SI231 - Matrix Computations, 2022 Fall

Homework Set #3

Prof. Yue Qiu

Acknowledgements:

1) Deadline: **2022-11-16 10:59:59**
2) **Late Policy details** can be found on piazza.
3) Submit your homework in **Homework 3** on **Gradescope**. Entry Code: **4V2N55**. **Make sure that you have correctly select pages for each problem.** If not, you probably will get 0 point.
4) No handwritten homework is accepted. You need to write LaTeX. (If you have difficulties in using LaTeX, you are allowed to use **MS Word or Pages** for the first and the second homework to accommodate yourself.)
5) Use the given template and give your solution in English. Solution in Chinese is not allowed.
6) Your homework should be uploaded in the PDF format, and the naming format of the file is not specified.

I. QR DECOMPOSITION VIA GRAM-SCHMIDT ORTHOGONALITY

Problem 1. (13 points + 7 points) Consider the matrix $\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & -1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$.

1) Compute its QR decomposition via Gram-Schmidt Orthogonality. You should write the derivation of finding the orthogonal matrix $\mathbf{Q}$ and upper triangular matrix $\mathbf{R}$.

   Hint: You are highly required to write down your solution procedures in detail. Only fractions are allowed, decimals are prohibited to use.

2) Solve least squares problems $\min \|\mathbf{Ax} - \mathbf{b}\|_2$ via QR decomposition, where $\mathbf{b} = [1, 0, 2, -1]^T$.

Solution:

1) $A = [a_1, a_2, a_3]$.

   $$\tilde{q}_1 = a_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, q_1 = \frac{\tilde{q}_1}{\|\tilde{q}_1\|} = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

   $$\tilde{q}_2 = a_2 - q_1^T a_2 q_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 \\ 3 \\ -3 \\ -1 \end{bmatrix}, q_2 = \frac{\tilde{q}_2}{\|\tilde{q}_2\|} = \frac{\sqrt{5}}{10}\begin{bmatrix} 1 \\ 3 \\ -3 \\ -1 \end{bmatrix}$$

   $$\tilde{q}_3 = a_3 - q_1^T a_3 q_1 - q_2^T a_3 q_2 = \begin{bmatrix} 1 \\ 4 \\ 1 \\ 0 \end{bmatrix} - \frac{3}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 \\ 3 \\ -3 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \end{bmatrix}, q_3 = \frac{\tilde{q}_3}{\|\tilde{q}_3\|} = \frac{1}{2}\begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

   Then we have $\mathbf{Q} = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{5}}{10} & -\frac{1}{2} \\ \frac{1}{2} & \frac{3\sqrt{5}}{10} & \frac{1}{2} \\ \frac{1}{2} & -\frac{3\sqrt{5}}{10} & \frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{5}}{10} & -\frac{1}{2} \end{bmatrix}$.(8 points)

   $r_{11} = a_1 \cdot q_1 = 2$, $r_{12} = a_2 \cdot q_1 = 1$, $r_{13} = a_3 \cdot q_1 = 3$,

   $r_{22} = a_2 \cdot q_2 = \sqrt{5}$, $r_{23} = a_3 \cdot q_2 = \sqrt{5}$,

   $r_{33} = a_3 \cdot q_3 = 2$,

   $\mathbf{R} = \begin{bmatrix} 2 & 1 & 3 \\ 0 & \sqrt{5} & \sqrt{5} \\ 0 & 0 & 2 \end{bmatrix}$. (5 points)

2) Project $\mathbf{b}$ onto $\mathcal{R}(\mathbf{A})$, $\mathbf{QRx} = \mathbf{QQ^Tb} \Rightarrow \mathbf{Rx} = \mathbf{Q}^T\mathbf{b} \Rightarrow \mathbf{x} = \mathbf{R}^{-1}\mathbf{Q}^T\mathbf{b}$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 3 \\ 0 & \sqrt{5} & \sqrt{5} \\ 0 & 0 & 2 \end{bmatrix}^{-1} \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{5}}{10} & -\frac{1}{2} \\ \frac{1}{2} & \frac{3\sqrt{5}}{10} & \frac{1}{2} \\ \frac{1}{2} & -\frac{3\sqrt{5}}{10} & \frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{5}}{10} & -\frac{1}{2} \end{bmatrix}^{T} \begin{bmatrix} 1 \\ 0 \\ 2 \\ -1 \end{bmatrix} \Rightarrow \mathbf{x} = [\tfrac{1}{5}, -\tfrac{9}{10}, \tfrac{1}{2}]^T.$$ (7 points)

Problem 2. (12 points + 13 points)

