E_E 505 Nonlinear System Theory
Homework 7

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Problem 4.27

Consider the system

$$\dot{x}_1 = -x_2x_3 + 1\dot{x}_2 = x_1x_3 - x_2, \qquad \dot{x}_3 = x_3^2(1 - x_3)$$

(a) Show that the system has a unique equilibrium point.

(b) Using linearization, show that the equilibrium point is asymptotically stable. Is it globally asymptotically stable?

Solution

(1) To find the equilibrium point(s), we need to set all the $\dot{x}_1$,$\dot{x}_2$ and $\dot{x}_2$ to zero.

$$0 = -x_2x_3 + 1 \;\; 0 = x_1x_3 - x_2 \;\; 0 = x_3^2(1 - x_3)$$

The third equation has two solutions: $x_3 = 0$, $x_3 = 1$, However, $x_3 = 0$ will not be consistent with equation 1. Therefore When $x_3 = 1$, $x_2 = 1$ and $x_1 = 1$. Therefore, the system has a unique equilibrium point at $(1, 1, 1)$.

(2) The Jacobian evaluated at $(1, 1, 1)$ is given by

$$\begin{pmatrix} 0 & -1 & -1 \\ 1 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix}$$

The eigenvalues are given by

$$\begin{pmatrix} -1 \\ -\frac{1}{2} - \frac{\sqrt{3}\,\mathrm{i}}{2} \\ -\frac{1}{2} + \frac{\sqrt{3}\,\mathrm{i}}{2} \end{pmatrix}$$

Which has negative real part. Therefore, the equilibrium point is Asymptotically stable but not globally Asymptotically stable because when $\dot{x}_3 = 0$ then, $\dot{x}_1 = 1$ and $\dot{x}_1(t)$ grows unbounded away from from the equilibrium point

Problem 4.28

Consider the system
$\dot{x_1} = -x_1 \;\; \dot{x_2} = (x_1x_2 - 1)x_2^3 + (x_1x_2 - 1 + x_1^2)x_2$

(a) Show that x = 0 is the unique equilibrium point.

(b) Show by linearization that x = 0 is asymptotically stable.

(c) Show that $\Gamma = x \in R^2 | x_1x_2 \geq 2$ is a positively invariant set.

(d) Is x = 0 globally asymptotically stable?

Solution

(a) we set $\dot{x_1} = \dot{x_2} = 0$ and solve for x_1 and x_2.We see that $x_1 = x_2 = 0$. Therefore, $x = 0$ is an equilibrium.

(b) The jacobian is given by

$$\left(\begin{array}{cc} -1 & 0 \\ x_2\,(2\,x_1 + x_2) + {x_2}^4 & 2\,x_1\,x_2 + x_1\,{x_2}^3 + {x_1}^2 + 3\,{x_2}^2\,(x_1\,x_2 - 1) - 1 \end{array} \right)_{x=0} = \left(\begin{array}{cc} -1 & 0 \\ 0 & -1 \end{array} \right)$$

with eigenvalues at $\left(\begin{array}{c} -1 \\ -1 \end{array} \right)$, Therefore the equilibrium is **Asymptotically stable**.

(c) Suppose that $x(t_0) \in \Gamma$, that is, $x_1(t_0)x_2(t_0) \geq 2$. Then, for $t \geq t_0$,Let

$$V = x_1x_2$$

$$\begin{aligned} \dot{V} &= x_2(t)\frac{d}{dt}x_1(t) + x_1(t)\frac{d}{dt}x_2(t) \\ &= x_2(t)(-x_1(t)) + x_1(t)((x_1(t)x_2(t) - 1)x_2(t)^3 + (x_1(t)x_2(t) - 1 + x_1(t)^2)x_2(t)) \\ &= x_1(t)^2x_2(t)^4 - x_2(t)^3 + x_1(t)^2x_2(t)^2 \\ &= 4x_2^2 > 0 \end{aligned}$$

where we used the fact that $x_1(t)x_2(t) = 2$. Therefore, Γ is a positively invariant set.

(d) The origin $x = 0$ is not globally Asymptotically stable, because trajectories starting in Γ does not go to the origin.

Problem 4.29

Consider the system
$\dot{x}_1 = x_1 - x_1^3 + x_2 \;\; \dot{x_2} = 3x_1 - x_2$

(a) Find all the equilibrium point of the system.

(b) Using linearization, study the stability of each equilibrium point.

(c) Using quadratic functions, estimate the region of attraction of each asymptotically stable equilibrium point. Try to make your estimate as large as possible.

(d) Construct the phase portrait of the system and show on it the exact regions of attraction as well as your estimates.

