SI231 Matrix Analysis and Computations
Eigenvalues, Eigenvectors, and Eigendecomposition

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Eigenvalues, Eigenvectors, and Eigendecomposition

- eigenvalue problem, facts about eigenvalues and eigenvectors
- eigendecomposition, the case of Hermitian & real symmetric matrices
- Schur decomposition
- variational characterizations of eigenvalues of Hermitian & real symmetric matrices
- similarity transformation
- power iteration, inverse iteration, Rayleigh quotient iteration
- orthogonal iteration
- LR iteration and QR iteration
- PageRank: a case study
- generalized and nonlinear eigenvalue problem

Notations and Conventions

- a square matrix $\mathbf{A}$ is said to be symmetric if $a_{ij} = a_{ji}$ for all i, j with $i \neq j$, or equivalently, if $\mathbf{A}^T = \mathbf{A}$
  - example:
  $$\mathbf{A} = \begin{bmatrix} 1 & 0 & 3 \\ 0 & -2 & 0.9 \\ 3 & 0.9 & 0 \end{bmatrix}$$
- a square matrix $\mathbf{A}$ is said to be Hermitian if $a_{ij} = a_{ji}^*$ for all i, j with $i \neq j$, or equivalently, if $\mathbf{A}^H = \mathbf{A}$
  - example:
  $$\mathbf{A} = \begin{bmatrix} 1 & 0 & 3 + \boldsymbol{j}0.7 \\ 0 & -2 & 0.9 - \boldsymbol{j} \\ 3 - \boldsymbol{j}0.7 & 0.9 + \boldsymbol{j} & 0 \end{bmatrix}$$
- we denote the set of all $n \times n$ real symmetric matrices by $\mathbb{S}^n$
- we denote the set of all $n \times n$ (complex) Hermitian matrices by $\mathbb{H}^n$

Notations and Conventions

- note the following subtleties:
  - by definition, a real symmetric matrix is also Hermitian
  - when we say that a matrix is Hermitian, we often imply that the matrix may be complex (at least for this course); a real Hermitian matrix is simply real symmetric
  - we can have a complex symmetric matrix, though we will not study it

Main Results

A matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$ (or $\mathbb{C}^{n\times n}$) is said to admit an eigendecomposition if there exists a nonsingular $\mathbf{V} \in \mathbb{C}^{n\times n}$ and a collection of scalars $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ such that

$$\mathbf{A} = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^{-1},$$

where $\mathbf{\Lambda} = \mathrm{Diag}(\lambda_1, \ldots, \lambda_n)$.

- the above $(\mathbf{V}, \mathbf{\Lambda})$ satisfies $\mathbf{A}\mathbf{v}_i = \lambda_i \mathbf{v}_i$ for $i = 1, \ldots, n$, which are eigen-equations
- $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are required to be linearly independent
- eigendecomposition *does not* always exist

Main Results

A real symmetric matrix $\mathbf{A} \in \mathbb{S}^n$ always admits an eigendecomposition

$$\mathbf{A} = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^T$$

where $\mathbf{V} \in \mathbb{R}^{n \times n}$ is orthogonal; $\mathbf{\Lambda} = \mathrm{Diag}(\lambda_1, \ldots, \lambda_n)$ with $\lambda_i \in \mathbb{R}$ for all i.

A Hermitian matrix $\mathbf{A} \in \mathbb{H}^n$ always admits an eigendecomposition

$$\mathbf{A} = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^H$$

where $\mathbf{V} \in \mathbb{C}^{n \times n}$ is unitary; $\mathbf{\Lambda} = \mathrm{Diag}(\lambda_1, \ldots, \lambda_n)$ with $\lambda_i \in \mathbb{R}$ for all i.

- differences: a Hermitian (or real symmetric matrix) always has
  - an eigendecomposition
  - real λ_i's
  - a $\mathbf{V}$ that is not only nonsingular but also unitary

Eigenvalues and Eigenvectors

We start with the basic definition of eigenvalues and eigenvectors.

