Online Lecture Notes

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1 Material for the Final Exam

The final exam will be about

1. All the material from the mid-exam (not so much focus)
2. Lecture 7: Linear Time-Varying Systems, Fundamental Solution, Periodic Orbits [+ Homework]
3. Lecture 8: Stability Analysis, Definition of Stability and Asymptotic Stability, Stability of Linear (Time-Invariant and Time-Varying/Periodic) and Nonlinear Systems (Lyapunov Theory) [+ Homework]
4. Lecture 11: Linear Quadratic Regulator (LQR), Discretization of Linear-Quadratic Control Problems, Dyanamic Prorgramming Solution (including the Discrete-Time Riccati Recursion), Continuous-Time LQR and Riccati Differential Equations, Infinite Horizon LQR (for $T \to \infty$); many examples! [+ Homework]

The final exam will take place online on June 18; 10:30-12:30. This is similar to the mid-term exam/homeworks, but details might be announced by academic affairs office. We submit the solutions via a "tencent meeting"—Yuxuan will send details.

1.1 Short Summary of Lecture 7

Key point of Lecture 7 on Linear Time-Varying Systems are

1. We introduced the so-called fundamental solution $G : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^{n_x \times n_x}$, which is defined to be the solution of the first order differential equation

$$\forall t, \tau \in \mathbb{R}, \qquad \frac{\partial}{\partial t} G(t,\tau) = A(t)G(t,\tau) \quad \text{with} \quad G(\tau,\tau) = I \ ,$$

which can be used to write out the solution of the LTV system

$$\dot{x}(t) = A(t)x(t) + b(t) \quad \text{with} \quad x(t_0) = x_0$$

in the form

$$x(t) = G(t,t_0)x_0 + \int_{t_0}^{t} G(t,\tau)b(\tau)\,\mathrm{d}\tau \ .$$

2. The three main properties of G are that

   (a) it generalizes matrix exponentials; that is $G(t,\tau) = e^{A(t-\tau)}$ if A is time-invariant,

   (b) we have $G(t_3,t_2)G(t_2,t_1) = G(t_3,t_1)$, and

   (c) the matrix-valued function G is invertible, $G(t,\tau)^{-1} = G(\tau,t)$. This is the same as "turning around the time direction".

3. The fundamental solution G is especially useful for analyzing periodic systems, where we usually can't find explicit solutions for the periodic orbits. Here, the key idea is to analyze the so-called monodromy matrix

$$G(T,0)$$

   at the period time $T > 0$ of the given periodic system. It turns out that if the eigenvalues of $G(T,0)$ are all different from 0, then our periodic system admits a periodic orbit and, if the eigenvalues of $G(T,0)$ are all contained in the open unit disc, then the trajectories of the periodic system converge to the periodic limit orbit.

1.2 Summary of Lecture 8: Stability of Linear and Nonlinear System

The most important things directly at the start:

Never mix up stability and convergence!!!

Stability refers to the property of the system that for every $\epsilon > 0$ there exists $\delta > 0$ such that the solution trajectory of the system

$$\dot{x}(t) = f(x(t)) \quad \text{with} \quad x(0) = x_0$$

satisfies $\|x(t)\| \leq \epsilon$ for all x_0 with $\|x_0\| \leq \delta$ and all $t \geq 0$. This is **NOT** equivalent to convergence,

$$\lim_{t\to 0} x(t) = 0 \ .$$

If a system has both properties; that is, if a system is both stable and convergent, then we say that this system is asymptotically stable,

Stability + Convergence = Asymptotic Stability

Other important results from Lecture 8:

1. An LTI system is asymptotically stable if and only if all eigenvalues of A have strictly negative real part.

2. A periodic LTV system is asymptotically stable if and only if all eigenvalues of $G(T,0)$ are contained in the open unit disc.

3. An LTI system is stable if and only if all eigenvalues of A have non-positive real part and all eigenvalues on the imaginary axis have a trivial Jordan normal block of dimension 1.

4. A periodic LTV system is stable if and only if all eigenvalues of $G(T, 0)$ are contained in the closed unit disc and all eigenvalues on the unit circle have a trivial Jordan normal block of dimension 1.

