E_E 505 Nonlinear System Theory
Homework 3

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Problem 2.30

$$\dot{x_1} = (\frac{\mu x_2}{k_m + x_2} - d) - x_1$$
$$\dot{x_2} = d(x_2 f - x_2) - (\frac{\mu x_1 x_2}{Y(k_m + x_2)})$$

let $\mu_m = 0.5, k_m = 0.1, Y = 0.4 and x_2 f = 4$

Solution

(a) Find all equilibrium points for $d > 0$ and determine the type of each point. The system has equilibrium points at $(0, 4), (\frac{82d-40}{50d-25}, \frac{d}{5-10d})$ and eigenvalues at $\begin{pmatrix} \frac{20}{41} - d \\ -d \end{pmatrix}$ and $\begin{pmatrix} -d \\ -82\,d^2 + 81\,d - 20 \end{pmatrix}$

(b) The system is stable when ($d < \frac{20}{41}$) and ($d > 0.5$); and unstable when ($d >= \frac{20}{41}$) and ($d <= 0.5$); The distance of second equilibrium point to the origin in shown in figure 1.

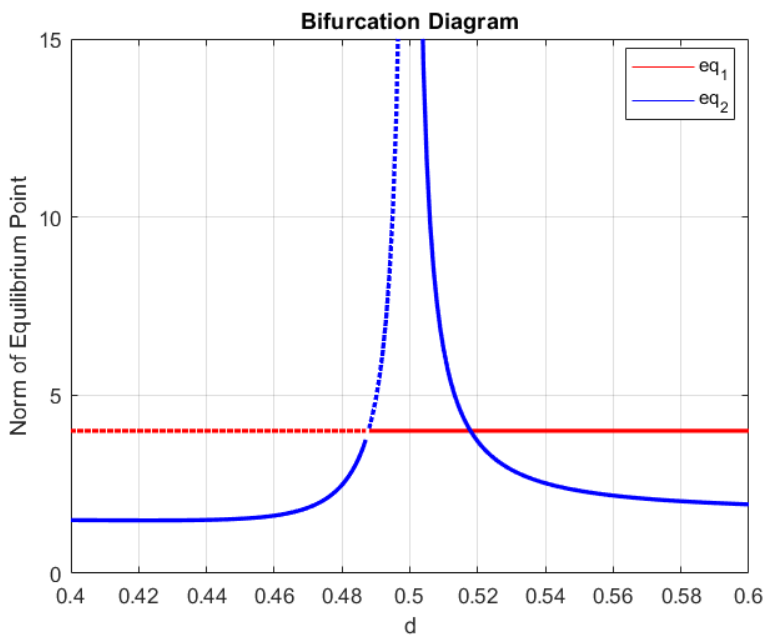

Figure 1

(c) The phase portrait of the system is given by figure 2

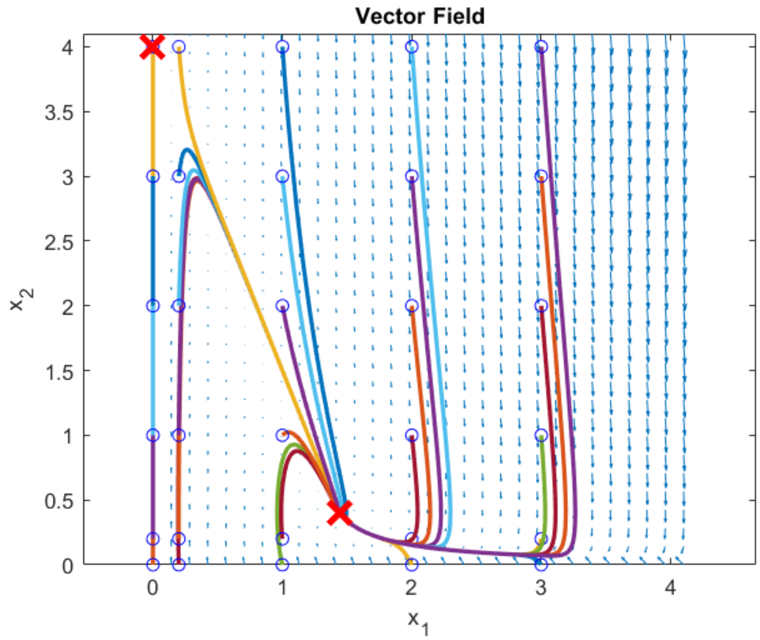

Figure 2

Problem 2.31

A biochemical reactor can be represented by the model

$$\dot{x_1} = \left(\frac{\mu_m x_2}{k_m + x_2 + k_1 x_2^2} - d\right) x_1 \tag{1}$$

