Mid-Term Take-Home Exam

Introduction to Control

Prof. Boris Houska

YOUR NAME: ________________________

Introduction

This is an "open-book" take-home exam. You are allowed to use any online or offline material you like, but you are neither allowed to share your solutions with others nor to receive help from others before the official end of the exam; that is, before the 24 hours are over. The reason for the latter rule is that you and your classmates are graded based on this exam.

I Scalar Differential Equation

Find the explicit solution $x(t)$ of the following scalar differential equations:

- $\dot{x}(t) = 2002 \quad \text{with} \quad x(0) = 1$ **(3 points)**

  Solution: $x(t) = 2002t + 1$

- $$\dot{x}(t) = \sum_{i=0}^{2022} \frac{t^i}{i!} \quad \text{with} \quad x(0) = 1$$

  (3 points)

  Solution:
  $$x(t) = \sum_{i=0}^{2023} \frac{t^i}{i!}$$

- $\dot{x}(t) = \dfrac{1}{1 + x(t)^2} \quad \text{with} \quad x(0) = 1$ **(3 points)**

  Solution: $\dfrac{\sqrt[3]{2}\left(\sqrt{9t^2 + 24t + 20} + 3t + 4\right)^{2/3} - 2}{2^{2/3}\sqrt[3]{\sqrt{9t^2 + 24t + 20} + 3t + 4}}$

- $\dot{x}(t) = -t^2 x(t) \quad \text{with} \quad x(0) = 1$ **(3 points)**

  Solution: $x(t) = \exp\left(-\dfrac{t^3}{3}\right)$

- $\dot{x}(t) = -\sin(x(t))^2 \tan(x(t)) \quad \text{with} \quad x(0) = \frac{1}{10}$ **(3 points)**

  Solution: $x(t) = \arcsin\left(\sqrt{\dfrac{1}{2t + 1/\sin(1/10)^2}}\right),\ t \in \left(\dfrac{1}{2} - \dfrac{1}{2\sin\left(\frac{1}{10}\right)},\ \infty\right)$

II Approximation of a Nonlinear Control Systems

Let us consider the non-linear control system

$$\dot{y}(t) = \cos(z(t))y(t), \quad \dot{z}(t) = \sin(y(t)) - 2z(t) + u(t) \; ; \qquad \text{with} \quad y(0) = 0 \quad \text{and} \quad z(0) = 0$$

with open-loop control input $u(t) = \frac{1}{10}\sin(t)$.

- Explain how to approximate the above nonlinear control system by linearizing it at the steady-state $y_s = z_s = u_s = 0$. **(5 points)**

Solution: This non-linear control system can be written in form of

$$[\dot{y}, \dot{z}]^\top = F(y, z) + Bu \quad \text{with} \quad F(y, z) = \begin{bmatrix} \cos(z)y \\ \sin(y) - 2z \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} .$$

To linearize the control system, let's take the first order Taylor expansion of F at $(0, 0)$,

$$A = \nabla F(y, z) \mid_{(0,0)} = \begin{bmatrix} \cos(z) & -\sin(z)y \\ \cos(y) & -2 \end{bmatrix} \Bigg|_{(0,0)} = \begin{bmatrix} 1 & 0 \\ 1 & -2 \end{bmatrix} .$$

With A and B defined as above, we can approximate this non-linear control system by

$$\begin{cases} \dot{y} = y \\ \dot{z} = y - 2z + u \end{cases}$$

with $x = [y, z]^\top$.

- Find an approximate solution for the open-loop trajectories $y(t)$ and $z(t)$ by using your linear approximation for the given open-loop control input, $u(t) = \frac{1}{10}\sin(t)$. **(10 points)**

Solution: In principle, we can use the matrix exponential with the explicit solution formula

$$x(t) = \exp(At)x_0 + \int_0^t \exp(A(t - \tau))Bu \; \mathrm{d}\tau .$$

to get the solution function directly. However, since we have $y(t) = 0$ for all t, we can simplify our analysis. This means that we only have to solve the second equation,

$$\dot{z} = -2z + u \implies z(t) = \int_0^t e^{-2(t-\tau)} \frac{\sin(\tau)}{10} \; \mathrm{d}\tau .$$

The integral can be computed by Euler's formula,

$$\int_0^t e^{2\tau} \sin(\tau) \mathrm{d}\tau = \mathrm{Im} \left(\int_0^t e^{2\tau} e^{i\tau} \mathrm{d}\tau \right) = \frac{1}{5} e^{2t} (2\sin(t) - \cos(t)) + \frac{1}{5}$$

