E_E 505 Nonlinear System Theory
Homework 2

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Tuesday 7^{th} February, 2023

Problem 2.2

For each of the system, find all equilibrium points and determine the type of each isolated equilibrium.

Sys 1

$$\dot{x_1} = x_2$$
$$\dot{x_2} = -x_1 + 1/16x_1^5 - x_2$$

The equilibrium points of the system are the roots of equations

$$0 = x_2$$
$$0 = -x_1 + 1/16x_1^5 - x_2$$
$$0 = -x_1 + 1/16x_1^5 - 0$$
$$-x_1(1 + \frac{1}{4}x_1^2)(1 + \frac{1}{2}x_1)(1 + \frac{1}{2}x_1)$$

The equilibrium points are $(x_1, x_2) = (0,0), (2,0) and (-2,0)$.
Linearizing and evaluating at $(0,0)$ we get

$$J = \begin{bmatrix} \frac{df_1}{dx_1} & \frac{df_1}{dx_2} \\ \frac{df_2}{dx_1} & \frac{df_2}{dx_2} \end{bmatrix} = \begin{bmatrix} 0\ 1 \\ -1\ \frac{5}{16}x_1^4\ 1 \end{bmatrix} \begin{bmatrix} 0\ 1 \\ -1\ 1 \end{bmatrix}$$

The eigenvalues $= -\frac{1}{2} \pm j\frac{\sqrt{3}}{2}$ The behavior is stable focus
Evaluating at $\pm 2, 0$

$$\begin{bmatrix} 0\ 1 \\ 4\ 1 \end{bmatrix}$$

The eigenvalues $= -\frac{1}{2} \pm j\frac{\sqrt{17}}{2}$ The behavior is a saddle point.

Sys 2

$$\dot{x_1} = 2x_1 - x_1x_2$$
$$\dot{x_2} = 2x_1^2 - x_2$$

The equilibrium points of $(x_1, x_2) = (0,0), (1,2) and (-1,2)$.
Linearizing and evaluating at the equilibrium,

$$J = \begin{bmatrix} 2 - x_2 & -x_2 \\ 4x_1 & -1 \end{bmatrix} = \begin{bmatrix} 2\ 0 \\ 0\ -1 \end{bmatrix} and \begin{bmatrix} 0 & -2 \\ 4 & -1 \end{bmatrix} and \begin{bmatrix} 0 & -2 \\ -4 & -1 \end{bmatrix}$$

With eigenvalues $(2, -1)$ and $(-\frac{1}{2} \pm j\sqrt{15/2}) and (-\frac{1}{2} \pm j\sqrt{15/2}$
The behavior of the system is Saddle point, stable focus and stable focus respectively.

Sys 3

$$\dot{x_1} = x_2$$
$$\dot{x_2} = -x_2 - \psi(x_1 - x_2)$$

where

$$\psi(y) = \begin{cases} y^3 + \frac{1}{2}y & |y| \leq 1, \\ 2y - \frac{1}{2}sgn(y) & |y| > 1 \end{cases}$$

setting f_1 and f_2 to zero

$$0 = x_2$$
$$0 = -x_2 - \psi(x_1 - x_2)$$

plugging in $0 = x_2, \psi(x_1) = 0$ plugging in $\psi(y)$ when it is ≤ 1

$$\dot{x_1} = x_2$$
$$\dot{x_2} = -x_2 - (x_1 - x_2)^3 - \frac{1}{2}(x_1 - x_2)$$

The jacobian evaluated at equilibrium points $0, 0$ is given by

$$J = \begin{bmatrix} 0\ 1 \\ -0.5\ \ -0.5 \end{bmatrix} =$$

The eigenvalues are

$$-\frac{1}{4} \pm j\frac{\sqrt{3}}{4}$$

The behavior of the system is stable focus

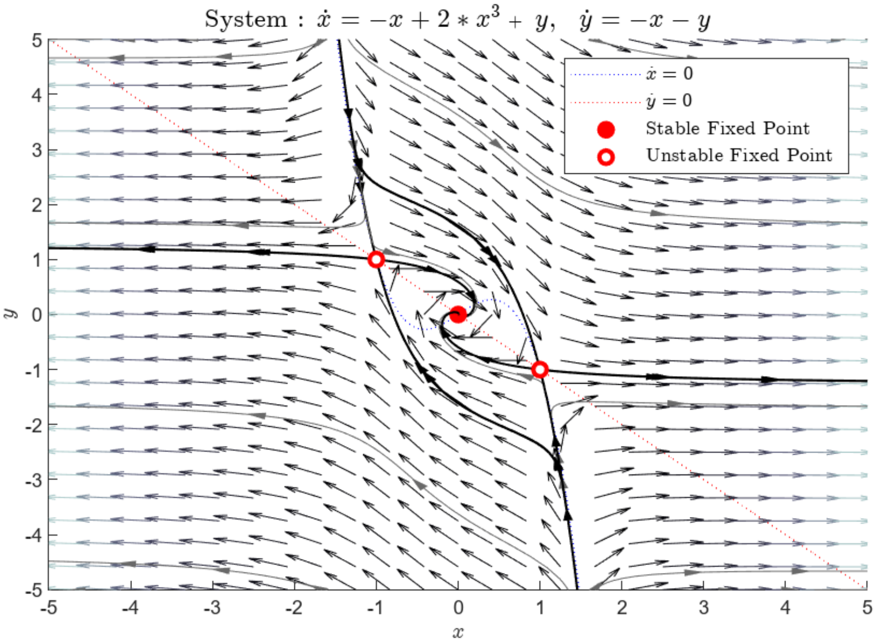

Figure 1

Problem 2.3

For each of the systems below, construct the phase portrait and discuss the qualitative behavior of the system

(a)

$$\dot{x_1} = -x_1 + 2x_1^3 + x_2$$
$$\dot{x_2} = -x_1 - x_2$$

This system has equilibrium at $(0, 0)$ with stable focus behavior, $(1, -1)$ and $(-1, 1)$ with saddle points behavior.

The eigenvalues are given as $\begin{pmatrix} -1 - \mathrm{i} \\ -1 + \mathrm{i} \end{pmatrix} \begin{pmatrix} 2 - 2\sqrt{2} \\ 2\sqrt{2} + 2 \end{pmatrix} \begin{pmatrix} 2 - 2\sqrt{2} \\ 2\sqrt{2} + 2 \end{pmatrix}$ respectively

The phase portrait is given by figure 1. Here we see that some trajectories converge to stable focus around the origin, while some diverge to infinity.

(b)

$$\dot{x_1} = x_1 + x_1 x_2$$
$$\dot{x_2} = -x_2 - x_2^2 + x_1 x_2 - x_1^3$$

This system has equilibrium at $(0, 0)$ with saddle behavior, $(0, 1)$ with unstable focus behavior and $(1, -1)$ with stable focus behavior.

The eigenvalues are given as $\begin{pmatrix} -1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \begin{pmatrix} -1 - \sqrt{3}\,\mathrm{i} \\ -1 + \sqrt{3}\,\mathrm{i} \end{pmatrix}$ respectively.

The phase portrait is given in figure 2. Here we see that trajectories on the right converge to stable focus, while trajectories on the left diverge to infinity.

