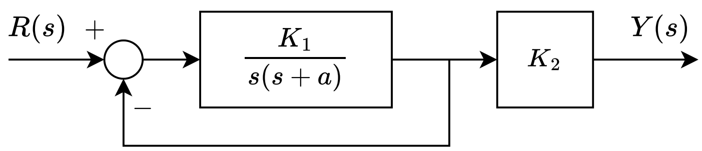
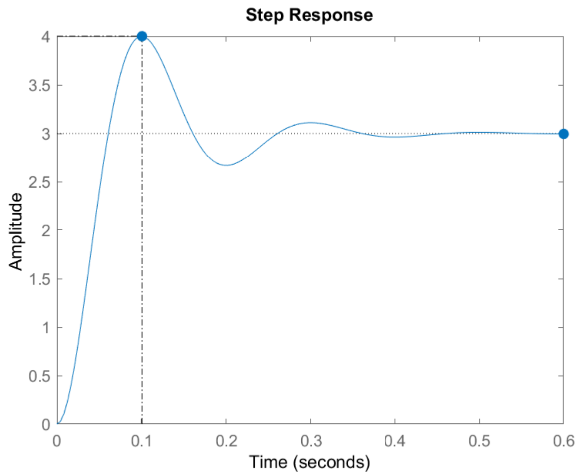
EE160 Homework 2 Solution

1. Consider the following 2nd-order system and its **unit step response**, some performance indices have been given in the figure, determine K_1, K_2 and a, respectively. **(9')**





Solution. According to the step response, we have

$$\begin{cases} y(\infty) &= 3 \\ T_p &= 0.1 \\ P.O. &= \frac{4-3}{3} \times 100\% = 33.3\% \end{cases} \qquad \mathbf{(2')}$$

The closed-loop transfer function is

$$\frac{Y(s)}{R(s)} = \frac{K_1 K_2}{s^2 + as + K_1} = K_2 \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \qquad \mathbf{(2')}$$

Similarly, we have

$$\begin{cases} T_p &= \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = 0.1 \\ P.O. &= 100e^{-\zeta\pi/\sqrt{1-\zeta^2}} = 33.3 \end{cases} \Rightarrow \begin{cases} \zeta &= 0.33 \\ \omega_n &= 33.28 \end{cases} \qquad \mathbf{(2')}$$

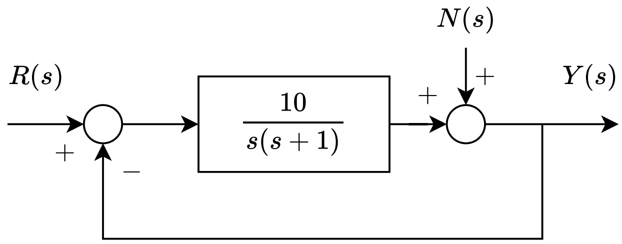
Compared with the coefficients of $\frac{Y(s)}{R(s)}$, we have

$$\begin{cases} K_1 &= \omega_n^2 = 1108 \\ a &= 2\zeta\omega_n = 22 \end{cases} \qquad \mathbf{(2')}$$

Moreover, since we consider the unit step response

$$y(\infty) = \lim_{s\to 0} sY(s) = \lim_{s\to 0} s \cdot \frac{K_1 K_2}{s^2 + as + K_1} \cdot \frac{1}{s} = K_2 = 3 \qquad \mathbf{(1')}$$

2. Consider the following 2nd-order system.



(a) When $r(t) = 1(t)$ (unit step signal), $n(t) = t$, calculate the steady-state error of the system. **(6')**

Solution. According to the block diagram

$$Y(s) = N(s) + [R(s) - Y(s)] \cdot \frac{10}{s(s+1)}$$

$$(1 + \frac{10}{s(s+1)}) \cdot Y(s) = \frac{10}{s(s+1)} \cdot R(s) + N(s) \quad \textbf{(2')}$$

$$Y(s) = \frac{10}{s^2+s+10} \cdot R(s) + \frac{s^2+s}{s^2+s+10} \cdot N(s)$$

Based on the definition of error

$$\begin{aligned} E(s) &= R(s) - Y(s) \\ &= \frac{s^2+s}{s^2+s+10} \cdot R(s) - \frac{s^2+s}{s^2+s+10} \cdot N(s) \end{aligned} \quad \textbf{(2')}$$

