfrak{c}. \arrow[u] \end{tikzcd}$$

The horizontal maps are surjective hence nonzero which implies that $R/\mathfrak{c} \longrightarrow S/\mathfrak{n}$ is separable by [ST14, Example 5.1]. □

6. Splitting prime

In this section, we give an explicit description of the largest uniformly perfectoid compatible ideal of a ring R and show that it detects the perfectoid purity of R, analogously to the splitting prime of Aberbach and Enescu [AE05]. We also generalize the idea to get a compatible core of an ideal, analogous to the Cartier core of [Bad21, Bro23, CRF24].

6.1. **The positive characteristic case.** We first start by expressing the positive characteristic splitting prime in terms of the perfection of R.

Definition 6.1 ([AE05])**.** Let $(R, \mathfrak{m})$ be a local ring of char $p > 0$. Suppose further that R is F-finite and let $\phi \in \operatorname{Hom}_R\left(R^{1/p^e}, R\right)$. The splitting prime of (R, ϕ) is

$$\beta(R, \phi) = \bigcap_{n>0} \left\{r \in R \mid \phi^n\left(r^{1/p^{en}} R^{1/p^{en}}\right) \subset \mathfrak{m}\right\}$$

where ϕ^n is defined as the composition

$$R^{1/p^{en}} \xrightarrow{\phi^{1/p^{e(n-1)}}} R^{1/p^{e(n-1)}} \xrightarrow{\phi^{1/p^{e(n-2)}}} R^{1/p^{e(n-2)}} \xrightarrow{\phi^{1/p^{e(n-3)}}} \cdots \xrightarrow{\phi} R.$$

In the special case $\phi(1) = 1$, we are able to express $\beta(R, \phi)$ in terms of R_{perf}. Indeed,

$$\beta(R, \phi) = \left\{r \in R \,\middle|\, \bigcup_{n>0} \phi^n\left(r^{1/p^{en}} R^{1/p^{en}}\right) \subset \mathfrak{m}\right\}.$$

Moreover, since

$$(r)_{\text{perf}} = \bigcup_{n>0} r^{1/p^{en}} R^{1/p^{en}},$$

letting ψ be the map from the proof of Lemma 2.18 constructed only from ϕ, we have

$$\beta(R, \phi) = \{r \in R \mid \psi((r)_{\text{perf}}) \subset \mathfrak{m}\}.$$

6.2. **The mixed characteristic case.** Let (R, ϕ) be a pair. The above discussion would lead us to try to define the splitting prime of (R, ϕ) as

$$\beta(R, \phi) \coloneqq \left\{r \in R \mid \phi\left((r)^A_\infty\right) \subset \mathfrak{m}\right\}.$$

Unfortunately, it is not clear to the author whether such an ideal is ϕ-compatible. This brings us to our actual definition.

Definition 6.2. Let (R, ϕ) be a pair. Let $\beta_0(R, \phi) \coloneqq \mathfrak{m}$ and

$$\beta_i(R, \phi) \coloneqq \left\{r \in R \mid \phi\left((r)^A_\infty\right) \subset \beta_{i-1}\right\}$$

for $i > 0$. We then define the splitting prime $\beta(R, \phi)$ as

$$\beta(R, \phi) \coloneqq \bigcap_{i>0} \beta_i(R, \phi).$$

Remark 6.3. When ϕ is surjective, we can show that $\beta_i(R, \phi) \supset \beta_{i+1}(R, \phi)$: let $r \notin \beta_i(R, \phi)$ and $x \in R^A_\infty$ such that $\phi(x) = 1$. Then, $r = \phi(rx) \in \phi\left((r)^A_\infty\right)$ so r is not in $\beta_{i+1}(R, \phi)$, which shows the desired inclusion. Although it is not clear whether this inclusion is strict or not, it explains why we take the intersection over all the β_is. Importantly, this inclusion is not strict in positive characteristic. In particular, this definition agrees with the positive characteristic one when ϕ is surjective.

Proposition 6.4. *With notation as in Definition 6.2, $\beta(R, \phi) \neq R$ if and only if ϕ is surjective and, in that case, $\beta(R, \phi)$ is the largest ϕ-compatible ideal of R.*

