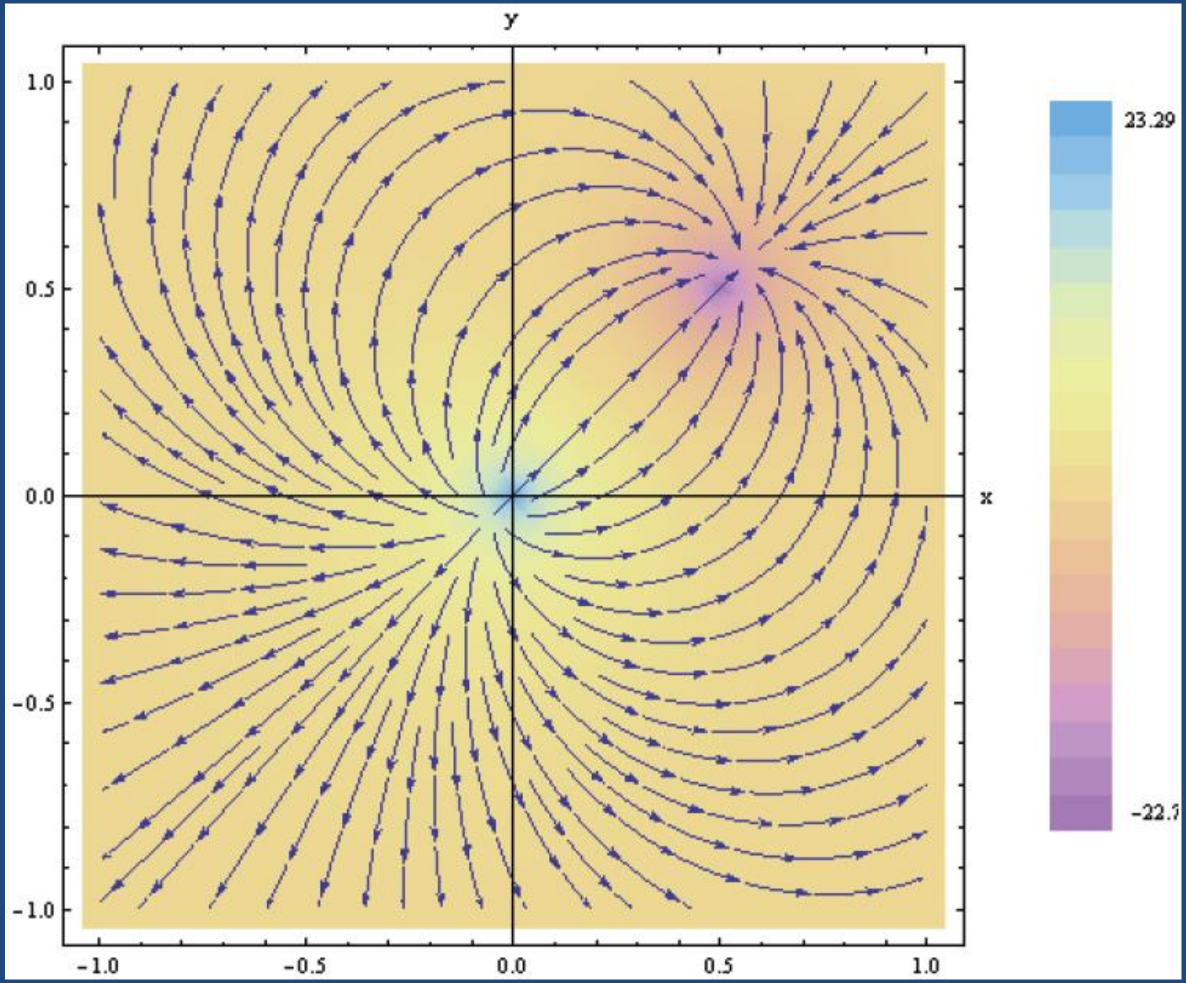


3. VECTOR ANALYSIS

7e Applied EM by Ulaby and Ravaioli

Chapter 3 Overview

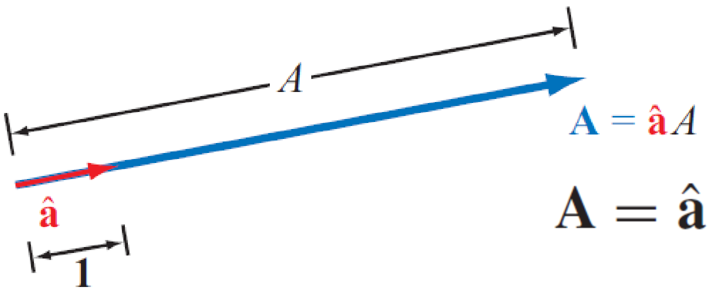
Chapter Contents

Objectives

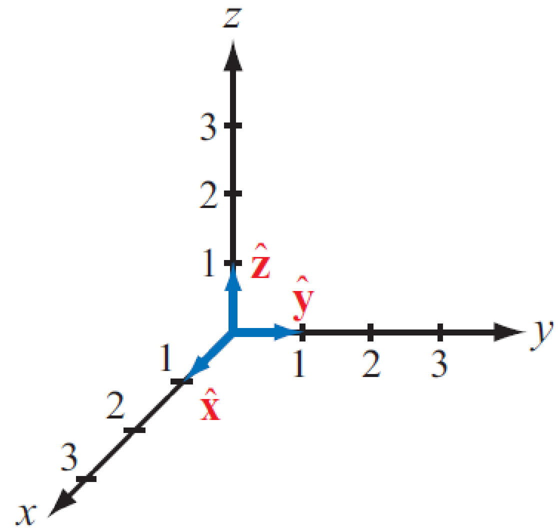
Upon learning the material presented in this chapter, you should be able to:

1. Use vector algebra in Cartesian, cylindrical, and spherical coordinate systems.
2. Transform vectors between the three primary coordinate systems.
3. Calculate the gradient of a scalar function and the divergence and curl of a vector function in any of the three primary coordinate systems.
4. Apply the divergence theorem and Stokes's theorem.

Laws of Vector Algebra



$$\mathbf{A} = \hat{\mathbf{a}}|\mathbf{A}| = \hat{\mathbf{a}}A$$



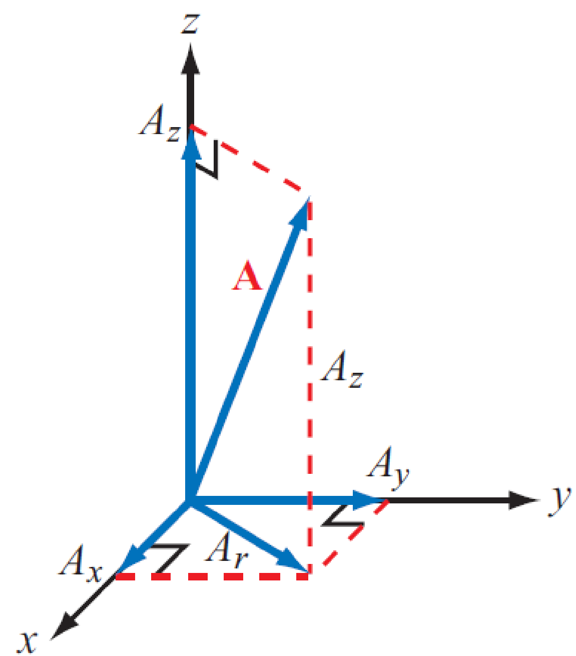
(a) Base vectors

^ to represent unit vector

$$\mathbf{A} = \hat{\mathbf{x}}A_x + \hat{\mathbf{y}}A_y + \hat{\mathbf{z}}A_z$$

$$A = |\mathbf{A}| = \sqrt[+]{A_x^2 + A_y^2 + A_z^2}$$

$$\hat{\mathbf{a}} = \frac{\mathbf{A}}{A} = \frac{\hat{\mathbf{x}}A_x + \hat{\mathbf{y}}A_y + \hat{\mathbf{z}}A_z}{\sqrt[+]{A_x^2 + A_y^2 + A_z^2}}$$



(b) Components of **A**

Properties of Vector Operations

Equality of Two Vectors

$$\mathbf{A} = \hat{\mathbf{a}}A = \hat{\mathbf{x}}A_x + \hat{\mathbf{y}}A_y + \hat{\mathbf{z}}A_z, \tag{3.6a}$$

$$\mathbf{B} = \hat{\mathbf{b}}B = \hat{\mathbf{x}}B_x + \hat{\mathbf{y}}B_y + \hat{\mathbf{z}}B_z, \tag{3.6b}$$

then $\mathbf{A} = \mathbf{B}$ if and only if $A = B$ and $\hat{\mathbf{a}} = \hat{\mathbf{b}}$, which requires that $A_x = B_x$, $A_y = B_y$, and $A_z = B_z$.

Equality of two vectors does not necessarily imply that they are identical; in Cartesian coordinates, two displaced parallel vectors of equal magnitude and pointing in the same direction are equal, but they are identical only if they lie on top of one another.

(1) Direction (2) magnitude

Commutative property

$$\mathbf{C} = \mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$$

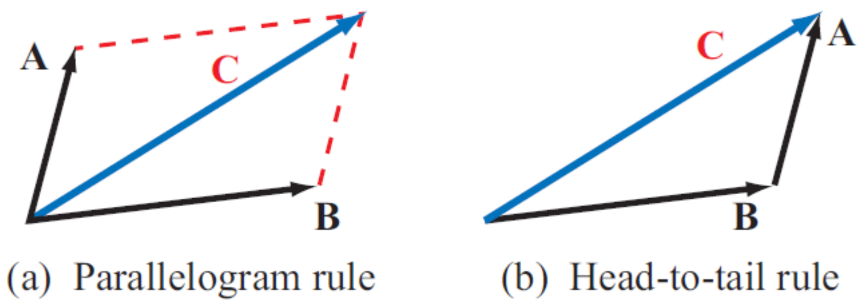


Figure 3-3: Vector addition by (a) the parallelogram rule and (b) the head-to-tail rule.

Position & Distance Vectors

Position Vector: From origin to point P

$\rightarrow$ to represent a general vector

$$\mathbf{R}_1 = \overrightarrow{OP_1} = \hat{\mathbf{x}}x_1 + \hat{\mathbf{y}}y_1 + \hat{\mathbf{z}}z_1$$

$$\mathbf{R}_2 = \overrightarrow{OP_2} = \hat{\mathbf{x}}x_2 + \hat{\mathbf{y}}y_2 + \hat{\mathbf{z}}z_2$$

Distance Vector: Between two points

$$\begin{aligned}\mathbf{R}_{12} &= \overrightarrow{P_1 P_2} \\ &= \mathbf{R}_2 - \mathbf{R}_1 \\ &= \hat{\mathbf{x}}(x_2 - x_1) + \hat{\mathbf{y}}(y_2 - y_1) + \hat{\mathbf{z}}(z_2 - z_1)\end{aligned}$$

Subscript 1 for Starting point
Subscript 2 for ending point

the distance d between P_1 and P_2 equals the magnitude of $\mathbf{R}_{12}$:

$$\begin{aligned}d &= |\mathbf{R}_{12}| \\ &= [(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]^{1/2}. \qquad (3.12)\end{aligned}$$

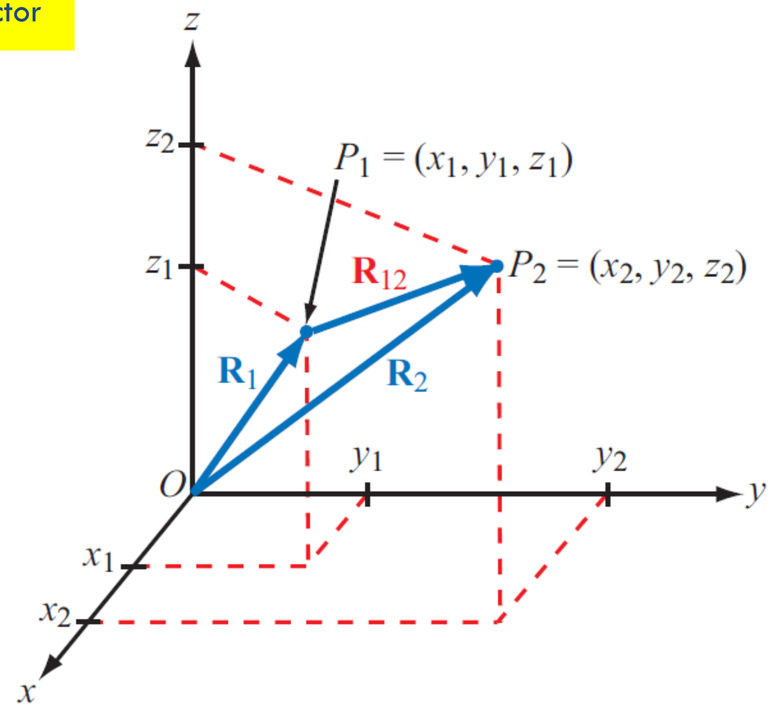


Figure 3-4: Distance vector $\mathbf{R}_{12} = \overrightarrow{P_1 P_2} = \mathbf{R}_2 - \mathbf{R}_1$, where $\mathbf{R}_1$ and $\mathbf{R}_2$ are the position vectors of points P_1 and P_2, respectively.

