Numerical Optimization, 2021 Fall
Homework 4 Solution

1 Lagrange

Please use Lagrange to give the dual problems of the following

1. [15pts]

$$\begin{aligned} \min \quad & 2x_1 - x_2 \\ \text{s.t.} \quad & 2x_1 - x_2 - x_3 \geq 3 \\ & x_1 - x_2 + x_3 \geq 2 \\ & x_i \geq 0, \quad i = 1, 2, 3. \end{aligned} \tag{1}$$

The Lagrangian is

$$\begin{aligned} \mathcal{L}(\boldsymbol{x}, \boldsymbol{\lambda}, \boldsymbol{\mu}) &= 2x_1 - x_2 + \lambda_1(3 - 2x_1 + x_2 + x_3) + \lambda_2(2 - x_1 + x_2 - x_3) - \mu_1 x_1 - \mu_2 x_2 - \mu_3 x_3 \\ &= (-2\lambda_1 - \lambda_2 - \mu_1 + 2)x_1 + (\lambda_1 + \lambda_2 - \mu_2 - 1)x_2 + (\lambda_1 - \lambda_2 - \mu_3)x_3 + (3\lambda_1 + 2\lambda_2) \end{aligned} \tag{2}$$

where $\boldsymbol{\lambda} = (\lambda_1, \lambda_2)^T \geq \mathbf{0}$ and $\boldsymbol{\mu} = (\mu_1, \mu_2, \mu_3)^T \geq \mathbf{0}$. The dual objective is

$$g(\boldsymbol{\lambda}, \boldsymbol{\mu}) = \min_{\boldsymbol{x}} \mathcal{L}(\boldsymbol{x}, \boldsymbol{\lambda}, \boldsymbol{\mu}) \tag{3}$$

Since we only have interests in the case that $g(\boldsymbol{\lambda}, \boldsymbol{\mu}) > -\infty$, each coefficient in front of primal variable x_i should be set as 0. Hence, the dual problem is

$$\begin{aligned} \max \quad & 3\lambda_1 + 2\lambda_2 \\ \text{s.t.} \quad & -2\lambda_1 - \lambda_2 - \mu_1 + 2 = 0 \\ & \lambda_1 + \lambda_2 - \mu_2 - 1 = 0 \\ & \lambda_1 - \lambda_2 - \mu_3 = 0 \\ & \lambda_i \geq 0, \ i = 1, 2 \\ & \mu_j \geq 0, \ j = 1, 2, 3 \end{aligned} \tag{4}$$

We can remove the redundant μ_j's. Therefore, the final form is

$$\begin{aligned} \max \quad & 3\lambda_1 + 2\lambda_2 \\ \text{s.t.} \quad & 2\lambda_1 + \lambda_2 \leq 2 \\ & \lambda_1 + \lambda_2 \geq 1 \\ & \lambda_1 - \lambda_2 \geq 0 \\ & \lambda_i \geq 0, \ i = 1, 2. \end{aligned} \tag{5}$$

2. [15pts]

$$
\begin{aligned}
\min \quad & 0 \cdot x_1 + 0 \cdot x_2 \\
\text{s.t.} \quad & -2x_1 + 2x_2 \le -1 \\
& 2x_1 - x_2 \le 2 \\
& -4x_2 \le 3 \\
& -15x_1 - 12x_2 \le -2 \\
& 12x_1 + 20x_2 \le -1.
\end{aligned} \tag{6}
$$

The Lagrangian is

$$
\begin{aligned}
\mathcal{L}(\boldsymbol{x}, \boldsymbol{\lambda}) &= \lambda_1(1 - 2x_1 + 2x_2) + \lambda_2(2x_1 - x_2 - 2) + \lambda_3(-4x_2 - 3) + \lambda_4(2 - 15x_1 - 12x_2) \\
&\quad + \lambda_5(1 + 12x_1 + 20x_2) \\
&= (-2\lambda_1 + 2\lambda_2 - 15\lambda_4 + 12\lambda_5)x_1 + (2\lambda_1 - \lambda_2 - 4\lambda_3 - 12\lambda_4 + 20\lambda_5)x_2 \\
&\quad + (\lambda_1 - 2\lambda_2 - 3\lambda_3 + 2\lambda_4 + \lambda_5)
\end{aligned} \tag{7}
$$

where $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5)^T \ge \mathbf{0}$. The dual objective is

$$
g(\boldsymbol{\lambda}) = \min_{\boldsymbol{x}} \mathcal{L}(\boldsymbol{x}, \boldsymbol{\lambda}) \tag{8}
$$

