SI231b: Matrix Computations

Fall 2020-21 - Midterm Exam

10:20 AM - 12:20 PM, Thursday, Dec. 17th, 2020

12 pages, 6 questions, and 120 points (20 points for bonus) in total

(NOTE: your exam grade will be counted by $\min\{100,$ "test paper grade"$\}$)

Problem 1 *(10 + 5 points)*

1) Given a matrix as follows:

$$\mathbf{A} = \begin{bmatrix} -3 & 1 & -2 \\ 1 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix},$$

derive the LU decomposition of $\mathbf{A}$.

(Hint: Methods for computing the LU decomposition are not restricted as long as necessary derivation steps are shown.)

Based on the LU decomposition, solve the following multiple linear systems

$$\mathbf{A}\mathbf{x}_i = \mathbf{b}_i \quad \text{for} \quad i = 1, 2, 3,$$

where

$$\mathbf{b}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad \mathbf{b}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

2) Given $\mathbf{A}$ as in 1), compute the condition number $\kappa_p(\mathbf{A})$ with respect to

- induced matrix 1-norm (i.e., $p = 1$)
- induced matrix ∞-norm (i.e., $p = \infty$) .

Your answer:

Reference solution:

1) **LU decomposition method**. Suppose that LU decomposition of $\mathbf{A}$ is given by $\mathbf{A} = \mathbf{LU}$, where

$$\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ \ell_{21} & 1 & 0 \\ \ell_{31} & \ell_{32} & 1 \end{bmatrix} \text{ and } \mathbf{U} = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix},$$

L: 2 points U: 2 points. Both of them are correct will get 5 points

Then we can have

$$\mathbf{LU} = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ \ell_{21}u_{11} & \ell_{21}u_{12} + u_{22} & \ell_{21}u_{13} + u_{23} \\ \ell_{31}u_{11} & \ell_{31}u_{12} + \ell_{32}u_{22} & \ell_{31}u_{13} + \ell_{32}u_{23} + u_{33} \end{bmatrix} = \begin{bmatrix} -3 & 1 & -2 \\ 1 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix},$$

so

$$\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{3} & 1 & 0 \\ -\frac{1}{3} & -\frac{1}{2} & 1 \end{bmatrix} \text{ and } \mathbf{U} = \begin{bmatrix} -3 & 1 & -2 \\ 0 & \frac{4}{3} & \frac{1}{3} \\ 0 & 0 & -\frac{1}{2} \end{bmatrix}.$$

Solving the lower triangular system $\mathbf{L}\mathbf{y}_i = \mathbf{b}_i$ firstly, we can get

$$\mathbf{y}_1 = \begin{bmatrix} 1 \\ \frac{1}{3} \\ \frac{1}{2} \end{bmatrix}, \quad \mathbf{y}_2 = \begin{bmatrix} 0 \\ 1 \\ \frac{1}{2} \end{bmatrix}, \quad \text{and} \quad \mathbf{y}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

and then solve the upper triangular system $\mathbf{U}\mathbf{x}_i = \mathbf{y}_i$,

$$\mathbf{x}_1 = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -1 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \quad \text{and} \quad \mathbf{x}_3 = \begin{bmatrix} \frac{3}{2} \\ \frac{1}{2} \\ -2 \end{bmatrix}.$$

Each result is 2 points. All of them are correct will get 5 points

2) Based on the result in 1) (or performing basic row operations on the augmented matrix $[\mathbf{A}, \mathbf{I}]$ [1]), we obtain the inverse of $\mathbf{A}$ as

$$\mathbf{A}^{-1} = [\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3] = \begin{bmatrix} \frac{1}{2} & 1 & \frac{3}{2} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ -1 & -1 & -2 \end{bmatrix}.$$

Thus we have

$$\|\mathbf{A}\|_1 = \max_j \sum_{i=1}^{3} |\mathbf{A}_{ij}| = 3 + 1 + 1 = 5,$$
$$\|\mathbf{A}\|_\infty = \max_i \sum_{j=1}^{3} |\mathbf{A}_{ij}| = 3 + 1 + 2 = 6.$$

Similarly, we have

$$\|\mathbf{A}^{-1}\|_1 = \frac{3}{2} + \frac{1}{2} + 2 = 4,$$
$$\|\mathbf{A}^{-1}\|_\infty = 1 + 1 + 2 = 4.$$

So the condition numbers with respect to 1-norm and ∞-norm are given by

$$\kappa_1(\mathbf{A}) = \|\mathbf{A}\|_1 \|\mathbf{A}^{-1}\|_1 = 5 \times 4 = 20,$$
$$\kappa_\infty(\mathbf{A}) = \|\mathbf{A}\|_\infty \|\mathbf{A}^{-1}\|_\infty = 6 \times 4 = 24.$$

Each result is 2 points. Both of them are correct will get 5 points

[1] Here $[\mathbf{A}, \mathbf{B}]$ denotes a new matrix combined by $\mathbf{A}$ and $\mathbf{B}$. For example, $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} b_{11} \\ b_{21} \end{bmatrix}$, then $[\mathbf{A}, \mathbf{B}] = \begin{bmatrix} a_{11} & a_{12} & b_{11} \\ a_{21} & a_{22} & b_{21} \end{bmatrix}$.

Problem 2 *(10 + 15 points)*

1) Given a matrix as follows:

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 2 \end{bmatrix},$$

derive the thin QR decomposition of $\mathbf{A}$.

(Hint: Methods for computing the QR decomposition are not restricted as long as necessary derivation steps are shown.)

