HOMEWORK 2
February 18^{th} 2016
beg2119

Question 1. *Using direct proof, show that the sum of two consecutive perfect squares is odd.*

Proof.

1. Suppose that the sum of two consecutive perfect squares is odd
2. The definition of consecutive squares is $(x)^2 + (x+1)^2$
3. $2x^2 + 2x + 1$
4. $2(x^2 + x) + 1$
5. Set k to the value of the parentheses,
6. $k = (x^2 + x)$
7. Thus, $2k + 1$
8. The definition of an odd number is $(\forall k \in \mathbb{Z})(2k+1))$
9. $\therefore$ The sum of two consecutive perfect squares is odd

$\square$

Question 2. *Use proof by contrapositive to prove the following proposition: If x^3 is odd, then x is odd.*

Proof.

1. Suppose that x^3 is odd and x is odd
2. the contrapositive must also be true: "if x is not odd, then x^3 is not odd"
3. OR, "if x is even, then x^3 is even"
4. The definition of an even number is $(\exists k \in \mathbb{Z})x = 2k$
5. Cube both sides,
6. $x^3 = 8k^3$
7. Let $m = 4k^3$ thus $x^3 = 2m$
8. $\therefore$ If x is even, then x^3 is even

$\square$

Question 3. *Prove that an integer is odd if and only if it is the sum of two consecutive integers.*

Proof.
1. Formally, $(\forall x \in \mathbb{Z})(2 \nmid (x + (x + 1)) \equiv 2x + 1$
2. The definition of an odd number is $2x + 1$
3. $\therefore$ If it is the sum of two consecutive integers then it is odd

$$* \ \boldsymbol{p \Leftrightarrow q} \ *$$

4. Suppose that x is odd,
5. $x = 2k + 1$
6. Can be expressed as $2k = k + k + 1$ which signifies consecutive integers such that $(\forall\ x \in \mathbb{Z})$
7. $\therefore$ If it is odd then it is it is the sum of two consecutive integers

$\square$

Question 4. *Prove or disprove the following proposition: any prime number greater than 2 can be expressed as 1 less than a power of 2. More formally, this means that for every prime number $p > 2$, there exists a natural number n such that $p = 2^n - 1$.*

Proof.
1. Let $\mathbb{P}$ denote the set of all prime numbers
2. $(\forall\ p > 2 \in \mathbb{P})(\exists\ n \in \mathbb{N})\ p = 2^n - 1$
3. Suppose that the previous proposition is true
4. Let $p = 11$, which clearly satisfies the condition
5. $11 = 2^n - 1$; $(2^3 = 8)$ & $(2^4 = 16)$
6. There is no natural value of n that satisfies this theorem when $p = 11$
7. $\therefore$ This proposition is disproven by counterexample

$\square$

Question 5. *Use direct proof for parts 1 and 2.*

1. *Let x be an integer. Prove the following proposition:*

   If $x \geq 3$, then $x^2 > 2x + 1$.

   Proof.
   (a) Suppose that $x \geq 3$
   (b) Multiply by x on both sides
   (c) $x^2 \geq 3x \equiv x^2 \geq 2x + x$

(d) From the original conditional clause we see $x^2 > 2x + 1$

(e) While $x \geq 3$,

(f) It must be the case that $2x + x > 2x + 1$

(g) $\therefore x^2 > 2x + 1$.

$\square$

2. *Let a and b be integers, and let c be a negative integer. Prove the following proposition:*

If $a > b$, then $a^2c^2 - 2abc^2 + b^2c^2$ is positive.

Proof. (a) $a^2c^2 - 2abc^2 + b^2c^2 = (ac - bc)^2$

(b) Take $a > b$

(c) Multiply by c, (a negative integer)

(d) $ac < bc$

(e) $ac - bc < 0$

(f) Multiply by negative (square)

(g) $(ac - bc)^2 > 0$

(h) $\therefore ((ac - bc)^2)$, or $(a^2c^2 - 2abc^2 + b^2c^2)$, is positive

$\square$

Question 6. *Let x and y be real numbers. Using proof by cases, show that the following property holds:*

$$|x + y| \leq |x| + |y|$$

Proof.

