Discrete Mathematics: Lecture 27

Shortest Paths and Djikstra’s Algorithm, Traveling Salesperson Problem, Planar Graph, Euler’s Formula, Kuratowski's Theorem

Xuming He
Associate Professor

School of Information Science and Technology
ShanghaiTech University

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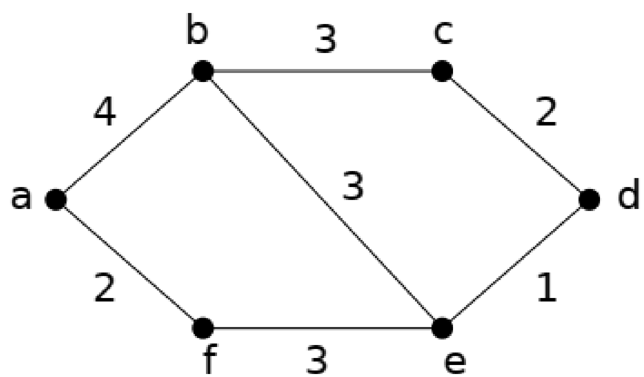
Notes by Prof. Liangfeng Zhang

Shortest Path Problem

Definition

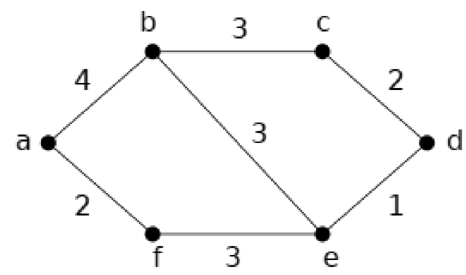
A **weighted graph** is a graph $G = (V, E)$ such that each edge is assigned with a strictly positive number.
The **length** of a path in weighted graph is the sum of the weights of the edges of this path.



a, b, c is a path of length 7 and b, e, d, c is a path of length 6

Remark: Observe that in a non-weighted graph the length of a path is the number of edges in the path!

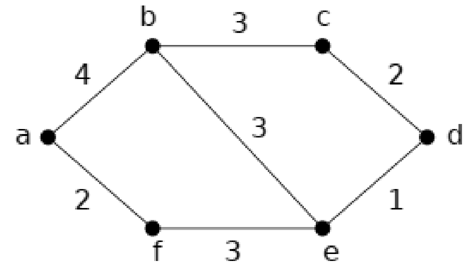
Djikstra’s Algorithm



1. Find the closest vertex to $a \rightsquigarrow$ analyse all the edges starting from a:
   a, b of length 4
   a, f of length 2
   $\Rightarrow$ f is the closest vertex to a. The shortest path from a to f has length 2.

2. Find the second closest vertex to $a \rightsquigarrow$ shortest paths from a to a vertex in $\{a, f\}$ followed by an edge from a vertex in $\{a, f\}$ to a vertex not in this set:
   a, b of length 4
   a, f, e of length 5
   $\Rightarrow$ b is the second closest vertex to a. The shortest path from a to b has length 4.

Djikstra's Algorithm



3. Find the third closest vertex to $a \rightsquigarrow$ shortest path from a to a vertex in $\{a, f, b\}$ followed by an edge from a vertex in $\{a, f, b\}$ to a vertex not in this set:
   a, b, c of length 7
   a, b, e of length 7
   a, f, e of length 5
   $\Rightarrow$ e is the third closest vertex to a. The shortest path from a to e has length 5.

4. Find the fourth closest vertex to $a \rightsquigarrow$ shortest path from a to a vertex in $\{a, f, b, e\}$ followed by an edge from a vertex in $\{a, f, b, e\}$ to a vertex not in this set:
   a, b, c of length 7
   a, f, e, d of length 6
   $\Rightarrow$ d is the fourth closest vertex to a. The shortest path from a to d has length 6.

Djikstra’s Algorithm

Goal: find the length of a shortest path from a to z with a series of iterations.

- A distinguished set of vertices is constructed by adding one vertex at each iteration.
- A labeling procedure is carried out at each iteration: a vertex w is labeled with the length of a shortest path from a to w that contains only vertices in the distinguished set.
- The vertex added to the distinguished set is one with minimal label among those vertices not already in the set.

Notations: $S_k :=$ distinguished set after k iterations, $L_k(v) :=$ length of a shortest path from a to v containing only vertices in S_k ("label" of v).

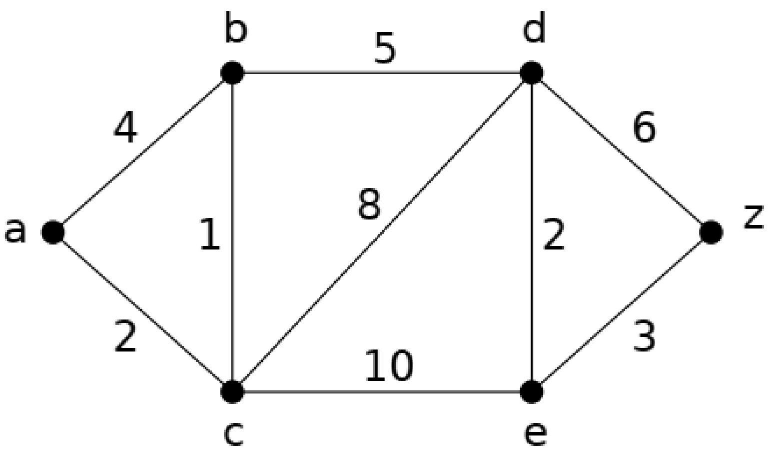
Initialization: $L_0(a) = 0$,
$L_0(v) = \infty$ for every vertex $v \neq a$,
$S_0 = \emptyset$.

kth iteration:

- S_k is formed from S_{k-1} by adding a vertex u not in S_{k-1} with smallest label,
- Update the labels of all vertices not in S_k so that $L_k(v)$ is the length of a shortest path from a to v containing only vertices in S_k, i.e.

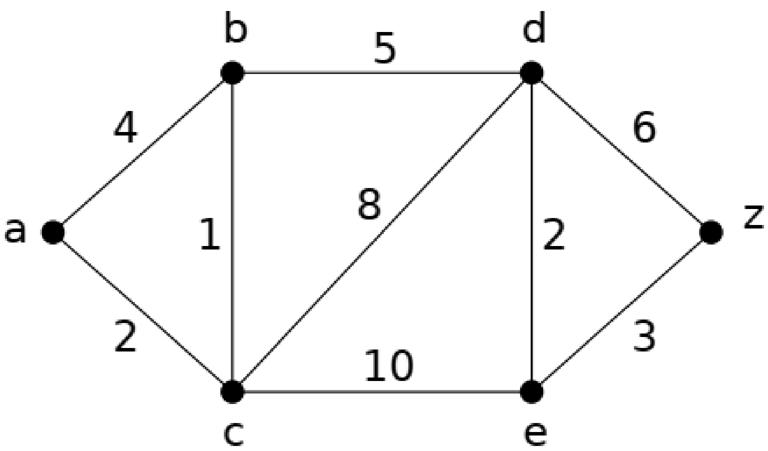
$L_k(v) = min\{L_{k-1}(v), L_{k-1}(u) + w(u,v)\}$ (with $w(u,v)$ length of the edge (u,v))

