

MLT Homework 5

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Question 1

Subquestion 1.1

Consider a hypothesis class $\mathcal{H} = \cup_{n=1}^{\infty} \mathcal{H}_n$, where for every $n \in \mathbb{N}$, \mathcal{H}_n is finite. Find a weighting function $w : \mathcal{H} \rightarrow [0, 1]$ such that $\sum_{h \in \mathcal{H}} w(h) \leq 1$ and so that for all $h \in \mathcal{H}$, $w(h)$ is determined by $|\mathcal{H}_{n(h)}|$.

Solution

Since we have a countably infinite union of finite sets, we know that the number of elements is countably infinite. Therefore, we can number them as:

$$h_1, h_2, \dots$$

If we pick weights as:

$$w(h_i) = \left(\frac{1}{2^{|\mathcal{H}_{n(h_i)}|}} \right)^i ; \quad i = 1, 2, \dots$$

the sum of the weights in the worst case would be, when $|\mathcal{H}_{n(h_i)}| = 1$ ($\forall i$):

$$\sum_{i=1}^{\infty} w(h_i) = \sum_{i=1}^{\infty} \left(\frac{1}{2} \right)^i = 1$$

Subquestion 1.2

Define such a function w when for all n \mathcal{H}_n is countable (possibly infinite).

Solution

Countably infinite union of countable sets is again a countable set, so we can choose the same weighted function as before.

Question 2

In this question we wish to show a No-Free-Lunch result for nonuniform learnability.

Subquestion 2.1

Let A be a nonuniform learner for class \mathcal{H} . For each $n \in \mathbb{N}$ define $\mathcal{H}_n^A = \{h \in \mathcal{H} : m^{NUL}(0.1, 0.1, h) \leq n\}$. Prove that each such class \mathcal{H}_n has a finite VC-dimension.

Solution

Because \mathcal{H} is nonuniform learnable, it is a union of agnostic PAC learnable hypothesis classes (Theorem 7.2). That is, $\mathcal{H} = \bigcup_{n \in \mathbb{N}} \mathcal{H}_n^A$, and \mathcal{H}_n^A is agnostic PAC learnable for all $n \in \mathbb{N}$. Because each \mathcal{H}_n^A is agnostic PAC learnable, it also has a finite VC-dimension, by the Fundamental Theorem of Statistical Learning (Theorem 6.7).

Subquestion 2.2

Prove that if class \mathcal{H} is nonuniformly learnable then there are classes \mathcal{H}_n so that $\mathcal{H} = \bigcup_{n \in \mathbb{N}} \mathcal{H}_n$ and, for every $n \in \mathbb{N}$, $VCdim(\mathcal{H}_n)$ is finite.

Solution

Let define:

$$\mathcal{H}_n = \{h \in \mathcal{H} : m^{NUL}(0.1, 0.1, h) \leq n\}$$

It is obvious that $\mathcal{H} = \bigcup_{n \in \mathbb{N}} \mathcal{H}_n$. From previous point we also know that $VCdim(\mathcal{H}_n)$ is finite.

Subquestion 2.3

Let \mathcal{H} be a class that shatters some infinite set. Then for every sequence of classes $(\mathcal{H}_n : n \in \mathbb{N})$ such that $\mathcal{H} = \bigcup_{n \in \mathbb{N}} \mathcal{H}_n$, there exists some n for which $VCdim(\mathcal{H}_n) = \infty$.

Solution

By assumption the hypothesis class \mathcal{H} shatters some infinite set K . Let $(\mathcal{H}_n : n \in \mathbb{N})$ be a set of hypothesis classes, each having a finite VC-dimension. We define subsets $K_n \subseteq K$ such that, for all n , $|K_n| > \text{VCdim}(\mathcal{H}_n)$, and all subsets are nonoverlapping, so $K_n \cap K_m = \emptyset$.

Now, for each K_n we pick a function $f_n : K_n \rightarrow \{0, 1\}$ so that no $h \in \mathcal{H}_n$ agrees with f_n on the domain K_n . Such a function exists for all K_n , because \mathcal{H}_n does not shatter K_n , since $|K_n| > \text{VCdim}(\mathcal{H}_n)$.

Next, we define the function $f : K \rightarrow \{0, 1\}$ by combining all f_n 's

$$f(k) = \begin{cases} f_1(k) & \text{if } k \in K_1 \\ \vdots \\ f_n(k) & \text{if } k \in K_n \end{cases}$$

where $k \in K$ are singletons in K .

We define $H_0 = \cup_{n \in \mathbb{N}} \mathcal{H}_n$. By the construction of all f_n 's it follows that f is not contained in H_0 . However, $f \in \mathcal{H}$, because \mathcal{H} shatters K by assumption. Put differently, $f \in (\mathcal{H} \setminus H_0)$.

This implies that $(\mathcal{H} \setminus H_0) \neq \emptyset$. And since we did not put any restrictions on \mathcal{H}_n 's except $\text{VCdim}(\mathcal{H}_n) < \infty$, $(\mathcal{H} \setminus H_0)$ must contain at least one hypothesis class that has infinite VC-dimension, which concludes the proof.

Subquestion 2.4

Construct a class \mathcal{H}_1 of functions from the unit interval $[0, 1]$ to $\{0, 1\}$ that is nonuniformly learnable but not PAC learnable.

