E_E 505 Nonlinear System Theory
Homework 6

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Tuesday 18th April, 2023

Problem 4.11

Using theorem 4.3, prove Lyapunov's first instability theorem:
For the system (4.1), if a continuously differentiable function $V_1(x)$ can be found in a neighborhood of the origin such that $V_1(0) = 0$, and $\dot{V}_1$ along trajectories of the system is positive definite, but V_1 itself is not negative definite or negative semidefinite arbitrarily near the origin, then the origin is unstable.

Solution

Theorem 4.3 states that If x = 0 is an equilibrium point for system (4.1) and $V : D \to R$ is C' such that V(0) = 0 and $V(x_0) > 0$ for some x_0 close to the origin, then $\dot{V}_(x) > 0$ in a set U, therefore x = 0 is unstable.

Therefore by this theorem, since V_1 itself is not negative definite or semidefinite and $\dot{V}_1 > 0$ and $\dot{V}_1(x)$ has a minimum over the compact set $\{x \in U\}$ and $\{V_1(x) \leq a\}$. Since $x(t)$ cannot stay forever in U because $V(x)$ is bounded on U and x_0 is in the neighborhood of the origin, then the origin is unstable.

Problem 4.12

Using Theorem 4.3, Prove Lyapunov's second instability theorem:
For the system (4.1), if in a neighborhood D of the origin, a continuously differentiable function $V_1(x)$ exists such that $V_1(0) = 0$ and $\dot{V}_1$ along the trajectories of the system is of the form $\dot{V}_1 = \lambda V_1 + W(x)$ where $\Lambda > 0$ and $W(x) \geq 0$ in D, and if $V_1(x)$ is not negative definite or negative semidefinite arbitrarily near the origin, then the origin is unstable.

Solution

By the theorem (4.3), $V_(0) = 0, V_(x_0) > 0$ for some x_0 in the neigborhood of the origin, Let $V = V_1$, then for all x inside the set U where $U = x \in B_r | V_1(x) \geq 0$,

Then $\dot{V}_1 = \lambda V_1 + W_{(x)} > 0$, as $t \to \infty, x(t)$ will eventually leave U, therefore the origin is unstable.

Problem 4.13

For each of the following systems, show that the origin is unstable:

(1)
$$\dot{x_1} = x_1^3 + x_1^2 x_2, \quad \dot{x_2} = -x_2 + x_2^2 + x_1 x_2 - x_1^3$$

(2)
$$\dot{x}_1 = -x_1^3 + x_2, \quad \dot{2} = x_1^6 - x_2^3$$

Hint: In part (2), show that $\Gamma = 0 \le x_1 \le 1 \cap x_2 \ge x_1^3 \cap x_2 \le x_1^2$ is a nonempty positively invariant set, investigate the behavior of the trajectories inside Γ

Solution

(1) Choose V function $V(x_1, x_2) = \frac{1}{2}(x_1^2 + x_2^2)$,

$$\begin{aligned}
\dot{V} &= x_1 \dot{x_1} + x_2 \dot{x_2} \\
&= x_1(x_1^3 + x_1^2 x_2) + x_2(-x_2 + x_2^2 + x_1 x_2 - x_1^3) \\
&= x_1^4 - x_1^2 x_2 + x_2^2(x_1 - 1) - x_1^3 \\
&= x_1^2(x_1^2 - x_2 - x_1) + x_2^2(x_1 - 1)
\end{aligned}$$

Choose a point such that $\dot{V} > 0$, i.e. $(x_1, x_2) = (1, \epsilon)$, for some small $\epsilon > 0$. We see that $\dot{V} > 0$ for this point, since:

$$\begin{aligned}
\dot{V}(1, \epsilon) &= (1)^2((1)^2 - \epsilon - 1) + (\epsilon)^2(1 - 1) \\
&= \epsilon(1 - \epsilon) > 0
\end{aligned}$$

Therefore, by Chetaev's theorem, the origin of the system is unstable.

