Week 11
Practical
MATH1012

Forms of Differential Equations

Separable:

$$\frac{dy}{dx} = f(x)g(y)$$

Other Examples:

$$y'(x) = x\sin(x)\sqrt{y}$$

$$y'(x) = \frac{x^2}{y^2}$$

$$y'(x) + y^2 = 0$$

e.g.

$$\frac{dy}{dx} = x^2 y$$

$$\frac{1}{y}dy = x^2 dx$$

$$\int \frac{1}{y}dy = \int x^2 dx$$

$$\ln(y) = \frac{x^3}{3} + c \qquad y = e^c e^{\frac{x^3}{3}}$$

$$y = e^{\frac{x^3}{3}+c} \qquad y = Ae^{\frac{x^3}{3}}$$

Forms of Differential Equations

Linear Differential Equation:

$$y' + p(x)y = f(x)$$

Method:

$$y = \frac{1}{r(x)} \int r(x) f(x) dx$$

$$r(x) = e^{\int p(x)dx}$$

Examples:

$$y' + 4y = e^{-x}$$

$$y' + \sin(x)\, y = 2x$$

$$xy' + (x^3 - 1)y = 2e^{-x}$$

$$y' + \sinh(x)\, y = e^{-x} + e^{x}$$

Linear – Brief Derivation

Standard Form for 1st Order Linear:

$$y' + p(x)y = f(x)$$

Integrating Factor Method:

$$r(x)y' + \boxed{r(x)p(x)}y = r(x)f(x)$$

$$\stackrel{?}{=}$$

$$\frac{d}{dx}[r(x)y]$$

$$= r(x)y' + \boxed{r'(x)}y$$

$$r'(x) = r(x)p(x)$$

$$\int \frac{r'(x)}{r(x)}dx = \int p(x)dx$$

$$\ln(r(x)) = \int p(x)dx$$

$$r(x) = e^{\int p(x)dx}$$

$$\frac{d}{dx}[r(x)y] = r(x)f(x)$$

$$y = \frac{1}{r(x)}\int r(x)f(x)dx$$

Forms of Differential Equations

Constant Coefficients (Homogenous)

$$ay'' + by' + cy = 0$$

Characteristic Equation:

$$ar^2 + br + c = 0$$

$$y'' - 2y' - 3y = 0$$

Guess $y = e^{rt}$

$$r^2e^{rt} - 2re^{rt} - 3e^{rt} = 0$$

$$r^2 - 2r - 3 = 0$$

$$r_1 = 3, \quad r_2 = -1$$

$$y = c_1e^{3t} + c_2e^{-t}$$

Real Roots

$$y'' - 2y' - 3y = 0$$

$$r^2 - 2r - 3 = 0$$

$$r = -1{,}3$$

$$y = c_1e^{-t} + c_2e^{3t}$$

$$y = c_1e^{r_1t} + c_2e^{r_2t}$$

Repeated Roots

$$y'' + 4y' + 4y = 0$$

$$r^2 + 4r + 4 = 0$$

$$r = -2$$

$$y = c_1e^{-2t} + c_2te^{-2t}$$

$$y = c_1e^{rt} + c_2te^{rt}$$

Complex Roots

$$y'' + 3y' + 4y = 0$$

$$r^2 + 3r + 4 = 0$$

$$y = c_1e^{\alpha t}\cos\beta t + c_2e^{\alpha t}\sin\beta t$$

Differential Equations (Additional Methods)

Method of Undetermined Coefficients

$$y'' - 2y' - 3y = \boxed{3e^{2t}}$$

1. Solve homogeneous equation *in general* y_h
2. Find *one* solution to full equation y_p
3. Add solutions together $y_h + y_p$

Non-homogeneity	Guess
e^{rt}	Ae^{rt}
$\sin(rt)$ or $\cos(rt)$	$A\sin(rt) + B\cos(rt)$
Degree n polynomial	$A_0 + A_1 t + \cdots + A_n t^n$

Guess $y_p = Ae^{2t}$

$$y'' - 2y' - 3y = 3e^{2t}$$

$$(Ate^{-t})'' - 2(Ate^{-t})' - 3(Ate^{-t}) = 3e^{-t}$$

$$(-2Ae^{-t} + \cancel{Ate^{t}}) - 2(Ae^{-t} - \cancel{Ate^{-t}}) - 3(\cancel{Ate^{-t}}) = 3e^{-t}$$

$$-4Ae^{-t} = 3e^{-t}$$

$$A = -3/4$$

Things I've been asked to mention:

Physics, Mathematics, and Computer Science (PMC) - Teaching Awards are open.

Valuable for Vindication/Progression of your good lecturers and tutors (And un-vindication of bad ones)

Link:

https://forms.office.com/r/jS4c63mQcy

Open until 3rd of November

One more Administrative Thing:

(Website)

Summary

Separable:

$$\frac{dy}{dx} = f(x)g(y)$$

Linear Differential Equation:

$$y' + p(x)y = f(x)$$

Method:

$$y = \frac{1}{r(x)} \int r(x) f(x) dx$$

$$r(x) = e^{\int p(x) dx}$$

Constant Coefficients:

$$ay'' + by' + cy = 0$$

Characteristic Equation:

$$ar^2 + br + c = 0$$

Real (Unique):

$$y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

Real Repeated:

$$y = c_1 e^{rt} + c_2 t e^{rt}$$

Complex:

$$y = c_1 e^{\alpha t} \cos \beta t + c_2 e^{\alpha t} \sin \beta t$$

Method of Undetermined Coefficients:

$$y'' - 2y' - 3y = \boxed{3e^{2t}}$$

1. Solve homogeneous equation *in general* y_h
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