# Final Exam

Introduction to Control

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# I Scalar Infinite Horizon Optimal Control

Solve the following scalar infinite horizon optimal control problems if a solution exists. If you think that there is no solution, prove that no solution exists.

- Problem 1:

$$\min_{x,u} \int_0^\infty x(t)^2 + u(t)^2 \,\mathrm{d}t \quad \text{s.t.} \quad \begin{cases} \forall t \in [0,\infty), \\ \dot{x}(t) = -x(t) + u(t) \\ x(0) = 1 \end{cases}$$

**(5 points)**

**Solution:** The algebraic Riccati equation

$$-2p_\infty + 1 - p_\infty^2 = 0$$

admits a positive solution $p_\infty = -1 + \sqrt{2}$. Hence, the optimal control gain is given by $K_\infty = 1 - \sqrt{2}$. The optimal control law has the form $\mu(x) = K_\infty x$.

- Problem 2:

$$\min_{x,u} \int_0^\infty u(t)^2 - x(t)^2 \,\mathrm{d}t \quad \text{s.t.} \quad \begin{cases} \forall t \in [0,\infty), \\ \dot{x}(t) = -x(t) \\ x(0) = 1 \end{cases}$$

**(5 points)**

**Solution:** The dynamics of state $x$ is independent of control input $u$,

$$\begin{cases} \dot{x}(t) = -x(t) \\ x(0) = 1 \end{cases} \quad \Rightarrow \quad x(t) = e^{-t} \quad \text{and} \quad \int_0^\infty x(t)^2 \,\mathrm{d}t = \frac{1}{2} \; .$$

Thus, the optimization problem has the form

$$\min_u \int_0^\infty u(t)^2 \,\mathrm{d}t = 0 \; .$$

Consequently, $u(t) = 0$ for $t \in [0,\infty)$ is the optimal solution.

- Problem 3:

$$\min_{x,u} \int_0^\infty x(t)^2 - u(t)^2 \,\mathrm{d}t \quad \text{s.t.} \quad \begin{cases} \forall t \in [0,\infty), \\ \dot{x}(t) = u(t) \\ x(0) = 1 \end{cases}$$

**(5 points)**

**Solution:** No solution exists. This optimization problem is equivalent to minimizing the integral

$$\int_0^\infty \left( x(t)^2 - \dot{x}(t)^2 \right) \,\mathrm{d}t \; .$$

However, this integral cannot be bounded from below. For example $x(t) = e^{\frac{t^2}{2}}, \dot{x}(t) = te^{\frac{t^2}{2}}$ yields

$$\int_0^\infty \left( x(t)^2 - \dot{x}(t)^2 \right) \,\mathrm{d}t = \int_0^\infty (1 - t^2) e^{\frac{t^2}{2}} \,\mathrm{d}t = -\infty \; .$$

# II Multivariate Infinite Horizon Optimal Control

Solve the infinite horizon optimal control problem in dependence on the parameter $y \in \mathbb{R}^2$:

$$\min_{x,u} \int_0^{\infty} \left[4x_1(t)^2 + 4x_2(t)^2 + u(t)^2\right] \, \mathrm{d}t$$

$$\text{s.t.} \quad \begin{cases} \forall t \in [0, \infty), \\ \dot{x}_1(t) = -x_1(t) + u(t) \\ \dot{x}_2(t) = -x_2(t) + u(t) \\ x_1(0) = y_1 \\ x_2(0) = y_2 \; . \end{cases}$$

What is the associated optimal feedback control law? **(15 points)**

**Solution:** Let us introduce the matrices

$$A = -I, \quad B = (1,\ 1)^{\intercal}, \quad Q = 4I, \quad R = 1, \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad P = \begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix} \; .$$

The associated algebraic Riccati equation can be written in the form

$$\begin{aligned} 0 &= A^\top P + PA + Q - PBR^{-1}B^\top P^\top \\ &= \begin{pmatrix} 4 - 2p_{11} - (p_{11} + p_{12})^2 & -2p_{12} - (p_{11} + p_{12})(p_{12} + p_{22}) \\ -2p_{12} - (p_{11} + p_{12})(p_{12} + p_{22}) & 4 - 2p_{22} - (p_{12} + p_{22})^2 \end{pmatrix} \; . \end{aligned}$$

