

# Lecture 6: Monte Carlo Simulation

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# Relevant Reading

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- Sections 15-1 – 15.4
- Chapter 16

# A Little History

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- Ulam, recovering from an illness, was playing a lot of solitaire
- Tried to figure out probability of winning, and failed
- Thought about playing lots of hands and counting number of wins, but decided it would take years
- Asked Von Neumann if he could build a program to simulate many hands on ENIAC

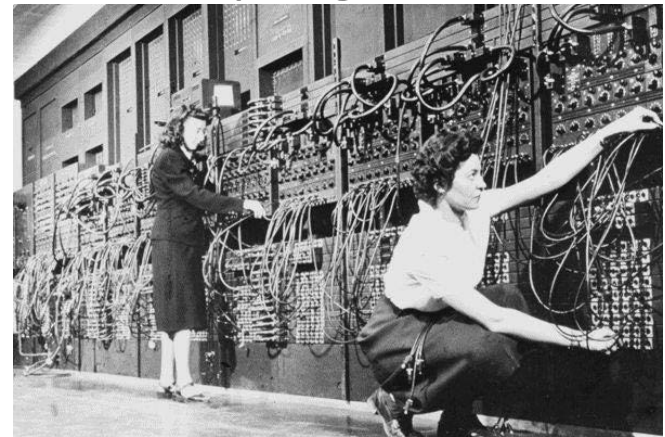


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# Monte Carlo Simulation

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- A method of estimating the value of an unknown quantity using the principles of inferential statistics
- **Inferential statistics**
  - *Population*: a set of examples
  - *Sample*: a proper subset of a population
  - Key fact: a *random sample* tends to exhibit the same properties as the population from which it is drawn
- Exactly what we did with random walks

# An Example

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- Given a single coin, estimate fraction of heads you would get if you flipped the coin an infinite number of times
- Consider one flip



How confident would you be about answering 1.0?

# Flipping a Coin Twice

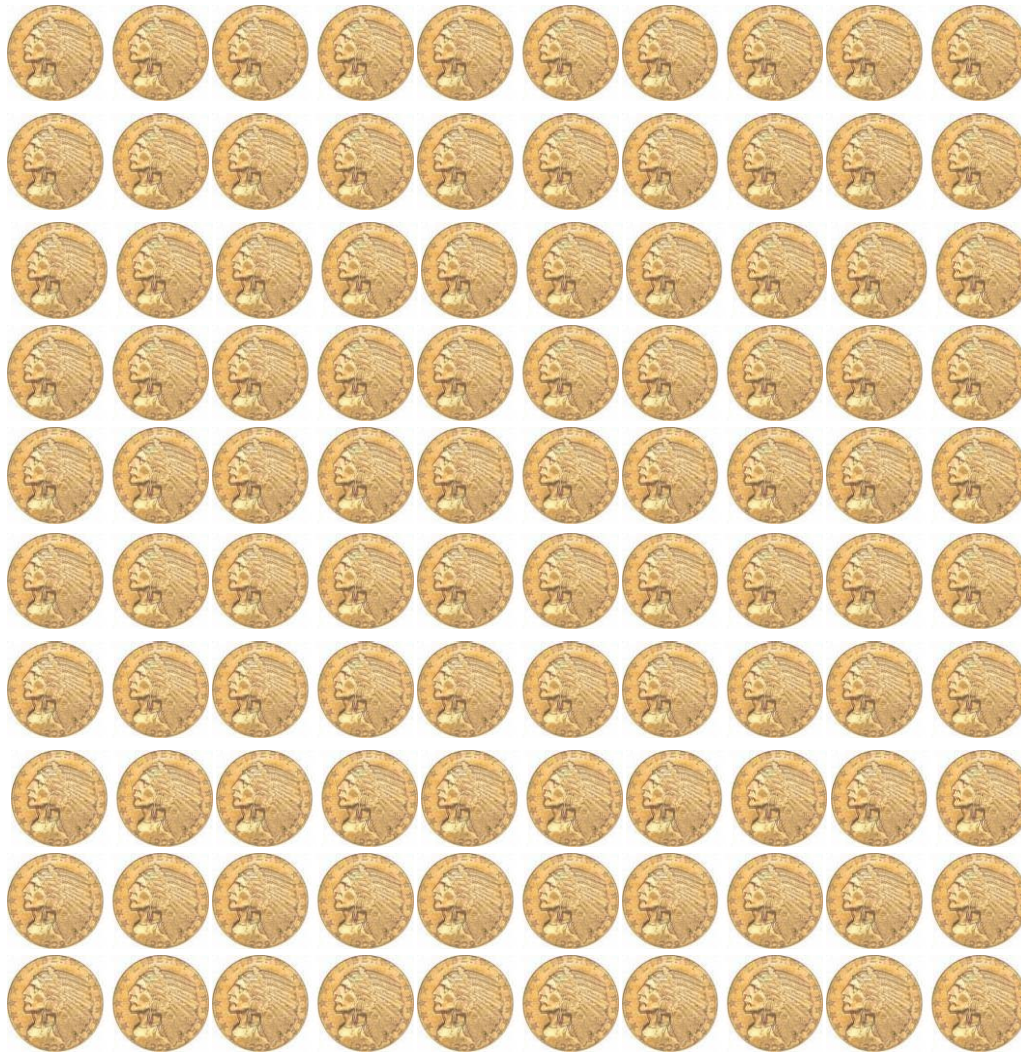
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Do you think that the next flip will come up heads?