Based on the QR decomposition, solve the following linear system

$$\mathbf{A}\mathbf{x} = \mathbf{b},$$

where

$$\mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}.$$

2) Consider two full-column rank matrices $\mathbf{A} \in \mathbb{R}^{m \times n_1}$ and $\mathbf{B} \in \mathbb{R}^{m \times n_2}$ with $n_1 < m$, $n_2 < m$. Let $\mathcal{R}(\mathbf{A})$ and $\mathcal{R}(\mathbf{B})$ denote the range spaces of $\mathbf{A}$ and $\mathbf{B}$, respectively, and suppose $\mathcal{R}(\mathbf{A})^\perp \cap \mathcal{R}(\mathbf{B})^\perp = \{\mathbf{0}\}$. Try to find a semi-orthogonal matrix $\mathbf{Q}$ (i.e., a matrix with orthonormal columns) based on QR decomposition methods, where the columns of $\mathbf{Q}$ form an orthonormal basis for the $\mathcal{R}(\mathbf{A}) \cap \mathcal{R}(\mathbf{B})$.

Hint:

- *Given a full-column rank matrix $\mathbf{M} \in \mathbb{R}^{m \times n}$ with $n < m$, its QR decomposition is given by*

$$\mathbf{M} = \mathbf{Q}\mathbf{R} = \begin{bmatrix} \mathbf{Q}_1, \mathbf{Q}_2 \end{bmatrix} \begin{bmatrix} \mathbf{R}_1 \\ \mathbf{0} \end{bmatrix}.$$

- *The $\mathcal{R}(\mathbf{A}) \cap \mathcal{R}(\mathbf{B})$ is orthogonal to its orthogonal compliment $(\mathcal{R}(\mathbf{A}) \cap \mathcal{R}(\mathbf{B}))^\perp = \mathcal{R}(\mathbf{A})^\perp + \mathcal{R}(\mathbf{B})^\perp$, where the notation $\mathcal{S}^\perp$ denotes the orthogonal complement of a subspace $\mathcal{S}$.*

Your answer:

Reference solution:

1) **QR decomposition method**. Suppose $\mathbf{A} = [\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2]$, applying the Gauss-Schmidt method, we have

- For $i = 1$, $\|\boldsymbol{\alpha}_1\|_2 = \sqrt{14}$, therefore $\boldsymbol{\beta}_1 = \frac{1}{\sqrt{14}} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. 2 points

- For $i = 2$, $\widetilde{\boldsymbol{\beta}}_2 = \boldsymbol{\alpha}_2 - (\boldsymbol{\beta}_1^T \boldsymbol{\alpha}_2)\boldsymbol{\beta}_1 = \frac{1}{7} \begin{bmatrix} -4 \\ -1 \\ 2 \end{bmatrix}$, therefore $\boldsymbol{\beta}_2 = \frac{1}{\sqrt{21}} \begin{bmatrix} -4 \\ -1 \\ 2 \end{bmatrix}$. 4 points

Therefore, we obtain the QR decomposition $\mathbf{A} = \mathbf{Q}\mathbf{R}$ with

$$\mathbf{Q} = \begin{bmatrix} \frac{1}{\sqrt{14}} & \frac{-4}{\sqrt{21}} \\ \frac{2}{\sqrt{14}} & \frac{-1}{\sqrt{21}} \\ \frac{3}{\sqrt{14}} & \frac{2}{\sqrt{21}} \end{bmatrix} \text{6 points and } \mathbf{R} = \mathbf{Q}^T\mathbf{A} = \begin{bmatrix} \sqrt{14} & \frac{8}{\sqrt{14}} \\ 0 & \frac{3}{\sqrt{21}} \end{bmatrix} .\text{8 points}$$

It is easy to check that $\mathbf{b} \notin \mathcal{R}(\mathbf{A})$, therefore, there is no solution for $\mathbf{A}\mathbf{x} = \mathbf{b}$. The least square solution is given by solving the upper triangular system

$$\mathbf{R}\mathbf{x}_{LS} = \mathbf{Q}^T\mathbf{b} = \begin{bmatrix} \frac{-5}{\sqrt{14}} \\ \frac{-8}{\sqrt{21}} \end{bmatrix} \quad \Longrightarrow \quad \mathbf{x}_{LS} = \begin{bmatrix} \frac{7}{6} \\ \frac{-8}{3} \end{bmatrix} .\text{10 points}$$

Full points are given only if you explicitly write $\mathbf{x}_{LS}$ or "No solution".

2) Denote the subspace

$$\mathcal{T} = (\mathcal{R}(\mathbf{A}) \cap \mathcal{R}(\mathbf{B}))^\perp = \mathcal{R}(\mathbf{A})^\perp + \mathcal{R}(\mathbf{B})^\perp.$$

