EE160 Introduction to Control: Homework 7

Prof. Boris Houska

Deadline: June 3, 2022

1. *(3 points) Adjoint time-varying differential equation.* Let $A : \mathbb{R} \to \mathbb{R}^{n_x \times n_x}$ be a given function and x the solution of the linear time-varying differential equation

$$\forall t \in [0, T], \quad \dot{x}(t) = A(t)x(t) \quad x(0) = x_0 \ .$$

Moreover, let λ denote the solutions of the associated adjoint differential equation, given by

$$\forall t \in [0, T], \quad \dot{\lambda}(t) = A(T-t)^\top \lambda(t) \quad \lambda(0) = \lambda_0 \ .$$

Prove that the equation

$$\lambda_0^\top x(T) = \lambda(T)^\top x_0$$

holds for all $T \in \mathbb{R}$.

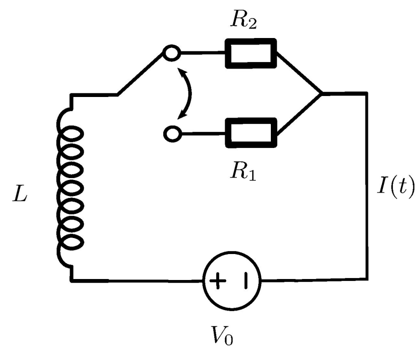
Solution: Let's introduce a scalar variable $z(t) = x(t)^\top \lambda(T-t)$, then $z(t)$ satisfies

$$\dot{z}(t) = \dot{x}(t)^\top \lambda(T-t) - x(t)^\top \dot{\lambda}(T-t) = x(t)^\top A(t)^\top \lambda(T-t) - x(t)^\top A(t)^\top \lambda(T-t) = 0 \ ,$$

so $z(t)$ is a constant function and

$$z(T) = z(0) \ \Rightarrow \ \lambda(T)^\top x_0 = \lambda_0^\top x(T) \ .$$

2. *(6 points) Electric circuit with periodic switch.* The electric circuit in the figure below consists of a battery with constant voltage $V_0 > 0$, an inductor with inductance $L > 0$, two resistors with resistance $R_1, R_2 > 0$, respectively, as well as a switch.



We assume that the switch changes its position every second. Thus, the period time is $T = 2s$. The current in the circuit at time t is denoted by $I(t)$. The following relations are known.

- The voltage V_0 at the battery is constant.
- The induced voltage at the inductor is given by $V_L(t) = L\dot{I}(t)$.
- The voltage at the resistors is $V_R(t) = R_1 I(t)$ if the switch is at time t at Position 1. Otherwise, if the switch is at Position 2, the voltage at the resistor is $V_R(t) = R_2 I(t)$.
- Due to Kirchhoff's voltage law, we have $V_L(t) + V_R(t) + V_0 = 0$.

(a) Derive a linear time-varying differential equation for the current $I(t)$.

(b) Find an explicit expression for the monodromy matrix $G(T,0)$ that is associated with the differential equation for the current $I(t)$.

(c) Work out an explicit expression for the periodic limit orbit $I_p(t)$ and prove that we have

$$\lim_{t\to\infty}(I(t)-I_p(t))=0$$

independent of the initial value $I(0)=I_0$.

Solution:

(a) WLOG, let

$$R(t)=\begin{cases} R_1, & t\in[2n,2n+1) \\ R_2, & t\in[2n+1,2n+2) \end{cases}, \ n=0,1,\dots$$

which is a periodic function with period time $T=2$. Then

$$V_R(t)=R(t)I(t) \quad\Rightarrow\quad \dot{I}(t)=-\frac{R(t)}{L}I(t)-\frac{V_0}{L}\,.$$

(b) For this scalar time-varying system, the fundamental matrix and monodromy matrix

$$G(t,\tau)=\exp\left(\int_\tau^t -\frac{R(s)}{L}\,\mathrm{d}s\right) \quad\Rightarrow\quad G(T,0)=\exp\left(-\frac{R_1+R_2}{L}\right)\,.$$

