Discrete Mathematics: Lecture 23

graph, vertex, edge, endpoints, directed, undirected, multiple edge, loop, complete graph, cycle, wheel, cube

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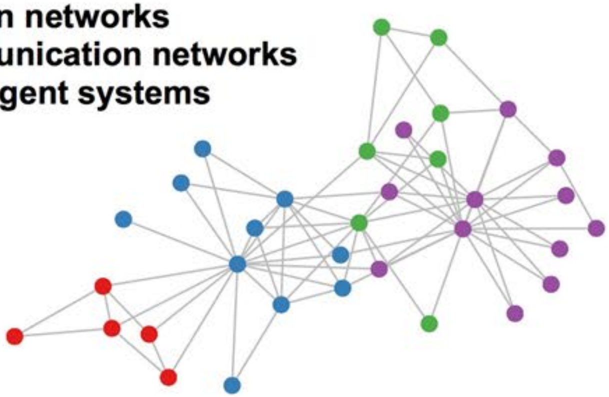
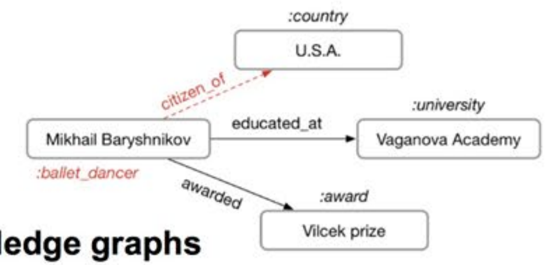
Notes by Prof. Liangfeng Zhang

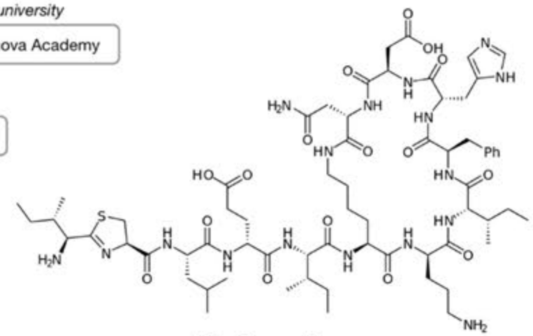
Real-world Graphs

Social networks
Citation networks
Communication networks
Multi-agent systems



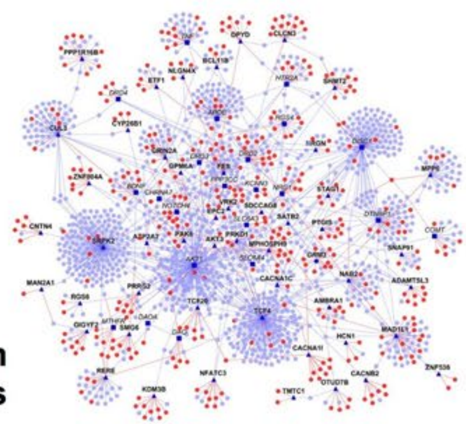


Knowledge graphs



Molecules

Protein interaction networks

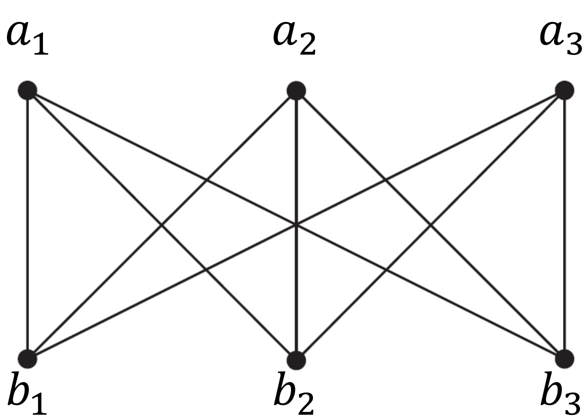


Road maps

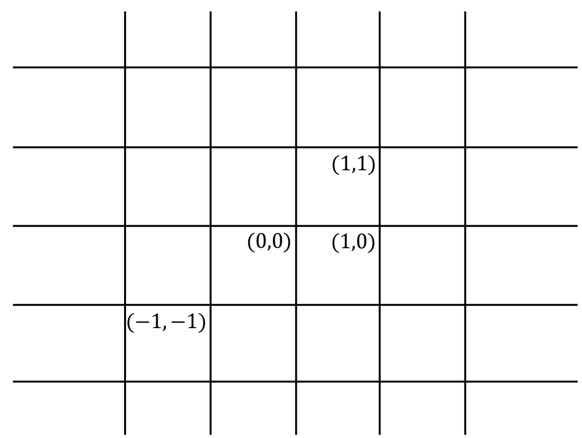
Graph

DEFINITION: A **graph** $G = (V, E)$ is defined by a nonempty set V of **vertices**顶点 and a set E of **edges**边, where each edge is associated with one or two vertices (called **endpoints**端点 of the edge).

- **Infinite Graph**无限图: $|V| = \infty$ or $|E| = \infty$
- **Finite Graph**有限图: $|V| < \infty$ and $|E| < \infty$; //$|V|$ is called the **order**阶数 of G



$V = \{a_1, a_2, a_3, b_1, b_2, b_3\}$
$E = \left\{\{a_i, b_j\}: i, j = 1,2,3\right\}$



$V = \{(i, j): i, j \in \mathbb{Z}\}$
$E = \{\{(a, b), (c, d)\}: |a - c| = 1 \text{ or } |b - d| = 1\}$

Graphs

- Loop & multiple edge

An edge with one endpoint is called a **loop**.
If there is more than one edge between two distinct vertices, it is called a **multiple edge**.

- Simple graph

A **simple graph** is a finite graph with no loops nor multiple edges.

- Weighted graph

A **weighted graph** is a graph $G = (V, E)$ such that each edge is assignated with a strictly positive number.

Graphs

- Directed graph

A **directed graph** $G = (V, E)$ consists of:

