

# Signals and Systems

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# Chapter 4: The Continuous-Time Fourier Transform

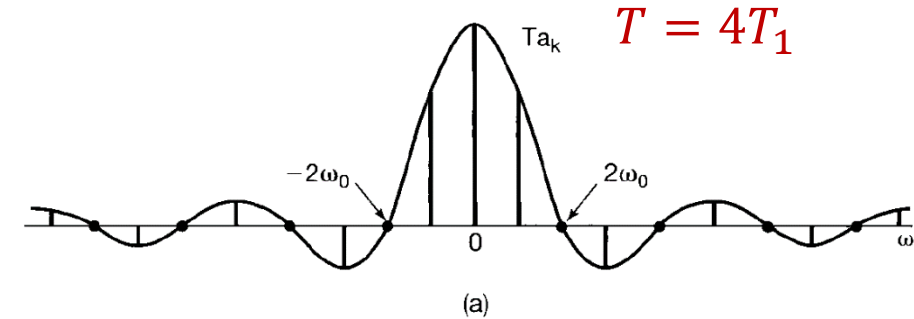
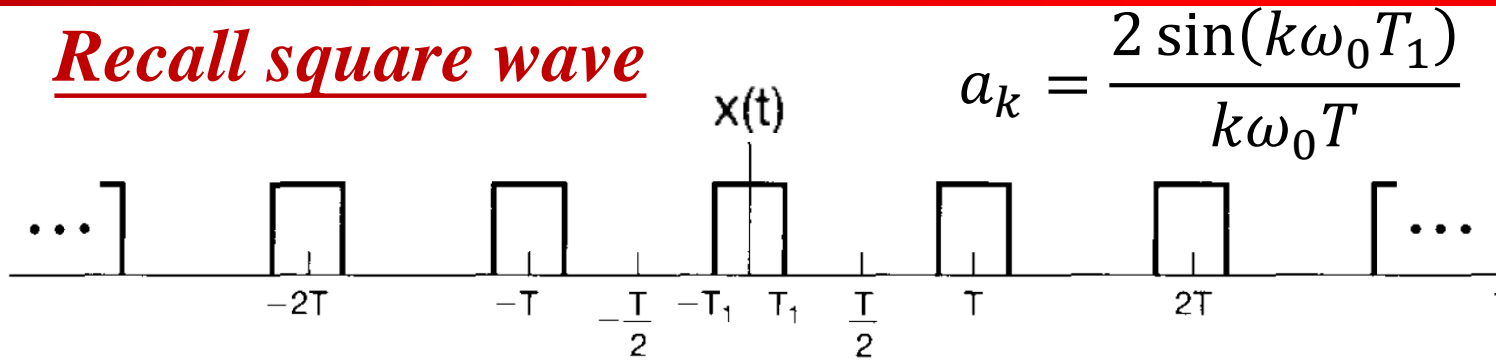
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- ❑ Representation of aperiodic signals- Continuous Fourier Transform
- ❑ Fourier transform for periodic signals
- ❑ Properties of continuous-time Fourier Transform
- ❑ The convolution property
- ❑ The multiplication property
- ❑ System characterized by differential equations



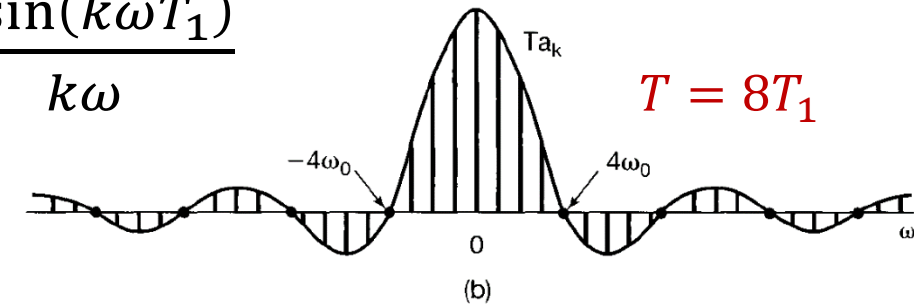
# Continuous Fourier Transform

Recall square wave



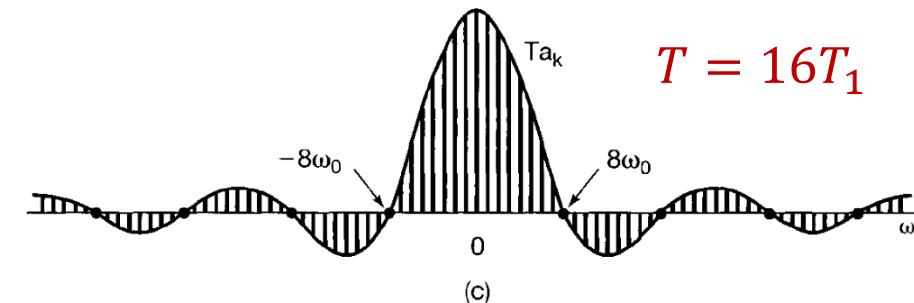
□  $Ta_k$ : Samples of an envelope function  $f(\omega) = \frac{2 \sin(k\omega T_1)}{k\omega}$

$$Ta_k = \left. \frac{2 \sin \omega T_1}{\omega} \right|_{\omega = k\omega_0}$$



□  $T \uparrow, \omega_0 \downarrow \Rightarrow$  the envelope is sampled with closer spacing

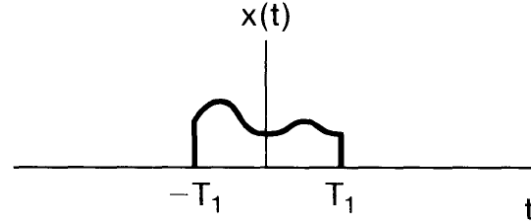
□  $T \rightarrow \infty, \Rightarrow Ta_k \rightarrow$  the envelope  $\frac{2 \sin(k\omega T_1)}{k\omega}$



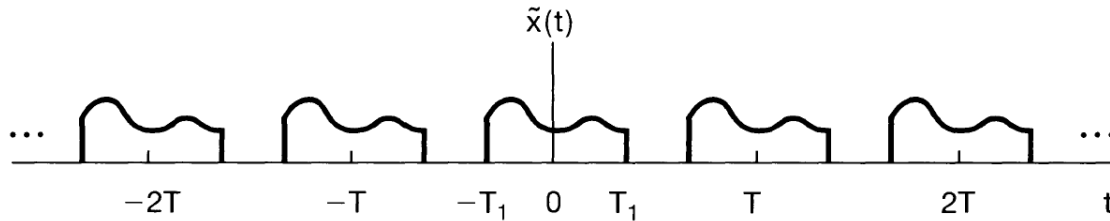
# Continuous Fourier Transform

## Development of FT

- Consider a signal of finite duration,  $x(t) = 0$  if  $|t| > T_1$



- Periodic extension of  $x(t)$  with  $T$



- FS representation of  $\tilde{x}(t)$

$$\tilde{x}(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$$

$$a_k = \frac{1}{T} \int_{-T/2}^{T/2} \tilde{x}(t) e^{-jk\omega_0 t} dt$$

# Continuous Four

## Development of FT

□ FS coefficients of  $\tilde{x}(t)$

$$a_k = \frac{1}{T} \int_{-T/2}^{T/2} \tilde{x}(t) e^{-jk\omega_0 t} dt$$

$$= \frac{1}{T} \int_{-T/2}^{T/2} x(t) e^{-jk\omega_0 t} dt = \frac{1}{T} \int_{-\infty}^{+\infty} x(t) e^{-jk\omega_0 t} dt = \frac{1}{T} X(jk\omega_0)$$

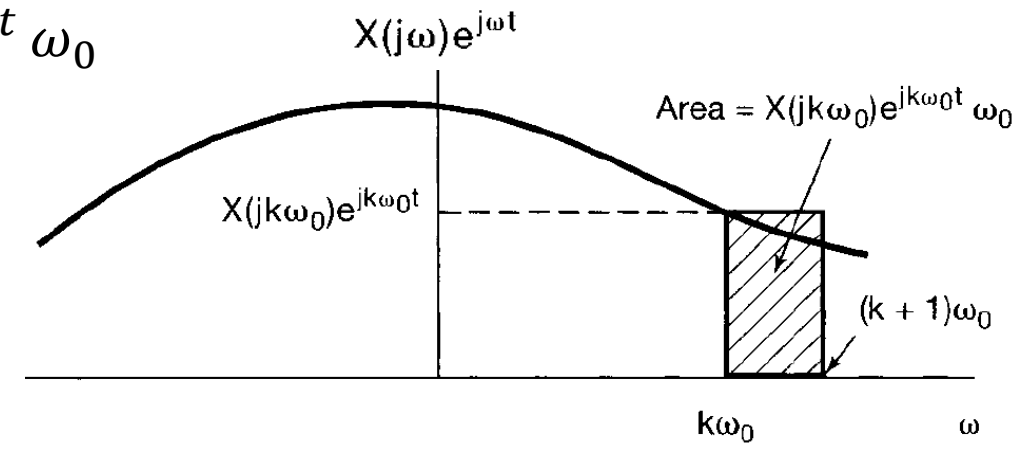
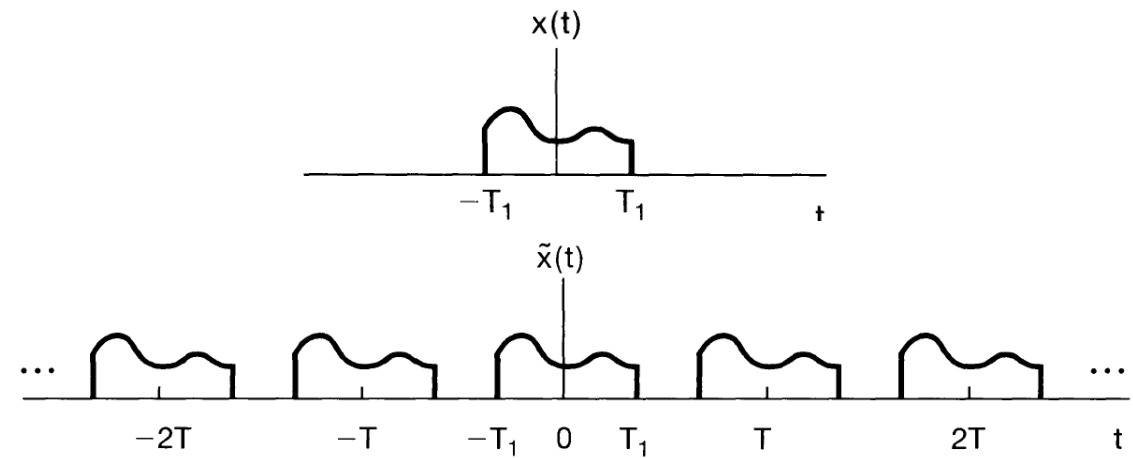
Define  $X(j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt$

□ FS of  $\tilde{x}(t)$

$$\tilde{x}(t) = \sum_{k=-\infty}^{+\infty} \frac{1}{T} X(jk\omega_0) e^{jk\omega_0 t} = \frac{1}{2\pi} \sum_{k=-\infty}^{+\infty} X(jk\omega_0) e^{jk\omega_0 t} \omega_0$$

□  $T \rightarrow \infty, \tilde{x}(t) \rightarrow x(t)$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) e^{j\omega t} d\omega$$



# Continuous Fourier Transform

## FT pairs

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt \quad \text{Fourier transform (FT)}$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) e^{j\omega t} d\omega \quad \text{Inverse Fourier transform}$$

