MLT Homework 10

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Question 1

Revisiting the AA

Subquestion 1.1

Non-uniform regret bound for AA with prior *Fix a prior distribution $\pi \in \Delta_K$. Consider the Aggregating Algorithm with prior π, defined by:*

$$w_t^k = \frac{\pi^k e^{-\sum_{s=1}^{t-1} l_s^k}}{\sum_{j=1}^K \pi^j e^{-\sum_{s=1}^{t-1} l_s^j}} \tag{1}$$

Show that for the mix-loss game, the regret w.r.t. each expert k is at most

$$R_T^k = \sum_{t=1}^T \left(\hat{l}_t - l_t^k\right) \leq -\ln \pi^k$$

Solution

In the final solution we needed the following fact:

$$\begin{aligned}
-\ln\left(\sum_{k=1}^K \pi^k e^{-\sum_{s=1}^T l_s^k}\right) &\leq \sum_{s=1}^T l_s^k - \ln \pi^k \\
\ln\left(\sum_{k=1}^K \pi^k e^{-\sum_{s=1}^T l_s^k}\right) &\geq -\sum_{s=1}^T l_s^k + \ln \pi^k \\
\sum_{k=1}^K \pi^k e^{-\sum_{s=1}^T l_s^k} &\geq e^{-\sum_{s=1}^T l_s^k + \ln \pi^k} \\
\sum_{k=1}^K \pi^k e^{-\sum_{s=1}^T l_s^k} &\geq \pi^k e^{-\sum_{s=1}^T l_s^k}; \quad \forall k
\end{aligned}$$

We also used:

$$\sum_{j=1}^{K} \pi^j = 1 \quad \Rightarrow \quad \ln\left(\sum_{j=1}^{K} \pi^j\right) = 0$$

So:

$$\begin{aligned}
\sum_{t=1}^{T} \hat{l}_t &= \sum_{t=1}^{T} -\ln\left(\sum_{k=1}^{K} w_t^k e^{-l_t^k}\right) \\
&= \sum_{t=1}^{T} -\ln\left(\sum_{k=1}^{K} \frac{\pi^k e^{-\sum_{s=1}^{t-1} l_s^k}}{\sum_{j=1}^{K} \pi^j e^{-\sum_{s=1}^{t-1} l_s^j}} e^{-l_t^k}\right) \\
&= \sum_{t=1}^{T} -\ln\left(\frac{\sum_{k=1}^{K} \pi^k e^{-\sum_{s=1}^{t} l_s^k}}{\sum_{j=1}^{K} \pi^j e^{-\sum_{s=1}^{t-1} l_s^j}}\right) \\
&= -\ln \prod_{t=1}^{T} \left(\frac{\sum_{k=1}^{K} \pi^k e^{-\sum_{s=1}^{t} l_s^k}}{\sum_{j=1}^{K} \pi^j e^{-\sum_{s=1}^{t-1} l_s^j}}\right) \\
&= -\ln\left(\frac{\sum_{k=1}^{K} \pi^k e^{-\sum_{s=1}^{T} l_s^k}}{\sum_{j=1}^{K} \pi^j}\right) \\
&= -\ln\left(\sum_{k=1}^{K} \pi^k e^{-\sum_{s=1}^{T} l_s^k}\right) + \ln\left(\sum_{j=1}^{K} \pi^j\right) \\
&\leq \sum_{s=1}^{T} l_s^k - \ln \pi^k
\end{aligned}$$

$$\Rightarrow \quad \sum_{t=1}^{T} \hat{l}_t - l_t^k \leq -\ln \pi^k$$

Subquestion 1.2

The AA does not exploit non-stationarity *Consider the Aggregating Algorithm in its incremental representation:*

$$w_1^k = \frac{1}{K} \quad \textit{and} \quad w_{t+1}^k = w_t^k e^{-l_t^k}.$$

At first sight, the AA adapts its weights sequentially to better fit the instantaneous losses. So maybe it can already compete with sequences of experts! Show that this is not the case, by constructing a sequence of losses such that the regret of the AA compared to the best sequence with one switch is arbitrarily high, even at a fixed horizon T.

