Chap 9— 3

隐函数和逆映射定理

9.3.1 隐函数的存在性和微商

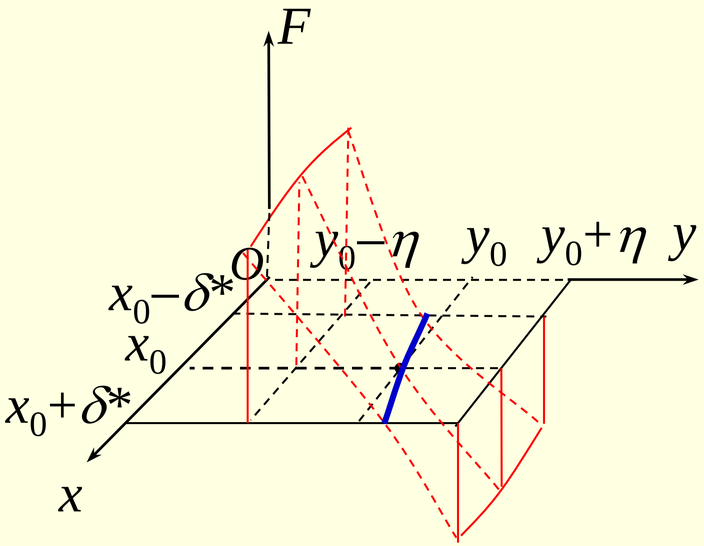
定理 设函数 F 在$M_0(x_0,y_0)$邻域内有连续偏导数, 且

$$F(x_0, y_0) = 0, \quad F'_y(x_0, y_0) \neq 0$$

则方程$F(x, y) = 0$在M_0某邻域内存在**唯一连续可导隐函数**$y = f(x)$, 满足

$$F(x, f(x)) \equiv 0, \ y_0 = f(x_0)$$

和 $$\boxed{\frac{\mathrm{d}y}{\mathrm{d}x} = f'(x) = -\frac{F'_x(x, y)}{F'_y(x, y)}}$$



例1 设方程 $\sin(x+y)+2x+y=0$ 在(0, 0)附近确定

隐函数$y=y(x)$, 求$\dfrac{\mathrm{d}y}{\mathrm{d}x}$ 和 $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$

例2 讨论方程 $F(x,y)=y^3-x=0$ 在(0, 0)附近确定

隐函数的情况.

例3 讨论方程 $F(x,y)=x^2+y^2-1=0$ 在(0, 1)和(1, 0)

附近确定隐函数的情况.

例4 设有方程 $e^{xy} - \cos x - \sin y = 0.$

(1) 证明: 在(0,0)点的某邻域内, 上述方程可确定唯一的隐函数$y = y(x)$满足$y(0) = 0$.

(2) 上述方程能否在(0,0)点的某邻域内, 确定隐函数$x = x(y)$? 为什么.

提示: $y'(x) = \dfrac{\sin x + y e^{xy}}{\cos y - x e^{xy}}, \quad y'(0) = 0$

$$y''(0) = \lim_{x \to 0} \frac{y'(x) - y'(0)}{x} = \lim_{x \to 0} \frac{\dfrac{\sin x}{x} + \dfrac{y}{x} e^{xy}}{\cos y - x e^{xy}} = 1$$

Ex. 设有方程 $x^2 + y + \sin xy = 0.$

(1) 证明: 在(0,0)点的某邻域内, 上述方程可确定唯一的隐函数$y = y(x)$满足$y(0) = 0.$

(2) 上述方程能否在(0,0)点的某邻域内, 确定隐函数$x = x(y)$?为什么.

提示: $y'(x) = -\dfrac{2x + y\cos xy}{1 + x\cos xy}, \quad y'(0) = 0$

$$y''(0) = \lim_{x\to 0}\frac{y'(x) - y'(0)}{x} = \lim_{x\to 0} - \frac{2 + \frac{y}{x}\cos xy}{1 + x\cos xy} = -2$$

■ 多元隐函数

定理 设函数F在$M_0(x_0,y_0,z_0)$的邻域内有连续偏导数, 且

$$F(x_0,y_0,z_0)=0,\quad F_z'(x_0,y_0,z_0)\neq 0$$

则方程$F(x,y,z)=0$在M_0某邻域内存在唯一具有连续偏导数的**隐函数**$z=f(x,y)$, 满足

$$F(x,y,f(x,y))\equiv 0,\quad z_0=f(x_0,y_0)$$

且有

$$\boxed{\frac{\partial z}{\partial x}=-\frac{F_x'}{F_z'},\ \frac{\partial z}{\partial y}=-\frac{F_y'}{F_z'}}$$

例5 设方程 $\mathrm{e}^z-xyz=0$ 确定隐函数$z=z(x,y)$, 求z'_x.