(c)

$$\dot{x_1} = [1 - x_1 - 2h(x)]x_1$$
$$\dot{x_2} = [2 - h(x)]x_2$$
$$where h(x) = x_2/(1 + x_1)$$

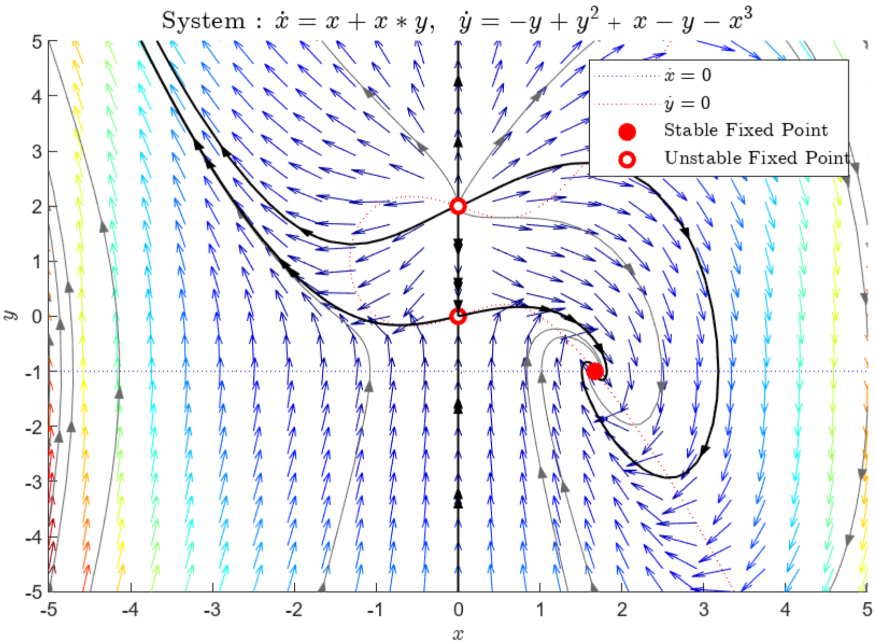

Figure 2

The system has equilibrium at $(0,0)(0,1)(1,-1)$and eigenvalue at $(-11),(12) and (-1-i\sqrt{3}-1+i\sqrt{3}$

The phase portrait is shown in figure 3, it shows that the system has unstable focus at the origin. It also has 2 limit circles,

(d)

$$\dot{x_1} = x_2$$
$$\dot{x_2} = -x_1 + x_2(1 - x_1^2 + 0.1x_1^4)$$

The system has equilibirum points $(0,0)$ with eigenvalues $\begin{pmatrix} \frac{1}{2} - \frac{\sqrt{3}\,\mathrm{i}}{2} \\ \frac{1}{2} + \frac{\sqrt{3}\,\mathrm{i}}{2} \end{pmatrix}$. From the phase portrait, we see that trajectories inside stable limit cycle converge while the others diverge to infinity

(e)

$$\dot{x_1} = (x_1 - x_2)(1 - x_1^2 - x_2^2)$$
$$\dot{x_2} = (x_1 + x_2)(1 - x_1^2 - x_1^2)$$

This system has equilibrium at $(0,0)$ with unit cycle, $(-1,0)$ with unstable focus behavior and (10) with stable focus behavior, with eigenvalues $\begin{pmatrix} 1-\mathrm{i} \\ 1+\mathrm{i} \end{pmatrix}$, $\begin{pmatrix} -2 \\ 0 \end{pmatrix}$ $\begin{pmatrix} -2 \\ 0 \end{pmatrix}$.

The phase vector field is given in figure 5. We see here that the trajectories converge to the unit cycle.

(f)

