Undergraduate Physics

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Aschola r

A foray into
undergrad physics
with a problem solving
approach,
without pulling any
mathematical wool
for an aspiring
physicist.
Source Code.

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Preface

This is a set of notes I am writing (as of now) in my high-school years — part of which I hope to publish one day. These are mostly meant to cover the whole of undergrad physics, covering, essentially newtonian mechanics(including special relativity), classical electromagnetism, waves and optics, thermodynamics and statistical mechanics, quantum mechanics and analytical mechanics. There will also be some introductory sections on more advanced graduate topics and generally the depth of material covered will be higher than the standard undergrad.

Out of these the sections that I wish to publish include everything but higher introductions, and analytical mechanics. I may write some sections on other topics as well and even in that case, most of them will be excluded.

Philosophy and Style

The book is particularly written for advanced high-schoolers, perhaps ones interested in physics olympiads who want to experience a broader coverage of physics in general. The whole text will also be certainly useful for undergrads with a strong background.

Frankly, I am not a master of the subjects I am writing on, all that I have written here is either borrowed from others and the perhaps only difference from each of them is in the way of exposition. The real catch is that it because it borrows from a lot of places — it is perhaps more complete than each of them, where I have tried to omit the weakest section of the books I have borrowed from, while keeping intact the wonderful ones.

These notes are meant to adhere to a particular style, they're meant to be rigorous, both physically and mathematically. I try not to pull any wool over your eyes, and we experience how to think, in essence, somewhat like a physicist. Tools such as approximations are particularly talked about so that you know how to use them.

This also means that some of the sections introduce terms you may not be typically introduced to in an introductory text. The math background required for the text is **not** covered in the text itself, calculus(both single and multi) is assumed a prerequisite. If you are not familiar with either — I would recommend going through a calculus course such as MIT OCW 18.01 and 18.02. The other prereqs, like probability theory for statistical mechanics and linear algebra for quantum mechanics, will be given an intro. They will be much better covered in the other set of notes I have, for undergraduate mathematics. You may look at them, if you're interested.

Hopefully this book gives you something new, and is innovative in some way or the other. If you like it, please let me know! It would mean a ton to me.

Good luck, and happy learning.

To the ones who believed in me.

Part A

Prerequisites

Chapter 1

Vectors

1.1 Vectors

Vectors are quantities with magnitude and direction, contrastingly to scalars which only have magnitude. For our purposes, vectors can be understood as directed line segments, although they're more rigorously defined as elements of a vector space, or can be understood through rotations. We'll talk about rotations later, but will not deal with vector spaces right now.

I shall use boldface to denote vectors, such as $\mathbf{A}$.

The utility of vectors stem from their independence from a particular co-ordinate system. They're directed *line segments*, only their length matters and we're not concerned with their end points. This means that a vector can be freely translated.

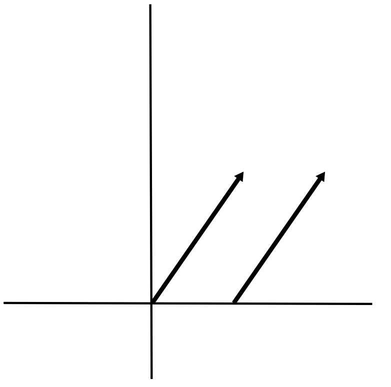


FIGURE 1.1. Identical Vectors

The magnitude of a vector, its norm, is its euclidean "length", we denote it by $\|\mathbf{A}\|$ or simply A.

A unit vector, $\hat{\mathbf{a}}$ ("a hat") is a vector whose magnitude is unity. We use the unit vector to often denote the direction of a vector by multiplying the unit vector with its magnitude. For instance, the unit vector that point is the direction of A, $\hat{\mathbf{A}}$ can be calculated as,

$$\hat{\mathbf{A}} = \frac{\mathbf{A}}{A}$$

where we divide by the magnitude of A to ensure that the magnitude of $\hat{\mathbf{a}}$ is 1.

1.2 Vector Algebra

There are two operations generally defined on vectors, *scalar multiplication* and *vector addition*.

1.2.1 Scalar Multiplication

Multiplying a vector $\mathbf{A}$ by a scalar c results in another vector, $c\mathbf{A}$ parallel or antiparallel(pointing in the opposite direction) to $\mathbf{A}$ scaled by a factor of $|c|$.

If c is positive, then the resultant vector is simply scaled by it. If we multiply a vector by -1, then the result is the simple reversal in direction of vectors. If c is negative, then the vector is reversed and scaled by $|c|$.

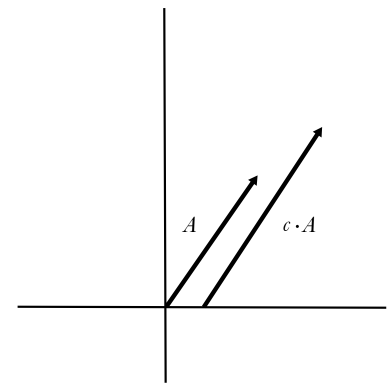


FIGURE 1.2. Scalar Multiplication. The vector, $\mathbf{A}$ multiplied by a scalar, c where $c > 1$.

1.2.2 Vector Addition

We define an operation called addition of vectors which produces another vector. To add two vectors, $\mathbf{A}$ and $\mathbf{B}$, we stick the tail of one vector with the head of another.

We can calculate the magnitude of their resultant vector, $\|\mathbf{A}+\mathbf{B}\|$ by using the cosine rule. In fig. 1.3 ϑ is the angle between $\mathbf{A}$ and $\mathbf{B}$.

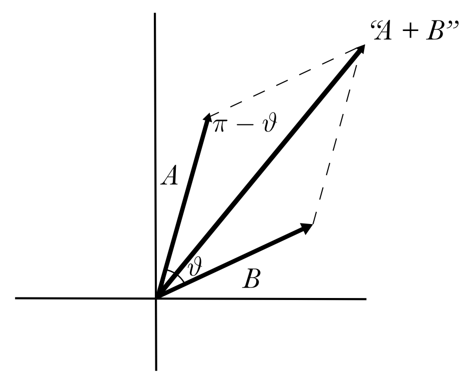


FIGURE 1.3. Vector Addition

$$
\begin{aligned}
\|\mathbf{A}+\mathbf{B}\|^2 &= A^2 + B^2 - 2AB\cos(\pi-\vartheta) \\
\|\mathbf{A}+\mathbf{B}\|^2 &= A^2 + B^2 + 2AB\cos\vartheta \\
\|\mathbf{A}+\mathbf{B}\| &= \sqrt{A^2 + B^2 + 2AB\cos\vartheta}
\end{aligned}
$$

To subtract two vectors, we simply reverse the vector and then add the resulting vector.

> **Example 1.2.1.** Consider the infinitesimal change vector, $d\mathbf{A} = \mathbf{A}(t+dt) - \mathbf{A}(t)$.
>
> We have,
>
> $$
\begin{aligned}
d\mathbf{A} &= A(t+dt)\hat{\mathbf{A}}(t+dt) - A(t)\hat{\mathbf{A}}(t) \\
d\mathbf{A} &= A(t+dt)\hat{\mathbf{A}}(t+dt) - A(t+dt)\hat{\mathbf{A}}(t) \\
&\quad + A(t+dt)\hat{\mathbf{A}}(t) - A(t)\hat{\mathbf{A}}(t) \\
d\mathbf{A} &= A(t+dt)(\hat{\mathbf{A}}(t+dt) - \hat{\mathbf{A}}(t)) + \hat{\mathbf{A}}(t)(A(t+dt) - A(t)) \\
d\mathbf{A} &= A\,d\hat{\mathbf{A}} + dA\hat{\mathbf{A}}
\end{aligned}
$$
>
> The first term represents a change in direction and the second represents a change in magnitude.

The magnitude of the first can be calculated as,

First consider fig. 1.4.

Now, let us first work out a geometric argument for $A\,d\hat{\mathbf{A}}$ which is perpendicular to $\mathbf{A}$, we can show its perpendicular because the other term, $dA\hat{\mathbf{A}}$ is parallel to A, and causes no change in direction.

The second term, thus must account for change in direction. We can also see that $A\,d\hat{\mathbf{A}}$ causes no reasonable change in the magnitude of $\mathbf{A}$. Therefore, it must be perpendicular to $\mathbf{A}$[1].

For now, let us consider $\Delta\mathbf{A}$ perpendicular to $\mathbf{A}$ as an approximation. We see that

1. This is because we are basically stating that $d\mathbf{A}$ can be broken into two parts, one parallel to $\mathbf{A}$ and one perpendicular to it. The perpendicular one causes no change in magnitude.

$$A(t_2) = \sqrt{A^2(t_1) + (\Delta A)^2}$$

Therefore, for small ΔA, the change is very small (it vanishes, essentially in the limiting case, $(dx)^2 = 0$ since it is a second order differential).

This allows us to consider a simple geometric argument which gives $\|\Delta\mathbf{A}\| = 2A\sin(\theta/2)$ where θ is the angle between $\mathbf{A}(t_1)$ and $\mathbf{A}(t_2)$.

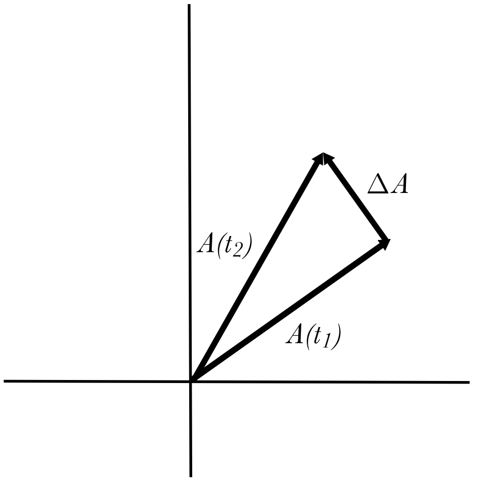


FIGURE 1.4. $\Delta\mathbf{A} \perp \mathbf{A}$

Since we're considering values of small θ, we may use the small angle approximation, $\sin(\theta) \approx \theta$. Therefore,

$$\|\boldsymbol{\Delta}\mathbf{A}\| \approx 2A \times \frac{\theta}{2} \tag{1.1}$$
$$\approx A \times \theta \tag{1.2}$$

Which is equivalent to $A\Delta\theta$. In the limit $\Delta\theta \to 0$,

$$dA_{\perp} = A\ d\theta \tag{1.3}$$

Since $dA_{\perp} = A\|\,d\hat{\mathbf{A}}\|$,

$$\|\,d\hat{\mathbf{A}}\| = d\theta \tag{1.4}$$

1.3 Dot Product

The *dot product* of two vectors results in a scalar and is defined as,

$$\mathbf{AB} = AB\cos\theta$$

where θ is the angle between $\mathbf{A}$ and $\mathbf{B}$.

Note that the quantity $\mathbf{A}\cdot\mathbf{B}$ is just A times the magnitude of the projection of $\mathbf{B}$ on $\mathbf{A}$. Similarly, $\mathbf{B}\cdot\mathbf{A}$ is magnitude of the projection of $\mathbf{A}$ on $\mathbf{B}$ times B.

We may note that,

$$\mathbf{A}\cdot\mathbf{A} = A^2$$

Or,

$$A = \sqrt{\mathbf{A}\cdot\mathbf{A}}$$

It can be easily shown that the scalar product distributes over vector addition. Another thing of importance is the identity(which is also easy to derive),

$$\frac{d\mathbf{A}\cdot\mathbf{B}}{dt} = \frac{d\mathbf{A}}{dt}\cdot\mathbf{B} + \frac{d\mathbf{B}}{dt}\cdot\mathbf{A}$$

1.4 Cross Product

The *cross product* is another product operation on vectors, this product produces a vector.

It is defined as,

$$\mathbf{A} \times \mathbf{B} = AB \sin\theta \hat{\mathbf{n}}$$

Where $\hat{\mathbf{n}}$ is the vector perpendicular to the plane containing $\mathbf{A}$ and $\mathbf{B}$. Since there are two directions that $\hat{\mathbf{n}}$ can have, we define the two vectors and their cross product to form a right-hand triple.

Place your fingers in the direction of $\mathbf{A}$ and curl them, along the smaller angle, towards $\mathbf{B}$. Then the direction in which your thumb is pointing is the direction of $\mathbf{A} \times \mathbf{B}$.

1.5 Vectors in Component form

Although we have worked without co-ordinate systems till now, to actually extract any meaning from vector operations, a coordinate system is necessary.

A co-ordinate system consists of three things — an origin, some axes and basis vectors. The concept of an origin and axes is already familiar to us, for instance, in the form of the cartesian plane. So our talk will be mostly centered around basis vectors.

Suppose we have a vector $\mathbf{C}$ which is the sum of two vectors $\alpha\mathbf{A}$ and $\beta\mathbf{B}$. We can then write $\mathbf{C}$ as,

$$\mathbf{C} = \alpha\mathbf{A} + \beta\mathbf{B}$$

We call $\mathbf{C}$ a linear combination of $\mathbf{A}$ and $\mathbf{B}$ and in essence, we can represent $\mathbf{C}$ in terms of $\mathbf{A}$ and $\mathbf{B}$.

The idea of basis vectors is based around this — we adopt a set of vectors in terms of which we can represent every other vector. Note, of course, that we have a bit of a problem here. We can't really adopt vectors that are linear combinations of others — otherwise we could simply replace them by their linear combination.

So basis vectors are vectors that are linearly independent. That is, their linear combination when equated to the n dimensional zero vector,

$$c_1\mathbf{e}_1 + c_2\mathbf{e}_2 + \cdots + c_n\mathbf{e}_n = \mathbf{0}$$

only has a trivial solution of $c_1 = c_2 = \cdots = c_n = 0$. Otherwise, we could have simply written the term $\mathbf{e}_i$ as the linear combination of other vectors whenever we needed to use it.

Basis vectors which are orthogonal — mutually perpendicular, form an *orthonormal* basis.

Although it might not be so obvious, all orthogonal sets of vectors are linearly independent. If we represent these as $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$, we can construct a set of basis vectors that are not orthogonal as, $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_{n-1}, \mathbf{e}_1 + \mathbf{e_2} + \cdots + \mathbf{e}_n\}$.

Which means that we don't always need the basis vectors to be orthogonal, but in practice we usually adopt an orthonormal basis.

In particular, we'll use the basis vectors which have magnitude of 1, i.e. unit vectors.

In cartesian co-ordinates, or when using the orthonormal cartesian basis, we use the unit vectors along the three axes, x, y and z. These are written as $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$ and $\hat{\mathbf{z}}$ respectively.