- V a non empty set of **vertices**,
- E a set of **directed edges**

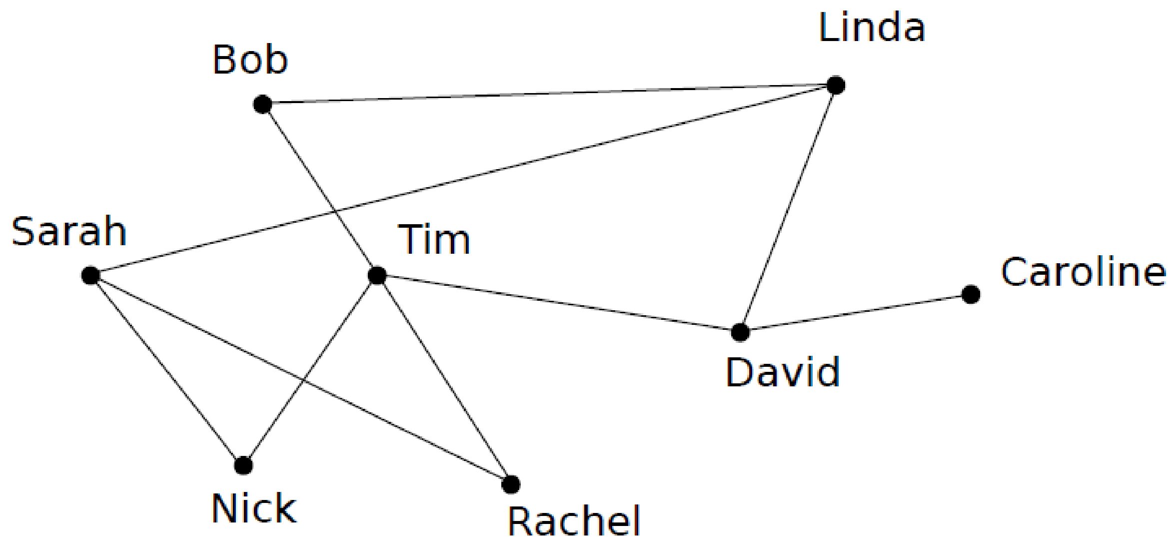
Each edge e is associated with an **ordered pair of vertices** (u, v), we say that e **starts at** u and **ends at** v.

- Subgraph

A **subgraph** of a graph $G = (V, E)$ is a graph $H = (W, F)$ where $W \subset V$, $F \subset E$. A subgraph H of G is a **proper subgraph** if $H \neq G$.

Graph Examples

Acquaintanceship Graph:



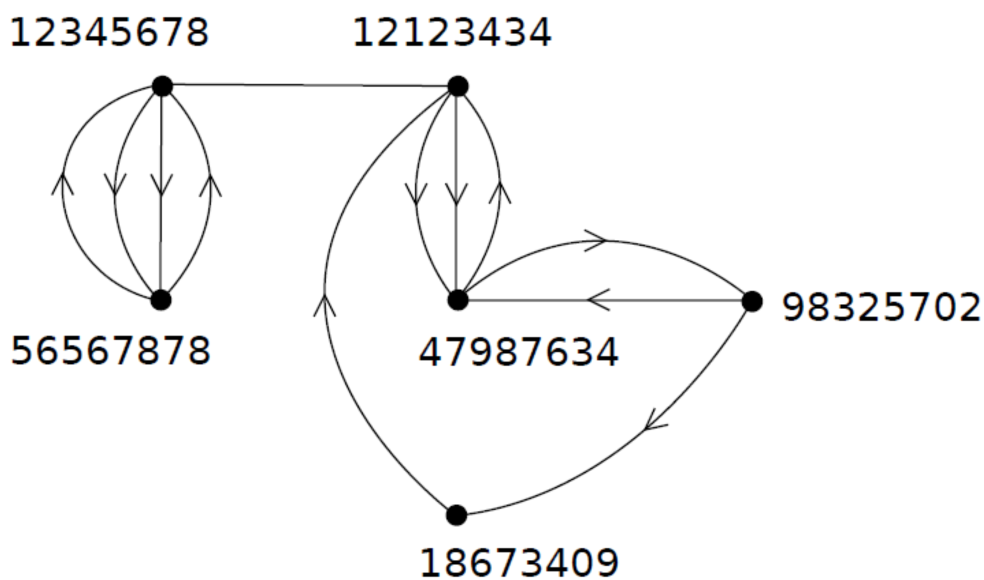
Tim knows Bob, David, Rachel and Nick. But Tim doesn't know Linda neither Caroline.

Simple graph, undirected

Graph Examples

Call Graphs: directed edges; the same edge may appear multiple times

- Vertices: telephone numbers
- Edges: there is an arc (u, v) if u called v
- AT&T experiment: calls during 20 days (290 million vertices and 4 billion edges)



