MLT Homework 11

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Question 1

Question 2

Question 3

Determine whether the following functions are exp-concave, and to what degree.

Subquestion 3.1

$x \mapsto (x-y)^2$ *on* $x, y \in [-Y, Y]$ *First consider a fixed* y *and then determine the worst-case over all* $y \in [-Y, Y]$.

Solution

Let's recall the definition of α-exp-concave function: A function f is α-exp-concave if $x \mapsto e^{-\alpha f(x)}$ is concave.

In our case we get:

$$\begin{aligned} x \mapsto e^{-\alpha(x-y)^2} \quad &\Rightarrow \quad g(x) = e^{-\alpha(x-y)^2} \\ &\Rightarrow \quad g'(x) = -2\alpha(x-y)e^{-\alpha(x-y)^2} \\ &\Rightarrow \quad g''(x) = -2\alpha e^{-\alpha(x-y)^2} + 4\alpha^2(x-y)^2 e^{-\alpha(x-y)^2} \end{aligned}$$

If g is concave then $g''(x) \leq 0$ for $\forall x$.

$$\begin{aligned}
-2\alpha e^{-\alpha(x-y)^2} + 4\alpha^2(x-y)^2 e^{-\alpha(x-y)^2} &\leq 0 \\
\underbrace{e^{-\alpha(x-y)^2}}_{\geq 0}(-2\alpha + 4\alpha^2(x-y)^2) &\leq 0 \\
2\alpha(-1 + 2\alpha(x-y)^2) &\leq 0 \\
-1 + 2\alpha(x-y)^2 &\leq 0 \\
\alpha &\leq \frac{1}{2(x-y)^2}
\end{aligned}$$

If we want to inequality to hold for $\forall x$ we need to observe when the fraction on the right is the smallest. So, when does $(x-y)^2$ reach maximum? It depends on y, but it is obvious that it is reached when x is Y or $-Y$, so:

$$\alpha = \frac{1}{2\max\{(Y-y)^2, (-Y-y)^2\}}$$

In the worst case the difference between x and y is $2Y$. From that it follows:

$$\alpha = \frac{1}{8Y^2}$$

Subquestion 3.2

$x \mapsto |x|^p$ on $x \in [-1, 1]$ for $p > 1$.

Solution

$$g(x) = e^{-\alpha|x|^p}$$

Since, g is not differentiable for $x = 0$, we are going analyse concavity on two intervals: $[-1, 0)$ and $(0, 1]$.

- $x \in [-1, 0)$

$$\begin{aligned}
g(x) &= e^{-\alpha(-x)^p} \\
g'(x) &= \alpha p(-x)^{p-1} e^{-\alpha(-x)^p} \\
g''(x) &= -\alpha p(p-1)(-x)^{p-2} e^{-\alpha(-x)^p} + \alpha p(-x)^{p-1} \alpha p(-x)^{p-1} e^{-\alpha(-x)^p}
\end{aligned}$$

$\Rightarrow$

$$\begin{aligned}
g''(x) &\leq 0 \\
-\alpha p(p-1)(-x)^{p-2}e^{-\alpha(-x)^p} + \alpha p(-x)^{p-1}\alpha p(-x)^{p-1}e^{-\alpha(-x)^p} &\leq 0 \\
-(p-1)(-x)^{-1} + (-x)^{p-1}\alpha p &\leq 0 \\
(-x)^{p-1}\alpha p &\leq \frac{p-1}{-x} \\
\alpha &\leq \frac{p-1}{p(-x)(-x)^{p-1}} \\
\alpha &\leq \frac{p-1}{p(-x)^p}
\end{aligned}$$

$\Rightarrow$

$$\begin{aligned}
\alpha &= \min_{x\in[-1,0)} \frac{p-1}{p(-x)^p} \\
\alpha &= \frac{p-1}{p}
\end{aligned}$$

- $x \in (0,1]$

$$\begin{aligned}
g(x) &= e^{-\alpha x^p} \\
g'(x) &= -\alpha p x^{p-1}e^{-\alpha x^p} \\
g''(x) &= -\alpha p(p-1)x^{p-2}e^{-\alpha x^p} + \alpha p x^{p-1}\alpha p x^{p-1}e^{-\alpha x^p}
\end{aligned}$$

$\Rightarrow$

$$\begin{aligned}
g''(x) &\leq 0 \\
-\alpha p(p-1)x^{p-2}e^{-\alpha x^p} + \alpha p x^{p-1}\alpha p x^{p-1}e^{-\alpha x^p} &\leq 0 \\
-(p-1)x^{-1} + x^{p-1}\alpha p &\leq 0 \\
x^{p-1}\alpha p &\leq \frac{p-1}{x} \\
\alpha &\leq \frac{p-1}{pxx^{p-1}} \\
\alpha &\leq \frac{p-1}{px^p}
\end{aligned}$$

$\Rightarrow$

$$\begin{aligned}
\alpha &= \min_{x\in(0,1]} \frac{p-1}{px^p} \\
\alpha &= \frac{p-1}{p}
\end{aligned}$$

We can conclude that g is $\frac{p-1}{p}$-exp-concave on intervals $[-1,0)$ and $(0,1]$. Since g is continuous function on $[-1,1]$, it is $\frac{p-1}{p}$-exp-concave on whole $[-1,1]$.

Question 4

Subquestion 4.1

Let $f(w) = g(\langle w, x\rangle)$ for some fixed x. Show that f is α-exp-concave when g is α-exp-concave.

Solution

Subquestion 4.2

Let $f(w, y) = g(\langle w, x\rangle, y)$ for some fixed x. Show that f is η-mixable when g is η-mixable.

Solution