# Undergraduate Mathematics

v.0.1.15.11.2024

Ascholar

A foray into
undergrad mathematics
with the philosophy
of a more careful
intuitive, and rigorous
introduction
for the curious
student.
Source Code.

# Contents

# List of Figures

Chapter 1

# Proofs

*Motivation: The basis of mathematics relies on proofs.*

Proofs are necessary for the study of higher mathematics. The notes for proofs are almost entirely based on [Vel06]. Mathematics itself is as a subject is based on *deductive* logic. We start with a set of axioms and end with theorem, conjectures, theorems, lemmas, etc.

The language of proofs is therefore necessary for the study of higher mathematics, and thus this chapter is included at the start instead of being chucked away at the distance.

*Part A*

# Calculus

Chapter 2

# Real Numbers

*"Give him threepence, since he must gain out of what he learns"*

Euclid *of Alexandaria*

The real numbers form the basis of all of the calculus we are going to study. Later, we'll look at expansions of them. To formulate a *deductive* system, we must define real numbers and build up everything over them. Well, in reality real numbers are arbitrary quantities that follow a certain set of axioms, every mathematical object that satisfies them is a real number.

## 2.1 REAL NUMBER AXIOMS

The axioms of real numbers can be divided broadly into three different types,

- The Field axioms
- The Order axioms
- The Continuity axiom

### 2.1.1 FIELD AXIOMS

Consider the set $\mathbb{F}$ along with two operations, $+, \cdot$. This structure, $(\mathbb{F}, +, \cdot)$ is called a *field* if it follows the following set of axioms.

Although we label the operations as symbols multiplication and addition, these can in effect be any two operations, provided they follow the axioms. A field is a general structure, and we'll talk much more about it in algebra.

These operations are such that for every $x, y \in \mathbb{F}$, we will have $x + y, xy \in \mathbb{F}$, which is what we call *closure*.

These sums and products are unique, which means we always have one and only have one $x+y, xy$ for $x, y \in \mathbb{F}$. We will talk about what these operations precisely are in a moment.

**A1.** *Commutativity*, $x + y = y + x$ and $xy = yx$.

**A2.** *Associativity*, $x + (y + z) = (x + y) + z$, and $x(yz) = (xy)z$.

**A3.** *Distributivity*, $x(y + z) = xy + xz$.

**A4.** *Identity*. There exist two *distinct* identity elements, 0 and 1 such that $x + 0 = x$ and $x \cdot 1 = x$.

**A5.** *Inverses*, for $\forall x \in \mathbb{R} \setminus \{0\}$ there exists a $y$ such that $xy = 1$ and $\forall x \in \mathbb{R}$, there exists a $z \in \mathbb{R}$, such that $x + z = 0$. These are called the reciprocal and negative of $x$ and are denoted as $1/x$ ($x^{-1}$) and $-x$.

First of all, we will show that the identity elements are unique, that is, if,

$$(\forall x),\ x + 0 = x \text{ and } (\forall x),\ x + 0' = x,\ \text{then } 0 = 0'$$

and a similar statement for 1 over multiplication.

**Proposition 2.1.** *The identity elements are unique.*

*Proof.* Let us assume they aren't and $0, 0'$ are two distinct identities.

Then,

$$0 = 0 + 0' = 0'.$$

But this leads to a contradiction! ↯ □

We use the symbol ↯ to show contradiction.

A similar argument for the multiplicative identity, 1 is left to the reader. Its not really difficult to do it, once you understand what we did in this proof, which is use first use that $0'$ is an identity for the first equality (so that $0 + 0' = 0$) and then 0 as the identity for the second equality.

Such $x$'s are also unique. We can show their uniqueness by showing that the inverses are unique.

**Proposition 2.2.** *The inverses are unique.*

*Proof.* The idea is this the identity elements are unique, and the multiplication or addition by an inverse results in an identity. Let the multiplicative inverse of $x$ be distinct reals, $y$ and $z$, then

$$y \cdot 1 = y \cdot (x \cdot z) = (y \cdot x) \cdot z = 1 \cdot z = 1.$$

What did we do here? the first equality follows from the definition of inverse, the second follows from Associativity and the third uses the definition of identity. This shows that $y = z$, but $y$ and $z$ were, in fact, distinct! ↯

□

Subtraction is treated as a shorthand for addition, $a+(-b) = a-b$ using inverses. Similarly, one writes $a \cdot b^{-1} = a/b$.