(2) Choose V function $V(x_1, x_2) = \frac{1}{2}(x_1^2 + x_2^2)$,

$$\begin{aligned}
\dot{V} &= x_1 \dot{x_1} + x_2 \dot{x_2} \\
&= x_1(-x_1^3 + x_2) + x_2((x_1^2)^3 - x_2^3)
\end{aligned}$$

Since $\Gamma = 0 \le x_1 \le 1 \cap x_2 \ge x_1^3 \cap x_2 \le x_1^2$ is a nonempty positively invariant set, trajectories cannot leave this set and all trajectories move to the equilibrium point, away from the origin. Therefore, the origin is unstable.

Problem 4.14

Consider the system

$$\dot{x_1} = x_2, \;\; \dot{x}_2 = -g(x_1)(x_1 + x_2)$$

Where g is locally Lipschitz and g(y) ≥ 1 for all $y \in R$. Verify that $V(x) = \int_0^{x_1} yg(y)dy + x_1x_2 + x_2^2$ is positive definite for all $x \in R^2$ and radially unbounded, and use it to show that the equilibrium point $x = 0$ is globally asymptotically stable.

Solution

To show that $V(x)$ is positive definite, we need to show that $V(x) > 0$. First, we have $V(0) = \int_0^0 yg(y)dy + 0 \cdot 0 + 0 = 0$ So $V(0) = 0$. When $x \neq 0$, $x_1 > 0$ and $x_2 \geq 0$, so $x_1x_2 + x_2^2 > 0$. Also, since $g(y) \geq 1$ for all $y \in R$, we have $\int_0^{x_1} yg(y)dy \geq \int_0^{x_1} ydy = \frac{1}{2}x_1^2 > 0$. Therefore, $V(x) > 0$ for all $x \neq 0$.

To show that the equilibrium point $x = 0$ is GAS, $V(0) = 0$. for $x \neq 0$,

$$\begin{aligned} V(x) &= \int_0^{x_1} yg(y)dy + x_1x_2 + x_2^2 \\ \dot{V}(x) &= x_2 \cdot x_2 + (x_1 + x_2)\left(-g(x_1)(x_1 + x_2)\right) + 2x_2 \cdot (-g(x_1)(x_1 + x_2)) \\ &= -g(x_1)(x_1 + x_2)^2 \leq 0 \end{aligned}$$

Since $g(y) \geq 1$ for all $y \in R$, we have $\dot{V}(x) \leq -(x_1 + x_2)^2 \leq 0$. Therefore, $\dot{V}(x)$ is negative definite for all $x \neq 0$, and so the equilibrium point $x = 0$ is globally asymptotically stable.

Problem 4.15

Consider the system

$$\dot{x_1} = x_2, \;\; \dot{x_2} = -h_1(x_1) - x_2 - h_2(x_3), \;\; \dot{x_3} = x_2 - x_3$$

where h_1 and h_2 are locally Lipschitz functions that satisfy $h_i(0) = 0$ and $yh_i(y) > 0$ for all $y \neq 0$

(a) Show that system has a unique equilibrium point at the origin.

(b) Show that $V(x) = \int_0^{x_1} h_1(y)dy + x_2^2/2 + \int_0^{x_3} h_2(y)dy$ is positive definite for all $x \in R^3$.

(c) Show that the origin is asymptotically stable.

(d) Under what condition on h_1 and h_2 can you show that the origin is globally asymptotically stable?

Solution

(a) When $\dot{x}_1 = 0$, $\dot{x}_2 = 0$ and $\dot{x}_3 = 0$, Then, $x_2 = 0, -h_1(x_1) - x_2 - h_2(x_3) = 0$, and $x_2 - x_3 = 0$. Therefore, $x_2 = 0, x_3 = 0, h_1(x_1) = 0, x_1 = 0$The system has only 1 unique equilibrium point at the origin.

(b) We know that $V(0) = 0$ and $V(x) > 0$ for all $x \neq 0$.
For $x_1 \neq 0$. Since $h_1(y) > 0$ for $y \neq 0$, we have $\int_0^{x_1} h_1(y)dy > 0$. Also, $x_2^2/2 \geq 0$ and $\int_0^{x_3} h_2(y)dy > 0$ (since $h_2(y) > 0$ for $y \neq 0$). Therefore, $V(x) > 0$ for $x_1 \neq 0$.

