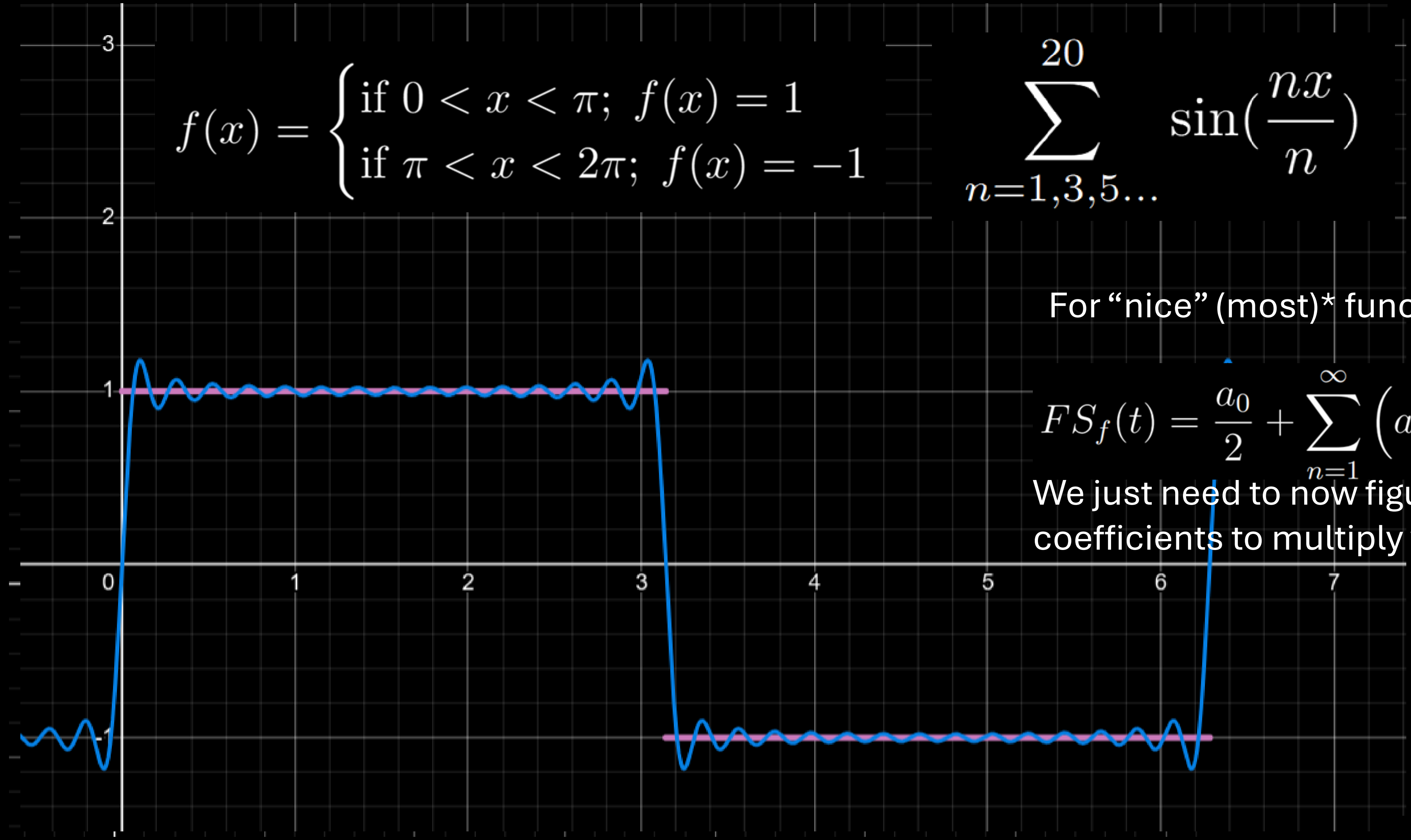
MATH1012
Practical (Wk 10)

$\boxed{\text{Fourier Series}}$

Fourier Series - Idea



$$f(x) = \begin{cases} \text{if } 0 < x < \pi; \ f(x) = 1 \\ \text{if } \pi < x < 2\pi; \ f(x) = -1 \end{cases}$$

$$\sum_{n=1,3,5\ldots}^{20} \sin\left(\frac{nx}{n}\right)$$

For “nice” (most)* functions

$$FS_f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi t}{L}\right) + b_n \sin\left(\frac{n\pi t}{L}\right) \right)$$

We just need to now figure out how to determine the coefficients to multiply with for each of these!

