ShanghaiTech University

EE 115B: Digital Circuits

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Lecture 8

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Boolean Algebra and Logic Simplification

- Boolean Algebra
- Logic Simplification
  - Karnaugh Map
  - Quine-McCluskey Method

Terminology

- Variable: the input
- Literal: a variable complemented or uncomplemented
- Implicant: a product term of literals obtained from the K-map such that when implicant = 1, F = 1
- Prime implicant (PI): The implicant which cannot be simplified further, i.e. no literal can be removed
- Essential prime implicant: The *prime implicants* which contain 1's that can only be grouped in 1 way
- Cover: A set of *prime implicants* which covers all 1's

Terminology Example

AB \ CD	00	01	11	10
00	1	1	0	0
01	1	1	1	0
11	0	0	1	1
10	1	1	0	0

- Variable: A, B, C and D
- Literal: A, B′, etc
- Implicant: A′BC′, AB′CD, A′D′, etc
- Prime implicant (PI): A′C′, ACD, ABD, BC′D, A′D′, etc
- Essential prime implicant: ACD, not ABD, not BC′D, etc
- Cover: ACD, A′D′, A′C′ and (BC′D or ABD)

Limitations of K-Map

- K-map is a very effective way to simplify functions with small number of variables
- When the number of variables is large, or when several functions need to be simplified, the use of a digital computer is desirable
- Quine-McCluskey (QM) method provides a systematic simplification procedure which can be readily programmed for a computer
- QM reduces the minterm expansion to obtain MSOP

Basic Two-level QM Minimization Steps

- The basic steps are the same for K-Map and Quine McCluskey (QM):

1. Form all prime implicants (PIs) (in a K-map we do it by grouping ones in as large a group as possible)
2. Determine a minimum cost set of PIs to cover all minterms (MTs)
   - Determine essential prime implicants (EPIs), include them in the final expression and delete all MTs they cover
   - Determine a min cost set of the remaining PIs to cover the remaining MTs (NP-hard)

Determination of Prime Implicants

- For each term
  - eliminate as many literals as possible by systematically XY + XY′ = X
  - where X represents a product of literals and Y is a single variable
- Two minterms will combine only if they differ in exactly one variable
- The resulting terms are called prime implicants

$$AB'CD' + AB'CD = AB'C$$

$$\underbrace{1\ 0\ 1}_{X}\ \underbrace{0}_{Y} + \underbrace{1\ 0\ 1}_{X}\ \underbrace{1}_{Y'} = \underbrace{1\ 0\ 1}_{X}\ -$$

Dash indicates a missing variable

$$A'BC'D + A'BCD'$$

$$0\ 1\ 0\ 1 + 0\ 1\ 1\ 0$$

Cannot be combined

Determination of Prime Implicants

- To find all PIs, all possible pairs of minterms should be compared and combined
- To reduce the number of comparisons, the minterms are sorted into groups according to the number of 1's in each term
- Consider an example $f(a,b,c,d)=\sum m(0,1,2,5,6,7,8,9,10,14)$

0	0000	Group 0
1	0001	Group 1
2	0010	
8	1000	
5	0101	Group 2
6	0110	
9	1001	
10	1010	
7	0111	Group 3
14	1110	

In the list, each term in group i has i 1's. E.g. all terms in group 2 have exactly two 1's

Determination of Prime Implicants

- Two terms can be combined if they differ in exactly one variable
- Non-adjacent groups differ in at least two variables and cannot be combined using $XY + XY' = X$
- Similarly, comparison of terms within a group is unnecessary
  - Since they have the same number of 1's, so they must differ in at least two variables
- Note that whenever the terms combine, they differ by a power of 2
- The terms are checked off whenever they are combined with another term
- Even when two terms have been checked and combined, they should still be compared and combined if possible - the resulting implicant may be used to form the minimum sum solution
- The redundant terms will be eliminated later

Determination of Prime Implicants

Group 0	0	√	0000
Group 1	1	√	0001
	2	√	0010
	8	√	1000
Group 2	5	√	0101
	6	√	0110
	9	√	1001
	10	√	1010
Group 3	7	√	0111
	14	√	1110

0, 1	000-
0, 2	00-0
0, 8	-000
1, 5	0-01
1, 9	-001
2, 6	0-10
2, 10	-010
8, 9	100-
8, 10	10-0
5, 7	01-1
6, 7	011-
6, 14	-110
10,14	1-10

Determination of Prime Implicants

- Note that the terms in the new column are also divided into groups according to the 1's present
- Again, the terms in this new column are compared and combined using $XY + XY' = XY$
- In order to combine now, the terms must
  - have the same variables i.e dashes should be in corresponding places and
  - differ in exactly one of these variables

Determination of Prime Implicants

Group 0	0	✓	0000
Group 1	1	✓	0001
	2	✓	0010
	8	✓	1000
Group 2	5	✓	0101
	6	✓	0110
	9	✓	1001
	10	✓	1010
Group 3	7	✓	0111
	14	✓	1110

0, 1	✓	000-
0, 2	✓	00-0
0, 8	✓	-000
1, 5		0-01
1, 9	✓	-001
2, 6	✓	0-10
2, 10	✓	-010
8, 9	✓	100-
8, 10	✓	10-0
5, 7		01-1
6, 7		011-
6, 14	✓	-110
10,14	✓	1-10