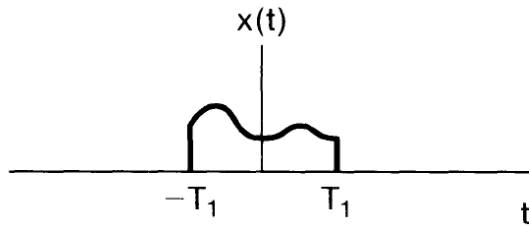
- ❑  $x(t)$  is a linear combination (specifically, an integral) of sinusoidal signals at different frequencies
- ❑  $X(j\omega)(d\omega/2\pi)$  is the weight for different frequencies
- ❑  $X(j\omega)$  is called the spectrum



# Continuous Fourier Transform

## FT vs. FS

### Fourier transform (FT)

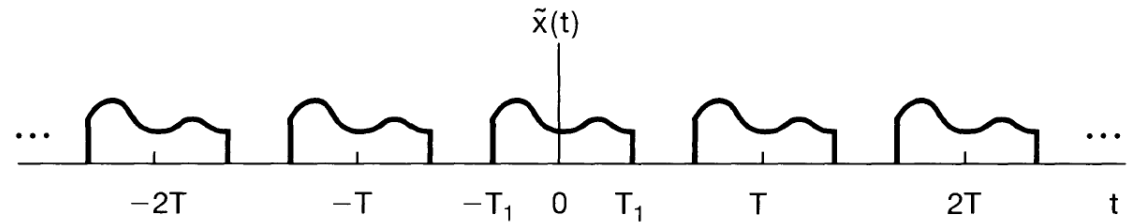


$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) e^{j\omega t} d\omega$$

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt$$

$$a_k = \frac{1}{T} X(j\omega) \text{ with } \omega = k\omega_0$$

### Fourier series (FS)



$$\tilde{x}(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$$

$$a_k = \frac{1}{T} \int_{-T/2}^{T/2} \tilde{x}(t) e^{-jk\omega_0 t} dt$$



# Continuous Fourier Transform

## Convergence of FT

□ Condition 1: Finite energy condition

$$\int_{-\infty}^{+\infty} |x(t)|^2 dt < \infty$$

□ Condition 2: Dirichlet condition

(1) Absolutely integrable  $\int_{-\infty}^{+\infty} |x(t)| dt < \infty$

(2) Finite maxima and minima in any finite interval

(3) Finite number of finite discontinuities in any finite interval





# Continuous Fourier Transform

## Examples

Consider the signal  $x(t) = e^{-at}u(t), a > 0$

Determine its FT

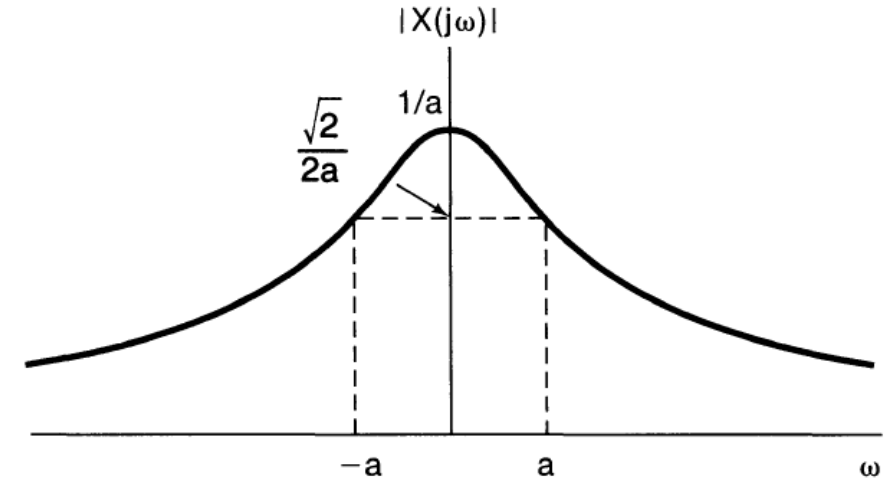
$$X(j\omega) = \int_0^{\infty} e^{-at} e^{-j\omega t} dt$$

$$= -\frac{1}{a + j\omega} e^{-(a+j\omega)t} \Big|_0^{\infty}$$

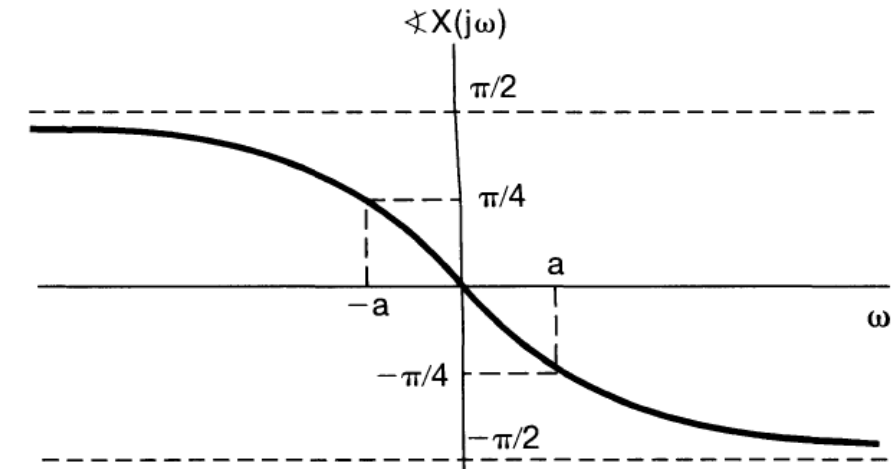
$$= \frac{1}{a + j\omega}, a > 0$$

$$|X(j\omega)| = \frac{1}{\sqrt{a^2 + \omega^2}}$$

$$\angle X(j\omega) = -\tan^{-1} \left( \frac{\omega}{a} \right)$$



(a)



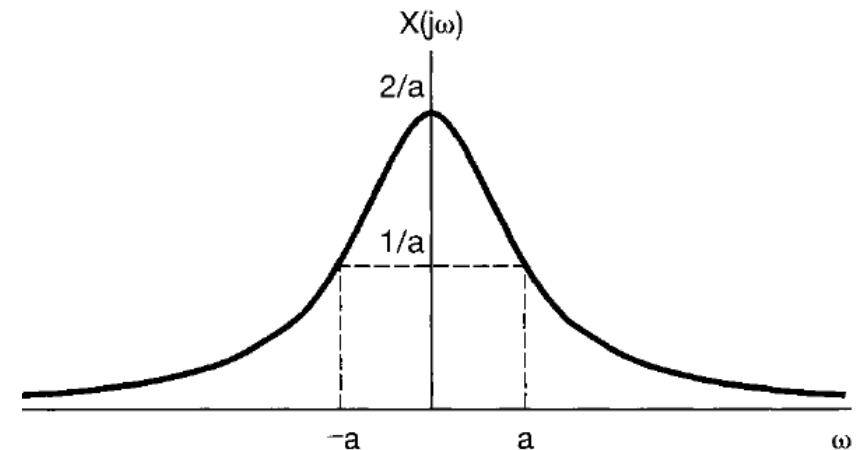
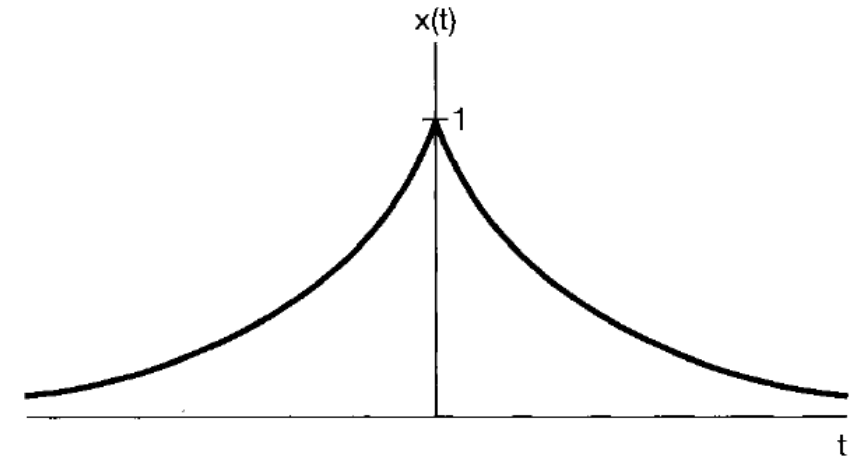
# Continuous Fourier Transform

## Examples

$$x(t) = e^{-a|t|}, a > 0 \quad X(j\omega) = ?$$

Solution

$$\begin{aligned} X(j\omega) &= \int_{-\infty}^{+\infty} e^{-a|t|} e^{-j\omega t} dt \\ &= \int_{-\infty}^0 e^{at} e^{-j\omega t} dt + \int_0^{+\infty} e^{-at} e^{-j\omega t} dt \\ &= \frac{1}{a - j\omega} + \frac{1}{a + j\omega} \\ &= \frac{2a}{a^2 + \omega^2} \end{aligned}$$

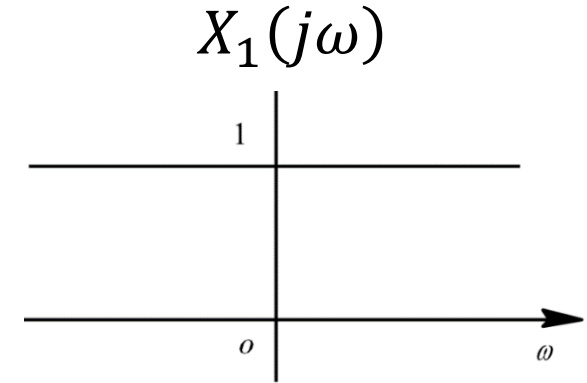
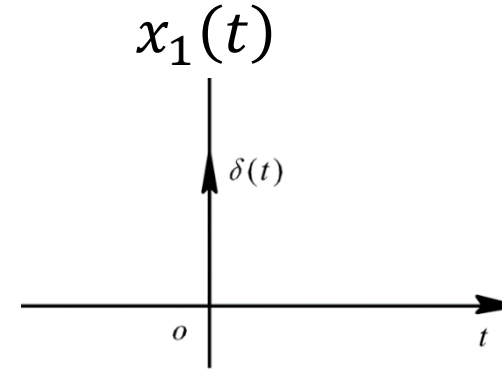


# Continuous Fourier Transform

## Examples

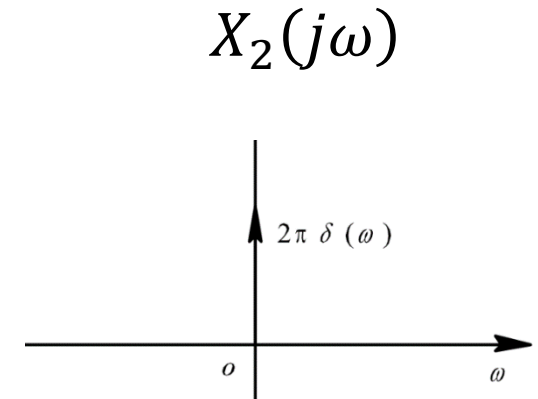
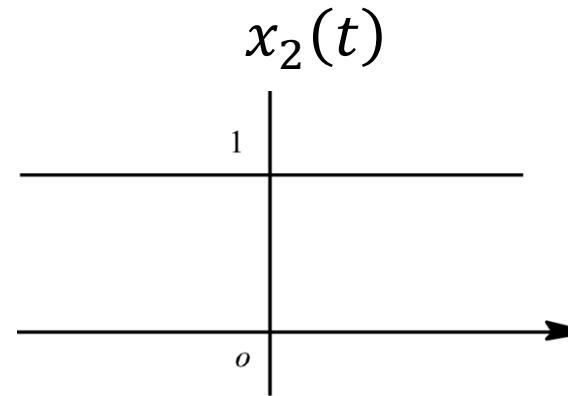
□  $x_1(t) = \delta(t) \quad X_1(j\omega) = ?$

$$X_1(j\omega) = \int_{-\infty}^{+\infty} \delta(t) e^{-j\omega t} dt = 1$$



□  $x_2(t) = 1 \quad X_2(j\omega) = ?$

$$X_2(j\omega) = \int_{-\infty}^{+\infty} e^{-j\omega t} dt = 2\pi\delta(\omega)$$