Proof. We first show that $\beta(R,\phi)$ is ϕ-compatible:

$$\begin{aligned}\phi\left((\beta(R,\phi))^A_\infty\right) &= \phi\left((\cap_{i>0}\beta_i(R,\phi))^A_\infty\right)\\ &\subset \phi\left(\cap_{i>0}\left(\beta_i(R,\phi)\right)^A_\infty\right)\\ &\subset \cap_{i>0}\phi\left(\left(\beta_i(R,\phi)\right)^A_\infty\right)\\ &\subset \cap_{i\geq 0}\beta_i(R,\phi)\\ &\subset \beta(R,\phi)\end{aligned}$$

where the first containment follows from Proposition 2.6. Let $\mathfrak{a} \subsetneq R$ be a ϕ-compatible ideal and suppose that $\mathfrak{a} \subset \beta_i(R,\phi)$. Then,

$$\phi\left(\mathfrak{a}^A_\infty\right) \subset \mathfrak{a} \subset \beta_i(R,\phi)$$

so $\mathfrak{a} \subset \beta_{i+1}$. Since $\mathfrak{a} \subset \mathfrak{m} = \beta_0(R,\phi)$, we must have

$$\mathfrak{a} \subset \bigcap_{i>0} \beta_i(R,\phi) = \beta(R,\phi)$$

so $\beta(R,\phi)$ is indeed the largest ϕ-compatible ideal. It remains to show that $\beta(R,\phi) \neq R$ if and only if ϕ is surjective. Notice that $\beta(R,\phi) \neq R$ if and only if $\beta_1(R,\phi) \neq R$. The backwards direction is clear so suppose that $\beta_1(R,\phi) = R$. Then, we must have $\beta_2(R,\phi) = R$ hence $\beta_3(R,\phi) = R$ and so on. Now, ϕ is surjective if and only if $\phi\left((1)^A_\infty\right) = R$ if and only if $1 \notin \beta_1(R,\phi)$ if and only if $\beta(R,\phi) \neq R$. □

This construction indeed defines a prime.

Corollary 6.5. *With notation as in Definition 6.2, if ϕ is surjective, then $\beta(R,\phi)$ is a prime ideal. In particular, it is the largest perfectoid pure center of (R,ϕ) and $R/\beta(R,\phi)$ is a normal domain.*

Proof. By Proposition 2.21, the pair $(R/\beta(R,\phi),\bar{\phi})$ has no $\bar{\phi}$-compatible ideals. By Proposition 5.2 $R/\beta(R,\phi)$ must be a normal local ring so it is a domain. □

Definition 6.6. Let R and $A \in \mathscr{A}$ be as in Notation 2.10. We define the splitting prime of R as

$$\beta(R) = \bigcap_{\phi \in \operatorname{Hom}_R(R^A_\infty, R)} \beta(R,\phi)$$

and write β when there is no confusion on the ring.

Proposition 6.7. *With notation as in Definition 6.6, $\beta(R) \neq R$ if and only if R is perfectoid pure, in which case it is the largest uniformly perfectoid compatible ideal of R. Moreover, R/β has no uniformly perfectoid compatible ideal and is a normal domain.*

Proof. It all readily follows from Proposition 6.4, Corollary 6.5, and Corollary 2.30. □

6.3. **Compatible cores and a test ideal.** We can can generalize the idea of splitting prime to find the largest compatible ideal contained inside a given ideal, similar to the Cartier core construction of [Bad21, Bro23, CRF24].

Definition 6.8. Let (R, ϕ) be a pair and $\mathfrak{a} \subset R$ be any ideal. We define $\upbeta_{\mathfrak{a}}(R, \phi)$, the compatible core of $\mathfrak{a}$ as

$$\upbeta_{\mathfrak{a}}(R, \phi) = \bigcap_{i>0} \upbeta_{\mathfrak{a},i}(R, \phi)$$

where $\upbeta_{\mathfrak{a},0}(R, \phi) = \mathfrak{a}$ and

$$\upbeta_{\mathfrak{a},i}(R, \phi) = \left\{ r \in R \mid \phi\left((r)^A_\infty\right) \subset \upbeta_{\mathfrak{a},i-1}(R, \phi) \right\}$$

for $i > 0$.

Proposition 6.9 (*cf.* [Bro23, Corollary 3.14], [CRF24, Proposition 4.5])**.** *Let (R, ϕ) be a pair and $\mathfrak{a} \subset R$ be a radical ideal. If for all minimal primes $\mathfrak{p}$ of $\mathfrak{a}$, $\mathrm{Im}(\phi) \not\subset \mathfrak{p}$, then $\upbeta_{\mathfrak{a}}(R, \phi)$ is the largest ϕ-compatible ideal contained in $\mathfrak{a}$.*

Proof. The proof is the same as Proposition 6.4. □

Proposition 6.10 (*cf.* [Bro23, Theorem 3.3], [CRF24, Proposition 4.14])**.** *Let (R, ϕ) be a pair. If $\mathscr{U} \coloneqq \operatorname{Spec} R \setminus V(\operatorname{Im} \phi)$ and $\mathfrak{p} \in \mathscr{U}$. Then, $\upbeta_{\mathfrak{p}}(R, \phi)$ is a prime ideal and the map $\upbeta_{(R,\phi)} \colon \mathscr{U} \longrightarrow \mathscr{U}$ given by*

$$\upbeta_{(R,\phi)} \colon \mathfrak{p} \mapsto \upbeta_{\mathfrak{p}}(R, \phi)$$

is continuous in the Zariski topology.

