

MLT Homework 4

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Question 1

We have shown that for a finite hypothesis class \mathcal{H} , $VCdim(\mathcal{H}) \leq \lfloor \log(|\mathcal{H}|) \rfloor$. However, this is just an upper bound. The VC-dimension of a class can be much lower than that.

Subquestion 1.1

Find an example of a class \mathcal{H} of functions over the real interval $\mathcal{X} = [0, 1]$ such that \mathcal{H} is infinite while $VCdim(\mathcal{H}) = 1$.

Solution

Let's define hypothesis class as:

$$\mathcal{H} = \{h_a : a \in [0, 1]\}$$

$$h_a(x) = \begin{cases} 1; & x \geq a \\ 0; & x < a \end{cases}$$

From definition we know $|\mathcal{H}| = \infty$. Now we need to prove that $VCdim(\mathcal{H}) = 1$.

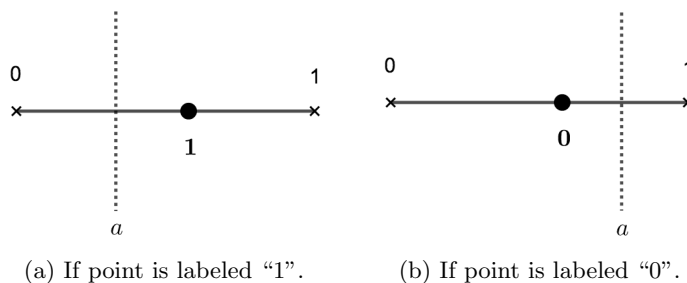


Figure 1: Proof that $VCdim(\mathcal{H}) \geq 1$.

- $\text{VCdim}(\mathcal{H}) \geq 1$: The proof we can see from the figure 1.
- $\text{VCdim}(\mathcal{H}) \leq 1$: From the figure 2 it is seen that hypothesis class \mathcal{H} does not shatter a set of two points (no matter how we position them).

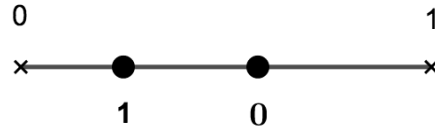


Figure 2: The problem we have when trying to shatter a set of two points.

Subquestion 1.2

Give an example of a finite hypothesis class \mathcal{H} over domain $\mathcal{X} = [0, 1]$, where $\text{VCdim}(\mathcal{H}) = \lfloor \log_2(|\mathcal{H}|) \rfloor$

Solution

Let's define hypothesis class:

$$\mathcal{H} = \{h_0, h_1\}$$

where $h_0(x) = 0$ ($\forall x$) and $h_1(x) = 1$ ($\forall x$). We would like to prove that $\text{VCdim}(\mathcal{H}) = \lfloor \log_2(|\mathcal{H}|) \rfloor = \lfloor \log_2(2) \rfloor = 1$.

- $\text{VCdim}(\mathcal{H}) \geq 1$: If we want to label a $x \in [0, 1]$ as 1, we pick h_1 as hypothesis, otherwise we pick h_0 . So, $\text{VCdim}(\mathcal{H}) \geq 1$.
- $\text{VCdim}(\mathcal{H}) \leq 1$: Let say that we have a set of two points. If we want to label one of the point with 1 and the other with 0, there does not exist a hypothesis in hypothesis class which can label two points differently.

We can conclude that $\text{VCdim}(\mathcal{H}) = 1$.

Question 2

6.8

Solution

Question 3

Let \mathcal{H} be the class of signed intervals, that is, $\mathcal{H} = \{h_{a,b,s} : a \leq b, s \in \{-1, 1\}\}$ where

$$h_{a,b,s}(x) = \begin{cases} s & \text{if } x \in [a, b] \\ -s & \text{if } x \notin [a, b] \end{cases}$$

Calculate $VCdim(\mathcal{H})$.

Solution

Claim: $VCdim(\mathcal{H}) = 3$.

- $VCdim(\mathcal{H}) \geq 3$: On figure 3 it is seen that set of three points can be shattered. Furthermore $VCdim(\mathcal{H}) \geq 3$.

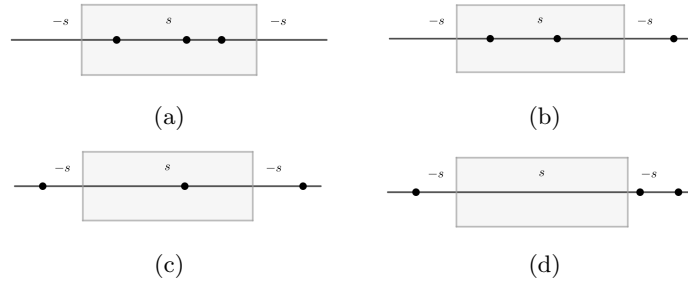


Figure 3: Proof that $VCdim(\mathcal{H}) \geq 3$.

- $VCdim(\mathcal{H}) \leq 3$: There does not exist a hypothesis $h \in \mathcal{H}$ that labels the situation on figure 4. From here we can conclude $VCdim(\mathcal{H}) \leq 3$.



Figure 4: Proof that $VCdim(\mathcal{H}) \leq 3$.

