SI211 Homework 4 & 5

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Soft-Deadline April 20, 2022
(we are aware of the current epidemic situation—if you need more time, it is completely fine if you submit later)

1. *Gram-Schmidt Algorithm.* Let $H = L_2[-1,1]$ be the Hilbert space of L_2 integrable functions on the interval $[-1,1]$ with scalar product

$$\langle f, g \rangle = \int_{-1}^{0} f(x)g(x)\mathrm{e}^{x}\mathrm{d}x + 2\int_{-1}^{0} f(x)g(x)\mathrm{e}^{-x}\mathrm{d}x\,.$$

   Apply the Gram-Schmidt algorithm to construct three linear independent polynomials q_0, q_1 with order ≤ 1, which satisfy:

$$\begin{aligned} &\langle q_0, q_1 \rangle = 0, \\ &\|q_0\|_H = \|q_1\|_H = 1\,. \end{aligned}$$

2. *Gauss Approximation.* Consider function $f(x) = x\mathrm{e}^{x}$. Use Gauss approximation to solve the least-squares optimization problem

$$\min_{p \in P_2} \int_{1}^{2} [f(x) - p(x)]^2\, \mathrm{d}x,$$

   where P_2 denotes the set of polynomials of order 2.

3. *Orthogonal Polynomials.* The Chebyshev polynomials of the first kind are given by

$$T_n(x) = \cos(n \cdot \arccos(x)), \quad n = 0, 1, 2, \ldots\,.$$

   Use the addition theorem for the cosine function to show that the functions T_n actually are polynomials of order n (hint: use induction!). Prove that these polynomials are orthogonal with respect to the scalar product

$$\langle f, g \rangle = \int_{-1}^{1} \frac{f(x)g(x)}{\sqrt{1-x^2}}\mathrm{d}x\,.$$

4. *Closed Newton-Cotes Interpolation.* Implement the closed Newton-Cotes formulas for $n = 2, 3, 4$ in any programming language of your choice and compute the integral

$$I = \int_{1}^{2} \frac{4}{7x^2}\mathrm{d}x$$

   with all three methods and compute the difference between your results and the exact value of the integral.

5. *Simpson's Rule on Infinite Intervals.* Introduce a suitable variable transformation and apply Simpson's rule to approximate the integral

$$I = \int_0^\infty x^2 \mathrm{e}^{-x^2} \mathrm{d}x.$$

Analyze the error bound of your approximation.

6. *2D-Simpson's Rule.* Apply the generalized Simpson's rule to approximate the double-integral

$$\int_X \mathrm{e}^{\sin x_1 \cos x_2} \mathrm{d}x$$

where $X = [0,1] \times [0,1]$. Here, we refer to the 2D generalization of Simpson's rule that we introduced in the lecture (see our lecture notes for details).