E_E 505 Nonlinear System Theory
Homework 4

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Problem 3.14

Consider the system

$$\dot{x_1} = -\frac{1}{\tau}x_1 tanh(\lambda x_1) - tanh(\lambda x_2)$$
$$\dot{x_2} = -\frac{1}{\tau}x_2 tanh(\lambda x_1) - tanh(\lambda x_2)$$

where λ and τ are positive constants

(a) Derive the sensitivity equation as λ and τ vary from nominal values λ_0 and τ_0

(b) Show that $r = \sqrt{x_1^2 + x_2^2}$ satisfies the differential inequality $\dot{r} \leq -\frac{1}{\tau}r + 2\sqrt{2}$

(c) Using the comparison lemma, show that the solution of the state equation satisfies the inequality $\|x(t)\|_2 \leq e^{-t/\tau}\|x(0)\|_2 + 2\sqrt{2}\tau(1 - e^{-t/\tau})$

Solution

(a) Taking the partial derivative with respect to x and τ We have

$$\frac{\partial f}{\partial x} = \begin{bmatrix} x_2\left(\tanh\left(\lambda\, x_2\right)^2 - 1\right) - x_1\left(\tanh\left(\lambda\, x_1\right)^2 - 1\right) & \frac{x_1}{\tau^2} \\ -x_1\left(\tanh\left(\lambda\, x_1\right)^2 - 1\right) - x_2\left(\tanh\left(\lambda\, x_2\right)^2 - 1\right) & \frac{x_2}{\tau^2} \end{bmatrix}$$
$$\frac{\partial f}{\partial \tau} = \begin{bmatrix} x_2\left(\tanh\left(\lambda\, x_2\right)^2 - 1\right) - x_1\left(\tanh\left(\lambda\, x_1\right)^2 - 1\right) & \frac{x_1}{\tau^2} \\ -x_1\left(\tanh\left(\lambda\, x_1\right)^2 - 1\right) - x_2\left(\tanh\left(\lambda\, x_2\right)^2 - 1\right) & \frac{x_2}{\tau^2} \end{bmatrix}$$

Evaluating the derivatives at nominal, λ_0 and τ_0

$$\frac{\partial f}{\partial x}|_{nominal} = \begin{bmatrix} -\lambda_0\left(\tanh\left(\lambda_0\, x_1\right)^2 - 1\right) - \frac{1}{\tau_0} & \lambda_0\left(\tanh\left(\lambda_0\, x_2\right)^2 - 1\right) \\ -\lambda_0\left(\tanh\left(\lambda_0\, x_1\right)^2 - 1\right) & -\lambda_0\left(\tanh\left(\lambda_0\, x_2\right)^2 - 1\right) - \frac{1}{\tau_0} \end{bmatrix}$$
$$\frac{\partial f}{\partial \lambda}|_{nominal} = \begin{bmatrix} x_2\left(\tanh\left(\lambda_0\, x_2\right)^2 - 1\right) - x_1\left(\tanh\left(\lambda_0\, x_1\right)^2 - 1\right) & \frac{x_1}{{\tau_0}^2} \\ -x_1\left(\tanh\left(\lambda_0\, x_1\right)^2 - 1\right) - x_2\left(\tanh\left(\lambda_0\, x_2\right)^2 - 1\right) & \frac{x_2}{{\tau_0}^2} \end{bmatrix}$$

$dotS = AS + B, s(0) = 0$

(b)

$$r = (x_1^2 + x_2^2)^{\frac{1}{2}}$$
$$\dot{r} = \frac{1}{2}(x_1^2 + x_2^2)^{-\frac{1}{2}} * (2x_1\dot{x_1} + 2x_2\dot{x_2})$$
$$\text{Where}(x_1^2 + x_2^2)^{-\frac{1}{2}}$$
$$= \frac{1}{\sqrt{x_1^2 + x_2^2}} = r^{-1}$$
$$\text{Therefore} r.\dot{r} = \sqrt{x_1^2 + x_2^2} * \frac{1}{2}\frac{1}{\sqrt{x_1^2 + x_2^2}} * (2x_1\dot{x_1} + 2x_2\dot{x_2})$$
$$r.\dot{r} = x_1\dot{x_1} + x_2\dot{x_2}$$

