SI231B - Matrix Computations, Spring 2022-23

Homework Set #4

`Prof. Ziping Zhao`

Acknowledgements:

1) Deadline: **2023-04-23 23:59:59**
2) Please submit your assignments via Gradescope.
3) You can write your homework using latex/word or you can write in handwriting and submit the scanned pdf.

Problem 1. (20 points)

1) Suppose $\mathbf{A}$ is a positive definite matrix. Prove that there exists a matrix $\mathbf{B}$ such that $\mathbf{A} = \mathbf{B}^2$. (10 points)
2) Prove that if we require $\mathbf{B}$ to be positive definite, then $\mathbf{B}$ is unique. (10 points)

Solution:

1) $\mathbf{A}$ is it diagonalisable by an orthogonal matrix. That is $\mathbf{A} = \mathbf{U}\mathbf{\Lambda}\mathbf{U}^T$, with $\mathbf{U}$ orthogonal and $\mathbf{\Lambda}$ diagonal. Since $\mathbf{A}$ is positive definite, all entries of $\mathbf{\Lambda}$ are positive hence have a square root. Denote the diagonal matrix of square roots by $\sqrt{\mathbf{\Lambda}}$. Let $\mathbf{B} = \mathbf{U}\sqrt{\mathbf{\Lambda}}\mathbf{U}^T$, then it is easy to check that $\mathbf{A} = \mathbf{B}^2$.
2) Suppose $\mathbf{B}\mathbf{x} = \lambda\mathbf{x}$.

   $(\mathbf{B}^2 - \mathbf{C}^2)\mathbf{x} = (\lambda^2\mathbf{I} - \mathbf{C}^2)\mathbf{x} = (\lambda\mathbf{I} + \mathbf{C})(\lambda\mathbf{I} - \mathbf{C})\mathbf{x} = \mathbf{0}$.

   $\lambda\mathbf{I} + \mathbf{C}$ is positive definite, so $(\lambda\mathbf{I} - \mathbf{C})\mathbf{x} = \mathbf{0}$.

   Therefore $\mathbf{B}$ and $\mathbf{C}$ have the same eigenvalues and eigenvectors.

   $\mathbf{B} = \mathbf{C}$.

Problem 2. (20 points)

For any graph G with vertex set $V = \{v_1, v_2, \dots, v_n\}$, a walk from vertex u to vertex v (not necessarily distinct) is a sequence of vertices, not necessarily distinct, such that w_{i-1} and w_i are adjacent, and $w_0 = u$ and $w_k = v$. In this case, the walk is of length k. $\mathbf{A}$ is the adjacency matrix of G. Prove that the (i, j) entry of $\mathbf{A}^k$ is the number of walks from v_i to v_j of length k.

(Hint: You can use mathematical induction.)

Solution: The proof is by induction on k. For $k = 1$, $a_{ij} = 1$ implies that $v_i \sim v_j$ and then clearly there is a walk of length $k = 1$ from v_i to v_j. If on the other hand, $a_{ij} = 0$ then v_i and v_j are not adjacent and then clearly there is no walk of length $k = 1$ from v_i to v_j. Now assume that the claim is true for some $k \geq 1$ and consider the number of walks of length $k+1$ from v_i to v_j. Any walk of length $k+1$ from v_i to v_j contains a walk of length k from v_i to a neighbour of v_j. If $v_p \in N(v_j)$ then by induction the number of walks of length k from v_i to v_p is

$$\sum_{v_p \in N(v_j)} \mathbf{A}^k(i, p) = \sum_{\ell=1}^{n} \mathbf{A}^k(i, \ell)\mathbf{A}(\ell, j) = \mathbf{A}^{k+1}(i, j).$$

Problem 3. (20 points)

Given

$$\mathbf{A} = \begin{bmatrix} 10 & 5 & 2 \\ 5 & 3 & 2 \\ 2 & 2 & 3 \end{bmatrix}.$$