Solution

Let denote:

$$\mathcal{H}_n = \{h_{a_1, \dots, a_n, b_1, \dots, b_n}; 0 \leq a_1 < b_1 \leq 1, \dots, 0 \leq a_n < b_n \leq 1\}$$

where:

$$h_{a_1, \dots, a_n, b_1, \dots, b_n}(x) = \begin{cases} 1; & a_1 \leq x \leq b_1 \text{ or } \dots \text{ or } a_n \leq x \leq b_n \\ 0; & \text{otherwise} \end{cases}$$

We claim that $\text{VCdim}(\mathcal{H}_n) = 2n$.

- $\text{VCdim}(\mathcal{H}_n) \geq 2n$: We would like to show that \mathcal{H}_n shatters a set of $2n$ points. Assume that $0 \leq x_1 < \dots < x_{2n} \leq 1$. Labeling for which we need the largest number of disjoint intervals is when x -es with even indexes are labeled 0 and others are labeled 1 (or vice versa). This is exactly n intervals that we need, so \mathcal{H}_n shatters $\{x_1, \dots, x_{2n}\}$.

- $\text{VCdim}(\mathcal{H}_n) \leq 2n$: Now, we are going to take a set of $2n + 1$ elements; $0 \leq x_1 < \dots < x_{2n+1} \leq 1$. If we want that \mathcal{H}_n shatters our set, there must exist a labeling function which can label the situation where x -es with odd indexes are labeled 1 and others are labeled 0. In order to do that we would need at least $n + 1$ intervals, which we do not have. So, \mathcal{H}_n does not shatter a set of $2n + 1$ elements.

Denote $\mathcal{H} = \bigcup_{i=1}^{\infty} \mathcal{H}_n$. From previous points we can conclude that \mathcal{H} is nonuniformly learnable. It is not PAC learnable due to $\text{VCdim}(\mathcal{H}) = \infty$.

Subquestion 2.5

Construct a class \mathcal{H}_2 of functions from the unit interval $[0, 1]$ to $\{0, 1\}$ that is not nonuniformly learnable.

Solution

Define \mathcal{H} as a set of all intervals over $[0, 1]$. It is obvious that \mathcal{H} shatters a set $S = \{\frac{1}{n}; n \in \mathbb{N}\}$. We also know that S is infinite. Therefore \mathcal{H} is not nonuniformly learnable (from previous points).

Question 3

Prove the Symmetrization Lemma (= double sample trick):

For any $\epsilon > 0$ such that $n\epsilon^2 \geq 2$,

$$\sup_{f \in \mathcal{F}} P_n [(R(f) - R_n(f)) \geq \epsilon] \leq P_{2n} \left[\sup_{f \in \mathcal{F}} (R'_n(f) - R_n(f)) \geq \frac{\epsilon}{2} \right]$$

where R is the risk, R_n empirical risk for the sample Z_1, \dots, Z_n and R'_n empirical risk for ghost sample Z'_1, \dots, Z'_n .

Solution

This question was solved with the help of [1]. Denote f^* a function that maximize $(R(f) - R_n(f))$. First, we would like to prove: If $(R(f^*) - R_n(f^*) \geq \epsilon)$ and $(R(f^*) - R'_n(f^*) \leq \frac{\epsilon}{2})$ then $(R'_n(f^*) - R_n(f^*) \geq \frac{\epsilon}{2})$.

$$\begin{aligned} \epsilon &< R(f^*) - R_n(f^*) \\ &= R(f^*) - R'_n(f^*) + R'_n(f^*) - R_n(f^*) \\ &\leq R'_n(f^*) - R_n(f^*) + \frac{\epsilon}{2} \end{aligned}$$

So, $(R'_n(f^*) - R_n(f^*)) \geq \frac{\epsilon}{2}$. If we write that with indicators:

$$I \left[R(f^*) - R_n(f^*) > \epsilon, R(f^*) - R'_n(f^*) \leq \frac{\epsilon}{2} \right] \leq I \left[R'_n(f^*) - R_n(f^*) \geq \frac{\epsilon}{2} \right]$$

Now, we are going to compute expected value over Z'_1, \dots, Z'_n :

$$I[R(f^*) - R_n(f^*) > \epsilon] P_n \left[R(f^*) - R'_n(f^*) \leq \frac{\epsilon}{2} \right] \leq P_n \left[R'_n(f^*) - R_n(f^*) \geq \frac{\epsilon}{2} \right]$$

With Chebishyev inequality we get:

$$P_n \left[R(f^*) - R'_n(f^*) \leq \frac{\epsilon}{2} \right] \geq 1 - \frac{4\text{Var}(f^*)}{n\epsilon^2} \geq 1 - \frac{1}{n\epsilon^2} \geq \frac{1}{2}$$

Therefore:

$$I[R(f^*) - R_n(f^*) \geq \epsilon] \leq 2P_n \left[R'_n(f^*) - R_n(f^*) \geq \frac{\epsilon}{2} \right]$$

Last, we again compute expectation but this time over Z_1, \dots, Z_n :

$$P_n [R(f^*) - R_n(f^*) \geq \epsilon] \leq 2P_{2n} \left[R'_n(f^*) - R_n(f^*) \geq \frac{\epsilon}{2} \right]$$

References

- [1] Han Liu John Lafferty and Larry Wasserman. Concentration of measure. www.stat.cmu.edu/~larry/=sml/Concentration.pdf. Online; accessed 14 October 2018.