Djikstra's Algorithm



- **k=0 (initialization):** $S_0 = \emptyset$, $L_0(a) = 0,\ L_0(b) = L_0(c) = L_0(d) = L_0(e) = L_0(z) = \infty$

Djikstra's Algorithm



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- **k=1:** $u := a \rightsquigarrow S_1 = \{a\}$,
  $L_0(a) + w(a,b) = 4 < L_0(b) \rightsquigarrow L_1(b) = 4$
  $L_0(a) + w(a,c) = 2 < L_0(c) \rightsquigarrow L_1(c) = 2$

- **k=2:** $u := c \rightsquigarrow S_1 = \{a, c\}$,
  $L_1(c) + w(c,b) = 3 < L_1(b) \rightsquigarrow L_2(b) = 3$
  $L_1(c) + w(c,d) = 10 < L_1(d) \rightsquigarrow L_2(d) = 10$
  $L_1(c) + w(c,e) = 12 < L_1(e) \rightsquigarrow L_2(e) = 12$

- **k=3:** $u := b \rightsquigarrow S_1 = \{a, c, b\}$,
  $L_2(b) + w(b,d) = 8 < L_2(d) \rightsquigarrow L_3(d) = 8$

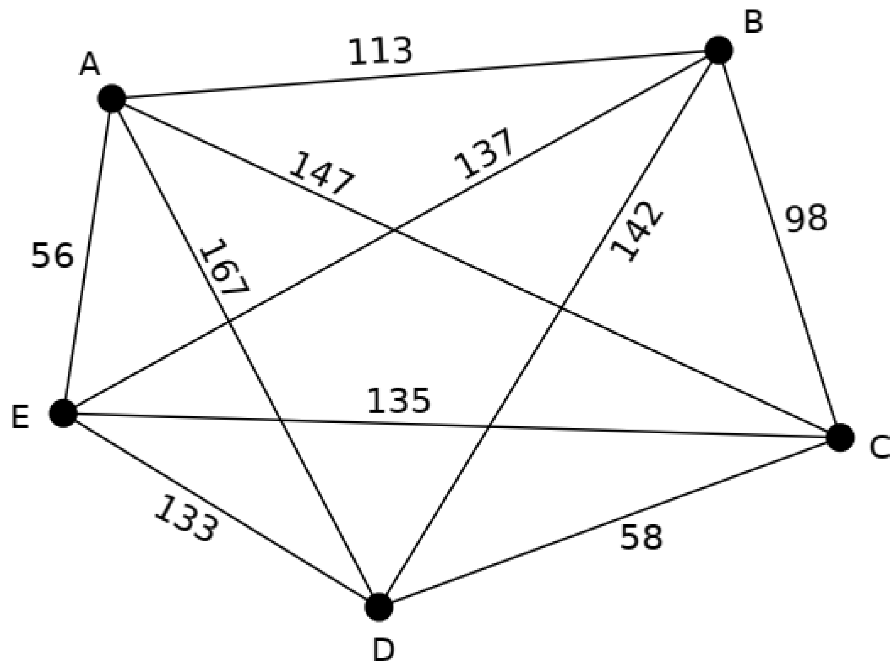
- **k=4:** $u := d \rightsquigarrow S_1 = \{a, c, b, d\}$,
  $L_3(d) + w(d,e) = 10 < L_3(e) \rightsquigarrow L_4(e) = 10$
  $L_3(d) + w(d,z) = 14 < L_3(z) \rightsquigarrow L_4(z) = 14$

- **k=5:** $u := e \rightsquigarrow S_1 = \{a, c, b, d, e\}$,
  $L_4(e) + w(e,z) = 13 < L_4(z) \rightsquigarrow L_5(z) = 13$

- **k=6:** $u := z \rightsquigarrow S_1 = \{a, c, b, d, z\}$,

- **return:** $L(z) = 13$

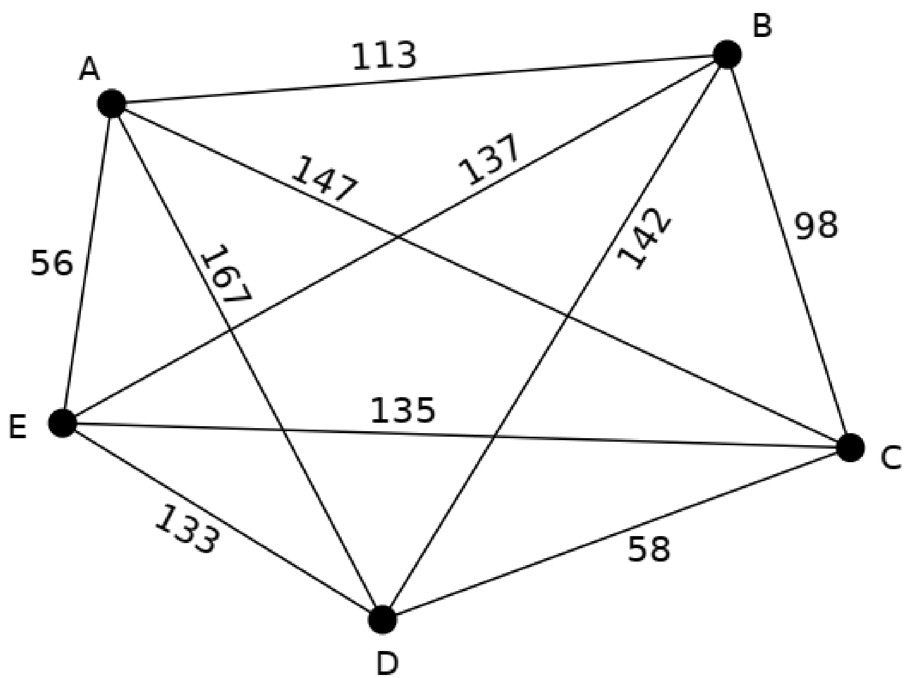
Traveling Salesperson Problem



Traveling salesperson problem: a traveling salesperson wants to visit each of the cities once and return to his starting point. In which order should he visit these cities to travel the minimum total distance?

