The coverage $c(\theta, P)$ is equivalent to the probability that $g_\theta(x) = 1$ for a sample $x \sim P$, we will define this event as a 'success'. Define:

$$\mathrm{Bin}(n, k, p) \triangleq \sum_{j=0}^{k} \binom{n}{j} p^j (1-p)^{n-j}, \tag{6}$$

$$\begin{aligned} b^* &\triangleq \overline{\mathrm{Bin}}(n, k, \delta) \\ &\triangleq \arg\min_b \left(\mathrm{Bin}(n, k, b) \leq 1 - \delta\right). \end{aligned} \tag{7}$$

Let E_k be the event that at most $k = m \cdot \hat{c}(\theta, S_m)$ samples x satisfy $g_\theta(x) = 1$ (or at most k 'successes'). Thus,

$$Pr\{E_k\} = \mathrm{Bin}(m, m \cdot \hat{c}(\theta, S_m), c(\theta, P)). \tag{8}$$

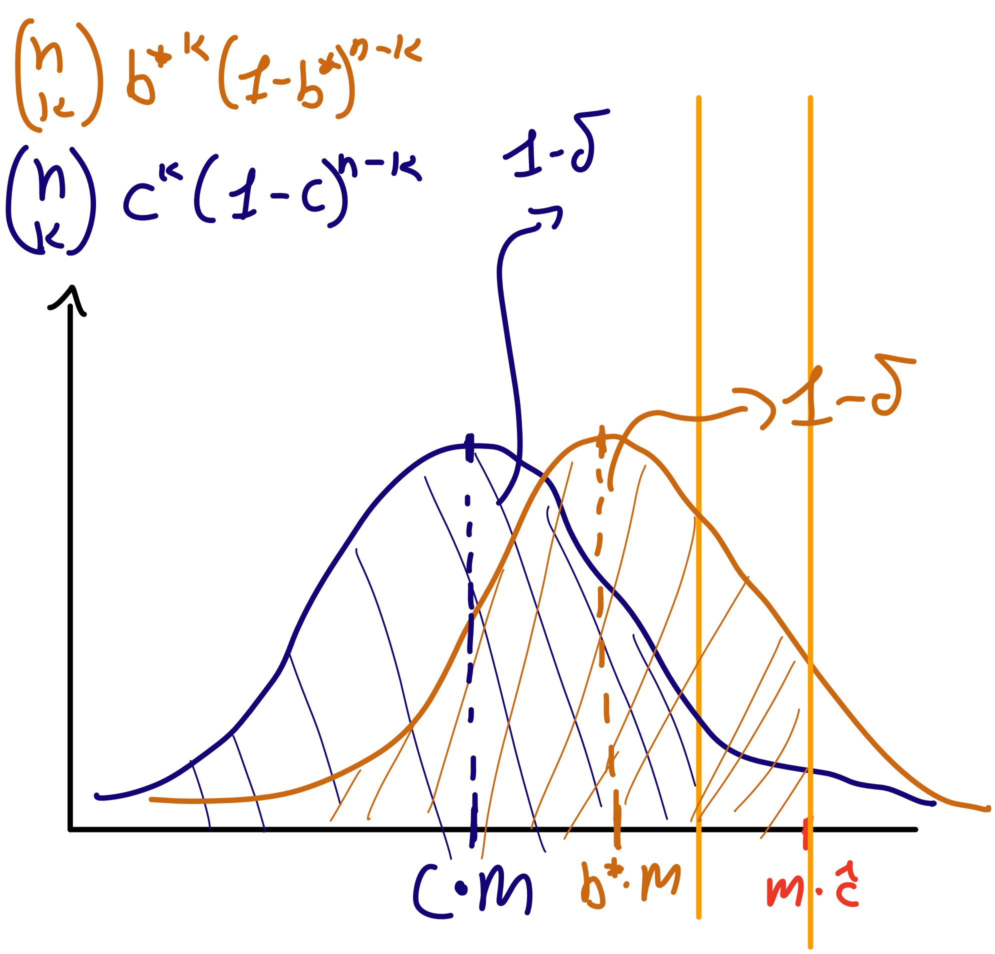
There is a probability of at most δ, that $k = m \cdot \hat{c}(\theta, S_m)$ 'successes' would fall into the right tail of the binomial of size δ (where the probability of a single success is $c(\theta, P)$). If this happens, then we get that the probability for at most $k = m \cdot \hat{c}(\theta, S_m)$ 'successes' is greater than $1 - \delta$. Mathematically,