Directed graph, multiple edges

Graph Examples

Precedence Graph

S_1 $a := 0$

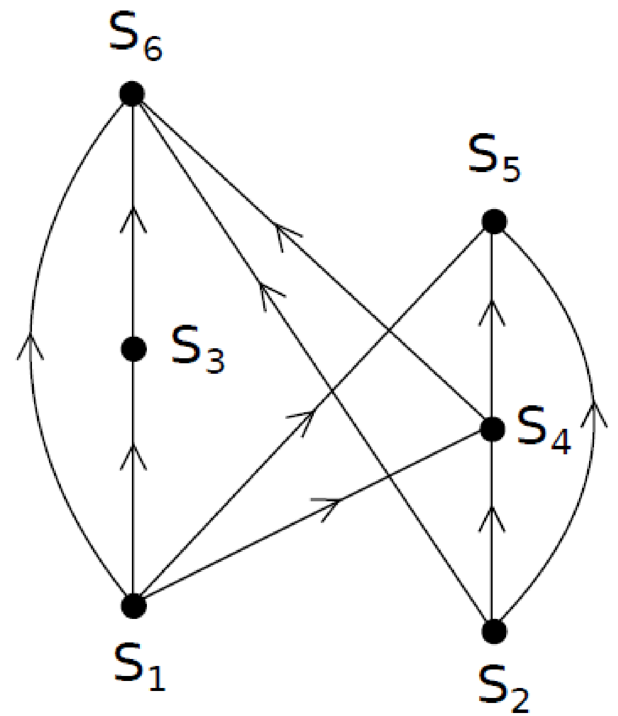
S_2 $b := 1$

S_3 $c := a + 1$

S_4 $d := b + a$

S_5 $e := d + 1$

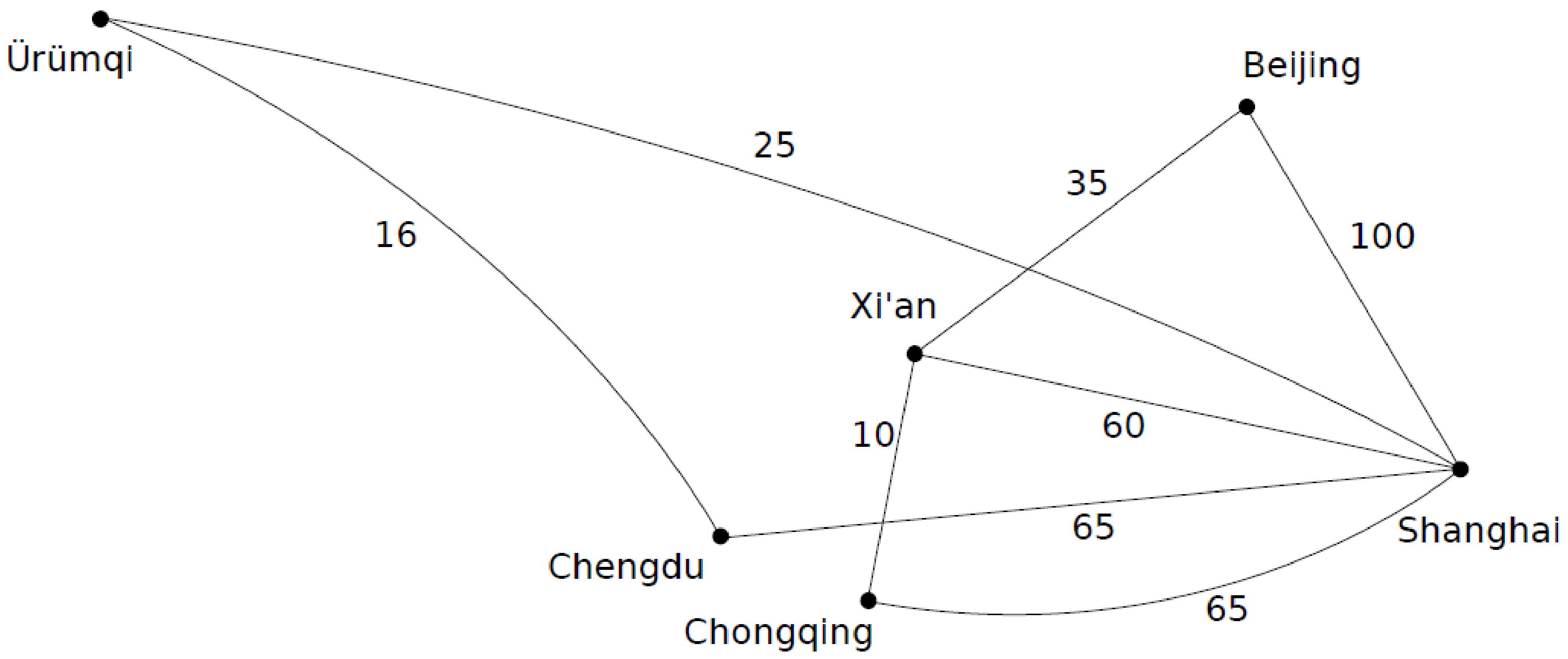
S_6 $f := c + d$



Directed simple graph

Graph Examples

Flights



Weighted graph

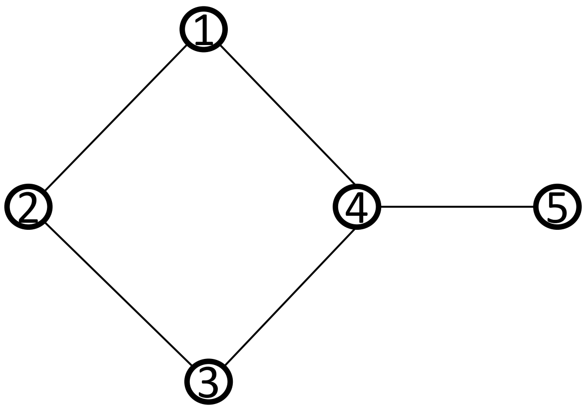
Types of Graphs

DEFINITION: Let $G = (V, E)$ be a graph with vertex set $V = \{v_1, \dots, v_n\}$.

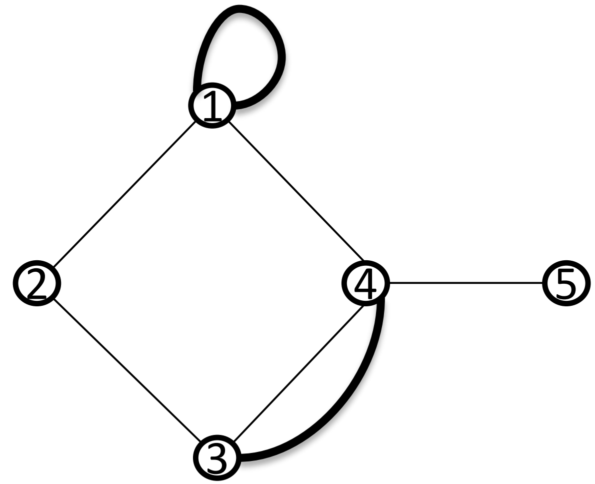
- **Question 1**: are the edges of G **directed**有向的?
  - No: G is an **undirected graph**无向图; the edge connecting v_i, v_j: $\{v_i, v_j\}$
  - Yes: G is a **directed graph**有向图; the edge starting at v_i and ending at v_j: (v_i, v_j)
- **Question 2**: are there **multiple edges**多重边connecting two different vertices v_i, v_j?
  - No: G is a **simple graph**简单图**;** Yes: G is a **multigraph**多重图
- **Question 3:** are there **loops**自环 connecting a vertex v_i to itself?
  - Yes: G is a **pseudograph**伪图

Type	Edges	Multiple Edges Allowed?	Loops Allowed?
Simple graph	undirected	No	No
Multigraph	undirected	Yes	No
Pseudograph	undirected	Yes	Yes
Simple directed graph	directed	No	No
Directed multigraph	directed	Yes	No
Mixed graph	undirected + directed	Yes	Yes

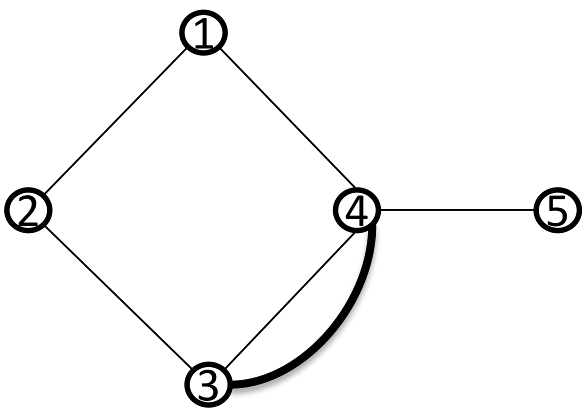
Types of Graphs



A Simple Graph (G_1)



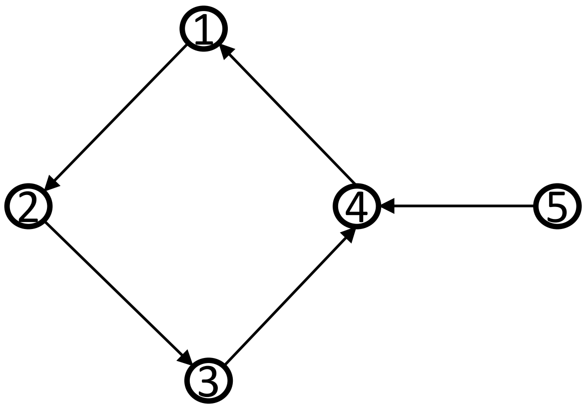
A Pseudograph (G_3)