$$\dot{x2} = d(x2f - x_2) - \frac{\mu_m x_1 x_2}{Y(K_m + x_2 + k_1 x_2^2)} \tag{2}$$

where the state variables are the nonnegative constants $d, \mu_m, k_m, k1, Y$, and $x2f$. are defined in Exercise 1.22. Let $\mu_m = 0.5, k_m = 0.1, k1 = 0.5, Y = 0.4$, and $x2f = 4$.

(a) Find all equilibrium points for $d > 0$ and determine the type of each point.

(b) Study bifurcation as d varies.

(c) Construct the phase portrait and discuss the qualitative behavior of the system when d = 0.1.

(d) Repeat part (c) when d = 0.25.

(e) Repeat part (c) when d = 0.5

Solution

(a) The equilibrium points occur at $\begin{pmatrix} (0,\ 4) \\ \frac{8}{5} - \frac{\sqrt{5}\,\sqrt{16\,d^2-20\,d+5}-10\,d+5}{25\,d},\ \frac{\sqrt{5}\,\sqrt{16\,d^2-20\,d+5}-10\,d+5}{10\,d} \\ \frac{8}{5} - \frac{10\,d+\sqrt{5}\,\sqrt{16\,d^2-20\,d+5}-5}{10\,d},\ \frac{10\,d+\sqrt{5}\,\sqrt{16\,d^2-20\,d+5}-5}{25\,d} \end{pmatrix}$

(b)

(c) The bifurcation diagram is shown figure 3 There is a saddle node at $d = 0.3$

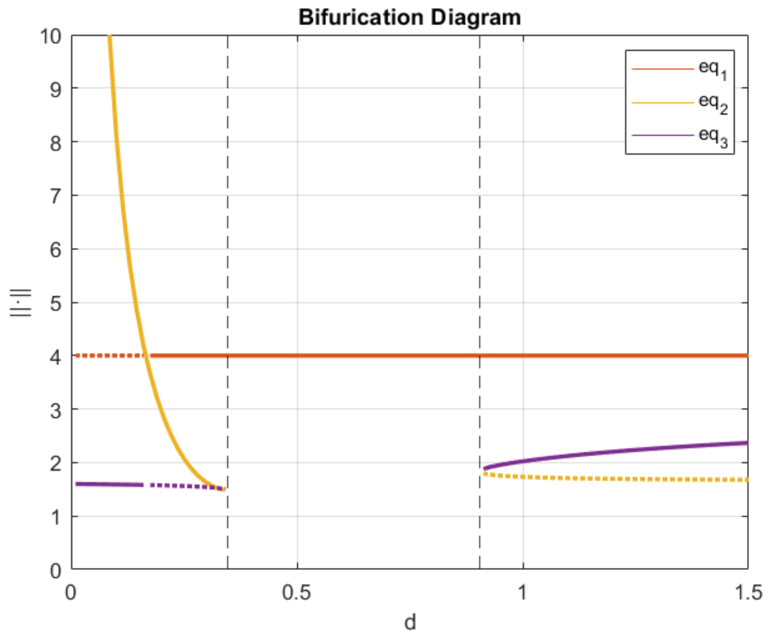

Figure 3

Problem 3.1

For each of the function $f(x)$ given next, find whether f is (a) continuously differentiable; (b) locally lipshcitz; (c) continuous;(d) globally Lipschitz.

