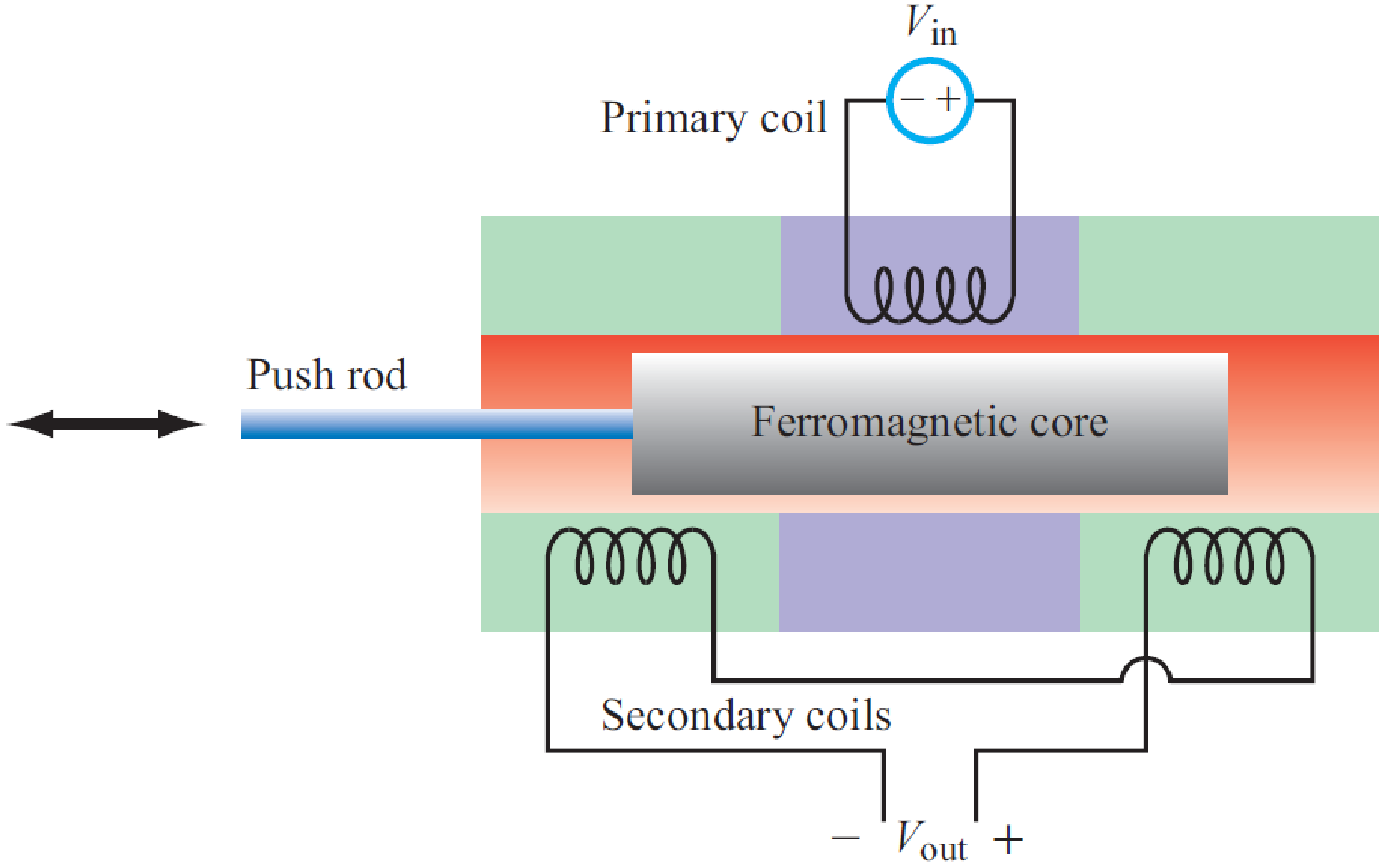
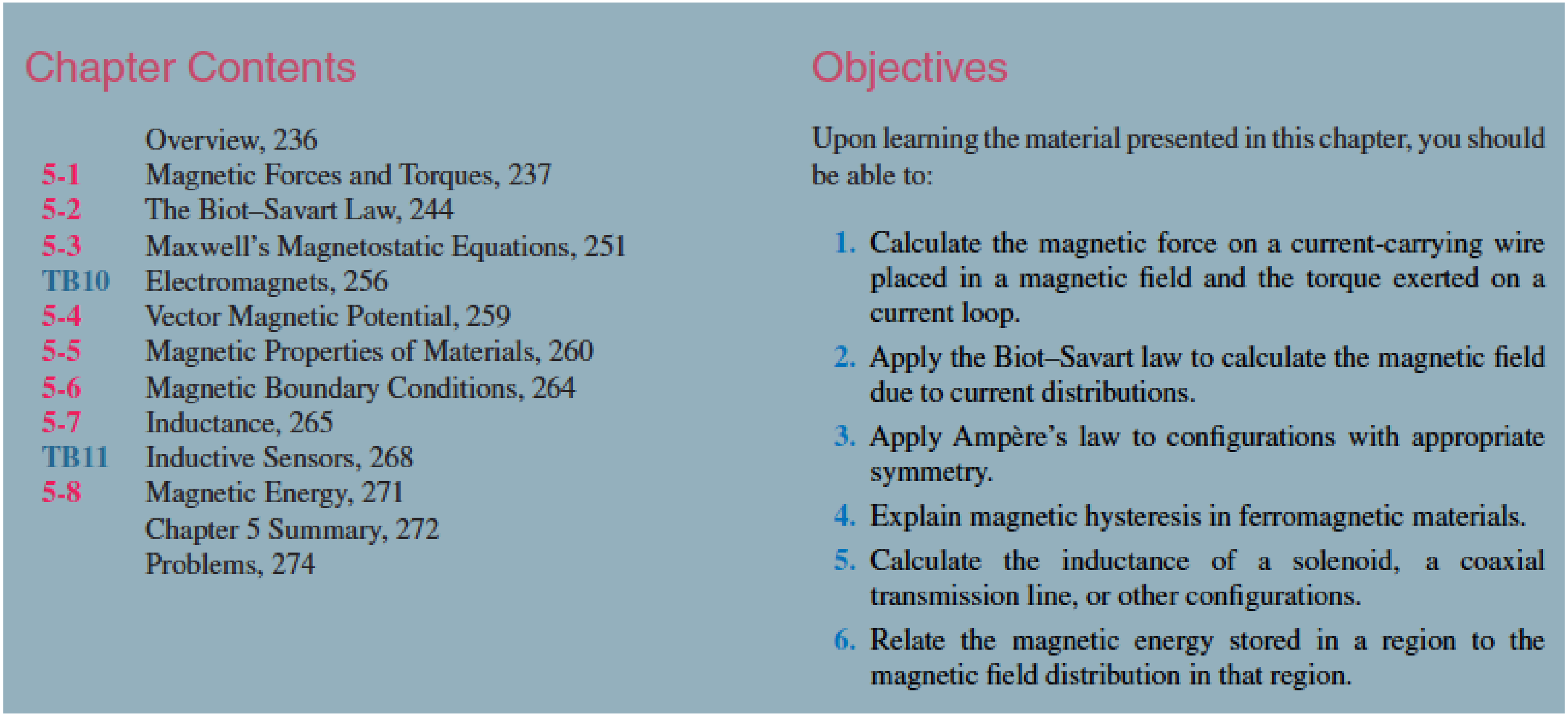


Figure TF11-1: Linear variable differential transformer (LVDT) circuit.

5. MAGNETOSTATICS

7e Applied EM by Ulaby and Ravaioli

Chapter 5 Overview



Chapter Contents

Objectives

Upon learning the material presented in this chapter, you should be able to:

1. Calculate the magnetic force on a current-carrying wire placed in a magnetic field and the torque exerted on a current loop.
2. Apply the Biot–Savart law to calculate the magnetic field due to current distributions.
3. Apply Ampère's law to configurations with appropriate symmetry.
4. Explain magnetic hysteresis in ferromagnetic materials.
5. Calculate the inductance of a solenoid, a coaxial transmission line, or other configurations.
6. Relate the magnetic energy stored in a region to the magnetic field distribution in that region.

Electric vs Magnetic Comparison

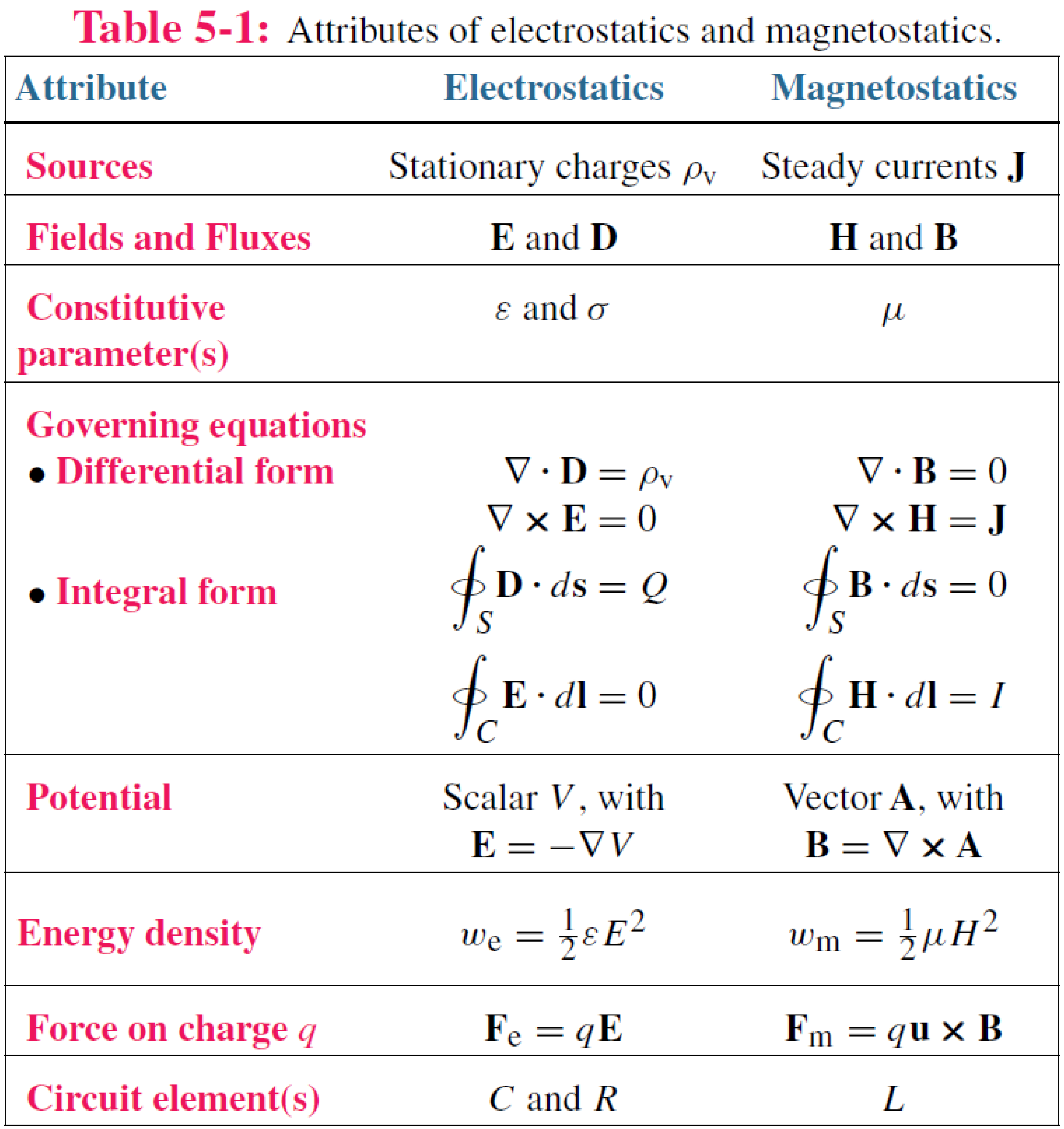


Table 5-1: Attributes of electrostatics and magnetostatics.

