Numerical analysis(SI211)$_{\textbf{Fall 2021-22}}$ Homework 3

Prof. Jiahua Jiang

Name: xxx Student No.: xxx E-mail: xxx@shanghaitech.edu.cn

Acknowledgements:

1. Deadline: **2021-12-24 11:59:00**, no late submission is allowed.

2. No handwritten homework is accepted. You should submit your homework in Blackboard with PDF format, we recommend you use LaTeX.

3. Giving your solution in English, solution in Chinese is not allowed.

4. Make sure that your codes can run and are consistent with your solutions, you can use any programming language.

5. Your PDF should be named as "your_student_id+HW3.pdf", package all your codes into "your_student_id+_Code3.zip" and upload. Don't put your PDF in your code file

6. All the results from your code should be shown in pdf but please do not inset your code into LaTeX.

7. Plagiarism is not allowed. Those plagiarized solutions and codes will get 0 point. If the results on the pdf are inconsistent with the results of code, your coding problem will get 0 point.

---

1. **Euler's Method**(20 points.)
For initial-value problem:

$$\begin{aligned} y'(x) &= ax + b \\ y(0) &= 0, \end{aligned} \tag{1}$$

use Euler's method and Taylor's method of order 2 to derive the approximation of y_{i+1} with step size h respectively. Besides, compare your results with the exact solution $y = \frac{1}{2}ax^2 + bx$ (i.e. compare y_{i+1} and $y(x_{i+1})$).
Solution:
Exact solution:

$$y(x_{i+1}) = \frac{1}{2}a(x_i+h)^2+b(x_i+h) = (\frac{1}{2}ax_i^2+bx_i)+(ax_i+b)h+\frac{1}{2}ah^2 = y_i+(ax_i+b)h+\frac{1}{2}ah^2$$

Euler's method:

$$y_{i+1} = y_i + y'(x_i)h = y_i + (ax_i + b)h$$

$$y(x_{i+1}) - y_{i+1} = \frac{1}{2}ah^2 = \mathcal{O}(h^2)$$

Taylor's method:

$$y_{i+1} = y_i + y'(x_i)h + \frac{1}{2}y''(x_i)h^2 = y_i + (ax_i + b)h + \frac{1}{2}ah^2$$

$$y(x_{i+1}) - y_{i+1} = 0$$

2. **Runge-Kutta Methods**(20 points.)
Prove the following Runge-kutta method is of order 3(i.e. has truncation error $\mathcal{O}(h^4)$)

$$\begin{aligned}
y_{i+1} &= y_i + \frac{h}{4}(K_1 + 3K_3) \\
K_1 &= f(x_i, y_i) \\
K_2 &= f(x_i + \frac{h}{3}, y_i + \frac{h}{3}K_1) \\
K_3 &= f(x_i + \frac{2}{3}h, y_i + \frac{2}{3}hK_2)
\end{aligned} \tag{2}$$

Solution:

$$\begin{aligned}
y'(x) &= f(x, y) \\
y''(x) &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y'(x) \\
y'''(x) &= \frac{\partial^2 f}{\partial x^2} + \frac{\partial f}{\partial x}\frac{\partial f}{\partial y}y'(x) + \frac{\partial \frac{\partial f}{\partial y}}{\partial x}y'(x) + \frac{\partial f}{\partial y}y''(x) \\
&= \frac{\partial^2 f}{\partial x^2} + \frac{\partial f}{\partial x}\frac{\partial f}{\partial y}y'(x) + (\frac{\partial f}{\partial y}\frac{\partial f}{\partial x} + \frac{\partial^2 f}{\partial y^2}y'(x))y'(x) + \frac{\partial f}{\partial y}y''(x) \\
&= \frac{\partial^2 f}{\partial x^2} + 2\frac{\partial f}{\partial x}\frac{\partial f}{\partial y}y'(x) + \frac{\partial^2 f}{\partial y^2}(y'(x))^2 + \frac{\partial f}{\partial y}y''(x) \\
K_1 &= f(x_i, y_i) = y'(x_i) \\
K_2 &= f(x_i + \frac{h}{3}, y_i + \frac{h}{3}K_1) = f(x_i, y_i) + [\frac{h}{3}\frac{\partial f}{\partial x} + \frac{h}{3}K_1\frac{\partial f}{\partial y}]\Big|_{x_i, y_i} + \mathcal{O}(h^2) \\
K_3 &= f(x_i + \frac{2}{3}h, y_i + \frac{2}{3}hK_2) = f(x_i, y_i) + [\frac{2h}{3}\frac{\partial f}{\partial x} + \frac{2h}{3}K_2\frac{\partial f}{\partial y}]\Big|_{x_i, y_i} + \\
&\quad \frac{1}{2}[\frac{4}{9}h^2\frac{\partial^2 f}{\partial x^2} + \frac{8}{9}h^2K_2\frac{\partial f}{\partial x}\frac{\partial f}{\partial y} + \frac{4}{9}h^2K_2^2\frac{\partial^2 f}{\partial y^2}]\Big|_{x_i, y_i} + \mathcal{O}(h^3) \\
&= y'(x_i) + \frac{2h}{3}[\frac{\partial f}{\partial x} + (y'(x) + \frac{h}{3}y''(x))\frac{\partial f}{\partial y}]\Big|_{x_i, y_i} \\
&\quad + \frac{2h^2}{9}[\frac{\partial^2 f}{\partial x^2} + 2y'(x)\frac{\partial f}{\partial x}\frac{\partial f}{\partial y} + (y'(x))^2\frac{\partial^2 f}{\partial y^2}]\Big|_{x_i, y_i} + \mathcal{O}(h^3) \\
&= y'(x_i) + \frac{2h}{3}y''(x_i) + \frac{2h^2}{9}[\frac{\partial^2 f}{\partial x^2} + 2y'(x)\frac{\partial f}{\partial x}\frac{\partial f}{\partial y} + (y'(x))^2\frac{\partial^2 f}{\partial y^2} + y''(x)\frac{\partial f}{\partial y}]\Big|_{x_i, y_i} + \mathcal{O}(h^3) \\
&= y'(x_i) + \frac{2h}{3}y''(x_i) + \frac{2h^2}{9}y''(x_i) + \mathcal{O}(h^3)
\end{aligned}$$

Above shows that

$$y_{i+1} = y_i + \frac{h}{4}(K_1 + 3K_3) = y_i + hy'(x_i) + \frac{h^2}{2}f''(x_i) + \frac{h^3}{6}f'''(x_i) + \mathcal{O}(h^4)$$

From Taylor expansion, we have

$$y(x_{i+1}) = y_i + hy'(x_i) + \frac{h^2}{2}f''(x_i) + \frac{h^3}{6}f'''(x_i) + \mathcal{O}(h^4),$$

therefore, We conclude that

$$y(x_{i+1}) - y_{i+1} = \mathcal{O}(h^4).$$

3. **Coding** **Runge-Kutta Order Four** (10 points.) Use Runge-Kutta Fourth-Order method to solve the following initial-value problem:

$$\begin{aligned} y'(x) &= x + y (0 \leq x \leq 1) \\ y(0) &= 1. \end{aligned} \tag{3}$$

The exact solution of the problem is $y(x) = -x - 1 + 2e^x$. With step size $h = 0.1$, give your predictions within the interval $x \in [0, 1]$. List the Runge-Kutta 4 method results and their errors in the following table.

x_i	Exact($y_i = y(x_i)$)	Runge-Kutta Order 4 (w_i)	Error($\lvert y_i - w_i \rvert$)
0.0	1.0	1.0	0
...	...	...	...

Solutions:

x_i	Exact($y_i = y(x_i)$)	Runge-Kutta Order 4 (w_i)	Error($\lvert y_i - w_i \rvert$)
0.0	1.0	1.0	0
0.1	1.11034184	1.11034167	1.69484629e-07
0.2	1.24280552	1.24280514	3.74618951e-07
0.3	1.39971762	1.39971699	6.21026931e-07
0.4	1.5836494	1.58364848	9.15121169e-07
0.5	1.79744254	1.79744128	1.26420658e-06
0.6	2.0442376	2.04423592	1.67659715e-06
0.7	2.32750541	2.32750325	2.16174740e-06
0.8	2.65108186	2.65107913	2.73040030e-06
0.9	3.01920622	3.01920283	3.39475376e-06
1.0	3.43656366	3.43655949	4.16864776e-06