Convex Sets

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Outline

$x \in \mathbb{R}^n$

$x \in \mathbb{B}^n$

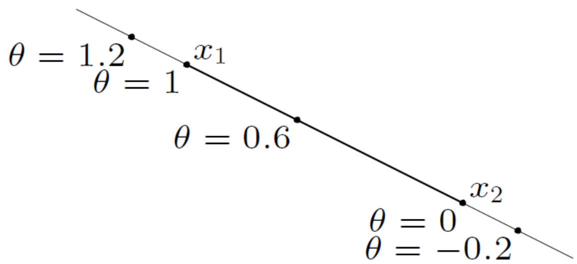
$\min \quad f(x)$

1. Affine and Convex Sets
2. Some Important Examples
3. Operations that Preserve Convexity
4. Generalized Inequalities
5. Separating and Supporting Hyperplanes

Definition of Affine Set

- **Line:** through $\boldsymbol{x}_1, \boldsymbol{x}_2$: all points

$$\boldsymbol{x} = \theta \boldsymbol{x}_1 + (1-\theta)\boldsymbol{x}_2 \quad (\theta \in \mathbb{R})$$



- **Affine set:** contains the line through any two distinct points in the set
- **Example:** solution set of linear equations $\{\boldsymbol{x} | \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}\}$
  (conversely, every affine set can be expressed as solution set of system of linear equations)

$x_1, x_2, \quad Ax_1 = b, \quad Ax_2 = b$

$$A[\theta x_1 + (1-\theta)x_2]$$
$$= \theta A x_1 + (1-\theta) A x_2 = b$$

$$Ax = b,$$

$$\boxed{Ax = b}$$

$$x = A^{-1}b$$

$$O(n^3)$$

$$A = \begin{bmatrix} a_1^T \\ a_2^T \\ \vdots \\ a_n^T \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$\begin{cases} a_1^T x = b_1, \\ \vdots \\ a_n^T x = b_n, \end{cases} \Rightarrow \begin{cases} a_1^T x - b_1 = 0 \ \checkmark \\ \vdots \qquad\qquad\qquad \checkmark \\ a_n^T x - b_n = 0 \ \checkmark \end{cases}$$

$$x \in C, \iff \{ x \mid Ax = b \}$$

$$\forall x_0 \in C,$$

$$C_0 = C - x_0 := \{ x - x_0 \mid x \in C \} \ \checkmark$$

$$\begin{cases} z_1, z_2 \in C_0, & \alpha, \beta \in \mathbb{R} \\ \alpha z_1 + \beta z_2 \in C_0 \end{cases}$$

✓

$C_0 \to C^{\perp}$,

$x \qquad \{a_1, a_2, \cdots a_n\}$

$$\begin{cases} a_1^T x = 0 \\ \vdots \\ a_n^T x = 0 \end{cases}$$

$x + x_0 \in C$

Definition of Convex Set

- **Line segment:** between $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$: all points

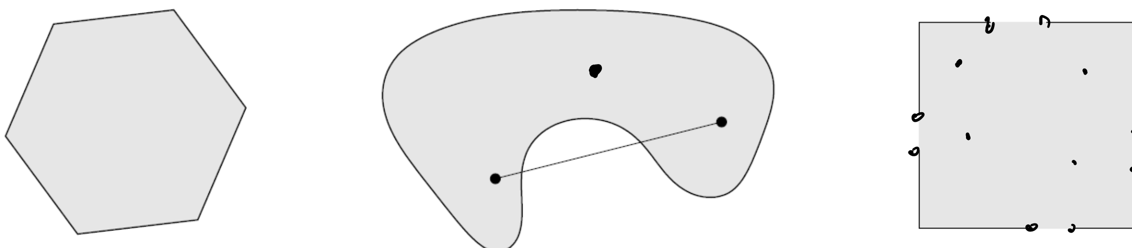
$$\boldsymbol{x} = \theta \boldsymbol{x}_1 + (1-\theta)\boldsymbol{x}_2$$