$$\dot{x_1} = -x_1^3 + x_2$$
$$\dot{x_2} = x_1 - x_2^3$$

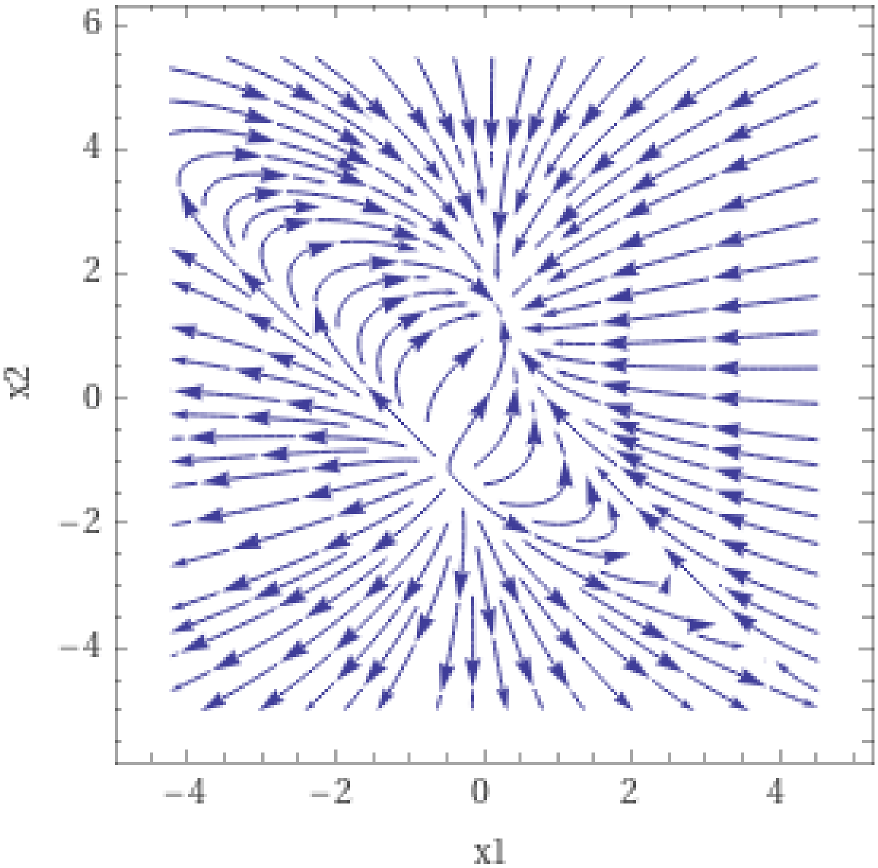

Figure 3

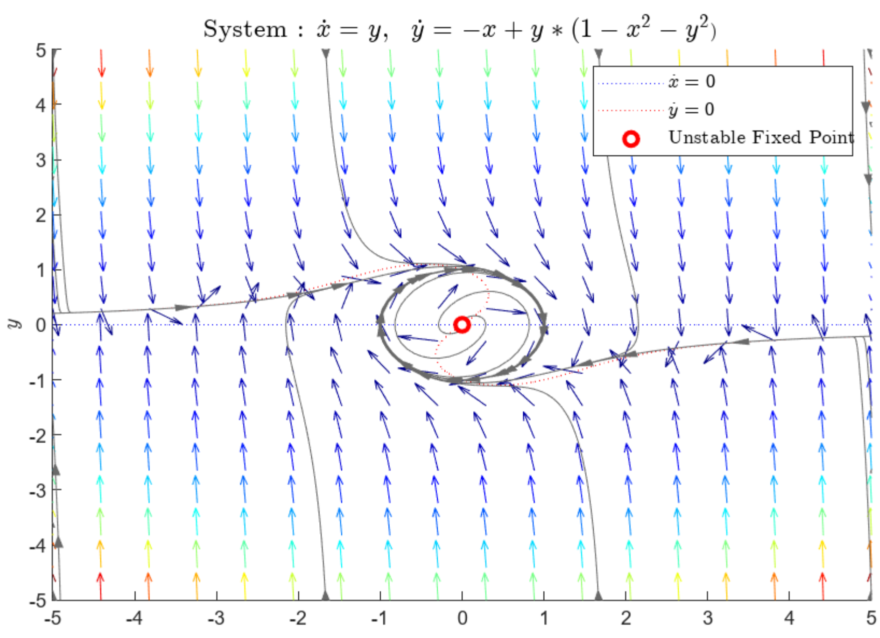

Figure 4

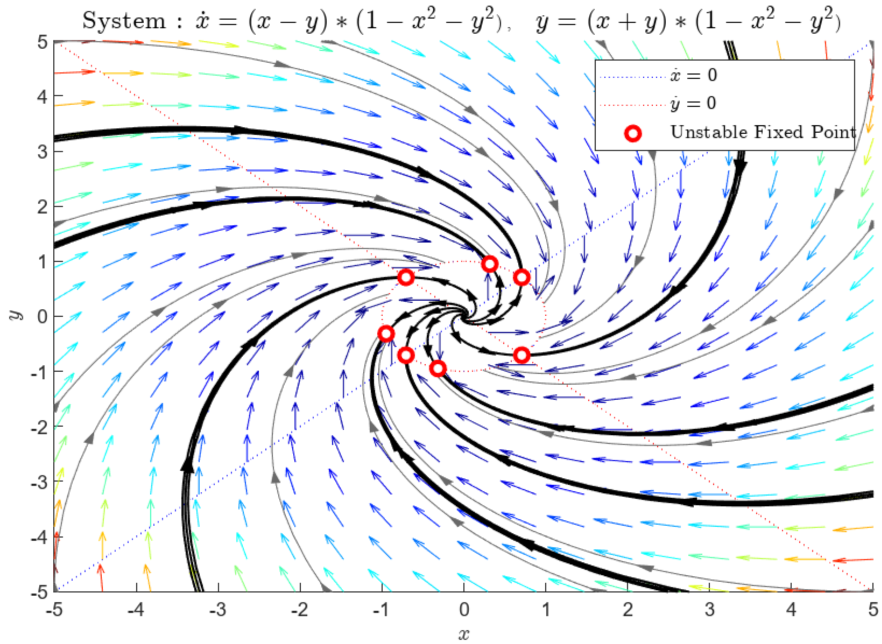

Figure 5

This system has equilibrium at $(0, 0)$ with saddle behavior, $(-1, -1)$ and $(1, 1)$ with stable modes behavior and $(1, -1)$ with stable focus behavior. The equilibrium points are $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$, $\begin{pmatrix} -4 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} -4 \\ -2 \end{pmatrix}$ respectively. The vector field shows the trajectories converge to the equilibrium points.

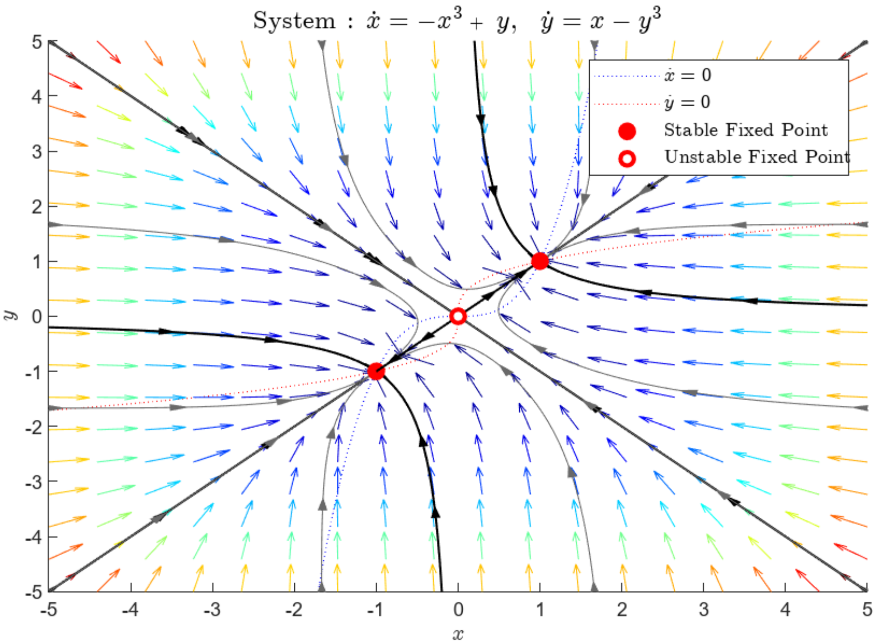

Figure 6

Problem 2.4

Mark the arrow of the phase portraits for the following systems and discuss the qualitative behavior

(a)

$$\dot{x_1} = x_2$$
$$\dot{x_2} = x_1 - 2tan^-1(x_1 + x_2)$$

This system has equilibrium at $(0, 0)$ and the jacobian is given by $\begin{pmatrix} 0 & 1 \\ 1 - \frac{2}{(x_1+x_2)^2+1} & -\frac{2}{(x_1+x_2)^2+1} \end{pmatrix}$ The system eigenvalues are $\begin{pmatrix} -1 \\ -1 \end{pmatrix}$ The phase portrait is given in figure 7 and the behavior of the system is a stable node. Almost all trajectories converge to the saddle point, while some outer trajectories diverge to infinitiy.

(b)

$$\dot{x_1} = 2x_1 - x_1x_2$$
$$\dot{x_2} = 2x_1^2 - x_2$$

This system has equilibrium at $(0, 0)$ with eigenvalue $-1, 2$, $(-1, 2)$ with eigenvalue $\begin{pmatrix} -\frac{1}{2} - \frac{\sqrt{15}\,\mathrm{i}}{2} \\ -\frac{1}{2} + \frac{\sqrt{15}\,\mathrm{i}}{2} \end{pmatrix}$ and $(1, 2)$ with eigenvalues $\begin{pmatrix} -\frac{1}{2} - \frac{\sqrt{15}\,\mathrm{i}}{2} \\ -\frac{1}{2} + \frac{\sqrt{15}\,\mathrm{i}}{2} \end{pmatrix}$. The behavior of the system at equilibrium points are saddle, and stable focus. The marked phase portrait is given by figure