5. A nonlinear system is asymptotically stable if we can find a Lyapunov function V such that

   (a) V is continuously differentiable and satisfies

   $$\forall x \neq 0, \qquad \nabla V(x)^\intercal f(x) < 0$$

   (b) V is radially unbounded, and

   (c) V is positive definite, meaning that $V(x) \geq 0$ for all x and $V(x) = 0$ if and only if $x = 0$.

6. A linear time invariant system is asymptotically stable if and only if one can find a quadratic Lyapunov function, which satisfies the above requirements. This function the form

   $$V(x) = x^\intercal P x,$$

   where P is the positive definite solution of the Lyapunov equation

   $$PA + A^\intercal P + Q = 0$$

   for a given positive definite matrix Q. Notice that this is exactly the algebraic Riccati equation for an LQR controller without control input! Recall that the algebraic Riccati equation for the infinite horizon LQR controller has the form

   $$PA + A^\intercal P + Q - PB^\intercal R^{-1} BP = 0 \; .$$

   This is an important connection between Lecture 8 and Lecture 11 !!! In this notation V corresponds to the cost-to-go function,

   $$V(x) = J_\infty(x) = \int_0^\infty x(t)^\intercal Q x(t) \, \mathrm{d}t \; .$$

1.3 Linear Quadratic Regulator

The solution to the discrete time linear-quadratic optimal control problem

$$J_0(z) = \min_{y,v} \sum_{k=0}^{N-1} \{ y_k^\intercal \mathcal{Q} y_k + v_k^\intercal \mathcal{R} v_k \} + y_N^\intercal \mathcal{P}_N y_N \quad \text{s.t} \quad \begin{cases} \forall k \in \{0, 1, \ldots, N-1\}, \\ x_{k+1} = \mathcal{A} y_k + \mathcal{B} v_k \\ x_0 = z \end{cases}$$

can be found by a so-called a dynamic programming recursion for the cost-to-function J_N, which can be used to show that

$$J_0(z) = z^\intercal \mathcal{P}_0 z$$

is a quadratic form, where the weighting matrix P_0 can be found by a backward Riccati recursion of the form

$$\mathcal{P}_i = \mathcal{A}^\intercal \mathcal{P}_{i+1} \mathcal{A} + \mathcal{Q} - \mathcal{A}^\intercal P_{i+1} \mathcal{B} [\mathcal{R} + B^\intercal \mathcal{P}_{i+1} \mathcal{B}]^{-1} \mathcal{B}^\intercal \mathcal{P}_{i+1} \mathcal{A}$$

If we take the limit for $h \to 0$, this becomes a differential equation of the form

$$-\dot{P}(t) = A^{\mathsf{T}} P(t) + P(t)A + Q - P(t)B^{\mathsf{T}} R^{-1} BP(t) \quad \text{with} \quad P(T) = \mathcal{P}_N$$

The corresponding continuous-time optimal control gain is given by

$$K(t) = -R^{-1} B^{\mathsf{T}} P(t)$$

This means that the solution trajectory can be found by a forward simulation,

$$\dot{x}(t) = [A + BK(t)]x(t) \quad \text{with} \quad x(0) = x_0 \; .$$

For the case that the time horizon $T \to \infty$ is tending to infinity, we obtain the so-called infinite horizon LQR controller, which can be solved by solving the alebraic Riccati equation

$$0 = A^{\mathsf{T}} P_\infty + P_\infty A + Q - P_\infty B^{\mathsf{T}} R^{-1} BP_\infty \quad \text{with} \quad P_\infty \succeq 0$$

The corresponding infinite horizon continuous-time optimal control gain is time-invariant

$$K_\infty = -R^{-1} B^{\mathsf{T}} P_\infty$$

This means that the closed-loop solution trajectory takes the simple form

$$x(t) = e^{(A+BK_\infty)t} x_0 \qquad \text{as well as} \qquad u(t) = K_\infty x(t) \; .$$

This is the solution to the infinite horizon continuous-time optimal control problem!