MLT Homework 4

Ana Borovac
Bas Haver

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Question 1

We have shown that for a finite hypothesis class $\mathcal{H}$, VCdim$(\mathcal{H}) \leq \lfloor \log(|\mathcal{H}|) \rfloor$. However, this is just an upper bound. The VC-dimension of a class can be much lower than that.

Subquestion 1.1

Find an example of a class $\mathcal{H}$ of functions over the real interval $\mathcal{X} = [0, 1]$ such that $\mathcal{H}$ is infinite while VCdim$(\mathcal{H}) = 1$.

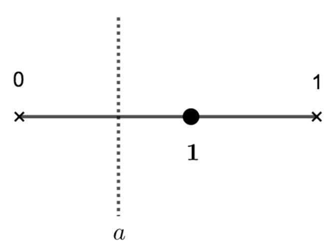
Solution

Let's define hypothesis class as:

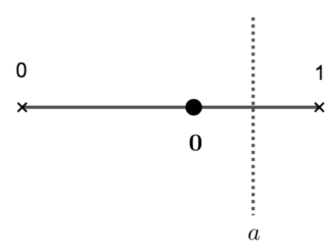
$$\mathcal{H} = \{h_a : \ a \in [0, 1]\}$$

$$h_a(x) = \begin{cases} 1; & x \geq a \\ 0; & x < a \end{cases}$$

From definition we know $|\mathcal{H}| = \infty$. Now we need to prove that VCdim$(\mathcal{H}) = 1$.



(a) If point is labeled "1".



(b) If point is labeled "0".

Figure 1: Proof that VCdim$(\mathcal{H}) \geq 1$.

- VCdim($\mathcal{H}$) ≥ 1: The proof we can see from the figure 1.
- VCdim($\mathcal{H}$) ≤ 1: From the figure 2 it is seen that hypothesis class $\mathcal{H}$ does not shatter a set of two points (no matter how we position them).

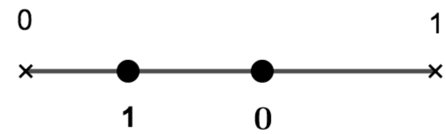


Figure 2: The problem we have when trying to shatter a set of two points.

Subquestion 1.2

Give an example of a finite hypothesis class $\mathcal{H}$ over domain $\mathcal{X} = [0, 1]$, where VCdim($\mathcal{H}$) $= \lfloor \log_2(|\mathcal{H}|) \rfloor$

Solution

Let's define hypothesis class:

$$\mathcal{H} = \{h_0, h_1\}$$

where $h_0(x) = 0$ $(\forall x)$ and $h_1(x) = 1$ $(\forall x)$. We would like to prove that VCdim($\mathcal{H}$) $= \lfloor \log_2(|\mathcal{H}|) \rfloor = \lfloor \log_2(2) \rfloor = 1$.

- VCdim($\mathcal{H}$) ≥ 1: If we want to label a $x \in [0, 1]$ as 1, we pick h_1 as hypothesis, otherwise we pick h_0. So, VCdim($\mathcal{H}$) ≥ 1.
- VCdim($\mathcal{H}$) ≤ 1: Let say that we have a set of two points. If we want to label one of the point with 1 and the other with 0, there does not exist a hypothesis in hypothesis class which can label two points differently.

We can conclude that VCdim($\mathcal{H}$) $= 1$.

Question 2

It is often the case that the VC-dimension of a hypothesis class equals (or can be bounded above by) the number of parameters one needs to set in order to define each hypothesis in the class. For instance, if $\mathcal{H}$ is the class of axis aligned rectangles in $\mathbb{R}^d$, then VCdim($\mathcal{H}$) $= 2d$, which is equal to the number of parameters used to define a rectangle in $\mathbb{R}^d$. Here is an example that shows that this is not always the case. We will see that a hypothesis class might be very complex and even not learneble, although it has a small number of parameters.
Consider the domain $\mathcal{X} = \mathbb{R}$, and the hypothesis class

$$\mathcal{H} = \{x \mapsto \lceil \sin(\theta x) \rceil : \theta \in \mathbb{R}\}$$

(here, we take $\lceil -1 \rceil = 0$). Prove that VCdim($\mathcal{H}$) $= \infty$.
Hint: There is more than one way to prove the required result. One option is by applying the following lemma: If $-.x_1x_2x_3\ldots$, is the binary expansion of $x \in (0,1)$, then for any natural number m, $\lceil \sin(2^m \pi x) \rceil = (1 - x_m)$, provided that $\exists k \geq m$ s.t. $x_k = 1$.

