SI152 Numerical Optimization
Quiz 1

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> Note:
>
> - Please provide enough calculation process to get full marks.

Exercise 1. (8-puzzle problem)

The eight-puzzle problem is a classical state search problem in AI research area. In the eight-puzzle problem, there are eight numbers on the chessboard of 3×3. There is also an empty space ,which is marked as NULL, on the chessboard. The pieces adjacent to empty space can be moved into it, i.e., NULL can exchange positions with other number of adjacent position.

Now consider the initial state as following Figure 1. We can encode the current state as the sequence of numbers' positions: $\{(1,1), (1, 2), (1, 3), (2, 1),(2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$. The positions are encoded from NULL, 1 to 8.

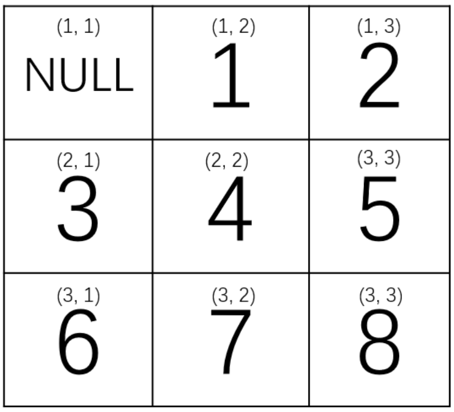

Figure 1: Initial state

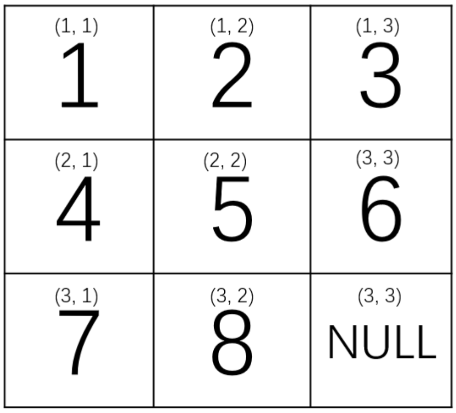

Figure 2: Final state

Suppose the final state is shown as following Figure 2. Now we want to find the move policy from initial state to achieve the final state in minimum exchange times. Please build an optimization model to solve this problem.

(Note: You can first briefly explain the state transition graph, and then model the shortest path problem of the graph as an optimization problem.)

Solution:

Step 1: Build the transition graph G. The state $\{(x, y), \cdots, (x, y + 1), \cdots\}$ is connected to the state $\{(x, y + 1), \cdots, (x, y), \cdots\}$. Similiar to state $\{(x, y), \cdots, (x, y - 1), \cdots\}$,$\{(x, y), \cdots, (x - 1, y), \cdots\}$,$\{(x, y), \cdots, (x + 1, y), \cdots\}$.

Step 2: From the above graph, we can build a 0-1 adjacent matrix W. We define a same size adjacent matrix X be the optimization variable. Suppose the initial state is s_i, the final state is s_f, the set of all available state is $\mathcal{S}$. Suppose that the set of all the connected edges can be written as $\mathcal{E} = \{(s_1, s_2)|W_{s_1 s_2} = 1\}$. Then we can develop the following optimization problem:

$$
\begin{aligned}
\min \quad & \sum_{e \in \mathcal{E}} W_e X_e \\
\text{s.t.} \quad & \sum_{s \in \mathcal{S}/\{s_i\}} X_{s_i s} = 1 \\
& \sum_{s \in \mathcal{S}/\{s_i\}} X_{s s_i} = 0 \\
& \sum_{s \in \mathcal{S}/\{s_f\}} X_{s_f s} = 0 \\
& \sum_{s \in \mathcal{S}/\{s_f\}} X_{s s_f} = 1 \\
& \sum_{s' \in \mathcal{S}/\{s\}} X_{s s'} = \sum_{s' \in \mathcal{S}/\{s\}} X_{s' s}, \forall s \in \mathcal{S}
\end{aligned}
$$

Exercise 2. (Basic feasible solution and vertex)

The main support evidence of simplex method is the special properties of basic feasible solution. So please prove that when P is the feasible region of a linear optimization problem, i.e., a nonempty polyhedron. Then a basic feasible solution $x \in P$ is also a vertex.

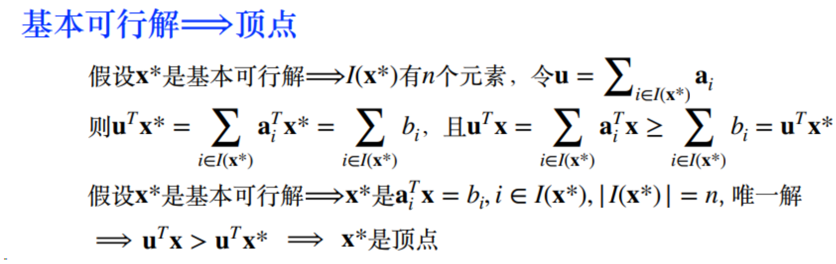

Figure 3:

Exercise 3. (Simplex method)

Now please use the simplex method to solve the following problem, please write down the simplex table every step, the optimal objective and the optimal variable value.

$$
\begin{aligned}
\min \quad & x_1 - 2x_2 + x_3 \\
\text{s.t.} \quad & x_1 + x_2 - 2x_3 + x_4 = 10 \\
& 2x_1 - x_2 + 4x_3 \leq 8 \\
& -x_1 + 2x_2 - 4x_3 \leq 4 \\
& x_i \geq 0, \quad i = 1, 2, 3, 4.
\end{aligned} \tag{1}
$$

Solution:

Step 1:

1	1	-2	1	0	0	10
2	-1	4	0	1	0	8
-1	$\boxed{2}$	-4	0	0	1	4
1	-2	1	0	0	0	0

Step 2:

$\frac{3}{2}$	0	0	1	0	-$\frac{1}{2}$	8
$\frac{3}{2}$	0	$\boxed{2}$	0	1	$\frac{1}{2}$	10
-$\frac{1}{2}$	1	-2	0	0	$\frac{1}{2}$	2
0	0	-3	0	0	1	-4

Step 3:

$\frac{3}{2}$	0	0	1	0	-$\frac{1}{2}$	8
$\frac{3}{4}$	0	1	0	$\frac{1}{2}$	$\frac{1}{4}$	5
1	1	0	0	1	1	12
$\frac{9}{4}$	0	0	0	-$\frac{3}{2}$	-$\frac{7}{4}$	-19