Next, in order to solve this equation, we introduce the shorthands

$$\alpha = p_{11} + p_{12} \quad \text{and} \quad \beta = p_{12} + p_{22} \; .$$

We know from the Riccati equation that

$$p_{11} = 2 - \frac{\alpha^2}{2}, \quad p_{12} = -\frac{\alpha\beta}{2}, \quad p_{22} = 2 - \frac{\beta^2}{2}$$

$$\Rightarrow \quad \alpha = 2 - \frac{\alpha^2}{2} - \frac{\alpha\beta}{2} \quad \text{and} \quad \beta = 2 - \frac{\beta^2}{2} - \frac{\alpha\beta}{2}$$

the difference of the two equations gives

$$\alpha - \beta = -\frac{\alpha^2 - \beta^2}{2} \quad \Rightarrow \quad \left(1 + \frac{\alpha + \beta}{2}\right)(\alpha - \beta) = 0$$

1. If $1 + \frac{\alpha + \beta}{2} = 0$, then
$$\alpha + \beta = -2 \quad \Rightarrow \quad \alpha = 2 - \alpha \; .$$
But this is a contradiction.

2. If $\alpha - \beta = 0$,
$$\alpha = \beta \quad \Rightarrow \quad \alpha = 2 - \alpha^2 = 0 \quad \Rightarrow \quad \alpha = 1 \text{ or } -2 \; ,$$
We need to set $\alpha = 1$ in order to ensure that the matrix $P$ is positive definite,
$$P = \begin{pmatrix} \frac{3}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{3}{2} \end{pmatrix}$$
The optimal control gain and feedback law are given by
$$K = -R^{-1}B^{\intercal}P = (-1 \;\; -1) \quad \text{and} \quad \mu(x) = Kx = -x_1 - x_2 \; .$$

# III Fundamental solution of a linear time-varying system

Let us consider the linear time-varying differential equation system

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) && (1) \\ \dot{x}_2(t) &= -x_1(t) - t \cdot x_2(t) && (2) \end{aligned}$$

with given parametric initial value $x(0) = [y_1, 0]^\intercal$.

- Write the above system in the form of a linear time-varying system in standard form, $\dot{x}(t) = A(t)x(t) + b(t)$. What are $A$ and $b$? **(2 points)**

  **Solution:** $A(t) = \begin{pmatrix} 0 & 1 \\ -1 & -t \end{pmatrix}$ and $b(t) = 0$.

- Work out the differential equation for the fundamental solution $G : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^{2\times 2}$ of the above linear system. **(4 points)**

  **Solution:** Let us introduce the notation

$$G(t,\tau) = \begin{pmatrix} G_{11}(t,\tau) & G_{12}(t,\tau) \\ G_{21}(t,\tau) & G_{22}(t,\tau) \end{pmatrix} ,$$

  where $G(t,\tau)$ needs to satisfy the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t} G(t,\tau) = A(t)G(t,\tau) \quad \text{with} \quad G(t,t) = I \ .$$

  The differential equation can be expanded explicitly as

$$\begin{pmatrix} \dot{G}_{11}(t,\tau) & \dot{G}_{12}(t,\tau) \\ \dot{G}_{21}(t,\tau) & \dot{G}_{22}(t,\tau) \end{pmatrix} = \begin{pmatrix} G_{21}(t,\tau) & G_{22}(t,\tau) \\ -G_{11}(t,\tau) - tG_{21}(t,\tau) & -G_{12}(t,\tau) - tG_{22}(t,\tau) \end{pmatrix}$$

- Try to find an explicit expression for $G$. You will get full points if you find explicit expressions for the left column of $G(t,0)$ (that is, the components $G_{11}$ and $G_{21}$). *[Hint: it might help to introduce the auxiliary function $\widetilde{G}_{11}(t,0) = tG_{11}(t,0)$.]*
  **(8 points)**

  **Solution:** Let us first solve the left column of the differential equation for $G(t,0)$:

$$\begin{cases} \dot{G}_{11}(t,0) = G_{21}(t,0) & \text{with} \quad G_{11}(0,0) = 1 \ , \\ \dot{G}_{21}(t,0) = -G_{11}(t,0) - tG_{21}(t,0) & \text{with} \quad G_{21}(0,0) = 0 \ . \end{cases}$$