# Flipping a Coin 100 Times

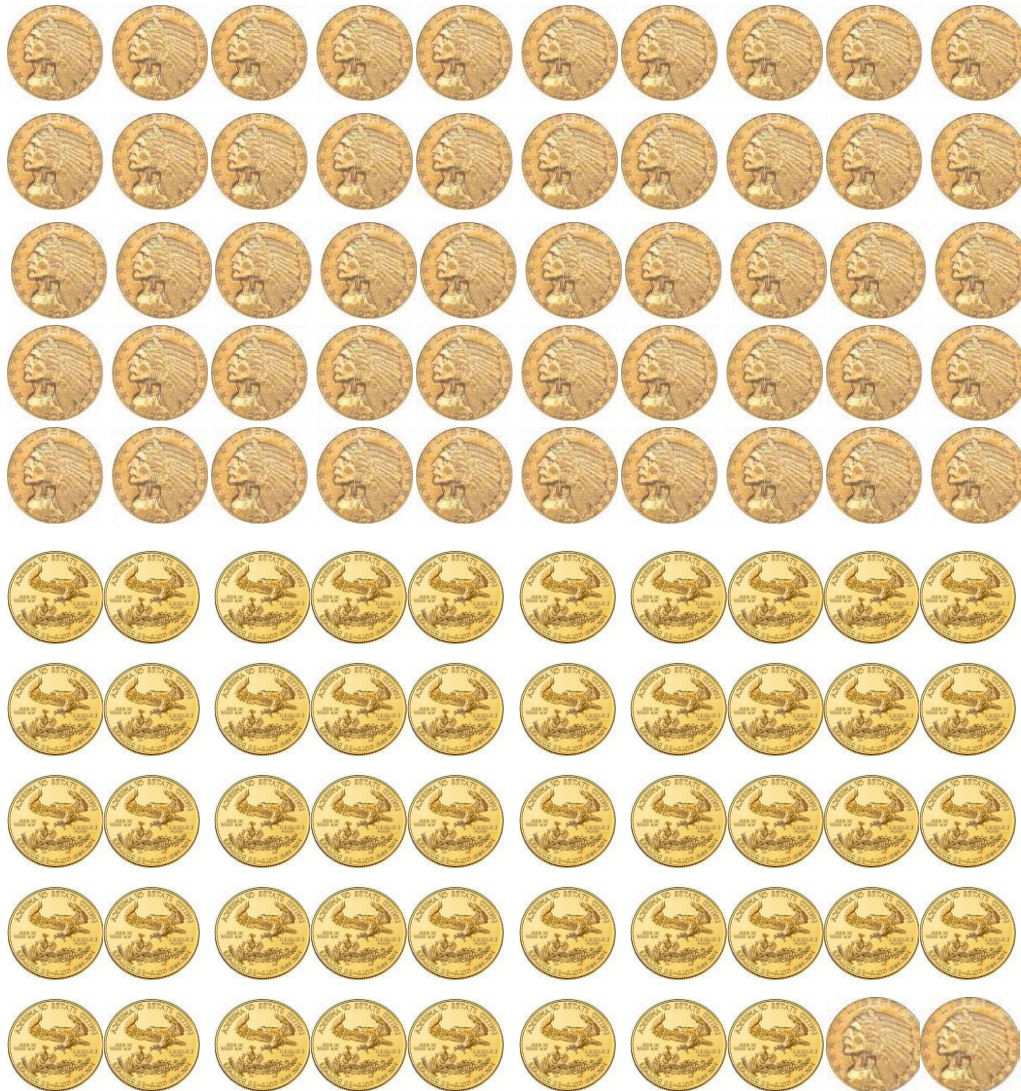
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Now do you  
think that the  
next flip will  
come up heads?