By the final value principle,

$$e(\infty) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} s\frac{s^2+s}{s^2+s+10}\left(\frac{1}{s} - \frac{1}{s^2}\right) = -\frac{1}{10} \quad \textbf{(2')}$$

(b) When $r(t) = 1(t)$ (unit step signal), $n(t) = 0$, calculate peak time T_p, settling time T_s and the percent overshoot $P.O.$ of the step response. **(7')**

Solution. The closed-loop transfer function is

$$\frac{Y(s)}{R(s)} = \frac{10}{s^2+s+10} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \quad \textbf{(2')}$$

Thus

$$\begin{cases} 2\zeta\omega_n &= 1 \\ \omega_n^2 &= 10 \end{cases} \Rightarrow \begin{cases} \zeta &= 0.158 \\ \omega_n &= \sqrt{10} \end{cases} \quad \textbf{(2')}$$

Peak time and the percent overshoot can be given

$$\begin{cases} T_p &= \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = 1.006s \\ T_s &= \frac{4}{\zeta\omega_n} = 8s \\ P.O. &= e^{-\zeta\pi/\sqrt{1-\zeta^2}} \times 100\% = 60.49\% \end{cases} \quad \textbf{(3')}$$

3. Consider the **unity feedback** third-order system with **open-loop** transfer function $G(s) = \frac{1}{s(0.1s+1)(0.5s+1)}$.

(a) Calculate the **poles** of the **closed-loop** transfer function. **(5')**

Solution. The closed-loop transfer function is

$$T(s) = \frac{G(s)}{1+G(s)} = \frac{20}{s^3+12s^2+20s+20} = \frac{10.24 \cdot 1.954}{(s+10.24)(s^2+1.763s+1.954)} \quad \textbf{(2')}$$

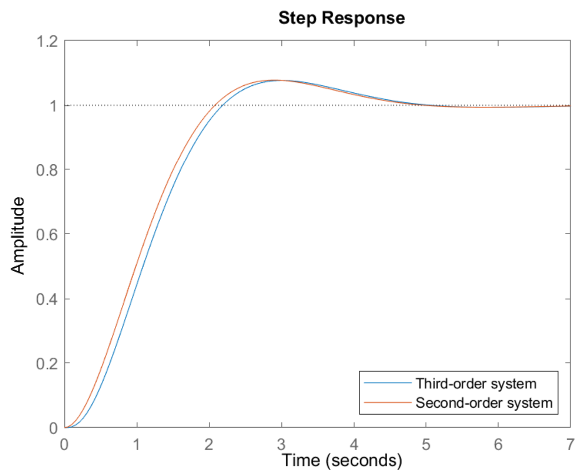
Thus the poles of the closed-loop system are $p_{1,2} = -0.8814 \pm 1.0848j$ and $p_3 = -10.24$. **(3')**

(b) Determine whether the response of the third-order system can be approximated by the dominant roots of the second-order system. If so, plot the **unit step response** of the **approximated second-order** system and the **original third-order** system on the **same** figure; if not, give your reasons. **(7')**

Solution. Since the real part of the dominant roots is less that one tenth of the real part of the third root, i.e.,

$$\frac{|-0.8814|}{|-10.24|} = 0.086 < \frac{1}{10} \quad \mathbf{(3')}$$

Then the response of the third-order system can be approximated by the second-order system. And the step response of the approximated second-order system $T_2(s) = \frac{1.954}{s^2+1.763s+1.954}$ and the original third-order system is given below. **(4')**



4. Consider two **unity feedback** systems, necessary information of the **open-loop** transfer function has been given in the table.