Attribute	Electrostatics	Magnetostatics
Sources	Stationary charges ρ_v	Steady currents $\mathbf{J}$
Fields and Fluxes	$\mathbf{E}$ and $\mathbf{D}$	$\mathbf{H}$ and $\mathbf{B}$
Constitutive parameter(s)	ε and σ	μ
Governing equations		
• **Differential form**	$\nabla \cdot \mathbf{D} = \rho_v$ $\nabla \times \mathbf{E} = 0$	$\nabla \cdot \mathbf{B} = 0$ $\nabla \times \mathbf{H} = \mathbf{J}$
• **Integral form**	$\oint_S \mathbf{D} \cdot d\mathbf{s} = Q$ $\oint_C \mathbf{E} \cdot d\mathbf{l} = 0$	$\oint_S \mathbf{B} \cdot d\mathbf{s} = 0$ $\oint_C \mathbf{H} \cdot d\mathbf{l} = I$
Potential	Scalar V, with $\mathbf{E} = -\nabla V$	Vector $\mathbf{A}$, with $\mathbf{B} = \nabla \times \mathbf{A}$
Energy density	$w_e = \frac{1}{2}\varepsilon E^2$	$w_m = \frac{1}{2}\mu H^2$
Force on charge q	$\mathbf{F}_e = q\mathbf{E}$	$\mathbf{F}_m = q\mathbf{u} \times \mathbf{B}$
Circuit element(s)	C and R	L

Electric & Magnetic Forces

Magnetic force

$$\mathbf{F}_{\mathrm{m}} = q\mathbf{u} \times \mathbf{B} \qquad \text{(N)}$$

Electromagnetic (Lorentz) force

$$\mathbf{F} = \mathbf{F}_{\mathrm{e}} + \mathbf{F}_{\mathrm{m}} = q\mathbf{E} + q\mathbf{u} \times \mathbf{B} = q(\mathbf{E} + \mathbf{u} \times \mathbf{B}).$$

Three differences between $\mathbf{F}_e$ and $\mathbf{F}_m$

1. Whereas the electric force is always in the direction of the electric field, the magnetic force is always perpendicular to the magnetic field.
2. Whereas the electric force acts on a charged particle whether or not it is moving, the magnetic force acts on it only when it is in motion.
3. Whereas the electric force expends energy in displacing a charged particle, the magnetic force does no work when a particle is displaced.

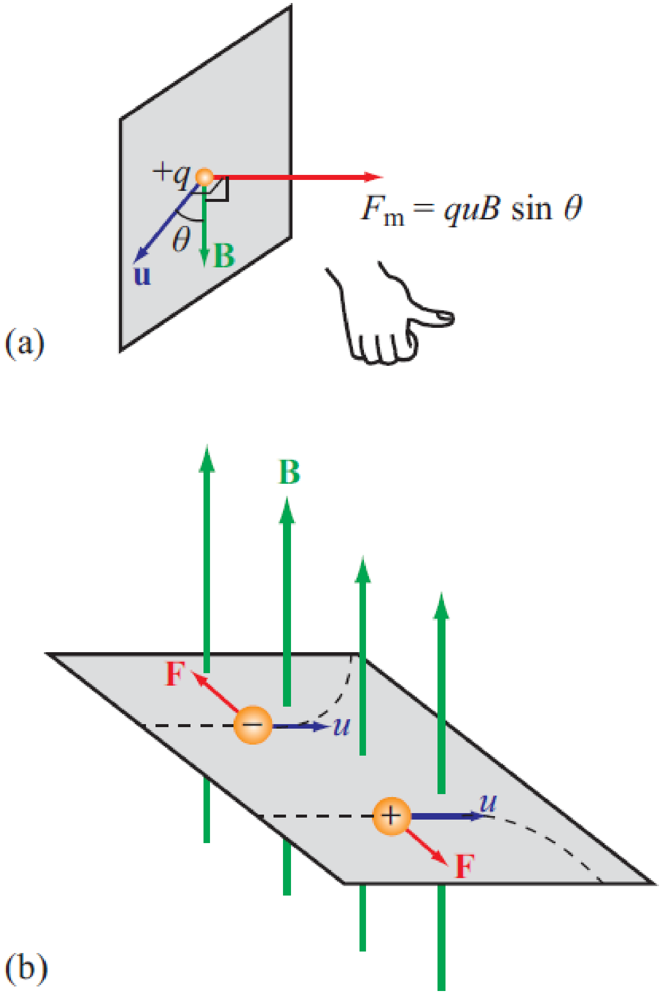


Figure 5-1: The direction of the magnetic force exerted on a charged particle moving in a magnetic field is (a) perpendicular to both **B** and **u** and (b) depends on the charge polarity (positive or negative).

Magnetic Force on a Current Element

Differential force dFm on a differential current I dl:

$$d\mathbf{F}_{\mathrm{m}} = I\, d\mathbf{l} \times \mathbf{B} \qquad \text{(N).} \qquad (5.9)$$

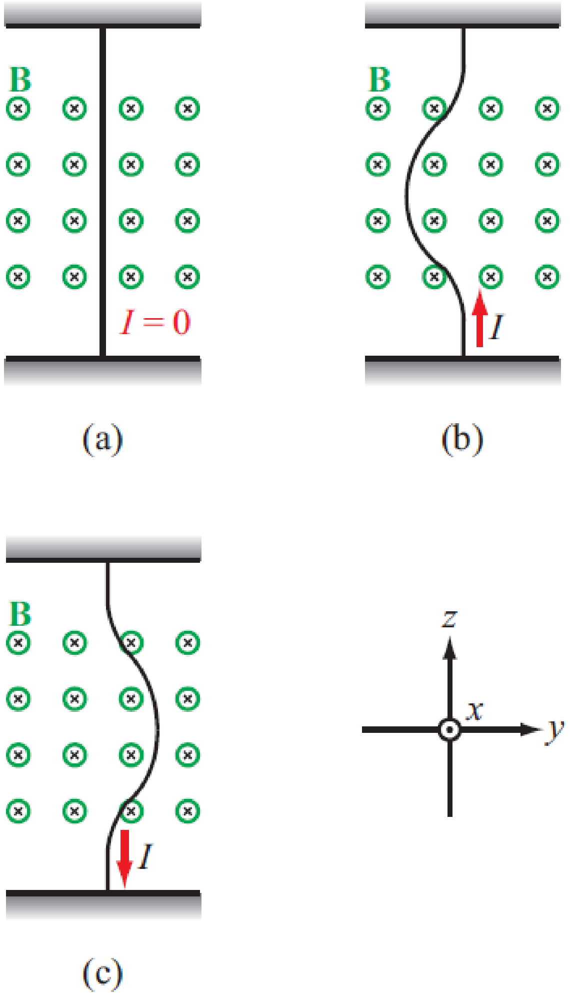
For a closed circuit of contour C carrying a current I, the total magnetic force is

$$\boxed{\mathbf{F}_{\mathrm{m}} = I \oint_C d\mathbf{l} \times \mathbf{B} \qquad \text{(N).} \qquad (5.10)}$$

If the closed wire shown in Fig. 5-3(a) resides in a uniform external magnetic field **B**, then **B** can be taken outside the integral in Eq. (5.10), in which case

$$\mathbf{F}_{\mathrm{m}} = I \left(\oint_C d\mathbf{l} \right) \times \mathbf{B} = 0. \qquad (5.11)$$

*This result, which is a consequence of the fact that the vector sum of the infinitesimal vectors d***l** *over a closed path equals zero, states that the total magnetic force on any closed current loop in a uniform magnetic field is zero.*



(a) (b) (c)

Figure 5-2: When a slightly flexible vertical wire is placed in a magnetic field directed into the page (as denoted by the crosses), it is (a) not deflected when the current through it is zero, (b) deflected to the left when I is upward, and (c) deflected to the right when I is downward.

Torque

$$\mathbf{T} = \mathbf{d} \times \mathbf{F} \qquad (\mathrm{N{\cdot}m})$$

d = moment arm
F = force
T = torque

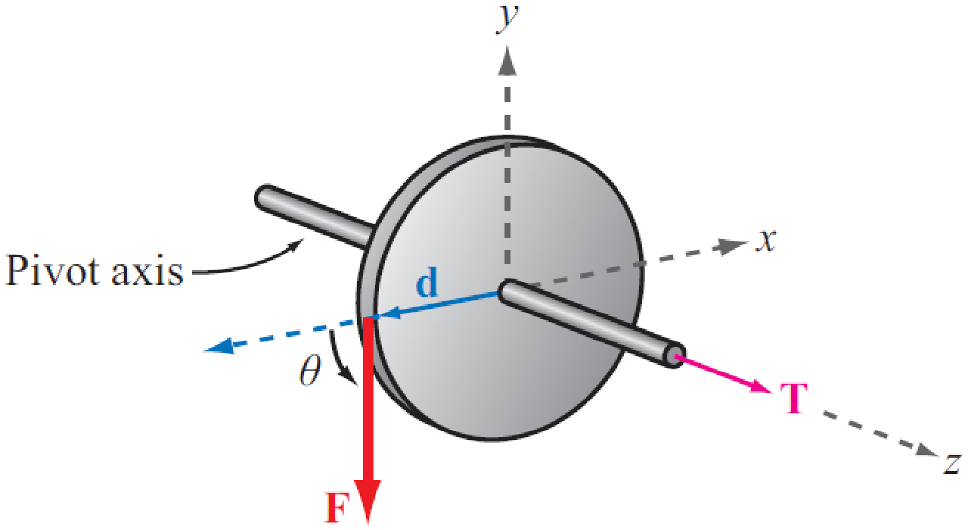


Figure 5-5: The force **F** acting on a circular disk that can pivot along the z-axis generates a torque $\mathbf{T} = \mathbf{d} \times \mathbf{F}$ that causes the disk to rotate.

