SI231B - Matrix Computations, Spring 2022-23

Homework Set #5

Prof. Ziping Zhao

Acknowledgements:

1) Deadline: **2023-05-06 23:59:59**
2) Please submit your assignments via Gradescope.
3) You can write your homework using latex/word or you can write in handwriting and submit the scanned pdf.

Problem 1. (20 points)

Consider a matrix $\mathbf{A} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \\ 2 & 1 \end{bmatrix}$.

1) Find the SVD of $\mathbf{A}$. (Both compact SVD and full SVD are correct.) (15 points)
2) Compute the pseudo-inverse of $\mathbf{A}$. (5 points)

Problem 2. (20 points)

The Frobenius norm of a matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$ is defined as $\|\mathbf{A}\|_F = \sqrt{Tr(\mathbf{A}^T\mathbf{A})}$.

1) Show that
$$\|\mathbf{A}\|_F = \left(\sum_{i,j} |\mathbf{A}_{ij}|^2\right)^{\frac{1}{2}}$$
(5 points)
2) Show that if $\mathbf{U}$ and $\mathbf{V}$ are orthogonal, then $\|\mathbf{UA}\|_F = \|\mathbf{AV}\|_F = \|\mathbf{A}\|_F$. (5 points)
3) Show that $\|\mathbf{A}\|_F = \sqrt{\sigma_1^2 + \cdots + \sigma_r^2}$, where $\sigma_1, \ldots, \sigma_r$ are the singular value of $\mathbf{A}$. (5 points)
4) Assume that $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r$. Show that $\sigma_1 \leq \|\mathbf{A}\|_F \leq \sqrt{r}\sigma_1$. (5 points)

Problem 3. (20 points)

Let $\mathbf{A} \in \mathbb{R}^{n\times n}$.

1) If $\kappa(\mathbf{A}) = 1$. Show that $\mathbf{A}$ is a multiple of an orthogonal matrix. (10 points)
2) If $\mathbf{A} = \gamma\mathbf{U}$, where $\mathbf{U}$ is an orthogonal matrix and $\gamma \in \mathbb{R}$. Show that $\kappa(\mathbf{A}) = 1$. (10 points)

Problem 4. (20 points)

Consider the problem of partitioning the vertex set $\mathcal{V}$ of a directed graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$ into two subsets $\mathcal{S}_1$ and $\mathcal{S}_2$ (i.e., finding a cut for a directed graph) so that the number of edges from $\mathcal{S}_1$ to $\mathcal{S}_2$ is maximized. This problem can be modeled in the following way. For each vertex i with $i = 1, \ldots, n$, we associate it with an indicator variable x_i, which equals to 1 if $i \in \mathcal{S}_1$ and 0 if $i \in \mathcal{S}_2$. The number of edges from $\mathcal{S}_1$ to $\mathcal{S}_2$ is given by

$$\sum_{i,j} a_{ij} x_i (1 - x_j) = \mathbf{x}^T \mathbf{A} (\mathbf{1} - \mathbf{x})$$

where $\mathbf{A} \in \mathbb{R}^{n \times n}$ is a 0-1 matrix called adjacency matrix with $a_{ij} = 1$ if there is an edge from i to j and $a_{ij} = 0$ otherwise. Then the problem is to obtain a 0-1 vector $\mathbf{x} \in \mathbb{R}^n$ by solving the problem

$$\begin{aligned} \underset{\mathbf{x}}{\text{maximize}} \quad & \mathbf{x}^T \mathbf{A} (\mathbf{1} - \mathbf{x}) \\ \text{subject to} \quad & x_i \in \{0, 1\}, \ \forall i. \end{aligned}$$

The problem is NP-hard. In practice an approximation solution is preferred, which is from the following problem:

$$\begin{aligned} \underset{\mathbf{x}}{\text{maximize}} \quad & \mathbf{x}^T \mathbf{A}_k (\mathbf{1} - \mathbf{x}) \\ \text{subject to} \quad & x_i \in \{0, 1\}, \ \forall i, \end{aligned}$$

where $\mathbf{A}_k = \sum_{i=1}^{k} \sigma_i \mathbf{u}_i \mathbf{v}_i^T$ denotes the truncated SVD for $\mathbf{A}$.

Prove that for any 0-1 vector $\mathbf{x}$ the following approximation bound holds:

$$\|\mathbf{x}^T \mathbf{A} (\mathbf{1} - \mathbf{x}) - \mathbf{x}^T \mathbf{A}_k (\mathbf{1} - \mathbf{x})\|_2 \leq \frac{n^2}{\sqrt{k+1}}.$$

Problem 5. (20 points)

Show that $\mathbf{A}\mathbf{A}^{\dagger}$ is the orthogonal projection onto the range space of $\mathbf{A}$, and $\mathbf{A}^{\dagger}\mathbf{A}$ is the orthogonal projection on the orthogonal complement of $\mathcal{N}(\mathbf{A})$.