# Flipping a Coin 100 Times

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Do you think that the probability of the next flip coming up heads is  $52/100$ ?

Given the data, it's your best estimate

But confidence should be low



# Why the Difference in Confidence?

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- Confidence in our estimate depends upon two things
- Size of sample (e.g., 100 versus 2)
- Variance of sample (e.g., all heads versus 52 heads)
- As the variance grows, we need larger samples to have the same degree of confidence

# Roulette

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No need to  
simulate, since  
answers obvious

Allows us to  
compare  
simulation results  
to actual  
probabilities

# Class Definition

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```
class FairRoulette():
    def __init__(self):
        self.pockets = []
        for i in range(1,37):
            self.pockets.append(i)
        self.ball = None
        self.pocketOdds = len(self.pockets) - 1
    def spin(self):
        self.ball = random.choice(self.pockets)
    def betPocket(self, pocket, amt):
        if str(pocket) == str(self.ball):
            return amt*self.pocketOdds
        else: return -amt
    def __str__(self):
        return 'Fair Roulette'
```

# Monte Carlo Simulation

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```
def playRoulette(game, numSpins, pocket, bet):
    totPocket = 0
    for i in range(numSpins):
        game.spin()
        totPocket += game.betPocket(pocket, bet)
    if toPrint:
        print(numSpins, 'spins of', game)
        print('Expected return betting', pocket, '=', \
              str(100*totPocket/numSpins) + '%\n')
    return (totPocket/numSpins)

game = FairRoulette()
for numSpins in (100, 1000000):
    for i in range(3):
        playRoulette(game, numSpins, 2, 1, True)
```

# 100 and 1M Spins of the Wheel

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100 spins of Fair Roulette  
Expected return betting 2 = -100.0%

100 spins of Fair Roulette  
Expected return betting 2 = 44.0%

100 spins of Fair Roulette  
Expected return betting 2 = -28.0%

1000000 spins of Fair Roulette  
Expected return betting 2 = -0.046%

1000000 spins of Fair Roulette  
Expected return betting 2 = 0.602%

1000000 spins of Fair Roulette  
Expected return betting 2 = 0.7964%

# Law of Large Numbers

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- In repeated independent tests with the same actual probability  $p$  of a particular outcome in each test, the chance that the fraction of times that outcome occurs differs from  $p$  converges to zero as the number of trials goes to infinity

Does this imply that if deviations from expected behavior occur, these deviations are likely to be ***evened out*** by opposite deviations in the future?

# Gambler's Fallacy

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- “On August 18, 1913, at the casino in Monte Carlo, black came up a record twenty-six times in succession [in roulette]. ... [There] was a near-panicky rush to bet on red, beginning about the time black had come up a phenomenal fifteen times.” -- Huff and Geis, *How to Take a Chance*
- Probability of 26 consecutive reds
- $1/67,108,865$
- Probability of 26 consecutive reds when previous 25 rolls were red
- $1/2$

# Regression to the Mean

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- Following an extreme random event, the next random event is likely to be less extreme
- If you spin a fair roulette wheel 10 times and get 100% reds, that is an extreme event (probability =  $1/1024$ )
- It is likely that in the next 10 spins, you will get fewer than 10 reds
  - But the expected number is only 5
- So, if you look at the average of the 20 spins, it will be closer to the expected mean of 50% reds than to the 100% of the first 10 spins



# Casinos Not in the Business of Being Fair

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# Two Subclasses of Roulette

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```
class EuRoulette(FairRoulette):  
    def __init__(self):  
        FairRoulette.__init__(self)  
        self.pockets.append('0')  
    def __str__(self):  
        return 'European Roulette'
```