These properties allow us to form laws of addition and multiplication — such as cancellation. We can form an outline of a proof of cancellation as,

*Proof.* We need to show that $a + b = a + c \iff b = c$. This can be done simply by adding $-a$ on both sides. Since addition is unique, and both numbers are equal, so must be their addition with another number.

By a formal manipulation,

$$\begin{aligned}
b &= b + (a - a) \\
b &= (b + a) - a \\
b &= (c + a) - a \\
b &= c + (a - a) \\
b &= c.
\end{aligned}$$

□

Let us prove a fundamental property of the additive inverse.

**Proposition 2.3.** *We have for all $a \in \mathbb{F}$,*

$$a \cdot 0 = 0.$$

*Proof.* How do we go about proving this? We use the fundamental property of $0$, as the additive inverse, $0 + 0 = 0$.

$$a \cdot 0 = a \cdot (0 + 0) = a \cdot 0 + a \cdot 0.$$

The left-hand side is the same as $0 + a \cdot 0$. Now, we have already justified cancellation for addition, so that cancelling $a \cdot 0$ from both sides, we have,

$$0 = a \cdot 0.$$

□

A similar procedure can be done for multiplication.

We see that this can also be used to prove that the product of any two negative numbers is positive. Let us first prove that $(-a)b = -ab$. This is rather easy as well, and follows from distributivity,

$$\begin{aligned}(-a)b &= (-a)b + ab - ab \\ (-a)b &= b[(-a) + a] - ab \\ (-a)b &= 0 \cdot b - ab \\ (-a)b &= -ab\end{aligned}$$

A similar proof can be used to show that $(-a)(-b) = ab$, as

$$\begin{aligned}(-a)(-b) &= (-a)(-b) + (-a)b - (-a)b \\ (-a)(-b) &= -a[(-b) + b] - (-ab) \\ (-a)(-b) &= -a \cdot 0 + ab \\ (-a)(-b) &= ab\end{aligned}$$

Note that none of our proofs are reliant on the fact that these numbers are real numbers. These properties hold for *any* field. In particular, they also hold for the real numbers, with the operations of addition and multiplication.

As such, real numbers form a *field*. These axioms permit us to perform our regular operations on the reals, namely addition, subtraction, multiplication and division by non-zero reals.

In the last proof, we use our previous result[1] to prove this one. Now it is obvious that for the reals, the product of any two negatives will be positive — however, let us first discuss what exactly "positive" and "negative" mean.

*1.* In the second line, $-(-a)b = -(-ab) = ab$. A proof for $-(-a) = a$ is left as an exercise to the reader.

## Exercises

**E1.** Complete the other half of the proofs for uniqueness of identity and inverses. [1]

**E2.** Show that $-(-a) = a$ and for $a \neq 0$, $(a^{-1})^{-1} = a$.

**E3.** Consider the equation $xy = 0$, show that it is only satisfied when $x = 0$ or $y = 0$. Hint: Consider the case when $x$ and $y$ are both not zero.

**E4.** Formulate a law for cancellation over multiplication. That is, given non-zero $c \in \mathbb{F}$ and,

$$ac = bc,$$

Show that $a = b$.

**E5.** We now consider the natural properties of division, or quotients. A quotient, $a/b$ is a shorthand for $a \cdot b^{-1}$. Prove the following familiar properties of quotients. Assume accordingly that the numbers having inverses are non-zero. (for instance both $a$ and $b$ are non-zero in 1).

1. $(ab)^{-1} = a^{-1}b^{-1}$. Hint: How is $(ab)^{-1}$ defined?
2. $\dfrac{a}{b} = \dfrac{ac}{bc}$.
3. $\dfrac{a}{b} + \dfrac{c}{d} = \dfrac{ad+bc}{bd}$.
4. $\dfrac{a}{b} \cdot \dfrac{c}{d} = \dfrac{ac}{bd}$.
5. $\dfrac{a}{b} \Big/ \dfrac{c}{d} = \dfrac{ad}{bc}$.
6. Show that $\dfrac{a}{b} = \dfrac{c}{d} \iff ad = bc$.

