Lemma 4.1. *Let P be any distribution and consider a selection function g_θ with a threshold θ whose coverage is $c(\theta, P)$. Let $0 < \delta < 1$ be given and let $\hat{c}(\theta, S_m)$ be the empirical coverage w.r.t. the set S_m, sampled i.i.d. from P. Let $b^*(m, m \cdot \hat{c}(\theta, S_m), \delta)$ be the solution of the following equation:*

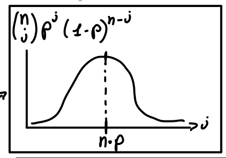
$$\underset{b}{arg\,min}\left(\sum_{j=0}^{m\cdot\hat{c}(\theta,S_m)}\binom{m}{j}b^j(1-b)^{m-j} \leq 1-\delta\right). \quad (4)$$

Then,

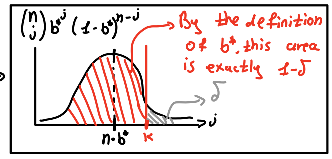
$$\boldsymbol{Pr}_{S_m}\{c(\theta,P) < b^*(m, m\cdot\hat{c}(\theta,S_m),\delta)\} < \delta. \quad (5)$$

Proof. We aim to establish an upper bound, b^*, for the coverage, $c(\theta, P)$, that holds with a probability of at most δ. The coverage $c(\theta, P)$ is equivalent to the probability that $g_\theta(x) = 1$ for a sample $x \sim P$, we will define this event as a 'success'. Define:

$$\text{Bin}(n,k,p) \triangleq \sum_{j=0}^{k}\boxed{\binom{n}{j}p^j(1-p)^{n-j}}, \quad (1)$$

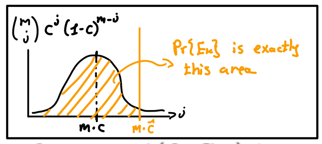


$$\boxed{\begin{aligned} b^* &\triangleq \overline{\text{Bin}}(n,k,\delta) \\ &\triangleq \underset{b}{\arg\min}\left(\text{Bin}(n,k,b) \leq 1-\delta\right). \end{aligned}} \quad (2)$$



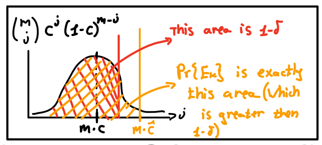
Note that given n, k, the function $\text{Bin}(n, k, p)$ is a monotonically decreasing function in p. Therefore, the solution b^* of Eq. 2 exists as a result of the intermediate value theorem.
Let E_k be the event that at most $k = m \cdot \hat{c}(\theta, S_m)$ samples x satisfy $g_\theta(x) = 1$ (or at most k 'successes'), when considering $c(\theta, P)$ to be the probability of a single success. Thus,

$$\boxed{Pr\{E_k\} = \text{Bin}(m, m\cdot\hat{c}(\theta,S_m), c(\theta,P)).} \xrightarrow[c\equiv c(\theta,P)]{\hat{c}\equiv\hat{c}(\theta,S_m)} \quad (3)$$



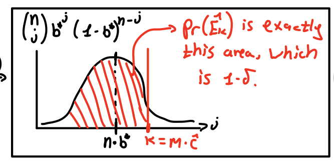
By definition, for every number between $0 < \tilde{\delta} < 1$, there is a probability of at most $\tilde{\delta}$, that $k = m \cdot \hat{c}(\theta, S_m)$ 'successes' would fall into the right tail of size $\tilde{\delta}$ of the binomial. If this happens, then we get that the probability for at most $k = m \cdot \hat{c}(\theta, S_m)$ 'successes' is greater than $1 - \tilde{\delta}$. We can apply this claim using the value $\tilde{\delta} = \delta$ introduced in the lemma, using $c(\theta, P)$ as the probability for a single success. Mathematically,

$$\boxed{Pr\{E_k\} > 1-\delta.} \longrightarrow \quad (4)$$



Let $\hat{E}_k$ be the event that at most $k = m \cdot \hat{c}(\theta, S_m)$ samples x satisfy $g_\theta(x) = 1$ (or at most k 'successes'), when the probability for a single success is b^*. It holds (2) that,

$$\boxed{Pr\{\hat{E}_k\} = \text{Bin}(m, m\cdot\hat{c}(\theta,S_m), b^*) = 1-\delta.} \xrightarrow{\hat{c}\equiv\hat{c}(\theta,S_m)} \quad (5)$$

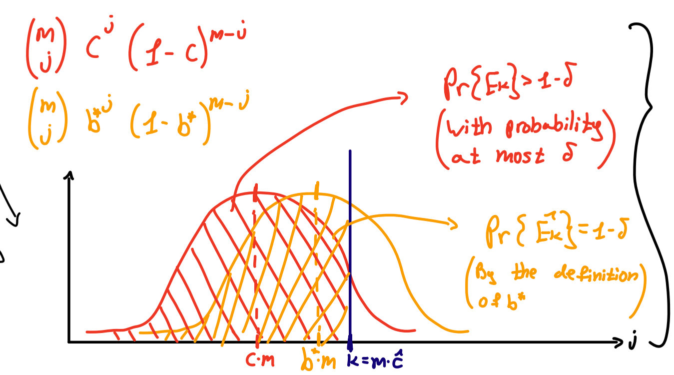


Therefore,

$$Pr\{E_k\} > Pr\{\hat{E}_k\}. \quad (6)$$

Consider the following statement,
let p_1, p_2 be two different probabilities for a single success. If the probability of at most k successes (out of n attempts) is more likely when considering p_1 as the probability of a single success, rather than p_2, we can conclude that $p_1 < p_2$.

Therefore, since the event of E_k occurs with higher probability then the event $\hat{E}_k$, see Eq. (6) , we can conclude that $c(\theta, P) < b^*$. This deduction is only true if we assume that $Pr\{E_k\} > 1 - \delta$, and this assumption is true with probability at most δ. Hence, we get that with probability at most δ, it holds that $c(\theta, P) < b^*$. □



$Pr\{E_k\} > Pr\{\hat{E}_k\} \Rightarrow c < b^*$