Advanced Macroeconomics

On business cycle theory (incomplete)

(Lecture Notes, Winter Term 2010/11, Thomas Steger, University of Leipzig)

1 A simple stochastic growth model

1.1 The model

Consider the following stochastic Ramsey model (Heer and Maussner, 2005, p. 35)

$$\max_{C_t}\{E_0\sum_{t=0}^{\infty}\beta^t u(C_t)\} \tag{1}$$

$$K_{t+1}=B_tK_t^{\alpha}(A_tL_t)^{1-\alpha}+(1-\delta)K_t-C_t \tag{2}$$

$$B_t=B_{t-1}^{\gamma}e^{\varepsilon_t},\quad A_t=A_{t-1}e^{\lambda} \tag{3}$$

$$K_0=K_{00},\quad A_0=A_{00} \tag{4}$$

$$\lim_{t\to\infty}\beta^t E_t\left(\mu_t K_t\right)=0 \tag{5}$$

where $t\in\mathbb{N}$ denotes the time index, E_0 the expectations conditional on information in period $t=0$,[1] $0<\beta<1$ a subjective discount factor, C_t consumption in period t, K_t the stock of capital, B_t a stochastic technology parameter, ε_t is an independently and identically normally distributed random variable with $E\left(\varepsilon_t\right)=\varepsilon$ and $V\left(\varepsilon_t\right)=\sigma_\varepsilon$, $0<\gamma<1$ a parameter that captures the degree of persistency of technology shocks, A_t captures (exogenous) technological progress (approximately) at rate $\lambda\geq 0$, L_t labor input (considered exogenous here), $\delta>0$ the depreciation rate, respectively.

The timing of events within every period is as follows: First, the shock materializes and then the agent decides on consumption. This problem differs from the deterministic model in two respects:

- Output at each period t depends not only on K_t but also on the realization of a

[1] The phrase "period t" describes the time interval $[t,t+1)$.

stochastic variable A_t. The timing of events then implies that, in period t, the agent decides upon C_t knowing the realization of A_t.

- As a result of uncertainty about the future, the agent decides only upon C_0 at $t = 0$ and postpones the decision on $\{C_1, C_2, ...\}$. At $t = 1$ the agent then decides upon C_1 knowing K_1 and B_1 and so on. An alternative solution strategy lies in the formulation of contingent plans (see Groth, 2010).

At $t = 0$, K_0 and B_0 are given and the agent decides upon C_0. To solve the above problem we employ the method of Lagrangian multipliers which requires to set up the following Lagrangian function (e.g., Chow, 1997)

$$\mathcal{L} := E_0\{\sum_{t=0}^{\infty} \beta^t u(C_t) + \sum_{t=0}^{\infty} \beta^t \mu_t \left[B_t K_t^{\alpha} (A_t L_t)^{1-\alpha} + (1-\delta)K_t - C_t - K_{t+1}\right]\}$$

The associated first-order conditions involve $\frac{\partial \mathcal{L}}{\partial C_0} = 0$ and $\frac{\partial \mathcal{L}}{\partial K_1} = 0$ (together with the dynamic budget constraint). To form these partials let us write the relevant parts of the Lagrangian explicitly as follows

$$\begin{aligned} \mathcal{L} \ : \ &= E_0\{\beta^0 u(C_0) + \beta^0 \mu_0 \left[B_0 K_0^{\alpha} (A_0 L_0)^{1-\alpha} + (1-\delta)K_0 - C_0 - K_1\right] \\ &+\beta^1 u(C_1) + \beta^1 \mu_1 \left[B_1 K_1^{\alpha} (A_1 L_1)^{1-\alpha} + (1-\delta)K_1 - C_1 - K_2\right] + ...\} \end{aligned}$$