---

### 2.1.2 ORDER AXIOMS

We now again give some axioms which allow us to *order* our field. This allows us to formulate notions of positive and negative.

**A1.** There exists a subset of our field, $\mathbb{F}^+$ such that $\forall x \in \mathbb{F} \setminus \{0\}$, either $x \in \mathbb{F}^+$ or $-x \in \mathbb{F}^+$.

**A2.** If $x, y \in \mathbb{F}^+$, both $xy$ and $x+y$ are also in $\mathbb{F}^+$.

**A3.** 0 is not in $\mathbb{F}^+$.

A field in which these axioms is true is called a *ordered field*.

We say that a number $x$ is positive if it is in $\mathbb{R}^+$. We also know define some symbols,

$$x > y \text{ means that } x - y \text{ is in } \mathbb{F}^+$$
$$x < y \text{ means that } y - x \text{ is in } \mathbb{F}^+$$

The second definition is unnecessary to be precise, $x < y$ may as well be interpreted as $y > x$, which is something we have already defined. In any case, we could equivalently say a field which is imbued with an *ordering relation* such as $>$ is called an *ordered field*. We will again, talk about relations and what this exactly means later.

The symbols $\geqslant$ and $\leqslant$ just imply that "…" is either greater/lesser or equal to "…". Note that weird as it may appear, statements like $6 \geqslant 6$ are completely valid. The real use of these symbols lie in terms of dealing with variables, or general statements.

Consider the following proposition,

**Proposition 2.4** (Trichotomy)**.** *If we have $x, y \in \mathbb{F}$, then one of the following must hold,*

1. $x < y$
2. $x > y$
3. $x = y$

This can be inferred rather directly from the order axioms. Clearly, by the order axioms we have that either $a = 0$, $a \in \mathbb{R}^+$ or $-a \in \mathbb{R}^+$. Replace $a$ by $x - y$ and we're done.

**Proposition 2.5.** *In an ordered field, we have,*

1. $a > b \iff a + c > b + c$, *for some* $c \in \mathbb{F}$.
2. $a > b \iff ac > bc$, *for* $c > 0$.
3. *For all* $a \neq 0$, $a^2 > 0$.
4. $1 > 0$.

*Proof.* 1. By definition, $x > y \iff x - y \in \mathbb{F}^+$. Now, note that,

$$x - y = x - y + c - c = (x + c) - (y + c) \in \mathbb{F}^+,$$

so we are done.

2. Similar to the previous proof, use the fact that $c \in \mathbb{F}^+$ and $x - y \in \mathbb{F}^+$ imply that $c(x - y) \in \mathbb{F}^+$ by the axioms.

3. If $a > 0$, then $a \cdot a = a^2 > 0$ (since $x, y > 0 \implies xy > 0$). If $a > 0$, then by the axioms $-a > 0$. Thus, $-a \cdot -a = a^2 > 0$. Since $(-a)(-b) = ab$.

4. Trivially follows from $1 = 1^2$. □

Let us actually discuss some examples of fields and ordered fields now.

> **Example 2.1.1.** The sets $\mathbb{Q}, \mathbb{R}$ along with multiplication and addition satisfy the field axioms. In fact, they also satisfy the order axioms. Thus, they're ordered fields.

> **Example 2.1.2.** Consider the set $\mathbb{F} = \{0, 1\}$. We define $0+0 = 0, 0+1 = 1+0 = 1, 1+1 = 0, 0 \cdot 0 = 0 \cdot 1 = 1 \cdot 0 = 0, 1 \cdot 1 = 1$. Then it follows the field axioms. However, we cannot have some ordering relation.
>
> This follows because (as we will show), $1 > 0$ in an ordered field, but $1+1 = 0 > 0+1 = 1$ ↯.

We are just adding 1 to both sides, as permitted by the axioms.

This also forms the basis of the idea of the absolute value, which is,

$$|a| = \begin{cases} a, & a \geqslant 0 \\ -a, & a \leqslant 0 \end{cases} \tag{2.1}$$

We see that this clearly a positive number. It is also referred to as the euclidean length of a number from 0 on the number line. Let us consider some serious properties now,

Consider the following two properties regarding the absolute value,

**Theorem 2.6.** *For $x, a \in \mathbb{R}$, $|x| \leqslant a$ if and only if $-a \leqslant x \leqslant a$.*

*Proof.* Let us suppose $|x| \leqslant a$. Then clearly, $-|x| \geqslant -a$. But $|x|$ can only take two values, and is obviously positive. Thus, $-|x| \leqslant x \leqslant |x|$. Thus, we have that, $-a \leqslant -|x| \leqslant x \leqslant |x| \leqslant a$ which proves one direction.

