2023Fall Probability & Mathematical Statistics 2023/11/09

Homework 6

Professor: Ziyu Shao & Dingzhu Wen *Due: 2023/11/19 10:59pm*

1. The *Beta distribution* with parameters $a = 3, b = 2$ has PDF

$$f(x) = 12x^2(1-x), \text{ for } 0 < x < 1.$$

Let X have this distribution.

(a) Find the CDF of X.

(b) Find $P(0 < X < 1/2)$.

(c) Find the mean and variance of X (without quoting results about the Beta distribution).

2. Let $U_1, \dots, U_n$ be i.i.d. $\text{Unif}(0,1)$, and $X = \max(U_1, \dots, U_n)$.

(a) What is the PDF of X?

(b) What is $E[X]$?

3. the *Laplace distribution* has PDF

$$f(x) = \frac{1}{2}e^{-|x|}$$

for all real x. The Laplace distribution is also called a *symmetrized Exponential* distribution. Explain this in the following two ways.

(a) Plot the PDFs and explain how they relate.

(b) Let $X \sim \text{Expo}(1)$ and S be a random sign (1 or -1, with equal probabilities), with S and X independent. Find the PDF of SX (by first finding the CDF), and compare the PDF of SX and the Laplace PDF.

4. The *Gumbel distribution* is the distribution of $-\log X$ with $X \sim \text{Expo}(1)$.

(a) Find the CDF of the Gumbel distribution.

(b) Let $X_1, X_2, \dots$ be i.i.d. $\text{Expo}(1)$ and let $M_n = \max(X_1, \dots, X_n)$. Show that $M_n - \log n$ converges in distribution to the Gumbel distribution, i.e., as $n \to \infty$ the CDF of $M_n - \log n$ converges to the Gumbel CDF.

5. Let $Z \sim \mathcal{N}(0, 1)$, and c be a nonnegative constant. Find $E[\max(Z - c, 0)]$, in terms of the standard Normal CDF Φ and PDF φ.

6. (Optional Challenging Problem) Suppose $X \sim \mathcal{N}(m, \sigma^2)$, where m is an integer and σ is a real number. Let $Y = \lfloor X \rfloor$ be the integer part of X.

   (a) Find the PMF of Y

   (b) Find $E(Y)$

   (c) Find $\mathrm{Var}(Y)$