Solution

Let T be an even number. We can let l_t be one for every $t \in [T/2]$ except for $l_t^1 = 0$ and for $t \in [T/2+1, \dots, T]$ be one everywhere again, but now except for $l_t^2 = 0$. Then the one-switch where we switch from arm 1 to arm 2 after time $T/2$ gives us a loss of zero. But for the mix-loss we have

$$\begin{aligned}
\sum_{t=1}^{T} \hat{l}_t &= \sum_{t=1}^{T} -\ln(\sum_{k=1}^{K} w_t^k e^{-l_t^k}) \\
&= -\ln(\sum_{k=1}^{K} e^{-\sum_{t=1}^{T} l_t^k}) \ln K \\
&= -\ln(2e^{-T/2} + (K-2)e^{-T}) + \ln K \\
&\geq -\ln(Ke^{-T/2}) + \ln K \\
&= -\ln e^{-T/2} = -T/2
\end{aligned}$$

Here we used the slides (lecture 8 slide 11) to find the second equality for the AA. So we find linearity and thus we do not learn at all.

Question 2

Collapsed Fixed Share Computation
In the Notes, we defined Fixed Share to predict with

$$w_t^k := \sum_{\xi \in [K]^T : \xi_t = k} w_t^{\xi} = \sum_{\xi \in [K]^T : \xi_t = k} \frac{\pi^{\xi} e^{-\sum_{s=1}^{t-1} l_s^{\xi}}}{\sum_{j \in [K]^T} \pi^j e^{-\sum_{s=1}^{t-1} l_s^j}}$$

where w_t^{ξ} are the weights of the AA (1) when run on experts $[K]^T$ with prior

$$\pi^{\xi} := \frac{1}{K} \prod_{t=2}^{T} \begin{cases} 1-\alpha & \text{if } \xi_{t-1} = \xi_t \\ \frac{\alpha}{K-1} & \text{if } \xi_{t-1} \neq \xi_t \end{cases}$$

and losses defined by $l_t^{\xi} = l_t^{\xi_k}$. In this exercise we find a way to maintain w_t^k directly in $O(K)$ time per round.

Subquestion 2.1

What is w_1^k?

Solution

$$\begin{aligned}
w_1^k &= \sum_{\xi \in [K]^T : \xi_1 = k} w_1^\xi \\
&= \sum_{\xi \in [K]^T : \xi_1 = k} \pi^\xi \\
&= \sum_{\xi \in [K]^T : \xi_1 = k} \frac{1}{K} \\
&= \left|\xi \in [K]^T : \xi_1 = k\right| \frac{1}{K}
\end{aligned}$$

Subquestion 2.2

Show that the Fixed Share weights satisfy the recurrence

$$w_{t+1}^k = (1-\beta)\frac{w_t^k e^{-l_t^k}}{\sum_j w_t^j e^{-l_t^j}} + \frac{\beta}{K}$$

where $\beta = \alpha \frac{K}{K-1}$.

Solution

First, let us prove the hint. So, we would like to se that the following holds:

$$\sum_{\xi : \xi_t = j, \xi_{t+1} = k} w_t^\xi = \begin{cases} (1-\alpha)\sum_{\xi : \xi_t = j} w_t^\xi; & j = k \\ \frac{\alpha}{K-1}\sum_{\xi : \xi_t = j} w_t^\xi; & j \neq k \end{cases}$$

For any $k \neq j$ it holds $\left(p(a,b,\xi) = \prod_{t=a}^b \begin{cases} 1-\alpha & \text{if } \xi_{t-1} = \xi_t \\ \frac{\alpha}{K-1} & \text{if } \xi_{t-1} \neq \xi_t \end{cases}\right)$:

$$\begin{aligned}
\frac{\sum_{\xi : \xi_t = j, \xi_{t+1} = j} w_t^\xi}{\sum_{\xi : \xi_t = j, \xi_{t+1} = k} w_t^\xi} &= \frac{\sum_{\xi : \xi_t = j, \xi_{t+1} = j} \frac{1}{K} p(2,t,\xi)(1-\alpha) p(t+2,T,\xi) e^{-\sum *}}{\sum_{\xi : \xi_t = j, \xi_{t+1} = j} \frac{1}{K} p(2,t,\xi) \frac{\alpha}{K-1} p(t+2,T,\xi) e^{-\sum *}} \\
&= \frac{1-\alpha}{\frac{\alpha}{K-1}}
\end{aligned}$$