Solution

(a) The equilibrium points of the system are given by $\begin{pmatrix} 0 & 0 \end{pmatrix}$, $\begin{pmatrix} -2 & -6 \end{pmatrix}$, $\begin{pmatrix} 2 & 6 \end{pmatrix}$

(b) The Jacobian is given by $\begin{pmatrix} 1 - 3{x_1}^2 & 1 \\ 3 & -1 \end{pmatrix}$ When it is evaluated at the equilibrium points $= \begin{pmatrix} 0 & 0 \end{pmatrix}$, it is unstable, while $\begin{pmatrix} -2 & -6 \end{pmatrix}$, $\begin{pmatrix} 2 & 6 \end{pmatrix}$ are **Asymptotically stable**.

(c) Linearizing the system about each equilibrium points, we see that $\begin{pmatrix} 0 & 0 \end{pmatrix}$ is unstable while $\begin{pmatrix} -2 & -6 \end{pmatrix}$, $\begin{pmatrix} 2 & 6 \end{pmatrix}$ are Asymptotically stable.
Using the Lyapunov function V(y) $= y^T P y$, $\dot{V} = y^T(PA + A^T P)y + 2y^T P g(y)$
We move the equilibrium point to the stable origin and perform a change of variables. so that $\dot{y} = \dot{x}$
Let $y_1 = x_1 = 2$ and $y_2 = x_2 = 6$, then

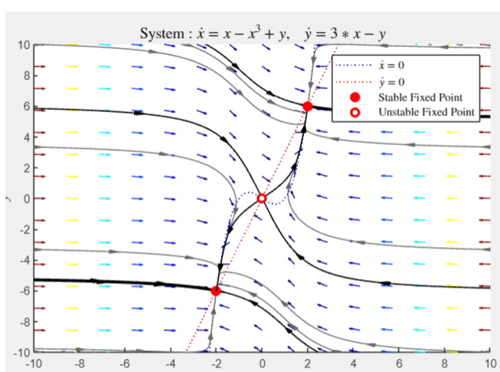
$$\dot{y} = -11y_1 + y_2 - 6y_1^2 - y_1^3$$
$$\dot{y_2} = 3y_1 - y_2$$

Solving the equation $PA + A^T P = -Q$, where Q = I, and $\dot{V} = -y^T y - 2y^T P g(y)$then we can get P=

$$\begin{bmatrix} 0.0938 \; 0.1771 \\ 0.1771 \; 0.6771 \end{bmatrix}$$

We can plot the region where $\dot{V} > 0$ and $V(x) \leq c$ We can choose c = 0.1

(d) The phase portrait can be seen below:



Problem 4.30

Repeat the previous exercise for the system

$$\dot{x}_1 = -\frac{1}{2}tan(\frac{\pi x_1}{2} + x_2)\dot{x}_2 = x_1 - \frac{1}{2}tan(\frac{\pi x_2}{2})$$

Solution

(a) The equilibrium points of the system are given by many, considering the region within $-3 < x, y < 3$ there are 5 equilibrium poins, given by: $[-2.672, -0.882, \;\; -0.5, -0.5 \;\; 0, 0 \;\; 0.5, 0.5 \;\; 2.672, 0.882]$

(b) Linearizing about the equilibrium points (0,0)

$$J = \begin{bmatrix} -\frac{\pi}{4}sec^2(\pi\frac{x_1}{2}) & 1 \\ 1 & -\frac{\pi}{4}sec^2(\pi\frac{x_2}{2}) \end{bmatrix}$$

with eigenvalues

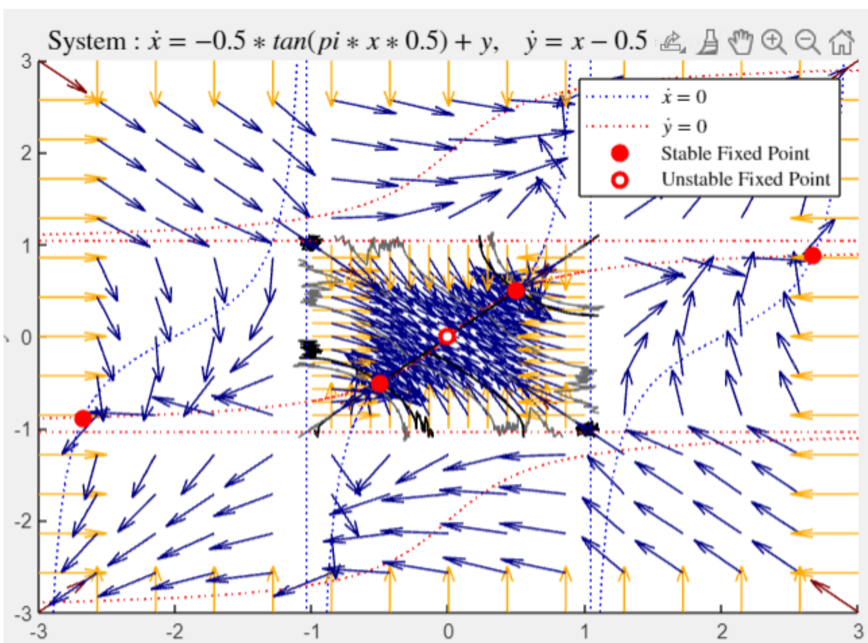
(c) Using the stable equilibrium and making a variable change,

