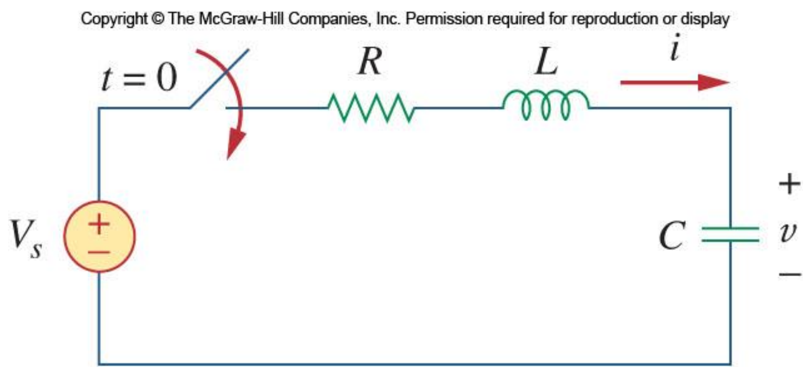
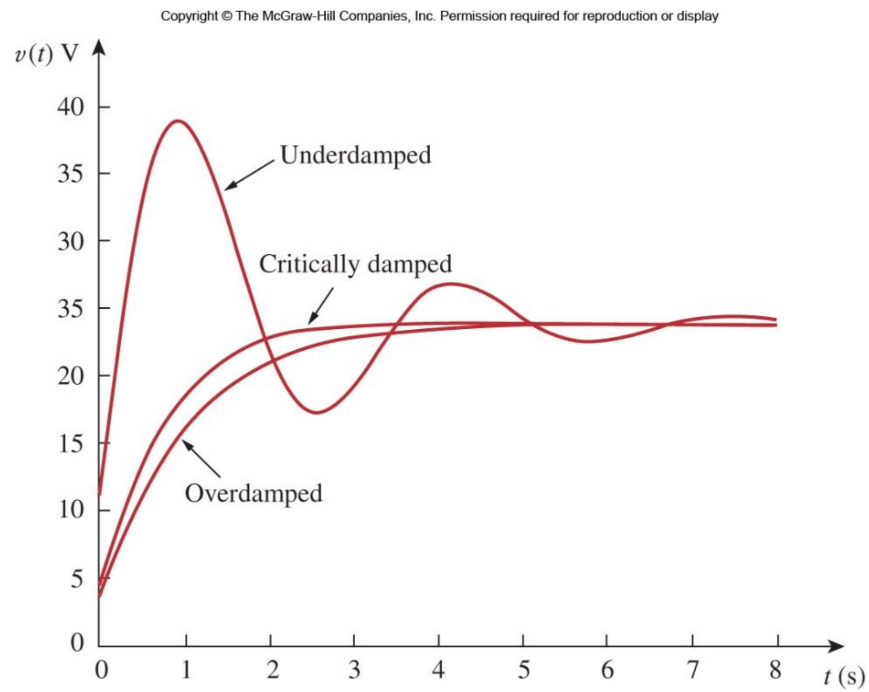
Lecture 13

- Frequency Response

Time Response of a Series RLC Circuit





$v(t) = V_s + (A_1 e^{s_1 t} + A_2 e^{s_2 t})$ (Overdamped)

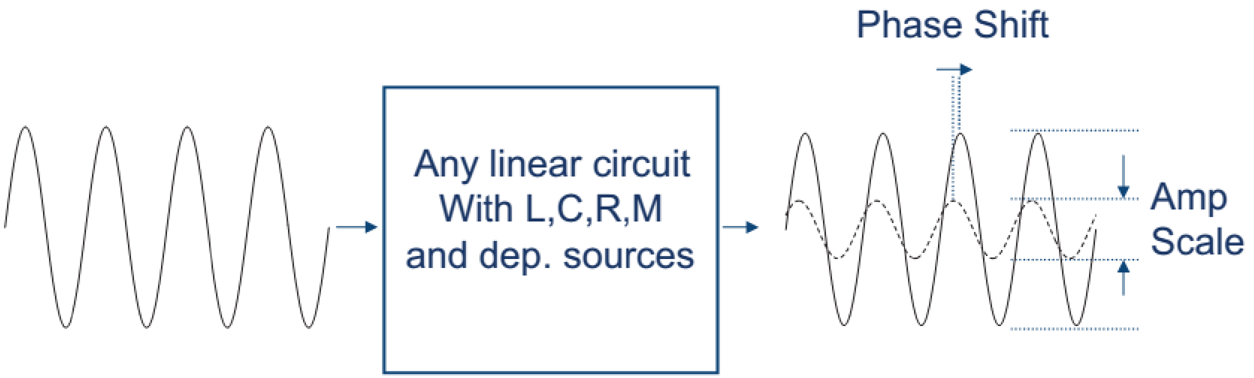
$v(t) = V_s + (A_1 + A_2 t)e^{-\alpha t}$ (Critically Damped)

$v(t) = V_s + (A_1 \cos\omega_d t + A_2 \sin\omega_d t)e^{-\alpha t}$ (Underdamped)

Outline

- Frequency response
- Transfer function
- Bode plots (or diagram)
- Resonance

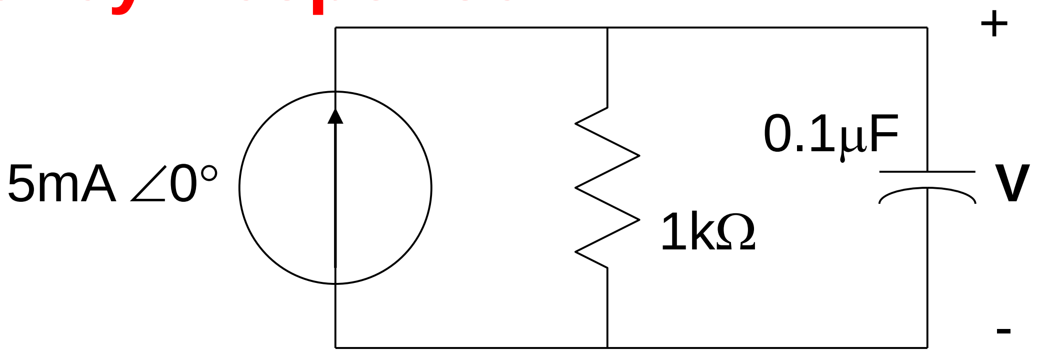
Frequency Response



- When a linear, time invariant (LTI) circuit is excited by a sinusoid, it’s output is a sinusoid at the *same* frequency.
  - Only the magnitude and phase of the output differ from the input.
- The “Frequency Response” is a characterization of the input-output response for sinusoidal inputs at all frequencies.
  - Significant for applications, esp. in communications and control systems.
- Filters play critical roles in blocking or passing specific frequencies or ranges of frequencies.
  - Widely used in radio, TV and phone systems.

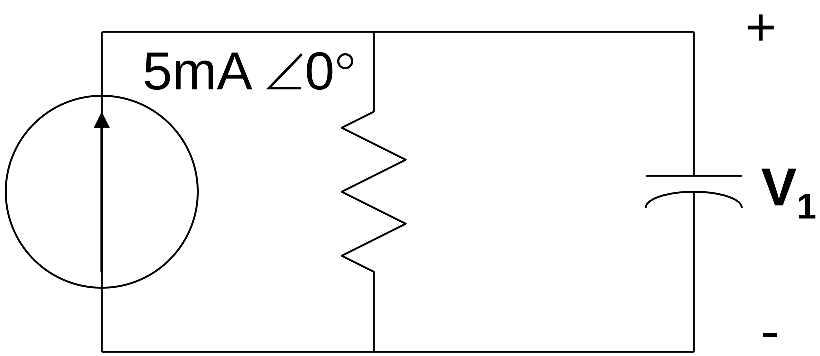
[Source: Berkeley]

Frequency Response

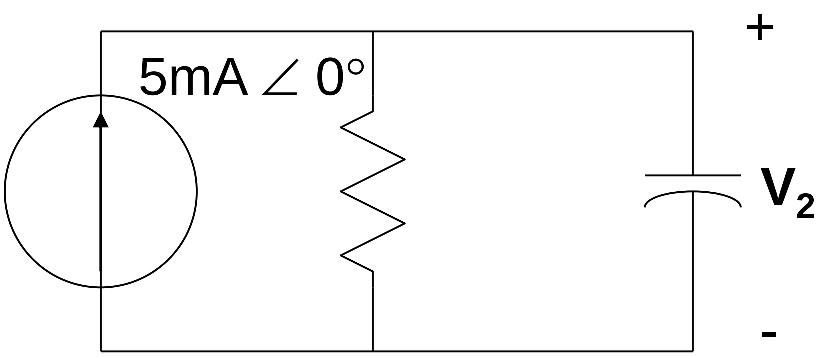


Find *v*

Case 1: ω=2π x 3000



Case 2: ω=2π x 455000



$$\mathbf{V}_1 = 2.34\angle -62.1°\mathrm{V}$$

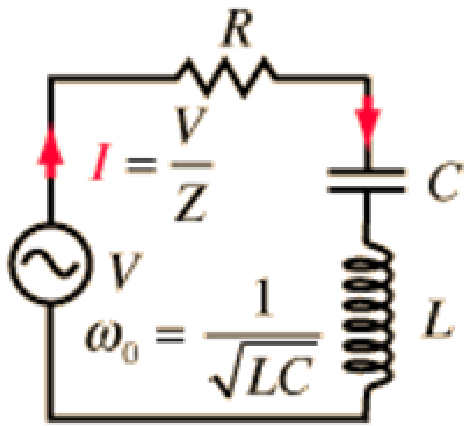
$$\mathbf{Z}_{eq} = 468.2\angle -62.1°\Omega$$