For $x_1 = 0$ and $x_3 \neq 0$. Since $h_2(y) > 0$ for $y \neq 0$, we have $\int_0^{x_3} h_2(y)dy > 0$. Also, $x_2^2/2 \geq 0$. Therefore, $V(x) > 0$ for $x_1 = 0$ and $x_3 \neq 0$.

For $x_1 = x_3 = 0$ and $x_2 \neq 0$. Since $x_2^2/2 > 0$, we have $V(x) > 0$ for $x_1 = x_3 = 0$ and $x_2 \neq 0$.

Therefore, $V(x) > 0$ for all $x \neq 0$, and $V(0) = 0$, and $V(x)$ is positive definite for all $x \in \mathbb{R}^3$.

(c) To show that the origin is asymptotically stable, we need to show that it is Lyapunov stable and that all solutions converge to the origin. Let $V(x) = \frac{1}{2}x_2^2 + \frac{1}{2}x_3^2$ be a Lyapunov function candidate.

Taking the derivative of V along the trajectories of the system gives:

$$\begin{aligned}
\dot{V} &= x_2\dot{x_2} + x_3\dot{x_3} \\
&= -x_2^2 - h_1(x_1)x_2 - h_2(x_3)x_2 + x_2x_3 - x_3^2 \\
&= -x_2^2 - h_1(x_1)x_2 - h_2(x_3)x_2 - x_3^2 + x_2x_3 - \frac{1}{2}(x_3^2 - 2x_2x_3) \\
&= -x_2^2 - h_1(x_1)x_2 - h_2(x_3)x_2 - x_3^2 + \frac{1}{2}(x_2 - x_3)^2 \\
&\leq 0
\end{aligned}$$

$\dot{V_(}x) = 0$ for all $x, y \in \mathbb{R}$ therefore, x = 0 is **Asymptotically stable.**

(d) For the system to be globally asymptotically stable, $V_(x)$ must be radially unbounded. *i.e* $\int_0^y h_i(z)dz \rightarrow \infty$ as $|y| \rightarrow \infty$ for i= 1, 2.

Problem 4.16

Show that the origin of

$$\dot{x}_1 = x_2, \ \ \dot{x}_2 = -x_1^3 - x_2^3$$

is globally asymptotically stable.

Solution

Let $V(x) = \frac{1}{2}(x_1^2 + x_2^2)$. Then,

$$\begin{aligned}\dot{V}(x) &= x_1 x_2 + x_2(-x_1^3 - x_2^3) \\ &= -x_2^4\end{aligned}$$

We see that $\dot{x}_2 = 0$ only if $x_1 = x_2 = 0$,Therefore, the origin of the given system is **Globally Asymptotically Stable.**

Problem 4.17

Consider Lienard's equation

$$\ddot{y} + h(y)\dot{y} + g(y) = 0$$

where g and h are continuously differentiable.

(a) Using $x_1 = y$ and $x_2 = \dot{y}$, Write that state equation and find conditions on g and h to ensure that the origin is an isolated equilibrium point.

(b) Using $V(x) = \int_0^{x_1} g(y)dy + (\frac{1}{2}x_2^2)$ as Lyapunov function candidate, find conditions on g and h to ensure that the origin is asymptotically stable.

(c) Repeat part (b) using $V(x) = (\frac{1}{2})[x_2 + \int_0^{x_1} h(y)dy]^2 + \int_0^{x_1} g(y)dy$

Solution

(a) The state equation for Lienard's equation can be written as

$$\begin{aligned}\dot{x_1} &= x_2\\ \dot{x_2} &= -h(x_1)x_2 - g(x_1)\end{aligned}$$

where $x_1 = y$ and $x_2 = \dot{y}$.

To determine the conditions on g and h that ensure the origin is an isolated equilibrium point, we need to examine the stability of the system near the origin.

First, let's find the equilibrium points of the system by setting $\dot{x_1} = \dot{x_2} = 0$. From $\dot{x_2} = 0$, we have

$$-h(x_1)x_2 - g(x_1) = 0$$

Since $x_2 = \dot{x_1} = 0$, we have $g(x_1) = 0$. Therefore, the equilibrium points are given by x_1 such that $g(x_1) = 0$.
Therefore, the conditions on g and h that ensure the origin is an isolated equilibrium point is for $g(x_1) = 0$ to have isolated root at the origin.