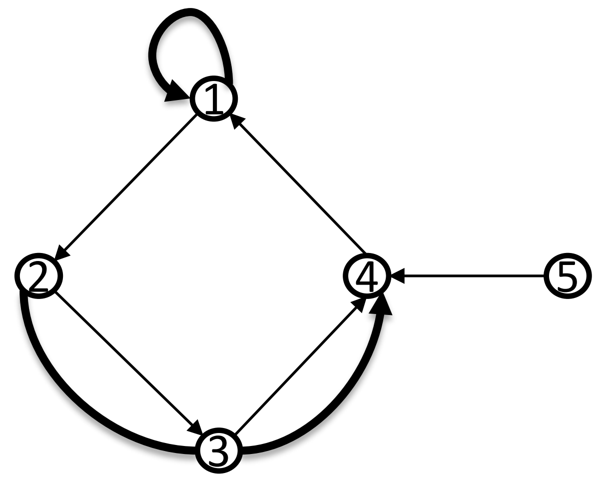
A Multigraph (G_2)

- Vertex set: $V = \{1,2,3,4,5\}$
- Edge set of G_1: $E = \{\{1,2\}, \{2,3\}, \{3,4\}, \{1,4\}, \{4,5\}\}$
- $\{4,5\}$ is an edge of the simple graph G_1
  - 4,5 are endpoints of the edge $\{4,5\}$
  - $\{4,5\}$ connects 4 and 5.
- $\{3,4\}$ is a multiple edge of the multigraph G_2
- There is a loop connecting 1 to itself in G_3

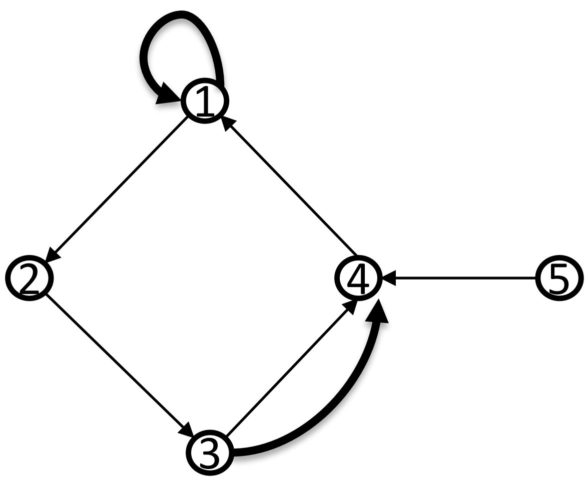
Types of Graphs



A Simple Directed Graph (G_4)



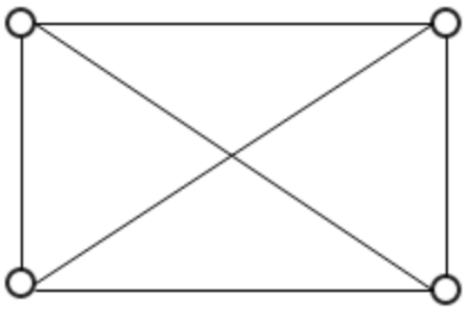
A Mixed Graph (G_6)



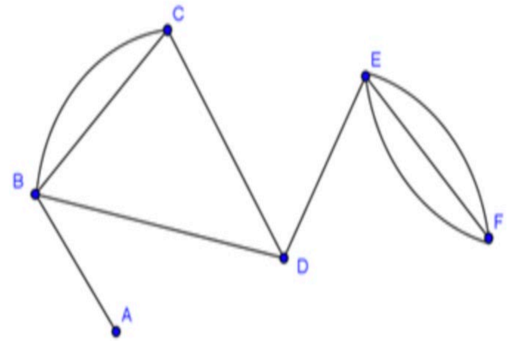
A Directed Pseudograph (G_5)

- Vertex set: $V = \{1,2,3,4,5\}$
- Edge set of G_4: $E = \{(1,2), (2,3), (3,4), (4,1), (5,4)\}$
  - $(5,4)$ is a directed edge
  - $(5,4)$ starts at 5 and ends at 4
- $(3,4)$ is a directed multiple edge in G_5
- There is a loop connecting 1 to itself in G_5

