Petri-Nets and Other Models

Summary

- Petri-Net Models
  - Definitions
  - Modeling protocols using Petri-Nets
  - Modeling Queueing Systems using Petri-Nets
- Max-Plus Algebra

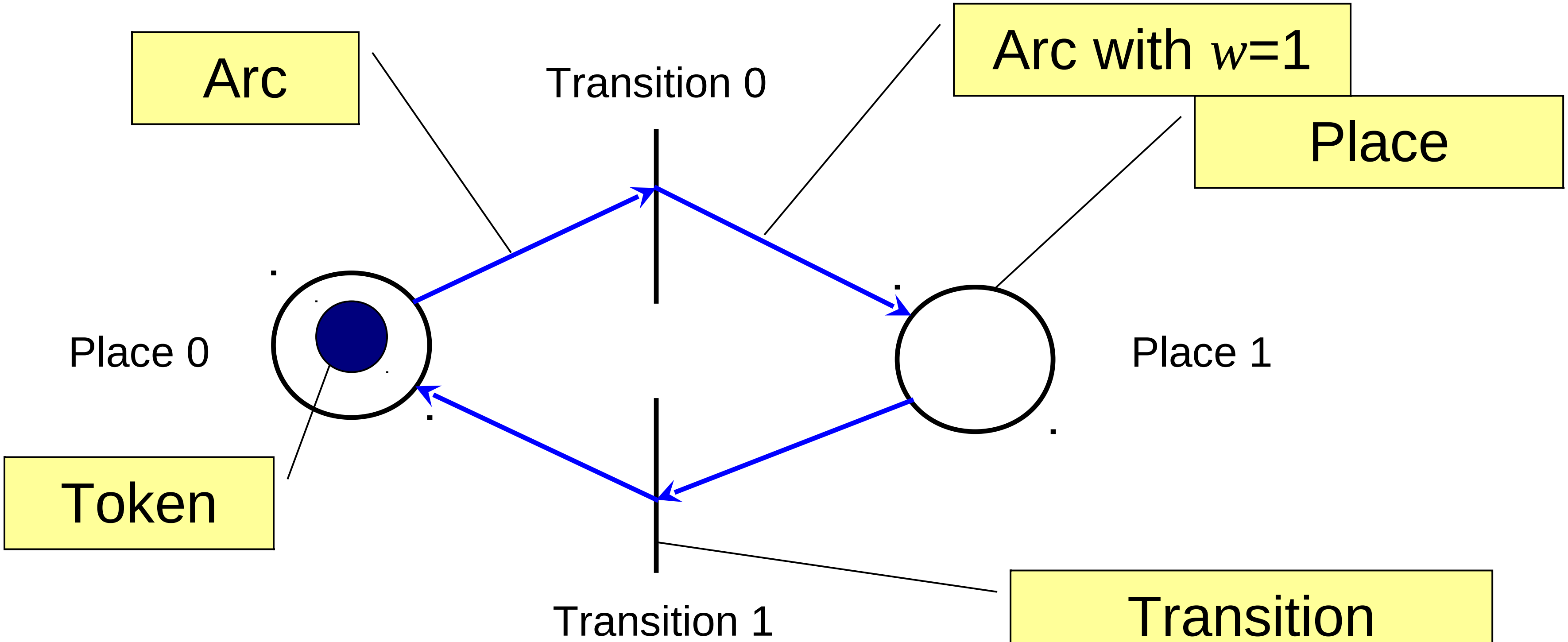
Marked Petri Net Graph

- A Petri net graph is a weighted bipartite graph

$$PN = (P, T, A, w, x)$$

- P is a finite set of places, $P=\{p_1,\ldots,p_n\}$
- T is a finite set of transitions, $T=\{t_1,\ldots,t_m\}$
- A is the set of arcs from places to transitions and from transitions to places
  - (p_i, t_j) or (t_j, p_i) represent the arcs
- w is the weight function on arcs
- x is the marking vector $x=[x_1,\ldots,x_n]$ represents the number of tokens in each place.

Petri Net Example



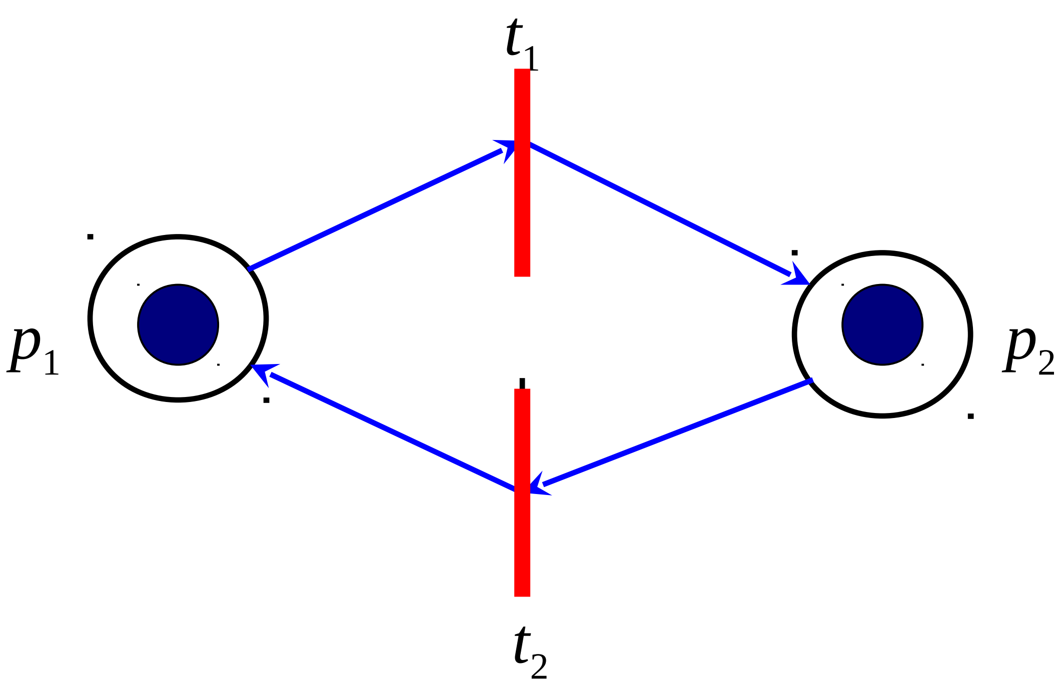
- $I(t_j) = \{p_i \in P : (p_i, t_j) \in A\}$
- $O(t_j) = \{p_i \in P : (t_j, p_i) \in A\}$
- $I(p_i) = \{t_j \in T : (t_j, p_i) \in A\}$
- $O(p_i) = \{t_j \in T : (p_i, t_j) \in A\}$

Petri Net Marking

- A transition $t_j \in T$ is enabled when each input place has at least a number of tokens equal to the weight of the arc, i.e.,

$$x_i \geq w(p_i, t_j) \text{ for all } p_i \in I(t_j)$$

- When a transition fires it removes a number of tokens (equal to the weight of each input arc) from each input place and deposits a number of tokens (equal to the weight of each output arc) to each output place.



