概率统计课件之4

第四章 正态分布

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正态分布

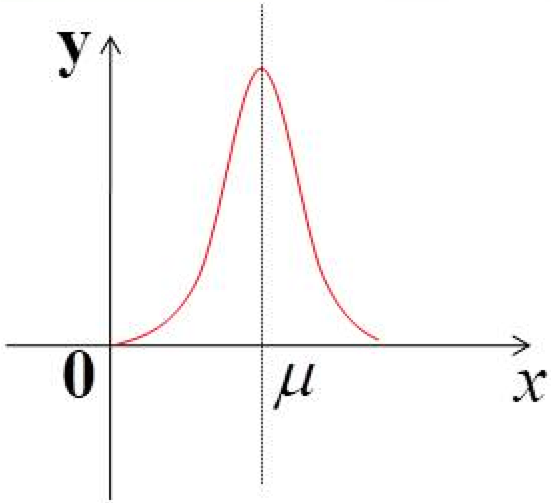
§4-1 正态分布

§4-2 正态随机变量的线性组合

§4-3 中心极限定理

一、正态分布

1、定义



设连续型随机变量X的概率密度为:

$$f(x)=\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

其中μ，σ（$0<\sigma<1$）为常数，则称X服从参数为

μ，σ^2的正态分布（高斯分布）.记作：$X\sim N(\mu,\sigma^2)$

2、性质

① 曲线关于$x=\mu$对称.

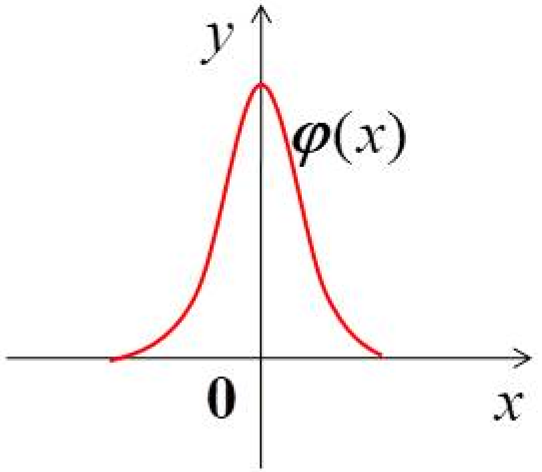
② 当$x=\mu$时取到最大值$f(\mu)=\dfrac{1}{\sqrt{2\pi}\sigma}$.

③ 以x轴为渐近线.

二、标准正态分布

1、定义

设连续型随机变量 的概率密度为:

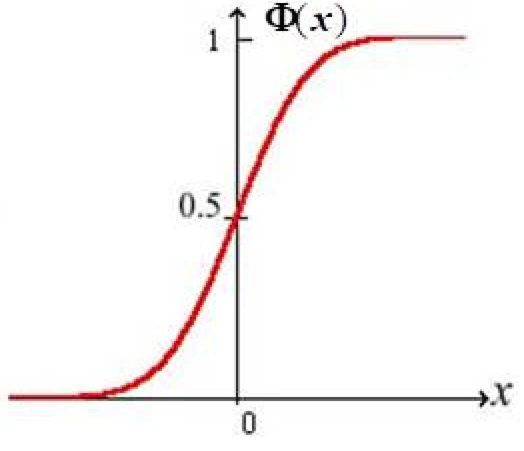


$$\varphi(x)=\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$$

则称X 服从标准正态分布. 记为: $X \sim N(0,1)$

分布函数为:

$$\Phi(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x}e^{-\frac{t^2}{2}}dt$$



2、性质

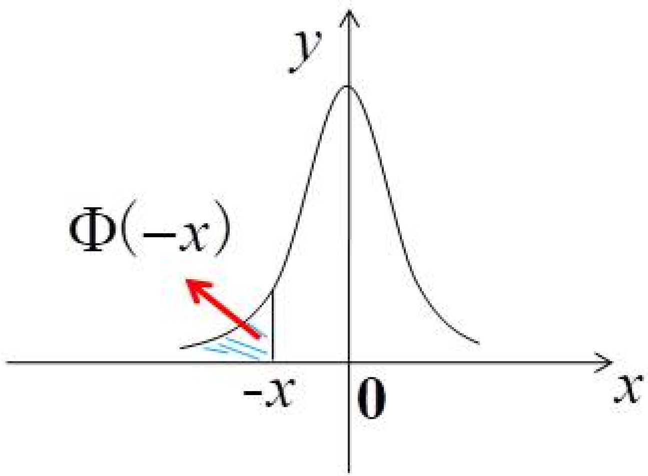
① 曲线关于y轴 对称.