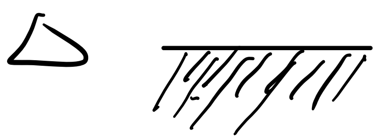
  with $0 \leq \theta \leq 1$

- **Convex set:** contains line segment between any two points in the set C

$$\boldsymbol{x}_1, \boldsymbol{x}_2 \in C, 0 \leq \theta \leq 1 \quad \Longrightarrow \quad \theta \boldsymbol{x}_1 + (1-\theta)\boldsymbol{x}_2 \in C$$

- **Examples** (one convex, two nonconvex sets)





$Ax = b$

$Ax \leq b$

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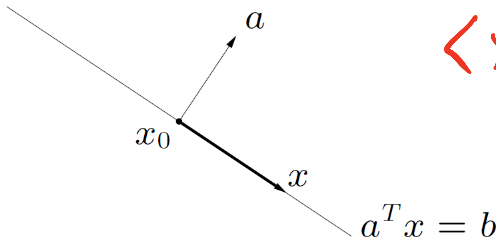
Examples: Hyperplanes and Halfspaces

- **Hyperplane:** set of the form $\{\boldsymbol{x} | \boldsymbol{a}^T\boldsymbol{x} = b\}(\boldsymbol{a} \neq \boldsymbol{0})$

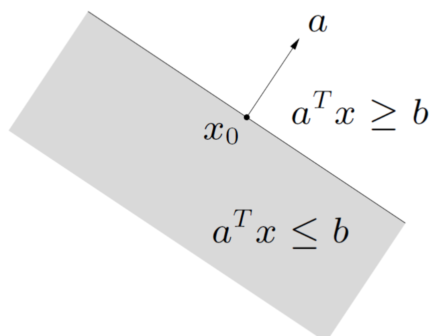


$\langle x - x_0, a \rangle = 0$

$a^T(x - x_0) = 0$

$a^T x = a^T x_0 := b$

- **Halfspace:** set of the form $\{\boldsymbol{x} | \boldsymbol{a}^T\boldsymbol{x} \leq b\}(\boldsymbol{a} \neq \boldsymbol{0})$



  - $\boldsymbol{a}$ is the normal vector
  - hyperplanes are affine and convex; halfspaces are convex

hyperplane: $a^T x_1 = b$, $a^T x_2 = b$,

$$a^T(\theta x_1 + (1-\theta) x_2) = b, \quad \theta \in \mathbb{R}$$

halfspace: $a^T x_1 \leq b$, $a^T x_2 \leq b$

$$a^T(\theta x_1 + (1-\theta) x_2) \leq b, \quad \theta \in [0, 1$$

Example: Polyhedra

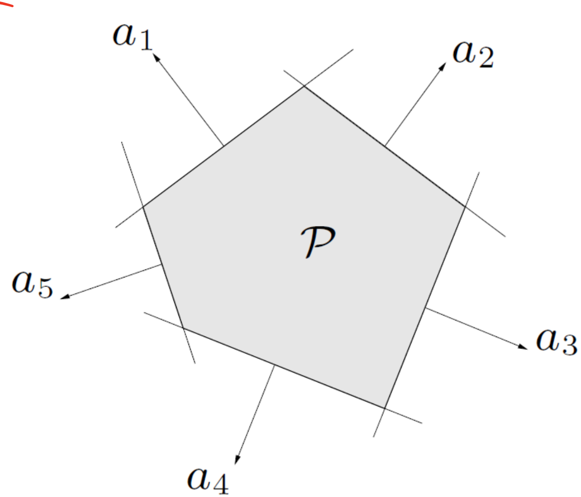
$$\begin{cases} Ax + y = b \\ y \succeq 0 \end{cases}$$

