MLT Homework 8

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Question 1

The Aggregating Algorithm plays $w_1^k = 1/K$ *and updates as*

$$w_{t+1}^k = \frac{w_t^k e^{-l_t^k}}{\sum_{j=1}^K w_t^j e^{-l_t^j}}$$

Let us define the Kullback-Leibler divergence *aka* relative entropy *(notion of distance between probability distributions) from* $p \in \Delta_K$ *to* $q \in \Delta_K$ *by*

$$KL(p.q) = \sum_{k=1}^K p_k \ln \frac{p_k}{q_k}$$

Fix $w_t \in \Delta_K$ *and* $l_t \in \mathbb{R}^K$. *Consider the minimisation problem*

$$\min_{w \in \Delta_K} w^T l_t + KL(w, w_t) \tag{1}$$

Subquestion 1.1

Show that the minimiser of problem (1) *is* w_{t+1}.

Solution

Subquestion 1.2

Show that the value of problem (1) *is the mix loss.*

Solution

Question 2

We saw in the lecture that the Hedge algorithm (for the dot-loss game) with learning rate $\eta = \sqrt{\frac{8 \ln K}{T}}$ *has regret after* T *rounds bounded by* $\sqrt{T/2 \ln K}$*. In practice, we may not know* T *in advance, or we may even desire an algorithm that has good guarantees for all* T *simultaneously, i.e. that keeps on operating forever.*
Consider the following exponential (base 3) restarting schedule to accomplish this. We run Hedge for 1 round, with η *tuned for 1 round. After that, we restart Hedge, and run it for 3 rounds with* η *tuned for 3 rounds. After that, we restart Hedge again for 9 rounds with* η *tuned for 9 rounds, and so on.*
Prove that the overall accumulated regret of Hedge with this scheme is bounded above by a universal constant times $\sqrt{T \ln K}$*. (Your argument should work for* T *that are not a power of 3).*

Solution

Question 3

Consider the $K = 2$ *expert version of the* T*-round dot loss game (Definition 2). In this exercise we will prove that the worst-case expected regret is at least of order* $\sqrt{T}$*. Consider an adversary that for each* $t = 1, \dots, T$ *assigns loss vector* $l_t = (0, 1)$ *or* $l_t = (1, 0)$ *i.i.d uniformly at random.*

Subquestion 3.1

Show that the expected loss of any learner is $T/2$*.*

Solution

We calculate the dot loss as:

$$\sum_{k=1}^{K} w_t^k l_t^k$$

Where, in our case:

$$w_t \in \{(0,0), (0,1), (1,0), (1,1)\} \text{ and } l_t \in \{(0,1), (1,0)\}$$

In the table below we can see all the possible values of $L_t = \sum_{k=1}^{2} w_t^k l_t^k$:

$\sum_{k=1}^{2} w_t^k l_t^k$	$(0,1)$	$(1,0)$
$(0,0)$	0	0
$(0,1)$	1	0
$(0,1)$	0	1
$(1,1)$	1	1

From that it follows:

$$P(L=0) = P(L=1) = \frac{1}{2}$$

And we can conclude:

$$\begin{aligned}\mathbb{E}\left[\sum_{t=1}^{T} L_t\right] &= \sum_{t=1}^{T} \mathbb{E}[L_t] \\ &= \sum_{t=1}^{T} \frac{1}{2} \cdot 0 + \frac{1}{2} \cdot 1 \\ &= \frac{T}{2}\end{aligned}$$

Subquestion 3.2

Show that $2(1/2 - l_t^k)$ is Rademacher for each $k \in \{1, 2\}$.

Solution

We know that l_t^k can take two values: 0 or 1. So:

$$\begin{aligned}2(1/2 - l_t^k) = 2(1/2 - 1) = -1 \\ 2(1/2 - l_t^k) = 2(1/2 - 0) = 1\end{aligned}$$

From that follows that $2(1/2 - l_t^k)$ takes values -1 or 1, therefore it is Rademacher.

Subquestion 3.3

Show that $\sum_{t=1}^{T}(1/2 - l_t^2) = -\sum_{t=1}^{T}(1/2 - l_t^1)$.