  For this aim, we introduce the auxiliary function $\tilde{G}_{11}(t,0) = tG_{11}(t,0)$ and use that

$$\frac{\mathrm{d}}{\mathrm{d}t}(\tilde{G}_{11}(t,0) + G_{21}(t,0)) = 0 \quad \Rightarrow \quad \tilde{G}_{11}(t,0) + G_{21}(t,0) = \tilde{G}_{11}(0,0) + G_{21}(0,0) = 0 \ ;$$

  that is,

$$tG_{11}(t,0) + G_{21}(t,0) = tG_{11}(t,0) + \dot{G}_{11}(t,0) = 0 \ .$$

This is a differential equation for $G_{11}(t,0)$, which can be solved by a separation of variables finding that

$$G_{11}(t,0) = e^{-\frac{t^2}{2}} \quad \text{and} \quad G_{21}(t,0) = -tG_{11}(t,0) = -te^{-\frac{t^2}{2}} \ .$$

Next, we solve the differential equation that is associated with the right column

$$\begin{cases} \dot{G}_{12}(t,0) = G_{22}(t,0) & \text{with} \quad G_{12}(0,0) = 0 \ , \\ \dot{G}_{22}(t,0) = -G_{12}(t,0) - tG_{22}(t,0) & \text{with} \quad G_{22}(0,0) = 1 \ . \end{cases}$$

The strategy is analogous:

$$\begin{aligned} & tG_{12}(t,0) + G_{22}(t,0) = 0G_{12}(0,0) + G_{22}(0,0) = 1 \\ \Rightarrow \quad & tG_{12}(t,0) + \dot{G}_{12}(t,0) = 1 \\ \Rightarrow \quad & (tG_{12}(t,0) + \dot{G}_{12}(t,0))e^{\frac{t^2}{2}} = e^{\frac{t^2}{2}} \\ \Rightarrow \quad & \frac{\mathrm{d}}{\mathrm{d}t}\left(G_{12}(t,0)e^{\frac{t^2}{2}}\right) = e^{\frac{t^2}{2}} \\ \Rightarrow \quad & G_{12}(t,0)e^{\frac{t^2}{2}} = c + \int_0^t e^{\frac{s^2}{2}}\,\mathrm{d}s \\ \Rightarrow \quad & G_{12}(t,0) = e^{-\frac{t^2}{2}} \int_0^t e^{\frac{s^2}{2}}\,\mathrm{d}s \\ \text{and} \quad & G_{22}(t,0) = 1 - tG_{12}(t,0) = 1 - te^{-\frac{t^2}{2}} \int_0^t e^{\frac{s^2}{2}}\,\mathrm{d}s \end{aligned}$$

In summary, we find that

$$G(t,0) = \begin{pmatrix} e^{-\frac{t^2}{2}} & e^{-\frac{t^2}{2}} \int_0^t e^{\frac{s^2}{2}}\,\mathrm{d}s \\ -te^{-\frac{t^2}{2}} & 1 - te^{-\frac{t^2}{2}} \int_0^t e^{\frac{s^2}{2}}\,\mathrm{d}s \end{pmatrix} \ .$$

- Explain how to find an explicit expression for the solution trajectories $x_1(t)$ and $x_2(t)$ in dependence on the parameter $y_1 \in \mathbb{R}$. **(6 points)**

**Solution:** Since the given the initial value is such that $x_2(0) = 0$, the left column of the fundamental matrix $G(t,0)$ is sufficient for finding an explicit expression for the solution trajectories,

$$\begin{cases} x_1(t) = G_{11}(t,0)x_1(0) = e^{-\frac{t^2}{2}} y_1 \ , \\ x_2(t) = G_{21}(t,0)x_1(0) = -te^{-\frac{t^2}{2}} y_1 \ . \end{cases}$$

# IV Gradient Flows

Let $V : \mathbb{R}^n \to \mathbb{R}$ be a non-negative smooth function that is radially unbounded and satisfies $V(0) = 0$ and $\nabla V(0) = 0$ and $\nabla V(x) \neq 0$ for all $x \neq 0$, where $\nabla V$ denotes the gradient of $V$. Prove that the nonlinear differential equation

$$\dot{y}(t) = -\nabla V(y(t))$$

for the state $y : \mathbb{R} \to \mathbb{R}^n$ is (globally) asymptotically stable at 0. **(10 points)**

**Solution:** The function $V$ is a Lyapunov function of the gradient flow that is strictly monotonuously descending as

$$\frac{\mathrm{d}}{\mathrm{d}t} V(y(t)) = \nabla V(y(t))^{\intercal} \cdot \dot{y}(t) = -\|\nabla V(y(t))\|_2^2 < 0 \quad \text{for} \quad y \neq 0 \, .$$

As $V$ is also continuous, positive definite, and radially unbounded, 0 must be the unique equilibrium point and the system is globally asymptotically stable.