Lecture 15

- Laplace Transform

The French Newton Pierre-Simon Laplace (Late 1700)

- Developed mathematics in astronomy, physics, and statistics
- Began work in calculus which led to the Laplace Transform
- Focused later on celestial mechanics
  - One of the first scientists to suggest the existence of black holes

https://en.wikipedia.org/wiki/Pierre-Simon_Laplace

What are Laplace Transforms?

$$F(s)=\int_{0_-}^{\infty} f(t)e^{-st}dt \qquad F(s) = \mathcal{L}[f(t)]$$

$$\Updownarrow$$

$$f(t)=\frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} F(s)e^{st}ds \qquad f(t)=\mathcal{L}^{-1}[F(s)]$$

- *t* is real, *s* is complex! $s=\sigma+j\omega$
- Assumes f(*t*) = 0 for all *t* < 0
- Note in f(*t*) $\rightarrow$ F(*s*), *t* is integrated and *s* is variable.
- Conversely, F(*s*) $\rightarrow$ f(*t*), *t* is variable and s is integrated.

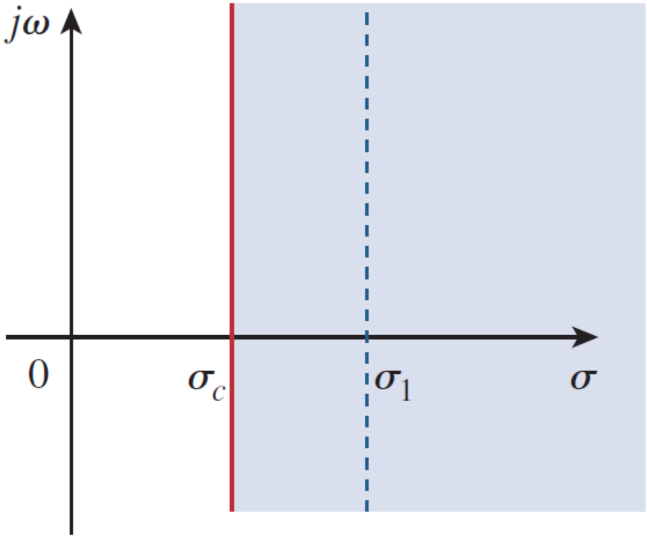
Restrictions

- There are two governing factors that determine whether Laplace transforms can be used:
  - f(t) must be at least piecewise continuous for $t \geq 0$
  - In order for f (t) to have a Laplace transform, the integral must converge to a finite value，The integral converges when

$$\int_{0^-}^{\infty} e^{-\sigma t} |f(t)| \, dt < \infty$$

Restrictions

- *F*(*s*) is undefined outside the region of convergence
- The region of convergence for the Laplace transform is $Re(s) = \sigma > \sigma_c$



Evaluating F(s) = L{f(t)}

- Straight Way – do the integral

$$f(t) = 1$$

$$F(s) = \int_0^{\infty} e^{-st} dt = -\frac{1}{s}(0-1) = \frac{1}{s}$$

$$f(t) = e^{-at}$$

$$F(s) = \int_0^{\infty} e^{-at} e^{-st} dt = \int_0^{\infty} e^{-(s+a)t} dt = \frac{1}{s+a}$$

$$f(t) = \sin t$$

$$F(s) = \int_0^{\infty} e^{-st} \sin(t) dt$$

Integrate by parts

$$f(t) = \sin t$$

$$F(s) = \int_0^\infty e^{-st} \sin(t) dt$$

Integrate by parts

$$\int u dv = uv - \int v du$$

$$\int_0^\infty e^{-st} \cos(t) dt =$$

$$= -e^{-st} 0 + s \int_0^\infty e^{-st} \sin(t) dt$$

$$\frac{1}{1+s^2}$$

$$\int_0^\infty e^{-st} \sin(t) dt = -e^{-st} \cos(t)|_0^\infty + s \int_0^\infty e^{-st} \cos(t) dt =$$

$$1 - s \int_0^\infty e^{-st} \cos(t) dt$$

Evaluating F(s) = L{f(t)}

This is the easy way ...