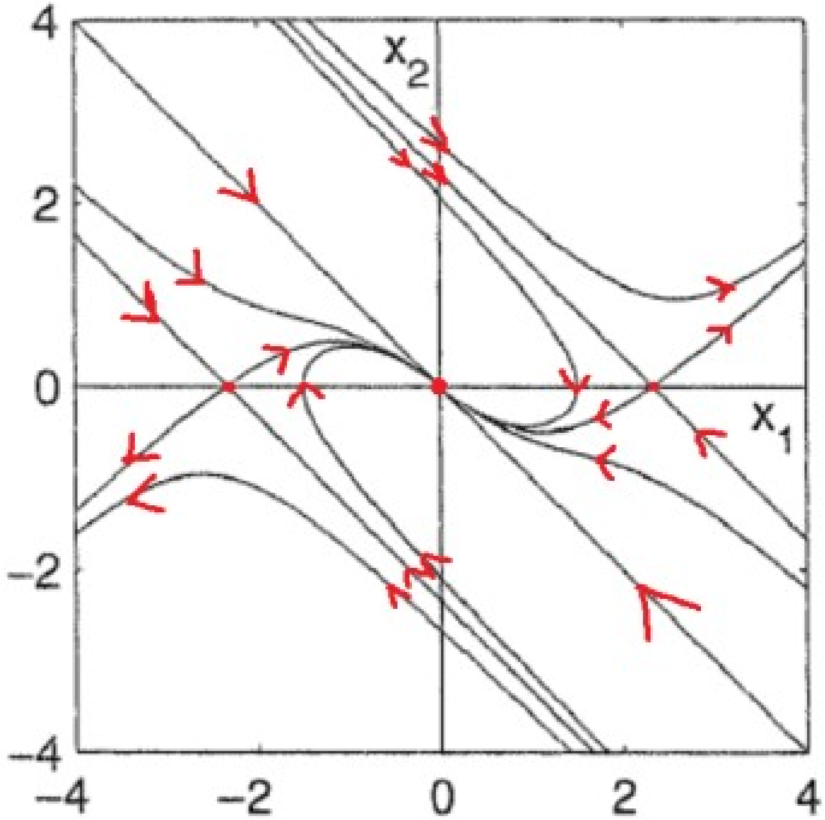

Figure 7

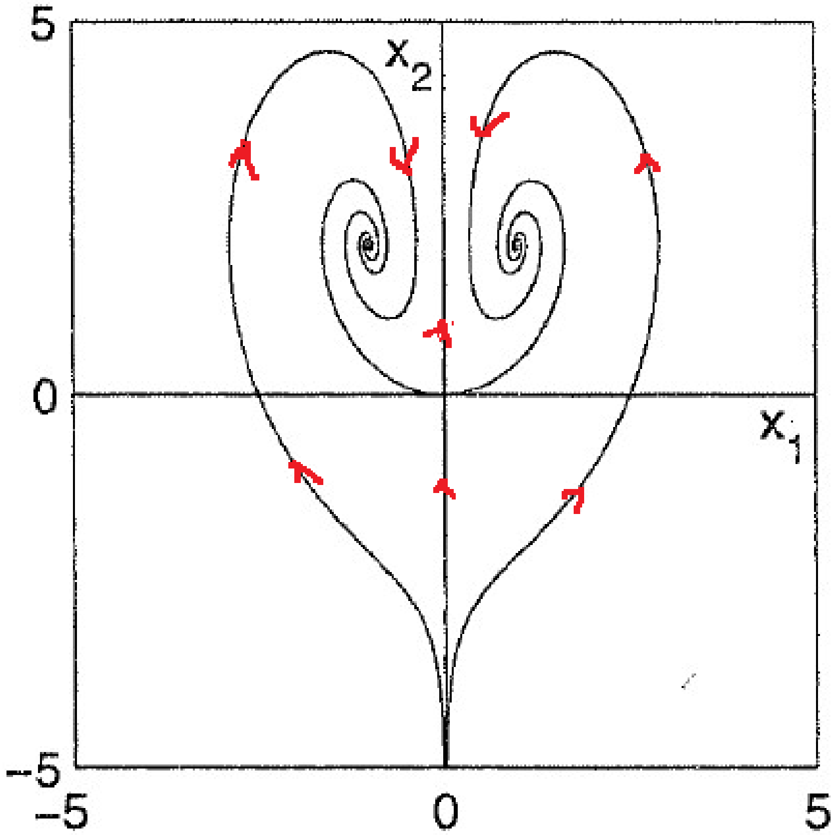

Figure 8

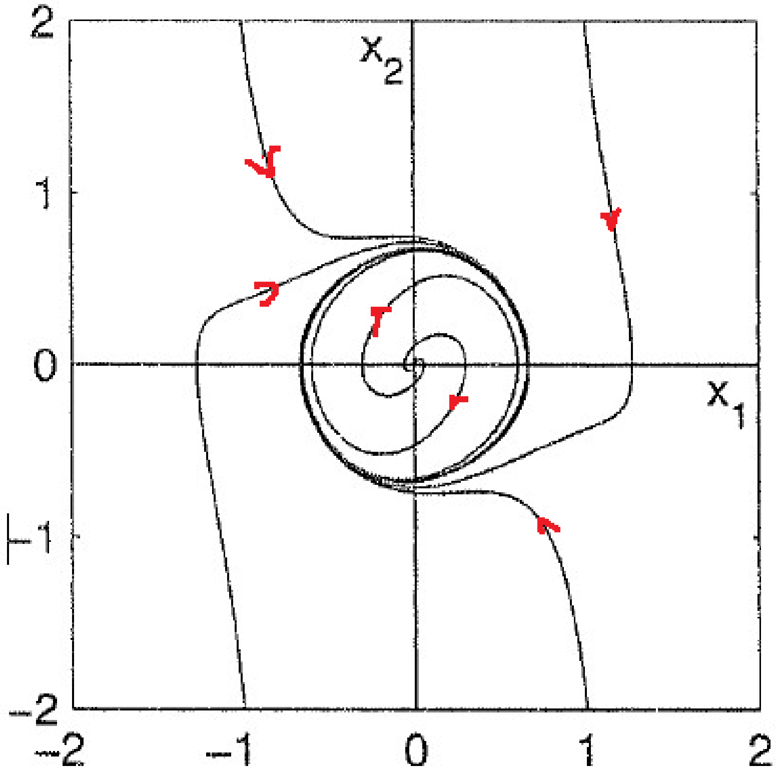

Figure 9

(c)

$$\dot{x_1} = x_2$$
$$\dot{x_2} = -x_+ x_2(1 - 3x_1^2 - 2x_2^2)$$

This system has equilibrium at $(0, 0)$ with eigenvalue $\begin{pmatrix} \frac{1}{2} - \frac{\sqrt{3}\,\mathrm{i}}{2} \\ \frac{1}{2} + \frac{\sqrt{3}\,\mathrm{i}}{2} \end{pmatrix}$ The behavior is unstable focus. This system has a limit cycle around the origin and all trajectory approach the limit cycle. The phase portrait is shown in figure 9.

(d)

$$\dot{x_1} = -(x_1 - x_1^2) + h(x)$$
$$\dot{x_2} = -(x_2 - x_2^2) + h(x)$$
$$h(x) = 1 - x_1 - x_2$$

This system has equilibrium points$(1, 0)$ and eigenvalues at $\begin{pmatrix} -\sqrt{2} - 1 \\ \sqrt{2} - 1 \end{pmatrix}$

This is a saddle. The second equilibrium point is $(0, 1)$ and eigenvalues at $\begin{pmatrix} -\sqrt{2} - 1 \\ \sqrt{2} - 1 \end{pmatrix}$ The behavior of the system is a saddle.

The third equilibrium is $\begin{pmatrix} \frac{3}{2} - \frac{\sqrt{5}}{2} & \frac{3}{2} - \frac{\sqrt{5}}{2} \end{pmatrix}$ with eigenvalues $\begin{pmatrix} 2 - \sqrt{5} \\ -\sqrt{5} \end{pmatrix}$ The behavior of the system is stable node The fourth equilibrium is given by $\begin{pmatrix} \frac{\sqrt{5}}{2} + \frac{3}{2} & \frac{\sqrt{5}}{2} + \frac{3}{2} \end{pmatrix}$ with eigenvalues $\begin{pmatrix} \sqrt{5} \\ \sqrt{5} + 2 \end{pmatrix}$ The

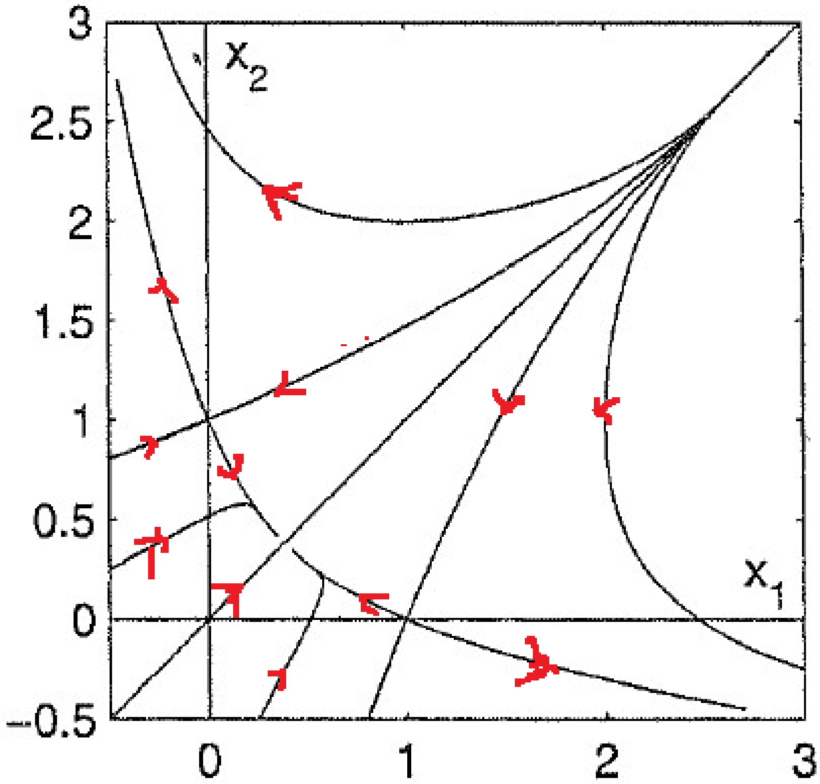

Figure 10

behaviour of the system is unstable node. The phase portrait is shown in figure 10