Vector Multiplication: Scalar Product or "Dot Product"

$$\mathbf{A} \cdot \mathbf{B} = AB\cos\theta_{AB}$$

Note the sequence (from A to B)

$$A = |\mathbf{A}| = \sqrt[+]{\mathbf{A} \cdot \mathbf{A}}$$

$$\theta_{AB} = \cos^{-1}\left[\frac{\mathbf{A} \cdot \mathbf{B}}{\sqrt[+]{\mathbf{A} \cdot \mathbf{A}}\ \sqrt[+]{\mathbf{B} \cdot \mathbf{B}}}\right]$$

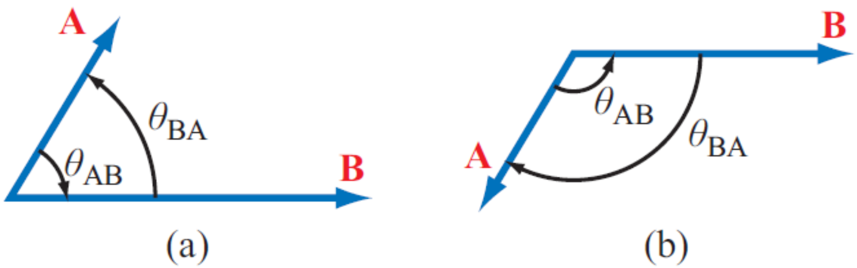


Figure 3-5: The angle θ_{AB} is the angle between **A** and **B**, measured from **A** to **B** between vector tails. The dot product is positive if $0 \le \theta_{AB} < 90°$, as in (a), and it is negative if $90° < \theta_{AB} \le 180°$, as in (b).

$$\hat{\mathbf{x}} \cdot \hat{\mathbf{x}} = \hat{\mathbf{y}} \cdot \hat{\mathbf{y}} = \hat{\mathbf{z}} \cdot \hat{\mathbf{z}} = 1,$$
$$\hat{\mathbf{x}} \cdot \hat{\mathbf{y}} = \hat{\mathbf{y}} \cdot \hat{\mathbf{z}} = \hat{\mathbf{z}} \cdot \hat{\mathbf{x}} = 0.$$

$$\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A} \quad \text{(commutative property)},$$

$$\mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) = \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{C} \quad \text{(distributive property)}$$

If $\mathbf{A} = (A_x, A_y, A_z)$ and $\mathbf{B} = (B_x, B_y, B_z)$, then

$$\mathbf{A} \cdot \mathbf{B} = (\hat{\mathbf{x}}A_x + \hat{\mathbf{y}}A_y + \hat{\mathbf{z}}A_z) \cdot (\hat{\mathbf{x}}B_x + \hat{\mathbf{y}}B_y + \hat{\mathbf{z}}B_z).$$

Hence:

$$\mathbf{A} \cdot \mathbf{B} = A_xB_x + A_yB_y + A_zB_z.$$

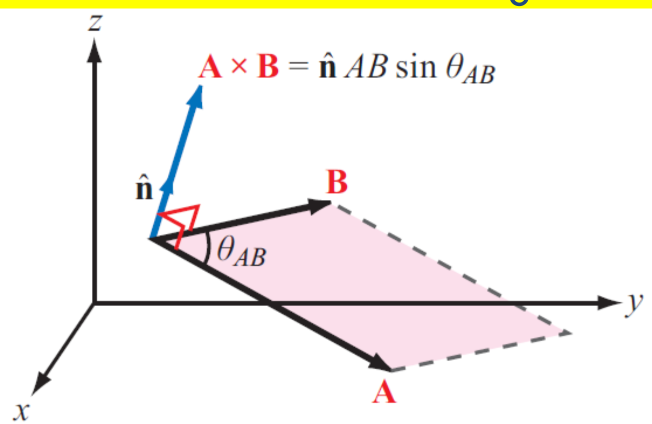
Vector Multiplication: Vector Product or "Cross Product"

$$\boxed{\mathbf{A} \times \mathbf{B} = \hat{\mathbf{n}}\, AB \sin\theta_{AB}}$$

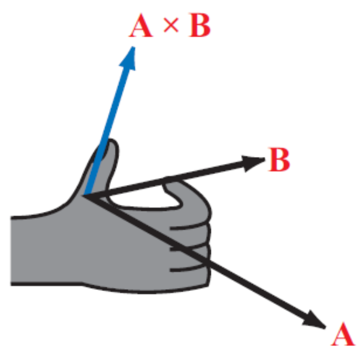
$$\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A} \qquad \text{(anticommutative)}$$

$$\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C} \qquad \text{(distributive)}$$

Determine direction and magnitude separately



(a) Cross product



(b) Right-hand rule

$$\mathbf{A} \times \mathbf{A} = 0$$

$$\boxed{\hat{\mathbf{x}} \times \hat{\mathbf{y}} = \hat{\mathbf{z}}, \qquad \hat{\mathbf{y}} \times \hat{\mathbf{z}} = \hat{\mathbf{x}}, \qquad \hat{\mathbf{z}} \times \hat{\mathbf{x}} = \hat{\mathbf{y}}. \qquad (3.25)}$$

Note the cyclic order $(xyzxyz\ldots)$. Also,

$$\boxed{\hat{\mathbf{x}} \times \hat{\mathbf{x}} = \hat{\mathbf{y}} \times \hat{\mathbf{y}} = \hat{\mathbf{z}} \times \hat{\mathbf{z}} = 0. \qquad (3.26)}$$

If $\mathbf{A} = (A_x, A_y, A_z)$ and $\mathbf{B} = (B_x, B_y, B_z)$,

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}.$$

Example 3-1: Vectors and Angles

In Cartesian coordinates, vector **A** points from the origin to point $P_1 = (2, 3, 3)$, and vector **B** is directed from P_1 to point $P_2 = (1, -2, 2)$. Find

(a) vector **A**, its magnitude A, and unit vector $\hat{\mathbf{a}}$,

(b) the angle between **A** and the y-axis,

(c) vector **B**,

(d) the angle θ_{AB} between **A** and **B**, and

(e) the perpendicular distance from the origin to vector **B**.

Solution: (a) Vector **A** is given by the position vector of $P_1 = (2, 3, 3)$ as shown in Fig. 3-7. Thus,

$$\mathbf{A} = \hat{\mathbf{x}}2 + \hat{\mathbf{y}}3 + \hat{\mathbf{z}}3,$$

$$A = |\mathbf{A}| = \sqrt{2^2 + 3^2 + 3^2} = \sqrt{22}\,,$$

$$\hat{\mathbf{a}} = \frac{\mathbf{A}}{A} = (\hat{\mathbf{x}}2 + \hat{\mathbf{y}}3 + \hat{\mathbf{z}}3)/\sqrt{22}\,.$$

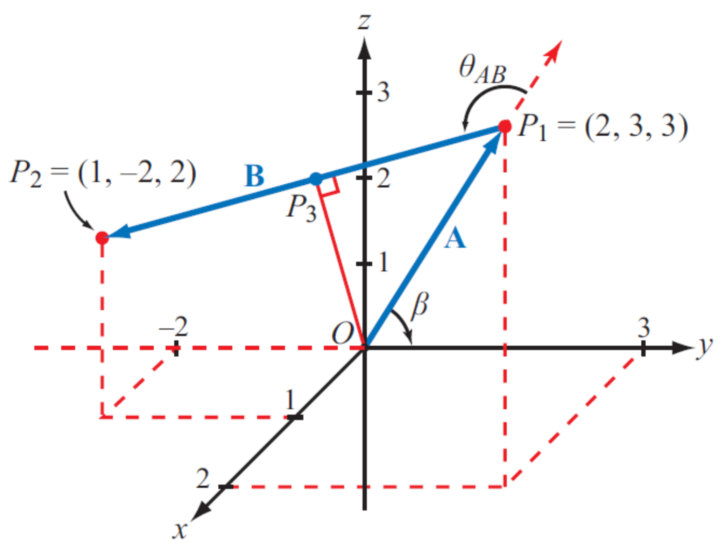


Figure 3-7: Geometry of Example 3-1.

(b) The angle β between **A** and the y-axis is obtained from

$$\mathbf{A} \cdot \hat{\mathbf{y}} = |\mathbf{A}||\hat{\mathbf{y}}| \cos\beta = A\cos\beta,$$

or

$$\beta = \cos^{-1}\left(\frac{\mathbf{A}\cdot\hat{\mathbf{y}}}{A}\right) = \cos^{-1}\left(\frac{3}{\sqrt{22}}\right) = 50.2^\circ.$$

(c)

$$\mathbf{B} = \hat{\mathbf{x}}(1-2) + \hat{\mathbf{y}}(-2-3) + \hat{\mathbf{z}}(2-3) = -\hat{\mathbf{x}} - \hat{\mathbf{y}}5 - \hat{\mathbf{z}}.$$

(d)

$$\theta_{AB} = \cos^{-1}\left[\frac{\mathbf{A}\cdot\mathbf{B}}{|\mathbf{A}||\mathbf{B}|}\right] = \cos^{-1}\left[\frac{(-2-15-3)}{\sqrt{22}\,\sqrt{27}}\right]$$
$$= 145.1^\circ.$$

(e) The perpendicular distance between the origin and vector **B** is the distance $|\overrightarrow{OP_3}|$ shown in Fig. 3-7. From right triangle OP_1P_3,

$$|\overrightarrow{OP_3}| = |\mathbf{A}|\sin(180^\circ - \theta_{AB})$$
$$= \sqrt{22}\,\sin(180^\circ - 145.1^\circ) = 2.68.$$

Triple Products

Scalar Triple Product

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A})$$

Trick: always move forward the last vector

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

Vector Triple Product

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B}),$$

which is known as the "bac-cab" rule.

Example 3-2: Vector Triple Product

Given $\mathbf{A} = \hat{\mathbf{x}} - \hat{\mathbf{y}} + \hat{\mathbf{z}}2$, $\mathbf{B} = \hat{\mathbf{y}} + \hat{\mathbf{z}}$, and $\mathbf{C} = -\hat{\mathbf{x}}2 + \hat{\mathbf{z}}3$, find $(\mathbf{A}\times\mathbf{B})\times\mathbf{C}$ and compare it with $\mathbf{A}\times(\mathbf{B}\times\mathbf{C})$.

Solution:

$$\mathbf{A}\times\mathbf{B} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ 1 & -1 & 2 \\ 0 & 1 & 1 \end{vmatrix} = -\hat{\mathbf{x}}3 - \hat{\mathbf{y}} + \hat{\mathbf{z}}$$

and

$$(\mathbf{A}\times\mathbf{B})\times\mathbf{C} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ -3 & -1 & 1 \\ -2 & 0 & 3 \end{vmatrix} = -\hat{\mathbf{x}}3 + \hat{\mathbf{y}}7 - \hat{\mathbf{z}}2.$$

A similar procedure gives $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \hat{\mathbf{x}}2 + \hat{\mathbf{y}}4 + \hat{\mathbf{z}}$.

Hence:

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) \neq (\mathbf{A}\times\mathbf{B})\times\mathbf{C}$$

Cartesian Coordinate System

Differential length vector

$$d\mathbf{l} = \hat{\mathbf{x}}\, dl_x + \hat{\mathbf{y}}\, dl_y + \hat{\mathbf{z}}\, dl_z = \hat{\mathbf{x}}\, dx + \hat{\mathbf{y}}\, dy + \hat{\mathbf{z}}\, dz, \quad (3.34)$$

where $dl_x = dx$ is a differential length along $\hat{\mathbf{x}}$, and similar interpretations apply to $dl_y = dy$ and $dl_z = dz$.

