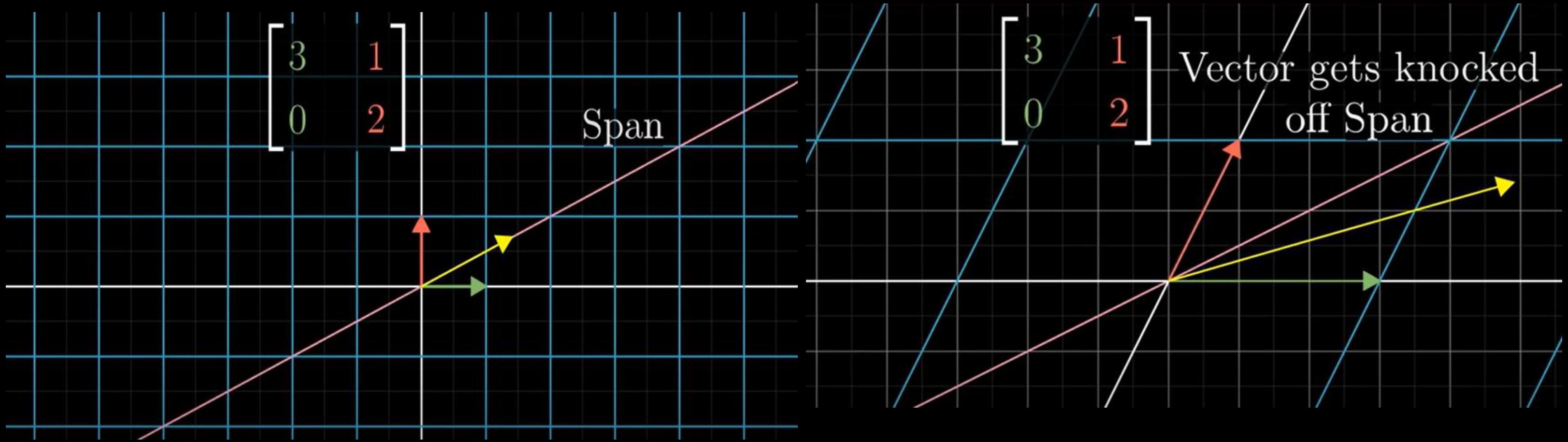
Week 6
MATH1012
Practical

Important Reminder:

Test 30th of August!
(Saturday)

This Week: Eigenvectors

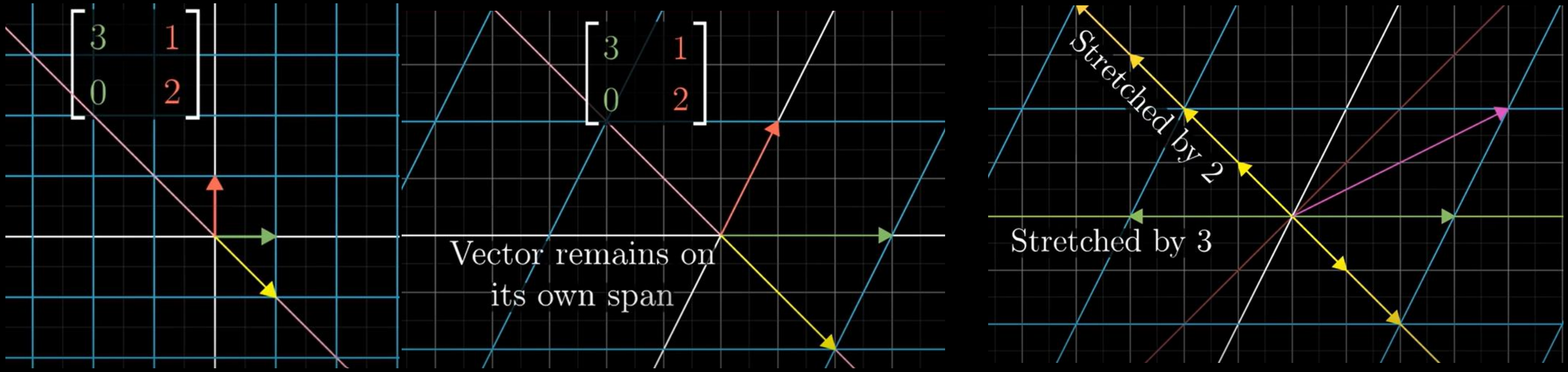
Most Vectors are knocked off their Span after a Linear Transformation:



$$\overbrace{A}^{\text{Transformation matrix}}\vec{\mathbf{v}} = \overbrace{\lambda}^{\text{Eigenvalue}}\vec{\mathbf{v}}$$