Petri Net Dynamics

- The state transition function f of a Petri net is defined for transition t_j if and only if

$$x_i \geq w(p_i, t_j) \text{ for all } p_i \in I(t_j)$$

- If $f(x, t_j)$ is defined, then we set

$$x'_i = x_i + w(t_j, p_i) - w(p_i, t_j) \quad \text{for all } i=1,\ldots,n$$

- **Define**
  - $\boldsymbol{u}_j=[0,\ldots0,1,0,\ldots0]$ where all elements are 0 except the j-th one.
  - Also define the matrix $\boldsymbol{A}=[a_{ji}]$ where

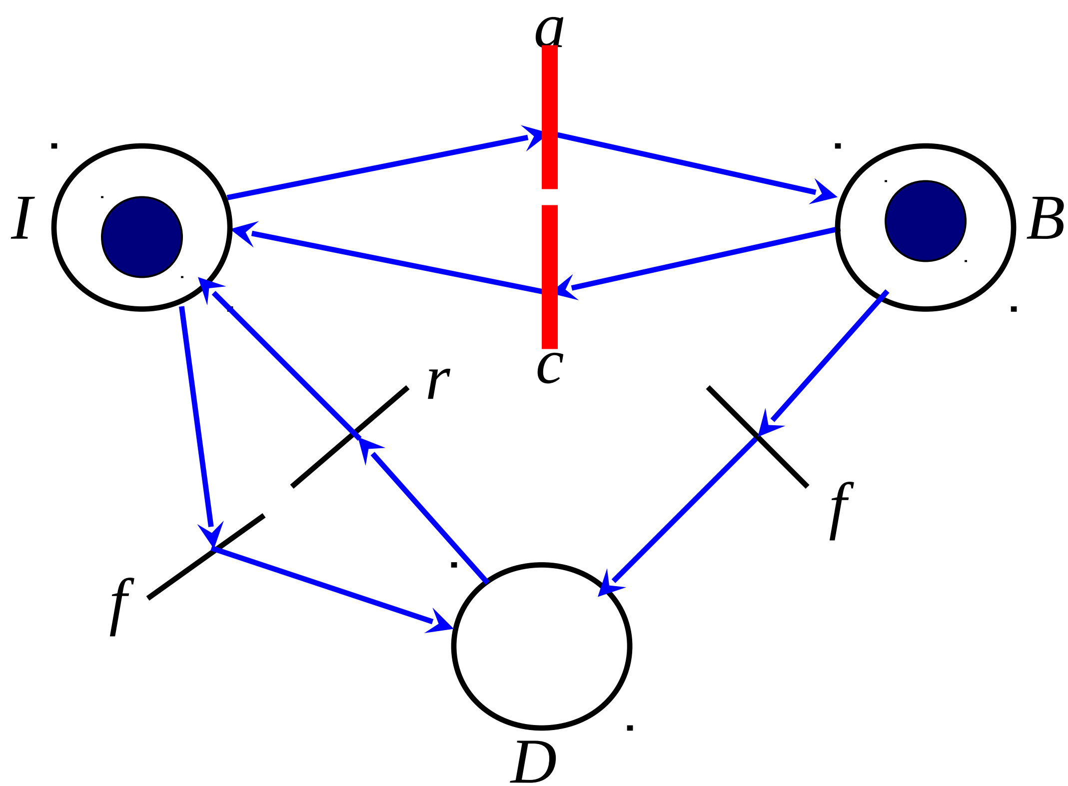
$$a_{ji} = w(t_j, p_i) - w(p_i, t_j)$$

- In vector form

$$\boldsymbol{x}' = \boldsymbol{x} + \boldsymbol{u}_j\boldsymbol{A}$$

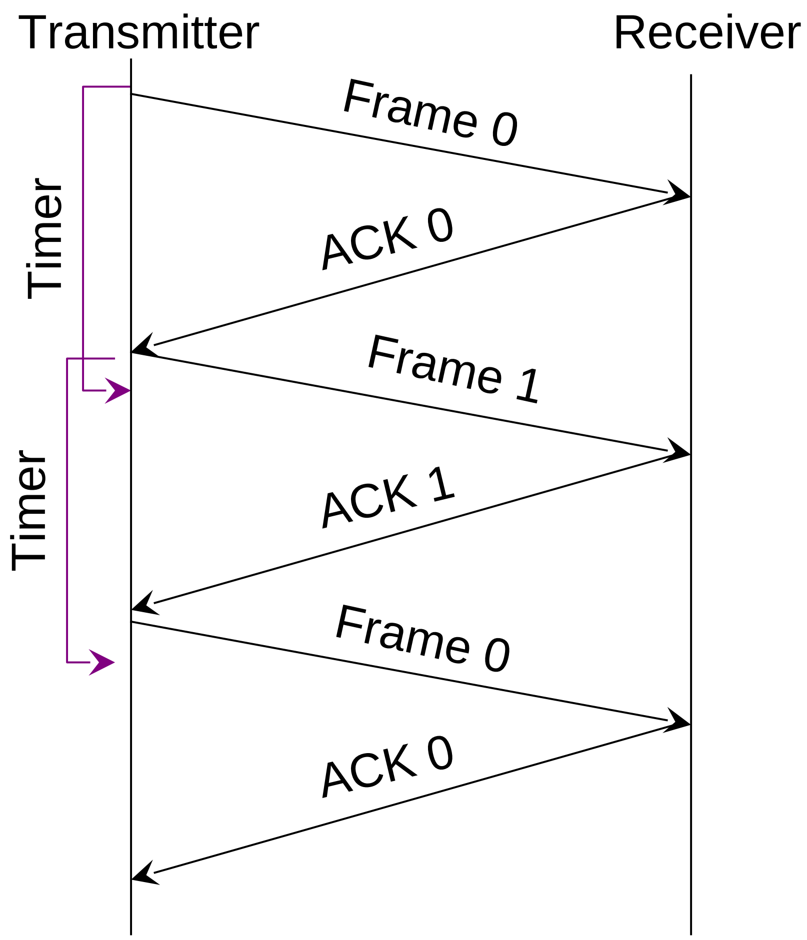
Example: Computer Basic Functions

- Design a Petri-net that imitates the basic behavior of a computer, i.e., being down, idle or busy

