Online Lecture Notes

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1 Examples for Linear Differential Equations

For instance if you get a example like

$$0 = 2x(t) + 5\dot{x}(t) + 2\ddot{x}(t) \qquad \Longrightarrow \qquad \ddot{x}(t) = -x(t) - \frac{5}{2}\dot{x}(t)$$

The first step is to write this differential equation in standard form. For this aim, we introduce the auxiliary state

$$y(t) = \begin{pmatrix} x(t) \\ \dot{x}(t) \end{pmatrix}$$

which satisfies the ODE in standard form:

$$\dot{y}(t) = \begin{pmatrix} y_2(t) \\ -y_1(t) - \frac{5}{2}y_2(t) \end{pmatrix} = Ay(t) \qquad \text{with} \qquad A = \begin{pmatrix} 0 & 1 \\ -1 & -\frac{5}{2} \end{pmatrix}$$

The solution of the ODE is given by

$$y(t) = e^{At}y_0 \qquad \text{with} \qquad y_0 \in \mathbb{R}^2 \; .$$

The eigenvalues of A are

$$\lambda_1 = -\frac{1}{2} \qquad \text{and} \qquad \lambda_2 = -2$$

(if you write this out in the exam, please write up the full path how you got there, also write out the charactersic polynomial...). The corresponding eigenvectors of A are given by

$$v_1 = \begin{pmatrix} 1 \\ -\frac{1}{2} \end{pmatrix} \qquad \text{and} \qquad v_2 = \begin{pmatrix} -\frac{1}{2} \\ 1 \end{pmatrix}$$

Thus, we set

$$T = [v_1, v_2] = \begin{pmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & -2 \end{pmatrix}$$

If you want to sure that you have to mistakes, check again that really

$$A = TDT^{-1}$$

(here this seem correct so far...). The corresponding matrix exponential is given by

$$e^{At} = Te^{Dt}T^{-1} = \begin{pmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} e^{-t/2} & 0 \\ 0 & e^{-2t} \end{pmatrix} \begin{pmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{pmatrix}^{-1} .$$

We leave the remaining steps as an exercise to you...