These directions are governed by the following ***right-hand rule:*** *when the thumb of the right hand points along the direction of the torque, the four fingers indicate the direction that the torque tries to rotate the body.*

Magnetic Torque on Current Loop

$$\mathbf{F}_1 = I(-\hat{\mathbf{y}}b) \times (\hat{\mathbf{x}}B_0) = \hat{\mathbf{z}}IbB_0,$$

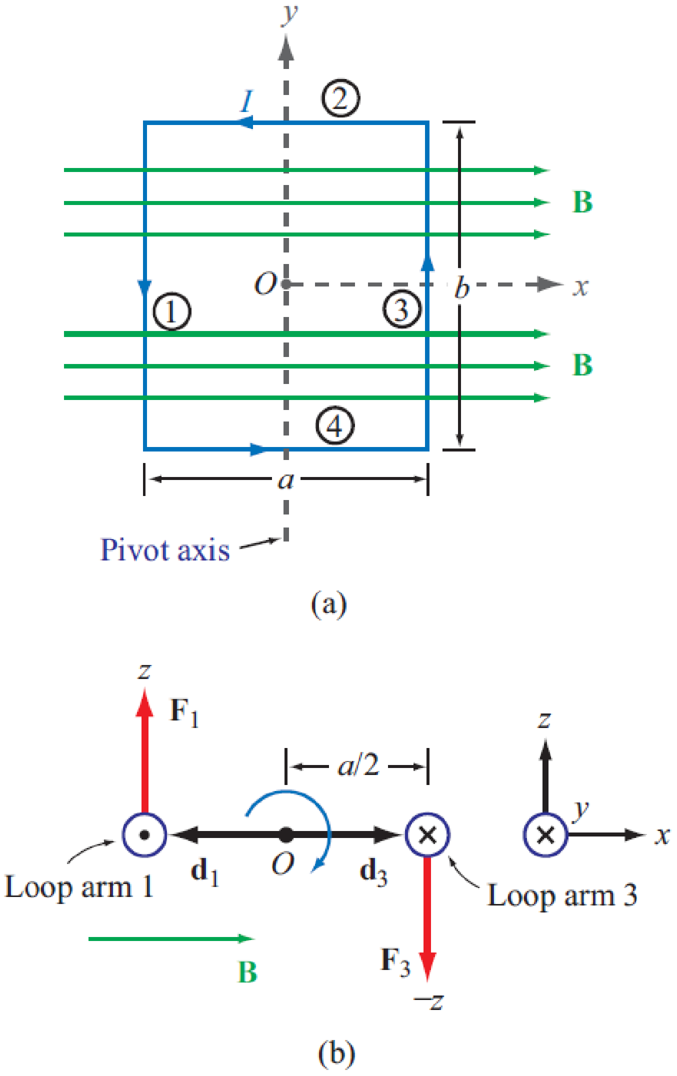
$$\mathbf{F}_3 = I(\hat{\mathbf{y}}b) \times (\hat{\mathbf{x}}B_0) = -\hat{\mathbf{z}}IbB_0.$$

No forces on arms 2 and 4 (because I and B are parallel, or anti-parallel)

Magnetic torque:

$$\begin{aligned}\mathbf{T} &= \mathbf{d}_1 \times \mathbf{F}_1 + \mathbf{d}_3 \times \mathbf{F}_3 \\ &= \left(-\hat{\mathbf{x}}\,\frac{a}{2}\right) \times (\hat{\mathbf{z}}IbB_0) + \left(\hat{\mathbf{x}}\,\frac{a}{2}\right) \times (-\hat{\mathbf{z}}IbB_0) \\ &= \hat{\mathbf{y}}IabB_0 = \hat{\mathbf{y}}I\underset{\uparrow}{A}B_0,\end{aligned}$$

Area of Loop



(a)

(b)

Figure 5-6: Rectangular loop pivoted along the y-axis: (a) front view and (b) bottom view. The combination of forces $\mathbf{F}_1$ and $\mathbf{F}_3$ on the loop generates a torque that tends to rotate the loop in a clockwise direction as shown in (b).

Inclined Loop

For a loop with *N* turns and whose surface normal is at angle theta relative to *B* direction:

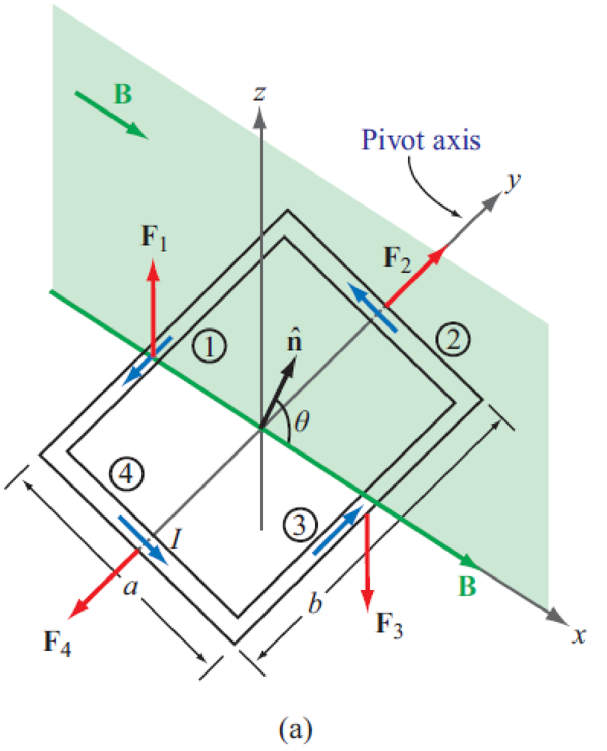
$$T = NIAB_0 \sin\theta. \qquad (5.18)$$

The quantity NIA is called the ***magnetic moment*** m of the loop. Now, consider the vector

$$\boxed{\mathbf{m} = \hat{\mathbf{n}}\, NIA = \hat{\mathbf{n}}\, m \qquad (\text{A}\cdot\text{m}^2), \qquad (5.19)}$$

where $\hat{\mathbf{n}}$ is the surface normal of the loop and governed by the following ***right-hand rule: when the four fingers of the right hand advance in the direction of the current*** I***, the direction of the thumb specifies the direction of*** $\hat{\mathbf{n}}$. In terms of $\mathbf{m}$, the torque vector $\mathbf{T}$ can be written as

$$\boxed{\mathbf{T} = \mathbf{m} \times \mathbf{B} \qquad (\text{N}\cdot\text{m}). \qquad (5.20)}$$



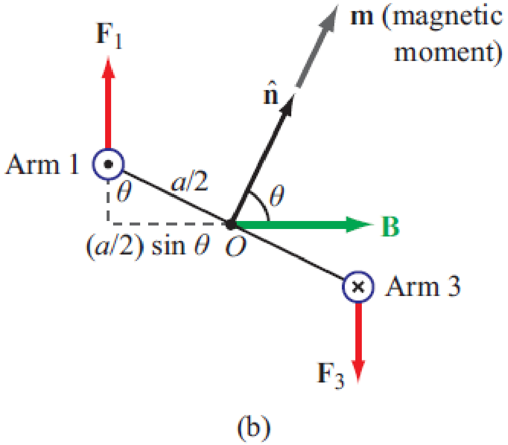


Figure 5-7: Rectangular loop in a uniform magnetic field with flux density **B** whose direction is perpendicular to the rotation axis of the loop, but makes an angle θ with the loop's surface normal $\hat{\mathbf{n}}$.

Biot-Savart Law

Magnetic field induced by a differential current:

$$d\mathbf{H} = \frac{I}{4\pi} \frac{d\mathbf{l} \times \hat{\mathbf{R}}}{R^2} \qquad \text{(A/m)}$$

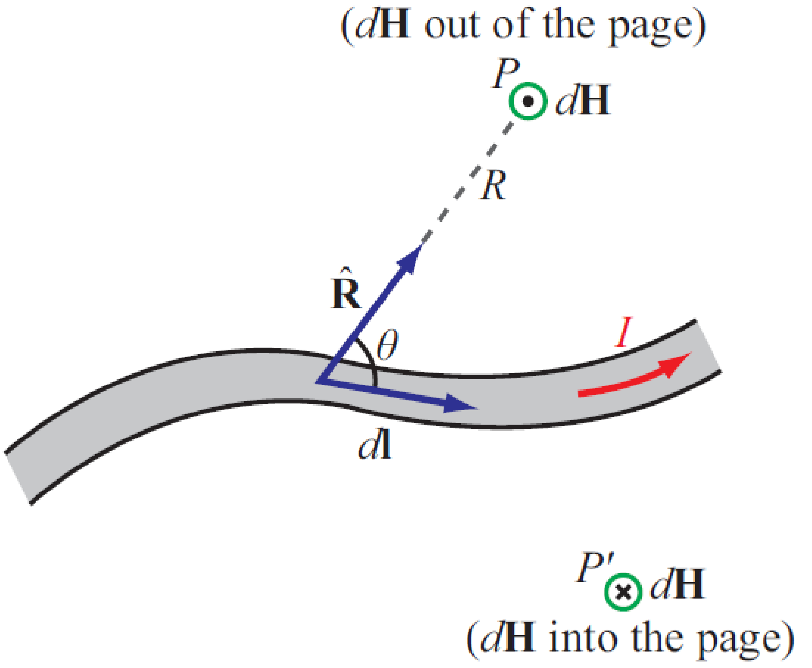


Figure 5-8: Magnetic field $d\mathbf{H}$ generated by a current element $I\ d\mathbf{l}$. The direction of the field induced at point P is opposite to that induced at point P'.

For the entire length:

$$\boxed{\mathbf{H} = \frac{I}{4\pi} \int_{l} \frac{d\mathbf{l} \times \hat{\mathbf{R}}}{R^2} \qquad \text{(A/m)}, \qquad (5.22)}$$

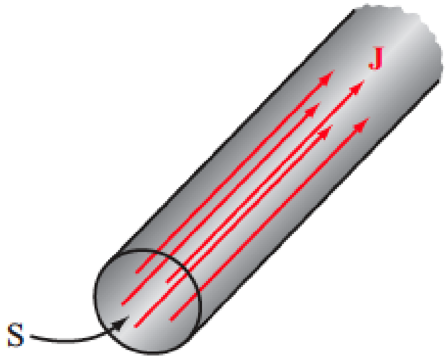
where l is the line path along which I exists.

Magnetic Field due to Current Densities

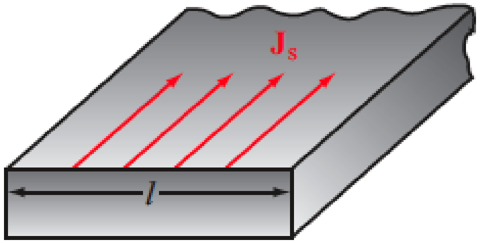
$$I\,d\mathbf{l} \quad \leftrightarrow \quad \mathbf{J}_s\,ds \quad \leftrightarrow \quad \mathbf{J}\,d\mathcal{V}$$

$$\mathbf{H} = \frac{1}{4\pi}\int_S \frac{\mathbf{J}_s \times \hat{\mathbf{R}}}{R^2}\,ds \quad \text{(surface current),}$$

$$\mathbf{H} = \frac{1}{4\pi}\int_{\mathcal{V}} \frac{\mathbf{J} \times \hat{\mathbf{R}}}{R^2}\,d\mathcal{V} \quad \text{(volume current).}$$



(a) Volume current density **J** in A/m^2



(b) Surface current density $\mathbf{J}_s$ in A/m

Figure 5-9: (a) The total current crossing the cross section S of the cylinder is $I = \int_S \mathbf{J} \cdot d\mathbf{s}$. (b) The total current flowing across the surface of the conductor is $I = \int_l J_s\, dl$.