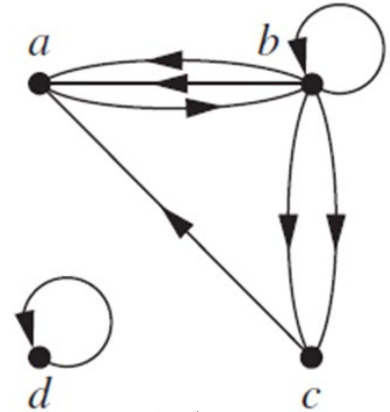
Bonus exercise



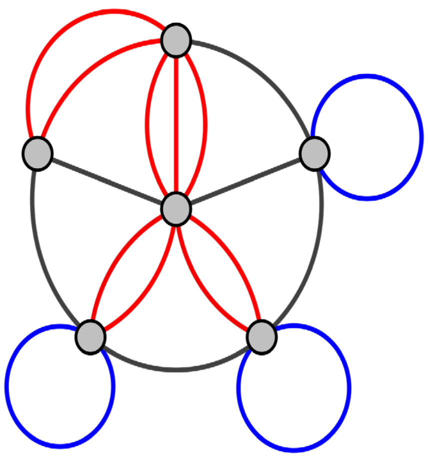
（1）



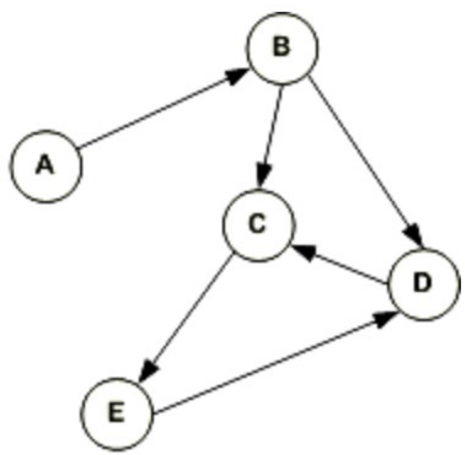
（3）



（5）

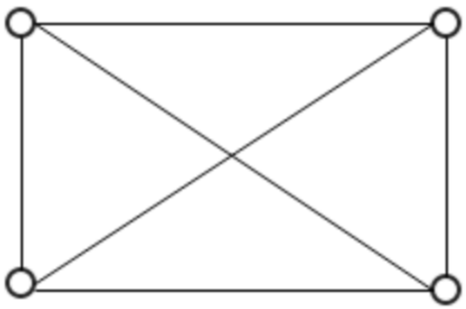


（2）

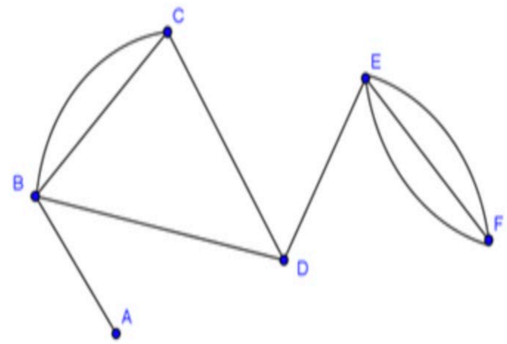


（4）

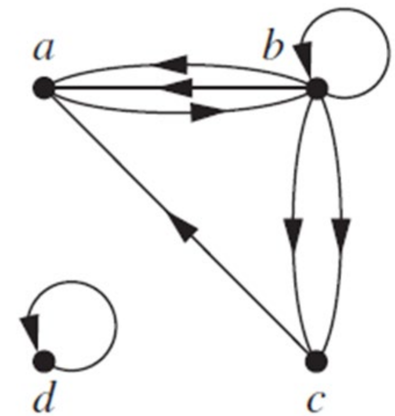
Bonus exercise



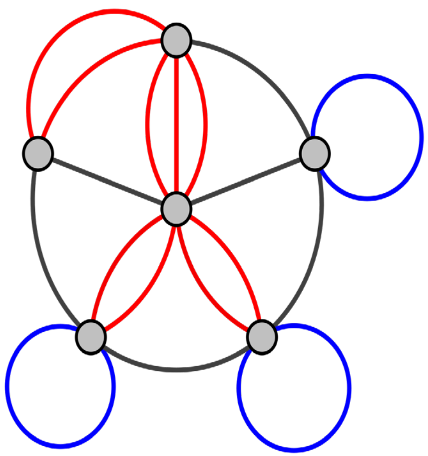
(1) simple graph



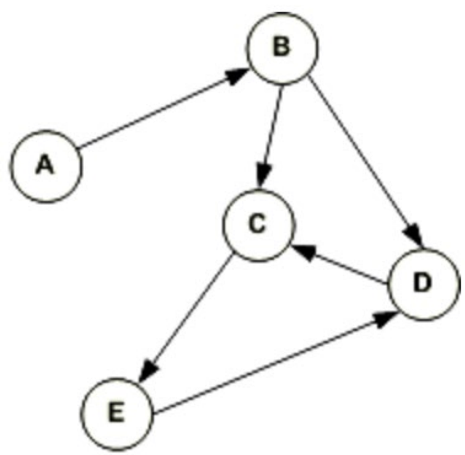
(3) multigraph



(5) directed pseudograph



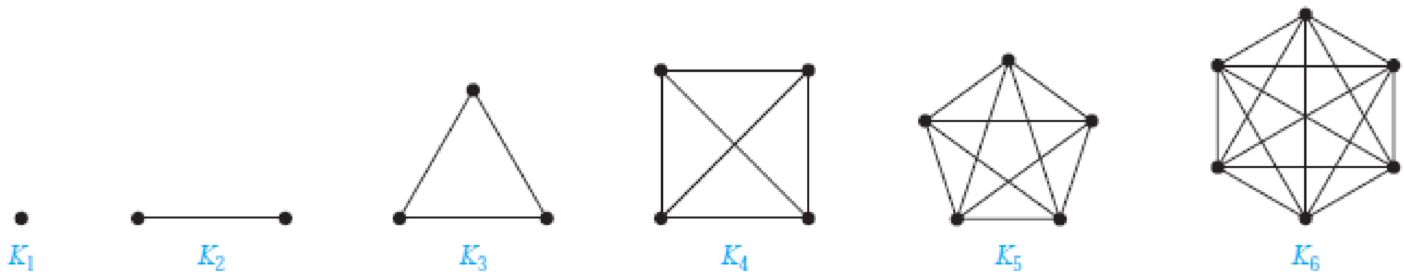
(2) pseudograph



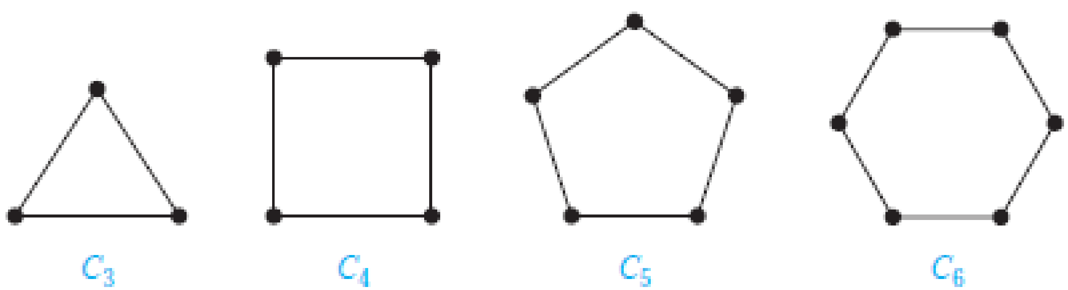
(4) simple directed graph

Special Simple Graphs

Complete Graph完全图 K_n: $V = \{v_1, \dots, v_n\}$; $E = \left\{\{v_i, v_j\}: 1 \le i \ne j \le n\right\}$

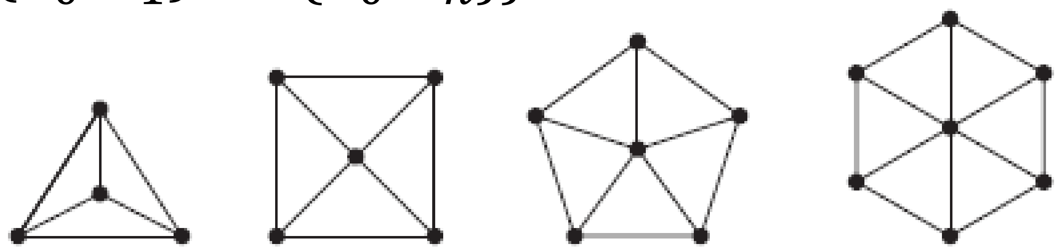


Cycle环,圈 C_n: $V = \{v_1, v_2, \dots, v_n\}$; $E = \left\{\{v_1 v_2\}, \{v_2, v_3\}, \dots, \{v_n, v_1\}\right\}$



Special Simple Graphs

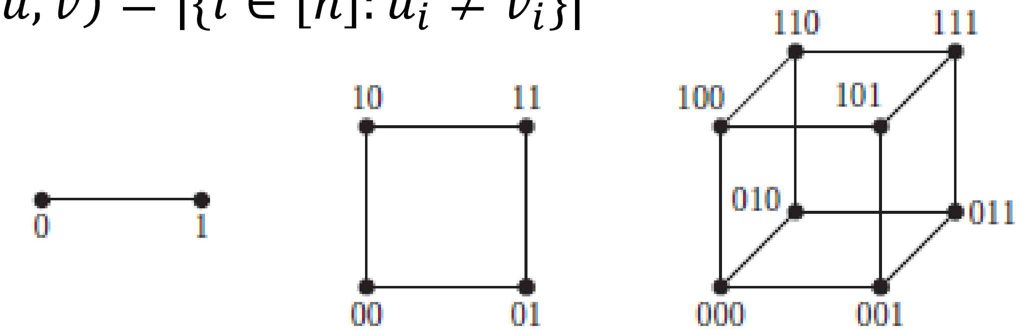
Wheel轮 W_n: $V = \{v_0, v_1, v_2, \dots, v_n\}$; $E = \{\{v_1, v_2\}, \dots, \{v_n, v_1\}\} \cup \{\{v_0, v_1\}, \dots, \{v_0, v_n\}\}$