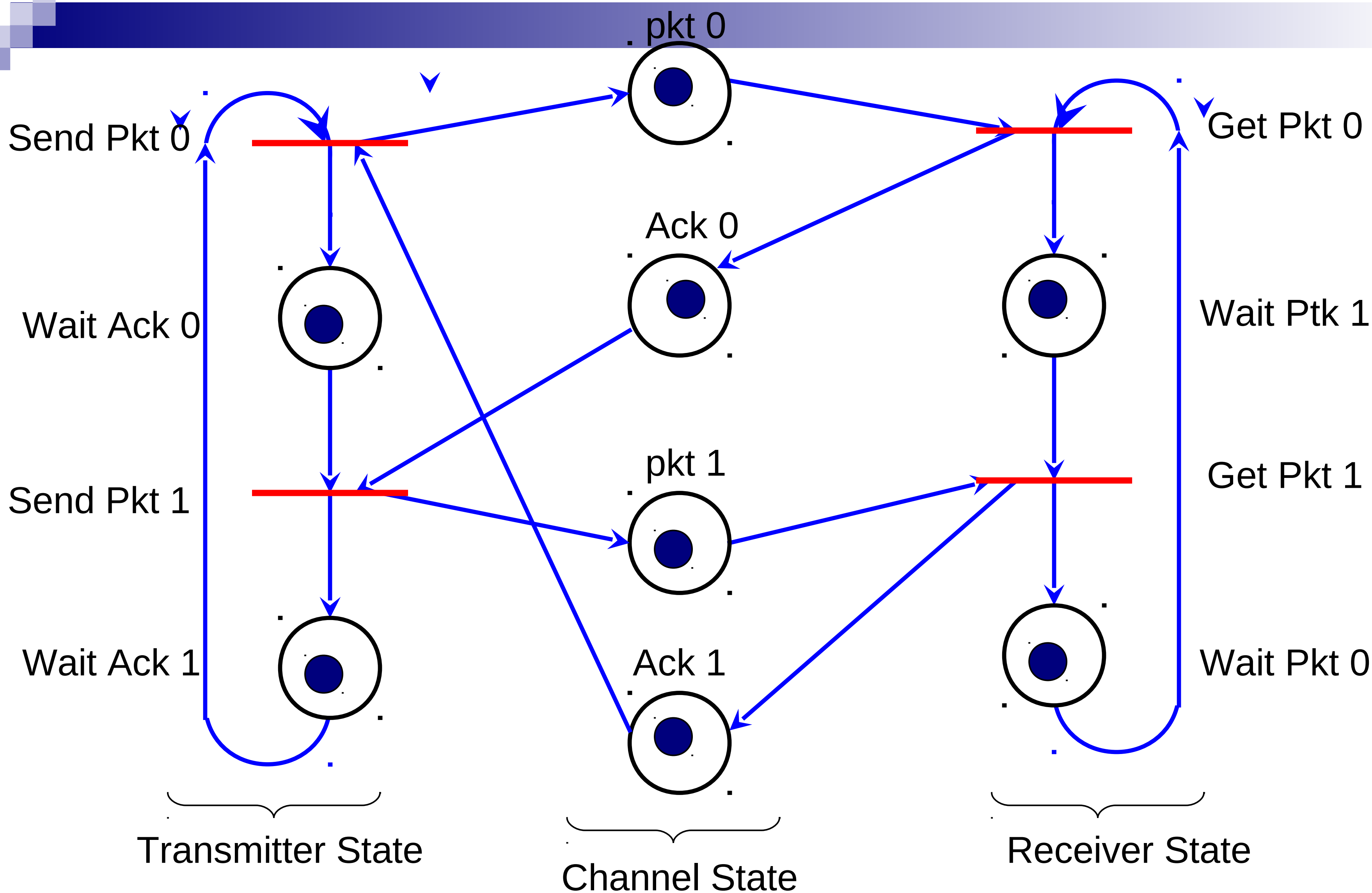


Modeling Protocols Using Petri Nets (Stop and Wait Protocol)

- The transmitter sends a frame and stops waiting for an acknowledgement from the receiver (ACK)
- Once the receiver correctly receives the expected packet, it sends an acknowledgement to let the transmitter send the next frame.
- When the transmitter does not receive an ACK within a specified period of time (timer) then it retransmits the packet.