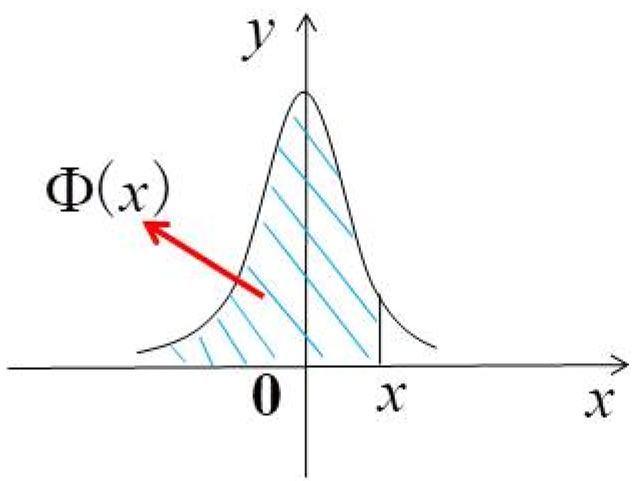
② 当 $x=0$ 时取到最大值 $\frac{1}{\sqrt{2\pi}}$.

③ 以横轴为渐近线.

④ $\Phi(-x)=1-\Phi(x)$

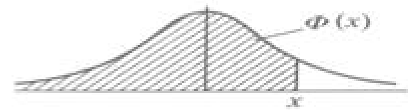
⑤ $\Phi(0)=0.5$





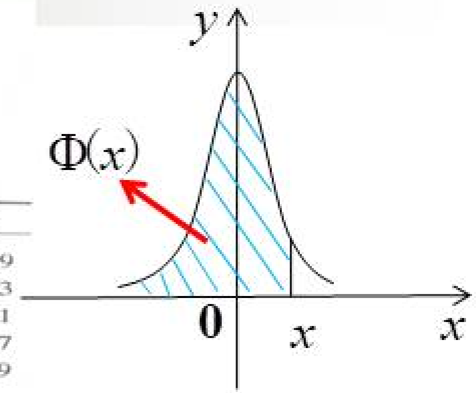
3、$\Phi(x)$ 函数分布表

标准正态分布表



$$\Phi(x)=\int_{-\infty}^{x}\frac{1}{\sqrt{2\pi}}e^{-t^2/2}\,dt$$

x	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7703	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9278	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9430	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9648	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9700	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9762	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9874	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998



