SI152 Numerical Optimization
Homework 1

Instructor: Professor Ye Shi

Due on 20 April 23:59 UTC+8

Note:

- Please provide enough calculation process to get full marks.
- Please submit your homework to Gradescope with entry code: **2KVGPJ**.
- Please check carefully whether the question number on the gradescope corresponds to each question.

Exercise 1. (Reformulate norm as LP problem) (3 pts)

From the lecture note, we learnt that the optimization problem contains ℓ_1-norm and ℓ_∞-norm can be formulated as a linear programming problem. Please *reformulate the following problem* as a linear programming problem.

$$\begin{aligned} \min_{x \in \mathbb{R}^{n\times 1}} \quad & ||\boldsymbol{Ax} - \boldsymbol{b}||_\infty \\ \text{s.t.} \quad & ||\boldsymbol{Cx} - \boldsymbol{d}||_1 \le e \end{aligned}$$

where $\boldsymbol{A} \in \mathbb{R}^{p\times n}, \boldsymbol{b} \in \mathbb{R}^{p\times 1}, \boldsymbol{C} \in \mathbb{R}^{q\times n}, \boldsymbol{d} \in \mathbb{R}^{q\times 1}, e \in \mathbb{R}$.

(Note1: The definition of ℓ_p-norm is $||\boldsymbol{x}||_p = \left(\sum_i |x_i|^p\right)^{\frac{1}{p}}$.)

(Note2: The problem does not need to be standard form, i.e., can contains some inequality constraints and some negative variables.)

Solution:

We first introduce $\boldsymbol{y} \in \mathbb{R}^{p\times 1}, \boldsymbol{z} \in \mathbb{R}^{q\times 1}$ to substitute $\boldsymbol{Ax} - \boldsymbol{b}$ and $\boldsymbol{Cx} - \boldsymbol{d}$.

$$\begin{aligned} \min_{\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}} \quad & ||\boldsymbol{y}||_\infty \\ \text{s.t.} \quad & ||\boldsymbol{z}||_1 \le e \\ & \boldsymbol{Ax} - \boldsymbol{b} = \boldsymbol{y} \\ & \boldsymbol{Cx} - \boldsymbol{d} = \boldsymbol{z} \end{aligned}$$

Then we cancel the norm in optimization problem

$$\begin{aligned} \min_{\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}^+,\boldsymbol{z}^-} \quad & t \\ \text{s.t.} \quad & \sum_i z_i^+ + z_i^- \le e \\ & y_i \le t, \quad i \in 1, \cdots, p \\ & \boldsymbol{Ax} - \boldsymbol{b} = \boldsymbol{y} \\ & \boldsymbol{Cx} - \boldsymbol{d} = \boldsymbol{z}^+ - \boldsymbol{z}^- \end{aligned}$$

which is the final solution.

Exercise 2. (No inequality constraint!) (2 pts)

The slack variables make the inequality constraints become equality constraints, which is an essential step in standard form formulation procedure. Please show that the extreme points in the following two sets are *one-to-one*.

$$S_1 = \{\boldsymbol{x} \in \mathbb{R}^n | \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}\},$$
$$S_2 = \{(\boldsymbol{x}, \boldsymbol{y}) \in \mathbb{R}^n \times \mathbb{R}^m | \boldsymbol{A}\boldsymbol{x} + \boldsymbol{y} = \boldsymbol{b}, \boldsymbol{x} \geq 0, \boldsymbol{y} \geq \boldsymbol{0}\}.$$

(Note: The "one-to-one" correspondence here means "given any extreme point of a set, we can construct an extreme point of the other set".)

Solution:
Consider an extreme point of set S_1 and suppose there are k active constraints satisfying $x_i = 0, i = 1, \cdots, k$, then there are $n-k$ active contraints satisfying $A_j^T x = b_j, j = 1, \cdots, n-k$. For set S_2, let $y_i = 0, i = 1, \cdots, n-k$ and $y_i = b_i - A_i^T x \geq 0$ for the remaining. Now the constraint $Ax + y = b$ is satisfied, which provides m active constraints, and there are k active constraints in $x \geq 0$ as well as $n-k$ active constraints in $y \geq 0$. Thus, there are $m + k + (n-k) = m + n$ active constraints in total, which constructs an extreme point of set S_2.
On the other hand, for an extreme point of set S_2, there are k active constraints satisfying $x_i = 0, i = 1, \cdots, k$ and there are $n-k$ active constraints satisfying $y_j = 0, j = 1, \cdots, n-k$. Hence, there are n active constraints satisfied in S_1, which makes up an extreme point of set S_1.

