Inverse Transforms

In principle, we can recover f from F via

$$f(t) = \frac{1}{2\pi j}\int_{\sigma - j\infty}^{\sigma + j\infty} F(S)e^{st}\,ds$$

Surprisingly, this formula isn't really useful!

What is more common/useful as follows:

$$F(s) = \frac{P(s)}{Q(s)} = \frac{b_m s^m + b_{m-1}s^{m-1} + \ldots + b_1 s + b_0}{a_n s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0}$$

Generally

$$F(s)=\frac{P(s)}{Q(s)}=\frac{b_m s^m+b_{m-1}s^{m-1}+\ldots+b_1 s+b_0}{a_n s^n+a_{n-1}s^{n-1}+\ldots+a_1 s+a_0}$$

a_{i} and b_{i} are real constants, and the exponents m,n are positive integers

- If m<n, proper rational function
- If m>n, improper rational function

Partial Fraction Expansion with Real Distinct Roots

- Let F(s) be proper rational function, then

$$F(s)=\frac{P(s)}{Q(s)}=\frac{b_m s^m+b_{m-1}s^{m-1}+\ldots+b_1 s+b_0}{a_n s^n+a_{n-1}s^{n-1}+\ldots+a_1 s+a_0}=\frac{P'(s)}{(s-p_1)(s-p_2)\cdots(s-p_n)}$$

Case I: If the roots are real, $p_i \neq p_j \;\; for \; \forall i \neq j$

$$F(s)=\frac{P(s)}{Q(s)}=\frac{K_1}{s-p_1}+\frac{K_2}{s-p_2}+\cdots+\frac{K_n}{s-p_n}=\sum_{j=1}^{n}\frac{K_j}{s-p_j}$$

p_j(j=1, 2, …, n) are ***the roots*** of equation $Q(s)=0$

K_j(j=1, 2, …, n) are unknown constants

Partial Fraction Expansion with Real Distinct Roots

$$F(s) = \frac{P(s)}{Q(s)} = \frac{K_1}{s - p_1} + \frac{K_2}{s - p_2} + \cdots + \frac{K_n}{s - p_n} = \sum_{j=1}^{n} \frac{K_j}{s - p_j}$$

Case I:

If the roots are real, $p_i \neq p_j \;\; for \; \forall i \neq j$

$$K_j = \lim_{s \to p_j} (s - p_j) F(s) = (s - p_j) F(s) \Big|_{s = p_j}$$

解:

$k_1 \quad k_2$

$$F(s) = \frac{1}{s-\underbrace{(-1)}} + \frac{5}{s-\underbrace{(-3)}}$$

$Q(s)=0$

$p_1 \,.\, p_2$

$$= \frac{1}{\underbrace{s+1}} + \frac{5}{s+3}$$

$u(t) \to \frac{1}{s}$

$e^{-t}u(t) \to \frac{1}{s+1}$

$u(t) \to \frac{1}{s}$

$e^{-3t}u(t) \to \frac{1}{s+3}$

$5 \cdot e^{-3t}u(t) \to \frac{5}{s+3}$

$$f(t) = (e^{-t} + 5e^{-3t}) \cdot u(t)$$

Exercise

$$F(s)=\frac{s^2+3s+5}{s^3+6s^2+11s+6}$$

$$F(s)=\frac{s^2+3s+5}{(s+1)(s+2)(s+3)}=\frac{K_1}{s+1}+\frac{K_2}{s+2}+\frac{K_3}{s+3}$$

For K_1:
$$\frac{(s^2+3s+5)\cancel{(s+1)}}{\cancel{(s+1)}(s+2)(s+3)}=\frac{K_1\cancel{(s+1)}}{\cancel{s+1}}+\frac{K_2(s+1)}{s+2}+\frac{K_3(s+1)}{s+3}$$
Set $s=-1$