Example 5-2: Magnetic Field of Linear Conductor

Solution: From Fig. 5-10, the differential length vector $d\mathbf{l} = \hat{\mathbf{z}}\, dz$. Hence, $d\mathbf{l} \times \hat{\mathbf{R}} = dz\, (\hat{\mathbf{z}} \times \hat{\mathbf{R}}) = \hat{\boldsymbol{\phi}} \sin\theta\, dz$, where $\hat{\boldsymbol{\phi}}$ is the azimuth direction and θ is the angle between $d\mathbf{l}$ and $\hat{\mathbf{R}}$. Application of Eq. (5.22) gives

$$\mathbf{H} = \frac{I}{4\pi} \int_{z=-l/2}^{z=l/2} \frac{d\mathbf{l} \times \hat{\mathbf{R}}}{R^2} = \hat{\boldsymbol{\phi}} \frac{I}{4\pi} \int_{-l/2}^{l/2} \frac{\sin\theta}{R^2}\, dz. \qquad (5.25)$$

Both R and θ are dependent on the integration variable z, but the radial distance r is not. For convenience, we will convert the integration variable from z to θ by using the transformations

$$R = r \csc\theta, \qquad (5.26a)$$

$$z = -r \cot\theta, \qquad (5.26b)$$

$$dz = r \csc^2\theta\, d\theta. \qquad (5.26c)$$

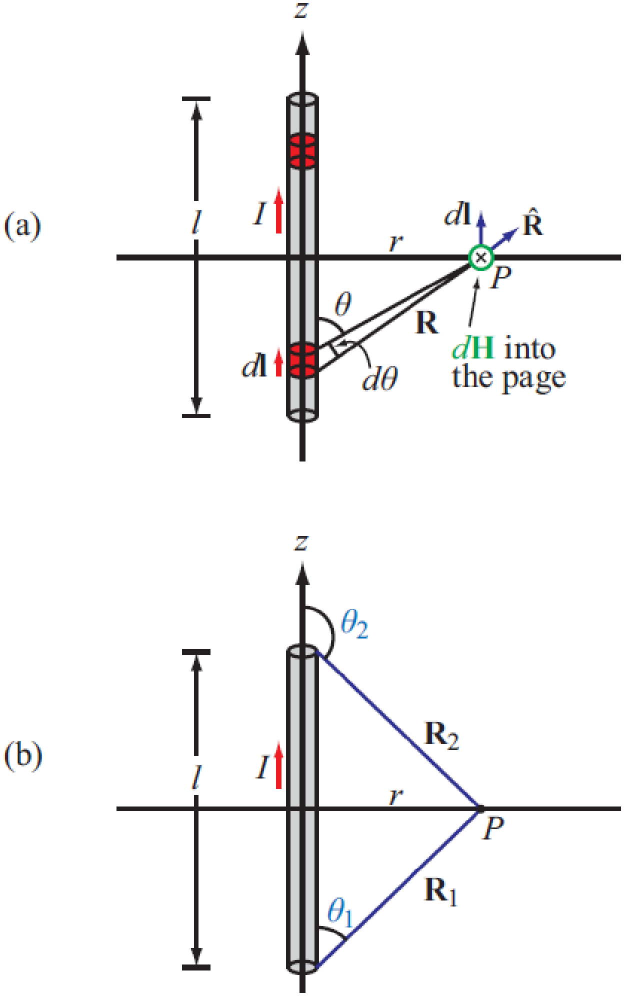


Figure 5-10: Linear conductor of length l carrying a current I. (a) The field $d\mathbf{H}$ at point P due to incremental current element $d\mathbf{l}$. (b) Limiting angles θ_1 and θ_2, each measured between vector $I\, d\mathbf{l}$ and the vector connecting the end of the conductor associated with that angle to point P (Example 5-2).

Cont.

Example 5-2: Magnetic Field of Linear Conductor (cont.)

$$\mathbf{H} = \hat{\boldsymbol{\phi}}\,\frac{I}{4\pi}\int_{\theta_1}^{\theta_2}\frac{\sin\theta\; r\csc^2\theta\; d\theta}{r^2\csc^2\theta}$$

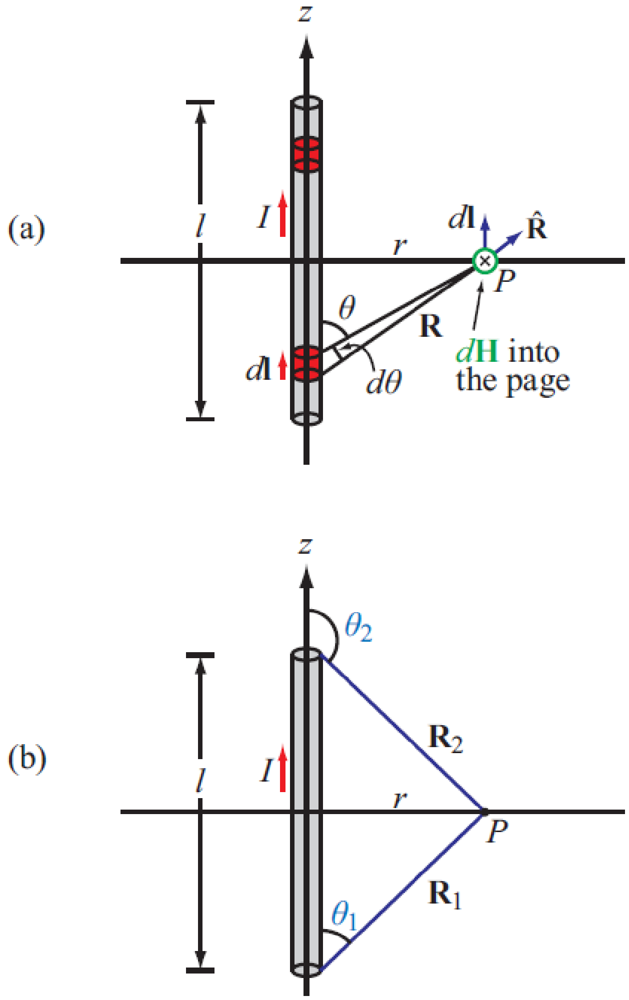
$$= \hat{\boldsymbol{\phi}}\,\frac{I}{4\pi r}\int_{\theta_1}^{\theta_2}\sin\theta\; d\theta$$

$$= \hat{\boldsymbol{\phi}}\,\frac{I}{4\pi r}(\cos\theta_1 - \cos\theta_2),$$

$$\cos\theta_1 = \frac{l/2}{\sqrt{r^2 + (l/2)^2}},$$

$$\cos\theta_2 = -\cos\theta_1 = \frac{-l/2}{\sqrt{r^2 + (l/2)^2}}$$

$$\mathbf{B} = \mu_0\mathbf{H} = \hat{\boldsymbol{\phi}}\,\frac{\mu_0 I l}{2\pi r\sqrt{4r^2 + l^2}}$$



$$\mathbf{B} = \hat{\boldsymbol{\phi}}\,\frac{\mu_0 I}{2\pi r} \quad \text{(infinitely long wire).}$$

Figure 5-10: Linear conductor of length l carrying a current I.
al current element
neasured between
l of the conductor
associated with that angle to point P (Example 5-2).

Example 5-3: Magnetic Field of a Loop

Magnitude of field due to dl is

$$dH = \frac{I}{4\pi R^2} |d\mathbf{l} \times \hat{\mathbf{R}}| = \frac{I\ dl}{4\pi(a^2 + z^2)}$$

d**H** is in the r–z plane , and therefore it has components dHr and dHz

z-components of the magnetic fields due to d**l** and dl' add because they are in the same direction, but their r-components cancel

Hence for element dl:

$$d\mathbf{H} = \hat{\mathbf{z}}\, dH_z = \hat{\mathbf{z}}\, dH \cos\theta = \hat{\mathbf{z}} \frac{I\cos\theta}{4\pi(a^2 + z^2)}\, dl$$

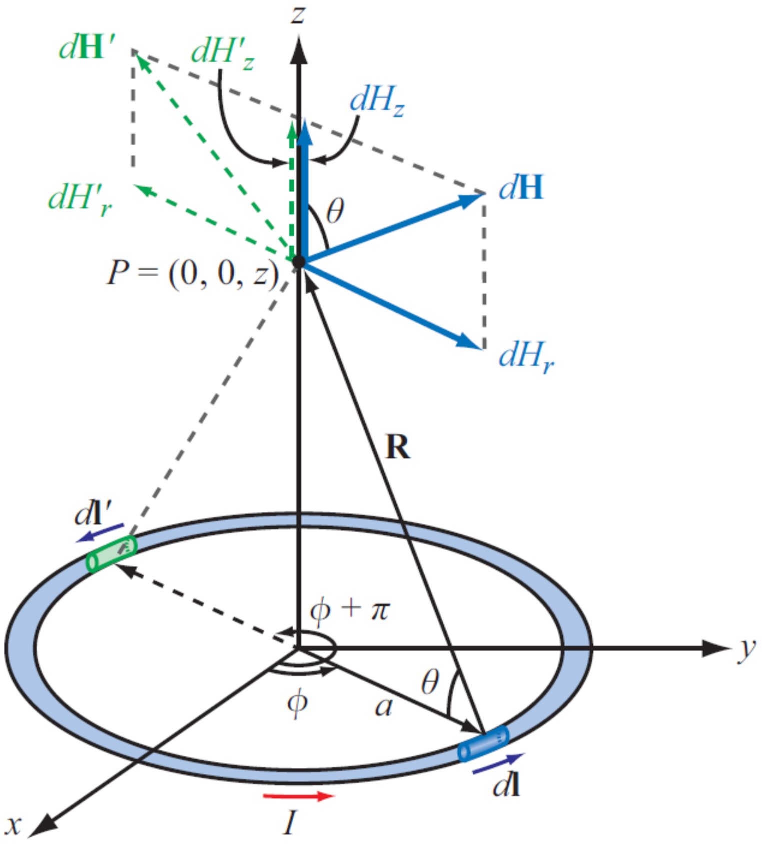


Figure 5-12: Circular loop carrying a current I (Example 5-3).

Cont.

Example 5-3:Magnetic Field of a Loop (cont.)

For the entire loop:

$$\mathbf{H} = \hat{\mathbf{z}}\,\frac{I\cos\theta}{4\pi(a^2+z^2)}\oint dl = \hat{\mathbf{z}}\,\frac{I\cos\theta}{4\pi(a^2+z^2)}(2\pi a). \qquad (5.33)$$

Upon using the relation $\cos\theta = a/(a^2+z^2)^{1/2}$, we obtain

$$\boxed{\mathbf{H} = \hat{\mathbf{z}}\,\frac{Ia^2}{2(a^2+z^2)^{3/2}} \qquad \text{(A/m)}.} \qquad (5.34)$$

At the center of the loop ($z = 0$), Eq. (5.34) reduces to

$$\boxed{\mathbf{H} = \hat{\mathbf{z}}\,\frac{I}{2a} \qquad (\text{at } z = 0),} \qquad (5.35)$$

and at points very far away from the loop such that $z^2 \gg a^2$, Eq. (5.34) simplifies to

$$\mathbf{H} = \hat{\mathbf{z}}\,\frac{Ia^2}{2|z|^3} \qquad (\text{at } |z| \gg a). \qquad (5.36)$$