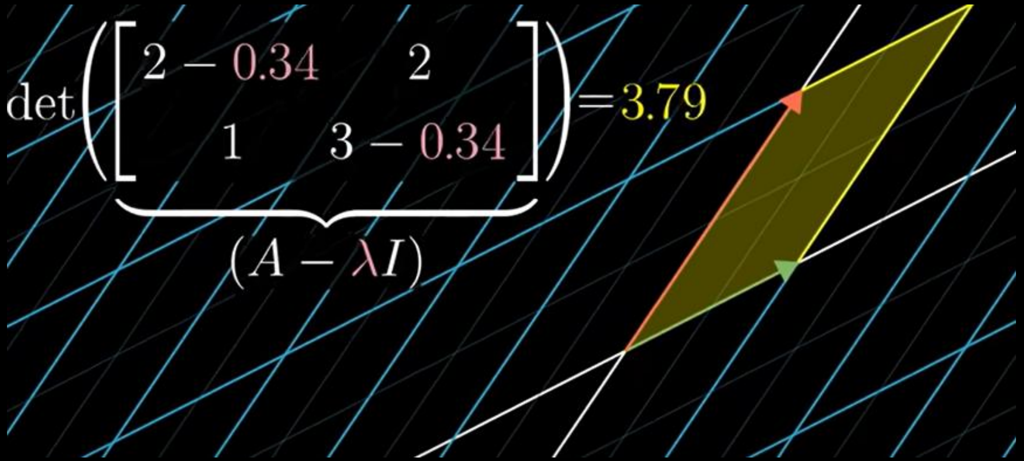
Eigenvector

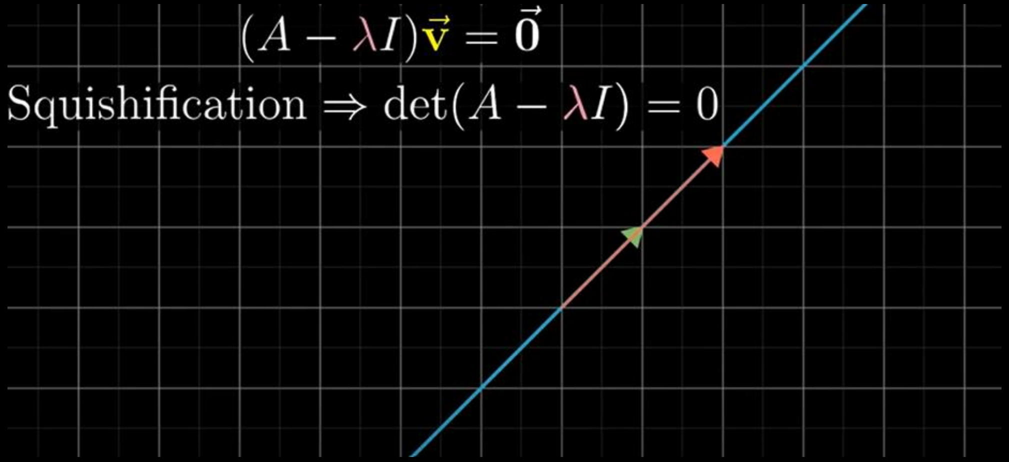
However, there are some special vectors which aren’t - these are Eigenvectors!



(Amount Stretched – Eigenvalue)

Eigenvectors (Continued)





Recipe:

1. Computing:

$$\det(A-\lambda I) = 0$$

(Solve for λ/ Eigenvalues)

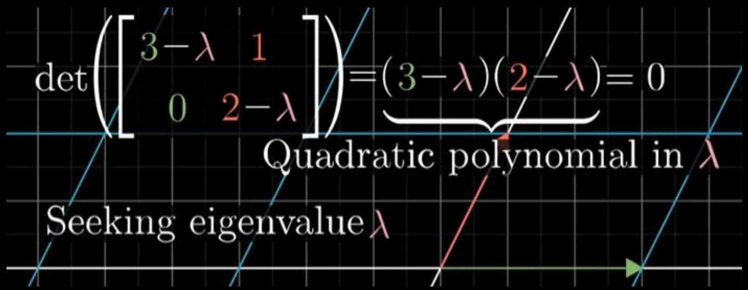
2. Substitute λ (our eigenvalue) into:

$$A-\lambda I$$

3. Solve parametrically to find eigenvectors (Av=0)

4. Repeat step 2, and 3 for all eigenvalues/values of λ

Example:



$\lambda = 2$

$$\begin{bmatrix} 3-2 & 1 \\ 0 & 2-2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

$$x_1 + x_2 = 0 \implies x_1 = -x_2$$

$$x_2 = x_2$$

$$\begin{cases} x_1 = -x_2 \\ x_2 = x_2 \end{cases} \implies \vec{v} = \begin{pmatrix} -x_2 \\ x_2 \end{pmatrix} = x_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\implies \vec{v} = \left\{ \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}$$

$$\begin{bmatrix} 3 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} x + \begin{bmatrix} 2 \\ -1 \end{bmatrix} y$$

This Week: Change of Basis

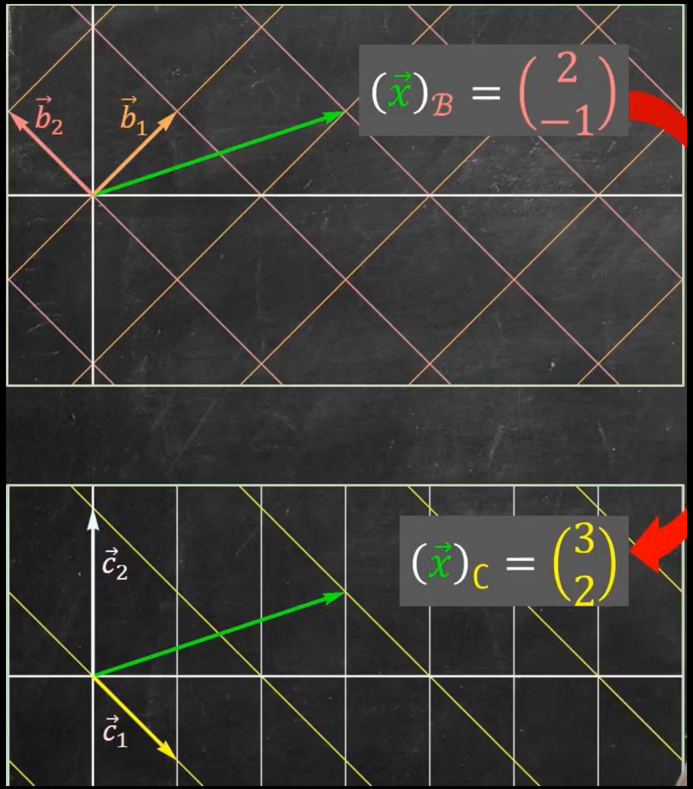
For a Basis $\mathcal{B} = \{\vec{b}_1, \vec{b}_2\}$
if $\vec{x} = c_1\vec{b}_1 + c_2\vec{b}_2$,
then $(\vec{x})_{\mathcal{B}} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$

$(\boldsymbol{x})_B$ - a vector $(\boldsymbol{x})$ written in terms of a basis B

$$\left\{ \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \end{bmatrix} \right\} \begin{bmatrix} 3 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} x + \begin{bmatrix} 2 \\ -1 \end{bmatrix} y$$

$(\boldsymbol{x})_S$ - a vector $(\boldsymbol{x})$ written in terms of the Standard Basis

$$\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} y$$



Two Bases: $\mathcal{B} = \{\vec{b}_1, \dots, \vec{b}_n\}$
$\mathrm{C} = \{\vec{c}_1, \dots, \vec{c}_n\}$

$$s_1\vec{b}_1 + \cdots + s_n\vec{b}_n = \vec{x} = t_1\vec{c}_1 + \cdots + t_n\vec{c}_n$$

$$s_1\vec{b}_1 + \cdots + s_n\vec{b}_n = (\vec{b}_1 \ \dots \vec{b}_n)\begin{pmatrix} s_1 \\ \vdots \\ s_n \end{pmatrix} = P_{\mathcal{B}}(\vec{x})_{\mathcal{B}}$$

$$t_1\vec{c}_1 + \cdots + t_n\vec{c}_n = (\vec{c}_1 \ \dots \vec{c}_n)\begin{pmatrix} t_1 \\ \vdots \\ t_n \end{pmatrix} = P_{\mathrm{C}}(\vec{x})_{\mathrm{C}}$$