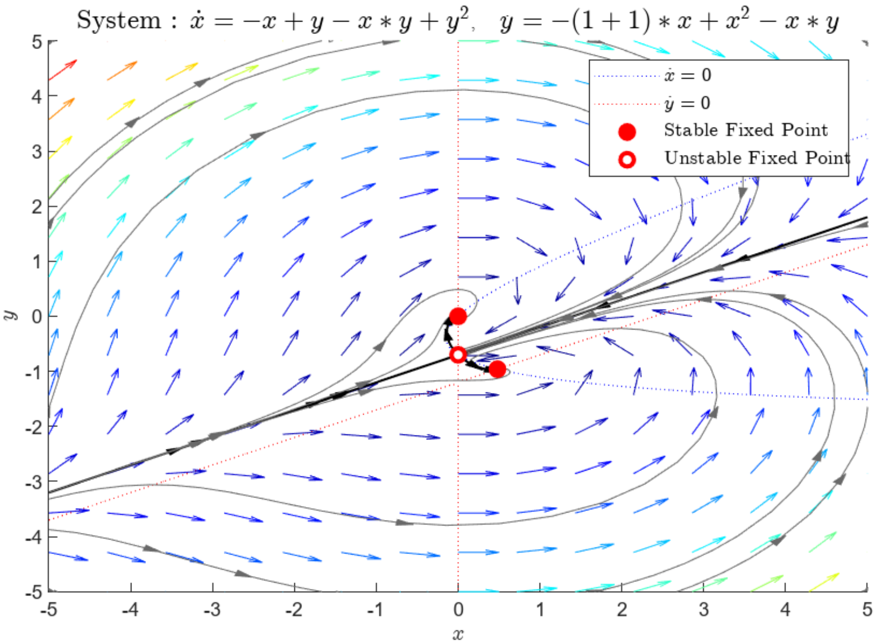

Figure 11

Problem 2.6

Mark the arrow of the phase portraits for the following systems and discuss the qualitative behavior

(a)

Find all the equilibrium points of the system

$$\dot{x_1} = -x_1 + ax_2 - bx_1x_2 + x_2^2 \quad (1)$$
$$\dot{x_2} = -(a+b)x_1 + bx_1^2 - x_1x_2 \quad (2)$$
$$(3)$$

The equilibrium points are $0, 0, \dfrac{b^2+ab}{b^2+1}$ and $0, a\ \dfrac{-(a+b)+ab}{b^2+1}$

(b)

Linearizing at equilibrium point,

at $[0, -a]$ eigenvalues are $-1, ab$, this is saddle if $b > 0$ and a stable node if $b < 0$ at $\dfrac{b^2+ab}{b^2+1}$

Eigenvalues: $-1 \pm \dfrac{\sqrt{(1-4b(a+b))}}{2}$ This is a stable focus if $4a(a+b) > 1$, a stable node if $O < 44(a+b) < 1$, and a saddle if $a(a+b) < 0$

For Equilibrium Points at $[0, 0]$, the Eigenvalues are $-\dfrac{\sqrt{(-4a^2-4ba+1)}}{2} - \frac{1}{2}$ and $-\dfrac{\sqrt{(-4a^2-4ba+1)}}{2} - \frac{1}{2}$

This is a stable focus if $4b(a+b) > 1$, a stable node if $0 < 4b(a+b) < 1$, and a saddle if $b(a+b) < 0$

(c)

when a=b=1 The phase portrait is shown in figure 11

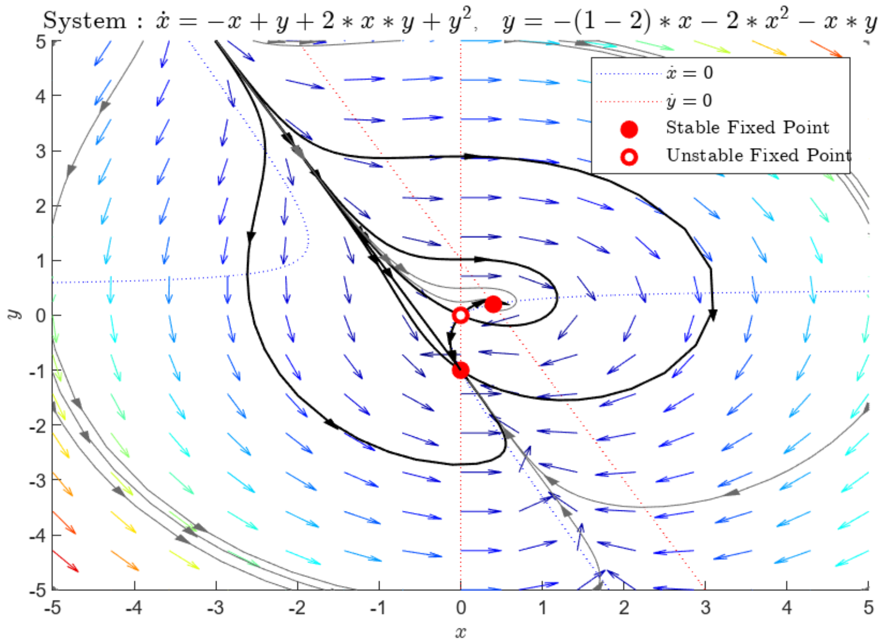

Figure 12

The equilibrium points are (0,0) (0,-1) and (1,-1). these are stable focus, saddle and stable focus respectively.

when $a = 1b = -2$ The phase portrait is given in figure 12

The equilibrium points are $(0,0)(0,-1), \dfrac{2}{5}, \dfrac{1}{5}$ These are saddle, stable node and stable focus respectively.

when $a = 1b = \frac{1}{-2}$ The phase portrait is shown in figure 13

The equilibrium points are $(0,0)(0,-1)\dfrac{-1}{5}, \dfrac{-2}{5}$ These are stable focus, stable node and saddle focus respectively

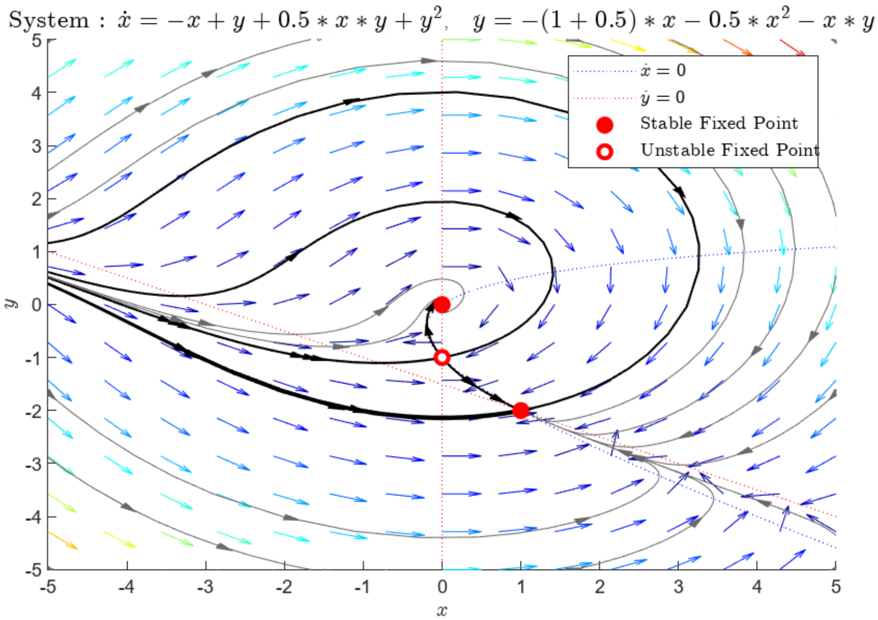

Figure 13

Problem 2.9

The cruise control of the longitudinal motion of a vehicle on a flat road can be modeled as

$$m\dot{v} = u - K_c sgn(v) - K_f v - K_a v^2$$
$$u = k_i \sigma + k_p(v_d - v)$$
$$\sigma = v_d - v,$$