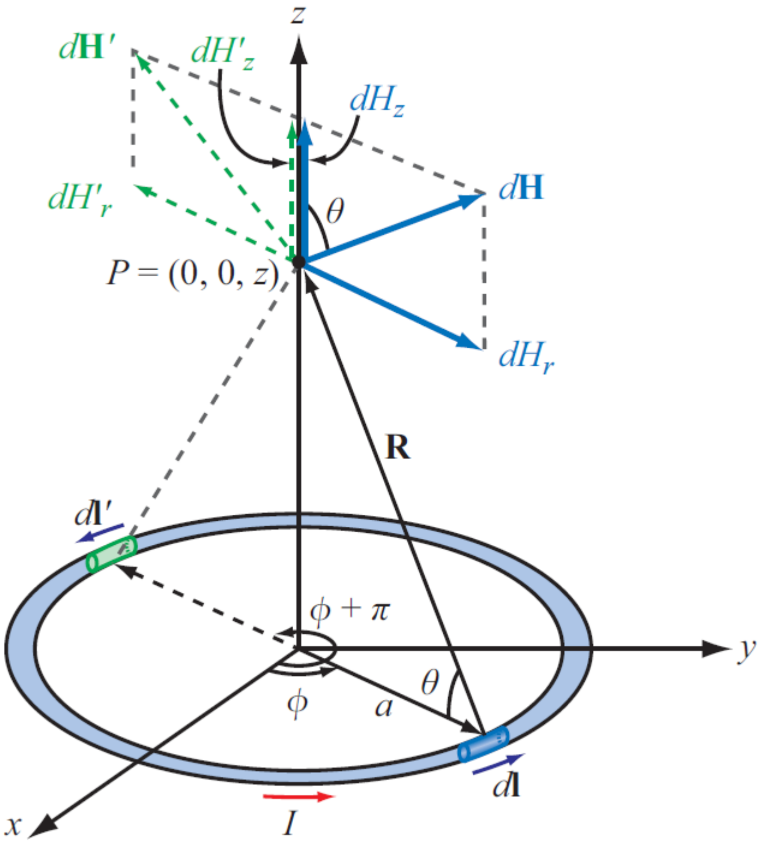
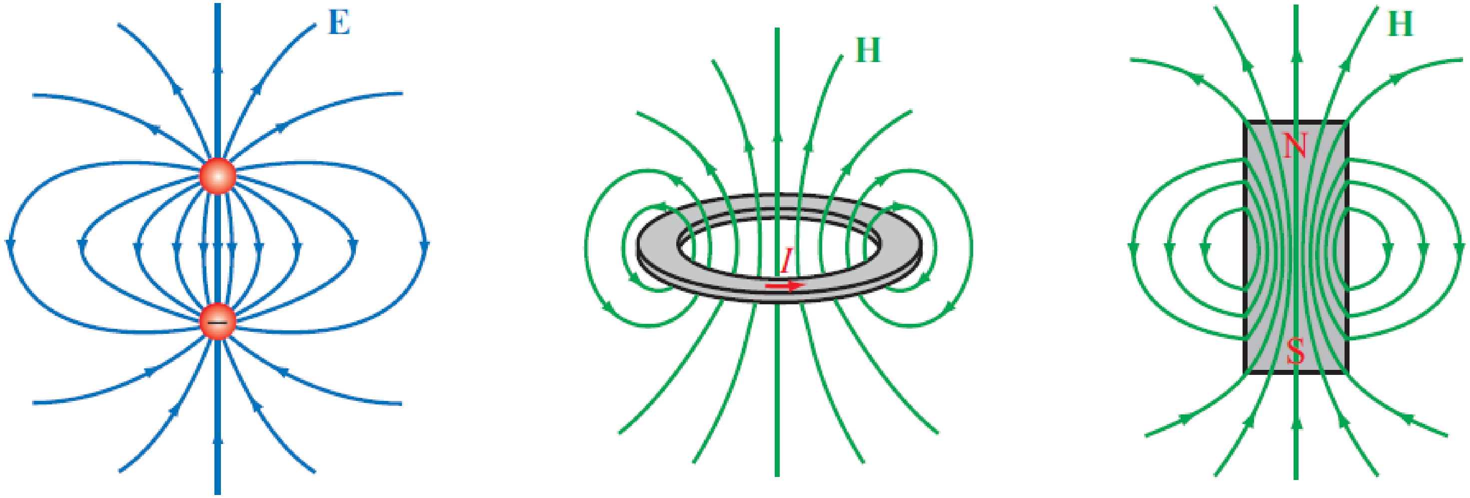


Figure 5-12: Circular loop carrying a current I (Example 5-3).

Magnetic Dipole



(a) Electric dipole (b) Magnetic dipole (c) Bar magnet

Figure 5-13: Patterns of (a) the electric field of an electric dipole, (b) the magnetic field of a magnetic dipole, and (c) the magnetic field of a bar magnet. Far away from the sources, the field patterns are similar in all three cases.

The dimension of the dipole is much smaller than the distance where you want to evaluate its field

Because a circular loop exhibits a magnetic field pattern similar to the electric field of an electric dipole, it is called a *magnetic dipole*

Forces on Parallel Conductors

$$\mathbf{B}_1 = -\hat{\mathbf{x}}\,\frac{\mu_0 I_1}{2\pi d}\,. \qquad (5.39)$$

The force $\mathbf{F}_2$ exerted on a length l of wire I_2 due to its presence in field $\mathbf{B}_1$ may be obtained by applying Eq. (5.12):

$$\begin{aligned}\mathbf{F}_2 = I_2 l\hat{\mathbf{z}} \times \mathbf{B}_1 &= I_2 l\hat{\mathbf{z}} \times (-\hat{\mathbf{x}})\,\frac{\mu_0 I_1}{2\pi d}\\ &= -\hat{\mathbf{y}}\,\frac{\mu_0 I_1 I_2 l}{2\pi d}\,, \qquad (5.40)\end{aligned}$$

and the corresponding force per unit length is

$$\mathbf{F}_2' = \frac{\mathbf{F}_2}{l} = -\hat{\mathbf{y}}\,\frac{\mu_0 I_1 I_2}{2\pi d}\,. \qquad (5.41)$$

A similar analysis performed for the force per unit length exerted on the wire carrying I_1 leads to

$$\mathbf{F}_1' = \hat{\mathbf{y}}\,\frac{\mu_0 I_1 I_2}{2\pi d}\,. \qquad (5.42)$$

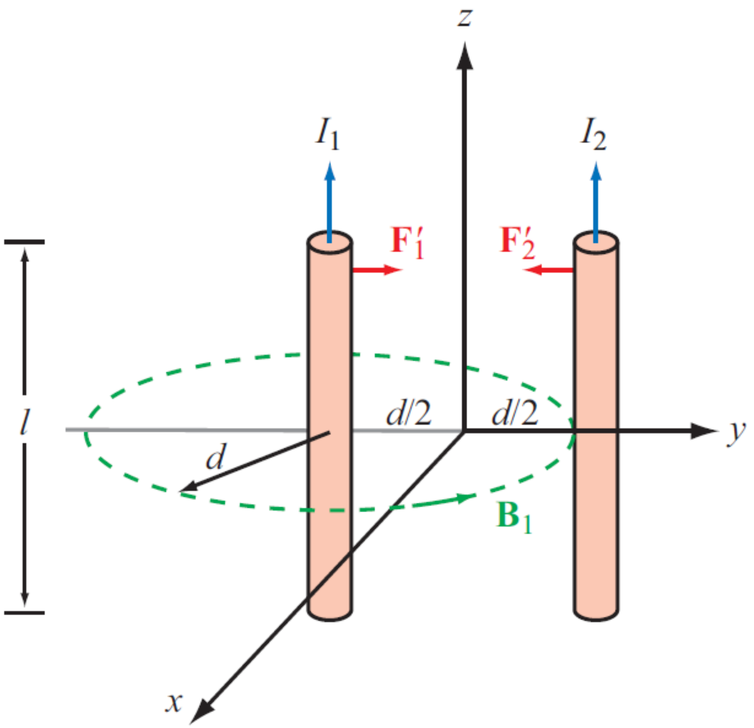


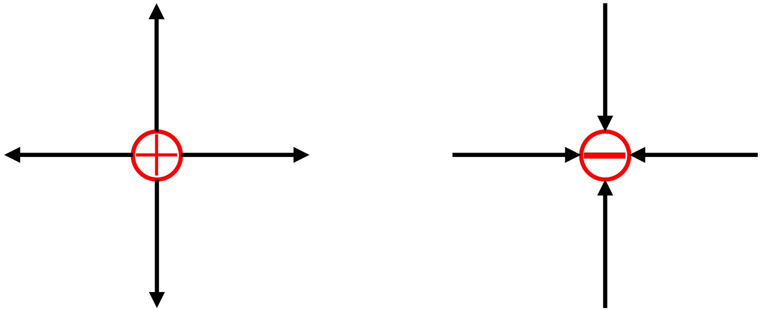
Figure 5-14: Magnetic forces on parallel current-carrying conductors.

Parallel wires attract if their currents are in the same direction, and repel if currents are in opposite directions

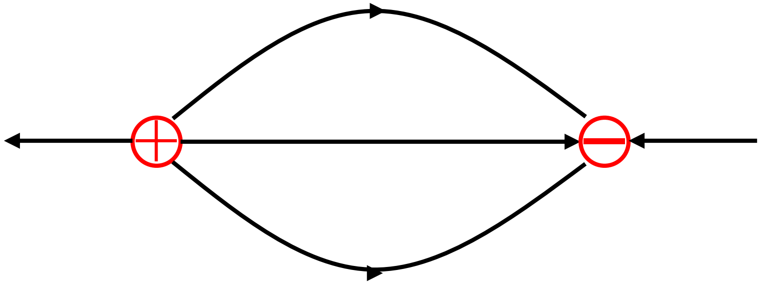
Gauss's Law for Magnetism

Gauss's Law for Electricity

$$\nabla \cdot \mathbf{D} = \rho_v \quad \longleftrightarrow \quad \oint_S \mathbf{D} \cdot ds = Q.$$



Electrostatics

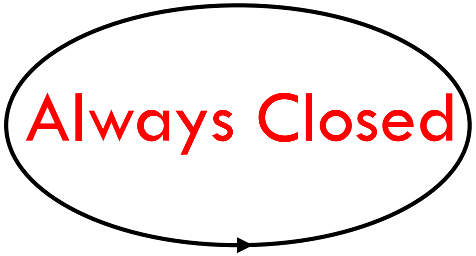


Gauss's Law for Magnetism

$$\nabla \cdot \mathbf{B} = 0 \quad \longleftrightarrow \quad \oint_S \mathbf{B} \cdot ds = 0.$$

No magnetic monopole

Magnetostatics



Always Closed

Ampère's Law

$$\nabla \times \mathbf{E} = 0 \quad \longleftrightarrow \quad \oint_C \mathbf{E} \cdot d\boldsymbol{\ell} = 0.$$

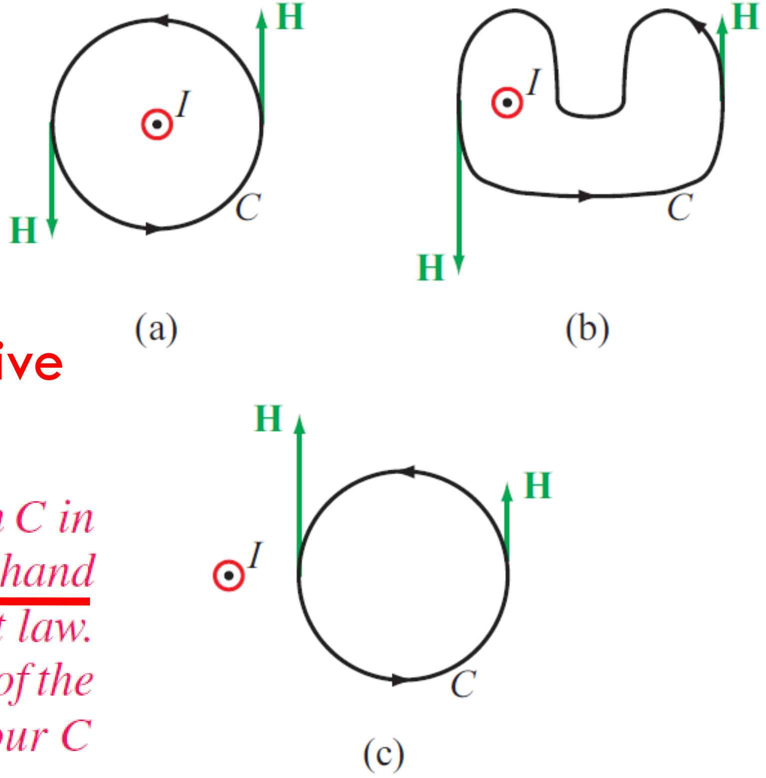
Conservative

$$\nabla \times \mathbf{H} = \mathbf{J} \quad \longleftrightarrow \quad \oint_C \mathbf{H} \cdot d\boldsymbol{\ell} = I$$

Not Conservative

unless $I = 0$

The sign convention for the direction of the contour path C in Ampère's law is taken so that I and **H** *satisfy the right-hand rule defined earlier in connection with the Biot–Savart law. That is, if the direction of I is aligned with the direction of the thumb of the right hand, then the direction of the contour C should be chosen along that of the other four fingers.*



(a) (b) (c)

Figure 5-16: Ampère's law states that the line integral of **H** around a closed contour C is equal to the current traversing the surface bounded by the contour. This is true for contours (a) and (b), but the line integral of **H** is zero for the contour in (c) because the current I (denoted by the symbol ⊙) is not enclosed by the contour C.