$$\dot{y} = \begin{bmatrix} -\frac{\pi}{2} & 1 \\ 1 & \frac{-\pi}{2} \end{bmatrix} y + \begin{bmatrix} -\frac{1}{2}tan(\frac{\pi}{2}y_1 + \frac{\pi}{4}) + \frac{\pi}{2}y_1 + \frac{1}{2} \\ -\frac{1}{2}tan(\frac{\pi}{2}y_2 + \frac{\pi}{4}) + \frac{\pi}{2}y_2 + \frac{1}{2} \end{bmatrix}$$

We choose $V = y^T P y$ where $PA + A^T P = -I$ Then $\dot{V} = -y^T y + 2y^T P g(y)$. We get

$$P = \begin{bmatrix} 0.5352 & 0.3407 \\ 0.3407 & 0.5352 \end{bmatrix}$$

(d) The phase portrait is given by



Problem 4.31

For each of the systems of Exercise 4.3, use linearization to show that the origin is asymptotically stable.

(4.3) For each of the following systems, use a quadratic Lyapunov candidate to show that the origin is asymptotically stable.

(1)
$$\dot{x}_1 = -x_1 + x_1 x_2, \qquad \dot{x}_2 = -x_2$$

(2)
$$\dot{x}_1 = -x_2 - x_1(1 - x_1^2 - x_2^2), \qquad \dot{x}_2 = x_1 - x_2(1 - x_1^2 - x_2^2)$$

(3)
$$\dot{x}_1 = x_2(1 - x_1^2) \qquad \dot{x}_2 = -(x_1 + x_2)(1 - x_1^2)$$

(4)
$$\dot{x}_1 = -x_1 - x_2 \qquad \dot{x}_2 = 2x_1 - x_2^3$$

Solution

(1) the Jacobian of the system is given by

$$\begin{pmatrix} x_2 - 1 & x_1 \\ 0 & -1 \end{pmatrix}$$

Linearizing around it's equilibrium $0, 0$

$$J = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

with eigenvalues at $-1, -1$ Therefore the equilibrium is **Asymptotically Stable.**

(2)
$$J = \begin{bmatrix} 3\,{x_1}^2 + {x_2}^2 - 1 & 2\,x_1\,x_2 - 1 \\ 2\,x_1\,x_2 + 1 & {x_1}^2 + 3\,{x_2}^2 - 1 \end{bmatrix}$$

Linearizing around it's equilibrium $0, 0$

$$\begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix} \lambda = \begin{bmatrix} -1 - \mathrm{i} \\ -1 + \mathrm{i} \end{bmatrix}$$

Therefore the equilibrium is **Asymptotically Stable.**

(3)
$$J = \begin{bmatrix} -2\,x_1\,x_2 & 1 - {x_1}^2 \\ 2\,x_1\,(x_1 + x_2) + {x_1}^2 - 1 & {x_1}^2 - 1 \end{bmatrix}$$

Linearizing around the equilibrium, $0, 0$ the eigenvalues are given as

$$\begin{bmatrix} -\frac{1}{2} - \frac{\sqrt{3}\,\mathrm{i}}{2} \\ -\frac{1}{2} + \frac{\sqrt{3}\,\mathrm{i}}{2} \end{bmatrix}$$

Therefore the equilibrium is **Asymptotically Stable.**

(4)
$$J = \begin{bmatrix} -1 & -1 \\ 2 & -3\,{x_2}^2 \end{bmatrix}$$

Linearizing around the equilibrium point $0, 0$ the eigenvalues are

$$\begin{bmatrix} -\frac{1}{2} - \frac{\sqrt{7}\,\mathrm{i}}{2} \\ -\frac{1}{2} + \frac{\sqrt{7}\,\mathrm{i}}{2} \end{bmatrix}$$