Solution

$$\begin{aligned}
\sum_{t=1}^{T}(1/2 - l_t^2) &= -\sum_{t=1}^{T}(1/2 - l_t^1)\\
\sum_{t=1}^{T}(1/2 - l_t^2) + \sum_{t=1}^{T}(1/2 - l_t^1) &= 0\\
\sum_{t=1}^{T}(1/2 - l_t^2 + 1/2 - l_t^1) &= 0\\
\sum_{t=1}^{T}(1/2 + 1/2 - (l_t^1 + l_t^2)) &= 0\\
\sum_{t=1}^{T}(1 - 1) &= 0\\
0 &= 0
\end{aligned}$$

Subquestion 3.4

Argue that the expected loss of the best expert is bounded above by $\mathbb{E}[\min_k \sum_{t=1}^{T} l_t^k] \leq T/2 - c\sqrt{T}$ *for some* $c > 0$. *You can use the following fact. Let* $X_1, \ldots, X_T$ *be i.i.d Rademacher random variables. Then*

$$\mathbb{E}\left[\sum_{t=1}^{T} X_t\right] \in \left[\sqrt{\frac{2(T-1)}{\pi}}, \sqrt{\frac{2(T+1)}{\pi}}\right].$$

Solution

We know, that $2(1/2 - l_t^k)$ is Rademacher, so it holds:

$$\begin{aligned}
\mathbb{E}\left[\sum_{t=1}^{T} 2(1/2 - l_t^2)\right] &\in \left[\sqrt{\frac{2(T-1)}{\pi}}, \sqrt{\frac{2(T+1)}{\pi}}\right]\\
\mathbb{E}\left[\sum_{t=1}^{T} 1/2 - l_t^2\right] &\in \left[\sqrt{\frac{(T-1)}{2\pi}}, \sqrt{\frac{(T+1)}{2\pi}}\right]
\end{aligned}$$

From the previous point:

$$\mathbb{E}\left[-\sum_{t=1}^{T} 1/2 - l_t^1\right] \in \left[\sqrt{\frac{(T-1)}{2\pi}}, \sqrt{\frac{(T+1)}{2\pi}}\right]$$
$$\mathbb{E}\left[\sum_{t=1}^{T} 1/2 - l_t^1\right] \in \left[-\sqrt{\frac{(T+1)}{2\pi}}, -\sqrt{\frac{(T-1)}{2\pi}}\right]$$

It follows:

$$\mathbb{E}\left[T/2 - \sum_{t=1}^{T} l_t^1\right] \in \left[-\sqrt{\frac{(T+1)}{2\pi}}, -\sqrt{\frac{(T-1)}{2\pi}}\right]$$
$$\mathbb{E}\left[T/2 - \sum_{t=1}^{T} l_t^2\right] \in \left[\sqrt{\frac{(T-1)}{2\pi}}, \sqrt{\frac{(T+1)}{2\pi}}\right]$$

$\Rightarrow$

$$\mathbb{E}\left[-\sum_{t=1}^{T} l_t^1\right] \in \left[-\sqrt{\frac{(T+1)}{2\pi}} - T/2, -\sqrt{\frac{(T-1)}{2\pi}} - T/2\right]$$
$$\mathbb{E}\left[-\sum_{t=1}^{T} l_t^2\right] \in \left[\sqrt{\frac{(T-1)}{2\pi}} - T/2, \sqrt{\frac{(T+1)}{2\pi}} - T/2\right]$$

$\Rightarrow$

$$\mathbb{E}\left[\sum_{t=1}^{T} l_t^1\right] \in \left[\sqrt{\frac{(T-1)}{2\pi}} + T/2, \sqrt{\frac{(T+1)}{2\pi}} + T/2\right]$$
$$\mathbb{E}\left[\sum_{t=1}^{T} l_t^2\right] \in \left[-\sqrt{\frac{(T+1)}{2\pi}} + T/2, -\sqrt{\frac{(T-1)}{2\pi}} + T/2\right]$$

From above we conclude $\mathbb{E}[\min_k \sum_{t=1}^{T} l_t^k] = \mathbb{E}[\sum_{t=1}^{T} l_t^2]$ and finally:

$$\mathbb{E}[\min_k \sum_{t=1}^{T} l_t^k] \leq T/2 - c\sqrt{T}$$