Two Bases: $\mathcal{B} = \{\vec{b}_1, \dots, \vec{b}_n\}$
$\mathrm{C} = \{\vec{c}_1, \dots, \vec{c}_n\}$

$$P_{\mathcal{B}}(\vec{x})_{\mathcal{B}} = \vec{x} = P_{\mathrm{C}}(\vec{x})_{\mathrm{C}}$$

$$\boxed{(\vec{x})_{\mathrm{C}} = P_{\mathrm{C}}^{-1}P_{\mathcal{B}}(\vec{x})_{\mathcal{B}}}$$

$$\boxed{(\vec{x})_{\mathcal{B}} = P_{\mathcal{B}}^{-1}P_{\mathrm{C}}(\vec{x})_{\mathrm{C}}}$$

Change of Basis Continued

For $\mathcal{B} = \left\{\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}-1\\1\end{pmatrix}\right\}$, $C = \left\{\begin{pmatrix}1\\-1\end{pmatrix}, \begin{pmatrix}0\\2\end{pmatrix}\right\}$

$$P_{\mathcal{B}} = \begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix} \qquad P_{C} = \begin{pmatrix}1 & 0\\ -1 & 2\end{pmatrix}$$

$$P_C^{-1}P_{\mathcal{B}} = \frac{1}{2}\begin{pmatrix}2 & 0\\ 1 & 1\end{pmatrix}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}$$

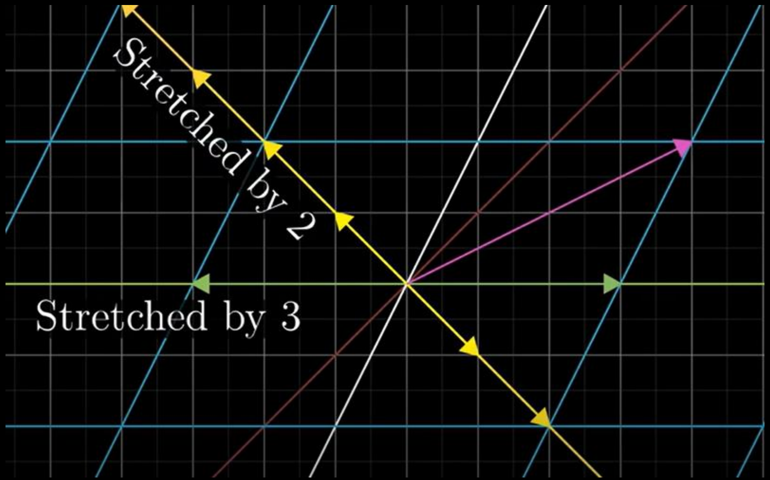
$$= \begin{pmatrix}1 & -1\\ 1 & 0\end{pmatrix}$$

$$\overbrace{\underbrace{\begin{bmatrix}2 & -1\\ 1 & 1\end{bmatrix}^{-1}}_{\text{Inverse change of basis matrix}}\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}\begin{bmatrix}2 & -1\\ 1 & 1\end{bmatrix}\begin{bmatrix}-1\\ 2\end{bmatrix}}^{\text{Transformed vector in her language}}$$

Eigenbasis

For certain transformations we have enough eigenvectors to form a basis:

(This is called an Eigenbasis)