We also know:

$$\sum_{k=1}^K \sum_{\xi : \xi_t = j, \xi_{t+1} = k} w_t^\xi = \sum_{\xi_t = j} w_t^\xi$$

From that it follows ($k \neq j$):

$$
\begin{aligned}
\sum_{\xi:\xi_t=j,\xi_{t+1}=j} w_t^\xi + (K-1) \sum_{\xi:\xi_t=j,\xi_{t+1}=k} w_t^\xi &= \sum_{\xi_t=j} w_t^\xi \\
\frac{1-\alpha}{\frac{\alpha}{K-1}} \sum_{\xi:\xi_t=j,\xi_{t+1}=k} w_t^\xi + (K-1) \sum_{\xi:\xi_t=j,\xi_{t+1}=k} w_t^\xi &= \sum_{\xi_t=j} w_t^\xi \\
\frac{(1-\alpha)(K-1)}{\alpha} \sum_{\xi:\xi_t=j,\xi_{t+1}=k} w_t^\xi + (K-1) \sum_{\xi:\xi_t=j,\xi_{t+1}=k} w_t^\xi &= \sum_{\xi_t=j} w_t^\xi \\
\left(\frac{(1-\alpha)(K-1)}{\alpha} + (K-1)\right) \sum_{\xi:\xi_t=j,\xi_{t+1}=k} w_t^\xi &= \sum_{\xi_t=j} w_t^\xi \\
\sum_{\xi:\xi_t=j,\xi_{t+1}=k} w_t^\xi &= \frac{\alpha}{K-1} \sum_{\xi_t=j} w_t^\xi
\end{aligned}
$$

And also:

$$
\begin{aligned}
\sum_{\xi:\xi_t=j,\xi_{t+1}=j} w_t^\xi &= -(K-1) \sum_{\xi:\xi_t=j,\xi_{t+1}=k} w_t^\xi + \sum_{\xi_t=j} w_t^\xi \\
\sum_{\xi:\xi_t=j,\xi_{t+1}=j} w_t^\xi &= -(K-1) \frac{\alpha}{K-1} \sum_{\xi_t=j} w_t^\xi + \sum_{\xi_t=j} w_t^\xi \\
\sum_{\xi:\xi_t=j,\xi_{t+1}=j} w_t^\xi &= (1-\alpha) \sum_{\xi_t=j} w_t^\xi
\end{aligned}
$$

Now, we can finally calculate w_{t+1}^k:

$$
\begin{aligned}
w_{t+1}^k &= \sum_{\xi:\xi_{t+1}=k} w_{t+1}^\xi \\
&\propto \sum_{\xi:\xi_{t+1}=k} w_t^\xi e^{-l_t^\xi} \\
&= \sum_{j=1}^K \left(\sum_{\xi:\xi_t=j,\xi_{t+1}=k} w_t^\xi \right) e^{-l_t^j} \\
&= \left(\sum_{\xi:\xi_t=k,\xi_{t+1}=k} w_t^\xi \right) e^{-l_t^k} + \sum_{j=1,j\neq k}^K \left(\sum_{\xi:\xi_t=j,\xi_{t+1}=k} w_t^\xi \right) e^{-l_t^j} \\
&= (1-\alpha) \left(\sum_{\xi:\xi_t=k} w_t^\xi \right) e^{-l_t^k} + \frac{\alpha}{K-1} \sum_{j=1,j\neq k}^K \left(\sum_{\xi:\xi_t=j} w_t^\xi \right) e^{-l_t^j} \\
&= (1-\alpha) \left(\sum_{\xi:\xi_t=k} w_t^\xi \right) e^{-l_t^k} + \frac{\alpha}{K-1} \sum_{j=1,j\neq k}^K \left(\sum_{\xi:\xi_t=j} w_t^\xi \right) e^{-l_t^j} \\
&\quad + \frac{\alpha}{K-1} \left(\sum_{\xi:\xi_t=k} w_t^\xi \right) e^{-l_t^k} - \frac{\alpha}{K-1} \left(\sum_{\xi:\xi_t=k} w_t^\xi \right) e^{-l_t^k} \\
&= \left(1-\alpha-\frac{\alpha}{K-1}\right) \left(\sum_{\xi:\xi_t=k} w_t^\xi \right) e^{-l_t^k} + \frac{\alpha}{K-1} \sum_{j=1}^K \left(\sum_{\xi:\xi_t=j} w_t^\xi \right) e^{-l_t^j} \\
&= \left(1-\alpha\frac{K}{K-1}\right) \left(\sum_{\xi:\xi_t=k} w_t^\xi \right) e^{-l_t^k} + \frac{\alpha}{K-1} \sum_{j=1}^K \left(\sum_{\xi:\xi_t=j} w_t^\xi \right) e^{-l_t^j} \\
&= (1-\beta)\, w_t^k e^{-l_t^k} + \frac{\beta}{K} \sum_{j=1}^K w_t^j e^{-l_t^j}
\end{aligned}
$$