Therefore, to find an orthonormal basis for $\mathcal{R}(\mathbf{A}) \cap \mathcal{R}(\mathbf{B})$ is equivalent to find an orthonormal basis for $\mathcal{T}^\perp$. Let the QR decomposition for $\mathbf{A}$ and $\mathbf{B}$ be

$$\mathbf{A} = \mathbf{Q}^{(\mathbf{A})}\mathbf{R}^{(\mathbf{A})} = \left[\mathbf{Q}_1^{(\mathbf{A})}, \mathbf{Q}_2^{(\mathbf{A})}\right] \begin{bmatrix} \mathbf{R}_1^{(\mathbf{A})} \\ \mathbf{0} \end{bmatrix}, \quad \mathbf{B} = \mathbf{Q}^{(\mathbf{B})}\mathbf{R}^{(\mathbf{B})} = \left[\mathbf{Q}_1^{(\mathbf{B})}, \mathbf{Q}_2^{(\mathbf{B})}\right] \begin{bmatrix} \mathbf{R}_1^{(\mathbf{B})} \\ \mathbf{0} \end{bmatrix},$$

where $\mathbf{Q}_2^{(\mathbf{A})} \in \mathbb{R}^{m \times (m-n_1)}$ and $\mathbf{Q}_2^{(\mathbf{B})} \in \mathbb{R}^{m \times (m-n_2)}$ are orthonormal basis for $\mathcal{R}(\mathbf{A})^\perp$ and $\mathcal{R}(\mathbf{B})^\perp$, respectively. Let $\mathbf{C} = [\mathbf{Q}_2^{(\mathbf{A})}, \mathbf{Q}_2^{(\mathbf{B})}] \in \mathbb{R}^{m \times (m-n_1+m-n_2)}$. First we have

$$\mathcal{T} = \mathcal{R}(\mathbf{C})\,. \qquad \text{(10 points)}$$

Second, since

$$\begin{aligned} \dim(\mathcal{T}) &= \dim(\mathcal{R}(\mathbf{A})^\perp + \mathcal{R}(\mathbf{B})^\perp) \\ &= \dim(\mathcal{R}(\mathbf{A})^\perp) + \dim(\mathcal{R}(\mathbf{B})^\perp) - \dim(\mathcal{R}(\mathbf{A})^\perp \cap \mathcal{R}(\mathbf{B})^\perp) \\ &= (m - n_1) + (m - n_2) - 0\,, \end{aligned}$$

columns of $\mathbf{C}$ are linearly independent and consist of a set of basis for $\mathcal{T}$, then let the QR decomposition for $\mathbf{C}$ be

$$\mathbf{C} = \mathbf{Q}^{(\mathbf{C})}\mathbf{R}^{(\mathbf{C})} = \left[\mathbf{Q}_1^{(\mathbf{C})}, \mathbf{Q}_2^{(\mathbf{C})}\right] \begin{bmatrix} \mathbf{R}_1^{(\mathbf{C})} \\ \mathbf{0} \end{bmatrix},$$

where $\mathbf{Q}_1^{(\mathbf{C})} \in \mathbb{R}^{m \times (2m-n_1-n_2)}$ is a set of orthonormal basis for $\mathcal{T}$. And $\mathbf{Q}_2^{(\mathbf{C})}$ is an orthonormal basis for $\mathcal{T}^\perp$, which is the desired $\mathbf{Q}$. (15 points)

Problem 3 *(10 points)*

Prove that $\mathbf{A}$ is a positive definite matrix if and only if it admits a unique Cholesky decomposition as $\mathbf{A} = \mathbf{G}\mathbf{G}^T$, where $\mathbf{G}$ is a nonsingular lower-triangular matrix.

(Hint: You may use this result: for any nonsingular symmetric matrix $\mathbf{A}$, it admits an LDL decomposition as $\mathbf{A} = \mathbf{L}\mathbf{D}\mathbf{L}^T$, where $\mathbf{L}$ is a lower triangular matrix and $\mathbf{D}$ is a diagonal matrix with positive diagonal elements.)

Your answer:

Reference solution: We will prove the theorem from two directions.

1) First we try to prove that if $\mathbf{A}$ is a positive definite matrix, then its Cholesky decomposition exists and it is unique. If $\mathbf{A}$ is PD, then it is nonsingular because let $\mathbf{x} \in \mathcal{N}(\mathbf{A})$, then $\mathbf{A}\mathbf{x} = \mathbf{0}$ and therefore $\mathbf{x}^T\mathbf{A}\mathbf{X} = 0$. Hence $\mathbf{x} = \mathbf{0}$ since $\mathbf{A}$ is PD, which implies that $\mathcal{N}(\mathbf{A}) = \{\mathbf{0}\}$ and consequently $\mathbf{A}$ is nonsingular. Then nonsingular symmetric matrix $\mathbf{A}$ must have LDL decomposition $\mathbf{A} = \mathbf{L}\mathbf{D}\mathbf{L}^T$. For any vector $\mathbf{x} \in \mathbb{R}^n$, there exists a vector $\mathbf{y} \in \mathbb{R}^n$ such that $\mathbf{y} = \mathbf{L}^T\mathbf{x}$. Since $\mathbf{A}$ is a positive definite matrix, we can derive that
$$\mathbf{y}^T\mathbf{D}\mathbf{y} = \mathbf{x}^T\mathbf{L}\mathbf{D}\mathbf{L}^T\mathbf{x} = \mathbf{x}^T\mathbf{A}\mathbf{x} > 0.$$
Hence, the diagonal entries of $\mathbf{D}$ are all positive. Let $\mathbf{G} = \mathbf{L}\mathbf{D}^{1/2}$ yields the Cholesky decomposition (3 points).
To prove the uniqueness, let
$$\mathbf{A} = \mathbf{G}_1\mathbf{G}_1^T = \mathbf{G}_2\mathbf{G}_2^T\,,$$
where $\mathbf{G}_1, \mathbf{G}_2$ are both nonsingular, then
$$(\mathbf{G}_2^T)^{-1}\mathbf{G}_1^T = \mathbf{G}_1^{-1}\mathbf{G}_2\,,$$
the left hand is upper triangular and the right hand is lower triangular. Let $\mathbf{G}_0 = \mathbf{G}_1^{-1}\mathbf{G}_2$, then $\mathbf{G}_0$ is a diagonal matrix. By $\mathbf{G}_1\mathbf{G}_1^T = \mathbf{G}_2\mathbf{G}_2^T$, we can also derive that
$$\mathbf{I} = \mathbf{G}_1^{-1}\mathbf{G}_2\mathbf{G}_2^T(\mathbf{G}_1^T)^{-1} = \mathbf{G}_0\mathbf{G}_0^T\,,$$
Hence, the diagonal entries of $\mathbf{G}_0$ must be 1 or -1. Since the diagonal entries of $\mathbf{G}_1$ and $\mathbf{G}_2$ are required to be positive, the diagonal entries of $\mathbf{G}_0$ can only be 1. Accordingly, $\mathbf{G}_1 = \mathbf{G}_2$, which concludes the uniqueness.
2) Second, we try to prove that if the Cholesky decomposition for $\mathbf{A}$ exists, i.e., there exists $\mathbf{G}$ such that $\mathbf{A} = \mathbf{G}\mathbf{G}^T$, then $\mathbf{A}$ is PD (3 points).
   a) First we have $\mathbf{A}^T = (\mathbf{G}\mathbf{G}^T)^T = \mathbf{G}\mathbf{G}^T = \mathbf{A}$, therefore $\mathbf{A}$ is symmetric (2 points).
   b) For any $\mathbf{x} \in \mathbb{R}^n$ with $\mathbf{x} \neq 0$, $\mathbf{x}^T\mathbf{A}\mathbf{x} = \mathbf{x}^T\mathbf{G}\mathbf{G}^T\mathbf{x} = \|\mathbf{G}^T\mathbf{x}\|_2^2 > 0$ (2 points).

