E_E 505 Nonlinear System Theory
Homework 5

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Problem 4.1

Consider a second-order autonomous system. For each of the following types of equilibrium points, classify whether the equilibrium point is stable, unstable, or asymptotically stable: (1) Stable node (2) unstable node (3) Stable focus (4) Unstable focus (5) Center (6) Saddle Justify your answer using phase portrait.

Solution

Considering the 6 phase portraits in order

(1) The phase portrait of **Stable node** given in fig 1 can be seen to be asymptotically stable. We can

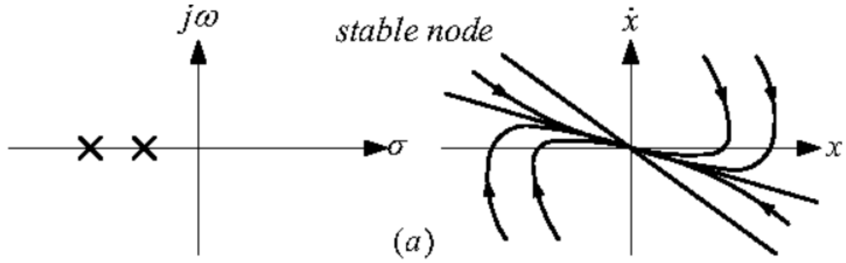

Figure 1

verify that trajectories starting close to the equilibrium is guaranteed to stay close to it and tends to it eventually

(2) The phase portrait of the **Unstable node** in fig 2 We can verify that the equilibrium point is unstable.

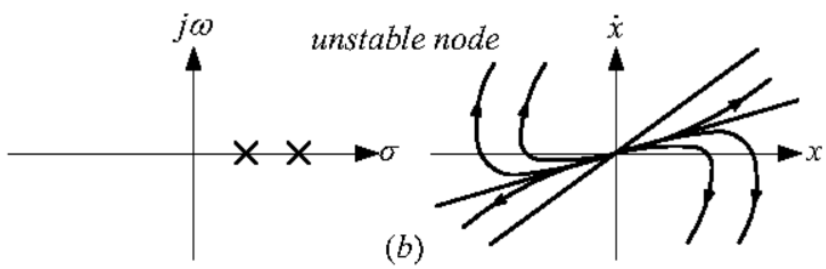

Figure 2

Trajectories starting in a ball of radius delta will escape to infinity.

(3) The phase portrait of the **Stable focus** can be seen to have asymptotically stable equilibrium. Trajectories starting at ball of radius delta, will eventually tend to zero.

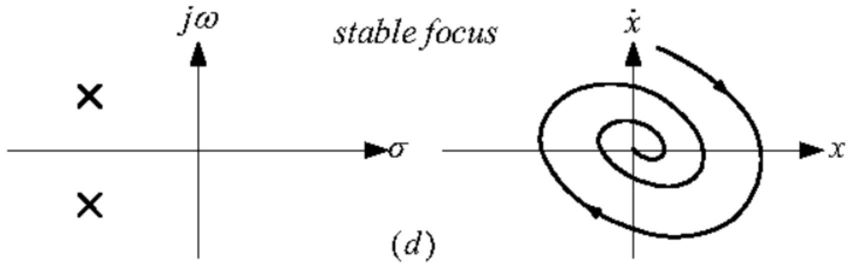

Figure 3

(4) The phase portrait of the **Unstable focus** can be seen to have unstable equilibrium. This is because

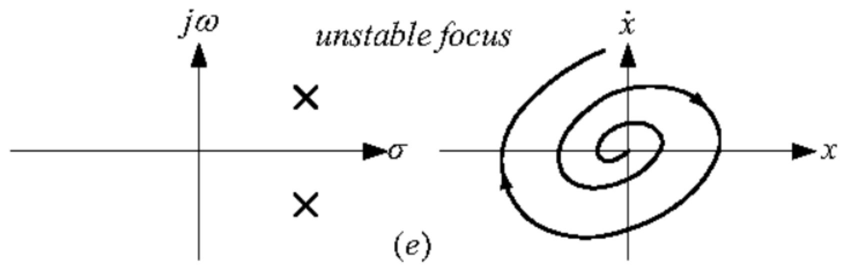

Figure 4

trajectories starting inside the ball delta does not stay in the ball epsilon, and it eventually escapes to zero.

