Discrete Mathematics: Lecture 25

Matching, path, connected, disconnected, connected component,

cut vertex, vertex cut, nonseparable, vertex connectivity, k-connected,

cut edge, edge cut, edge connectivity

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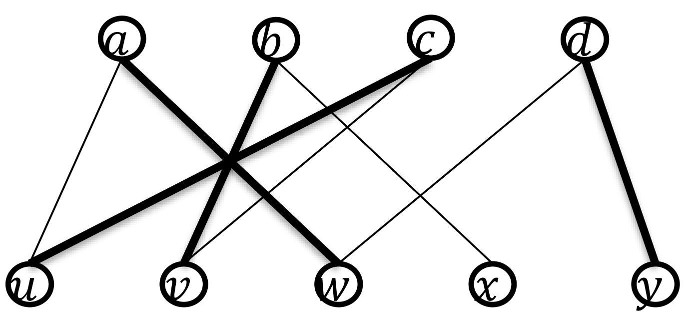
Matching

DEFINITION: Let $G = (V, E)$ be a simple graph. $M \subseteq E$ is a **matching**匹配 if $e \cap e' = \emptyset$ for every $e, e' \in M$. A vertex $v \in V$ is **matched** in M if $\exists e \in M$ such that $v \in e$, otherwise, v is **not matched.**

- **maximum matching**最大匹配**:** a matching with largest number of edges.
- In a bipartite graph $G = (A \cup B, E)$, $M \subseteq E$ is a **complete matching**完全匹配 from A to B if every $u \in A$ is matched.



- $V = \{a, b, c, d, u, v, w, x, y\}$
- $V_1 = \{a, b, c, d\}$;
- $V_2 = \{u, v, w, x, y\}$
- $E = \{au, aw, bv, bx, cu, cv, dw, dy\}$

- $M = \{au, bv\}$ is a matching
  - a, b, u, v are matched in M
  - c, d, x, y are not matched in M
  - M is not a maximum matching
- $M' = \{aw, bv, cu, dy\}$ is a maximum matching
- M' is a complete matching from V_1 to V_2

Hall's Theorem

EXAMPLE: Marriage on an Island

- There are m boys $X = \{x_1, \dots, x_m\}$ and n girls $Y = \{y_1, \dots, y_n\}$
- $G = \left(X \cup Y, E = \left\{\{x_i, y_j\}: x_i \text{ and } y_j \text{ are willing to get married}\right\}\right)$
- What is the largest number of couples that can be formed?

THEOREM (Hall 1935): A bipartitie graph $G = (X \cup Y, E)$ has a complete matching from X to Y iff $|N(A)| \geq |A|$ for any $A \subseteq X$.

- $\Rightarrow$: Let $\{\{x_1, y_1\}, \dots, \{x_m, y_m\}\}$ be a complete matching from X to Y
  - For any $A = \{x_{i_1}, \dots, x_{i_s}\} \subseteq X$, $N(A) \supseteq \{y_{i_1}, \dots, y_{i_s}\}$
    - $|N(A)| \geq s = |A|$
- $\Leftarrow$: suppose that $|N(A)| \geq |A|$ for any $A \subseteq X$. Find a complete matching M.
  - By induction on $|X|$
  - $|X| = 1$: Let $X = \{x\}$.
    - $|N(X)| \geq 1$
      - $\exists y \in Y$ such that $e = \{x, y\} \in E$.
        - $M = \{e\}$ is a complete matching from X to Y

Hall's Theorem

- **Induction hypothesis**: "$\forall A \subseteq X, |N(A)| \geq |A| \Rightarrow \exists$ complete matching" is true when $|X| \leq k$
- Prove that "$\forall A \subseteq X, |N(A)| \geq |A| \Rightarrow \exists$ complete matching" when $|X| = k + 1$
  - Let $X = \{x_1, \dots, x_k, x_{k+1}\}$.
  - **Case 1:** $\forall\ A \subseteq X$ with $1 \leq |A| \leq k$, $|N_G(A)| \geq |A| + 1$
    - $N_G(A)$: A's neighborhood in G
    - Say $y_{k+1} \in N_G(\{x_{k+1}\})$.
    - Let $V' = (X \setminus \{x_{k+1}\}) \cup (Y \setminus \{y_{k+1}\})$; $E' = \{e \in E : e \subseteq V' \times V'\}$
    - Let $G' = (V', E') = G - \{x_{k+1}\} - \{y_{k+1}\}$.
      - $\forall\ A \subseteq \{x_1, \dots, x_k\}, |N_{G'}(A)| \geq |N_G(A)| - |\{y_{k+1}\}| \geq |A| + 1 - 1 = |A|$
        - $\exists$ a complete matching M' from $X - \{x_{k+1}\}$ to $Y - \{y_{k+1}\}$ in G' (IH)
    - $M = M' \cup \left\{\{x_{k+1}, y_{k+1}\}\right\}$ is a complete matching from X to Y in G