$$Pr\{E_k\} > 1 - \delta. \tag{9}$$

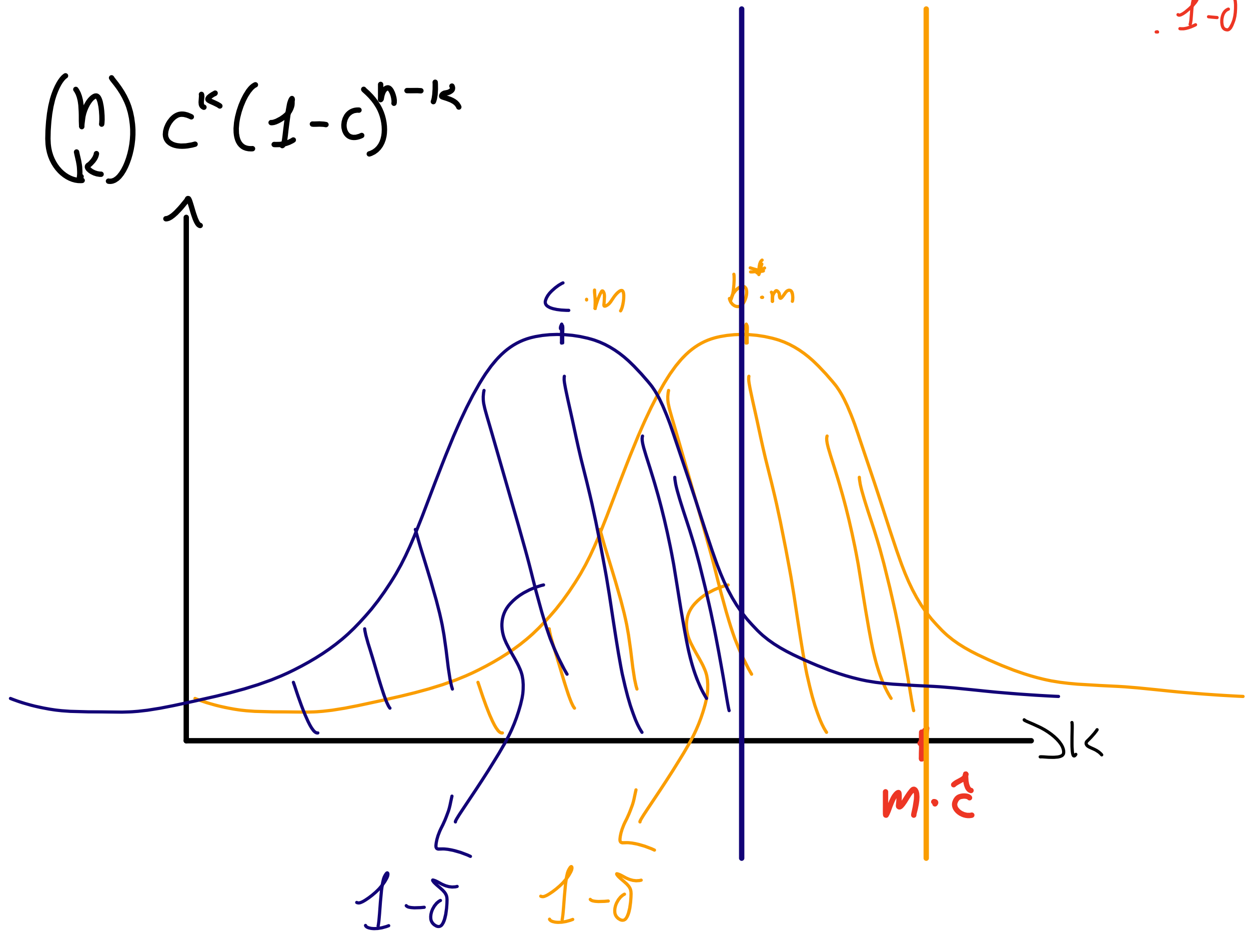
When considering b^* to be the probability of a single success, it holds (7) that

