Online Lecture Notes

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1 Announcements

We have four more lecture slots: this week Tue/Thu and next Tue/Thu before the lecture completes. The final exam will be announced officially by the university in the coming days—the tentative is June 18 (10:30 - 12:30), but wait for the official confirmation. Since we have not much time left, we will

1. skip Lecture 9 and 10; the corresponding material will not be part of the final exam.
2. this week and Tue in one week we will complete the Lecture 11 on LQR Control.
3. on Thursday next we will offer a review / Q & A session
4. the final exam will, in principle, be about all the material that we covered in the lecture, but we will focus on the second half. This means that we will focus on

   (a) Linear Time Varying System, Fundamental Solution, Periodic Systems (Lecture 7; Homework 7)

   (b) Stability Analysis including stability of linear systems and Lyapunov theory for nonlinear system (Lecture 8, Homework 8)

   (c) LQR Control (Lecture 11 + Homework 9): this will be the main focus!

   The final exam will be online, too. Same as before, but this time we have only two ours—we will announce more details next week.

2 Linear Quadratic Regulator

The goal of this lecture is to design linear controllers by minimizing quadratic objective functions. For simplicity of presentation, we will initially focus on linear time-invariant systems of the form

$$\dot{x}(t) = Ax(t) + Bu(t) \qquad \text{with} \qquad x(0) = x_0$$

with $A \in \mathbb{R}^{n_x \times n_x}$ and $B \in \mathbb{R} n_x \times n_u$ being given matrices. We would like to choose the control input such that the integral

$$\int_0^T \{x(\tau)^\intercal Q x(\tau) + u(\tau)^\intercal R u(\tau)\} \, \mathrm{d}\tau + x(T)^\intercal P x(T)$$

takes the smallest possible value for a given control horizon length T. Formally, we may also consider the case $T = \infty$, but we will discuss this later, as we first need to ensure that the integral exist. The function

$$\ell(x, u) = x^\intercal Q x + u^\intercal R u$$

is called the quadratic stage-cost function of the LQR controller. In this lecture we will assume that the matrices $Q \in \mathbb{R}^{n_x \times n_x}$ and $R \in \mathbb{R}^{n_u \times n_u}$ are symmetric and positive definite. In general, these matrices can be used to “tune” the objective function. If we choose a “large” Q and a small R, we want to bring the state to 0 as quickly as possible. Otherwise, if we choose R larger than Q, this means that we have high control cost. Similarly, the quadratic terminal cost function,

$$m(x) = x^\intercal P x,$$

can be tuned by choosing the symmetric and positive matrix $P \in \mathbb{R}^{n_x \times n_x}$. Notice that the above assumption on the positive definiteness of the matrix Q, R, and P is not strictly needed—we could also consider the more general case where these matrices are positive semi-definite (although we usually need at least that R is positive definite).

The minimization problem that we wish to solve can be written in the form

$$\begin{aligned}
\min_{x,u} \quad & \int_0^T \{x(\tau)^\intercal Q x(\tau) + u(\tau)^\intercal R u(\tau)\} \, \mathrm{d}\tau + x(T)^\intercal P x(T) \\
\text{s.t.} \quad & \left\{ \begin{array}{lcl} \forall t \in [0, T], & & \\ \dot{x}(t) & = & Ax(t) + Bu(t) \\ x(0) & = & x_0 \, . \end{array} \right.
\end{aligned}$$

This optimization is called a linear-quadratic *optimal control problem.* The optimization variables are the integrable functions x and u, which are infinite dimensional. The analysis of such infinite dimensional optimization problems is in general not trivial. Thus, in order to solve this optimal control, we will pass through several steps:

1. In the first step, we will discretize the above optimal control by approximating the control input by a piecewise constant input and using a Euler discretization scheme (also see Lectures 2 and 4 for details). This will lead to a finite dimensional approximation of the above optimal control problem, a so called discrete-time linear quadratic control problem.

2. In the second step, we will introduce a so called *dynamic programming* strategy for solving the discrete-time linear control problem.

3. And finally, in the third step, we will take the limit for vanishing discretization parameters going back from discrete to continuous time in order to recover the optimal solution of the above infinite dimensional optimal control problem.

2.1 Step 1: Piecewise constant control function approximation and Euler's method

In order to do the first step, we introduce a step size $h > 0$, for instance by breaking the time horizon interval $[0, T]$ into N pieces,

$$h = \frac{T}{N}$$

for a large number $N \in \mathbb{N}$ of discrete time interval. The corresponding discrete time points are given by

$$t_0 = 0,\ t_1 = h,\ t_2 = 2h,\ \ldots\ , t_N = Nh = T\ .$$

Notice that the solution of the original continuous-time system can be written in the form