Solution set of finitely many linear inequalities and equalities

$$Ax \overset{+y}{\preceq} b, \qquad Cx = d$$

$$C = \begin{bmatrix} c_1^T \\ \vdots \\ c_p^T \end{bmatrix}$$

($A \in \mathbb{R}^{m\times n}$, $C \in \mathbb{R}^{p\times n}$, $\preceq$ is componentwise inequality)



$$\begin{cases} c_1^T x = d_1 \\ \vdots \\ c_p^T x = d_p \end{cases}$$

polyhedron is intersection of finite number of halfspaces and hyperplanes

Examples: Euclidean Balls and Ellipsoids

- **(Euclidean) Ball** with center $\boldsymbol{x}_c$ and radius r:

$$B(\boldsymbol{x}_c, r) = \{\boldsymbol{x} \mid \|\boldsymbol{x} - \boldsymbol{x}_c\|_2 \leq r\} = \{\boldsymbol{x}_c + r\boldsymbol{u} \mid \|\boldsymbol{u}\|_2 \leq 1\}$$

$x - x_c = r \cdot \vec{u}$, $\quad \|r u\|_2 \leq r \Rightarrow \|u\|_2 \leq 1$

- **Ellipsoid:** set of the form

$$\begin{aligned} E(\boldsymbol{x}_c, \boldsymbol{P}) &= \{\boldsymbol{x} \mid (\boldsymbol{x} - \boldsymbol{x}_c)^T \boldsymbol{P}^{-1} (\boldsymbol{x} - \boldsymbol{x}_c) \leq 1\} \\ &= \{\boldsymbol{x}_c + \boldsymbol{A}\boldsymbol{u} \mid \|\boldsymbol{u}\|_2 \leq 1\} \end{aligned}$$

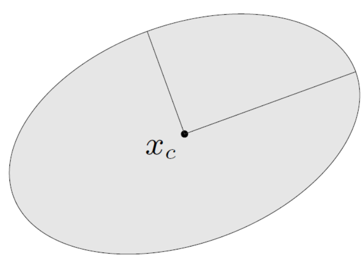
with $\boldsymbol{P} \in \mathbb{S}^n_{++}$ (i.e., $\boldsymbol{P}$ symmetric positive definite), $\boldsymbol{A}$ square and nonsingular

$P = Q^T \Lambda Q$, $\quad \Lambda = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$

$P^{-1} = Q^T \Lambda^{-1} Q$

$x - x_c = Au$.

$A = P^{\frac{1}{2}}$



$u^T A^T P^{-1} A u \leq 1 \Rightarrow u^T \cdot I \cdot u \leq 1$

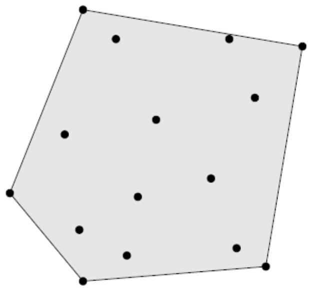
Convex Combination and Convex Hull

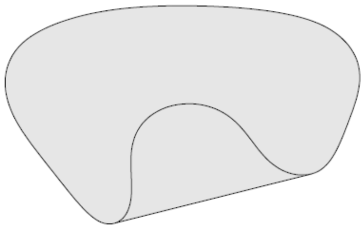
- **Convex combination** of $\boldsymbol{x}_1, \cdots, \boldsymbol{x}_k$: any point $\boldsymbol{x}$ of the form

$$\boldsymbol{x} = \theta_1 \boldsymbol{x}_1 + \theta_2 \boldsymbol{x}_2 + \cdots + \theta_k \boldsymbol{x}_k$$

  with $\theta_1 + \cdots + \theta_k = 1, \theta_i \geq 0$

- **Convex hull** $\mathsf{conv}\, S$: set of all convex combinations of points in S