Stop and Wait Protocol: Normal Operation



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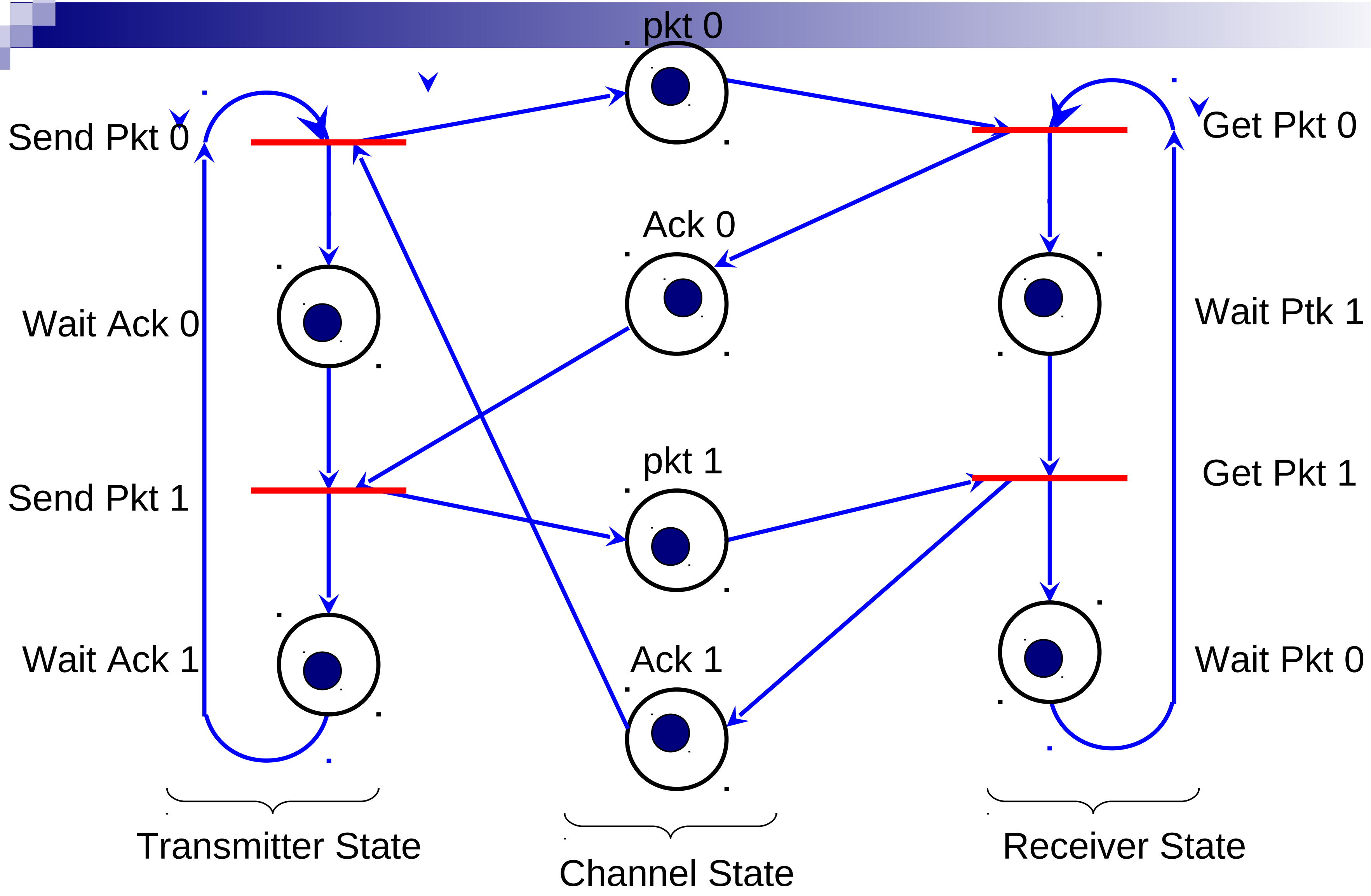
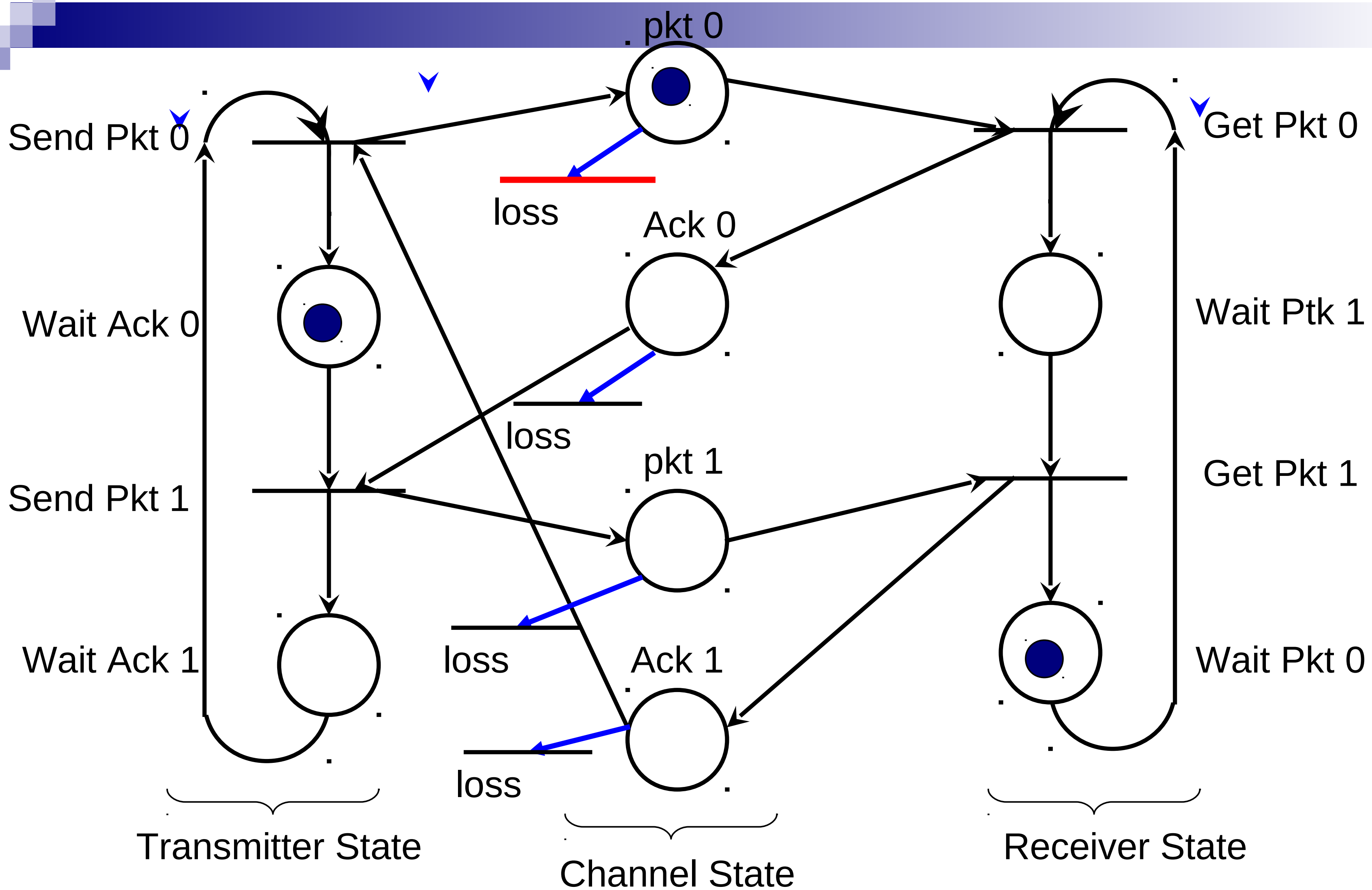
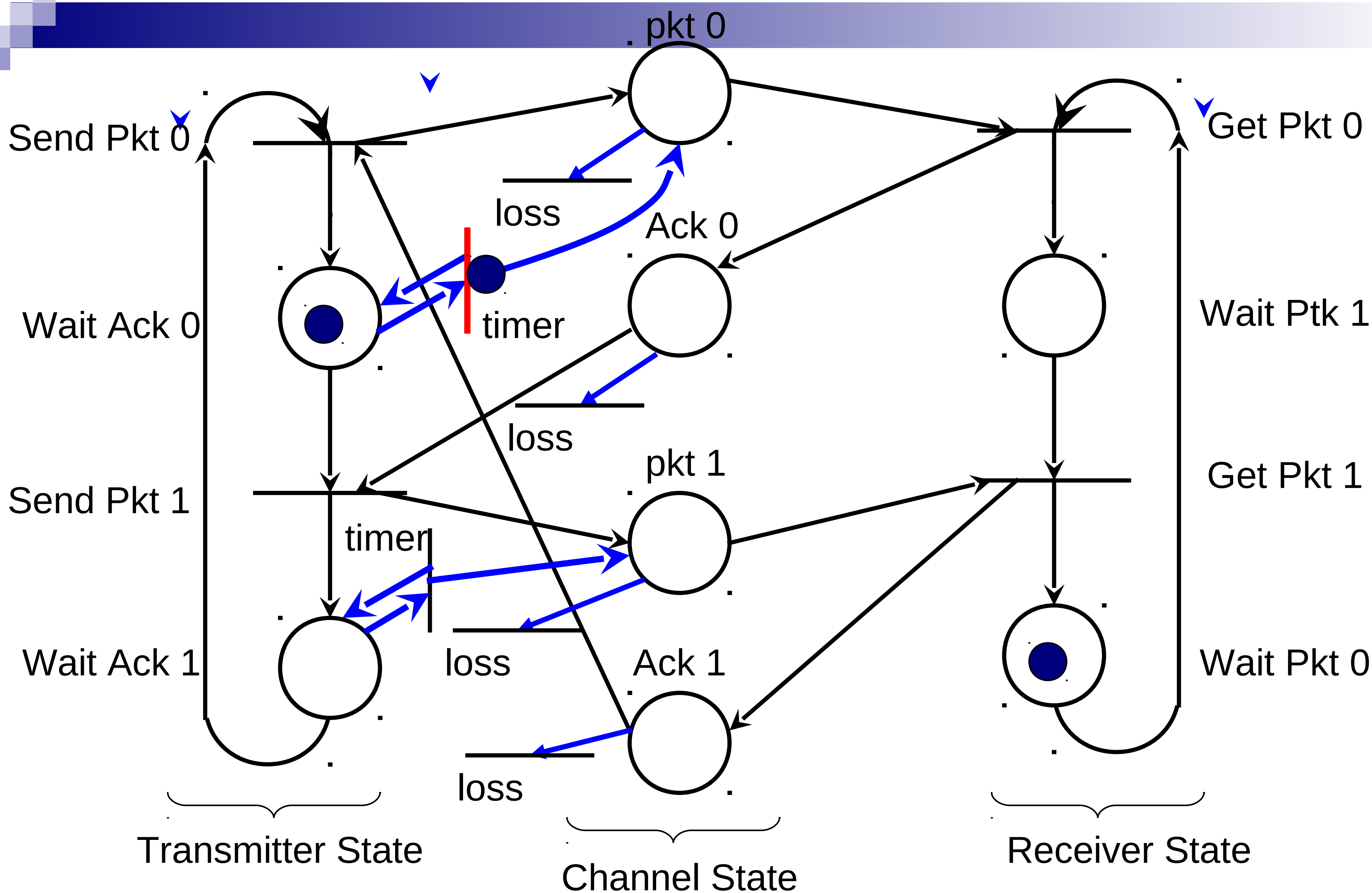