$$\Phi(1.04)=0.8508$$

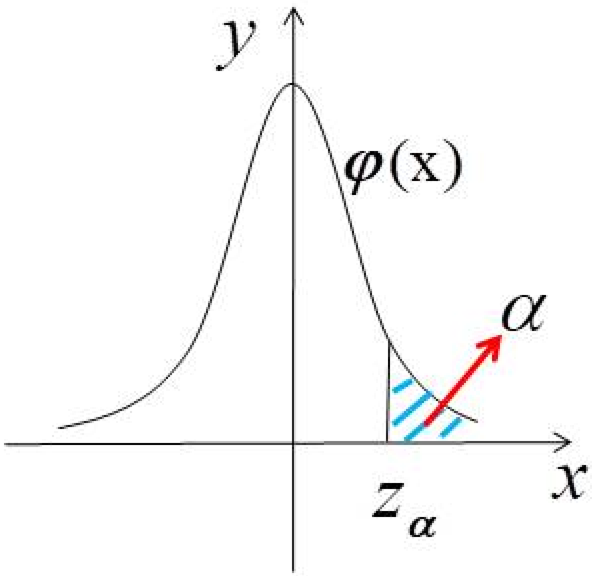
$$\Phi(1.82)=0.9656$$

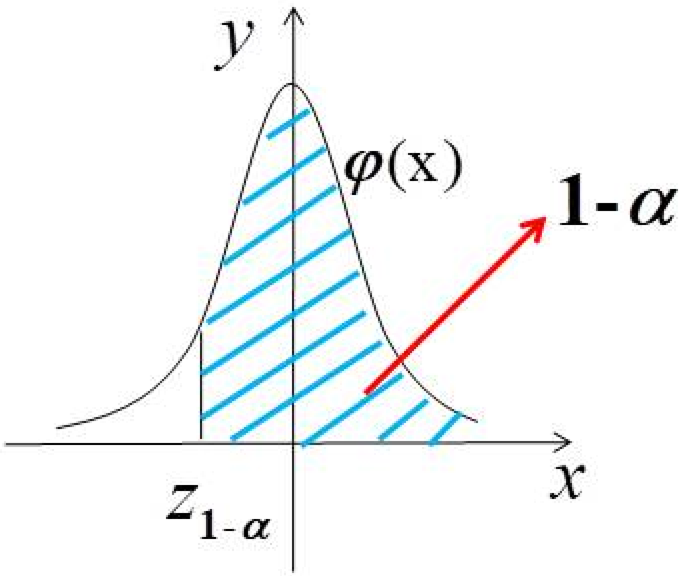
4、标准正态分布的上α分位数（点）

定义：设X~N(0,1)，若对 $\forall\alpha\ (0<\alpha<1)$有

$$P\{X>z_\alpha\}=\int_{z_\alpha}^{+\infty}\varphi(x)dx=\alpha$$

则称 z_α 为标准正态分布的 上α分位数(点)。

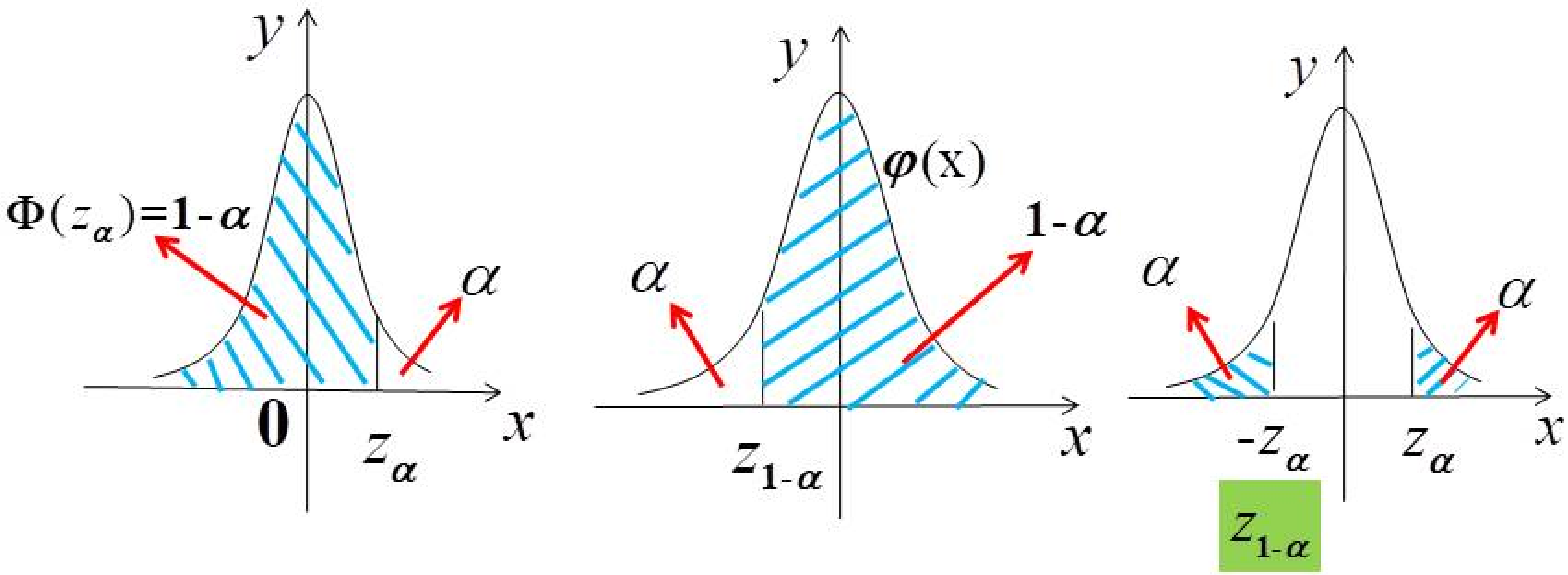




注意：

(1) $\Phi(z_\alpha) = P\{X \le z_\alpha\} = 1 - P\{X > z_\alpha\} = 1 - \alpha$，故对给定的 α，由 $\Phi(x)$ 函数分布表，可求得 z_α；

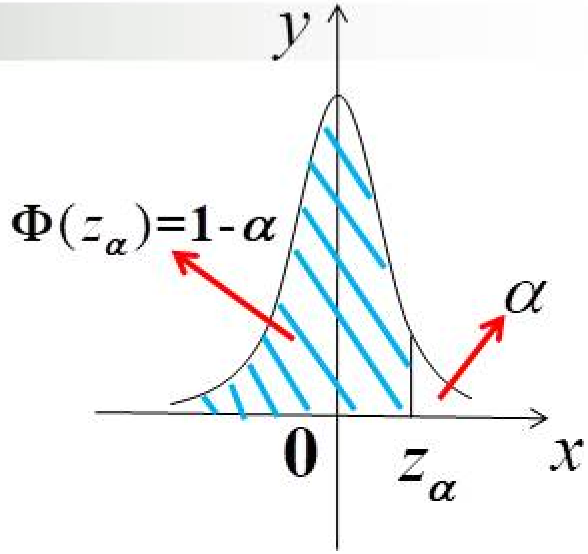
(2) $z_{1-\alpha} = -z_\alpha$



例：(1)若 α=0.05 ，求 z_α