Internal Magnetic Field of Long Conductor

For $r < a$

$$\oint \mathbf{H}_1 \cdot d\mathbf{l}_1 = I_1,$$

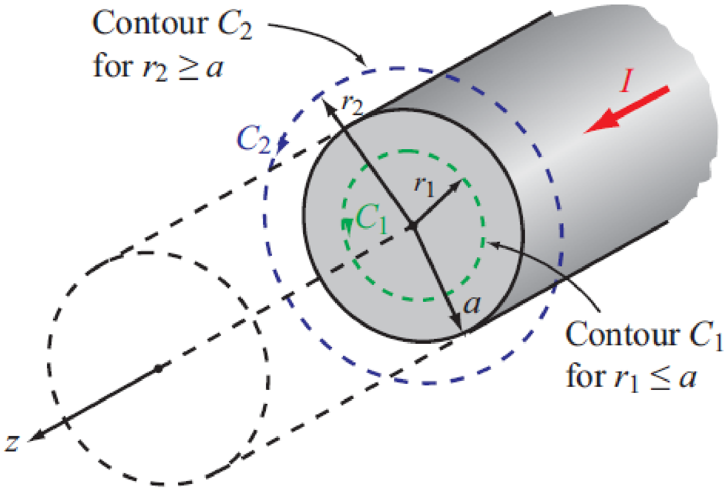
$$\oint_{C_1} \mathbf{H}_1 \cdot d\mathbf{l}_1 = \int_0^{2\pi} H_1(\hat{\boldsymbol{\phi}} \cdot \hat{\boldsymbol{\phi}}) r_1 \, d\phi = 2\pi r_1 H_1 .$$

The current I_1 flowing through the area enclosed by C_1 is equal to the total current I multiplied by the ratio of the area enclosed by C_1 to the total cross-sectional area of the wire:

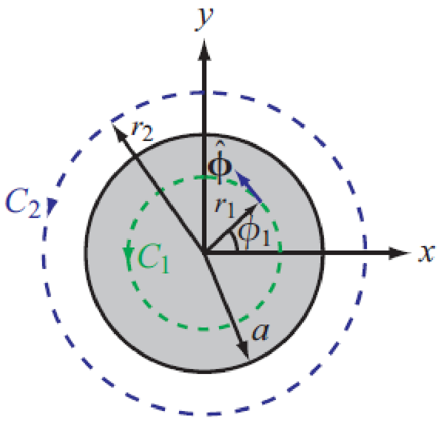
$$I_1 = \left(\frac{\pi r_1^2}{\pi a^2}\right) I = \left(\frac{r_1}{a}\right)^2 I .$$

Equating both sides of Eq. (5.48) and then solving for $\mathbf{H}_1$ yields

$$\boxed{\mathbf{H}_1 = \hat{\boldsymbol{\phi}} H_1 = \hat{\boldsymbol{\phi}} \frac{r_1}{2\pi a^2} I \qquad (\text{for } r_1 \le a).} \qquad (5.49\text{a})$$



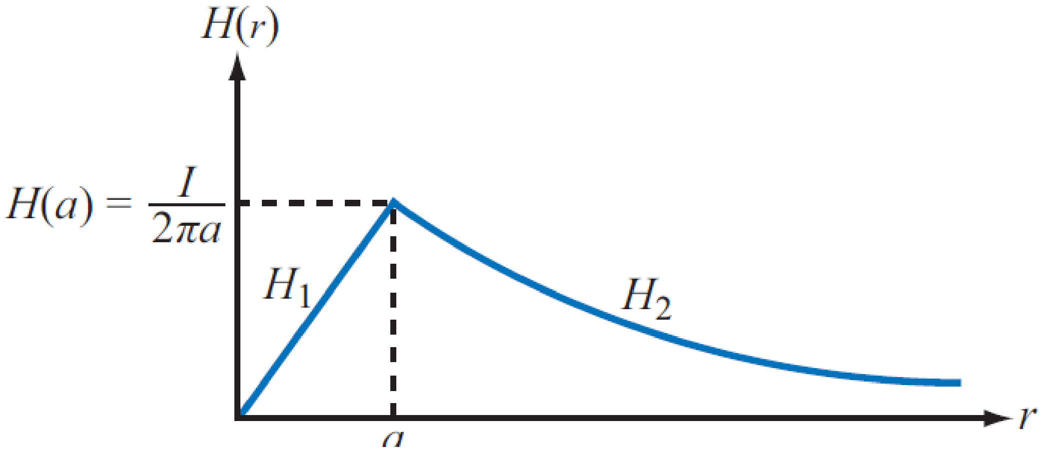
(a) Cylindrical wire



(b) Wire cross section

Cont.

External Magnetic Field of Long Conductor



For $r > a$

(b) For $r = r_2 \geq a$, we choose path C_2, which encloses all the current I. Hence, $\mathbf{H}_2 = \hat{\boldsymbol{\phi}}\, H_2$, $d\boldsymbol{\ell}_2 = \hat{\boldsymbol{\phi}}\, r_2\, d\phi$, and

$$\boxed{\oint_{C_2} \mathbf{H}_2 \cdot d\mathbf{l}_2 = 2\pi r_2 H_2 = I,}$$

which yields

$$\mathbf{H}_2 = \hat{\boldsymbol{\phi}} H_2 = \hat{\boldsymbol{\phi}}\, \frac{I}{2\pi r_2} \qquad (\text{for } r_2 \geq a). \qquad (5.49\text{b})$$

Magnetic Field of Toroid

Applying Ampere's law over contour C:

$$\oint_C \mathbf{H} \cdot d\boldsymbol{\ell} = I$$

Ampere's law states that the line integral of **H** around a closed contour C *is equal to the current traversing the* surface bounded by the contour.

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = \int_0^{2\pi} (-\hat{\boldsymbol{\phi}} H) \cdot \hat{\boldsymbol{\phi}} r \, d\phi = -2\pi r H = -NI.$$

Hence, $H = NI/(2\pi r)$ and

$$\mathbf{H} = -\hat{\boldsymbol{\phi}} H = -\hat{\boldsymbol{\phi}} \frac{NI}{2\pi r} \qquad \boxed{(\text{for } a < r < b).}$$

The magnetic field outside the toroid is zero. Why?

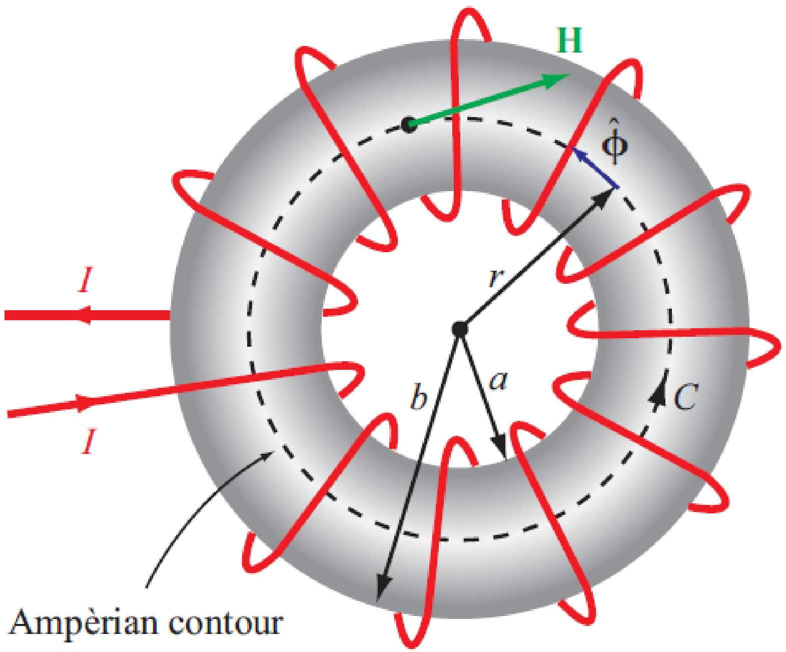
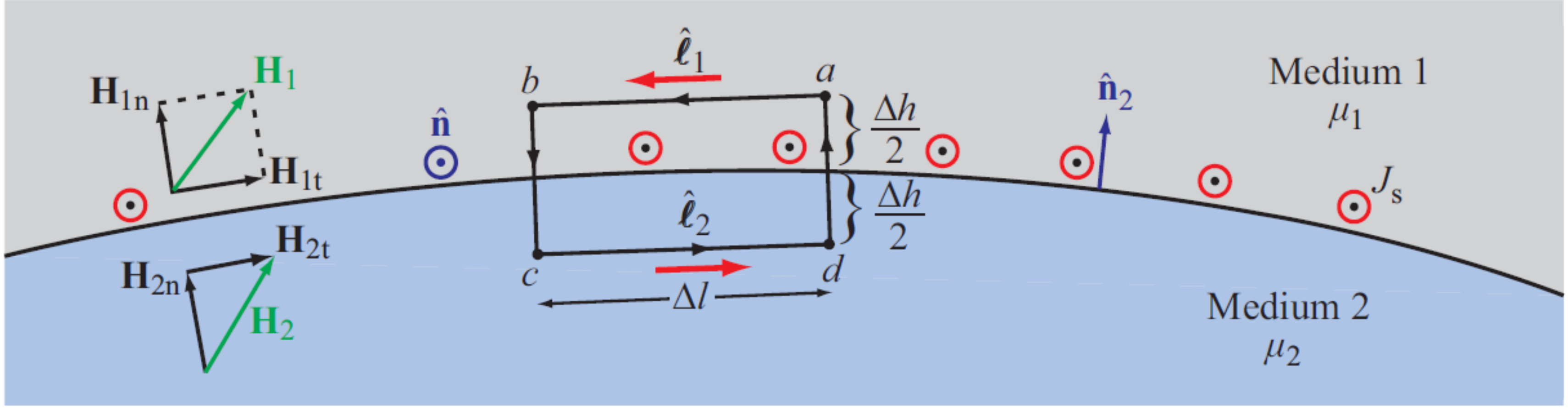


Figure 5-18: Toroidal coil with inner radius a and outer radius b. The wire loops usually are much more closely spaced than shown in the figure (Example 5-5).