Thus, for the given open-loop control input u the trajectory can be written in the explicit form

$$\begin{cases} y(t) = 0 \\ z(t) = \frac{1}{50} \left(2\sin(t) - \cos(t) + e^{-2t} \right) \end{cases} .$$

III Multivariate Linear Differential Equations

- Find all twice differentiable functions $x : \mathbb{R} \to \mathbb{R}$ that satisfy the differential equation

$$x(t) + \dot{x}(t) + \ddot{x}(t) = 0 \ .$$

(6 points)

Solution: Let us set $y = [x, \dot{x}]^\intercal$ and consider the equivalent first order differential equations

$$\dot{y} = Ay \quad \text{with} \quad A = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} .$$

Since A is diagonalizable with eigenvalues $\lambda_{1,2} = \frac{1}{2}(-1 \pm \sqrt{3}i)$, the components of $\exp(At)$ must be a linear combinations of $\exp(\lambda_1 t)$ and $\exp(\lambda_2 t)$; that is

$$e^{-\frac{t}{2}} \left[\cos\left(\frac{\sqrt{3}}{2} t \right) \pm \sin\left(\frac{\sqrt{3}}{2} t \right) i \right] .$$

Thus, all solution functions must have the form

$$x(t) = c_1 e^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}}{2} t \right) + c_2 e^{-\frac{t}{2}} \sin\left(\frac{\sqrt{3}}{2} t \right) ,$$

where c_1, c_2 are arbitrary constants.

- Solve the linear differential equation system

$$\begin{aligned} \dot{x}_1(t) &= -x_1(t) - x_2(t) + x_3(t) & \qquad x_1(0) &= 0 \\ \dot{x}_2(t) &= x_3(t) - 2x_2(t) & \qquad x_2(0) &= 0 \\ \dot{x}_3(t) &= -2x_3(t) & \qquad x_3(0) &= 1 \end{aligned}$$

(9 points)

Solution: Similar to II(b), it is possible find x_1, x_2, x_3 by using matrix exponentials. Alternatively, we can solve the three ODEs recursively. For this aim, we start with the third equation,

$$\begin{cases} \dot{x}_3(t) = -2x_3(t) \\ x_3(0) = 1 \end{cases} \Rightarrow x_3(t) = e^{-2t} ,$$

and the second equation becomes

$$\begin{cases} \dot{x}_2(t) = -2x_3(t) + e^{-2t} \\ x_2(0) = 0 \end{cases} \Rightarrow x_2(t) = \int_0^t e^{-2(t-\tau)} e^{-2\tau} \mathrm{d}\tau = t e^{-2t} .$$

Finally, the first equation can be solved, too,

$$\begin{cases} \dot{x}_1(t) = -x_1(t) - t e^{-2t} + e^{-2t} \\ x_1(0) = 0 \end{cases} \Rightarrow x_1(t) = \int_0^t e^{-(t-\tau)} (e^{-2\tau} - \tau e^{-2\tau}) \mathrm{d}\tau = t e^{-2t} .$$

In summary, the solution is given by

$$\begin{aligned} x_1(t) &= t e^{-2t} \\ x_2(t) &= t e^{-2t} \\ x_3(t) &= e^{-2t} . \end{aligned}$$

IV PID Control of Linear System

Let us consider the linear control system

$$\begin{aligned}\dot{x}_1(t) &= x_1(t) + x_2(t)\\ \dot{x}_2(t) &= -x_1(t) + x_2(t) + u(t)\\ y(t) &= x_1(t)\ .\end{aligned}$$

We would like to design a PID controller of the form

$$u(t) = Ky(t) + K_{\mathrm{I}} \int_0^t y(\tau)\,\mathrm{d}\tau + K_{\mathrm{D}}\dot{y}(t)$$

with control gains $K, K_{\mathrm{I}}, K_{\mathrm{D}} \in \mathbb{R}$.