But how can we write an arbitrary vector $\mathbf{A}$ in terms of these unit vectors? The answer lies in projecting the vector along the axes. We use the dot product for it.

$$A_x = \mathbf{A} \cdot \hat{\mathbf{x}} = A\cos\alpha \tag{1.5}$$
$$A_y = \mathbf{A} \cdot \hat{\mathbf{y}} = A\cos\beta \tag{1.6}$$
$$A_z = \mathbf{A} \cdot \hat{\mathbf{z}} = A\cos\gamma \tag{1.7}$$

Where α, β, and γ are the angles the vector makes with the x, y and z axes respectively. The cosine of these angles are called *directional cosines*.

Thus, we write $\mathbf{A}$ in terms of it elements, in matrix form, as,

$$\mathbf{A} = \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} \tag{1.8}$$

Now scalar multiplication is simply,

$$c\mathbf{A} = \begin{pmatrix} cA_x \\ cA_y \\ cA_z \end{pmatrix}$$

And vector addition is,

$$\mathbf{A} + \mathbf{B} = \begin{pmatrix} A_x + B_x \\ A_y + B_y \\ A_z + B_z \end{pmatrix}$$

For the dot product, representing $\mathbf{A}$ and $\mathbf{B}$ in terms of the basis vectors,

$$\mathbf{A} \cdot \mathbf{B} = (A_x\hat{\mathbf{x}} + A_y\hat{\mathbf{y}} + A_z\hat{\mathbf{z}}) \cdot (B_x\hat{\mathbf{x}} + B_y\hat{\mathbf{y}} + B_z\hat{\mathbf{z}})$$
$$\mathbf{A} \cdot \mathbf{B} = A_xB_x + A_yB_y + A_zB_z$$

Where we use the distributive property of dot product and the fact that $\hat{\mathbf{x}} \cdot \hat{\mathbf{y}} = \hat{\mathbf{y}} \cdot \hat{\mathbf{z}} = \hat{\mathbf{z}} \cdot \hat{\mathbf{x}} = 0$.

We can find the angle between two vectors, θ, as,

$$\cos\theta = \frac{\mathbf{A}\cdot\mathbf{B}}{AB}$$

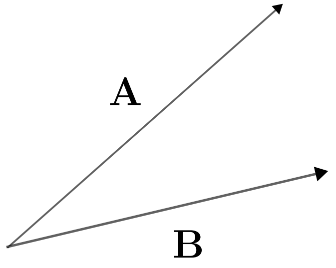


FIGURE 1.5. Two arbitrary vectors

Note that now we can calculate the magnitude of $\mathbf{A}$ as,

$$\mathbf{A}\cdot\mathbf{A} = A^2 = A_x^2 + A_y^2 + A_z^2$$

which gives the magnitude as $A = \sqrt{A_x^2 + A_y^2 + A_z^2}$.

Also, a very neat observation from the dot product is that,

$$\mathbf{A}\cdot\mathbf{B} \leqslant AB$$

This is a rather simple idea since $\cos\theta \in [-1, 1]$.

Now if generalize this to n dimensions, we will find the inner product, $\mathbf{A}\cdot B = AB\cos\theta$ with each vector lying in an n dimensional space with n axis components. This has similar properties to that of the dot product and has the same but extended computation for the component form. Let $(a_1, a_2, \ldots, a_n)$ be the components of $\mathbf{A}$ and $(b_1, b_2, \ldots, b_n)$ be the components of $\mathbf{B}$, then,

$$\mathbf{A}\cdot\mathbf{B} \leqslant AB$$

Using the magnitude formula we derived earlier and that the inner product has the same computation(sum of the product of same axis components),

$$a_1b_1 + a_2b_2 + \cdots + a_nb_n \leqslant \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}\sqrt{b_1^2 + b_2^2 + \ldots b_n^2} \tag{1.9}$$

which is the *Cauchy-Shwartz* inequality. Note that the equality is when $\cos\theta = 0$. Thus, when $\mathbf{A}$ lies parallel to $\mathbf{B}$. This can also be interpreted as $\mathbf{A}$ being being a scaling of $\mathbf{B}$ by some real number $\lambda = B/A$.

Given two vectors $\mathbf{A}$ and $\mathbf{B}$, we can find the projection of one along the *direction* of the other by using the dot product.

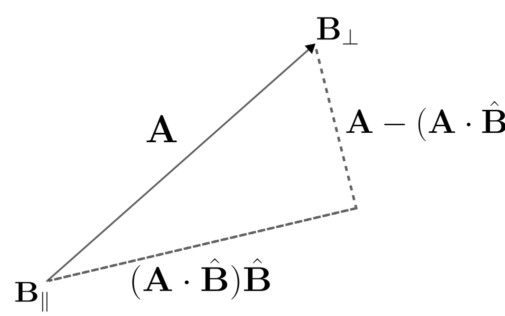


FIGURE 1.6. $\mathbf{A}$ projected along directions tangential and perpendicular to $\mathbf{B}$

We use the fact that the vector $\mathbf{A}$ along $\mathbf{B}$ is simply $\mathbf{A}\cdot\hat{\mathbf{B}}$. Since this is the magnitude, we now simply multiply this by $\hat{\mathbf{B}}$.

The projection of $\mathbf{A}$ along the direction perpendicular to $\mathbf{B}$ is simply $\mathbf{A} - (\mathbf{A}\cdot\hat{\mathbf{B}})\hat{\mathbf{B}}$.

The two directions can be seen in fig. 1.6.

> **Example 1.5.1.** Consider the unit vectors along the incident, reflected, and normal ray. Given that the unit vectors along the incident ray and normal ray are $\hat{\mathbf{I}}$ and $\hat{\mathbf{N}}$ respectively, find the reflected unit vector.
>
> *Solution.* The projection of $\hat{\mathbf{I}}$ perpendicular and tangential to the normal are $\hat{\mathbf{I}} - (\hat{\mathbf{I}}\cdot\hat{\mathbf{N}})\hat{\mathbf{N}}$ and $(\hat{\mathbf{I}}\cdot\hat{\mathbf{N}})\hat{\mathbf{N}}$.

Clearly, for the reflected unit vector, the direction of the perpendicular will remain the same while of the tangential will be flipped. Thus, the components of $\hat{\mathbf{R}}$ along the normal are, $\hat{\mathbf{I}} - (\hat{\mathbf{I}} \cdot \hat{\mathbf{N}})\hat{\mathbf{N}}$ and $-(\hat{\mathbf{I}} \cdot \hat{\mathbf{N}})\hat{\mathbf{N}}$. Adding these vectors gives us $\hat{\mathbf{R}}$,

$$\hat{\mathbf{R}} = \hat{\mathbf{I}} - 2(\hat{\mathbf{I}} \cdot \hat{\mathbf{N}})\hat{\mathbf{N}}$$

□

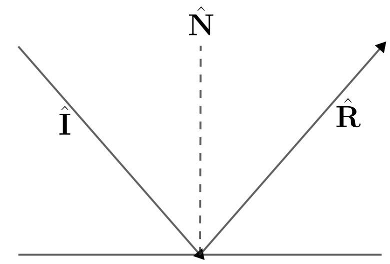


FIGURE 1.7. Incidence, normal and reflected unit vectors.

Finally, the cross product is a little difficult to compute. It can be computed either as the determinant,

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

Or using the matrix notation,

$$\begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} \times \begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix} = \begin{pmatrix} A_yB_z - A_zB_y \\ A_zB_x - A_xB_z \\ A_xB_y - A_yB_x \end{pmatrix}$$

To compute a particular component, cover up that particular row. Then, multiply the component of $\mathbf{A}$ in the row below (loop to the top if necessary) with the diagonal component of $\mathbf{B}$ and subtract it by the product of the component of $\mathbf{A}$ two rows below, and the diagonal component of $\mathbf{B}$. For example, for the y-component,

$$\begin{pmatrix} A_x \\ - \\ A_z \end{pmatrix} \times \begin{pmatrix} B_x \\ - \\ B_z \end{pmatrix}$$

And the term simply is $A_zB_x - A_xb_z$ by looping through the top.

If $\mathbf{A} \times \mathbf{B} = 0$, then using the component form,

$$\begin{aligned} A_yB_z - A_zB_y &= 0 \\ A_zB_x - A_xB_z &= 0 \\ A_xB_y - A_yB_x &= 0 \end{aligned}$$

Solving for them leads to,

$$\frac{A_x}{B_x} = \frac{A_y}{B_y} = \frac{A_z}{B_z}$$

1.6 Some geometric results regarding vectors

1.6.1 Lami's Theorem

Consider three vectors, $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3$ such that $\mathbf{A}_1 + \mathbf{A}_2 + \mathbf{A}_3 = 0$. If the angles θ_1, θ_2 and θ_3 are chosen according to fig. 1.8, then,

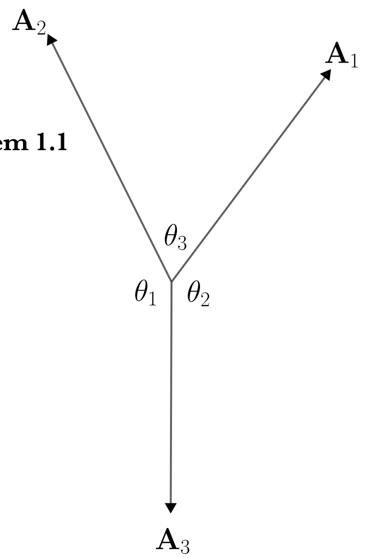


FIGURE 1.8. Lami's Theorem

Lami's Theorem — **Theorem 1.1**

$$\frac{A_1}{\theta_1} = \frac{A_2}{\theta_2} = \frac{A_3}{\theta_3}$$

The proof follows from arranging the vectors as sides of a triangle and then applying the sine law. Note that the angles between the sides would be $\pi - \theta_i$. Since $\sin(\pi - \theta) = \sin\theta$, we can simply replace them.

1.6.2 Internal and External Segments

Consider a point P between the points A and B which divides the line segment $\overline{AB}$ into two parts, $\overline{AP}$ and $\overline{PB}$ where

$$\frac{AP}{PB} = \frac{m}{n}$$

Theorem 1.2

Given that the position vector of A is $\mathbf{r}_A$ and of B is $\mathbf{r}_B$, the position vector of P will be,

$$\mathbf{r}_P = \frac{n\mathbf{r}_\mathbf{A} + m\mathbf{r}_\mathbf{B}}{m+n} \tag{1.10}$$

The proof is almost trivial if you know the section theorem. Then we can just apply the theorem for all components of $\mathbf{r}_{A,B}$ and add them to get the desired result.

The same result if P lies on the extension of $\overline{AB}$ and

$$\frac{AP}{BP} = \frac{m}{n}$$

is,

$$\boxed{\mathbf{r}_P = \frac{n\mathbf{r}_\mathbf{A} - m\mathbf{r}_\mathbf{B}}{m-n}} \tag{1.11}$$

This can also be derived geometrically by consider two vector triangles.

Another important result, the component wise proof of which we'll take as granted from co-ordinate geometry is of the centroid of a triangle. Given position vectors of the vertices $\mathbf{r}_A$, $\mathbf{r}_B$ and $\mathbf{r_C}$, the position vector of the centroid of the triangle is,

$$\boxed{\mathbf{r}_C = \frac{\mathbf{r}_A + \mathbf{r}_B + \mathbf{r}_C}{3}} \tag{1.12}$$

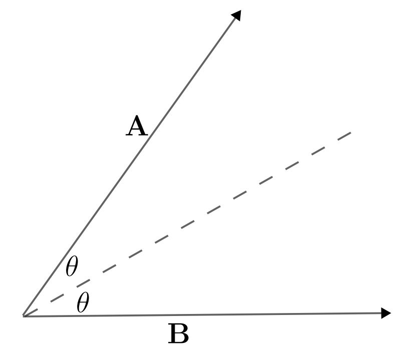


FIGURE 1.9. Angle bisector of 2θ.

1.6.3 Angle Bisectors

Given two vectors $\mathbf{A}$ and $\mathbf{B}$, we can find out the direction of the angle bisector of the angle between them.

A natural instinct might be to construct something like a parallelogram. Well, that doesn't quite work since the resultant does not co-incide with the angle bisector.

But we can use a parallelogram of $\hat{\mathbf{A}}$ and $\hat{\mathbf{B}}$ to construct a rhombus — whose diagonal, the resultant of these vectors lies along the angle bisector! Constructing and shifting parallelly the external angle bisector, we find it lies along the other diagonal, the difference of the two unit vectors.

Note that $\hat{\mathbf{A}} + \hat{\mathbf{B}}$ is not a unit vector, we will need to find the unit vector along that direction.

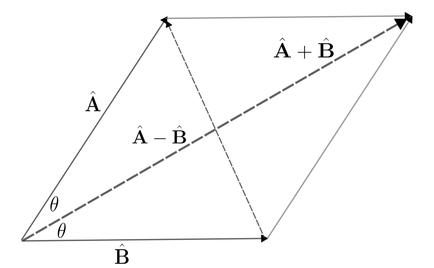


FIGURE 1.10. The internal angle bisector lies along $\hat{\mathbf{A}} + \hat{\mathbf{B}}$ and the external lies along $\hat{\mathbf{A}} - \hat{\mathbf{B}}$.

1.6.4 Loci

A set of points that fulfill some particular condition are called a locus. Some loci that will be of our use are given.

Circle

A point lies on a circle with center at (h, k) if it is at some fixed distance, r, the radius from the center. Thus, if the point has co-ordinated x, y it must obey the equation,

$$(x-h)^2 + (y-k)^2 = r^2. \tag{1.13}$$

The area of such a circle is πr^2 and its perimeter is $2\pi r$.

Sphere

A natural extension of the circle into 3 dimensions is the sphere. If the center of the sphere is at h, k, f, then the co-ordinate of any point on its surface will be determined by,

$$(x-h)^2 + (y-k)^2 + (z-f)^2 = r^2 \tag{1.14}$$

For some fixed z, the x and y co-ordinates just form a circle. Thus, a sphere can be thought of being made up these circles. The idea is useful in differential analysis.