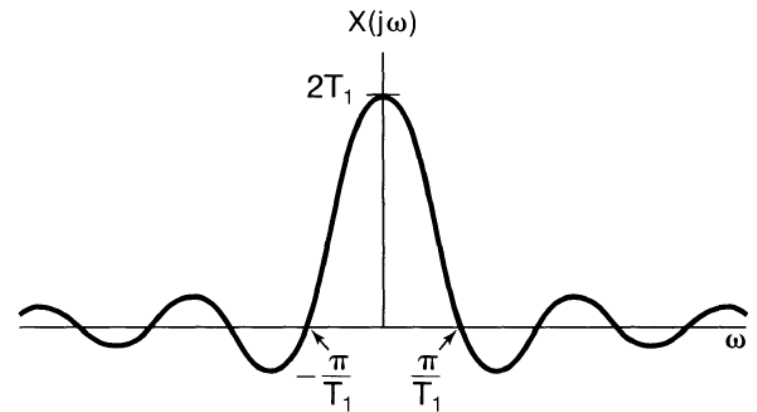
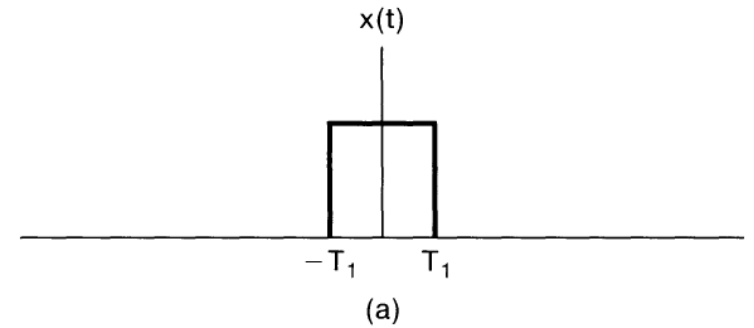
# Continuous Fourier Transform

## Examples

$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & |t| > T_1 \end{cases} \quad X(j\omega) = ?$$

Solution

$$X(j\omega) = \int_{-T_1}^{T_1} e^{-j\omega t} dt = 2 \frac{\sin \omega T_1}{\omega}$$



# Continuous Fourier Transform

## Examples

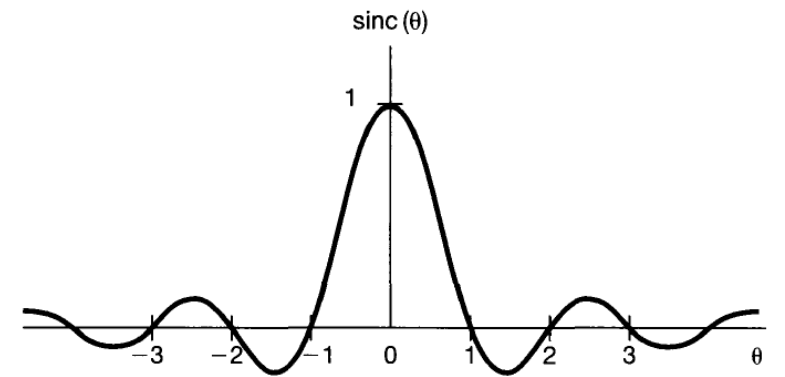
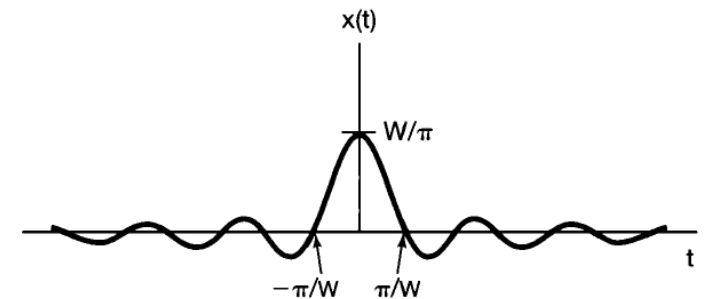
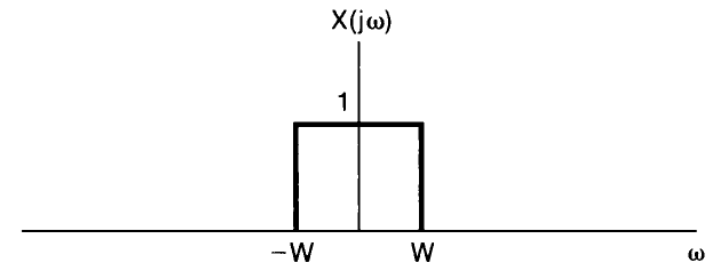
$$X(j\omega) = \begin{cases} 1, & |\omega| < W \\ 0, & |\omega| > W \end{cases} \quad x(t) = ?$$

Solution

$$\begin{aligned} x(t) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) e^{j\omega t} d\omega \\ &= \frac{1}{2\pi} \int_{-W}^W e^{j\omega t} d\omega = \frac{\sin Wt}{\pi t} \end{aligned}$$

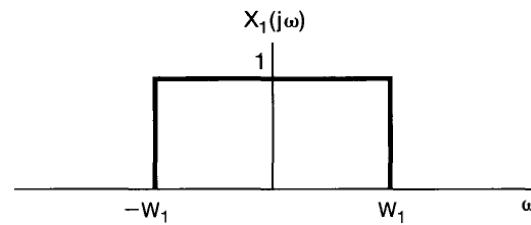
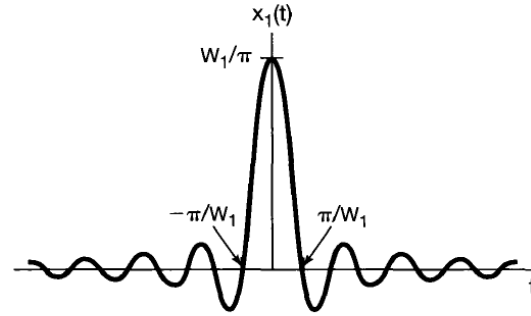
$$\text{sinc}(\theta) = \frac{\sin \pi\theta}{\pi\theta}$$

$$\frac{\sin Wt}{\pi t} = \frac{W}{\pi} \frac{\sin Wt}{Wt} = \frac{W}{\pi} \text{sinc}\left(\frac{Wt}{\pi}\right)$$

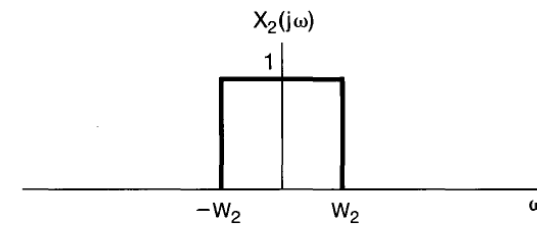
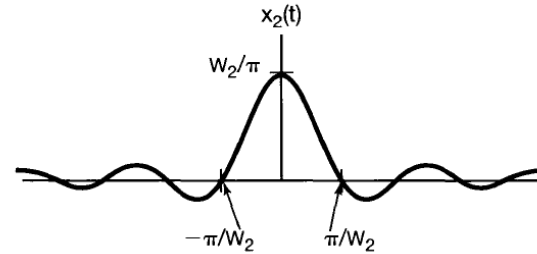


# Continuous Fourier Transform

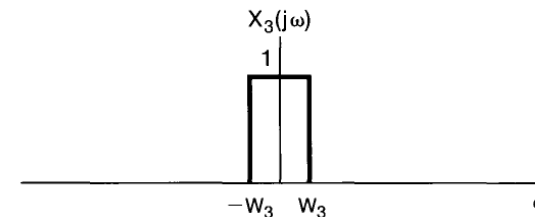
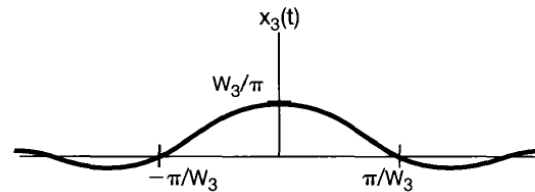
## Examples



(a)



(b)



(c)



# Chapter 4: The Continuous-Time Fourier Transform

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- ❑ Representation of aperiodic signals- Continuous Fourier Transform
- ❑ Fourier transform for periodic signals
- ❑ Properties of continuous-time Fourier Transform
- ❑ The convolution property
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- ❑ System characterized by differential equations



# Fourier transform for periodic signals

□ A period signal can be represented by a FS, but also a FT

$$x(t) = \sum_{K=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) e^{j\omega t} d\omega$$

□ The relationship between  $a_k$  and  $X(j\omega)$ ?

➤ Consider  $x_1(t) = a_k e^{jk\omega_0 t}$ , the FT of  $x_1(t)$ :  $X_1(j\omega)=?$

$$x_1(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X_1(j\omega) e^{j\omega t} d\omega = a_k e^{jk\omega_0 t} \quad \Rightarrow \quad X_1(j\omega) = 2\pi a_k \delta(\omega - k\omega_0)$$

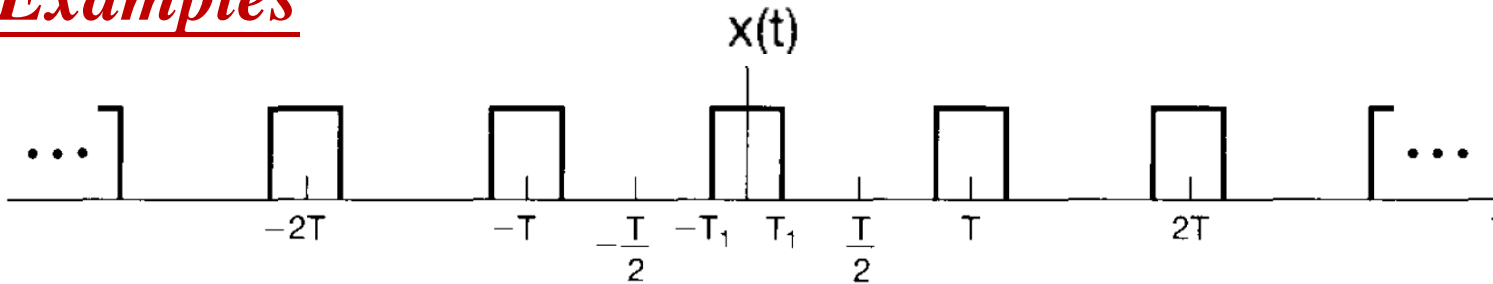
$$\text{➤ For } x(t) = \sum_{K=-\infty}^{\infty} a_k e^{jk\omega_0 t} \quad X(j\omega) = \sum_{K=-\infty}^{\infty} a_k 2\pi \delta(\omega - k\omega_0)$$