Proof. Since $\mathfrak{p} \in \mathscr{U}$, $\upbeta_{\mathfrak{p}}(R, \phi)$ is radical. Indeed, $\sqrt{\upbeta_{\mathfrak{p}}(R, \phi)} \subset \mathfrak{p}$ and is ϕ-compatible by Proposition 2.34 and Proposition 2.37. Now, all the the minimal primes of $\upbeta_{\mathfrak{p}}(R, \phi)$ are also compatible and at least one of them must be contained in $\mathfrak{p}$. By Proposition 6.9, this implies that $\upbeta_{\mathfrak{p}}(R, \phi)$ must be prime. To show that it is continuous in the Zariski topology, we follow the proof of [Bro23, Theorem 3.23]. Let $\mathfrak{a} \subset R$ be an ideal. We show that the inverse image of $V \coloneqq V(\mathfrak{a}) \cap \mathscr{U}$ under $\upbeta_{(R,\phi)}$ is also closed. In fact, we claim

$$\upbeta^{-1}_{(R,\phi)}(V) = V(\mathfrak{b}) \cap \mathscr{U}$$

where

$$\mathfrak{b} \coloneqq \bigcap_{\mathfrak{p} \in \mathscr{U}, \upbeta_{\mathfrak{p}}(R,\phi) \in V} \upbeta_{(R,\phi)}(\mathfrak{p}).$$

Indeed, if $\mathfrak{p} \in \upbeta^{-1}_{(R,\phi)}(V)$, $\upbeta_{\mathfrak{p}}(R, \phi) \subset \mathfrak{p}$ and since $\mathfrak{b} \subset \upbeta_{\mathfrak{p}}(R, \phi)$, $\mathfrak{b} \subset \mathfrak{p}$. On the other hand, since $\mathfrak{a} \subset \mathfrak{b}$, if $\mathfrak{p} \in V(\mathfrak{b}) \cap \mathscr{U}$ then $\mathfrak{a} \subset \mathfrak{b} \subset \upbeta_{\mathfrak{p}}(R, \phi) \subset \mathfrak{p}$ so $\mathfrak{p} \in V(\mathfrak{a}) \cap \mathscr{U}$. □

Not only is this map continuous but we can actually describe the fibers explicitly when ϕ is surjective. However, we first need to define one more object. Let (R, ϕ) be a pair with ϕ surjective. By Corollary 2.39, there are only finitely many ϕ-compatible ideals. In particular, if $\mathfrak{p}$ is a compatible prime, there are only finitely many ϕ-compatible ideals not contained in that prime. The intersection of all these is therefore a nonzero ideal which is the smallest ϕ-compatible ideal not contained in $\mathfrak{p}$. This leads us to our next definition.

Definition 6.11 (*cf.* [Tak10, Smo20])**.** Let (R, ϕ) be a pair with ϕ surjective. Let $\mathfrak{p}$ be a compatible prime of (R, ϕ). We define $\uptau_{\mathfrak{p}}(R, \phi)$ to be the smallest ϕ-compatible ideal not contained in $\mathfrak{p}$ and call it the test ideal along $\mathfrak{p}$. If $\mathfrak{p} = 0$, we write $\uptau(R, \phi)$ and call it the test ideal of (R, ϕ). If R is not a domain, then we let $\uptau(R, \phi)$ be the smallest ϕ-compatible ideal not contained in any of the minimal primes of R.

Proposition 6.12 (*cf.* [CRF24, Proposition 4.20])**.** *Let (R, ϕ) be a pair with ϕ surjective and let $\mathfrak{p} \in \operatorname{Spec} R$ be a ϕ-compatible prime. Then, $\upbeta^{-1}_{(R,\phi)}(\mathfrak{p}) = V(\uptau_{\mathfrak{p}}(R, \phi))^{\mathsf{C}} \cap V(\mathfrak{p})$.*

Proof. Let $\mathfrak{q} \in \beta^{-1}_{(R,\phi)}(\mathfrak{p})$. If $\tau_{\mathfrak{p}}(R,\phi) \subset \mathfrak{q}$, then by Proposition 6.9,

$$\tau_{\mathfrak{p}}(R,\phi) \subset \beta_{\mathfrak{q}}(R,\phi) = \beta_{\mathfrak{p}}(R,\phi),$$

a contradiction. On the other hand, if $\mathfrak{q} \in V(\tau_{\mathfrak{p}}(R,\phi))^{\mathsf{C}} \cap V(\mathfrak{p})$ then $\mathfrak{p} \subset \beta_{\mathfrak{q}}(R,\phi)$. If this were a strict inequality, we would have $\beta_{\mathfrak{q}}(R,\phi) \supset \tau_{\mathfrak{p}}(R,\phi)$, a contradiction. $\square$

As usual, this can also be done for uniformly perfectoid compatible ideals.