1) For positive definite matrix $\mathbf{H}$, there exist a lower triangular matrix $\mathbf{L}$ with strictly positive real entries on its main diagonal that satisfies $\mathbf{H} = \mathbf{L}\mathbf{L}^*$. Please prove the uniqess of the lower triangular matrix $\mathbf{L}$.
   Hint: The notation $\mathbf{L}^*$ means the hermitian transpose of matrix $\mathbf{L}$.
2) Let $\mathbf{A}$ be a nonsingular square matrix and let $\mathbf{A} = \mathbf{Q}\mathbf{R}$ and $\mathbf{A}^*\mathbf{A} = \mathbf{U}^*\mathbf{U}$ be QR and cholesky factorizations, respectively, with the usual normalization $r_{jj}, u_{jj} > 0$. Is it true or false that $\mathbf{R} = \mathbf{U}$? Prove your answer.
   Hint: Here $\mathbf{U}$ is an upper triangular matrix.

Solution:

1) Suppose that there another decomposition $\mathbf{H} = \mathbf{M}\mathbf{M}^*$, then we have $\mathbf{L}\mathbf{L}^* = \mathbf{M}\mathbf{M}^*$, and $\mathbf{M}^{-1}\mathbf{L} = \mathbf{M}^*(\mathbf{L}^*)^{-1}$.
   Since $\mathbf{L}$ and $\mathbf{M}$ are lower triangular, $\mathbf{M}^{-1}\mathbf{L}$ is lower triangular.
   Since $\mathbf{M}^*$ and $(\mathbf{L}^*)^{-1}$ are upper triangular, $\mathbf{M}^*(\mathbf{L}^*)^{-1}$ is upper triangular.
   The lower triangular matrix $\mathbf{M}^{-1}\mathbf{L}$ can be equal to the upper triangular matrix $\mathbf{M}^*(\mathbf{L}^*)^{-1}$ only if both matrices are diagonal. Therefore, $\mathbf{M}^{-1}\mathbf{L} = \mathbf{D} = \mathbf{M}^*(\mathbf{L}^*)^{-1}$. (6 points)
   Since $(\mathbf{D}^*)^{-1} = \left(\left(\mathbf{M}^*(\mathbf{L}^*)^{-1}\right)^*\right)^{-1} = \left(\mathbf{M}^{-1}\mathbf{L}\right)^{-1} = \mathbf{D}$, $\mathbf{D}\mathbf{D}^* = \mathbf{I}$.
   Morever, they need to satisfy the constraint $\mathbf{M}\mathbf{D} = \mathbf{L}$, where the diagonal entries of both $\mathbf{M}$ and $\mathbf{L}$ are real and strictly positive. Hence $\mathbf{D} = \mathbf{I}$, $\mathbf{L} = \mathbf{M}$. (6 points)
2) It is true.
   Since $\mathbf{A}$ is nonsingular, it will have a unique QR factorization with $r_{jj} > 0$. Then we have $\mathbf{A}^*\mathbf{A} = \mathbf{R}^*\mathbf{Q}^*\mathbf{Q}\mathbf{R} = \mathbf{R}^*\mathbf{R}$ because $\mathbf{Q}$ is a unitary matrix. (5 points)
   On the other hand, $\mathbf{A}x \neq 0$ for any $x \neq 0$ because $\mathbf{A}$ is nonsingular.
   $x^*\mathbf{A}^*\mathbf{A}x = (\mathbf{A}x)^*(\mathbf{A}x) = \|\mathbf{A}x\|_2^2 > 0 \Longrightarrow \mathbf{A}^*\mathbf{A}$ is positive definite. (5 points)
   Based on the above conclusion, positive definite matrix $\mathbf{A}^*\mathbf{A}$ has a unique cholesky factorization $\mathbf{A}^*\mathbf{A} = \mathbf{U}^*\mathbf{U}$. Therefore, we have $\mathbf{R} = \mathbf{U}$. (3 points)

II. HOUSEHOLDER REFLECTION

Problem 3. (10 points + 7 points + 8 points)

Let $\mathbf{v}$ be a nonzero n-dimensional vector. The hyperplane $\mathcal{H}$ normal to $\mathbf{v}$ is the $(n-1)$-dimensional subspace of all vectors $\mathbf{z}$ such that $\mathbf{v}^{\mathrm{T}}\mathbf{z} = 0$. A *reflector* is a linear transformation $\mathbf{R}$ such that $\mathbf{Rx} = -\mathbf{x}$ if $\mathbf{x}$ is a scalar multiple of $\mathbf{v}$, and $\mathbf{Rx} = \mathbf{x}$ if $\mathbf{v}^{\mathbf{T}}\mathbf{x} = \mathbf{0}$. Thus, the hyperplane acts as a mirror: for any vector, its component within the hyperplane is invariant, whereas its component orthogonal to the hyperplane is reversed.