(a)

using $x_1 = \sigma$ and $x_2 = v$ as state variables, find the state model of the system

$$\dot{x_2} = \frac{k_i}{m}x_i + \frac{k_p}{m}(v_d - x_2) - \frac{k_c}{m}sgn(x_2) - \frac{k_f}{m}x_2 - \frac{k_c}{m}x_2^2$$
$$\dot{x_1} = -x_2 + v_d$$

(b)

Let V_d be a positive constant, find all equilibrium points and determine the type of each point. The equilibrium point are

$$0 = -x_2 + v_d, x_d = v_d$$
$$0 = k_i x_1 - k_c sgn(k_c) - k_f x_2 - K_a x_2^2$$
$$x_1 = \frac{1}{k_i}(K_c + k_f v_d + k_a v_d^2)$$
$$x_2 = v_d$$

The eigenvalues are

$$\frac{-\frac{1}{m}(K_f + 2K_a v_d + K_p \pm \sqrt{\frac{1}{m^2}(K_f + 2K_a v_d + K_p)^2 - \frac{1}{m}4K_t)}}{2}$$

Figure 14

Figure 15

The system has stable node when $(K_f + 2K_a v_d + K_p)^2 > 4mK$ and stable focus when $(K_f + 2K_a v_d + K_p)^2 < 4mK$

(c)

the phase portrait is shown in figure 14

With the constants $m = 1500; Kc = 110; Kf = 2.5; Ka = 1; Ki = 15; Kp = 500; vd = 30;$ the system has eigenvalues at $\begin{pmatrix} -\frac{\sqrt{161}}{80} - \frac{3}{16} \\ \frac{\sqrt{161}}{80} - \frac{3}{16} \end{pmatrix}$ This is a stable node. From the phase portrait, we see that the trajectories converge to the stable node.

(d)

When $Ki = 150$ The phase plane is shown in figure 15. Here the equilibrium point is stable focus and all trajectories approach the stable focus.

(d)

When saturation is used to limit u to range $0 \leq u \leq 1800N$ The behavior of the system does not change

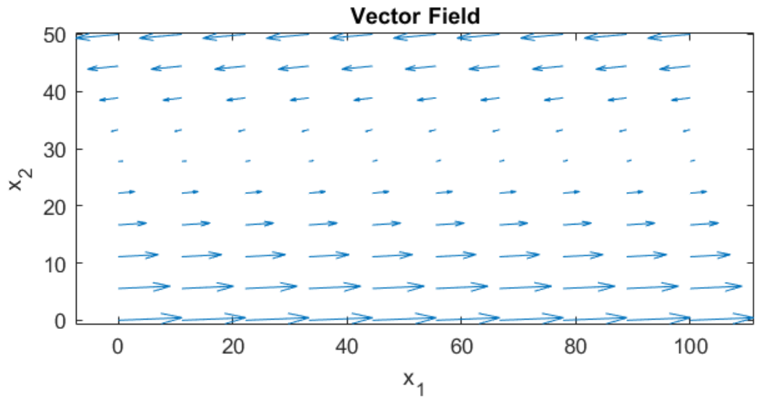

Figure 16

Problem 2.10

Consider the tunnel diode circuit, find the equilibrium points and determine the type at each point.

Solution

Here R and E are taken as $E = 0.2V$ and $R = 0.2k\omega$

$$\dot{x_1} = 0.5(-h(x_1) + x_2)$$
$$\dot{x_1} = 0.2(-x_1 - 0.2x_2 + 0.2)$$

Solving with matlab, the equilibrium points are $\begin{pmatrix} 0.057 \\ 0.32 + 0.29\,\mathrm{i} \\ 0.32 - 0.29\,\mathrm{i} \\ 1.0 + 0.34\,\mathrm{i} \\ 1.0 - 0.34\,\mathrm{i} \end{pmatrix}$ and $\begin{pmatrix} 0.72 \\ -0.6 - 1.5\,\mathrm{i} \\ -0.6 + 1.5\,\mathrm{i} \\ -4.0 - 1.7\,\mathrm{i} \\ -4.0 + 1.7\,\mathrm{i} \end{pmatrix}$ However the only real equilibrium are $\begin{pmatrix} 0.057 & 0.72 \end{pmatrix}$ with eigenvalues $\begin{pmatrix} -0.065 \\ -4.0 \end{pmatrix}$
This is a stable node.

b

the phase portrait is shown in figure 17

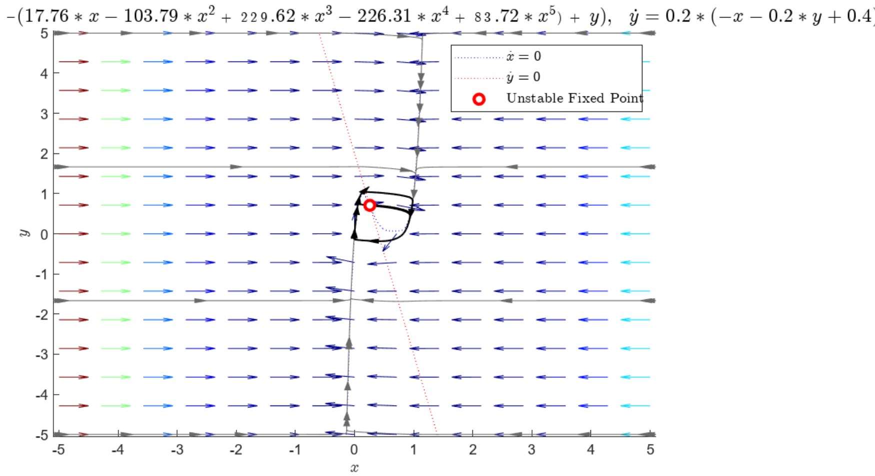

Figure 17

Problem 2.16

A prey -predatory system may be modeled as

$$\dot{x_1} = x_1(1 - x_1 - ax_2)$$
$$\dot{x_1} = bx_2(x_1 - x_2)$$

where $x_1 and x_2$ are dimensionless variables proportional to the prey and predator populations, respectively, and a and b are positive constant

a

Find all equilibrium points determine the type of each point. solving with matlab, we see that the system has 3 point equilibrium points $\begin{pmatrix} 1 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} \frac{1}{a+1} & \frac{1}{a+1} \end{pmatrix}$ with eigenvalues 0, $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, This is a saddle, and the last eigenvalue $[-(b - sqrt(b^2 - 4 * a * b - 2 * b + 1) + 1)/(2 * (a + 1)); -(b + sqrt(b^2 - 4 * a * b - 2 * b + 1) + 1)/(2 * (a + 1))]$ is a stable node if $sqrt(b^2 - 4 * a * b - 2 * b + 1) > 0$ and stable focus if $sqrt(b^2 - 4 * a * b - 2 * b + 1) < 0$

b

When $a = 1$ and $b = 0.5$ The phase portrait is shown as From the portrait, we see that trajectories on the top right corner converge to the stable focus.