解：$\Phi(z_{0.05})=1-0.05=0.95=\Phi(1.645)$

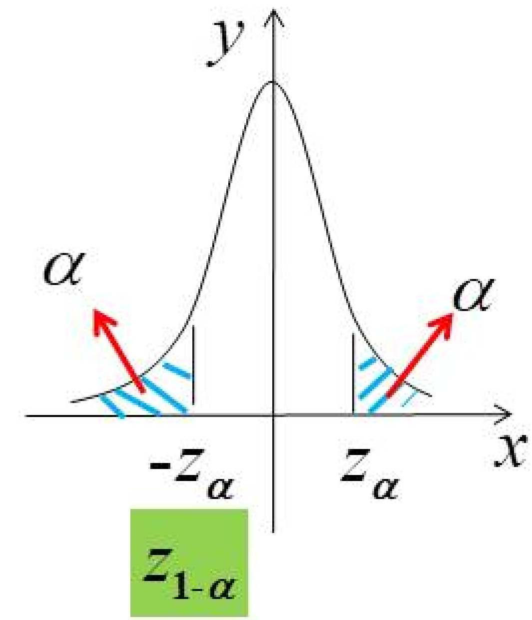
$\therefore z_{0.05}=1.645$



(2)若α=0.025，求 z_α

解：$\Phi(z_{0.025})=1-0.025=0.975=\Phi(1.96)$

$\therefore z_{0.025}=1.96$



(3) 若α=0.975，求 z_α

解： $z_{0.975}=-z_{0.025}=-1.96$

三、正态分布与标准正态分布的关系

1、若 $X \sim N(\mu, \sigma^2)$，则

(1) $Y = aX + b \sim N(a\mu + b, (a\sigma)^2)$，其中 $a \neq 0$

(2) $Y = \dfrac{X - \mu}{\sigma} \sim N(0,1)$

证明: (1) $\because X \sim N(\mu, \sigma^2)$ $\therefore f_X(x) = \dfrac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

