A Math Notebook

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PROOFS 0

Motivation: The basis of mathematics relies on proofs.

Proofs are necessary for the study of higher mathematics. The notes for proofs are almost entirely based on [Vel06]. Mathematics itself is as a subject is based on *deductive* logic. We start with a set of axioms and end with theorem, conjectures, theorems, lemmas, etc.

The language of proofs is therefore necessary for the study of higher mathematics, and thus this chapter is included at the start instead of being chucked away at the distance.

PART A

Calculus

CHAPTER

REAL NUMBERS 1

The real numbers form the basis of all of the calculus we are going to study. Later, we'll look at expansions of them. To formulate a *deductive* system, we must define real numbers and build up everything over them. Well, in reality real numbers are arbitrary quantities that follow a certain set of axioms, every mathematical object that satisfies them is a real number.

1.1 Real Number Axioms

The axioms of real numbers can be divided broadly into three different types,

- The Field axioms
- The Order axioms
- The Continuity axiom

1.1.1 Field Axioms

The field axioms are something like this. We consider the set of reals, $\mathbb{R}$ along with the operations $(+, \cdot)$. Broadly, the *field* is $(\mathbb{R}, +, \cdot)$. I'll define what a field is after calculus. These operations are such that for every $x, y \in \mathbb{R}$, we will have $x + y, xy \in \mathbb{R}$. These sum and product are unique, which means we always have one and only have one $x + y, xy$ for $x, y \in \mathbb{R}$. Then,

A1. *Commutativity*, $x + y = y + x$ and $xy = yx$.

A2. *Associativity*, $x + (y + z) = (x + y) + z$, and $x(yz) = (xy)z$.

A3. *Distributivity*, $x(y + z) = xy + xz$.

A4. *Identity*. There exist two *distinct* identity elements, 0 and 1 such that $x + 0 = x$ and $x \cdot 1 = x$.

A5. *Inverses*, for $\forall x \in \mathbb{R} \setminus \{0\}$ there exists a y such that $xy = 1$ and $\forall x \in \mathbb{R}$, there exists a $z \in \mathbb{R}$, such that $x + z = 0$. These are called the reciprocal and negative of x and are denoted as $1/x$ (x^{-1}) and $-x$.

It is possible to formulate subtraction and division from these axioms, if $x + a = b$, then it is denoted as $b - a$. And if $xa = b$, it is denoted as ba^{-1}.

These are also unique. It is not difficult to show that, by the uniqueness of inverses.

It is also possible to treat subtraction as a shorthand for addition, $a + (-b) = a - b$ using inverses. Division can be treated similarly.

These properties allow us to form laws of addition and multiplication — such as cancellation. We can form an outline of a proof of cancellation as,

Proof. We need to show that $a + b = a + c \iff b = c$. This can be done simply by adding $-a$ on both sides. Since addition is unique, and both numbers are equal, so must be their addition with another number. $\square$

A similar procedure can be done for multiplication.

We see that this can also be used to prove that the product of any two negative numbers is positive. Let us first prove that $(-a)b = -ab$. This is rather easy as well, and follows from distributivity,

$$\begin{aligned} (-a)b + ab &= b[(-a) + a] \\ (-a)b + ab &= 0 \\ (-a)b + ab - ab &= -ab \\ (-a)b &= -ab \end{aligned}$$

A similar proof can be used to show that $(-a)(-b) = ab$, as

$$\begin{aligned} (-a)(-b) + (-a)b &= -a[(-b) + b] \\ (-a)(-b) - ab &= 0 \\ (-a)(-b) - ab + ab &= ab \\ (-a)(-b) &= ab \end{aligned}$$

Here we use our previous result to prove this one. Now it is obvious that the product of any two negatives will be positive — however, let us first discuss what exactly “positive” and negative mean.

1.1.2 Order Axioms

The real numbers are ordered, we now present some axioms to look at what we mean by order.

A1. There exists a subset of real numbers, $\mathbb{R}^+$ such that $\forall x \in \mathbb{R} \setminus \{0\}$, either $x \in \mathbb{R}^+$ or $-x \in \mathbb{R}^+$.