- <u>CASE 1</u>
  Let both x & y be positive
  $|x + y| \leq |x| + |y|$
  $x + y \leq x + y$
  - TRUE

- <u>CASE 2</u>
  Let x be positive and y be negative $|x - y| \leq |x| + |-y|$
  $x - y \leq x + y$
  - TRUE

- <u>CASE 3</u>
  Let both x & y be negative
  $|-x - y| \leq |-x| + |-y|$
  $|-(x + y)| \leq x + y$
  $x + y \leq x + y$
  - TRUE

- <u>CASE 4</u>
  Case 4:
  Let x be 0 and y be positive
  $|0 + y| \leq |0| + |y|$
  $|y| \leq y$
  $y \leq y$
  - TRUE

$\square$

Question 7. *Using proof by contradiction, show that there are no integers x, y that satisfy the equation $5x + 25y = 1723$.*

Proof.

1. Suppose that $(x, y \in \mathbb{Z})5x + 25y = 1723$
2. $5(x + 5y) = 1723$
3. $x + 5y = 1723/5$
4. The result is not in the set of integers, and two integers cannot add to create something outside of that set
5. $\therefore$ There are no integers x & y that satisfy the equation $5x + 25y = 1723$

$\square$

Question 8. *Prove, using any method you'd like, that the sum of any three consecutive integers is divisible by 3.*

Proof.

1. Assume that the sum of any three consecutive integers is divisible by 3
2. $(x, y, z \in \mathbb{Z})\ x < y < z$
3. As x, y, z are consecutive, $y = x + 1, z = x + 1 + 1$
4. Thus, $x + y + z \equiv x + (x + 1) + (x + 1 + 1)$
5. $3x + 3 = 3(x + 1) = 3y$
6. Because $(y \in \mathbb{Z})$, $3|3y$
7. $\therefore$ The sum of any three consecutive integers is divisible by 3

$\square$

Question 9. *Prove, using any method you'd like, that the difference between distinct, nonconsecutive perfect squares is composite. Recall that an integer x is composite if and only if there exists some integer y such that $1 < y < x$ and $y|x$. In other words, x is composite if it has some positive factor other than 1 and itself, i.e. x is not prime.*

Proof. 1. Let a & b be nonconsecutive perfect squares

2. $(\exists\, x, y \in \mathbb{Z}^+)$ such that $a = x^2$ & $b = y^2$
3. x & y are non consecutive, so $(x - y) \neq 1$
4. The difference between distinct, nonconsecutive perfect squares should be composite
5. $x^2 - y^2 = (x + y)(x - y)$
6. Neither factor $(x + y)$ nor factor $(x - y)$ equals 1 or $x^2 - y^2$
7. $\therefore$ The difference between distinct, nonconsecutive perfect squares is composite

$\square$

Question 10. *Convert the following statements into the formal notation of propositional logic (i.e. using variables and logical operators). Make sure to explain what each variable you introduce represents.*

- *Whenever we add a rational number and an irrational number, the sum is irrational.*
  Let $\mathbb{R}$ represent the set of all real numbers, $\mathbb{Q}$ represent the set of all rational numbers, & $\mathbb{R}\backslash\mathbb{Q}$ represent the set of all irrational numbers
  $Q + \mathbb{R}\backslash\mathbb{Q} \Rightarrow \mathbb{R}\backslash\mathbb{Q}$
- *Two integers are odd only if their sum is even.*
  Let $\mathbb{Z}$ represent the set of all integers and $(x, y, z \in \mathbb{Z})$
  $2|x, y$ but $2 \nmid z$
  $x + y \Rightarrow z$
- *It is necessary that $a|(b + c)$ be true for $a|b$ and $a|c$ to be true.*
  $a|(b + c) \Rightarrow a|b \wedge a|c$
- *For $(ac)|(bd)$ to be true , it is sufficient that $a|b$ and $c|d$.*
  $a|b \wedge c|d \Rightarrow (ac)|(bd)$
- *For x to be an odd number, it is necessary and sufficient that $x-1$ is even.*
  $(x \in \mathbb{Z})$
  $2|(x - 1) \Rightarrow x$
- *An integer is even if and only if its square is even.*
  $(x \in \mathbb{Z})$
  $2|x \Leftrightarrow 2|x^2$

BONUS

Question 11. *Let x and y be two numbers. State whether the following proposition is true or not:*

$$\text{If } x > y \text{ and } x < y, \text{ then } x = y.$$

True. Contradictions imply everything. (on the contrary tautologies are implied by everything)