- Explain how to derive a linear differential equation that can be used to analyze the state trajectories of the closed-loop control system. Find an explicit expression for the closed-loop system matrix in dependence on the parameters K, K_{I}, and K_{D}. **(6 points)**

  Solution: By substituting the PID control law for u in the second differential equation we find

$$\begin{aligned}\dot{x}_2(t) &= -x_1(t) + x_2(t) + Kx_1(t) + K_{\mathrm{I}} \int_0^t x_1(\tau)\,\mathrm{d}\tau + K_{\mathrm{D}}\dot{x}_1(t)\\ &= (K + K_D - 1)x_1(t) + (K_D + 1)x_2(t) + K_{\mathrm{I}} \int_0^t x_1(\tau)\,\mathrm{d}\tau\ .\end{aligned}$$

  Next, we introduce an auxiliary state $x_3(t) = \int_0^t x_1(\tau)\,\mathrm{d}\tau$, such that the dynamics for the state $x = [x_1, x_2, x_3]^\top$ can be written in the form

$$\begin{aligned}\dot{x}_1 &= x_1 + x_2\\ \dot{x}_2 &= (K + K_{\mathrm{D}} - 1)x_1 + (K_{\mathrm{D}} + 1)x_2 + K_{\mathrm{I}}x_3\\ \dot{x}_3 &= x_1\ .\end{aligned}$$

  Thus, the closed-loop system matrix is given by

$$A_{\mathrm{cl}} = \begin{bmatrix} 1 & 1 & 0\\ K + K_D - 1 & K_D + 1 & K_{\mathrm{I}}\\ 1 & 0 & 0 \end{bmatrix}\ .$$

- Do you need all three control gains in order to asymptotically stabilize the system? Answer this question by answering the following three questions separately:

  1. Is it possible to asymptotically stabilize the system for $K = 0$, $K_{\mathrm{I}} \neq 0$, and $K_{\mathrm{D}} \neq 0$?
  2. Is it possible to asymptotically stabilize the system for $K \neq 0$, $K_{\mathrm{I}} = 0$, and $K_{\mathrm{D}} \neq 0$?
  3. Is it possible to asymptotically stabilize the system for $K \neq 0$, $K_{\mathrm{I}} \neq 0$, and $K_{\mathrm{D}} = 0$?

  Justify your answers! **(3x3 = 9 points)**

  Solution: In order to asymptotically stabilize the linear system, we have to ensure that all the eigenvalues of A_{cl} have strictly negative real part. The characteristic equation is

$$\det(\lambda I - A_{\mathrm{cl}}) = \lambda^3 - (K_D + 2)\lambda^2 + (2 - K)\lambda - K_I = 0\ .$$

We adjust the location of all roots of this equation by tuning the parameters $K, K_{\mathrm{I}}, K_{\mathrm{D}}$. However, the question is if we can do this with only two parameters adjustable. Before starting our analysis, we specify the relation between a cubic polynomial's roots and its coefficients:

$$\begin{aligned} & (\lambda-\lambda_1)(\lambda-\lambda_2)(\lambda-\lambda_3) \\ = \; & \lambda^3-(\lambda_1+\lambda_2+\lambda_3)\lambda^2+(\lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_2\lambda_3)\lambda-\lambda_1\lambda_2\lambda_3\ . \end{aligned}$$

1. If we set $K=0$, the cubic equation takes the form

   $$\lambda^3-(K_D+2)\lambda^2+2\lambda-K_I\ .$$

   We would like the three roots λ_1, λ_2 and λ_3 to satisfy

   $$\lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_2\lambda_3=2\ .$$

   One possible choice is

   $$\lambda_1=-1,\ \lambda_{2,3}=-\frac{1}{2}\pm\sqrt{3}i \quad \text{with control gain} \quad K_{\mathrm{D}}=-4,\ K_{\mathrm{I}}=-1\ .$$

2. If $K_I=0$, we don't need the auxiliary state x_3 and the quadratic characteristics equation is

   $$\lambda^2-(K_{\mathrm{D}}+2)\lambda+2-K=0\ .$$

   In this case, we can tune K and K_{D} such that

   $$K_{\mathrm{D}}+2<0 \quad \text{and} \quad 2-K>0$$

   in order to asymptotically stabilize the closed-loop system.

3. If $K_D=0$, the cubic equation is given by

   $$\lambda^3-2\lambda^2+(2-K)\lambda-K_{\mathrm{I}}=0\ .$$

   But the sum of three roots with negative real part cannot be equal to 2.

In summary, there is no need to tune all three control gains if our only goal to stabilize the asymptotically stabilize the system. The proportional-differential or integral-differential controller are sufficient to asymptotically stabilize this particular linear control system.