W_3 W_4 W_5 W_6

n-**Cubes**方体 Q_n: $V = \{0,1\}^n$; $E = \{\{u, v\}: d(u, v) = 1\}$

- $d(u, v) = |\{i \in [n]: u_i \neq v_i\}|$

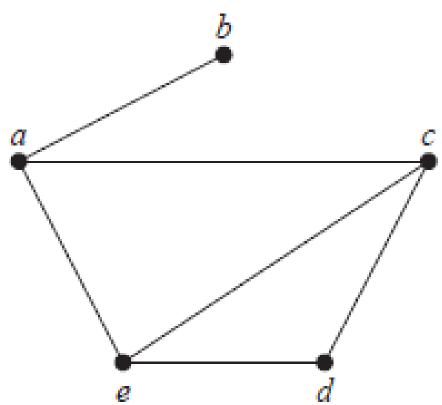


Q_1 Q_2 Q_3

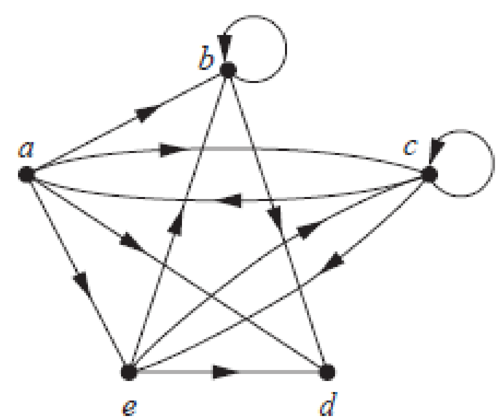
Adjacency List

DEFINITION: Let $G = (V, E)$ be a graph with no multiple edges. The **adjacency list**邻接表 of G is a list the vertices of the graph and all adjacent vertices

- $v_i, v_j \in V$ are **adjacent**相邻的 if $\{v_i, v_j\}$ or (v_i, v_j) is an edge



a	b, c, e
b	a
c	a, d, e
d	c, e
e	a, c, d

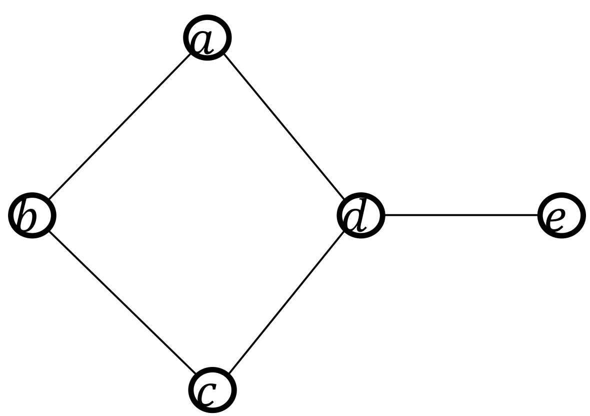


a	b, c, d, e
b	b, d
c	a, c, e
d	
e	b, c, d

Adjacency Matrix

DEFINITION: Let $G = (V = \{v_1, \dots, v_n\}, E)$ be a simple graph. The **adjacency matrix**邻接矩阵 of G is an $n \times n$ matrix $A = (a_{ij})$, where

$$a_{ij} = \begin{cases} 1 & \{v_i, v_j\} \in E \\ 0 & \{v_i, v_j\} \notin E \end{cases}$$

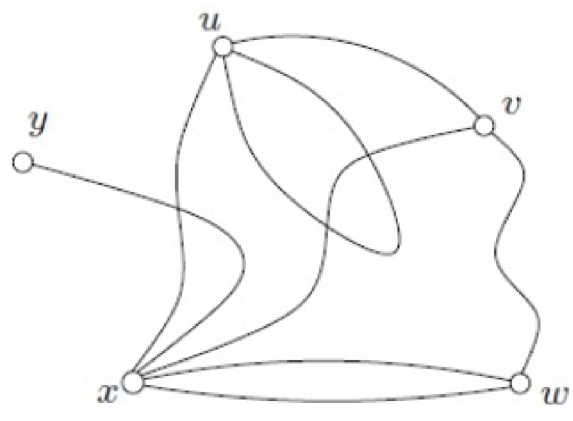


	a	b	c	d	e
a	0	1	0	1	0
b	1	0	1	0	0
c	0	1	0	1	0
d	1	0	1	0	1
e	0	0	0	1	0

Adjacency Matrix

DEFINITION: Let $G = (V = \{v_1, \dots, v_n\}, E)$ be an undirected graph. The **adjacency matrix** of G is an $n \times n$ matrix $A = (a_{ij})$, where

- $a_{ij} =$ **multiplicity**重数 of $\{v_i, v_j\}$ when $i \neq j$
- $a_{ii} = 1$ if $\exists$ a loop from v_i to itself; $a_{ii} = 0$, otherwise.

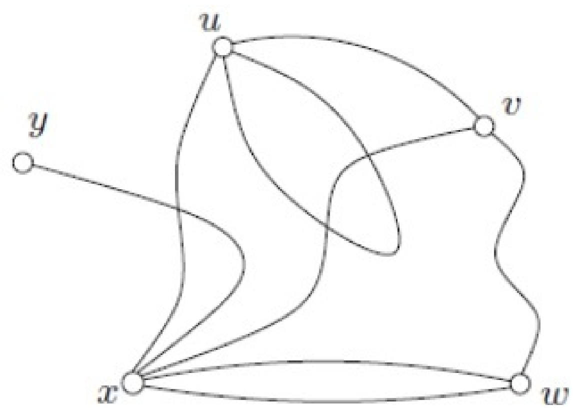


	u	v	w	x	y
u	1	1	0	1	0
v	1	0	1	1	0
w	0	1	0	2	0
x	1	1	2	0	1
y	0	0	0	1	0

Adjacency Matrix

DEFINITION: Let $G = (V = \{v_1, \dots, v_n\}, E)$ be an undirected graph. The **adjacency matrix** of G is an $n \times n$ matrix $A = (a_{ij})$, where

- $a_{ij} =$ **multiplicity**重数 of $\{v_i, v_j\}$ when $i \neq j$
- $a_{ii} = 1$ if $\exists$ a loop from v_i to itself; $a_{ii} = 0$, otherwise.



