

# MLT Homework 4

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## Question 1

We have shown that for a finite hypothesis class  $\mathcal{H}$ ,  $VCdim(\mathcal{H}) \leq \lfloor \log(|\mathcal{H}|) \rfloor$ . However, this is just an upper bound. The VC-dimension of a class can be much lower than that.

### Subquestion 1.1

Find an example of a class  $\mathcal{H}$  of functions over the real interval  $\mathcal{X} = [0, 1]$  such that  $\mathcal{H}$  is infinite while  $VCdim(\mathcal{H}) = 1$ .

### Solution

Let's define hypothesis class as:

$$\mathcal{H} = \{h_a : a \in [0, 1]\}$$

$$h_a(x) = \begin{cases} 1; & x \geq a \\ 0; & x < a \end{cases}$$

From definition we know  $|\mathcal{H}| = \infty$ . Now we need to prove that  $VCdim(\mathcal{H}) = 1$ .

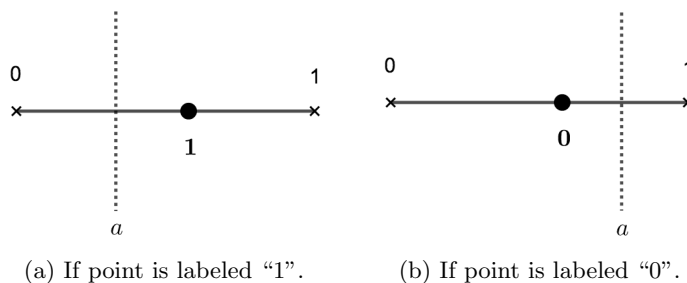


Figure 1: Proof that  $VCdim(\mathcal{H}) \geq 1$ .

- $\text{VCdim}(\mathcal{H}) \geq 1$ : The proof we can see from the figure 1.
- $\text{VCdim}(\mathcal{H}) \leq 1$ : From the figure 2 it is seen that hypothesis class  $\mathcal{H}$  does not shatter a set of two points (no matter how we position them).

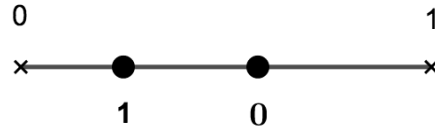


Figure 2: The problem we have when trying to shatter a set of two points.

### Subquestion 1.2

Give an example of a finite hypothesis class  $\mathcal{H}$  over domain  $\mathcal{X} = [0, 1]$ , where  $\text{VCdim}(\mathcal{H}) = \lfloor \log_2(|\mathcal{H}|) \rfloor$

#### Solution

Let's define hypothesis class:

$$\mathcal{H} = \{h_0, h_1\}$$

where  $h_0(x) = 0$  ( $\forall x$ ) and  $h_1(x) = 1$  ( $\forall x$ ). We would like to prove that  $\text{VCdim}(\mathcal{H}) = \lfloor \log_2(|\mathcal{H}|) \rfloor = \lfloor \log_2(2) \rfloor = 1$ .

- $\text{VCdim}(\mathcal{H}) \geq 1$ : If we want to label a  $x \in [0, 1]$  as 1, we pick  $h_1$  as hypothesis, otherwise we pick  $h_0$ . So,  $\text{VCdim}(\mathcal{H}) \geq 1$ .
- $\text{VCdim}(\mathcal{H}) \leq 1$ : Let say that we have a set of two points. If we want to label one of the point with 1 and the other with 0, there does not exist a hypothesis in hypothesis class which can label two points differently.

We can conclude that  $\text{VCdim}(\mathcal{H}) = 1$ .

### Question 2

6.8

#### Solution

### Question 3

Let  $\mathcal{H}$  be the class of signed intervals, that is,  $\mathcal{H} = \{h_{a,b,s} : a \leq b, s \in \{-1, 1\}\}$  where

$$h_{a,b,s}(x) = \begin{cases} s & \text{if } x \in [a, b] \\ -s & \text{if } x \notin [a, b] \end{cases}$$

Calculate  $VCdim(\mathcal{H})$ .

### Solution

Claim:  $VCdim(\mathcal{H}) = 3$ .

- $VCdim(\mathcal{H}) \geq 3$ : On figure 3 it is seen that set of three points can be shattered. Furthermore  $VCdim(\mathcal{H}) \geq 3$ .

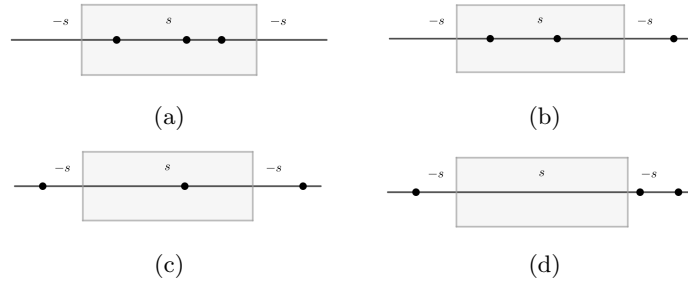


Figure 3: Proof that  $VCdim(\mathcal{H}) \geq 3$ .

- $VCdim(\mathcal{H}) \leq 3$ : There does not exist a hypothesis  $h \in \mathcal{H}$  that labels the situation on figure 4. From here we can conclude  $VCdim(\mathcal{H}) \leq 3$ .



Figure 4: Proof that  $VCdim(\mathcal{H}) \leq 3$ .