Therefore the equilibrium is **Asymptotically Stable.**

4.32 For each for the following systems, investigate whether the origin is stable, asymptotically stable, or unstable:

(1)
$$\begin{aligned}\dot{x}_1 &= -x_1 + x_1^2\\ \dot{x}_2 &= -x_2 + x_3^2\\ \dot{x}_3 &= x_3 - x_1^2\end{aligned}$$

(2)
$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -\sin x_3 + x_1[-2x_3 - \mathrm{sat}(y)]^2\\ \dot{x}_3 &= -2x_3 - \mathrm{sat}(y)\\ &\text{where } y = -2x_1 - 5x_2 + 2x_3\end{aligned}$$

(3)
$$\begin{aligned}\dot{x}_1 &= -2x_1 + x_1^3\\ \dot{x}_2 &= -x_2 + x_1^2\\ \dot{x}_3 &= -x_3\end{aligned}$$

(4)
$$\begin{aligned}\dot{x}_1 &= -x_1\\ \dot{x}_2 &= -x_1 - x_2 - x_3 - x_1x_3\\ \dot{x}_3 &= (x_1+1)x_2\end{aligned}$$

Figure 1

Problem 4.32

Solution

(1)

$$J = \begin{bmatrix} 2\,x_1 - 1 & 0 & 0 \\ 0 & -1 & 2\,x_3 \\ -2\,x_1 & 0 & 1 \end{bmatrix}$$

Linearizing around the equilibrium $(0,0,0)$, $\lambda_{1,2,3} = -1, -1, 1$
Therefore the origin is **Not stable**

(2) Near the origin, sat(y) = y, taking the Jacobian,

$$J = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 2 & 5 & -4 \end{bmatrix}$$

$\lambda_{1,2,3} = -1, -1, -2$. Therefore the origin is **Asymptotically Stable**

(3)

$$J = \begin{bmatrix} 3\,{x_1}^2 - 2 & 0 & 0 \\ 2\,x_1 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Linearizing around the equilibrium $(0,0,0)$, $\lambda_{1,2,3} = -2, -1, -1$
Therefore the origin is **Asymptotically Stable**

(4)

$$\begin{bmatrix} -1 & 0 & 0 \\ -x_3 - 1 & -1 & -x_1 - 1 \\ x_2 & x_1 + 1 & 0 \end{bmatrix}$$

Linearizing around the equilibrium $(0,0,0)$, $\lambda_{1,2,3} = -1 - \frac{1}{2} - \frac{\sqrt{3}\,\mathrm{i}}{2}, -\frac{1}{2} + \frac{\sqrt{3}\,\mathrm{i}}{2}$ Therefore the origin is **Asymptotically Stable**

Problem 4.33

Consider the second-order system $\dot{x} = f(x)$, where $f(0) = 0$ and $f(x)$ is twice continuously differentiable in some neighborhood of the origin. Suppose $[\partial f/\partial x](0) = -B$, where B be Hurwitz. let P be the positive definite solution of the Lyapunov equation $PB + B^T P = -I$ and take $V(x) = x^T Px$. Show that there exists $c^* > 0$ such that, for every $0 < c < c*$, the surface $V(x) = c$ is closed and $[\partial V/\partial x]f(x) > 0$ for all $x \in V(x) = c$

Solution

Since f(x) is twice differentiable, then for any c ¿ 0, $V(x) = x^T Px = c$ is closed curve. And

$$\dot{V} = -x^T(PB + B^T P)x + 2x^T Pg(x) = x^T x + 2x^T Pg(x)$$

Taking the P2 norm,

$$\|\dot{V}\|_2 \geq \|x\|_2^2 + 2\|x^T Pg(x)\|_2 \geq \|x\|_2^2 - 2\|x\|_2\|P\|_2\|g(x)\|_2$$

For $\dot{V} > 0$ we require $\|g(x)\|_2 \to 0$ faster than $\|x\|_2 \to 0$ and

$$\|g(x)\|_2 \leq \gamma\|x\|_2^2$$

is inside a ball with radius c^* for all $\gamma > 0$. The radius is given by solving

$$\|x\|_2 - 2\|P\|_2\|g(x)\|_2 \geq \|x\|_2 - 2\|P\|_2\gamma\|x\|_2^2 > 0$$

and

$$c^* = \|x\|_2 < \frac{1}{2\gamma\|P\|_2}$$

Since f(x) is a second order twice differentiable function, $\|g(x)\|_2 \leq \gamma\|x\|_2^2$ and we can decompose $f(x)$ into $g(x) + Ax$ We know that g(x) is a second order twice differentiable function. therefore by Mean Value Theorem, we have $|g_i(x)| \leq k\|x\|^2 \leq \gamma\|x\|^2$

Problem 4.34

Prove Lemma 4.2 which states that
Let α_1 and α_2 be class $\mathcal{K}$ functions on $[0, a), \alpha_3$ and α_4 be class $\mathcal{K}_\infty$ functions, and β be a class $\mathcal{KL}$ function. Denote the inverse of α_i by α_i^{-1}, Then

. α_i^{-1} is defined on $[0, \alpha_i(a))$ and belongs to class $\mathcal{K}$.