A2. If $x, y \in \mathbb{R}^+$, both xy and $x + y$ are also in $\mathbb{R}^+$.

A3. 0 is not in $\mathbb{R}^+$.

We say that a number x is positive if it is in $\mathbb{R}^+$. We also know define some symbols,

$$x > y \text{ means that } x - y \text{ is in } \mathbb{R}^+$$
$$x < y \text{ means that } y - x \text{ is in } \mathbb{R}^+$$

The symbols $\geqslant$ and $\leqslant$ just imply that "..." is either greater/lesser or equal to "...". Note that weird as it may appear, statements like $6 \geqslant 6$ are completely valid. The real use of these symbols lie in terms of dealing with variables, or general statements.

Consider the following proposition,

Proposition 1.1 (Trichotomy). *If we have $x, y \in \mathbb{R}$, the one of the following must hold,*

1. $x < y$
2. $x > y$
3. $x = y$

This can be inferred rather directly from the order axioms. Clearly, by the order axioms we have that either $a = 0$, $a \in \mathbb{R}^+$ or $-a \in \mathbb{R}^+$. Replace a by $x - y$ and we're done.

This also forms the basis of the idea of the absolute value, which is,

$$|a| = \begin{cases} a & a \geqslant 0 \\ -a & a \leqslant 0 \end{cases} \tag{1.1}$$

We see that this clearly a positive number. It is also referred to as the euclidean length of a number from 0 on the number line. Let us consider some serious properties now,

Consider the following two properties regarding the absolute value,

Theorem 1.2. *For $x, a \in \mathbb{R}$, $|x| \leqslant a$ if and only if $-a \leqslant x \leqslant a$.*

Proof. Let us suppose $|x| \leqslant a$. Then clearly, $-|x| \geqslant -a$. But $|x|$ can only take two values, and is obviously positive. Thus, $-|x| \leqslant x \leqslant |x|$. Thus, we have that, $-a \leqslant -|x| \leqslant x \leqslant |x| \leqslant a$ which proves one direction.

For the other direction, suppose $-a \leqslant x \leqslant a$. Then, if $x \geqslant 0$, $|x| = x \leqslant a$. If $x < 0$, $|x| = -x \leqslant a$ since $x \geqslant -a \implies -x \leqslant a$. In both cases, $|x| \leqslant a$. □

$$\int dx = \frac{\partial f}{\partial y} = \partial f$$

PART B

Linear Algebra

CHAPTER

VECTOR SPACES 2

This part of linear algebra is almost entirely based on [Axl24].

2.1 Lists

In most of the chapters, I'll adopt the convention that $\mathbb{F}$ is a field. Some specific results require $\mathbb{F} = \mathbb{C}$ or $\mathbb{F} = \mathbb{R}$ but they'll be explicitly mentioned.

Elements of $\mathbb{F}$ are called *scalars*. We call these scalars to form a distinction with vectors, which will be defined soon.

Cartesian products of $\mathbb{F}$ with itself n times are denoted as $\mathbb{F}^n$. For something like $\mathbb{F}^3$ this is equivalent to saying,

$$\mathbb{F}^3 = \{(x, y, z) \mid x, y, z \in \mathbb{F}\}$$

It is the set of all ordered triples of the elements of $\mathbb{F}$. If we have $\mathbb{F} = \mathbb{R}$, for instance, $\mathbb{R}^2$ can be though to be a plane while $\mathbb{R}^3$ can be thought to be ordinary space.

To generalise these results to n dimensions, we use the concept of n-tuples or *lists*.

Definition 2.1

Lists

For a non-negative integer, n, an n-tuple or a list of length n is an ordered collection of n elements.

A list $(a_1, a_2, \ldots, a_m) = (b_1, b_2, \ldots, b_n)$ if and only if $n = m$ and $a_1 = b_1, a_2 = b_2, \ldots, a_m = b_n$.