Since we only have interests in the case that $g(\boldsymbol{\lambda}) > -\infty$, each coefficient in front of primal variable x_i should be set as 0. Hence, the dual problem is

$$
\begin{aligned}
\max \quad & \lambda_1 - 2\lambda_2 - 3\lambda_3 + 2\lambda_4 + \lambda_5 \\
\text{s.t.} \quad & -2\lambda_1 + 2\lambda_2 - 15\lambda_4 + 12\lambda_5 = 0 \\
& 2\lambda_1 - \lambda_2 - 4\lambda_3 - 12\lambda_4 + 20\lambda_5 = 0 \\
& \lambda_i \ge 0, \ i = 1, 2, \dots, 5.
\end{aligned} \tag{9}
$$

2 Primal-Dual Feasibility

From the lecture we know that the primal and dual of an LP problem may be both infeasible, please write a specific example of this situation and then briefly explain why are both problems infeasible. [20pts]
The following LP problem has no feasible solution

$$
\begin{aligned}
\min \quad & x_1 - 2x_2 \\
\text{s.t.} \quad & x_1 - x_2 \ge 2 \\
& -x_1 + x_2 \ge -1 \\
& x_1, x_2 \ge 0
\end{aligned} \tag{10}
$$

and neither does its dual

$$
\begin{aligned}
\min \quad & 2\lambda_1 - \lambda_2 \\
\text{s.t.} \quad & \lambda_1 - \lambda_2 \le 1 \\
& -\lambda_1 + \lambda_2 \le -2 \\
& \lambda_1, \lambda_2 \ge 0
\end{aligned} \tag{11}
$$

Reason: The two constraints of each problem can't be true at the same time.

3 Auxiliary Problem

Consider the auxiliary problem in the first phase of *Simplex Method*

$$\begin{aligned} \min_{\boldsymbol{x}\in\mathbb{R}^n,\boldsymbol{y}\in\mathbb{R}^m} \quad & \sum_{i=1}^{m} y_i \\ \text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x}+\boldsymbol{y}=\boldsymbol{b} \\ & \boldsymbol{x}\geq \mathbf{0} \\ & \boldsymbol{y}\geq \mathbf{0}. \end{aligned} \tag{12}$$

where $\boldsymbol{A}\in\mathbb{R}^{m\times n}$, $\boldsymbol{c}\in\mathbb{R}^n$ and $\boldsymbol{b}\geq\mathbf{0}\in\mathbb{R}^m$ are given.

1. Please write the dual of problem (12) and show how you get that answer. [15pts]

   We use the Lagrange method to write out the dual problem. First of all, the Lagrangian is defined as

$$\begin{aligned} \mathcal{L}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}) &= \boldsymbol{e}^T\boldsymbol{y}+\boldsymbol{\alpha}^T(\boldsymbol{b}-\boldsymbol{A}\boldsymbol{x}-\boldsymbol{y})-\boldsymbol{\beta}^T\boldsymbol{x}-\boldsymbol{\gamma}^T\boldsymbol{y} \\ &= (\boldsymbol{e}-\boldsymbol{\alpha}-\boldsymbol{\gamma})^T\boldsymbol{y}-(\boldsymbol{A}^T\boldsymbol{\alpha}+\boldsymbol{\beta})^T\boldsymbol{x}+\boldsymbol{\alpha}^T\boldsymbol{b} \end{aligned} \tag{13}$$