Differential area vectors

$$d\mathbf{s}_x = \hat{\mathbf{x}}\, dl_y\, dl_z = \hat{\mathbf{x}}\, dy\, dz \qquad (y\text{–}z \text{ plane}), \qquad (3.35a)$$

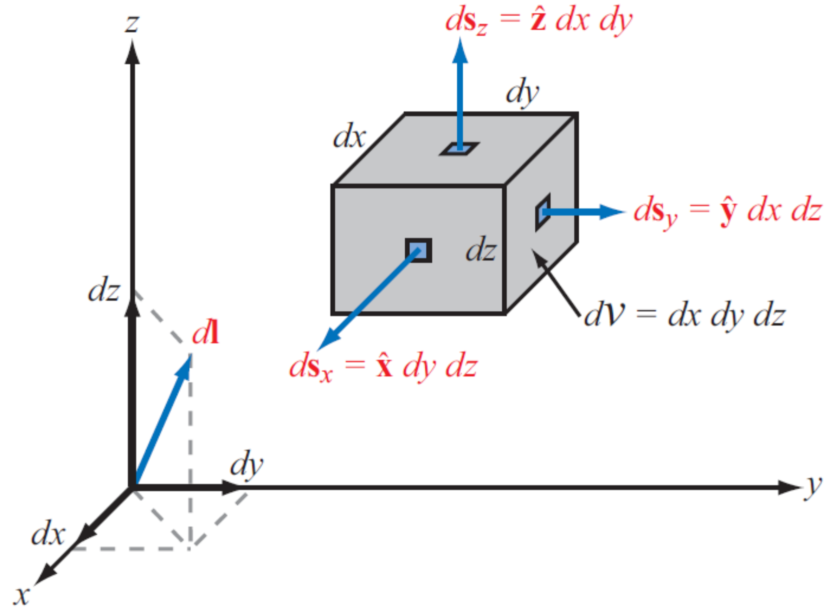
with the subscript on $d\mathbf{s}$ denoting its direction. Similarly,

$$d\mathbf{s}_y = \hat{\mathbf{y}}\, dx\, dz \qquad (x\text{–}z \text{ plane}), \qquad (3.35b)$$

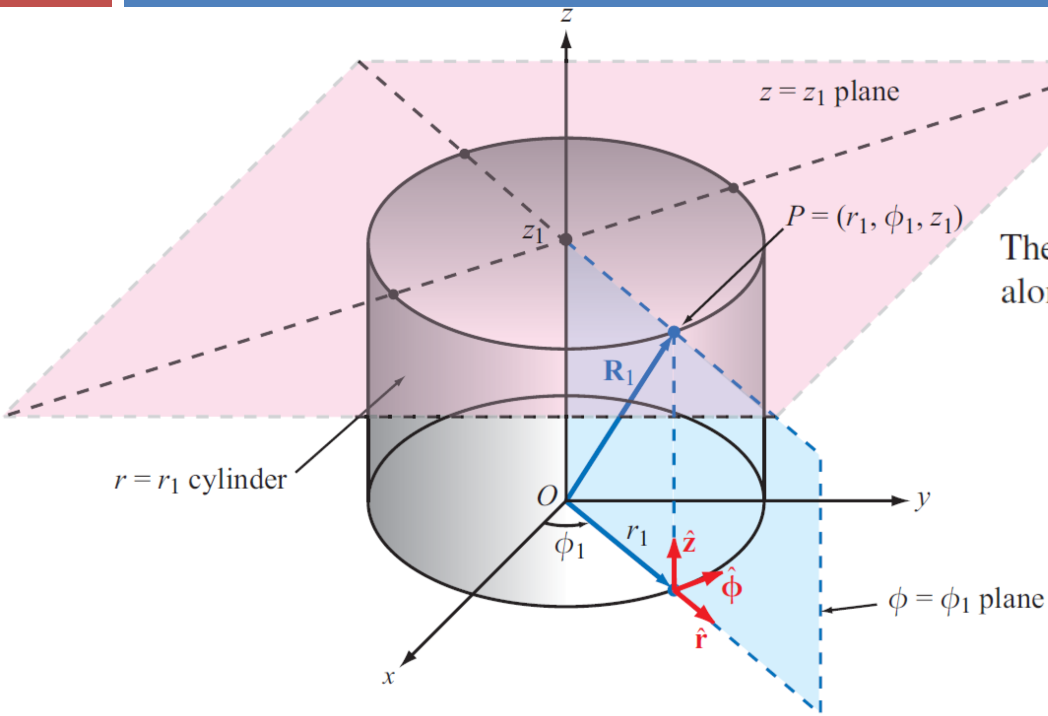
$$d\mathbf{s}_z = \hat{\mathbf{z}}\, dx\, dy \qquad (x\text{–}y \text{ plane}). \qquad (3.35c)$$

A ***differential volume*** equals the product of all three differential lengths:

$$dV = dx\, dy\, dz. \qquad (3.36)$$



Cylindrical Coordinate System



The position vector $\overrightarrow{OP}$ shown in Fig. 3-9 has components along r and z only. Thus,

$$\mathbf{R}_1 = \overrightarrow{OP} = \hat{\mathbf{r}}r_1 + \hat{\mathbf{z}}z_1. \tag{3.40}$$

The mutually perpendicular base vectors are $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\phi}}$, and $\hat{\mathbf{z}}$, with $\hat{\mathbf{r}}$ pointing away from the origin along r, $\hat{\boldsymbol{\phi}}$ pointing in a direction tangential to the cylindrical surface, and $\hat{\mathbf{z}}$ pointing along the vertical. Unlike the Cartesian system, in which the base vectors $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ are independent of the location of P, in the cylindrical system both $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\phi}}$ are functions of ϕ.

Cylindrical Coordinate System

The base unit vectors obey the following right-hand cyclic relations:

$$\hat{\mathbf{r}} \times \hat{\boldsymbol{\phi}} = \hat{\mathbf{z}}, \qquad \hat{\boldsymbol{\phi}} \times \hat{\mathbf{z}} = \hat{\mathbf{r}}, \qquad \hat{\mathbf{z}} \times \hat{\mathbf{r}} = \hat{\boldsymbol{\phi}}, \qquad (3.37)$$

Trick: always move forward the last vector

and like all unit vectors, $\hat{\mathbf{r}} \cdot \hat{\mathbf{r}} = \hat{\boldsymbol{\phi}} \cdot \hat{\boldsymbol{\phi}} = \hat{\mathbf{z}} \cdot \hat{\mathbf{z}} = 1$, and $\hat{\mathbf{r}} \times \hat{\mathbf{r}} = \hat{\boldsymbol{\phi}} \times \hat{\boldsymbol{\phi}} = \hat{\mathbf{z}} \times \hat{\mathbf{z}} = 0$.

In cylindrical coordinates, a vector is expressed as

$$\mathbf{A} = \hat{\mathbf{a}}|\mathbf{A}| = \hat{\mathbf{r}}A_r + \hat{\boldsymbol{\phi}}A_\phi + \hat{\mathbf{z}}A_z, \qquad (3.38)$$

$$dl_r = dr, \qquad dl_\phi = r\, d\phi, \qquad dl_z = dz. \qquad (3.41)$$

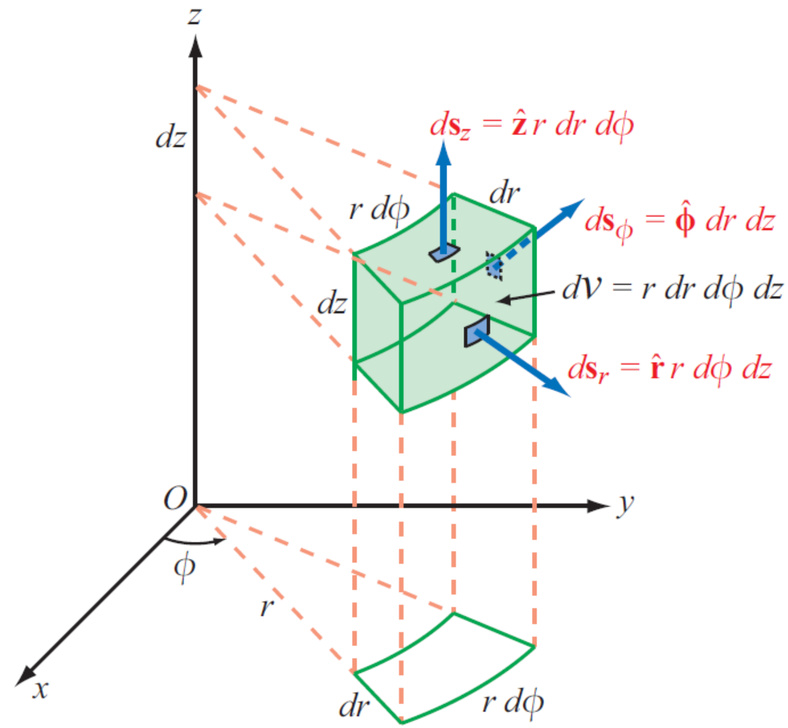


Figure 3-10: Differential areas and volume in cylindrical coordinates.

Note that the differential length along $\hat{\boldsymbol{\phi}}$ is $r\,d\phi$, not just $d\phi$. The differential length $d\mathbf{l}$ in cylindrical coordinates is given by

$$d\mathbf{l} = \hat{\mathbf{r}}\, dl_r + \hat{\boldsymbol{\phi}}\, dl_\phi + \hat{\mathbf{z}}\, dl_z = \hat{\mathbf{r}}\, dr + \hat{\boldsymbol{\phi}} r\, d\phi + \hat{\mathbf{z}}\, dz. \qquad (3.42)$$

Example 3-3: Distance Vector in Cylindrical Coordinates

Find an expression for the unit vector of vector **A** shown in Fig. 3-11 in cylindrical coordinates.

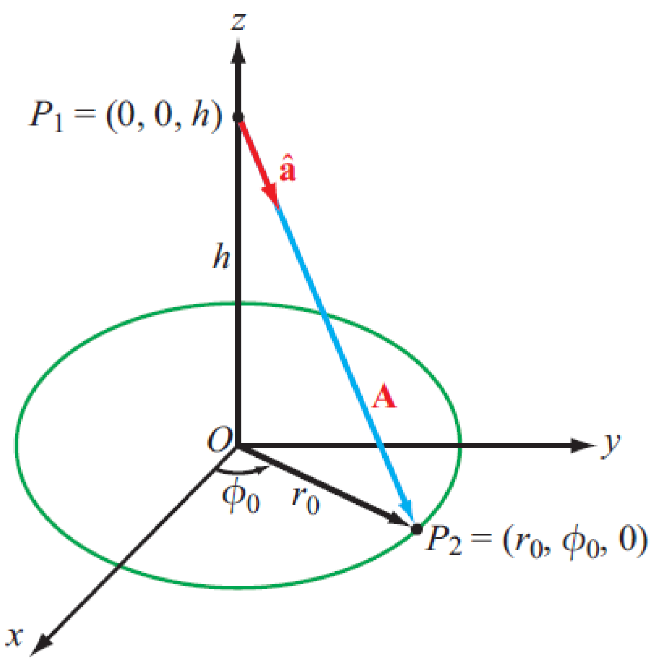


Figure 3-11: Geometry of Example 3-3.

Solution: In triangle OP_1P_2,

$$\overrightarrow{OP_2} = \overrightarrow{OP_1} + \mathbf{A}.$$

Hence,

$$\mathbf{A} = \overrightarrow{OP_2} - \overrightarrow{OP_1} = \hat{\mathbf{r}}r_0 - \hat{\mathbf{z}}h,$$

and

$$\hat{\mathbf{a}} = \frac{\mathbf{A}}{|\mathbf{A}|} = \frac{\hat{\mathbf{r}}r_0 - \hat{\mathbf{z}}h}{\sqrt{r_0^2 + h^2}}.$$

We note that the expression for **A** is independent of ϕ_0. That is, all vectors from point P_1 to any point on the circle defined by $r = r_0$ in the x–y plane are equal in the cylindrical coordinate system. The ambiguity can be eliminated by specifying that **A** passes through a point whose $\phi = \phi_0$.

Example 3-4: Cylindrical Area

Find the area of a cylindrical surface described by $r = 5$, $30° \leq \phi \leq 60°$, and $0 \leq z \leq 3$ (Fig. 3-12).

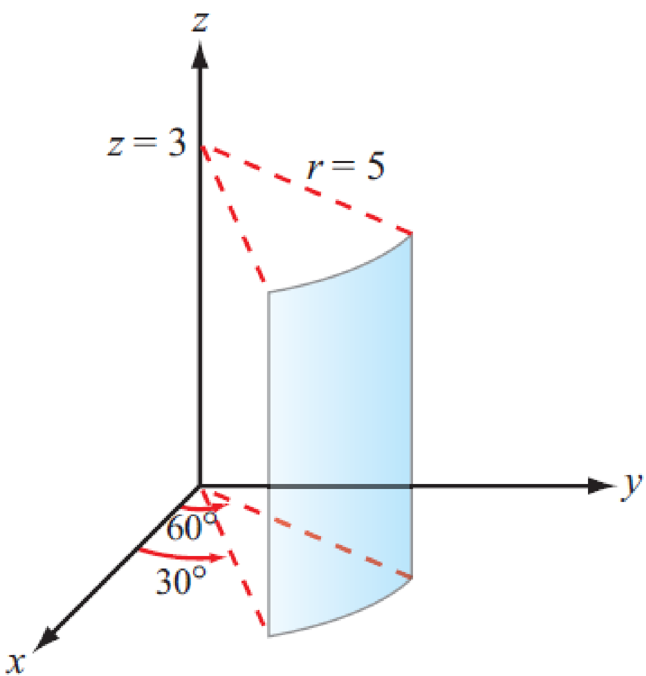


Figure 3-12: Cylindrical surface of Example 3-4.

Solution: The prescribed surface is shown in Fig. 3-12. Use of Eq. (3.43a) for a surface element with constant r gives

$$S = r \int_{\phi=30°}^{60°} d\phi \int_{z=0}^{3} dz$$

$$= 5\phi \Big|_{\pi/6}^{\pi/3} \; z \Big|_{0}^{3}$$

$$= \frac{5\pi}{2} .$$

Note that ϕ had to be converted to radians before evaluating the integration limits.