Solution

Assume VCdim($\mathcal{H}$) $= k < \infty$. We will reach towards a contradiction by finding $k+1$ elements which are shattered by $\mathcal{H}$, from which we can conclude that VCdim($\mathcal{H}$) $= \infty$. All these $k+1$ elements will be found in the interval $(0,1)$, such that they can be written as a binary expansion. We choose our elements to have all different combinations of zeros and ones for every entry of the x_m. This can of course be done in multiple way, we just choose one. For example x_1 can be chosen to be one for all elements, x_2 one for all elements except for the first element, where we choose x_2 to be zero, x_3 one for all elements except for the second one, where we choose x_3 zero. We proceed this way until we have $x_{2^{k+1}}$ all zero. Now place a one for $x_{2^{k+1}+1}$ in order to always be able to apply the lemma as stated in the hint. Note now that all these elements are different, so that we really have constructed $k+1$ elements. Now the lemma as in the hint gives us, when we choose $\theta = 2^m\pi$, that we label $\lceil \sin(\theta x) \rceil = 1 - x_m$. So if $x_m = 1$ we label it zero and if $x_m = 0$ we label it one. Now for any combination of the $k+1$ elements we can fix an arbitrarily chosen labeling. This labeling can then be constructed by choosing m such that x_m would give the "opposite" labeling (which is such that $1 - x_m$ is the admired labeling). This m exists by construction. Since our labeling was fixed arbitrarily, we can construct any labeling. Therefore our $k+1$ elements are shattered by $\mathcal{H}$.
Since we assumed VCdim($\mathcal{H}$) $= k < \infty$ and found $k+1$ shattered elements, we have a contradiction and conclude VCdim($\mathcal{H}$) $= \infty$.

Question 3

Let $\mathcal{H}$ be the class of signed intervals, that is, $\mathcal{H} = \{h_{a,b,s} : a \leq b, s \in \{-1,1\}\}$ where

$$h_{a,b,s}(x) = \begin{cases} s & \text{if } x \in [a,b] \\ -s & \text{if } x \notin [a,b] \end{cases}$$

Calculate VCdim($\mathcal{H}$).

Solution

Claim: VCdim($\mathcal{H}$) $= 3$.

- VCdim($\mathcal{H}$) ≥ 3: On figure 3 it is seen that set of three points can be shattered. Furthermore VCdim($\mathcal{H}$) ≥ 3.

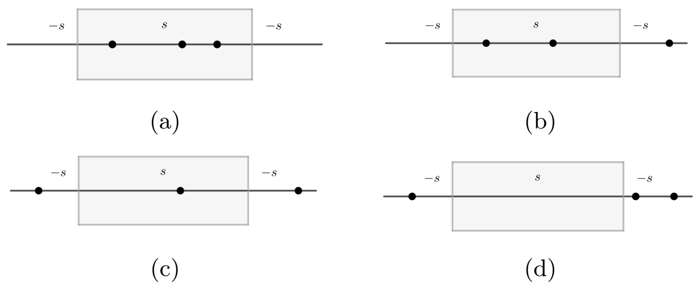


Figure 3: Proof that VCdim($\mathcal{H}$) ≥ 3.

- VCdim($\mathcal{H}$) ≤ 3: There does not exist a hypothesis $h \in \mathcal{H}$ that lables the situation on figure 4. From here we can conclude VCdim($\mathcal{H}$) ≤ 3.

Figure 4: Proof that VCdim($\mathcal{H}$) ≤ 3.

Question 4

VC of union: *Let $\mathcal{H}_1, \ldots, \mathcal{H}_r$ be hypothesis classes over some fixed domain set $\mathcal{X}$. Let $d = \max_i$ VCdim($\mathcal{H}_i$) and assume for simplicity that $d \geq 3$.*

Subquestion 4.1

Prove that

$$VCdim(\cup_{i=1}^{r} \mathcal{H}_i) \leq 4d \log_2 \left(\frac{2d}{\ln 2} \right) + 2 \log_2(r).$$

Solution

First, denote a union class as $\mathcal{H}_\cup = \cup_{i=1}^{r} \mathcal{H}_i$. Second, assume that VCdim($\mathcal{H}_\cup$) $=$ k and therefore $\mathcal{H}_\cup$ shatters a set of k elements . Furthermore, the union class can produce all 2^k possible labelings on these elements.