Problem: given a $\mathbf{A} \in \mathbb{R}^{n\times n}$ (or $\mathbb{C}^{n\times n}$), find a vector $\mathbf{v} \in \mathbb{C}^n$ with $\mathbf{v} \neq \mathbf{0}$ such that

$$\mathbf{A}\mathbf{v} = \lambda\mathbf{v}, \qquad \text{for some } \lambda \in \mathbb{C} \tag{$*$}$$

or, equivalently,

$$\begin{aligned} \text{find} \quad & \mathbf{v} \in \mathbb{C}^n \backslash \{\mathbf{0}\} \text{ and } \lambda \in \mathbb{C} \\ \text{s.t.} \quad & \mathbf{A}\mathbf{v} = \lambda\mathbf{v} \end{aligned}$$

- $(*)$ is called an (ordinary) eigenvalue problem, eigen-problem, or eigen-equation
- let $(\mathbf{v}, \lambda)$ be a solution to $(*)$. We call
  - $(\mathbf{v}, \lambda)$ an eigen-pair of $\mathbf{A}$
  - λ an eigenvalue of $\mathbf{A}$ (always finite); $\mathbf{v}$ an eigenvector of $\mathbf{A}$ associated with λ
- if $(\mathbf{v}, \lambda)$ is an eigen-pair of $\mathbf{A}$, $(\alpha\mathbf{v}, \lambda)$ is also an eigen-pair for any $\alpha \in \mathbb{C}, \alpha \neq 0$
- unless specified, we will assume $\|\mathbf{v}\|_2 = 1$ in the sequel (negl. phase ambiguity)

Eigenvalues and Eigenvectors

Problem: given a $\mathbf{A} \in \mathbb{R}^{n \times n}$ (or $\mathbb{C}^{n \times n}$), find a vector $\mathbf{v} \in \mathbb{C}^n$ with $\mathbf{v} \neq \mathbf{0}$ such that

$$\mathbf{A}\mathbf{v} = \lambda \mathbf{v}, \qquad \text{for some } \lambda \in \mathbb{C} \tag{$*$}$$

- from $(*)$, action of matrix $\mathbf{A}$ on a subspace $\mathcal{V}_\lambda \subseteq \mathbb{C}^n$ sometimes is equivalent to scalar multiplication

$$\mathcal{V}_\lambda = \{\mathbf{v} \in \mathbb{C}^n \mid \mathbf{A}\mathbf{v} = \lambda \mathbf{v}, \text{ for some } \lambda \in \mathbb{C}\}$$

- the subspace $\mathcal{V}_\lambda$ satisfying $(*)$ is called the eigenspace of $\mathbf{A}$ associated with λ, and any $\mathbf{v} \in \mathcal{V}_\lambda$ with $\mathbf{v} \neq \mathbf{0}$ is an eigenvector
  - $\mathcal{V}_\lambda = \mathcal{N}(\mathbf{A} - \lambda \mathbf{I})$ since $\mathbf{A}\mathbf{v} = \lambda \mathbf{v} \iff (\mathbf{A} - \lambda \mathbf{I})\mathbf{v} = \mathbf{0}$
  - $\mathcal{V}_\lambda$ is an invariant subspace of $\mathbf{A}$, i.e., $\mathbf{A}\mathcal{V}_\lambda \subseteq \mathcal{V}_\lambda$

- the set of all eigenvalues of $\mathbf{A}$, denoted by $\sigma(\mathbf{A}) \subseteq \mathbb{C}$, is called the spectrum of $\mathbf{A}$

Eigenvalues and Eigenvectors

Fact: Every $\mathbf{A} \in \mathbb{R}^{n\times n}$ (or $\mathbb{C}^{n\times n}$) has n eigenvalues.

- from the eigenvalue problem we see that
$$\begin{aligned}\mathbf{Av} = \lambda\mathbf{v} \text{ for some } \mathbf{v} \neq \mathbf{0} \quad &\Longleftrightarrow \quad (\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = \mathbf{0} \text{ for some } \mathbf{v} \neq \mathbf{0} \\ &\Longleftrightarrow \quad \det(\mathbf{A} - \lambda\mathbf{I}) = 0\end{aligned}$$
- let $p(\lambda) = \det(\mathbf{A} - \lambda\mathbf{I})$, called the characteristic polynomial of $\mathbf{A}$; eigenvalues (eigenvectors) are sometimes called characteristic values (characteristic vectors)
- from the determinant def., it can be shown that $p(\lambda)$ is a polynomial of degree n, viz., $p(\lambda) = \alpha_0 + \alpha_1\lambda + \alpha_2\lambda^2 + \ldots + \alpha_n\lambda^n$ where α_i's depend on $\mathbf{A}$
  - specifically, $\alpha_0 = \det(\mathbf{A})$, . . . , $\alpha_{n-1} = (-1)^{n-1}\mathrm{tr}(\mathbf{A})$, $\alpha_n = (-1)^n$
- as $p(\lambda)$ is a polynomial of degree n, it can be factored as $p(\lambda) = \prod_{i=1}^n(\lambda_i - \lambda)$ where $\lambda_1, \ldots, \lambda_n$ are the roots of $p(\lambda)$
  - specifically, $\alpha_0 = \prod_{i=1}^n \lambda_i$, . . . , $\alpha_{n-1} = (-1)^{n-1}\sum_{i=1}^n \lambda_i$, $\alpha_n = (-1)^n$
- we have the characteristic equation $\det(\mathbf{A} - \lambda\mathbf{I}) = 0 \Longleftrightarrow \lambda \in \{\lambda_1, \ldots, \lambda_n\}$
- the spectrum is hence defined by $\sigma(\mathbf{A}) = \{\lambda \in \mathbb{C} \mid \det(\mathbf{A} - \lambda\mathbf{I}) = 0\}$