system	type number	zeros	poles	high frequency gain
i.	1	-1	$-4, -1 \pm j$	7
ii.	2	$-\frac{1}{2}, -\frac{1}{4}$	$-1 \pm 3j$	10

(a) Determine the **open-loop** transfer function $G(s)$ of each system. **(4')**

Solution.

i. $G(s) = \frac{7(s+1)}{s(s+4)(s^2+2s+2)}$ **(2')**

ii. $G(s) = \frac{10(s+\frac{1}{2})(s+\frac{1}{4})}{s^2(s^2+2s+10)}$ **(2')**

(b) Calculate the position error constant K_p, velocity error constant K_v and acceleration error constant K_a of each system, respectively. **(6')**

Solution.

i.

$$K_p = \lim_{s\to 0} G(s) = \infty$$
$$K_v = \lim_{s\to 0} sG(s) = \frac{7}{8} \quad \mathbf{(3')}$$
$$K_a = \lim_{s\to 0} s^2 G(s) = 0$$

ii.

$$K_p = \lim_{s\to 0} G(s) = \infty$$
$$K_v = \lim_{s\to 0} sG(s) = \infty \quad \mathbf{(3')}$$
$$K_a = \lim_{s\to 0} s^2 G(s) = \frac{1}{8}$$

(c) Based on the error constants derived in (b), give the steady-state error e_{ss} when the input signal $r(t) = 1(t)$ (unit step signal), $r(t) = t$ and $r(t) = t^2$ of each system, respectively. **(6')**

Solution.

i.

$$\begin{aligned} r(t) = 1(t) \quad &\Rightarrow e_{ss} = \frac{1}{1+K_p} = 0 \\ r(t) = t \quad &\Rightarrow e_{ss} = \frac{1}{K_v} = \frac{8}{7} \qquad \mathbf{(3')} \\ r(t) = t^2 \quad &\Rightarrow e_{ss} = \infty \end{aligned}$$

ii.

$$\begin{aligned} r(t) = 1(t) \quad &\Rightarrow e_{ss} = 0 \\ r(t) = t \quad &\Rightarrow e_{ss} = 0 \qquad \mathbf{(3')} \\ r(t) = t^2 \quad &\Rightarrow e_{ss} = \frac{2}{K_a} = 16 \end{aligned}$$

5. Using the Routh-Hurwitz criterion, determine if the system with the following characteristic equation is stable. **(10')**

(a) $s^4 + 2s^3 + s^2 + 2s + 1 = 0$

(b) $s^5 + 8s^4 + 15s^3 + 36s^2 + 42s + 11 = 0$

Solution.

(a) Unstable

$$\begin{array}{llll} s^4 & 1 & 1 & 1 \\ s^3 & 2 & 2 & \\ s^2 & \varepsilon & 1 & \\ s^1 & 2 - \frac{2}{\varepsilon} & & \\ s^0 & 1 & & \end{array}$$

(b) Stable

$$\begin{array}{llll} s^5 & 1 & 15 & 42 \\ s^4 & 8 & 36 & 11 \\ s^3 & 21/2 & 325/8 & \\ s^2 & 106/21 & 11 & \\ s & 7523/424 & & \\ s^0 & 11 & & \end{array}$$

6. Determine the range of K for a stable system **(10')**

(a) a system with a characteristic equation $s^3 + Ks^2 + (1+K)s + 6 = 0$

(b) A unit negative feedback system with a open-loop transfer function

$$G(s) = \frac{K}{(s+2)(s+3)(s+5)}$$

Solution.