0, 1, 8, 9	-00-
0, 2, 8, 10	-0-0
~~0, 8, 1, 9~~	~~-00-~~
~~0, 8, 2, 10~~	~~-0-0~~
2, 6, 10, 14	--10
2, 10, 6, 14	--10

Determination of Prime Implicants

- Duplicate terms are deleted
- The remaining implicants are again grouped and compared, but since no combination is possible, the procedure is terminated
- Grouping, comparison and combination continues until no more terms can be combined
- Terms which are not checked off in any level are prime implicants
  - Can also be in the first table itself

Prime Implicants

Group 0	0	✓	0000
Group 1	1	✓	0001
	2	✓	0010
	8	✓	1000
Group 2	5	✓	0101
	6	✓	0110
	9	✓	1001
	10	✓	1010
Group 3	7	✓	0111
	14	✓	1110

0, 1	✓ 000-
0, 2	✓ 00-0
0, 8	✓ -000
1, 5	0-01
1, 9	✓ -001
2, 6	✓ 0-10
2, 10	✓ -010
8, 9	✓ 100-
8, 10	✓ 10-0
5, 7	01-1
6, 7	011-
6, 14	✓ -110
10,14	✓ 1-10

0, 1, 8, 9	-00-
0, 2, 8, 10	-0-0
2, 6, 10, 14	--10

Prime Implicants

1, 5	0-01	a′c′d
5, 7	01-1	a′bd
6, 7	011-	a′bc
0, 1, 8, 9	-00-	b′c′
0, 2, 8, 10	-0-0	b′d′
2, 6, 10, 14	--10	cd′

Prime Implicant Cover

- Since every minterm is included in at least one of the prime implicants, the function is equal to the sum of its prime implicants. Therefore,

$$f = \underset{(1,5)}{a'c'd} + \underset{(5,7)}{a'bd} + \underset{(6,7)}{a'bc} + \underset{(0,1,8,9)}{b'c'} + \underset{(0,2,8,10)}{b'd'} + \underset{(2,6,10,14)}{cd'}$$

- In this expression, each term has a minimum number of literals (prime implicant), but the number of terms is not minimum
- Can be minimized by using consensus theorem

$$AB + A'C + BC = AB + A'C$$

$$f = a'c'd + a'bd + a'bc + b'c' + b'd' + cd'$$

$$f = a'c'd + a'bd + a'bc + b'c' + cd'$$

$$f = a'c'd + ((a'b)d + (a'b)c + cd') + b'c'$$

$$f = a'c'd + (a'b)d + cd' + b'c'$$

$$f = (a'd)c' + (a'd)b + b'c' + cd'$$

$$f = a'db + b'c' + cd'$$

Prime Implicant Cover

- The Quine-McCluskey algorithm finds all the product term implicants of a function
- The non-prime terms are checked off
- The remaining terms are all prime implicants
- An MSOP expression consists of a sum of some of the prime implicants, but not necessarily all
- Deriving minimal cover by algebraically manipulating is not always easy
- The prime implicant chart provides a systematic way to derive MSOP

Prime Implicant Chart

- Second part of the Quine McCluskey algorithm - selects a minimum set of prime implicants
- Minterms of the functions listed across the top, implicants on the side
- A prime implicant is equal to a sum of minterms
  - It is said to cover these minterms
- If a prime implicant covers a minterm, an X is placed at the intersection of the minterm and prime implicant
- If a minterm column contains only one X, then the corresponding row is an essential prime implicant
  - This implies that we need that implicant in order to cover that minterm
  - Always choose essential prime implicants first!
- After an implicant is chosen, the corresponding row is crossed out
- All the minterms covered by this implicant are also crossed out

Prime Implicant Chart

Minterms	Prime Implicants	0	1	2	5	6	7	8	9	10	14
1, 5	$a'c'd$		X		X						
5, 7	$a'bd$				X		X				
6, 7	$a'bc$					X	X				
0, 1, 8, 9	$b'c'$	X	X					X	X		
0, 2, 8, 10	$c'd'$	X		X				X		X	
2, 6, 10, 14	cd'			X		X				X	X

- If a minterm is covered by only one implicant, the implicant is called essential prime implicant

Prime Implicant Chart

Minterms	Prime Implicants	0	1	2	5	6	7	8	9	10	14
1, 5	*a′c′d*		X		X						
5, 7	*a′bd*				X		X				
6, 7	*a′bc*					X	X				
0, 1, 8, 9	*b′c′*	X	X					X	X		
0, 2, 8, 10	*c′d′*	X		X				X		X	
2, 6, 10, 14	*cd′*			X		X				X	X

- Chosen implicants: b′c′ and cd′
- Two minterms remaining: 5 and 7 – include *a′bd*

Minimal Cover

- In total, 3 prime implicants are needed to cover the function *f*

$$f = a'bd + b'c' + cd'$$

- Note that *a'bd* is not an essential prime implicant
  - m(5) and m(7) are also covered in other PIs
  - *a'c'd* and *a'bc* could have also been chosen
  - But then, we won't have obtained minimum SOP