	x	y	u	v	w
x	0	1	1	1	2
y	1	0	0	0	0
u	1	0	1	1	0
v	1	0	1	0	1
w	2	0	0	1	0

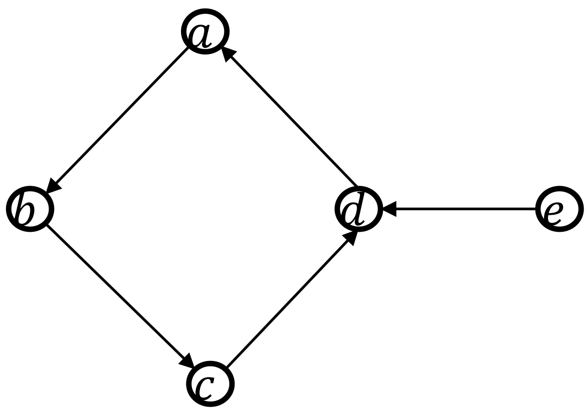
REMARKs: features of the adjacency matrices of undirected graphs

- The adjacency matrix depends on the ordering of the vertices
- The adjacency matrix of a simple graph is always symmetric
- The (i, j) entry counts the multiplicity of $\{v_i, v_j\}$, $i \neq j$

Adjacency Matrix

DEFINITION: Let $G = (V = \{v_1, \dots, v_n\}, E)$ be a simple directed graph. The **adjacency matrix** of G is an $n \times n$ matrix $A = (a_{ij})$, where

$$a_{ij} = \begin{cases} 1 & (v_i, v_j) \in E \\ 0 & (v_i, v_j) \notin E \end{cases}$$



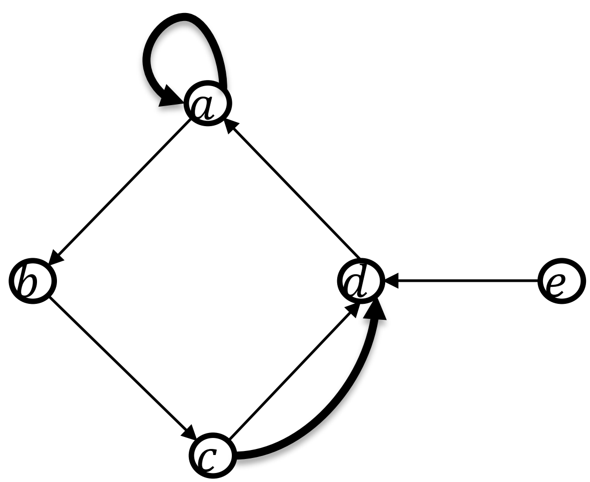
	a	b	c	d	e
a	0	1	0	0	0
b	0	0	1	0	0
c	0	0	0	1	0
d	1	0	0	0	0
e	0	0	0	1	0

REMARKS: The adjacency matrix is no longer symmetric

Adjacency Matrix

DEFINITION: Let $G = (V = \{v_1, \dots, v_n\}, E)$ be a directed multigraph. The **adjacency matrix** of G is an $n \times n$ matrix $A = (a_{ij})$, where

$$a_{ij} = \begin{cases} \text{multiplicity of } (v_i, v_j) & (v_i, v_j) \in E \\ 0 & (v_i, v_j) \notin E \end{cases}$$



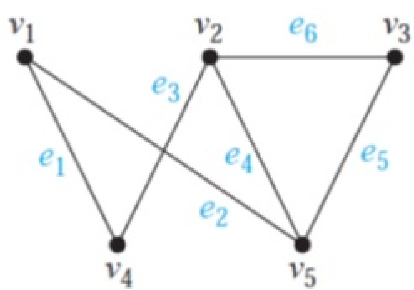
	a	b	c	d	e
a	1	1	0	0	0
b	0	0	1	0	0
c	0	0	0	2	0
d	1	0	0	0	0
e	0	0	0	1	0

Incidence Matrix

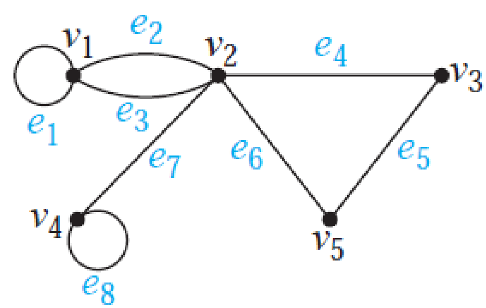
DEFINITION: Let $G = (V = \{v_1, \dots, v_n\}, E = \{e_1\ , \dots, e_m\})$ be undirected. The **incidence matrix**关联矩阵 of G is an $n \times m$ matrix $B = (b_{ij})$, where

$$b_{ij} = \begin{cases} 1 & \text{if } e_j \text{ is incident with } v_i \\ 0 & \text{otherwise} \end{cases}$$

- e_j **incident with** v_i: v_i is an endpoint of e_j



$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array}
\begin{array}{c} \begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{array} \\
\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

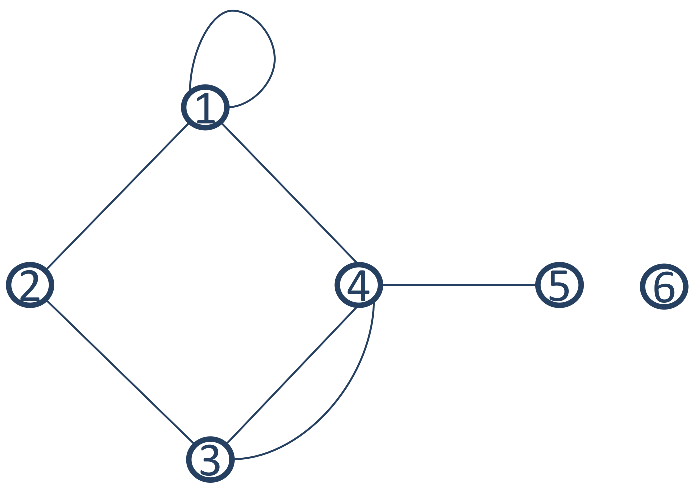


$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array}
\begin{array}{c} \begin{array}{cccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \end{array} \\
\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

Degree

DEFINITION: Let $G = (V, E)$ be an <u>undirected</u> graph. We say that two vertices $u, v \in V$ are **adjacent**相邻的 (or **neighbors**邻居) if $\{u, v\} \in E$.