$\Rightarrow$ **Hamiltonian circuit with minimum total weight in the complete graph.**

Traveling Salesperson Problem



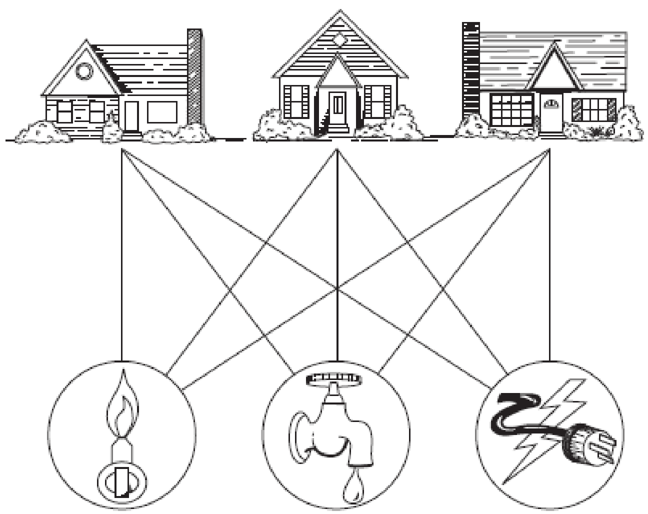
Route	Tot. dist.
A, B, C, D, E, A	610
A, B, C, E, D, A	516
A, B, E, D, C, A	588
A, B, E, C, D, A	458
A, B, D, E, C, A	540
A, B, D, C, E, A	504
A, D, B, C, E, A	598
A, D, B, E, C, A	576
A, D, E, B, C, A	682
A, D, C, B, E, A	646
A, C, D, B, E, A	670
A, C, B, D, E, A	728

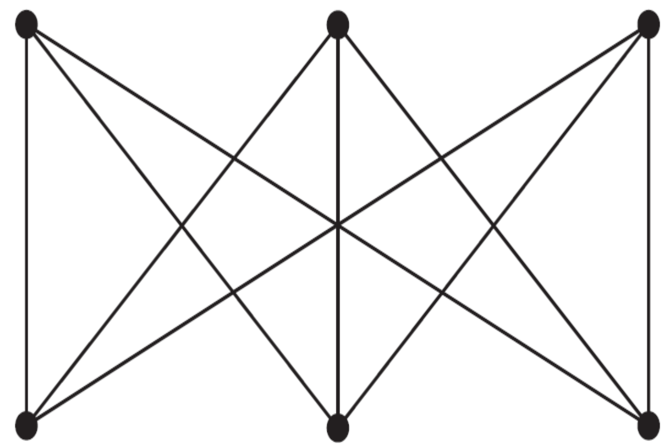
Traveling salesperson problem: a traveling salesperson wants to visit each of the cities once and return to his starting point. In which order should he visit these cities to travel the minimum total distance?
$\Rightarrow$ **Hamiltonian circuit with minimum total weight in the complete graph.**

Planar Graph

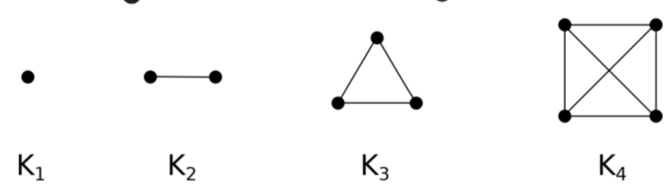
DEFINITION: Let $G = (V, E)$ be an undirected graph. G is called a **planar graph**平面图 if it can be drawn in the plane without any edges crossing.

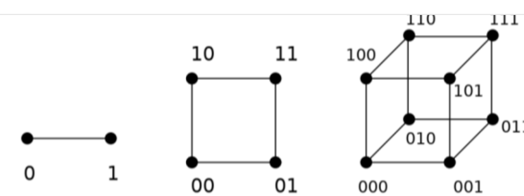
- Crossing of edges: an intersection other than endpoints (vertices)
- **planar representation**平面表示: a drawing w/o edge crossing; **nonplanar**非平面的





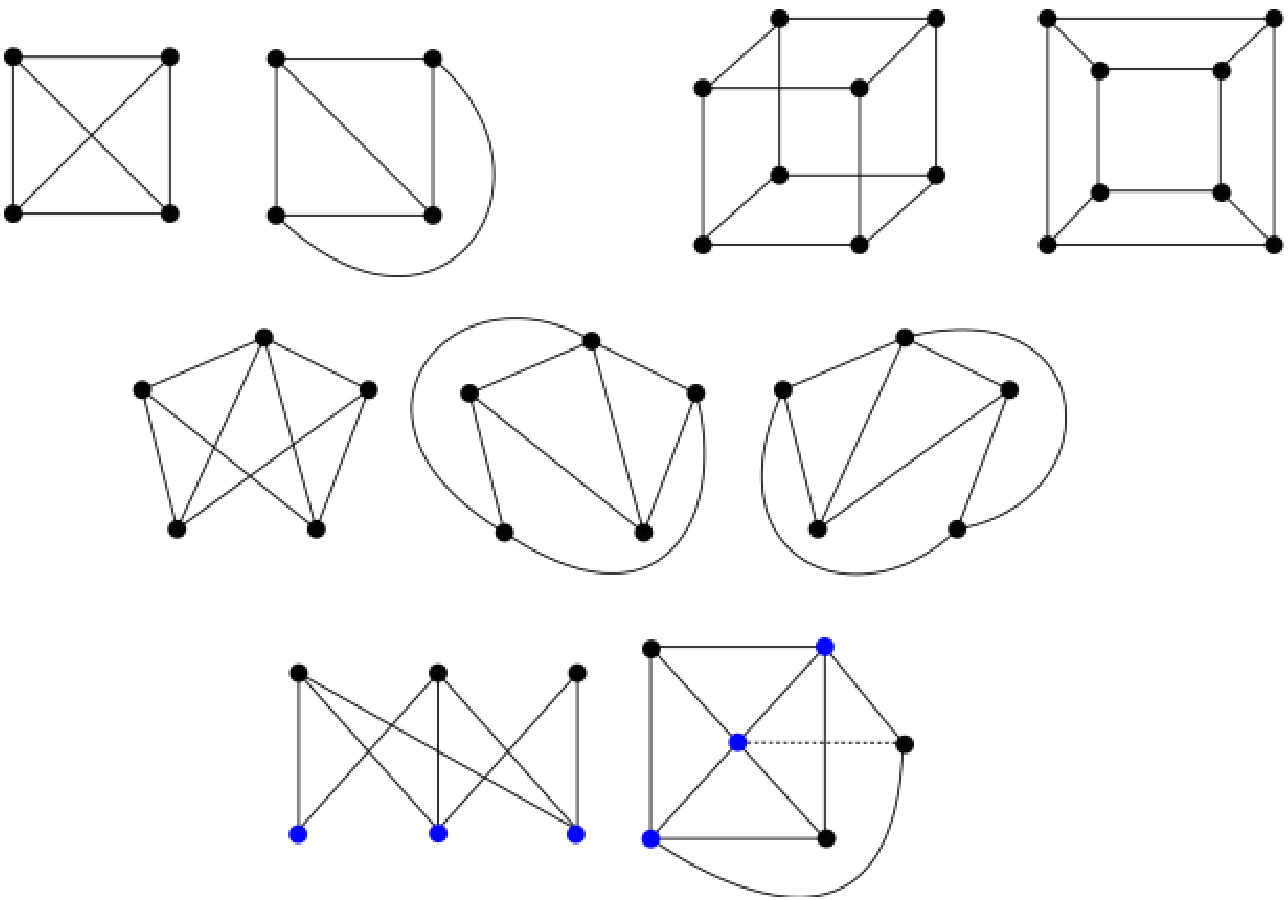
- K_1, K_2, K_3, K_4 are planar graphs
- $K_{1,n}, K_{2,n}$ are planar graphs
- $C_n\ (n \geq 3), W_n (n \geq 3)$ are planar graphs
- Q_1, Q_2, Q_3 are planar graphs