Fourier Series – The Coefficients

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nt + \sum_{n=1}^{\infty} b_n \sin nt$$

$$\int_0^{2\pi} \sin(mt)\sin(nt)dt = \begin{cases} 0, & m \neq n \\ \pi, & m = n \end{cases}$$

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(t)dt,$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(t)\,\cos\left(\frac{n\pi t}{L}\right)dt,$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(t)\,\sin\left(\frac{n\pi t}{L}\right)dt.$$

$$\int_0^{2\pi} f(t) \sin mt\, dt$$

$$= \int_0^{2\pi} \left(\cancelto{0}{\frac{a_0}{2}} + \sum_{n=1}^{\infty} a_n \cancelto{0}{\cos nt} + \sum_{n=1}^{\infty} b_n \sin nt\right) \sin mt\, dt$$

$$= \int_0^{2\pi} b_m \sin mt \sin mt\, dt = b_m \pi$$

$$\int_0^{2\pi} f(t) \cos mt\, dt$$

$$= \int_0^{2\pi} \left(\cancelto{0}{\frac{a_0}{2}} + \sum_{n=1}^{\infty} a_n \cos nt + \sum_{n=1}^{\infty} b_n \cancelto{0}{\sin nt}\right) \cos mt\, dt$$

$$= \int_0^{2\pi} a_m \cos mt \cos mt\, dt = a_m \pi$$

$$\int_0^{2\pi} f(t) 1 dt$$

$$= \int_0^{2\pi} \left(\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cancelto{0}{\cos nt} + \sum_{n=1}^{\infty} b_n \cancelto{0}{\sin nt}\right) 1 dt$$

$$= a_0 \pi$$

$$b_m = \frac{1}{\pi}\int_0^{2\pi} f(t) \sin mt\, dt$$

$$a_m = \frac{1}{\pi}\int_0^{2\pi} f(t) \cos mt\, dt$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(t) 1 dt$$

Fourier Series (Walkthrough Computation)

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(t)dt,$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos\left(\frac{n\pi t}{L}\right)dt,$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(t)\sin\left(\frac{n\pi t}{L}\right)dt.$$

Ex: $f(t) = \begin{cases} 1, & 0 < x < \pi \\ 0, & \pi < x < 2\pi \end{cases}$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(t)1dt$$

$$a_m = \frac{1}{\pi}\int_0^{2\pi} f(t)\cos mt\, dt$$

$$b_m = \frac{1}{\pi}\int_0^{2\pi} f(t)\sin mt\, dt$$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} 1\cdot 1dt$$

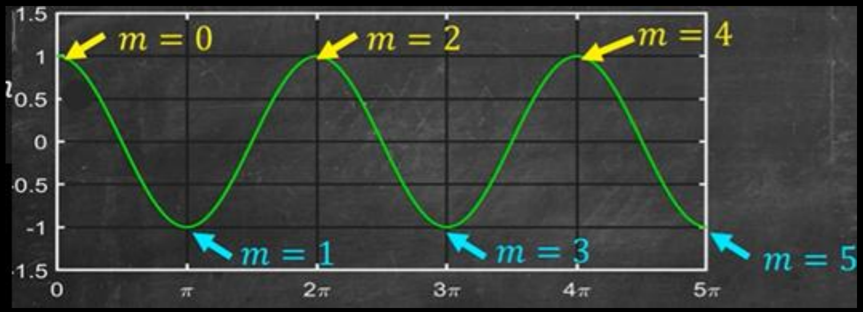
$$a_m = \frac{1}{\pi}\int_0^{\pi} 1\cos mt\, dt$$

$$b_m = \frac{1}{\pi}\int_0^{\pi} 1\sin mt\, dt$$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} 1\cdot 1dt = 1$$

$$a_m = \frac{1}{\pi}\int_0^{\pi} 1\cos mt\, dt = \frac{1}{\pi}\cdot\frac{1}{m}\sin mt\Big|_0^{\pi} = 0$$

$$b_m = \frac{1}{\pi}\int_0^{\pi} 1\sin mt\, dt = \frac{1}{\pi}\cdot\frac{-1}{m}\cos mt\Big|_0^{\pi}$$



$$b_m = \frac{1}{\pi}\cdot\frac{-1}{m}\cos mt\Big|_0^{\pi}$$

$$= \frac{1}{\pi}\cdot\frac{-1}{m}[\cos m\pi - 1]$$

$$= \frac{1}{\pi}\cdot\frac{-1}{m}[(-1)^m - 1]$$

$$= \begin{cases} 0, & m \text{ even} \\ \dfrac{2}{\pi m}, & m \text{ odd} \end{cases}$$