$$F_Y(y) = P\{Y \le y\} = P\{aX + b \le y\}$$

$$= \begin{cases} P\{X \le \dfrac{y-b}{a}\} = F_X(\dfrac{y-b}{a}) & a > 0 \\ P\{X \ge \dfrac{y-b}{a}\} = 1 - F_X(\dfrac{y-b}{a}) & a < 0 \end{cases}$$

$$f_Y(y)=\begin{cases} f_X\left(\dfrac{y-b}{a}\right)\cdot\dfrac{1}{a} & a>0 \\ -f_X\left(\dfrac{y-b}{a}\right)\cdot\dfrac{1}{a} & a<0 \end{cases} = f_X\left(\frac{y-b}{a}\right)\cdot\frac{1}{|a|}$$

$$=\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{\left(\frac{y-b}{a}-\mu\right)^2}{2\sigma^2}}\cdot\frac{1}{|a|}=\frac{1}{\sqrt{2\pi}\,|a|\,\sigma}e^{-\frac{[y-(a\mu+b)]^2}{2(|a|\sigma)^2}}$$

$$\therefore\quad Y\sim N((a\mu+b),a^2\sigma^2)$$

(2) 令 $a=\dfrac{1}{\sigma}$, $b=-\dfrac{\mu}{\sigma}$ ，则

当前无法显示此图像。

即 $\dfrac{X-\mu}{\sigma}\sim N(0,1)$

2、若 $X \sim N(\mu, \sigma^2)$，则 $\dfrac{X-\mu}{\sigma} \sim N(0,1)$

(1)

$$F(x) = P\{X \le x\} = P\{\frac{X-\mu}{\sigma} \le \frac{x-\mu}{\sigma}\} = \Phi\left(\frac{x-\mu}{\sigma}\right)$$

$$f(x) = F'(x) = \frac{1}{\sigma}\varphi\left(\frac{x-\mu}{\sigma}\right)$$

(2) 故对 $\forall[a, b]$ 有

$$P\{a \le X \le b\} = F(b) - F(a) = \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)$$

因此，由标准正态分布的分布函数即能求出任意正态 *R.V.* 落入任意区间的概率.

3、概率的计算

(1) $X\sim N(0,1)$时， $P\{a<X\le b\}=\Phi(b)-\Phi(a)$

例1：已知X ~ N(0,1)，求：

(1) $P\{X<2.35\}$ ；(2) $P\{X<-1.24\}$；(3) $P\{|X|<1.54\}$

解：(1) $P\{X<2.35\}=\Phi(2.35)=0.9904$

(2) $P\{X<-1.24\}=\Phi(-1.24)=1-\Phi(1.24)=1-0.8975=0.1075$

(3) $P\{|X|<1.54\}=P\{-1.54<X<1.54\}=\Phi(1.54)-\Phi(-1.54)$

$=2\Phi(1.54)-1=2\times 0.9382-1=0.8764$

(2) $X \sim N(\mu, \sigma^2)$时，$P\{a \le X \le b\} = \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)$

例2：已知X～N(1.5,4)，求：

(1) $P\{X < 3.5\}$ (2) $P\{X > 2\}$

(3) $P\{X < -4\}$ (4) $P\{|X| < 3\}$

解：(1) $P\{X<3.5\}=F(3.5)=\Phi\left(\frac{3.5-1.5}{2}\right)=\Phi(1)=0.8413$

(2) $P\{X>2\}=1-F(2)=1-\Phi\left(\frac{2-1.5}{2}\right)=1-\Phi(0.25)=0.4013$

(3) $P\{X<-4\}=\mathrm{F}(-4) \quad =\Phi(\frac{-4-1.5}{2})=\Phi(-2.75)$

$$=1-\Phi(2.75)=1-0.9970=0.003$$

(4) $P\{|X|<3\}=P\{-3<\mathrm{X}<3\}=\mathrm{F}(3)-\mathrm{F}(-3)$

$$=\Phi(\frac{3-1.5}{2})-\Phi(\frac{-3-1.5}{2})=\Phi(0.75)-\Phi(-2.25)$$

$$=\Phi(0.75)+\Phi(2.25)-1=0.7612$$

(2) $X \sim N(\mu, \sigma^2)$时，$P\{a \le X \le b\} = \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)$

