MLT Homework 4

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Question 1

Verify the following lemma: If $0.x_1x_2x_3\dots$ *is the binary expansion of* $x \in (0,1)$, *then for any natural number* m, $\lceil \sin(2^m\pi) \rceil$, *provided that* $\exists k \geq m$ *s.t.* $x_k = 1$

Solution

First we will see that the first $m-1$ zeros and ones do not contribute in the value of $\lceil \sin(2^m\pi x) \rceil$. This is because

$$2^m \sum_{i=1}^{m-1} \frac{x_i}{2^i} = \sum_{i=1}^{m-1} 2^{m-i} x_i \equiv 0 \mod 2.$$

By the periodicity of the sine now makes that it gives no contribution to the outcome of $\sin(2^m\pi x)$. Furthermore we have

$$2^m \sum_{i=m+1}^{\infty} = \sum_{i=m+1}^{\infty} 2^{m-i} x_i = \sum i = 1^\infty 2^{-i} x_i,$$

which is in $(0,1)$ if any of the x_i equals 1 for $i > m$. Now we combine this to get

$$\begin{aligned} 2^m \sum_{i=1}^{\infty} \frac{x_i}{2^i} &= 2^m \Big(\sum_{i=1}^{m-1} \frac{x_i}{2^i} + \frac{x_m}{2^m} + \sum_{i=m+1}^{\infty} \frac{x_i}{2^i}\Big) \\ &= 2^m \sum_{i=1}^{m-1} \frac{x_i}{2^i} + x_m + 2^m \sum_{i=m+1}^{\infty} \frac{x_i}{2^i} \end{aligned}$$

Where the first term does not contribute in the sine and the last term will be in $(0,1)$, with the exception that $x_m = 1$ and $x_i = 0$ for all $i > m$. But then $\lceil \sin(2^m\pi x) \rceil = \lceil \sin(\pi) \rceil = 0 - 1 - x_m$, so the lemma still holds.
Thus, suppose $\sum_{i=m+1}^{\infty} \frac{x_i}{2^i} \in (0,1)$. Now for $x_m = 0$ we have that

$$2^m \sum_{i=1}^{m-1} \frac{x_i}{2^i} + x_m + 2^m \sum_{i=m+1}^{\infty} \frac{x_i}{2^i} \in ((0,1) \mod 2),$$

so that

$$\lceil \sin(2^m \pi x) \rceil = 1 = 1 - x_m.$$

Now for $x_m = 1$ we have that

$$2^m \sum_{i=1}^{m-1} \frac{x_i}{2^i} + x_m + 2^m \sum_{i=m+1}^{\infty} \frac{x_i}{2^i} \in ((0,1) \mod 2),$$

so that

$$\lceil \sin(2^m \pi x) \rceil = 0 = 1 - x_m.$$

Therefore we can conclude that the lemma holds.