(5) The phase portrait of **Center point** can be seen to have stable equilibrium, however it is not asymp-

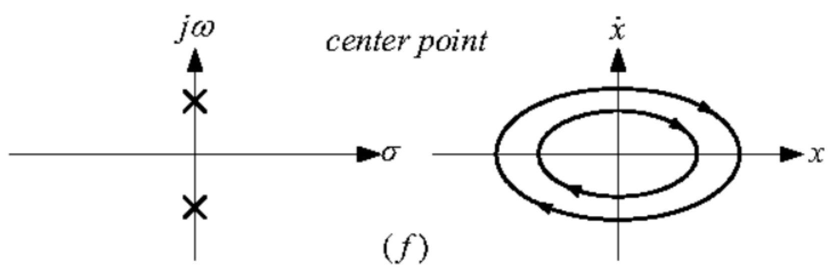

Figure 5

totically stable. This is because trajectories starting outside of a ball of radius delta tends to a limit set. They do not tend to zero

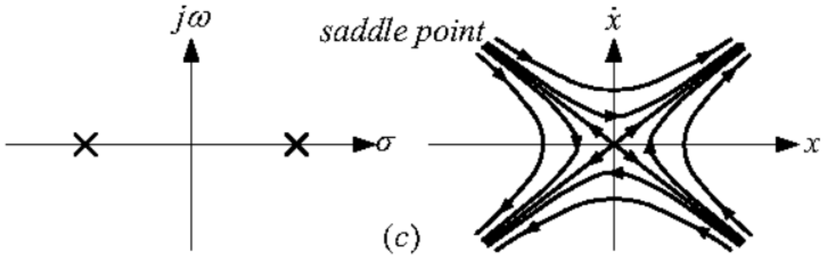

Figure 6

(6) The phase portrait of the **Saddle point** can be seen to have unstable equilibrium. We can confirm that some trajectories starting in a ball of radius delta will eventually leave.

Problem 4.2

1. Consider the scalar system $\dot{x} = ax^p + g(x)$, where p is a positive integer and $g(x)$ satisfies $|g(x)| \leq k|x|^{p+1}$ in some neighborhood of the origin x = 0. show that the origin is asymptotically stable if p is odd and a < 0. Show that it is unstable if p is odd and a > 0 or p is even and a ≠ 0

<u>Solution</u>

$$\begin{aligned} \text{Let} V(x) &= \frac{1}{p+1} x^{p+1}. \\ \dot{V(x)} = x^p \dot{x} \quad &= ax^{2P} + x^P g(x)) \\ |\dot{V}(x)| &\leq a|x|^{2P} + |x|^p |g(x)| \\ \leq a|x|^2 P + k|x|^{2P} \quad &= (a+k)|x|^{2P} \end{aligned}$$

Therefore, from the above we see that

- If p is odd and a < 0, the origin is **asymptotically stable**
  If p is odd and a > 0, the origin is **unstable**.
  If p is even and a ≠ 0, the origin is **unstable**.

Problem 4.3

For each of the following systems, use a quadratic Lyapunov function candidate to show that the origin is asymptotically stable.

(1)
$$\dot{x_1} = -x_1 + x_1x_2, \qquad \dot{x_2} = -x_2$$

(2)
$$\dot{x_1} = -x_2 - x_1(1 - x_1^2 - x_2^2), \qquad \dot{x_2} = x_1 - x_2(1 - x_1^2 - x_2^2)$$

(3)
$$\dot{x_1} = x_2(1 - x_1^2), \qquad \dot{x_2} = -(x_1 + x_2)(1 - x_1^2)$$

(4)
$$\dot{x_1} = -x_1 - x_2,, \qquad \dot{x_2} = 2x_1 - x_2^3$$

<u>Solution</u>

(1) For the system
$$\dot{x_1} = -x_1 + x_1x_2, \qquad \dot{x_2} = -x_2$$

Let $V(x_1, x_2) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2$ be a Lyapunov function candidate.