1) Show that $\mathbf{A}$ is positive definite. (10 points)
2) Find the Cholesky factorization of $\mathbf{A}$. (10 points)

Solution:

1) Consider all leading principal minors of $\mathbf{A}$, we have

$$\begin{aligned}
&\det({}^{(1)}\mathbf{A}) = 10 > 0 \\
&\det({}^{(2)}\mathbf{A}) = \begin{vmatrix} 10 & 5 \\ 5 & 3 \end{vmatrix} = 5 > 0 \\
&\det({}^{(3)\mathbf{A}}) = \det(\mathbf{A}) = 3 > 0
\end{aligned}$$

2) Follow the Gaussian elimination,

$$\mathbf{A} = \begin{bmatrix} 10 & 5 & 2 \\ 5 & 3 & 2 \\ 2 & 2 & 3 \end{bmatrix} \longrightarrow \begin{bmatrix} 10 & 5 & 2 \\ 0 & \frac{1}{2} & 1 \\ 0 & 1 & \frac{13}{5} \end{bmatrix} \longrightarrow \begin{bmatrix} 10 & 5 & 2 \\ 0 & \frac{1}{2} & 1 \\ 0 & 0 & \frac{3}{5} \end{bmatrix} = \mathbf{U}$$

Then, $\mathbf{A} = \mathbf{LU} = \mathbf{LDU} = \mathbf{LD}^{\frac{1}{2}}\mathbf{D}^{\frac{1}{2}}\mathbf{U} = \left(\mathbf{D}^{\frac{1}{2}}\mathbf{U}\right)^T \left(\mathbf{D}^{\frac{1}{2}}\mathbf{U}\right) = \mathbf{G}^T\mathbf{G}$, and

$$G = \begin{bmatrix} \sqrt{10} & \frac{5}{\sqrt{10}} & \frac{2}{\sqrt{10}} \\ 0 & \frac{1}{\sqrt{2}} & \sqrt{2} \\ 0 & 0 & \frac{\sqrt{3}}{\sqrt{5}} \end{bmatrix}$$

Problem 4. (20 points)

Prove the following propositions:

1) Suppose matrix $\mathbf{A} \in \mathbb{R}^n$ is positive definite, show that all diagonal entries $a_{ii} > 0$. (6 points)
2) Let $\mathbf{A} = \begin{bmatrix} 1 & a \\ a & b \end{bmatrix}$ and $a^2 < b$. Show that $\mathbf{A}$ is positive definite. (7 points)
3) Let matrix $\mathbf{A} \in \mathbb{R}^n, \mathbf{B} \in \mathbb{R}^m$ and $\mathbf{A}, \mathbf{B}$ is positive definite, show that $\begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{B} \end{bmatrix}$ is positive definite. (7 points)

Solution:

1) Since $\mathbf{A}$ is positive definite, $\mathbf{x}^T\mathbf{A}\mathbf{x} > 0$. Let $\mathbf{x} = \mathbf{e}_i$ for the i-th diagonal entries, we have

$$0 < \mathbf{x}^T\mathbf{A}\mathbf{x} = \mathbf{e}_i^T\mathbf{A}\mathbf{e}_i = a_{ii}$$

Thus $a_{ii} > 0, i = 1, \ldots, n$.

2) The determinant of the first leading principal minors $\det({}^{(1)}\mathbf{A}) = 1 > 0$, the determinant of the second leading principal minors $\det({}^{(2)}\mathbf{A}) = \det(\mathbf{A}) = b - a^2 > 0$. Thus, $\mathbf{A}$ is positive definite.

