SI251 - Convex Optimization Quiz

September 5, 2022

1. (15 pt) Given a function
$$f(x) = \frac{x_1 + x_2}{3 + x_1^2 + x_2^2 + x_1 x_2}, \tag{1}$$
determine the minimal point x.

2. (15 pt) In order to minimize $f(x)$ where $x \in \mathbb{R}$, we takes the following iteration:
$$x_{k+1} = x_k + \alpha_k p_k, \tag{2}$$
where $p_k = H_k \nabla f(x_k)$ and $\alpha_k \to 0^+$. What kind of H_k can guarantee that p_k is a descent direction?

3. (20 pt) Given a detailed proof of the following statements:
   - For matrices $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{m\times n}$, prove that $\mathbf{rank}(A+B) \le \mathbf{rank}(A) + \mathbf{rank}(B)$;
   - For matrices $A \in \mathbb{R}^{s\times n}$, $B \in \mathbb{R}^{n\times m}$, prove that $\mathbf{rank}(A) + \mathbf{rank}(B) - n \le \mathbf{rank}(AB)$.

4. (20 pt) Kullback-Leibler (KL) divergence can be expressed as
$$kl(x, y) = f(x) - f(y) - \nabla f(y)^T (x - y), \tag{3}$$
where $f(y) = \sum_{i=1}^{n} y_i \log y_i$ is the negative entropy of y.

   Please prove: $kl(x, y) \ge 0$ for all x, $y \in \mathbb{R}^n_{++}$, also show that $kl(x, y) = 0$ iff $x = y$.

5. (30 pt) Determine whether the following statements are true or false, and explain the reason for your judgement.
   - Suppose A and B is $m \times n$ and $n \times m$ matrix respectively ($n \ge m$). Then the non-zero eigenvalues of BA and AB are identical, and $|I_m - AB| = |I_n - BA|$.
   - If A is symmetric and has r non-zero eigenvalues, then $\mathbf{rank}(A) = r$.
   - Give two matrices with the same shape, suppose $A \succ 0$ (positive definite) and $B \succeq 0$ (positive semidefinite), then
$$|A + B| \ge |A|, \tag{4}$$
with equality iff $B = \mathbf{0}$.