Hall's Theorem

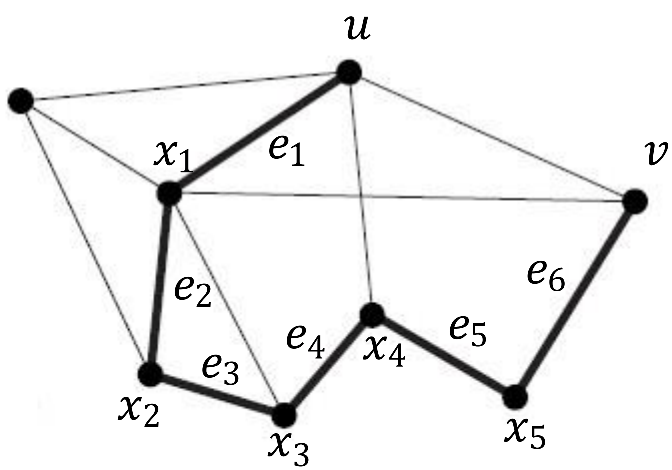
- **Case 2:** $\exists A \subseteq X, 1 \leq |A| \leq k$ such that $|N_G(A)| = |A|$
  - Say $A = \{x_1, \dots, x_j\}$ and $N_G(A) = \{y_1, \dots, y_j\}$, where $1 \leq j \leq k$
  - Let $V' = A \cup N_G(A), E' = \{e \in E: e \subseteq V' \times V'\}$ and $G' = (V', E')$
    - $\forall A' \subseteq A,\ |N_{G'}(A')| = |N_G(A')| \geq |A'|$
    - There is a complete matching M' from A to $N_G(A)$ in G' (IH)
  - Let $V'' = (X \setminus A) \cup \left(Y \setminus N_G(A)\right), E'' = \{e \in E: e \subseteq V'' \times V''\}$,
  - Let $G'' = (V'', E'') = G - A - N_G(A)$
    - Then $\forall\ A'' \subseteq X \setminus A, |N_{G''}(A'')| \geq |A''|$.
      - Otherwise, $|N_G(A'' \cup A)| = |N_{G''}(A'')| + |N_G(A)| < |A''| + |A|$
        - $\exists$ a complete matching M'' from $X \setminus A$ to $Y \setminus N_G(A)$ (IH)
  - $M = M' \cup M''$ is a complete matching from X to Y

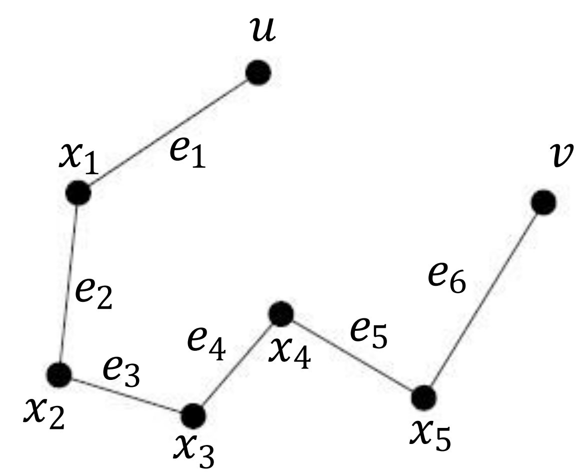
Path (Undirected)

DEFINITION: Let $G = (V, E)$ be an undirected graph and let $k \in \mathbb{N}$. A **path**路径 **of length** $\boldsymbol{k}$ from u to v in G is a sequence of k edges $e_1, \dots, e_k$ of G for which there exist vertices $x_0 = u, x_1, \dots, x_{k-1}, x_k = v$ such that $e_i = \{x_{i-1}, x_i\}$ for every $i \in [k]$.

- The path is **circuit**回路 if $u = v$ and $k > 0$
- The path **passes through**经过 $x_1, \dots, x_{k-1}$
- The path **traverses**遍历 $e_1, e_2, \dots, e_k$
- The path is **simple**简单 if it doesn't contain an edge more than once.
- If G is simple, the path can be denoted as $x_0, x_1, \dots, x_k$