Eigenvalues and Eigenvectors

Let $\lambda_1, \ldots, \lambda_n$ denote the n eigenvalues of $\mathbf{A}$. We write

$$\mathbf{A}\mathbf{v}_i = \lambda_i \mathbf{v}_i, \qquad i = 1, \ldots, n,$$

where $\mathbf{v}_i$ denotes an eigenvector of $\mathbf{A}$ associated with λ_i.

- we should be careful about the meaning of n eigenvalues: *they are defined as the n roots of the characteristic polynomial* $p(\lambda) = \det(\mathbf{A} - \lambda\mathbf{I})$
- example: consider

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

  - from the original definition $\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$, one can verify that $\lambda = 1$ is the only eigenvalue of $\mathbf{A}$
  - from the characteristic polynomial, which is $p(\lambda) = (1 - \lambda)^2$, we see two roots $\lambda_1 = \lambda_2 = 1$ as two eigenvalues (with repetitions)
  - every matrix has at least one eigenvalue, and every eigenvalue appears at least once in $p(\lambda)$

Eigenvalues and Eigenvectors

Fact: an eigenvalue can be complex even if $\mathbf{A}$ is real.

- a polynomial $p(\lambda) = \alpha_0 + \alpha_1\lambda + \alpha_2\lambda^2 + \ldots + \alpha_n\lambda^n$ with real coefficients α_i's can have complex roots
- example: consider

$$\mathbf{A} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

  - we have $p(\lambda) = \lambda^2 + 1$, so $\lambda_1 = \boldsymbol{j}$, $\lambda_2 = -\boldsymbol{j}$
- similarily, an eigenvalue can be real even if $\mathbf{A}$ is complex

Fact: complex eigenvalues of real matrices appear in conjugate pairs (cf. Complex Conjugate Root Theorem) (e.g., eigenvalues of orthogonal matrices satisfy $|\lambda_i| = 1$)

Fact: if $\mathbf{A}$ is real and there exists a real eigenvalue λ of $\mathbf{A}$, the associated eigenvector $\mathbf{v}$ can be taken as real.

- obviously, when $\mathbf{A} - \lambda\mathbf{I}$ is real we can define $\mathcal{V}_\lambda = \mathcal{N}(\mathbf{A} - \lambda\mathbf{I})$ on $\mathbb{R}^n$
- or, if $\mathbf{v} \in \mathbb{C}^n$ is a complex eigenvector of a real $\mathbf{A}$ associated with a real λ, we can write $\mathbf{v} = \Re(\mathbf{v}) + \boldsymbol{j}\Im(\mathbf{v})$, where $\Re(\mathbf{v}), \Im(\mathbf{v}) \in \mathbb{R}^n$. It is easy to verify that $\Re(\mathbf{v})$ and $\Im(\mathbf{v})$ are eigenvectors associated with λ.

Further Discussion: Repeated Eigenvalues

- w.l.o.g., order $\lambda_1, \ldots, \lambda_n$ such that $\{\lambda_1, \ldots \lambda_k\}$, $k \leq n$, is the set of all distinct eigenvalues of $\mathbf{A}$; i.e., $\lambda_i \neq \lambda_j$ for all $i, j \in \{1, \ldots, k\}$, $i \neq j$; $\lambda_i \in \{\lambda_1, \ldots \lambda_k\}$ for all $i \in \{1, \ldots, n\}$