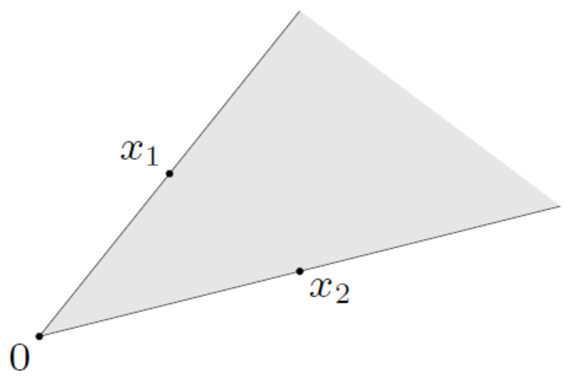


Conic Combination and Convex Cone

- **Conic (nonnegative) combination** of $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$: any point of the form

$$\boldsymbol{x} = \theta_1 \boldsymbol{x}_1 + \theta_2 \boldsymbol{x}_2$$

  with $\theta_1 \geq 0, \theta_2 \geq 0$



- **Convex cone:** set that contains all conic combinations of points in the set

Convex Cones: Norm Balls and Norm Cones

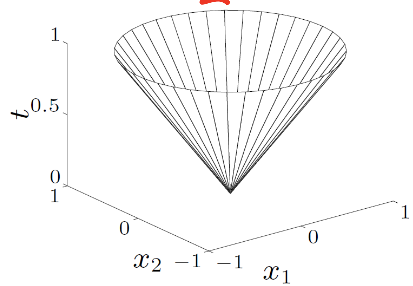
- **Norm:** a function $\|\cdot\|$ that satisfies
  - $\|\boldsymbol{x}\| \geq 0$; $\|\boldsymbol{x}\| = 0$ if and only if $\boldsymbol{x} = \mathbf{0}$
  - $\|t\boldsymbol{x}\| = |t|\|\boldsymbol{x}\|$ for $t \in \mathbb{R}$
  - $\|\boldsymbol{x} + \boldsymbol{y}\| \leq \|\boldsymbol{x}\| + \|\boldsymbol{y}\|$

$\langle x, y \rangle \leq \|x\| \cdot \|y\|$

notation: $\|\cdot\|$ general (unspecified) norm; $\|\cdot\|_{\text{symb}}$ a particular norm

- **Norm ball** with center $\boldsymbol{x}_c$ and radius r: $\{\boldsymbol{x} \mid \|\boldsymbol{x} - \boldsymbol{x}_c\| \leq r\}$

- **Norm cone:** $\{(\boldsymbol{x}, t) \in \mathbb{R}^{n+1} \mid \|\boldsymbol{x}\| \leq t\}$



Euclidean norm cone or second-order cone (aka ice-cream cone)

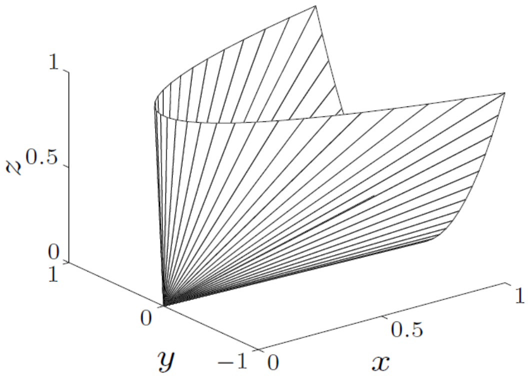
Positive Semidefinite Cone

- **Notation**
  - $\mathbb{S}^n$ is set of symmetric $n \times n$ matrices
  - $\mathbb{S}^n_+ = \{\boldsymbol{X} \in \mathbb{S}^n | \boldsymbol{X} \succeq 0\}$: positive semidefinite $n \times n$ matrices

$$\boldsymbol{X} \in \mathbb{S}^n_+ \quad \Longleftrightarrow \quad \boldsymbol{z}^\top \boldsymbol{X} \boldsymbol{z} \geq 0 \text{ for all } \boldsymbol{z}$$

    $\mathbb{S}^n_+$ is a convex cone
  - $\mathbb{S}^n_{++} = \{\boldsymbol{X} \in \mathbb{S}^n | \boldsymbol{X} \succ 0\}$: positive definite $n \times n$ matrices

