Probability & Statistics for EECS **Name (Print):** ____________________
Spring 2023
Final Exam
June 01, 2023
Time Limit: 180 Minutes Advisor Name ____________________

This exam contains 11 pages (including this cover page) and 9 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

Try to answer as many problems as you can. The following rules apply:

- **Mysterious or unsupported answers will NOT receive full credit**. Correct answers, unsupported by calculations, explanation, or algebraic work will receive no credit. Incorrect answers supported by substantially correct calculations and explanations might still receive partial credit.

Do not write in the table to the right.

Problem	Points	Score
1	5	
2	10	
3	15	
4	20	
5	10	
6	10	
7	10	
8	10	
9	10	
Total:	100	

1. (5 points) Please describe the pros and cons of *Bayesian statistical inference* and *Classical statistical inference*. Then explain why conjugate priors are important for Bayesian statistical inference.

2. (10 points) Let X and Y be two continuous random variables with joint PDF

$$f_{X,Y}(x,y) = \begin{cases} cx^2y & \text{if } 0 \le y \le x \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

   (a) (5 points) Find the value of constant c.

   (b) (5 points) Find the conditional probability $P(Y \le \frac{X}{4} \big| Y \le \frac{X}{2})$.

3. (15 points) Let X and Y be two integer random variables with joint PMF

$$P_{X,Y}(x,y) = \begin{cases} \frac{1}{6 \cdot 2^{\min(x,y)}} & \text{if } x, y \geq 0, |x-y| \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

(a) (5 points) Find the marginal distributions of X and Y. Are X and Y independent?

(b) (5 points) Find $P(X = Y)$.

(c) (5 points) Find $E[X|Y = 2]$ and $\mathrm{Var}[X|Y = 2]$.

4. (20 points) Let Z_1, Z_2 be two *i.i.d.* random variables satisfying standard normal distributions, *i.e.*, $Z_1, Z_2 \sim \mathcal{N}(0,1)$. Define

$$X = \sigma_X Z_1 + \mu_X;$$
$$Y = \sigma_Y \left(\rho Z_1 + \sqrt{1-\rho^2} Z_2\right) + \mu_Y,$$

where $\sigma_X > 0$, $\sigma_Y > 0$, $-1 < \rho < 1$.

(a) (5 points) Show that X and Y are bivariate normal.

(b) (5 points) Find the correlation coefficient between X and Y, *i.e.*, $\text{Corr}(X, Y)$.

(c) (5 points) Find the joint PDF of X and Y.

(d) (5 points) Find $E[Y|X]$ and $\text{Var}[Y|X]$.

5. (10 points) Let the random variable $X \sim \mathcal{N}(\mu, \tau^2)$. Given $X = x$, random variables $Y_1, Y_2, \ldots, Y_n$ are *i.i.d.* and have the same conditional distribution, *i.e.*, $Y_i|X = x \sim \mathcal{N}(x, \sigma^2)$. Define the sample mean $\bar{Y}$ as follows:

$$\bar{Y} = \frac{Y_1 + \cdots + Y_n}{n}.$$

   (a) (4 points) Find the posterior PDF of X given $\bar{Y}$.

   (b) (3 points) Find the MAP (Maximum a Posterior Probability) estimates of X given $\bar{Y}$.

   (c) (3 points) Find the MMSE estimates of X given $\bar{Y}$. (We know that the MMSE of X given Y is given by $g(Y) = E[X|Y]$).

6. (10 points) Let $X_1 \sim \text{Expo}(\lambda_1)$, $X_2 \sim \text{Expo}(\lambda_2)$ and $X_3 \sim \text{Expo}(\lambda_3)$ be independent.
   (a) (5 points) Find $E(X_1 + X_2 + X_3 | X_1 > 1, X_2 > 2, X_3 > 3)$ in terms of $\lambda_1, \lambda_2, \lambda_3$.
   (b) (5 points) Find $P(X_1 = \min(X_1, X_2, X_3))$.

7. (10 points) Let $X \sim \text{Expo}(\lambda)$, $Y \sim \text{Expo}(\lambda)$; X and Y are independent.

   (a) (5 points) Find $E(X|X+Y)$.

   (b) (5 points) Find $E(X^2|X+Y)$.

8. (10 points) Instead of predicting a single value for the parameter, we give an interval that is likely to contain the parameter: A $1-\delta$ confidence interval for a parameter p is an interval $[\hat{p}-\epsilon, \hat{p}+\epsilon]$ such that $P\left(p \in [\hat{p}-\epsilon, \hat{p}+\epsilon]\right) \geq 1-\delta$. Now we toss a coin with probability p landing heads and probability $1-p$ landing tails. The parameter p is unknown and we need to estimate its value from experiment results. We toss such coin N times. Let $X_i = 1$ if the ith result is head, otherwise 0. We estimate p by using $\hat{p} = \frac{X_1+\cdots+X_N}{N}$. Find the $1-\delta$ confidence interval for p, then discuss the impacts of δ and N. **Hint**: You can use the following Hoeffding bound: Let the random variables $X_1, X_2, \ldots, X_n$ be independent with $E(X_i) = \mu$, $a \leq X_i \leq b$ for each $i = 1, \ldots, n$, where a, b are constants. Then for any $\epsilon \geq 0$,

$$P\left(\left|\frac{1}{n}\sum_{i=1}^{n} X_i - \mu\right| \geq \epsilon\right) \leq 2\exp\left(-\frac{2n\epsilon^2}{(b-a)^2}\right).$$

9. (10 points) Show the following inequalities.