# Fourier transform for periodic signals

## Examples

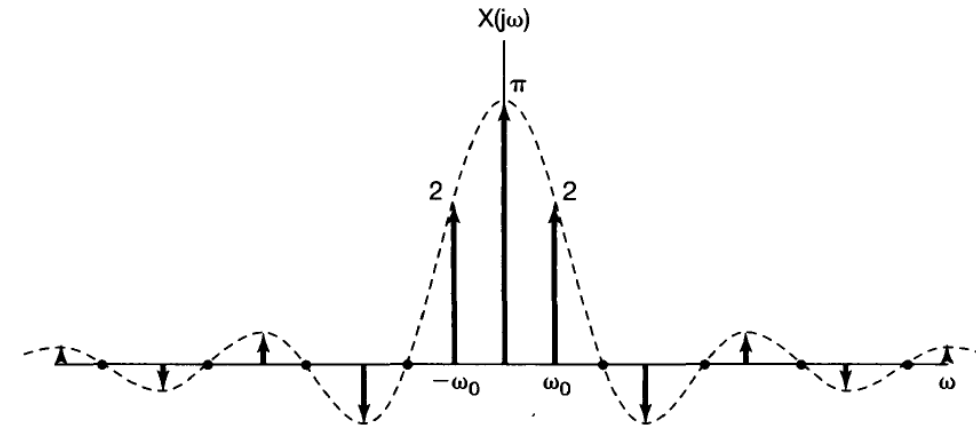


$X(j\omega)?$

## Solution

$$x(t) = \sum_{K=-\infty}^{\infty} a_k e^{jk\omega_0 t} \quad a_k = \frac{\sin(k\omega_0 T_1)}{\pi k}$$

$$\begin{aligned} X(j\omega) &= \sum_{K=-\infty}^{\infty} a_k 2\pi \delta(\omega - k\omega_0) \\ &= \sum_{K=-\infty}^{\infty} \frac{2 \sin(k\omega_0 T_1)}{k} \delta(\omega - k\omega_0) \end{aligned}$$

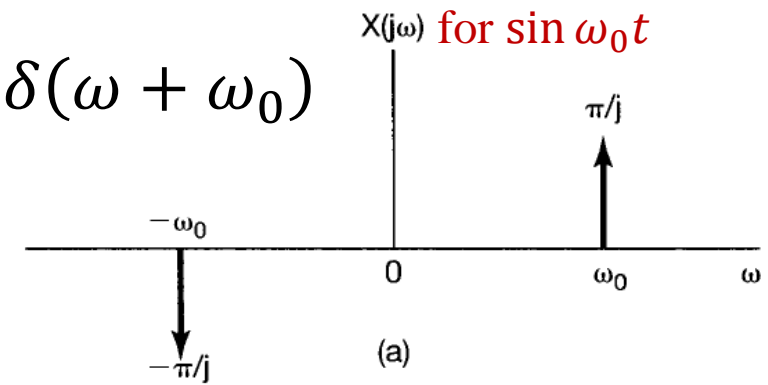


# Fourier transform for periodic signals

## Examples

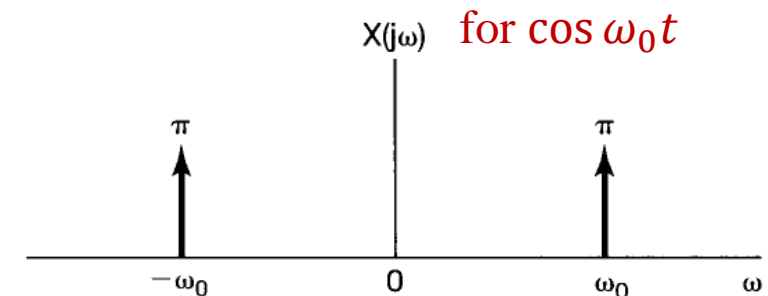
$$x_1(t) = \sin \omega_0 t \quad a_1 = 1/2j \quad a_{-1} = -1/2j \quad a_k = 0, k \neq \pm 1$$

$$X_1(j\omega) = \sum_{K=-\infty}^{\infty} a_k 2\pi \delta(\omega - k\omega_0) = \frac{\pi}{j} \delta(\omega - \omega_0) - \frac{\pi}{j} \delta(\omega + \omega_0)$$



$$x_2(t) = \cos \omega_0 t \quad a_k = 1/2, k = \pm 1, a_k = 0, k \neq \pm 1$$

$$X_1(j\omega) = \pi \delta(\omega - \omega_0) + \pi \delta(\omega + \omega_0)$$



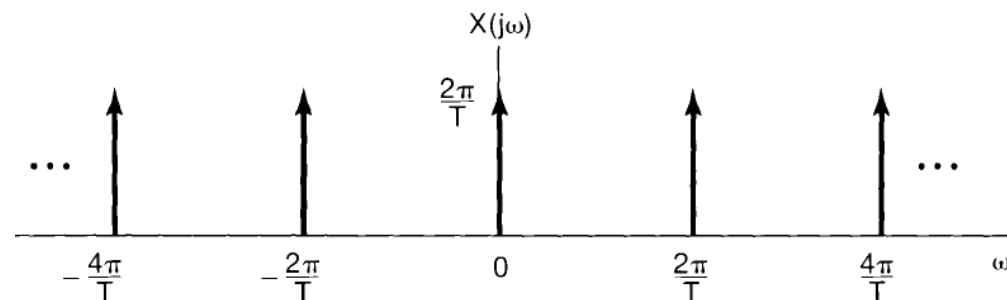
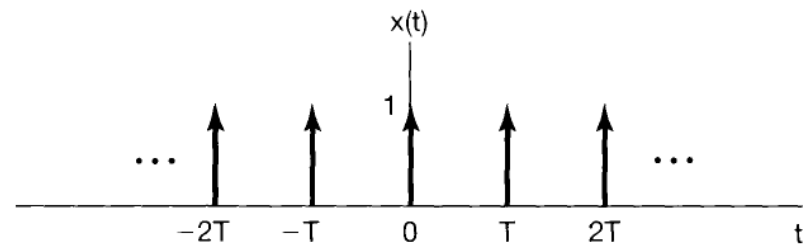
# Fourier transform for periodic signals

## Examples

$$x(t) = \sum_{K=-\infty}^{\infty} \delta(t - kT)$$

$$a_k = \frac{1}{T} \int_{-T/2}^{T/2} \delta(t) e^{-jk\omega_0 t} dt = \frac{1}{T}$$

$$\begin{aligned} X(j\omega) &= \frac{2\pi}{T} \sum_{K=-\infty}^{\infty} \delta(\omega - k\omega_0) \\ &= \frac{2\pi}{T} \sum_{K=-\infty}^{\infty} \delta\left(\omega - \frac{2k\pi}{T}\right) \end{aligned}$$



# Chapter 4: The Continuous-Time Fourier Transform

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# Properties of continuous-time Fourier Transform

## Short notation for FT pairs

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) e^{j\omega t} d\omega \quad X(j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt$$

$$x(t) \xleftrightarrow{\mathcal{F}} X(j\omega)$$

$$X(j\omega) = \mathcal{F}\{x(t)\}$$

$$x(t) = \mathcal{F}^{-1}\{X(j\omega)\}$$



# Properties of continuous-time Fourier Transform

## Linearity

$$x(t) \xleftrightarrow{\mathcal{F}} X(j\omega) \quad y(t) \xleftrightarrow{\mathcal{F}} Y(j\omega)$$



$$ax(t) + bx(t) \xleftrightarrow{\mathcal{F}} aX(j\omega) + bX(j\omega)$$



# Properties of continuous-time Fourier Transform

## Time shifting

$$x(t) \xleftrightarrow{\mathcal{F}} X(j\omega) \Rightarrow x(t - t_0) \xleftrightarrow{\mathcal{F}} e^{-j\omega t_0} X(j\omega)$$

□ Proof

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) e^{j\omega t} d\omega$$

$$x(t - t_0) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) e^{j\omega(t-t_0)} d\omega = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-j\omega t_0} X(j\omega) e^{j\omega t} d\omega$$

$$\mathcal{F}\{x(t)\} = X(j\omega) = |X(j\omega)| e^{j\angle X(j\omega)}$$

$$\mathcal{F}\{x(t - t_0)\} = e^{-j\omega t_0} X(j\omega) = |X(j\omega)| e^{j[\angle X(j\omega) - \omega t_0]}$$

□ A time shift on a signal introduces a phase shift into its FT,  $-\omega t_0$ , which is a linear function of  $\omega$ .



# Properties of continuous-time Fourier Transform

## Examples

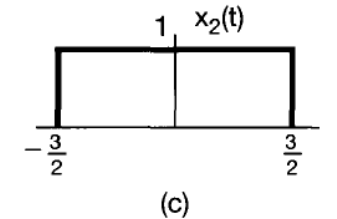
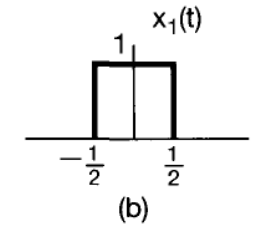
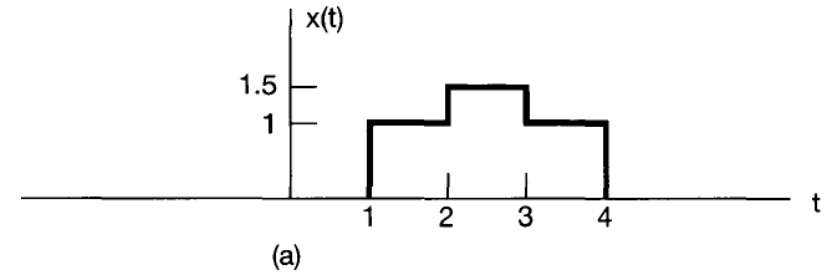
□  $x(t)$  can be expressed as

$$x(t) = \frac{1}{2}x_1(t - 2.5) + x_2(t - 2.5)$$

$$X_1(j\omega) = 2 \frac{\sin \omega T_1}{\omega} = 2 \frac{\sin \omega/2}{\omega}$$

$$X_2(j\omega) = 2 \frac{\sin 3\omega/2}{\omega}$$

$$X(j\omega) = e^{-j5\omega/2} \left( \frac{\sin \omega/2 + 2 \sin 3\omega/2}{\omega} \right)$$





# Properties of continuous-time Fourier Transform

## Conjugation and Conjugate Symmetry

□ Conjugation property  $x(t) \xleftrightarrow{\mathcal{F}} X(j\omega) \Rightarrow x^*(t) \xleftrightarrow{\mathcal{F}} X^*(-j\omega)$

$$X^*(j\omega) = \left[ \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt \right]^* = \int_{-\infty}^{+\infty} x^*(t) e^{j\omega t} dt$$

$$X^*(-j\omega) = \int_{-\infty}^{+\infty} x^*(t) e^{-j\omega t} dt = \mathcal{F}\{x^*(t)\}$$

## □ Conjugation Symmetry

$$X(-j\omega) = X^*(j\omega) \quad [x(t) \text{ real}]$$

For a real-valued signal, the FT need only to be specified for positive frequencies.