Exercise 3. (Reduced cost) (2 pts)

If the reduced cost $r_j > 0$ for every j corresponding to a variable x_j that is not basic, show that the corresponding basic feasible solution is the *unique optimal* solution.

Solution:
We first show that the BFS is optimal and then show its uniqueness.

1. Because $r_j \geq 0$ for every j corresponding to a nonbasic variable, the corresponding basic feasible solution is an optimal solution by the optimal criterion listed on lecture slides. (Lecture 3 Page 11).

2. Since $r_j > 0$, change the basis will strictly increase the objective function value. Therefore, the optimal solution is unique.

Exercise 4. (Two-phase simplex method) (3 pts)

Please *use the two-phase simplex procedure* to solve the following problem.

$$
\begin{aligned}
\min \quad & -3x_1 + x_2 + 3x_3 - x_4 \\
\text{s.t.} \quad & x_1 + 2x_2 - x_3 + x_4 = 0 \\
& 2x_1 - 2x_2 + 3x_3 + 3x_4 = 9 \\
& x_1 - x_2 + 2x_3 - x_4 = 6 \\
& x_i \geq 0, \quad i = 1, 2, 3, 4.
\end{aligned}
\tag{1}
$$

Solution:

First stage: Initialization

$$
\begin{array}{ccccccc|c}
\hline
1 & 2 & -1 & 1 & 1 & 0 & 0 & 0 \\
2 & -2 & 3 & 3 & 0 & 1 & 0 & 9 \\
1 & -1 & 2 & -1 & 0 & 0 & 1 & 6 \\
\hline
0 & 0 & 0 & 0 & 1 & 1 & 1 & -15 \\
\hline\hline
\end{array}
$$

$$
\begin{array}{ccccccc|c}
\boxed{1} & 2 & -1 & 1 & 1 & 0 & 0 & 0 \\
2 & -2 & 3 & 3 & 0 & 1 & 0 & 9 \\
1 & -1 & 2 & -1 & 0 & 0 & 1 & 6 \\
\hline
-4 & 1 & -4 & -3 & 0 & 0 & 0 & -15 \\
\hline\hline
\end{array}
$$

$$
\begin{array}{ccccccc|c}
1 & 2 & -1 & 1 & 1 & 0 & 0 & 0 \\
0 & -6 & \boxed{5} & 1 & -2 & 1 & 0 & 9 \\
0 & -3 & 3 & -2 & -1 & 0 & 1 & 6 \\
\hline
0 & 9 & -8 & 1 & 4 & 0 & 0 & -15 \\
\hline\hline
\end{array}
$$

$$
\begin{array}{ccccccc|c}
1 & \frac{4}{5} & 0 & \frac{6}{5} & \frac{3}{5} & \frac{1}{5} & 0 & \frac{9}{5} \\
0 & -\frac{6}{5} & 1 & \frac{1}{5} & -\frac{2}{5} & \frac{1}{5} & 0 & \frac{9}{5} \\
0 & \boxed{\frac{3}{5}} & 0 & -\frac{13}{5} & \frac{1}{5} & -\frac{3}{5} & 1 & \frac{3}{5} \\
\hline
0 & -\frac{3}{5} & 0 & \frac{13}{5} & \frac{4}{5} & \frac{8}{5} & 0 & -\frac{3}{5} \\
\hline\hline
\end{array}
$$

$$
\begin{array}{ccccccc|c}
1 & 0 & 0 & \frac{14}{3} & \frac{1}{3} & 1 & -\frac{4}{3} & 1 \\
0 & 0 & 1 & -5 & 0 & -1 & 2 & 3 \\
0 & 1 & 0 & -\frac{13}{3} & \frac{1}{3} & -1 & \frac{5}{3} & 1 \\
\hline
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
\hline
\end{array}
$$

Second Stage: Simplex

$$
\begin{array}{cccc|c}
\hline
1 & 0 & 0 & \frac{14}{3} & 1 \\
0 & 0 & 1 & -5 & 3 \\
0 & 1 & 0 & -\frac{13}{3} & 1 \\
\hline
-3 & 1 & 3 & -1 & 0 \\
\hline\hline
\end{array}
$$

$$
\begin{array}{cccc|c}
1 & 0 & 0 & \frac{14}{3} & 1 \\
0 & 0 & 1 & -5 & 3 \\
0 & 1 & 0 & -\frac{13}{3} & 1 \\
\hline
0 & 0 & 0 & \frac{97}{3} & 7 \\
\hline
\end{array}
$$

so $x_1 = 1, x_2 = 1, x_3 = 3x_4 = 0$, $obj = (-3) \times 1 + 1 \times 1 + 3 \times 3 + 0 = 7$