$$\mathrm{Bin}(m, m \cdot \hat{c}(\theta, S_m), b^*) = 1 - \delta. \tag{10}$$

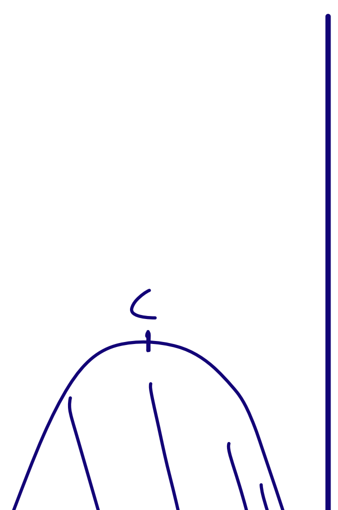
Since E_k is the event of at most $k = m \cdot \hat{c}(\theta, S_m)$ 'successes', where a probability for each success is $c(\theta, P)$, and it is more likely than the event of at most k successes where the probability for a single success is b^*, we can conclude that $c(\theta, P) < b^*$. This deduction is only true if we assume that $Pr\{E_k\} > 1 - \delta$, and this assumption is true with probability at most δ. Hence, we get that with probability at most δ, it holds that $c(\theta, P) < b^*$.



$m\cdot\hat{c}$ קרוב. b^* הוא כזה שהסיכוי $\underbrace{\text{שלכל היותר } m\cdot\hat{c}}_{\text{(שטעויות בקבוצה)}}$ הצלחות הוא לכל היותר $1-\delta$.



אם $c<b^*$ אז הסיכוי $\overbrace{\text{לכל היותר } m\cdot\hat{c}}^{\text{שטעויות בקבוצה}}$ ר
גדול מ- $1-\delta$. לכן הסיכוי שיהיה $\underbrace{\text{יותר מ- } m\cdot\hat{c}}_{\text{שגויות בקבוצה}}$ קטן מ- δ.



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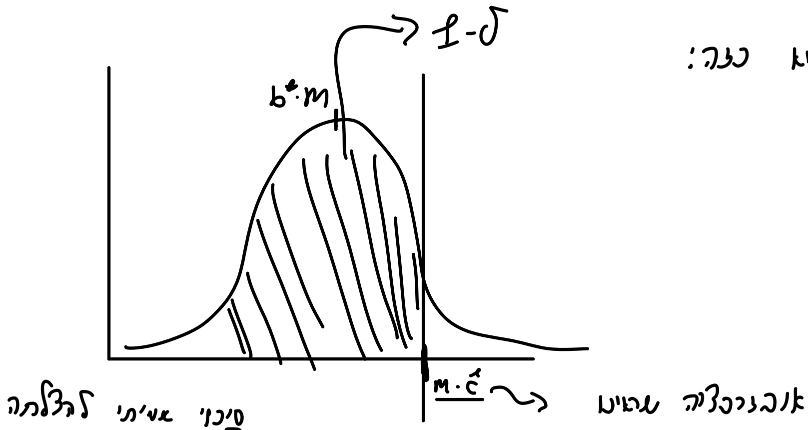
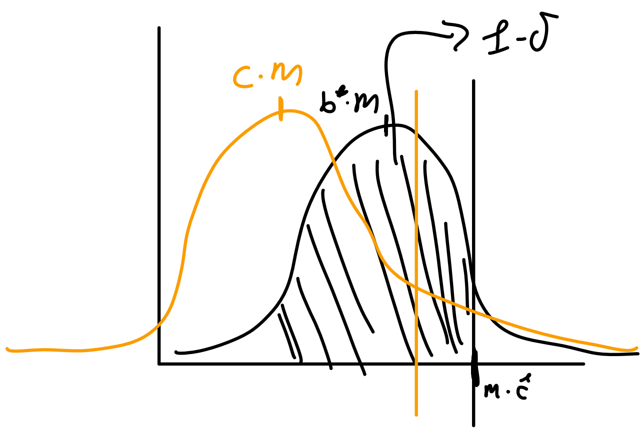
b^* הוא כזה:



אם ההתפלגות האמיתית הוא משמאל לזו, אז $c<b^*$



אם אכן $c<b^*$, אז $m\cdot\hat{c}$ נפל בזנב δ של ההתפלגות האמיתית.
הסיכוי שהוא נפל בזנב הוא δ. לכן כולל ש- $m\cdot\hat{c}$ הוא מאורע שקרה,
הסיכוי שההתפלגות האמיתית מקיימת $c<b^*$ היא δ.