Definition 6.13. Let R be as in Notation 2.10 and fix $A \in \mathscr{A}$. Let $\mathfrak{a} \subset R$ be any ideal. We define $\beta_{\mathfrak{a}}(R)$, the compatible core of $\mathfrak{a}$ as

$$\beta_{\mathfrak{a}}(R) = \bigcap_{\phi \in \operatorname{Hom}_R(R^A_\infty, R)} \beta_{\mathfrak{a}}(R,\phi)$$

Proposition 6.14 (*cf.* [Bro23, Corollary 3.14])**.** *Let R be as in Notation 2.10 and fix $A \in \mathscr{A}$. Let $\mathfrak{a} \subset R$ be a radical ideal. If for all minimal primes $\mathfrak{p}$ of $\mathfrak{a}$ and $\phi \in \operatorname{Hom}_R\left(R^A_\infty, R\right)$, $\operatorname{Im}(\phi) \not\subset \mathfrak{p}$, then $\beta_{\mathfrak{a}}(R)$ is the largest ϕ-compatible ideal contained in $\mathfrak{a}$.*

Proof. This follows from Proposition 6.9. $\square$

Proposition 6.15 (*cf.* [Bro23, Theorem 3.3])**.** *Let R be as in Notation 2.10 and fix $A \in \mathscr{A}$. Let*

$$\mathscr{U} := \bigcup_{\phi \in \operatorname{Hom}_R(R^A_\infty, R)} \operatorname{Spec} R \setminus V(\operatorname{Im}\phi)$$

and $\mathfrak{p} \in \mathscr{U}$. Then, $\beta_{\mathfrak{p}}(R)$ is a prime ideal and the map $\beta_R \colon \mathscr{U} \longrightarrow \mathscr{U}$ given by

$$\beta_R \colon \mathfrak{p} \mapsto \beta_{\mathfrak{p}}(R)$$

is continuous in the Zariski topology.

Proof. Same as Proposition 6.10 $\square$

Definition 6.16 (*cf.* [Tak10, Smo20])**.** Let R be as in Notation 2.10 and assume further that it is perfectoid pure. Let $\mathfrak{p}$ be a uniformly perfectoid compatible prime of R. We define $\tau_{\mathfrak{p}}(R)$ to be the smallest uniformly perfectoid compatible ideal not contained in $\mathfrak{p}$ and call it the test ideal along $\mathfrak{p}$. If $\mathfrak{p} = 0$, we write $\tau(R)$ and call it the test ideal of R. If R is not a domain, then we let $\tau(R)$ be the smallest uniformly perfectoid compatible ideal not contained in any of the minimal primes of R.

Proposition 6.17 (*cf.* [CRF24, Proposition 4.20])**.** *Let R be as in Notation 2.10 and assume further that it is perfectoid pure. Let $\mathfrak{p}$ be a uniformly perfectoid compatible prime of R. Then, $\beta_R^{-1}(\mathfrak{p}) = V(\tau_{\mathfrak{p}}(R))^{\mathsf{C}} \cap V(\mathfrak{p})$.*

Proof. Same as Proposition 6.12. $\square$

Lemma 6.18. *Let $A \subset R$ be a Noether normalization and let $h, g \in A$ be arbitrary. Let $\lambda_{g,A}(R) := \sum_{\phi \in \operatorname{Hom}_R(R^A_\infty, R)} \phi\left((g)^A_\infty\right)$. Let $B := A[\![y]\!]$ and make R and A into B-algebras by sending y to h. Then, $\lambda_{g,A}(R) = \lambda_{g,B}(R) := \sum_{\phi \in \operatorname{Hom}_R(R^B_\infty, R)} \phi\left((g)^A_\infty\right)$*

Proof. By Lemma 2.28, we have a surjection $\operatorname{Hom}_R(R^B_\infty, R) \longrightarrow \operatorname{Hom}_R(R^A_\infty, R)$ and the inclusion $\subset$ follows. For the other inclusion, fix $\psi \in \operatorname{Hom}_R(R^B_\infty, R)$, $x \in (g)^A_\infty$ and $z \in R^B_\infty$. Let $\phi \colon R^A_\infty \longrightarrow R$ be the composition of the maps $\times z \colon R^A_\infty \longrightarrow R^B_\infty$ and ψ. Then, $\psi(xz) = \phi(x) \in \lambda_{g,A}$. This implies that $\psi((g)^A_\infty R^B_\infty) \subset \lambda_{g,A}$. By Lemma 2.22 and Proposition 2.9 we get $\psi((g)^B_\infty) \subset \lambda_{g,A}$ as desired. $\square$