Ellipse

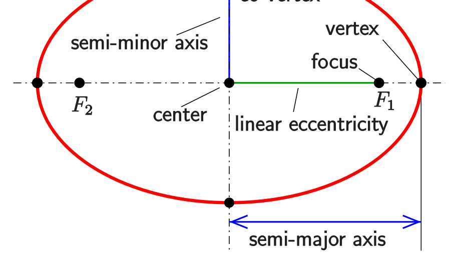


FIGURE 1.11. Ellipse by Ag2gaeh CC BY-SA 4.0, https://commons.wikimedia.org/w/index.php?curid=57497218

A ellipse is a weirder shape. It is a sort of stretched circle, and the stretching is non-uniform across the two dimensions. In fig. 1.11, the length of the semi-minor axis is commonly denoted as b and that of the semi major axis is a. Then any point on the ellipse must have co-ordinates (x, y) such that,

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1 \tag{1.15}$$

The distance of the foci from the center is its linear eccentricity, $c = \sqrt{a^2 - b^2}$. Overall, the ellipse and other conic sections are characterized by their eccentricity,

$$e = \sqrt{1 - \frac{b^2}{a^2}}$$

The eccentricity of the ellipse ranges from 0 which is just a circle, to 1 where it becomes a parabola. The area of the ellipse is πab. However, its perimeter only has a solution through an infinite series integral, so we'll leave it out for now.

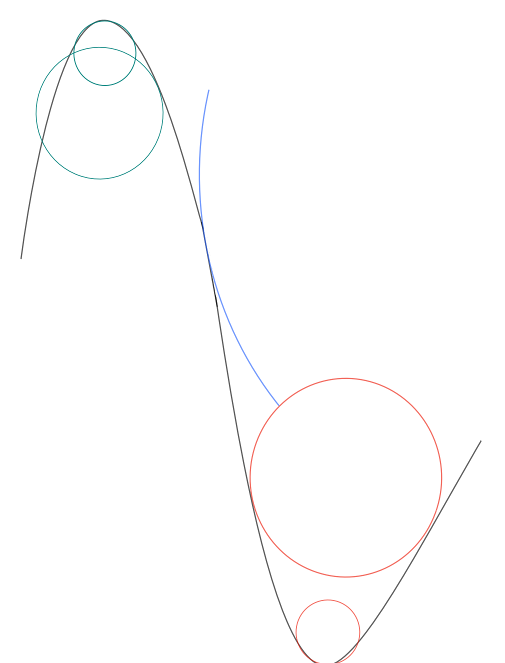
An ellipse can extended to three dimensions to form a shape called ellipsoid. Its locus is given by,

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} + \frac{(z-f)^2}{c^2} = 1 \tag{1.16}$$

Its volume is given by $\frac{4}{3}\pi abc$.

1.7 Radius of Curvature

The *radius of curvature* is the radius of the circle which best approximates a curve at any given point. The talk of a circle at any given point can be non-rigorously understood as the circle containing the arc from $(x, f(x))$ to $(x + dx, f(x + dx))$.



We can create a formula of the radius of curvature by first noting that the radius at the point, R_c will be,

$$R_c = \frac{d\ell}{d\theta}$$

Where $d\ell = \sqrt{dx^2 + dy^2} = \sqrt{1 + (f'(x))^2}\ dx$ is uniquely described by that point.

Now we just need to figure out $d\theta$. We can do so by first noting that if the curve is the graph of some differentiable function, f, then,

$$\tan\theta = f'(x)$$

We can differentiate this again with respect to x to get,

$$\sec^2\theta \frac{d\theta}{dx} = f''(x)$$

Using the identity $1 + \tan^2\theta = \sec^2\theta$ and $\tan^2\theta = f'(x)$,

$$(1 + (f'(x))^2)\frac{d\theta}{dx} = f''(x)$$

And,

$$d\theta = \frac{f''(x)}{1 + (f'(x)^2)}\, dx$$

Substituting both of these, we get,

$$\boxed{R_c = \frac{[1 + (f'(x))^2]^{3/2}}{f''(x)}} \tag{1.17}$$

The radius of curvature allows us to say much about $f''(x)$. In (1.17) clearly the numerator is positive. Thus, the sign of the radius of curvature is uniquely determined by $f''(x)$.

What exactly does the negative sign mean? Well it basically tells whether the circle of that radius lies above or below our curve. In fig. 1.12, we can easily find out where $f''(x)$ is negative or positive depending on where the circle lies. The blue circle of infinite radius is at the *inflection point*, a point where $f''(x)$ is 0.

An inflection point occurs here since $f''(x)$ changes signs from negative to positive and thus, must become 0 at some point.

Given something like a position time graph, we can easily find where the acceleration is positive by just well, drawing out circles. Nice.

Example 1.7.1. Find the roc of ellipse at its top most and right most points, $(0, b)$ and $(a, 0)$ centered at $(0, 0)$

Solution. We will here figure out the radius at $(0, b)$ and argue by symmetry for $(a, 0)$. In well actuality, the roc at $(a, 0)$ needs use of parametrized forms and so does the general roc equation.

However, for $(0, b)$ we may note that

$$\left.\frac{dy}{dx}\right|_{(0,b)} = 0.$$

And then implicitly derivate the ellipse equation wrt x. We then obtain,

$$\left.\frac{d^2y}{dx^2}\right|_{(0,b)} = -\frac{b}{a^2}$$

By using (1.17) we get,

$$\left.R_c\right|_{(0,b)} = \frac{-a^2}{b} \tag{1.18}$$

By a bit of an ad-hoc symmetric argument, the roc at $(a, 0)$ is b^2/a. $\square$

.

It's much better evaluated using the parametric equations.

Chapter 2

Unit and Dimensions

Units are dimension are in many ways what give the abstract ideas of physics some ground in the real life. The dimensions of a physical quantity represent what kind of physical quantity is, while the unit is a measure of that physical quantity.

The distinction is appreciable to know, just not always useful.

2.1 Dimensions

There are seven fundamental quantities which are assigned specific dimensions. It is not necessary to take these as fundamental, in some perfectly fine formulations like the Planck quantities this is not the case. But we'll not be working in these for now.

Quantity	Unit Name	Unit symbol	Dimension
Mass	kilogram	kg	M
Length	meter	m	L
Time	second	s	T
Electric Current	ampere	A	I
Luminous Intensity	candela	cd	J
Temperature	kelvin	K	Θ
Amount of substance	mole	mol	N

Table 2.1. The 7 fundamental quantities.

For any physical quantity A, we use the notation $[A]$ to denote its dimension. For instance,

$$[F] = MLT^{-2}$$

Evaluating the dimensions has a really neat use, it allows us to check our answers, and establish some relations through the use of *dimensional* analysis.

2.2 Rules regarding dimensions

There are particle constrains that help us utilise dimensions quite a lot. For one, the dimensions on both sides of an equation must be homogenous, i.e the same. It is nonsense to ask if 1 second is equal to 1 kelvin.

Another constraint is that the operations of addition and subtraction can only be performed on quantities of the same dimension, and indeed you cannot add a metre to a kilogram.

Thus, we arrive at a third derived constraint, namely that the functions such as sin, cos, exp, log must all have dimensionless constraints.

This because for instance, let us take the taylor expansion of e^x,

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \dots$$

We note that if we x had a dimension, for example $[x] = L$, then we would be adding L to L^2, L^3 and so on. But this is not permitted, we cannot add a metre to a square metre or a square metre to a cubic metre and so on. Thus, we have that their arguments must be dimensionless.

2.3 Dimensional Analysis

Suppose that we know that the rest energy of a particle depends on it mass and the speed of light. If we were asked to find a relation determining its rest energy we could use the idea of dimensional analysis.

We see that,

$$E \propto m, c$$

We could note that the dimesion on both side of the equation must have the same dimesions. Thus we need to combine them in such a manner that we get the dimensions of energy.

Since $[c] = LT^{-1}$, $[m] = M$ and $[E] = ML^{-2}T^{-2}$, we can form a guess and say that,

$$E = mc^2,$$

since the dimensions are same across both quantities. In fact this the correct formula!

In general, for some quantity A dependent on parameters $Q_1, Q_2, \dots, Q_n$,

$$A = Q_1^{\alpha_1} \cdot Q_2^{\alpha_2} \cdot \dots \cdot Q_n^{\alpha_n}$$

where each of $\alpha_1, \alpha_2, \dots, \alpha_n$ are constants. Then we simply use dimensional analysis to compute the result.

There are catches, however, suppose that we were asked to find the kinetic energy of the particle which depends on its mass and velocity. We might guess by dimensional analysis alone that it is mv^2, which is obviously false. What dimensional analysis gives us is that,

$$E \propto mv^2$$

And we must attach a constant for a definite answer,

$$E = kmv^2$$

The constants comes out to be $1/2$, but we can't determine it with dimensional analysis alone.

Let us take a particular example now, the pendulum: its period depends on L, g, and the amplitude θ_0.

We note that the amplitude and angles in general are dimensionless since they're arguments of the trig functions. If we use dimensional analysis, we run into a bit of a problem.

If we use dimensional analysis, we see that we can use any combination of θ_0, since they're all dimensionless. In particular, we can use any function of θ_0.

If we ignore θ_0 since its dimensionless, we get that

$$T \propto \sqrt{\frac{L}{g}}$$

We hence attach a dimensionless group, a function of θ_0 to get the period of a pendulum as $T = f(\theta_0)\sqrt{L/g}$.

In general this can be done for any dimensionless group, for instance the resistance of conductor depends on its resistivity, the cross-sectional area, and its length. Say we know their dimensions and get that,

$$R = \rho \frac{L}{A}$$

We could have also formulated by dimensional analysis, that $R = \rho/\sqrt{A}$, the dimensions are the same. In fact, we could multiply it by any function of L^2/A since it is dimensionless.

Buckingham Pi Theorem **Theorem 2.1**

Dimensional analysis can't always pin down the form of the answer. If one has N quantities with D independent dimensions, then one can form $N - D$ independent dimensionless quantities. Dimensional analysis can't say how the answer depends on them.

This is one of the limitations of dimensional analysis.

2.4 Limiting Cases

Sometimes we can verify our answers using limiting cases.

Suppose that we don't remember if the acceleration of a block on a smooth, non-accelerating wedge is $g \sin \theta$ or $g \cos \theta$.

We can then consider the limiting cases of θ. For instance, if $\theta \to 0$ then the block rests on a horizontal plane and experiences no acceleration, which is only satisfied by $g \sin \theta$. We could have also looked at $\theta \to \pi/2$ as another limiting case.

Part B

Mechanics

Chapter 3

Kinematics

Kinematics is the study of motion without concerning its cause. It allows us to calculate and find out the evolution of a body. Everything in our world undergoes motion, one way or the other. To begin our study of motion, let us first concern ourselves with some definitions.

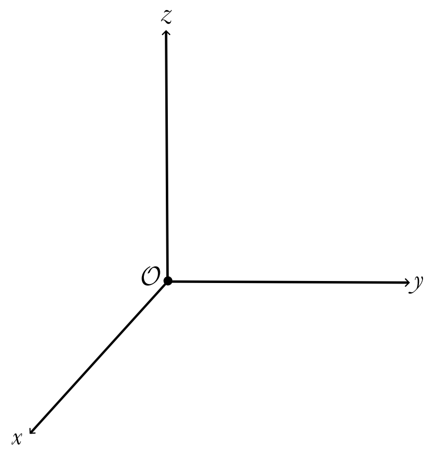


FIGURE 3.1. A Reference Frame

3.1 Reference Frame and Point Particle

Motion of any object is considered to relative to an observer. The observer defines a particular co-ordinate system called the reference frame

We use a reference frame according to our convenience. Most of the times we will work in a right handed cartesian co-ordinate system, the one in fig. 3.1 (or a rotation of it).

Reference Frame — **Definition 3.1**

A *reference frame* is a co-ordinate frame relative to which motion of a particle is considered.

We shall, until we encounter rigid body systems consider the motion of a point particle. Well, why? It is done so to simplify the calculation of our system. The motion of a (rigid) body is discussed later.

Point Particle — **Definition 3.2**

A particle whose size is negligible in the study of its motion is called a *point particle*.

To discuss the motion, state of our body, it is also necessary to setup some time co-ordinate. It is the reading of the clock in the observer's frame. In general, we setup our clocks at 0 at the start of an event.

One thing of importance is that a position vector of a particle is a function of time. We may denote this as $r(t)$ to show that is a function.

3.2 Position Vector and Displacement

3.2.1 Position Vector

We define the position of a particle relative to a reference frame using a *position vector*. One end of the position vector is the origin of our reference frame. The other is the particle itself. Consider fig. 3.2, the vector $\mathbf{r}_A$ from $\mathcal{O}$ to A. It describes the position of A relative to the co-ordinate frame. Let $A = (x, y, z)$. Then we denote $\mathbf{r}_A$ as $\mathbf{r}(x, y, z)$.

The position to any point is written as,

$$\mathbf{r} = (x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

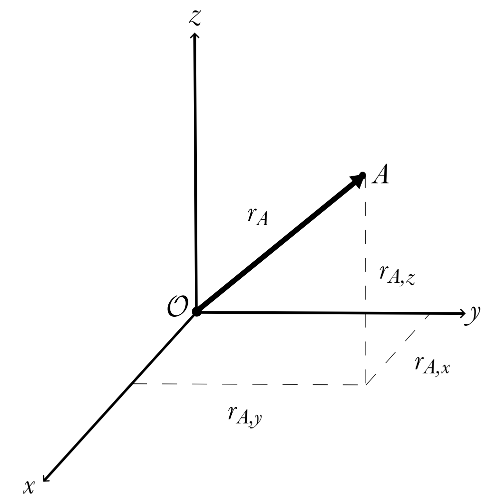


FIGURE 3.2. A position vector, $\mathbf{r}$. *Note that the position vector is not a* true *vector. It is tied to particular reference frame.*

3.2.2 Displacement

Consider the movement of a particle from $A = (x_1, y_1, z_1)$ to $B = (x_2, y_2, z_2)$. The *displacement* is the change in the position vector, and defines a true vector $\mathbf{s} = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$. $\mathbf{s}$ is called the displacement vector.

Note that it contains no information about the individual positions, but only about the relative position of each. Thus, our choice of reference frame, and thus position vector does not matter when we're concerned with displacement. In fig. 3.3, the vector $\mathbf{S}_{AB}$ defines a displacement from A to B.

The difference between displacement and distance is that of the path they describe. The distance covered is the length of the actual path, while the magnitude of the displacement is simply the length of the vector between the final and initial positions.