(1) $f(x) = x^2 + |x|$

(2) $f(x) = x + \mathrm{sgn}(x)$

(3) $f(x) = \sin(x)\mathrm{sgn}(x)$

(4) $f(x) = -x + a\sin(x)$

(5) $f(x) = -x + 2|x|$

(6) $f(x) = \tan(x)$

(7) $f(x) = \begin{bmatrix} ax_1 + \tanh(bx1) - \tanh(bx2) \\ ax_2 + \tanh(bx1) - \tanh(bx2) \end{bmatrix}$

(8) $f(x) = \begin{bmatrix} -x_1 + a|x_2| \\ -(a+b)x_1 + bx_1^2 - x_1x_2 \end{bmatrix}$

Solution

3.1.1

(1) $f(x) = x^2 + |x|$ (a)$f(x)$ is not continuously differentiable at $x = 0$
(b) Is it Locally Lipschitz? Since x^2 is continuously differentiable, it is locally lipschitz, also $|x|$ is locally lipschitz which follows from the definition $|f(x) - f(y)| = |x - y|$
(c) LL, hence C.
(d) It is not Globally Lipschitz, because $\dfrac{d}{dx}x^2 = 2x$ is not bounded

3.1.2

$f(x) = x + sgn(x)$
(a) $\neq$ continous because $sgn(x)$ is discontinous at $x = 0$ therefore f(x) is not continuously Differentiable
(b) since f(x) $\neq$ continuous at $x = 0$ It is not LL
(c) f(x) $\neq$ continuous at $x = 0$
(d) since f(x) $\neq$ continuous at $x = 0$ It is not GL.

3.1.3

$f(x) = sin(x)sgn(x)$
(a) Not CD at $x = 0$
(b) LL, proof:$f(x)$ is continuous except at $x = 0$
$|f(x_1) - f(x_2)| \leq L|x_1 - x_2|$ Using the mean value theorem
$|f(x_1) - f(x_2)| = |sgn(x_1)sin(x_1) - sgn(x_2)sin(x_2)|$
$|sin(x_1) - sin(x_2)| = |cos(x)(x_1 - x_2)| \leq L|(x_1 - x_2)|$
Taking L = 1 and the fact that $|cos(x)| \leq 1$ Thus, f(X) is locally lipschitz
(c) GL, hence it is continuous
(d) GL proof: Consider when $x \geq 0 \geq y$ and $x \geq y \geq 0$
First in $x \geq y \geq 0$, $|f(x) - f(y)| = |sin(x) - sin(y)| = 2|sin\frac{x-y}{2}cos\frac{x+y}{2}| \leq 2|sin\frac{x-y}{2}| \leq |x - y|$ since $Cos(x) \leq 1 and |sin(x)| \leq |x|$
Also, for $x \geq 0 \geq y$ $|f(x) - f(y)| = |sin(x) + sin(y)| = |sin(x) + sin(y)| \leq |x + y| \leq |x - y|$. Therefore, $f(x)$ is GL.

3.1.4

$f(x) = -x + asin(x)$ (a) f(x) is CD
(b) f(x) is LL
(c) f(x) is Continuous
(d) f(x) is GL since $|\partial \frac{f}{dx}| = |-1 + acos(x)| \leq |1 + a|$ is bounded

3.1.5

$f(x) = -x + 2|x|$
(a) f(x) is not CD at x = 0
(b) f(x) is GL, therefore it is LL
(c) f(x) is LL therefore it is Continuous
(d) f(x) is GL since the $-x$ is globally lipschitz

3.1.6

$f(x) = tanx$
(a) = f(x) is CD over the $\frac{-\pi}{2} < x < \frac{\pi}{2}$
(b) = f(x) is LL in D
(c) = f(x) is Continuous in the above domain
(d) f(x) is not GL in this Domain since $\partial \frac{f}{dx} = sec^2 x$ is not bounded near x = 0.