For K_2:
$$\frac{(s^2+3s+5)\cancel{(s+2)}}{(s+1)\cancel{(s+2)}(s+3)}=\frac{K_1(s+2)}{s+1}+\frac{K_2\cancel{(s+2)}}{\cancel{s+2}}+\frac{K_3(s+2)}{s+3}$$

Set $s=-2$: --

$$F(s)=\frac{1.5}{s+1}+\frac{-3}{s+2}+\frac{2.5}{s+3}$$

$\hookrightarrow \mathcal{L}^{-1}$: $f(t)=(1.5e^{-t}-3\cdot e^{-2t}+2.5\cdot e^{-3t})\cdot u(t)$

Partial Fraction Expansion with Multiple Roots

- Case II:
- If Q(s) has multiple roots

$$F(s) = \boxed{\frac{K_{11}}{s-p_1} + \frac{K_{12}}{(s-p_1)^2} + \cdots + \frac{K_{1r}}{(s-p_1)^r}} + \boxed{\frac{K_{r+1}}{s-p_{r+1}} \cdots + \frac{K_n}{s-p_n}}$$

$$K_{1r} = (s-p_1)^r F(s)\Big|_{s=p_1}$$

$$K_{1(r-1)} = \frac{d}{ds}\left[(s-p_1)^r F(s)\right]_{s=p_1}$$

$$K_{1(r-2)} = \frac{1}{2!}\frac{d^2}{ds^2}\left[(s-p_1)^r F(s)\right]_{s=p_1}$$

$$\vdots$$

$$K_{11} = \frac{1}{(r-1)!}\frac{d^{r-1}}{ds^{r-1}}\left[(s-p_1)^r F(s)\right]_{s=p_1}$$

$$F(s) = \boxed{\frac{K_{11}}{s-p_1} + \frac{K_{12}}{(s-p_1)^2}}$$

For K_{12}:

$$F(s)\cdot(s-P_1)^2 = \frac{K_{11}(S-P_1)^{\cancel{2}}}{\cancel{S-P_1}} + \frac{K_{12}\cancel{(S-P_1)^2}}{\cancel{(S-P_1)^2}}$$

Set $S = P_1$

$K_{12} = \cdots$

For k_{11}: $F(s)\cdot\cancel{(s-P_1)} = \frac{k_{11}\cancel{(s-P_1)}}{\cancel{s-P_1}} + \cancel{\frac{k_{12}(s-P_1)}{(s-P_1)^2}}$

set $s=P_1$

For k_{11}:

$$F(s)\cdot(s-P_1)^2 = \frac{k_{11}(s-P_1)^2}{s-P_1} + \frac{k_{12}(s-P_1)^2}{(s-P_1)^2}$$

$$\left[F(s)(s-P_1)^2\right]' = \left[k_{11}(s-P_1) + k_{12}\right]'$$

$$\therefore \quad k_{11} = \left[F(s)(s-P_1)^2\right]'\Big|_{s=P_1}$$

$$\left(\frac{u}{v}\right)' = \frac{u'v - v'u}{v^2}$$

Exercise

$$F(s) = \frac{10s^2 + 4}{s(s+1)(s+2)^2}$$

$$F(s) = \frac{\overset{1}{K_{11}}}{s} + \frac{\overset{-14}{K_{21}}}{s+1} + \frac{\overset{13}{K_{31}}}{s+2} + \frac{\overset{22}{K_{32}}}{(s+2)^2}$$

$$f(t) = \left[1 - 14e^{-t} + (13 + 22t)e^{-2t}\right]u(t)$$

$$\left[\frac{(10s^2+4)\cancel{(s+2)^2}}{s(s+1)\cancel{(s+2)^2}}\right]' = \left[\frac{K_{11}(s+2)^2}{s} + \frac{K_{21}(s+2)^2}{s+1} + \frac{K_{31}(s+2)^{\cancel{2}}}{\cancel{s+2}} + \frac{K_{32}\cancel{(s+2)^2}}{\cancel{(s+2)^2}}\right]'$$