- **Example:** $\begin{bmatrix} x & y \\ y & z \end{bmatrix} \in \mathbb{S}^2_+$



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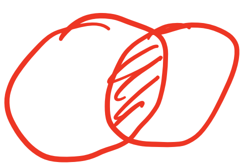
Operations that Preserve Convexity

How to establish the convexity of a given set C

- Apply the definition (can be cumbersome)

$$\boldsymbol{x}_1, \boldsymbol{x}_2 \in C, 0 \le \theta \le 1 \quad \Longrightarrow \quad \theta \boldsymbol{x}_1 + (1-\theta)\boldsymbol{x}_2 \in C$$

- Show that C is obtained from simple convex sets(hyperplanes, halfspaces, norm balls,$\cdots$) by operations that preserve convexity
  - intersection
  - affine functions
  - perspective function
  - linear-fractional functions

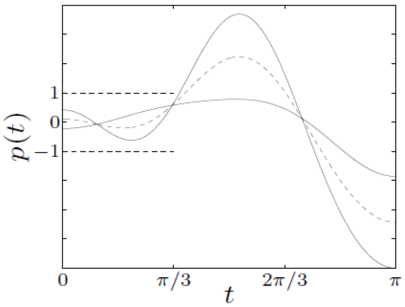


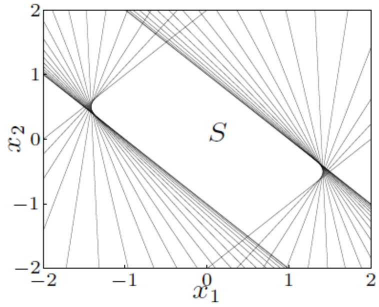
Intersection

- **Intersection:** if $S_1, S_2, \ldots, S_k$ are convex, then $S_1 \cap S_2 \cap \cdots \cap S_k$ is convex (k can be any positive integer)

- Example 1: a polyhedron is the intersection of halfspaces and hyperplanes

- Example 2:
$$S = \{\boldsymbol{x} \in \mathbb{R}^m | |p(t)| \leq 1 \text{ for } |t| \leq \pi/3\}$$
where $p(t) = x_1 \cos t + x_2 \cos 2t + \cdots + x_m \cos mt$





for $m = 2$

Affine Function

suppose $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is affine ($f(\boldsymbol{x}) = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}$ with $\boldsymbol{A} \in \mathbb{R}^{m \times n}$, $\boldsymbol{b} \in \mathbb{R}^m$)

- the image of a convex set under f is convex
$$S \subseteq \mathbb{R}^n \text{ convex} \quad \Longrightarrow \quad f(S) = \{f(\boldsymbol{x}) | \boldsymbol{x} \in S\} \text{ convex}$$
- the inverse image $f^{-1}(C)$ a convex set under f is convex
$$C \subseteq \mathbb{R}^m \text{ convex} \quad \Longrightarrow \quad f^{-1}(C) = \{\boldsymbol{x} \in \mathbb{R}^n | f(\boldsymbol{x}) \in C\} \text{ convex}$$

Examples

- scaling, translation, projection
- solution set of linear matrix inequality $\{\boldsymbol{x} | x_1\boldsymbol{A}_1 + \cdots + x_m\boldsymbol{A}_m \preceq \boldsymbol{B}\}$ (with $\boldsymbol{A}_i, \boldsymbol{B} \in \mathbb{S}^p$)
- $\{(\boldsymbol{x}, t) \in \mathbb{R}^{n+1} | \|\boldsymbol{x}\| \le t\}$ is convex, so is
$$\{\boldsymbol{x} \in \mathbb{R}^n | \|\boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}\| \le \boldsymbol{c}^T\boldsymbol{x} + d\}$$

$$f(x) = \begin{pmatrix} A \\ c^T \end{pmatrix} x + \begin{pmatrix} b \\ d \end{pmatrix}$$