Spherical Coordinate System

Trick: always move forward the last vector

$$\hat{\mathbf{R}} \times \hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\phi}}, \quad \hat{\boldsymbol{\theta}} \times \hat{\boldsymbol{\phi}} = \hat{\mathbf{R}}, \quad \hat{\boldsymbol{\phi}} \times \hat{\mathbf{R}} = \hat{\boldsymbol{\theta}}. \tag{3.45}$$

A vector with components A_R, A_θ, and A_ϕ is written as

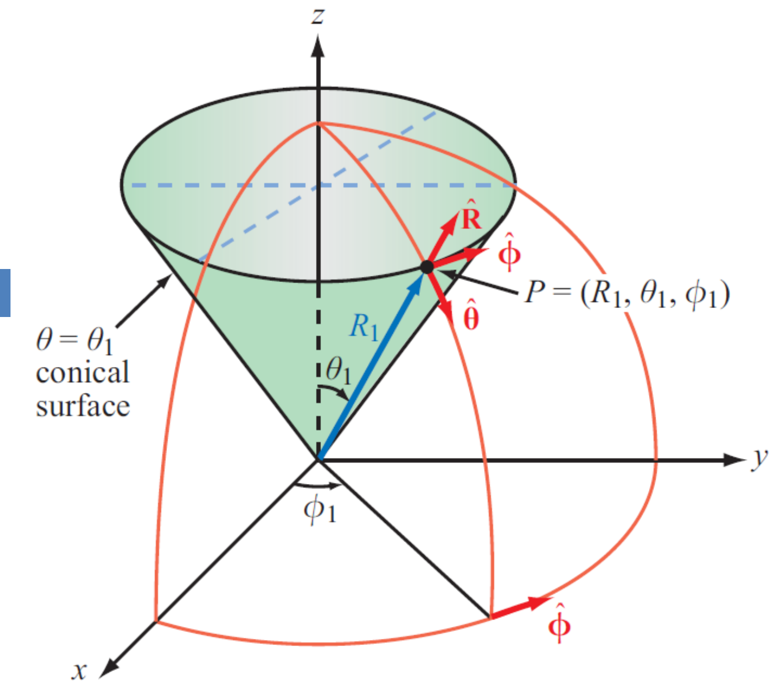
$$\mathbf{A} = \hat{\mathbf{a}}|\mathbf{A}| = \hat{\mathbf{R}}A_R + \hat{\boldsymbol{\theta}}A_\theta + \hat{\boldsymbol{\phi}}A_\phi, \tag{3.46}$$

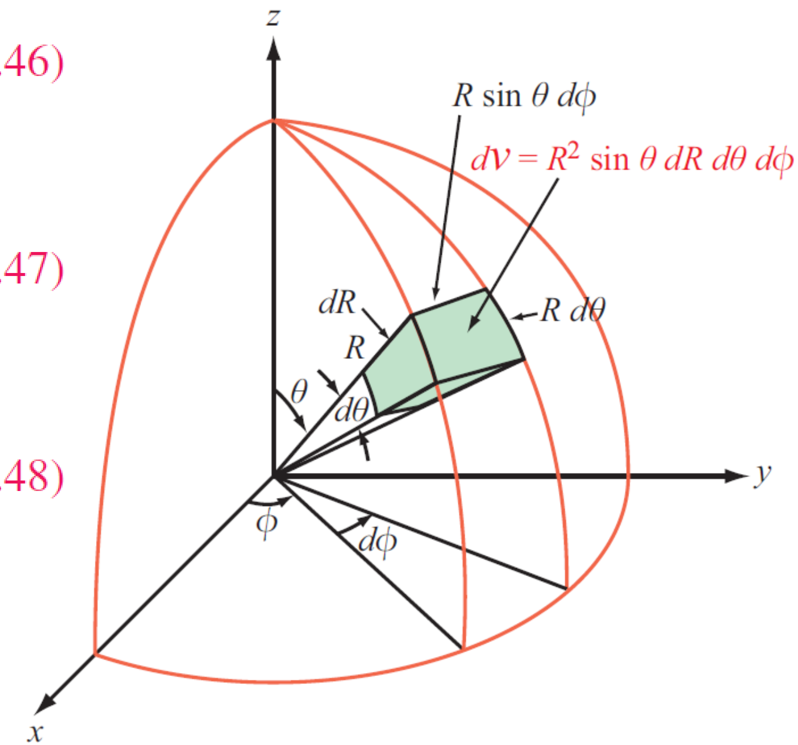
and its magnitude is

$$|\mathbf{A}| = \sqrt[+]{\mathbf{A}\cdot\mathbf{A}} = \sqrt[+]{A_R^2 + A_\theta^2 + A_\phi^2}. \tag{3.47}$$

The position vector of point $P = (R_1, \theta_1, \phi_1)$ is simply

$$\mathbf{R}_1 = \overrightarrow{OP} = \hat{\mathbf{R}}R_1, \tag{3.48}$$





Example 3-5: Surface Area in Spherical Coordinates

The spherical strip shown in Fig. 3-15 is a section of a sphere of radius 3 cm. Find the area of the strip.

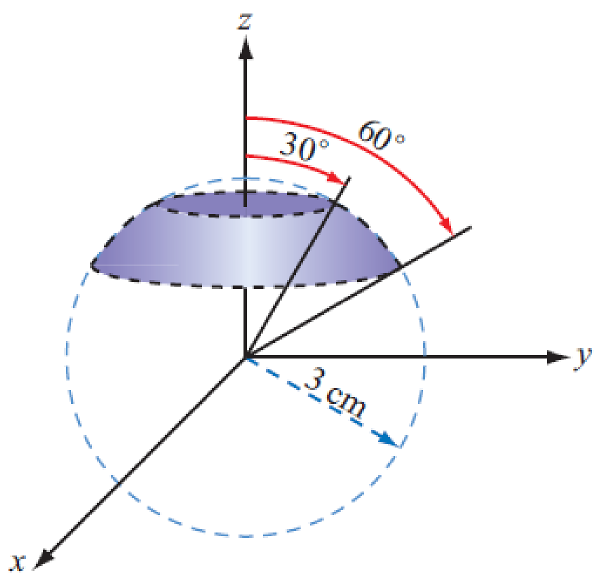


Figure 3-15: Spherical strip of Example 3-5.

Solution: Use of Eq. (3.50b) for the area of an elemental spherical area with constant radius R gives

$$S = R^2 \int_{\theta=30^\circ}^{60^\circ} \sin\theta \, d\theta \int_{\phi=0}^{2\pi} d\phi$$

$$= 9(-\cos\theta)\Big|_{30^\circ}^{60^\circ} \; \phi\Big|_0^{2\pi} \qquad (\text{cm}^2)$$

$$= 18\pi(\cos 30^\circ - \cos 60^\circ) = 20.7 \text{ cm}^2.$$

Example 3-6: Charge in a Sphere

A sphere of radius 2 cm contains a volume charge density ρ_v given by

$$\rho_v = 4\cos^2\theta \qquad (\text{C/m}^3).$$

Find the total charge Q contained in the sphere.

Solution:

$$Q = \int_{\mathcal{V}} \rho_v \, d\mathcal{V}$$

$$= \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \int_{R=0}^{2\times 10^{-2}} (4\cos^2\theta) R^2 \sin\theta \, dR \, d\theta \, d\phi$$

$$= 4 \int_0^{2\pi} \int_0^{\pi} \left(\frac{R^3}{3}\right)\Big|_0^{2\times 10^{-2}} \sin\theta \cos^2\theta \, d\theta \, d\phi$$

$$= \frac{32}{3} \times 10^{-6} \int_0^{2\pi} \left(-\frac{\cos^3\theta}{3}\right)\Big|_0^{\pi} d\phi$$

$$= \frac{64}{9} \times 10^{-6} \int_0^{2\pi} d\phi$$

$$= \frac{128\pi}{9} \times 10^{-6} = 44.68 \qquad (\mu\text{C}).$$

Table 3-1: Summary of vector relations.

	Cartesian Coordinates	Cylindrical Coordinates	Spherical Coordinates
Coordinate variables	x, y, z	r, ϕ, z	R, θ, ϕ
Vector representation $\mathbf{A} =$	$\hat{\mathbf{x}}A_x + \hat{\mathbf{y}}A_y + \hat{\mathbf{z}}A_z$	$\boxed{\hat{\mathbf{r}}A_r + \hat{\boldsymbol{\phi}}A_\phi + \hat{\mathbf{z}}A_z}$	$\boxed{\hat{\mathbf{R}}A_R + \hat{\boldsymbol{\theta}}A_\theta + \hat{\boldsymbol{\phi}}A_\phi}$
Magnitude of A $\quad \lvert\mathbf{A}\rvert =$	$\sqrt[+]{A_x^2 + A_y^2 + A_z^2}$	$\sqrt[+]{A_r^2 + A_\phi^2 + A_z^2}$	$\sqrt[+]{A_R^2 + A_\theta^2 + A_\phi^2}$
Position vector $\overrightarrow{OP_1} =$	$\hat{\mathbf{x}}x_1 + \hat{\mathbf{y}}y_1 + \hat{\mathbf{z}}z_1$, for $P = (x_1, y_1, z_1)$	$\boxed{\hat{\mathbf{r}}r_1 + \hat{\mathbf{z}}z_1, \text{ for } P = (r_1, \phi_1, z_1)}$	$\boxed{\hat{\mathbf{R}}R_1, \text{ for } P = (R_1, \theta_1, \phi_1)}$
Base vectors properties	$\hat{\mathbf{x}}\cdot\hat{\mathbf{x}} = \hat{\mathbf{y}}\cdot\hat{\mathbf{y}} = \hat{\mathbf{z}}\cdot\hat{\mathbf{z}} = 1$ $\hat{\mathbf{x}}\cdot\hat{\mathbf{y}} = \hat{\mathbf{y}}\cdot\hat{\mathbf{z}} = \hat{\mathbf{z}}\cdot\hat{\mathbf{x}} = 0$ $\hat{\mathbf{x}}\times\hat{\mathbf{y}} = \hat{\mathbf{z}}$ $\hat{\mathbf{y}}\times\hat{\mathbf{z}} = \hat{\mathbf{x}}$ $\hat{\mathbf{z}}\times\hat{\mathbf{x}} = \hat{\mathbf{y}}$	$\hat{\mathbf{r}}\cdot\hat{\mathbf{r}} = \hat{\boldsymbol{\phi}}\cdot\hat{\boldsymbol{\phi}} = \hat{\mathbf{z}}\cdot\hat{\mathbf{z}} = 1$ $\hat{\mathbf{r}}\cdot\hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}}\cdot\hat{\mathbf{z}} = \hat{\mathbf{z}}\cdot\hat{\mathbf{r}} = 0$ $\hat{\mathbf{r}}\times\hat{\boldsymbol{\phi}} = \hat{\mathbf{z}}$ $\hat{\boldsymbol{\phi}}\times\hat{\mathbf{z}} = \hat{\mathbf{r}}$ $\hat{\mathbf{z}}\times\hat{\mathbf{r}} = \hat{\boldsymbol{\phi}}$	$\hat{\mathbf{R}}\cdot\hat{\mathbf{R}} = \hat{\boldsymbol{\theta}}\cdot\hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\phi}}\cdot\hat{\boldsymbol{\phi}} = 1$ $\hat{\mathbf{R}}\cdot\hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\theta}}\cdot\hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}}\cdot\hat{\mathbf{R}} = 0$ $\hat{\mathbf{R}}\times\hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\phi}}$ $\hat{\boldsymbol{\theta}}\times\hat{\boldsymbol{\phi}} = \hat{\mathbf{R}}$ $\hat{\boldsymbol{\phi}}\times\hat{\mathbf{R}} = \hat{\boldsymbol{\theta}}$
Dot product $\quad \mathbf{A}\cdot\mathbf{B} =$	$A_xB_x + A_yB_y + A_zB_z$	$A_rB_r + A_\phi B_\phi + A_zB_z$	$A_RB_R + A_\theta B_\theta + A_\phi B_\phi$
Cross product $\quad \mathbf{A}\times\mathbf{B} =$	$\begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$	$\begin{vmatrix} \hat{\mathbf{r}} & \hat{\boldsymbol{\phi}} & \hat{\mathbf{z}} \\ A_r & A_\phi & A_z \\ B_r & B_\phi & B_z \end{vmatrix}$	$\begin{vmatrix} \hat{\mathbf{R}} & \hat{\boldsymbol{\theta}} & \hat{\boldsymbol{\phi}} \\ A_R & A_\theta & A_\phi \\ B_R & B_\theta & B_\phi \end{vmatrix}$
Differential length $\quad d\mathbf{l} =$	$\hat{\mathbf{x}}\,dx + \hat{\mathbf{y}}\,dy + \hat{\mathbf{z}}\,dz$	$\hat{\mathbf{r}}\,dr + \hat{\boldsymbol{\phi}}r\,d\phi + \hat{\mathbf{z}}\,dz$	$\hat{\mathbf{R}}\,dR + \hat{\boldsymbol{\theta}}R\,d\theta + \hat{\boldsymbol{\phi}}R\sin\theta\,d\phi$
Differential surface areas	$d\mathbf{s}_x = \hat{\mathbf{x}}\,dy\,dz$ $d\mathbf{s}_y = \hat{\mathbf{y}}\,dx\,dz$ $d\mathbf{s}_z = \hat{\mathbf{z}}\,dx\,dy$	$d\mathbf{s}_r = \hat{\mathbf{r}}r\,d\phi\,dz$ $d\mathbf{s}_\phi = \hat{\boldsymbol{\phi}}\,dr\,dz$ $d\mathbf{s}_z = \hat{\mathbf{z}}r\,dr\,d\phi$	$d\mathbf{s}_R = \hat{\mathbf{R}}R^2\sin\theta\,d\theta\,d\phi$ $d\mathbf{s}_\theta = \hat{\boldsymbol{\theta}}R\sin\theta\,dR\,d\phi$ $d\mathbf{s}_\phi = \hat{\boldsymbol{\phi}}R\,dR\,d\theta$
Differential volume $dV =$	$dx\,dy\,dz$	$r\,dr\,d\phi\,dz$	$R^2\sin\theta\,dR\,d\theta\,d\phi$

Technology Brief 5: GPS

Figure TF5-1: iPhone map feature.