Substituting for $\dot{x_1}, \dot{x_2}$

$$\begin{aligned}
&x_1(-\frac{1}{\tau}x_1 tanh(\lambda x_1) - tanh(\lambda x_2)) + x_2(-\frac{1}{\tau}x_2 tanh(\lambda x_1) - tanh(\lambda x_2)) \\
&= -\frac{1}{\tau}(x_1^2 + x_2^2) + x_1 tanh(\lambda x_1) - tanh(\lambda x_2) + x_2 tanh(\lambda x_1) - tanh(\lambda x_2)) \\
&(x_1^2 + x_2^2) = r^2 \\
&-(1/\tau)r^2 + rcos(\theta)(tanh(\lambda x_1) - tanh(\lambda x_2)) + rsin(\theta))tanh(\lambda x_1) + tanh(\lambda x_1)) \\
&\leq -(1/\tau)r^2 + 2r(|cos(\theta)|) + |sin(\theta)| \\
&\leq -(1/\tau)r^2 + 2\sqrt{2r}
\end{aligned}$$

(c) Let

$$\begin{aligned}
\dot{v} \leq \frac{-1}{\tau}v + 2\sqrt{2}, v = \sqrt{x_1^2 + x_2^2} = \|x\|_2 \\
\dot{u} = \frac{-1}{\tau}u + 2\sqrt{2}, u_0 = v_0 = \|x_0\| \\
\|x\|_2 \leq \exp^{-t/\tau} \|x_0\|_2 + 2\sqrt{2}\tau(1 - e^{-t/\tau}) \\
v \leq e^{-t/\tau}v_0 + 2\sqrt{2}\tau(1 - e^{-t/\tau}) \\
\text{Solving} \quad \dot{u} = \frac{-1}{\tau}u + 2\sqrt{2} \\
u(s) = \frac{\|x_0\|}{s + 1/\tau} + \frac{2\sqrt{2}}{s(s + 1/\tau)} \\
u(t) = \|x_0\|_2 e^{-t/\tau} + 2\sqrt{2}\tau(1 - e^{-t/\tau})
\end{aligned}$$

Problem 3.15

Using the comparison lemma, show that the solution of the state equation
$\dot{x_1} = -x_1 + \frac{2x_2}{1+x_2^2}$,
$\dot{x_2} = -x_2 + \frac{2x_1}{1+x_1^2}$
satisfies the inequality
$\|x(t)\|_{\le} e^{-t}\|x(0)\|_2 + \sqrt{2}(1-e^{-t})$

<u>Solution</u>

By comparison $V(x)$ is continuous and positive definite for all $x \in \mathbb{R}^2$. $\dot{V}(x) \le -kV(x)$ for some positive constant k and for all $x \in \mathbb{R}^2$. Let $V(x) = |x|^2$. Note that $V(x)$ is positive definite for all $x \in \mathbb{R}^2$. $V(x)$ along the trajectory of the system: let

$$
\begin{aligned}
\dot{V}(x) &= 2x^T\dot{x} \\
= 2(x_1\ x_2) &\begin{pmatrix} -x_1 + \frac{2x_2}{1+x_2^2} \\ -x_2 + 1 + x_1^2 \end{pmatrix} \\
&\le -2(x_1^2 + x_2^2) + 4(x_1^2 + x_2^2) = 2(x_1^2 + x_2^2) \\
&= -2(x_1^2 + x_2^2) + 4(x_1^2 + x_2^2) \\
&= 2(x_1^2 + x_2^2)
\end{aligned}
$$

We use the fact that

$$
\begin{aligned}
&\frac{2x_2}{1+x_2^2} \le \sqrt{2}, \\
&\frac{2x_1}{1+x_1^2} \le \sqrt{2} \text{ and} \\
&x_1^2 + x_2^2 = |x(t)|_2^2
\end{aligned}
$$

By comparison lemma,

$$\|x(t)\|_2^2 \le e^{2t}\|x(0)\|_2^2$$

Taking the square root of both sides, we get

$$\|x(t)\|_2 \le e^t\|x(0\|_2)$$

Using the inequality,

$$\|x(t)\|_2 \le \|x(t)\|_1 \le \sqrt{2}$$

Therefore

$$\|x(t)\|_2 \le \sqrt{2}\|x(t)\|_2 \le \sqrt{2e^t}\|x(0)\|_2$$

Problem 3.16

Using the comparison lemma, find an upper bound on the solution of the scalar equation
$\dot{x} = -x + \frac{sint}{1+x^2}, x(0) = 2$

Solution

By comparison lemma, we need to find a function
$V(x)$ that is continuous and positive definite for all $x \in [0, \infty)$. Let

$$\begin{aligned}
&V(x) = 1 + x^2 \\
&\dot{V} = 2x\dot{x} \\
&= 2x(-x + \frac{sint}{1+x^2}) \\
&= -2x^2 - 2x\frac{sint}{1+x^2} \leq -2x^2 + 2x = -2x(x-1) \text{ Since } \qquad \frac{sint}{1+x^2} \leq 1
\end{aligned}$$

and $x(t)$ is positive $\forall t \geq 0$, then

$$0 \leq x(t) \leq x(0) = 2 \forall t \geq 0.$$

Therefore

$$0 \leq x(t) - 1 \leq 1 \forall t \geq 0.$$

This implies that

$$\begin{aligned}
-2x(x-1) \leq -4x(x-1) \forall x \in [0, \infty) 0 \leq x(t) \leq x(0) = 2, \\
0 \leq x(t) - 1 \leq 1, \\
-2x(x-1) \leq -4x(x-1)
\end{aligned}$$