Planar Graph

Examples

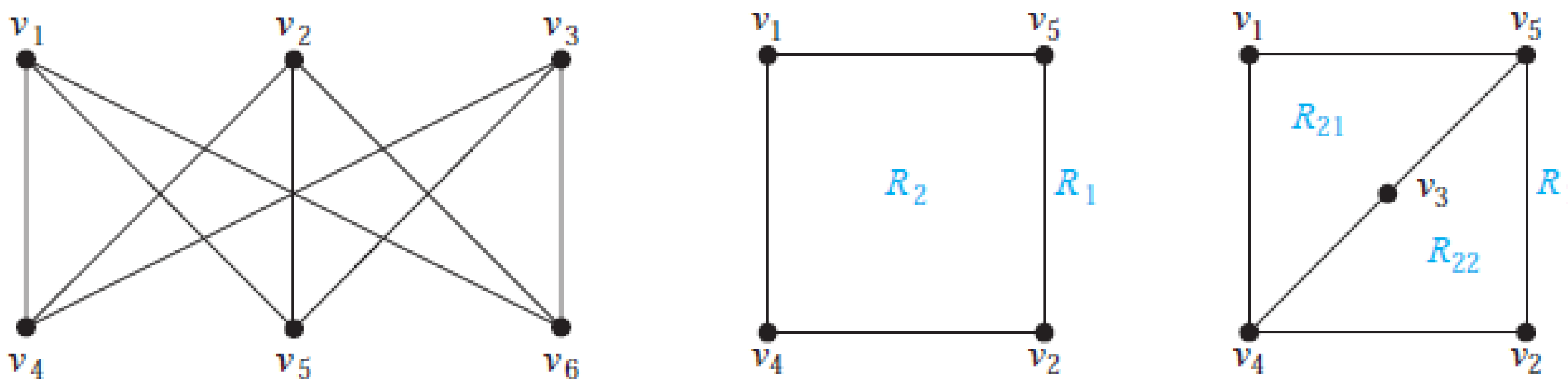


A graph may be planar even if it is usually drawn with crossings, because it may be possible to draw it in a different way without crossings.

Nonplanar Graph

Jordan Curve Theorem: Every **simple closed planar curve** Γ separates the plane into a bounded interior region and an unbounded exterior region. Any planar curve connecting the two regions must intersect Γ.

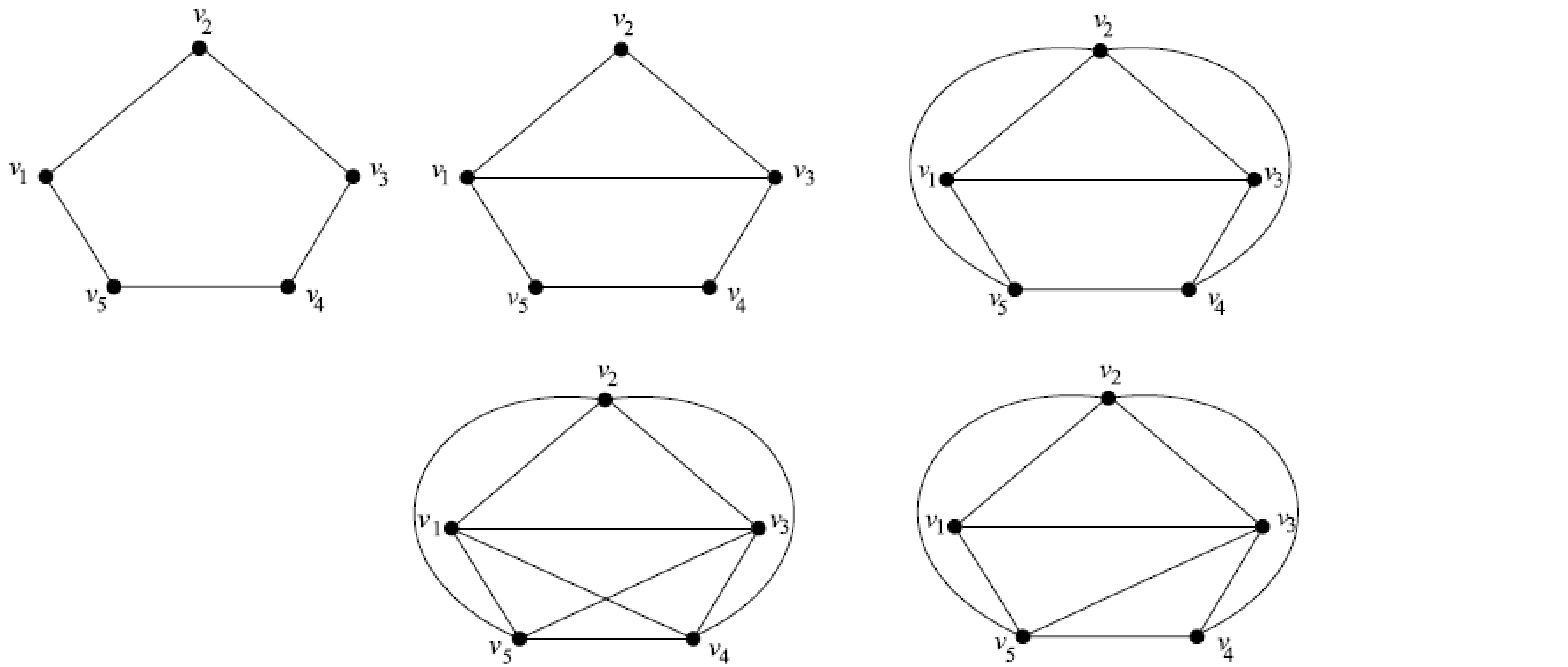
EXAMPLE: The bipartite graph $K_{3,3}$ is not planar.



- choose a simple circuit v_1, v_5, v_2, v_4, v_1 in $K_{3,3}$
- If $K_{3,3}$ is a planar, then the circuit forms a simple closed planar curve
- Add v_3, v_6 and the edges incident with them.
  - Intersection occurs (due to the Jordan curve Theorem).

Nonplanar Graph

EXAMPLE: The complete graph K_5 is not planar.

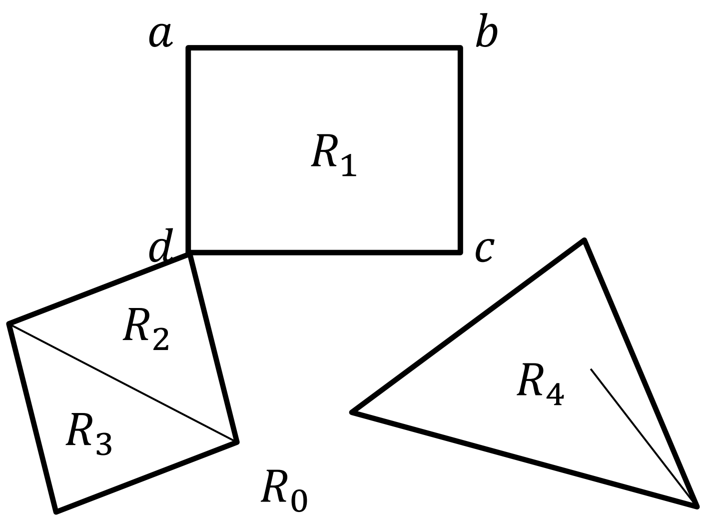


- $v_1, v_2, v_3, v_4, v_5, v_1$ is a simple closed curve in the planar representation of K_5
- Every remaining edge is in the interior region or in the exterior region
  - at least one is in the interior region
- No matter how you draw the remaining edges, crossing occurs.