Example





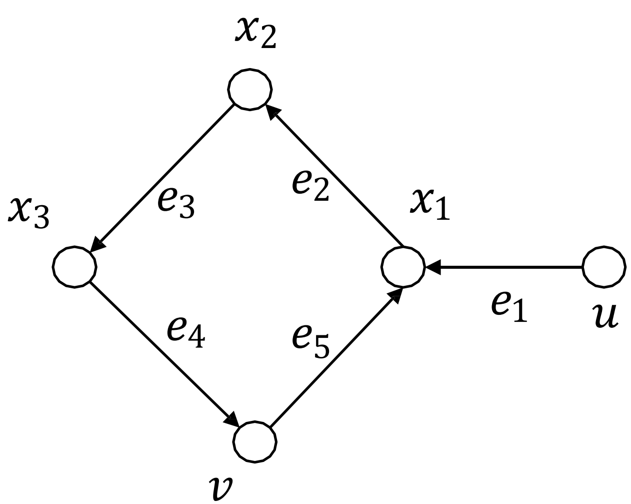
- The right-hand side graph is a path from u to v
- The path is $e_1, e_2, e_3, e_4, e_5, e_6$
- The path is simple
- The path can be denoted by $u, x_1, x_2, x_3, x_4, x_5, v$
- The path passes through x_1, x_2, x_3, x_4, x_5
- The path traverses $e_1, e_2, e_3, e_4, e_5, e_6$
- $e_1, e_2, e_3, e_4, e_5, e_6, e_7 = \{v, u\}$ is a (simple) circuit

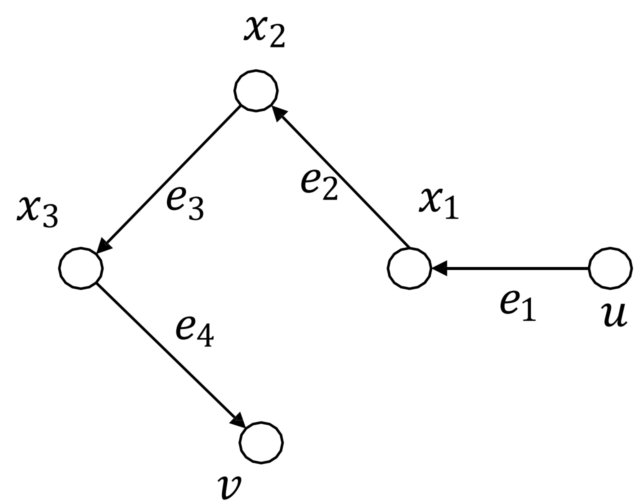
Path (Directed)

DEFINITION: Let $G = (V, E)$ be a directed graph and let $k \in \mathbb{N}$. A **path of length k** from u to v in G is a sequence of k edges $e_1, \ldots, e_k$ of G for which there exist vertices $x_0 = u, x_1, \ldots, x_{k-1}, x_k = v$ such that $e_i = (x_{i-1}, x_i)$ for every $i \in [k]$.

- The path is a **circuit** if $u = v$ and $k > 0$
- The path **passes through** $x_1, \ldots, x_{k-1}$
- The path **traverses** $e_1, e_2, \ldots, e_k$
- The path is **simple** if it doesn't contain an edge more than once.
- If G has no multiple edges, the path can be denoted as $x_0, \ldots, x_k$

Example



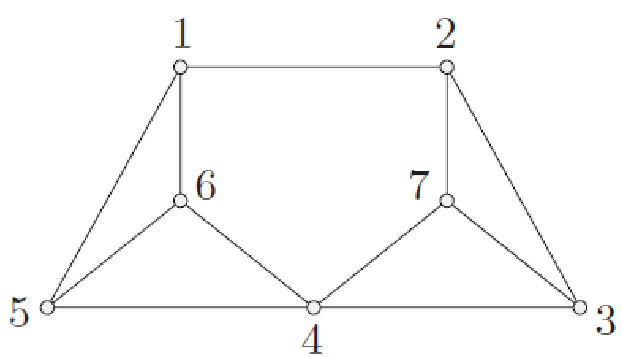


- e_1, e_2, e_3, e_4 is a path
- The path is simple
- The path can be denoted by u, x_1, x_2, x_3, v
- The path passes through x_1, x_2, x_3
- The path traverses e_1, e_2, e_3, e_4
- e_2, e_3, e_4, e_5 is a (simple) circuit

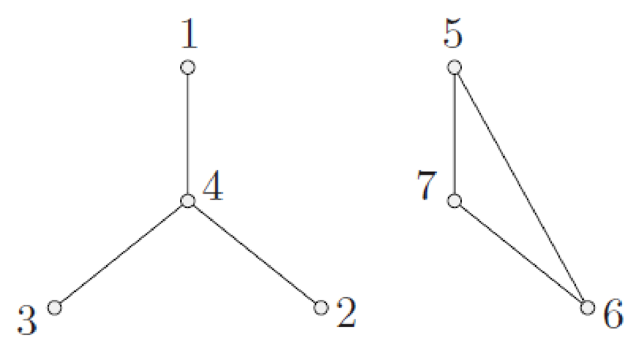
Connectivity

DEFINITION: An undirected graph G is said to be **connected**连通的 if there is a path between any pair of distinct vertices.