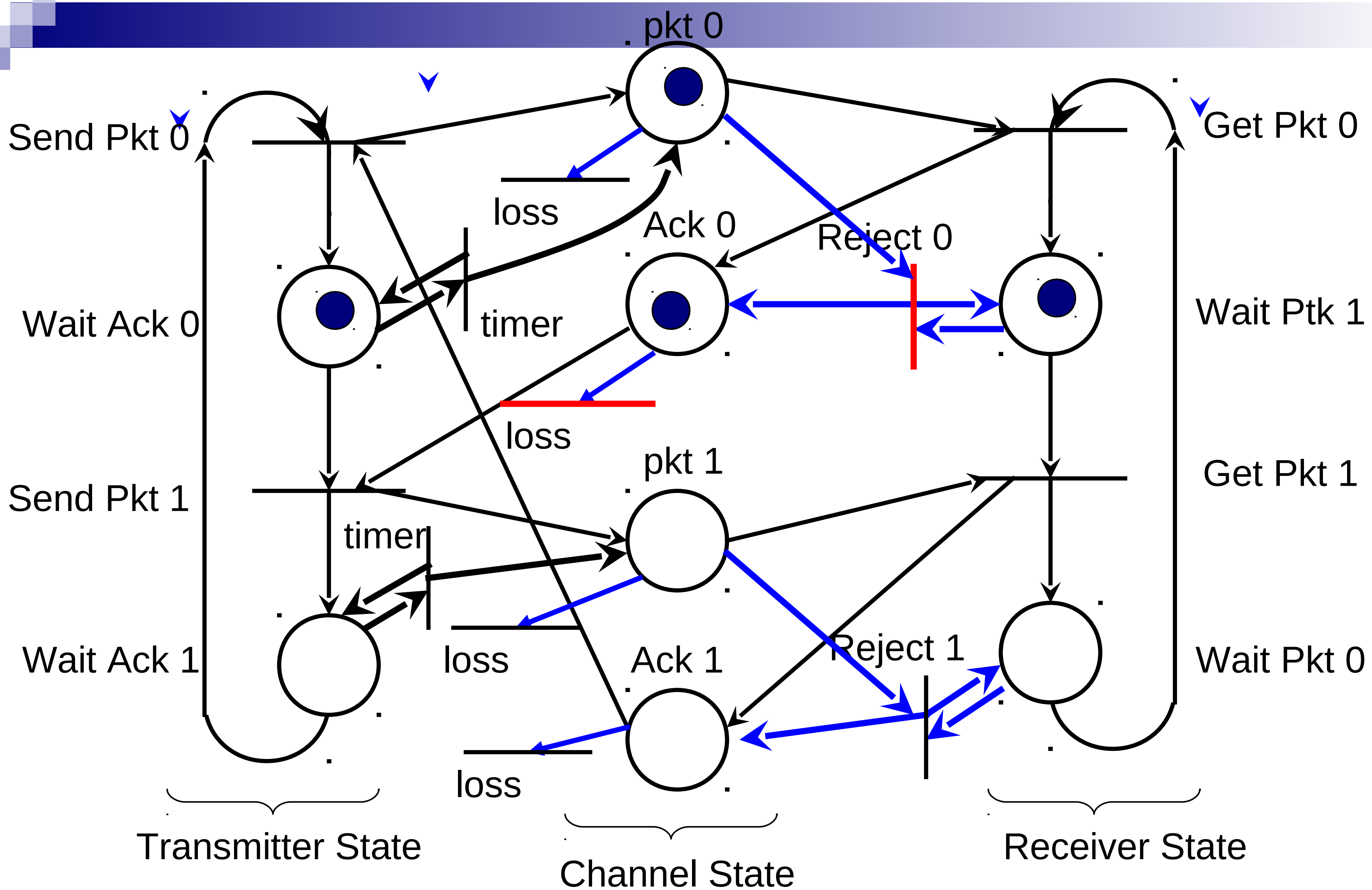
Stop and Wait Protocol: Deadlock



Stop and Wait Protocol: Deadlock Avoidance Using Timers

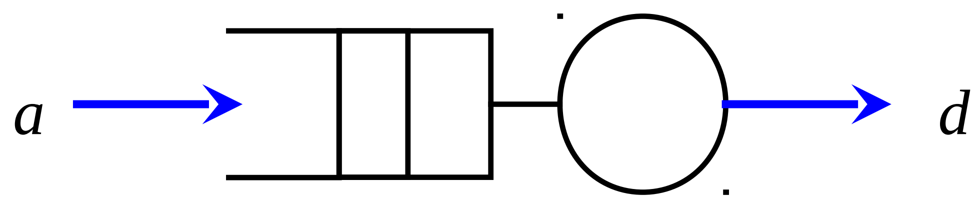


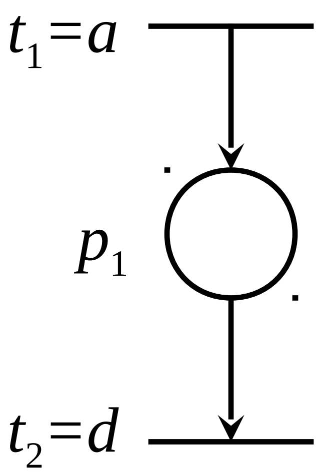
Stop and Wait Protocol: Loss of Acknowledgements