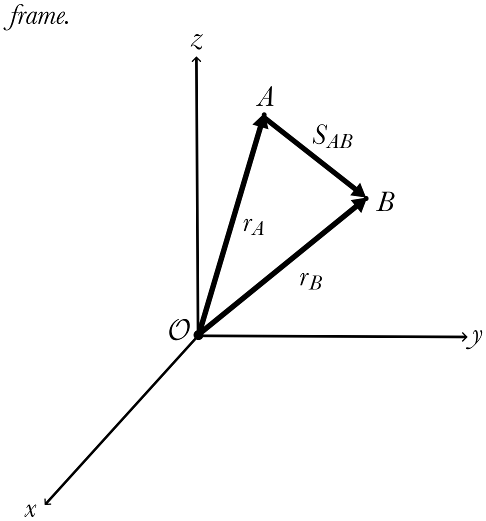


FIGURE 3.3.

3.3 Velocity and Acceleration

We are already familiar with the notion of speed. In elementary terms, speed is distance/time. It tells us nothing about the direction of the object whose speed we are talking into consideration. Thus, it is a *scalar* quantity.

The Velocity of a particle, contrastingly, contains information about both the speed and direction of the object in consideration.

It is a vector. Similar to position, it is also a function of time $v(t)$, though it may also be dependent on the position $v(x)$. Even when that is the case, it can still be written as function of time.

Velocity **Definition 3.3**

Velocity is defined as,

$$\mathbf{v} \equiv \dot{\mathbf{r}} \tag{3.1}$$

The symbol $\equiv$ stands for 'defined as'.

In particular, velocity is a function that is parametrized by time, it is a map,

$$\mathbf{v} : \mathbb{R} \to \mathbb{R}^3,$$

which acts on the parameter t as,

$$t \mapsto (v_x(t), v_y(t), v_z(t)) = \mathbf{v}(t).$$

The same holds for position as well,

$$\mathbf{x} : \mathbb{R} \to \mathbb{R}^3.$$

One thing of note is that velocity can also be expressed as the derivative of displacement, which is just $\mathbf{r} - \mathbf{r}_0$. Since the latter term is constant, we simply get that $\dot{\mathbf{s}} = \dot{\mathbf{r}}$.

From this definition, we may gather that,

$$\int_{r_0}^{r} d\mathbf{r} = \int_0^t \mathbf{v}\ dt,$$

Or,

$$\mathbf{r} = \mathbf{r}_0 + \int_0^t \mathbf{v}\ dt. \tag{3.2}$$

Sometimes, we do not need to compute the derivative of the position vector, and simply need an average estimate, or in the case of uniformly accelerated motion, it is not required that we computer derivatives. In such a case we use the concept of **average velocity**,

$$\overline{\mathbf{v}}_{12} = \frac{\Delta\mathbf{s}}{t_2 - t_1}$$

The dot over r in $\dot{\mathbf{r}}$ tells us that r has been differentiated with respect to time.

Velocity is also referred to as 'instantaneous velocity'. This is done to remind of the difference with average velocity. We will not refer to it that way since by velocity we shall always mean $\mathbf{v}$ as defined in definition 3.3. The same goes for speed. Notably, speed is not defined as $d|r|/\ dt$.

While the average speed is very well that, speed is actually the magnitude of the velocity of that point. Consider the distance between a and b as $a \to b$. The path between them approaches a straight line. Thus, the distance and displacement become the same in the limiting case. Therefore, the *magnitude* of velocity is the speed at that instant.

Therefore,

$$\text{speed} = \left|\frac{d\mathbf{r}}{dt}\right|$$

3.4 Acceleration

If a body does not move with uniform velocity, it accelerates. Acceleration is also a vector and is the rate of change of velocity.

Definition 3.4

Acceleration

Acceleration is defined as,

$$\mathbf{a} \equiv \dot{\mathbf{v}} = \ddot{\mathbf{r}} \tag{3.3}$$

Much like the velocity definition, we have,

$$\mathbf{v} = \mathbf{v}_0 + \int_0^t \mathbf{a}\ dt \tag{3.4}$$

There is also another way to represent acceleration, as,

$$\mathbf{a} = \mathbf{v}\frac{d\mathbf{v}}{dx}$$

Such a representation is particularly useful when the velocity of a particle is a function of its position. In such a case, differentiating with time is quite a hassle.

3.5 Uniformly Accelerated Motion

When the acceleration of a particle, a is constant, it undergoes uniformly accelerated motion. We use special cases of integrals of a and v with respect to time and position for deriving the equations of importance.

Theorem 3.5

For a particle with initial velocity u undergoing uniformly accelerated motion of acceleration a,

$$v = u + at \tag{3.5}$$

$$v^2 = u^2 + 2as \tag{3.6}$$

$$s = ut + \frac{1}{2}at^2 \tag{3.7}$$

$$s_n = u + \frac{a}{2}(2n-1) \tag{3.8}$$

The derivation of these is quite easy, integrate $\int a\ dx$ for the first, $\int a\ dt$ for the second (both with constant acceleration), substitute from the former into the latter for the third, and finally for n^{th} second just do $s_n - s_{n-1}$.

Example 3.5.1. A car is at distance d from a boy. It starts accelerating at a m/s^2. What is the minimum velocity that the boy should have to catch up with the car?

Solution. Consider separation of boy and car, $s_{c,b}$. Using the equations of motion, we have $s_b = vt$, $s_c = \frac{1}{2}at^2$. Thus,

$$s_{c,b} = d + \frac{1}{2}at^2 - vt \tag{3.9}$$

From an inspection of the co-effecients of the x, x^2 terms and the constant, we see that if a real solution to this exists, it must be positive (if this may not be apparent, recall vieta's relation and note a, v, d are all positive).

So they must always meet if this has a real solution!

Therefore, from the simple equation, (3.9), we must have, for real solutions, $b^2 - 4ac \geqslant 0$ for the equation $at^2 + bt + c$. Solving for this by substituting values from (3.9), we have,

$$v \geqslant \sqrt{2ad} \tag{3.10}$$

□

Example 3.5.2. A body is dropped at $t = 0$, after time $t = t_0$, another body is thrown downwards with velocity u m/s. Assuming first body reaches ground first, plot graph of separation.

Solution. At instant t_0, displacement of first particle = $\frac{1}{2}gt_0^2$. Note that here we set up co-ordinates such that positive y is downwards from point of drop. The displacement of first body at $t = t$ after t_0 but before $\boxed{\text{reaching ground}}$ is

$$s_1 = \frac{1}{2}gt_0^2 + \frac{1}{2}g(t - t_0)^2 \tag{3.11}$$

While for second body is,

$$s_2 = ut + \frac{1}{2}g(t - t_0)^2 \tag{3.12}$$

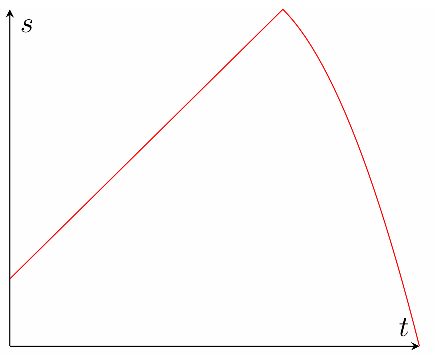
Thus,

$$s_{1,2} = \frac{1}{2}g(t_0)^2 + t(gt_0 - u) \tag{3.13}$$

Thus, before reaching ground, $s - t$ graph is linear. However, after first body reaches ground,

$$s_{1,2} = ut + \frac{1}{2}g(t)^2 \tag{3.14}$$

which is a parabola. Thus, overall graph is,



□

3.5.1 Motion in higher dimensions

We can mostly safely extend our results from one dimension to higher dimensions. So for instances, velocity of a particle whose trajectory is defined as $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ can simply be computed as,

$$v = \dot{x}\mathbf{i} + \dot{y}\mathbf{j} + \dot{z}\mathbf{k}$$

Such a thing is not true for non-cartesian co-ordinates, as we say in the case of polar forms. Due to the orthogonality of the cartesian unit vectors, and using dot products we simply have,

$$v_x = \dot{x}, v_y = \dot{y}, v_z = \dot{z}$$

And the same is true for all other vectors. This gives us the power to resolve our vector components into components alongside unit vectors and consider independent one dimensional motion for each of them.

3.5.2 Free Fall

When an object is dropped downwards freely, it undergoes *free fall*. We ignore air resistance, until we're told not to and in general also talk about cases of the object being thrown instead of dropped. An object dropped from any height undergoes accelerations $\mathbf{a}$ where $\|\mathbf{a}\| = g$. The direction and sign of acceleration is entirely dependent on us.

In general, it is a good idea to setup our axes such that the direction above ground is positive and towards the ground is negative. The acceleration during free fall is always towards the ground. Therefore, for these cases $\mathbf{a} = -g\mathbf{j}$.

Some important points of note are that "g" does not become positive or negative or whatever, this is a common misconception. Instead, we just decide where exactly the positive direction is and setup g according to that.

Example 3.5.3. Two balls are thrown in intervals of 2 seconds with velocity u each. Find the time when they collide.

Solution. Note that when they collide, their displacement from ground must be the same! Setting the origin at ground, and direction of away from ground as positive, we have,

For the first ball, $h_1 = ut - \frac{1}{2}gt^2$. Since the second ball is thrown 2 seconds later, $h_2 = u(t-2) - \frac{1}{2}g(t-2)^2$. Now, we just have $h_1 = h_2$ and the calculation is trivial. □

The thing to note in this example is that you do not need to setup two cases, one in which g is negative and one in which g is positive since it is in the direction of velocity. Just note that these equations are vectorial and we are done.

3.5.3 Projectile Motion

Projectile motion is one of the most important motion to consider in 2 dimensions. It is the motion of a projected body. Generally, we only care about the motion of a body projected from ground under the influence of gravity.

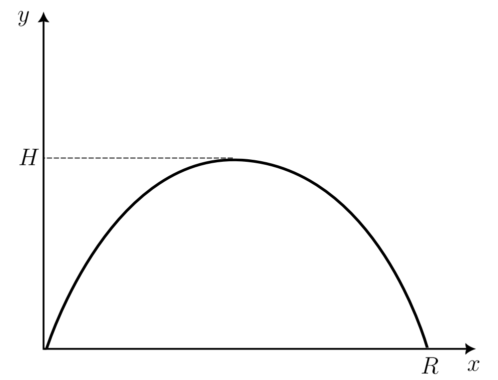


FIGURE 3.4. Trajectory of a projectile

Let a body be projected at an angle θ from the ground at velocity $\mathbf{v}$ along θ. Then, by resolving $\mathbf{v}$ into $v_x\mathbf{i} + v_y\mathbf{j}$, we have,

$$x = v_x t\mathbf{i}$$
$$y = v_y t - \frac{1}{2}gt^2\mathbf{j}$$

Doing some manipulations, we may note that $v_x = v\sin(\theta)$, $v_y = v\cos(\theta)$. Also, substituting $t = x/v\sin(\theta)$, we have,

$$\boxed{y = x\tan(\theta) - \frac{gx^2}{2v^2\cos^2(\theta)}} \tag{3.15}$$

We can also use the identity $1/\cos^2\theta = \tan^2\theta + 1$ and get another and perhaps nicer form of the equations,

$$\boxed{y = x\tan\theta\left(1 - \frac{x}{R}\right)} \tag{3.16}$$

after substituting with R[1].

1. calculated just a bit later.

The path of the projectile is parabolic as shown in fig. 3.4.

Here, H is the maximum height of the particle. Clearly, when the projectile is at H, $v_y = 0$. We have, $v_y^2 = 2gH$,

$$H = \frac{u^2\sin^2(\theta)}{2g} \tag{3.17}$$

The time period of the projectile can be simply calculated as well, since at the final time, T, $y = v_y t - 1/2gt^2 = 0$ and,

$$T = \frac{2v\sin(\theta)}{g} \tag{3.18}$$

R is the range the particle covers horizontally in time T and is clearly,

$$R = \frac{v^2\sin(2\theta)}{g} \tag{3.19}$$

Note that if the particle is projected at $\theta = \pi/4$, the range is maximum and is $u^2/2g$. Otherwise, for $0 < \theta < \pi/2$, $\sin(2\theta)$ attains the same value for two θ_1, θ_2. These θ_1, θ_2 are complementary by trigonometry.

More generally, if we consider the projectile to have an initial position of (x_0, y_0), then, by substituting for time, we find that the vertex of the parabola is $\left(y_0 + \frac{u^2\sin^2\theta}{2g}, x_0 + \frac{u^2\sin 2\theta}{g}\right)$.

The range is a little more difficult to computer here, as the particle will transverse to the ground which is lower than y_0. Freely setting up our co-ordinates as $(h, 0)$, we get by the equation of trajectory,

$$0 = h + R\tan(\theta) - \frac{gx^2}{2v^2\cos^2(\theta)}$$

Derivating this with respect to theta to find the maxima, and then using the attained value, we have,

$$R = \frac{u}{g}\sqrt{u^2 + 2gh} \tag{3.20}$$

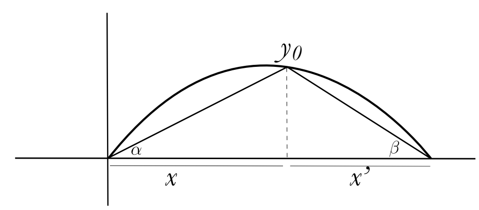


FIGURE 3.5. Projectile grazing a wedge.

Example 3.5.4. A projectile starting from one vertex of a wedge, grazes the other vertex and ends upon the other vertex, with an initial velocity v_0 making an angle θ with the horizontal. If the angles of the vertices are α and β, figure out a relation between $\tan\theta$, $\tan\alpha$ and $\tan\beta$.

Solution. Divide the range of the projectile into two lengths, x and x' such that $x + x' = R$. Now, clearly,

$$\tan\alpha = \frac{y_0}{x} \qquad \tan\beta = \frac{y_0}{x'}$$

Also, by the alternate equation of trajectory,

$$y_0 = x\tan\theta\left(1 - \frac{x}{R}\right) = x\tan\theta\left(1 - \frac{x}{x + x'}\right)$$

Which gives us,

$$\tan\theta = y_0\left(\frac{x+x'}{xx'}\right) = \frac{y_0}{x} + \frac{y_0}{x'}$$

Finally substituting the values, we get,

$$\tan\theta = \tan\alpha + \tan\beta \tag{3.21}$$

Pretty nice and useful result.

□

Some peculiar things about projectile motion is that, a) it is reversible, and b) the time to attain some vertical displacement can be found in terms of the maximum height alone.