אם הסיכוי שנדחה את האלטרנטיבה $\hat{c}\cdot m$ קטן מ-δ.
לכן, בהינתן שדחינו את האלטרנטיבה $\hat{c}\cdot m$, הסיכוי ש- $c<b^*$ קטן מ-δ.

נרצה להראות חסם b^* עליון ל-$c(\theta,\ell)$

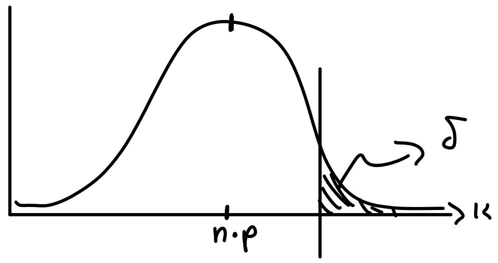
Lemma 4.1. *Let P be any distribution and consider a selection function g_θ with a threshold θ whose coverage is $c(\theta, P)$. Let $0 < \delta < 1$ be given and let $\hat{c}(\theta, S_m)$ be the empirical coverage w.r.t. the set S_m, sampled i.i.d. from P. Let $b^*(m, m \cdot \hat{c}(\theta, S_m), \delta)$ be the solution of the following equation:*

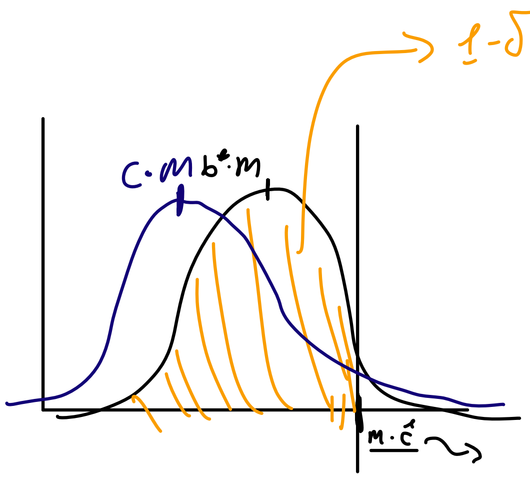
$$\underset{b}{arg\,min}\left(\sum_{j=0}^{m\cdot\hat{c}(\theta,S_m)}\binom{m}{j}b^j(1-b)^{m-j} \le 1-\delta\right). \quad (4)$$

Then,

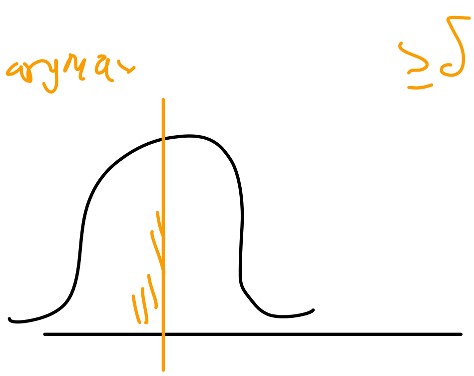
$$\boldsymbol{Pr}_{S_m}\{c(\theta, P) < b^*(m, m \cdot \hat{c}(\theta, S_m), \delta)\} < \delta. \quad (5)$$

$$\binom{n}{k} p^k (1-p)^{n-k} = f(k,p)$$





אובזרבציה שראינו



Proof. The coverage $c(\theta, P)$ is equivalent to the probability that $g_\theta(x) = 1$ for a sample $x \sim P$, we will define this event as a 'success'. Define:

$$\text{Bin}(n, k, p) \triangleq \sum_{j=0}^{k} \binom{n}{j} p^j (1-p)^{n-j}, \quad (1)$$

$$\begin{aligned} b^* &\triangleq \overline{\text{Bin}}(n, k, \delta) \\ &\triangleq \underset{b}{\arg\min}\,(\text{Bin}(n, k, b) \le 1 - \delta). \end{aligned} \quad (2)$$

נכניס ל-(4) למה?

Let E_k be the event that at most $k = m \cdot \hat{c}(\theta, S_m)$ samples x satisfy $g_\theta(x) = 1$ (or at most k 'successes'), when considering $c(\theta, P)$ to be the probability of a single success. Thus,

להוסיף ציור

$$\underline{Pr\{E_k\} = \text{Bin}(m, m \cdot \hat{c}(\theta, S_m), c(\theta, P))}. \quad (3)$$

of size δ

להכניס להתחלה את ההגדרה?

