A JEE Notebook

Part 1: on Mathematics

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Contents

Preface

§1 The Philosophy

First, before jumping to any content, I shall discuss the philosophy behind these notes. These notes are made because I did not like pen-paper notes, these have version control, can be easily rewritten are more accessible(on phone/pc and if needed, on print).

Another thing to note is that these notes follow a specific philosophy, in that they try to be expository. They are meant to explain things. I have noticed that books often talk circularly, create redundancies, and etc.

To combat this, I have tried to clearly lay out everything here. Also, because of how books are made, generally in various volumes, content often goes repeated. These notes will try to remove that.

Lastly, I have often added some sort of additional non-JEE relevant content in many places, because 1) JEE is not the only thing I care for, 2) I am particularly interested in Mathematics/Physics so I study extra stuff, 3) They often help in exposition and better understanding. These are, however, mentioned in the text, and you can easily skip them.

Happy learning!

Part I

Algebra

CHAPTER 1

Sets

"You don't have to be great to start, but you have to start to be great."

Zig Ziglar

§1.1 Naive Set Theory

Motivation: There is little Mathematics that does not use the language of sets.

Sets are perhaps the most important structures in all of Mathematics. Although an indepth discussion of sets is beyond these notes, and their axiomation is complicated, we shall not let this affect us and discuss a less rigorous formalisation of sets. Namely, *Naive Set Theory*.

These definitions are superficious We have not and will not define what we mean by collection or elements. That is what makes this theory *naive*.

Definition 1.1.1

A set is any collection of elements.
It is denoted as $\{a, b, c\}$ where a, b, c are elements of the set. This representation of sets is called *Roster form*.

Well, what do we mean by any collection? To give a few examples, the set of flowers would be $\{$roses, lilies, sunflowers, $\ldots\}$ The trailing $\ldots$are used to show that the elements continue on *ad infinitum*. Hence, any collection of objects is a set.

Often, instead of manually writing out $\{$roses, lilies, sunflowers, $\ldots\}$, we will use a condition that each element in the set follows. Here, the condition is that all elements all flowers. We denoted this by saying that S is the set of all x such that x is a flower. We represent this in what is called *Set-Builder* notation. Thus,

$$S = \{x : P(x)\} \quad \text{or} \quad S = \{x \mid P(x)\}$$

Here, $P(x)$ represents the condition. We use the different symbols depending on the context, mostly for ensuring that the statements are clear.

Do not be overwhelmed by terms like "set-builder". They're only formal names and not something to remember.

Remark 1.1.2. Notice the equality symbol between S and $\{x : P(x)\}$. Generally, we denote sets as capital latin letters $A, B, C, \ldots$, and their elements by little latin letters $a, b, c, \ldots$.

Example 1.1.3

The sets of all Natural numbers is denoted by $\mathbb{N}$. Generally, it is either used to represent $\{0, 1, 2, \ldots\}$ or $\{1, 2, 3, \ldots\}$. It matters little. We will let $\mathbb{N} = \{1, 2, 3, \ldots\}$ and $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$ in this text

Example 1.1.4

The other common sets are similarly denoted by *blackboard bold* letters as well.

1. $\mathbb{Z}$ is the set of all integers.
2. $\mathbb{Q}$ is the set of all rationals.
3. $\mathbb{R}$ is the set of all reals.
4. $\mathbb{C}$ is set of all complex numbers.

The symbol $\in$ in $x \in S$ is used to show that x is an element of S. It may be read as 'belongs to'.

Example 1.1.5

The cartesian plane is also a set. It defined as

$$\{(x, y) \mid x, y \in \mathbb{R}\}$$

Extra

§1.1.1 Why Naive?

One interesting formulation possible in naive set-theory is the "Russel's Paradox". It goes as follows. Let S be a set $\{x \mid x \notin S\}$. Basically, S is the set of all elements not in S. This leads to a paradox because for x to be in S, it has to follow the condition that it is not in S! And any element not in S must be in S.

There is no particular solution to this using naive set-theory. This is one of the reasons why this discussion of sets is so naive.

§1.1.2 Types of Sets

The set $\{\varnothing\}$,is not a null set! In-fact it is the set that contains the empty or null set.

