Numerical Optimization, 2021 Fall
Homework 2 Solution

1 LP Modeling

A businessman is considering an investment project. The project has a lifetime of 4 years, with cash flows of $-\$100,000$, $+\$50,000$, $+\$70,000$, and $+\$30,000$ in each of the 4 years, respectively. At any time he may borrow funds at the rates of 12%, 22%, and 34% (total) for 1, 2, or 3 periods, respectively. He may loan funds at 10% per period. He calculates the present value of a project as the maximum amount of money he would pay now, to another party, for the project, assuming that he has no cash on hand and must borrow and lend to pay the other party and operate the project while maintaining a nonnegative cash balance after all debts are paid. Formulate the project valuation problem in a linear programming framework. [40pts]

Solution:

Note: As long as your modeling makes sense, you will get full points. Following is one possible solution.

Decision variables:

- x_{ij}: the amount of money borrowed in the ith year and returned in the jth year;
- y_i: the amount of money loaned in the ith year for one period;
- z: the amount of money he would pay now.

The project valuation problem can be formulated as follows:

$$\begin{aligned}
\max \quad & z \\
\text{s.t.} \quad & x_{12} + x_{13} + x_{14} - z - y_1 - 100,000 \geq 0 \\
& x_{23} + x_{24} + 1.1y_1 + 50,000 - y_2 - 1.12x_{12} \geq 0 \\
& x_{34} + 1.1y_2 + 70,000 - y_3 - 1.22x_{13} - 1.12x_{23} \geq 0 \\
& 30,000 + 1.1y_3 - 1.34x_{14} - 1.22x_{24} - 1.12x_{34} \geq 0 \\
& x_{12}, x_{13}, x_{14}, x_{23}, x_{24}, x_{34}, y_1, y_2, y_3 \geq 0.
\end{aligned} \tag{1}$$

2 LP Conversion

Convert the following problem to a linear program in standard form [30pts]

$$\begin{aligned}
\min \quad & |x| + |y| + |z| \\
\text{s.t.} \quad & x + y \leq 1 \\
& 2x + z = 3.
\end{aligned} \tag{2}$$

Solution 1:

For any number a, we can use two nonnegative numbers $a^+ := \max\{a, 0\}$, $a^- := \max\{-a, 0\}$ to denote it as $a = a^+ - a^-$. We also have $|a| = a^+ + a^-$. So we can convert the above program into the following linear program (s is the slack variable)

$$\begin{aligned} \min \quad & x^+ + x^- + y^+ + y^- + z^+ + z^- \\ \text{s.t.} \quad & x^+ - x^- + y^+ - y^- + s = 1 \\ & 2x^+ - 2x^- + z^+ - z^- = 3 \\ & x^+, x^-, y^+, y^-, z^+, z^-, s \geq 0. \end{aligned} \tag{3}$$

Solution 2:

Since minimizing $|x|$ is equivalent to minimizing x' satisfying $|x| < x'$, we can reformulate the problem as

$$\begin{aligned} \min \quad & x' + y' + z' \\ \text{s.t.} \quad & x + y \leq 1 \\ & 2x + z = 3 \\ & x \leq x' \\ & -x \leq x' \\ & y \leq y' \\ & -y \leq y' \\ & z \leq z' \\ & -z \leq z'. \end{aligned} \tag{4}$$

and by introducing the slack variables, we have the following standard form:

$$\begin{aligned} \min \quad & x' + y' + z' \\ \text{s.t.} \quad & x + y + s_1 = 1 \\ & 2x + z = 3 \\ & x + s_2 = x' \\ & -x + s_3 = x' \\ & y + s_4 = y' \\ & -y + s_5 = y' \\ & z + s_6 = z' \\ & -z + s_7 = z' \\ & s_1, s_2, s_3, s_4, s_5, s_6, s_7 \geq 0. \end{aligned} \tag{5}$$

3 LP Conversion

A class of piecewise linear functions can be represented as $f(\boldsymbol{x}) = \text{Maximum}\ (\boldsymbol{c_1^T x} + d_1, \boldsymbol{c_2^T x} + d_2, \ldots, \boldsymbol{c_p^T x} + d_p)$. For such a function f, consider the problem

$$\begin{aligned} \min \quad & f(\boldsymbol{x}) \\ \text{s.t.} \quad & \boldsymbol{Ax = b, x \geq 0}. \end{aligned} \tag{6}$$

Show how to convert this problem to a linear programming problem. [30pts]

Solution:

Similar to the technique we use for absolute values, we can formulate the problem as follows.

$$\begin{aligned} \min \quad & z \\ \text{s.t.} \quad & \boldsymbol{c_i^T x} + d_i \leq z, \quad 1 \leq i \leq p \\ & \boldsymbol{Ax} = \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0}. \end{aligned} \tag{7}$$

Reference

[1] David G. Luenberger and Yinyu Ye, *Linear and Nonlinear Programming*. Springer International Publishing Switzerland, 2016.