EE160 Introduction to Control: Homework 1

(Grading: max 10 points per exercise)

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Deadline: Mar 4, 2022

Please submit your homework as pdf file.

1. *Scalar linear differential equations* Find explicit expressions for the differential equations

   (a) $\dot{x}(t) = -2x(t) + 4 \quad \text{with} \quad x(0) = 0$

   (b) $\dot{x}(t) = x(t) - 2 \quad \text{with} \quad x(1) = 1$

   (c) $\dot{x}(t) = -x(t) + 2 \quad \text{with} \quad x(-1) = 1.$

2. *Functional equation of the exponential function*

   (a) Let $x(t)$ be the solution of a scalar linear differential equation of the form

   $$\dot{x}(t) = ax(t) \quad \text{with} \quad x(0) = 1 \tag{1}$$

   Show that x satisfies $x(t_1 + t_2) = x(t_1)x(t_2)$ for all $t_1, t_2 \in \mathbb{R}$.

   (b) Can you show the reverse statement? That is, can we show that if a differentiable function x satisfied $x(t_1 + t_2) = x(t_1)x(t_2)$ for all $t_1, t_2 \in \mathbb{R}$ as well as $x(0) = 1$, then x satisfies a scalar linear differential equation of the form

   $$\dot{x}(t) = ax(t) \quad \text{with} \quad x(0) = 1$$

   Either prove that this is possible or construct a counter example.

3. *Heat transfer through a wall.* Let us consider two neighbouring rooms that are separated by a wall.The constant heat capacities of the rooms are denoted by C1 and C2, respectively, such that the energy of the rest room is $E_1(t) = C_1T_1(t)$ while the energy of the second room is $E_2(t) = C_2T_2(t)$, where $T_1(t)$ and $T_2(t)$ denote the temperatures of the rooms at time t. We assume that the rooms are isolated in such a way that the total energy $E_1(t) + E_2(t) = E$ is constant, but the wall between the rooms is not isolated. The heat transfer through this wall is proportional to the temperature difference between the rooms such that

   $$\dot{E}_1(t) = k(T_2(t) - T_1(t)),$$

   where k is the heat transfer coefficient of the wall.

   (a) Derive a scalar differential equation for the temperature $T_1(t)$ under the assumption that the initial room temperatures $T_1(0)$ and $T_2(0)$ are given. (*Hint: you may use that the total energy* $C_1T_1(t) + C_2T_2(t) = C_1T_1(0) + C_2T_2(0)$ *is invariant in order to eliminate* $T_2(t)$.)

   (b) Due to the physical interpretation of $C1, C2$, and k, we may assume that these constants are all strictly positive. Prove that the room temperatures T_1 and T_2 satisfy

   $$\lim_{t\to\infty} T_1(t) = \lim_{t\to\infty} T_2(t) = T_s,$$

   where $T_s = \dfrac{C_1T_1(0) + C_2T_2(0)}{C_1 + C_2}$ is the steady-state temperature.