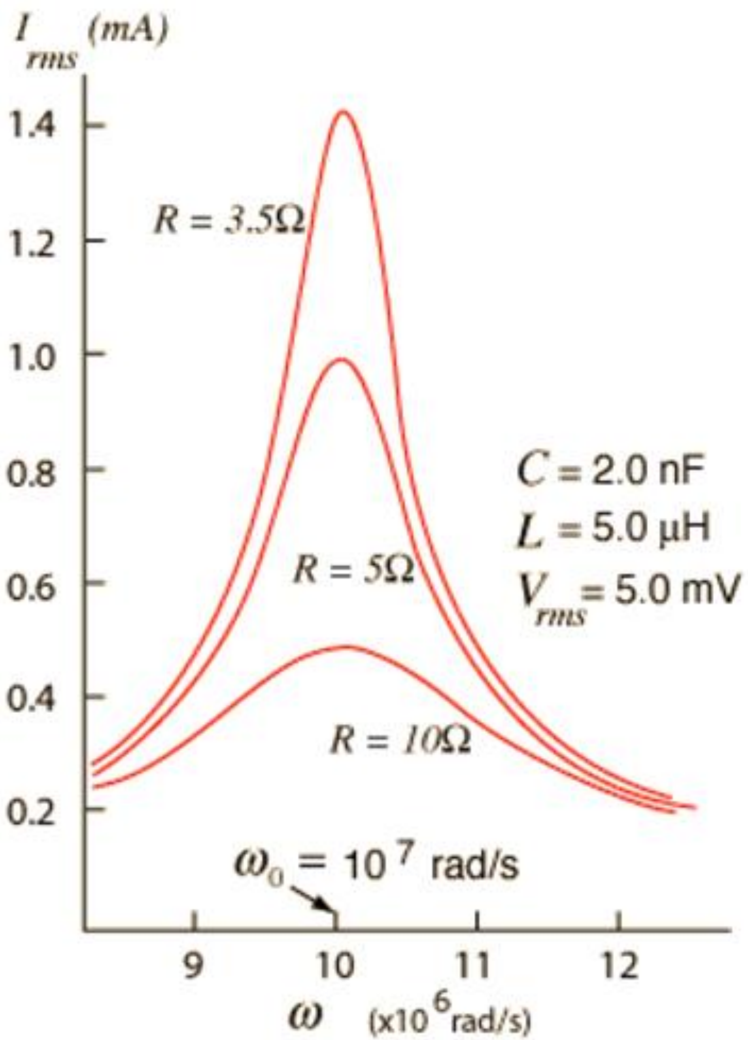
$$\mathbf{V}_2 = 17.5\angle -89.8°\mathrm{mV}$$

$$\mathbf{Z}_{eq} = 3.5\angle -89.8°\Omega$$

[Source: Berkeley]

Frequency Response of an RLC Network





[Source: Georgia State U]

Frequency Ranges of Common Signals

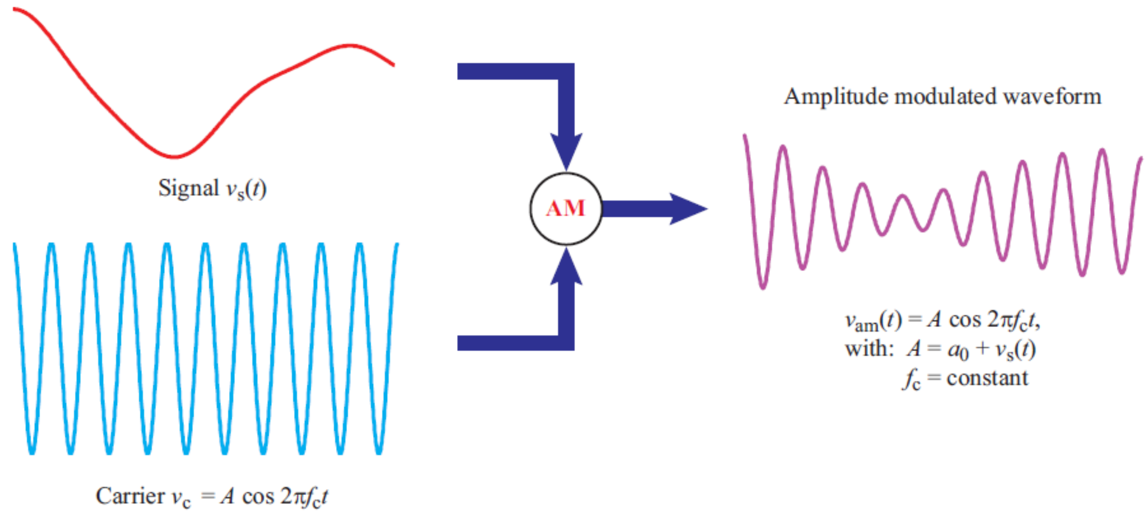
- When we listen to music, our ears respond differently to the various frequency components: some pleasing, whereas others are not.

Frequency Ranges of Selected Signals

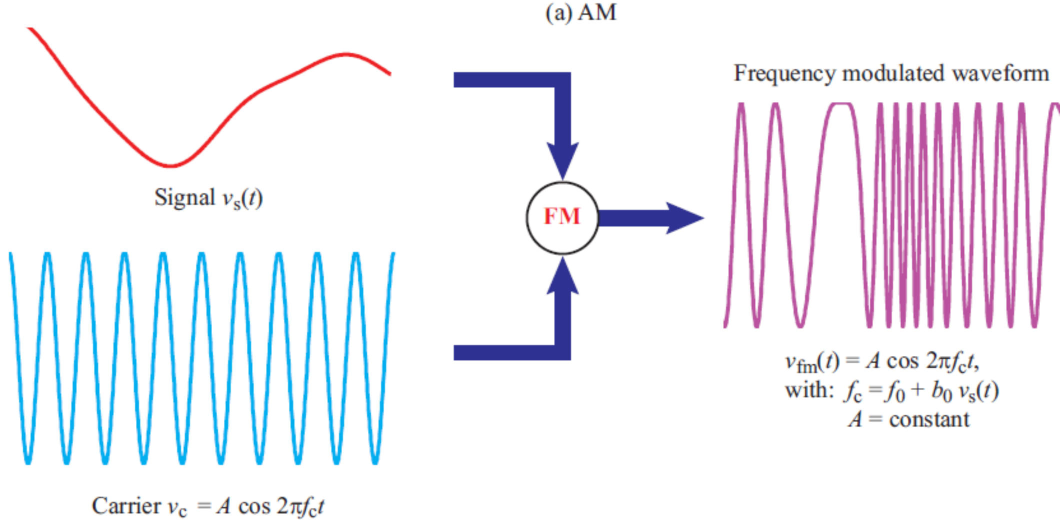
Electrocardiogram	0.05 to 100 Hz
Audible sounds	20 Hz to 15 kHz
AM radio broadcasting	540 to 1600 kHz
HD component video signals	Dc to 25 MHz
FM radio broadcasting	88 to 108 MHz
Cellular phone	824 to 894 MHz and 1850 to 1990 MHz
Satellite television downlinks (C-band)	3.7 to 4.2 GHz
Digital satellite television	12.2 to 12.7 GHz

[Allan, 6th ed.]

Signal Modulation



(a) AM



(b) FM

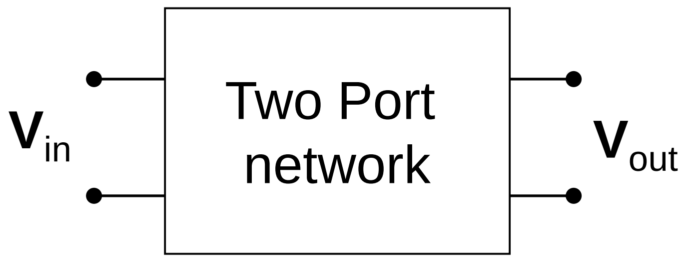
[Source: Berkeley]

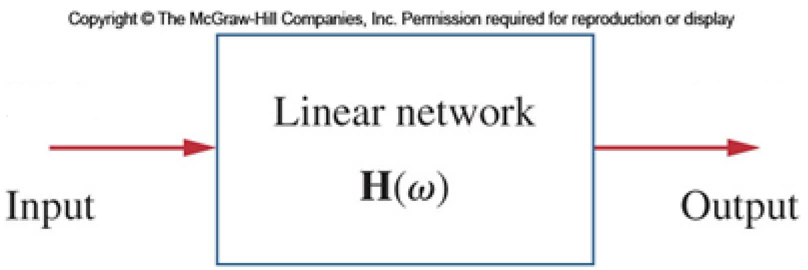
Outline

- Frequency response
- Transfer function
- Bode plots (or diagram)
- Resonance

Transfer Function – Voltage Gain

- One useful way to analyze the frequency response of a circuit is the concept of the transfer function.
  - Complex quantity
  - Both magnitude and phase are function of frequency



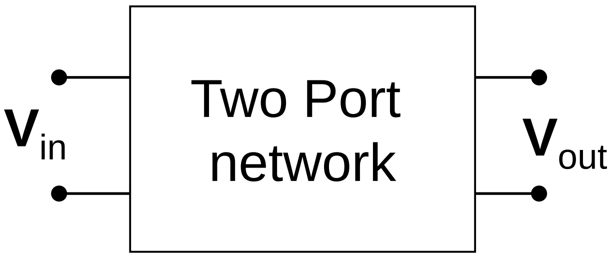


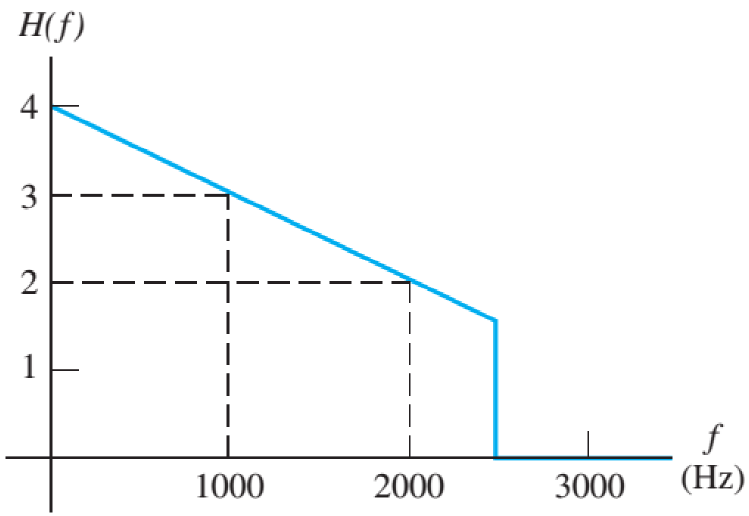
$$\mathbf{H}(f) = \frac{\mathbf{V_{out}}}{\mathbf{V_{in}}} = \frac{V_{out}}{V_{in}} \angle\left(\theta_{out} - \theta_{in}\right)$$

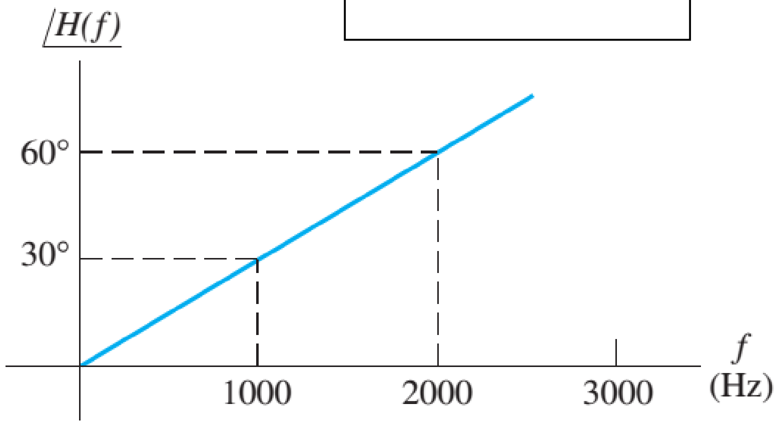
$$\mathbf{H(f)} = H(f)\angle\theta$$

Example

- The transfer function $H(f)$ is shown below. If the input signal is $v_{in}(t) = 2\cos(2000\pi t + 40°)$, find an expression as a function of time for the output.