(c) **Method 1**

First for any time $t=NT+t'$, $t\in[0,2)$, fundamental matrix

$$\begin{aligned} G(t,0) &= G(T,0)^N G(t',0) \quad \text{and} && (1)\\ G(t',0) &= \begin{cases} e^{-R_1t'/L}, & t'\in[0,1) \\ e^{-R_1/L}e^{-R_2(t'-1)/L}, & t'\in[1,2) \end{cases} && (2) \end{aligned}$$

With the above expressions,

$$\begin{aligned} I_p(t+T) &= G(T,0)I_p(t)+\int_t^{t+T} G(t+T,\tau)\left(-\frac{V_0}{L}\right)\,\mathrm{d}\tau \\ &= G(T,0)x_p(t)-\frac{V_0}{L}\int_0^T G(t+T,t+s)\,\mathrm{d}s \\ &= G(T,0)x_p(t)-\frac{V_0}{L}G(t+T,t)\int_0^T G(t,t+s)\,\mathrm{d}s \end{aligned}$$

Now discuss the integral term for $t\in[0,1)$ and $t\in[1,2)$ separately.

- When $t\in[0,1)$,

$$\begin{aligned} &\int_0^T G(t,t+s)\,\mathrm{d}s \\ =&\int_0^{1-t} G(t,t+s)\,\mathrm{d}s+\int_{1-t}^{2-t} G(t,t+s)\,\mathrm{d}s+\int_{2-t}^{2} G(t,t+s)\,\mathrm{d}s \\ =&\int_0^{1-t}\exp(sR_1/L)\,\mathrm{d}s+\int_{1-t}^{2-t}\exp(((1-t)R_1+(t+s-1)R_2)/L)\,\mathrm{d}s \\ &+\int_{2-t}^{2}\exp(((s-1)R_1+R_2)/L)\,\mathrm{d}s \\ =&\frac{e^{(1-t)R_1/L}-1}{R_1/L}+\frac{e^{(1-t)R_1/L}(e^{R_2/L}-1)}{R_2/L}+\frac{e^{(R_1+R_2)/L}(1-e^{-tR_1/L})}{R_1/L} \\ =&\left(\frac{L}{R_1}-\frac{L}{R_2}\right)\left(e^{R_1/L}-e^{(R_1+R_2)/L}\right)e^{-tR_1/L}+\frac{L}{R_1}\left(e^{(R_1+R_2)/L}-1\right) \end{aligned}$$

and

$$
\begin{aligned}
&G(t+T,t)\int_0^T G(t,t+s)\,\mathrm{d}s \\
&= \left(\frac{L}{R_1}-\frac{L}{R_2}\right)\left(e^{-R_2/L}-1\right)e^{-tR_1/L}+\frac{L}{R_1}\left(1-e^{-(R_1+R_2)/L}\right)
\end{aligned}
$$

- When $t \in [1,2)$

$$
\begin{aligned}
&\int_0^T G(t,t+s)\,\mathrm{d}s \\
= \;& \int_0^{2-t} G(t,t+s)\,\mathrm{d}s + \int_{2-t}^{3-t} G(t,t+s)\,\mathrm{d}s + \int_{3-t}^{2} G(t,t+s)\,\mathrm{d}s \\
= \;& \int_0^{2-t} \exp(sR_2/L)\,\mathrm{d}s + \int_{2-t}^{3-t} \exp(((2-t)R_2+(t+s-2)R_1)/L)\,\mathrm{d}s \\
& + \int_{3-t}^{2} \exp(((2-t)R_2+R_1+(t+s-3)R_2)/L)\,\mathrm{d}s \\
= \;& \frac{e^{(2-t)R_2/L}-1}{R_2/L}+\frac{e^{(2-t)R_2/L}(e^{R_1/L}-1)}{R_1/L}+\frac{e^{(R_1+R_2)/L}-e^{R_2(2-t)/L}e^{R_1/L}}{R_2/L} \\
= \;& \left(\frac{L}{R_2}-\frac{L}{R_1}\right)(e^{R_2/L}-e^{(R_1+R_2)/L})e^{(1-t)R_2/L}+\frac{L}{R_2}\left(e^{(R_1+R_2)/L-1}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
&G(t+T,t)\int_0^T G(t,t+s)\,\mathrm{d}s \\
&= \left(\frac{L}{R_2}-\frac{L}{R_1}\right)\left(e^{-R_1/L}-1\right)e^{(1-t)R_1/L}+\frac{L}{R_2}\left(1-e^{-(R_1+R_2)/L}\right)\ .
\end{aligned}
$$