■ 隐映射存在定理

若函数 $F(x,y,u,v),G(x,y,u,v)$ 在点 $P_0(x_0,y_0,u_0,v_0)$ 某一邻域内有连续的偏导数, 且 $F(x_0,y_0,u_0,v_0)=0$, $G(x_0,y_0,u_0,v_0)=0$, **Jacobi行列式**

$$J_0=\left.\frac{\partial(F,G)}{\partial(u,v)}\right|_{P_0}=\left.\begin{vmatrix}\frac{\partial F}{\partial u} & \frac{\partial F}{\partial v}\\ \frac{\partial G}{\partial u} & \frac{\partial G}{\partial v}\end{vmatrix}\right|_{P_0}\neq 0,$$

则 $\begin{cases}F(x,y,u,v)=0\\ G(x,y,u,v)=0\end{cases}$ 可唯一确定**隐映射** $\begin{cases}u=u(x,y)\\ v=v(x,y)\end{cases}$

满足此方程组及 $\begin{cases}u_0=u(x_0,y_0)\\ v_0=v(x_0,y_0)\end{cases}$ 且有连续偏导数

$$\frac{\partial u}{\partial x}=-\frac{1}{J}\frac{\partial(F,G)}{\partial(x,v)}=-\frac{\begin{vmatrix}F_x & F_v\\ G_x & G_v\end{vmatrix}}{\begin{vmatrix}F_u & F_v\\ G_u & G_v\end{vmatrix}}\qquad \frac{\partial u}{\partial y}=-\frac{1}{J}\frac{\partial(F,G)}{\partial(y,v)}=-\frac{\begin{vmatrix}F_y & F_v\\ G_y & G_v\end{vmatrix}}{\begin{vmatrix}F_u & F_v\\ G_u & G_v\end{vmatrix}}$$

$$\frac{\partial v}{\partial x}=-\frac{1}{J}\frac{\partial(F,G)}{\partial(u,x)}=-\frac{\begin{vmatrix}F_u & F_x\\ G_u & G_x\end{vmatrix}}{\begin{vmatrix}F_u & F_v\\ G_u & G_v\end{vmatrix}}\qquad \frac{\partial v}{\partial y}=-\frac{1}{J}\frac{\partial(F,G)}{\partial(u,y)}=-\frac{\begin{vmatrix}F_u & F_y\\ G_u & G_y\end{vmatrix}}{\begin{vmatrix}F_u & F_v\\ G_u & G_v\end{vmatrix}}$$

例6 设函数$\begin{cases}u=u(x,y)\\ v=v(x,y)\end{cases}$由方程组$\begin{cases}x^2+y^2-uv=0\\ xy-u^2+v^2=0\end{cases}$确定,

求 $\dfrac{\partial u}{\partial x},\dfrac{\partial v}{\partial y}$.

$$\boxed{\frac{\partial u}{\partial x}=\frac{4xv+yu}{2(u^2+v^2)},\qquad \frac{\partial v}{\partial y}=\frac{4yu-xv}{2(u^2+v^2)}}$$

9.3.2 从微分看隐函数定理

由 $F(x, y(x)) \equiv 0$, 两端取微分得

$$\mathrm{d}F = F_x'\mathrm{d}x + F_y'\mathrm{d}y = 0$$

导出

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{F_x'(x, y(x))}{F_y'(x, y(x))}$$

想一想 两个三元方程的方程组 $\begin{cases} F(x, y, z) = 0 \\ G(x, y, z) = 0 \end{cases}$ 情形?

