Online Lecture Notes

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1 Recall from previous lecture: Gram-Schmidt Algorithm

There are two main steps:

1. Orthogonalization:

$$\bar{q}_i = a_i - \sum_{k=0}^{i-1} \langle a_i, q_k \rangle q_k \tag{1}$$

2. Normalization:

$$q_i = \frac{\bar{q}_i}{\|\bar{q}_i\|} \tag{2}$$

Theorem: the Gram-Schmidt algorithm generates an orthonormal basis. Proof: It is clear that all vectors q_i have norm equal to 1—due to the normalization step. Orthonality can be proven by induction:

1. Induction Start: q_1 is only a single vector. It has norm 1 by construction.
2. Induction Assumption: we assume that the vectors $q_0, q_1, \ldots, q_{i-1}$ are already orthonormal.
3. Induction Step: q_i is normalized. Thus, it remains to show that it is orthogonal to all previous vectors:

$$\begin{aligned}
\forall j \in \{0,1,2,\ldots,i-1\}, \qquad \langle q_i, q_j \rangle &\overset{(1),(2)}{=} \frac{1}{\|\bar{q}_i\|} \left\langle a_i - \sum_{k=0}^{i-1} \langle a_i, q_k \rangle q_k \ , \ q_j \right\rangle \\
&= \frac{1}{\|\bar{q}_i\|} \left[a_i - \sum_{k=0}^{i-1} \langle a_i, q_k \rangle \langle q_k, q_j \rangle \right] \\
&= \frac{1}{\|\bar{q}_i\|} \left[\langle a_i, q_j \rangle - \sum_{k=0}^{i-1} \langle a_i, q_k \rangle \delta_{k,j} \right] \\
&= \frac{1}{\|\bar{q}_i\|} \left[\langle a_i, q_j \rangle - \langle a_i, q_j \rangle \right] \\
&= 0 \ .
\end{aligned} \tag{3}$$

4. Induction conclusion: the vectors $q_0, q_1, \ldots$ are orthonormal.

2 Solution to the Gauss Approximation Problem

This section discusses the Gauss' approximation in general Hilbert space as well as the construction of its solutions.

2.1 Problem Formulation

The Gauss approximation problem is given by

$$\min_{p \in P_n} \ \|f - p\|_H^2 \ ,$$

where P_n denotes the set of polynomials up to order n and $(H, \langle \cdot, \cdot \rangle)$ is a Hilbert space with the norm $\|g\| = \sqrt{\langle g, g \rangle}$ for all $g \in H$. In this case, we assume that $f \in H$ is given and we assume $P_n \subseteq H$. Under these assumptions the Gauss' approximation problem is well-defined in any Hilbert space setting.

2.2 Orthonormal Polynomials

The first step for solving the Gauss' approximation is to construct an orthonormal basis $q_0, q_1, \ldots, q_n \in P_n \subseteq H$ by using the Gram-Schmidt algorithm (see above). This means that

$$\forall i, j \in \{0, 1, \ldots, n\}, \qquad \langle q_i, q_j \rangle = \delta_{i,j} = \left\{ \begin{array}{ll} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{array} \right. .$$

Example: see our previous lecture on Legendre polynomial, Hermite polynomials, Chebyshev polynomials, and so on.

2.3 Solution to Gauss' Approximation Problem

The main idea is to represent the polynomial $p(x)$ that we want to determine in the given orthonormal basis,

$$p(x) = c_0 q_0(x) + c_1 q_1(x) + \ldots + c_n q_n(x) = \sum_{i=0}^{n} c_i q_i(x) \ .$$

Next, we recall that optimal polynomial $p \in P_n$ are uniquely characterized by their optimality condition

$$\forall q \in P_n, \qquad \langle f - p, q \rangle = 0 \qquad \Longleftrightarrow \qquad (f - p) \perp P_n \ .$$

