Tame ramification and centers of F-purity

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Abstract: I introduce a notion of tame ramification for general finite covers. When specialized to the separable case, it extends to higher dimensions the classical notion of tame ramification for Dedekind domains and curves. However, when applied to the Frobenius map, it naturally yields the notion of centers of F-purity, which lets us describe how these behave under finite covers. This is joint work with Javier Carvajal-Rojas

Notation and assumptions:

1. $\theta\colon R \to S$ is a finite ring extension and $\mathfrak{T}$ is in $\mathrm{Hom}_R(S, R)$.
2. If R is a ring of characteristic $p > 0$ then $F\colon R \to R$ is the Frobenius morphism, F^e its e^{th} iteration.
3. $F^e_* R$ is an R-module which is R as a set but R acts on it via F^e as follows: $rF^e_* r' = F^e_* r^{p^e} r'$.
4. All rings are Noetherian and those of char. $p > 0$ are *F-finite* i.e. $F^e_* R$ is a finite R module for all e.

Definition: Let $\mathfrak{a} \subset R$, then

$$\mathfrak{a}^{\mathfrak{T}} := \{s \in S \mid \mathfrak{T}(sS) \subset \mathfrak{a}\} \subset S$$

is the largest ideal of S whose image under $\mathfrak{T}$ is in $\mathfrak{a}$.

Proposition:

1. $\mathfrak{a}^{\mathfrak{T}}$ is a proper ideal if and only if $\mathfrak{T}(S) \not\subseteq \mathfrak{a}$.
2. If $\mathfrak{p} \in \operatorname{Spec} R$ and $\mathfrak{T}(S) \not\subseteq \mathfrak{p}$ then $\mathfrak{p}^{\mathfrak{T}} \cap R = \mathfrak{p}$. Moreover if $\mathfrak{q} \in \operatorname{Ass}(S/\mathfrak{p}^{\mathfrak{T}})$ then $\mathfrak{q} \cap R = \mathfrak{p}$.

Definition: S/R is *tamely $\mathfrak{T}$-ramified* over $\mathfrak{p} \in \operatorname{Spec} R$ if
$$\mathfrak{p}^{\mathfrak{T}} = \sqrt{\mathfrak{p}S}.$$

Theorem: R and S Dedekind domains, $\mathfrak{T} = \mathrm{Tr}$ the trace.

$$\left\{\begin{matrix} S/R \text{ is tamely} \\ \mathfrak{T}\text{-ramified over } \mathfrak{p} \end{matrix}\right\} \iff \left\{\begin{matrix} S/R \text{ is tamely} \\ \text{ramified over } \mathfrak{p} \end{matrix}\right\}$$

Definition: R of characteristic $p > 0$, $\mathfrak{a} \subset R$ an ideal, and $\phi \in \mathrm{Hom}(F^e_* R, R)$ surjective. We say $\mathfrak{a}$ is *ϕ-compatible* if
$$\phi(F^e_*\mathfrak{a}) \subset \mathfrak{a}.$$

Facts:

1. (R, ϕ) is F-regular $\iff$ (0) and (1) are the only ϕ-compatible ideals.
2. The set of ϕ-compatible ideals tells us where the singularities of (R, ϕ) are the most severe.
3. These ideals are radical and closed under taking minimal primes $\implies$ can focus on the prime ones, the *centers of F-purity* (CFPs).
4. $\mathfrak{a} \subset R$ is ϕ-compatible $\iff$ $\mathfrak{a}^{\phi} = \mathfrak{a}$ in $F^e_* R$.

Definition [3]: For $\mathfrak{p} \in \operatorname{Spec} R$ a CFP, $\tau_{\mathfrak{p}}(R, \phi)$ is the smallest ϕ-compatible ideal $\mathfrak{a}$ of R such that $\mathfrak{a} \not\subseteq \mathfrak{p}$.

Definition [2]: A *transposition of pairs* $(\theta, \mathfrak{T}) : (R, \phi) \to (S, \psi)$ is a commutative diagram

$$\begin{tikzcd} F^e_* S \arrow[r, "\psi"] \arrow[d, "F^e_*\mathfrak{T}"'] & S \arrow[d, "\mathfrak{T}"] \\ F^e_* R \arrow[r, "\phi"] & R \end{tikzcd}$$

with (R, ϕ) as above, char $S = p$, and ψ surjective.

Remark: For $\mathfrak{p} \in \operatorname{Spec} R$, following the previous diagram from the bottom right gives

$$\begin{tikzcd} (F^e_*\mathfrak{p}^{\phi})^{F^e_*\mathfrak{T}} = F^e_*(\mathfrak{p}^{\mathfrak{T}})^{\psi} \arrow[r, "\psi"] \arrow[d, "F^e_*\mathfrak{T}"'] & \mathfrak{p}^{\mathfrak{T}} \arrow[d, "\mathfrak{T}"] \\ F^e_*\mathfrak{p}^{\phi} \arrow[r, "\phi"] & \mathfrak{p} \end{tikzcd}$$

when $\mathfrak{p} = \mathfrak{p}^{\phi}$, $F^e_*\mathfrak{p}^{F^e_*\mathfrak{T}} = F^e_*(\mathfrak{p}^{\mathfrak{T}})^{\psi}$ so $\mathfrak{p}^{\mathfrak{T}} = (\mathfrak{p}^{\mathfrak{T}})^{\psi} \implies \mathfrak{p}^{\mathfrak{T}}$ is ψ-compatible.

Theorem: $(\theta, \mathfrak{T}) : (R, \phi) \to (S, \psi)$ a transposition of pairs, $\mathfrak{p}$ a CFP for (R, ϕ), TFAE:

1. All $\mathfrak{q} \in \operatorname{Spec} S$ lying above $\mathfrak{p}$ are CFPs for (S, ψ) and $\mathfrak{T}(\tau_{\mathfrak{q}}(S, \psi)) = \tau_{\mathfrak{p}}(R, \phi)$ for all such $\mathfrak{q}$.
2. S/R is tamely $\mathfrak{T}$-ramified over $\mathfrak{p}$

Remark: The theorem generalizes to the case ψ is not surjective by updating the definition of CFPs to $\mathfrak{q} \in \operatorname{Spec} S$ is a CFP if $\psi(F^e_*\mathfrak{q}) \subset \mathfrak{q}$ and $\psi_{\mathfrak{q}}$ is surjective.

References:

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