Question 4

VC of union: Let $\mathcal{H}_1, \dots, \mathcal{H}_r$ be hypothesis classes over some fixed domain set \mathcal{X} . Let $d = \max_i VCdim(\mathcal{H}_i)$ and assume for simplicity that $d \geq 3$.

Subquestion 4.1

Prove that

$$VCdim(\cup_{i=1}^r \mathcal{H}_i) \leq 4d \log_2 \left(\frac{2d}{\ln 2} \right) + 2 \log_2(r).$$

Solution

First, denote a union class as $\mathcal{H}_\cup = \cup_{i=1}^r \mathcal{H}_i$. Second, assume that $\text{VCdim}(\mathcal{H}_\cup) = k$ and therefore \mathcal{H}_\cup shatters a set of k elements. Furthermore, the union class can produce all 2^k possible labelings on these elements.

Let's recall Sauer's lemma: Let \mathcal{H} be a hypothesis class with $\text{VCdim}(\mathcal{H}) \leq d < \infty$. Then for all m ,

$$\Pi_{\mathcal{H}}(m) \leq m^d$$

From our assumption it follows:

$$\Pi_{\mathcal{H}_\cup}(k) = 2^k$$

The definition of shatter function gives as the following inequality:

$$\Pi_{\mathcal{H}_\cup}(k) \leq \Pi_{\mathcal{H}_1}(k) + \dots + \Pi_{\mathcal{H}_r}(k)$$

Now, we can use Sauer's lemma on each summand:

$$2^k = \Pi_{\mathcal{H}_\cup}(k) \leq \Pi_{\mathcal{H}_1}(k) + \dots + \Pi_{\mathcal{H}_r}(k) \leq \underbrace{k^d + \dots + k^d}_r = rk^d$$

If we use \log_2 on the inequality, we get:

$$k \leq d \log_2 k + \log_2 r \quad (1)$$

In the next step we are going to use Lemma A.2 from the book, which says: Let $a \geq 1$ and $b > 0$. Then $x \geq 4a \log_2 \left(\frac{2a}{\ln 2} \right) + 2b \implies x \geq a \log_2(x) + b$.

Let's assume that $\text{VCdim}(\mathcal{H}_\cup) > 4d \log_2 \left(\frac{2d}{\ln 2} \right) + 2 \log_2(r)$. From our first assumption we get:

$$k > 4d \log_2 \left(\frac{2d}{\ln 2} \right) + 2 \log_2(r)$$

Now, we can use Lemma A.2 (where $a = d \geq 3$, $b = \log_2 r > 0$):

$$k > d \log_2 k + \log_2 r$$

We got into a contradiction with (1), this means that our assumption was not correct and it holds:

$$\text{VCdim}(\mathcal{H}_\cup) \leq 4d \log_2 \left(\frac{2d}{\ln 2} \right) + 2 \log_2(r)$$

Subquestion 4.2

Prove that for $r = 2$ it holds that

$$\text{VCdim}(\mathcal{H}_1 \cup \mathcal{H}_2) \leq 2d + 1$$

Solution

This question was solved with the help of [1].

As same as before:

$$\Pi_{\mathcal{H}_1 \cup \mathcal{H}_2}(m) \leq \Pi_{\mathcal{H}_1}(m) + \Pi_{\mathcal{H}_2}(m)$$

Now we use Sauer's lemma:

$$\Pi_{\mathcal{H}_1 \cup \mathcal{H}_2}(m) \leq \Pi_{\mathcal{H}_1}(m) + \Pi_{\mathcal{H}_2}(m) \leq \sum_{i=0}^d \binom{m}{i} + \sum_{i=0}^d \binom{m}{i}$$

If we use the fact $\binom{m}{i} = \binom{m}{m-i}$, we get:

$$\begin{aligned} \sum_{i=0}^d \binom{m}{i} + \sum_{i=0}^d \binom{m}{i} &= \sum_{i=0}^d \binom{m}{i} + \sum_{i=0}^d \binom{m}{m-i} \\ &= \sum_{i=0}^d \binom{m}{i} + \sum_{i=m-d}^m \binom{m}{i} \\ &= \underbrace{\binom{m}{0} + \cdots + \binom{m}{d}}_{d+1} + \underbrace{\binom{m}{m-d} + \cdots + \binom{m}{m}}_{d+1} \end{aligned}$$

If $m > 2d + 1$:

$$\binom{m}{0} + \cdots + \binom{m}{d} + \binom{m}{m-d} + \cdots + \binom{m}{m} \leq \sum_{i=0}^m \binom{m}{i} - \binom{m}{d+1} < 2^m$$

Let's sum up what we just calculated:

$$\Pi_{\mathcal{H}_1 \cup \mathcal{H}_2}(m) < 2^m$$

So, if $m > 2d + 1$ the set with m elements can not be shattered, therefore:

$$\text{VCdim}(\mathcal{H}_1 \cup \mathcal{H}_2) \leq 2d + 1$$

References

- [1] Mehryar Mohri. Solution assignment 2. cs.nyu.edu/~mohri/ml/ml10/so12.pdf, 2010. Online; accessed 7 October 2018.