Remark 6.19. The hope would be that $\tau(R)$ is equal to other mixed characteristic test ideals (see [MS18, HLS24, BMP+23a, BMP+23b, Mur23, Rob22, ST21, PRG21]). In this generality, this is far beyond the scope of this paper. However, if R is normal $\mathbb{Q}$-Gorenstein and $A \subset R$ is a Noether normalization, by [CLM+22, Lemma 5.1.6], we are are able to describe the BCM-test ideal $\tau_{R^A_\infty}(R)$ as

$$\tau_{R^A_\infty}(R) = \sum_{\phi \in \mathrm{Hom}_R(R^A_\infty, R)} \phi\left((g)^A_\infty\right)$$

where $g \in A$ is such that $A[g^{-1}] \longrightarrow R[g^{-1}]$ is étale. It is not clear to the author if such an ideal is compatible. Interestingly, by Lemma 6.18, for a fixed $g \in R$, the ideal $\sum_{\phi \in \mathrm{Hom}_R(R^A_\infty, R)} \phi\left((g)^A_\infty\right)$ does not depend on the choice of $A \in \mathscr{A}$ which is hinting at it being uniformly perfectoid compatible. Let $\tau_1(R, A) \coloneqq \tau_{R^A_\infty}(R)$,

$$\tau_i(R, A) \coloneqq \sum_{\phi \in \mathrm{Hom}_R(R^A_\infty, R)} \phi\left(\tau_{i-1}(R, A)^A_\infty\right)$$

for $i > 1$ and

$$J \coloneqq \sum_{i,A} \tau_i(R, A)$$

then J is a nonzero uniformly perfectoid compatible ideal. It is equal to $\tau(R)$ if and only if there is $x \in \tau(R)$ with $A[x^{-1}] \longrightarrow R[x^{-1}]$ étale. Since for any $x \in \tau(R)$, xg has that property, we see that $J = \tau(R)$. In particular, $\tau_{R^A_\infty}(R) \subset \tau(R)$ and a normal $\mathbb{Q}$-Gorenstein BCM-regular ring has no uniformly perfectoid compatible ideals.

7. Behavior under étale morphisms

In this section, we show that compatible ideals behave well under étale morphisms. This relies heavily on the almost purity theorem of Bhatt–Scholze, which we now state. Other versions of the almost purity theorem can be found in [Fal02, Sch12, KL15, And18b].

Theorem 7.1. [BS22, Theorem 10.9] *Let R be a perfectoid ring and $\mathfrak{a}$ a finitely generated ideal of R. Let S be a finitely presented finite R-algebra such that* $\operatorname{Spec} S \longrightarrow \operatorname{Spec} R$ *is finite étale outside $V(\mathfrak{a})$. Then, S_{perfd} is discrete and a perfectoid ring and the map $S \longrightarrow S_{\mathrm{perfd}}$ is an isomorphism away from $V(\mathfrak{a})$. In particular, a finite étale cover of a perfectoid ring is perfectoid.*

Now, let (R, ϕ) be as in Notation 2.10 and $(S, \mathfrak{n})$ be an R-algebra such that $R \longrightarrow S$ is finite étale. By stability under base change, $R^A_\infty \otimes_R S$ is finite étale over $R^A_\infty \otimes_R R \cong R^A_\infty$, in particular it is perfectoid. In fact, it is isomorphic to S^A_∞ by the universal properties of perfectoidization and tensor product. This gives us the following setting.

Setting 7.2. Let $(R, \mathfrak{m})$ be a complete local Noetherian ring. Let $(S, \mathfrak{n})$ be an R-algebra such that $R \longrightarrow S$ is finite étale. By [BMP+24, Lemma 4.6, Lemma 4.15], R is perfectoid pure if and only if S is. Take any unramified regular local ring A such that R (and therefore S) is a module finite A-algebra. Fix $\phi \in \mathrm{Hom}_R(R^A_\infty, R)$. Let $\psi \coloneqq \phi \otimes_R S \colon S^A_\infty \longrightarrow S$ so ψ is an extension of ϕ to S^A_∞. We have a commutative diagram

$$\begin{tikzcd} S^A_\infty \arrow[r, "\psi"] & S \\ R^A_\infty \arrow[u] \arrow[r, "\phi"] & R \arrow[u] \end{tikzcd} \tag{7.2.1}$$

where the vertical arrows are inclusions. The data $(R,\phi) \longrightarrow (S,\psi)$ as above is what we now call an étale morphism of pairs.