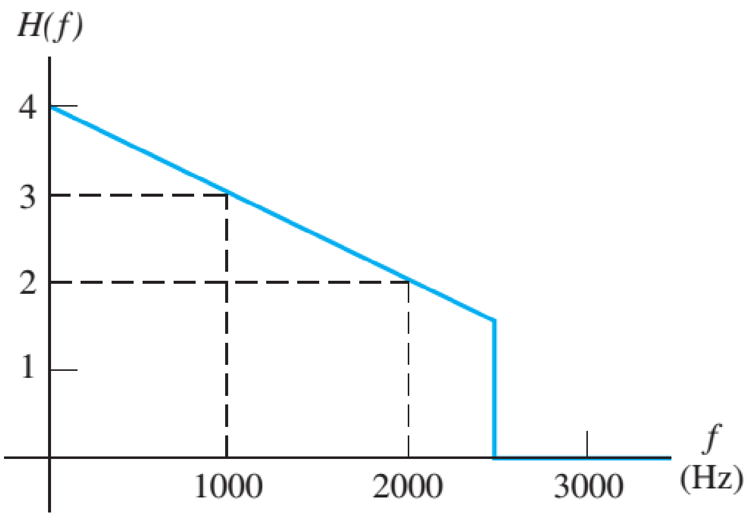


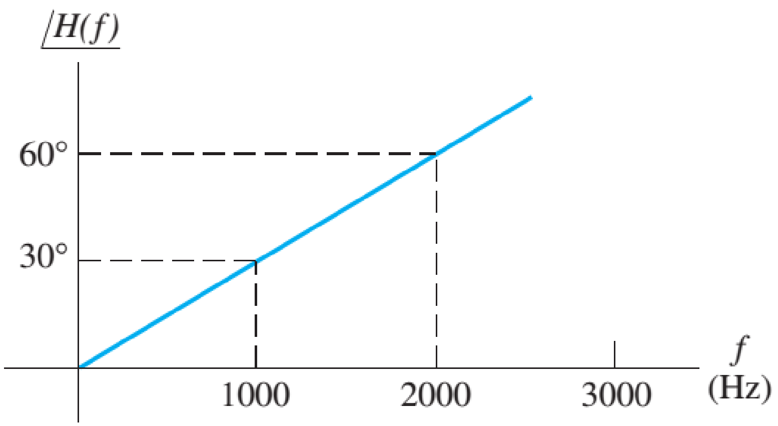
Example

- If the input signal is

$$v_{in}(t) = 3 + 2\cos(2000\pi t) + \cos(4000\pi t - 70°),$$

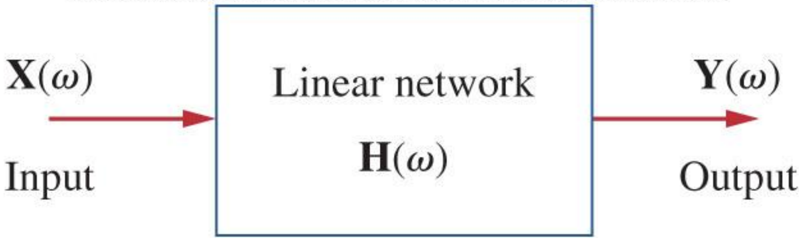
find an expression as a function of time for the output.





Transfer Function – More General Definition

- The transfer function *H(ω)* is the frequency-dependent ratio of a forced function *Y(ω)* to the forcing function *X(ω)*.



$$H(\omega) = \frac{Y(\omega)}{X(\omega)}$$

$$H(\omega) = \text{Voltage gain} = \frac{V_o(\omega)}{V_i(\omega)}$$

$$H(\omega) = \text{Current gain} = \frac{I_o(\omega)}{I_i(\omega)}$$

$$H(\omega) = \text{Transfer impedance} = \frac{V_o(\omega)}{I_i(\omega)}$$

$$H(\omega) = \text{Transfer admittance} = \frac{I_o(\omega)}{V_i(\omega)}$$

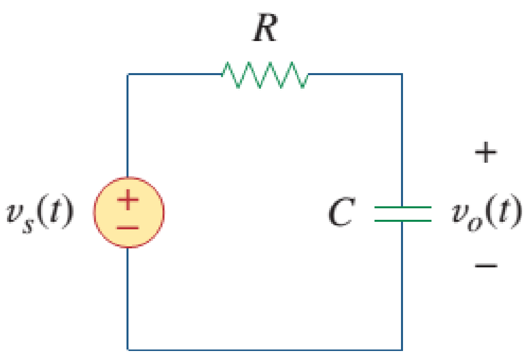
Zeros and Poles

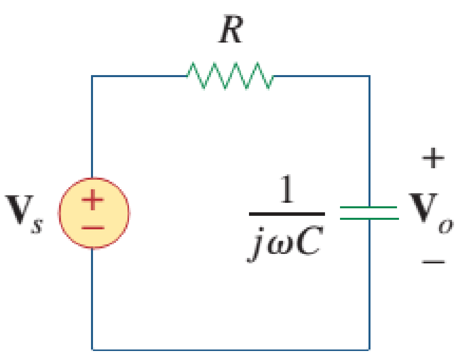
The transfer function $\mathbf{H}(\omega)$ can be expressed in terms of its numerator polynomial $\mathbf{N}(\omega)$ and denominator polynomial $\mathbf{D}(\omega)$ as

$$\boxed{\mathbf{H}(\omega) = \frac{\mathbf{N}(\omega)}{\mathbf{D}(\omega)}} \qquad \mathbf{(14.3)}$$

where $\mathbf{N}(\omega)$ and $\mathbf{D}(\omega)$ are not necessarily the same expressions for the input and output functions, respectively. The representation of $\mathbf{H}(\omega)$ in Eq. (14.3) assumes that common numerator and denominator factors in $\mathbf{H}(\omega)$ have canceled, reducing the ratio to lowest terms. The roots of $\mathbf{N}(\omega) = 0$ are called the *zeros* of $\mathbf{H}(\omega)$ and are usually represented as $j\omega = z_1, z_2, \ldots$. Similarly, the roots of $\mathbf{D}(\omega) = 0$ are the *poles* of $\mathbf{H}(\omega)$ and are represented as $j\omega = p_1, p_2, \ldots$.

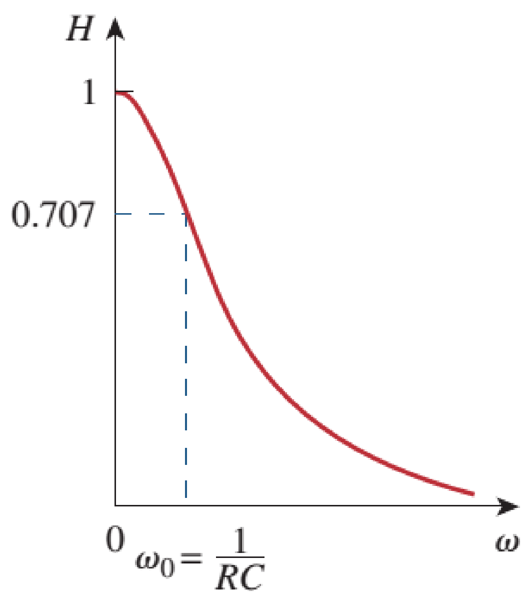
Example

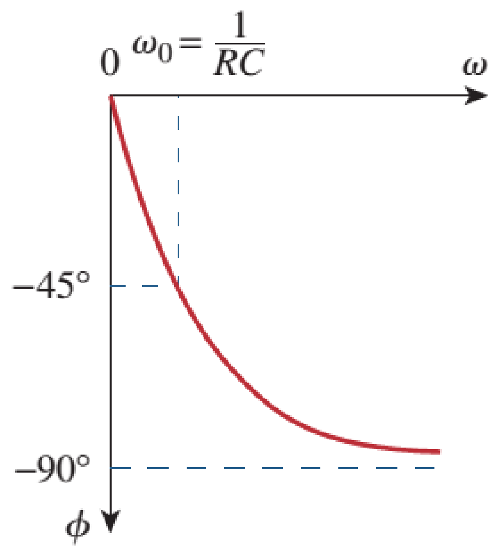




$$H = \frac{1}{\sqrt{1 + (\omega/\omega_0)^2}}, \qquad \phi = -\tan^{-1}\frac{\omega}{\omega_0}$$

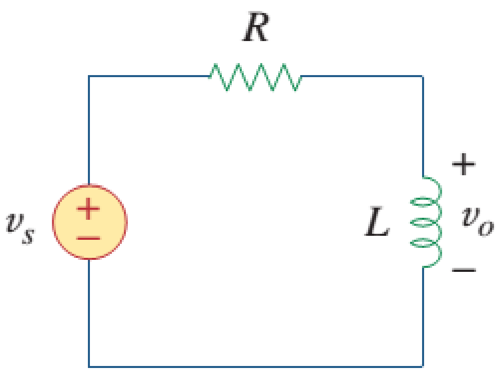
ω/ω_0	H	ϕ	ω/ω_0	H	ϕ
0	1	0	10	0.1	$-84°$
1	0.71	$-45°$	20	0.05	$-87°$
2	0.45	$-63°$	100	0.01	$-89°$
3	0.32	$-72°$	∞	0	$-90°$