For the other direction, suppose $-a \leqslant x \leqslant a$. Then, if $x \geqslant 0$, $|x| = x \leqslant a$. If $x < 0$, $|x| = -x \leqslant a$ since $x \geqslant -a \implies -x \leqslant a$. In both cases, $|x| \leqslant a$. $\square$

We will not look at the least upper bound axiom as of now. Note we still haven't shown the existence of radicals, which you must take for granted for now.

## 2.2 FUNCTIONS

The idea of a function is very simple. You have an input, and you want to get an output. For instance, consider the standard operations we defined on the reals, $+, \cdot$. These take two inputs, and produce the appropriate output, the sum and the product, respectively.

More precisely, functions are maps. They map an element of a set, to another element of a different set. The basis of such a mapping is the definition of a function.

For instance, we could consider a mapping from $\mathbb{R}$ to $\mathbb{R}$, $f$

$$x \overset{f}{\mapsto} x + 1,$$

which maps an element from the set of real numbers, to its successor.

We also denote this mapping by writing $(x, x+1) \in f$. The first element is our input, which we map to second element.

There are certain restriction, however, that we must impose, for such an object to be useful. In the case of functions we impose the restriction that the mappings are *unique*.

That is, if

$$(a, b) \in f \text{ and } (a, c) \in f,$$

then, $b = c$.

Now, we are finally ready to give the formal definition.

**Definition 2.7**

A *function* from $A \to B$ is a set such that,

$$f \subseteq \{A \times B\} \mid (a, b), (a, c) \in f \iff b = c. \tag{2.2}$$

Now we can define the operations of multiplication and addition rigorously,

$$\begin{aligned} +\colon\ & \mathbb{R}^2 \to \mathbb{R} \mid (x, y) \to x + y, \\ \times\colon\ & \mathbb{R}^2 \to \mathbb{R} \mid (x, y) \to x \times y. \end{aligned}$$

We usually write $xy$ or $x \cdot y$ instead of $x \times y$.

$$\frac{\partial \varphi}{\partial x^i}\, dx^i = \nabla \varphi = \boldsymbol{\varphi} \tag{2.3}$$

*Part B*

# Linear Algebra

Chapter 3

# Vector Spaces

This part of linear algebra is almost entirely based on [Axl24].

## 3.1 LISTS

In most of the chapters, I'll adopt the convention that $\mathbb{F}$ is a field. Some specific results require $\mathbb{F} = \mathbb{C}$ or $\mathbb{F} = \mathbb{R}$ but they'll be explicitly mentioned.

Elements of $\mathbb{F}$ are called *scalars*. We call these scalars to form a distinction with vectors, which will be defined soon.

Cartesian products of $\mathbb{F}$ with itself $n$ times are denoted as $\mathbb{F}^n$. For something like $\mathbb{F}^3$ this is equivalent to saying,

$$\mathbb{F}^3 = \{(x, y, z) \mid x, y, z \in \mathbb{F}\}$$

It is the set of all ordered triples of the elements of $\mathbb{F}$. If we have $\mathbb{F} = \mathbb{R}$, for instance, $\mathbb{R}^2$ can be though to be a plane while $\mathbb{R}^3$ can be thought to be ordinary space.

To generalise these results to $n$ dimensions, we use the concept of $n$-tuples or *lists*.

**Definition 3.1**

**Lists**

For a non-negative integer, $n$, an $n$-tuple or a list of length $n$ is an ordered collection of $n$ elements.

A list $(a_1, a_2, \ldots, a_m) = (b_1, b_2, \ldots, b_n)$ if and only if $n = m$ and $a_1 = b_1, a_2 = b_2, \ldots, a_m = b_n$.

Thus, two lists are equal iff their length is equal and they have the same elements in the same order. A list has a finite number of elements, thus, even though $(a_1, a_2, \ldots, a_m)$ is a list, $(a_1, a_2, \ldots)$ is not. The list of length 0 is denoted by $()$.