Lemma 7.3. *Let $R \longrightarrow S$ be as in Setting 7.2 and $\mathfrak{a} \subset R$ be an ideal. If $\mathfrak{a}^{A,R}_\infty$ is the perfectoidization of $\mathfrak{a}$ in R^A_∞, then $\mathfrak{a}^{A,R}_\infty \otimes_R S =: \mathfrak{a}^{A,S}_\infty$ is the perfectoidization of $\mathfrak{a}$ in S^A_∞.*

Proof. Note that $R^A_\infty/\mathfrak{a}^{A,R}_\infty \otimes_R S$ is finite étale over $R^A_\infty/\mathfrak{a}^{A,R}_\infty \otimes_R R = R/\mathfrak{a}^{A,R}_\infty$ so is itself perfectoid. In particular, $\mathfrak{a}^{A,R}_\infty \otimes_R S$ is a perfectoid ideal of S^A_∞. □

Proposition 7.4. *Let $(R,\phi) \longrightarrow (S,\psi)$ be as in Setting 7.2 and assume further that ϕ is surjective. Then, $\beta(R,\phi) = \beta(S,\psi) \cap R$ and $\beta(S,\psi) = \beta(R,\phi)S$. More generally, if $\mathfrak{a}$ is any radical ideal of S, then $\beta_{\mathfrak{a}\cap R}(R,\phi) = \beta_{\mathfrak{a}}(S,\psi) \cap R$. Moreover, for all radical ideals $\mathfrak{b} \subset R$, $\beta_{\mathfrak{b}}(R,\phi)S = \sqrt{\beta_{\mathfrak{b}}(R,\phi)S} = \beta_{\sqrt{\mathfrak{b}S}}(S,\psi) = \beta_{\mathfrak{b}S}(S,\psi)$.*

Proof. The inclusion $\beta_{\mathfrak{a}\cap R}(R,\phi) \supset \beta_{\mathfrak{a}}(S,\psi) \cap R$ follows directly from (7.2.1) since the contraction of a ψ-compatible ideal must be ϕ-compatible. For "$\subset$", we proceed by induction. Let $r \in R$. By Lemma 7.3, $R^A_\infty/(r)^{A,R}_\infty \otimes_R S$ is the perfectoidization of (r) in S^A_∞. If $r \in \beta_{(\mathfrak{a}\cap R),1}(R,\phi)$ for an ideal $\mathfrak{a}$ of S. Then,

$$\psi\left((r)^{A,R}_\infty \otimes_R S\right) = \phi\left((r)^{A,R}_\infty\right) \otimes_R S \subset (\mathfrak{a} \cap R) \otimes_R S \subset \mathfrak{a}.$$

This implies that $r \in \beta_{\mathfrak{a},1}(S,\psi)$ hence $\beta_{(\mathfrak{a}\cap R),1}(R,\phi) \subset \beta_{\mathfrak{a},1}(S,\psi)$. Suppose that we know that $\beta_{(\mathfrak{a}\cap R),i}(R,\phi) \subset \beta_{\mathfrak{a},i}(S,\psi)$ and let $r \in \beta_{(\mathfrak{a}\cap R),i+1}(R,\phi)$. Then,

$$\psi\left((r)^{A,R}_\infty \otimes_R S\right) = \phi\left((r)^{A,R}_\infty\right) \otimes_R S \subset \beta_{(\mathfrak{a}\cap R),i}(R,\phi) \otimes_R S \subset \beta_{\mathfrak{a},i}(S,\psi).$$

This implies that $r \in \beta_{\mathfrak{a},i+1}(S,\psi)$ hence $\beta_{(\mathfrak{a}\cap R),i+1}(R,\phi) \subset \beta_{\mathfrak{a},i+1}(S,\psi)$. The implication to the last statement follows as in [CRF24, Definition-Proposition 5.5] but we write it here for the sake of completeness. The first and third equalities follow from the fact that an extension of a radical ideal under an étale morphism is radical. We therefore only need to show the middle equality. Let $\mathfrak{b} \subset R$ be a radical ideal. We have

$$\beta_{\sqrt{\mathfrak{b}S}}(R,\phi) = \bigcap_{\mathfrak{b}\subset\mathfrak{q}\cap R} \beta_{\mathfrak{q}}(S,\psi)$$

and

$$\sqrt{\beta_{\mathfrak{b}}(R,\phi)S} = \bigcap_{\beta_{\mathfrak{b}(R,\phi)\subset\mathfrak{q}\cap R}} \mathfrak{q}$$