Regions

DEFINITION: Let $G = (V, E)$ be a planar graph. Then the plane is divided into several **regions**面 by the edges of G.

- The infinite region is **exterior region**外部面. The others are **interior regions**内部面.
- The **boundary**边界 of a region is a subset of E.
- The **degree**度数 of a region is the number of edges on its boundary.
  - If an edge is shared by R_i, R_j, then it contributes 1 to $\deg(R_i), \deg(R_j)$
  - If an edge is on the boundary of a single region R_i, then it contributes 2 to $\deg(R_i)$



- The plane is divided into 5 regions R_0, R_1, R_2, R_3, R_4
  - R_0 is the exterior region
  - R_1, R_2, R_3, R_4 are interior regions
- The boundary of R_1; $\deg(R_1) = 4$
- There are 4 edges on the boundary of R_4
  - $\deg(R_4) = 1 + 1 + 1 + 2 = 5$ because one of the edges contribute 2 to $\deg(R_4)$
- $\deg(R_0) = 11, \deg(R_1) = 4, \deg(R_2) = 3, \deg(R_3) = 3, \deg(R_4) = 5$

Euler’s Formula

THEOREM: Let $G = (V, E)$ be a connected planar simple graph with e edges and v vertices. Let r be the number of regions in a planar representation of G. Then $r = e - v + 2$.

THEOREM: Let G be a planar simple graph with p connected components. Then $|V(G)| - |E(G)| + |R(G)| = p + 1$.

- Let $G_1, G_2, \ldots, G_p$ be the connected components of G.
  - By Euler’s formula, $|R(G_i)| = |E(G)_i| - |V(G_i)| + 2$ for all $i \in [p]$
- $|V(G)| = |V(G_1)| + |V(G_2)| + \cdots + |V(G_p)|$
- $|E(G)| = |E(G_1)| + |E(G_2)| + \cdots + |E(G_p)|$
- $|R(G)| = |R(G_1)| + |R(G_2)| + \cdots + \left|R\left(G_p\right)\right| - p + 1$
- $|V(G)| - |E(G)| + |R(G)| = \sum_{i=1}^{p}(|V(G_i)| - |E(G_i)| + |R(G_i)|) - p + 1$

$$= 2p - p + 1 = p + 1$$

Euler's Formula: Proof*

Proof of Euler's formula by induction on the number e of edges

• A simple connected planar graph with 0 edges has only one vertex and one face (unbounded). The relation $f = e - v + 2$ is satisfied.

• Suppose the relation is satisfied for all simple connected planar graphs with k edges.
Consider a simple connected planar graph G with $k + 1$ edges, $k \geq 0$. This graph can be seen as a simple connected planar graph G' with k edges (satisfying the relation by induction hypothesis) to which we add one edge. There are two ways to add an edge to G' to get G:

- either the two endpoints of the edge are already in G': in this case, adding the edge adds also one face,
- either only one of the endpoint is already in G': in this case, adding the edge adds also one vertex but no other face.

In both cases, the relation $f = e - v + 2$ is satisfied by G.

Application

THEOREM: Let G be a connected planar simple graph. If every region has degree $\geq l$ in a planar representation of G, then

then $|E(G)| \leq \frac{l}{l-2}(|V(G)| - 2)$.

- Let $R_1, \ldots R_t$ be the regions given by a planar representation of G $\quad //t = |R(G)|$
  - $\deg(R_i) \geq l$ for every $i = 1,2, \ldots, t$
- Let $r = \deg(R_1) + \deg(R_2) + \cdots + \deg(R_t)$. Then $r = 2|E(G)|$.
  - Every edge contributes 2 to r
    - If $e \in E$ is on the boundary of a single region R_i, then e contributes 2 to $\deg(R_i)$;
    - If $e \in E$ is shared by R_i and R_j, then e contributes 1 to $\deg(R_i)$ and 1 to $\deg(R_j)$;
- $2|E(G)| = r = \deg(R_1) + \deg(R_2) + \cdots + \deg(R_t) \geq lt = l|R(G)|$
- $|R(G)| = |E(G)| - |V(G)| + 2$
- Hence, $|E(G)| \leq \frac{l}{l-2}(|V(G)| - 2)$

Application

COROLLARY: Let G be a connected planar simple graph. If $|V(G)| \geq 3$, then $|E(G)| \leq 3|V(G)| - 6$.

- Every region has degree ≥ 3 in a planar representation of G
- Let $l = 3$ in the previous theorem
  - $|E(G)| \leq \frac{3}{3-2}(|V(G)| - 2) = 3|V(G)| - 6.$

EXAMPLE: The complete graph K_5 is not planar.

- $|E(K_5)| = \binom{5}{2} = 10, |V(K_5)| = 5, K_5$ is connected simple and of order ≥ 3
- $|E(K_5)| > 3|V(K_5)| - 6$
  - Hence, K_5 cannot be planar

COROLLARY: Let G be a connected planar simple graph. Then G has a vertex of degree ≤ 5.

- $|V(G)| < 3$: the statement is true.
- $|V(G)| \geq 3$: $\forall u \in V(G), \deg(u) \geq 6 \Rightarrow 2|E(G)| = \sum_u \deg(u) \geq 6|V(G)|$
  - G cannot be planar

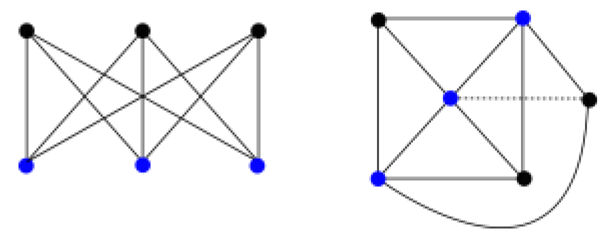
Application

COROLLARY: Let G be a connected planar simple graph. If $|V(G)| \geq 3$ and there is no circuits of length 3 in G, then $|E(G)| \leq 2|V(G)| - 4$.

- Let $R_1, \ldots R_t$ be the regions given by a planar representation of G $\quad //t = |R(G)|$
  - $\deg(R_i) \geq 4$ for every $i = 1,2, \ldots, t$
- Hence, $|E(G)| \leq \frac{4}{4-2}(|V(G)| - 2) = 2|V(G)| - 4$

EXAMPLE: The complete bipartite graph $K_{3,3}$ is not planar.