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)\,\mathrm{d}\tau\ .$$

This means that at the discrete-time points we have

$$\begin{aligned}
x(t_i) &= e^{At_i}x_0 + \int_0^{t_i} e^{A(t_i-\tau)}Bu(\tau)\,\mathrm{d}\tau \\
&= e^{A(t_{i-1}+h)}x_0 + \int_0^{t_{i-1}+h} e^{A(t_{i-1}+h-\tau)}Bu(\tau)\,\mathrm{d}\tau \\
&= e^{Ah}e^{At_{i-1}}x_0 + e^{Ah}\int_0^{t_{i-1}} e^{A(t_{i-1}-\tau)}Bu(\tau)\,\mathrm{d}\tau + \int_{t_{i-1}}^{t_{i-1}+h} e^{A(t_{i-1}+h-\tau)}Bu(\tau)\,\mathrm{d}\tau \\
&= e^{Ah}x(t_{i-1}) + \int_0^h e^{A(h-\tau')}Bu(t_{i-1}+\tau')\,\mathrm{d}\tau' \qquad (1)
\end{aligned}$$

Next, we can use Taylor series of the matrix exponential

$$\forall \tau' \in [0, h], \qquad e^{A(h-\tau')} = I + O(h),$$

which yields

$$\begin{aligned}
\int_0^h e^{A(h-\tau')}Bu(t_{i-1}+\tau')\,\mathrm{d}\tau' &= \int_0^h Bu(t_{i-1}+\tau')\,\mathrm{d}\tau' + O(h^2) \\
&= B\int_0^h u(t_{i-1}+\tau')\,\mathrm{d}\tau' + O(h^2) \\
&= hBv_{i-1} + O(h^2) \qquad (2)
\end{aligned}$$

where we have introduced the average values

$$v_{i-1} \stackrel{\text{def}}{=} \frac{1}{h} \int_0^h u(t_{i-1} + \tau') \, \mathrm{d}\tau'$$

Notice that these average values exist for any integrable function u!!! We do not any assumption about the continuity of the input function u. The above discretization argument is completely general—it works for any integrable function u. If we substitute (2) in (1), we find that

$$x(t_i) = e^{Ah} x(t_{i-1}) + hBv_{i-1}$$

We can further simplify this expression by substituting

$$e^{Ah} = I + hA + O(h^2),$$

which yields the linear discrete system

$$\forall i \in \{1, \ldots, N\}, \qquad y_i = \mathcal{A} y_{i-1} + \mathcal{B} v_{i-1} \quad \text{with} \quad \mathcal{A} = I + hA \quad \text{and} \quad \mathcal{B} = hB$$

started with $y_0 = x_0$. Notice that this corresponds to Euler's discretization (but our derivation was more general and works for any integrable function u). Notice that in the latter expression we have introduced the shorthands

$$\forall i \in \{0, 1, \ldots, N\}, \qquad y_i = x(t_i) + O(h)$$

for the discrete time states. Moreover, we can discretize the integral in the objective function, too.

$$\begin{aligned}
\int_0^T \{x(\tau)^\intercal Q x(\tau) + u(\tau)^\intercal R u(\tau)\} \, \mathrm{d}\tau &= \sum_{i=0}^{N-1} \int_{t_i}^{t_i+h} \{x(\tau)^\intercal Q x(\tau) + u(\tau)^\intercal R u(\tau)\} \, \mathrm{d}\tau \\
&= \sum_{i=0}^{N-1} \int_{t_i}^{t_i+h} \{y_i^\intercal Q y_i + v_i^\intercal R v_i\} \, \mathrm{d}\tau + O(h) \\
&= \sum_{i=0}^{N-1} \{y_i^\intercal \mathcal{Q} y_i + v_i^\intercal \mathcal{R} v_i\} + O(h), \qquad (3)
\end{aligned}$$

where we have introduced the shorthands

$$\mathcal{Q} = hQ \qquad \text{and} \qquad \mathcal{R} = hR \, .$$

This means that in summary we have found a discrete-time approximation of our original infinite dimensional optimization problem, which can be written in the form

$$\begin{aligned}
\min_{y,v} \quad & \sum_{i=0}^{N-1} \{y_i^\intercal \mathcal{Q} y_i + v_i^\intercal \mathcal{R} v_i\} + y_N^\intercal P y_N \\
\text{s.t.} \quad & \left\{ \begin{array}{l} \forall i \in \{0, \ldots, N-1\}, \\ \quad y_{i+1} \; = \; \mathcal{A} y_i + \mathcal{B} v_i \\ \quad y_0 \; = \; x_0 \, . \end{array} \right.
\end{aligned}$$

The above optimization problem is called a linear-quadratic discrete-time optimal control problem. This is a finite dimensional optimization problem with optimization variables

$$y = \left(\, y_0^\intercal,\ y_1^\intercal,\ \ldots,\ y_N^\intercal \,\right)^\intercal \qquad \text{and} \qquad v = \left(\, v_0^\intercal,\ v_1^\intercal,\ \ldots,\ v_{N-1}^\intercal \,\right)^\intercal$$

Notice that $y \in \mathbb{R}^{(N+1)n_x}$ and $v \in \mathbb{R}^{Nn_u}$, since we have $N+1$ grid points but only N control intervals. Notice that the above discretization is accurate in the sense that all approximation errors are of order $O(h)$. This means that we can later take the limit for $h \to 0$ without making mistakes (this will be discussed in more detail later). However, our next step will be to solve the above discrete-time optimization problem explicitly. Our next lecture will be about the details about how to solve this optimization recursively by using dynamic programming.