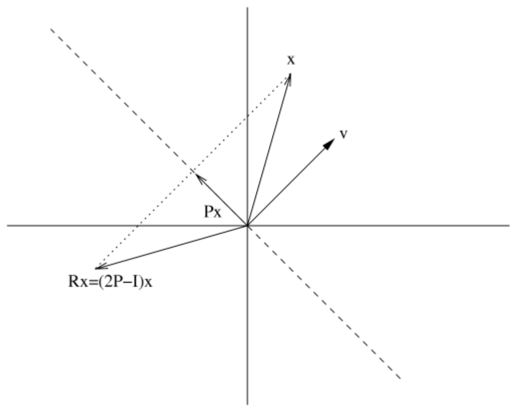

Figure 1: Reflector

1) Show that $\mathbf{R} = \mathbf{2P} - \mathbf{I}$, where $\mathbf{P}$ is the orthogonal projector onto the hyperplane normal to $\mathbf{v}$.
2) Show that $\mathbf{R}$ is symmetric and orthogonal.
3) Show that the Householder transformation

$$\mathbf{H} = \mathbf{I} - 2\frac{\mathbf{v}\mathbf{v}^T}{\mathbf{v}^T\mathbf{v}}$$

is a reflector.

Solution:

1) We can obtain the reflection $\mathbf{Rx}$ of a vector $\mathbf{x}$ with respect to a hyperplane through the origin by adding to $\mathbf{x}$ twice the vector from $\mathbf{x}$ to $\mathbf{Px}$, where $\mathbf{Px}$ is the projection of $\mathbf{x}$ onto the same hyperplane . Thus

$$\mathbf{Rx} = \mathbf{x} + 2(\mathbf{Px} - \mathbf{x}) = (2\mathbf{Px} - \mathbf{x}) = (2\mathbf{P} - \mathbf{I})\mathbf{x}$$

Since this has to hold for all $\mathbf{x}$ we have $\mathbf{R} = \mathbf{2P} - \mathbf{I}$.

An alternative way to derive the same result is to observe that the projection $\mathbf{Px}$ lies halfway between $\mathbf{x}$ and its reflection $\mathbf{Rx}$. Therefore

$$\frac{1}{2}(\mathbf{x} + \mathbf{Rx}) = \mathbf{Px} \Rightarrow \mathbf{Rx} = (2\mathbf{Px} - \mathbf{x}) = (2\mathbf{P} - \mathbf{I})\mathbf{x}$$

which leads to the same result.(10 points)

2) Since $\mathbf{P}$ is the orthogonal projector,then $\mathbf{P} = \mathbf{P}^T \Longrightarrow \mathbf{R} = 2\mathbf{P} - \mathbf{I} = 2\mathbf{P}^T - \mathbf{I} = \mathbf{R}^T$.

   To show $\mathbf{R} = 2\mathbf{P} - \mathbf{I}$ is orthogonal:(since $\mathbf{P} = \mathbf{P}^2$)

$$\mathbf{R}\mathbf{R}^T = (2\mathbf{P} - \mathbf{I})(2\mathbf{P} - \mathbf{I})^T = 4\mathbf{P}^2 - 4\mathbf{P} + \mathbf{I} = \mathbf{I}$$

   (7 points)

3) The Householder matrix reflects all vectors in the direction of $\mathbf{v}$

$$\mathbf{H}(\alpha\mathbf{v}) = \left(\mathbf{I} - 2\frac{\mathbf{v}\mathbf{v}^{\mathbf{T}}}{\mathbf{v}^{\mathrm{T}}\mathbf{v}}\right)(\alpha\mathbf{v}) = \alpha\mathbf{v} - \mathbf{2}\alpha\frac{\mathbf{v}\left(\mathbf{v}^{\mathrm{T}}\mathbf{v}\right)}{\mathbf{v}^{\mathrm{T}}\mathbf{v}} = \alpha(\mathbf{v} - \mathbf{2}\mathbf{v}) = -(\alpha\mathbf{v})$$

   and leaves all vectors $\mathbf{x}$ with $\mathbf{v}^{\mathbf{T}}\mathbf{x} = \mathbf{0}$ invariant

$$\mathbf{H}\mathbf{x} = \left(I - 2\frac{\mathbf{v}^T}{\mathbf{v}^T\mathbf{v}}\right)\mathbf{x} = \mathbf{x} - 2\frac{\mathbf{v}\left(\mathbf{v}^T\mathbf{x}\right)}{\mathbf{v}^T\mathbf{v}} = \mathbf{x}$$

   therefore,$\mathbf{H}$ is a reflector about the hyperplane $\left\{\mathbf{x} : \mathbf{v}^{\mathbf{T}}\mathbf{x} = \mathbf{0}\right\}$.(8 points)

III. QR DECOMPOSITION VIA GIVENS ROTATION

Problem 4. (12 points + 12 points + 6 points)

Given a dense matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & -1 \\ -1 & 3 & 2 \\ 1 & -2 & 3 \end{bmatrix} \tag{1}$$

and a sparse matrix

$$\mathbf{B} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 2 & 0 & 2 & 0 \\ 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 & 2 \end{bmatrix} \tag{2}$$

1) Give the QR decomposition of $\mathbf{A}$ with $\mathbf{Q}$ being square.
2) Give the QR decomposition of $\mathbf{B}$ with $\mathbf{Q}$ being square.
3) Discuss when Givens rotation is better than Householder reflection and when Householder reflection is better than Givens rotation.