   Hence $\mathbf{A}$ is PD.

Problem 4 *(10 + 10 + 10 points)*

1) Prove that a real symmetric matrix with a positive diagonal entry has at least one positive eigenvalue.
2) A real symetric matrix is given as follows:

$$\mathbf{S} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

Compute its eigenvalues, eigenvectors, and eigendecomposition.

3) A matrix is given as follows:

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

Based on the realtion between SVD and the eigendecomposition for a real symmetric matrix, figure out the SVD of $\mathbf{A}$ as $\mathbf{A} = \mathbf{U\Sigma V}^T$ where $\mathbf{U} \in \mathbb{R}^{2\times 2}$, $\mathbf{\Sigma} \in \mathbb{R}^{2\times 3}$, and $\mathbf{V} \in \mathbb{R}^{3\times 3}$.

Your answer:

Reference solution:

1). (**Method 1**) Let $\mathbf{A}$ be a $n \times n$ real symmetric matrix. Its Rayleigh quotient is defined as $R(\mathbf{A}, \mathbf{v}) = \frac{\mathbf{v}^T\mathbf{A}\mathbf{v}}{\mathbf{v}^T\mathbf{v}}$. (2 points)

- For any $i \in \{1, 2, \dots, n\}$,

$$\mathbf{A}_{ii} = \mathbf{e}_i^T\mathbf{A}\mathbf{e}_i = \frac{\mathbf{e}_i^T\mathbf{A}\mathbf{e}_i}{\mathbf{e}_i^T\mathbf{e}_i} = R(\mathbf{A}, \mathbf{e}_i).\text{(3points)}$$

- Let λ_1 be the largest eigenvalue of $\mathbf{A}$. Since $\mathbf{v}^T\mathbf{A}\mathbf{v} \leq \mathbf{v}^T(\lambda_1\mathbf{v}) = \lambda_1 \cdot \mathbf{v}^T\mathbf{v}$, we have $R(\mathbf{A}, \mathbf{v}) \leq \lambda_1$ with equality attained when $\mathbf{v}$ is chosen as the eigenvector corresponding to λ_1. (**Rayleigh-Ritz Theorem or maximum principle can be directly used here.**) (4 points)

Then we have $\lambda_1 \geq \mathbf{A}_{ii}$ for any $i \in \{1, 2, \dots, n\}$. If there is a positive diagonal entry of $\mathbf{A}$, λ_1 must be therefore positive, which concludes the proof. (1 points)

(**Method 2**) We prove this by contradiction. For a real symmetric matrix $\mathbf{A}$, let $\mathbf{A} = \mathbf{V}^T\mathbf{\Lambda}\mathbf{V}$ denote its eigendecomposition, where $\mathbf{\Lambda} = \mathrm{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$ (λ_1 is the largest eigenvalue of $\mathbf{A}$) and $\mathbf{V} = [\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n]$. Assume that $\mathbf{A}$ has no positive eigenvalue, *i.e.*, $\forall i \in \{1, 2, \dots, n\}$, $\lambda_i \leq 0$. (2 points) For all $i \in \{1, 2, \dots, n\}$,

$$\begin{aligned} \mathbf{A}_{ii} &= \mathbf{e}_i^T\mathbf{A}\mathbf{e}_i = \mathbf{e}_i^T\mathbf{V}^T\mathbf{\Lambda}\mathbf{V}\mathbf{e}_i \text{ (3 points)} \\ &= \mathbf{v}_i^T \cdot \mathbf{\Lambda} \cdot \mathbf{v}_i \\ &\leq \mathbf{v}_{ii}^2 \cdot \lambda_1 \leq 0, \text{ (5 points)} \end{aligned}$$

which contradicts to that $\mathbf{A}$ has at least one positive diagonal entry. Hence, we conclude that a real symmetric matrix with a positive entry has at least one positive eigenvalue.