例3：已知X～N(500,100)

(1) 若 $P\{X < x\} = 0.9$ ，求x；

(2) 若 $P\{X > y\} = 0.04$ ，求y

解: (1) $P\{X < x\} = F(x) = \Phi(\frac{x-500}{10}) = 0.90$

$\because \Phi(1.28) \approx 0.90 \quad \therefore \frac{x-500}{10} \approx 1.28 \quad \therefore x \approx 512.8$

(2) $P\{X > y\} = 1 - \Phi(\frac{y-500}{10}) = 0.04 \Rightarrow \Phi(\frac{y-500}{10}) = 0.96$

$\because \Phi(1.75) \approx 0.96 \quad \therefore \frac{y-500}{10} \approx 1.75 \quad \therefore y \approx 517.5$

四、正态分布R.V.的期望与方差

1、标准正态R.V.的期望与方差

设 $X \sim N(0,1)$, $\varphi(x) = \dfrac{1}{\sqrt{2\pi}} e^{-x^2/2} \quad -\infty < x < \infty$

$$E(X) = \int_{-\infty}^{+\infty} x \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx = 0$$

$$D(X) = E(X^2) - [E(X)]^2 = \int_{-\infty}^{+\infty} x^2 \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-x^2/2} dx = \frac{1}{\sqrt{2\pi}} \cdot \sqrt{2} \int_{-\infty}^{+\infty} e^{-(x/\sqrt{2})^2} d\frac{x}{\sqrt{2}}$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \sqrt{2} \cdot \sqrt{\pi} = 1$$