及n个$m+n$元方程的方程组 $\boldsymbol{F}(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{0}$ 情形? 其中

$$\boldsymbol{F} : D \subset \mathbb{R}^{m+n} \to \mathbb{R}^n, \boldsymbol{x} = (x_1, x_2, \cdots, x_m), \boldsymbol{y} = (y_1, y_2, \cdots, y_n)$$

- 求隐函数所有偏导数时，**微分法**比较简便

例7 设$z = f(x, y)$由方程 $z = x + y - xe^{z}$ 确定，求z'_x, z'_y.

例8 函数 $y = y(x), z = z(x)$由方程组

$$\begin{cases} z = x + \varphi(x+y) \\ F(x, y, z) = 0 \end{cases}$$

确定，其中φ, F均可微，$F_y + \varphi' F_z' \neq 0$，求 $\dfrac{dy}{dx}, \dfrac{dz}{dx}$.

$$\boxed{\frac{dy}{dx} = -\frac{F_x' + (1+\varphi')F_z'}{F_y' + \varphi' F_z'}, \qquad \frac{dz}{dx} = -\frac{\varphi' F_x' - (1+\varphi')F_y'}{F_y' + \varphi' F_z'},}$$

9.3.3 逆映射存在定理

设函数 $u=u(x,y), v=v(x,y)$ 在 $B(P_0(x_0,y_0))$ 有连续偏导数, 且 $u_0=u(x_0,y_0)$, $v_0=v(x_0,y_0)$, Jacobi行列式

$$J_0=\left.\frac{\partial(u,v)}{\partial(x,y)}\right|_{P_0}\neq 0,$$

则 $\begin{cases} u=u(x,y) \\ v=v(x,y) \end{cases}$ 在$B(u_0, v_0)$存在**逆映射** $\begin{cases} x=x(u,v) \\ y=y(u,v) \end{cases}$

满足 $\begin{cases} x_0=x(u_0,v_0) \\ y_0=y(u_0,v_0) \end{cases}$ 且有连续偏导数

$$\boxed{\begin{aligned}&\frac{\partial x}{\partial u}=\frac{1}{J}\cdot\frac{\partial v}{\partial y}, && \frac{\partial x}{\partial v}=-\frac{1}{J}\cdot\frac{\partial u}{\partial y}\\ &\frac{\partial y}{\partial u}=-\frac{1}{J}\cdot\frac{\partial v}{\partial x}, && \frac{\partial y}{\partial v}=\frac{1}{J}\frac{\partial u}{\partial x}\end{aligned}}$$

推论 同前定理条件, 则有

$$\boxed{\frac{\partial(x,y)}{\partial(u,v)}\cdot\frac{\partial(u,v)}{\partial(x,y)}=1}$$

证 记

$$\boldsymbol{f}=\begin{pmatrix}u\\ v\end{pmatrix}=\begin{pmatrix}u(x,y)\\ v(x,y)\end{pmatrix},\ \text{逆映射}\quad \boldsymbol{f}^{-1}=\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}x(u,v)\\ y(u,v)\end{pmatrix}$$

则复合映射

$$\boldsymbol{f}^{-1}\circ\boldsymbol{f}:\begin{pmatrix}x\\y\end{pmatrix}\to\begin{pmatrix}x\\y\end{pmatrix}$$ 即恒等映射 $\boldsymbol{I}:\begin{cases}x=x\\y=y\end{cases}$

由 $\boldsymbol{f}^{-1}\circ\boldsymbol{f}=\boldsymbol{I}$，两端求Jacobi矩阵, 得 $\boldsymbol{J}\left(\boldsymbol{f}^{-1}\right)\cdot\boldsymbol{J}(\boldsymbol{f})=\boldsymbol{E}$

取行列式得 $\left|\boldsymbol{J}\left(\boldsymbol{f}^{-1}\right)\cdot\boldsymbol{J}(\boldsymbol{f})\right|=\left|\boldsymbol{J}\left(\boldsymbol{f}^{-1}\right)\right|\cdot\left|\boldsymbol{J}(\boldsymbol{f})\right|=\left|\boldsymbol{E}\right|=1$

即
$$\boxed{\frac{\partial(x,y)}{\partial(u,v)}\cdot\frac{\partial(u,v)}{\partial(x,y)}=1}$$

例9 求极坐标变换 $x=r\cos\theta, y=r\sin\theta$ 的逆变换偏微商.