- **neighborhood**邻域 of v in G: $N(v) = \{u \in V : \{u, v\} \in E\}$
  - $N(A) = \cup_{v \in A} N(v)$ for $A \subseteq V$
- the **degree**度 $\deg(v)$ of $v \in V$ in G, is the number of edges incident with v
  - every loop from v to v contributes 2 to $\deg(v)$
- v is **isolated**孤立的 if $\deg(v) = 0$; v is **pendant**悬挂的 if $\deg(v) = 1$

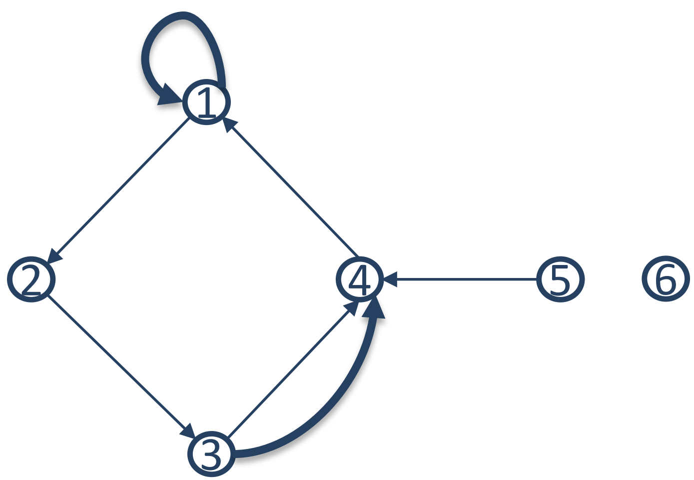


- 4 and 5 are adjacent
- $\{4,5\}$ is incident with 4 and 5
- $N(4) = \{1,3,5\}$; $N(\{1,4\}) = \{1,2,3,4,5\}$
- $\deg(1) = 4, \deg(2) = 2, \deg(3) = 3, \deg(4) = 4, \deg(5) = 1$
- 6 is isolated; 5 is pendant

Degree

DEFINITION: Let $G = (V, E)$ be a <u>directed</u> graph. If $(u, v) \in E$, we say that u is **adjacent to** v and v is **adjacent from** u.

- u is the **initial vertex**起始点 of (u, v); v is the **terminal vertex**终点 of (u, v)
  - $u = v$: u is the initial vertex and the terminal vertex
- **in-degree**入度 $\deg^-(v)$: the number of edges where v is the terminal vertex
- **out-degree**出度 $\deg^+(v)$: the number of edges where v is the initial vertex
  - $u = v$: the loop contributes 1 to $\deg^-(v)$ and 1 to $\deg^+(v)$



- 5 is adjacent to 4; 4 is adjacent from 5
- 5 is the initial vertex of (5,4)
- 4 is the terminal vertex of (5,4)
- 1 is the initial and terminal vertex of a loop
- $\deg^-(1) = 2$; $\deg^+(1) = 2$
- $\deg^-(4) = 3$; $\deg^+(4) = 1$

Handshaking Theorem

THEOREM: Let $G = (V, E)$ be an <u>undirected</u> graph. Then

$2|E| = \sum_{v \in V} \deg(v)$ and $|\{v \in V: \deg(v) \text{ is odd}\}|$ is even.

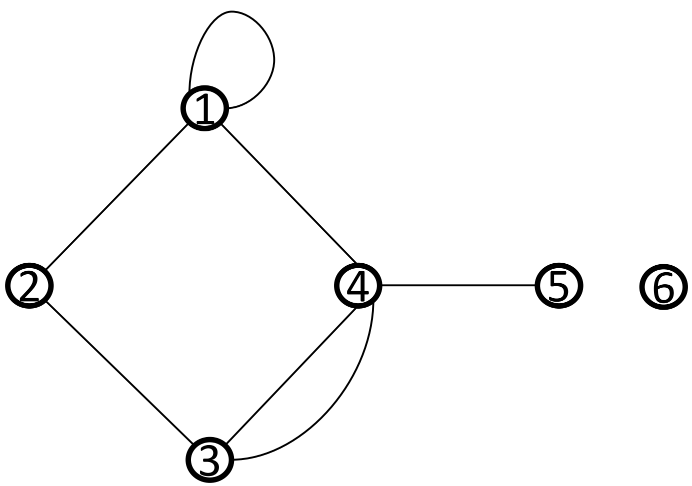
- Any edge $e \in E$ contribute 2 to the sum $\sum_{v \in V} \deg(v)$
  - $e = \{v_i, v_j\}$: e contributes 1 to $\deg(v_i)$ and 1 to $\deg(v_j)$
  - $e = \{v_i\}$: e contributes 2 to $\deg(v_i)$
- The m edges contribute $2|E|$ to $\sum_{v \in V} \deg(v)$.
  - Hence, $\sum_{v \in V} \deg(v) = 2|E|$
- $\sum_{v \in V} \deg(v) = \sum_{v \in V: 2|\deg(v)} \deg(v) + \sum_{v \in V: 2\nmid\deg(v)} \deg(v)$
  - $2|\sum_{v \in V} \deg(v)$; $2|\sum_{v \in V: 2|\deg(v))} \deg(v)$
    - $2|\sum_{v \in V: 2\nmid\deg(v)} \deg(v)$
      - $|\{v \in V: \deg(v) \text{ is odd}\}|$ must be even

Handshaking Theorem

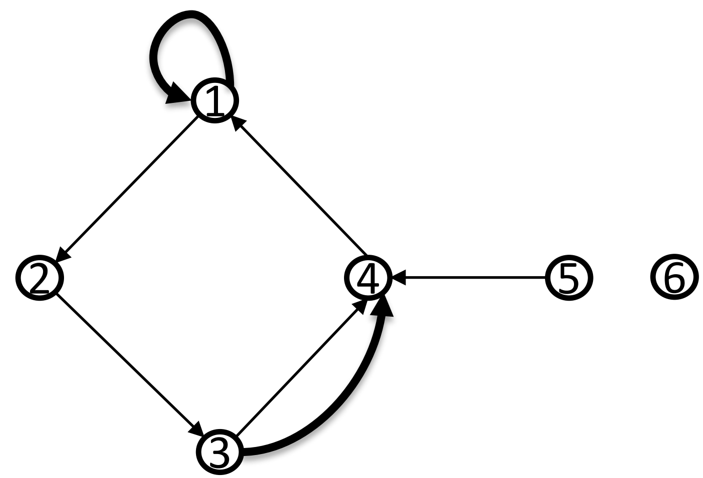
THEOREM: Let $G = (V, E)$ be a <u>directed</u> graph. Then

$$\sum_{v \in V} \deg^-(v) = \sum_{v \in V} \deg^+(v) = |E|.$$

- Every edge $e \in E$ contributes 1 to $\sum_{v \in V} \deg^-(v)$
  - $e = (v_i, v_j)$ contributes 1 to $\deg^-(v_i)$
- Hence, $\sum_{v \in V} \deg^-(v) = |E|$



v	1	2	3	4	5	6
$\deg(v)$	4	2	3	4	1	0



v	1	2	3	4	5	6
$\deg^-(v)$	2	1	1	3	0	0
$\deg^+(v)$	2	1	2	1	1	0