Figure TF5-2: GPS nominal satellite constellation. Four satellites in each plane, 20,200 km altitudes, 55° inclination.

How does a GPS receiver determine its location?

GPS: Three Segments

- Space segment: 24 satellites (now 32) on six orbital planes at 20200 km altitude, circling Earth every 12 hours, transmitting continuous coded signals at 1.57542 and 1.22760 GHz
- User segment: receivers, receiving and processing signals from multiple satellite signals, have many channels to monitor multiple satellites and distinguish them via special codes
- Control segment: distributed around the world, monitoring the satellites and updating their precise orbital information

GPS: Minimum of 4 Satellites Needed

Unknown: location of receiver (x_0, y_0, z_0)
Also unknown: time offset of receiver clock t_0

Quantities known with high precision: locations of satellites and their atomic clocks (satellites use expensive high precision clocks, whereas receivers do not)

Solving for 4 unknowns requires at least 4 equations (four satellites)

$$d_1^2 = (x_1 - x_0)^2 + (y_1 - y_0)^2 + (z_1 - z_0)^2 = c\,[(t_1 + t_0)]^2\,,$$
$$d_2^2 = (x_2 - x_0)^2 + (y_2 - y_0)^2 + (z_2 - z_0)^2 = c\,[(t_2 + t_0)]^2\,,$$
$$d_3^2 = (x_3 - x_0)^2 + (y_3 - y_0)^2 + (z_3 - z_0)^2 = c\,[(t_3 + t_0)]^2\,,$$
$$d_4^2 = (x_4 - x_0)^2 + (y_4 - y_0)^2 + (z_4 - z_0)^2 = c\,[(t_4 + t_0)]^2\,.$$

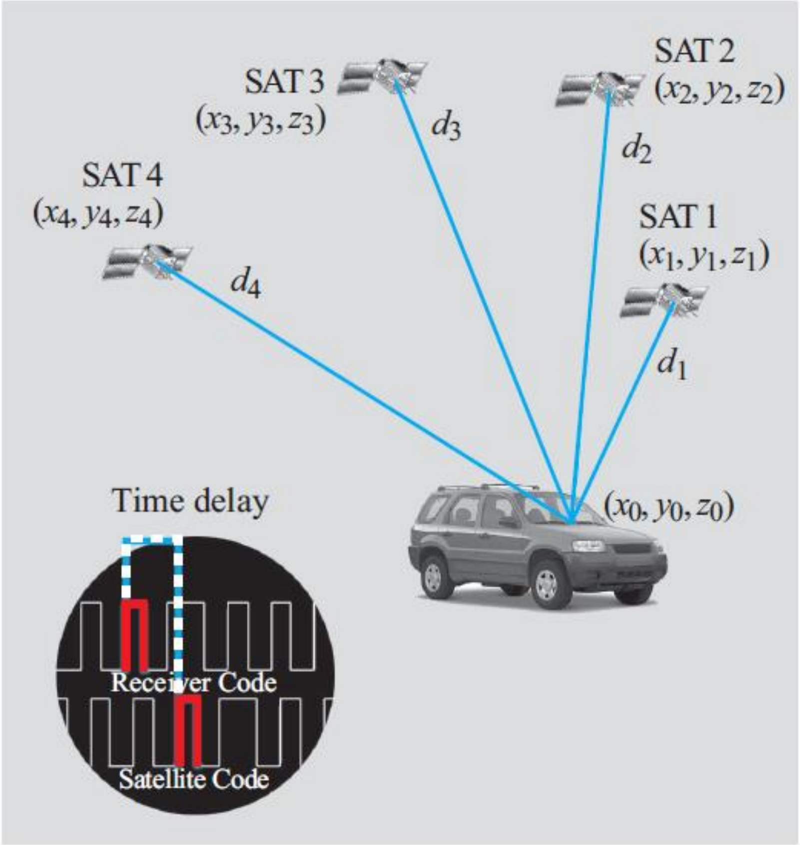
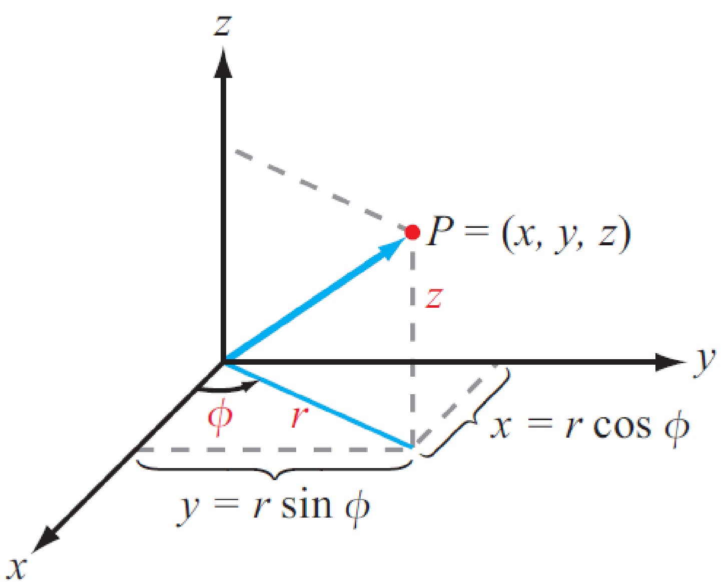


Figure TF5-3: Automobile GPS receiver at location (x_0, y_0, z_0).

Coordinate Transformations: Coordinates

- To solve a problem, we select the coordinate system that best fits its geometry
- Sometimes we need to transform between coordinate systems



$$r = \sqrt[+]{x^2 + y^2}, \qquad \phi = \tan^{-1}\left(\frac{y}{x}\right),$$

and the inverse relations are

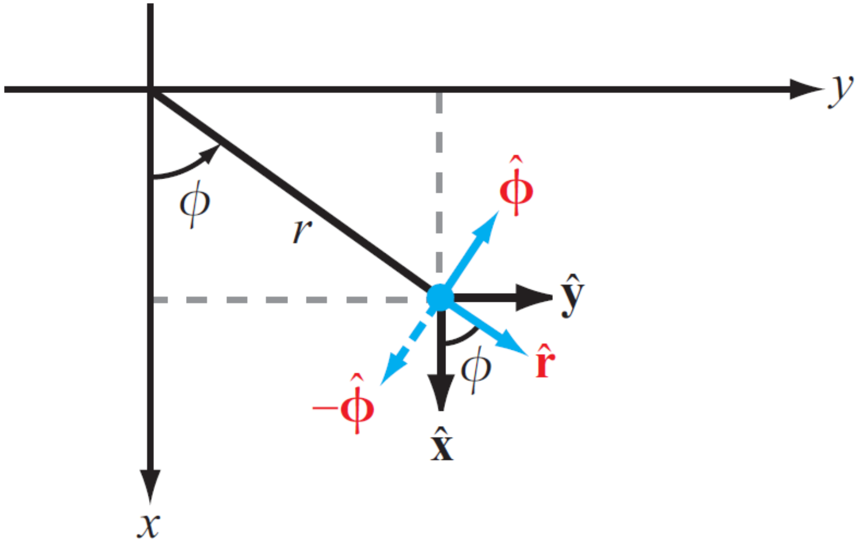
$$x = r\cos\phi, \qquad y = r\sin\phi.$$

Figure 3-16: Interrelationships between Cartesian coordinates (x, y, z) and cylindrical coordinates (r, ϕ, z).

Coordinate Transformations: Unit Vectors

$$\hat{\mathbf{r}} \cdot \hat{\mathbf{x}} = \cos\phi, \qquad \hat{\mathbf{r}} \cdot \hat{\mathbf{y}} = \sin\phi,$$
$$\hat{\boldsymbol{\phi}} \cdot \hat{\mathbf{x}} = -\sin\phi, \qquad \hat{\boldsymbol{\phi}} \cdot \hat{\mathbf{y}} = \cos\phi.$$

$$\hat{\mathbf{r}} = \hat{\mathbf{x}}\cos\phi + \hat{\mathbf{y}}\sin\phi.$$
$$\hat{\boldsymbol{\phi}} = -\hat{\mathbf{x}}\sin\phi + \hat{\mathbf{y}}\cos\phi.$$



$$\hat{\mathbf{x}} = \hat{\mathbf{r}}\cos\phi - \hat{\boldsymbol{\phi}}\sin\phi,$$
$$\hat{\mathbf{y}} = \hat{\mathbf{r}}\sin\phi + \hat{\boldsymbol{\phi}}\cos\phi.$$

Table 3-2: Coordinate transformation relations.

Transformation	Coordinate Variables	Unit Vectors	Vector Components
Cartesian to cylindrical	$r = \sqrt[+]{x^2 + y^2}$ $\phi = \tan^{-1}(y/x)$ $z = z$	$\hat{\mathbf{r}} = \hat{\mathbf{x}} \cos\phi + \hat{\mathbf{y}} \sin\phi$ $\hat{\boldsymbol{\phi}} = -\hat{\mathbf{x}} \sin\phi + \hat{\mathbf{y}} \cos\phi$ $\hat{\mathbf{z}} = \hat{\mathbf{z}}$	$A_r = A_x \cos\phi + A_y \sin\phi$ $A_\phi = -A_x \sin\phi + A_y \cos\phi$ $A_z = A_z$
Cylindrical to Cartesian	$x = r\cos\phi$ $y = r\sin\phi$ $z = z$	$\hat{\mathbf{x}} = \hat{\mathbf{r}} \cos\phi - \hat{\boldsymbol{\phi}} \sin\phi$ $\hat{\mathbf{y}} = \hat{\mathbf{r}} \sin\phi + \hat{\boldsymbol{\phi}} \cos\phi$ $\hat{\mathbf{z}} = \hat{\mathbf{z}}$	$A_x = A_r \cos\phi - A_\phi \sin\phi$ $A_y = A_r \sin\phi + A_\phi \cos\phi$ $A_z = A_z$
Cartesian to spherical	$R = \sqrt[+]{x^2 + y^2 + z^2}$ $\theta = \tan^{-1}[\sqrt[+]{x^2 + y^2}/z]$ $\phi = \tan^{-1}(y/x)$	$\hat{\mathbf{R}} = \hat{\mathbf{x}} \sin\theta\cos\phi + \hat{\mathbf{y}} \sin\theta\sin\phi + \hat{\mathbf{z}} \cos\theta$ $\hat{\boldsymbol{\theta}} = \hat{\mathbf{x}} \cos\theta\cos\phi + \hat{\mathbf{y}} \cos\theta\sin\phi - \hat{\mathbf{z}} \sin\theta$ $\hat{\boldsymbol{\phi}} = -\hat{\mathbf{x}} \sin\phi + \hat{\mathbf{y}} \cos\phi$	$A_R = A_x \sin\theta\cos\phi + A_y \sin\theta\sin\phi + A_z \cos\theta$ $A_\theta = A_x \cos\theta\cos\phi + A_y \cos\theta\sin\phi - A_z \sin\theta$ $A_\phi = -A_x \sin\phi + A_y \cos\phi$
Spherical to Cartesian	$x = R\sin\theta\cos\phi$ $y = R\sin\theta\sin\phi$ $z = R\cos\theta$	$\hat{\mathbf{x}} = \hat{\mathbf{R}} \sin\theta\cos\phi + \hat{\boldsymbol{\theta}} \cos\theta\cos\phi - \hat{\boldsymbol{\phi}} \sin\phi$ $\hat{\mathbf{y}} = \hat{\mathbf{R}} \sin\theta\sin\phi + \hat{\boldsymbol{\theta}} \cos\theta\sin\phi + \hat{\boldsymbol{\phi}} \cos\phi$ $\hat{\mathbf{z}} = \hat{\mathbf{R}} \cos\theta - \hat{\boldsymbol{\theta}} \sin\theta$	$A_x = A_R \sin\theta\cos\phi + A_\theta \cos\theta\cos\phi - A_\phi \sin\phi$ $A_y = A_R \sin\theta\sin\phi + A_\theta \cos\theta\sin\phi + A_\phi \cos\phi$ $A_z = A_R \cos\theta - A_\theta \sin\theta$
Cylindrical to spherical	$R = \sqrt[+]{r^2 + z^2}$ $\theta = \tan^{-1}(r/z)$ $\phi = \phi$	$\hat{\mathbf{R}} = \hat{\mathbf{r}} \sin\theta + \hat{\mathbf{z}} \cos\theta$ $\hat{\boldsymbol{\theta}} = \hat{\mathbf{r}} \cos\theta - \hat{\mathbf{z}} \sin\theta$ $\hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}}$	$A_R = A_r \sin\theta + A_z \cos\theta$ $A_\theta = A_r \cos\theta - A_z \sin\theta$ $A_\phi = A_\phi$
Spherical to cylindrical	$r = R\sin\theta$ $\phi = \phi$ $z = R\cos\theta$	$\hat{\mathbf{r}} = \hat{\mathbf{R}} \sin\theta + \hat{\boldsymbol{\theta}} \cos\theta$ $\hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}}$ $\hat{\mathbf{z}} = \hat{\mathbf{R}} \cos\theta - \hat{\boldsymbol{\theta}} \sin\theta$	$A_r = A_R \sin\theta + A_\theta \cos\theta$ $A_\phi = A_\phi$ $A_z = A_R \cos\theta - A_\theta \sin\theta$

Example 3-7: Cartesian to Cylindrical Transformations

Given point $P_1 = (3, -4, 3)$ and vector $\mathbf{A} = \hat{\mathbf{x}}2 - \hat{\mathbf{y}}3 + \hat{\mathbf{z}}4$, defined in Cartesian coordinates, express P_1 and $\mathbf{A}$ in cylindrical coordinates and evaluate $\mathbf{A}$ at P_1.