# Properties of continuous-time Fourier Transform

Time reversing

$$x(t) \xleftrightarrow{\mathcal{F}} X(j\omega) \Rightarrow x(-t) \xleftrightarrow{\mathcal{F}} X(-j\omega)$$

□  $x(t)$  even  $\Rightarrow X(j\omega) = X(-j\omega)$ ,  $x(t)$  real  $\Rightarrow X(-j\omega) = X^*(j\omega)$

□  $x(t)$  real and even  $\Rightarrow X(j\omega)$  real and even

□  $x(t)$  real and odd  $\Rightarrow X(j\omega)$  purely imaginary and odd

□ If  $x(t)$  real

$$\left. \begin{aligned} x(t) &= x_e(t) + x_o(t) \\ \mathcal{F}\{x(t)\} &= \mathcal{F}\{x_e(t)\} + \mathcal{F}\{x_o(t)\} \end{aligned} \right\} \Leftrightarrow \begin{cases} E_v\{x(t)\} \xleftrightarrow{\mathcal{F}} R_e\{X(j\omega)\} \\ O_d\{x(t)\} \xleftrightarrow{\mathcal{F}} j \cdot I_m\{X(j\omega)\} \end{cases}$$



# Properties of continuous-time Fourier Transform

## Example

□ For  $a > 0$

$$e^{-at}u(t) \xleftrightarrow{\mathcal{F}} 1/(a + j\omega)$$

$$e^{-a|t|} \xleftrightarrow{\mathcal{F}} 2a/(a^2 + \omega^2)$$

□ use FT properties

$$e^{-a|t|} = e^{-at}u(t) + e^{at}u(-t) = 2E_v\{e^{-at}u(t)\}$$

$$E_v\{e^{-at}u(t)\} \xleftrightarrow{\mathcal{F}} R_e \left\{ \frac{1}{a + j\omega} \right\}$$

$$\mathcal{F}\{e^{-a|t|}\} = 2R_e \left\{ \frac{1}{a + j\omega} \right\} = \frac{2a}{a^2 + \omega^2}$$



# Properties of continuous-time Fourier Transform

## Differential and integration

$$x(t) \xleftrightarrow{\mathcal{F}} X(j\omega) \Rightarrow \boxed{\frac{dx(t)}{dt} \xleftrightarrow{\mathcal{F}} j\omega X(j\omega)} \quad \boxed{\int_{-\infty}^t x(\tau) d\tau \xleftrightarrow{\mathcal{F}} \frac{1}{j\omega} X(j\omega) + \pi X(0)\delta(\omega)}$$

□ Proof

$$\frac{dx(t)}{dt} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) \frac{d(e^{j\omega t})}{dt} d\omega = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) \cdot j\omega \cdot e^{j\omega t} d\omega$$

$$\int_{-\infty}^t x(\tau) d\tau = \int_{-\infty}^t \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) e^{j\omega\tau} d\omega d\tau = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) \int_{-\infty}^t e^{j\omega\tau} d\tau d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{X(j\omega)}{j\omega} e^{j\omega t} d\omega, \omega \neq 0$$

$\pi X(0)\delta(\omega)$       DC components



# Properties of continuous-time Fourier Transform

Example FT of unit step  $x(t) = u(t)$

$$g(t) = \delta(t) \xleftrightarrow{\mathcal{F}} G(j\omega) = 1 \quad x(t) = u(t) = \int_{-\infty}^t g(\tau) d\tau$$

□ Use integration property

$$X(j\omega) = \frac{1}{j\omega} G(j\omega) + \pi X(0) \delta(\omega) = \frac{1}{j\omega} + \pi \delta(\omega)$$

□ Recover  $G(j\omega)$  by differential property

$$\delta(t) = \frac{du(t)}{dt} \xleftrightarrow{\mathcal{F}} j\omega \left[ \frac{1}{j\omega} + \pi \delta(\omega) \right] = 1$$

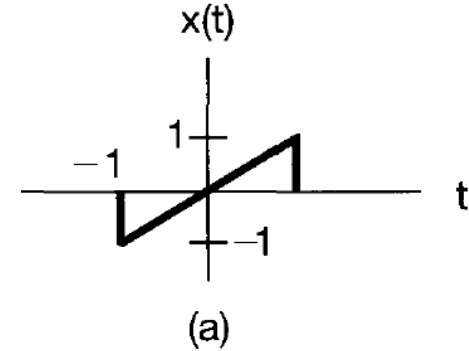


# Properties of continuous-time Fourier Transform

## Example

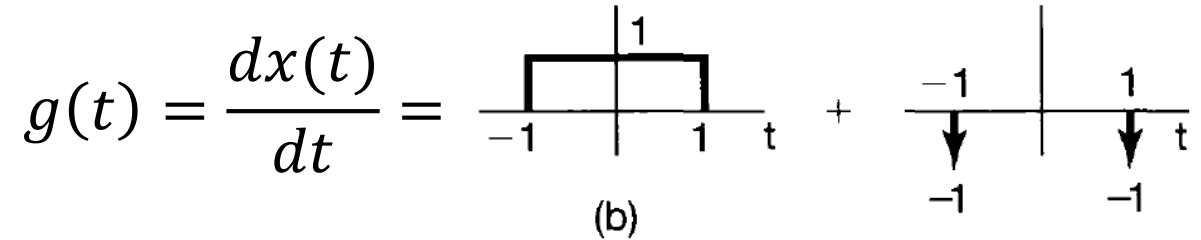
Determine the FT of  $x(t)$

## □ Solution



$$g(t) = \frac{dx(t)}{dt}$$

$$G(j\omega) = \frac{2 \sin \omega}{\omega} - e^{j\omega} - e^{-j\omega}$$



use integration property

$$X(j\omega) = \frac{1}{j\omega} G(j\omega) + \pi G(0) \delta(\omega) = \frac{2 \sin \omega}{j\omega^2} - \frac{2 \cos \omega}{j\omega}$$



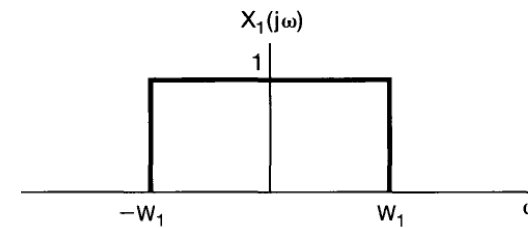
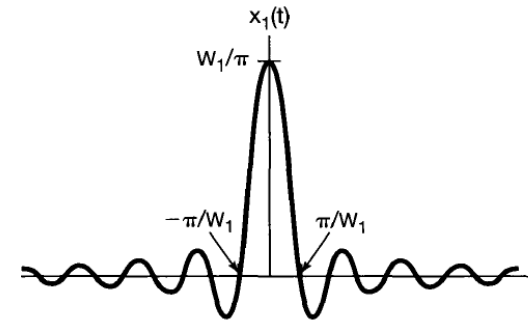
# Properties of continuous-time Fourier Transform

## Time and frequency scaling

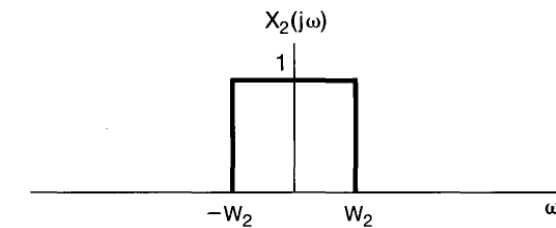
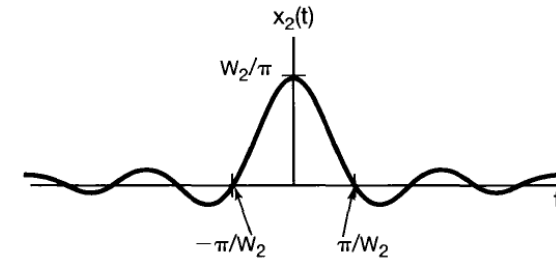
$$x(t) \xleftrightarrow{\mathcal{F}} X(j\omega) \Rightarrow \boxed{x(at) \xleftrightarrow{\mathcal{F}} \frac{1}{|a|} X\left(\frac{j\omega}{a}\right)} \quad a \neq 0$$

$$\mathcal{F}\{x(at)\} = \int_{-\infty}^{\infty} x(at) e^{-j\omega t} dt$$

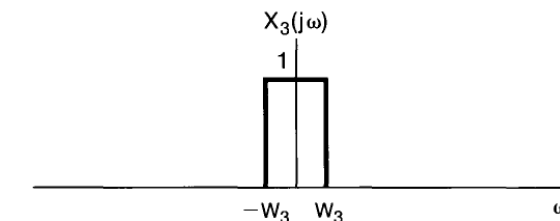
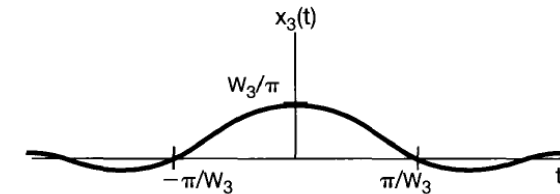
$$\mathcal{F}\{x(at)\} = \begin{cases} \frac{1}{a} \int_{-\infty}^{+\infty} x(\tau) e^{-j(\omega/a)\tau} d\tau, & a > 0 \\ -\frac{1}{a} \int_{-\infty}^{+\infty} x(\tau) e^{-j(\omega/a)\tau} d\tau, & a < 0 \end{cases}$$



(a)



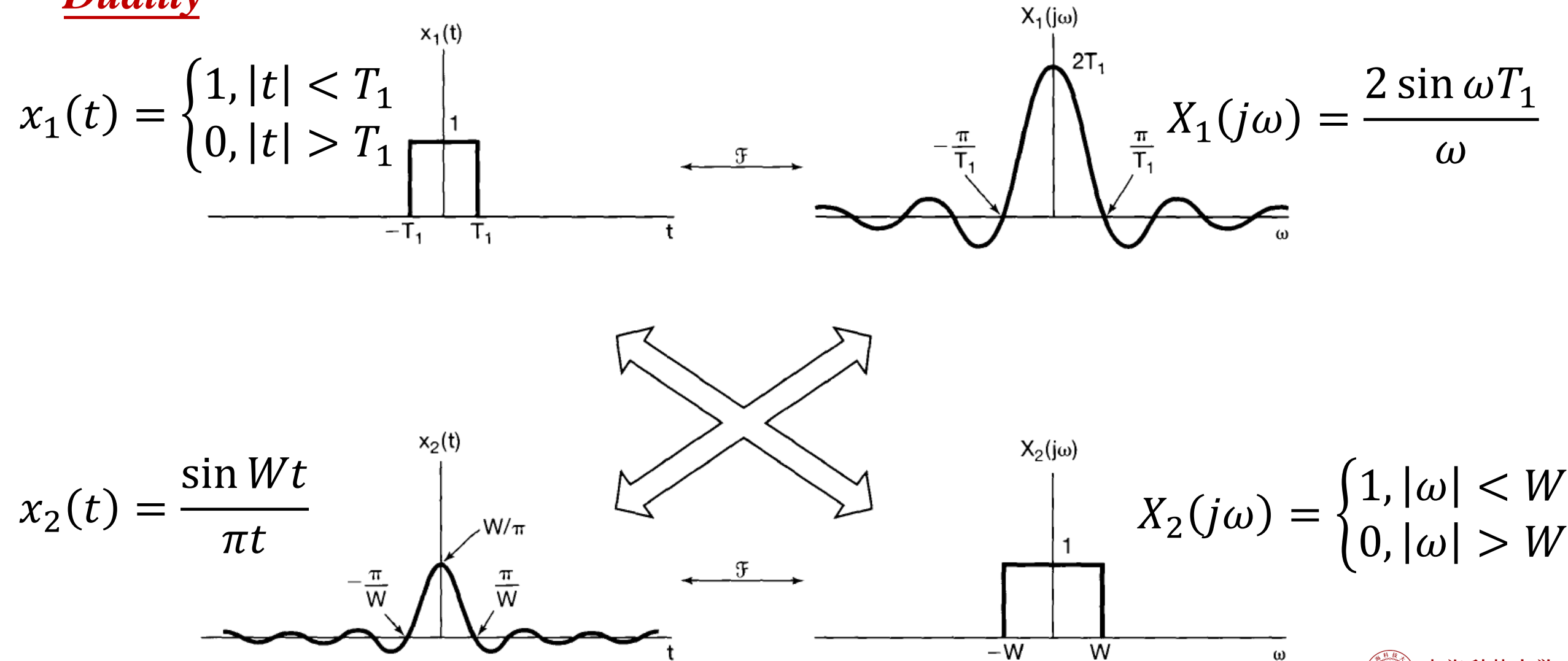
(b)



