MLT Homework 8

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Question 1

The Aggregating Algorithm plays $w_1^k = 1/K$ *and updates as*

$$w_{k+1}^k = \frac{w_t^k e^{-l_t^k}}{\sum_{j=1}^K w_t^j e^{-l_t^j}}$$

Let us define the Kullback-Leibler divergence *aka* relative entropy *(notion of distance between probability distributions) from* $p \in \Delta_K$ *to* $q \in \Delta_K$ *by*

$$KL(p.q) = \sum_{k=1}^K p_k \ln \frac{p_k}{q_k}$$

Fix $w_t \in \Delta_K$ *and* $l_t \in \mathbb{R}^K$*. Consider the minimisation problem*

$$\min_{w \in \Delta_K} w^T l_t + KL(w, w_t) \tag{1}$$

Subquestion 1.1

Show that the minimiser of problem (1) *is* w_{t+1}*.*

Solution

Subquestion 1.2

Show that the value of problem (1) *is the mix loss.*

Solution

Question 2

We saw in the lecture that the Hedge algorithm (for the dot-loss game) with learning rate $\eta = \sqrt{\frac{8 \ln K}{T}}$ *has regret after* T *rounds bounded by* $\sqrt{T/2 \ln K}$*. In practice, we may not know* T *in advance, or we may even desire an algorithm that has good guarantees for all* T *simultaneously, i.e. that keeps on operating forever.*
Prove that the overall accumulated regret of Hedge with this scheme is bounded above by a universal constant times $\sqrt{T \ln K}$*. (Your argument should work for* T *that are not a power of 3).*

Solution

Question 3

Consider the $K = 2$ *expert version of the* T*-round dot loss game (Definition 2). In this exercise we will prove that the worst-case expected regret is at least of order* $\sqrt{T}$*. Consider an adversary that for each* $t = 1, \ldots, T$ *assigns loss vector* $l_t = (0, 1)$ *or* $l_t = (1, 0)$ *i.i.d uniformly at random.*

Subquestion 3.1

Show that the expected loss of any learner is $T/2$*.*

Solution

Subquestion 3.2

Show that $2(1/2 - l_t^k)$ *is Rademacher for each* $k \in \{1, 2\}$*.*

Solution

Subquestion 3.3

Show that $\sum_{t=1}^{T}(1/2 - l_t^2) = -\sum_{t=1}^{T}(1/2 - l_t^1)$*.*

Solution

Subquestion 3.4

Argue that the expected loss of the best expert is bounded above by $\mathbb{E}[\min_k \sum_{t=1}^{T} l_t^k] \leq T/2 - c\sqrt{T}$ *for some* $c > 0$*. You can use the following fact. Let* $X_1, \ldots, X_T$ *be* *i.i.d Rademacher random variables. Then*

$$\mathbb{E}\left[\sum_{t=1}^{T} X_t\right] \in \left[\sqrt{\frac{2(T-1)}{\pi}}, \sqrt{\frac{2(T+1)}{\pi}}\right].$$

Solution