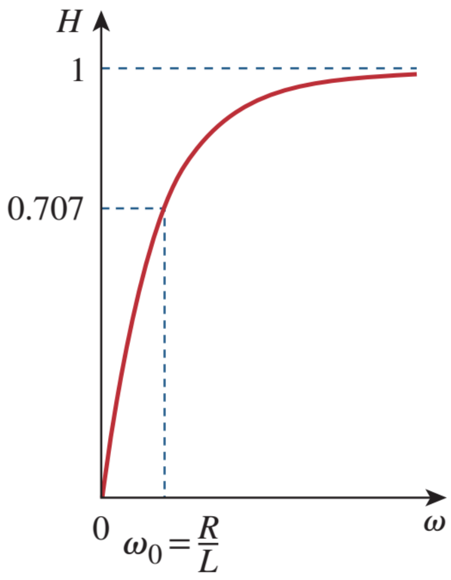


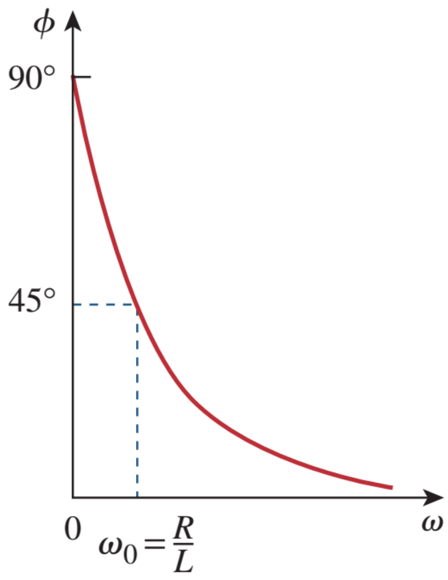


Exercise

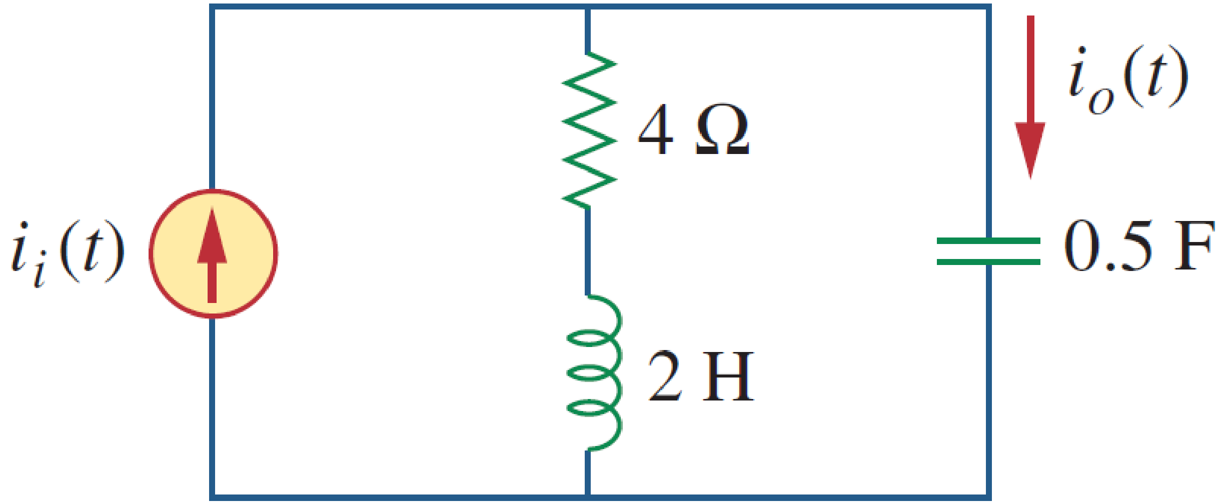
- Obtain the transfer function $\mathbf{V_o}/\mathbf{V_s}$ of the RL circuit. Assuming $v_s = V_m \cos \omega t$.







For the circuit in Fig. 14.6, calculate the gain $\mathbf{I}_o(\omega)/\mathbf{I}_i(\omega)$ and its poles and zeros.



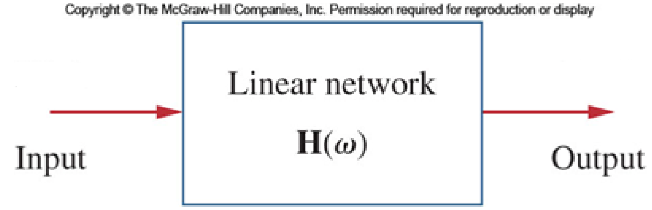
Outline

- Frequency response
- Transfer function
- Bode plots (or diagram)
- Resonance

Decibel Scale

- The transfer function can be seen as an expression of gain, which is typically expressed in log form.
  - in bels, or more commonly decibels

$$G_{dB} = 10\log_{10}\frac{P_2}{P_1}$$



1. $\log P_1P_2 = \log P_1 + \log P_2$
2. $\log P_1/P_2 = \log P_1 - \log P_2$
3. $\log P^n = n\log P$
4. $\log 1 = 0$

- We will soon discuss Bode plots, which are based on logarithmic scales.

Bel and Decibel (dB)

- A **bel** (symbol **B**) is a unit of measure of ratios of power levels, i.e. relative power levels.
  - The name was coined in the early 20th century in honor of Alexander Graham Bell, a telecommunication pioneer.
  - Definition of bel:

$$B = \log_{10}(P_1/P_2)$$

  where P_1 and P_2 are power levels.

- The bel is too large for everyday use, so the **decibel (dB)**, equal to 0.1B, is more commonly used.

$$dB = 10 \log_{10}(P_1/P_2)$$

  - used to measure electric power, gain or loss of amplifiers, etc.

dB for Power

- To express a power in terms of decibels, one starts by choosing a reference power, $P_{reference}$, and write

$$\text{Power P in decibels} = 10 \log_{10}(P/P_{reference})$$

- Exercise: Express a power of 50 mW in decibels relative to 1 watt and 1mW.

  P (dB) =

- dBm to express absolute values of power relative to a milliwatt.

$$\text{dBm} = 10 \log_{10} (\text{power in milliwatts} / 1 \text{ milliwatt})$$

  - 100 mW = dBm
  - 10 mW = dBm

[Source: Berkeley]

dB for Voltage or Current

- We can similarly relate the reference voltage or current to the reference power, as

$$P_{\text{reference}} = (V_{\text{reference}})^2/R \text{ or } P_{\text{reference}} = (I_{\text{reference}})^2 R$$

Hence,

$$\text{Voltage, V in decibels} = 20\log_{10}(V/V_{\text{reference}})$$
$$\text{Current, I, in decibels} = 20\log_{10}(I/I_{\text{reference}})$$

Question: How many decibels larger is the voltage of a 9-volt transistor battery than that of a 1.5-volt AA battery?

Question: The voltage gain of an amplifier with input = 0.2 mV and output = 0.5 V is ?

[Source: Berkeley]

Summary

If G is defined as the power gain,

$$G = \frac{P}{P_0},$$

then the corresponding gain in dB is defined as

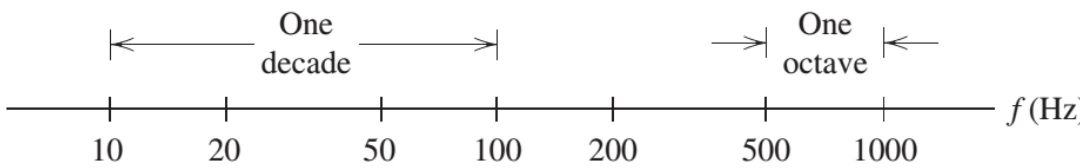
$$G\ [\text{dB}] = 10 \log G = 10 \log \left(\frac{P}{P_0} \right) \qquad (\text{dB}).$$

$$G\ [\text{dB}] = 10 \log \left(\frac{\frac{1}{2}|\mathbf{V}|^2/R}{\frac{1}{2}|\mathbf{V}_0|^2/R} \right) = 20 \log \left(\frac{|\mathbf{V}|}{|\mathbf{V}_0|} \right)$$

$\dfrac{P}{P_0}$	dB
10^N	$10N$ dB
10^3	30 dB
100	20 dB
10	10 dB
4	$\simeq 6$ dB
2	$\simeq 3$ dB
1	0 dB
0.5	$\simeq -3$ dB
0.25	$\simeq -6$ dB
0.1	-10 dB
10^{-N}	$-10N$ dB

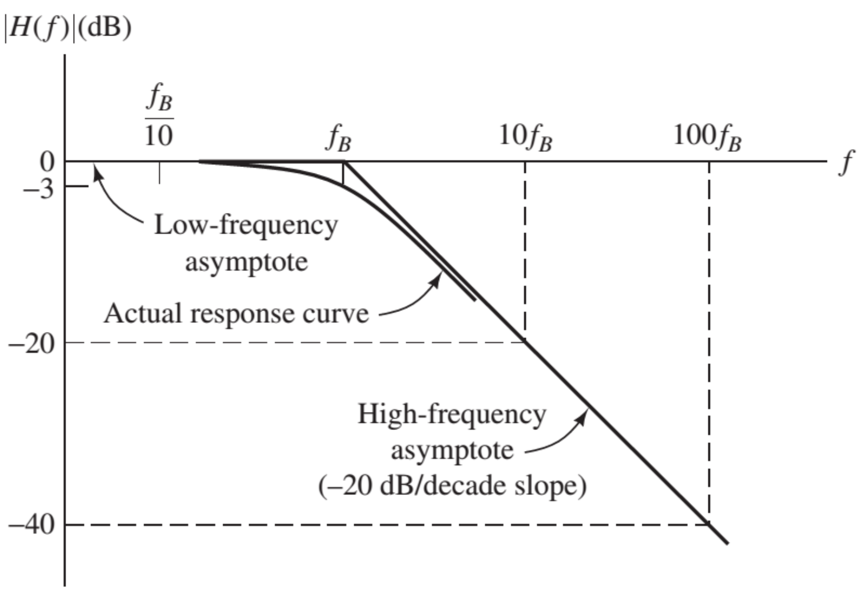
$\left\lvert \dfrac{\mathbf{V}}{\mathbf{V}_0} \right\rvert$ or $\left\lvert \dfrac{\mathbf{I}}{\mathbf{I}_0} \right\rvert$	dB
10^N	$20N$ dB
10^3	60 dB
100	40 dB
10	20 dB
4	$\simeq 12$ dB
2	$\simeq 6$ dB
1	0 dB
0.5	$\simeq -6$ dB
0.25	$\simeq -12$ dB
0.1	-20 dB
10^{-N}	$-20N$ dB

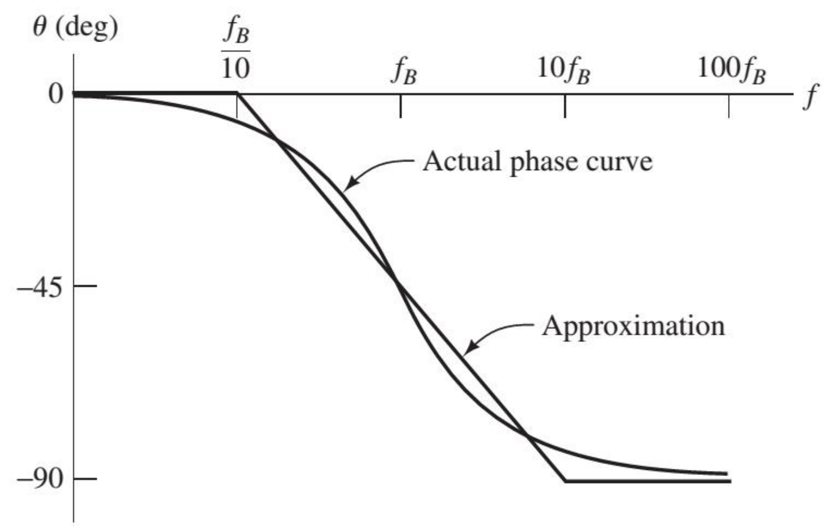
Bode Plots



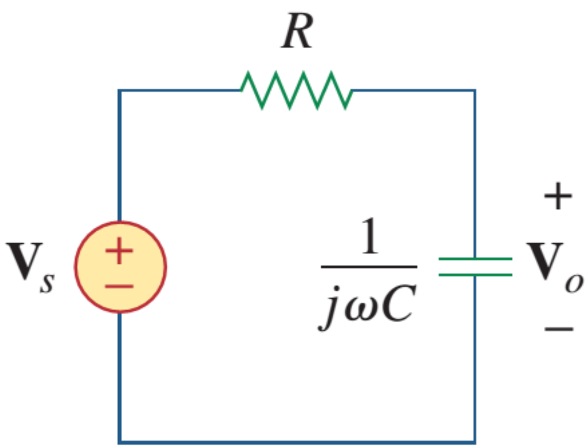
Plotting the frequency response, magnitude or phase, on plots with