- Recognize a few different transforms
- Learn a few different properties
- Do a little math

Homogeneity and Additivity

$$\mathcal{L}[a_1 f_1(t)] = a_1 \mathcal{L}[f_1(t)] = a_1 F_1(s)$$

$$\mathcal{L}[a_1 f_1(t) + a_2 f_2(t)] = a_1 \mathcal{L}[f_1(t)] + a_2 \mathcal{L}[f_2(t)] = a_1 F_1(s) + a_2 F_2(s)$$

here a_1 and a_2 are constants

Important implication:

$$\sum_{k=1}^{k} i_k(t) = 0 \Longleftrightarrow \sum_{k=1}^{k} I_k(s) = 0$$

$$\sum_{k=1}^{k} u_k(t) = 0 \Longleftrightarrow \sum_{k=1}^{k} U_k(s) = 0$$

Differentiation

$$\mathcal{L}\left[\frac{d}{dt} f(t)\right] = sF(s) - f(0_-)$$

$$\mathcal{L}[f^{(n)}(t)] = s^n F(s) - s^{n-1} f(0_-) - s^{n-2} f^{(1)}(0_-) - \cdots - f^{(n-1)}(0_-)$$

$$= s^n F(s) - \sum_{k=0}^{n-1} s^k f^{(n-1-k)}(0_-)$$

Integration

$$\mathcal{L}\left[\int_{0_-}^{t} f(\tau)d\tau\right] = \frac{1}{s}F(s)$$

Properties: Scaling in Time

$$L\{f(at)\} = \frac{1}{a}F(\frac{s}{a})$$

$$L\{f(at)\} =$$

$$\int_0^{\infty} f(at)e^{-st}dt =$$

$$u = at, t = \frac{u}{a}, dt = \frac{1}{a}du$$

$$\frac{1}{a}\int_0^{\frac{\infty}{a}} f(u)e^{-(\frac{s}{a})u}du =$$

$$\frac{1}{a}F(\frac{s}{a})$$

Example :

$$L\{\sin(\omega t)\}$$

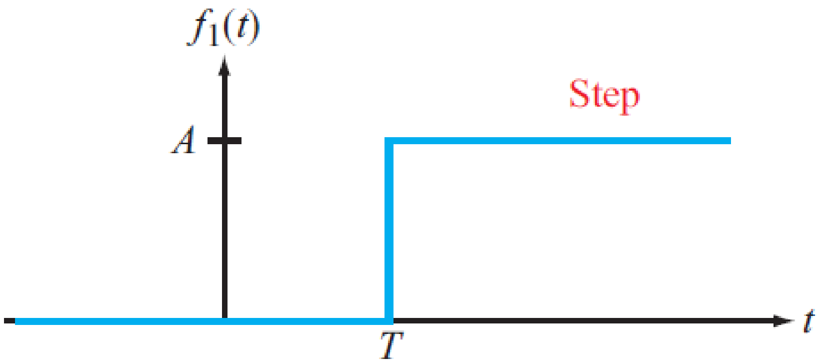
$$\frac{1}{\omega}(\frac{1}{(s/\omega)^2} + 1) =$$

$$\frac{1}{\omega}(\frac{\omega^2}{s^2 + \omega^2}) =$$

$$\frac{\omega}{s^2 + \omega^2}$$

Time Shift

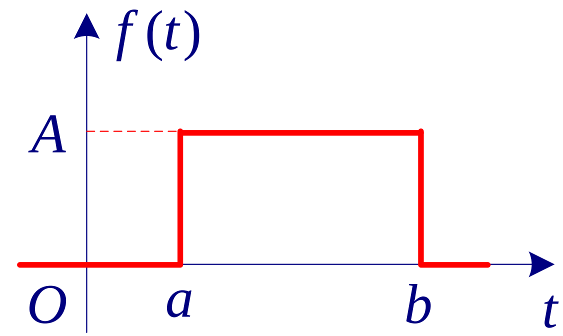
$$\mathcal{L}[f(t-\tau)] = e^{-s\tau} F(s)$$