2、正态R.V.的期望与方差

设 $Y \sim N(\mu, \sigma^2)$，则$Y = \mu + \sigma X$

$$E(Y) = E(\mu + \sigma X) = \mu + \sigma E(X) = \mu$$

$$D(Y) = D(\mu + \sigma X) = \sigma^2 D(X) = \sigma^2$$

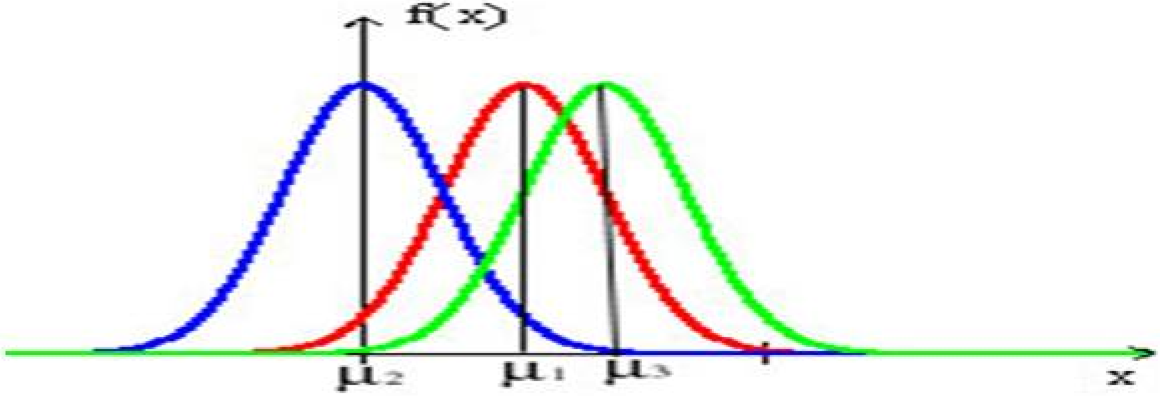
注意：

(1) 若 $X \sim N(\mu, \sigma^2)$，则 $E(X) = \mu$， $D(X) = \sigma^2$

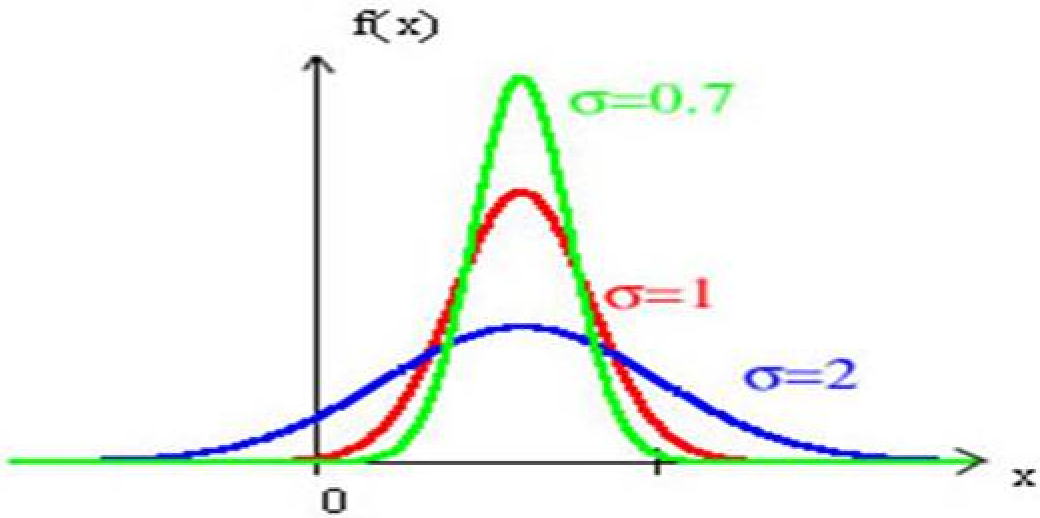
μ决定了图形的中心位置，σ决定了图形中峰的陡峭程度.

(2) 正态分布的性质

① 如果固定σ，改变的μ值，则图形沿 ox轴平移，而不改变形状.



②如果固定μ，当σ越小时图形变得越尖；当σ越大时图形变得越平坦.



③ "3σ"法则

当$X \sim N(0,1)$时

$$P(|X| \leqslant 1) = 2\Phi(1) - 1 = 0.6826$$

$$P(|X| \leqslant 2) = 2\Phi(2) - 1 = 0.9954$$

$$P(|X| \leqslant 3) = 2\Phi(3) - 1 = 0.9974$$

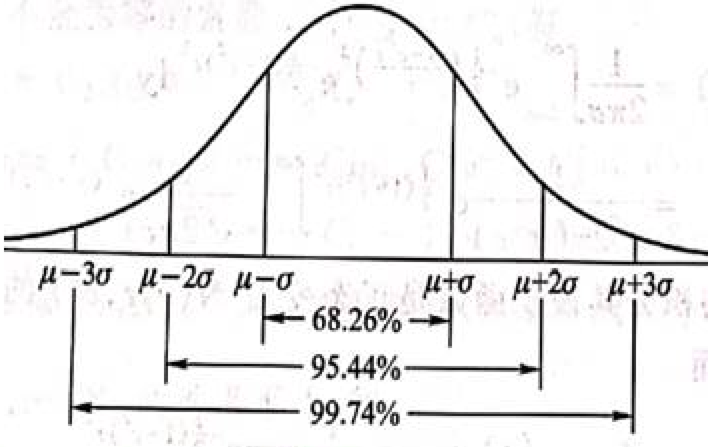
这说明：X的取值几乎全部集中在$[-3,3]$区间内，超出这个范围的可能性仅占不到0.03%.

当$X \sim N(\mu, \sigma^2)$时

$$P\{|X-\mu| < \sigma\} = 2\Phi(1) - 1 = 0.6826$$

$$P\{|X-\mu| < 2\sigma\} = 2\Phi(2) - 1 = 0.9544$$

$$P\{|X-\mu| < 3\sigma\} = 2\Phi(3) - 1 = 0.9974$$



这说明：X的取值几乎全部集中在区间$[\mu-3\sigma, \mu+3\sigma]$

这在统计学上称作"3σ准则".

- 正态分布
  - 正态分布
    - 定义
    - 性质
  - 标准正态分布
    - 定义
    - 性质
    - 分布函数表
    - 上 α 分位数
  - 二者的关系
    - 分布函数之间的关系
    - 概率密度函数之间的关系
    - 正态随机变量落入任意区间概率的计算
  - 正态分布的期望与方差