3.1.7

$f(x) = \begin{bmatrix} ax_1 + tanh(bx_1) - tanh(bx_2) \\ ax_2 + tanh(bx_1) + tanh(bx_2) \end{bmatrix}$

The partial derivative of f(x) is given by

$$\begin{bmatrix} a + bsech^2(bx_1) & -bsech^2(bx^2) \\ bsech^2(bx_1) & a + bsech^2(bx_2) \end{bmatrix}$$

From the partial derivative, we see that f(x) is continuously differentiable hence it is locally Lipschitz.

(a) f(x) is CD
(b) f(x) is LL
(c) f(x) is Continuous
(d) f(x) is GL since the partial derivative is globally bounded

3.1.8

$f(x) = \begin{bmatrix} -x_1 + a|x_2| \\ -(a+b)x_1 + bx_1^2 - x_1x_2 \end{bmatrix}$
The partial derivative of f(x) is given by

$$\begin{pmatrix} -1 & a \\ 2\,b\,x_1 - b - x_2 - a & -x_1 \end{pmatrix}$$

We see that
(a) f(x) is not CD because f_1 is not CD
(b) f_1 and f_2 is LL so $f(x)$ is LL
(c) f(x) is Continuous
(d) f(x) is no GL since the partial derivative of f_2 is not globally bounded.

Problem 3.2

Let $D_r = x\epsilon R^n|||x|| < r$. For each of the following systems, represented as $\dot{x} = f(t,x)$, find whether (a) f is locally Lipschitz in x on D_r for sufficiently small r; (b) f is locally Lipschitz in x on D_r, for any finite $r > 0$ (c) f is globally Lipschitz in x:

(1) The pendulum equation with friction and constant input torque (section 1.2.1).

(2) The tunnel-diode circuit (Example 2.1)

(3) The mass-spring equation with linear spring, linear viscous damping, Coulumb friction, and zero external force (section 1.2.3)

(4) The Van der pol oscillator (Example 2.6)

(5) The closed-loop equation of a third-other adaptive control system (Section 1.2.5).

(6) The system $\dot{x} = Ax - B\psi(Cx)$ where A, B, and C are $nxn, nx1, and 1xn$ respectively and $\psi(.)$ is the dead zone nonlinearlity of figure 1.10(c)

Solution

(1) The pendulum equation is given by

$$f(x) = \begin{bmatrix} x2 \\ -\frac{g}{l}sinx_1 - \frac{k}{m}x_2 + \frac{1}{ml^2}T \end{bmatrix}$$

Taking the partial derivative of f(x)

$$\begin{pmatrix} 0 & 1 \\ -\frac{g\ \cos(x_1)}{l} & -\frac{k}{m} \end{pmatrix}$$

The partial derivative is globally bounded and therefore globally lipschitz. Since it is GL, it is also locally lipschitz for any $r > 0$

(2) The tunnel diode circuit equation is given by

$$f(x) = \begin{bmatrix} \frac{1}{C}(-h(x_1) + x_2) \\ \frac{1}{L}(-x_1 - Rx_2 + u) \end{bmatrix}$$

Taking the partial derivative of f(x), we get $\begin{pmatrix} -\frac{h}{C} & \frac{1}{C} \\ -\frac{1}{L} & -\frac{R}{L} \end{pmatrix}$

The partial derivative is continuous, and bounded on a set D_r. Therefore, it is LL on D_r for $r > 0$. However, it is not globally Lipschitz because $h'(x)$ is not bounded.