2). The characteristic equation is $\det(\mathbf{S} - \lambda\mathbf{I}) = 0$, i.e.,

$$(1-\lambda)^3 + (\lambda - 1) = \lambda(\lambda - 2)(1 - \lambda) = 0,$$

so that $\lambda = 0, 1, 2$.

Correct simplified characteristic equation is 1 point, each eigenvalue is 1 points.

Now we obtain the normalized eigenvectors:

- for $\lambda = 0$,
$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Longrightarrow \mathbf{v}_0 = \begin{bmatrix} \sqrt{2}/2 \\ 0 \\ -\sqrt{2}/2 \end{bmatrix} \; or \; \begin{bmatrix} -\sqrt{2}/2 \\ 0 \\ \sqrt{2}/2 \end{bmatrix};$$
- for $\lambda = 1$,
$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \Longrightarrow \mathbf{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix};$$
- for $\lambda = 2$,
$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 2v_1 \\ 2v_2 \\ 2v_3 \end{bmatrix} \Longrightarrow \mathbf{v}_2 = \begin{bmatrix} \sqrt{2}/2 \\ 0 \\ \sqrt{2}/2 \end{bmatrix}.$$

  Then
$$\mathbf{V} = [\mathbf{v}_2, \mathbf{v}_1, \mathbf{v}_0] = \begin{bmatrix} \sqrt{2}/2 & 0 & \sqrt{2}/2 \\ 0 & 1 & 0 \\ \sqrt{2}/2 & 0 & -\sqrt{2}/2 \end{bmatrix} \; or \; \begin{bmatrix} \sqrt{2}/2 & 0 & -\sqrt{2}/2 \\ 0 & 1 & 0 \\ \sqrt{2}/2 & 0 & \sqrt{2}/2 \end{bmatrix}.$$

  The eigendecompostion of $\mathbf{S}$ is give by
$$\mathbf{S} = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^T,$$
  where $\mathbf{\Lambda} = \text{Diag}(2, 1, 0)$.

Each correct eigenvector is 2 points, each non-normalized eigenvector will penalize 1 point.

3). Let the SVD be $\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^T$ with $\mathbf{U} \in \mathbb{R}^{3\times3}$, $\mathbf{\Sigma} \in \mathbb{R}^{3\times2}$ and $\mathbf{V} \in \mathbb{R}^{2\times2}$.

- For $\mathbf{A}^T\mathbf{A}$ (this computes $\mathbf{V}$ and $\mathbf{\Sigma}$),
$$\mathbf{A}^T\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \mathbf{S}. \text{1 point.}$$

  So based on the results in 2), the matrix $\mathbf{V}$ is given as
$$\mathbf{V} = \begin{bmatrix} \sqrt{2}/2 & 0 & \sqrt{2}/2 \\ 0 & 1 & 0 \\ \sqrt{2}/2 & 0 & -\sqrt{2}/2 \end{bmatrix} \; or \; \begin{bmatrix} \sqrt{2}/2 & 0 & -\sqrt{2}/2 \\ 0 & 1 & 0 \\ \sqrt{2}/2 & 0 & \sqrt{2}/2 \end{bmatrix}.$$

  Each correct eigenvector is 1 point.

- For $\mathbf{A}\mathbf{A}^T$ (this computes $\mathbf{U}$),

$$\mathbf{A}\mathbf{A}^T = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \text{1 points.}$$

which is a diagonal matrix, and the singular values are $\sigma_1 = \sqrt{2}$ and $\sigma_2 = 1$. So the matrix $\mathbf{U}$ is the identity matrix and

$$\mathbf{\Sigma} = \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} . \text{3 points.}$$

Each correct eigenvector is 1 point.

- Therefore, the SVD of $\mathbf{A}$ is

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \sqrt{2}/2 & 0 & \sqrt{2}/2 \\ 0 & 1 & 0 \\ \sqrt{2}/2 & 0 & -\sqrt{2}/2 \end{bmatrix} or \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \sqrt{2}/2 & 0 & \sqrt{2}/2 \\ 0 & 1 & 0 \\ -\sqrt{2}/2 & 0 & \sqrt{2}/2 \end{bmatrix} .$$

Problem 5 *(10 + 10 points)*

For any matrix $\mathbf{A} \in \mathbb{R}^{m\times n}$, we can define a unique pseudo-inverse of $\mathbf{A}$ as $\mathbf{A}^\dagger \in \mathbb{R}^{n\times m}$.

Hint: You can directly use the following results for this problem:

- *Given the SVD of* $\mathbf{A}$ *with rank*$(\mathbf{A}) = r$ *as*
$$\mathbf{A} = \begin{bmatrix}\mathbf{U}_1 & \mathbf{U}_2\end{bmatrix}\begin{bmatrix}\tilde{\mathbf{\Sigma}} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\mathbf{V}_1^T\\ \mathbf{V}_2^T\end{bmatrix},$$
where $\tilde{\mathbf{\Sigma}} \in \mathbb{R}^{r\times r}$, *we have*
$$\mathbf{A}^\dagger = \mathbf{V}_1\tilde{\mathbf{\Sigma}}^{-1}\mathbf{U}_1^T.$$
- $\mathbf{A}^\dagger$ *satisfies* $\mathbf{A}\mathbf{A}^\dagger\mathbf{A} = \mathbf{A}$, $\mathbf{A}^\dagger\mathbf{A}\mathbf{A}^\dagger = \mathbf{A}^\dagger$, $(\mathbf{A}\mathbf{A}^\dagger)^T = \mathbf{A}\mathbf{A}^\dagger$, *and* $(\mathbf{A}^\dagger\mathbf{A})^T = \mathbf{A}^\dagger\mathbf{A}$.

