Online Lecture Notes

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1 Periodic Orbits of Time-Varying Differential Equations

Let us consider the time-varying differential equation

$$\dot{x}(t) = A(t)x(t) + b(t)$$

with periodic coefficient functions $A : \mathbb{R} \to \mathbb{R}^{n_x \times n_x}$ and $b : \mathbb{R} \to \mathbb{R}^{n_x}$ such that

$$A(t+T) = A(t) \quad \text{and} \quad b(t+T) = b(t) \; .$$

This means that the corresponding fundamental solution $G(t,\tau)$ is periodic, too:

$$G(t+T, \tau+T) = G(t,\tau) \; .$$

Here, the variable $T > 0$ denotes the period time of the system. In order to generalize the concept of steady-states, we introduce so-called periodic orbits $x_{\mathrm{p}} : \mathbb{R} \to \mathbb{R}^{n_x}$ satisfying

$$\forall t \in [0,T], \qquad \dot{x}_{\mathrm{p}}(t) = A(t)x_{\mathrm{p}}(t) + b(t) \quad \text{and} \quad x_{\mathrm{p}}(0) = x_{\mathrm{p}}(T) \; .$$

We learned in the last lecture that the solution of the linear time varying linear differential equation can be expressed by using the fundamental solution. In this case, this means that

$$\forall t \in [0,T], \qquad x_{\mathrm{p}}(t) = G(t,0)x_{\mathrm{p}}(0) + \int_0^t G(t,\tau)b(\tau)\,\mathrm{d}\tau \; .$$

In particular, we can evaluate this expression at the time $t = T$, which yields

$$x_{\mathrm{p}}(T) = G(T,0)x_{\mathrm{p}}(0) + \int_0^T G(T,\tau)b(\tau)\,\mathrm{d}\tau = x_{\mathrm{p}}(0) \; .$$

The solution of the above linear with respect to $x_{\mathrm{p}}(0)$ is formally given by

$$x_{\mathrm{p}}(0) = [I - G(0,T)]^{-1} \int_0^T G(T,\tau)b(\tau)\,\mathrm{d}\tau \; .$$

Clearly, this expression is only correct if the so called *monodromy matrix* $G(0,T)$ has not eigenvalues that are equal 1. Notice that the periodic orbit itself is then given by

$$x_{\mathrm{p}}(t) = G(t,0)[I - G(0,T)]^{-1} \int_0^T G(T,\tau)b(\tau)\,\mathrm{d}\tau + \int_0^t G(t,\tau)b(\tau)\,\mathrm{d}\tau$$

1.1 General solution of periodic time-varying systems

In the next step, we would like to analyze the general solution of the differential equation

$$\dot{x}(t) = A(t)x(t) + b(t) \qquad \text{with} \qquad x(0) = x_0$$

with periodic coefficient functions A and b, as above. We assume that we have already found a periodic orbit $x_{\mathrm{p}}(t)$ satisfying

$$\forall t \in [0, T], \qquad \dot{x}_{\mathrm{p}}(t) = A(t)x_{\mathrm{p}}(t) + b(t) \quad \text{and} \quad x_{\mathrm{p}}(0) = x_{\mathrm{p}}(T) \; .$$

The idea now is to analyze the difference between the function $x(t)$ and the periodic orbit $x_{\mathrm{p}}(t)$. We denote this difference by

$$y(t) = x(t) - x_{\mathrm{p}}(t) \; .$$

Notice that this difference function satisfies the differential equation

$$\begin{aligned} \dot{y}(t) &= \dot{x}(t) - \dot{x}_{\mathrm{p}}(t) \\ &= A(t)x(t) + b(t) - [A(t)x_{\mathrm{p}}(t) + b(t)] \\ &= A(t)[x(t) - x_{\mathrm{p}}(t)] \\ &= A(t)y(t) \\ \text{with} \qquad y(0) &= x_0 - x_{\mathrm{p}}(0) = y_0 \end{aligned} \tag{1}$$

Notice that the solution trajectory $y(t)$ of this differential equation can be written in the form

$$y(t) = G(t, 0)y_0 \; .$$

If we want to analyze this expression for $y(t)$ for large times $t \to \infty$, we can use the periodicity property. Let us assume that $t = NT + t'$ with $t' \in [0, T]$ and $N \in \mathbb{N}$. We have

$$\begin{aligned} G(t, 0) &= G(TN + t', 0) \\ &= G(NT + t', NT)G(NT, 0) \\ &= G(t', 0)G(NT, 0) \\ &= G(t', 0)G(NT, (N-1)T)G((N-1)T, (N-2)T) \ldots G(T, 0) \\ &= G(t', 0) \underbrace{G(T, 0)G(T, 0) \ldots G(T, 0)}_{N \text{ times}} \\ &= G(t', 0)G(T, 0)^N \; . \end{aligned} \tag{2}$$

Analyzing this expression for large times $t \to \infty$ is equivalent to analyzing this exression for $N \to \infty$. Thus, the only question is what happens to the expression

$$G(T, 0)^N$$

for large $N \to \infty$. Let $G(T, 0) = T(D + M)T^{-1}$ be a Jordan normal decomposition of the monodromy matrix with D being a diagonal matrix, M being a nil-potent matrix, and $DM = MD$, then

$$
\begin{aligned}
G(T,0)^N &= \left[T(D+M)T^{-1}\right]^N \\
&= T\left[D+M\right]^N T^{-1} \\
&= T\left[\sum_{i=0}^{N}\binom{N}{i}M^{N-i}D^i\right]T^{-1} \qquad (3)
\end{aligned}
$$

Here, we have used that D and M commute. Moreover, we know that M is nil-potent. Thus,

$$\forall i < N - n_x, \qquad M^{N-i} = 0$$

This means that

$$
\begin{aligned}
G(T,0)^N &= T\left[\sum_{i=0}^{N}\binom{N}{i}M^{N-i}D^i\right]T^{-1} \\
&= T\left[\sum_{i=N-n_x}^{N}\binom{N}{i}M^{N-i}D^i\right]T^{-1} \\
&= T\left[\sum_{j=0}^{n_x}\binom{N}{N-n_x+j}M^{N-(N-n_x+j)}D^{N-n_x+j}\right]T^{-1} \\
&= TD^{N-n_x}\underbrace{\left[\sum_{j=0}^{n_x}\binom{N}{N-n_x+j}M^{n_x-j}D^j\right]}_{O(N^{n_x})}T^{-1}
\end{aligned}
\qquad (4)
$$

If the eigenvalues of the monodromy matrix $G(T,0)$ are all in the open unit disc, we have

$$\lim_{N\to\infty} D^{N-n_x} = 0$$

exponentially. More in detail, if $|D_{i,i}| < \kappa < 1$, then

$$\|G(t,0)\| \leq \kappa^{N-n_x}\left\|\left[\sum_{j=0}^{n_x}\binom{N}{N-n_x+j}M^{n_x-j}D^j\right]\right\|$$

By using this expression we see that

$$\lim_{N\to\infty}\|G(t,0)\| = 0\,,$$

since the exponential terms of the form κ^{N-n_x} overpower the polynomial growing terms. This is equivalent to stating the following:

Theorem: If the eigenvalues of the monodromy matrix $G(T, 0)$ are all in the open unit disc, then

$$\lim_{t \to 0} G(t, 0) = 0 \ ,$$

which implies that

$$\lim_{t \to \infty} \|x(t) - x_{\mathrm{p}}(t)\| = 0 \ .$$

This is the same as saying that the trajectory $x(t)$ converges to the periodic orbit $x_{\mathrm{p}}(t)$.