(a) $K > 0$, $\frac{K(K+1)-6}{K} > 0$, $K > 2$

$$\begin{array}{lll} s^3 & 1 & \text{K+1} \\ s^2 & \text{K} & 6 \\ s^1 & \frac{K(K+1)-6}{K} & 0 \\ s^0 & 6 & \end{array}$$

(b)

$$\frac{K}{s^3+10s^2+31s+K+30}$$

$,280 > K > -30$

$$\begin{array}{lcc} s^3 & 1 & 31 \\ s^2 & 10 & K+30 \\ s^1 & 28-K/10 & 0 \\ s^0 & K+30 & 0 \end{array}$$

7. Consider the feedback control system shown in the following Figure. The controller and model transfer functions are given by

$$G_c(s) = K \quad \text{and} \quad G(s) = \frac{s+40}{s(s+10)}$$

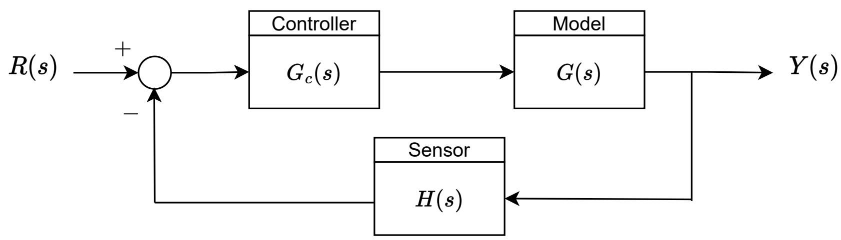
and the sensor transfer function is

$$H(s) = \frac{1}{s+20}$$

.(15')

(a) Determine the limiting value of gain K for a stable system.

(b) Let $K=3$, determine whether the relative stability of the system is smaller than -1 ($\sigma < -1$).



(a)

$$\frac{K\left(s^2+60s+800\right)}{40K+(200+K)s+30s^2+s^3}$$

$$\begin{array}{lcc} s^3 & 1 & K+200 \\ s^2 & 30 & 40\,K \\ s^1 & 200-\frac{K}{3} & 0 \\ s^0 & 40\,K & 0 \end{array}$$

$$600 > K > 0$$

(b) No.

$$40K+(200+K)(s-\sigma)+30(s-\sigma)^2+(s-\sigma)^3$$

Let $K=3, \sigma=1$

$$s^3+27\,s^2+146\,s-54$$

$$\begin{array}{lcc} s^3 & 1 & 146 \\ s^2 & 27 & -54 \\ s^1 & 148 & 0 \\ s^0 & -54 & 0 \end{array}$$

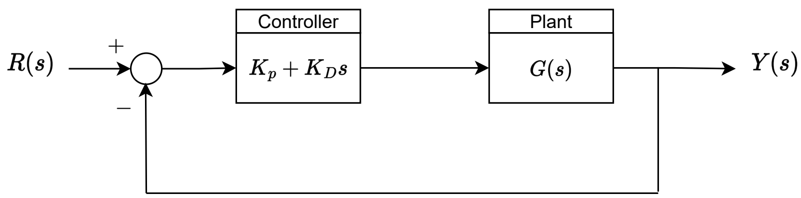
8. Consider a spacecraft attitude control problem. The control system structure is shown below. Suppose we only consider the pitch-axis rotation of the spacecraft, which has a plant transfer function

$$G(s) = \frac{1}{Is^2} + \frac{L/I}{s^2+2\zeta\omega s+\omega^2}$$

where $I = 77,076kg/m^2$ is the spacecraft pitch inertia, $L = -1.387$ the telescope structure gain in the pitch axis, $\omega = 14.068$rad/s the structure natural frequency, and ζ the passive damping ratio assumed as 0.005. A proportional plus derivative controller is used in the system, where

$$G_c(s) = K_p + K_D s$$

and where $K_p > 0$ and $K_D > 0$. Please give a selection of K_p and K_d via trial and error to stabilize the system and proof the stability. **(15')**



Solution.

$$G = \frac{D}{N} = \frac{(L+1)s^2 + 2\zeta\omega s + \omega^2}{Is^2(s^2 + 2\zeta\omega s + \omega^2)}$$

$$G_p = \frac{G * G_c}{G * G_c + 1} = \frac{D * G_c}{D * G_c + N}$$

$$D * G_c + N = ((L+1)s^2 + 2\zeta\omega s + \omega^2) * (K_p + K_D s) + Is^2(s^2 + 2\zeta\omega s + \omega^2)$$

I write a script to generate the routh table and verify the system stability given K_p and K_d