1) Prove that $\mathbf{A}^\dagger\mathbf{A}$ is the orthogonal projector onto $\mathcal{N}(\mathbf{A})^\perp$, or equivalently $\mathcal{R}(\mathbf{A}^T)$.
   (Hint: You need to prove that $\mathbf{A}^\dagger\mathbf{A}$ *is an orthogonal projector first. A matrix* $\mathbf{P}$ *is an orthogonal projector if it is symmetric and idempotent (i.e.,* $\mathbf{P}^2 = \mathbf{P}$*).)*
2) Let $\mathbf{A} \in \mathbb{R}^{m\times n}$ and $\mathbf{b} \in \mathbb{R}^n$, the least squares problem is given by
$$\min_{\mathbf{x}\in\mathbb{R}^n} \|\mathbf{A}\mathbf{x} - \mathbf{b}\|_2^2, \tag{LS}$$
   the solution to which has the form of
$$\mathbf{x}^\star = \mathbf{A}^\dagger\mathbf{b} + \boldsymbol{\eta}, \quad \text{with} \quad \boldsymbol{\eta} \in \mathcal{N}(\mathbf{A}).$$
   Prove that $\mathbf{A}^\dagger\mathbf{b}$ is the solution to (LS) of minimum 2-norm.

Your answer:

Reference solution:

1) First, $\mathbf{A}^\dagger\mathbf{A}$ is an orthogonal projection since
$$(\mathbf{A}^\dagger\mathbf{A})(\mathbf{A}^\dagger\mathbf{A}) = (\mathbf{A}^\dagger\mathbf{A}\mathbf{A}^\dagger)(\mathbf{A}) = \mathbf{A}^\dagger\mathbf{A}\,,$$
   and $(\mathbf{A}^\dagger\mathbf{A})^T = \mathbf{A}^\dagger\mathbf{A}$. (5 points)
   To see $\mathbf{A}^\dagger\mathbf{A}$ is the orthogonal projection onto the orthogonal complement of $\mathcal{N}(\mathbf{A})$, we simply show that $\mathcal{R}(\mathbf{A}^\dagger\mathbf{A}) = \mathcal{N}(\mathbf{A})^\perp$, which is equivalent to show $\mathcal{N}((\mathbf{A}^\dagger\mathbf{A})^T) = \mathcal{N}(\mathbf{A}) = \mathcal{N}(\mathbf{A}^\dagger\mathbf{A})$. First we have
$$\mathbf{x} \in \mathcal{N}(\mathbf{A}) \Rightarrow \mathbf{A}\mathbf{x} = \mathbf{0} \Rightarrow \mathbf{A}^\dagger\mathbf{A}\mathbf{x} = \mathbf{0} \Rightarrow \mathbf{x} \in \mathcal{N}(\mathbf{A}^\dagger\mathbf{A}) \Rightarrow \mathcal{N}(\mathbf{A}) \subseteq \mathcal{N}(\mathbf{A}^\dagger\mathbf{A})\,,$$
   next we have
$$\mathbf{x} \in \mathcal{N}(\mathbf{A}^\dagger\mathbf{A}) \Rightarrow \mathbf{A}^\dagger\mathbf{A}\mathbf{x} = \mathbf{0} \Rightarrow \mathbf{A}\mathbf{A}^\dagger\mathbf{A}\mathbf{x} = \mathbf{0} \Rightarrow \mathbf{A}\mathbf{x} = \mathbf{0} \Rightarrow \mathbf{x} \in \mathcal{N}(\mathbf{A}) \Rightarrow \mathcal{N}(\mathbf{A}^\dagger\mathbf{A}) \subseteq \mathcal{N}(\mathbf{A})\,.$$
   Therefore, $\mathcal{N}(\mathbf{A}^\dagger\mathbf{A}) = \mathcal{N}(\mathbf{A})$ and consequently $\mathcal{R}(\mathbf{A}^\dagger\mathbf{A}) = \mathcal{N}(\mathbf{A})^\perp$. Hence, $\mathbf{A}^\dagger\mathbf{A}$ is the orthogonal projection onto the orthogonal complement of $\mathcal{N}(\mathbf{A})$. (5 points)
2) Any solution to the LS problem can be written as
$$\mathbf{x} = \mathbf{A}^\dagger\mathbf{b} + \boldsymbol{\eta}\,, \quad \boldsymbol{\eta} \in \mathcal{N}(\mathbf{A})\,.$$

For any vector $\mathbf{z} \in \mathbb{R}^n$, by the conclusion of 1), the orthogonal projection onto $\mathcal{N}(\mathbf{A})$ is given by

$$\Pi_{\mathcal{N}(\mathbf{A})} = \mathbf{I} - \mathbf{A}^\dagger \mathbf{A}\,,$$

and therefore we can rewrite $\mathbf{x}$ as

$$\mathbf{x} = \mathbf{A}^\dagger \mathbf{b} + (\mathbf{I} - \mathbf{A}^\dagger \mathbf{A})\mathbf{z}\,, \quad \mathbf{z} \in \mathbb{R}^n\,.$$