- frequency X in log scale
- Y scale in dB (for magnitude) or degree (for phase)





Bode Plot of First-Order Lowpass Filter



Magnitude Plot

$$|H(f)| = \frac{1}{\sqrt{1+(f/f_B)^2}}$$

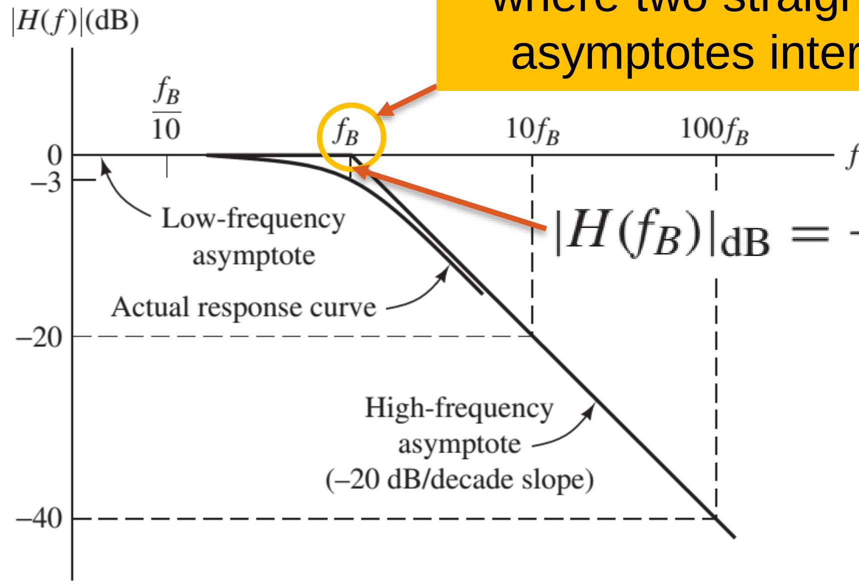
Convert magnitude to decibels, we have

$$|H(f)|_{dB} = 20\log|H(f)| = -10\log\left[1+\left(\frac{f}{f_B}\right)^2\right]$$

- When $f \ll f_B$, i.e., low frequency, $|H(f)|_{dB} \cong 0$;
- When $f \gg f_B$, $|H(f)|_{dB} \cong -20\log\left(\frac{f}{f_B}\right)$.

$$-20\log\left(\frac{f}{f_B}\right)$$

f	$\lvert H(f)\rvert_{dB}$
f_B	0
$2f_B$	−6
$10f_B$	−20
$100f_B$	−40
$1000f_B$	−60



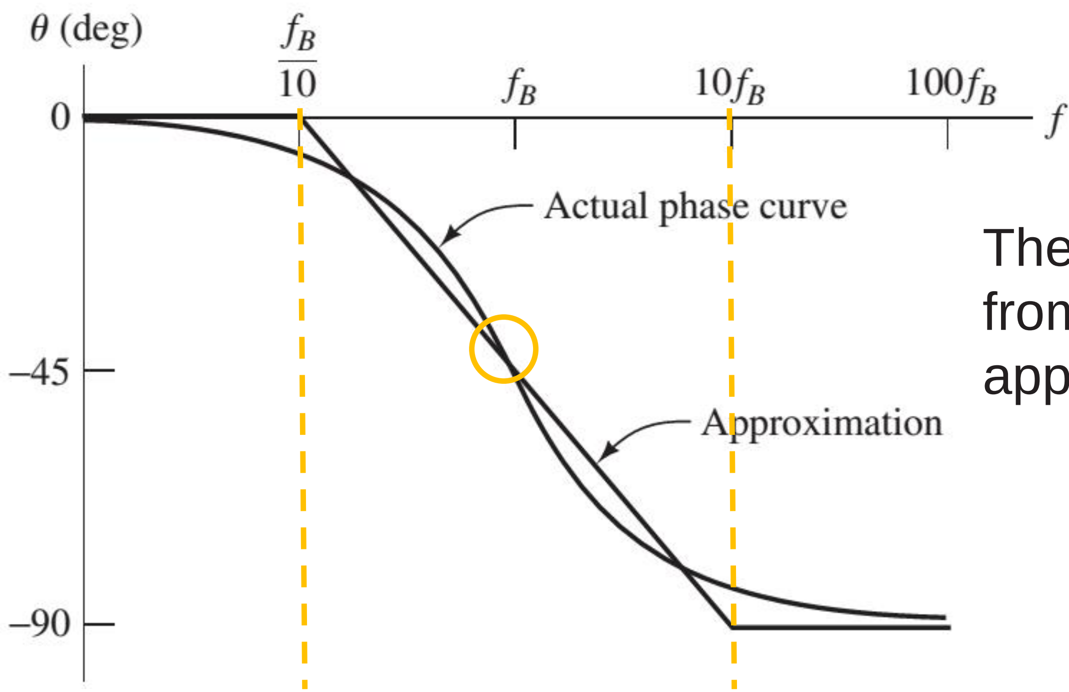
Corner (break) frequency where two straight-line asymptotes intersect

$$|H(f_B)|_{dB} = -3\text{ dB}$$

Phase Plot

$$\angle H(f) = -\arctan\left(\frac{f}{f_B}\right)$$

- When $f < f_B/10$, i.e., low frequency, $\angle H(f) \cong 0$;
- When $f > 10f_B$, $|H(f)|_{dB} \cong -90°$.

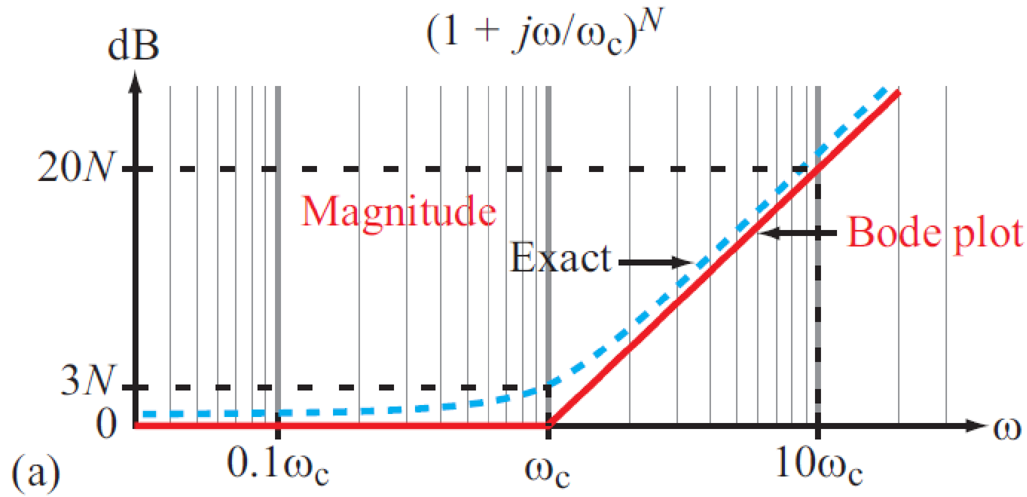


The actual phase curve departs from these straight-line approximations by less than 6°.

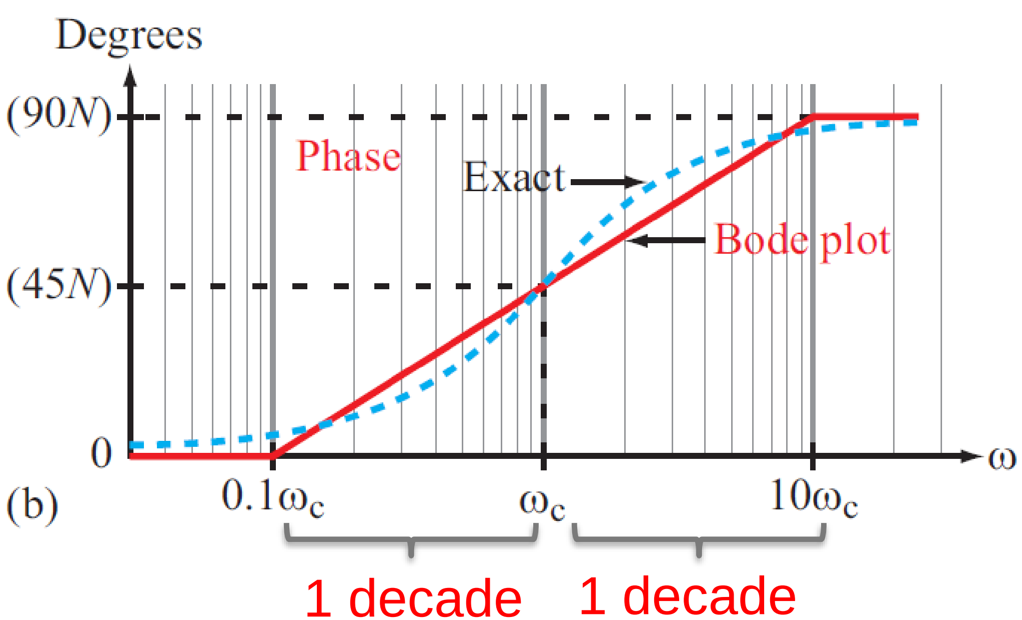
More Example

Simple zero:

$$\mathbf{H} = (1 + j\omega/\omega_c)^N$$



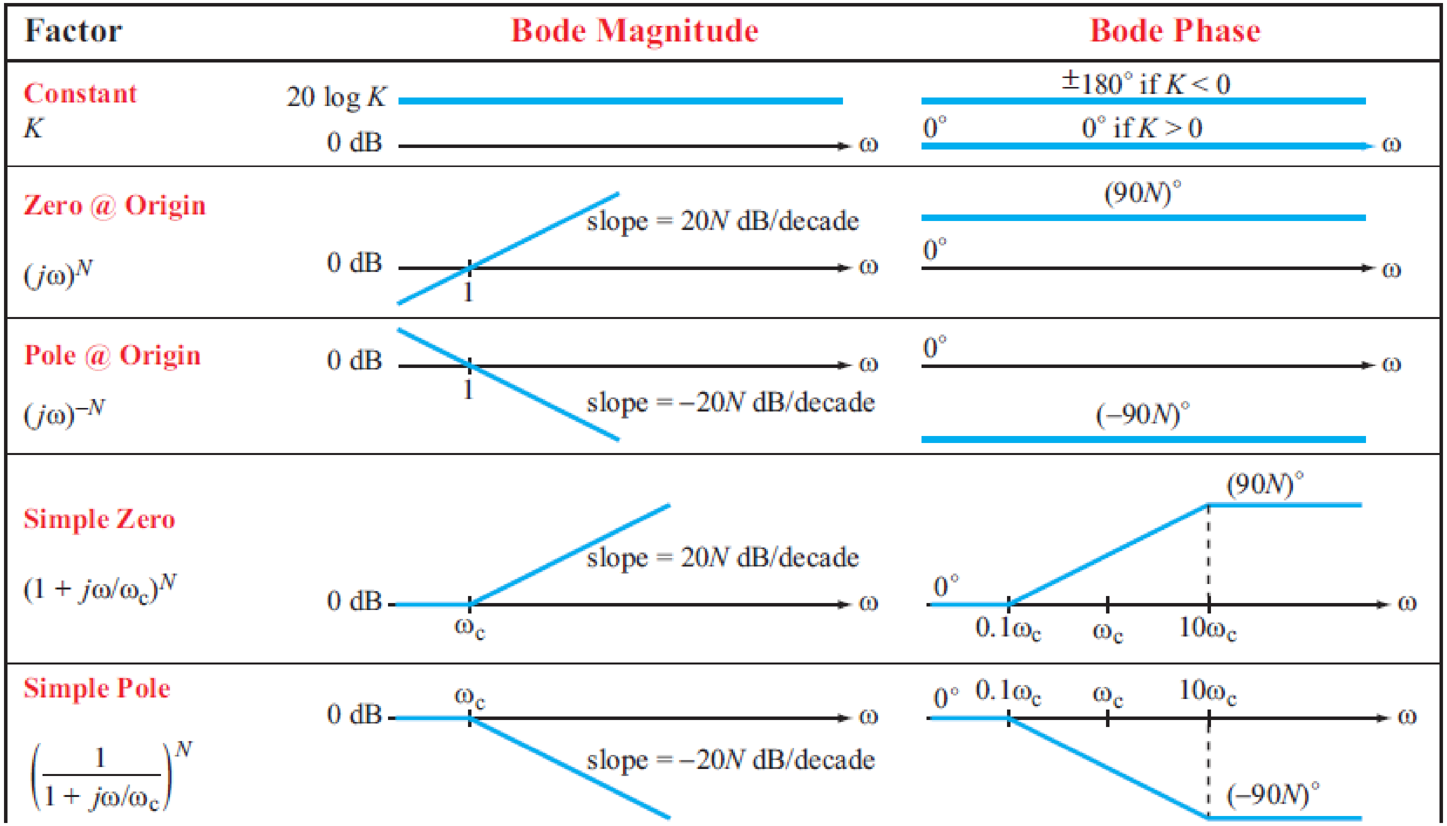
Bode Magnitude Slope = 20N dB per decade



Bode Phase Slope
= 45N degrees per decade

Exercises

- $H(\omega) = K$
- $H(\omega) = (j\omega)^N$
- $H(\omega) = 1/(j\omega)^N$



[Source: Berkeley]

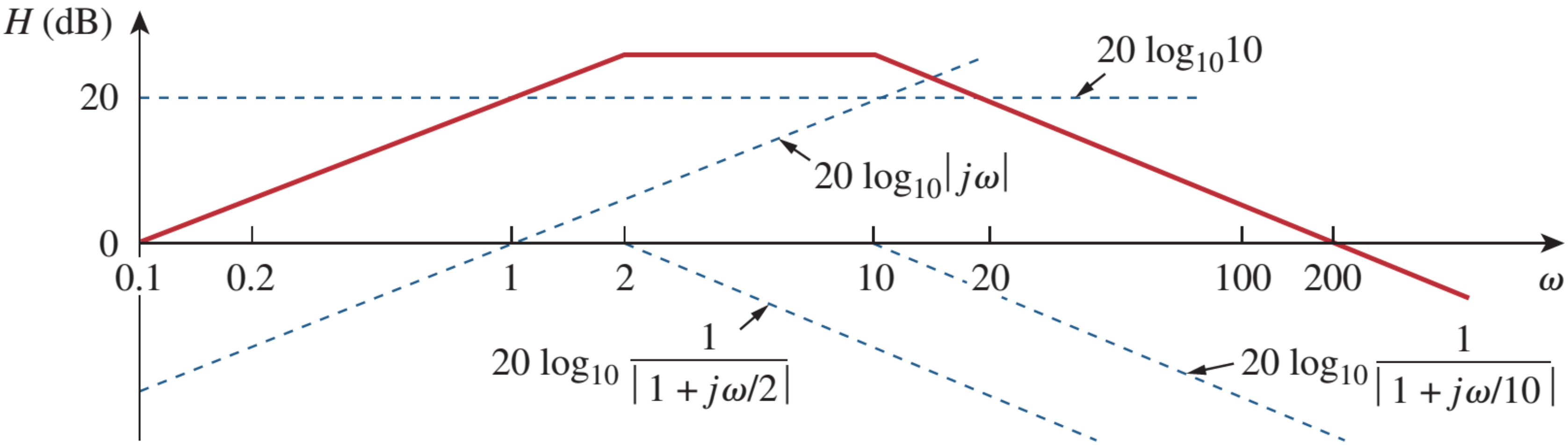
Standard Form

$$\mathbf{H}(\omega) = \frac{200\,j\omega}{(j\omega + 2)(j\omega + 10)}$$

$$\mathbf{H}(\omega) = \frac{10\,j\omega}{(1 + j\omega/2)(1 + j\omega/10)}$$

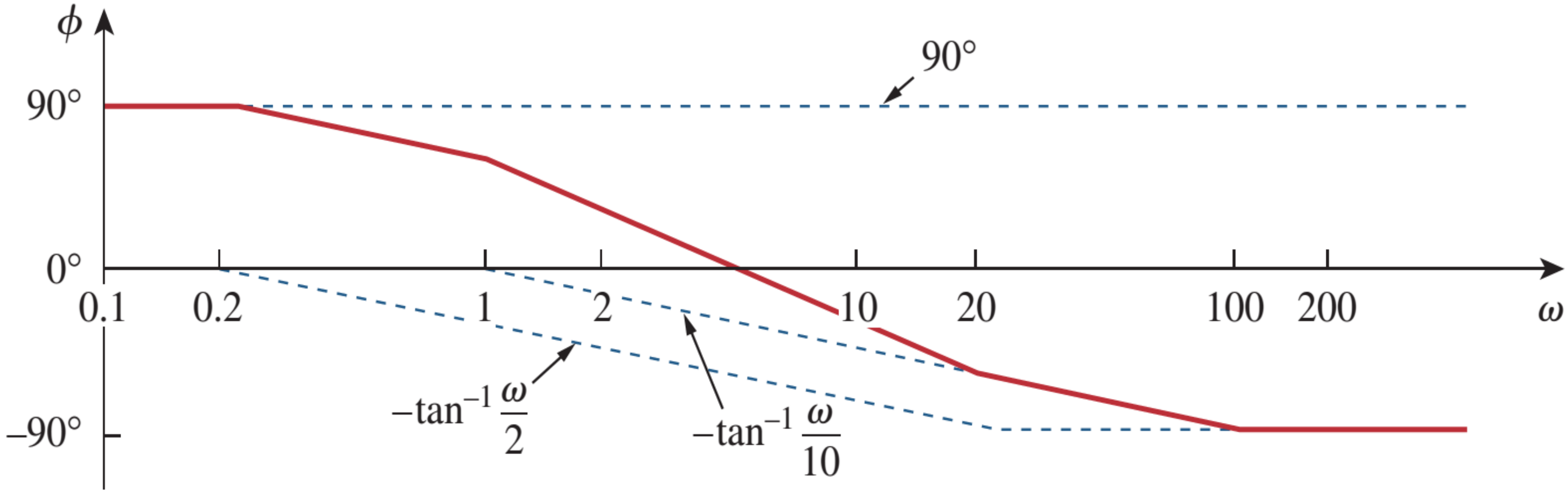
$$= \frac{10|j\omega|}{|1 + j\omega/2||1 + j\omega/10|} \underline{/90^\circ - \tan^{-1}\omega/2 - \tan^{-1}\omega/10}$$

Example - Magnitude



Example - Phase

$$\phi = 90° - \tan^{-1}\frac{\omega}{2} - \tan^{-1}\frac{\omega}{10}$$



Exercises

- $$\mathbf{H}(\omega) = \frac{(20 + j4\omega)^2}{j40\omega(100 + j2\omega)}$$

- $$\mathbf{H}(\omega) = \frac{(j10\omega + 30)^2}{(300 - 3\omega^2 + j90\omega)}$$

Outline

- Frequency response
- Transfer function
- Bode plots (or diagram)
- Resonance

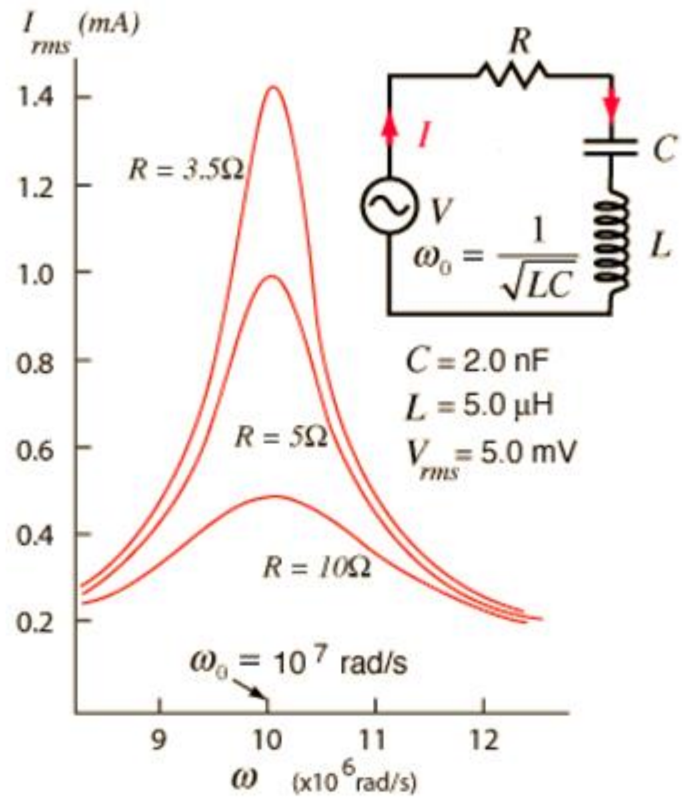
Series Resonance

- A series resonant circuit consists of an inductor and capacitor in series.

$$H(\omega) = \frac{I}{V} = \frac{1}{R + j\omega L + \frac{1}{j\omega C}}$$

$$\mathbf{Z} = R + j\left(\omega L - \frac{1}{\omega C}\right)$$

  - Resonance occurs when the imaginary part of *Z* is zero.
  - The value of ω that satisfies this is called the resonant frequency.