Thus, two lists are equal iff their length is equal and they have the same elements in the same order. A list has a finite number of elements, thus, even though $(a_1, a_2, \ldots, a_m)$ is a list, $(a_1, a_2, \ldots)$ is not. The list of length 0 is denoted by $()$.

We denote lists surrounded by paratheses and their elements separated by commas.

Example 2.1.1 (Lists vs sets). The lists $(3, 5)$ and $(5, 3)$ are not equal but $\{3, 5\} = \{5, 3\}$. The list $(4, 4, 4) \neq (4, 4) \neq (4)$ but $\{4, 4, 4\} = \{4, 4\} = \{4\}$.

2.2 Higher Products

Generalising the nth cartesian product of $\mathbb{F}$ with itself,

$$\mathbb{F}^n = \underbrace{\mathbb{F} \times \mathbb{F} \times \cdots \times \mathbb{F}}_{n\text{-times}}$$

We may say,

Definition 2.2

$\mathbb{F}^n$ is the set of all lists of length n such that the elements of the list are in $\mathbb{F}$,

$$\mathbb{F}^n = \{(a_1, a_2, \ldots, a_n) \mid a_1, a_2, \ldots, a_n \in \mathbb{F}\}$$

Where $n \in \mathbb{Z}_{\geqslant 0}$.

Definition 2.3

Co-ordinate

If $(a_1, a_2, \ldots, a_n) \in \mathbb{F}^n$ and $1 \leqslant i \leqslant n$, x_i is called the ith co-ordinate of $(a_1, a_2, \ldots, a_n)$.

While $\mathbb{R}^2$ can be visualised as a plane, and $\mathbb{R}^3$ as space, it is not possible to visualise them for $n \geqslant 4$. Similarly, $\mathbb{C}^1$ can be thought of as a plane, but cannot be visualised for $n \geqslant 2$.

However, we can perform algebraic operations on the lists of some arbitrary length n which may even be very great.

Definition 2.4

Addition in $\mathbb{F}^n$

Addition in $\mathbb{F}^n$ is defined as,

$$(x_1, x_2, \ldots, x_n) + (y_1, y_2, \ldots, y_n) = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$$

To avoid the cumbersome notation of writing out $(x_1, x_2, \ldots, x_n)$ we will adopt the notation that $x = (x_1, x_2, \ldots, x_n)$.

Proposition 2.5. $x + y = y + x$

Proof. $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n)$. Thus,

$$\begin{aligned} x + y &= (x_1, x_2, \ldots, x_n) + (y_1, y_2, \ldots, y_n) \\ &= (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n) \\ &= (y_1 + x_1, y_2 + x_2, \ldots, x_n + y_n) \\ &= y + x \end{aligned}$$

□

The proof is based on the commutativity of reals and definition 2.4. The elements of $\mathbb{F}^2$, x, y can be thought of as points or vectors. Disregarding the axes, vectors may independently be thought as some objects.

We define two more things in $\mathbb{F}^n$,

Additive Inverse in $\mathbb{F}^n$ — **Definition 2.6**

The additive inverse of $x \in \mathbb{F}^n$ is $-x \in \mathbb{F}^n$ where,

$$x + (-x) = 0$$

And if $x = (x_1, x_2, \ldots, x_n)$,

$$-x = (-x_1, -x_2, \ldots, -x_n)$$

The additive inverse of x in $\mathbb{R}^2$ is the vector of equal length but opposite direction.

The final operation is *scalar multiplication*

Scalar multiplication — **Definition 2.7**

For a scalar λ, and $x \in \mathbb{R}^n$, their product is defined as

$$\lambda\,(x_1, x_2, \ldots, x_n) = (\lambda x_1, \lambda x_2, \ldots, \lambda x_n)$$

2.3 Vector Space

The definition of a vector space comes off from the properties we have kind of discussed above for $\mathbb{R}^n$. Defining our operations for any vector space,

Operators on a vector space — **Definition 2.8**

Addition *on V is a function $+ : V^2 \to V$ where $\mathbf{v}+\mathbf{w} \in V$ for any $\mathbf{v}, \mathbf{w} \in V$.*

Scalar multiplication *on V is a function $\bullet : V \times \mathbb{R} \to V$ where $\lambda\mathbf{v} \in V$ for any $\mathbf{v} \in V$ and $\lambda \in \mathbb{R}$.*

Now, let us formally define V.