so to show that $\sqrt{\beta_{\mathfrak{b}}(R,\phi)S} \subset \beta_{\sqrt{\mathfrak{b}S}}(R,\phi)$, it suffices to show that $\mathfrak{b} \subset \mathfrak{q} \cap R$ for some $\mathfrak{q} \in \operatorname{Spec} S$ implies $\beta_{\mathfrak{b}}(R,\phi) \subset \beta_{\mathfrak{q}}(S,\psi) \cap R$. But $\mathfrak{b} \subset \mathfrak{q} \cap R$ implies

$$\beta_{\mathfrak{b}}(R,\phi) \subset \beta_{\mathfrak{q}\cap R}(R,\phi) = \beta_{\mathfrak{q}} \cap R.$$

which is what we wanted. For the other containment, it suffices to show this in case $\mathfrak{b} = \mathfrak{p}$ is a prime since the β construction commutes with taking intersections and we are working in a flat extension. Then, let $\mathfrak{q} \in \operatorname{Spec} S$ such that $\beta_{\mathfrak{p}}(R,\phi) \subset \mathfrak{q} \cap R$. We want to show that $\beta_{\sqrt{\mathfrak{p}S}}(S,\psi) \subset \mathfrak{q}$. By going-down, there is $\mathfrak{q}' \subset \mathfrak{q}$ lying over $\beta_{\mathfrak{p}}$ and by going up there is $\mathfrak{q}'' \supset \mathfrak{q}'$ lying over $\mathfrak{p}$ so that $\sqrt{\mathfrak{p}S} \subset \mathfrak{q}''$. We have

$$\beta_{\mathfrak{q}''}(S,\psi) \cap R = \beta_{\mathfrak{q}'' \cap R}(R,\phi) = \beta_{\mathfrak{p}}(R,\phi) = \beta_{\beta_{\mathfrak{p}}(R,\phi)}(R,\phi) = \beta_{\mathfrak{q}' \cap R}(R,\phi) = \beta_{\mathfrak{q}'}(S,\psi) \cap R.$$

Since $\mathfrak{q}' \subset \mathfrak{q}''$, $\beta_{\mathfrak{q}'}(S,\psi) \subset \beta_{\mathfrak{q}''}(S,\psi)$ so by incomparability these must be equal. Using now that $\sqrt{\mathfrak{p}S} \subset \mathfrak{q}''$, we get $\beta_{\sqrt{\mathfrak{p}S}}(S,\psi) \subset \mathfrak{q}' \subset \mathfrak{q}$, which is what we wanted. □

Corollary 7.5. *Let $(R,\phi) \longrightarrow (S,\psi)$ be as in Setting 7.2 and assume further that they are domains. Let $\tau(R,\phi), \tau(S,\psi)$ be the respective nonzero smallest compatible ideals. If ϕ is surjective, then $\tau(R,\phi) = \tau(S,\psi) \cap R$ and $\tau(R,\phi)S = \tau(S,\psi)$.*

Proof. $\tau(S,\psi) \cap R$ is compatible so it contains $\tau(R,\phi)$. Extending to (S,ψ) gives

$$\tau(R,\phi)S \subset (\tau(S,\psi) \cap R)S \subset \tau(S,\psi)$$

but since these are all compatible ideals, we must have

$$\tau(R,\phi)S \supset \tau(S,\psi)$$

so this is an equality.

To show that $\tau(R,\phi) = \tau(S,\psi) \cap R$, note that $\beta_R^{-1}(0_R) \subset \beta_S^{-1}(0_S) \cap R$. Indeed, let $\mathfrak{q} \in \operatorname{Spec} S$ be such that $\mathfrak{q} \cap R = \mathfrak{p}$ with $\beta_{\mathfrak{p}}(R,\phi) = 0$. By Proposition 7.4, $\beta_{\mathfrak{q}}(S,\psi) \cap R = \beta_{\mathfrak{q} \cap R}(R,\phi) = 0$ so $\beta_{\mathfrak{q}}(S,\psi) = 0$. Then, by Proposition 6.12, $V(\tau(R,\phi)) \subset V(\tau(S,\psi)) \cap R$ which implies $\tau(S,\psi) \cap R \subset \tau(R,\phi)$. Since the other inclusion is automatic, $\tau(S,\psi) \cap R = \tau(R,\phi)$. □

As usual, these also hold for uniformly perfectoid compatible ideals.