   where $\boldsymbol{e}\in\mathbb{R}^m$ denotes the vector whose elements are all 1s, $\boldsymbol{\alpha}\in\mathbb{R}^m$, $\boldsymbol{\beta}\in\mathbb{R}^n_+$ and $\boldsymbol{\gamma}\in\mathbb{R}^n_+$ are multipliers. Then, the dual objective is

$$\begin{aligned} g(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}) &= \min_{\boldsymbol{x},\boldsymbol{y}} \mathcal{L}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}) \\ &= \min_{\boldsymbol{x},\boldsymbol{y}} \left[(\boldsymbol{e}-\boldsymbol{\alpha}-\boldsymbol{\gamma})^T\boldsymbol{y}-(\boldsymbol{A}^T\boldsymbol{\alpha}+\boldsymbol{\beta})^T\boldsymbol{x}+\boldsymbol{\alpha}^T\boldsymbol{b}\right] \end{aligned} \tag{14}$$

   Note that we are only interested in the case that $g(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma})>-\infty$, This means $\boldsymbol{e}-\boldsymbol{\alpha}-\boldsymbol{\gamma}=\mathbf{0}$ and $\boldsymbol{A}^T\boldsymbol{\alpha}+\boldsymbol{\beta}=\mathbf{0}$. Therefore, we have the dual problem as follows

$$\begin{aligned} \max \quad & \boldsymbol{\alpha}^T\boldsymbol{b} \\ \text{s.t.} \quad & \boldsymbol{A}^T\boldsymbol{\alpha}\leq\mathbf{0} \\ & \boldsymbol{\alpha}\leq\boldsymbol{e}. \end{aligned} \tag{15}$$

2. Does the dual problem have optimal solutions? Why? [20pts]

   Yes, the dual problem (15) has an optimal solution.

   It should be noticed that 0 is served as a lower bound of the primal problem (12), and hence, we must have an optimal solution $(\hat{\boldsymbol{x}},\hat{\boldsymbol{y}})$ of (12), which can be obtained by applying the Simplex Method. Hence, by the strong duality theorem, the dual problem (15) must have an optimal solution, say $\hat{\boldsymbol{\alpha}}$, and the optimums of (12) and (15) have the relation $\hat{\boldsymbol{\alpha}}^T\boldsymbol{b}=\boldsymbol{e}^T\hat{\boldsymbol{y}}$.

4 Self-Dual

Consider the linear program of the form

$$\begin{aligned} \min \quad & \boldsymbol{q}^T\boldsymbol{z} \\ \text{s.t.} \quad & \boldsymbol{M}\boldsymbol{z}\geq-\boldsymbol{q} \\ & \boldsymbol{z}\geq\mathbf{0}. \end{aligned} \tag{16}$$

in which the matrix M is *skew symmetric*; that is, $M = -M^T$. Please show that the problem (16) and its dual are the same. [15pts]

The dual problem can be written as

$$\begin{aligned} \max \quad & -\boldsymbol{q}^T \boldsymbol{w} \\ \text{s.t.} \quad & \boldsymbol{w}^T \boldsymbol{M} \leq \boldsymbol{q}^T \\ & \boldsymbol{w} \geq \boldsymbol{0}. \end{aligned} \tag{17}$$

Changing the maximization to minimization,

$$\max \quad -\boldsymbol{q}^T \boldsymbol{w} \qquad \longrightarrow \qquad \min \quad \boldsymbol{q}^T \boldsymbol{w}$$

Using the skew-symmetry and eliminating minus signs where possible

$$\boldsymbol{w}^T \boldsymbol{M} \leq \boldsymbol{q}^T \qquad \longrightarrow \qquad \boldsymbol{M}\boldsymbol{w} \geq -\boldsymbol{q}$$

The resulted problem is

$$\begin{aligned} \min \quad & \boldsymbol{q}^T \boldsymbol{w} \\ \text{s.t.} \quad & \boldsymbol{M}\boldsymbol{w} \geq -\boldsymbol{q} \\ & \boldsymbol{w} \geq \boldsymbol{0}. \end{aligned} \tag{18}$$

which is exactly the primal problem, except the trivial naming of the variables.