$$\begin{aligned}\dot{V} &= x_1\dot{x_1} + x_2\dot{x_2} \\ &= x_1(-x_1 + x_1x_2) + x_2(-x_2) \\ &= -x_1^2 + x_1^2x_2 - x_2^2 \\ &= -(x_1^2 - x_1^2x_2 + x_2^2) \\ &= -\frac{1}{4}(2x_1 - x_2)^2 - \frac{3}{4}x_2^2\end{aligned}$$

We see that $\dot{V}$ is negative semi-definite everywhere except at the equilibrium points. Therefore, by the Lyapunov stability theorem, we conclude that $(0, 0)$ is an **Asymptotically stable** equilibrium.

(2) For the system

$$\dot{x_1} = -x_2 - x_1(1 - x_1^2 - x_2^2), \qquad \dot{x_2} = x_1 - x_2(1 - x_1^2 - x_2^2)$$

Let $V(x) = x_1^2 + x_2^2$. be a quadratic Lyapunov function,which is positive definite for all non-zero values of x_1 and x_2.

$$\begin{aligned}\dot{V} &= 2x_1\dot{x_1} + 2x_2\dot{x_2} \\ &= -2x1^2(1 - x_1^2 - x_2^2) - 2x_2^2(1 - x_1^2 - x_2^2) \\ &= -2(1 - x_1^2 - x_2^2)(x_1^2 + x_2^2)\end{aligned}$$

We see that V(x) is positive definite, and its derivative is negative definite, we can conclude that the equilibrium point (0,0) is **Asymptotically stable.**

(3) For the system
$$\dot{x_1} = x_2(1 - x_1^2), \qquad \dot{x_2} = -(x_1 + x_2)(1 - x_1^2)$$

Let $V(x) = x_1^2 + x_2^2$. be a quadratic Lyapunov function,which is positive definite for all non-zero values of x_1 and x_2.

$$\begin{aligned}\dot{V}(x) &= 2x_1\dot{x_1} + 2x_2\dot{x_2} \\ &= 2x_1x_2(1 - x_1^2) - 2x_2(x_1 + x_2)(1 - x_1^2) \\ &= -2x_2^2(1 - x_1^2) \\ &\leq 0\end{aligned}$$

We can conclude that the equilibrium point $(0, 0)$ is **Asymptotically stable**.

(4) For the system

$$\dot{x_1} = -x_1 - x_2 \qquad \dot{x_2} = 2x_1 - x_2^3$$

Let $V(x_1, x_2) = \frac{1}{2}x_1^2 + \frac{1}{4}x_2^2$

$$\begin{aligned}\dot{V} &= x_1(-x_1 - x_2) + \frac{1}{2}x_2(2x_1 - x_2^3) \\ &= -x_1^2 - \frac{1}{2}x_2^2 - \frac{1}{2}x_2^4 \\ &= -x_1^2 - \frac{1}{2}x_2^2(1 + \frac{x_2^2}{2})\end{aligned}$$

Since x_2^2 is non-negative, we have $1 + x_2^2/2 > 1$. Therefore, $\dot{V}$ is negative definite for all non-zero values of x, because the quadratic term in x_2 is always negative. $\dot{V}$ is zero only at the origin, since that is the only equilibrium of the system, therefore the equilibrium is **Asymptotically stable**.

Problem 4.4

(151) Euler equations for a rotating rigid spacecraft are given by

$$\begin{aligned} J_1\dot{\omega}_1 &= (J_2 - J_3)\omega_2\omega_3 + u_1 \\ J_2\dot{\omega}_2 &= (J_3 - J_1)\omega_3\omega_1 + u_2 \\ J_3\dot{\omega} &= (J_1 - J_2)\omega_1\omega_2 + u_3 \end{aligned}$$

Where ω_1 to ω_3 are the components of the angular velocity ω along the principle axes, u_1 to u_3 are torque inputs applied about the principal axes, and J_1 to J_3 are the principal moments of inertia.

(a) Show that with $u_1 = u_2 = u_3 = 0$ the origin$\omega = 0$ is stable. Is it asymptotically stable?