Perspective and Linear-fractional Function I

$$\left[\underbrace{\frac{x_1}{x_{n+1}} \cdots \frac{x_n}{x_{n+1}}}\right] \frac{x_{n+1}}{x_{n+1}} \qquad 1$$

- **Perspective function** $P : \mathbb{R}^{n+1} \to \mathbb{R}^n$

$$P(\boldsymbol{x}, t) = \boldsymbol{x}/t, \quad \mathrm{dom}P = \{(\boldsymbol{x}, t) | t > 0\}$$

images and inverse images of convex sets under perspective are convex

- **Linear-fractional function** $f : \mathbb{R}^n \to \mathbb{R}^m$

$$f(\boldsymbol{x}) = \frac{\boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}}{\boldsymbol{c}^T \boldsymbol{x} + d}, \quad \mathrm{dom}f = \{\boldsymbol{x} | \boldsymbol{c}^T \boldsymbol{x} + d > 0\}$$

images and inverse images of convex sets under linear-fractional functions are convex

If C is convex, then $\underbrace{P^{-1}(C)}$ $=\{(x,t)\in R^{n+1} \mid \underbrace{x/t}\in C,\ t>0\}$ is convex

suppose $(x,t)\in P^{-1}(C)$, $(y,s)\in P^{-1}(C)$, $\theta\in[0,1]$

We need to show

$$\theta(x,t)+(1-\theta)(y,s)\in P^{-1}(C)$$

$$\frac{\theta x+(1-\theta)y}{\theta t+(1-\theta)s}=\underbrace{u\left(\frac{x}{t}\right)}+\underbrace{(1-u)\cdot\frac{y}{s}}$$

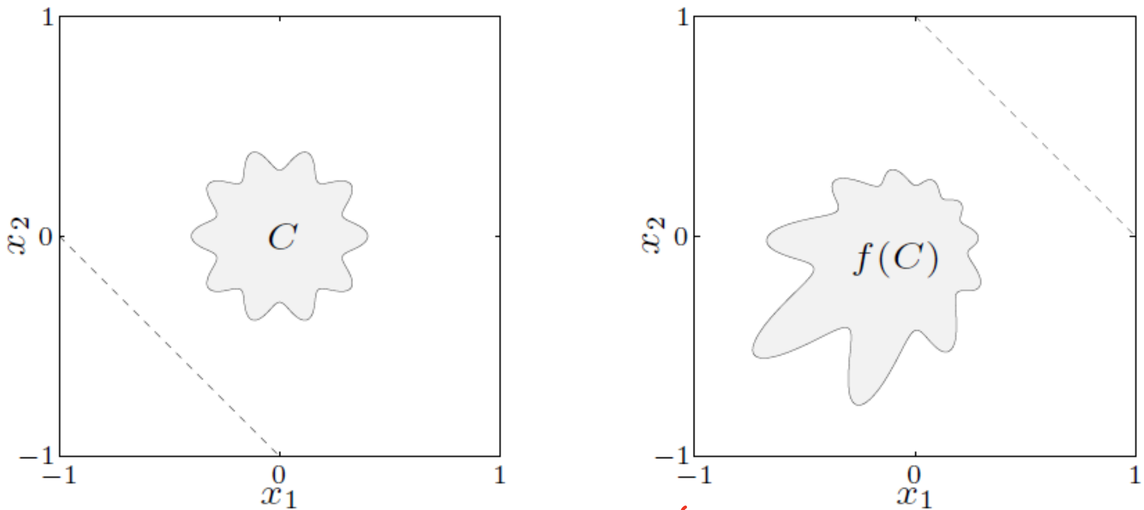
$$u=\frac{\theta t}{\theta t+(1-\theta)s}\in[0,1]$$

Perspective and Linear-fractional Function II

- **Examples** of a linear-fractional function

$$f(\boldsymbol{x}) = \frac{1}{\underbrace{x_1 + x_2 + 1}_{}}\boldsymbol{x}$$

$c^T x + d$.