. α_3^{-1} is defined on $[0, \infty)$ and belongs to class $\mathcal{K}_\infty$

. $\alpha_1.\alpha_2$ belongs to class $\mathcal{K}$

. $\alpha_3.\alpha_4$ belongs to class $\mathcal{K}_\infty$

. $\sigma(r, s) = \alpha_1(\beta(\alpha_2(r), s))$ belongs to class $\mathcal{KL}$

Solution

- By definition, a function α is in class $\mathcal{K}$ if it is continuous, strictly increasing, $\alpha(0) = 0$, and $\lim_{r\to\infty} \alpha(r) = \infty$. Since α_i is in class $\mathcal{K}$, it is continuous, strictly increasing, and $\alpha_i(0) = 0$. Since α_i is continuous and strictly increasing, Therefore, its inverse α_i^{-1} is also continuous and strictly increasing. $\alpha_i^{-1}(0) = 0$: Since $\alpha_i(0) = 0$, we have $\alpha_i^{-1}(0) = 0$.

- By similar argument, Since α_3 is in class $\mathcal{K}_\infty$, it is continuous, strictly increasing, and $\alpha_3(0) = 0$, therefore α_3^{-1} is continuous and strictly increasing,and $\alpha_3^{-1}(\infty) = \infty$ therefore α_3^{-1} is a class $\mathcal{K}_\infty$

- Since α_1 and α_2 are both in class $\mathcal{K}$, they are both continuous, strictly increasing, $\alpha_1(0) = \alpha_2(0) = 0$, and $\lim_{r\to\infty} \alpha_i(r) = \infty$. Therefore, their composition $\alpha_1.\alpha_2$ is also continuous, strictly increasing, $(\alpha_1.\alpha_2)(0) = 0$, and $\lim_{r\to\infty}(\alpha_1.\alpha_2)(r) = \infty$. Thus, $\alpha_1.\alpha_2$ belongs to class $\mathcal{K}$.

- Since α_3 is in class $\mathcal{K}\infty$ and α_4 is in class $\mathcal{K}\infty$, they are both continuous, strictly increasing, $\alpha_3(0) = \alpha_4(0) = 0$, and $\lim_{r\to\infty} \alpha_i(r) = \infty$. Therefore, their composition $\alpha_3.\alpha_4$ is also continuous, strictly increasing, $(\alpha_3.\alpha_4)(0) = 0$, and $\lim_{r\to\infty}(\alpha_3.\alpha_4)(r) = \infty$.Thus, $\alpha_3.\alpha_4$ belongs to class $\mathcal{K}_\infty$.

- For fixed s, the function is strictly increasing with respect to r, and zero when r = 0. For fixed r, the function $\beta(., s) \to$ as $s \to \infty$. Thus, the function belongs to class $\mathcal{KL}$

Problem 4.35

Let α be a class $\mathcal{K}$ function on [0,a). Show that

$$\alpha(r_1 + r_2) \leq \alpha(2r_1) + \alpha(2r_2), \forall r_1, r_2 \in [0, \frac{0}{2}]$$

Solution

Since α is a class $\mathcal{K}$ function on $[0, a)$, we know that it is non-decreasing and continuous, and that $\alpha(0) = 0$.
Let $r_1, r_2 \in [0, \frac{a}{2}]$

1. $\alpha(0) = 0$;
2. α is continuous and non-decreasing;
3. $\lim_{r \to a} \alpha(r) = \infty$.

Let $r_1, r_2 \in [0, \frac{a}{2}]$ be arbitrary. Then, we have

$$\begin{aligned} \alpha(r_1 + r_2) &\leq \alpha(a - \frac{a}{2} + a - \frac{a}{2}) \\ = \alpha(2a - \frac{a}{2} - \frac{a}{2}) &\leq \alpha(2a - \frac{a}{2}) + \alpha(2a - \frac{a}{2}) \\ &= \alpha(2r_1) + \alpha(2r_2), \end{aligned}$$

Thus $\alpha r_1 + \alpha r_2 \leq \alpha(2r_1) + \alpha(2r_2)$ is always satisfied.

Problem 4.36

Is the origin of the scalar system $\dot{x} = \frac{-x}{(t+1)}, t \leq 0$, uniformly asymptotcally stable?