(3) The mass spring oscillator is given by

$$f(x) = \begin{bmatrix} x_2 & -\frac{k}{m}x_1 - \frac{c}{m}x_2 - \frac{1}{m}\eta(x_1, x_2) \end{bmatrix}$$

The function f(x) is discontinuous at $x_2 = 0$ Therefore it is not Locally lipschitz, and hence not Globally lipschitz

(4) The Van der pol oscillator is given by
$f(x) = \begin{bmatrix} x_2 & -x_1 + \epsilon(1 - x_1^2)x_2 \end{bmatrix}$ Taking the partial derivative

$\partial d/dx = \begin{bmatrix} 0\ 1 \\ -1-2\epsilon x_1 x_2\ \epsilon(1-x_1^2) \end{bmatrix}$
(1) f(x) is continuously differentiable, and continuous in a domain $D_r$therefore it is locally lipschitz (2) it is continuously differentiable (3) It is not globally lipschitz since it partial derivative is not globally bounded.

(5) The state space model of the system is as follows:
$f(t,x)\ =\ \begin{bmatrix} a_m x_0 + k_p x1r(t) + k_p x_2(x0 + y_m(t)) \\ -\gamma x_0 r(t) \\ -\gamma x0(x0 + y_m(t)) \end{bmatrix}$ Taking the partial derivative, we have
$\frac{\partial f}{\partial x}\ =\ \begin{bmatrix} a_m + k_p x_2 & k_p r(t) & k_p(x_0 + y_m(t)) \\ -\gamma r(t) & 0 & 0 \\ -\gamma(2x_0 + y_m(t)) & 0 & 0 \end{bmatrix}$
$\frac{\partial f}{\partial x}$ is continuously differentiable and bounded in D_r for a bounded r(t) and $y_m(t)$, therefore, f is locally lipschitz. (b) The jacobian of f is not globally bounded, therefore f is not globally lipschitz.

(6) The state space equation is given by
$f(x) = Ax + B\psi(Cx)$
where $\psi(.)$ is the standard dead zone non-linearity. (a) the term $\psi(x)$ is lipschitz continuous, and therefore is globally lipschitz, (b) since it is GL it is LL

Problem 3.5

Let $\| \, . \, \|_\alpha$ *and* $\| \, . \, \|_\beta$ be two different p - norms on R^n. Show that $f : R^n \to R^m$ is Lipschtz in $\| \, . \, \|_\alpha$ if and only if it is Lipschitz in $\| \, . \, \|_\beta$

Solution

By the equivalence of p-norms in finite dimensional vector spaces, we know If f is Lipschitz in $\| \cdot \|_\beta$, then it is Lipschitz in $\| \cdot \|_\alpha$. and vice versa

If f is Lipschitz in $\| \cdot \|_\alpha$. This means that there exists a constant $L > 0$ such that for all $x, y \in \mathbb{R}^n$,

$$\|f(x) - f(y)\|_\alpha \leq L_\alpha \|x - y\|_\alpha.$$

Also, since $\| \cdot \|_\alpha$ and $\| \cdot \|_\beta$ are different p-norms, there exist positive constants c_1 and c_2 such that for all $x \in \mathbb{R}^n$,

$$c_1 \|x|_\beta \leq \|x\|_\alpha \leq c_2 \|x\|_\beta.$$

Using these inequalities, we have

$$\|f(x) - f(y)\|_\beta \leq \frac{1}{c_1} \|f(x) - f(y)\|_\alpha \leq \frac{L}{c_1} \|x - y\|_\alpha \leq \frac{Lc_2}{c_1} \|x - y\|_\beta,$$

This is verified by the inequality $\|x - y\|_\alpha \leq c_2 \|x - y\|_\beta$. Similarly, If f is Lipschitz in $\| \cdot \|_\beta$, following from above argument, we see that it is also Lipschitz in $\| \cdot \|_\alpha$.

Problem 3.6

Let $f(t,x)$ be piecewise continuous in t, locally Lipschitz in x, and

$$\|f(t,x)\| \leq k_1 + k_2\|x\|, \ \forall(t,x) \in [t_0,\infty) \text{ x } R^n$$

(a) Show that the solution of (3.1) satisfies

$$\|x(t)\| \leq \|x_0\|exp[k_2(t-t_0)] + \frac{k_1}{k_2}\{exp[k_2(t-t_0)] - 1\}$$

for all $t \geq t_0$ for which the solution exists.