The first order conditions $\frac{\partial \mathcal{L}}{\partial C_0} = 0$ and $\frac{\partial \mathcal{L}}{\partial K_1} = 0$ can now be expressed as follows

$$\frac{\partial \mathcal{L}}{\partial C_0} = E_0\{u'(C_0) - \mu_0\} = 0$$

$$\frac{\partial \mathcal{L}}{\partial K_1} = E_0\{-\mu_0 + \beta^1 \mu_1 \left[\alpha B_1 K_1^{\alpha-1} (A_1 L_1)^{1-\alpha} + 1 - \delta\right]\} = 0$$

Since C_0, K_1 and, hence, the multiplier μ_0 are non-stochastic we may write

$$u'(C_0) = \mu_0$$

$$\mu_0 = \beta^1 E_0\{\mu_1 \left[\alpha B_1 K_1^{\alpha-1} (A_1 L_1)^{1-\alpha} + 1 - \delta\right]\}$$

At $t = 1$, K_1 and B_1 are given and the agent decides upon C_1. To solve this problem we set up the following Lagrangian function

$$\mathcal{L} := E_1\{\sum_{t=1}^{\infty} \beta^{t-1} u(C_t) + \sum_{t=1}^{\infty} \beta^{t-1} \mu_t \left[B_t K_t^{\alpha} (A_t L_t)^{1-\alpha} + (1-\delta) K_t - C_t - K_{t+1} \right]\}$$

The associated first-order conditions involve $\frac{\partial \mathcal{L}}{\partial C_1} = 0$ and $\frac{\partial \mathcal{L}}{\partial K_2} = 0$ (together with the dynamic budget constraint). To form these partials let us write the relevant parts of the Lagrangian explicitly as follows

$$\begin{aligned} \mathcal{L} \ : \ &= E_1\{\beta^0 u(C_1) + \beta^0 \mu_1 \left[B_1 K_1^{\alpha} (A_1 L_1)^{1-\alpha} + (1-\delta) K_1 - C_1 - K_2 \right] \\ &+ \beta^1 u(C_2) + \beta^1 \mu_2 \left[B_2 K_2^{\alpha} (A_2 L_2)^{1-\alpha} + (1-\delta) K_2 - C_2 - K_3 \right] + ...\} \end{aligned}$$

The first order conditions $\frac{\partial \mathcal{L}}{\partial C_1} = 0$ and $\frac{\partial \mathcal{L}}{\partial K_2} = 0$ can now be expressed as follows

$$\frac{\partial \mathcal{L}}{\partial C_1} = E_1\{u'(C_1) - \mu_1\} = 0$$

$$\frac{\partial \mathcal{L}}{\partial K_2} = E_1\{-\mu_1 + \beta^1 \mu_2 \left[\alpha B_2 K_2^{\alpha-1} (A_2 L_2)^{1-\alpha} + 1 - \delta \right]\} = 0$$

Since C_1, K_2 and, hence, the multiplier μ_1 are non-stochastic we may write

$$u'(C_1) = \mu_1$$

$$\mu_1 = \beta^1 E_1\{\mu_2 \left[\alpha B_2 K_2^{\alpha-1} (A_2 L_2)^{1-\alpha} + 1 - \delta \right]\}$$

Continuing this way, one finds that, since K_t must be optimal at t, the plan for choosing $\{C_0, C_1, ..\}$ and $\{K_0, K_1, ..\}$ must solve the system

$$u'(C_t) = \mu_t$$

$$\mu_t = \beta E_t\{\mu_{t+1} \left[\alpha B_{t+1} K_{t+1}^{\alpha-1} (A_{t+1} L_{t+1})^{1-\alpha} + 1 - \delta \right]\}$$

$$K_{t+1} = B_t K_t^{\alpha} (A_t L_t)^{1-\alpha} + (1-\delta) K_t - C_t$$

Eliminating the shadow price μ_t yields

$$u'(C_t) = \beta E_t\{u'(C_{t+1}) \left[\alpha B_{t+1} K_{t+1}^{\alpha-1} (A_{t+1} L_{t+1})^{1-\alpha} + 1 - \delta\right]\}$$