```
class AmRoulette(EuRoulette):  
    def __init__(self):  
        EuRoulette.__init__(self)  
        self.pockets.append('00')  
    def __str__(self):  
        return 'American Roulette'
```

# Comparing the Games

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Simulate 20 trials of 1000 spins each

Exp. return for Fair Roulette = 6.56%

Exp. return for European Roulette = -2.26%

Exp. return for American Roulette = -8.92%

Simulate 20 trials of 10000 spins each

Exp. return for Fair Roulette = -1.234%

Exp. return for European Roulette = -4.168%

Exp. return for American Roulette = -5.752%

Simulate 20 trials of 100000 spins each

Exp. return for Fair Roulette = 0.8144%

Exp. return for European Roulette = -2.6506%

Exp. return for American Roulette = -5.113%

Simulate 20 trials of 1000000 spins each

Exp. return for Fair Roulette = -0.0723%

Exp. return for European Roulette = -2.7329%

Exp. return for American Roulette = -5.212%

# Sampling Space of Possible Outcomes

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- Never possible to guarantee perfect accuracy through sampling
- Not to say that an estimate is not precisely correct
- Key question:
  - How many samples do we need to look at before we can have justified confidence on our answer?
- Depends upon variability in underlying distribution

# Quantifying Variation in Data

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$$\text{variance}(X) = \frac{\sum_{x \in X} (x - \mu)^2}{|X|}$$

$$\sigma(X) = \sqrt{\frac{1}{|X|} \sum_{x \in X} (x - \mu)^2}$$

- Standard deviation simply the square root of the variance
- Outliers can have a big effect
- Standard deviation should always be considered relative to mean

# For Those Who Prefer Code

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```
def getMeanAndStd(X):  
    mean = sum(X)/float(len(X))  
    tot = 0.0  
    for x in X:  
        tot += (x - mean)**2  
    std = (tot/len(X))**0.5  
    return mean, std
```

# Confidence Levels and Intervals

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- Instead of estimating an unknown parameter by a single value (e.g., the mean of a set of trials), a confidence interval provides a range that is likely to contain the unknown value and a confidence that the unknown value lays within that range
- “The return on betting a pocket 10k times in European roulette is -3.3%. The margin of error is  $\pm 3.5\%$  with a 95% level of confidence.”
- What does this mean?
- If I were to conduct an infinite number of trials of 10k bets each,
  - My expected average return would be -3.3%
  - My return would be between roughly -6.8% and +0.2% 95% of the time

# Empirical Rule

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- Under some assumptions discussed later
  - ~68% of data within one standard deviation of mean
  - ~95% of data within 1.96 standard deviations of mean
  - ~99.7% of data within 3 standard deviations of mean