By definition, there is a probability of at most δ, that $k = m \cdot \hat{c}(\theta, S_m)$ 'successes' would fall into the right tail of the binomial ~~of size δ~~ (where the probability of a single success is $c(\theta, P)$). If this happens, then we get that the probability for at most $k = m \cdot \hat{c}(\theta, S_m)$ 'successes' is greater than $1 - \delta$. Mathematically,

$$Pr\{E_k\} > 1 - \delta. \quad (4)$$

Let $\hat{E}_k$ be the event that at most $k = m \cdot \hat{c}(\theta, S_m)$ samples x satisfy $g_\theta(x) = 1$ (or at most k 'successes'), when the probability for a single success is b^*. It holds (2) that,

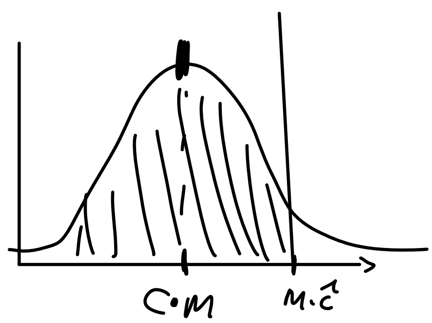
$$Pr\{\hat{E}_k\} = \text{Bin}(m, m \cdot \hat{c}(\theta, S_m), b^*) = 1 - \delta. \quad (5)$$

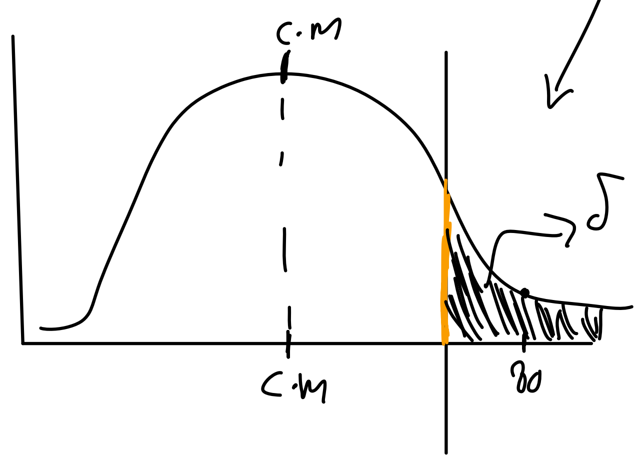
Therefore,

$$Pr\{E_k\} > Pr\{\hat{E}_k\}. \quad (6)$$

Consider the following statement,
let p_1, p_2 be two different probabilities for a single success. If the probability of at most k successes (out of n attempts) is more likely when considering p_1 as the probability of a single success, rather than p_2, we can conclude that $p_1 < p_2$.

Therefore, since the event of E_k occurs with higher probability then the event $\hat{E}_k$, see Eq. (6) , we can conclude that $c(\theta, P) < b^*$. This deduction is only true if we assume that $Pr\{E_k\} > 1 - \delta$, and this assumption is true with probability at most δ. Hence, we get that with probability at most δ, it holds that $c(\theta, P) < b^*$. □





Lemma 4.1. *Let P be any distribution and consider a selection function g_θ with a threshold θ whose coverage is $c(\theta, P)$. Let $0 < \delta < 1$ be given and let $\hat{c}(\theta, S_m)$ be the empirical coverage w.r.t. the set S_m, sampled i.i.d. from P. Let $b^*(m, m \cdot \hat{c}(\theta, S_m), \delta)$ be the solution of the following equation:*

$$\underset{b}{\arg\min} \left(\sum_{j=0}^{m \cdot \hat{c}(\theta, S_m)} \binom{m}{j} b^j (1-b)^{m-j} \le 1 - \delta \right). \quad (4)$$

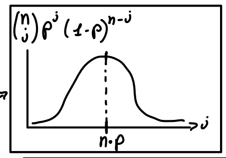
Then,

$$\boldsymbol{Pr}_{S_m}\{c(\theta, P) < b^*(m, m \cdot \hat{c}(\theta, S_m), \delta)\} < \delta. \quad (5)$$