(b) Suppose the torque inputs apply the feedback control $u_i = -k_i\omega_i$, where k_i to k_3 are positive constants. Show that the origin of the closed-loop system is globally asymptotically stable.

Solution

Let $V(\omega) = \frac{1}{2}(j_1\omega_1^2 + j_2\omega_2^2 + j_3\omega_3^2)$ be a Lyapunov function candidate
Then

$$\begin{aligned} \dot{V}(\omega) &= J_1\omega_1\dot{\omega}_1 + J_2\omega_2\dot{\omega}_2 + J_3\omega_3\dot{\omega}_3 \\ &= (J_2 - J_3)\omega_2\omega_3\omega_1 + (J_3 - J_1)\omega_3\omega_1\omega_2 + (J_1 - J_2)\omega_1\omega_2\omega_3 \\ &= 0 \end{aligned}$$

(a) The equilibrium point is zero since $\dot{V} = 0$ Therefore the equilibrium point is **Stable** but not AS

(b) Using the same Lyapunov function candidate as in part (a),

$$\begin{aligned} \dot{V}(\omega) &= \frac{d}{dt}\Big[\frac{1}{2}(J_1\omega_1^2 + J_2\omega_2^2 + J_3\omega_3^2)\Big] \\ &= J_1\omega_1\dot{\omega}_1 + J_2\omega_2\dot{\omega}_2 + J_3\omega_3\dot{\omega}_3 \\ &= J_1\omega_1[(J_2 - J_3)\omega_2\omega_3 - k_1\omega_1] + J_2\omega_2[(J_3 - J_1)\omega_3\omega_1 - k_2\omega_2] + J_3\omega_3[(J_1 - J_2)\omega_1\omega_2 - k_3\omega_3] \\ &= -(k_1J_1)\omega_1^2 - (k_2J_2)\omega_2^2 - (k_3J_3)\omega_3^2 < 0 \end{aligned}$$

we get

$$\dot{V} = -k_1\omega_1^2 - k_2\omega_2^2 - k_3\omega_3 < 0$$

Therefore the equilibrium point is globally **Asymptotically stable**

Problem 4.5

Let g(x) be a map from R^n into R^n. Show that g(x) is the gradient vector of scalar function V: $R^n \to R$ if and only if

$$\frac{\partial g_i}{\partial x_j} = \frac{\partial g_j}{\partial x_i}, \forall i.j = 1, 2..., n$$

Solution

We know that a gradient system is twice continuously differentiable. Let $g = \nabla V$, then $g_i(x) = \frac{\partial V}{\partial x_i}$

$$\frac{\partial g_i}{\partial x_j} = \frac{\partial}{\partial x_j}\frac{\partial V}{\partial x_i} = \frac{\partial^2 V}{\partial x_i \partial x_j}$$

and

$$\frac{\partial g_j}{\partial x_i} = \frac{\partial}{\partial x_i}\frac{\partial V}{\partial x_j} = \frac{\partial^2 V}{\partial x_i \partial x_j}$$

We see that $\frac{\partial g_i}{\partial x_j} = \frac{\partial g_j}{\partial x_i}$ for all $i, j = 1, 2, ..., n$ We can prove this in reverse direction:
Let V =

$$\int_0^{x_1} g_1(y_1, 0..., 0)dy_1 + \int_0^{x_2} g_1(x_1, y_2, ..., 0)dy_2 + ... + \int_0^{x_n} g_1(x_1, x_2, ...y_n,)dy_n$$

Taking a derivative with respect to x_1

$$\nabla v(x) = g_1(x_1, 0, ...0) + \int_0^{x_2} \frac{partial}{partial x_1} g_2(x_1, y_2, ..., 0)dy_2 + ... + \int_0^{x_n} \frac{partial}{partial x_1} g_2(x_1, x_2, ..., y_n)dy_n$$
$$= g_1(x_1, x_2, ..., x_n) = g_1(x)$$

We also verify that $\frac{\partial V}{\partial x_i} = \frac{\partial V}{\partial x_1} = g_i(x)$ for all $i = 1, 2, ..., n$, hence $\nabla V(x) = g(x)$.