- Graph of order 1 is connected; the complete graph K_n is connected
- **disconnected**非连通的: not connected
- **disconnect** G**:** remove vertices or edges to produce a disconnected subgraph



A Connected Graph



A Disconnected Graph

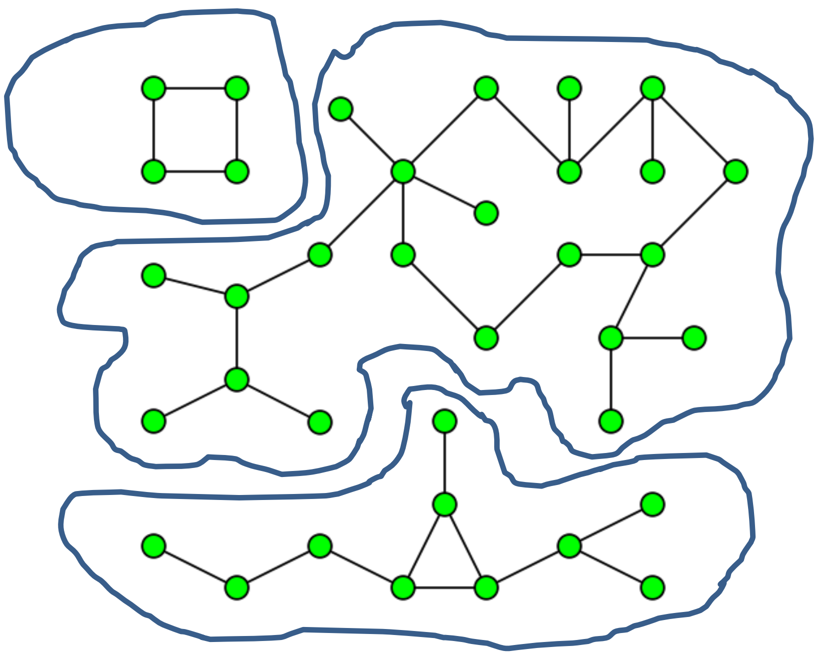
Connectivity

THEOREM: Let $G = (V, E)$ be a connected undirected graph. Then there is a simple path between any pair of distinct vertices.

- Let $u, v \in V$ and $u \neq v$. Find a simple path from u to v.
- G is connected $\Rightarrow$ there are paths from u to v.
  - Let $x_0 = u, x_1, \dots, x_{k-1}, x_k = v$ be one that has least length k.
    - This path must be simple.
      - otherwise, the path contains some edge more than once
        - $\exists i, j \in \{0,1,\dots,k\}$, say $i < j$, such that $x_i = x_j$
          - $x_0, x_1, \dots, x_{i-1}, x_j, \dots, x_k$ is a shorter path from u to v
      - The contradiction shows that the path must be simple

Connected Component

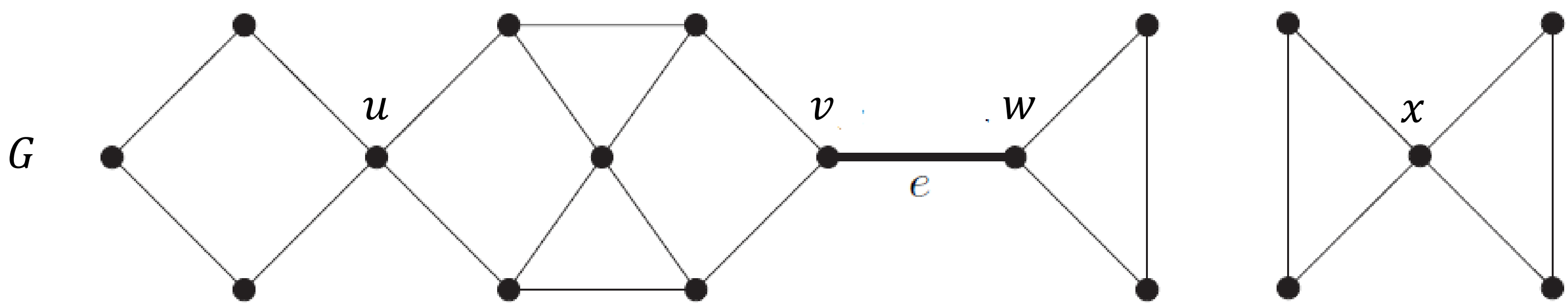
DEFINITION: A **connected component**连通分支 of a graph $G = (V, E)$ is a connected subgraph of G that is not a proper subgraph of a connected subgraph of G. //i.e., maximal极大 connected subgraph



Connected Component

DEFINITION: A **connected component**连通分支 of a graph $G = (V, E)$ is a connected subgraph of G that is not a proper subgraph of a connected subgraph of G. //i.e., maximal极大 connected subgraph

- $v \in V$ is a **cut vertex**割点 if $G - v$ has more connected components than G
- $e \in E$ is a **cut edge**割边、**bridge**桥 if $G - e$ has more connected components than G