$$K_{t+1} = B_t K_t^{\alpha} (A_t L_t)^{1-\alpha} + (1-\delta) K_t - C_t$$

In summary, the evolution of the economy is governed by the following dynamic system

$$u'(C_t) = \beta E_t\{u'(C_{t+1}) \left[\alpha B_{t+1} K_{t+1}^{\alpha-1} (A_{t+1} L_{t+1})^{1-\alpha} + 1 - \delta\right]\}$$

$$K_{t+1} = B_t K_t^{\alpha} (A_t L_t)^{1-\alpha} + (1-\delta) K_t - C_t$$

$$B_t = B_{t-1}^{\gamma} e^{\varepsilon_t}, \quad A_t = A_{t-1} e^{\lambda}$$

$$K_0 = K_{00}, \quad A_0 = A_{00}, \quad B_0 = B_{00}, \quad \lim_{t\to\infty} \beta^t E_t \left(\mu_t K_t\right) = 0$$

1.2 Formation of first order conditions: general remark

Observe that the set of first order conditions results from

$$\mathcal{L} := E_0\{\sum_{t=0}^{\infty} \beta^t u(C_t) + \sum_{t=0}^{\infty} \beta^t \mu_t \left[B_t K_t^{\alpha} (A_t L_t)^{1-\alpha} + (1-\delta) K_t - C_t - K_{t+1}\right]\}$$

together with (where we assume that labor is endogenous)

$$\frac{\partial \mathcal{L}}{\partial C_t} = 0$$

$$\frac{\partial \mathcal{L}}{\partial L_t} = 0$$

$$\frac{\partial \mathcal{L}}{\partial K_{t+1}} = 0$$

$$\frac{\partial \mathcal{L}}{\partial (\beta^t \mu_t)} = 0$$

This statement suppresses the expectations operator!

1.3 Deterministic dynamic system and pseudo steady state

Let us ignore shocks for the moment, i.e. assume that $\varepsilon_t = \varepsilon \, \forall \, t$. Assuming furthermore $u'(C_t) = \ln C_t$, the corresponding deterministic system can then be written as follows:

$$\frac{C_{t+1}}{C_t} = \beta \left[\alpha B_{t+1} K_{t+1}^{\alpha-1} (A_{t+1} L_{t+1})^{1-\alpha} + 1 - \delta \right]$$

$$K_{t+1} = B_t K_t^{\alpha} (A_t L_t)^{1-\alpha} + (1-\delta) K_t - C_t$$

$$B_t = B_{t-1}^{\gamma} e^{\varepsilon}, \quad A_t = A_{t-1} e^{\lambda}$$

Next we turn to the steady state growth rate. Since $0 < \gamma < 1$ the technology parameter B exhibits a stationary solution $\tilde{B} = e^{\frac{\varepsilon}{1-\gamma}}$. The growth rate of A_t is given by $\hat{A} = e^{\lambda} - 1$. It can be easily shown that the steady state growth rate is characterized by

$$\hat{Y} = \hat{K} = \hat{C} = \hat{A} = e^{\lambda} - 1$$

Next we define normalized variables $k_t := \frac{K_t}{A_t}$ $(K_t = A_t k_t)$ and $c_t := \frac{C_t}{A_t}$ $(C_t = A_t c_t)$ such that the Euler equation becomes

$$\frac{A_{t+1} c_{t+1}}{A_t c_t} = \beta \left[\alpha B_{t+1} (A_{t+1} k_{t+1})^{\alpha-1} (A_{t+1} L_{t+1})^{1-\alpha} + 1 - \delta \right]$$

$$\Longrightarrow \quad \frac{c_{t+1}}{c_t} = e^{-\lambda} \beta \left[\alpha B_{t+1} (k_{t+1})^{\alpha-1} L_{t+1}^{1-\alpha} + 1 - \delta \right] \tag{6}$$