Let's recall Sauer's lemma: Let $\mathcal{H}$ be a hypothesis class with VCdim($\mathcal{H}$) $\leq d < \infty$. Then for all m,

$$\Pi_{\mathcal{H}}(m) \leq m^d$$

From our assumption it follows:

$$\Pi_{\mathcal{H}_\cup}(k) = 2^k$$

The definition of shatter function gives as the following inequality:

$$\Pi_{\mathcal{H}_\cup}(k) \leq \Pi_{\mathcal{H}_1}(k) + \cdots + \Pi_{\mathcal{H}_r}(k)$$

Now, we can use Sauer's lemma on each summand:

$$2^k = \Pi_{\mathcal{H}_\cup}(k) \leq \Pi_{\mathcal{H}_1}(k) + \cdots + \Pi_{\mathcal{H}_r}(k) \leq \underbrace{k^d + \cdots k^d}_{r} = rk^d$$

If we use $\log_2$ on the inequality, we get:

$$k \leq d \log_2 k + \log_2 r \tag{1}$$

In the next step we are going to use Lemma A.2 from the book, which says: Let $a \geq 1$ and $b > 0$. Then $x \geq 4a \log_2 \left(\frac{2a}{\ln 2}\right) + 2b \implies x \geq a \log_2(x) + b$.

Let's assume that $\text{VCdim}(\mathcal{H}_\cup) > 4d \log_2 \left(\frac{2d}{\ln 2}\right) + 2\log_2(r)$. From our first assumption we get:

$$k > 4d \log_2 \left(\frac{2d}{\ln 2}\right) + 2\log_2(r)$$

Now, we can use Lemma A.2 (where $a = d \geq 3$, $b = \log_2 r > 0$):

$$k > d \log_2 k + \log_2 r$$

We got into a contradiction with (1), this means that our assumption was not correct and it holds:

$$\text{VCdim}(\mathcal{H}_\cup) \leq 4d \log_2 \left(\frac{2d}{\ln 2}\right) + 2\log_2(r)$$

Subquestion 4.2

Prove that for $r = 2$ it holds that

$$\mathit{VCdim}(\mathcal{H}_1 \cup \mathcal{H}_2) \leq 2d + 1$$

Solution

This question was solved with the help of [1].

As same as before:

$$\Pi_{\mathcal{H}_1 \cup \mathcal{H}_2}(m) \leq \Pi_{\mathcal{H}_1}(m) + \Pi_{\mathcal{H}_2}(m)$$

Now we use Sauer's lemma:

$$\Pi_{\mathcal{H}_1 \cup \mathcal{H}_2}(m) \leq \Pi_{\mathcal{H}_1}(m) + \Pi_{\mathcal{H}_2}(m) \leq \sum_{i=0}^{d} \binom{m}{i} + \sum_{i=0}^{d} \binom{m}{i}$$

If we use the fect $\binom{m}{i} = \binom{m}{m-i}$, we get:

$$\begin{aligned}
\sum_{i=0}^{d}\binom{m}{i} + \sum_{i=0}^{d}\binom{m}{i} &= \sum_{i=0}^{d}\binom{m}{i} + \sum_{i=0}^{d}\binom{m}{m-i} \\
&= \sum_{i=0}^{d}\binom{m}{i} + \sum_{i=m-d}^{d}\binom{m}{i} \\
&= \underbrace{\binom{m}{0} + \cdots + \binom{m}{d}}_{d+1} + \underbrace{\binom{m}{m-d} + \cdots + \binom{m}{m}}_{d+1}
\end{aligned}$$

If $m > 2d + 1$:

$$\binom{m}{0} + \cdots + \binom{m}{d} + \binom{m}{m-d} + \cdots + \binom{m}{m} \leq \sum_{i=0}^{m}\binom{m}{i} - \binom{m}{d+1} < 2^m$$

Let's sum up what we just calculated:

$$\Pi_{\mathcal{H}_1 \cup \mathcal{H}_2}(m) < 2^m$$

So, if $m > 2d + 1$ the set with m elements can not be shattered, therefore:

$$\mathrm{VCdim}(\mathcal{H}_1 \cup \mathcal{H}_2) \leq 2d + 1$$

References

[1] Mehryar Mohri. Solution assignment 2. `cs.nyu.edu/~mohri/ml/ml10/sol2.pdf`, 2010. Online; accessed 7 October 2018.