Remark: 1) and 2) just give one example solution respectively, and other correct solutions will also be ok.

Solution:

1)

$$J_{12} = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad A^{(1)} = \begin{bmatrix} \sqrt{2} & -\frac{\sqrt{2}}{2} & -\frac{3\sqrt{2}}{2} \\ 0 & \frac{5\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ 1 & -2 & 3 \end{bmatrix}$$

$$J_{13} = \begin{bmatrix} \frac{\sqrt{6}}{3} & 0 & \frac{\sqrt{3}}{3} \\ 0 & 1 & 0 \\ -\frac{\sqrt{3}}{3} & 0 & \frac{\sqrt{6}}{3} \end{bmatrix} \qquad A^{(2)} = \begin{bmatrix} \sqrt{3} & -\sqrt{3} & 0 \\ 0 & \frac{5\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ 0 & -\frac{\sqrt{6}}{2} & \frac{3\sqrt{6}}{2} \end{bmatrix}$$

$$J_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{5\sqrt{7}}{14} & -\frac{\sqrt{21}}{14} \\ 0 & \frac{\sqrt{21}}{14} & \frac{5\sqrt{7}}{14} \end{bmatrix} \qquad R = A^{(3)} = \begin{bmatrix} \sqrt{3} & -\sqrt{3} & 0 \\ 0 & \sqrt{14} & -\frac{\sqrt{14}}{7} \\ 0 & 0 & 4\frac{\sqrt{42}}{7} \end{bmatrix}$$

For $J_{23}J_{13}J_{12}A = R$, sinceJ_{23}, J_{13}, J_{12} are orthogonal, we can get

$$(J_{23}J_{13}J_{12})^{-1} = J_{12}^T J_{13}^T J_{23}^T = \begin{bmatrix} \frac{\sqrt{3}}{3} & 3\frac{\sqrt{14}}{14} & \frac{-\sqrt{42}}{42} \\ -\frac{\sqrt{3}}{3} & \frac{\sqrt{14}}{7} & 2\frac{\sqrt{42}}{21} \\ \frac{\sqrt{3}}{3} & -\frac{\sqrt{14}}{14} & 5\frac{\sqrt{42}}{42} \end{bmatrix}$$

2)

$$J_{13} = \begin{bmatrix} \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -\frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad B^{(1)} = \begin{bmatrix} \sqrt{2} & 0 & -\frac{\sqrt{2}}{2} & 0 & \sqrt{2} \\ 0 & 2 & 0 & 2 & 0 \\ 0 & 0 & -\frac{\sqrt{2}}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 & 2 \end{bmatrix}$$

$$J_{25} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{\sqrt{2}}{2} & 0 & 0 & \frac{\sqrt{2}}{2} \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & -\frac{\sqrt{2}}{2} & 0 & 0 & \frac{\sqrt{2}}{2} \end{bmatrix} \qquad B^{(2)} = \begin{bmatrix} \sqrt{2} & 0 & -\frac{\sqrt{2}}{2} & 0 & \sqrt{2} \\ 0 & 2\sqrt{2} & 0 & \sqrt{2} & \sqrt{2} \\ 0 & 0 & -\frac{\sqrt{2}}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -\sqrt{2} & \sqrt{2} \end{bmatrix}$$

$$J_{45} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \qquad R = B^{(3)} = \begin{bmatrix} \sqrt{2} & 0 & -\frac{\sqrt{2}}{2} & 0 & \sqrt{2} \\ 0 & 2\sqrt{2} & 0 & \sqrt{2} & \sqrt{2} \\ 0 & 0 & -\frac{\sqrt{2}}{2} & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2} & -\sqrt{2} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

For $J_{45}J_{25}J_{13}B = R$, sinceJ_{23}, J_{25}, J_{25} are orthogonal, we can get

$$(J_{45}J_{25}J_{13})^{-1} = J_{13}^T J_{25}^T J_{45}^T = \begin{bmatrix} \frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} & 0 & 0 \\ 0 & \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & \frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} & 0 \end{bmatrix}$$

3) Givens rotation is more efficient when the matrix is sparse, while Householder reflection is more efficient when the matrix is dense. (While setting some entry to be zero, Givens rotation only modifies two rows. So it is possible to apply sets of rotation in parallel and make it more efficient than Householder reflection. This answer will also be given points but it is not the main purpose of this problem.)