- denote μ_i as the number of repeated eigenvalues of λ_i, $i = 1, \ldots, k$
  - i.e., μ_i is the multiplicity of λ_i as the root of $p(\lambda)$ (λ_i is simple if $\mu_i = 1$ and is repeated/degenerate if $\mu_i > 1$)
  - μ_i is called the algebraic multiplicity of the eigenvalue λ_i

- every λ_i can have more than one eigenvector (scaling not counted)
  - denote $\gamma_i = \dim \mathcal{V}_{\lambda_i} = \dim \mathcal{N}(\mathbf{A} - \lambda_i \mathbf{I}) = n - \dim \mathcal{R}(\mathbf{A} - \lambda_i \mathbf{I})$, $i = 1, \ldots, k$
  - i.e., we can find γ_i linearly independent $\mathbf{v}_i$'s in $\mathcal{V}_{\lambda_i}$
  - γ_i is called the geometric multiplicity of the eigenvalue λ_i

Property 1. We have $\mu_i \geq \gamma_i$ for all $i = 1, \ldots, k$ (not trivial, requires a proof)

- **Implication:** # of repeated eigenvalues $\geq$ # of linearly indep. eigenvectors

Further Discussion: Repeated Eigenvalues

- eigenvalue λ_i is called defective if $\mu_i > \gamma_i$, and called nondefective if $\mu_i = \gamma_i$

- a matrix is called defective if it has one or more defective eigenvalues, and called nondefective if it has no defective eigenvalue

  – example: consider

$$\mathbf{A}_1 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \qquad \mathbf{A}_2 = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}.$$

Eigenvalues and Eigenvectors

We have, or can easily prove, the following properties:

- $\det(\mathbf{A}) = \prod_{i=1}^{n} \lambda_i$

- $\mathrm{tr}(\mathbf{A}) = \sum_{i=1}^{n} \lambda_i$

- the eigenvalues of $\mathbf{A}^k$ are $\lambda_1^k, \ldots, \lambda_n^k$

- $\mathrm{rank}(\mathbf{A}) < n$ (i.e., $\mathbf{A}$ is rank-deficient) if and only if 0 is one eigenvalue of $\mathbf{A}$ (proof: $\mathrm{nullity}(\mathbf{A}) = \dim \mathcal{N}(\mathbf{A}) > 0$ and $\mathrm{rank}(\mathbf{A}) + \mathrm{nullity}(\mathbf{A}) = n$)

- $\mathrm{rank}(\mathbf{A}) \geq$ number of nonzero eigenvalues (with repetitions) of $\mathbf{A}$

Eigenvalues and Eigenvectors

- The linearly independent eigenvectors $\mathbf{v}_i$ with nonzero eigenvalues form a basis (not necessarily orthonormal) for all possible products $\mathbf{Ax}$, for $\mathbf{x} \in \mathbb{C}^n$, which is the column space of the $\mathbf{A}$.

- The number of linearly independent eigenvectors $\mathbf{v}_i$ with nonzero eigenvalues is equal to the rank of the matrix $\mathbf{A}$, and also the dimension of the column space.

- The linearly independent eigenvectors $\mathbf{v}_i$ with an eigenvalue of zero form a basis for the null space of $\mathbf{A}$.

Right and Left Eigenvectors

- right eigenvector or eigenvector associated with λ

$$\mathbf{A}\mathbf{v} = \lambda \mathbf{v} \quad \text{for} \quad \mathbf{v} \neq \mathbf{0}$$

- left eigenvector associated with λ

$$\mathbf{u}^T \mathbf{A} = \lambda \mathbf{u}^T \quad \text{for} \quad \mathbf{u} \neq \mathbf{0}$$

(unless specified, eigenvectors are commonly referred to right eigenvectors)

  - left eigenvectors of $\mathbf{A}$ are nothing else but the right eigenvectors of $\mathbf{A}^T$
  - while the eigenvalues of $\mathbf{A}$ and $\mathbf{A}^T$ are the same, the sets of left- and right-eigenvectors may be different in general

Eigendecomposition

A matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$ (or $\mathbb{C}^{n\times n}$) is said to be diagonalizable, or admit an eigendecomposition, if there exists a nonsingular $\mathbf{V} \in \mathbb{C}^{n\times n}$ such that

$$\mathbf{A} = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^{-1},$$

where $\mathbf{\Lambda} = \mathrm{Diag}(\lambda_1, \ldots, \lambda_n)$. (diagonalization of matrix $\mathbf{A}$: $\mathbf{V}^{-1}\mathbf{A}\mathbf{V} = \mathbf{\Lambda}$)