Solution

Let

$$\begin{aligned} V(x) &= x^2 \\ \dot{V}(x) &= \frac{d}{dt}(x^2) \\ &- 2x\frac{x}{t+1} = -\frac{2x^2}{t+1} \end{aligned}$$

For any $x \neq 0$, we have $\frac{d}{dt}V(x) < 0$ for all $t \leq 0$. This means that $V(x)$ is a decreasing function along the trajectories of the system, except at $x = 0$ where it is constant. Therefore it is stable. As $t \to \infty$.

$$\begin{aligned} V(x(t)) &= x_0 exp(\int_{t0}^{t}) - \frac{1}{\tau+1}d\tau \\ &= x(t_0)\frac{1+t_0}{1+t} \end{aligned}$$

As $t \to \infty$, we have $x(t) \to 0$ for all non-zero initial conditions x_0. This means that the trajectories of the system approach the origin as $t \to \infty$ and do so at an exponential rate. Therefore, the origin of the system is **Uniformly Asymptotically stable.**

Problem 4.37

For each of the linear systems, use a quadratic Lyapunov function to show that the origin is exponentially stable.

(1)
$$\dot{x} = \begin{bmatrix} -1 & \alpha(t) \\ \alpha(t) & -2 \end{bmatrix} x, |\alpha(t)| \leq 1$$

(2)
$$\dot{x} = \begin{bmatrix} -1 & \alpha(t) \\ -\alpha(t) & -2 \end{bmatrix} x$$

(3)
$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & -\alpha(t) \end{bmatrix} x, \alpha(t) \geq 2$$

(4)
$$\dot{x} = \begin{bmatrix} -1 & 0 \\ \alpha(t) & -2 \end{bmatrix} x$$

Solution

(1) Choose $V(x) = \frac{1}{2}(x_1^2 + x_2^2)$

$$\begin{aligned}
\dot{V}(x) &= x_1(-x_1 - 2x_2\alpha(t)) + x_2(-2x_1\alpha(t) - 2x_2) \\
&= -x_1^2 - 2x_1x_2\alpha(t) - 2x_2^2 \\
&= -\frac{1}{2}(x_1^2 + 2x_1x_2\alpha(t) + 2x_2^2) - \frac{1}{2}(x_1^2 + x_2^2) \\
&\leq -\frac{1}{2}(x_1^2 + x_2^2) = -V(x)
\end{aligned}$$

Since $|\alpha(t)| \leq 1$ hence we can obtain the inequality. Therefore, $\dot{V}(x) \leq -\frac{1}{2}V(x)$, where $\lambda = \frac{1}{2}$ Thus, the origin of the system is **Exponentially Stable**

(2) Choose $V(x) = \frac{1}{2}(x_1^2 + x_2^2)$

$$\begin{aligned}
\dot{V}(x) &= x_1\dot{x_1} + x_2\dot{x_2} \\
&= x_1(-1x_1 + \alpha(t)x_2) + x_2(-\alpha(t)x_1 - 2x_2) \\
&= -x_1^2 - 2x_2^2 - \alpha(t)x_1x_2
\end{aligned}$$

Therefore, the origin is **Exponentially Stable**

(3)
$$\dot{V}(x) = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T \begin{bmatrix} 0 & 1 \\ -1 & -\alpha(t) \end{bmatrix} \begin{bmatrix} x_1 \ x_2 \end{bmatrix} = -\alpha(t)x_2^2$$

Since $\alpha(t) \geq 2$, $\dot{V}(x) \leq -2x_2^2$ for all $x \neq 0$. Therefore, the origin is **Exponentially Stable**

(4) Choose $V(x) = \frac{1}{2}(x^T P x)$

$$\dot{V} = -x_1^2 P_{11} - 2x_1x_2P_{12} - x_2^2P_{22} - \alpha(t)x_1x_2P_{21}$$

Choose P to be positive definite, then $\dot{V} \leq 0$ herefore, the origin is **Exponentially Stable**

4.38 ([95]) An RLC circuit with time-varying elements is represented by

$$\dot{x}_1 = \frac{1}{L(t)}x_2, \qquad \dot{x}_2 = -\frac{1}{C(t)}x_1 - \frac{R(t)}{L(t)}x_2$$

Suppose that $L(t)$, $C(t)$, and $R(t)$ are continuously differentiable and satisfy the inequalities $k_1 \le L(t) \le k_2$, $k_3 \le C(t) \le k_4$, and $k_5 \le R(t) \le k_6$ for all $t \ge 0$, where k_1, k_3, and k_5 are positive. Consider a Lyapunov function candidate

$$V(t,x) = \left[R(t) + \frac{2L(t)}{R(t)C(t)}\right]x_1^2 + 2x_1x_2 + \frac{2}{R(t)}x_2^2$$

(a) Show that $V(t,x)$ is positive definite and decrescent.