Problem 4.6

1. Consider the system
   $\dot{x_1} = x_2,\ \ \dot{x_2} = -(x_1 + x_2) - h(x_1 + x_2)$
   Where h is continuously differentiable and zh(z) ¿ 0 for all z $\neq$ 0. Using the variable gradient method, find a Lyapunov function that shows that the origin is globally asymptotically stable.

Solution

To apply the variable gradient method, we first write the system in a matrix form:

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 - h(x_1 + x_2) & -1 - h(x_1 + x_2) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Let$\nabla V = g(x) =$

$$\begin{bmatrix} \alpha x_1 + \beta x_2 \\ \gamma x_1 + \delta x_2 \end{bmatrix}$$

and we know that $\frac{\partial g_1}{\partial x_2} = \frac{\partial g_2}{\partial x_1}$ therefore $\beta = \gamma$
We choose a Lyapunov function candidate $V(x) = \frac{1}{2}(x_1^2 + x_2^2)$

$$\begin{aligned}\dot{V}(x) = \nabla V(x)^T \dot{x} &= \begin{bmatrix} \alpha x_1 + \beta x_2 & \gamma x_1 + \delta x_2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 - h(x_1 + x_2) & -1 - h(x_1 + x_2) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= -(\alpha + \gamma)x_1^2 - (\beta + \delta + 2h(\alpha + \gamma))x_1 x_2 - (\alpha + \beta + 2h(\alpha + \gamma))x_2^2\end{aligned}$$

For $\dot{V}(x)$ to be negative definite, we want the following inequalities to hold for all $x \neq 0$:

(i) $\alpha + \gamma > 0$ (to ensure the negative definiteness of the first term)

(ii) $\alpha\delta - \beta\gamma > 0$ (to ensure the negative definiteness of the determinant of $g(x)$)

(iii) $\beta + \delta + 2h(\alpha + \gamma) > 0$ (to ensure the negative definiteness of the cross term)

(iv) $\alpha + \beta + 2h(\alpha + \gamma) > 0$ (to ensure the negative definiteness of the second term)

Let $\alpha = 1$, $\beta = -1$, $\gamma = 1$, and $\delta = 1$ to satisfy these conditions. Then $\dot{V}(x) = -2h(x_1 + x_2)^2 < 0$ for all $x \neq 0$. Therefore, the origin is globally asymptotically stable.

Problem 4.7

Consider the system $\dot{x} = -Q\Phi(x)$, where Q is a symmetric positive definite matrix and $\psi(x)$ is a continuously differentiable function for which the ithe component of ψ_i depends only on x_i, that is $\Phi_i(x) = \psi_i(x_i)$. Assume that $\psi_i(0)$ and $y\psi_i(y) > 0$ in some neighborhood of y = 0, for all $1 \leq i \leq n$

1. (a) Using the variable gradient method, find a Lyapunov function that shows that the origin is asymptotically stable.
2. (b) Under what conditions will it be globally asymptotically stable?
3. (c) Apply to the case

$$n = 2, \psi_1 = x_1 - x_1^2, \psi_2(x_2) = x_2 + x_2^3, Q = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

Solution

(a) The variable gradient method involves finding a Lyapunov function of the form $V(x) = \frac{1}{2}x^T Px$, where P is a symmetric positive definite matrix. The derivative of V along the system trajectory is given by:

$$\dot{V}(x) = x^T P\dot{x} \quad = -x^T PQ\Phi(x) < 0$$
$$= \sum_{i=1}^{n} \psi_i(x_i)x_i^T Qx_i$$

Where Q is positive definite and $\psi_i(x_i) > 0$ for $x_i \neq 0$
if $\dot{V}(x) = 0$ for some $x \neq 0$, then $\Phi(x) = 0$, also $x_i = 0$ for all i. Thus, $\dot{V}(x) = 0$ only at the origin.
Since Q is positive definite and $\dot{V}(x) < 0$ for all x in a neighborhood of the origin, except for $x = 0$. The origin is asymptotically stable.

(b) The origin is globally asymptotically stable if $\psi_i(x_i) > 0$ for all i and $x_i \neq 0$ in a neighborhood of $x = 0$.