$$c = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ \vdots \end{bmatrix} \quad d = 1$$

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Generalized Inequalities I

- A convex cone $K \subseteq \mathbb{R}^n$ is a **proper cone** if
  - K is closed (contains its boundary)
  - K is solid (has nonempty interior)
  - K is pointed (contains no line)

min $c^T x$

s.t. $Ax = b$

$Gx \le h$

$Gx + s = h$

$s \ge 0$

- **Examples**
  - nonnegative orthant

$$K = \mathbb{R}^n_+ = \{\boldsymbol{x} \in \mathbb{R}^n | x_i \ge 0, i = 1, \cdots, n\}$$

  - positive semidefinite cone

$$K = \mathbb{S}^n_+ = \{\boldsymbol{X} \in \mathbb{R}^{n\times n} | \boldsymbol{X} = \boldsymbol{X}^T \succeq \mathbf{0}\}$$

  - nonnegative polynomials on $[0,1]$:

$$K = \{\boldsymbol{x} \in \mathbb{R}^n | x_1 + x_2 t + x_3 t^2 + \cdots + x_n t^{n-1} \ge 0 \text{ for } t \in [0,1]\}$$

$$\min_{x \in \mathbb{R}^n} f(x)$$

$$\min_{\substack{x \in \mathbb{R}^n \\ s \in \mathbb{R}_+}} c^T d$$

$$Ax = b$$

Generalized Inequalities II

Generalized inequality defined by a proper cone K:

$$\boldsymbol{y} \succeq_K \boldsymbol{x} \quad \Longleftrightarrow \quad \boldsymbol{y} - \boldsymbol{x} \succeq_K \mathbf{0} \text{ or } \boldsymbol{y} - \boldsymbol{x} \in K$$

Examples

- Componentwise inequality ($K = \mathbb{R}^n_+$)

$$\boldsymbol{y} \succeq_{\mathbb{R}^n_+} \boldsymbol{x} \quad \Longleftrightarrow \quad y_i \geq x_i, \quad i = 1, \cdots, n$$

- Matrix inequality ($K = \mathbb{S}^n_+$)

$$\boldsymbol{Y} \succeq_{\mathbb{S}^n_+} \boldsymbol{X} \quad \Longleftrightarrow \quad \boldsymbol{Y} - \boldsymbol{X} \text{ positive semidefinite}$$

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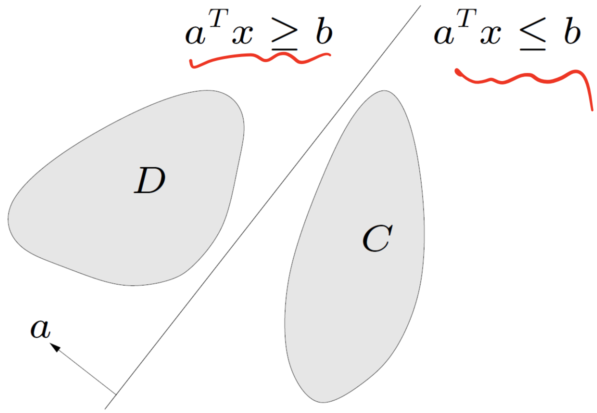
Separating Hyperplane Theorem

If C and D are nonempty disjoint convex sets, there exist $\boldsymbol{a} \neq \boldsymbol{0}$ and b, such that

$$\boldsymbol{a}^T\boldsymbol{x} \leq b \text{ for } \boldsymbol{x} \in C, \quad \boldsymbol{a}^T\boldsymbol{x} \geq b \text{ for } \boldsymbol{x} \in D$$



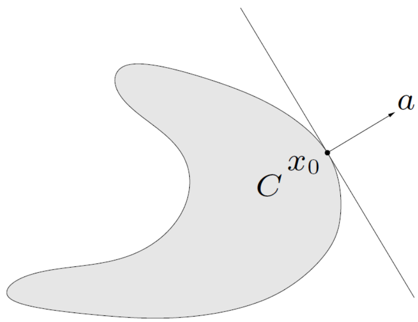
the hyperplane $\{\boldsymbol{x} | \boldsymbol{a}^T\boldsymbol{x} = b\}$ separates C and D