Ex: $f(t) = \begin{cases} 1, & 0 < x < \pi \\ 0, & \pi < x < 2\pi \end{cases}$

$a_0 = 1$

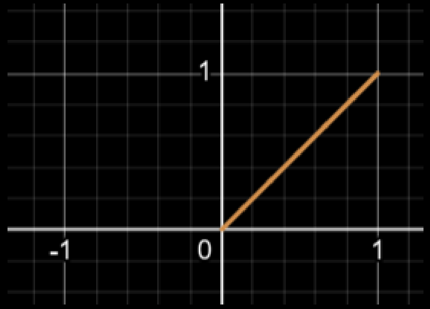
$a_m = 0$

$$b_m = \begin{cases} 0, & m \text{ even} \\ \dfrac{2}{\pi m}, & m \text{ odd} \end{cases}$$

$$f(t) = \frac{1}{2} + \sum_{n=1,3,5\ldots}^{\infty} \frac{2}{\pi n}\sin nt$$

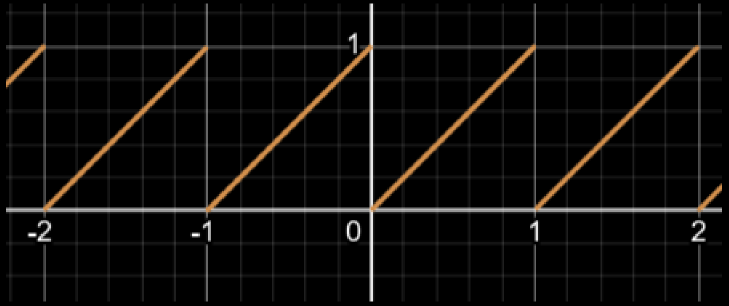
Periodic Extension, Fourier Series Graphing

Periodic Function:

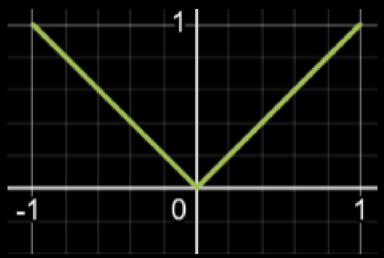


i.e. f(x) = x
$\{0<x<1\}$

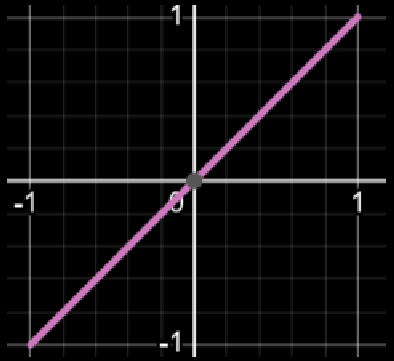
Periodic Extension:



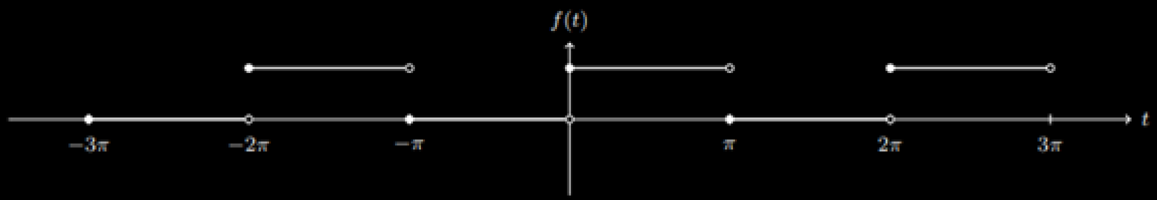
Even Extension:

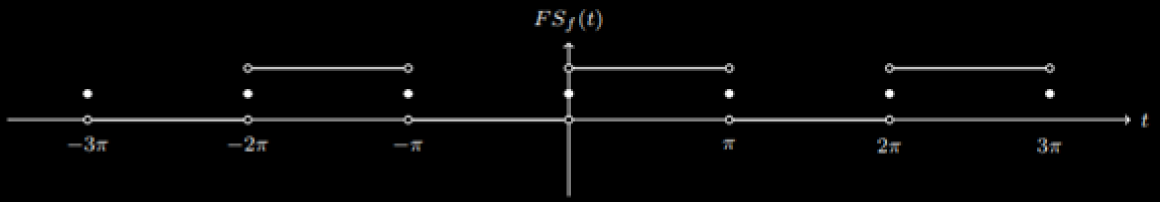


Odd Extension:



At Discontinuities Fourier Series converges to midpoint of left & right-hand side values.





Final Tip/Summary

Fourier Series:

$$FS_f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi t}{L}\right) + b_n \sin\left(\frac{n\pi t}{L}\right)\right)$$

Coefficients:

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(t)dt,$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos\left(\frac{n\pi t}{L}\right)dt,$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(t)\sin\left(\frac{n\pi t}{L}\right)dt.$$

If your function is even or odd you can automatically exclude the odd (Sine) or, even term (Cosine) – as Odd functions will only be composed of odd Fourier terms, and vice versa.

(Likewise, if the average term for the periodic extension is 0; a_0)

Sources

Images (Grey Background) for Slide 3, 4 – Trefor Bazett