Definition 2.9

Vector Space

A set V along with the operations of addition and scalar multiplication is a vector space if the following properties hold for $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and $a, b \in \mathbb{R}$.

A1. *Commutativity,* $\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$.

A2. *Associativity,* $(\mathbf{u} + \mathbf{w}) + \mathbf{v} = \mathbf{u} + (\mathbf{w} + \mathbf{v})$ *and* $a(b6v) = (ab)\mathbf{v}$.

A3. *Additive Identity, there an element* $0 \in V$ *such that* $\mathbf{v} + 0 = \mathbf{v}$.

A4. *Additive Inverse, there exist an element* $-\mathbf{v} \in V$ *for each* $\mathbf{v}$ *such that* $\mathbf{v} + (-\mathbf{v}) = 0$.

A5. *Multiplicative Identity,* $1\mathbf{v} = \mathbf{v}$

A6. *Distributive Property,* $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$ *and* $\mathbf{v}(a + b) = \mathbf{v}a + \mathbf{v}b$.

Example 2.3.1. The set of functions, $\mathbb{F}^S$, such that

$$\mathbb{F}^S = \{f \mid f : S \to \mathbb{F}\}$$

is a vector space over $\mathbb{F}$ with addition being defined as,

$$(f + g)(x) = f(x) + g(x)$$

And scalar multiplication as,

$$c \cdot f = c \cdot f(x)$$

The properties are easy to verify. Note that $f(x), g(x) \in \mathbb{F}^n$, so their sum is commutative, since addition is commutative in $\mathbb{F}^n$. Similarly, we can use the properties of $\mathbb{F}^n$ to prove that $\mathbb{F}^S$ is really a vector space.

We can also use this general notation for the set of functions, for $\mathbb{F}^n$. We can think of $\mathbb{F}^n$ as $\mathbb{F}^{\{1,2,\dots\}}$. So, it is the set of functions from $\{1, 2, \dots\}$. We can think of each co-ordinate, x_i corresponding to the value of the function at that point, $f(i)$.

Let us prove some general results regarding vector spaces.

Proposition 2.10. *The additive identity is unique.*

Proof. This is easy enough to show. Let $\mathbf{0}, \mathbf{0}'$ be two additive identities. Then,

$$\mathbf{0} = \mathbf{0} + \mathbf{0}' = \mathbf{0}'$$

And we're done. □

Proposition 2.11. *Every* $\mathbf{v} \in V$ *has a unique additive inverse.*

Proof. Since the existence is guaranteed by the axioms, we focus on uniqueness. Let $\mathbf{w}$ and $\mathbf{u}$ be two additive inverses of $\mathbf{v}$. Then,

$$\mathbf{w} = \mathbf{w} + \mathbf{0} = \mathbf{w} + (\mathbf{v} + \mathbf{u}) = (\mathbf{w} + \mathbf{v}) + \mathbf{u} = \mathbf{0} + \mathbf{u} = \mathbf{u}$$

□

Since additive inverses are unique, we can use particular symbol to denote them. We denote the additive inverse of $\mathbf{v}$ by $-\mathbf{v}$.

Proposition 2.12. $0\mathbf{v} = \mathbf{0}$

Proof. Be careful by what we mean by the two zero's. The proposition is that the product of the scalar zero times the vector $\mathbf{v}$ is the vector zero.

Now,

$$0\mathbf{v} = (0 + 0)\mathbf{v} = 0\mathbf{v} + 0\mathbf{v}$$

Adding the additive inverse, $-0\mathbf{v}$ on both sides, we get $0\mathbf{v} = \mathbf{0}$. □

Exercises

1. Show that $a\mathbf{0} = \mathbf{0}$ for all $a \in \mathbb{F}$. [1]

2. Prove that $-(-\mathbf{v}) = \mathbf{v}$. [1]

PART C

Appendices

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