The capital accumulation equation can be written as

$$A_{t+1} k_{t+1} = B_t (A_t k_t)^{\alpha} (A_t L_t)^{1-\alpha} + (1-\delta) A_t k_t - A_t c_t$$

$$A_{t+1} k_{t+1} = B_t A_t k_t^{\alpha} L_t^{1-\alpha} + (1-\delta) A_t k_t - A_t c_t$$

$$\Longrightarrow \quad k_{t+1} = e^{-\lambda} \left[B_t k_t^{\alpha} L_t^{1-\alpha} + (1-\delta) k_t - c_t \right] \tag{7}$$

Turning to the determination of the steady state we first notice that the long run value of B, denoted as $\tilde{B}$, results from $\tilde{B} = \tilde{B}^{\,\gamma} e^{\varepsilon}$ to read $\tilde{B} = e^{\frac{\varepsilon}{1-\gamma}}$. Moreover, the steady state of system (6) and (7) is defined by $\tilde{k} = k_t = k_{t-1}$ and $\tilde{c} = c_t = c_{t-1}$. Applying this steady state condition yields

$$\frac{c_{t+1}}{c_t} = 1 \implies \tilde{k} = \left(\frac{\alpha \tilde{B} L_{t+1}^{1-\alpha}}{e^{\lambda}/\beta - 1 + \delta} \right)^{\frac{1}{1-\alpha}} \tag{8}$$

$$\tilde{k} = e^{-\lambda} \left[\tilde{B} \tilde{k}^{\alpha} L_t^{1-\alpha} + (1-\delta)\tilde{k} - c_t \right] \implies \tilde{c} = \tilde{B} \tilde{k}^{\alpha} L_t^{1-\alpha} + (1-\delta)\tilde{k} - \tilde{k} e^{\lambda} \tag{9}$$

2 A basic RBC model

Consider the following basic RBC model (cf. Brunner and Strulik, 2004, p. 82)

$$\max_{C_t, L_t} \{ E_0 \sum_{t=0}^{\infty} \beta^t \left[\ln C_t + \eta(1 - L_t) \right] \} \tag{10}$$

$$K_{t+1} = B_t K_t^{\alpha} (A_t L_t)^{1-\alpha} + (1-\delta) K_t - C_t \tag{11}$$

$$B_t = B_{t-1}^{\gamma} e^{\varepsilon_t}, \quad A_t = A_{t-1} e^{\lambda} \tag{12}$$

$$K_0 = K_{00}, \quad A_0 = A_{00} \tag{13}$$

$$\lim_{t \to \infty} \beta^t E_t \left(\mu_t K_t \right) = 0 \tag{14}$$

To solve this dynamic optimization problem we set up the following Lagrangian function

$$\mathcal{L} := E_0 \{ \sum_{t=0}^{\infty} \beta^t \left[\ln C_t + \eta(1 - L_t) \right] + \sum_{t=0}^{\infty} \beta^t \mu_t \left[B_t K_t^{\alpha} (A_t L_t)^{1-\alpha} + (1-\delta) K_t - C_t - K_{t+1} \right] \}$$

The set of first order conditions must now be complemented by a static efficiency condition of the form $\frac{\partial u(C_t, L_t)}{\partial L_t} = \mu_t \frac{\partial F(K_t, L_t)}{\partial L_t}$ or

$$\eta = \mu_t (1-\alpha) B_t K_t^{\alpha} (A_t L_t)^{-\alpha} A_t$$

The complete set of first order conditions then reads as follows

$$\frac{1}{C_t} = \mu_t$$

$$\eta = \mu_t(1-\alpha)B_t K_t^{\alpha}(A_t L_t)^{-\alpha} A_t$$

$$\mu_t = \beta E_t\{\mu_{t+1}\left[\alpha B_{t+1} K_{t+1}^{\alpha-1}(A_{t+1}L_{t+1})^{1-\alpha} + 1 - \delta\right]\}$$

