Online Lecture Notes

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1 Linear Time-Varying Differential Equations

In this lecture, we analyze linear time-varying ODEs of the form

$$\dot{x}(t) = A(t)x(t) + b(t) \quad \text{with} \quad x(0) = x_0 \ .$$

Here, $A : \mathbb{R} \to \mathbb{R}^{n_x \times n_x}$ and $b : \mathbb{R} \to \mathbb{R}^{n_x}$ are integrable coefficients functions.

1.1 Uniqueness of Solutions

If A is bounded, $\|A(t)\|_2 \leq \sigma$ for a given upper bound $\sigma < \infty$, then the solution of the linear time-varying ODE is unique. We have two ways to prove this.

1. The first way to prove is to use the Picard-Lindelöf theorem: since A is bounded we have

   $$\|A(t)x(t)+b(t)-(A(t)y(t)+b(t))\|_2 \ \leq \ \|A(t)(x(t)-y(t))\|_2 \leq \sigma \|x(t)-y(t)\|_2 \ .$$

   Thus, the ODE is Lipschitz continuous, which means that the solution exists and is unique.

2. The other more direct proof is to start to solutions x_1 and x_2 of the time-varying ODE, such that

   $$y(t) = x_1(t) - x_2(t) \qquad \text{satisfies} \qquad \dot{y}(t) = A(t)y(t) \quad \text{with} \quad y(0) = 0 \ .$$

   Next, we introduce the auxiliary function

   $$v(t) = e^{-2\sigma|t|}\|y(t)\|_2^2 \ .$$

   Let us work out the time derivative of v, which is given by

   $$\dot{v}(t) = -2\sigma \mathrm{sgn}(t) e^{-2\sigma|t|}\|y(t)\|_2^2 + e^{-2\sigma|t|}\frac{\mathrm{d}}{\mathrm{d}t}\|y(t)\|_2^2$$

   In order to simplify this expression further, we first have a look at the term

   $$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\|y(t)\|_2^2 &= \frac{\mathrm{d}}{\mathrm{d}t} y(t)^\intercal y(t) \\ &= \dot{y}(t)^\intercal y(t) + y(t)^\intercal \dot{y}(t) \\ &= y(t)^\intercal A(t)^\intercal y(t) + y(t)^\intercal A(t) y(t) \\ &= 2y(t)^\intercal A(t) y(t) \ , \end{aligned} \tag{1}$$

since we can exploit the symmetry of the Euclidean scalar product, using that $y(t)^\intercal A(t)^\intercal y(t) = (A(t)y(t))^\intercal y(t) = y(t)^\intercal(A(t)y(t)) = y(t)^\intercal A(t)y(t)$. By substituting this result, we arrive at the expression

$$\begin{aligned}
\dot{v}(t) &= -2\sigma\mathrm{sgn}(t)e^{-2\sigma|t|}\|y(t)\|_2^2 + e^{-2\sigma|t|}\frac{\mathrm{d}}{\mathrm{d}t}\|y(t)\|_2^2 \\
&= -2\sigma\mathrm{sgn}(t)e^{-2\sigma|t|}\|y(t)\|_2^2 + e^{-2\sigma|t|}2y(t)^\intercal A(t)y(t) \\
&= -2e^{-2\sigma|t|}y(t)^\intercal\left[A(t) - \mathrm{sgn}(t)\sigma I\right]y(t) \qquad (2)
\end{aligned}$$

Let us discuss two cases separately:

- If $t > 0$, then we have

$$\dot{v}(t) = -2e^{-2\sigma|t|}y(t)^\intercal\left[A(t) - \sigma I\right]y(t) \ \leq \ 0$$

- If $t < 0$, then we have

$$\dot{v}(t) = -2e^{-2\sigma|t|}y(t)^\intercal\left[A(t) + \sigma I\right]y(t) \ \geq \ 0$$

We now know that the function v satisfies

(a) $v(t) \geq 0$,

(b) $v(0) = 0$,

(c) $\dot{v}(t) \leq 0$ for $t > 0$, and

(d) $\dot{v}(t) \geq 0$ for $t < 0$.

But the only function v, which satisfies all four properties is given by $v(t) = 0$. Thus, we have

$$0 = v(t) = e^{-2\sigma|t|}\|y(t)\|_2^2 \quad \Longrightarrow \quad y(t) = 0\ .$$

Thus, $x_1(t) = x_2(t)$ and the solution must be unique. This completes out explicit proof of uniqueness.