(c)

# Properties of continuous-time Fourier Transform

## Duality





# Properties of continuous-time Fourier Transform

## Duality

$$x(t) \xleftrightarrow{\mathcal{F}} X(j\omega) \Rightarrow X(t) \xleftrightarrow{\mathcal{F}} 2\pi x(-j\omega)$$

## □ Proof

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) e^{j\omega t} d\omega$$

$$x(-t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) e^{-j\omega t} d\omega$$

## Interchange $\omega$ and $t$

$$x(-j\omega) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(t) e^{-j\omega t} dt$$

$$2\pi x(-j\omega) = \int_{-\infty}^{+\infty} X(t) e^{-j\omega t} dt \Rightarrow X(t) \xleftrightarrow{\mathcal{F}} 2\pi x(-j\omega)$$

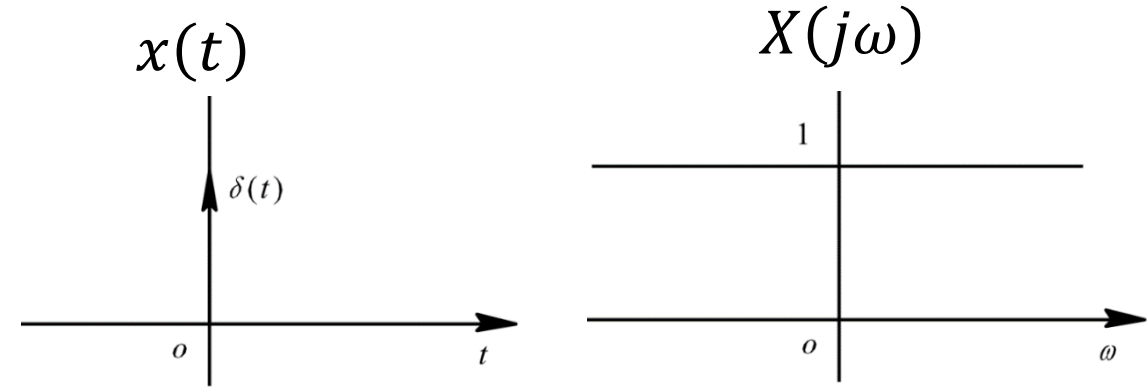


# Properties of continuous-time Fourier Transform

## Example

$$x(t) = \delta(t) \quad X(j\omega) = 1$$

$$x(t) = 1 \quad X(j\omega) = 2\pi\delta(\omega)$$

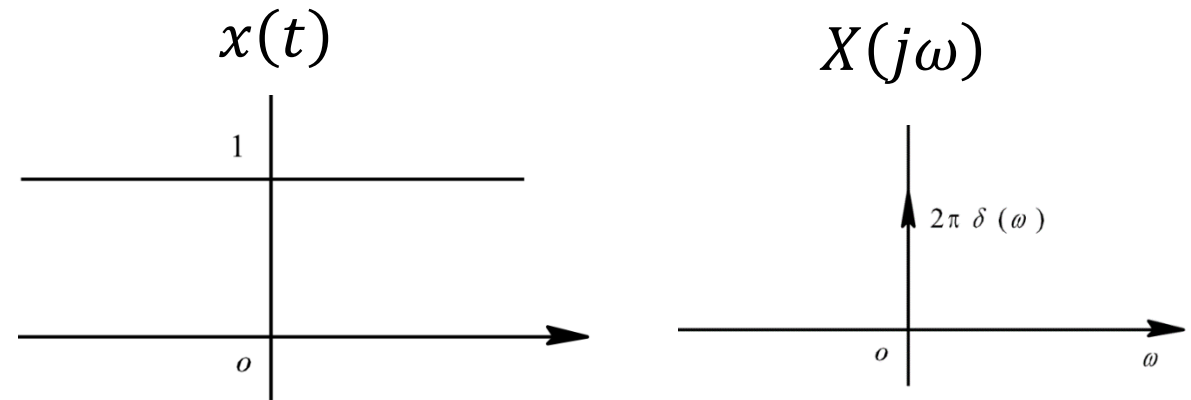


## Principle

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) \cdot e^{j\omega t} d\omega$$

$$x(j\omega) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(t) \cdot e^{j\omega t} dt$$

$$x(-j\omega) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(t) \cdot e^{-j\omega t} dt$$



# Properties of continuous-time Fourier Transform

Example

$$g(t) = \frac{2}{1+t^2} \quad G(j\omega) = ?$$

Solution: calculate  $G(j\omega)$  is difficult; use duality property

$$e^{-a|t|} \xleftrightarrow{\mathcal{F}} 2a/(a^2 + \omega^2)$$

$$e^{-|t|} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{2}{1+\omega^2} \cdot e^{j\omega t} d\omega$$

$$t \longrightarrow -t$$

$$e^{-|t|} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{2}{1+\omega^2} \cdot e^{-j\omega t} d\omega$$

$$2\pi e^{-|\omega|} = \int_{-\infty}^{+\infty} \frac{2}{1+t^2} \cdot e^{-j\omega t} dt$$

$$\therefore G(j\omega) = 2\pi e^{-|\omega|}$$



# Properties of continuous-time Fourier Transform

## Example

Duality property can determine or suggest other FT properties

$$\frac{dx(t)}{dt} \xleftrightarrow{\mathcal{F}} j\omega X(j\omega).$$

 $\Leftrightarrow$ 

$$-jtx(t) \xleftrightarrow{\mathcal{F}} \frac{dX(j\omega)}{d\omega}.$$

$$\int_{-\infty}^t x(\tau) d\tau \xleftrightarrow{\mathcal{F}} \frac{1}{j\omega} X(j\omega) + \pi X(0) \delta(\omega).$$

 $\Leftrightarrow$ 

$$-\frac{1}{jt} x(t) + \pi x(0) \delta(t) \xleftrightarrow{\mathcal{F}} \int_{-\infty}^{\omega} x(\eta) d\eta.$$

$$x(t - t_0) \xleftrightarrow{\mathcal{F}} e^{-j\omega t_0} X(j\omega).$$

 $\Leftrightarrow$ 

$$e^{j\omega_0 t} x(t) \xleftrightarrow{\mathcal{F}} X(j(\omega - \omega_0))$$



# Properties of continuous-time Fourier Transform

## Parseval's relation

$$\int_{-\infty}^{+\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |X(j\omega)|^2 d\omega$$

$$\begin{aligned} \int_{-\infty}^{+\infty} |x(t)|^2 dt &= \int_{-\infty}^{+\infty} x(t)x^*(t) dt \\ &= \int_{-\infty}^{+\infty} x(t) \left[ \frac{1}{2\pi} \int_{-\infty}^{+\infty} X^*(j\omega) e^{-j\omega t} d\omega \right] dt \\ &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} X^*(j\omega) \left[ \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt \right] d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} |X(j\omega)|^2 d\omega \end{aligned}$$



# Properties of continuous-time Fourier Transform

## Parseval's relation

Fourier transform

$$\int_{-\infty}^{+\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |X(j\omega)|^2 d\omega$$

Fourier series

$$\frac{1}{T} \int_T |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |a_k|^2$$



# Chapter 4: The Continuous-Time Fourier Transform

- ❑ Representation of aperiodic signals- Continuous Fourier Transform
- ❑ Fourier transform for periodic signals
- ❑ Properties of continuous-time Fourier Transform
- ❑ The convolution property
- ❑ The multiplication property
- ❑ System characterized by differential equations



# The convolution property

$$y(t) = x(t) * h(t) \xleftrightarrow{\mathcal{F}} Y(j\omega) = H(j\omega)X(j\omega)$$

## □ Proof

$$\begin{aligned} Y(j\omega) &= \mathcal{F}\{y(t)\} = \int_{-\infty}^{+\infty} \left[ \int_{-\infty}^{+\infty} x(\tau) h(t - \tau) d\tau \right] e^{-j\omega t} dt \\ &= \int_{-\infty}^{+\infty} x(\tau) \left[ \int_{-\infty}^{+\infty} h(t - \tau) e^{-j\omega t} dt \right] d\tau \\ &= \int_{-\infty}^{+\infty} x(\tau) e^{-j\omega \tau} H(j\omega) d\tau = H(j\omega) \int_{-\infty}^{+\infty} x(\tau) e^{-j\omega \tau} d\tau = H(j\omega)X(j\omega) \end{aligned}$$

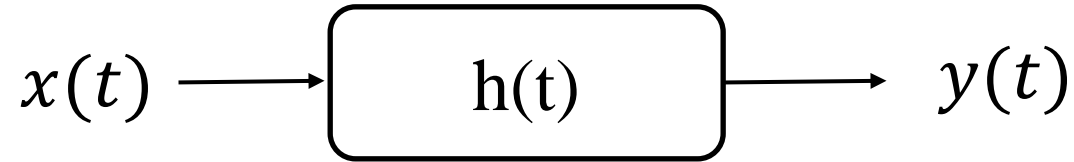
- $H(j\omega)$ : Frequency response; important for analyzing LTI systems
- Only stable continuous-time LTI systems have  $H(j\omega)$
- Non-stable continuous-time LTI system: Laplace transform





# The convolution property

## Example



□ Assume  $h(t) = \delta(t - t_0)$ ,  $\mathcal{F}\{x(t)\} = X(j\omega)$ , determine  $Y(j\omega)$

□ Solution 1

$$H(j\omega) = e^{-j\omega t_0} \quad Y(j\omega) = H(j\omega)X(j\omega) = e^{-j\omega t_0}X(j\omega)$$

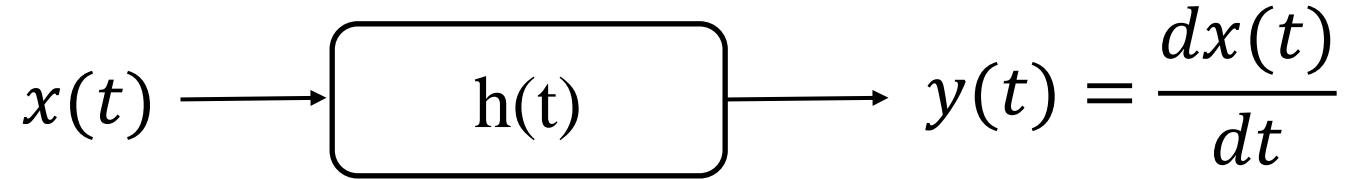
□ Solution 2

$$y(t) = x(t - t_0) \quad Y(j\omega) = e^{-j\omega t_0}X(j\omega)$$



# The convolution property

## Example



□ Differentiation property  $\Rightarrow Y(j\omega) = j\omega X(j\omega)$

□ Convolution property  $\Rightarrow Y(j\omega) = H(j\omega)X(j\omega)$

□ Therefore,  $H(j\omega) = j\omega$



# The convolution property

## Example

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t) = \int_{-\infty}^t x(\tau) d\tau \quad Y(j\omega) = ?$$