Note that

$$[(\mathbf{I} - \mathbf{A}^\dagger \mathbf{A})\mathbf{z}]^T(\mathbf{A}^\dagger \mathbf{b}) = \mathbf{z}^T[\mathbf{I} - (\mathbf{A}^\dagger \mathbf{A})^T](\mathbf{A}^\dagger \mathbf{b}) = \mathbf{z}^T \mathbf{A}^\dagger \mathbf{b} - \mathbf{z}^T \mathbf{A}^\dagger \mathbf{A} \mathbf{A}^\dagger \mathbf{b} = \mathbf{0}\,,$$

which means $\mathbf{A}^\dagger \mathbf{b} \perp \tilde{\mathbf{z}}$, therefore we have

$$\|\mathbf{x}\|_2^2 = \|\mathbf{A}^\dagger \mathbf{b} + (\mathbf{I} - \mathbf{A}^\dagger \mathbf{A})\mathbf{z}\|_2^2 = \|\mathbf{A}^\dagger \mathbf{b}\|_2^2 + \|(\mathbf{I} - \mathbf{A}^\dagger \mathbf{A})\mathbf{z}\|_2^2 \geq \|\mathbf{A}^\dagger \mathbf{b}\|_2^2\,.$$

Equality holds if and only if $\mathbf{z} = \mathbf{0}$. So $\mathbf{A}^\dagger \mathbf{b}$ is the unique minimum norm solution.(10 points)

Bonus Problem *(10 points + 10 points)*

1) Consider the following problem

$$\min_{\mathbf{Q}\in\mathbb{R}^{m\times m}, \mathbf{Q}^T\mathbf{Q}=\mathbf{I}} \|\mathbf{QA}-\mathbf{B}\|_F^2,$$

where $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{m\times n}$ and $\|\cdot\|_F$ denotes the Frobenius norm defined as $\|\mathbf{X}\|_F = \sqrt{\sum_{i,j}|x_{ij}|^2} = \sqrt{\mathrm{tr}(\mathbf{X}^T\mathbf{X})}$. Show that an optimal solution to the above problem is given by

$$\mathbf{Q}^\star = \mathbf{UV}^T,$$

where $\mathbf{U\Sigma V}^T$ represents the SVD of $\mathbf{BA}^T$.

2) Consider

$$\min_{\mathbf{X}} \|\mathbf{Y}-\mathbf{X}\|_F^2 + \sum_{i=1}^{\min\{m,n\}} w_i\sigma_i(\mathbf{X}),$$

where $\mathbf{X}, \mathbf{Y} \in \mathbb{R}^{m\times n}$, $w_1 \geq w_2 \geq ... \geq w_n > 0$, and $\sigma_i(\mathbf{X})$ denotes the ith singular value of the matrix $\mathbf{X}$. Let $\mathbf{Y} = \mathbf{U\Sigma V}^T$ be the SVD of $\mathbf{Y}$. Show that an optimal solution to the above problem is

$$\mathbf{X}^\star = \mathbf{UDV}^T,$$

where $\mathbf{D} \in \mathbb{R}^{m\times n}$ has $d_{ij} = 0$ for all $i \neq j$, and

$$d_{ii} = \mathcal{T}_{w_i}(\sigma_i(\mathbf{Y})), \qquad i = 1, 2, ..., \min\{m, n\},$$

where the operator $\mathcal{T}_w$ is defined as

$$\mathcal{T}_w(y) = \arg\min_x (y-x)^2 + w|x|.$$

Hint: You can directly use the following inequality without proof to solve the above problems.

$$\mathrm{tr}\left(\mathbf{A}^T\mathbf{B}\right) \leq \sum_{i=1}^{\min\{m,n\}} \sigma_i(\mathbf{A})\sigma_i(\mathbf{B}), \quad \textit{where } \mathbf{A}, \mathbf{B} \in \mathbb{R}^{m\times n}.$$

Your answer:

Reference solution (Note: one point will be deducted if there are missing or incorrect details):

1) (3 points) First, we have

$$\begin{aligned}\|\mathbf{QA}-\mathbf{B}\|_F^2 &= \mathrm{tr}[(\mathbf{QA}-\mathbf{B})^T(\mathbf{QA}-\mathbf{B})] = \mathrm{tr}[(\mathbf{A}^T\mathbf{Q}^T-\mathbf{B}^T)(\mathbf{QA}-\mathbf{B})] \\ &= \mathrm{tr}[\mathbf{A}^T\underbrace{\mathbf{Q}^T\mathbf{Q}}_{\mathbf{I}}\mathbf{A} - \mathbf{A}^T\mathbf{Q}^T\mathbf{B} - \mathbf{B}^T\mathbf{QA} + \mathbf{B}^T\mathbf{B}] \\ &= \mathrm{tr}(\mathbf{A}^T\mathbf{A} + \mathbf{B}^T\mathbf{B}) - 2\,\mathrm{tr}(\mathbf{A}^T\mathbf{Q}^T\mathbf{B}).\end{aligned}$$

(1 points) Therefore, for fixed $\mathbf{A}$ and $\mathbf{B}$, minimizing $\|\mathbf{QA}-\mathbf{B}\|_F^2$ is equivalent to maximizing $\mathrm{tr}(\mathbf{A}^T\mathbf{Q}^T\mathbf{B})$.