(b) $V(0) = 0$
Next, we compute $\dot{V}(x)$.

$$\begin{aligned}\dot{V}(x) &= g(x_1)x_2 - h(x_1)x_2^2 - x_2 g(x_1)\\ &= -h(x_1)x_2^2\\ &= -\dot{y}h(y)\dot{y}\end{aligned}$$

$\dot{V}(x)$ is negative definite if $h(y)$ is positive for all y. Therefore, if $h(y) > 0$ for all y, then the origin is asymptotically stable.

(c)

$$\begin{aligned}V(x) &= (\frac{1}{2})[x_2 + \int_0^{x_1} h(y)dy]^2 + \int_0^{x_1} g(y)dy\\ \dot{V}(x) &= \left[x_2 + \int_0^{x_1} h(y)d_y\right][\dot{x}_2 + h(x_1)\dot{x}_1] + \dot{x}_1 g(x_1)\\ \dot{V}(x) &= -g(x_1)\int_0^{x_2} h(y)d_y\end{aligned}$$

Since $\dot{V} \leq 0$, $g(x) = 0, x_1 = x_2 = 0$
By LaSSalle's theorem, the origin is asymptotically stable.

Problem 4.18

The mass - spring system of Exercise 1.12 is modelled by

$$M\ddot{y} = Mg - ky - c_1\dot{y}v - c_2\dot{y}|\dot{y}|$$

Show that the system has a globally asymptotically stable equilibrium point.

Solution

The state equation is

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= g - (k/m)x_1 - (c_1/m)x_2 - (c_2/m)x_2|x_2|\end{aligned}$$

Choose $V(x) = 0.5x_1^2 + 0.5x_2^2$

$$\begin{aligned}\dot{V}(x) &= x_1\ddot{x}_1 + x_2\ddot{x}_2\\ &= -c_1x_2^2 - c_2x_2^2|x_2| \leq 0\end{aligned}$$

Since $\dot{V} \leq 0$ therefore the origin is globally Asymptotically stable

Problem 4.19

Consider the equations of motion of m-link robot, described in Exercise 1.4. Assume that $P_(q)$ is a positive definite function of q and g(q) = 0 has an isolated root at q = 0

(a) With u = 0, use the total energy $V(q, \dot{q}) = \frac{1}{2}\dot{q}^T M(q)\dot{q} + P(q)$ as a Lyapunov function candidate to show that the origin (q = 0, $\dot{q}$ = 0) is stable.

(b) with $u = -K_d\dot{q}$ where K_d is a positive diagonal matrix, show that the origin is asymptotically stable.

(c) With u =$g(q) - K_p(q - q^*) - K_d\dot{d}$ where K_p and K_d are positive diagonal matrices and q^* is a desired robot position in R^m,Show that the point$(q = q^*, q = 0)$ is an asymptotically stable equilibrium point.

Solution

The system can be written as

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + D\dot{q} + g = u$$

$$\ddot{q} = M^{-1}(u - C\dot{q} - D\dot{q} - q)$$

(1) When U = 0, using the Lyapunov candidate function $V(q, \dot{q}) = \frac{1}{2}\dot{q}^T M(q)\dot{q} + P(q)$

$$\begin{aligned}\dot{V}(q, \dot{q}) &= \frac{d}{dt}\left(\frac{1}{2}\dot{q}^T M(q)\dot{q} + P(q)\right) \\ &= \frac{1}{2}\dot{q}^T M(q)\ddot{q} + \frac{dP(q)}{dq}\dot{q} \\ &= \frac{1}{2}\dot{q}^T(M - 2C\dot{q})\dot{q} - \dot{q}^T D\dot{q} \leq \dot{q}^T D\dot{q} \leq 0\end{aligned}$$