1.2 Construction of solutions in the scalar case

Let us consider the general scalar linear time-varying differential equation

$$\dot{x}(t) = a(t)x(t) + b(t) \quad \text{with} \quad x(0) = x_0\ .$$

Our goal is to show that the function

$$x(t) = \exp\left(\int_0^t a(\tau)\mathrm{d}\tau\right)x_0 + \int_0^t \exp\left(\int_\tau^t a(\tau')\mathrm{d}\tau'\right)b(\tau)\,\mathrm{d}\tau \qquad (3)$$

is the solution of the above scalar ODE. In order to show this, we work out the time derivative of the above explicit expression, which satisfies

$$\begin{aligned}
\dot{x}(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\left[\exp\left(\int_0^t a(\tau)\mathrm{d}\tau\right)x_0 + \int_0^t \exp\left(\int_\tau^t a(\tau')\mathrm{d}\tau'\right)b(\tau)\,\mathrm{d}\tau\right] \\
&= a(t)\exp\left(\int_0^t a(\tau)\mathrm{d}\tau\right)x_0 + \exp\left(\int_t^t a(\tau')\mathrm{d}\tau'\right)b(t) \\
&\quad + \int_0^t a(t)\exp\left(\int_\tau^t a(\tau')\mathrm{d}\tau'\right)b(\tau)\,\mathrm{d}\tau \\
&= a(t)\left[\exp\left(\int_0^t a(\tau)\mathrm{d}\tau\right) + \int_0^t \exp\left(\int_\tau^t a(\tau')\mathrm{d}\tau'\right)b(\tau)\,\mathrm{d}\tau\right]x_0 + b(t) \\
&= a(t)x(t) + b(t) \qquad (4)
\end{aligned}$$

Thus, our differential equation is satisfied; the initial value condition

$$x(0) = \exp\left(\int_0^0 a(\tau)\mathrm{d}\tau\right)x_0 + \int_0^0 \exp\left(\int_\tau^0 a(\tau')\mathrm{d}\tau'\right)b(\tau)\,\mathrm{d}\tau \;=\; x_0\,.$$

This completes our proof.

1.3 Warning!

If the function $A(t)$ is uniformly commuting, $A(t)A(t') = A(t')A(t)$ for all $t, t' \in \mathbb{R}$, we do have that

$$x(t) = \exp\left(\int_0^t A(\tau)\mathrm{d}\tau\right)x_0 + \int_0^t \exp\left(\int_\tau^t A(\tau')\mathrm{d}\tau'\right)b(\tau)\,\mathrm{d}\tau\,.$$

BUT, if A is not uniformly commuting with itself, then this formula is wrong in general.

1.4 Fundamental Solutions

The main motivation for introducting fundamental solutions of linear-time varying ODEs is the following. For the scalar case we have

$$\begin{aligned}
x(t) &= \exp\left(\int_0^t a(\tau)\mathrm{d}\tau\right)x_0 + \int_0^t \exp\left(\int_\tau^t a(\tau')\mathrm{d}\tau'\right)b(\tau)\,\mathrm{d}\tau \\
&= G(t,0)x_0 + \int_0^t G(t,\tau)b(\tau)\,\mathrm{d}\tau\,, \qquad (5)
\end{aligned}$$

where we have introduced the auxiliary function

$$G(t,\tau) = \exp\left(\int_\tau^t a(\tau')\mathrm{d}\tau'\right)\,.$$

The function G satisfies the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}G(t,\tau) = a(t)G(t,\tau) \qquad \text{and} \qquad G(\tau,\tau) = 1\,.$$

Now, the good news that this idea generalizes for the vector-valued case. For this aim, we define the matrix-valued function $G(t, \tau)$ as the solution of the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t} G(t, \tau) = A(t) G(t, \tau) \qquad \text{and} \qquad G(\tau, \tau) = I \ .$$

Now, it turns out that we have

$$x(t) = G(t, 0) x_0 + \int_0^t G(t, \tau) b(\tau) \, \mathrm{d}\tau \ .$$

This means that the function G generalize the matrix exponential! In the next lecture, we will prove that this claim holds.