Example: Queueing Model

- What are some possible Petri nets that can model the simple FIFO queue

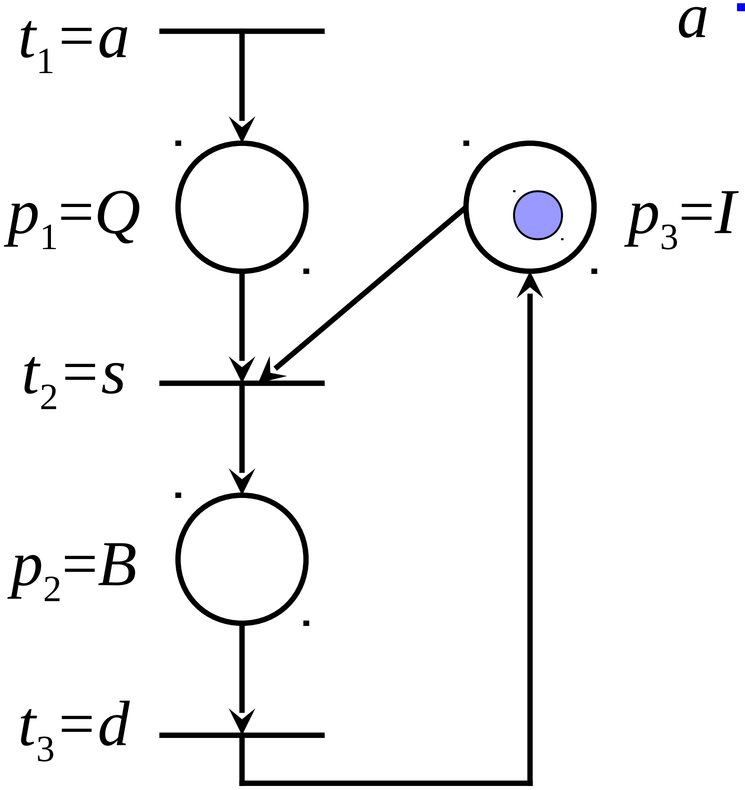


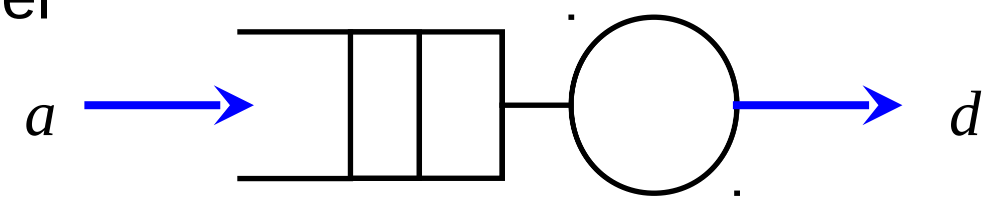


$$x' = x + u\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Example: Queueing Model

- A “richer” Petri-net model



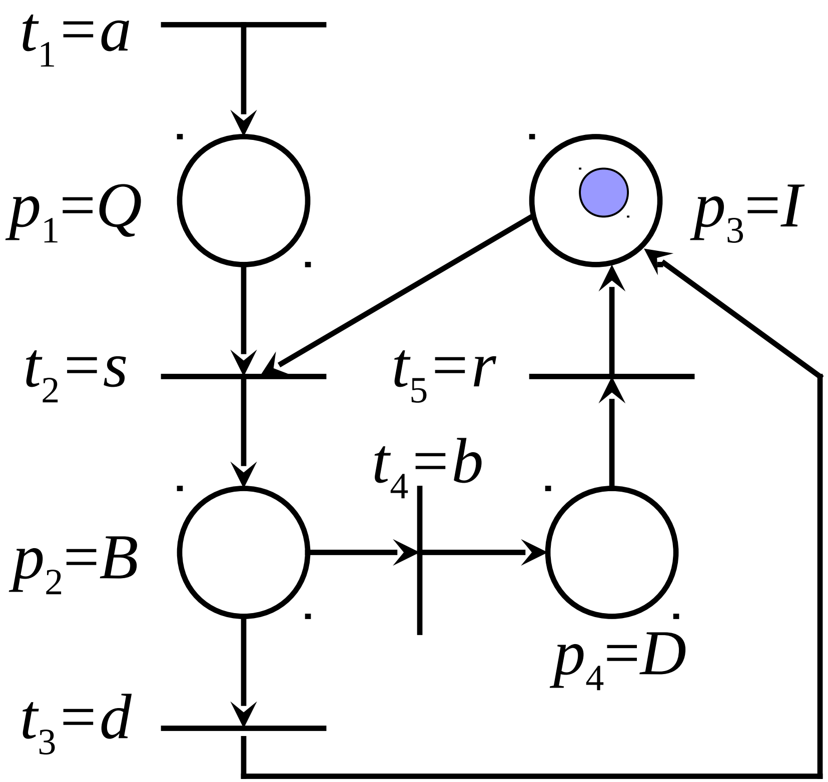


$$x' = x + u \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$x_0 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$$

Example: Queueing Model with Server Breakdown

- How does the previous model change if server may break down when it serves a customer?

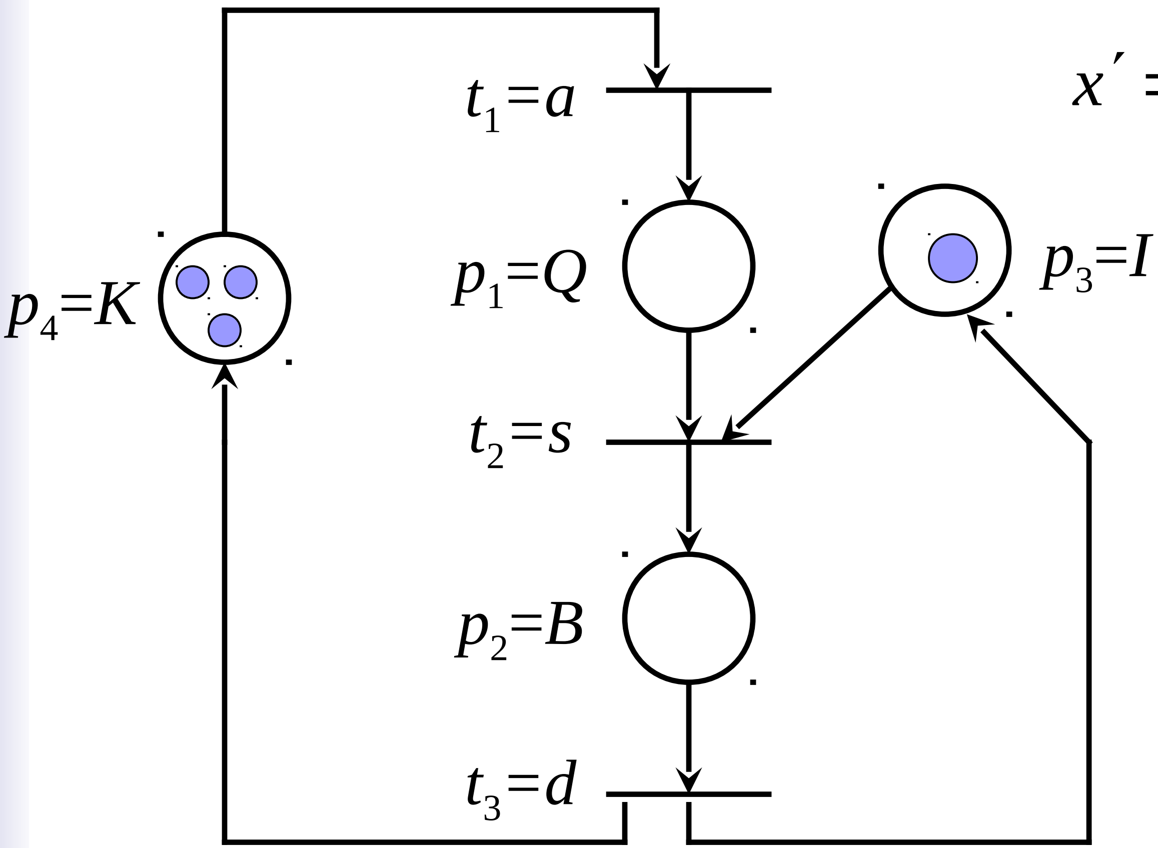


$$x' = x + u \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$$x_0 = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}$$

Example: Finite Capacity Queueing Model

- What is a Petri net model for a finite capacity queue?