Therefore the origin is stable

(2) When $u = -K_d\dot{q}$,

$$\ddot{q} = M^{-1}(-K_d\dot{q} - C\dot{q} - D\dot{q} - q)$$

using the Lyapunov candidate function $V(q, \dot{q}) = \frac{1}{2}\dot{q}^T M(q)\dot{q} + P(q)$

$$\dot{V} = -\dot{q}^T K_d\dot{q} - \dot{q}^T Dq$$

which is < 0 Therefore, the origin is **Asymptotically stable**

(3) When u =$g(q) - K_p(q - q^*) - K_d\dot{d}$

$$\begin{aligned}M(q)\ddot{q} + C(q, \dot{q})\dot{q} + D\dot{q} + g &= g(q) - K_p(q - q^*) - K_d\dot{d} \\ M(q)\ddot{q} + C(q, \dot{q})\dot{q} + D\dot{q} + K_p(q - q^*) + K_d\dot{d} &= 0\end{aligned}$$

Let y =$q - q^*$, $\dot{y} = \dot{q}$, and $\ddot{y} = \ddot{q}$ then

$$M\ddot{y} + C\dot{y} + D\dot{y} + K_p y + K_d\dot{y} = 0$$

We can choose the Lyapunov candidate $V = \frac{1}{2}M\dot{y}^2 + \frac{1}{2}K_p y^2$.

$$\begin{aligned}\dot{V} &= -(\dot{y})^2(C + D) - K_p y^2 \\ &\leq -C(\dot{y})^2 - K_p y^2 < 0\end{aligned}$$

Therefore, the origin $(\dot{q}, 0)$ is **Asymptotically Stable.**

Problem 4.20

Suppose the set M in LaSalle's theorem consist of a finite number of isolated points. Show that $lim_{t \to \infty} x(t)$ exists and equals one of these points

Solution

By LaSSalle's theorem, Ω is a positively invariant bounded and compact set that contains M, trajectories $x(t)$ is a solution in ω approaches M as $T \to \infty$. Since $\dot{V}(x) \leq 0$ in ω, $V(x)$ is continuous in ω, $V(x(t))$ is a decreasing function of t and bounded from below on ω. By these facts, $V(x(t))$ has a limit as $t \to \infty$. Considering a positive set L^+ in ω, for any $p \in L^+$ there is a sequence t_n with $t_n \to \infty$ and $x(t_n) \to p$ as $n \to \infty$. In other words $L^+ \subset M \subset E \subset \omega$. Since x(t) is bounded, x(t) approaches L^+ as $t \to \infty$. Hence, x(t) approaches M as $t \to \infty$

Problem 4.21

A gradient system is a dynamical system of the form $\dot{x} = -\nabla V(x)$, where $\nabla V(x) = [\frac{\partial V}{\partial x}]^T$ and V: $D \complement R^n \to R$ is twice continuously differentiable.

(a) Show that $\dot{V}(x) \leq 0$ for all $x \in D$, and $\dot{V}(x) =$ if and only if x is an equillibrium point

(b) Take $D = R^n$. Suppose that set $\Omega_c = x \in R^n | V_(x) \leq c$ is compact for every $c \in R$. Show that every solution of the system us defined for all $t \geq 0$

(c) Continuing with part(b), suppose $\nabla V(x) \leq 0$, except for a finite number of points $p_1, ..., p_r$ Show that for every solution $x(t), limt_{t \to \infty} x(t)$ exists and equals one of the points $p_1, ..., p_r$

Solution

(a) Taking the derivative of V

$$\dot{V_(}x) = \frac{\partial V}{\partial x} \frac{\partial x}{\partial t}$$
$$= \nabla V(x)^T \dot{x} = -\nabla V^T \nabla V \leq 0$$

$-\nabla V(x)^T \nabla V(x) \leq 0$ for all $\forall x \in D$ and $\dot{V}(x) = 0$ if and only if $\nabla V(x) = 0$.

If $\nabla V(x) = 0$, then $\dot{x} = 0$, which means that x is an equilibrium point. Conversely, if x is an equilibrium point, then $\dot{x} = 0$, which implies that $\nabla V(x) = 0$. Therefore, we have shown that $\dot{V}(x) = 0$ if and only if x is an equilibrium point.

(b) Since $\dot{V} \leq$ we see that Ω_c is positively invariant and bounded. therefore by theorem 3.3, a unique solution exists $\forall t \geq 0$

(c) From the proof in 4:20, Since Ω_c is an invariant set, we see that points exists and trajectories will approach one of the limit points as $t \to \infty$