EE160 Homework 2 Solution

1. *(3 points) Scalar linear control systems.*

   (a) $x(t) = \frac{t^2}{2}e^{-t}$.

   (b) $x(t) = \begin{cases} 1 - e^{-t} & \text{if } 0 < t < 1 \\ (1 - e^{-1})e^{-t+1} & \text{if } t > 1. \end{cases}$

   (c) $x(t) = \begin{cases} 0 & \text{if } t \leq 1 \\ e^{-t+1} & \text{if } t > 1. \end{cases}$

2. *(3 points) Highly oscillatory input functions.*

   (a) For such non-zero initial state control system, we can 1) shift state first and use linear superposition principle or 2) calculate the expression directly by using solution of scalar linear control system. Here, we give the former method.
   Let $y(t) = x(t) - x_0$, differential equation of $y(t)$ is

   $$\dot{y}(t) = ay(t) + \underbrace{ax_0 + bu(t)}_{v(t)} \quad \text{with initial state} \quad y(0) = 0$$

   linear superposition principle can be used by considering control input as $v(t) = ax_0 + b\sin(\omega t)$ and

   $$y(t) = \psi_1(t) + \psi_2(t)$$

   where ψ_1, ψ_2 are response functions of $\phi_1(t) = ax_0$, $\phi_2(t) = b\sin(\omega t)$ of linear control system

   $$\dot{y}(t) = ay(t) + v(t)$$

   here ϕ_1 can also be considered as response of initial value. With the results in the lecture notes

   $$\psi_1(t) = x_0(e^{at} - 1) \quad \text{and} \quad \psi_2(t) = \frac{\omega b e^{at}}{a^2 + \omega^2} - \frac{b}{\sqrt{a^2 + \omega^2}}\sin(\omega t + \theta)$$

   the explicit expression is obtained as

   $$x(t) = \psi_1(t) + \psi_2(t) + x_0 = \left(x_0 + \frac{\omega b}{a^2 + \omega^2}\right)e^{at} - \frac{b}{\sqrt{a^2 + \omega^2}}\sin(\omega t + \theta)$$

   where phase shift $\theta = \arccos\left(\frac{a}{\sqrt{a^2+\omega^2}}\right)$.

   (b) Maximal amplitude of $x(t)$

   $$A = \frac{|b|}{\sqrt{a^2 + \omega^2}}$$

   is very small when frequency ω is large, this can be viewed as a low-pass filter.

3. *(4 points) RC-Circuit with AC input.*

   Given the input function and values of parameters,

   $$\dot{V}_C(t) = -100V_C + 1000\sin(50t)$$

   solution of this differential equation is

   $$\begin{aligned} V_C(t) &= e^{-100t}V_C(0) + 40(e^{-100t} - \cos(50t) + 2\sin(50t)) \\ &= e^{-100t}(V_C(0) + 40) - 40\sqrt{5}\sin(50t + \theta) \end{aligned}$$

   when t is very large, exponential term vanishes and $V_C(t) \approx -40\sqrt{5}\sin(50t + \theta)$ where phase shift $\theta = \arccos\left(-\frac{2}{\sqrt{5}}\right)$. Amplitude of output function $A_{\text{out}} = 40\sqrt{5}$ and amplitude amplification factor

   $$\frac{A_{\text{out}}}{A_{\text{in}}} = \frac{40\sqrt{5}}{100} \approx 0.89.$$