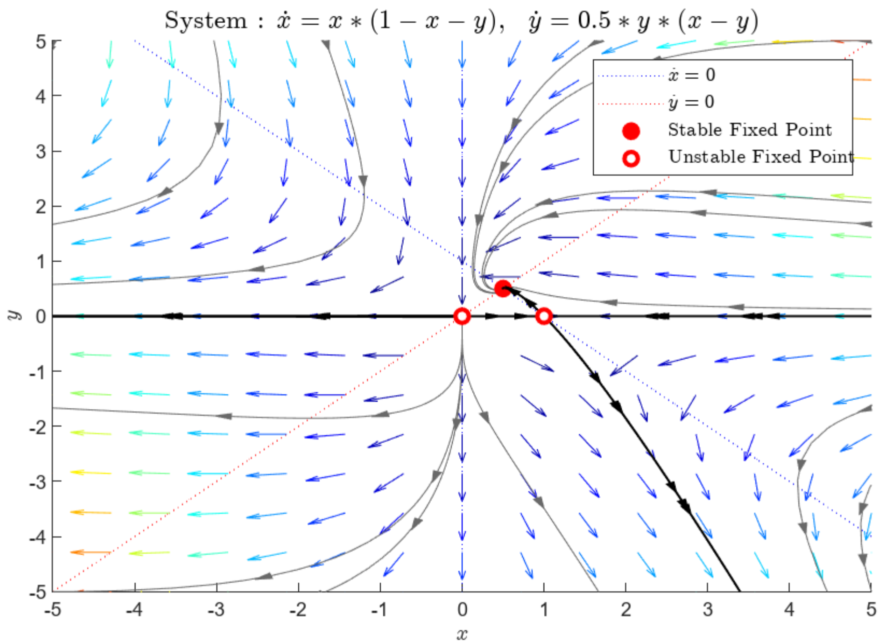

Figure 18

Problem 2.17

Use Poincare-Bendixson's criterion to show that the system has periodic orbit

Solution

Poincare-Bendixson criterion says that considering M as a closed bounded subset in a plane, when M has no equilibrium point or 1 equilibrium point, and the eigenvalues are real and positive, and also if all trajectory starting in m stays in M for all future time, then M has a periodic orbit. To use this, considering a vector field $f(x)$ and a closed curve defined by the equation $V(x) = c$, when the inner product of $f(x) and \nabla V$ is negative, the vector field point inward. When the inner product of $f(x) and \nabla V$ is positive, the vector field point outward. Trajectories can leave a set when the vector field points outward. With this basic definition, we investigate periodic orbit of the following system.

a

System 1 can be written as

$$\dot{x_1} = x_2$$

$$\dot{x_2} = \epsilon x_2(1 - x_1^2 - x_2^2) - x_1$$

This system has equilibrium at the origin. The Jacobian around the origin is $\begin{pmatrix} 0 & 1 \\ -1 & \varepsilon \end{pmatrix}$ and the eigenvalues (λ) are $\begin{pmatrix} \frac{\varepsilon}{2} - \frac{\sqrt{(\varepsilon-2)\,(\varepsilon+2)}}{2} \\ \frac{\varepsilon}{2} + \frac{\sqrt{(\varepsilon-2)\,(\varepsilon+2)}}{2} \end{pmatrix}$ This shows that the origin is unstable node or unstable focus.

$$f(x) * \nabla V(x) = 2x_1x_2 + 2\epsilon x_2^2(1 - (x_1^2 + x_2^2)) - 2x_1x_2 = 2\epsilon x_2^2(1 - c)$$

Choosing

$$V(x) = x_1^2 + x_2^2 = c$$

$$f(x) * \nabla V(x) = 2x_2^2(1 - c) \leq 0$$

Therefore, for any $c \geq 1$ The system has a periodic orbit. The vector field in figure 19 confirms this.

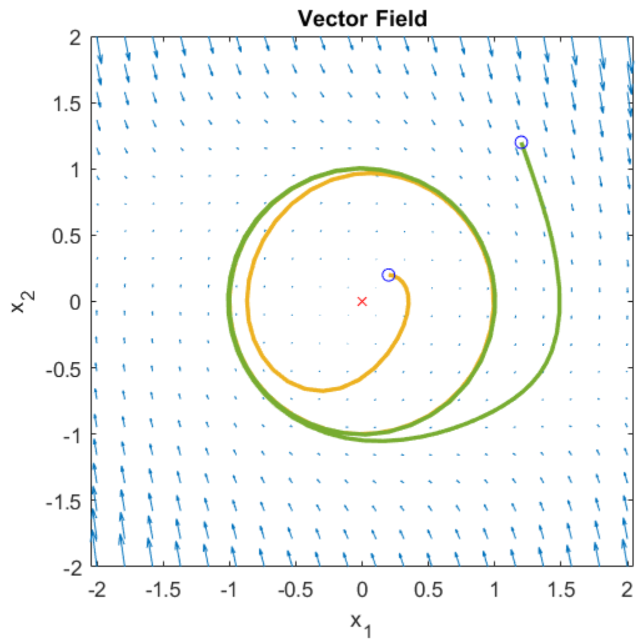

Figure 19

b

$$\dot{x_1} = x_2$$
$$\dot{x_2} = -x_1 + x_2(2 - 3x_1^2 - 2x_2^2)$$

This system has equilibrium at the origin and the Jacobian is $\begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}$ with positive real eigenvalues $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ Choosing

$$V(x) = x_1^2 + x_2^2 = c$$
$$f(x) * \nabla V(x) = 2 - 2c - x_1^2$$

Therefore, for any $c \geq 1$ The system has a periodic orbit.

c

$$\dot{x_1} = x_2$$
$$\dot{x_2} = -x_1 + x_2 - 2(x_1 + 2x_2^2)x2^2$$

This system has equilibrium at the origin and the Jacobian is $\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$ The eigenvalues are $\begin{pmatrix} \frac{1}{2} - \frac{\sqrt{3}\,\mathrm{i}}{2} \\ \frac{1}{2} + \frac{\sqrt{3}\,\mathrm{i}}{2} \end{pmatrix}$.

Choosing

$$V(x) = x_1^2 + x_2^2 = c$$
$$f(x) * \nabla V(x) = 2x_2^2(1 - 2x_1 + 4x_1^2 - 4C)$$

Therefore, for any $c \geq 1$ The system has a periodic orbit.

d

$$\dot{x_1} = x_1 + x_2 - x_1 h(x)$$
$$\dot{x_2} = -2x_2 + x_2 - x_2 h(x)$$

Where $h(x) = max|x_1|, x_2|$ This system has equilibrium at the origin and eigenvalues $(\lambda) = 1 \pm j\sqrt{2}$. The equilibrium is unstable focus. following from Example 2.8 in the textbook, Choosing

$$V(x) = x_1^2 + x_2^2$$
$$f(x) * \nabla V(x) = 2x_1(x_1 + x_2 - x_1 h(x)) + 2x_2(-2x_1 + x_2 - x_2 h(x)) = 3c - 2c^2$$

Therefore, for any $c \geq 1.5$ The system has a periodic orbit.

Problem 2.20

For each of the following systems, show that the system has no limit cycles.To confirm that there is no limit cycle, we use the Bendixson criterion to check that $\frac{\partial f_1}{\partial x_1} + \frac{\partial f_1}{\partial x_1}$ is not identically zero and does not change sign

a

$$\dot{x_1} = -x_1 + x_2$$
$$\dot{x_2} = g(x_1) + ax_2 a \neq 1$$
$$\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} = -1 + a$$

This is $\neq 0$ and does not change sign, therefore, no limit cycle exists.

b

$$\dot{x_1} = -x_1 + x_1^3 + x_1 x_2^2$$
$$\dot{x_2} = -x_2 + x_2^3 + x_1^2 x_2$$
$$\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} = 4(x_1^2 + x_2^2) - 2$$

There is no limit cycle in this case because the expression $x_1^2 + x_2^2$ is always positive, therefore with Bendixson criterion, no limit cycle exists.

c

$$\dot{x_1} = 1 - x_1 x_2^2$$
$$\dot{x_2} = x_2$$
$$\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} = -x_2^2$$

When $x_2 \neq 0$ no limit cycle exists.

d

$$\dot{x_1} = x_1 x_2$$
$$\dot{x_2} = x_2$$
$$\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} = x_2 + 1$$

when $x_2 \geq 1$ there is no limit cycle.

e

$$\dot{x_1} = x_2 cos(x_1)$$
$$\dot{x_2} = sin x_1$$
$$\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} = -x_2 sin x_1$$

when $-x_2 sin x_1 = 0$ no limit cycle exists.