Solution: For point P_1, $x = 3$, $y = -4$, and $z = 3$. Using Eq. (3.51), we have

$$r = \sqrt[+]{x^2 + y^2} = 5, \quad \phi = \tan^{-1}\frac{y}{x} = -53.1^\circ = 306.9^\circ,$$

and z remains unchanged. Hence, $P_1 = (5, 306.9^\circ, 3)$ in cylindrical coordinates.

The cylindrical components of vector $\mathbf{A} = \hat{\mathbf{r}}A_r + \hat{\boldsymbol{\phi}}A_\phi + \hat{\mathbf{z}}A_z$ can be determined by applying Eqs. (3.58a) and (3.58b):

$$A_r = A_x \cos\phi + A_y \sin\phi = 2\cos\phi - 3\sin\phi,$$

$$A_\phi = -A_x \sin\phi + A_y \cos\phi = -2\sin\phi - 3\cos\phi,$$

$$A_z = 4.$$

Hence,

$$\mathbf{A} = \hat{\mathbf{r}}(2\cos\phi - 3\sin\phi) - \hat{\boldsymbol{\phi}}(2\sin\phi + 3\cos\phi) + \hat{\mathbf{z}}4.$$

At point P, $\phi = 306.9^\circ$, which gives

$$\mathbf{A} = \hat{\mathbf{r}}3.60 - \hat{\boldsymbol{\phi}}0.20 + \hat{\mathbf{z}}4.$$

Example 3-8: Cartesian to Spherical Transformation

Express vector $\mathbf{A} = \hat{\mathbf{x}}(x + y) + \hat{\mathbf{y}}(y - x) + \hat{\mathbf{z}}z$ in spherical coordinates.

Solution: Using the transformation relation for A_R given in Table 3-2, we have

$$\begin{aligned}A_R &= A_x \sin\theta\cos\phi + A_y \sin\theta\sin\phi + A_z\cos\theta \\ &= (x + y)\sin\theta\cos\phi + (y - x)\sin\theta\sin\phi + z\cos\theta.\end{aligned}$$

Using the expressions for x, y, and z given by Eq. (3.61c), we have

$$\begin{aligned}A_R &= (R\sin\theta\cos\phi + R\sin\theta\sin\phi)\sin\theta\cos\phi \\ &\quad + (R\sin\theta\sin\phi - R\sin\theta\cos\phi)\sin\theta\sin\phi + R\cos^2\theta \\ &= R\sin^2\theta\,(\cos^2\phi + \sin^2\phi) + R\cos^2\theta \\ &= R\sin^2\theta + R\cos^2\theta = R.\end{aligned}$$

Similarly,

$$\begin{aligned}A_\theta &= (x + y)\cos\theta\cos\phi + (y - x)\cos\theta\sin\phi - z\sin\theta, \\ A_\phi &= -(x + y)\sin\phi + (y - x)\cos\phi,\end{aligned}$$

and following the procedure used with A_R, we obtain

$$\begin{aligned}A_\theta &= 0, \\ A_\phi &= -R\sin\theta.\end{aligned}$$

Hence,

$$\mathbf{A} = \hat{\mathbf{R}}A_R + \hat{\boldsymbol{\theta}}A_\theta + \hat{\boldsymbol{\phi}}A_\phi = \hat{\mathbf{R}}R - \hat{\boldsymbol{\phi}}R\sin\theta.$$

Using the relations:

$$\begin{aligned}x &= R\sin\theta\cos\phi, \\ y &= R\sin\theta\sin\phi, \\ z &= R\cos\theta.\end{aligned}$$

Leads to:

$$A_\theta = 0$$

$$A_\phi = -R\sin\theta$$

$$\mathbf{A} = \hat{\mathbf{R}}A_R + \hat{\boldsymbol{\theta}}A_\theta + \hat{\boldsymbol{\phi}}A_\phi = \hat{\mathbf{R}}R - \hat{\boldsymbol{\phi}}R\sin\theta$$

Distance Between 2 Points

$$d = |\mathbf{R}_{12}|$$
$$= [(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]^{1/2}. \quad (3.66)$$

$$d = \Big[(r_2 \cos\phi_2 - r_1 \cos\phi_1)^2$$
$$+ (r_2 \sin\phi_2 - r_1 \sin\phi_1)^2 + (z_2 - z_1)^2\Big]^{1/2}$$
$$= \Big[r_2^2 + r_1^2 - 2r_1 r_2 \cos(\phi_2 - \phi_1) + (z_2 - z_1)^2\Big]^{1/2}$$
$$\text{(cylindrical).} \quad (3.67)$$

$$d = \Big\{R_2^2 + R_1^2 - 2R_1 R_2[\cos\theta_2 \cos\theta_1$$
$$+ \sin\theta_1 \sin\theta_2 \cos(\phi_2 - \phi_1)]\Big\}^{1/2}$$
$$\text{(spherical).} \quad (3.68)$$

Gradient of A Scalar Field

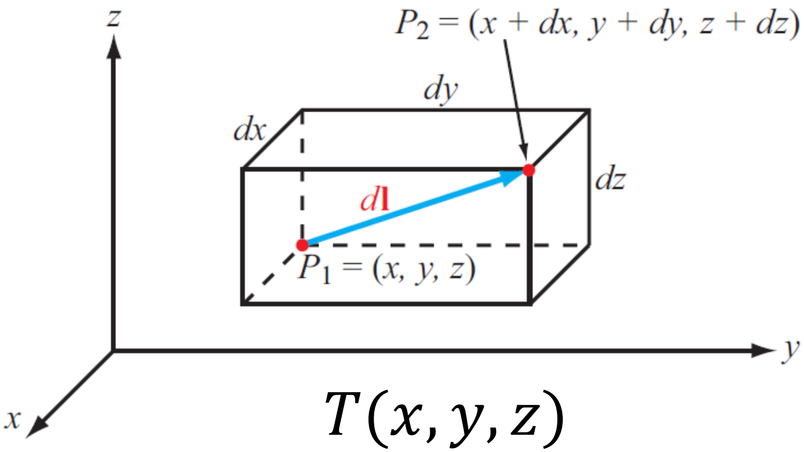


Figure 3-19: Differential distance vector $d\mathbf{l}$ between points P_1 and P_2.

From differential calculus, the temperature difference between points P_1 and P_2, $dT = T_2 - T_1$, is

$$dT = \frac{\partial T}{\partial x}\,dx + \frac{\partial T}{\partial y}\,dy + \frac{\partial T}{\partial z}\,dz. \qquad (3.70)$$

Because $dx = \hat{\mathbf{x}} \cdot d\mathbf{l}$, $dy = \hat{\mathbf{y}} \cdot d\mathbf{l}$, and $dz = \hat{\mathbf{z}} \cdot d\mathbf{l}$, Eq. (3.70) can be rewritten as

$$\begin{aligned} dT &= \hat{\mathbf{x}}\frac{\partial T}{\partial x} \cdot d\mathbf{l} + \hat{\mathbf{y}}\frac{\partial T}{\partial y} \cdot d\mathbf{l} + \hat{\mathbf{z}}\frac{\partial T}{\partial z} \cdot d\mathbf{l} \\ &= \left[\hat{\mathbf{x}}\frac{\partial T}{\partial x} + \hat{\mathbf{y}}\frac{\partial T}{\partial y} + \hat{\mathbf{z}}\frac{\partial T}{\partial z}\right] \cdot d\mathbf{l}. \end{aligned} \qquad (3.71)$$

Define

$$\boxed{\nabla T = \text{grad } T = \hat{\mathbf{x}}\frac{\partial T}{\partial x} + \hat{\mathbf{y}}\frac{\partial T}{\partial y} + \hat{\mathbf{z}}\frac{\partial T}{\partial z}\,. \qquad (3.72)}$$

Equation (3.71) can then be expressed as

$$dT = \nabla T \cdot d\mathbf{l}. \qquad (3.73)$$

The symbol ∇ is called the ***del*** or ***gradient operator*** and is defined as

$$\boxed{\nabla = \hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z} \qquad \text{(Cartesian).} \qquad (3.74)}$$

Also known as: nabla operator

Gradient (cont.)

With $d\mathbf{l} = \hat{\mathbf{a}}_l\, dl$, where $\hat{\mathbf{a}}_l$ is the unit vector of $d\mathbf{l}$, the ***directional derivative*** of T along $\hat{\mathbf{a}}_l$ is

$$\boxed{\frac{dT}{dl} = \nabla T \cdot \hat{\mathbf{a}}_l. \qquad (3.75)}$$

$$\frac{dT}{dl} = \nabla T \cdot \hat{a}_l = |\nabla T| \cos\theta$$

The directional derivative dT/dl is the projection of the gradient on $\hat{a}_l$. So the directional derivative gets its maximum value when the direction $\hat{a}_l$ is the same as the direction of the gradient. Put this in another way, the magnitude of the gradient is the maximum value of the directional derivative and the function varies fastest on its direction

Example 3-9: Directional Derivative

Find the directional derivative of $T = x^2 + y^2 z$ along direction $\hat{\mathbf{x}}2 + \hat{\mathbf{y}}3 - \hat{\mathbf{z}}2$ and evaluate it at $(1, -1, 2)$.

Solution: First, we find the gradient of T:

$$\nabla T = \left(\hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}\right)(x^2 + y^2 z)$$
$$= \hat{\mathbf{x}}2x + \hat{\mathbf{y}}2yz + \hat{\mathbf{z}}y^2.$$

We denote $\mathbf{l}$ as the given direction,

$$\mathbf{l} = \hat{\mathbf{x}}2 + \hat{\mathbf{y}}3 - \hat{\mathbf{z}}2.$$

Its unit vector is

$$\hat{\mathbf{a}}_l = \frac{\mathbf{l}}{|\mathbf{l}|} = \frac{\hat{\mathbf{x}}2 + \hat{\mathbf{y}}3 - \hat{\mathbf{z}}2}{\sqrt{2^2 + 3^2 + 2^2}} = \frac{\hat{\mathbf{x}}2 + \hat{\mathbf{y}}3 - \hat{\mathbf{z}}2}{\sqrt{17}}.$$

Application of Eq. (3.75) gives

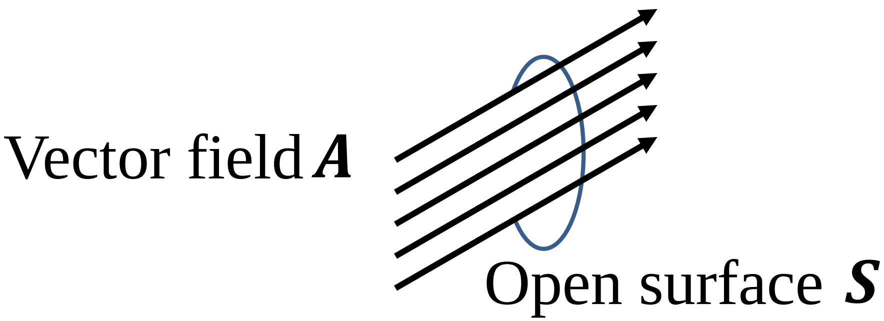
$$\frac{dT}{dl} = \nabla T \cdot \hat{\mathbf{a}}_l = (\hat{\mathbf{x}}2x + \hat{\mathbf{y}}2yz + \hat{\mathbf{z}}y^2) \cdot \left(\frac{\hat{\mathbf{x}}2 + \hat{\mathbf{y}}3 - \hat{\mathbf{z}}2}{\sqrt{17}}\right)$$
$$= \frac{4x + 6yz - 2y^2}{\sqrt{17}}.$$