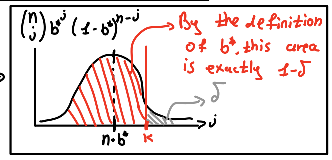
Proof. We aim to establish an upper bound, b^*, for the coverage, $c(\theta, P)$, that holds with a probability of at most δ. The coverage $c(\theta, P)$ is equivalent to the probability that $g_\theta(x) = 1$ for a sample $x \sim P$, we will define this event as a 'success'. Define:

non vuccus, since
Bin is amonotonically decreasing
function

$$\mathrm{Bin}(n, k, p) \triangleq \sum_{j=0}^{k} \boxed{\binom{n}{j} p^j (1-p)^{n-j}}, \quad (1)$$



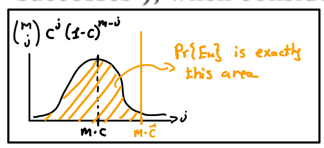
$$\boxed{\begin{aligned} b^* &\triangleq \overline{\mathrm{Bin}}(n, k, \delta) \\ &\triangleq \underset{b}{\arg\min}\,(\mathrm{Bin}(n, k, b) \le 1 - \delta)\,. \end{aligned}} \quad (2)$$



Let E_k be the event that at most $k = m \cdot \hat{c}(\theta, S_m)$ samples x satisfy $g_\theta(x) = 1$ (or at most k 'successes'), when considering $c(\theta, P)$ to be the probability of a single success. Thus,

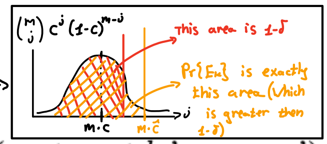
for every number between 0-1...
.. and also for delta.

$$\boxed{Pr\{E_k\} = \mathrm{Bin}(m, m \cdot \hat{c}(\theta, S_m), c(\theta, P)).} \xrightarrow[c \equiv c(\theta, P)]{\hat{c} \equiv \hat{c}(\theta, S_m)} \quad (3)$$



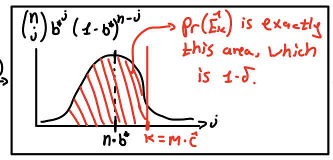
By definition, there is a probability of at most δ, that $k = m \cdot \hat{c}(\theta, S_m)$ 'successes' would fall into the right tail of size δ of the binomial (where the probability of a single success is $c(\theta, P)$). If this happens, then we get that the probability for at most $k = m \cdot \hat{c}(\theta, S_m)$ 'successes' is greater than $1 - \delta$. Mathematically,

$$\boxed{Pr\{E_k\} > 1 - \delta.} \quad (4)$$



Let $\hat{E}_k$ be the event that at most $k = m \cdot \hat{c}(\theta, S_m)$ samples x satisfy $g_\theta(x) = 1$ (or at most k 'successes'), when the probability for a single success is b^*. It holds (2) that,

$$\boxed{Pr\{\hat{E}_k\} = \mathrm{Bin}(m, m \cdot \hat{c}(\theta, S_m), b^*) = 1 - \delta.} \xrightarrow{\hat{c} \equiv \hat{c}(\theta, S_m)} \quad (5)$$



Therefore,

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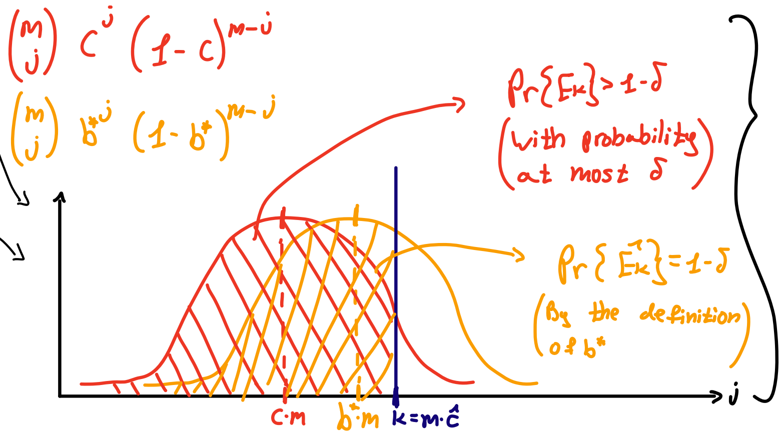