(b) Find conditions on $\dot{L}(t)$, $\dot{C}(t)$, and $\dot{R}(t)$ that will ensure exponential stability of the origin.

Problem 4.38

Solution

(a) To show that V is decrescent, we use the bounds on the time varying elements

$$V \le \left[k_6 + \tfrac{2K_2}{k_3k_5}\right]x_1^2 + 2x_1x_2 + \frac{2}{k_5}x_2^2$$

Next we show that V is bounded below by some class $\mathcal{K}$ function.

$$V \ge \left[k_5 + \tfrac{2k_1}{k_4k_6}\right]x_1^2 + 2x_1x_2 + \frac{2}{k_6}x_2^2 = \begin{bmatrix} p_{11} & 1 \\ 1 & p_{22} \end{bmatrix}$$

is positive definite when $p_{11}p_{22} > 1$ or equivalently,
$\frac{2k_5}{k_6} + \frac{4k_1}{k_4k_6^2} > 1$

$$\begin{aligned}
\dot{V} &= \left[R + \tfrac{2L}{RC}\right]2x_1\dot{x}_1 + \left[\dot{R} + \tfrac{2RC\dot{L} - 2L\dot{R}C - 2LR\dot{C}}{R^2C^2}\right]x_1^2 + 2x_1\dot{x}_2 \\
&+ 2\dot{x}_1x_2 + \frac{4}{R}x_2\dot{x}_2 - \frac{2}{R^2}x_2^2\dot{R} \\
&= x_1^2\left[\dot{R} + \tfrac{2R\dot{C}\dot{L} - 2L\dot{R}C}{R^2C^2} - \tfrac{2}{C}\right] + x_2^2\left[\tfrac{2}{L} - \tfrac{4}{L} - \tfrac{2}{R^2}\dot{R}\right] \\
&= x_1^2\left[\dot{R} + \tfrac{2\dot{L}}{RC} - \tfrac{2L\dot{R}}{R^2C} - \tfrac{2L\dot{C}}{RC^2} - \tfrac{2}{C}\right] + x_2^2\left[\tfrac{-2}{L} - \tfrac{-2}{L} - \tfrac{2}{R^2\dot{R}}\right] \\
&= x_1^2\left(\frac{-2}{C}\right)\left[1 + \dot{R}\left(\tfrac{-C}{2} + \tfrac{L}{R^2}\right) + \tfrac{L\dot{C}}{RC} - \tfrac{\dot{L}}{R}\right] + \frac{-2}{L}\left[1 + \tfrac{L\dot{R}}{R^2}\right]x_2 \\
-\dot{V} &= \frac{2}{C}C_1x_1^2 + \frac{2}{L}C_2X_2^2 \ge \frac{2C_1}{k_3}x_1^2 + \frac{2C_2}{k_1}x_2^2
\end{aligned}$$

(b) For $\dot{L}(t)$,$\dot{C}(t)$, and $\dot{R}(t)$ to be exponentially stable, -$\dot{V}$ is bounded above by some class $\mathcal{K}$ function of the same order as the bounds on V, taking a derivative and factoring, we get

$$-\dot{V} = \frac{2c_1}{C}x_1^2 + \frac{2c_2}{L}x_2^2 \ge \frac{2C_1}{K_3}x_1^2 + \frac{2C_2}{k_1}x_2^2$$

where

$$c_1 = 1 + \dot{R}\left(\frac{L}{R^2} - \frac{C}{2}\right) + \frac{L\dot{C}}{RC} - \frac{\dot{L}}{R} > 0$$

and $C_2 = 1 + \frac{L\dot{R}}{R^2} > 0$ which makes the origin **UES**.

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4.39 ([154]) A pendulum with time-varying friction is represented by

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -\sin x_1 - g(t)x_2$$

Suppose that $g(t)$ is continuously differentiable and satisfies

$$0 < a < \alpha \leq g(t) \leq \beta < \infty \quad \text{and} \quad \dot{g}(t) \leq \gamma < 2$$

for all $t \geq 0$. Consider the Lyapunov function candidate

$$V(t,x) = \tfrac{1}{2}(a\sin x_1 + x_2)^2 + [1 + ag(t) - a^2](1 - \cos x_1)$$

(a) Show that $V(t,x)$ is positive definite and decrescent.