## Question 4

**VC of union:** Let  $\mathcal{H}_1, \dots, \mathcal{H}_r$  be hypothesis classes over some fixed domain set  $\mathcal{X}$ . Let  $d = \max_i VCdim(\mathcal{H}_i)$  and assume for simplicity that  $d \geq 3$ .

### Subquestion 4.1

Prove that

$$VCdim(\cup_{i=1}^r \mathcal{H}_i) \leq 4d \log_2 \left( \frac{2d}{\ln 2} \right) + 2 \log_2(r).$$

### Solution

First, denote a union class as  $\mathcal{H}_\cup = \cup_{i=1}^r \mathcal{H}_i$ . Second, assume that  $\text{VCdim}(\mathcal{H}_\cup) = k$  and therefore  $\mathcal{H}_\cup$  shatters a set of  $k$  elements. Furthermore, the union class can produce all  $2^k$  possible labelings on these elements.

Let's recall Sauer's lemma: Let  $\mathcal{H}$  be a hypothesis class with  $\text{VCdim}(\mathcal{H}) \leq d < \infty$ . Then for all  $m$ ,

$$\Pi_{\mathcal{H}}(m) \leq m^d$$

From our assumption it follows:

$$\Pi_{\mathcal{H}_\cup}(k) = 2^k$$

The definition of shatter function gives as the following inequality:

$$\Pi_{\mathcal{H}_\cup}(k) \leq \Pi_{\mathcal{H}_1}(k) + \dots + \Pi_{\mathcal{H}_r}(k)$$

Now, we can use Sauer's lemma on each summand:

$$2^k = \Pi_{\mathcal{H}_\cup}(k) \leq \Pi_{\mathcal{H}_1}(k) + \dots + \Pi_{\mathcal{H}_r}(k) \leq \underbrace{k^d + \dots + k^d}_r = rk^d$$

If we use  $\log_2$  on the inequality, we get:

$$k \leq d \log_2 k + \log_2 r \quad (1)$$

In the next step we are going to use Lemma A.2 from the book, which says: Let  $a \geq 1$  and  $b > 0$ . Then  $x \geq 4a \log_2 \left( \frac{2a}{\ln 2} \right) + 2b \implies x \geq a \log_2(x) + b$ .

Let's assume that  $\text{VCdim}(\mathcal{H}_\cup) > 4d \log_2 \left( \frac{2d}{\ln 2} \right) + 2 \log_2(r)$ . From our first assumption we get:

$$k > 4d \log_2 \left( \frac{2d}{\ln 2} \right) + 2 \log_2(r)$$

Now, we can use Lemma A.2 (where  $a = d \geq 3$ ,  $b = \log_2 r > 0$ ):

$$k > d \log_2 k + \log_2 r$$

We got into a contradiction with (1), this means that our assumption was not correct and it holds:

$$\text{VCdim}(\mathcal{H}_\cup) \leq 4d \log_2 \left( \frac{2d}{\ln 2} \right) + 2 \log_2(r)$$

### Subquestion 4.2

Prove that for  $r = 2$  it holds that

$$\text{VCdim}(\mathcal{H}_1 \cup \mathcal{H}_2) \leq 2d + 1$$

## Solution

This question was solved with the help of [1].

As same as before:

$$\Pi_{\mathcal{H}_1 \cup \mathcal{H}_2}(m) \leq \Pi_{\mathcal{H}_1}(m) + \Pi_{\mathcal{H}_2}(m)$$

Now we use Sauer's lemma:

$$\Pi_{\mathcal{H}_1 \cup \mathcal{H}_2}(m) \leq \Pi_{\mathcal{H}_1}(m) + \Pi_{\mathcal{H}_2}(m) \leq \sum_{i=0}^d \binom{m}{i} + \sum_{i=0}^d \binom{m}{i}$$

If we use the fact  $\binom{m}{i} = \binom{m}{m-i}$ , we get:

$$\begin{aligned} \sum_{i=0}^d \binom{m}{i} + \sum_{i=0}^d \binom{m}{i} &= \sum_{i=0}^d \binom{m}{i} + \sum_{i=0}^d \binom{m}{m-i} \\ &= \sum_{i=0}^d \binom{m}{i} + \sum_{i=m-d}^m \binom{m}{i} \\ &= \underbrace{\binom{m}{0} + \cdots + \binom{m}{d}}_{d+1} + \underbrace{\binom{m}{m-d} + \cdots + \binom{m}{m}}_{d+1} \end{aligned}$$

If  $m > 2d + 1$ :

$$\binom{m}{0} + \cdots + \binom{m}{d} + \binom{m}{m-d} + \cdots + \binom{m}{m} \leq \sum_{i=0}^m \binom{m}{i} - \binom{m}{d+1} < 2^m$$

Let's sum up what we just calculated:

$$\Pi_{\mathcal{H}_1 \cup \mathcal{H}_2}(m) < 2^m$$

So, if  $m > 2d + 1$  the set with  $m$  elements can not be shattered, therefore:

$$\text{VCdim}(\mathcal{H}_1 \cup \mathcal{H}_2) \leq 2d + 1$$

## References

- [1] Mehryar Mohri. Solution assignment 2. [cs.nyu.edu/~mohri/ml/ml10/so12.pdf](http://cs.nyu.edu/~mohri/ml/ml10/so12.pdf), 2010. Online; accessed 7 October 2018.