Retarded kernels for longitudinal survival analysis and dynamic prediction

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— Model B: Code Details —

1. Model B

In the Supplementary Material for 'Retarded kernels for longitudinal survival analysis' Davies, Coolen and Galla (2021) we define the following integral

$$\mathcal{I}_{ij}[\{a_\mu,\tau_\mu\}] = \int_0^{\min(s_j,T_i)} \sum_{\mu=1}^{p} \beta_\mu(T_i,t',s_j) z_\mu^j(t')\,\mathrm{d}t', \tag{S1}$$

that appears in the negative log likelihood equation and the survival probability equation for retarded kernel models. We then derive, for Model B with nearest neighbour step function interpolation,

$$\mathcal{I}_{ij}^{(\mathrm{B})}[\{a_\mu,\tau_\mu\}] = \sum_{\mu=1}^{p}\sum_{\ell=1}^{n_j} a_\mu z_\mu^j(t_\ell)\theta(T_i-U_{j\ell})\Bigg\{\mathrm{e}^{-T_i/\tau_\mu}\Big(\mathrm{e}^{\min(T_i,U_{j\ell+1})/\tau_\mu}-\mathrm{e}^{U_{j\ell}/\tau_\mu}\Big) + \Big(\min(T_i,U_{j\ell+1})-U_{j\ell}\Big)\Bigg(\frac{\mathrm{e}^{-T_i/\tau_\mu}}{\min(s_j,T_i)}+\theta(T_i-s_j)\frac{1-\mathrm{e}^{(s_j-T_i)/\tau_\mu}}{s_j}\Bigg)\Bigg\}. \tag{S2}$$

This equation is used to write the Model B `C++` codes provided in the GitHub repository `https://github.com/AnnieDavies/Supplement_Davies_Coolen_Galla_2021`. In particular this equation is used in the functions `mu_sum1`, `mu_sum2` and `EXP_B`. For the interested reader who may wish to follow the source codes in detail, the form of Eq (S2) used to write the codes is as follows:

$$\mathcal{I}_{ij}^{(\mathrm{B})}[\{a_\mu,\tau_\mu\}] = \sum_{\mu=1}^{p}\sum_{\ell=1}^{n_j} a_\mu z_\mu^j(t_{j\ell})\theta(T_i-U_{j\ell})\Bigg[\frac{(\min(U_{j\ell+1},T_i)-U_{j\ell})}{\min(s_j,T_i)} + \Big(\mathrm{e}^{-(T_i-\min(U_{j\ell+1},T_i))/\tau_\mu}-\mathrm{e}^{-(T_i-U_{j\ell})/\tau_\mu}\Big) - \frac{1}{\min(s_j,T_i)}\Big(\mathrm{e}^{-(T_i-\min(s_j,T_i))/\tau_\mu}-\mathrm{e}^{-T_i/\tau_\mu}\Big)\Big(\min(U_{j\ell+1},T_i)-U_{j\ell}\Big)\Bigg] \tag{S3}$$

Below we show that Eqs (S3) and (S2) are equivalent. First we rearrange Eq (S3) to obtain

$$\mathcal{I}_{ij}^{(\mathrm{B})}[\{a_\mu,\tau_\mu\}] = \sum_{\mu=1}^{p}\sum_{\ell=1}^{n_j} a_\mu z_\mu^j(t_{j\ell})\theta(T_i-U_{j\ell})\Bigg[\mathrm{e}^{-T_i/\tau_\mu}\Big(\mathrm{e}^{\min(U_{j\ell+1},T_i)/\tau_\mu}-\mathrm{e}^{U_{j\ell}/\tau_\mu}\Big) + \Big(\min(U_{j\ell+1},T_i)-U_{j\ell}\Big)\Bigg(\frac{1}{\min(s_j,T_i)}-\frac{1}{\min(s_j,T_i)}\Big(\mathrm{e}^{(\min(s_j,T_i)-T_i)/\tau_\mu}-\mathrm{e}^{-T_i/\tau_\mu}\Big)\Bigg)\Bigg]. \tag{S4}$$

Now if $T_i > s_j$ such that $\min(s_j,T_i)=s_j$, Eq (S4) becomes

$$\mathcal{I}_{ij}^{(\mathrm{B})}[\{a_\mu,\tau_\mu\}] = \sum_{\mu=1}^{p}\sum_{\ell=1}^{n_j} a_\mu z_\mu^j(t_{j\ell})\theta(T_i-U_{j\ell})\Bigg[\mathrm{e}^{-T_i/\tau_\mu}\Big(\mathrm{e}^{\min(U_{j\ell+1},T_i)/\tau_\mu}-\mathrm{e}^{U_{j\ell}/\tau_\mu}\Big) + \Big(\min(U_{j\ell+1},T_i)-U_{j\ell}\Big)\Bigg(\frac{\mathrm{e}^{-T_i/\tau_\mu}}{s_j}+\frac{1-\mathrm{e}^{(s_j-T_i)/\tau_\mu}}{s_j}\Bigg)\Bigg]. \tag{S5}$$

Otherwise ($T_i \le s_j$ such that $\min(s_j,T_i)=T_i$) we find

$$\mathcal{I}_{ij}^{(\mathrm{B})}[\{a_\mu,\tau_\mu\}] = \sum_{\mu=1}^{p}\sum_{\ell=1}^{n_j} a_\mu z_\mu^j(t_{j\ell})\theta(T_i-U_{j\ell})\Bigg[\mathrm{e}^{-T_i/\tau_\mu}\Big(\mathrm{e}^{\min(U_{j\ell+1},T_i)/\tau_\mu}-\mathrm{e}^{U_{j\ell}/\tau_\mu}\Big) + \Big(\min(U_{j\ell+1},T_i)-U_{j\ell}\Big)\Bigg(\frac{\mathrm{e}^{-T_i/\tau_\mu}}{T_i}\Bigg)\Bigg]. \tag{S6}$$

Since $s_j = \min(s_j, T_i)$ in Eq (S5) and $T_i = \min(s_j, T_i)$ in Eq (S6), the only difference between these two equations is the term the extra term $\frac{1-\mathrm{e}^{(s_j-T_i)/\tau_\mu}}{s_j}$ in Eq (S5) when $T_i > s_j$. Therefore, using the step function $\theta(T_i - s_j)$ we can combine Eqs (S5) and (S6) to obtain Eq (S2):

$$\mathcal{I}_{ij}^{(\mathrm{B})}[\{a_\mu, \tau_\mu\}] = \sum_{\mu=1}^{p} \sum_{\ell=1}^{n_j} a_\mu z_\mu^j(t_{j\ell})\theta(T_i - U_{j\ell}) \Bigg[\mathrm{e}^{-T_i/\tau_\mu} \left(\mathrm{e}^{\min(U_{j\ell+1}, T_i)/\tau_\mu} - \mathrm{e}^{U_{j\ell}/\tau_\mu} \right) + \Big(\min(U_{j\ell+1}, T_i) - U_{j\ell} \Big) \left(\frac{\mathrm{e}^{-T_i/\tau_\mu}}{\min(s_j, T_i)} + \theta(T_i - s_j) \frac{1 - \mathrm{e}^{(s_j - T_i)/\tau_\mu}}{s_j} \right) \Bigg]. \quad \text{(S7)}$$