$$K_{t+1} = B_t K_t^{\alpha}(A_t L_t)^{1-\alpha} + (1-\delta)K_t - C_t$$

Eliminating the shadow price one gets

$$\eta = \frac{1}{C_t}(1-\alpha)B_t K_t^{\alpha}(A_t L_t)^{-\alpha} A_t$$

$$\frac{1}{C_t} = \beta E_t\{\frac{1}{C_{t+1}}\left[\alpha B_{t+1} K_{t+1}^{\alpha-1}(A_{t+1}L_{t+1})^{1-\alpha} + 1 - \delta\right]\}$$

$$K_{t+1} = B_t K_t^{\alpha}(A_t L_t)^{1-\alpha} + (1-\delta)K_t - C_t$$

2.1 Deterministic dynamic system and pseudo steady state

Let us ignore shocks for the moment, i.e. assume that $\varepsilon_t = \varepsilon \ \forall \ t$. The corresponding deterministic system can then be written as follows:

$$\eta = \frac{1}{C_t}(1-\alpha)B_t K_t^{\alpha}(A_t L_t)^{-\alpha} A_t$$

$$\frac{C_{t+1}}{C_t} = \beta\left[\alpha B_{t+1} K_{t+1}^{\alpha-1}(A_{t+1}L_{t+1})^{1-\alpha} + 1 - \delta\right]$$

$$K_{t+1} = B_t K_t^{\alpha}(A_t L_t)^{1-\alpha} + (1-\delta)K_t - C_t$$

$$B_t = B_{t-1}^{\gamma} e^{\varepsilon}, \quad A_t = A_{t-1}e^{\lambda}$$

As before, the steady state growth rate is given by $\hat{Y} = \hat{K} = \hat{C} = \hat{A} = e^{\lambda} - 1$. Defining normalized variables $k_t := \frac{K_t}{A_t}$ ($K_t = A_t k_t$) and $c_t := \frac{C_t}{A_t}$ ($C_t = A_t c_t$), the first order condition $\frac{\partial u(C_t,L_t)}{\partial L_t} = \mu_t \frac{\partial F(K_t,L_t)}{\partial L_t}$ may be expressed as

$$\eta = \frac{1}{A_t c_t}(1-\alpha)B_t(A_t k_t)^{\alpha}(A_t L_t)^{-\alpha}A_t$$

$$\eta = \frac{1}{c_t}(1-\alpha)B_t k_t^{\alpha} L_t^{-\alpha}$$

The steady state in terms of normalized variables is then determined by

$$\eta = \frac{1}{c}(1-\alpha)\tilde{B}k^{\alpha}L^{-\alpha} \tag{15}$$

$$1 = e^{-\lambda}\beta\left[\alpha\tilde{B}(k)^{\alpha-1}L^{1-\alpha} + 1 - \delta\right] \tag{16}$$

$$k = e^{-\lambda}\left[\tilde{B}k^{\alpha}L^{1-\alpha} + (1-\delta)k - c\right] \tag{17}$$

Solving (16) for L gives

$$L = \left(\frac{e^{\lambda}/\beta - 1 + \delta}{\alpha\tilde{B}}\right)^{\frac{1}{1-\alpha}} k \tag{18}$$

Plugging this result into (15) gives

$$\eta = \frac{1}{c}(1-\alpha)\tilde{B}k^{\alpha}\left(\frac{e^{\lambda}/\beta - 1 + \delta}{\alpha\tilde{B}}\right)^{\frac{-\alpha}{1-\alpha}} k^{-\alpha} \tag{19}$$

$$\eta = \frac{1}{c}(1-\alpha)\tilde{B}\left(\frac{e^{\lambda}/\beta - 1 + \delta}{\alpha\tilde{B}}\right)^{\frac{-\alpha}{1-\alpha}} \tag{20}$$

$$c = \frac{(1-\alpha)\tilde{B}}{\eta}\left(\frac{e^{\lambda}/\beta - 1 + \delta}{\alpha\tilde{B}}\right)^{\frac{-\alpha}{1-\alpha}} \tag{21}$$

References

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