Online Lecture Notes

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1 Properties of Matrix Exponentials

1. We have
$$e^0 = \sum_{i=0}^{\infty} \frac{1}{i!}(t \cdot 0)^i = \frac{1}{0!} I = I$$
This simply follows by the definition of the matrix power $A^0 = I$ for any matrix A. This holds in particular for the zero-matrix $A = 0$, $0^0 = I$.

2. The derivative of the matrix exponential function $X(t) = e^{At}$ is given by
$$\dot{X}(t) = Ae^{At} = AX(t)\ .$$
This is simply the standard "chain rule". We can also check that this propery holds, by using the series expansion:
$$\begin{aligned}\dot{X}(t) &= \frac{\mathrm{d}}{\mathrm{d}t} \sum_{i=0}^{\infty} \frac{1}{i!} t^i A^i = \sum_{i=0}^{\infty} \frac{1}{i!} A^i \frac{\mathrm{d}}{\mathrm{d}t} t^i = \sum_{i=1}^{\infty} \frac{i}{i!} A^i t^{i-1} = \sum_{i=1}^{\infty} \frac{1}{(i-1)!} A^i t^{i-1} \\ &= A \left[\sum_{i=1}^{\infty} \frac{1}{(i-1)!} A^{i-1} t^{i-1} \right] = A \left[\sum_{i=0}^{\infty} \frac{1}{i!} A^i t^i \right] = Ae^{At} = X(t)\ .\end{aligned}$$
Notice that this means that the function $X(t)$ satisfies the matrix differential equation
$$\dot{X}(t) = AX(t) \quad \text{and} \quad X(0) = I\ . \tag{1}$$
This can be understood as for vector-valued ODEs. In principle, we could also introduce the state $x(t) = \mathrm{vec}(X(t))$ and resort all the coefficients of the matrix A into a larger matrix.

3. In the above derivation, we could also bracket out the matrix A to the right side. This means that
$$Ae^{At} = A \left[\sum_{i=0}^{\infty} \frac{1}{i!} t^i A^i \right] = \sum_{i=0}^{\infty} \frac{1}{i!} t^i A^{i+1} = \left[\sum_{i=0}^{\infty} \frac{1}{i!} t^i A^i \right] A = e^{At} A\ .$$
Thus, in summary, the matrix-valued function $X(t) = e^{At}$ commutes with A; that is
$$AX(t) = X(t)A\ .$$

4. Recall that the scalar exponential function satisfies the addition theorem

$$e^{at_1+at_2} = e^{at_1}e^{at_2} \ .$$

The same property still holds for matrix exponentials if we use the same matrix in both expressions. This means that

$$e^{At_1+At_2} = e^{At_1}e^{At_2} = e^{At_2}e^{At_1} \ .$$

In more general, if A and B are any matrices that commute, $AB = BA$, then the addition theorem for the matrix exponential holds,

$$e^{A+B} = e^A e^B = e^B e^A \ .$$

This can be proven by using the series expansion and resort terms

$$\begin{aligned} e^{A+B} &= \sum_{i=0}^{\infty} \frac{1}{i!}(A+B)^i = \sum_{i=0}^{\infty} \frac{1}{i!} \sum_{j=0}^{\infty} \binom{i}{j} A^j B^{i-j} = \left[\sum_{i=0}^{\infty} \frac{1}{i!} A^i \right] \left[\sum_{i=0}^{\infty} \frac{1}{i!} B^i \right] \\ &= e^A e^B = e^B e^A \ . \end{aligned}$$

Notice that this derivation is in complete analogy to the proof of the addition theorem for the scalar e-function—the above infinite sum can be resorted in exactly the same way.

5. If we rewrite the addition theorem for the matrix e-function for the function $X(t) = e^{At}$, this means that

$$X(t_1 + t_2) = e^{At_1+At_2} = e^{At_1}e^{At_2} = X(t_1)X(t_2) \ .$$

6. Another thing that we know about the scalar e-function is that it can always be inverted, $e^{-at} = \frac{1}{e^{at}}$ for $a \in \mathbb{R}$. A completely analogous property holds for the matrix e-function,

$$X(-t) = e^{-At} = \left[e^{At} \right]^{-1} = X(t)^{-1} \ . \tag{2}$$

For the proof, we can consider two differential equations for the functions $X(t) = e^{At}$ and $Y(t) = e^{-At}$, which are given by

$$\begin{aligned} \dot{X}(t) &= AX(t) \quad \text{and} \quad X(0) = I & (3) \\ \dot{Y}(t) &= -AY(t) \quad \text{and} \quad Y(0) = I \ . & (4) \end{aligned}$$

Next, we introduce the product $Z(t) = X(t)Y(t)$, which satisfies,

$$\dot{Z}(t) = \dot{X}(t)Y(t) + X(t)\dot{Y}(t) \quad \text{and} \quad Z(0) = X(0)Y(0) = I \ .$$

The expression for $\dot{Z}(t)$ can be found by substituting the above differential equations for X and Y, which yields

$$\dot{Z}(t) = \dot{X}(t)Y(t) + X(t)\dot{Y}(t) \overset{(3),(4)}{=} AX(t)Y(t) - X(t)AY(t) \ .$$

We know that A commutes with $X(t)$. Thus, we get

$$\dot{Z}(t) = AX(t)Y(t) - AX(t)Y(t) = 0 \ .$$

This means that Z must be constant. Since we have $Z(0) = I$, this means that

$$\forall t \in \mathbb{R}, \qquad I = Z(t) = X(t)Y(t) \qquad \Longrightarrow \qquad X(t)^{-1} = Y(t)\ .$$

This proves that (2) holds.