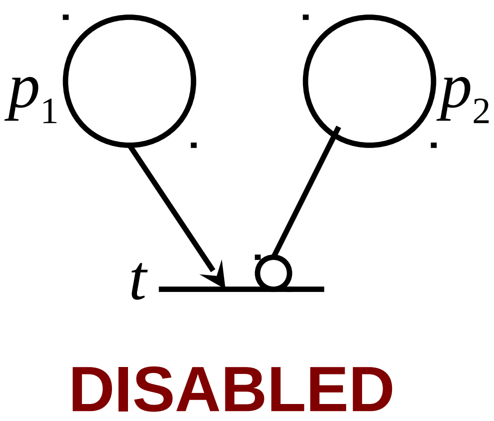


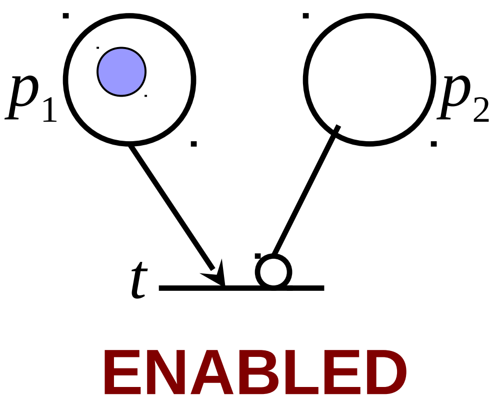
$$x' = x + u \begin{bmatrix} 1 & 0 & 0 & -1 \\ -1 & 1 & -1 & 0 \\ 0 & -1 & 1 & 1 \end{bmatrix}$$

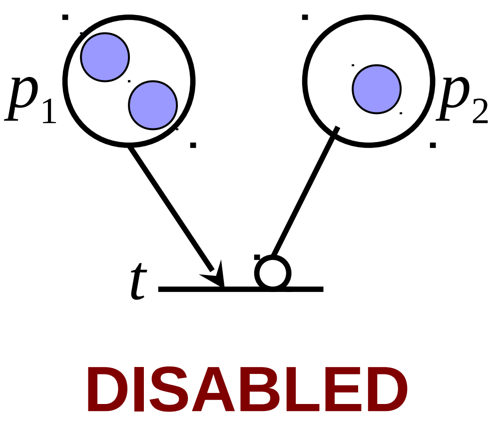
$$x_0 = \begin{bmatrix} 0 & 0 & 1 & K \end{bmatrix}$$

Other Petri Net Variations

- Inhibitor Arcs: A transition with an inhibitor arc is enabled when
  - All input places connected to normal arcs (arrows) have a number of tokens at least equal to the weight of the arcs and
  - All input places connected to inhibitor arcs (circles) have no tokens.







Other Petri Net Variations

- Colored Petri Nets
  - In this case, tokens have various properties associated with them. This can be an attribute or an entire data structure. For example,
    - Priority
    - Class
    - Etc.

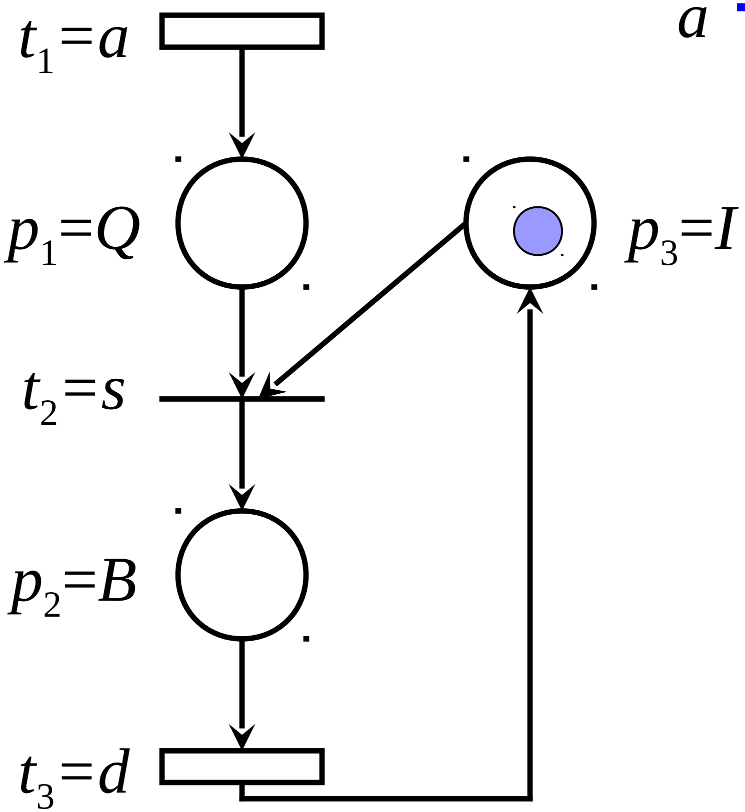
Timed Petri Net Graph

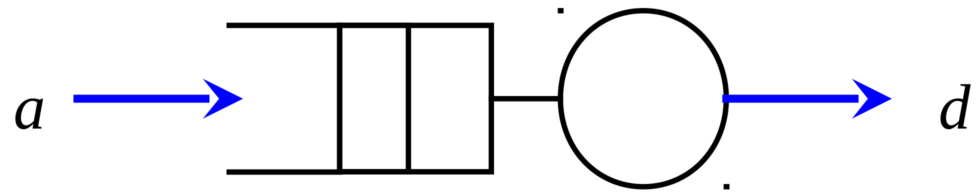
- In the previous discussion, the Petri net models had no time dimension. In other words, we did not consider the time when a transition occurred.

$$PN = (P, T, A, w, x, V)$$

- Timed Petri nets are similar to Petri nets with the addition of a clock structure associated with each timed transition
- A timed transition t_j (denoted by a rectangle) once it becomes enabled fires after a delay v_{jk}.

Example: Timed Petri Net





- Transitions t_1 and t_3 fire after a delay given by the model clock structure
- Transition t_2 fires immediately after it becomes enabled

Petri Net Timing Dynamics

- Notation
  - x is the current state
  - e is the transition that caused the Petri net into state x
  - t is the time that the corresponding event occurred

  - e' is the next transition to fire (*firing* transition)
  - t' is the next time the transition fires
  - x' is the next state given by $x' = f(x, e')$.

  - N'_i is the next score of transition i
  - y'_i is the next clock value of transition i (after e' occurs)

The Event Timing Dynamics

- **Step 1**: Given x evaluate which transitions are enabled
- **Step 2**: From the clock value y_i of all enabled transitions (denoted by $\Gamma(x)$) determine the minimum clock value

$$y^* = \min_{i \in \Gamma(x)} \{y_i\}$$

- **Step 3**: Determine the firing transition

$$e' = \arg\min_{i \in \Gamma(x)} \{y_i\}$$

- **Step 4**: Determine the next state

$$x' = f(x, e')$$

  - where $f()$ is the state transition function.