We denote lists surrounded by paratheses and their elements separated by commas.

**Example 3.1.1** (Lists vs sets)**.** The lists $(3, 5)$ and $(5, 3)$ are not equal but $\{3, 5\} = \{5, 3\}$. The list $(4, 4, 4) \neq (4, 4) \neq (4)$ but $\{4, 4, 4\} = \{4, 4\} = \{4\}$.

## 3.2 HIGHER PRODUCTS

Generalising the $n$th cartesian product of $\mathbb{F}$ with itself,

$$\mathbb{F}^n = \underbrace{\mathbb{F} \times \mathbb{F} \times \cdots \times \mathbb{F}}_{n\text{-times}}$$

We may say,

**Definition 3.2**

$\mathbb{F}^n$ is the set of all lists of length $n$ such that the elements of the list are in $\mathbb{F}$,

$$\mathbb{F}^n = \{(a_1, a_2, \ldots, a_n) \mid a_1, a_2, \ldots, a_n \in \mathbb{F}\}$$

Where $n \in \mathbb{Z}_{\geqslant 0}$.

**Definition 3.3**

**Co-ordinate**

If $(a_1, a_2, \ldots, a_n) \in \mathbb{F}^n$ and $1 \leqslant i \leqslant n$, $x_i$ is called the $i$th co-ordinate of $(a_1, a_2, \ldots, a_n)$.

While $\mathbb{R}^2$ can be visualised as a plane, and $\mathbb{R}^3$ as space, it is not possible to visualise them for $n \geqslant 4$. Similarly, $\mathbb{C}^1$ can be thought of as a plane, but cannot be visualised for $n \geqslant 2$.

However, we can perform algebraic operations on the lists of some arbitrary length $n$ which may even be very great.

**Definition 3.4**

**Addition in $\mathbb{F}^n$**

Addition in $\mathbb{F}^n$ is defined as,

$$(x_1, x_2, \ldots, x_n) + (y_1, y_2, \ldots, y_n) = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$$

To avoid the cumbersome notation of writing out $(x_1, x_2, \ldots, x_n)$ we will adopt the notation that $x = (x_1, x_2, \ldots, x_n)$.

**Proposition 3.5.** $x + y = y + x$

*Proof.* $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n)$. Thus,

$$\begin{aligned} x + y &= (x_1, x_2, \ldots, x_n) + (y_1, y_2, \ldots, y_n) \\ &= (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n) \\ &= (y_1 + x_1, y_2 + x_2, \ldots, x_n + y_n) \\ &= y + x \end{aligned}$$

□

The proof is based on the commutativity of reals and definition 3.4. The elements of $\mathbb{F}^2$, $x, y$ can be thought of as points or vectors. Disregarding the axes, vectors may independently be thought as some objects.

We define two more things in $\mathbb{F}^n$,

**Definition 3.6**

**Additive Inverse in $\mathbb{F}^n$**

The additive inverse of $x \in \mathbb{F}^n$ is $-x \in \mathbb{F}^n$ where,

$$x + (-x) = 0$$

And if $x = (x_1, x_2, \ldots, x_n)$,

$$-x = (-x_1, -x_2, \ldots, -x_n)$$

The additive inverse of $x$ in $\mathbb{R}^2$ is the vector of equal length but opposite direction.

The final operation is *scalar multiplication*

**Definition 3.7**

**Scalar multiplication**

For a scalar $\lambda$, and $x \in \mathbb{R}^n$, their product is defined as

$$\lambda (x_1, x_2, \ldots, x_n) = (\lambda x_1, \lambda x_2, \ldots, \lambda x_n)$$

## 3.3 VECTOR SPACE

The definition of a vector space comes off from the properties we have kind of discussed above for $\mathbb{R}^n$. Defining our operations for any vector space,

**Operators on a vector space** — **Definition 3.8**

*Addition* on $V$ is a function $+ : V^2 \to V$ where $\mathbf{v} + \mathbf{w} \in V$ for any $\mathbf{v}, \mathbf{w} \in V$.

*Scalar multiplication* on $V$ is a function $\bullet : V \times \mathbb{R} \to V$ where $\lambda \mathbf{v} \in V$ for any $\mathbf{v} \in V$ and $\lambda \in \mathbb{R}$.