Proposition 7.6. *Let $R \longrightarrow S$ be as in Setting 7.2 and assume R (equivalently S) is perfectoid pure. Let $\mathfrak{b} \subset S$ be a radical ideal. Then, $\beta_{\mathfrak{b}}(S) \cap R = \beta_{\mathfrak{b} \cap R}(R)$. Moreover, if $\mathfrak{a} \subset R$ is a radical ideal then, $\beta_{\mathfrak{a}}(R)S = \beta_{\mathfrak{a}S}(S)$.*

Proof. Take any unramified regular local ring A such that R (and therefore S) is a module finite A-algebra. Let $\mathfrak{b} \subset S$ be a radical ideal and let $\mathfrak{a} := \mathfrak{b} \cap R$. Then, $\beta_{\mathfrak{b}}(S) \cap R$ is a uniformly perfectoid compatible ideal of R: if $\phi \in \operatorname{Hom}_R(R^A_\infty, R)$ then $\phi \otimes_R S \in \operatorname{Hom}_S(S^A_\infty, S)$ and

$$\phi\left((\beta_{\mathfrak{b}}(S) \cap R)^A_\infty\right) \subset \left(\phi\left((\beta_{\mathfrak{b}}(S) \cap R)^A_\infty\right) \otimes_R S\right) \cap R \subset \beta_{\mathfrak{b}}(S) \cap R$$

This shows $\beta_{\mathfrak{b}}(S) \cap R \subset \beta_{\mathfrak{a}}(R)$. For the other inclusion, we proceed by induction. Let $r \in \beta_{\mathfrak{a},1}(R)$. Then, for any $x \in (r)^{A,R}_\infty$ the composition

$$\operatorname{Hom}_R\left(R^A_\infty, R\right) \xrightarrow{\operatorname{Hom}_R(\times x, R)} \operatorname{Hom}_R\left(R^A_\infty, R\right) \longrightarrow \operatorname{Hom}_R(R,R) \cong R \longrightarrow R/\mathfrak{a}$$

is zero. Tensoring with S and using the fact that $\mathfrak{a}S \subset \mathfrak{b}$, we get that the composition

$$\operatorname{Hom}_S\left(S^A_\infty, S\right) \xrightarrow{\operatorname{Hom}_S(\times x, S)} \operatorname{Hom}_S\left(S^A_\infty, S\right) \longrightarrow \operatorname{Hom}_S(S,S) \cong S \longrightarrow S/\mathfrak{b}$$

is zero. Precomposing with multiplication by any element of S^A_∞ on S^A_∞ would keep the composition 0 so for any $\psi \in \operatorname{Hom}_S(S^A_\infty, S)$, $\psi((rS)^A_\infty) \subset \mathfrak{b}$ hence $r \in \beta_{\mathfrak{b},1}(S) \cap R$. Now, suppose that we know that $\beta_{\mathfrak{a},i}(R) \subset \beta_{\mathfrak{b},i}(S) \cap R$ and let $r \in \beta_{\mathfrak{a},i+1}(R)$ and $x \in (r)^{A,R}_\infty$. The composition

$$\operatorname{Hom}_R\left(R^A_\infty, R\right) \xrightarrow{\operatorname{Hom}_R(\times x, R)} \operatorname{Hom}_R\left(R^A_\infty, R\right) \longrightarrow \operatorname{Hom}_R(R,R) \cong R \longrightarrow R/\beta_{\mathfrak{a},i}(R)$$

is zero so tensoring with S and using the fact that $\beta_{\mathfrak{a},i}(R)S \subset \beta_{\mathfrak{b},i}(S)$, we get that the composition

$$\operatorname{Hom}_S\left(S_\infty^A, S\right) \xrightarrow{\operatorname{Hom}_S(\times x, S)} \operatorname{Hom}_S\left(S_\infty^A, S\right) \longrightarrow \operatorname{Hom}_S(S,S) \cong S \longrightarrow S/\beta_{\mathfrak{b},i}(S)$$

is zero. Precomposing with multiplication by any element of S_∞^A on S_∞^A would keep the composition 0 so for any $\psi \in \operatorname{Hom}_S(S_\infty^A, S)$, $\psi((rS)_\infty^A) \subset \beta_{\mathfrak{b},i}(S)$ hence $r \in \beta_{\mathfrak{b},i+1}(S) \cap R$. This shows $\beta_{\mathfrak{a}}(R) = \beta_{\mathfrak{b}}(S) \cap R$. The implication about β under extensions follows as in Proposition 7.4. $\square$

Corollary 7.7. *Let $R \longrightarrow S$ be as in Setting 7.2 and assume further that they are domains. Let $\tau(R), \tau(S)$ be the respective nonzero smallest compatible ideals. If R (equivalently S) is perfectoid pure, then $\tau(R) = \tau(S) \cap R$ and $\tau(R)S = \tau(S)$.*

Proof. Same as Corollary 7.5. $\square$

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