Probability & Statistics
Fall 2021
Midterm
2021/11/09
Time Limit: 100 Minutes

Name (Print): _____

Advisor Name _____

This exam contains 9 pages (including this cover page) and 8 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

Try to answer as many problems as you can. The following rules apply:

- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.

Do not write in the table to the right.

Problem	Points	Score
1	10	
2	10	
3	10	
4	10	
5	20	
6	10	
7	10	
8	20	
Total:	100	

1. (10 points) A six-sided fair dice is rolled three times independently. What is more likely: a sum of 11 or a sum of 12? (You need to compute the corresponding probability for each case).

2. (10 points) A family has two children. We assume each child has a black hair with probability p, independent of each other and of gender. What is the probability that both children are girls given that at least one is a black-hair girl?

3. (10 points) Consider a coin that comes up heads with probability p and tails with probability $1 - p$. Let q_n be the probability that after n independent tosses, there have been an even number of heads. Find q_n.

4. (10 points) Suppose that there are N distinct types of coupons and that, independently of past types collected, each new one obtained is type j with probability $p_j = \frac{1}{N}$. Find the expected value and variance of the number of different types of coupons that appear among the first n collected.

5. (20 points) Alvin's database of friends contains n entries, but due to a software bug, the addresses correspond to the names in a totally random fashion. Alvin writes a holiday card to each of his friends and sends it to the (software-corrupted) address. Let X denote the number of friends of him who will get the correct card.
   (a) (5 points) Find $E(X)$.
   (b) (5 points) Find $Var(X)$.
   (c) (5 points) Find the PMF of X.
   (d) (5 points) When $n \to \infty$, show that the distribution of X converges to a Poisson distribution.

6. (10 points) In Monty Hall problem, now suppose the car is not placed randomly with equal probability behind the three doors. Instead, the car is behind door one with probability p_1, behind door two with probability p_2, and behind door three with probability p_3. Here $p_1 + p_2 + p_3 = 1$ and $p_1 \geq p_2 \geq p_3 > 0$. You are to choose one of the three doors, after which Monty will open a door he knows to conceal a goat. Monty always chooses randomly with equal probability among his options in those cases where your initial choice is correct. What strategy should you follow?

7. (10 points) At a movie theater, there is a line (queue) for buying the ticket. A manager announces that she will give a free ticket to the first person in line whose birthday is the same as someone who has already bought ticket. Now You are a VIP of this theater, and you are given the opportunity to choose any position in line. Assuming that you do not know anyone else's birthday and all birthdays are distributed randomly throughout the year (365 days in a year), which position in line gives you the largest chance of getting the free ticket?

8. (20 points) Suppose a fair coin with probability 1/2 for heads is tossed repeatedly, and we obtain a sequence of H and T (H denotes Head and T denotes Tail). Let N denote the number of toss to observe the first occurrence of the pattern "HHH".
   (a) (10 points) Find the recursive equation for the probability of event "N=n".
   (b) (5 points) Find $E(N)$.
   (c) (5 points) Find $Var(N)$.