$Pr\{E_k\} > Pr\{\hat{E}_k\} \Rightarrow c < b^*$

תשובה שגויה, קל לשבר

Lemma 4.1. *Let P be any distribution and consider a selection function g_θ with a threshold θ whose coverage is $c(\theta, P)$. Let $0 < \delta < 1$ be given and let $\hat{c}(\theta, S_m)$ be the empirical coverage w.r.t. the set S_m, sampled i.i.d. from P. Let $b^*(m, m \cdot \hat{c}(\theta, S_m), \delta)$ be the solution of the following equation:*

$$\underset{b}{arg\,min}\left(\sum_{j=0}^{m\cdot\hat{c}(\theta,S_m)}\binom{m}{j}b^j(1-b)^{m-j} \leq 1-\delta\right). \quad (4)$$

Then,

$$\boldsymbol{Pr}_{S_m}\{c(\theta, P) < b^*(m, m \cdot \hat{c}(\theta, S_m), \delta)\} < \delta. \quad (5)$$

Proof. We aim to establish an upper bound, b^*, for the coverage, $c(\theta, P)$, that holds with a probability of at most δ. The coverage $c(\theta, P)$ is equivalent to the probability that $g_\theta(x) = 1$ for a sample $x \sim P$, we will define this event as a 'success'. Define:

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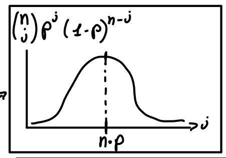
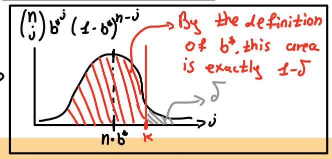


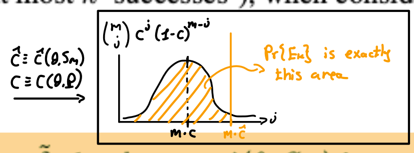
$$\boxed{\begin{aligned} b^* &\triangleq \overline{\mathrm{Bin}}(n, k, \delta) \\ &\triangleq \underset{b}{\arg\min}\left(\mathrm{Bin}(n, k, b) \leq 1 - \delta\right). \end{aligned}} \quad (2)$$



Note that given n, k, the function $\mathrm{Bin}(n, k, p)$ is a monotonically decreasing function in p. Therefore, the solution b^* of Eq. 2 exists as a result of the Intermediate value theorem.

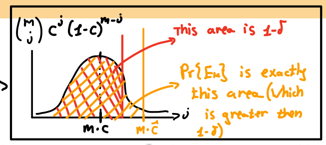
Let E_k be the event that at most $k = m \cdot \hat{c}(\theta, S_m)$ samples x satisfy $g_\theta(x) = 1$ (or at most k 'successes'), when considering $c(\theta, P)$ to be the probability of a single success. Thus,

$$\boxed{Pr\{E_k\} = \mathrm{Bin}(m, m \cdot \hat{c}(\theta, S_m), c(\theta, P)).} \quad (3)$$



By definition, for every number between $0 < \tilde{\delta} < 1$, there is a probability of at most $\tilde{\delta}$, that $k = m \cdot \hat{c}(\theta, S_m)$ 'successes' would fall into the right tail of size $\tilde{\delta}$ of the binomial. If this happens, then we get that the probability for at most $k = m \cdot \hat{c}(\theta, S_m)$ 'successes' is greater than $1 - \tilde{\delta}$. We can apply this claim using the value $\tilde{\delta} = \delta$ introduced in the lemma, using $c(\theta, P)$ as the probability for a single success. Mathematically,

$$\boxed{Pr\{E_k\} > 1 - \delta.} \quad (4)$$



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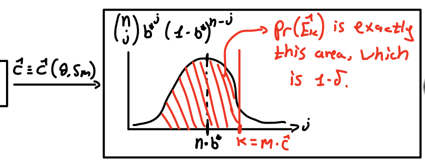