At $(1, -1, 2)$,

$$\left.\frac{dT}{dl}\right|_{(1,-1,2)} = \frac{4 - 12 - 2}{\sqrt{17}} = \frac{-10}{\sqrt{17}}.$$

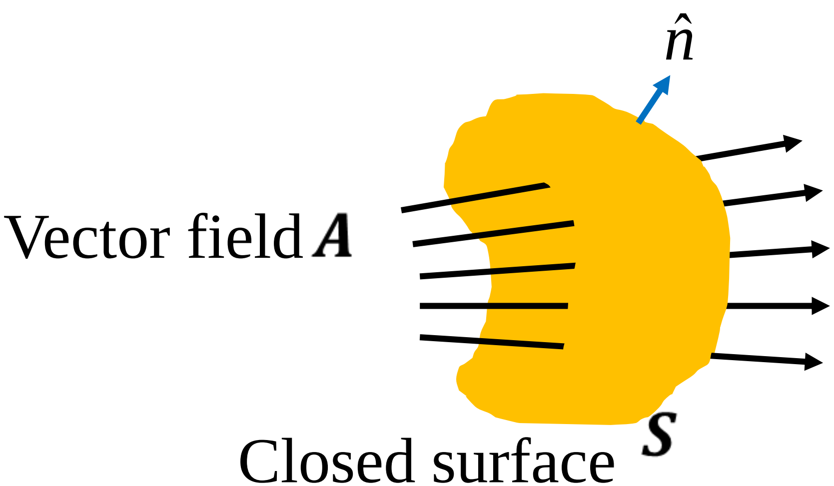
Flux

- Define flux for a vector field



Total flux flowing through the surface S

$$\Phi = \int_S \boldsymbol{A} \cdot d\boldsymbol{S}$$



$$\Phi = \oiint_S \boldsymbol{A} \cdot d\boldsymbol{S}$$

Divergence of a Vector Field

$$\text{div}\boldsymbol{E} = \nabla \cdot \boldsymbol{E} = \lim_{\Delta V \to 0} \frac{\oiint_S \boldsymbol{E} \cdot d\boldsymbol{S}}{\Delta V}$$

$$\nabla \cdot \mathbf{E} = \text{div } \mathbf{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \qquad (3.96)$$

$$F_1 = \int_{\text{Face 1}} \mathbf{E} \cdot \hat{\mathbf{n}}_1 \, ds$$

Flux on Face 1

$$= \int_{\text{Face 1}} (\hat{\mathbf{x}} E_x + \hat{\mathbf{y}} E_y + \hat{\mathbf{z}} E_z) \cdot (-\hat{\mathbf{x}}) \, dy \, dz$$

$$\approx -E_x(1) \, \Delta y \, \Delta z,$$

$$F_2 = E_x(2) \, \Delta y \, \Delta z$$

Flux on Face 2

$$E_x(2) = E_x(1) + \frac{\partial E_x}{\partial x} \Delta x \qquad F_1 + F_2 = \frac{\partial E_x}{\partial x} \Delta x \, \Delta y \, \Delta z$$

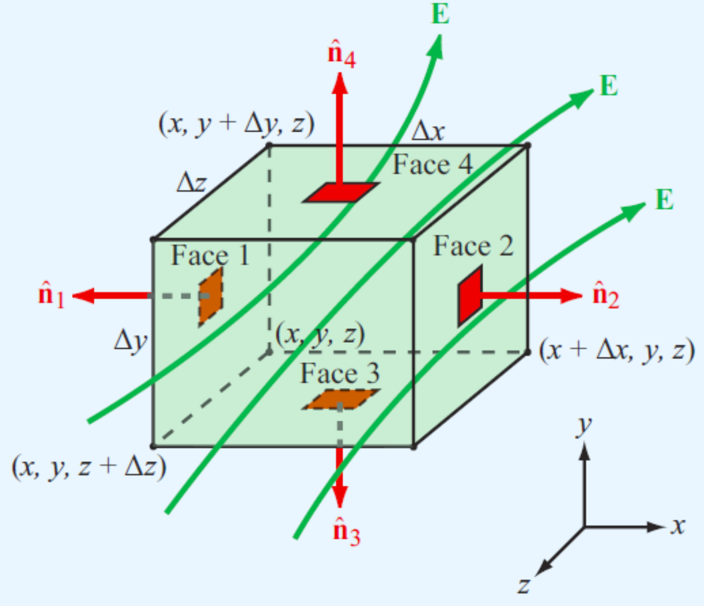


Figure 3-21 Flux lines of a vector field **E** passing through a differential rectangular parallelepiped of volume $\Delta v = \Delta x \, \Delta y \, \Delta z$.

Derivation of Divergence

$$F_3 + F_4 = \frac{\partial E_y}{\partial y}\,\Delta x\,\Delta y\,\Delta z,$$

$$F_5 + F_6 = \frac{\partial E_z}{\partial z}\,\Delta x\,\Delta y\,\Delta z.$$

$$\oint_S \mathbf{E}\cdot d\mathbf{s} = \left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}\right)\Delta x\,\Delta y\,\Delta z$$

$$= (\text{div}\,\mathbf{E})\,\Delta \mathcal{V},$$

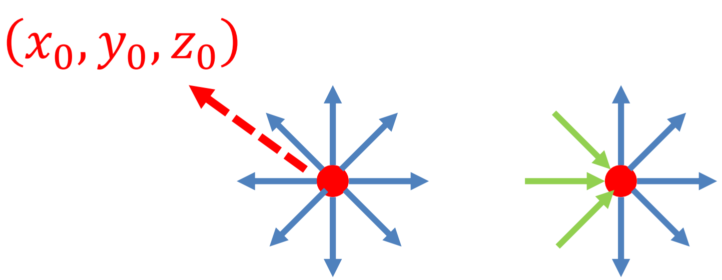
$$\Delta \mathcal{V} = \Delta x\,\Delta y\,\Delta z$$

$$\text{div}\,\mathbf{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}$$

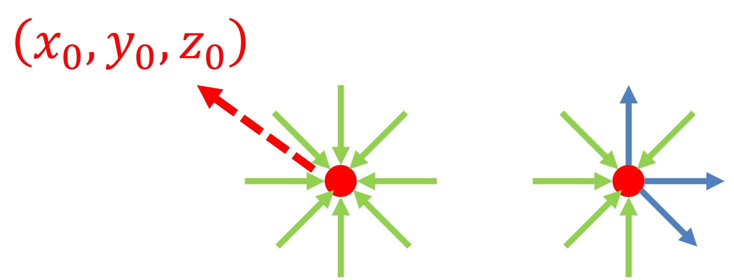
Physical Meaning of Divergence

From the definition of the divergence of **E** *given by Eq. (3.95), field* **E** *has positive divergence if the net flux out of surface S is positive, which may be "viewed" as if volume ΔV contains a* ***source*** *of field lines. If the divergence is negative, ΔV may be viewed as containing a* ***sink*** *of field lines because the net flux is into ΔV. For a uniform field* **E***, the same amount of flux enters ΔV as leaves it; hence, its divergence is zero and the field is said to be* ***divergenceless****.*

$$\nabla \cdot \boldsymbol{A}(x_0, y_0, z_0) > 0$$



$$\nabla \cdot \boldsymbol{A}(x_0, y_0, z_0) < 0$$



Divergence Theorem

$$\int_V \nabla \cdot \mathbf{E}\, dV = \oint_S \mathbf{E} \cdot d\mathbf{s} \qquad \text{(divergence theorem).}$$

Obtained from the definition of divergence (3.98)

Useful tool for converting integration over a volume to one over the surface enclosing that volume, and vice versa

Example 3-11: Calculating the Divergence

Determine the divergence of each of the following vector fields and then evaluate them at the indicated points:

(a) $\mathbf{E} = \hat{\mathbf{x}}3x^2 + \hat{\mathbf{y}}2z + \hat{\mathbf{z}}x^2 z$ at $(2, -2, 0)$;

(b) $\mathbf{E} = \hat{\mathbf{R}}(a^3 \cos\theta / R^2) - \hat{\boldsymbol{\theta}}(a^3 \sin\theta / R^2)$ at $(a/2, 0, \pi)$.

Solution:

(a)
$$\begin{aligned}\nabla \cdot \mathbf{E} &= \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \\ &= \frac{\partial}{\partial x}(3x^2) + \frac{\partial}{\partial y}(2z) + \frac{\partial}{\partial z}(x^2 z) \\ &= 6x + 0 + x^2 \\ &= x^2 + 6x.\end{aligned}$$

At $(2, -2, 0)$, $\nabla \cdot \mathbf{E}\Big|_{(2,-2,0)} = 16$.

(b) From the expression given on the inside of the back cover of the book for the divergence of a vector in spherical coordinates, it follows that

$$\begin{aligned}\nabla \cdot \mathbf{E} &= \frac{1}{R^2}\frac{\partial}{\partial R}(R^2 E_R) + \frac{1}{R\sin\theta}\frac{\partial}{\partial\theta}(E_\theta \sin\theta) \\ &\quad + \frac{1}{R\sin\theta}\frac{\partial E_\phi}{\partial\phi} \\ &= \frac{1}{R^2}\frac{\partial}{\partial R}(a^3\cos\theta) + \frac{1}{R\sin\theta}\frac{\partial}{\partial\theta}\left(-\frac{a^3\sin^2\theta}{R^2}\right) \\ &= 0 - \frac{2a^3\cos\theta}{R^3} \\ &= -\frac{2a^3\cos\theta}{R^3}.\end{aligned}$$

At $R = a/2$ and $\theta = 0$, $\nabla \cdot \mathbf{E}\Big|_{(a/2,0,\pi)} = -16$.

An Example

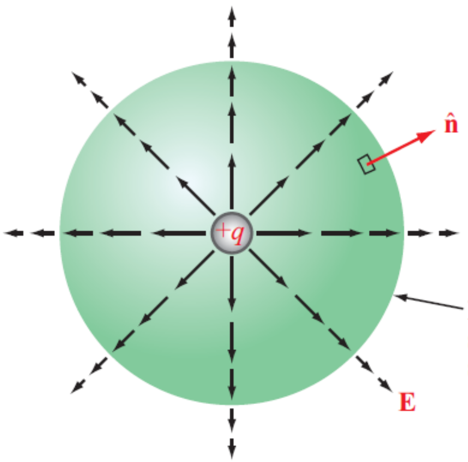


Figure 3-20: Flux lines of the electric field **E** due to a positive charge q.

$$\nabla \psi = \hat{\mathbf{a}}_r \frac{\partial \psi}{\partial r} + \hat{\mathbf{a}}_\theta \frac{1}{r} \frac{\partial \psi}{\partial r} + \hat{\mathbf{a}}_\phi \frac{1}{r \sin\theta} \frac{\partial \psi}{\partial \phi}$$

$$\nabla \cdot \mathbf{A} = \frac{1}{r^2} \frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r \sin\theta} \frac{\partial}{\partial \theta}(\sin\theta A_\theta) + \frac{1}{r \sin\theta} \frac{\partial A_\phi}{\partial \phi}$$

$$\nabla \times \mathbf{A} = \frac{\hat{\mathbf{a}}_r}{r \sin\theta} \left[\frac{\partial}{\partial \theta}(A_\phi \sin\theta) - \frac{\partial A_\theta}{\partial \phi} \right] + \frac{\hat{\mathbf{a}}_\theta}{r} \left[\frac{1}{\sin\theta} \frac{\partial A_r}{\partial \phi} - \frac{\partial}{\partial r}(r A_\phi) \right] + \frac{\hat{\mathbf{a}}_\phi}{r} \left[\frac{\partial}{\partial r}(r A_\theta) - \frac{\partial A_r}{\partial \theta} \right]$$

$$\mathbf{E} = \hat{\mathbf{R}} \frac{q}{4\pi\varepsilon R^2}$$

$$\nabla \cdot \boldsymbol{E} = \frac{q}{4\pi\varepsilon} \frac{1}{R^2} \frac{\partial}{\partial R}\left(R^2 \frac{1}{R^2} \right) = 0 \qquad (R \neq 0)$$

$$\int_V \nabla \cdot \mathbf{E} \, dV = \oint_S \mathbf{E} \cdot d\mathbf{s}$$

(divergence theorem)

$$(\nabla \cdot \boldsymbol{E}) V = |\boldsymbol{E}| 4\pi R^2 = \frac{q}{\varepsilon} \qquad \nabla \cdot \boldsymbol{E} = \frac{q}{\varepsilon V} = \frac{\rho_v}{\varepsilon}$$

$$\boxed{\nabla \cdot \mathbf{D} = \rho_{\mathrm{v}}} \qquad (R = 0)$$

Another Example

From the last example, one may consider that a vector field with only R component and only varying with R purely has nonzero divergence at R = 0. Is this true?

$$\mathbf{A} = \hat{R}\frac{1}{R}$$

$$\nabla \cdot \mathbf{A} = \frac{1}{R^2}\frac{\partial}{\partial R}\left(\frac{R^2}{R}\right) = \frac{1}{R^2} \neq 0$$