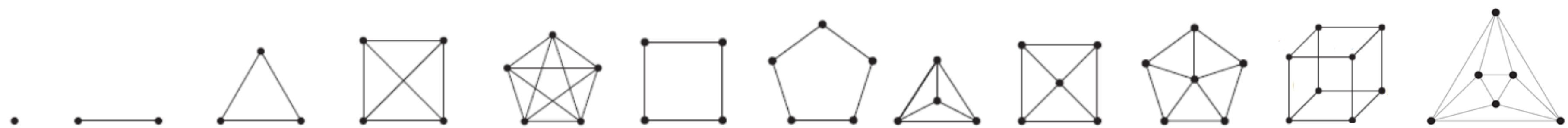
- There are 2 connected components in the graph G
- cut vertices: u, v, w, x
- cut edge: e

Vertex Connectivity

DEFINITION: A connected undirected graph $G = (V, E)$ is said to be **nonseparable**不可分的 if G has no cut vertex.

DEFINITION: Let $G = (V, E)$ be a connected simple graph.

- **vertex cut**点割集: A subset $V' \subseteq V$ such that $G - V'$ is disconnected
- **vertex connectivity**点连通度 $\kappa(G)$: the minimum number of vertices whose removal disconnect G or results in K_1; equivalently,
  - if G is disconnected, $\kappa(G) = 0$; //additional definition
  - if $G = K_n, \kappa(G) = n - 1$ //K_n has no vertex cut
  - else, $\kappa(G)$ is the minimum size of a vertex cut of G



0 1 2 3 4 2 2 3 3 3 3 4

These graphs are all nonseparable

Vertex Connectivity

THEOREM: Let $G = (V, E)$ be a simple graph of order n. Then

- $0 \leq \kappa(G) \leq n - 1$
  - Removing $n - 1$ vertices gives K_1
    - $\kappa(G) \leq n - 1$
- $\kappa(G) = 0$ iff G is disconnected or $G = K_1$
  - trivial
- $\kappa(G) = n - 1$ iff $G = K_n (n \geq 2)$
  - If: obvious
  - Only if:
    - $n = 2$: $\kappa(G) = 1 \Rightarrow G = K_2$
    - $n \geq 3$: Prove by contradiction. Suppose that $G \neq K_n$.
      - There exist distinct $u, v \in V$ such that $u \neq v$ and $\{u, v\} \notin E$
        - Let $X = V - \{u, v\}$. Then $G - X$ is disconnected.
          - $\kappa(G) \leq |X| = n - 2 < n - 1.$
            - This contradicts the condition $\kappa(G) = n - 1.$

Vertex Connectivity

DEFINITION: A simple graph $G = (V, E)$ is called $\boldsymbol{k}$**-connected**k连通的 (**$\boldsymbol{k}$-vertex-connected**) k点连通的 if $\kappa(G) \geq k$.

THEOREM: Let $G = (V, E)$ be a simple graph of order n. Then

- G is 1-connected iff G is connected and $G \neq K_1$.
  - **Only if**: G disconnected or $G = K_1 \Rightarrow \kappa(G) = 0$
  - **If** : $G \neq K_1 \Rightarrow n \geq 2$; G is connected $\Rightarrow$ removing 0 vertex cannot disconnect G or give $K_1 \Rightarrow \kappa(G) \geq 1$
- G is 2-connected iff G is nonseparable and $n \geq 3$.
  - **Only if**: $n \leq 2 \Rightarrow \kappa(G) \leq 1$; G not nonseparable $\Rightarrow$ G has cut vertex $\Rightarrow \kappa(G) \leq 1$.
  - **If**: $n \geq 3 \Rightarrow$ removing ≤ 1 vertex cannot result in K_1; G nonseparable$\Rightarrow$ removing ≤ 1 vertex cannot disconnect G; Hence. $\kappa(G) \geq 2$.
- G is k-connected iff G is j-connected for all $j \in \{0,1, \dots, k\}$
  - **Only if**: $\kappa(G) \geq k \Rightarrow \kappa(G) \geq j$ for all $j \in \{0,1, \dots, k\} \Rightarrow G$ is j connected
  - **If**: G is obviously k-connected

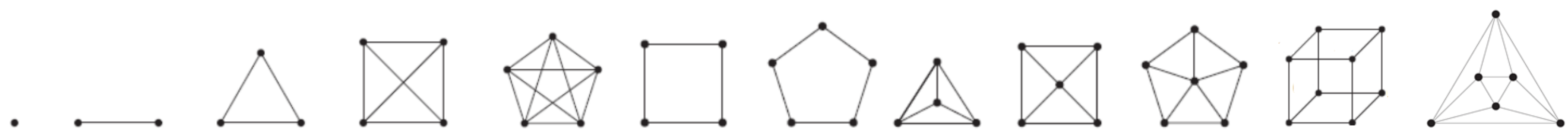
Edge Connectivity

DEFINITION: Let $G = (V, E)$ be a connected simple graph. $E' \subseteq E$ is an **edge cut**边割集 of G if $G - E'$ is disconnected.