- $\left|E\left(K_{3,3}\right)\right| = 3 \times 3 = 9, \left|V\left(K_{3,3}\right)\right| = 3 + 3 = 6 \geq 3$
- $K_{3,3}$ is connected, simple and of order ≥ 3.
- There is no circuits of length 3 in $K_{3,3}$
- $\left|E\left(K_{3,3}\right)\right| = 9 > 8 = 2\left|V\left(K_{3,3}\right)\right| - 4$
- Hence, $K_{3,3}$ cannot be planar

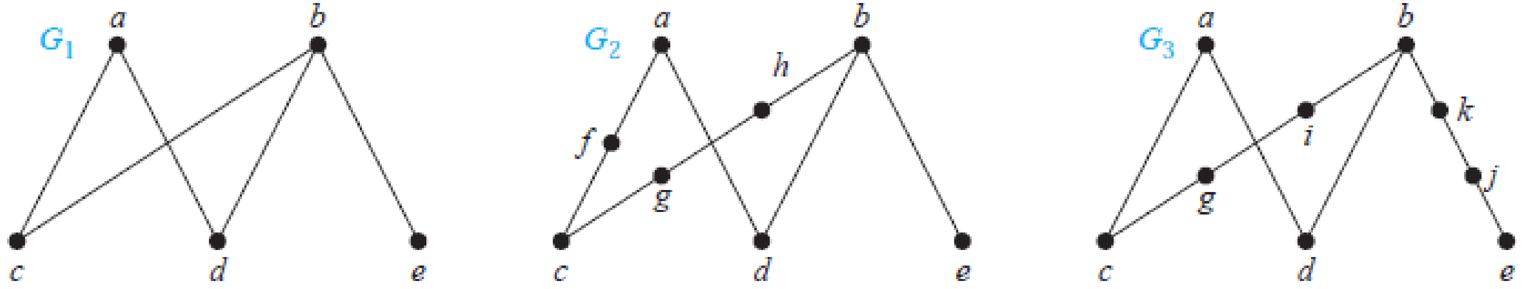


REMARKS: K_5 and $K_{3,3}$ are fundamental nonplanar graphs.

Homeomorphic

DEFINITION: Let $G = (V, E)$ be a graph and $\{u, v\} \in E$.

- **elementary subdivision**初等细分: $G' = (V \cup \{w\}, E - \{u, v\} + \{u, w\} + \{v, w\})$
- Two graphs are **homeomorphic**同胚的 if they can be obtained from the same graph via elementary subdivisions

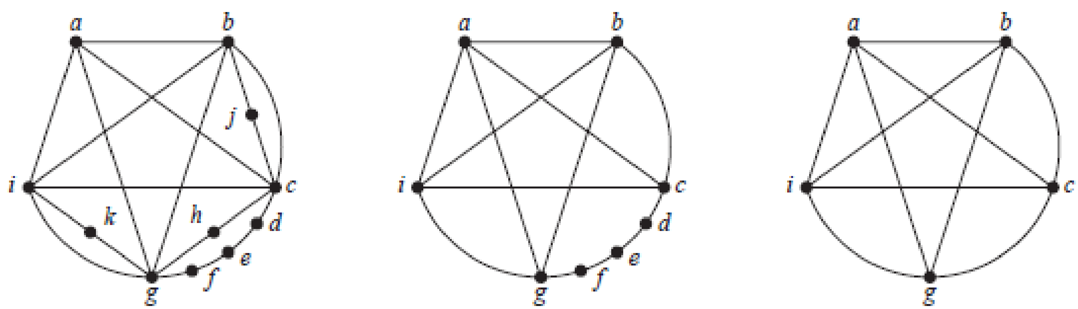


G_2 and G_3 are homeomorphic

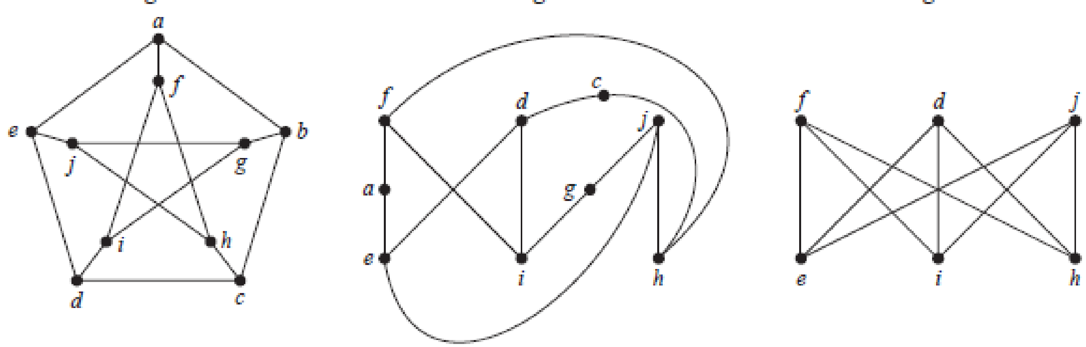
Kuratowski's Theorem

THEOREM: A graph G is nonplanar if and only if it has a subgraph homeomorphic to $K_{3,3}$ or K_5.

EXAMPLE: The following graph is nonplanar.