Consider the vertical position of the particle to be s at some time t. Then,

$$s = u_y t - \frac{1}{2}gt^2$$

Also, note that if the maximum height is h,

$$u_y = \pm\sqrt{2gh}$$

Where the positive value is during the ascent and the negative is during the descent. Then,

$$s = \pm\sqrt{2gh}t - \frac{1}{2}gt^2$$

Solving for t gives us,

$$t = \pm\sqrt{\frac{2h}{g}} \pm \sqrt{\frac{2(h-s)}{g}}$$

If the initial velocity is positive and $s > 0$,

$$t = \sqrt{\frac{2h}{g}} \pm \sqrt{\frac{2(h-s)}{g}} \tag{3.22}$$

the two values correspond to the fact that the particle will have the same vertical displacement at two instances of time.

If the initial velocity is positive but $s < 0$, i.e the particle starts at some height but then transverses below it,

$$t = \sqrt{\frac{2h}{g}} + \sqrt{\frac{2(h-s)}{g}} \tag{3.23}$$

is the only reasonable solution.

If the initial velocity is negative, then clearly the particle will attain a lower position. The maximum height in this case is simply the initial height of the particle.

$$t = -\sqrt{\frac{2h}{g}} + \sqrt{\frac{2(h-s)}{g}} \tag{3.24}$$

Somewhat qualitatively, projectile motion is independent of its horizontal range. If some projectiles have same maximum height, their time of flight *will be the same*.

Exercises

E1. *Projectile with vectors.* We can integrate the kinematic equations vectorially, to get

$$\mathbf{v}(t) = \mathbf{v}_0 + \mathbf{g}t,$$

and for the displacement,

$$\mathbf{s}(t) = \mathbf{v}_0 t + \frac{1}{2}\mathbf{g}t^2.$$

(a) Show that these two equations are, in fact true. Hint: How can we integrate a vector? Try to use its projections. Refer to §3.5.1 Motion in higher dimensions if needed.

(b) Note that the vectors $\mathbf{v_0}t$, $\mathbf{g}t^2/2$ are parallel to vector $\mathbf{v_0}$ and $\mathbf{g}$ respectively. Use this to find the range and time of flight. Hint: You will find fig. 3.6 useful.

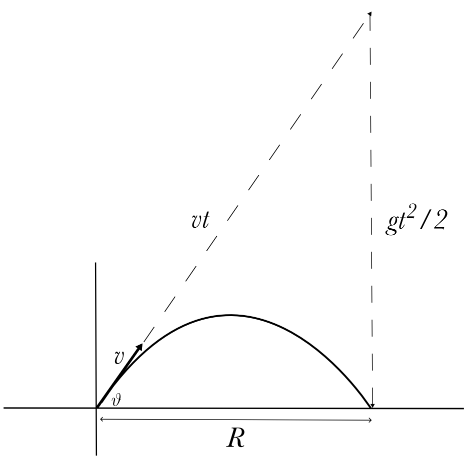


FIGURE 3.6. Scaled up $\mathbf{v}_0$ and $\mathbf{g}$ vectors in projectile motion.
[3]

E2. *Projectile Motion in tilted axes.* Consider the case of a projectile launched along a wedge, as in, fig. 3.7. If we were to figure out d, we could use a number of ways. The equation of the projectile is, [3]

$$y = x\tan(\theta+\alpha) - \frac{gx^2}{2v_0^2\cos^2(\theta+\alpha)}$$

and of the wedge is,

$$y = x\tan\theta$$

We could simply equate them to figure out the distance d, which is the range of the projectile. However, let us look at alternate way of doing this.

(a) Consider fig. 3.8, we have rotated the axes by the angle of inclination of the wedge, θ. Now, find the kinematic equations along the two axes.

(b) Solve these to find the time at which the projectile hits the wedge, and the distance it travels.

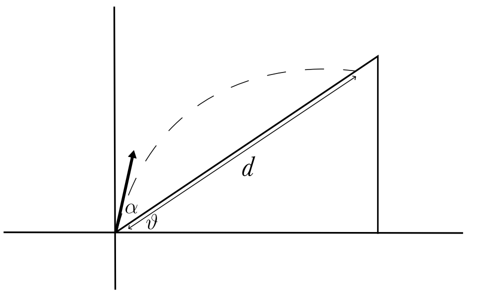


FIGURE 3.7. Projectile along a wedge

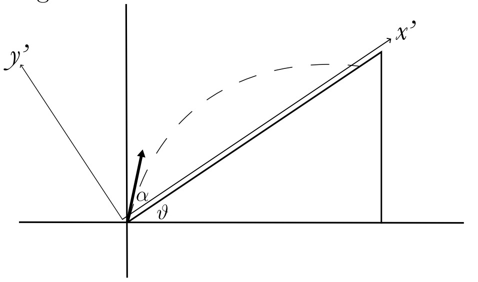


FIGURE 3.8. The cartesian axes rotated by θ.

---

and we may solve each of these to get any desired result. Due to the dependence on both θ and α, it is a bit difficult to things like maximum range and the loci method maybe preferred.

Example 3.5.5. Consider a projectile thrown with a velocity v_0 making an angle α with a wedge of inclination θ as shown in the fig. 3.9. Find out the angle α such that the projectile has maximum range.

Solution. The locus of the projectile is,

$$y = x\tan\alpha - \frac{gx^2}{2v_0^2\cos^2\alpha}$$

and of the wedge is,

$$y = -x\tan\theta$$

At the range, they will be equal,

$$x\tan\alpha - \frac{gx^2}{2v_0^2\cos^2\alpha} = -x\tan\theta \implies \tan\alpha + \tan\theta = \frac{gx}{2v_0^2\cos^2\alpha}$$

which gives us the desired range,

$$x = \frac{2v_0^2(\tan\alpha + \tan\theta)\cos^2\alpha}{g}$$

So the condition for maximum range is simply, $dx/\,d\alpha = 0$. This is also kind of illustrates a nice idea, since we need to get the derivative 0, we can safely discard any multiplicative or additive constants right away which will either have a zero differentiation, or can be divided by since the other side of the equation is simply 0.

We get that,

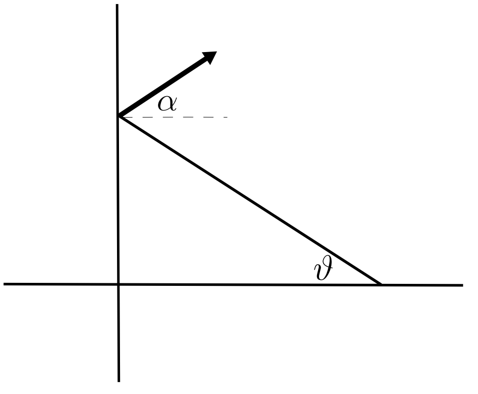


FIGURE 3.9. Projectile thrown from a wedge.

$$\frac{d\{(\tan\alpha + \tan\theta)\cos^2\alpha\}}{d\alpha} = 0$$
$$\frac{d\{\sin 2\alpha/2 + \tan\theta\cos^2\alpha\}}{d\alpha} = 0$$
$$\cos 2\alpha - \tan\theta \sin 2\alpha = 0$$
$$\tan 2\alpha = \cot\theta$$
$$\tan 2\alpha = \tan(\pi/2 - \theta)$$
$$\alpha = \frac{\pi}{4} - \frac{\theta}{2}$$

And we're done. □

3.6 Motion in Polar Co-ordinates

The polar co-ordinate axes are the two-dimesional subset of the 3-d cylindrical co-ordinates. Any point in polar co-ordinates is depicted by a system of two unit vectors, $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$. However, each of these are dependent on the position of the particle and maybe written explicitly as $\hat{\mathbf{r}}(\theta)$ and $\hat{\boldsymbol{\theta}}(\theta)$.

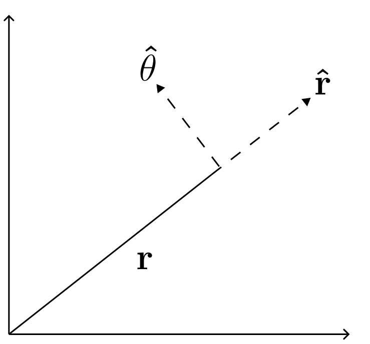


FIGURE 3.10. Unit vectors in polar co-ordinates

Here, $\hat{\mathbf{r}}$ points in the direction of increasing radius(along the radial vector) $\hat{\boldsymbol{\theta}}$ points in the direction of increasing θ (tangent to the radial vector). These two unit vectors are *orthogonal* at any point.

In fig. 3.10 if $\mathbf{r}$ makes an angle θ with the horizontal, then,

$$\hat{\mathbf{r}} = \cos\theta\mathbf{i} + \sin\theta\mathbf{j} \tag{3.25}$$
$$\hat{\boldsymbol{\theta}} = -\sin\theta\mathbf{i} + \cos\theta\mathbf{j} \tag{3.26}$$

Now let us formulate kinematics in polar co-ordinates. The position $\mathbf{r}$ can be written as

$$\mathbf{r} = r\hat{\mathbf{r}} \tag{3.27}$$

Velocity follows normally,

$$\mathbf{v} = \dot{\mathbf{r}} = \dot{r}\hat{\mathbf{r}} + r\frac{d\hat{\mathbf{r}}}{dt} \tag{3.28}$$

or does it? What even is $\dfrac{d\hat{\mathbf{r}}}{dt}$? Well the answer lies in the time-derivative of the unit vectors. Using the definitions in (3.25), we get,

$$\frac{d\hat{\mathbf{r}}}{dt} = \dot{\theta}\hat{\boldsymbol{\theta}} \tag{3.29}$$
$$\frac{d\hat{\boldsymbol{\theta}}}{dt} = -\dot{\theta}\hat{\mathbf{r}} \tag{3.30}$$

Using these results, we get that

$$\boxed{\begin{aligned} \mathbf{v} &= \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}} \qquad (3.31) \\ \mathbf{a} &= (\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{r}} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\boldsymbol{\theta}} \qquad (3.32) \end{aligned}}$$

The most interesting result are the various terms of acceleration in (3.32). First let us consider the terms along $\hat{\mathbf{r}}$. $\ddot{r}$ is the *radial* acceleration. It is the change in the radial speed(and thus the radius). The second term $-r\dot{\theta}^2$ is the *centripetal* acceleration, this is term responsible for change in the direction of the tangential velocity. In essence, it is the component which accounts for the rotation of the particle.

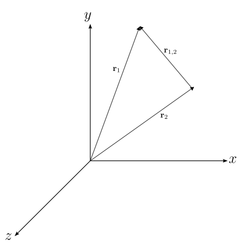
Now let us consider the terms along $\hat{\boldsymbol{\theta}}$. The $r\ddot{\theta}$ term is the tangential acceleration resulting from change in tangential speed, i.e if the radial vector is of constant magnitude, and speed v is changing,

$$a_t = r\ddot{\theta} = \dot{v}.$$

The other term $2\dot{r}\dot{\theta}$ is the *coriolis* acceleration. I shall discuss it when discussing rotating frames.

3.7 Reference Frames

There is a very neat technique to solve questions of kinematics, it is to switch our reference frames. Let the base reference frame be that of earth, then,



If $\mathbf{r}_1$ evolves with time, $r_1(t)$. And so does $r_2(t)$. Then,

$$\begin{aligned} \mathbf{r}_1 &= \mathbf{r}_{1,2} + \mathbf{r}_2 \\ \dot{\mathbf{r}}_1 &= \dot{\mathbf{r}}_{1,2} + \dot{\mathbf{r}}_2 \\ \mathbf{v}_1 &= \mathbf{v}_{1,2} + \mathbf{v}_2 \\ \mathbf{v}_{1,2} &= \mathbf{v}_1 - \mathbf{v}_2 \end{aligned}$$

Obviously if the vectors face the same direction in the same plane, then, the components subtract. If they face opposite direction, then the components add.

$\mathbf{v}_{1,2}$ is called the velocity of *1* w.r.t *2*. When doing problems, try to setup a reference frame such that velocity of any single object vanishes. This allows for a very neat simplification of problems.

3.8 Drag Forces

In a more realistic scenario, a particle undergoing motion in any media experiences a *drag force*, anti-parallel to its velocity. In particular,

$$\mathbf{a} = -\alpha v \hat{\mathbf{v}} - \beta v^2 \hat{\mathbf{v}}$$

where the linear term arises from a property of the media, the viscosity(we will talk about it in the later sections.)

The quadratic term is a result of the collision of the particles with atoms and molecules during its motion. In general, the quadratic term dominates for higher velocities.

It is very difficult to analyse motion with drag, but we shall have a look at some particular cases of linear drag.

§4.2.2 Newton's Second Law

Example 3.8.1. Consider a projectile of mass m thrown upwards with initial velocity v_0. It undergoes a linear drag force, $\langle F, =, - \rangle\, kv\hat{\mathbf{v}}$. Find the velocity of the particle as a function of time.

Solution. Using newton's second law, also by noting that it also undergoes gravitation, $\mathbf{a} = -g - kv/m\hat{\mathbf{v}}$.

Now, we note that,

$$\frac{dv}{dt} = -g - \frac{kv}{m}$$

$$\frac{dv}{g + kv/m} = -\, dt$$

$$\int_{v_0}^{v_f} \frac{dv}{g + kv/m} = \int_0^t dt$$

$$\frac{m}{k} \log(g + kv/m)\Bigg|_{v_0}^{v_f} = t$$

Which finally gives us,

$$v(t) = e^{-kt/m} v_0 + \frac{mg}{k}(e^{-kt/m} - 1) \tag{3.33}$$

□

The same idea can be extended to projectile — 2d motion by considering the components of the accelerations,

$$\mathbf{a} = -kv_x\mathbf{i} - (g + kv_y)\mathbf{j},$$

and now considering the equations $\dot{\mathbf{v}} = -kv_x$ and the same for the y co-ordinate.

Ulitmately, this leads to the equations,

$$v_x = u_x e^{-kt} \tag{3.34}$$

$$v_y = \left(u_y + \frac{g}{k}\right)e^{-kt} - \frac{g}{k} \tag{3.35}$$

3.9 A bag of techniques

Although almost everything can be found with a bash-y mathematical solution, sometimes it is just nicer to use a particularly slick idea to solve problems. Some of these are discussed below.

3.9.1 Fermat's Principle

Given that in a medium, the body can move relative to it with some fixed velocity v_1. Ultimately, it has to cross to a point in another medium, in which it can move with velocity v_2 with respect to the first one. If it has to reach some fixed point in the other medium, what could be its *optimal* trajectory?