(3 points) Second, since $\mathbf{BA}^T = \mathbf{U\Sigma V}^T$, the objective can be further derived as

$$\mathrm{tr}(\mathbf{A}^T\mathbf{Q}^T\mathbf{B}) = \mathrm{tr}(\mathbf{Q}^T\mathbf{BA}^T) = \mathrm{tr}(\mathbf{Q}^T\mathbf{U\Sigma V}^T) = \mathrm{tr}(\mathbf{Q}^T\mathbf{UV}^T\mathbf{\Sigma}).$$

Let $\mathbf{X} = \mathbf{Q}^T\mathbf{U}\mathbf{V}^T$, $\mathbf{X}$ is orthogonal since

$$\mathbf{X}^T\mathbf{X} = \mathbf{V}^T\mathbf{U}^T\mathbf{Q}\mathbf{Q}^T\mathbf{U}\mathbf{V}^T = \mathbf{I}\,, \quad \mathbf{X}\mathbf{X}^T = \mathbf{Q}^T\mathbf{U}\mathbf{V}^T\mathbf{V}\mathbf{U}^T\mathbf{Q} = \mathbf{I}\,.$$

(3 points) Observe that

$$\sum_{j=1}^{m} x_{ij}^2 = 1\,, \quad \text{for } i = 1, \dots, m\,,$$

then we must have $-1 \le x_{ij} \le 1$ for $1 \le i, j \le m$. Therefore,

$$\mathrm{tr}(\mathbf{X}\mathbf{\Sigma}) = \sum_{i=1}^{m} x_{ii}\sigma_i \le \sum_{i=1}^{m} \sigma_i\,.$$

Observe that the equality holds when $\mathbf{X}_{ii} = 1$ for $i = 1, \dots, m$, which indicates that $\mathbf{X} = \mathbf{I}$, i.e., $\mathbf{Q}^\star = \mathbf{U}\mathbf{V}^T$.

2) (4 points) We can conduct the following transformation:

$$\begin{aligned}\|\mathbf{Y} - \mathbf{X}\|_F^2 &= \mathrm{tr}(\mathbf{Y} - \mathbf{X})^T(\mathbf{Y} - \mathbf{X}) = \mathrm{tr}(\mathbf{Y}^T\mathbf{Y} + \mathbf{X}^T\mathbf{X}) - 2\mathrm{tr}(\mathbf{X}^T\mathbf{Y}) \\ &= \left[\sum_{i=1}^{\min\{m,n\}} \sigma_i^2(\mathbf{X}) + \sigma_i^2(\mathbf{Y})\right] - 2\mathrm{tr}(\mathbf{X}^T\mathbf{Y}).\end{aligned}$$

According to the inequality in the Hint, it can be derived that

$$\|\mathbf{Y} - \mathbf{X}\|_F^2 + \sum_{i=1}^{\min\{m,n\}} w_i\sigma_i(\mathbf{X}) \ge \sum_{i=1}^{\min\{m,n\}} \left[\sigma_i^2(\mathbf{X}) + \sigma_i^2(\mathbf{Y}) - 2\sigma_i(\mathbf{X})\sigma_i(\mathbf{Y}) + w_i\sigma_i(\mathbf{X})\right]\,,$$

in which the equality holds if the SVD of $\mathbf{X}$ shares the same left and right singular vectors with $\mathbf{Y}$, i.e., for $\mathbf{Y} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^T$, then the SVD for $\mathbf{X}$ is given by $\mathbf{X} = \mathbf{U}\mathbf{D}\mathbf{V}^T$ for some $\mathbf{D}$.

(2 points) So far, we convert the original optimization problem to

$$\min_{\mathbf{X}} \sum_{i=1}^{\min\{m,n\}} \sigma_i^2(\mathbf{X}) - 2\sigma_i(\mathbf{X})\sigma_i(\mathbf{Y}) + \sigma_i(\mathbf{Y})^2 + w_i\sigma_i(\mathbf{X})\,, \tag{1}$$

denote $d_i = \mathbf{D}_{ii}$ for $i = 1, \dots, p$, where $p = \min\{m, n\}$. Then (1) is equivalent to

$$\begin{aligned}\min_{d_1,\dots,d_p \ge 0} \sum_{i=1}^{p} d_i^2 - 2d_i\sigma_i(\mathbf{Y}) + \sigma_i(\mathbf{Y})^2 + w_id_i &= \min_{d_1,\dots,d_p \ge 0} \sum_{i=1}^{p} (d_i - \sigma_i(\mathbf{Y}))^2 + w_id_i \\ &= \sum_{i=1}^{p} \min_{d_i \ge 0} (d_i - \sigma_i(\mathbf{Y}))^2 + w_id_i\,,\end{aligned}$$

therefore the optimal $\{d_1, \dots, d_p\}$ can be found by

$$d_i^* = \arg\min_{d_i \ge 0} (d_i - \sigma_i(\mathbf{Y}))^2 + w_id_i\,.$$

(4 points) Next, consider the operator $\mathcal{T}_w(y) = \arg\min_x (y - x)^2 + w|x|$ in the case of $w > 0$ and $y \ge 0$. Let $f(x) = (y - x)^2 + w|x|$, and for any negative value $\hat{x} < 0$, we have

$$f(\hat{x}) = (y - \hat{x})^2 + w|\hat{x}| \ge y^2 + 0 = f(0).$$

therefore the minimizer of f cannot be negative value in this case. Hence,

$$d_i^* = \arg\min_{d_i \ge 0} (d_i - \sigma_i(\mathbf{Y}))^2 + w_id_i = \arg\min_{d_i} (d_i - \sigma_i(\mathbf{Y}))^2 + w_i|d_i| = \mathcal{T}_{w_i}(\sigma_i(\mathbf{Y}))\,.$$

Therefore $\mathbf{X}^* = \mathbf{U}\mathbf{D}\mathbf{V}^T$ is an optimal solution.