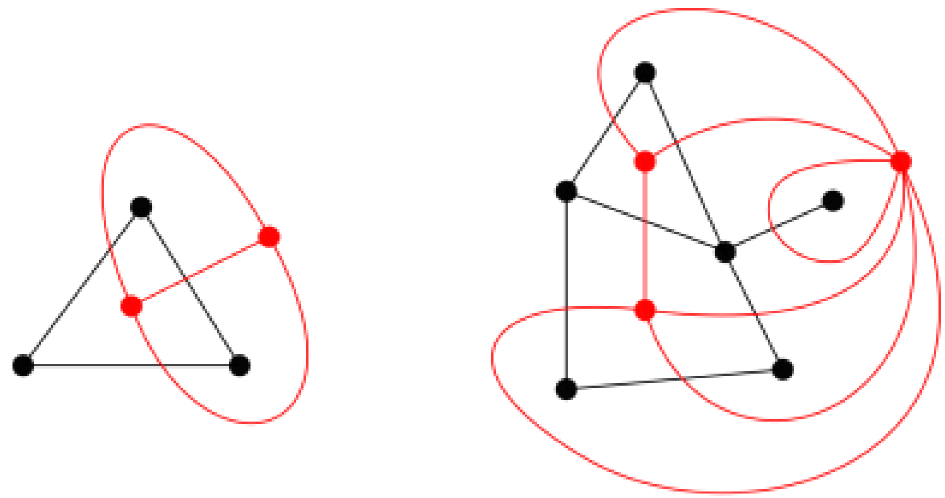
There is a subgraph homoemorphic to K_5



There is a subgraph homoemorphic to $K_{3,3}$

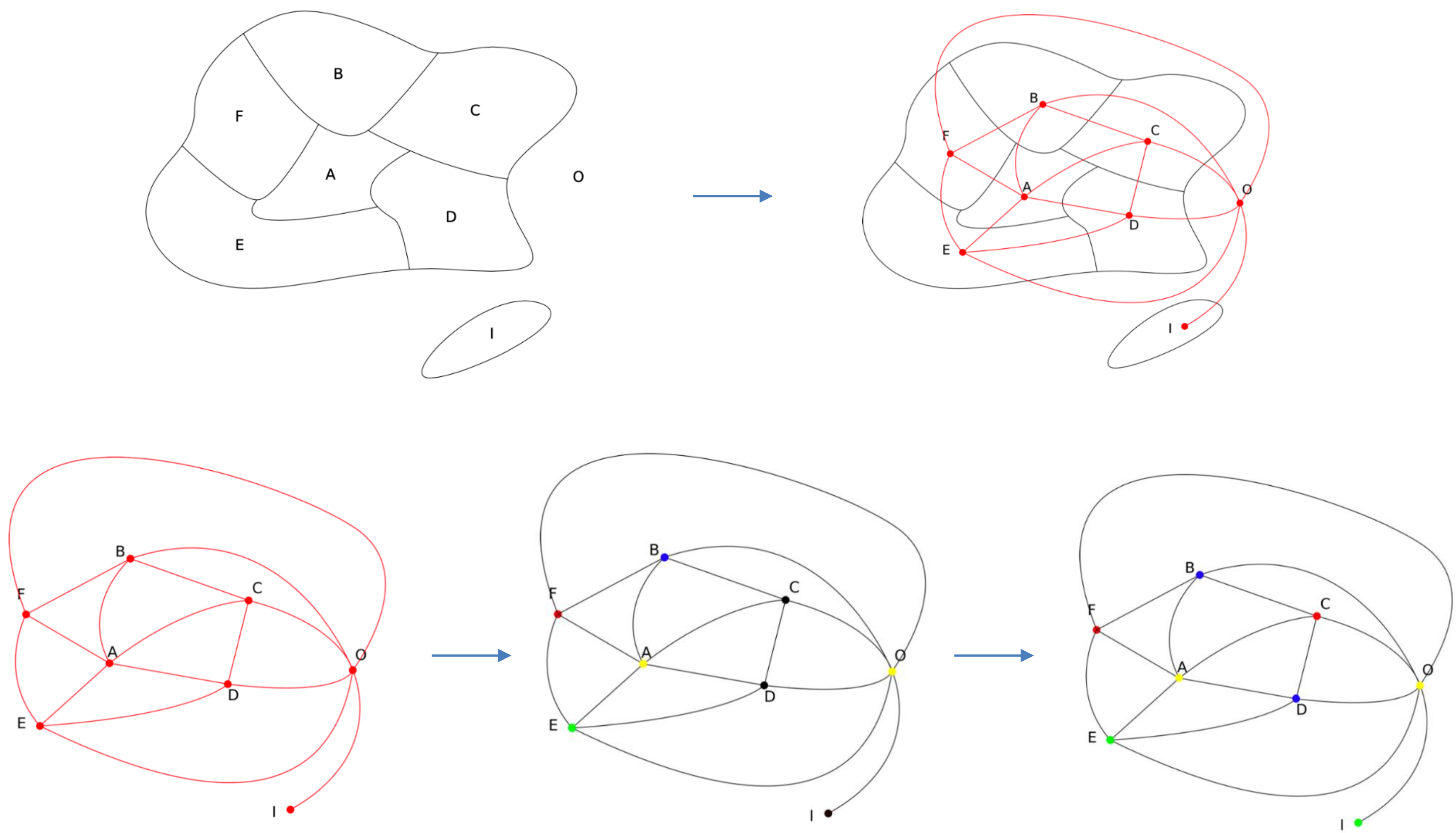
Dual Graph

Let G be a planar graph and assume we take a planar representation of G that we denote also G. The **dual of** G is the graph G^* that has a vertex for each face of G and an edge connecting two vertices if the corresponding faces in G have a common edge in their boundary.



Remark: The dual of a planar simple graph is not necessarily simple.

Coloring a Map

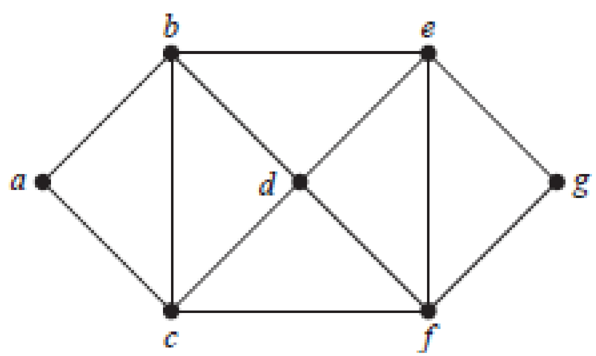


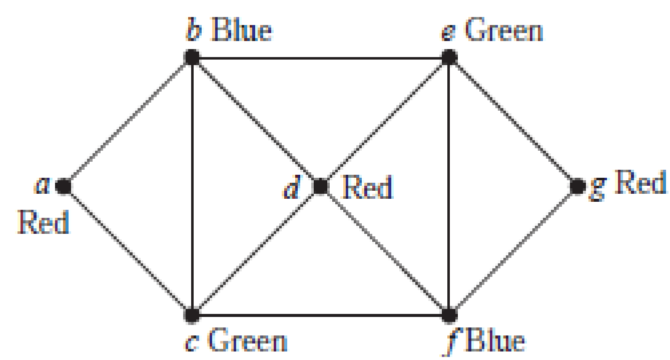
Coloring regions of the map $\Leftrightarrow$ Coloring vertices of the dual graph

Graph Coloring

DEFINITION: Let $G=(V,E)$ be a simple graph. A ***k*-coloring**$_{k\text{-着色}}$ of G is a map $f\colon V\to[k]$ such that $f(u)\neq f(v)$ whenever $\{u,v\}\in E$.

- **chromatic number ($\chi(G)$)**色数**:** the least k s.t. G has a k-coloring.





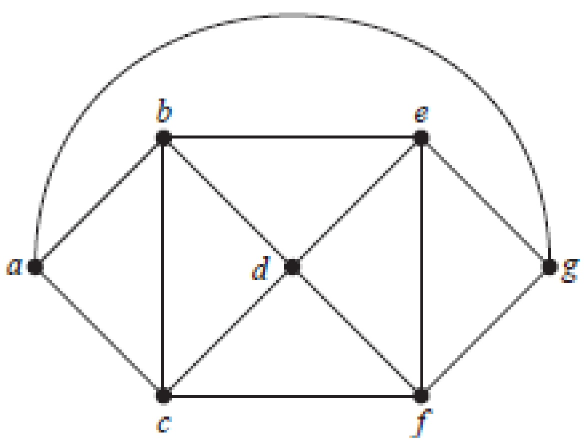
$\chi(G)=3$

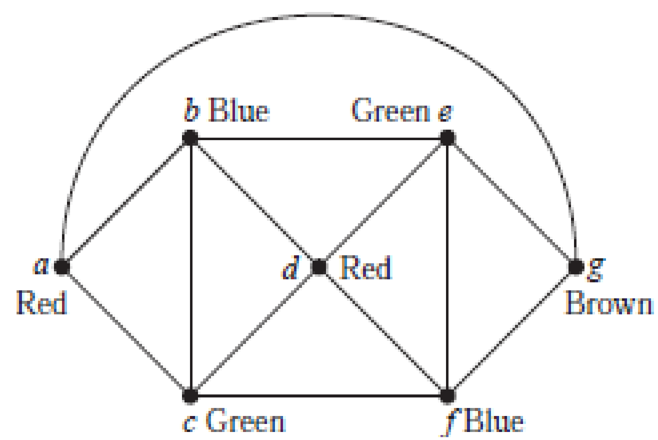
The chromatic number is at least 3 because *a; b; c* is a circuit of length 3

Graph Coloring

DEFINITION: Let $G = (V, E)$ be a simple graph. A ***k*-coloring**$_{k\text{-着色}}$ of G is a map $f: V \to [k]$ such that $f(u) \neq f(v)$ whenever $\{u, v\} \in E$.