# Applying Empirical Rule

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```
resultDict = {}
games = (FairRoulette, EuRoulette, AmRoulette)
for G in games:
    resultDict[G().__str__()] = []
for numSpins in (100, 1000, 10000):
    print('\nSimulate betting a pocket for', numTrials,
          'trials of', numSpins, 'spins each')
    for G in games:
        pocketReturns = findPocketReturn(G(), 20,
                                           numSpins, False)
        mean, std = getMeanAndStd(pocketReturns)
        resultDict[G().__str__()].append((numSpins,
                                           100*mean,
                                           100*std))
    print('Exp. return for', G(), '=',
          str(round(100*mean, 3))
          + '%, ', '+/- ' + str(round(100*1.96*std, 3))
          + '% with 95% confidence')
```

# Results

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**Simulate betting a pocket for 20 trials of 1000 spins each**

**Exp. return for Fair Roulette = 3.68%, +/- 27.189% with 95% confidence**

**Exp. return for European Roulette = -5.5%, +/- 35.042% with 95% confidence**

**Exp. return for American Roulette = -4.24%, +/- 26.494% with 95% confidence**

**Simulate betting a pocket for 20 trials of 100000 spins each**

**Exp. return for Fair Roulette = 0.125%, +/- 3.999% with 95% confidence**

**Exp. return for European Roulette = -3.313%, +/- 3.515% with 95% confidence**

**Exp. return for American Roulette = -5.594%, +/- 4.287% with 95% confidence**

**Simulate betting a pocket for 20 trials of 1000000 spins each**

**Exp. return for Fair Roulette = 0.012%, +/- 0.846% with 95% confidence**

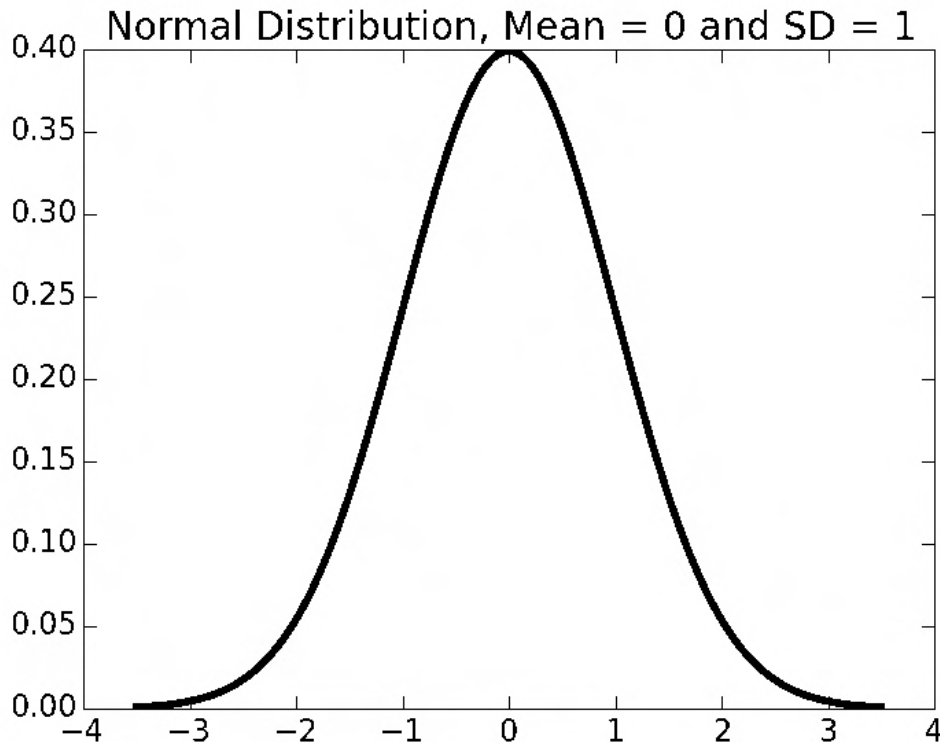
**Exp. return for European Roulette = -2.679%, +/- 0.948% with 95% confidence**

**Exp. return for American Roulette = -5.176%, +/- 1.214% with 95% confidence**

# Assumptions Underlying Empirical Rule

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- The mean estimation error is zero
- The distribution of the errors in the estimates is normal



# Defining Distributions

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- Use a probability distribution
- Captures notion of relative frequency with which a random variable takes on certain values
  - Discrete random variables drawn from finite set of values
  - Continuous random variables drawn from reals between two numbers (i.e., infinite set of values)
- For **discrete variable**, simply list the probability of each value, must add up to 1
- **Continuous case** trickier, can't enumerate probability for each of an infinite set of values

# PDF's

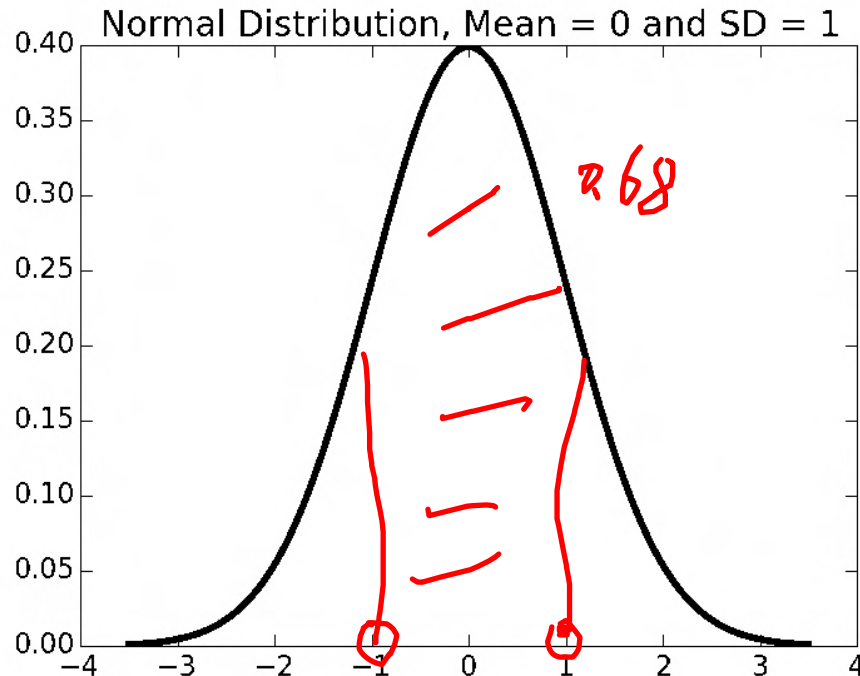
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- Distributions defined by *probability density functions* (PDFs)
- Probability of a random variable lying between two values
- Defines a curve where the values on the x-axis lie between minimum and maximum value of the variable
- Area under curve between two points, is probability of example falling within that range

# Normal Distributions

$$P(x) = \frac{1}{\sigma\sqrt{2\pi}} * e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$e = \sum_{n=0}^{\infty} \frac{1}{n!}$$



~68% of data within one standard deviation of mean

~95% of data within 1.96 standard deviations of mean

~99.7% of data within 3 standard deviations of mean

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