(c) We see that $Q = Q^T > 0, \psi_i(x) = \psi_i(x_i)$, and $x_i\psi_i(x_i) > 0$ in the neighborhood of the origin. However, $x_i\psi_i(x_i) = x_1^2 - x_1^3$ is not negative definite for all possible x_1 the origin is locally asymptotically stable, but not global AS.

Problem 4.8

Consider the second-order system

$$\dot{x} = \frac{-6x_1}{u^2} + 2x_2, \quad \dot{x_2} = \frac{-2(x_1 + x_2)}{u^2}$$
$$\text{where} u = 1 + x_1^2.$$

Let

$$V(x) = \frac{x_1^2}{1 + x_1^2 + x_2^2}$$

(a) Show that $V(x) > 0$ and $\dot{V}(x) < 0$ for all $x \in R^2 - \{0\}$

(b) Consider the hyperbola $x_2 = \frac{2}{(x_1 - \sqrt{2})}$ Show by investigating the vector field on the boundary of this hyperbola, that trajectories to the right of the branch in the first quadrant cannot cross that branch

(c) Show that the origin is not globally asymptotically stable.
Hint: In part (b), show that $\frac{\dot{x_2}}{x_1} = \frac{-1}{1+2\sqrt{2}x_1+2x_1^2}$ on the hyperbola, and compare with the slope of tangents to the hyperbola.

Solution

(a) To show that $\dot{V}(x) < 0$ for all $x \in \mathbb{R}^2 - \{0\}$, we need to compute $\dot{V}(x)$ and show that it is negative.

$$\begin{aligned}
\dot{V}(x) &= \frac{\partial V}{\partial x_1}\dot{x_1} + \frac{\partial V}{\partial x_2}\dot{x_2} \\
&= \frac{2x_1}{(1 + x_1^2 + x_2^2)^2}\left(\frac{-6x_1}{u^2}\right) + \frac{2x_2}{(1 + x_1^2 + x_2^2)^2}\left(\frac{-2(x_1 + x_2)}{u^2}\right)
\end{aligned}$$

Next, we can substitute the expression for u to obtain

$$\begin{aligned}
\dot{V}(x) &= \frac{-12x_1^2}{(1 + x_1^2)^2(1 + x_1^2 + x_2^2)} - \frac{4x_2(x_1 + x_2)}{(1 + x_1^2)^2(1 + x_1^2 + x_2^2)} \\
&= -\frac{4}{(1 + x_1^2)^2(1 + x_1^2 + x_2^2)}\left(3x_1^2 + x_2(x_1 + x_2)\right).
\end{aligned}$$

For $\dot{V}(x) < 0$ for all $x \in \mathbb{R}^2 - \{0\}, x_1$ and x_2 must have the opposite sign.

(b) To investigate the behavior of trajectories on the hyperbola
$x_2 = \frac{2}{(x_1 - \sqrt{2})}$, we can substitute $x_2 = \frac{2}{(x_1 - \sqrt{2})}$ into the differential equations and simplify as follows:

$$\begin{aligned}
\dot{x_1} &= \frac{-6x_1}{u^2} + 2\left(\frac{2}{x_1 - \sqrt{2}}\right) = \frac{-6x_1}{1 + x_1^2)^2} + \frac{4(x_1 + \sqrt{2})}{(x_1 - \sqrt{2})^2(1 + x_1^2)} < 0 \\
\dot{x_2} &= \frac{-2(x_1 + x_2)}{u^2} = \frac{-2(x_1 + \frac{2}{(x_1 - \sqrt{2})})}{(1 + x_1^2)^2} \\
&= \frac{-2(x_1 + \sqrt{2})}{(1 + x_1^2)^2} < 0.
\end{aligned}$$

Therefore, $\dot{V}(x) < 0$ for all x on the hyperbola, since $\dot{V}(x) = \frac{\partial V}{\partial x_1}\dot{x_1} + \frac{\partial V}{\partial x_2}\dot{x_2}$, and both terms in this expression are negative on the hyperbola. This shows that trajectories starting to the right of the hyperbola in the first quadrant cannot cross that branch, since $\dot{V}(x)$ is negative along the trajectories and decreases as they approach the hyperbola.