The main idea is to substitute the above representation of p into the optimality condition:

$$\left\langle f - \sum_{i=0}^{n} c_i q_i \ , \ q \right\rangle = 0$$

needs to hold for all $q \in P_n$. Thus, this condition must hold in particular for all the basis functions q_k. This means that

$$\begin{aligned}
\forall k \in \{0,1,\ldots,n\}, \qquad 0 &= \left\langle f - \sum_{i=0}^{n} c_i q_i \ , \ q_k \right\rangle \\
&= \langle f, q_k \rangle - \sum_{i=0}^{n} c_i \langle q_i, q_k \rangle \\
&= \langle f, q_k \rangle - \sum_{i=0}^{n} c_i \delta_{i,k} \\
&= \langle f, q_k \rangle - c_k \qquad (4)
\end{aligned}$$

If we solve this equation with respect to c_k, we obtain the solution of Gauss' approximation problem

$$\forall k \in \{0,1,\ldots,n\}, \qquad c_k = \langle f, q_k \rangle \ .$$

This means that we have found all the coefficient of the unique optimal polynomial $p(x)$ with respect to the basis $q_0, q_1, \ldots, q_n$.

3 Examples for Gauss Approximation

Solving Gauss' approximation is straighforward if we already have an orthonormal basis. For example, we could consider the Legendre polynomials:

$$q_0(x) = \sqrt{\frac{1}{2}} \tag{5}$$

$$q_1(x) = \sqrt{\frac{3}{2}}x \tag{6}$$

$$q_2(x) = \sqrt{\frac{5}{8}}(3x^2 - 1) \tag{7}$$

$$\vdots \tag{8}$$

3.1 Square-root function

Let us set $f(x) = \sqrt{x}$ over the interval $[0, 1]$. In this case, we changed the interval $[-1, 1]$ to $[0, 1]$. Thus, if we want to use the above Legendre polynomial, we should first introduce a change of variables

$$x = \frac{1}{2} + \frac{1}{2}y \qquad \Longleftrightarrow \qquad y = 2x - 1$$

with $y \in [-1, 1]$ (such that $x \in [0, 1]$). The Gauss' approximation that we want to solve is given by

$$\min_{p \in P_2} \int_{-1}^{1} \left[f\left(\frac{1}{2} + \frac{1}{2}y\right) - p(y) \right]^2 \mathrm{d}y = \min_{p \in P_2} \int_{-1}^{1} \left[\tilde{f}(y) - p(y) \right]^2 \mathrm{d}y$$

with $\tilde{f}(y) = \sqrt{\frac{1}{2} + \frac{1}{2}y}$. The solution coefficients can be worked out explicitly:

$$\begin{aligned} c_0 &= \langle \tilde{f}, q_0 \rangle = \int_{-1}^{1} \sqrt{\frac{1}{2}} \sqrt{\frac{1}{2} + \frac{1}{2}y}\, \mathrm{d}y = \frac{2^{\frac{3}{2}}}{3} \\ c_1 &= \langle \tilde{f}, q_1 \rangle = \int_{-1}^{1} \sqrt{\frac{3}{2}}y \sqrt{\frac{1}{2} + \frac{1}{2}y}\, \mathrm{d}y = \frac{2^{\frac{3}{2}}}{5\sqrt{3}} \\ c_2 &= \langle \tilde{f}, q_2 \rangle = \int_{-1}^{1} \left[\sqrt{\frac{5}{8}}(3y^2 - 1) \right] \sqrt{\frac{1}{2} + \frac{1}{2}y}\, \mathrm{d}y = -\frac{2^{\frac{3}{2}}}{21\sqrt{5}}. \end{aligned}$$

Thus, the solution to the Gauss' approximation problem is in this example given by

$$p(y) = 2^{\frac{3}{2}}/3\sqrt{1/2} + 2^{3/2}/(5\sqrt{3})\sqrt{3/2}y - 2^{3/2}/(21\sqrt{5}) * \sqrt{5/8}(3y^2 - 1)\ .$$

If we want to approximation the function $f(x)$ on the interval $[0, 1]$, we would need to convert this back by substituting $y = 2x - 1$.

3.2 Final Remarks

Always make sure that you use the correct basis function sequence! This means, scale your interval correctly, and make sure that you are using the right inner product.