DEFINITION: Let $G = (V, E)$ be a simple graph.

The **edge connectivity**边连通度 *($\lambda(G)$)* of G is defined as below:

- G disconnected*:* $\lambda(G) = 0$
- G connected*:*
  - $|V| = 1$: $\lambda(G) = 0$
  - $|V| > 1$: $\lambda(G)$ is the minimum size of edge cuts of G.



0 1 2 3 4 2 2 3 3 3 3 4

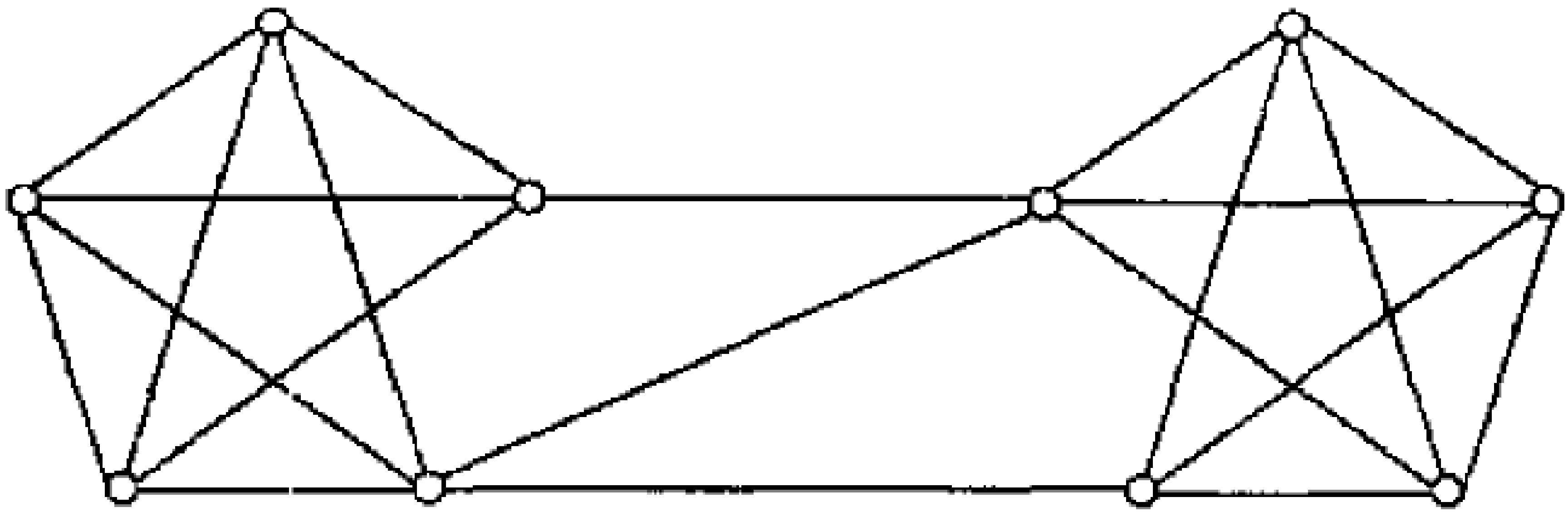
Edge Connectivity

THEOREM: Let $G = (V, E)$ be a simple graph of order n. Then

- $0 \leq \lambda(G) \leq n - 1$
  - $n = 1$: $G = K_1$ and $\lambda(G) = 0$
  - $n > 1$: $\deg(u) \leq n - 1$ for every $u \in V$
    - By removing $\{\{u, x\}: \{u, x\} \in E\}$, we can disconnect G.
      - Hence, $\lambda(G) \leq n - 1$.
- $\lambda(G) = 0$ iff G is disconnected or $G = K_1$
  - Only if: $n > 1$ and G connected $\Rightarrow \lambda(G) \geq 1$;
  - If: definition
- $\lambda(G) = n - 1$ iff $G = K_n$ $(n \geq 2)$
  - Only if: if $G \neq K_n$, then $\deg(u) < n - 1$ for some $u \in V$.
    - Remove $\left\{\{u, x\}: \{u, x\} \in E\right\}$. Then G is disconnected. $\lambda(G) < n - 1$
  - If: $\lambda(K_n) \geq \kappa(K_n) = n - 1$. (see the next theorem)

Connectivity

THEOREM: Let $G = (V, E)$ be a simple graph. Then $\kappa(G) \leq \lambda(G) \leq \delta(G)$, where $\delta(G) = \min_{v \in V} \deg(v)$ is the least degree of $G's$ vertices.