□ Assume  $\mathcal{F}\{x(t)\} = X(j\omega)$

$$h(t) = \int_{-\infty}^t \delta(\tau) d\tau = u(t)$$

□ Frequency response  $H(j\omega) = \frac{1}{j\omega} + \pi\delta(\omega)$

□ Convolution property  $Y(j\omega) = H(j\omega)X(j\omega)$

$$Y(j\omega) = \frac{1}{j\omega} X(j\omega) + \pi X(0)\delta(\omega)$$

□ Consistent with integration property



# The convolution property

## The Fourier Transform of the step function

$$u(t) \xleftrightarrow{\mathcal{F}} H(j\omega) = \frac{1}{j\omega} + \pi\delta(\omega)$$

□ Proof

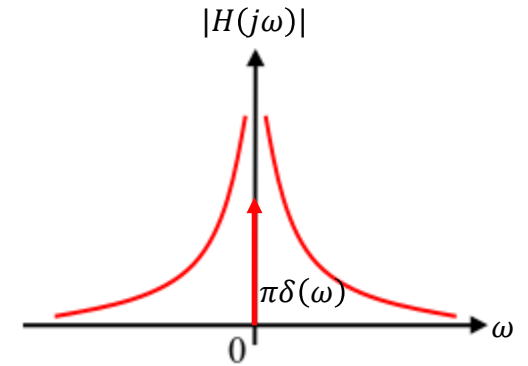
Duality

$$u(t) = \frac{1}{2} + \frac{1}{2} \operatorname{sgn}(t) \xleftrightarrow{\mathcal{F}} H(j\omega) = \frac{1}{2} [2\pi\delta(\omega) + \mathcal{F}[\operatorname{sgn}(t)]]$$

$\int_{-\infty}^{+\infty} \operatorname{sgn}(t) e^{-j\omega t} dt$  is not absolutely integrable.  $\mathcal{F}[\operatorname{sgn}(t)]$  cannot be found directly.

Noted that  $\operatorname{sgn}(t) = p(t) = \lim_{a \rightarrow 0} e^{-a|t|} \operatorname{sgn}(t) = \lim_{a \rightarrow 0} [e^{-at} u(t) - e^{at} u(-t)]$

$$\mathcal{F}[\operatorname{sgn}(t)] = P(j\omega) = \int_{-\infty}^{\infty} (\lim_{a \rightarrow 0} [e^{-at} u(t) - e^{at} u(-t)]) e^{-j\omega t} dt$$



# The convolution property

## The Fourier Transform of the step function

[Check here](https://www.tutorialspoint.com/fourier-transform-of-signum-function)

<https://www.tutorialspoint.com/fourier-transform-of-signum-function>

$$= \lim_{a \rightarrow 0} \left[ \int_{-\infty}^{\infty} e^{-at} u(t) e^{-j\omega t} dt - \int_{-\infty}^{\infty} e^{at} u(-t) e^{-j\omega t} dt \right]$$

$$= \lim_{a \rightarrow 0} \left[ \int_0^{\infty} e^{-(a+j\omega)t} dt - \int_{-\infty}^0 e^{(a-j\omega)t} dt \right] = \lim_{a \rightarrow 0} \left[ \int_0^{\infty} e^{-(a+j\omega)t} dt - \int_0^{\infty} e^{-(a-j\omega)t} dt \right]$$

$$= \lim_{a \rightarrow 0} \left\{ \left[ \frac{e^{-(a+j\omega)t}}{-(a+j\omega)} \right]_0^{\infty} - \left[ \frac{e^{-(a-j\omega)t}}{-(a-j\omega)} \right]_0^{\infty} \right\} = \lim_{a \rightarrow 0} \left\{ \left[ \frac{e^{-\infty} - e^0}{-(a+j\omega)} \right] - \left[ \frac{e^{-\infty} - e^0}{-(a-j\omega)} \right] \right\}$$

$$= \lim_{a \rightarrow 0} \left\{ \frac{1}{(a+j\omega)} - \frac{1}{(a-j\omega)} \right\} = \frac{2}{j\omega}$$

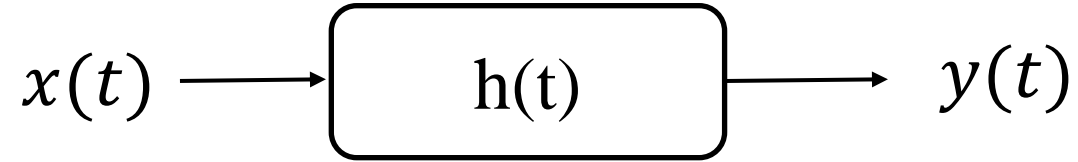
Hence

$$H(j\omega) = \frac{1}{2} [2\pi\delta(\omega) + P(j\omega)] = \frac{1}{j\omega} + \pi\delta(\omega)$$



# The convolution property

## Example



$$h(t) = e^{-at}u(t), a > 0 \quad x(t) = e^{-bt}u(t), b > 0 \quad y(t) = ?$$

□ Solution  $b \neq a$

$$H(j\omega) = \frac{1}{a + j\omega} \quad X(j\omega) = \frac{1}{b + j\omega} \quad Y(j\omega) = \frac{1}{(a + j\omega)(b + j\omega)}$$

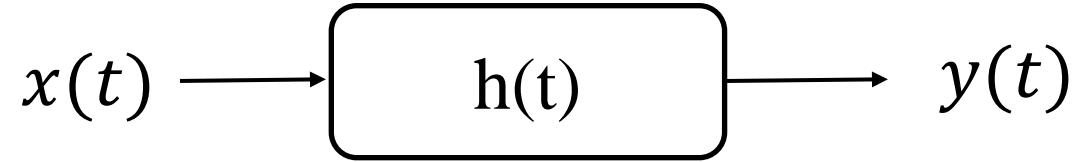
$$Y(j\omega) = \frac{A}{a + j\omega} + \frac{B}{b + j\omega} \quad A = \frac{1}{b - a} = -B$$

$$Y(j\omega) = \frac{1}{b - a} \left( \frac{1}{a + j\omega} - \frac{1}{b + j\omega} \right) \quad y(t) = \frac{1}{b - a} [e^{-at} - e^{-bt}]u(t), b \neq a$$



# The convolution property

## Example



$$h(t) = e^{-at}u(t), a > 0 \quad x(t) = e^{-bt}u(t), b > 0 \quad y(t) = ?$$

□ Solution  $b = a$

$$Y(j\omega) = \frac{1}{(a + j\omega)^2} = j \frac{d}{d\omega} \left[ \frac{1}{a + j\omega} \right]$$

$$e^{-at}u(t) \xleftrightarrow{\mathcal{F}} 1/(a + j\omega)$$

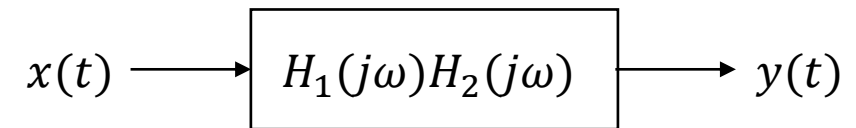
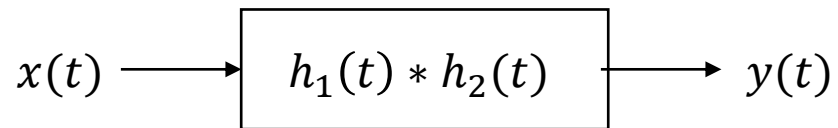
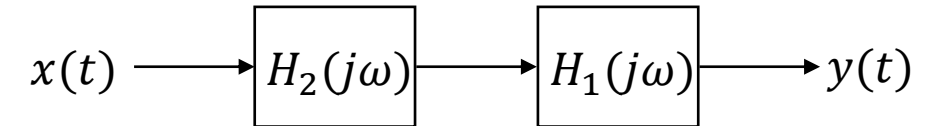
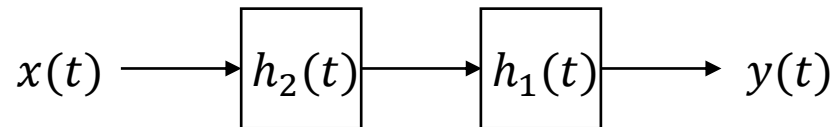
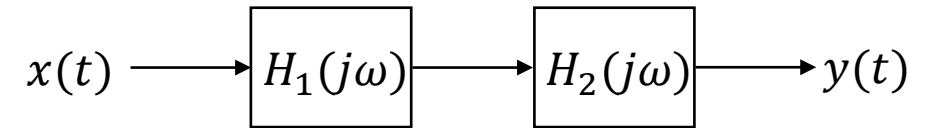
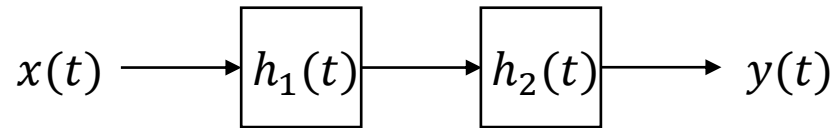
$$te^{-at}u(t) \xleftrightarrow{\mathcal{F}} j \frac{d}{d\omega} \left[ \frac{1}{a + j\omega} \right]$$

$$\therefore y(t) = te^{-at}u(t)$$



# The convolution property

They are equivalent:





# Chapter 4: The Continuous-Time Fourier Transform

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- ❑ Representation of aperiodic signals- Continuous Fourier Transform
- ❑ Fourier transform for periodic signals
- ❑ Properties of continuous-time Fourier Transform
- ❑ The convolution property
- ❑ The multiplication property
- ❑ System characterized by differential equations



# The multiplication property

$$r(t) = s(t)p(t) \xleftrightarrow{\mathcal{F}} R(j\omega) = \frac{1}{2\pi} [S(j\omega) * P(j\omega)]$$

□ Multiplication of two signals is often referred to as *Amplitude Modulation*

$$\begin{aligned} s(t)p(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} S(j\theta) e^{j\theta t} d\theta \frac{1}{2\pi} \int_{-\infty}^{\infty} P(j\omega') e^{j\omega' t} d\omega' \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} S(j\theta) P(j\omega') e^{j(\theta+\omega')t} d\theta d\omega' \\ &\quad \omega' = \omega - \theta \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} S(j\theta) P(j(\omega - \theta)) e^{j\omega t} d\theta d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \boxed{\frac{1}{2\pi} \int_{-\infty}^{\infty} S(j\theta) P(j(\omega - \theta)) d\theta} e^{j\omega t} d\omega \end{aligned}$$