3) Let $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{y} \in \mathbb{R}^m$, $\mathbf{z} = \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} \in \mathbb{R}^{n+m}$. Since $\mathbf{A}, \mathbf{B}$ are positive definite, we have

$$\mathbf{z}^T \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{B} \end{bmatrix} \mathbf{z} = \begin{bmatrix} \mathbf{x}^T & \mathbf{y}^T \end{bmatrix} \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{B} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} = \mathbf{x}^T\mathbf{A}\mathbf{x} + \mathbf{y}^T\mathbf{B}\mathbf{y} > 0.$$

Problem 5. (20 points)

1) Given a real symmetric matrix $\mathbf{M} \in \mathbb{S}^n$ and suppose $\mathbf{M}$ satisfies the following condition

$$m_{ii} \geq \sum_{j \neq i} |m_{ij}| \quad \text{for all } i, \tag{1}$$

prove that $\mathbf{M}$ is PSD. (7 points)

2) Given a real symmetric PSD matrix $\mathbf{M} \in \mathbb{S}^n$, does $\mathbf{M}$ always satisfy the condition in (1)? *(Note: Necessary explanations are needed.)* (5 points)

3) Given a real symmetric PSD matrix $\mathbf{M} \in \mathbb{S}^n$, Prove that $\mathbf{M}$ always satisfies the following condition

$$\sum_{i=1}^{n} m_{ii} \geq \frac{2}{n-1} \sum_{i=1}^{n} \sum_{j=1}^{i-1} m_{ij}.$$

(8 points)

Solution:

1) For any $\mathbf{x} \in \mathbb{R}^n$, we have

$$\begin{aligned}\mathbf{x}^T\mathbf{M}\mathbf{x} &= \sum_{i=1}^{n}\sum_{j=1}^{n} m_{ij}x_ix_j = \sum_{i=1}^{n} m_{ii}x_i^2 + \sum_{i=1}^{n}\sum_{j=1}^{i-1} 2m_{ij}x_ix_j \geq \sum_{i=1}^{n}(\sum_{j\neq i}|m_{ij}|)x_i^2 - \sum_{i=1}^{n}\sum_{j=1}^{i-1} 2|m_{ij}||x_i||x_j| \\ &= \sum_{i=1}^{n}\sum_{j=1}^{i-1} |m_{ij}|(x_i^2 + x_j^2 - 2|x_i||x_j|) = \sum_{i=1}^{n}\sum_{j=1}^{i-1} |m_{ij}|(|x_i| - |x_j|)^2 \geq 0\end{aligned}$$

2) No. A counterexample is the following real symmetric PSD matrix

$$\mathbf{I}_4 + \mathbf{1}_4\mathbf{1}_4^T = \begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix}.$$

So the condition in (1) is just a sufficient condition for a real symmetric matrix $\mathbf{M} \in \mathbb{S}^n$ to be PSD.

3) For a real symmetric PSD matrix $\mathbf{M} \in \mathbb{S}^n$, we have

$$\text{tr}(\mathbf{M}) = \sum_{i=1}^{n} \lambda_i(\mathbf{M}) \geq \lambda_{\max}(\mathbf{M}) = \max_{\|\mathbf{x}\|_2=1} \mathbf{x}^T\mathbf{M}\mathbf{x} \geq \mathbf{x}^T\mathbf{M}\mathbf{x}, \ \forall \|\mathbf{x}\|_2 = 1. \tag{2}$$

Define $\mathbf{x} = \frac{1}{\sqrt{n}}\mathbf{1}_n$, which satisfies $\|\mathbf{x}\|_2 = 1$. We have

$$\mathbf{x}^T\mathbf{M}\mathbf{x} = \frac{1}{n}(\sum_{i=1}^{n} m_{ii} + \sum_{i=1}^{n}\sum_{j=1}^{i-1} 2m_{ij}).$$

Based on (2), we have

$$\text{tr}(\mathbf{M}) = \sum_{i=1}^{n} m_{ii} \geq \frac{1}{n}(\sum_{i=1}^{n} m_{ii} + \sum_{i=1}^{n}\sum_{j=1}^{i-1} 2m_{ij}),$$

which implies $\sum_{i=1}^{n} m_{ii} \geq \frac{2}{n-1} \sum_{i=1}^{n} \sum_{j=1}^{i-1} m_{ij}$.