Then solving the equation

$$
I_p(t+T)=I_p(t)\ ,
$$

we could obtain

$$
I_p(t)=\begin{cases}
\left(\frac{V_0}{R_1}-\frac{V_0}{R_2}\right)\frac{e^{-R_2/L}-1}{e^{-(R_1+R_2)/L}-1}e^{-tR_1/L}-\frac{V_0}{R_1}, & t\in(0,1) \\
\left(\frac{V_0}{R_2}-\frac{V_0}{R_1}\right)\frac{e^{-R_1/L}-1}{e^{-(R_1+R_2)/L}-1}e^{(1-t)R_2/L}-\frac{V_0}{R_2}, & t\in(1,2)\ .
\end{cases} \tag{3}
$$

Method 2

This is a piece-wise scalar linear time-invariant system whose solution function can be determined by the initial state I_0,

$$
\begin{aligned}
t\in[0,1],\quad \dot{I}(t)&=-\frac{R_1}{L}I(t)-\frac{V_0}{L}\ \Rightarrow\ I(t)=e^{-R_1t/L}\left(I_0+\frac{V_0}{R_1}\right)-\frac{V_0}{R_1} \\
t\in(1,2),\quad \dot{I}(t)&=-\frac{R_2}{L}I(t)-\frac{V_0}{L}\ \Rightarrow\ I(t)=e^{-R_2t/L}\left(I_1+\frac{V_0}{R_2}\right)-\frac{V_0}{R_2}
\end{aligned}
$$

From the continuity and periodicity of $I(t)$, we know

$$
\begin{aligned}
I_1=I(1)&=e^{-R_1/L}\left(I_0+\frac{V_0}{R_1}\right)-\frac{V_0}{R_1}\quad\text{and} \\
I_0=I(2)&=e^{-R_2/L}\left(I_1+\frac{V_0}{R_2}\right)-\frac{V_0}{R_2}
\end{aligned}
$$

Solving the equation,

$$I_0 = \frac{e^{-(R_1+R_2)/L}/R_1 + e^{-R_2/L}(1/R_2 - 1/R_1) - 1/R_2}{1 - e^{-(R_1+R_2)/L}}$$

this gives the same expression as (3).

Because R_1, R_2, L are positive, monodromy matrix $G(t, 0)$ in (1) goes to 0 as $t \to \infty$, which means

$$\lim_{t\to\infty} (I(t) - I_p(t)) = 0$$

for any initial value $I(0)$.

3. *(6 points) Stability analysis of dynamical systems.* Determine the equilibrium points and their stability properties of the following dynamical system for $t > 0$,

   (a) $\dot{x} = x(x - 1)$

   (b) $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

   (c) $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & e^{2t} \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

Solution:

(a) The scalar system has two equilibrium points: 0 and 1 and 0 is stable, 1 is unstable.

(b) The unique equilibrium point at $(0, 0)$ is asymptotically stable.

(c) Explicit solution for this time-varying ODEs is

$$\begin{cases} x_1(t) = \frac{1}{2}e^{-t}\left(e^{2t}x_2(0) + 2x_1(0) - x_2(0)\right) \\ x_2(t) = e^{-t}x_2(0) \ , \end{cases}$$

so the origin is unstable.