# The multiplication property

## Example

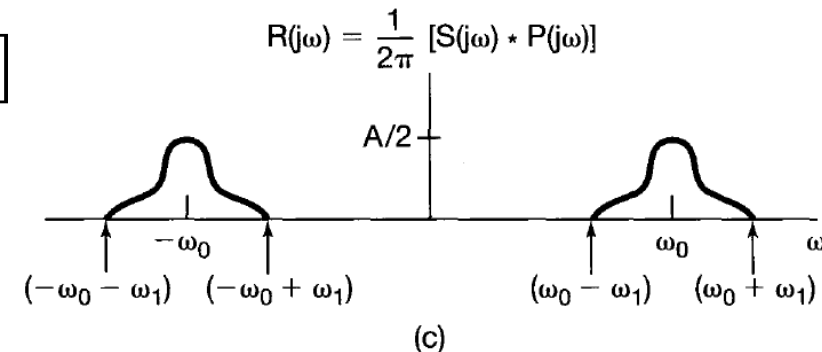
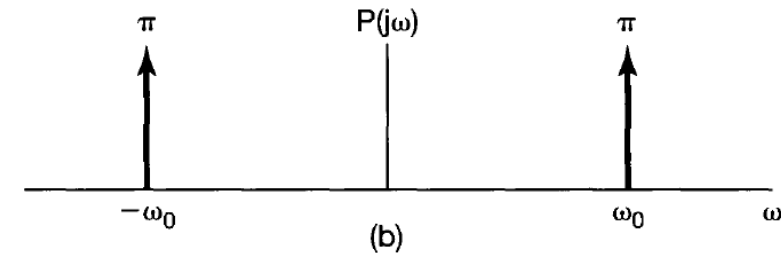
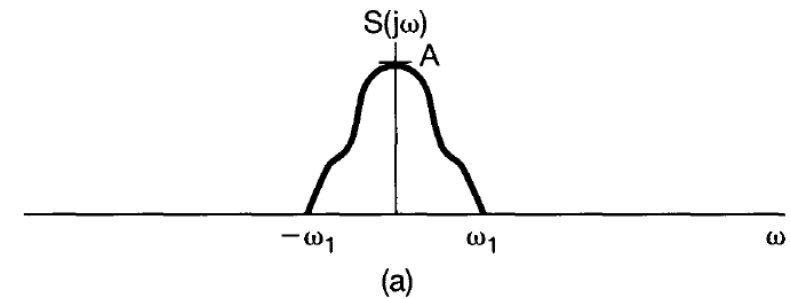
Consider a signal  $p(t) = \cos \omega_0 t$  and a signal  $s(t)$  with spectrum  $S(j\omega)$ , determine the FT of  $r(t) = p(t)s(t)$

□ **Solution**  $P(j\omega) = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$

$$R(j\omega) = 1/2\pi \cdot S(j\omega) * P(j\omega)$$

$$= 1/2\pi \cdot S(j\omega) * [\pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)]$$

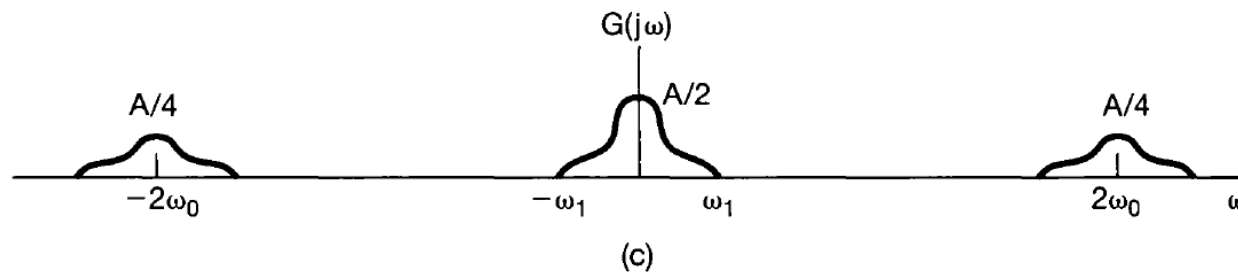
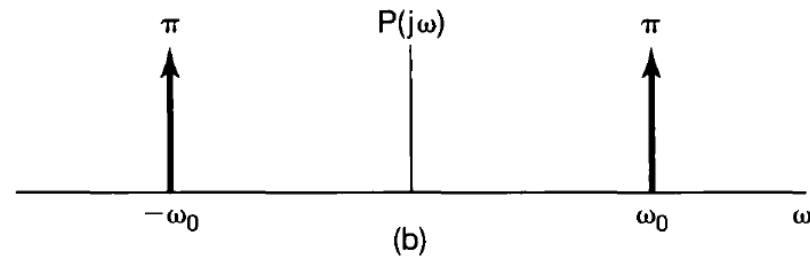
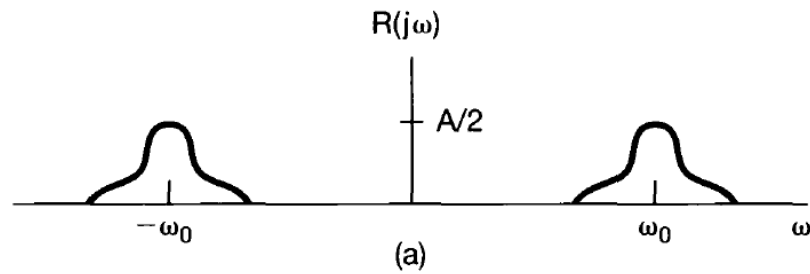
$$= 1/2[S[j(\omega - \omega_0)] + S[j(\omega + \omega_0)]]$$



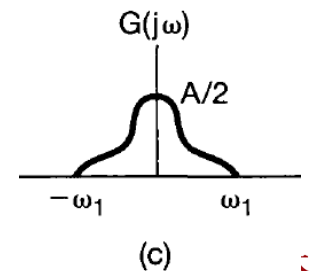
# The multiplication property

## Example

$$g(t) = r(t)p(t) \quad G(j\omega) = ?$$



Low-pass filter



# The multiplication property

## Example

$$x(t) = \frac{\sin(t) \sin(t/2)}{\pi t^2} \quad X(j\omega) = ?$$

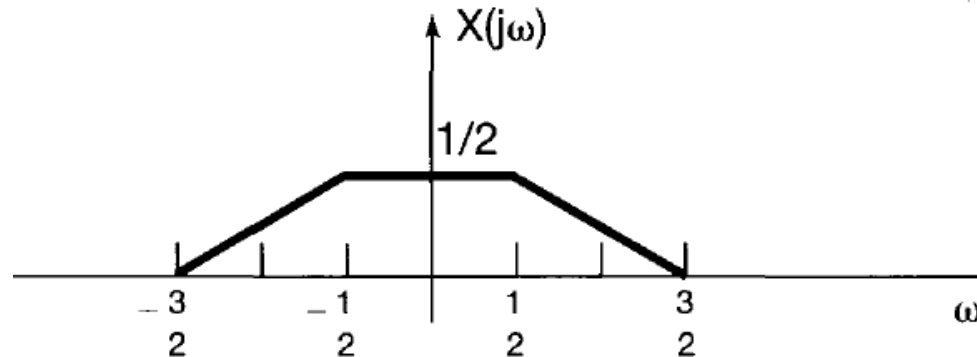
## □ Solution

$$x(t) = \pi \frac{\sin(t)}{\pi t} \cdot \frac{\sin(t/2)}{\pi t}$$

$$X(j\omega) = \frac{1}{2} \mathcal{F} \left\{ \frac{\sin(t)}{\pi t} \right\} * \mathcal{F} \left\{ \frac{\sin(t/2)}{\pi t} \right\}$$

$$\tilde{X}(j\omega) = \begin{cases} 1, & |\omega| < W \\ 0, & |\omega| > W \end{cases}$$

$$\tilde{x}(t) = \frac{\sin Wt}{\pi t}$$



# Chapter 4: The Continuous-Time Fourier Transform

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- ❑ Representation of aperiodic signals- Continuous Fourier Transform
- ❑ Fourier transform for periodic signals
- ❑ Properties of continuous-time Fourier Transform
- ❑ The convolution property
- ❑ The multiplication property
- ❑ System characterized by differential equations



# System characterized by differential equations

□ Differential equation

$$\sum_{k=0}^N a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^M b_k \frac{d^k x(t)}{dt^k}$$

$$Y(j\omega) = H(j\omega)X(j\omega) \Rightarrow H(j\omega) = \frac{Y(j\omega)}{X(j\omega)}$$

$$\mathcal{F} \left\{ \sum_{k=0}^N a_k \frac{d^k y(t)}{dt^k} \right\} = \mathcal{F} \left\{ \sum_{k=0}^M b_k \frac{d^k x(t)}{dt^k} \right\} \Rightarrow \sum_{k=0}^N a_k \mathcal{F} \left\{ \frac{d^k y(t)}{dt^k} \right\} = \sum_{k=0}^M b_k \mathcal{F} \left\{ \frac{d^k x(t)}{dt^k} \right\}$$

$$\sum_{k=0}^N a_k \cdot (j\omega)^k Y(j\omega) = \sum_{k=0}^M b_k \cdot (j\omega)^k X(j\omega) \Rightarrow Y(j\omega) \sum_{k=0}^N a_k \cdot (j\omega)^k = X(j\omega) \sum_{k=0}^M b_k \cdot (j\omega)^k$$

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{\sum_{k=0}^M b_k \cdot (j\omega)^k}{\sum_{k=0}^N a_k \cdot (j\omega)^k}$$



# System characterized by differential equations

## Example

$$\frac{dy(t)}{dt} + ay(t) = x(t), \quad a > 0$$

$$\mathcal{F}\left\{\frac{dy(t)}{dt} + ay(t)\right\} = \mathcal{F}\{x(t)\}$$

$$j\omega Y(j\omega) + aY(j\omega) = X(j\omega)$$

$$H(j\omega) = \frac{1}{j\omega + a} \quad \Rightarrow \quad h(t) = e^{-at}u(t), \quad a > 0$$





# System characterized by differential equations

## Example

$$\frac{d^2 y(t)}{dt} + 4 \frac{dy(t)}{dt} + 3y(t) = \frac{dx(t)}{dt} + 2x(t)$$

$$\mathcal{F} \left\{ \frac{d^2 y(t)}{dt} + 4 \frac{dy(t)}{dt} + 3y(t) \right\} = \mathcal{F} \left\{ \frac{dx(t)}{dt} + 2x(t) \right\}$$

$$(j\omega)^2 Y(j\omega) + 4(j\omega)Y(j\omega) + 3Y(j\omega) = (j\omega)X(j\omega) + 2X(j\omega)$$

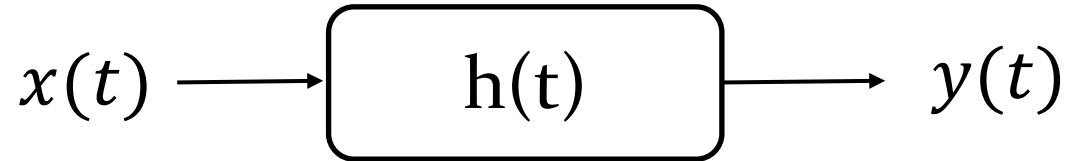
$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{(j\omega) + 2}{(j\omega)^2 + 4(j\omega) + 3} = \frac{1/2}{j\omega + 1} + \frac{1/2}{j\omega + 3}$$

$$h(t) = \frac{1}{2} e^{-t} u(t) + \frac{1}{2} e^{-3t} u(t)$$



# System characterized by differential equations

## Example



$$x(t) = e^{-t}u(t) \quad \frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 3y(t) = \frac{dx(t)}{dt} + 2x(t) \quad y(t) = ?$$

## □ Solution

$$Y(j\omega) = H(j\omega)X(j\omega) = \left[ \frac{j\omega + 2}{(j\omega + 1)(j\omega + 3)} \right] \left[ \frac{1}{j\omega + 1} \right] = \frac{j\omega + 2}{(j\omega + 1)^2(j\omega + 3)}$$

$$Y(j\omega) = \frac{A_{11}}{j\omega + 1} + \frac{A_{12}}{(j\omega + 1)^2} + \frac{A_{21}}{j\omega + 3} \quad A_{11} = \frac{1}{4}, \quad A_{12} = \frac{1}{2}, \quad A_{21} = -\frac{1}{4}$$

$$Y(j\omega) = \frac{1/4}{j\omega + 1} + \frac{1/2}{(j\omega + 1)^2} - \frac{1/4}{j\omega + 3} \Rightarrow y(t) = \left( \frac{1}{4}e^{-t} + \frac{1}{2}te^{-t} - \frac{1}{4}e^{-3t} \right) u(t)$$

