# Reference Solution of SI231b: Matrix Computations of 2021 Fall

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**Problem 1.**

To show $\mathcal{T}$ is also a subspace, one needs to show

1. $0 \in \mathcal{T}$
2. $\forall x,\ y \in \mathcal{T},\ x + y \in \mathcal{T}$
3. $\forall \alpha \in \mathbb{R},\ \forall x \in \mathcal{T},\ \alpha x \in \mathcal{T}$

Item 1 accounts for 2 points, each of the second and third item accounts for 3 points.

One set of basis of $\mathcal{T}$ is $Tv_1,\ Tv_2,\ \cdots\cdots, Tv_n$. Next, one needs to show any vector from $\mathcal{T}$ can be represented by this set of basis, and the linear independence of these vectors need to be shown.

Give the set of basis vectors accounts for 1 point; Show any vector from $\mathcal{T}$ can be represented by this set of vectors accounts for 3 points; Show the linear independence of these vectors account for 3 points.

**Problem 2.**

(1) To show the uniqueness of this LU factorization, one can assume that there exists two distinct factorization which gives

$$A = L_1 U_1 = L_2 U_2.$$

This gives $L_2^{-1} L_1 = U_2 U_1^{-1}$. Since the product of lower-triangular matrices are lower-triangular, and the product of upper-triangular matrices are upper triangular. This means that $L_2^{-1} L_1 = U_2 U_1^{-1}$ equals a diagonal matrix.

Since $L_1$ and $L_2$ are lower-triangular matrices with unit diagonals, it's easy to verify that $L_2^{-1}$ also has unit diagonals and also $L_2^{-1} L_1$, this in turn states that

$$L_2^{-1} L_1 = U_2 U_1^{-1} = I,$$

which gives $L_1 = L_2$ and $U_1 = U_2$

Only giving $L_2^{-1} L_1 = U_2 U_1^{-1} = I$ without explanation gets 3 points.

(2) $L = \begin{bmatrix} 1 & & \\ -\frac{1}{3} & 1 & \\ -\frac{1}{3} & -\frac{1}{2} & 1 \end{bmatrix}$, $U = \begin{bmatrix} -3 & 1 & -2 \\ & \frac{4}{3} & \frac{1}{3} \\ & & -\frac{1}{2} \end{bmatrix}$

Both correct $L$ and $U$ gets 5 points; 1 correct gets 3 points; none correct, but the results are lower-triangular and upper triangular gets 1 point.

(3) $x_1 = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -1 \end{bmatrix}$, $x_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$, $x_3 = \begin{bmatrix} \frac{3}{2} \\ \frac{1}{2} \\ -2 \end{bmatrix}$.

All correct gets 5 points; 2 correct answers get 4 points; 1 correct answer gets 2 points.

(4) According to (3), $A^{-1} = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 1 & \frac{3}{2} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ -1 & -1 & -2 \end{bmatrix}$.

$\|A\|_1 = 5$, $\|A\|_\infty = 6$, $\|A^{-1}\|_1 = 4$, $\|A^{-1}\|_\infty = 4$.

Therefore,

$$\kappa_1(A) = \|A\|_1 \|A^{-1}\|_1 = 20,$$
$$\kappa_\infty(A) = \|A\|_\infty \|A^{-1}\|_\infty = 24.$$

Correct $A^{-1}$ gets 2 points, each correct norm for both $A$ and $A^{-1}$ gets 1 point, and the 2 correct condition numbers get 1 point.

**Problem 3.**

(1) The orthonormal basis for $\mathcal{R}(A)$ is

$$q_1 = \frac{a_1}{\|a_1\|_2} = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}.$$

$$q_2 = a_2 - < a_2, q_1 > q_1 = \begin{bmatrix} 2 \\ 2 \\ -2 \\ 2 \end{bmatrix}, \quad q_2 = \frac{q_2}{\|q_2\|_2} = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}.$$

Correct $q_1$ gets 2 points, correct $q_2$ gets 3 points. Wrong $q_1$ or $q_2$ but correct formula gets 1 point each.

(2) $Q = [q_1,\ q_2] = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & -1 \\ -1 & 1 \end{bmatrix}$, $R = \begin{bmatrix} 2 & 2 \\ & 4 \end{bmatrix}$.

Correct $Q$ gets 2 points and $R$ gets 3 points.

(3) The least square solution is given by solving

$$QRx = QQ^Tb,$$

i.e., $x = R^{-1}Q^Tb = \begin{bmatrix} \frac{3}{2} \\ \frac{3}{2} \end{bmatrix}$.

If numerical result of $x$ is wrong but correct formula of $x = R^{-1}Q^Tb$ or $x = (A^TA)^{-1}A^Tb$ is given, this case gets 2 points.