- $\kappa(G) = 2$
- $\lambda(G) = 3$
- $\delta(G) = 4$

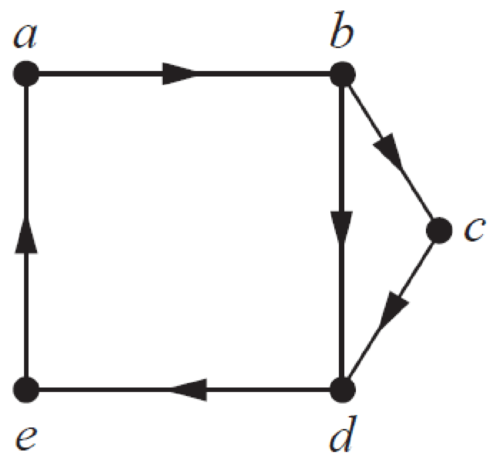
https://cp-algorithms.com/graph/edge_vertex_connectivity.html

http://www.math.caltech.edu/~2014-15/2term/ma006b/05%20connectivity%201.pdf

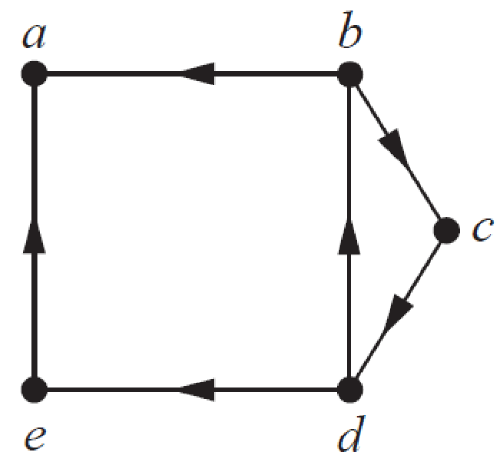
Connected Directed Graphs

DEFINITION: Let $G = (V, E)$ be a directed graph. G is said to be **strongly connected** if there is a path from u to v and a path from v to u for all $u, v \in V$ $(u \neq v)$.

- **weakly connected***:* the graph is connected if we remove the directions of all direct edges.



Strongly connected

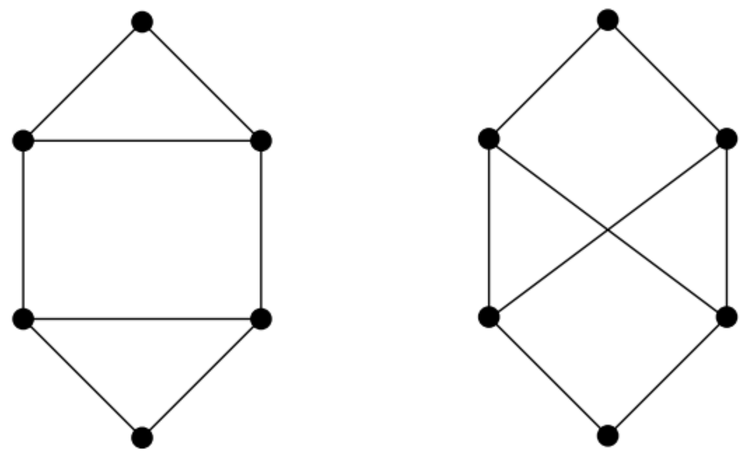


Weakly connected

Paths and Isomorphism

Theorem

The existence of a simple circuit of length k, $k \geq 3$ is an isomorphism invariant for simple graphs.



G_1 G_2

6 vertices, 8 edges

Degree sequence: 3, 3, 3, 3, 2, 2

Paths and Isomorphism*

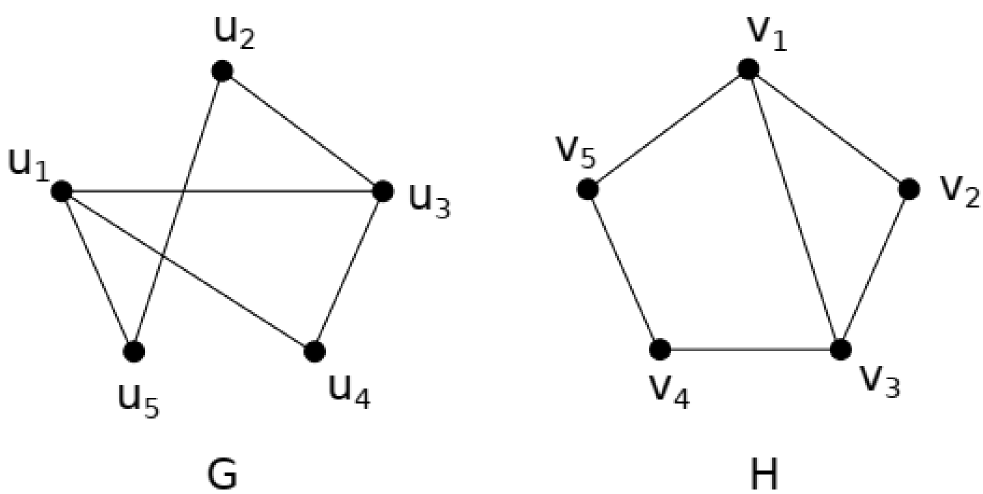
Theorem

The existence of a simple circuit of length k, $k \geq 3$ is an isomorphism invariant for simple graphs.