- a.k.a. eigenvalue decomposition, spectral decomposition
- in defining diagonalizability, we didn't say that $(\mathbf{v}_i, \lambda_i)$ has to be an eigen-pair of $\mathbf{A}$. But

$$\begin{aligned} \mathbf{A} = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^{-1} &\iff \mathbf{A}\mathbf{V} = \mathbf{V}\mathbf{\Lambda},\ \mathbf{V} \text{ nonsingular} \\ &\iff \mathbf{A}\mathbf{v}_i = \lambda_i \mathbf{v}_i,\ i = 1, \ldots, n,\ \mathbf{V} \text{ nonsingular (hence, } \mathbf{v}_i \neq \mathbf{0}) \end{aligned}$$

  Also, $\lambda_1, \ldots, \lambda_n$ must be the n eigenvalues of $\mathbf{A}$; this can be seen from the characteristic polynomial $\det(\mathbf{A} - \lambda\mathbf{I}) = \det(\mathbf{\Lambda} - \lambda\mathbf{I}) = \prod_{i=1}^n (\lambda_i - \lambda)$

Theorem 1. (the sufficient and necessary condition for the existence of eigendec.) For $\mathbf{A} \in \mathbb{R}^{n\times n}$ (or $\mathbb{C}^{n\times n}$), it admits an eigendcomposition if and only if there exist n linearly independent eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ of $\mathbf{A}$

Eigendecomposition

If $\mathbf{A}$ admits an eigendecomposition, the following properties can be shown (easily):

- $\det(\mathbf{A}) = \prod_{i=1}^{n} \lambda_i$
- $\mathrm{tr}(\mathbf{A}) = \sum_{i=1}^{n} \lambda_i$
- the eigenvalues of $\mathbf{A}^k$ are $\lambda_1^k, \ldots, \lambda_n^k$
- $\mathrm{rank}(\mathbf{A}) =$ number of nonzero eigenvalues (with repetitions) of $\mathbf{A}$ (Quiz)
- suppose that $\mathbf{A}$ is also nonsingular. Then, $\mathbf{A}^{-1} = \mathbf{V}\mathbf{\Lambda}^{-1}\mathbf{V}^{-1}$

Note: the first three properties can be shown to be valid for any $\mathbf{A}$ (do not depend on the existence of eigendec.); the fourth property may not be valid when $\mathbf{A}$ does not admit an eigendecomposition (see the example next page); the third and fifth properties can be used for effcient computations of matrix powers and inversions

Eigendecomposition

Question: Does every $\mathbf{A} \in \mathbb{R}^{n\times n}$ (or $\mathbb{C}^{n\times n}$) admit an eigendecomposition?

- the answer is no.
- counter example: consider

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

  - the characteristic polynomial is $p(\lambda) = -\lambda^3$, so $\lambda_1 = \lambda_2 = \lambda_3 = 0$
  - it is easy to see that

$$\mathcal{V}_{\lambda_1} = \mathcal{N}(\mathbf{A} - \lambda_1 \mathbf{I}) = \mathcal{N}(\mathbf{A}) = \mathcal{R}(\mathbf{A}^T)^\perp = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

  - any selection of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in \mathcal{N}(\mathbf{A})$ is linearly dependent
- $\mathbf{A}$ does not admit an eigendecomposition if $\mu_i > \gamma_i$ for some $i \in \{1, \ldots, k\}$, (i.e., $\mathbf{A}$ is defective).

Eigendecomposition

- there exist matrix subclasses in which eigendecomposition is guaranteed to exist
  - one example is the circulant matrix subclass, as seen in Least Squares Topic; Given $\mathbf{h} = [h_0, h_1, \ldots, h_{n-1}]^T$ and a circulant matrix

$$\mathbf{A} = \mathrm{Circ}(\mathbf{h}) = \begin{bmatrix} h_0 & h_{n-1} & \ldots & \ldots & h_1 \\ h_1 & h_0 & h_{n-1} & \ldots & h_2 \\ h_2 & h_1 & h_0 & \ldots & h_3 \\ \vdots & & & & \vdots \\ \vdots & & & & \vdots \\ h_{n-1} & \ldots & \ldots & h_1 & h_0 \end{bmatrix} \in \mathbb{R}^{n \times n},$$

    its eigendecomposition is $\mathbf{A} = \mathbf{\Psi}^H \mathbf{D} \mathbf{\Psi}$, where $\mathbf{\Psi}$ is the unitary DFT matrix; $\mathbf{D} = \mathrm{Diag}(\sqrt{n}\mathbf{\Psi}\mathbf{h}) = \mathrm{Diag}(d_1, \ldots, d_n)$; $d_i = \sum_{k=0}^{n-1} h_k e^{-\boldsymbol{j}2\pi k(i-1)/n}$.
  - another example is the Hermitian matrix subclass, as we will see later

Question: under which conditions of μ_i and γ_i, can a matrix admit an eigendec.?