   (a) (5 points) Let $X \sim \text{Pois}(\lambda)$. If there exists a constant $a > \lambda$, then

$$P(X \geq a) \leq \frac{e^{-\lambda}(e\lambda)^a}{a^a}.$$

   (b) (5 points) Let X be a random variable with finite variance σ^2. Then for any constant $a > 0$,

$$P(|X - \mathbb{E}[X]| \geq a) \leq \frac{2\sigma^2}{\sigma^2 + a^2}.$$

	Y discrete	**Y continuous**
X discrete	$P(Y=y\|X=x)=\frac{P(X=x\|Y=y)P(Y=y)}{P(X=x)}$	$f_Y(y\|X=x)=\frac{P(X=x\|Y=y)f_Y(y)}{P(X=x)}$
X continuous	$P(Y=y\|X=x)=\frac{f_X(x\|Y=y)P(Y=y)}{f_X(x)}$	$f_{Y\|X}(y\|x)=\frac{f_{X\|Y}(x\|y)f_Y(y)}{f_X(x)}$

	Y discrete	**Y continuous**
X discrete	$P(X=x)=\sum_y P(X=x\|Y=y)P(Y=y)$	$P(X=x)=\int_{-\infty}^{\infty} P(X=x\|Y=y)f_Y(y)dy$
X continuous	$f_X(x)=\sum_y f_X(x\|Y=y)P(Y=y)$	$f_X(x)=\int_{-\infty}^{\infty} f_{X\|Y}(x\|y)f_Y(y)dy$

Figure 1: Bayes' Rule & LOTP.

Table of distributions

Name	**Param.**	**PMF or PDF**	**Mean**	**Variance**
Bernoulli	p	$P(X=1)=p, P(X=0)=q$	p	pq
Binomial	n, p	$\binom{n}{k}p^k q^{n-k}$, for $k\in\{0,1,\dots,n\}$	np	npq
FS	p	pq^{k-1}, for $k\in\{1,2,\dots\}$	$1/p$	q/p^2
Geom	p	pq^k, for $k\in\{0,1,2,\dots\}$	q/p	q/p^2
NBinom	r, p	$\binom{r+n-1}{r-1}p^r q^n, n\in\{0,1,2,\dots\}$	rq/p	rq/p^2
HGeom	w, b, n	$\frac{\binom{w}{k}\binom{b}{n-k}}{\binom{w+b}{n}}$, for $k\in\{0,1,\dots,n\}$	$\mu=\frac{nw}{w+b}$	$\left(\frac{w+b-n}{w+b-1}\right)n\frac{\mu}{n}\left(1-\frac{\mu}{n}\right)$
Poisson	λ	$\frac{e^{-\lambda}\lambda^k}{k!}$, for $k\in\{0,1,2,\dots\}$	λ	λ
Uniform	$a<b$	$\frac{1}{b-a}$, for $x\in(a,b)$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$
Normal	μ, σ^2	$\frac{1}{\sigma\sqrt{2\pi}}e^{-(x-\mu)^2/(2\sigma^2)}$	μ	σ^2
Log-Normal	μ, σ^2	$\frac{1}{x\sigma\sqrt{2\pi}}e^{-(\log x-\mu)^2/(2\sigma^2)}, x>0$	$\theta=e^{\mu+\sigma^2/2}$	$\theta^2(e^{\sigma^2}-1)$
Expo	λ	$\lambda e^{-\lambda x}$, for $x>0$	$1/\lambda$	$1/\lambda^2$
Gamma	a, λ	$\Gamma(a)^{-1}(\lambda x)^a e^{-\lambda x}x^{-1}$, for $x>0$	a/λ	a/λ^2
Beta	a, b	$\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}x^{a-1}(1-x)^{b-1}$, for $0<x<1$	$\mu=\frac{a}{a+b}$	$\frac{\mu(1-\mu)}{a+b+1}$
Chi-Square	n	$\frac{1}{2^{n/2}\Gamma(n/2)}x^{n/2-1}e^{-x/2}$, for $x>0$	n	$2n$
Student-t	n	$\frac{\Gamma((n+1)/2)}{\sqrt{n\pi}\Gamma(n/2)}(1+x^2/n)^{-(n+1)/2}$	0 if $n>1$	$\frac{n}{n-2}$ if $n>2$