(b) Can the solution have a finite escape time?

Solution

Using the integral form of the differential equation $\|x(t)\| \leq \|x_0 + \int_{t_0}^t f(s,x(s))ds\|$
$\|x(t)\| \leq \|x_0 + k_1(t-t_0) + \int_{t_0}^t k_2\|x_0\|ds\|$

Applying the Bellman inequality:
$\|x(t)\| \leq \|x_0 + k_1(t-t_0) + \int_{to}^t (\|x_0\| + k1(s-t_0))k_2 e^{k2(t-s)}ds\|$
(b) $\|x(t)\|$ finite for all times and cannot escape to infinity.

Problem 3.8

Show that the state equation

$$\dot{x_1} = -x_1 + \frac{2x_2}{1+x_2^2}, \quad x_1(0) = a$$
$$\dot{x_2} = -x_2 + \frac{2x_1}{1+x_1^2}, \quad x_2(0) = b$$

has a unique solution defined for all $t \geq 0$

Solution

Taking the partial derivative of the equation, we see that it is continuously differentiable, $\forall\ \mathbf{x} \in \mathbb{R}^2$.

$$= \begin{pmatrix} -1 & \frac{4x_2}{(1+x_2^2)^2} & \frac{4x_1}{(1+x_1^2)^2} & -1 \end{pmatrix}$$

Therefore it is locally Lipschitz.

We also not that $\|f(x)\| \leq k_1 + k_2\|x\|$

Applying the Gronwall-Bellman lemma, which states that if $\mathbf{f}$ is Lipschitz continuous with constant L and locally bounded, then the initial value problem $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ with initial condition $\mathbf{x}(0) = \mathbf{x}_0$ has a unique solution defined for all $t \geq 0$.

Therefore, the system of differential equations given in the problem has a unique solution defined for all $t \geq 0$.

Problem 3.9

Suppose that the second-order system $\dot{x} = f(x)$, with a locally Lipschitz $f(x)$, has a limit cycle. Show that any solution that starts in the region enclosed by the limit cycle cannot have a finite escape time.

Solution

For the second-order system $\dot{x} = f(x)$, if W is the limit cycle of $f(x)$ where $f(x)$ is locally Lipschitz, any trajectory that starts in W stays in W for all time.

By theorem 3.3, Let $f(t, x)$ be piecewise continuous in t and locally Lipschitz in x for all t $\geq t_0$ and all x in a domain D $\subset$ R^n, and W is a compact subset of D, x_0, $\in$W, then every solution of
$\dot{x} = f(t, x), x(t_0) = x_0,$
lies entirely in W. There is a unique solution that is defined for all t $\geq t_0$

Problem 3.10

Derive the sensitivity equations for the tunnel-diode circuit of example 2.1 as L and C vary from their nominal values.

Solution

The equation is given as
$\dot{x_1} = \frac{1}{C}[-h(x_1) + x_2]$
$\dot{x_2}\frac{1}{L}(-x_1 - Rx_2 + u)$
Taking the partial derivative with respect to x and λ where λ represent the parameters of the system namely L and C
$\frac{\partial f}{\partial x} = \begin{bmatrix} -\frac{1}{C}h'(x_1) & \frac{1}{c} \\ -\frac{1}{L} & \frac{R}{L} \end{bmatrix}$
$\frac{\partial f}{\partial \lambda} = \begin{bmatrix} -\frac{1}{C^2}[-h(x_1) + x_2] & 0 \\ 0 & -\frac{1}{L^2}(-x_1 - Rx_2 + u) \end{bmatrix}$ Let S = $\frac{\partial x}{\partial \lambda} = \begin{bmatrix} x_3 & x_5 \\ x_4 & x_6 \end{bmatrix} = \begin{bmatrix} \frac{\partial x_1}{\partial C} & \frac{\partial x_1}{\partial L} \\ \frac{\partial x_2}{\partial C} & \frac{\partial x_2}{\partial L} \end{bmatrix}$ Therefore
$\dot{x_1} = 0.5[-h(x_1) + x_2]$
$\dot{x_2} = 0.2(-x_1 - 1.5x_2) + 1.2$
$\dot{x_3} = 0.5[-h'(x_1)x_3 + x_4] - 0.25[-h(x_1) + x_2]$
$\dot{x_4} = 0.2(-x_2 - 1.5x_4)$
$\dot{x_5} = 0.5[-h'(x_1)x_1x_5 + x_6]$
$\dot{x_6} = 0.2(-x_5 - 1.5x_6) - 0.04(1.2 - x_1 - 1.5x_2)$
Where $C = 2, L = 5. x_1(0) = x_10,\ x_2(0) = x_20,\ x_3(0) = 0 x_4(0) = 0,\ x_5(0) = 0 x_6(0) = 0$