Now, let us formally define $V$.

**Vector Space** — **Definition 3.9**

A set $V$ along with the operations of addition and scalar multiplication is a *vector space* if the following properties hold for $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and $a, b \in \mathbb{R}$.

**A1.** Commutativity, $\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$.

**A2.** Associativity, $(\mathbf{u} + \mathbf{w}) + \mathbf{v} = \mathbf{u} + (\mathbf{w} + \mathbf{v})$ and $a(b\mathbf{v}) = (ab)\mathbf{v}$.

**A3.** Additive Identity, there is an element $\mathbf{0} \in V$ such that $\mathbf{v} + \mathbf{0} = \mathbf{v}$.

**A4.** Additive Inverse, there exist an element $-\mathbf{v} \in V$ for each $\mathbf{v}$ such that $\mathbf{v} + (-\mathbf{v}) = 0$.

**A5.** Multiplicative Identity, $1\mathbf{v} = \mathbf{v}$

**A6.** Distributive Property, $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$ and $\mathbf{v}(a + b) = \mathbf{v}a + \mathbf{v}b$.

**Example 3.3.1.** The set of functions, $\mathbb{F}^S$, such that

$$\mathbb{F}^S = \{f \mid f : S \to \mathbb{F}\}$$

is a vector space over $\mathbb{F}$ with addition being defined as,

$$(f + g)(x) = f(x) + g(x)$$

And scalar multiplication as,

$$(c \cdot f)(x) = c \cdot f(x)$$

The properties are easy to verify. Note that $f(x), g(x) \in \mathbb{F}^n$, so their sum is commutative, since addition is commutative in $\mathbb{F}^n$. Similarly, we can use the properties of $\mathbb{F}^n$ to prove that $\mathbb{F}^S$ is really a vector space.

We can also use this general notation for the set of functions, for $\mathbb{F}^n$. We can think of $\mathbb{F}^n$ as $\mathbb{F}^{\{1,2,\dots\}}$. So, it is the set of functions from $\{1, 2, \dots\}$. We can think of each co-ordinate, $x_i$ corresponding to the value of the function at that point, $f(i)$.

Let us prove some general results regarding vector spaces.

**Proposition 3.10.** *The additive identity is unique.*

*Proof.* This is easy enough to show. Let $\mathbf{0}$, $\mathbf{0}'$ be two additive identities. Then,

$$\mathbf{0} = \mathbf{0} + \mathbf{0}' = \mathbf{0}'$$

And we're done. □

**Proposition 3.11.** *Every* $\mathbf{v} \in V$ *has a unique additive inverse.*

*Proof.* Since the existence is guaranteed by the axioms, we focus on uniqueness. Let $\mathbf{w}$ and $\mathbf{u}$ be two additive inverses of $\mathbf{v}$. Then,

$$\mathbf{w} = \mathbf{w} + \mathbf{0} = \mathbf{w} + (\mathbf{v} + \mathbf{u}) = (\mathbf{w} + \mathbf{v}) + \mathbf{u} = \mathbf{0} + \mathbf{u} = \mathbf{u}$$

□

Since additive inverses are unique, we can use particular symbol to denote them. We denote the additive inverse of $\mathbf{v}$ by $-\mathbf{v}$.

**Proposition 3.12.** $0\mathbf{v} = \mathbf{0}$

*Proof.* Be careful by what we mean by the two zero's. The proposition is that the product of the scalar zero times the vector $\mathbf{v}$ is the vector zero.

Now,

$$0\mathbf{v} = (0 + 0)\mathbf{v} = 0\mathbf{v} + 0\mathbf{v}$$

Adding the additive inverse, $-0\mathbf{v}$ on both sides, we get $0\mathbf{v} = \mathbf{0}$. □

## Exercises

**E6.** Show that $a\mathbf{0} = \mathbf{0}$ for all $a \in \mathbb{F}$. [1]

**E7.** Prove that $-(-\mathbf{v}) = \mathbf{v}$. [1]

---

*Part C*

# Appendices

# Index

# Bibliography

[1] Sheldon Axler. *Linear Algebra Done Right*. 2024.

[2] Daniel J. Velleman. *How to Prove it, A Structured Approach, 2nd Edition*. Cambridge Press, 2006.