Boundary Conditions



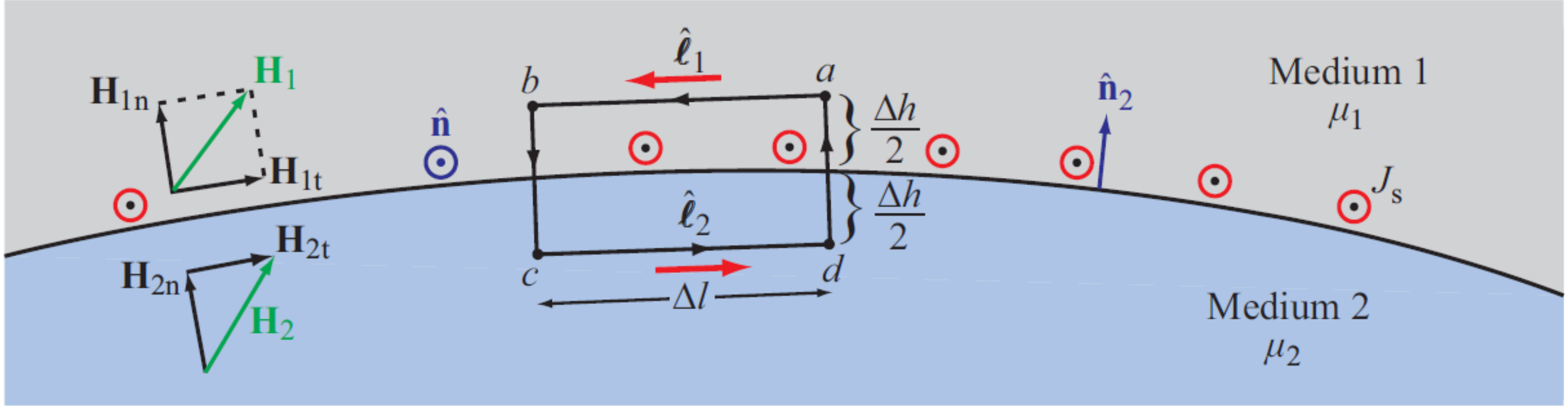
$$\oint_S \mathbf{D} \cdot d\mathbf{s} = Q \quad \rightarrow \quad \boxed{D_{1n} - D_{2n} = \rho_s.} \qquad (5.78)$$

By analogy, application of Gauss's law for magnetism, as expressed by Eq. (5.44), leads to the conclusion that

$$\oint_S \mathbf{B} \cdot d\mathbf{s} = 0 \quad \rightarrow \quad \boxed{B_{1n} = B_{2n}.} \qquad (5.79)$$

Thus the normal component of ***B*** *is continuous across the boundary between two adjacent media.*

Boundary Conditions



$$\Delta h \rightarrow 0$$

$$\oint_C \mathbf{H}\cdot d\mathbf{l} = \int_a^b \mathbf{H}_1\cdot\hat{\boldsymbol{\ell}}_1\,d\ell + \int_c^d \mathbf{H}_2\cdot\hat{\boldsymbol{\ell}}_2\,d\ell = I,$$

$$(\mathbf{H}_1 - \mathbf{H}_2)\cdot\hat{\boldsymbol{\ell}}_1\;\Delta l = \mathbf{J}_s\cdot\hat{\mathbf{n}}\;\Delta l.$$

$$\hat{\boldsymbol{\ell}}_1 = \hat{\mathbf{n}}\times\hat{\mathbf{n}}_2$$

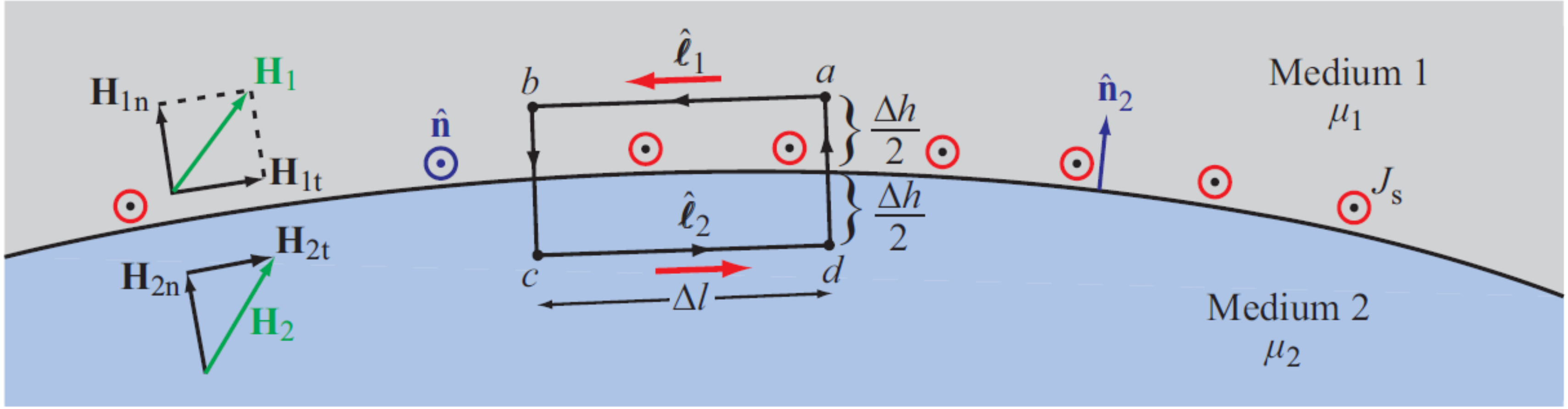
$$\hat{\mathbf{n}}\cdot[\hat{\mathbf{n}}_2\times(\mathbf{H}_1 - \mathbf{H}_2)] = \mathbf{J}_s\cdot\hat{\mathbf{n}}$$

$$\boxed{\hat{\mathbf{n}}_2\times(\mathbf{H}_1 - \mathbf{H}_2) = \mathbf{J}_s.}$$

Surface currents can exist only on the surfaces of perfect conductors and superconductors. Hence, ***at the interface between media with finite conductivities,*** $\mathbf{J}_s = 0$ and

$$\boxed{H_{1t} = H_{2t}. \qquad (5.85)}$$

Boundary Condition for Finite σ



$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} = \sigma \vec{E} + \frac{\partial \vec{D}}{\partial t}$$

$$\vec{J}_s = \int_{h_1}^{h_2} \vec{J}_v dh \approx \frac{\Delta h}{2} \vec{J}_v$$

$$\oint_C \vec{H} \cdot d\vec{l} = \sigma \oiint_S \vec{E} \cdot d\vec{s} + \frac{\partial}{\partial t} \oiint_S \vec{D} \cdot d\vec{s}$$

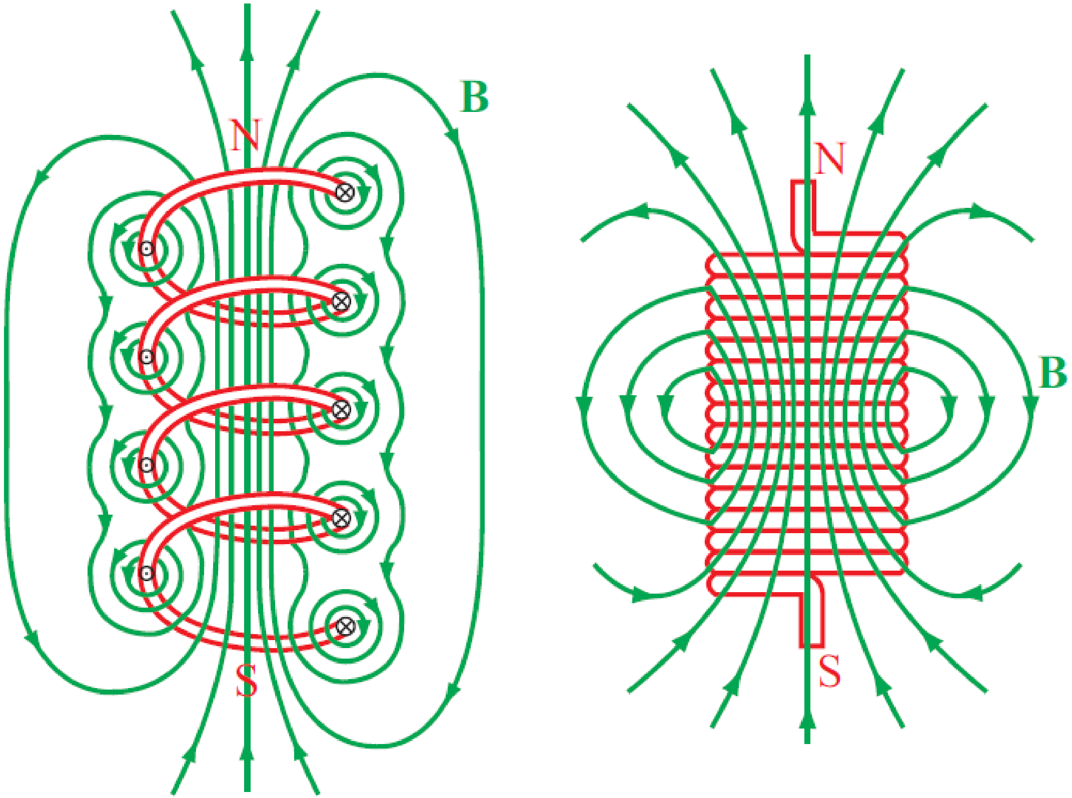
$$\Delta h \rightarrow 0$$

$$\vec{J}_s = 0$$

$$\Delta h \rightarrow 0 \qquad \oint_C \vec{H} \cdot d\vec{l} = 0$$

Continuous

Solenoid



(a) Loosely wound solenoid

(b) Tightly wound solenoid

Inside the solenoid:

$$\mathbf{B} \simeq \hat{\mathbf{z}}\mu n I = \frac{\hat{\mathbf{z}}\mu N I}{l} \qquad \text{(long solenoid with } l/a \gg 1\text{)}$$

Inductance

Magnetic Flux

$$\Phi = \int_S \mathbf{B} \cdot d\mathbf{s} \qquad \text{(Wb).}$$

Flux Linkage

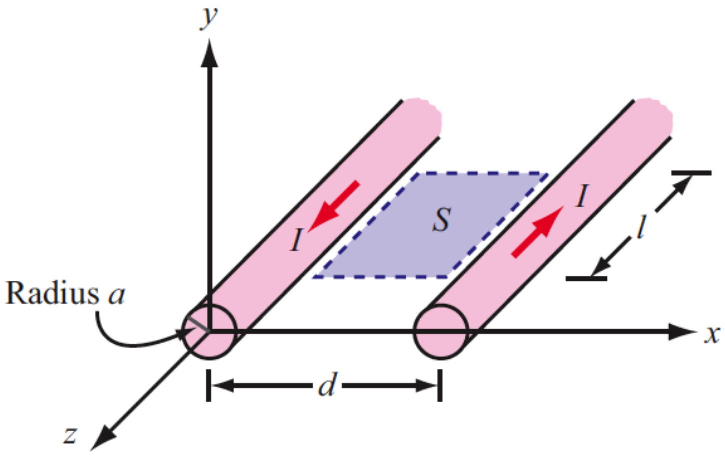
$$\Lambda = N\Phi = \mu \frac{N^2}{l} I S \qquad \text{(Wb)}$$

Inductance

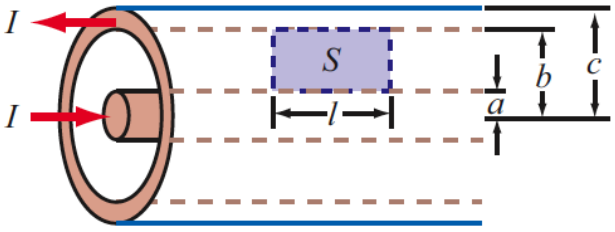
$$L = \frac{\Lambda}{I} = \frac{\Phi}{I} = \frac{1}{I} \int_S \mathbf{B} \cdot d\mathbf{s}. \qquad (5.96)$$

Solenoid

$$L = \mu \frac{N^2}{l} S \qquad \text{(solenoid),} \qquad (5.95)$$



(a) Parallel-wire transmission line



(b) Coaxial transmission line

Figure 5-27: To compute the inductance per unit length of a two-conductor transmission line, we need to determine the magnetic flux through the area S between the conductors.