The Event Timing Dynamics

- **Step 5**: Determine

$$t' = t + y^*$$

- **Step 6**: Determine the new clock values

$$y_i' = \begin{cases} y_i - y^* & \text{if } i \neq e' \text{ and } i \in \Gamma(x) \\ v_{i,N_i+1} & \text{if } i = e' \text{ or } i \notin \Gamma(x) \end{cases}, \quad i \in \Gamma(x')$$

- **Step 7**: Determine the new transition scores

$$N_i' = \begin{cases} N_i + 1 & \text{if } i = e' \text{ or } i \notin \Gamma(x) \\ N_i & \text{Otherwise} \end{cases}, \quad i \in \Gamma(x')$$

Two Operation (Dioid) Algebras

- The operation of timed automata or timed Petri nets can be captured with two simple operations:
  - Addition $a \oplus b \equiv \max\{a, b\}$
  - Multiplication $a \otimes b \equiv a + b$
- This is also called the max-plus algebra

Basic Properties of Max-Plus Algebra

- Commutativity

$$a \oplus b = \max\{a, b\} = \max\{b, a\} = b \oplus a$$

$$a \otimes b = a + b = b + a = b \otimes a$$

- Associativity:

$$(a \oplus b) \oplus c = \max\{\max\{a, b\}, c\} = \max\{a, \max\{b, c\}\} = a \oplus (b \oplus c)$$

$$(a \otimes b) \otimes c = a(b \otimes c)$$

- Distribution of addition over multiplication

$$(a \oplus b) \otimes c = \max\{a, b\} + c = \max\{a + c, b + c\}$$

$$= (a \otimes c) \oplus (b \otimes c)$$

- Null element

$$a \oplus \eta = a \qquad a \otimes \eta = \eta$$

For example let $\eta = -\infty$

Example

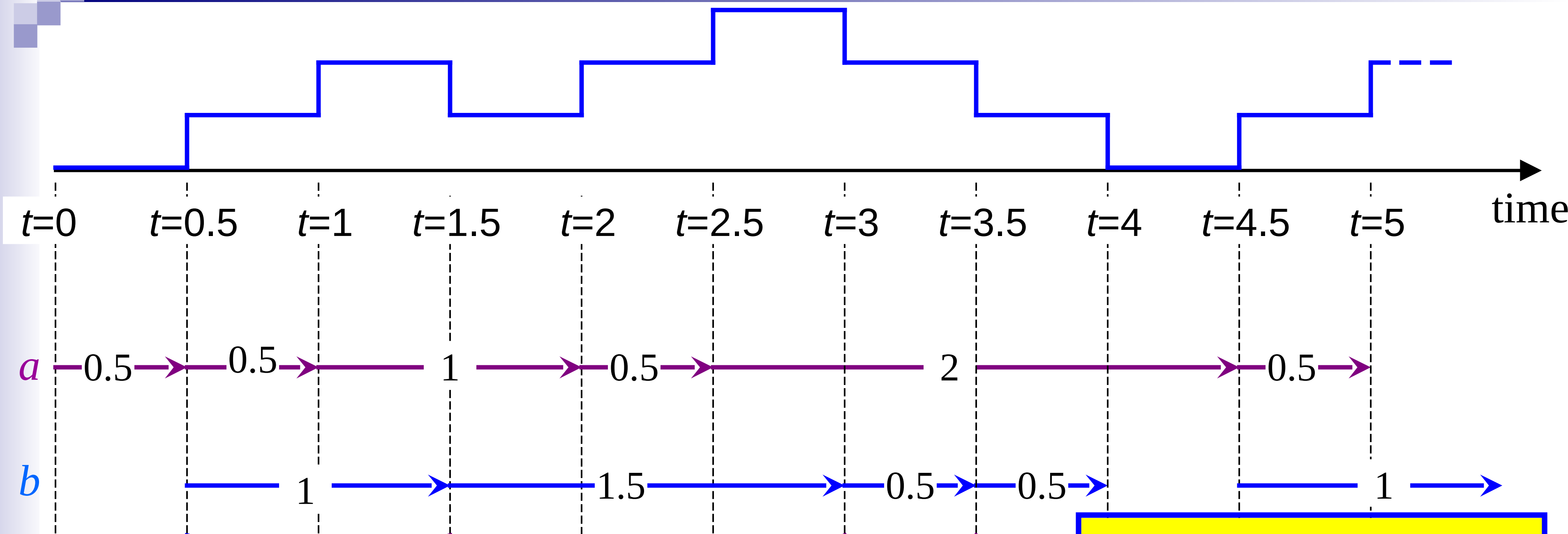
$$\begin{bmatrix} 1 & 0 \\ 2 & -2 \end{bmatrix} \otimes \begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \max\{1+2, 0+3\} & \max\{1-1, 0+1\} \\ \max\{2+2, -2+3\} & \max\{2-1, -2+1\} \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 1 \\ 4 & 1 \end{bmatrix}$$

$$a \begin{bmatrix} 3 & 1 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} a+3 & a+1 \\ a+4 & a+1 \end{bmatrix}$$

Queueing Models and Max-Plus Algebra



$$a_1 = v_{a,1}$$

$$a_2 = a_1 + v_{a,2}$$

$$a_3 = a_2 + v_{a,3}$$

$$a_k = a_{k-1} + v_{a,k}$$

$$d_1 = a_1 + v_{d,1}$$

$$d_2 = d_1 + v_{d,2}$$

$$d_3 = d_2 + v_{d,3}$$

$$d_k = \max\{a_k, d_{k-1}\} + v_{d,k}$$

Queueing Dynamics

- Let a_k be the arrival time of the k-th customer and d_k its departure time, k=1,…,K, then

$$a_k = a_{k-1} + v_{a,k}$$

$$d_k = \max\{a_k, d_{k-1}\} + v_{d,k}$$

$$= \max\{a_{k-1} + v_{a,k}, d_{k-1}\} + v_{d,k} \qquad k=1,2,\ldots,\ a_0=0,\ d_0=0$$

- In matrix form, let $x_k=[a_k, d_k]^T$ then

$$x_{k+1} = \begin{bmatrix} v_{a,k} & -L \\ v_{a,k} + v_{d,k+1} & v_{d,k+1} \end{bmatrix} \begin{bmatrix} a_k \\ d_k \end{bmatrix} = \mathbf{A}_k x_k \qquad x_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

where $-L$ is sufficiently small such that $\max\{a_k+v_{ak}, d_k-L\}=a_k+v_{ak}$

Example

- Determine the sample path of the FIFO queue when
  - v_a={0.5, 0.5, 1.0, 0.5, 2.0, 0.5, …}
  - v_d={1.0, 1.5, 0.5, 0.5, 1.0, …}

$$x_{k+1} = \begin{bmatrix} a_{k+1} \\ d_{k+1} \end{bmatrix} = \begin{bmatrix} v_{a,k} + a_k \\ \max\{a_k + v_{a,k}, d_k\} + v_{d,k} \end{bmatrix} \qquad x_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x_1 = \begin{bmatrix} a_1 \\ d_1 \end{bmatrix} = \begin{bmatrix} v_{a,0} + a_0 \\ \max\{a_0 + v_{a,0}, d_0\} + v_{d,0} \end{bmatrix} = \begin{bmatrix} 0 + 0.5 \\ \max\{0 + 0.5, 0\} + 1 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 1.5 \end{bmatrix}$$

$$x_2 = \begin{bmatrix} 0.5 + 0.5 \\ \max\{1, 1.5\} + 1.5 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \qquad x_3 = \begin{bmatrix} 1.0 + 1.0 \\ \max\{2, 3\} + 0.5 \end{bmatrix} = \begin{bmatrix} 2 \\ 3.5 \end{bmatrix} \quad \cdots$$

Communication Link

- How would you model a transmission link that can transmit packets at a rate G packets per second and has a propagation delay equal to 100ms?