Although this division of sets is superfluous at best, we will still categorise them for a reference.

- **Finite Set** : A set having a finite number of elements is called a finite set.
- **Infinite Set** : A set which has infinite elements is called an infinite set.
- **Null Set** : A set which has no elements is called a null set. It is represented as $\{\}$ or $\varnothing$.
- **Singleton Set** : A set containing a single element is called a singleton set.

§1.2 Equality and Subsets

§1.2.1 Sets of Sets

Remember that we referred to sets as a collection of *any* objects? That might lead to the question that can we also make sets of sets? In-fact, that is exactly right, we can! A set of sets maybe represented as $\{\{a, b, c\}, \{d, e, f\}, \{g, h, i\}\}$. All of the elements of this set are themselves sets. One thing to note here is that we cannot form a set of all sets, and we will see why in a moment.

§1.2.2 Subsets

Consider the sets $\mathbb{N}$ and $\mathbb{Z}$. You might see that all the elements of $\mathbb{N}$ are also present in $\mathbb{Z}$. If we were to describe it, we could say that $\mathbb{N}$ is contained in $\mathbb{Z}$. Such a relation between sets is expressed by saying $\mathbb{N}$ is a subset of $\mathbb{Z}$.

Definition 1.2.1

A set A is a subset of B if all elements of A are also elements of B.
This is represented as

$$A \subseteq B$$

Example 1.2.2

$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$.

Example 1.2.3

Let $A = \{x \mid x \text{ is prime and } x \text{ is odd}\}$, and $B = \{x \mid x \text{ is prime}\}$, then $A \subseteq B$.

One thing to note here is that for A to be a subset of B, it does not need to have less elements than B. Infact, A is a subset of A itself because all elements of A are in A.

To show that each element of A is contained in B but all elements of B are not contained in A, we say that A is a proper subset of B.

Note that A is not a proper subset of A.

Definition 1.2.4

A is a proper subset of B if all elements of A are in B but all elements of B are not in A.
We represent this as

$$A \subsetneq B$$

§1.2.3 Superset

If A is a subset of B, then B is a superset of A. It is represented as

$$B \supseteq A$$

This is another way of saying the same thing. The definition of proper superset is also the same.

§1.2.4 Equality

We may say that two sets are equal if they have the same elements. However, there is one thing to consider. When discussing about sets, we do not care how many times an element appears. The sets $\{1, 1, 1, 1, \ldots\}$ is the same as $\{1\}$. Therefore, we always ignore the repeating terms. Thus, two sets are equal if they have the same number of elements, **regardless of the frequency they appear in.** The same thing applies for subsets and supersets, **the frequency of elements does not matter.**

We can phrase the equality of sets in another manner, in terms of subsets.

Refer to Appendix A for $\iff$

Definition 1.2.5 (Equality in sets)

Two sets A and B are equal if and only if $A \subseteq B$ and $B \subseteq A$.
Therefore,

$$A = B \iff A \subseteq B \text{ and } B \subseteq A$$

§1.2.5 Power-set and Cardinality

meow

§1.2.6 Cartesian Product

meow meow

CHAPTER 2

Elementary Prereqs

§2.1 Intervals

The notation][is French, and is less commonly used in India, USA, etc.

Definition 2.1.1

For $a, b, x \in \mathbb{R}$ and $a < b$,

1. The subset $\{x \mid a < x < b\}$ of $\mathbb{R}$ is called the open interval of a, b and is represented as (a, b) or $]a, b[$.
2. The subset $\{x \mid a \leq x \leq b\}$ of $\mathbb{R}$ is called the closed interval of a, b and is represented as $[a, b]$.