We used the inequality

$$\begin{aligned}
&\frac{sint}{1+x^2} \leq 1 \\
&0 \leq x(t) \leq x(0) = 0 \leq x(t) - 1 \leq 1 \\
&-2x(x-1) \leq -4x(x-1)
\end{aligned}$$

Hence, we have

$$\dot{V}(x) \leq -4x(x-1) \forall x \in [0, \infty)$$

Problem 3.17

Consider the initial value problem(3.1) and let $D \subset R^n$ be a domain that contains x = 0. Suppose x(t), the solution of (3.1), belongs to D for all t ≥ 0 and

$$\|f(t,x)\|_2 \leq L\|x\|_2 \text{on}[t_0,\infty)xD$$

show that

(a)

$$\frac{d}{dt}[x^T(t)]| \leq 2L\|x(t)\|_2^2$$

(b)

$$\|x_0\|_2 exp[-L(t-t_0)] \leq \|x(t)\|_2 \leq \|x_0\|_2 exp[L(t-t_0)]$$
$$\dot{x} = f(t,x),\ x(t_0) = x_0$$

Solution

(a)

$$\frac{d}{dt}[x^T(t)] = \dot{x}^T(t)x(t) + x^T(t)\dot{x}(t)$$
$$f^T(t,x(t))x(t) + x^T(t)f(t,x(t)) = 2x^T(t)f(t,x(t))$$

From cauchy-Schwarz inequality, we have

$$|2x^T(t)f(t,x(t))| \leq 2|x(t)|_2.|f(t,x(t))|_2 \leq 2L|x(t)|_2^2$$

Therefore

$$|\frac{d}{dt}[x^T(t)]| \leq 2L|x(t)|_2^2$$

(b) Let Z(t) = $x^T(t)x(t)$ and $V_0 = x_0^T x_0$ We can show that

$$-2LZ(t) \leq \dot{Z}(t) \leq 2LZ(t)$$

Integrating both sides

$$-2L(t-t_0) \leq In(\frac{dV}{V_0}) \leq 2L(t-t_0)$$
$$V_0 e^{[-2L(t-t_0)]} \leq V(t) \leq V_0 e^{[2L(t-t_0)]}$$

taking the square root

$$\|x_0\|_2 e^{-L(t-t_0)} \leq \|x_0\|2e^(L(t-t_0)) \ \square$$

Problem 3.18

Let $y(t)$ be a nonnegative scalar function that satisfies the inequality

$$y(t) \leq k_1 e^{-\alpha(t-t_0)} + \int_{t_0}^{t} \qquad e^{-\alpha(t-\tau)}[k_2 y(\tau) + k_3]d\tau$$

Where k_1, k_2 and k_3 are non negative constants and α is a positive constant that satisfies $\alpha \geq k_2$. Using the Gronwall-Bellman inequality, show that

$$y(t) \leq k_1 e^{-\alpha(t-t_0)} + \frac{k_3}{\alpha - k_2}[1 - e^{-(\alpha-k_2)(t-t_0)}]$$

Hint: Take

$$z(t) = y(t)e^{\alpha(t-t_0)}$$

and find the inequality satisfied by z.

$$y(t) \leq k_1 e^{-\alpha(t-t_0)} + \int_{t_0}^{t} e^{-\alpha(t-\tau)}[k_2 y(\tau) + k_3]d\tau$$

Where k_1, k_2 and k_3 are non negative constants and α is a positive constant that satisfies $\alpha \geq k_2$. Using the Gronwall-Bellman inequality, show that

$$y(t) \leq k_1 e^{-\alpha(t-t_0)} + \frac{k_3}{\alpha - k_2}[1 - e^{-(\alpha-k_2)(t-t_0)}]$$

Hint: Take $z(t) = y(t)e^{\alpha(t-t_0)}$ and find the inequality satisfied by z.