- there exist simple sufficient conditions under which eigendecomposition exists

Eigendecomposition

Property 2. Let $\mathbf{A} \in \mathbb{R}^{n\times n}$ (or $\mathbb{C}^{n\times n}$), and suppose that λ_i's are ordered such that $\{\lambda_1, \dots, \lambda_k\}$ is the set of all distinct eigenvalues of $\mathbf{A}$. Also, let $\mathbf{v}_i$ be *any* eigenvector associated with λ_i. Then $\mathbf{v}_1, \dots, \mathbf{v}_k$ must be linearly independent.
(requires a proof)

- Implication: (a sufficient condition for existence of eigendec.) if all the eigenvalues of $\mathbf{A}$ are distinct (somtimes called unique), i.e.,

$$\lambda_i \neq \lambda_j, \qquad \text{for all } i, j \in \{1, \dots, n\} \text{ with } i \neq j,$$

  then $\mathbf{A}$ admits an eigendecomposition
  - to have all the eigenvalues to be distinct is not that hard, as we will see later
  - in this case, the eigendecomp. is unique up to an ordering for the eigenvalues

Theorem 2. (the sufficient and necessary condition for existence of eigendec.) $\mathbf{A}$ admits an eigendcomposition if and only if $\mu_i = \gamma_i$ for all i (i.e., $\mathbf{A}$ is nondefective)

- can be proved by considering Property 1
- A matrix $\mathbf{A} \in \mathbb{C}^{n\times n}$ has an eigendecomposition if and only if the sum of the dimensions of the eigenspaces is n.
- except orderings for the eigenvalues, eigendecomp. is generally nonunique since any orthonormal basis for the eigenspace of a degenerate eigenvalue can be used

Eigendecomposition for Hermitian Matrices

Consider the Hermitian matrix subclass.

Property 3. Let $\mathbf{A} \in \mathbb{H}^n$.

1. the eigenvalues $\lambda_1, \ldots, \lambda_n$ of $\mathbf{A}$ are real

2. suppose that λ_i's are ordered such that $\{\lambda_1, \ldots, \lambda_k\}$ is the set of all distinct eigenvalues of $\mathbf{A}$. Also, let $\mathbf{v}_i$ be *any* eigenvector associated with λ_i. Then $\mathbf{v}_1, \ldots, \mathbf{v}_k$ must be orthonormal.

(requires a proof)

- the above results apply to real symmetric matrices; recall $\mathbf{A} \in \mathbb{S}^n \Longrightarrow \mathbf{A} \in \mathbb{H}^n$
- Corollary: for a real symmetric matrix, all eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ can be chosen as real
- implication: for a Hermitian $\mathbf{A}$ with all its eigenvalues being distinct, then $\mathbf{A}$ admit an eigendecomposition with unitary $\mathbf{V}$.
- In fact, a Hermitian $\mathbf{A}$ always admits an eigendecomposition with unitary $\mathbf{V}$, i.e., is always unitarily diagonalizable (or orthogonally diagonalizable for the real case)!

Eigendecomposition for Hermitian Matrices

Theorem 3. Every $\mathbf{A} \in \mathbb{H}^n$ admits an eigendecomposition

$$\mathbf{A} = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^H,$$

where $\mathbf{V} \in \mathbb{C}^{n\times n}$ is unitary; $\mathbf{\Lambda} = \mathrm{Diag}(\lambda_1, \ldots, \lambda_n)$ with $\lambda_i \in \mathbb{R}$ for all i. Also, if $\mathbf{A} \in \mathbb{S}^n$, $\mathbf{V}$ can be taken as real orthogonal.

- $\mathbf{A} = \sum_{i=1}^n \lambda_i \mathbf{v}_i \mathbf{v}_i^H$
- Corollary: if $\mathbf{A}$ is Hermitian (or real symmetric), $\mu_i = \gamma_i$ for all i
- Proof? a consequence of a more powerful decomposition, namely, the Schur decomposition