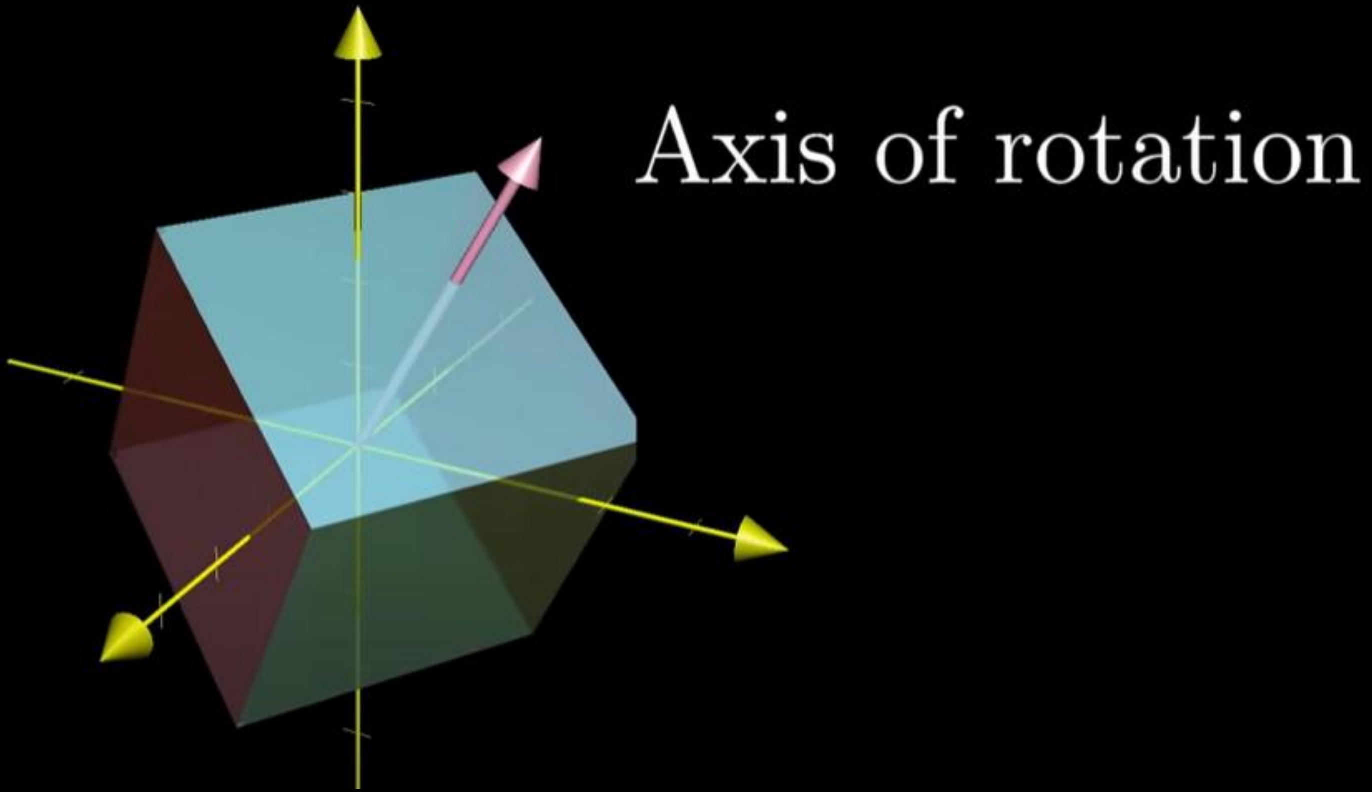
When this matrix exists – this can simplify the difficulty of a Linear Algebra problem significantly (by diagonalization):

$$\underbrace{\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \cdots\cdots \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}}_{\text{100 times}} \begin{bmatrix} x \\ y \end{bmatrix}$$

We can use a change of basis matrix to transform our original matrix to an eigenbasis

$$\underbrace{\begin{bmatrix} 3 & 4 \\ 1 & 1 \end{bmatrix} \cdots\cdots \begin{bmatrix} 3 & 4 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 3 & 4 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 3 & 4 \\ 1 & 1 \end{bmatrix}}_{\text{100 times}} \begin{bmatrix} x \\ y \end{bmatrix}$$

Example



Final Notes

Algebraic Multiplicity – Number of Times a Root Appears in a Characteristic Equation:

e.g. For $(\lambda - 2)^2(\lambda - 3) = 0$:

$\lambda = 2$ has multiplicity 2,

$\lambda = 3$ has multiplicity 1

Geometric Multiplicity – Dimension of Eigenvectors for a given Eigenvalue Span.

If $\lambda = -3$ was associated with eigenvectors with $span\{(1,0,1), (3,1,4\}$ it would have geometric multiplicity of 2.

Know Eigenvalue Rules such as:

If λ is an eigenvalue of A:

then λ^n is an eigenvalue of A^n

then λ is an eigenvalue of A^T

then $\lambda + n$ is an eigenvalue of $A + nI$

Sources

Images –

3Blue1Brown/Grant Sanderson: Slides - 3,4,7,8

Trefor Bazett: Slides – 5,6