$$\mathbf{F}_1(\mathbf{s}) = \int_{0^-}^{\infty} f_1(t)\, e^{-\mathbf{s}t}\, dt = \int_{0^-}^{\infty} A\, u(t-T)\, e^{-\mathbf{s}t}\, dt$$

$$= A \int_{T}^{\infty} e^{-\mathbf{s}t}\, dt = -\frac{A}{\mathbf{s}}\, e^{-\mathbf{s}t}\Big|_{T}^{\infty} = \frac{A}{\mathbf{s}}\, e^{-\mathbf{s}T}.$$

Example



$$f(t) = A[u(t-a) - u(t-b)]$$

$$F(s) = A\mathcal{L}[u(t-a) - u(t-b)] = \frac{A}{s}(e^{-as} - e^{-bs})$$

Frequency Shift

$$\mathcal{L}[\ e^{\alpha t} f(t)\ ] = F(s-\alpha)$$

- Example

$$\mathcal{L}\ [e^{-\alpha t} \sin \omega t] = \frac{\omega}{(s+\alpha)^2 + \omega^2}$$

TABLE 12.1 An Abbreviated List of Laplace Transform Pairs

Type	$f(t)$ $(t > 0-)$	$F(s)$
(impulse)	$\delta(t)$	1
(step)	$u(t)$	$\frac{1}{s}$
(ramp)	t	$\frac{1}{s^2}$
(exponential)	e^{-at}	$\frac{1}{s+a}$
(sine)	$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
(cosine)	$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
(damped ramp)	te^{-at}	$\frac{1}{(s+a)^2}$
(damped sine)	$e^{-at}\sin \omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$
(damped cosine)	$e^{-at}\cos \omega t$	$\frac{s+a}{(s+a)^2+\omega^2}$

TABLE 12.2 An Abbreviated List of Operational Transforms

Operation	$f(t)$	$F(s)$
Multiplication by a constant	$Kf(t)$	$KF(s)$
Addition/subtraction	$f_1(t) + f_2(t) - f_3(t) + \cdots$	$F_1(s) + F_2(s) - F_3(s) + \cdots$
First derivative (time)	$\frac{df(t)}{dt}$	$sF(s) - f(0^-)$
Second derivative (time)	$\frac{d^2f(t)}{dt^2}$	$s^2F(s) - sf(0^-) - \frac{df(0^-)}{dt}$
nth derivative (time)	$\frac{d^nf(t)}{dt^n}$	$s^nF(s) - s^{n-1}f(0^-) - s^{n-2}\frac{df(0^-)}{dt} - s^{n-3}\frac{df^2(0^-)}{dt^2} - \cdots - \frac{d^{n-1}f(0^-)}{dt^{n-1}}$
Time integral	$\int_0^t f(x)\,dx$	$\frac{F(s)}{s}$
Translation in time	$f(t-a)u(t-a), a > 0$	$e^{-as}F(s)$
Translation in frequency	$e^{-at}f(t)$	$F(s+a)$
Scale changing	$f(at), a > 0$	$\frac{1}{a}F\left(\frac{s}{a}\right)$
First derivative (s)	$tf(t)$	$-\frac{dF(s)}{ds}$
nth derivative (s)	$t^nf(t)$	$(-1)^n\frac{d^nF(s)}{ds^n}$
s integral	$\frac{f(t)}{t}$	$\int_s^\infty F(u)\,du$

15.6 Find $F(s)$ given that

$$f(t) = \begin{cases} 5t, & 0 < t < 1 \\ -5t, & 1 < t < 2 \\ 0, & \text{otherwise} \end{cases}$$

15.9 Determine the Laplace transforms of these functions:

(a) $f(t) = (t - 4)u(t - 2)$

(b) $g(t) = 2e^{-4t}u(t - 1)$

(c) $h(t) = 5\cos(2t - 1)u(t)$

(d) $p(t) = 6[u(t - 2) - u(t - 4)]$

The Inverse Laplace Transform

- $F(s) = \frac{N(s)}{D(s)}$
- Steps to Find the Inverse Laplace Transform:
  - 1. Decompose $F(s)$ into simple terms using partial fraction expansion.
  - 2. Find the inverse of each term by matching entries in the table.