Proof: Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be isomorphic graphs: there is a bijective function $f : V_1 \to V_2$ respecting adjacency conditions.
Assume G_1 has a simple circuit of length k: $u_0, u_1, \ldots, u_k = u_0$, with $u_i \in V_1$ for $0 \leq i \leq k$. Let's denote $v_i = f(u_i)$, for $0 \leq i \leq k$.

$$(u_i, u_{i+1}) \in E_1 \Rightarrow (f(u_i), f(u_{i+1})) = (v_i, v_{i+1}) \in E_2, \text{ for } 0 \leq i \leq k-1.$$

So $v_0, \ldots, v_k$ is a path of length k in G_2.
It is a circuit because $v_k = f(u_k) = f(u_0) = v_0$.
It is simple: if not, at least one edge is traversed more than once, so it would mean that there exist $0 \leq i \neq j \leq k-1$ such that $(v_i, v_{i+1}) = (v_j, v_{j+1})$. But this implies $(u_i, u_{i+1}) = (u_j, u_{j+1})$ by bijectivity of f. This is impossible because $u_0, u_1, \ldots, u_k$ is simple.



5 vertices, 6 edges
Degree sequence: 3, 3, 2, 2, 2
1 simple circuit of length 3,
1 simple circuit of length 4,
1 simple circuit of length 5.

Isomorphic graphs ?

If there is an iso $f : V_G \to V_H$, the simple circuit of length 5 u_1, u_4, u_3, u_2, u_5 must be sent to the simple circuit of length 5 in H, respecting the degrees of vertices.
Check that $f(u_1) = v_1, f(u_4) = v_2, f(u_3) = v_3, f(u_2) = v_4, f(u_5) = v_5$ is an isomorphism by writing adjacency matrices.

Counting Paths Between Vertices

Theorem

Let G be a graph with adjacency matrix A with respect to the ordering of vertices $v_1, \ldots, v_n$. The number of different paths of length $r \geq 1$ from v_i to v_j equals the (i, j) entry of the matrix A^r.

Proof: By induction

• $r = 1$: the number of paths of length 1 from v_i to v_j is equal to the (i, j) entry of A by definition of A, as it corresponds to the number of edges from v_i to v_j.

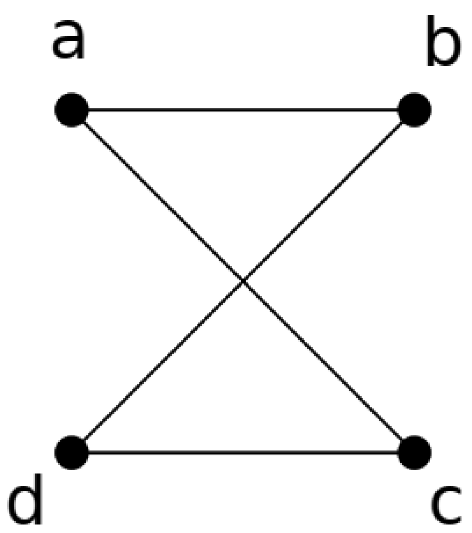
• Assume the (i, j) entry of the matrix A^r is the number of different paths of length r from v_i to v_j.
We can write $A^{r+1} = A^r A$
Let's denote $A^r = (b_{ij})_{1 \leq i,j \leq n}$, and $A = (a_{ij})_{1 \leq i,j \leq n}$. The (i, j) entry of A^{r+1} is given by:

$$\sum_{k=1}^{n} b_{ik} a_{kj} = b_{i1} a_{1j} + b_{i2} a_{2j} + \cdots + b_{in} a_{nj} \tag{1}$$

By hypothesis: b_{ik} equals the number of paths of length r from v_i to v_k.
"Path of length $r + 1$ from v_i to v_j = path of length r from v_i to any vertex v_k + an edge from v_k to v_j."
This is equal to the sum (1).

Example

How many paths of length four are there from a to d in the simple graph G



G

with ordering of vertices (a, b, c, d, e):

$$A_G = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

$$A_G^2 = \begin{pmatrix} 2 & 0 & 0 & 2 \\ 0 & 2 & 2 & 0 \\ 0 & 2 & 2 & 0 \\ 2 & 0 & 0 & 2 \end{pmatrix} \quad A_G^3 = \begin{pmatrix} 0 & 4 & 4 & 0 \\ 4 & 0 & 0 & 4 \\ 4 & 0 & 0 & 4 \\ 0 & 4 & 4 & 0 \end{pmatrix} \quad A_G^4 = \begin{pmatrix} 8 & 0 & 0 & 8 \\ 0 & 8 & 8 & 0 \\ 0 & 8 & 8 & 0 \\ 8 & 0 & 0 & 8 \end{pmatrix}$$