(4) The orthogonal projector onto $\mathcal{R}(A)$ is given by $P = QQ^T$. Since

$$\mathcal{R}(A) + \mathcal{N}(A^T) = \mathbb{R}^4, \quad \mathcal{R}(A) \perp \mathcal{N}(A^T).$$

The complementary projector $I - P$ is the orthogonal projector that projects onto $\mathcal{N}(A^T)$, which is

$$I - P = \frac{1}{2}\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Without stating $\mathcal{N}(A^T)$ being the orthogonal complement of $\mathcal{R}(A)$ but just give $I-P$ gets 6 points; Explain the above logic in detail without the final numerical results gets 8 points; All correct gets 10 points.

**Problem 4.**

(1) Assume $\lambda_1$ and $\lambda_2$ are two distinct eigenvalues of $A$ with corresponding eigenvectors $v_1$ and $v_2$, then

$$Av_1 = \lambda_1 v_1, \quad Av_2 = \lambda_1 v_2.$$

This gives

$$\lambda_1 v_2^T v_1 = v_2^T A v_1, \quad \lambda_2 v_1^T v_2 = v_1^T A v_2,$$

i.e., $(\lambda_1 - \lambda_2)v_2^T v_1 = 0$ due to the symmetry of the real matrix $A$. Since $\lambda_1$ and $\lambda_2$ are distinct, therefore, $v_1$ and $v_2$ are orthogonal.

Using the Schur decomposition to prove this result is also possible.

(2) The eigenpair is given by $(\lambda + d^T v, v)$.

(3) $|\lambda(A)| \le \|A\|_1 = \frac{2}{3}$, therefore, $\lim_{k\to\infty} A^k = 0$.

Using $|\lambda(A)| \le \|A\|_\infty = \frac{2}{3}$ to give the result is also correct; Or by computing the eigenvalues of $A$ to give the result is also right.

(4) QR iteration with Hessenberg reduction can preserve the upper Hessenberg structure for each iteration, therefore it reduces the computational complexity of each step from $\mathcal{O}(n^3)$ to $\mathcal{O}(n^2)$.

Without stating that the upper Hessenberg structure preserving at each iteration loses 2 points; Without quantifying the complexity loses 1 points.

**Problem 5**

(1) The SVD of $A$ given by $A = U\Sigma V^T$ with $U \in \mathbb{R}^{m\times m}$, $\Sigma \in \mathbb{R}^{m\times n}$, $V \in \mathbb{R}^{n\times n}$, then

$$\begin{aligned}
\|Ax\|_2^2 &= x^T A^T A x \\
&= x^T V \Sigma^T \Sigma \underbrace{V^T x}_{y} \\
&= y^T \Sigma^T \Sigma y \\
&= y^T \begin{bmatrix} \sigma_1^2 & & & \\ & \sigma_2^2 & & \\ & & \cdots & \\ & & & \sigma_n^2 \end{bmatrix} y
\end{aligned}$$

Associating with $\|y\|_2 = \|x\|_2 = 1$, one can finish this proof.

(2) It's easy to show that

$$\begin{bmatrix} & A^T \\ A & \end{bmatrix} = \begin{bmatrix} & V \\ U & \end{bmatrix} \begin{bmatrix} \Sigma & \\ & \Sigma \end{bmatrix} \begin{bmatrix} V^T & \\ & U^T \end{bmatrix}$$

To show this form is an SVD, one needs to show the orthogonality of the matrix $\begin{bmatrix} & V \\ U & \end{bmatrix}$ and $\begin{bmatrix} V^T & \\ & U^T \end{bmatrix}$, one also needs to point out that $\begin{bmatrix} \Sigma & \\ & \Sigma \end{bmatrix}$ is a diagonal matrix with positive diagonal entries.

Only giving this form gets 7 points; each statement of the orthogonality and positiveness of the diagonal matrix gets 1 point

**Bonus**

To show the non-singularity of the matrix $I - A$, one can show that $\mathcal{N}(I - A) = \{0\}$.

$\forall x \in \mathcal{N}(I - A)$, $(I - A)x = 0$, i.e., $x = Ax$.

Next, one can/need to show that $x \in \mathcal{R}(I - A^T)$. It's easy to see that $x = \frac{1}{2}(I - A^T)x$ due to the fact $x = Ax$ and $A = -A^T$. This gives $x \in \mathcal{R}(I - A^T)$.

Together one gets $x \in \mathcal{R}(I - A^T) \cap \mathcal{N}(I - A) = \{0\}$, which means $\mathcal{N}(I - A) = \{0\}$.

To show the orthogonality of the matrix $(I - A)^{-1}(I + A)$, one need to show

$(I-A)^{-1}(I+A)((I-A)^{-1}(I+A))^T = I$, which is easy because

$$\begin{aligned}
(I-A)^{-1}(I+A)((I-A)^{-1}(I+A))^T &= (I-A)^{-1}(I+A)(I+A^T)(I-A^T)^{-1} \\
&= (I-A)^{-1}(I+A)(I-A)(I+A)^{-1} \\
&= (I-A)^{-1}(I-A^2)(I+A)^{-1} \\
&= (I-A)^{-1}(I-A)(I+A)(I+A)^{-1} \\
&= I
\end{aligned}$$

Showing the non-singularity gets 5 points and the orthogonality gets 5 points.

There are different approaches to solve this problem, here we only give the easy one.