Simple Poles

$$F(s) = \frac{N(s)}{(s+p_1)(s+p_2)\cdots(s+p_n)}$$

$p_i \neq p_j$ for all $i \neq j$

$$F(s) = \frac{k_1}{s+p_1} + \frac{k_2}{s+p_2} + \cdots + \frac{k_n}{s+p_n}$$

$$(s+p_1)F(s) = k_1 + \frac{(s+p_1)k_2}{s+p_2} + \cdots + \frac{(s+p_1)k_n}{s+p_n}$$

$$k_i = (s+p_i)F(s)\,\big|_{s=-p_i}$$

$$f(t) = (k_1 e^{-p_1 t} + k_2 e^{-p_2 t} + \cdots + k_n e^{-p_n t})u(t)$$

Repeated Poles

- Suppose *F*(*s*) has *n* repeated poles

$$F(s) = \frac{k_n}{(s+p)^n} + \frac{k_{n-1}}{(s+p)^{n-1}} + \cdots + \frac{k_2}{(s+p)^2} + \frac{k_1}{s+p} + F_1(s)$$

Determine coefficient k_n

$$k_n = (s+p)^n F(s)\,|_{s=-p}$$

$$k_{n-1} = \frac{d}{ds}[(s+p)^n F(s)]\,|_{s=-p}$$

$$\mathcal{L}^{-1}\left[\frac{1}{(s+a)^n}\right] = \frac{t^{n-1}e^{-at}}{(n-1)!}u(t)$$

$$k_{n-m} = \frac{1}{m!}\frac{d^m}{ds^m}[(s+p)^n F(s)]\,|_{s=-p}$$

$$f(t) = \left(k_1 e^{-pt} + k_2 t e^{-pt} + \frac{k_3}{2!}t^2 e^{-pt} + \cdots + \frac{k_n}{(n-1)!}t^{n-1}e^{-pt}\right)u(t) + f_1(t)$$

Complex Poles

$$F(s) = \frac{A_1 s + A_2}{s^2 + as + b} + F_1(s)$$

$$s^2 + as + b = (s + \alpha)^2 + \beta^2$$

$$A_1 s + A_2 = A_1(s + \alpha) + B_1\beta$$

$$F(s) = \frac{A_1(s + \alpha)}{(s + \alpha)^2 + \beta^2} + \frac{B_1\beta}{(s + \alpha)^2 + \beta^2} + F_1(s)$$

$$f(t) = (A_1 e^{-\alpha t}\cos\beta t + B_1 e^{-\alpha t}\sin\beta t)u(t) + f_1(t)$$

Exercise

$$F(s)=\frac{s^2+3s+5}{s^3+6s^2+11s+6}$$

$$F(s)=\frac{s^2+3s+5}{(s+1)(s+2)(s+3)}=\frac{K_1}{s+1}+\frac{K_2}{s+2}+\frac{K_3}{s+3}$$

Exercise

$$F(s)=\frac{10s^2+4}{s(s+1)(s+2)^2}$$

$$F(s)=\frac{K_{11}}{s}+\frac{K_{21}}{s+1}+\frac{K_{31}}{s+2}+\frac{K_{32}}{(s+2)^2}$$

$$\mathrm{f(t)}=[1-14e^{-t}+(13+22t)e^{-2t}]\varepsilon(t)$$

The Convolution Integral

$$y(t) = \int_{-\infty}^{\infty} x(\lambda)h(t-\lambda)\,d\lambda$$

or simply

$$y(t) = x(t) * h(t)$$

$$F(s) = \mathcal{L}[f_1(t) * f_2(t)] = F_1(s)F_2(s)$$