[Source: Georgia State U]

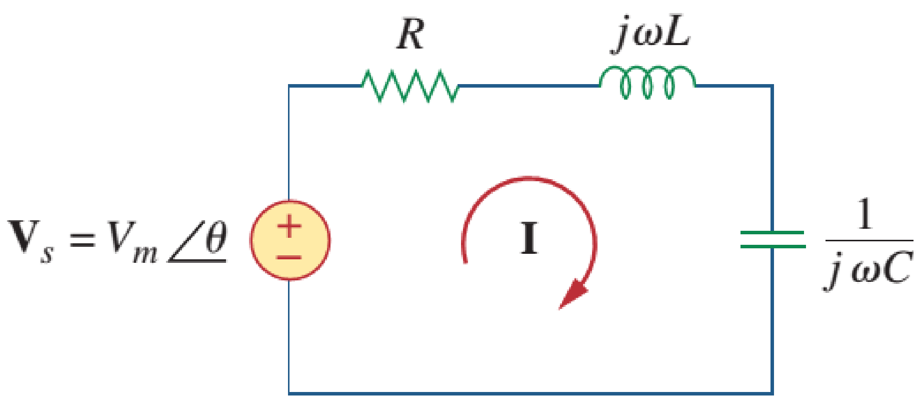
$$\omega_0 = \frac{1}{\sqrt{LC}} \text{ rad/s}$$

$$f_0 = \frac{1}{2\pi\sqrt{LC}} \text{ Hz}$$

Series Resonance

$$H(\omega) = \frac{I}{V} = \frac{1}{R + j\omega L + \frac{1}{j\omega C}}$$

- At resonance:
  - The impedance is purely resistive
  - The voltage V_s and the current I are in phase
  - The magnitude of the transfer function is minimum
  - The inductor and capacitor voltages can be much more than the source



$$|\mathbf{V}_L| = \frac{V_m}{R}\omega_0 L$$

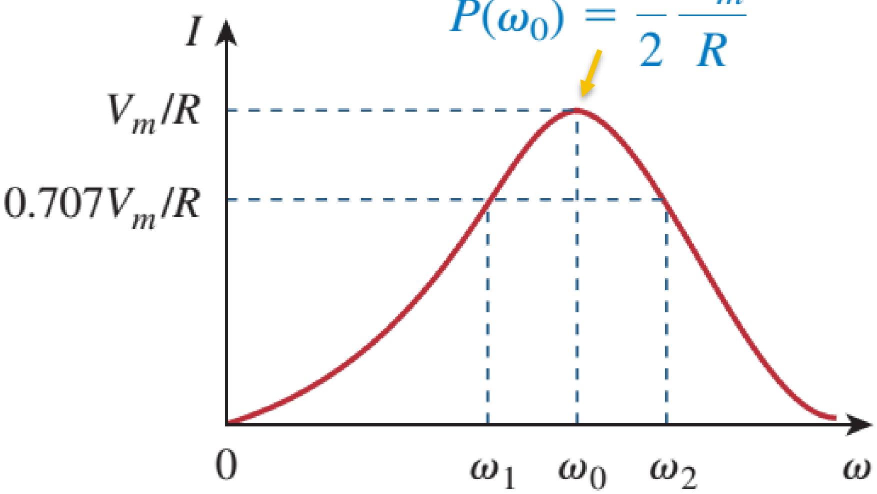
$$|\mathbf{V}_C| = \frac{V_m}{R}\frac{1}{\omega_0 C}$$

Half-Power Frequencies

- The frequency response of the current magnitude:

$$I = |\mathbf{I}| = \frac{V_m}{\sqrt{R^2 + (\omega L - 1/\omega C)^2}}$$

$$P(\omega_0) = \frac{1}{2}\frac{V_m^2}{R}$$



$$\omega_0 = \frac{1}{\sqrt{LC}} \text{ rad/s}$$

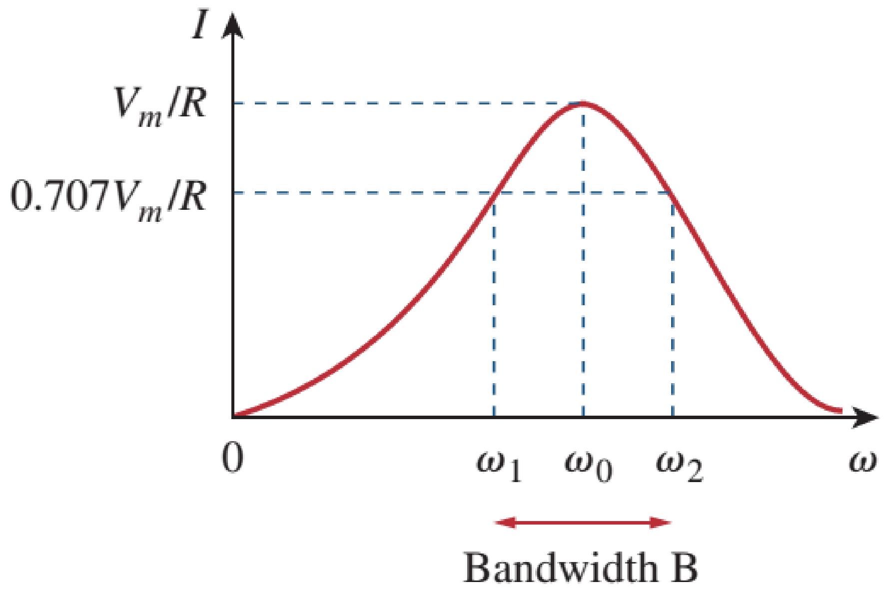
$$P(\omega_1) = P(\omega_2) = \frac{1}{2}P(\omega_0)$$

$$\omega_1 = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}}$$

$$\omega_2 = \frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}}$$

$$\omega_0 = \sqrt{\omega_1\omega_2}$$

Bandwidth



$$\omega_1 = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}}$$

$$\omega_2 = \frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}}$$

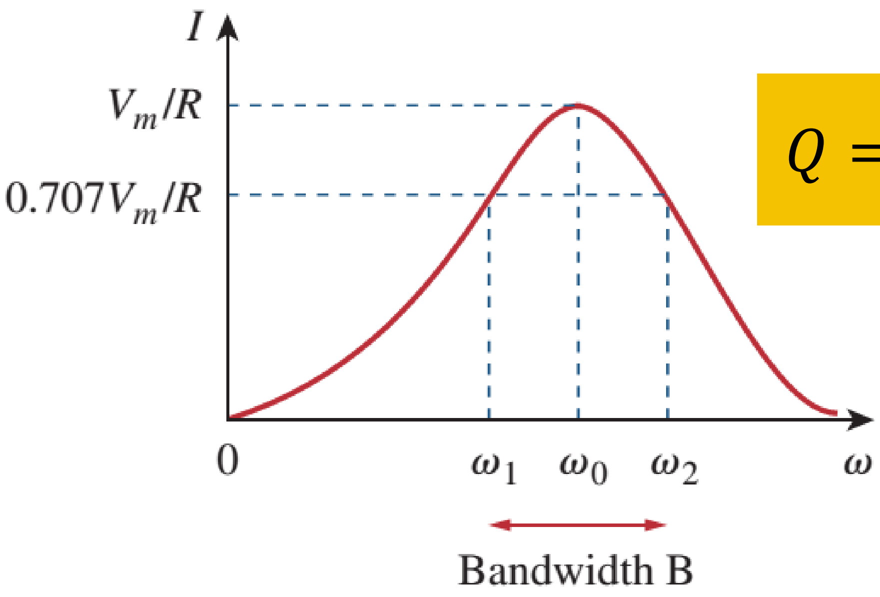
$$B = \omega_2 - \omega_1 = \frac{R}{L}$$

- Bandwidth: the difference between the two half-power frequencies

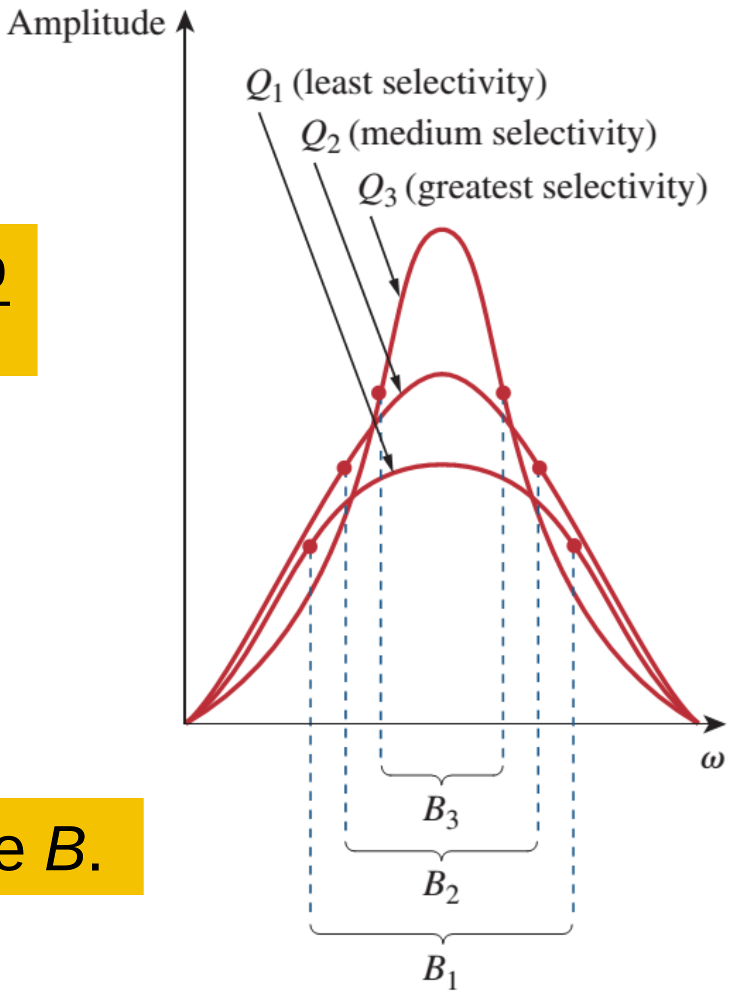
Quality Factor *Q*

$$\omega_0 = \frac{1}{\sqrt{LC}} \qquad B = \omega_2 - \omega_1 = \frac{R}{L}$$

- Quality factor *Q*: measure the "sharpness" of the resonance.