7. We have $e^{A^\intercal t} = \left[e^{At}\right]^\intercal$. Proof:

$$e^{A^\intercal t} = \sum_{i=0}^{\infty} \frac{1}{i!} \left[A^\intercal\right]^i t^i = \left[\sum_{i=0}^{\infty} \frac{1}{i!} A^i t^i\right]^\intercal = \left[e^{At}\right]^\intercal$$

2 Examples for Matrix Eponentials

The goal of this section is to develop a basic intuition about matrix exponential and their properties.

2.1 Diagonal matrices

We have have that

$$e^{\mathrm{diag}(d)t} = \mathrm{diag}(\ \exp(d_1 t),\ \exp(d_2 t),\ \ldots,\ \exp(d_n t)\ )$$

for any diagonal vector $d \in \mathbb{R}^n$. (see Lecture from last Tuesday).

2.2 Nil-potent matrix examples

The easiest example for a nil-potent is given by

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} ,$$

which satisfies $A^2 = 0$, since

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} .$$

It follows that $A^3 = A^2 A = 0$, $A^4 = A^2 A^2 = 0$, and so on. That is, $A^i = 0$ for all $i \geq 2$. This means that the corresponding matrix exponential function

$$X(t) = e^{At} = \sum_{i=0}^{\infty} \frac{1}{i!} t^i A^i = \sum_{i=0}^{1} \frac{1}{i!} t^i A^i = I + tA$$

We can also write this out component-wise, which yields

$$e^{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} t} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + t \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} .$$

We can generalize this example by constructing higher order recursive integrators. This means that we analyze the matrix

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & 1 \\ 0 & 0 & 0 & 0 & \ldots & 0 \end{pmatrix} \in \mathbb{R}^{n \times n}$$

Now, we can work out the matrix powers

$$A^2 = \begin{pmatrix} 0 & 1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & 1 \\ 0 & 0 & 0 & 0 & \ldots & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & 1 \\ 0 & 0 & 0 & 0 & \ldots & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 0 & 1 & \ldots & 0 \\ 0 & 0 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & 0 & \ldots & 0 \end{pmatrix}$$

and so on—the corresponding higher order power are looking very similar: "the 1s are moving up to right corner". Until we get $A^n = 0$ and then of course also $A^i = 0$ for all $i \geq 0$. This means that

$$e^{At} = I + tA + \frac{1}{2!}tA^2 + \ldots + \frac{1}{(n-1)!}t^{n-1}A^{n-1}$$

This is a polynomial function! All components of this function are polynomials in t. In detail,

$$e^{At} = \begin{pmatrix} 1 & t & \frac{t^2}{2} & \frac{t^6}{2} & \ldots & \frac{t^{n-1}}{(n-1)!} \\ 0 & 1 & t & \frac{t^2}{2} & \ldots & \frac{t^{n-2}}{(n-2)!} \\ 0 & 0 & 1 & t & \ldots & \frac{t^{n-3}}{(n-3)!} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 1 & \ldots & t \\ 0 & 0 & 0 & 0 & \ldots & 1 \end{pmatrix}$$

2.3 Example for Matrix Exponential Products

Let us once more consider the matrix $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ as well as its transpose $A^\intercal = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. We have

$$e^{At} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad e^{A^\intercal t} = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$$

Notice that the matrices A and $A^\intercal$ do not commute,

$$AA^\intercal = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

but

$$A^\intercal A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \qquad \Longrightarrow \qquad AA^\intercal \neq A^\intercal A$$

For this example, it turns out that the addition theorem fails to holds. Let us check this! On the one side, we have

$$e^{At}e^{A^\intercal t} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} = \begin{pmatrix} 1+t^2 & t \\ t & 1 \end{pmatrix}.$$

On the other side we have

$$\begin{aligned} e^{At+A^\intercal t} &= \sum_{i=0}^{\infty} \frac{1}{i!}t^i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^i = \sum_{i=0}^{\infty} \frac{1}{(2i)!}t^{2i} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \sum_{i=0}^{\infty} \frac{1}{(2i+1)!}t^{2i+1} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ &= \cosh(t) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \sinh(t) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \cosh(t) & \sinh(t) \\ \sinh(t) & \cosh(t) \end{pmatrix}. \end{aligned} \tag{5}$$

What we see in this example is that

$$e^{At}e^{A^\intercal t} \neq e^{At+A^\intercal t} .$$

This is an example, where the addition theorem for the matrix exponential fails to hold! This is because the matrices A and $A^\intercal$ do not commute.

3 Implementation in Julia

In Julia matrix exponentials can be implemented by using the standard overloaded exponential function `exp`,

1. `exp([0 1; 0 0])` would give us the matrix exponetial.
2. BUT: `exp.([0 1; 0 0])` would give us the componentwise exponetial.