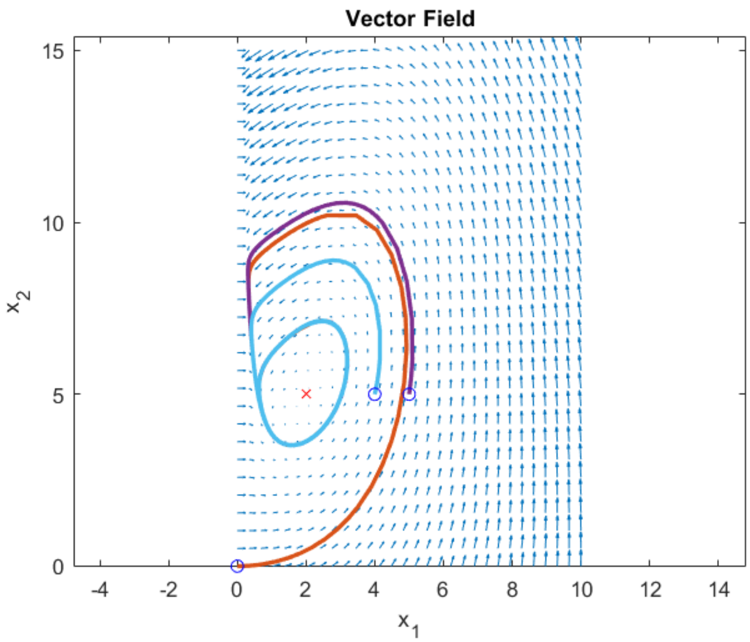

Figure 20

Problem 2.29

For the checmical oscillator model given by

$$\dot{x_1} = a - x_1 - \frac{4x_1x_2}{1+x_1^2}$$
$$\dot{x_2} = bx_1(1 - \frac{x_2}{1+x_1^2})$$

a

Using Poincare-Bendixson criterion, show that the system has a periodic orbit when $b < \frac{3a}{5} - \frac{25}{a}$ This system has equilibrium points at

$$(x_1, x_2) = (a/5, (a/5)^2 + 1)$$

Linearizing around the equilibrium point, we get $J = \frac{25}{(a/5)^2+1}\begin{pmatrix} 3\,a^2 - 125 & -20\,a \\ 2\,a^2\,b & -5\,a\,b \end{pmatrix}$ The real part of the eigenvalues can be written as $(\lambda 1, 2)3a^2 - 5ab - 125$ when $b < \frac{3a}{5} - \frac{25}{a}, (\lambda 1, 2) > 0$ The equilibrium point will be unstable node or unstable focus. The phase portrait is shown as figure 20. We can see that along the rectagular region, between points $(0, 0)$ and $(a, 1 + a^2)$ the vector fields points inward. The trajectory remain in the region therefore we can conclude that a periodic orbit exists

b

The phase portrait is given for $a = 10 and b = 2$ by figure 21. All trajectories in the first quadrant approach the limit cycle.

c

when $a = 10 and b = 4$ in the phase portrait in figure 22, all trajectories in the first quadrant approach the limit cycle.

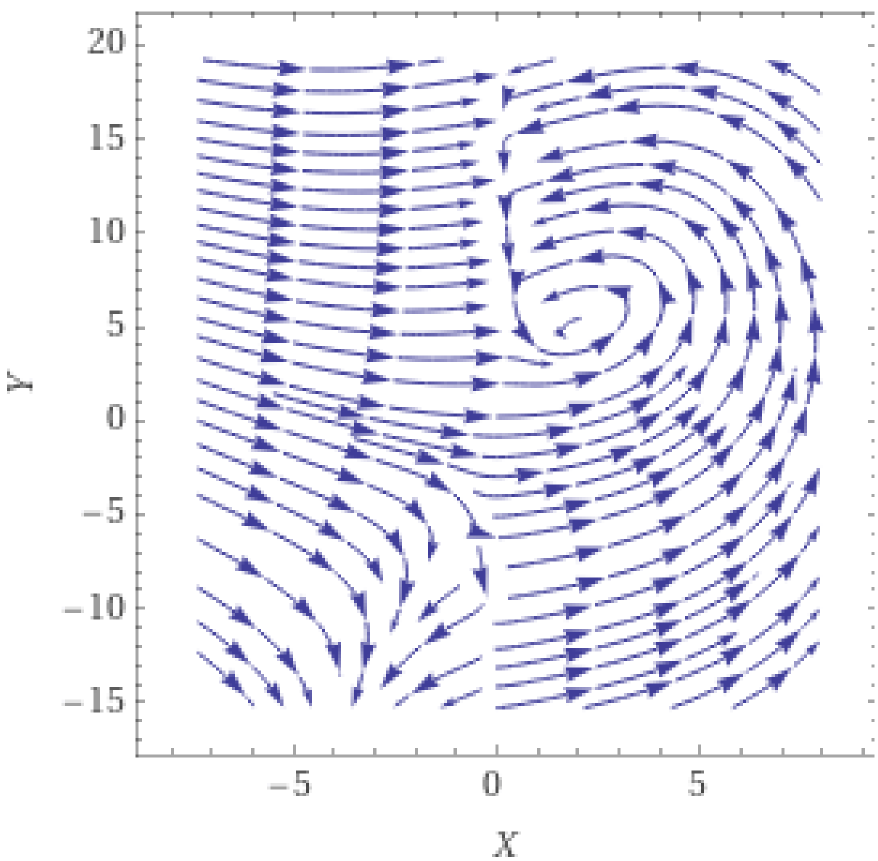

Figure 21

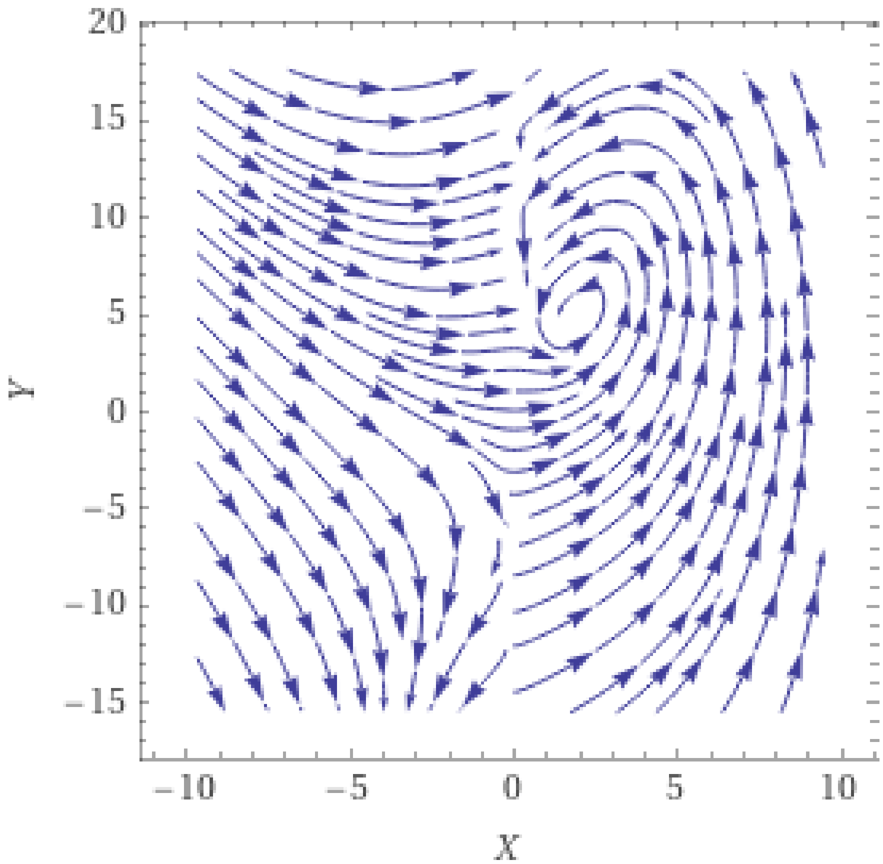

Figure 22

d

As b varies while a is fixed, the eigenvalues move from the right half plane to the left half plane. This is known as Hopf bifurcation.