Problem 3.11

Derive the sensitivity equations for the Van der Pol oscillator of the example 2.6 as ϵ varies from its nominal value. Use the state equation in the x-coordinates.

Solution

The van der pol equation is given by $\dot{x_1} = x_2$
$\dot{x_2} = -x_1 + \epsilon(1 - x_1^2)x_2$ Taking the partial derivative with respect to x and ϵ We have $\frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 \\ -1 - 2\epsilon x_1 x_2 & \epsilon(1 - x_1^2) \end{bmatrix}$
and the jacobian with respect to ϵ is given by
$\frac{\partial f}{\partial \epsilon} = \begin{bmatrix} 0 \\ (1 - x_1^2)x_2 \end{bmatrix}$
Let S $= \frac{\partial f}{\partial \epsilon} = \begin{bmatrix} x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \frac{\partial x_1}{\partial \epsilon} \\ \frac{\partial x2}{\partial \epsilon} \end{bmatrix}$ Therefore
$\dot{x_1} = x_2$
$\dot{x_2} = -x_1 + \epsilon_0(1 - x_1^2)x_2$
$\dot{x_3} = x_4$
$\dot{x_4} = -[1 + 2\epsilon_0 x_1 x_2]x_3 + \epsilon_0(1 - x_1^2)x_4 + (1 - x_1^2)x_2$
Where $x_1(0) = x_1 0,\ x_2(0) = x_2 0,\ x_3(0) = 0 x_4(0) = 0,$

Problem 3.13

Derive the sensitivity equations for the system

$$\dot{x_1} = tan^{-1}(ax_1) - x_1x_2,$$
$$\dot{x_2} = bx_1^2 - cx_2$$

as the parameters a, b, c vary from their nominal values $a_0 = 1, b_0 = 0$ and $c_0 = 1$.

Solution

Taking the partial derivatives, $\frac{df}{dx}$ and $\frac{df}{d\lambda}$ we get

$\frac{df}{dx} = \begin{pmatrix} \frac{a}{a^2\,{x_1}^2+1} - x_2 & -x_1 \\ 2\,b\,x_1 & -c \end{pmatrix}$

$\frac{df}{d\lambda} = \begin{bmatrix} \frac{df}{da} & \frac{df}{db} & \frac{df}{dc} \end{bmatrix} = \begin{bmatrix} \frac{x_1}{a^2\,{x_1}^2+1} & 0 & 0 \\ 0 & {x_1}^2 & -x2 \end{bmatrix}$

evaluating the Jacobian at nominal parameters

Let S $= \begin{bmatrix} x_3 & x_5 & x_7 \\ x_4 & x_6 & x_8 \end{bmatrix}$ The augmented equation (3.7) is given by