Supporting Hyperplane Theorem

Supporting hyperplane to set C at boundary point $\boldsymbol{x}_0$:

$$\{\boldsymbol{x} | \boldsymbol{a}^T\boldsymbol{x} = \boldsymbol{a}^T\boldsymbol{x}_0\}$$

where $\boldsymbol{a} \neq \boldsymbol{0}$ and $\boldsymbol{a}^T\boldsymbol{x} \leq \boldsymbol{a}^T\boldsymbol{x}_0$ for all $\boldsymbol{x} \in C$



Supporting hyperplane theorem: if C is convex, then there exists a supporting hyperplane at every boundary point of C

Dual Cones and Generalized Inequalities

- **Dual cone** of a cone K:

$$K^* = \{\boldsymbol{y} | \boldsymbol{y}^T \boldsymbol{x} \geq 0 \text{ for all } \boldsymbol{x} \in K\}$$

- Examples
  - $K = \mathbb{R}^n_+$: $K^* = \mathbb{R}^n_+$
  - $K = \mathbb{S}^n_+$: $K^* = \mathbb{S}^n_+$
  - $K = \{(\boldsymbol{x}, t) | \|\boldsymbol{x}\|_2 \leq t\}$: $K^* = \{(\boldsymbol{x}, t) | \|\boldsymbol{x}\|_2 \leq t\}$
  - $K = \{(\boldsymbol{x}, t) | \|\boldsymbol{x}\|_1 \leq t\}$: $K^* = \{(\boldsymbol{x}, t) | \|\boldsymbol{x}\|_\infty \leq t\}$

  First three examples are **self-dual** cones

- Dual cones of proper cones are proper, hence define generalized inequalities:

$$\boldsymbol{y} \succeq_{K^*} \mathbf{0} \quad \Longleftrightarrow \quad \boldsymbol{y}^T \boldsymbol{x} \geq 0 \text{ for all } \boldsymbol{x} \succeq_K \mathbf{0}$$

$X, \underbrace{Y \in S^n}$ $\underbrace{Tr(X, Y) \geq 0}$ for all $\underbrace{X \geq 0}$.

$\Longleftrightarrow$ $\underbrace{Y \in S_+^n}$

Proof: Suppose $\underbrace{Y \notin S_+^n}$. there exists $q \in \mathbb{R}^n$

$q^T Y q < 0 \Leftrightarrow Tr(q q^T Y) < 0$

$$q^T Y q = Tr(\underbrace{q^T Y}_{\delta} q) < 0$$

$$= Tr(\underbrace{q \cdot q^T}Y) < 0$$

Let $X = q \cdot q^T \geq 0$, $\underbrace{Tr(XY) < 0}$

$\Rightarrow Y \notin (S_+^n)^*$.

$$X = \sum_{i=1}^{n} \lambda_i \underbrace{q_i} \underbrace{q_i^T}, \quad \lambda_i \geq 0$$

$$Tr(YX) = Tr(Y \cdot \sum_{i=1}^{n} \lambda_i q_i q_i^T).$$

$$= \underbrace{\sum_{i=1}^{n} \lambda_i q_i^T \cdot x\, q_i}\; \geq 0$$

$$\min_{x \in \mathbb{R}^n} f(x) = x^T Y x$$

$$= Tr(x \cdot x^T \cdot Y)$$

$$\Leftrightarrow \min_{W \in S_+^n} f(W) = Tr(W \cdot Y)$$

$$\text{s.t. } rank(W) = 1$$

Reference

Chapter 2 of:

- Stephen Boyd and Lieven Vandenberghe, *Convex Optimization*. Cambridge, U.K.: Cambridge University Press, 2004.