(c) The origin is not globally **Asymptotically Stable** as trajectories to the right of the parabola can never reach the origin.

Problem 4.9

In checking radial unboundedness of a positive definite function $V(x)$, it may appear that it is sufficient to examine $V(x)$ as $\|x\| \to \infty$ along the principal axes. This is not true, as shown by the function

$$V(x) = \frac{(x_1 + x_2)^2}{1 + (x_1 + x_2)^2} + (x_1 - x_2)^2$$

(a) show that $V(x)$ as $\|x\| \to \infty$ along the lines $x_1 = 0 or x_2 = 0$

(b) Show that $V(x)$ is not radially unbounded

Solution

(a) For

$$V = \frac{x_2^2}{1 + x_2^2} + x_2^2$$

when $x_1 = 0, \|x\| \to \infty$
For

$$V = \frac{x_2^2}{1 + x_1^2} + x_1^2$$

when $x_2 = 0, \|x\| \to \infty$

(b) For

$$V = \frac{2\gamma^2}{1 + (2\gamma)^2}$$

When $x_1 = x_2 = \gamma, \|x\| \to \infty$ and $V \to 1$. Therefore, V is not radially bounded.

Problem 4.10

(Krasovskii's Method) Consider the system $\dot{x} = f(x)$ with $f(0) = 0$. Assume that $f(x)$ is continuously differentiable and its Jacobian $[\frac{\partial f}{\partial x}]$ satisfies

$$P\left[\frac{\partial f}{\partial x}(x)\right] + \left[\frac{\partial f}{\partial x}(x)\right]^T P \leq -I, \forall x \in R^n, where P = p^T > 0$$

(a) Using the representation $f(x) = \int_0^1 \frac{\partial f}{\partial x}(\sigma x) x d\sigma$ Show that
$x^T P f(x) + f^T(x) P x \leq -x^T x, \ \ \forall x \in R^n$

(b) Show that V(x) = $f^T(x)Pf(x)$ is positive definite for all $x \in R^n$ and radially unbounded.

(c) Show that the origin is globally asymptotically stable.

Solution

(a)

$$\begin{aligned} x^T P f(x) + f^T(x) P x =& x^T P \int_0^1 \frac{\partial f}{\partial x}(\sigma x) x d\sigma + \int_0^1 x^T \frac{\partial f^T}{\partial x}(\sigma x) d\sigma P x \\ &= x^T \int_0^1 \left(P\frac{\partial f}{\partial x}(\sigma x) + \frac{\partial f^T}{\partial x}(\sigma x) P \right) d\sigma x \\ &\left(P\frac{\partial f}{\partial x}(\sigma x) + \frac{\partial f^T}{\partial x}(\sigma x) P \right) \leq -I \leq -x^T x \end{aligned}$$

(b) To show that $V(x)$ is positive definite for all $x \in \mathbb{R}^n$, let $x \neq 0$ and consider

$$V(x) = f^T(x)Pf(x) \ \text{ and } \ \|f(x)\|_2^2 > 0,$$

since P is positive definite and $f(x) \neq 0$. Let $V(0) = f^T(0)Pf(0) = 0$, and $V(x)$ be a continuous function, so $V(x)$ is positive definite for all $x \in \mathbb{R}^n$.

For $V(x)$ to be radially unbounded, let $|x| \to \infty$. Since $f(x)$ is continuous and $f(0) = 0$, there exists $r > 0$ such that $\|f(x)\| \geq r$ for all $|x| \geq 1$. Then

$$V(x) = f^T(x)Pf(x) \ \ = \|f(x)\|_2^2 \geq r^2 \|P\|_2,$$

as $\|x\| \to \infty, V(x)$ is radially unbounded

(c) Taking the derivative of $V(x)$

$$\begin{aligned} &\dot{V}(x) = 2f^T(x)Pf'(x) = 2f^T(x)Pf'(x) + 2f^T(x)f'(x)^T Pf(x) \\ &= 2f^T(x)Pf'(x) + f'(x)^T Pf(x) \\ &< -2f^T(x)f(x) < 0 \forall x \in R^n \end{aligned}$$

This shows that the origin is globally asymptotically stable