Add figures for all

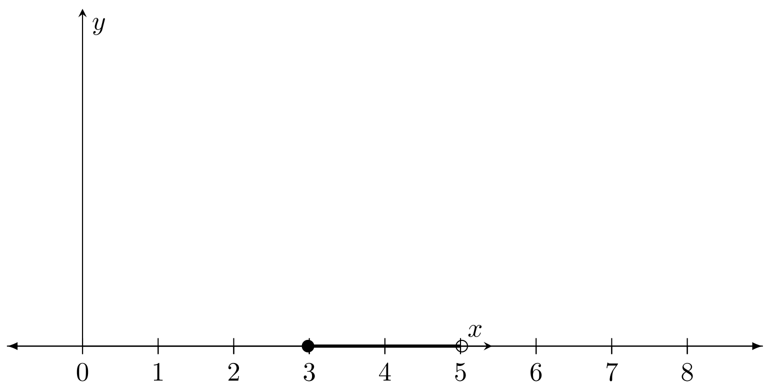

Figure 2.1: Closed-Open Interval

We can make all kind of closed-open $(a \leq x < b)$ or open-closed $(a < x \leq b)$ intervals, but there is no point in giving them a name.

§2.2 Inequalities

Real numbers and all its subsets have a common property. They're ordered under the $<, >, =$ relations. Thus, for any two real numbers, we can always compare them and have the trichotomy :

Claim 2.2.1 — For $a, b \in \mathbb{R}$ we will always have,

1. $a < b$,
2. $a = b$, or
3. $a > b$.

This why inequalities are so useful for reals! We can always form inequalities for equations. Now, let us propose the following properties of inequalities.

Proposition 2.2.2 — For $a, b \in \mathbb{R}$,

1. $a < b \iff a \pm k < b \pm k$
2. $a < b, c < d \iff a + c < b + d$
3. $a < b, c < d \iff a - d < b - c$
4. $a < b, k > 0 \implies ka < kb$
5. $a < b, k < 0 \implies ka > kb$
6. $a < b, b < c \implies a < c$
7. $0 < a < b, r > 0 \implies a^r < b^r$
8. $0 < a < b, r < 0 \implies a^r > b^r$
9. $a > 0 \iff a + \dfrac{1}{a} \geq 2$
10. $a < 0 \iff a + \dfrac{1}{a} \leq 2$

§2.2.1 Inequalities concerning squares

One thing of note to consider is how do the inequalities of x from x^2 follow? Consider $a \leq x \leq b$. You might expect that for $a \leq x \leq b$, $a^2 \leq x \leq b^2$. But that is simply incorrect! What we instead get is that x^2 lies between 0 and $\max(a^2, b^2)$. Here, max represents that x^2 lies between 0 and whichever of the a^2, b^2 is greater.

In-fact, if $a \leq x \leq b \iff a^2 \leq x^2 \leq b^2$, then a, b and hence x must be ≥ 0!

Also, if $x^2 \leq a^2$, then $x \in [-a, a]$
And, if $x^2 \geq a^2$, then $x \in (-\infty, -a] \cup [a, \infty)$

Example 2.2.3
For $-5 \leq x \leq 4$, we have $0 \leq x^2 \leq \max(-5^2, 4^2)$. Therefore, $0 \leq x^2 \leq 25$.

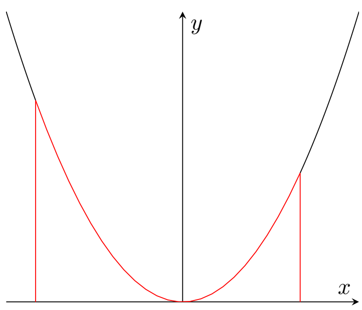

Figure 2.2: x^2 for $-5 \leq x \leq 4$

Part II

Calculus

CHAPTER 3

Functions and Relations

§3.1 Relation

A relation between two sets A, B is the subset of their *cartesian product* $A \times B$.

Cartesian Product §1.2.6

Since, a cartesian product between A and B is also a set of the ordered pairs, so is a relation R,

$$R \subseteq \{(a, b) \mid a \in A, b \in B\}$$

There need not be any specific relation between the two elements. But often, it is useful to talk about relations which employ a specific relationship between the two elements. Hence, we can think of a relation as somehow connecting the two elements.

A relation on a set A is simply the subset of $A \times A$.

Example 3.1.1

Consider sets $A = \{2, 4, 6, 8, \dots\}$ and $B = \{1, 3, 5, 7, 9, \dots\}$. Let R be the relation such that,

$$R = \{(a, b) \mid a, b \text{ are divisible by 3 and}, a \in A, b \in B\}$$

This relation is simply the set of all multiples of 3.