Solution

Let $z(t) = y(t)e^{\alpha(t-t_0)}$. Then, we have

$$z(t) = y(t)e^{\alpha(t-t_0)} \leq k_1 e^{-\alpha(t-t_0)} e^{\alpha(t-t_0)} + k_1 \int_{t_0}^{t} e^{-\alpha(t-\tau)+\alpha(t-t_0)} e^{\alpha(\tau-t_0)}[k_2 y(\tau) + k_3]d\tau$$

$$= k_1 + k_2 \int_{t_0}^{t} e^{-\alpha(t-\tau)}[z(\tau)d\tau + \frac{k_3}{\alpha}(e^{-\alpha(t-t_0)} - 1)]$$

Apply the Gronwall-Bellman inequality

$$z(t) \leq k_1 + k_2 \int_{t_0}^{t} e^{-\alpha(t-\tau)} k_1 e^{-\alpha(\tau-t_0)} + \frac{k_3}{\alpha} e(^{\alpha(t-t_0)} - 1)d\tau + \frac{k_3}{\alpha} e^{\alpha(t-t_0)} - 1)$$

$$= k_1 e^{-\alpha(t-t_0)} + \frac{k_3}{\alpha}(e^{-\alpha(t-t_0)} - 1) + k_1 k_2 \int_{t_0}^{t} e^{-\alpha(t-\tau)} e^{-\alpha(\tau-t_0)d\tau} + \frac{k_2 k_3}{\alpha} \int_{t_0}^{t} e^{-\alpha(t-\tau)} e^{-\alpha(\tau-t_0-1}d\tau$$

$$= k_1 e^{-\alpha(t-t_0)-1} + \frac{k_1 k_2}{\alpha}(e^{-\alpha(t-t_0)} - 1) + \frac{k_2 k_3}{\alpha^2}(e^{-\alpha(t-t_0)-1}$$

Then, $z(t) \leq k_1 e^{\alpha(t-t_0)} + \frac{k_3 + k_2 k_1}{\alpha}(e^{\alpha(t-t_0)} - 1 + \frac{k_2 k_3}{\alpha^2}(e^{\alpha(t-t_0)} - 1)$ therefore
$y(t) \leq k_1 e^{-}(\alpha - k_2)(t - t_0) + \frac{k_3}{(\alpha - k_2)}[1 - e^{-}(\alpha - k_2)(t - t_0)]$

Problem 3.19

Let $f : R^n \to R^n$ be locally Lipschitz in a domain $D \subset R^n$. Let $S \subset D$ be a compact set. Show that there is a positive constant L such that for all x,y $\in S$

$$\|f(x) - f(y)\| \leq L\|x - y\|$$

Hint: The set s can be covered by a finite number of neighborhoods

$$S \subset N(a_1, r_1) \cup N(a_2, r_2) \cup ... \cup N(a_k, r_k)$$

Consider the following two cases separately:

(a)

$$x, y \in S \cap N(a_i, r_i) \quad \text{for some i}$$

$$x, y \notin S \cap N(a_i, r_i)$$

for any i; in this case

$$\|x - y\| \leq min_i r_i$$

In the second case, use the fact that $f(x)$is uniformly bounded on S

Solution

Since f is locally Lipschitz in D, for each i, there exists a Lipschitz constant $L_i > 0$ such that for any

$$x, y \in N(a_i, r_i) \cap S, \text{there is} |f(x) - f(y)| \leq L_i |x - y|$$

Let

$$L = \max_{i=1}^{k} L_i.$$

Then for any $x, y \in S$, there exist $i, j \in 1, 2, \ldots, k$ such that

$$x \in N(a_i, r_i) \text{and} y \in N(a_j, r_j)$$

Case 1:

$$x, y \in S \cap N(a_i, r_i)$$

for some i Then,

$$|x - y| \leq |x - a_i| + |a_i - a_j| + |a_j - y| \leq r_i + |a_i - a_j| + r_j$$

follows from the triangle inequality.Moreover, we have

$$f(x) \in N(a_i, r_i) andf(y) \in N(a_j, r_j)$$

so we have

$$|f(x) - f(y)| \leq L_j |f(x) - f(y)| \leq L_j |x - y| \leq L_j (r_i + |a_i - a_j| + r_j)$$

Then,

$$|f(x) - f(y)| \leq \max L_i, L_j (r_i + |a_i - a_j| + r_j) \leq L(r_i + |a_i - a_j| + r_j)$$

where we have used the fact that

$$L = \max_{i=1}^{k} L_i$$

Case 2:

$$x, y \notin S \cap N(a_i, r_i) for any i; in this case |x - y| \leq \min_i r_i$$

In this case, we can use the fact that f is uniformly bounded on S, that is, there exists a constant $L > 0$ such that $|f(x)| \leq M$ for all $x \in S$. Then, we have

$$\|f(x) - f(y)\| \leq L_i \|x - y\|$$
$$\text{whenever} \|x - y\| \geq min_i r_i, \|f(x) - f(y)\| \leq \frac{C}{min_i r_i} \|x - y\|$$