Example 5-7: Inductance of Coaxial Cable

The magnetic field in the region S between the two conductors is approximately

$$\mathbf{B} = \hat{\boldsymbol{\phi}} \frac{\mu I}{2\pi r}$$

Total magnetic flux through S:

$$\Phi = l \int_a^b B \, dr = l \int_a^b \frac{\mu I}{2\pi r} \, dr = \frac{\mu I l}{2\pi} \ln\left(\frac{b}{a}\right)$$

Inductance per unit length:

$$L' = \frac{L}{l} = \frac{\Phi}{lI} = \frac{\mu}{2\pi} \ln\left(\frac{b}{a}\right).$$

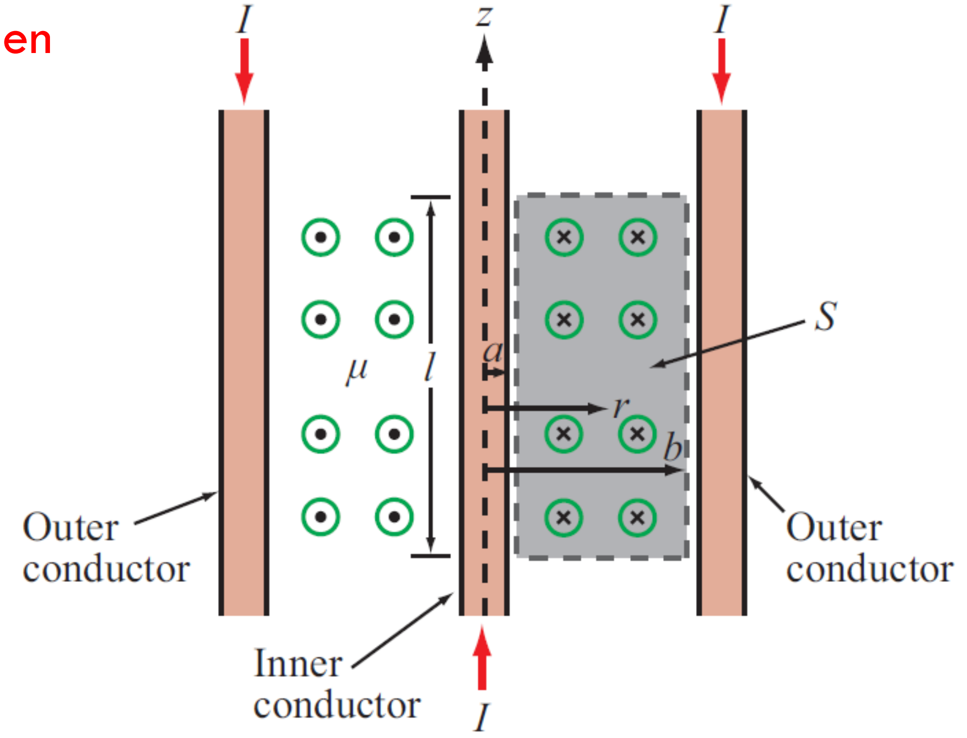
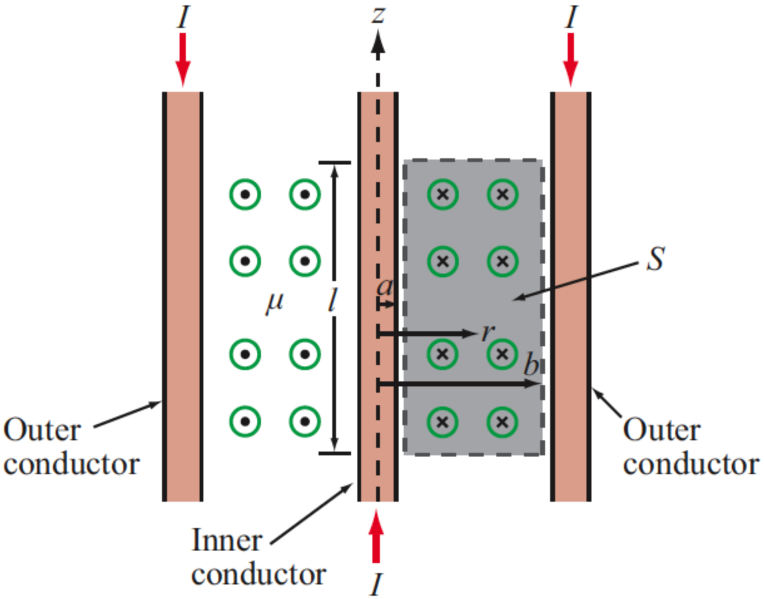


Figure 5-28: Cross-sectional view of coaxial transmission line (Example 5-7).

Magnetic Energy

Example 5-8: **Magnetic Energy in a Coaxial Cable**



Magnetic field in the insulating material is

$$H = \frac{B}{\mu} = \frac{I}{2\pi r}$$

The magnetic energy stored in the coaxial cable is

$$W_\mathrm{m} = \frac{1}{2} \int_\mathcal{V} \mu H^2 \, d\mathcal{V} = \frac{\mu I^2}{8\pi^2} \int_\mathcal{V} \frac{1}{r^2} \, d\mathcal{V}$$

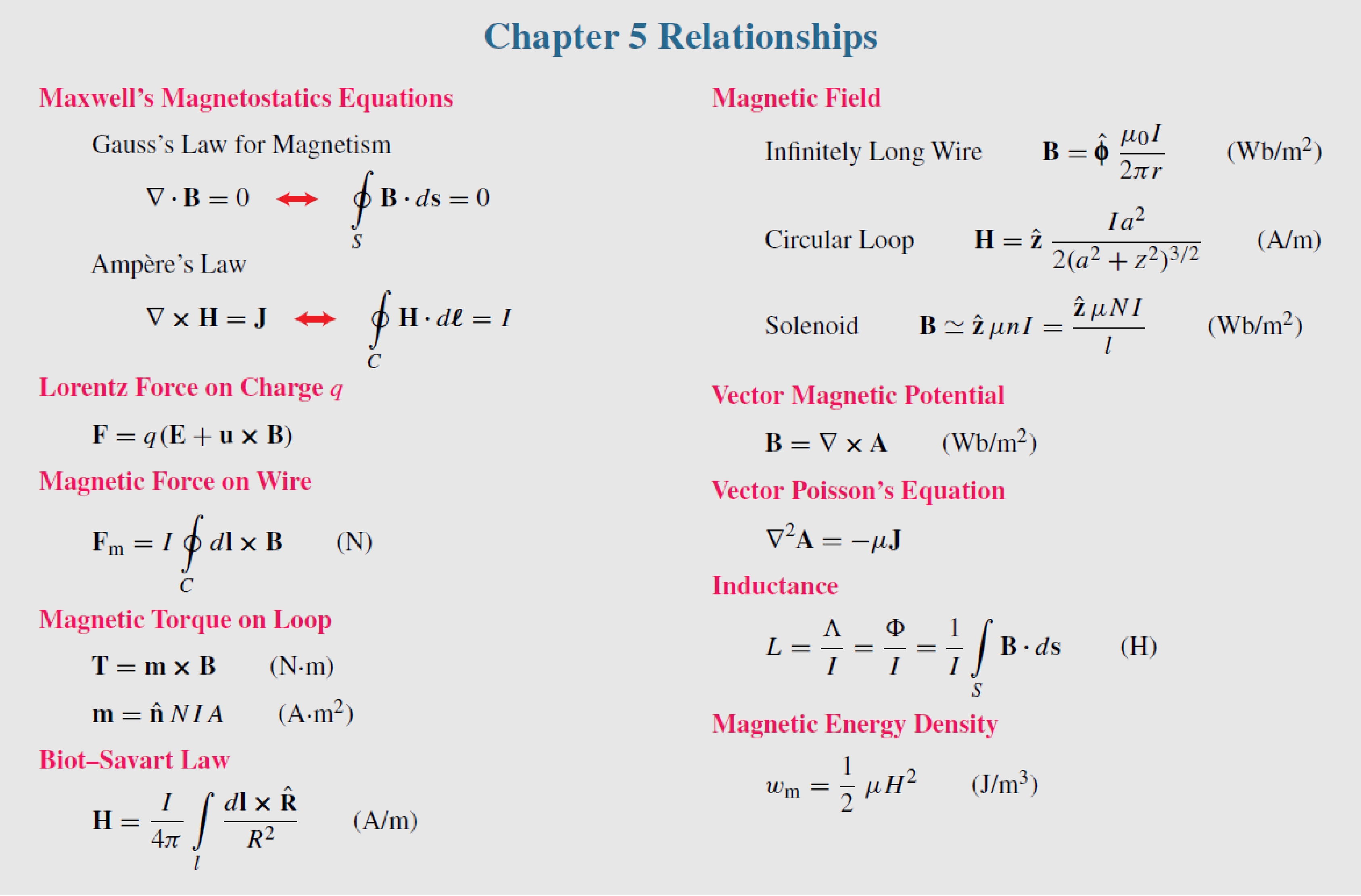
$$\boxed{w_\mathrm{m} = \frac{W_\mathrm{m}}{\mathcal{V}} = \frac{1}{2} \mu H^2 \qquad (\mathrm{J/m^3}).}$$

$$W_\mathrm{m} = \frac{\mu I^2}{8\pi^2} \int_a^b \frac{1}{r^2} \cdot 2\pi r l \, dr$$

$$= \frac{\mu I^2 l}{4\pi} \ln\left(\frac{b}{a}\right)$$

$$= \frac{1}{2} L I^2 \qquad (\mathrm{J}),$$

Summary



Chapter 5 Relationships

Maxwell's Magnetostatics Equations

Gauss's Law for Magnetism

$$\nabla \cdot \mathbf{B} = 0 \quad \longleftrightarrow \quad \oint_S \mathbf{B} \cdot d\mathbf{s} = 0$$

Ampère's Law

$$\nabla \times \mathbf{H} = \mathbf{J} \quad \longleftrightarrow \quad \oint_C \mathbf{H} \cdot d\boldsymbol{\ell} = I$$

Lorentz Force on Charge q

$$\mathbf{F} = q(\mathbf{E} + \mathbf{u} \times \mathbf{B})$$

Magnetic Force on Wire

$$\mathbf{F}_\mathrm{m} = I \oint_C d\mathbf{l} \times \mathbf{B} \qquad \text{(N)}$$

Magnetic Torque on Loop

$$\mathbf{T} = \mathbf{m} \times \mathbf{B} \qquad \text{(N·m)}$$

$$\mathbf{m} = \hat{\mathbf{n}}\, NIA \qquad \text{(A·m}^2\text{)}$$

Biot–Savart Law

$$\mathbf{H} = \frac{I}{4\pi} \int_l \frac{d\mathbf{l} \times \hat{\mathbf{R}}}{R^2} \qquad \text{(A/m)}$$

Magnetic Field

Infinitely Long Wire $\quad \mathbf{B} = \hat{\boldsymbol{\phi}}\, \dfrac{\mu_0 I}{2\pi r} \qquad \text{(Wb/m}^2\text{)}$

Circular Loop $\quad \mathbf{H} = \hat{\mathbf{z}}\, \dfrac{Ia^2}{2(a^2 + z^2)^{3/2}} \qquad \text{(A/m)}$

Solenoid $\quad \mathbf{B} \simeq \hat{\mathbf{z}}\, \mu n I = \dfrac{\hat{\mathbf{z}}\, \mu N I}{l} \qquad \text{(Wb/m}^2\text{)}$

Vector Magnetic Potential

$$\mathbf{B} = \nabla \times \mathbf{A} \qquad \text{(Wb/m}^2\text{)}$$

Vector Poisson's Equation

$$\nabla^2 \mathbf{A} = -\mu \mathbf{J}$$

Inductance

$$L = \frac{\Lambda}{I} = \frac{\Phi}{I} = \frac{1}{I} \int_S \mathbf{B} \cdot d\mathbf{s} \qquad \text{(H)}$$

Magnetic Energy Density

$$w_\mathrm{m} = \frac{1}{2}\, \mu H^2 \qquad \text{(J/m}^3\text{)}$$