$$
\Rightarrow \qquad w_{t+1}^k = (1-\beta) \frac{w_t^k e^{-l_t^k}}{\sum_{j=1}^K w_t^j e^{-l_t^j}} + \frac{\beta}{K}
$$

Question 3

Switching for the Dot-Loss Game

Recall how in Lecture 6 we obtained the Hedge algorithm for the dotloss game with bounded losses $l_t \in [0,1]^K$ by running the Aggregating Algorithm on the scaled losses η_t^k, where $\eta > 0$ is called the learning rate parameter.

In this exercise we apply the same reduction to obtain a non-stationarity algorithm for the dot-loss game, i.e. we apply Fixed Share with switching rate α to the scaled losses ηl_t^k. Let us call this algorithm (η, α)-Fixed Share.

Subquestion 3.1

Show that for any expert sequence $\xi \in [K]^T$ with B blocks, (η, α)-Fixed Share guarantees

$$R_T^\xi \leq \frac{\ln K + (B-1)\ln(K-1) - (B-1)\ln\alpha - (T-B)\ln(1-\alpha)}{\eta} + \frac{T\eta}{8}$$

Hint: Use Hoeffding's Lemma and the Fixed Share mix loss regret bound.

Solution

$$\begin{aligned} R_t^\xi &= \sum_{t=1}^T \hat{l}_t - l_t^{\xi_t} \\ &= \sum_{t=1}^T < w_t, l_t > - l_t^{\xi_t} \end{aligned}$$

From lectures 8, we know:

$$\begin{aligned} \sum_{t=1}^T < w_t, l_t > &\leq \sum_{t=1}^T \left(-\frac{1}{\eta} \ln \left(\sum_\xi w_t^\xi e^{-\eta l_t^\xi} \right) + \frac{\eta}{8} \right) \\ &= \sum_{t=1}^T \left(-\frac{1}{\eta} \ln \left(\sum_\xi w_t^\xi e^{-\eta l_t^\xi} \right) \right) + \frac{T\eta}{8} \end{aligned}$$

And also (combined with the first question and lectures 10):

$$\begin{aligned} \sum_{t=1}^T \left(-\frac{1}{\eta} \ln \left(\sum_\xi w_t^\xi e^{-\eta l_t^\xi} \right) \right) &\leq \sum_{t=1}^T l_t^{\xi_t} - \frac{\ln \pi^\xi}{\eta} \\ &\leq \sum_{t=1}^T l_t^{\xi_t} + \frac{\ln K + (B-1)\ln(K-1) - (B-1)\ln\alpha - (T-B)\ln(1-\alpha)}{\eta} \end{aligned}$$

$$\Rightarrow \quad R_T^\xi \leq \frac{\ln K + (B-1)\ln(K-1) - (B-1)\ln\alpha - (T-B)\ln(1-\alpha)}{\eta} + \frac{T\eta}{8}$$

Subquestion 3.2

Now fix B up front. Compute the optimal tuning of α and η in terms of B and T. Show that with this tuning (η, α)-Fixed Share guarantees for every ξ with B blocks

$$R_T^\xi \leq \sqrt{\frac{T}{2}\left((B-1)\ln(K-1) + \ln K + TH\left(\frac{B-1}{T-1}\right)\right)}$$