Curl of a Vector Field

$$\text{Circulation} = \oint_C \mathbf{B} \cdot d\mathbf{l}.$$

$$\nabla \times \mathbf{B} = \text{curl } \mathbf{B}$$

$$= \lim_{\Delta s \to 0} \frac{1}{\Delta s} \left[\hat{\mathbf{n}} \oint_C \mathbf{B} \cdot d\mathbf{l} \right]_{\max} . \quad (3.103)$$

Right-hand rule

Thus, curl **B** *is the circulation of* **B** *per unit area, with the area Δs of the contour C being oriented such that the circulation is maximum.*

$$\nabla \times \mathbf{B} = \hat{\mathbf{x}} \left(\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} \right) + \hat{\mathbf{y}} \left(\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x} \right) + \hat{\mathbf{z}} \left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} \right)$$

$$= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ B_x & B_y & B_z \end{vmatrix} .$$

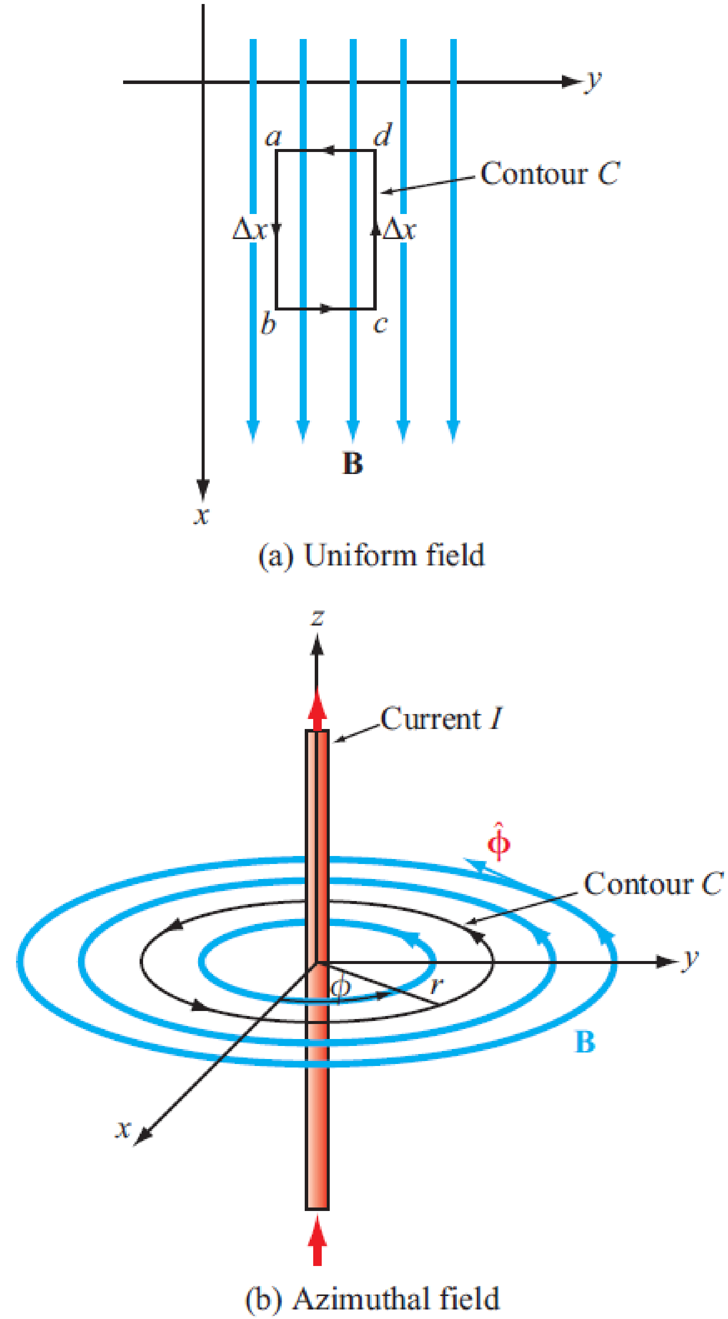
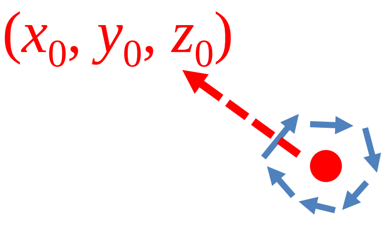


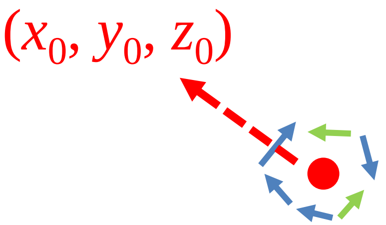
Figure 3-22: Circulation is zero for the uniform field in (a), but it is not zero for the azimuthal field in (b).

Physical Meaning of Curl

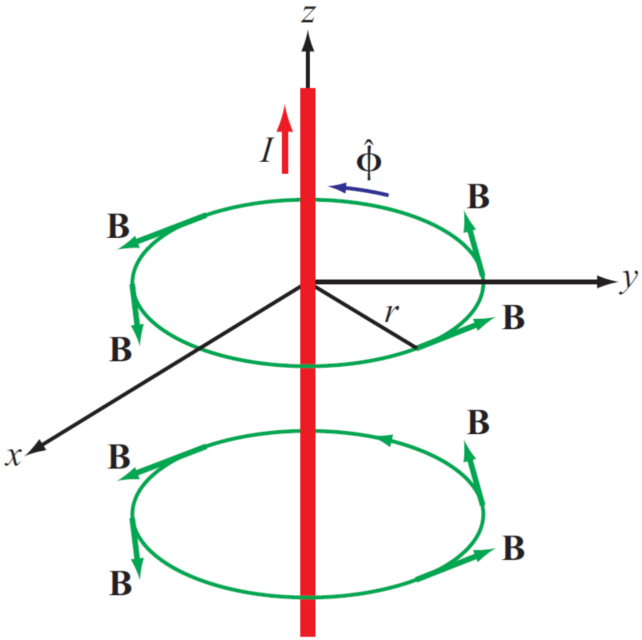
$$\nabla \times \boldsymbol{A}(x_0, y_0, z_0) \neq 0$$



If curl of a vector field $\boldsymbol{A}$ at a point is not zero, this means the field $\boldsymbol{A}$ has an overall rotational feature at this point. The direction of the curl is defined by the rotation direction



An Example



$$\nabla\psi = \hat{\mathbf{a}}_\rho \frac{\partial \psi}{\partial \rho} + \hat{\mathbf{a}}_\phi \frac{1}{\rho}\frac{\partial \psi}{\partial \phi} + \hat{\mathbf{a}}_z \frac{\partial \psi}{\partial z}$$

$$\nabla \cdot \mathbf{A} = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho A_\rho\right) + \frac{1}{\rho}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times \mathbf{A} = \hat{\mathbf{a}}_\rho \left(\frac{1}{\rho}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right) + \hat{\mathbf{a}}_\phi \left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho}\right) + \hat{\mathbf{a}}_z \left(\frac{1}{\rho}\frac{\partial \left(\rho A_\phi\right)}{\partial \rho} - \frac{1}{\rho}\frac{\partial A_\rho}{\partial \phi}\right)$$

$$\mathbf{B} = \hat{\boldsymbol{\phi}}\, \frac{\mu_0 I}{2\pi r}$$

$$\begin{cases} \nabla \times \mathbf{B} = \hat{z}\frac{\mu_0 I}{2\pi}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{1}{r}\right) = 0 & (r \neq 0) \\ \nabla \times \mathbf{B} = \frac{1}{\Delta s}\hat{z}|\mathbf{B}|2\pi r = \hat{z}\frac{1}{\Delta s}\mu_0 I = \hat{z}\mu_0 J & (r = 0) \end{cases}$$

$$\nabla \times \mathbf{H} = \mathbf{J}$$

Another Example

From the last example, one may consider that the location where a magnetic field can get nonzero curl is at the place with a current source. And if no source at a location, then the curl here must be 0. Is this true?

$$\mathbf{B} = \hat{\phi}\frac{1}{R^2}$$

$$\nabla \times \mathbf{B} = \frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{1}{R^2}\right) = -\frac{1}{R^3} \neq 0$$

Stokes's Theorem

Stokes's theorem *converts the surface integral of the curl of a vector over an open surface S into a line integral of the vector along the contour C bounding the surface S.*

For the geometry shown in Fig. 3-23, ***Stokes's theorem*** states

$$\int_S (\nabla \times \mathbf{B}) \cdot d\mathbf{s} = \oint_C \mathbf{B} \cdot d\mathbf{l} \qquad \text{(Stokes's theorem)}, \tag{3.107}$$

Obtained from the definition of curl

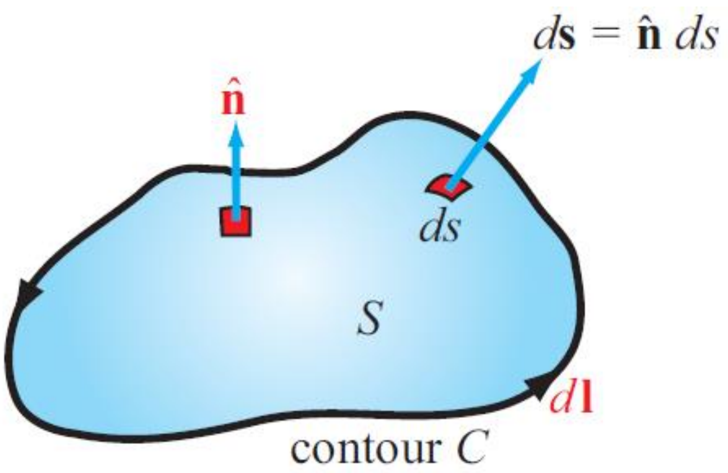


Figure 3-23: The direction of the unit vector $\hat{\mathbf{n}}$ is along the thumb when the other four fingers of the right hand follow $d\mathbf{l}$.

Laplacian Operator

Laplacian of a Scalar Field

$$\nabla^2 V = \nabla \cdot (\nabla V) = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} . \qquad (3.110)$$

Laplacian of a Vector Field

$$\begin{aligned} \nabla^2 \mathbf{E} &= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \mathbf{E} \\ &= \hat{\mathbf{x}}\, \nabla^2 E_x + \hat{\mathbf{y}}\, \nabla^2 E_y + \hat{\mathbf{z}}\, \nabla^2 E_z \end{aligned}$$

Useful Relation

$$\nabla^2 \mathbf{E} = \nabla(\nabla \cdot \mathbf{E}) - \nabla \times (\nabla \times \mathbf{E}). \qquad (3.113)$$

Units

- How does the unit change after performing the operation of gradient, divergence and curl?

$$\nabla = \hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}$$

$$\nabla \cdot \mathbf{E} = \operatorname{div}\mathbf{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}$$

$$\nabla \times \mathbf{B} = \hat{\mathbf{x}}\left(\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z}\right) + \hat{\mathbf{y}}\left(\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x}\right) + \hat{\mathbf{z}}\left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}\right)$$

Faraday's Law

$$\boxed{\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t},}$$

$$\frac{\mathrm{V}}{\mathrm{m}}\frac{1}{\mathrm{m}} = \frac{\mathrm{H}}{\mathrm{m}}\frac{\mathrm{A}}{\mathrm{m}}\frac{1}{\mathrm{s}} \Rightarrow \mathrm{V} = \frac{\mathrm{HA}}{\mathrm{s}} \Rightarrow \Omega = \frac{\mathrm{H}}{\mathrm{s}}$$

Chapter 3 Relationships

Distance Between Two Points

$$d = [(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]^{1/2}$$

$$d = \left[r_2^2 + r_1^2 - 2r_1 r_2 \cos(\phi_2 - \phi_1) + (z_2 - z_1)^2\right]^{1/2}$$

$$d = \left\{R_2^2 + R_1^2 - 2R_1 R_2[\cos\theta_2 \cos\theta_1 + \sin\theta_1 \sin\theta_2 \cos(\phi_2 - \phi_1)]\right\}^{1/2}$$

Coordinate Systems

Table 3-1

Coordinate Transformations

Table 3-2

Vector Products

$$\mathbf{A} \cdot \mathbf{B} = AB \cos\theta_{AB}$$

$$\mathbf{A} \times \mathbf{B} = \hat{\mathbf{n}}\, AB \sin\theta_{AB}$$

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})$$

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$$

Divergence Theorem

$$\int_V \nabla \cdot \mathbf{E}\, dV = \oint_S \mathbf{E} \cdot d\mathbf{s}$$

Vector Operators

$$\nabla T = \hat{\mathbf{x}}\frac{\partial T}{\partial x} + \hat{\mathbf{y}}\frac{\partial T}{\partial y} + \hat{\mathbf{z}}\frac{\partial T}{\partial z}$$

$$\nabla \cdot \mathbf{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}$$

$$\nabla \times \mathbf{B} = \hat{\mathbf{x}}\left(\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z}\right) + \hat{\mathbf{y}}\left(\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x}\right) + \hat{\mathbf{z}}\left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}\right)$$

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

(see back cover for cylindrical and spherical coordinates)

Stokes's Theorem

$$\int_S (\nabla \times \mathbf{B}) \cdot d\mathbf{s} = \oint_C \mathbf{B} \cdot d\mathbf{l}$$