(b) Show that $\dot{V} \leq -(\alpha - a)x_2^2 - a(2-\gamma)(1-\cos x_1) + O(\|x\|^3)$, where $O(\|x\|^3)$ is a term bounded by $k\|x\|^3$ in some neighborhood of the origin.

(c) Show that the origin is uniformly asymptotically stable.

Problem 4.39

Solution

(a) We notice that $1 + ag(t) - a^2 > 1$ since g(t)¿a then

$$V \leq \frac{1}{2}(asinx_1 + x_2)^2$$

so V is positive definite.

$$g(t) \leq \beta$$

and

$$1 - cosx_1 \leq x_1^2$$

$$V \leq \frac{1}{2}(asinx_1 + x_2)^2 + (1 + a\beta - a^2)x_1^2$$

Therefore, V is bounded above by a class $\mathcal{K}$ function.

(b)

$$\begin{aligned}
V =& \frac{1}{2}(asinx_1 + x_2)^2 + [1 + ag(t) - a^2](1 - cosx_1) \\
\dot{V} =& a^2 sinx_1 cosx_1 x_2 + asinx_1(-sinx_1 - g(t)x_2) + ax_2^2 cosx_1 + \\
& x_2(-sinx_1 - g(t)x_2) + x_2 sinx_1 + ag(t)x_2 sinx_1 - a^2 x_2 sinx_1 + a\dot{g}(t) - a\dot{g}(t)cosx_1 \\
&= x_2^2(acosx_1 - g(t)) + a(1 - cosx_1)\dot{g}(t) - asin^2x_1 - a^2(1 - cosx_1)x_2 sinx_1 \leq x_2^2(a - \alpha) + a\gamma(1 - cosx_1) - asin^2x + 0(\|x\|^3) \\
&= -x_2^2(\alpha - a) - a(1 - cosx_1)(2 - \gamma) + a(2(1 - cosx_1) - sin^2x_1) + 0(\|x\|^3)
\end{aligned}$$

where

$$(1 - cosx_1)x_2 sinx_1 \text{and} 2(1 - cosx_1) - sin^2x_1 \text{ are } o(\|x\|^3)$$

(c) The origin is AS because V is bounded above and below by a class $\mathcal{K}$ function, and -$\dot{V}$ is bounded below by a class $\mathcal{K}$ function.

Problem 4.40

(Floquet theory) Consider the linear system $\dot{x} = A(t)x$, where A(t) $= A(t+T)$ Let $\phi(.,.)$ be the state transition matrix. Define a constant matrix B via the equation $exp(BT) =$
$Phi(T, 0)$, and let $P(t) = exp(Bt)\Phi(0, t)$. Show that

(1) P(t+T) = P(t).

(2) $\Phi(t, \tau) = P^{-1}(t)exp[(t-\tau)B]P(\tau)$

(3) The origin of $\dot{x} = A(t)x$ is exponentially stable if and only if B is Hurwitz.

Solution

(a) To show that $P(t+T) = P(t)$, we use the periodicity of $A(t)$, which implies that $\Phi(t+T, s) = \Phi(t, s)$. Then,

$$\begin{aligned} P(t+T) &= \exp(B(t+T))\Phi(0, t+T) \\ &= \exp(BT)\exp(Bt)\Phi(0, t+T) \\ &= \exp(Bt)\Phi(T, 0)\Phi(0, t+T) \\ &= \exp(Bt)\Phi(T, t+T) \\ &= \exp(Bt)\Phi(0, t) \\ &= P(t) \end{aligned}$$

(b)

$$\begin{aligned} &P(t) = exp(Bt)\phi(0, t) \\ &\phi(0, t) = exp(-Bt)P(t) \\ &\phi(t, 0) = P^{-1}(t)exp(Bt) \\ &\phi(t, \tau) = \phi(t, 0)\phi(0, \tau) = P^{-1}(t)exp(B(t-\tau))P(\tau) \end{aligned}$$

(c) The solution to the differential equation is given by

$$\begin{aligned} &x(t) = \phi(t, t_0)x(t_0). \\ &\|x(t)\| \leq \|\phi(t, t_0)\|\|x(t_0)\| \\ &\leq \|P^{-1(t)}\|\|exp(B(t-t_0))\|\|P(t_0)\|\|x(t_0\|) \\ &\leq c_1 c_2\|exp(B(t-t_0))\|\|x(t_0)\| \end{aligned}$$

Since P(t) and $P^{-1}(t)$ are bounded for all $t \geq 0$. If B is Hurwitz, then $\|x(t)\| \to 0$ exponentially and the origin is exponentially stable. The origin is Uniformly Exponentially sable if and only if B is Hurwitz.