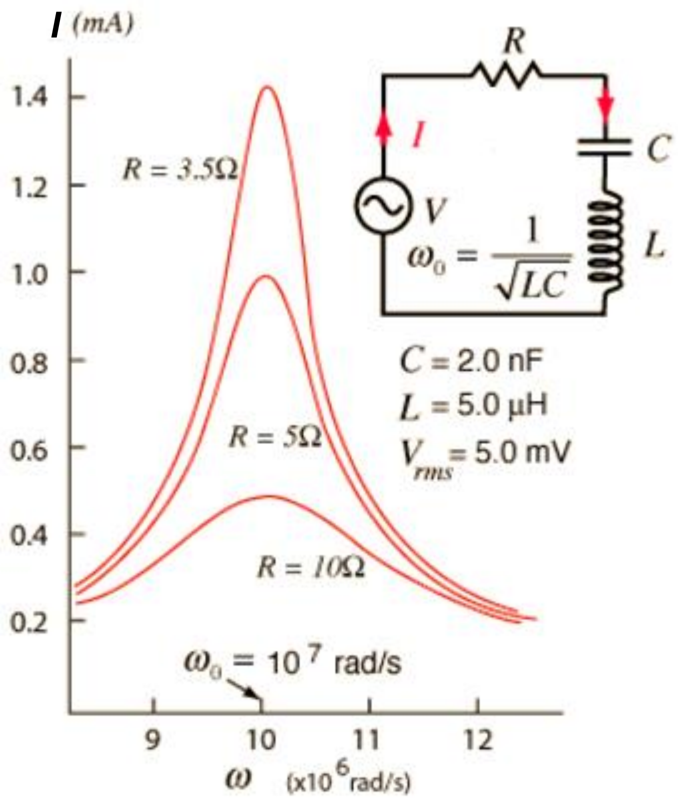


$$Q = \frac{\omega_0}{B}$$



The higher the *Q*, the smaller the *B*.

Quality Factor *Q* – From Energy Perspective



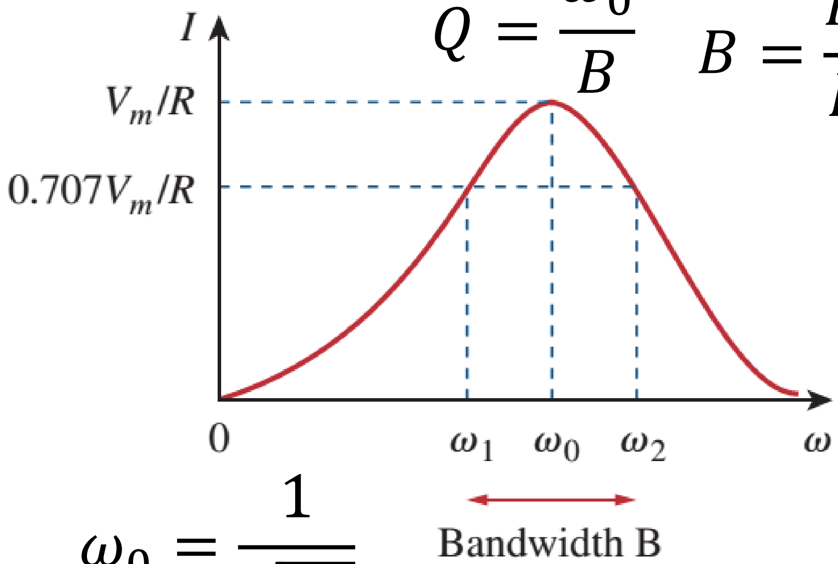
[Source: Georgia State U]

$$Q = 2\pi \frac{\text{Peak energy stored in the circuit}}{\text{Energy dissipated by the circuit in one period at resonance}}$$

$$\boxed{Q = \frac{\omega_0 L}{R} = \frac{1}{\omega_0 C R}}$$

Approximation of Half-Power Frequencies

$$Q = \frac{\omega_0}{B} \qquad B = \frac{R}{L}$$



$$\omega_0 = \frac{1}{\sqrt{LC}}$$

$$\omega_1 = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}}$$

$$\omega_2 = \frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}}$$

$$\frac{\omega_1}{\omega_0} = -\frac{1}{2Q} + \sqrt{1 + \frac{1}{4Q^2}}$$

$$\frac{\omega_2}{\omega_0} = \frac{1}{2Q} + \sqrt{1 + \frac{1}{4Q^2}}$$

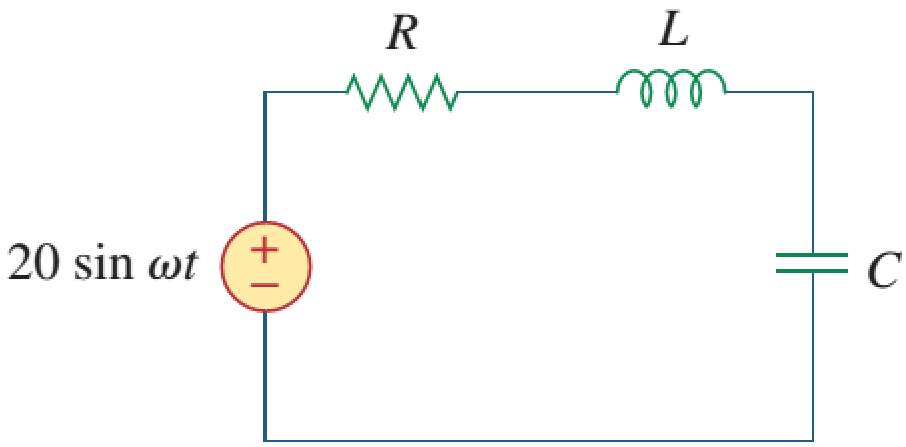
- For high-Q ($Q \geq 10$) circuits, half-power frequencies can be approximated as

$$\omega_1 \simeq \omega_0 - \frac{B}{2}, \qquad \omega_2 \simeq \omega_0 + \frac{B}{2}$$

Example

In the circuit, $R = 2\Omega$, $L = 1\text{mH}$ and $C = 0.4\mu\text{F}$

- Find resonant frequency ω_0.
- Find half-power frequencies.
- Calculate *Q* and bandwidth *B*.
- Determine the amplitude of the current at $\omega_{0,}$ ω_1 and ω_2.



$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{10^{-3} \times 0.4 \times 10^{-6}}} = 50 \text{ krad/s}$$

$$B = \omega_2 - \omega_1 = 2 \text{ krad/s}$$

$$Q = \frac{\omega_0}{B} = \frac{50}{2} = 25$$

At $\omega = \omega_0$,

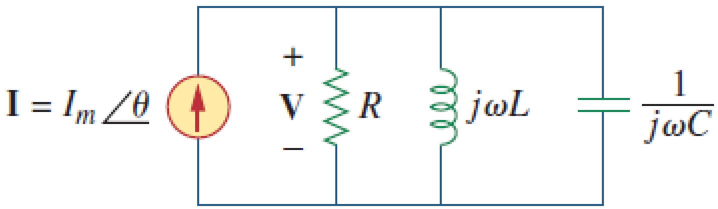
$$I = \frac{V_m}{R} = \frac{20}{2} = 10 \text{ A}$$

At $\omega = \omega_1, \omega_2$,

$$I = \frac{V_m}{\sqrt{2}R} = \frac{10}{\sqrt{2}} = 7.071 \text{ A}$$

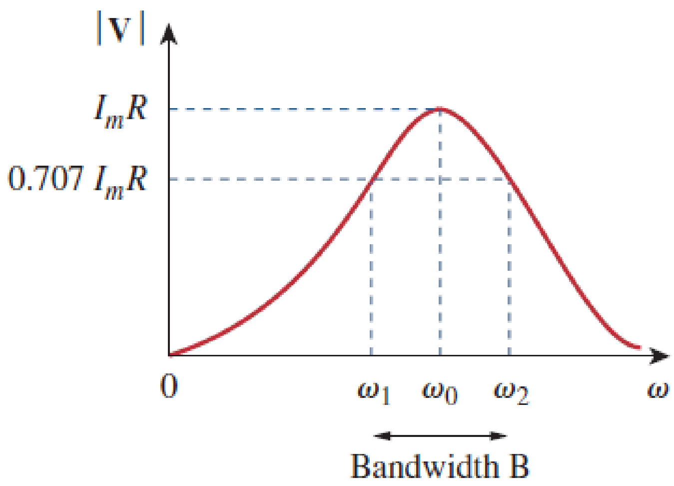
Parallel Resonance

$$\omega_0 = \frac{1}{\sqrt{LC}} \text{ rad/s}$$



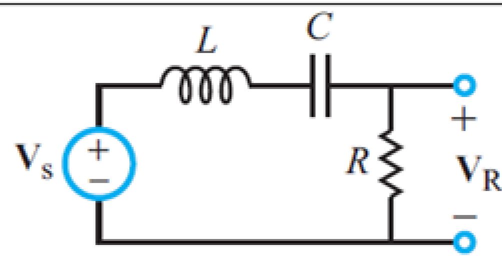
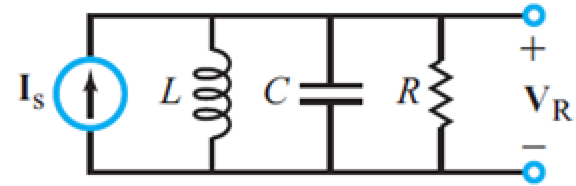
$$\omega_1 = -\frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 + \frac{1}{LC}}$$

$$\omega_2 = \frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 + \frac{1}{LC}}$$



$$B = \omega_2 - \omega_1 = \frac{1}{RC}$$

$$Q = \frac{\omega_0}{B} = \omega_0 RC = \frac{R}{\omega_0 L}$$

RLC Circuit		
Transfer Function	$\mathbf{H} = \dfrac{\mathbf{V}_R}{\mathbf{V}_s}$	$\mathbf{H} = \dfrac{\mathbf{V}_R}{\mathbf{I}_s}$
Resonant Frequency, ω_0	$\dfrac{1}{\sqrt{LC}}$	$\dfrac{1}{\sqrt{LC}}$
Bandwidth, B	$\dfrac{R}{L}$	$\dfrac{1}{RC}$
Quality Factor, Q	$\dfrac{\omega_0}{B} = \dfrac{\omega_0 L}{R}$	$\dfrac{\omega_0}{B} = \dfrac{R}{\omega_0 L}$
Lower Half-Power Frequency, ω_1	$\left[-\dfrac{1}{2Q} + \sqrt{1 + \dfrac{1}{4Q^2}}\right]\omega_0$	$\left[-\dfrac{1}{2Q} + \sqrt{1 + \dfrac{1}{4Q^2}}\right]\omega_0$
Upper Half-Power Frequency, ω_2	$\left[\dfrac{1}{2Q} + \sqrt{1 + \dfrac{1}{4Q^2}}\right]\omega_0$	$\left[\dfrac{1}{2Q} + \sqrt{1 + \dfrac{1}{4Q^2}}\right]\omega_0$

Notes: (1) The expression for Q of the series RLC circuit is the inverse of that for Q of the parallel circuit. (2) For $Q \geq 10$, $\omega_1 \simeq \omega_0 - \frac{B}{2}$, and $\omega_2 \simeq \omega_0 + \frac{B}{2}$.

[Source: Berkeley]