$\dot{x_1} = tan^{-}(x_1) - x_1x_2$

$\dot{x_2} = -x_2$

$\dot{x_3} = (\frac{1}{1+x_1^2} - x_2)x_3 - x_1x_4 + \frac{x_1}{1+x_1^2}$

$\dot{x_4} = -x_4$

$\dot{x_5} = (\frac{1}{1+x_1^2} - x_2)x_5 - x_1x_6$

$\dot{x_6} = -x_6 + x_1^2$

$\dot{x_7} = (\frac{1}{1+x_1^2} - x_2)x_7 - x_1x_8$

$\dot{x_8} = -x_8 - x_2$

with initial conditions

$x_1(0) = x_{10}, x_2(0) = x_{20}, x_3(0) = x_4(0) = x_5(0) = x_6(0) = x_7(0) = x_8(0)$

Problem 3.20

Show that if $f : R^n \to R^n$ is Lipschitz on $W \subset R^n$, then $f(x)$ is uniformly continuous on W.

Solution

To show this relationship, we use the epsilon delta argument. For $f : \mathbb{R}^n \to \mathbb{R}^n$ to be Lipschitz on $W \subset \mathbb{R}^n$ with Lipschitz constant L, for all $x, y \in W$, we have $|f(x) - f(y)| \leq L\,|x - y|$.

To show that $f(x)$ is uniformly continuous on W, we need to show that for any $\epsilon > 0$, there exists a $\delta > 0$ such that for any $x, y \in W$ with $|x - y| < \delta$, we have $|f(x) - f(y)| < \epsilon$.

When $\epsilon > 0$. We can choose $\delta = \frac{\epsilon}{L}$, then $\|f(x) - f(y)\| \leq \epsilon \leq L\delta$

Since $x, y \in W$ $|x - y| < \delta = \frac{\epsilon}{L}$. Then we have:

$$|f(x) - f(y)| \leq L\,|x - y| \qquad \text{(by Lipschitz condition)} \ < L \cdot \frac{\epsilon}{L} = \ \epsilon.$$

This shows that $f(x)$ is uniformly continuous on W.

Problem 3.21

For any $x \in R^n - 0$ and any $p \in [1, \infty)$, define $y \in R^n$ by

$$y_i = \frac{x_1^{p-1}}{\|x\|_p^{p-1}} sign(x_i^p)$$

Show that $y^T x = \|x\|_p$ and $\|x\|_q = 1$ where $q \in (1, \infty)$ is determined from $\frac{1}{P} + \frac{1}{q} = 1$ For $p = \infty$ find a vector y such that $y^T x = \|x\|_\infty$ and $\|y\|_1 = 1$.

Solution

(a) To show that $y^T x = \|x\|p$, we have:

$$\begin{aligned} y^T x &= \sum i = 1^n y_i x_i \\ &= \sum_{i=1}^{n} \frac{x_1^{p-1}}{|x|_p^{p-1}} sign(x_i^p) x_i \\ &= \frac{x_1^{p-1}}{\|x\|p^{p-1}} \sum i = 1^n |x_i|^p \quad = \frac{x_1^{p-1}}{\|x\|_p^{p-1}} \|x\|_p^p \quad = \|x\|_p \end{aligned}$$

(b) To show that $\|y\|q = 1$, with $\frac{1}{p} + \frac{1}{q} = 1$ we have:

$$\begin{aligned} \|y\|_q^q = \frac{|x_1|^{(p-1)q} + ... + |x_n|^{(p-1)q}}{\|x\|_p^{(p-1)q}} \\ = \frac{|x_1|^p + ... + |x_n|^p}{\|x\|_p^p} \\ = \frac{\|x\|_p^p}{\|x\|_p^p} = 1 \end{aligned}$$

This shows that $\|y\|_q^q = 1$

(c) when $p = \infty$ and q = 1 $\|x\|_\infty = \max(x_1, ...x_n)$ and $\|y\|_1 = |y| + ... + |y_n|$
when $y_i = \begin{cases} 1, & i = minargmax|x_i| \\ 0 & \text{otherwise} \end{cases}$
$y^T x = \|x\|_\infty$ and $\|y\|_1 = 1$