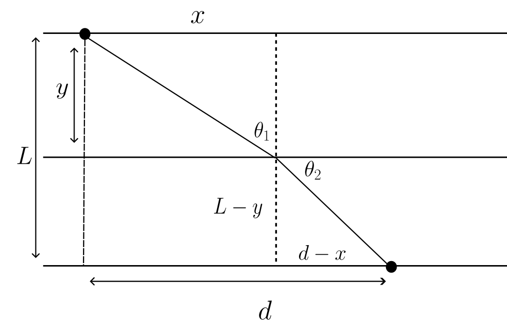


FIGURE 3.11. The optimal trajectory in the two media.

Let the optimal trajectory be as in fig. 3.11. The time in the first media is

$$t_1 = \frac{\sqrt{x^2 + y^2}}{v_1},$$

and in the other media is

$$t_2 = \frac{\sqrt{(L-y)^2 + (d-x)^2}}{v_2}$$

.

Note that y is fixed (since it is the vertical distance between the media.), Thus, the way to get minimum time is, to get for $t = t_1 + t_2$ to be such that,

$$\frac{dt}{dx} = 0$$

Now, differentiating the whole equation, we will get the final result,

$$\frac{-2x}{\sqrt{(x^2+y^2)}v_1} + \frac{-2(d-x)}{\sqrt{(L-y)^2 + (d-x)^2}v_2} = 0 \tag{3.36}$$

Note however, that,

$$\sin\theta_1 = \frac{x}{\sqrt{(x^2+y^2)}} \qquad \sin\theta_2 = \frac{(d-x)}{\sqrt{(L-y)^2+(d-x)^2}}$$

Replacing both of these and factoring out -2, we get that,

$$\boxed{\frac{\sin\theta_1}{v_1} = \frac{\sin\theta_2}{v_2}} \tag{3.37}$$

This is a very neat result and allows us to greatly simplify everything. Note here that both the velocities, v_1 and v_2 are relative to any common inertial frame. We can use some frame moving at some velocity to observe this and it wouldn't change a thing.

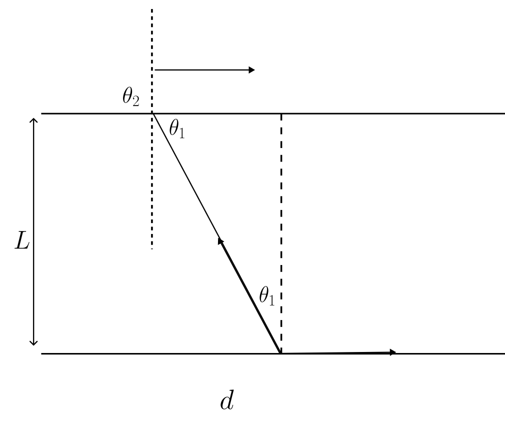


FIGURE 3.12. Shortest time to reach opposite point in a stream.

Example 3.9.1. If a man can swim relative to water with a velocity v_0, and can walk on the ground with velocity $3v_0$ and the velocity of the stream is $2v_0$, find out minimum time in which the man can reach a point directly opposite to him. The width of the river is L. Look at fig. 3.12 for clarity

Solution. The man could walk in any direction on the ground, but since we want to minimize the time, we consider him walking in the same direction as the river current to get the velocity of man on ground relative to water being $5v_0$. The velocity of him in the river relative to the stream is v_0.

From fermat's principle, the most optimal trajectory obeys

$$\frac{\sin\theta_1}{v_1} = \frac{\sin\theta_2}{v_2}$$

where the angles are taken relative to the normal.

Since $\theta_2 = \frac{\pi}{2}$, $v_1 = v_0$, and $v_2 = 5v_0$,

$$sin\theta_1 = \frac{1}{5}$$

After which we can easily find the time as by a simple angle chase, θ_1 is the angle between the velocity of man and the vertical. From it we can find the resultant velocity with water (using components) and subsequently the drift and time required. □

Chapter 4

Netwon's Laws

In this chapter we shall look at newton's laws of motion. These form the basis of *dynamics* which is the study of a motion's cause. In general, we solve mechanical problems by first considering its dynamics and then reduce it to a purely kinematical problem.

First let us concern ourselves with the concept of linear momentum.

4.1 Linear Momentum

Definition 4.1 The *Linear Momentum*, $\mathbf{p}$ of a particle of mass m moving with velocity $\mathbf{v}$ is defined as,

$$\mathbf{p} = m\mathbf{v}$$

The total momentum of a system of particles is simply the sum of momentum of each particle. Momentum is a useful quantity because it is conserved, we'll look at that latter. But for now, it has certain unique properties that make the concept of momentum extremely important.

4.2 Newton's Three Laws

4.2.1 Newton's First law

A1. Objects tend to continue in a state of constant velocity unless acted upon by a net external force.

The first law actually is much better stated as the assertion that *inertial frames* exist.

It defines inertial frames as such reference frames where objects move at constant velocity without the action of an external force. It also makes an empirical observation that such frames exist. Thus, its part definition and part experimental fact.

In fact an inertial frame may also be stated as a frame which has no acceleration. Here, the fact that we can say that it has no acceleration refers to the newtonian concept that acceleration is absolute. That in any frame of reference, we may know that a body is accelerating, even when velocity is relative and an object may be at rest in certain frames and moving in others.

We may measure the acceleration in any reference frame by placing an accelerometer and the acceleration must come out the same no matter what frame we choose.

Since the inertial frames are non-accelerating, they travel with some constant velocity relative to each other. Thus, a frame that travels with constant velocity to an inertial frame is also a reference frame.

Another way to talk about inertial frames is that they are the frames in which newton's laws hold. Certainly, if our frame is non-inertial, none of the laws will hold in this frame.

4.2.2 Newton's Second Law

A very nice reference which discusses interaction in detail is [KK14]

Axiom 2. In an inertial frame, the net external force on a body is proportional to its change in momentum.

Thus, our claim is that

$$\sum \mathbf{F} = \dot{\mathbf{P}} = m\ddot{\mathbf{r}},$$

if the mass is constant. We now see the importance of the first law, it sets up the frame in which the second law works, and unlike what one could deduce from its earlier understanding, is not a special case of the second law.

Let us take an interlude here and talk about what mass and force really are.

Mass

Assume that there are two bodies, of masses m_1 and m_2. If they say, undergo an "equal amount" of physical interaction, essentially they undergo the same force. In this example, in particular we can consider a very elementary idea of the deformation in springs.

By experimental conjecture, we claim that the deformation in spring is proportional to the force applied on it. Consider a setup in which we connect a carriage of negligible mass connected to a spring by a rod to another cart.

Each of it on an air track, which is a one-dimensional track that has holes blowing out air to ensure that the effect of forces such as friction is negligible on the body.

If now we accelerate the cart, and thus the carriage, the spring experiences a particular deformation. We now measure when the spring stops compressing. And carefully mark the length ℓ_0 and the acceleration in which we achieved for the cart of mass m_1, a_1.

Now, we can repeat this experiment for a cart of mass m_2 and get some acceleration a_2 such that by adjusting the acceleration such that the final length when it stops compressing becomes ℓ_0.

Since we claimed that the force experienced by spring is proportional to its compression, the force experienced when a mass m_1 is accelerated at a_1 is equal to the force experienced when m_2 is accelerated at a_2. So, $m_1 a_1 = m_2 a_2$. And taking the m_1 as the unit mass, we define,

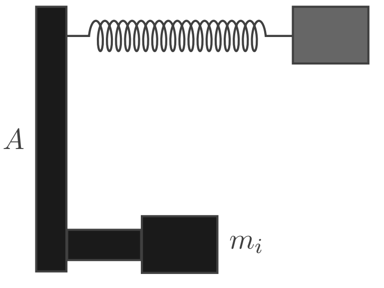


FIGURE 4.1. The carriage connected to a spring, A is the rod connected to another cart. As we accelerate the cart, we also accelerate the carriage and compress the spring until we achieve our desired compression length.

$$m_2 \equiv m_1 \frac{a_1}{a_2}.$$

We define the mass of the first body relative to the other. The accelerations can be absolutely measured using an accelerometer. In such a manner we define the acceleration of any ith body relative to the first one as,

$$m_i \equiv m_1 \frac{a_1}{a_i}$$

We define some standard mass as 1 unit and then define the other masses relative to this. Such a definition is an *operational definition*.

Force

We say a *force* is produced by a physical interaction. We dont talk about what forces "are". Merely, how they come to be. Whenever we interact physically with an object, we apply a force on it.

It might be tempting to ask in case of forces where such an interaction is not visible, say gravity, does this really hold? In fact, it does the interaction is mediated through the use of fields. We'll have a look at them later.

A layman may think of forces as things that cause acceleration, that is not however, how a physicist views them. Thus, named forces like the centripetal force, are not really forces by our definition. What we do is we define what forces can act in a physical model. Gravity, electromagnetic force, weak and strong force being the fundamental ones.

Sometimes we may also be concerned with forces arising due to macroscopic effects of small interactions. We do not thus, consider them by first principle and repeatedly apply the laws of our fundamental forces, but instead make empirical laws to avoid complexity, like the normal, tension, spring, etc. forces.

Thus, whatever acceleration is obtained, is obtained through these forces, centripetal forces are the components of these forces along the radial part of polar co-ordinates and not a force itself.

4.2.3 Newton's Third Law

The final law describes how forces come in pairs and result of a physical interaction.

Axiom 3. If a body A applies a force on a body B, then the body B applies an equal and opposite force on A. These forces are called *action-reaction pairs*.

That is to say,

$$\mathbf{F}_{AB} = -\mathbf{F}_{BA}.$$

The third law is also known as the weak law of action and reaction — in direct contrast with the strong law which requires the equal and opposite forces to lie on the same line of action. Most forces such as gravity and the normal force in fact obey the strong law of action and reaction.

4.3 Trajectory and Spaces

If you do not know about differential equations, I recommend getting a read on them before continuing this section (or skipping it for now).

The preceding discussion on forces tells us an interesting thing, the force is a real physical interaction — embedded into our physical model. So, a particle does not undergo a force just because it has a specific acceleration. It undergoes some acceleration because it is under the influence of a force.

Generally, forces are actually functions of position, since in essence we care mostly about how particles interact with each other. Such interactions are independent of time, we are interested in how these interactions are affected by the position of the particles in space.

So, if we denote the force as function of position, by the second law,

$$m\ddot{\mathbf{r}} = \mathbf{F}(\mathbf{r}).$$

A little neat fact to note is that the trajectory is a double differentiable function of time. While this maybe apparent from the existence of forces, we may also write it out as a law, if we wish to, of course.

This is a differential equation, the solution to which is function, $\mathbf{r}(t)$. This is called the *trajectory* of the particle. So when we solve the equation given by the second law, we find for a function $\mathbf{r}(t)$, according to which the particle evolved with time (basically which governs what path it follows in space). Of course, we only really get a general solution by this, a class of functions, to be precise.

To get the particular trajectory of a particle, we need some additional information, the initial (boundary) conditions, which in most cases are the initial velocity and position.

To be a little more precise, the trajectory is a solution of an *equation of motion*, which is a second order differential equation. Newton's second law is one such way to obtain an equation of motion, but as you may have guessed, there are other ways to do so too. So the second law is not the only possible way to obtain the trajectory of the particle, we'll discuss more about this in analytical mechanics.

4.3.1 Configuration space

To describe the motion of a particle, we some *degrees of freedom*. These are the amount of co-ordinates we need to describe the motion of a particle.

For instance in the case of point particles, since they do not have any dimensions themselves, we only need to describe their position and as such, we need three co-ordinates. This is different for, for instance a rigid body, whose *orientation*(because we can rotate it) as well as its position matter, so we need six degrees of freedom.

For a system of two point particles — we need 6 degrees of freedom, to describe the positions of both particles. For a system of n particles, we need $3n$ degrees of freedom.

An *instantaneous configuration* of some particles is any possible arrangement of those particles in space. The set of these configurations, $\mathcal{Q}$ is called the *configuration space*.

Consider a system of two particles in free space — we need 6 co-ordinates (degrees of freedom) to describe the arrangement, the position(location) of two particles. So, $\mathcal{Q}$ consists of six co-ordinates, and is the set $\mathbb{R}^6$. Similarly, we may extend this to a system of n particles in free space, where,

$$\mathcal{Q} = \mathbb{R}^{3n}$$

If two particles were instead confined to move along a sphere(which is denoted $\mathbf{S}$), then $\mathcal{Q} = \mathbf{S}^2$.

4.4 Some Phenomenological Forces

4.4.1 The Normal Force

The contact force on a body can be resolved to two mutually perpendicular forces, the *normal force*, perpendicular to the object, and *friction*, tangential to the object.

These contact forces arise due to atomic interactions. If we say keep a book on a table, the book pushes against the table. On a microscopic scale, the molecules push against each other. After a certain point, the molecules start repulsing each other and the table applies a normal force, perpendicular to the body, to stop it.

Since no object is perfectly rigid, there always occurs a deformation on its surface. However, this is usually negligible, and we may safely assume that the object is perfectly rigid.

4.4.2 Friction

Friction much like the normal force, arises from molecular interactions. It is, however, way too complex to analyse. Thus, our study of friction is pure phenomenological. Here, by friction we refer to dry friction, and not as is, drag.

Friction opposes the relative motion of objects — and this is in fact the most we can say safely, and generally. It is independent of the contact area.

This might feel a little weird, but consider the following argument — say we have an object who has contact area with another object. This contact area is actually a very small fraction of its overall surface area. It is, also, proportional to the pressure.

If we double the surface area of the object, let us assume that the contact area doubles as well. If we keep the normal force constant, the pressure halves. Friction, however, the product of the now half pressure, and the double contact area, remains the same.

This is a very ideal approximation of friction, but we'll make do with it now.

We differentiate between *static friction* and *kinetic friction*. When we apply a force on an object, and yet it does not undergo motion, the static friction assumes a value to balance out the force. This friction is actually related to the normal by $|f_s| \leqslant \mu_s N$. Where μ_s is the coefficient of static friction.

However, it can not keep the object at rest forever, after a certain amount of force is applied, the object undergoes motion and experiences kinetic friction. It is approximately constant and related to the normal by $|f_k| = \mu_k N$, where μ_k is the coefficient of kinetic friction.

Generally, $\mu_k < \mu_s$ and they're both less than 1. Though for some exceptionally rough surfaces, they may have a value greater than 1.

Example 4.4.1. Consider a person along the wall standing inside vertical drum, of radius r, in which the floor is slowly falling away. If the coefficient of friction between the wall and person is μ, find the smallest value of the angular speed, ω such that the person does not fall away.