- **chromatic number ($\chi(G)$)**$_{色数}$**:** the least k s.t. G has a k-coloring.



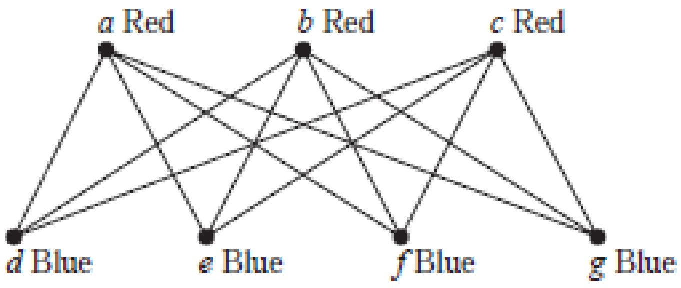


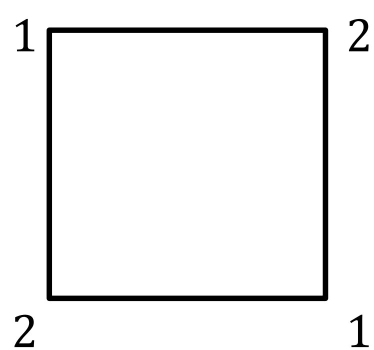
$\chi(G) = 4$

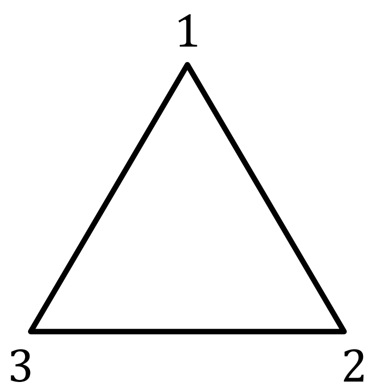
Graph Coloring

THEOREM: Let $G = (V, E)$ be a simple graph.

- $1 \leq \chi(G) \leq |V|$
- $\chi(G) = 1$ iff $E = \emptyset$
- $\chi(G) = 2$ iff G is bipartite and $|E| \geq 1$.
- $\chi(K_n) = n$ for every integer $n \geq 1$.
    - $\chi(G) \geq n$ if G has a subgraph isomorphic to K_n
- $\chi(C_n) = 2$ if $2|n$; $\chi(C_n) = 3$ if $2|(n-1)$; $(n \geq 3)$
- $\chi(G) \leq \Delta(G) + 1$, where $\Delta(G) = \max\{\deg(v): v \in V\}$.

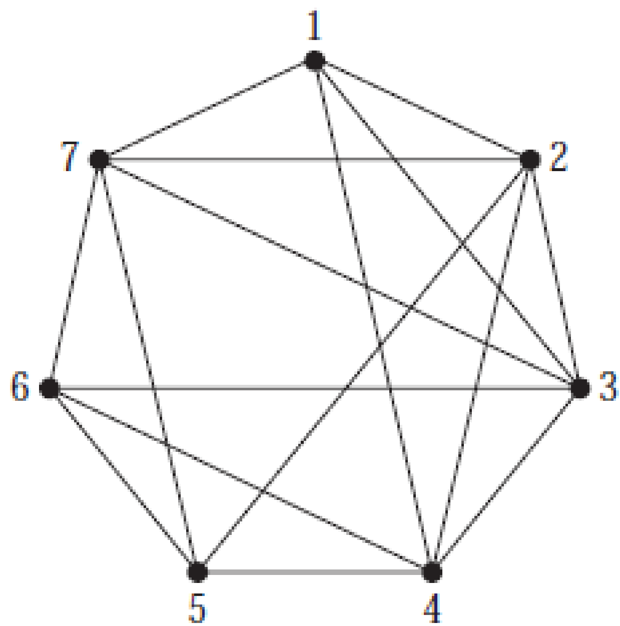


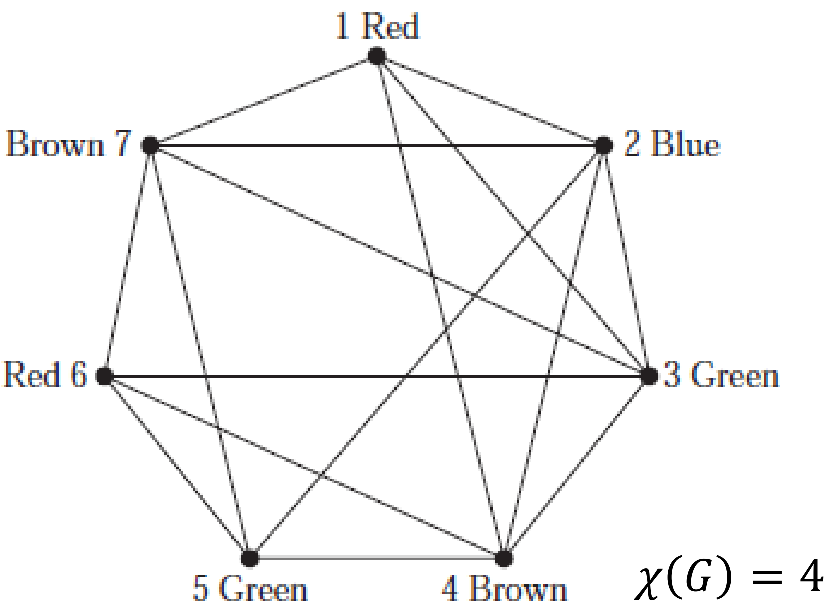




Application

PROBLEM: How can the final exams at a university be scheduled so that no student has two exams at the same time?





- There are 7 different courses, they are vertices of a graph.
- Two courses are adjacent if there is a student registered both courses.
- Choose time slots for the courses such that no two adjacent courses take place at the same time.
  - $\chi(G)$ time slots is needed.

$1 \leq \chi(G) \leq 7$

$1 \leq \chi(G) \leq \Delta(G) + 1 = 6$

$\chi(G) \geq 4$: G has a subgraph isomorphic to K_4

4-coloring Theorem

Theorem (Four coloring Theorem)

The chromatic number of a simple planar graph is no greater than 4.

Remarks: The proof of the 4-coloring Theorem depends on a computer. The two previous theorems are true for planar graphs only. A non planar graph can have an arbitrarily large chromatic number.