Example 3.1.2

Let $A = \{1, 2, 8\}$, $B = \{3, 5, 6\}$. Let R be the relation,

$$R = \{(a, b) \mid a > b, a \in A, b \in B\}$$

Then R contains the ordered pairs $(8, 3)$, $(8, 5)$ and $(8, 6)$

We show that $(a, b) \in R$ by saying aRb.

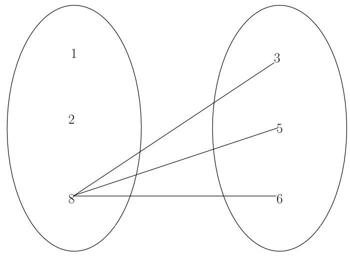

Figure 3.1: Pictorial representation of Example 3.1.2

§3.1.1 Domain

The domain of a relation $R \subseteq A \times B$ is the set of all first elements of the ordered pairs of the relation, that are in set A.

$$D_R = \{a \mid (a, b) \in R\}$$

§3.1.2 Range and Co-domain

Range

Range of R is the set of all second elements of the ordered pairs in the relation, that are in set B.

$$Range_R = \{b \mid (a, b) \in R\}$$

Co-domain

You may note that $Range_R \subsetneq Co - domain_R$

Co-domain of $R \subseteq A \times B$ is the set B itself.

§3.2 Types Of Relations

Although it might seem like discussing the type of relations would be as trivial as sets, there are actually a couple really important type of relations. I'll mention the ones we really need to know about.

§3.2.1 Trivial Types

Mainly, these are trivial in the way that they're of very low importance to us. The ones that are useful later, have been especially marked.

Empty Relation

The relation R is an empty relation if it does not contain any elements. $R = \varnothing \subsetneq A \times B$.

Universal Relation

Yes, it is the relation that contains all the elements. A universal relation R on A is $R = A \times A$.

Inverse Relation

Let R be the relation from A to B. Then R^{-1} is the relation from B to A such that,

$$R^{-1} = \{(b, a) \mid (a, b) \in R\}$$

Therefore, we may have $(a, b) \in R \iff (b, a) \in R^{-1}$.

Identity Relation

The identity relation I_A on A is defined as

$$I_A = \{(a, a) \mid a \in A\}$$

§3.2.2 Equivalence Relation

Reflexive relation

A relation R is a reflexive relation on A, if for every $a \in A$, $(a, a) \in R$.

All identity relations are reflexive relations but all reflexive relations are not identity relations. See Example 3.2.1.

Example 3.2.1

The relation $I_A = \{(1,1),(2,2),(3,3)\}$ is an identity relation on the set $A = \{1,2,3\}$. However, the relation $R = \{(1,1),(2,3)(3,2),(2,2),(3,3)\}$ is not an identity relation on A, but rather a reflexive relation. But $I_A \subsetneq R$.

Here, R is a reflexive relation but not an identity relation.

Symmetric Relation

A relation R on A is symmetric if $(a, b) \in R \iff (b, a) \in R$.

Example 3.2.2

Let R be the relation on $\mathbb{R}$ such that

$$R = \{(x, y) \mid x + y = 5\}$$

This relation is symmetric, because suppose any two reals x and $y \in R$. Then, $x + y = 5$. But since addition is commutative, $x + y = y + x = 5$. Thus, $(x, y) \in R \iff (y, x) \in R$.

Transitive Relation

A relation R on A is transitive if $(a, b) \in R$ and $(b, c) \in R \implies (a, c) \in R$.

Example 3.2.3

The relation $<$ is a transitive relation on $\mathbb{R}$. For instance, if $a < b$, $b < c$, then $a < c$.

Equivalence Relation

An equivalence relation on A is a relation that is transitive, reflexive, and symmetric.

If you haven't already noticed, I have deliberately chosen examples that are non conventional. I have done it in order to showcase that relations can be anything!

Example 3.2.4

The relation $=$ is an equivalence relation on $\mathbb{R}$. It is symmetric as $a = b \iff b = a$. It is reflexive as $a = a$. It is transitive as $a = b, b = c \implies a = c$.

Part III

Appendices

APPENDIX A

Logic and Proofs

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