Solution. This is a particularly famous amusement park ride, called the spinning terror. Using polar co-ordinates, and that the friction must be equal to the weight to prevent slipping,

$$\begin{aligned} f &= mg \\ N &= m\omega^2 r \end{aligned}$$

Since $f \leqslant \mu N$,

$$g \leqslant \mu\omega^2 r \iff \omega \geqslant \sqrt{\frac{g}{\mu r}}$$

□

4.4.3 Tension

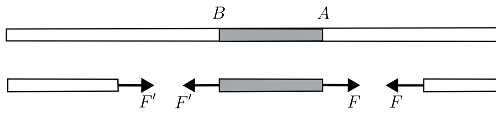


FIGURE 4.2. F, the tension at A, and F' the tension at B.

The tension is the force a piece of string applies on another. Such a force is applied due to molecular interactions, when we stretch a string, the attraction between molecules opposes such a stretch and each piece of string applies tension on one another.

If the string could be compressible however, we would have had another layer of complexity, the fact that we can bring the molecules close till only a certain distance after which they will start repelling each other and tension would oppose the compression.

So tension at any point is really the magnitude of force acting between adjacent sections of a rope.

We will shortly consider pulleys as well. If the pulleys are connected rigidly to a body, and a string is wrapped around it, the tension experienced by the string applies on the pulley as well. And since the pulley is connected through a rigid support to a body, it applies on the body as well. Look at fig. 4.3. In particular note that the tension on the body is directed towards the extension of the strings.

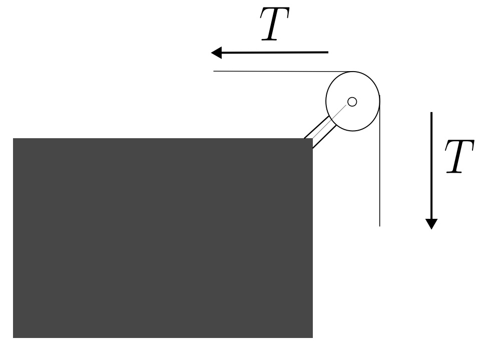


FIGURE 4.3. Tension applied on a body connected rigidly to a pulley.

Example 4.4.2. Let the rope of uniform mass density, λ and length L hang from a tree. Write the tension as a function of position from the bottom.

Solution. Consider an infinitesimal section of the rope, of length dx. Since the rope isn't accelerating, the net tension on the rope must balance its weight. Thus,

$$T(x + dx) - T(x) = dT = dmg$$

Using $\lambda = dm/\,dx \iff dm = \lambda\, dx$,

$$\int_T^{T_0} dT = \int_0^{L-x} \lambda g\, dx$$

Which is a differential equation. To solve, it we need to find T_0, a boundary value condition. However, note that at $x = 0$, the tension must balance the rope's weight, hence, $T_0 = mg = \lambda L g$.

So we get the desired expression,

$$T = \lambda x g$$

□

4.5 Free body Diagrams

The crux of this discussion will be on how do we solve problems? We're given some initial condition, perhaps a force of some sort is applied. We know it must be related to the acceleration of the particle by Netwon's second Law. But ultimately, how are we meant to solve it?

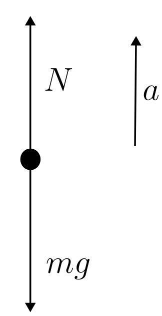


FIGURE 4.4. Free body diagram of the man.

The first step into solving a problem of dynamics, is to setup a free body diagram. Consider for instance, a man moving in an elevator, accelerating upwards with acceleration a. What is the force experienced by man?

Essentially, we're here only concerned with the forces that act on the man. So disregarding all complexity, we can draw a simple force diagram, that allows us to readily assess the problem without losing any information.

We see that there must be only two forces being applied on the body, the normal force and its own weight[1].

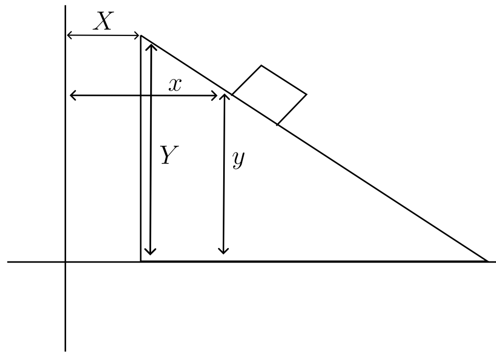
So, the net force on the object must be, $N - mg$. Taking great care of the directions, by the second law,

1. It is often a source of confusion whether the normal and weight are action-reaction pairs of Newton's third law. They aren't, the weight of the body is the gravitational force it experiences due to the Earth, its opposing force is the force experienced by Earth due to the body. However, due to the huge mass of the Earth, the acceleration provided by

$$N - mg = ma \iff N = m(g + a)$$

And the man experiences a force greater than his weight — an easily verifiable situation found to be true.

4.6 Constraints



Sometimes the equation $F = ma$ doesn't have all the information we need to solve a problem. This is often when there are constraints involved in the problem which give additional information.

Example 4.6.1. Consider a block placed on a wedge. Let the wedge accelerate at some horizontal acceleration. Then find the relation between the accelerations of the block and the wedge.

Solution. If the position of the front end of the block is (x, y) and that of the wedge is (X, Y),

$$\tan\theta = \frac{Y - y}{x - X} \iff (X - x)\tan\theta = Y - y$$

Differentiating this twice with respect to time,

$$\ddot{y} = (\ddot{X} - \ddot{x})\tan\theta \tag{4.1}$$

where $\ddot{X}$ is the horizontal acceleration of the wedge. □

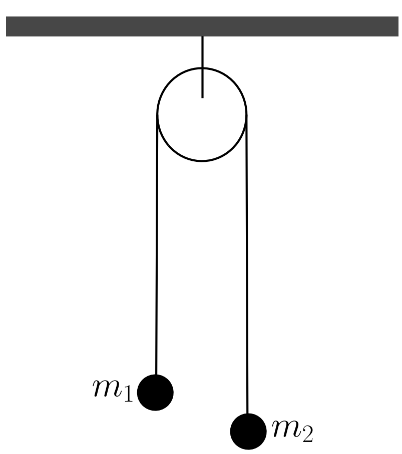


FIGURE 4.5. Classic Atwood's machine

Now we will consider a rather classic system for the use of constraints, Atwood's machine. As show in fig. 4.5, it is a system consisting of a pulley and two masses.

If we were to try and find the acceleration of the two masses, we would encounter three unknowns, a_1, a_2 and T and two equations from the second law.

To solve such a system we can consider the vertical positions of the blocks. The idea here is to use the fact that a string has a fixed length(an inextensible one, anyway, but this is quite a forgiving approximation). If the height of the pulley is h, their vertical accelerations y_1, y_2 and the length of string ℓ, then,

$$h - y_1 + h - y_2 = \ell$$

Differentiating twice with respect to time,

$$-a_1 - a_2 = 0 \iff a_1 = -a_2$$

Hence, now our unknowns reduce from three to two and the system becomes solvable. If the pulley itself was accelerating, its height was changing,

$$\ddot{h} - a_1 + \ddot{h} - a_2 = 0 \iff \ddot{h} = \frac{a_1 + a_2}{2}$$

And generally a pulley's position is the average of the position of the masses attached to it. If the system of pulleys consists of a lot of pulleys, it is difficult to use the fact that the length of the string is constant (often called the conservation of the string.). So we use a result from energy, that we will not derive quite yet, that is,

$$\sum Tx = \sum Tv = 0$$

the velocity sum is just a corollary of the position one. In a lot of scenarios, this can be extended to accelerations as well, but sometimes, especially when changing angles are involved, one needs to be careful. When such angles are not involved, we can easily figure out the relation between accelerations.

For instance in our previous example,

$$Ta_1 + Ta_2 = 0 \iff a_1 = -a_2$$

which is the desired result.

4.6.1 Rigid Body constraint

The constraint we will be discussing in this section in particular is the rigid body constraint. A rigid body is any body that follows a particular constraint the distance between any two points on the body is always equal. This approximately carries over to strings as well — since we will mostly be dealing with in-extensible ones anyway.

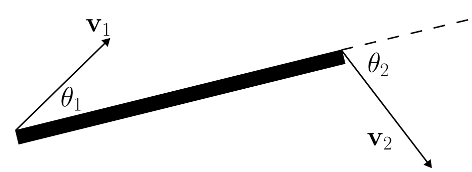


FIGURE 4.6. Using the fixed length constraint of a rigid body.

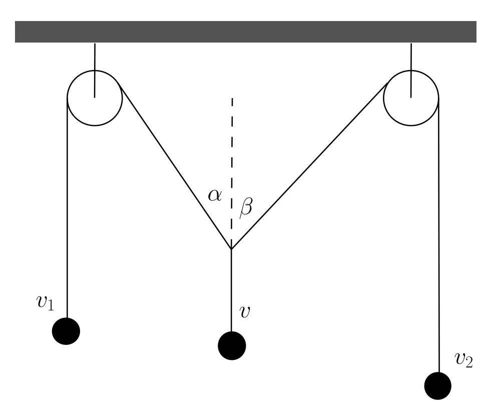
Let the velocity of one end of a rod be $\mathbf{v}_1$ making an angle θ_1 with the rod and the velocity of the other end be $\mathbf{v}_2$ making an angle θ_2 with the rod.

Since the two points on the rod must be at a fixed distance, the relative velocities of the component of these velocities along the rod must be equal,

$$\boxed{v_1 \cos\theta_1 = v_2 \cos\theta_2} \tag{4.2}$$

This particularly allows us to sidestep many annoyances we may get if we instead tried to solve problems by considering components of vectors. The cautionary thing about them is that the component of the vector is always, obviously, of lesser magnitude than the original vector.

If we have two vectors, at some angle such that we know one is the component of another, we can not be sure which one is the component of the other, at least not unless we know their magnitudes. This isn't normally a problem, but can readily become one.



Example 4.6.2. Consider the system shown in fig. 4.7. Relate v with v_1 and v_2.

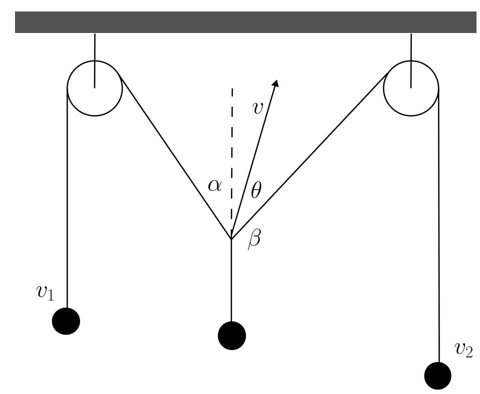
Solution. Let the velocity v make an angle θ with the string as shown. Using the rigid body constraint,

$$v \cos\theta = v_2$$

And,

$$v \cos(\alpha + \beta - \theta) = v_1$$

And substituting for $\cos\theta$, we get the desired relation. □



4.6.2 Problems of Friction

The problems of friction are in particular annoying. This is mostly due to the fact that friction on a body assumes whatever magnitude the force applied on it has (the body is at rest in such a case), until a certain threshold, after which its magnitude is constant, essentially when there is relative motion.

When a body is not moving, $f \leqslant \mu N$ where equality is at the limiting case, just when the body is about to move. We will mostly find this annoying when dealing with problems dealing with stacks of blocks.

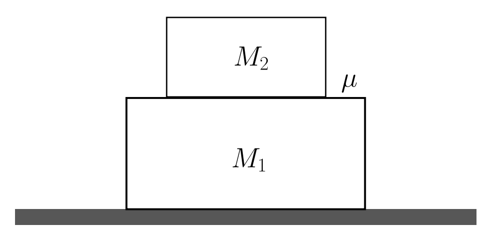


FIGURE 4.8. A block of mass M_2 on the top of a block of mass M_1. The coefficient of friction between the two blocks is μ and the surface on which the blocks are kept is smooth.

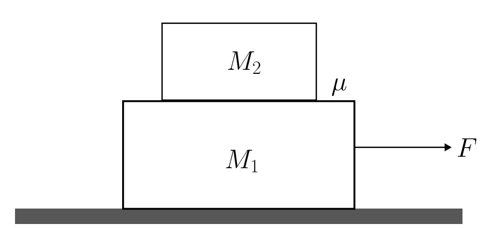
Consider the setup shown in fig. 4.8. The maximum magnitude the friction between the two blocks can assume is $\mu N = \mu M_2 g$. What if we now apply a force on the block?

There are two possible cases — either the friction assumes its maximum values, that is so when the upper block moves with respect to the lower block — and since their initial velocities are zero, if the acceleration of the upper block differs from the lower one.

Let's first consider that the blocks move together, i.e. there is no relative motion between the bodies. Then if we consider the two blocks as a whole system, friction is an internal force and plays no part in determining the acceleration of the bodies.

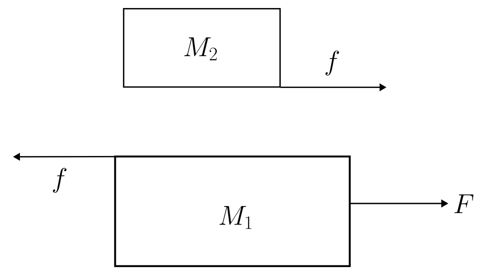
Since their acceleration is the same,

$$F = (M_1 + M_2)a$$



Gives us the acceleration. Since for the upper block, the friction is the force that provides the acceleration, we can find that as well.

Now let's consider the second case — that they do not move together. This can only happen if friction assumes its maximum value. Drawing the force diagrams of the two blocks, We can find the acceleration of the two blocks since we know both $f(\mu M_2 g)$ and F.



So how do we know which acceleration is right?

Call the acceleration of the upper block in the first case common acceleration, and of the second maximum acceleration. If the common acceleration is lower than the maximum one, the upper block will move with the common acceleration. Why is that so?

Well let's assume the block moves with maximum acceleration when the common acceleration is lower. Note, however, that the lower block moves with common acceleration, lesser than the maximum one.

So, the upper block moves faster than the lower one, but that cannot be, since the very reason friction was applied in the forward direction (which grants in the maximum acceleration), is because the lower block moved faster, and the upper one slid back!

We thus arrive at a contradiction. So the block must move with common acceleration, and the two blocks move together.

In the case when the common acceleration is greater, we see that the friction, which provides acceleration cannot assume any value to provide the common acceleration — it has a max value and the max acceleration is lower than the common acceleration. So the bodies do not move together and the upper one moves with maximum acceleration.