Therefore,

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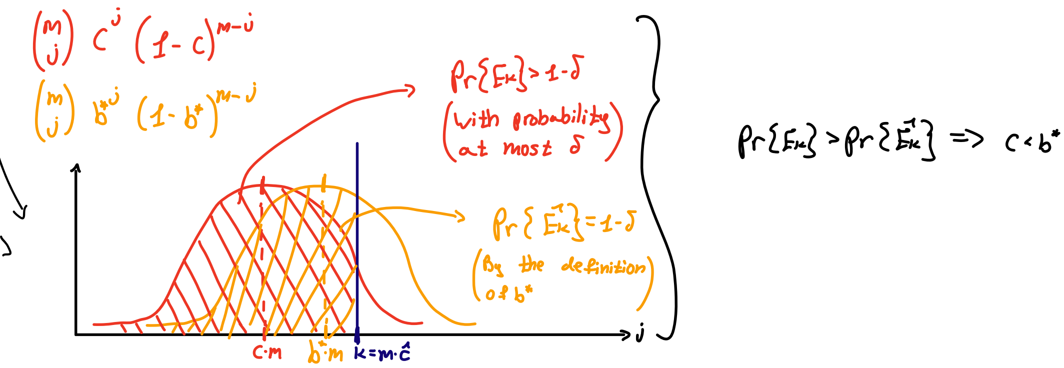

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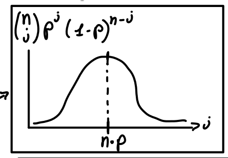

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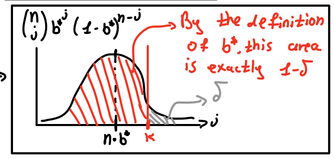

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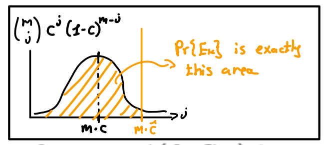
$$\boxed{Pr\{E_k\} = \mathrm{Bin}(m, m \cdot \hat{c}(\theta, S_m), c(\theta, P)).} \xrightarrow[c \equiv c(\theta,P)]{\hat{c} \equiv \hat{c}(\theta,S_m)} \quad (3)$$



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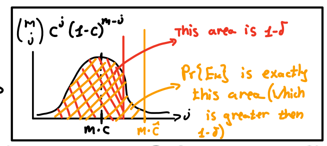
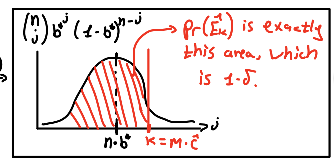
$$\boxed{Pr\{E_k\} > 1 - \delta.} \quad (4)$$



Let $\hat{E}_k$ be the event that at most $k = m \cdot \hat{c}(\theta, S_m)$ samples x satisfy $g_\theta(x) = 1$ (or at most k 'successes'), when the probability for a single success is b^*. It holds (2) that,

$$\boxed{Pr\{\hat{E}_k\} = \mathrm{Bin}(m, m \cdot \hat{c}(\theta, S_m), b^*) = 1 - \delta.} \xrightarrow{\hat{c} \equiv \hat{c}(\theta,S_m)} \quad (5)$$

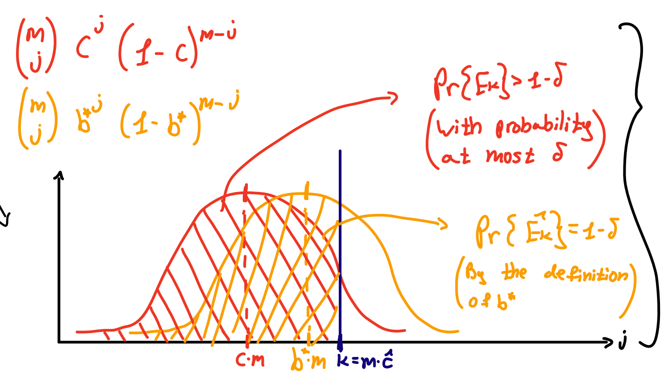


Therefore,

$$Pr\{E_k\} > Pr\{\hat{E}_k\}. \quad (6)$$

Consider the following statement,
let p_1, p_2 be two different probabilities for a single success. If the probability of at most k successes (out of n attempts) is more likely when considering p_1 as the probability of a single success, rather than p_2, we can conclude that $p_1 < p_2$.

Therefore, since the event of E_k occurs with higher probability then the event $\hat{E}_k$, see Eq. (6) , we can conclude that $c(\theta, P) < b^*$. This deduction is only true if we assume that $Pr\{E_k\} > 1 - \delta$, and this assumption is true with probability at most δ. Hence, we get that with probability at most δ, it holds that $c(\theta, P) < b^*$. □



$Pr\{E_k\} > Pr\{\hat{E}_k\} \Rightarrow c < b^*$