Our above argument can be summarized as follows,

> The body will move with whichever of the common or maximum acceleration is lower.

Add spring force and we're done with chapter :)

Chapter 5

Energy and Momenta

Motivation: While we could very well stop our discussion of mechanics with newton's laws, it is often difficult to describe physical phenomenon with just them. Thus, we develop an equivalent way of describing motion — using the concept of energy.

5.1 Momentum

Newton's laws essentially hold for point particles. However, we have, and without justification, often used for instance the second law, on an extended system of particles — a *body*.

If we were to consider the motion of such a body, the only way seems to be to account for all interactions of each particle of the body. This, however, is a very tedious task. Hope is not lost, however, and the problem is much simpler than what we might think!

If we apply some external force on the body, the total force is actually the sum of the external force, plus the internal forces between the particles themselves. Let us denote the force on some particle i by j as,

$$\mathbf{f}_{ij}$$

This is actually the usual convention when writing forces, $\mathbf{F}_{ij}$ is the force on i due to (interaction between i and) j.

The total internal force on i is thus,

$$\sum_{j \neq i} \mathbf{f}_{ij}$$

And the total net force on the system is,

$$\sum_{i} \sum_{j \neq i} \mathbf{f}_{ij}$$

The neat thing here is that, this sum sums over over all pairs $\mathbf{f}_{ij}$, $\mathbf{f}_{ji}$, which, by the third law is zero and thus,

$$\sum_{i} \sum_{j \neq i} \mathbf{f}_{ij} = \sum (\mathbf{f}_{ij} + \mathbf{f}_{ji}) = \mathbf{0} \tag{5.1}$$

So when considering the force on a system, we only need to care about external force! Cool, but considering the interaction is still a tedious task.

The force on the system by the second law must be,

$$\mathbf{F} = \mathbf{F}_{ext} = \sum m_i \ddot{\mathbf{r}}_i = \sum \dot{\mathbf{p}}_i$$

Now, we can simplify the equation a little by abbreviating the sum as,

$$\mathbf{P} = \sum \mathbf{p}_i \tag{5.2}$$

So our equation becomes,

$$\mathbf{F} = \dot{\mathbf{P}} \tag{5.3}$$

This looks very similar to the case we had for a single particle. Consider the mass to be constant, can we push this analogy to $F = ma$? Well, we can try,

$$\mathbf{F} = \dot{\mathbf{P}} = M\ddot{\mathbf{R}}$$

where $M = \sum m_i$ is the total mass of the system. Then,

$$M\ddot{\mathbf{R}} = \sum m_i \ddot{\mathbf{r}}_\mathbf{i} \iff \mathbf{R} = \frac{\sum m_i \mathbf{r}_i}{M}$$

The vector $\mathbf{R}$ locates the center of mass of the system, the point about which, the *weighted*[1] position vectors of the points of the body sum to zero. This is easy to verify. In essence, we can reduce the whole system to a particle, of the mass of the whole body, located at its center of mass.

1. Weighted using the mass at that point

Definition 5.1

The *center of mass* of the system is located at the position $\mathbf{R}$ from origin, where,

$$\mathbf{R} \equiv \frac{\sum m_i \mathbf{r}_i}{\sum m_i} \tag{5.4}$$

If the mass distribution is continuous, them,

$$\mathbf{R} = \frac{\int \mathbf{r}\ dm}{\int\ dm} \tag{5.5}$$

Let us calculate the center of mass of a few bodies,

> **Example 5.1.1.** Calculate the center of mass of a rod of length L and mass M, with uniform mass density, λ.

Solution. Setup the origin at one end of the rod. Since the mass density is uniform,

$$\lambda = \frac{M}{L}$$

Now, note that $dm = \lambda\ dx$, if x is distance from the origin. Thus,

$$\int_0^L \mathbf{r}\ dm = \int_0^L x\hat{\mathbf{x}}\ \lambda\ dx$$

Using the value of λ, the integral becomes

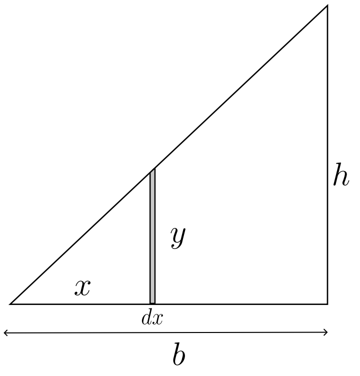
$$\frac{M}{L}\hat{\mathbf{x}} \int_0^L x\ dx = \frac{ML}{2}\hat{\mathbf{x}}$$

Thus, the com is at,

$$\mathbf{R} = \frac{L}{2} \tag{5.6}$$

□

A better way to do this is choose a nice origin, about which the body is symmetrical. For instance, the rod is symmetrical about its midpoint. Now for two points equal distance apart from the mid point, their masses are equal, but their position vectors are opposite and thus $\sum m_i \mathbf{r_1}$ evaluates to $\mathbf{0}$, and our center of mass is at origin.



Lets consider another example, this time we will show a neat idea on avoiding double integrals.

Example 5.1.2. Consider a right-angled triangular plate of height h, and breadth b. If the mass density of the disk, σ is uniform, find the center of mass of the disk.

Solution. Let us divide the plate into thin vertical strips, of breadth dx. If the horizontal and vertical distancea of the strip from the origin, at one of the non-perpendicular vertices, are x and y, then

$$\frac{x}{y} = \frac{b}{h}$$

The center of mass of this strip is located at the midpoint of this disk, at $y/2$. So, we can reduce each of these strips into their com and then calculate the center of mass of the plate.

$$\int \mathbf{r}\ dm = \int x\hat{\mathbf{x}} + y/2\hat{\mathbf{y}}\ dm$$

But, $dm = \sigma\ dA$ and $dA = y\ dx$.

However, we also have, $x/y = b/h \iff y = xh/b$. Thus,

$$dm = \sigma \frac{xh}{b}\ dx$$

And our integral is,

$$\int_0^b x\hat{\mathbf{x}}\sigma\frac{xh}{b}\ dx + \int_0^b \frac{xh}{2b}\sigma\hat{\mathbf{y}}\frac{xh}{b}\ dx$$

Also, $\sigma = 2M/bh$, substituting and evaluating the integral, we have,

$$\mathbf{R} = \frac{2b}{3}\hat{\mathbf{x}} + \frac{h}{3}\hat{\mathbf{y}} \tag{5.7}$$

□

Part C

Electricity and Magnetism

Chapter 6

Electrostatics

Motivation: The world around is very well governed largely at day to day scales by electromagnetic interactions. Tensions, friction and a lot more are results of electrical forces between microscopic particles. So a study of electromagnetism is of vital importance to us.

In electrostatics we will deal with the study of stationary charges. We will discuss Coulomb's law and further alternate versions of it for high symmetries like Gauss's law. This forms the basis of all of electromagnetic theory we will encounter.

6.1 Electric Charge

The theory of electromagnetism is centered around the idea of charge. The fundamental property of charge is their duality — there are two particular types of charges, *positive, and negative*.

The basic *independent* unit of charge is the electron, the charge of which, we denote $-e$ where the negative sign represents that it is negative[1].

1. This is just convention, we could have as well denoted it positive, and it wouldn't have made a difference.

In contrast, the positive charge of equal magnitude is found in proton, whose charge is e. This nice set of duality is rather interesting and maybe linked to a more fundamental concept which we'll see in a moment.

Charges in particular follow two nice set of properties — *conservation* and *quantization*.

6.1.1 Conservation of Charges

Any isolated system of charges, which basically means that no matter is allowed to cross into the boundary of our system; is conserved. What we mean is that the algebraic sum of the charges is always constant.

The isolation isn't exactly the most general criteria, we can let a photon cross over into our system since it carries no charges and observe a very interesting phenomenon, of *pair creation*.

A photon when exposed to gamma rays may end its existence and create an electron and *positron*, the anti-particle of electron which carries a charge of e but is same in all other respects to an electron[2]. Note that this still ensures conservation, the algebraic sum is still the same. Thus, it follows the "law of charge conservation", which we'll take as something fundamental.

2. In contrast to proton which has a different mass and so on.

Taking this as more fundamental also maybe allows us to answer why the proton has equal magnitude of charge as an electron. Some certain theories suggest that a proton *may* decay into a positron and some uncharged particles. This event must follow the law of charge conservation and thus implies that a proton has the same charge as that of a positron. However, no one has observed with certainty such a decay.

6.1.2 Quantization of Charge

Charge in its fundamental form is only present as in an electron or proton, so a system of charges must have a charge of $\pm ne$, where $n \in \mathbb{Z}$. Quarks that exist inside the proton contain a charge in multiples of $\pm e/3$.

In particular in the neutron two quarks consist of charge $-e/3$ and one with charge $2e/3$, rendering the overall charge neutral. Quarks, however, have yet to be found singularly, thus we have not observed such fractional charges and the basic unit of charges remains e.

6.2 Coulomb's Law

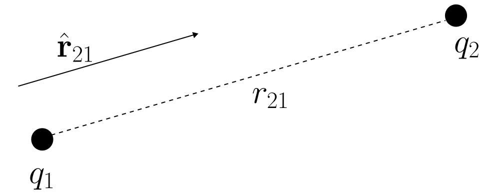


FIGURE 6.1. Two charges q_1 and q_2 and the radial vector between them.

Coulomb's law, which we take as an empirical law, states that for point charges of charge q_1, q_2, the electrical force on the charge 2 by 1 is,

$$\boxed{\mathbf{F}_{21} = k\frac{q_1 q_2}{\mathscr{r}}\hat{\boldsymbol{\mathscr{r}}}} \tag{6.1}$$

where according to standard convention, $\hat{\mathbf{r}}_{21}$ is the unit vector in the direction of the line from charge 1 to 2 and $\mathbf{F}_{21}$ is the force on 2 by 1.

We could also argue that the line joining the two charges is the only possible direction the force could act at all. After-all, for stationary charges, there is no other *unique* direction[3].

3. Note of course, that this is not true for moving charges and in fact one component of the force acts along the direction $\mathbf{r} \times (\mathbf{r} \times \mathbf{a})$.

The constant k here evaluates to $8.988 \times 10^5\ \mathrm{N\,m^2/C^2}$. For reasons involving easier calculations, k is often written as,

$$k \equiv \frac{1}{4\pi\epsilon_0} \iff \boxed{\epsilon_0 = \frac{1}{4\pi k} = 8.854 \times 10^{-12}\mathrm{C^2/(N\,m^2)}} \tag{6.2}$$

We will see its use when talking about Gauss's law.

Note that for two alike charges, $q_1 q_2 > 0$ and the electrical force points along the direction of $\mathbf{r}_{21}$ or from the charge 1 towards 2. Thus, it is repulsive in nature. The converse is clearly true for particles of unlike charge.

Part D

Appendices

Appendix A

Solutions

Solutions to the Exercises of Section 3.5

Solution 1. We see that the velocity at any further moment is just the resultant of a scaled gravitational acceleration vector and the initial velocity vector and the displacement is the resultant of the scaled velocity and gravitational acceleration vectors. [3]

Since both of them are multiplied by a positive scalar the time t or that squared, we simply scale both of the vectors to find the resultant.

In fig. 3.6, we can see that since $\mathbf{v}_0$ is tangent to the curve at the initial point, so is $\mathbf{v}_0 t$. Now, we can calculate the time taken to reach the following point since the displacement at that instant, R, is going to be the resultant of the two vectors as shown in fig. 3.6.

Thus,

$$\sin\theta = \frac{gt^2}{2v_0 t} \implies t = \frac{2v_0 \sin\theta}{g}$$

which is the desired flight time. We can also use,

$$\cos\theta = \frac{R}{v_0 t} \implies R = v_0 t \cos\theta$$

by plugging in the expression for time we just got, we also get the range of the projectile.

Solution 2. The acceleration along the y axis is simply $g\cos\theta$ and along the x axis is $g\sin\theta$. Thus, our equations in each direction are, [3]

$$v_x(t) = v_0 \cos\alpha - g\sin\theta t$$
$$v_y(t) = v_0 \sin\alpha - g\cos\theta t$$

We can solve this by noting that the displacement in y direction must be 0 at the time it hits the wedge. So we integrate and the velocity to get,

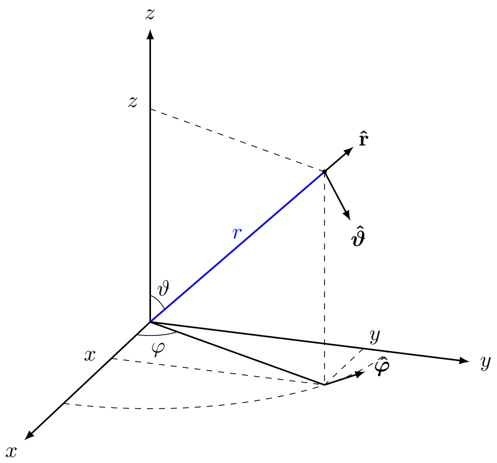
$$y(t) = v_0 \sin\alpha t - \frac{1}{2} g\cos\theta t = 0 \implies t = \frac{2v_0 \sin\alpha}{g\cos\theta}$$

Plug this in the equation obtained by integrating the velocity in the x direction to get the range.

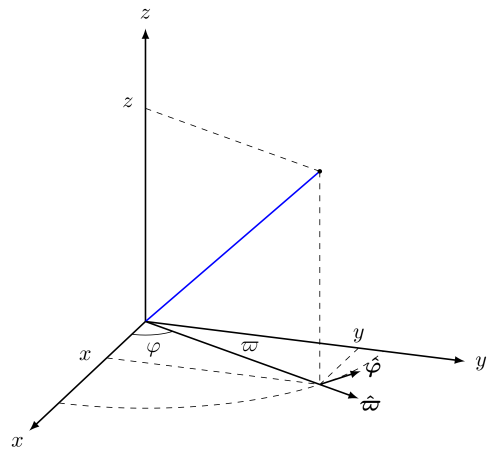
Appendix B

Curvilinear Co-ordinate Systems

Aside from the familiar cartesian co-ordinate system, we primarily deal with two *curvilinear* co-ordinate systems — *spherical* and, *cylindrical* co-ordinate systems.



(a) Spherical Co-ordinates



(b) Cylindrical Co-ordinates

FIGURE B.1. Unit vectors in curvilinear co-ordinate systems.

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Bibliography

[1] Robert Kolenkow **and** Daniel Kleppner. *An Introduction to Mechanics*. Cambridge University Press, 2014.