Discrete Mathematics: Lecture 22 (I)

logic equivalence, tautological implication, building arguments

Xuming He
Associate Professor

School of Information Science and Technology
ShanghaiTech University

Spring Semester, 2022

Notes by Prof. Liangfeng Zhang

Review: Types of WFFs (Proposition)

Tautology(重言式): a WFF whose truth value is **T** for all truth assignment

- $p \vee \neg p$ is a tautology

Contradiction(矛盾式): a WFF whose truth value is **F** for all truth assignment

- $p \wedge \neg p$ is a contradiction

Contingency(可能式): neither tautology nor contradiction

- $p \rightarrow \neg p$ is a contingency

Satisfiable(可满足的):a WFF is satisfiable if it is true for at least one truth assignment

Rule of Substitution: (代入规则) Let B be a formula obtained from a tautology A by substituting a propositional variable in A with an arbitrary formula. Then B must be a tautology.

- $p \vee \neg p$ is a tautology: $(q \wedge r) \vee \neg(q \wedge r)$ is a tautology as well.

Review: Types of WFFs (Predicate)

DEFINITION: A WFF is **logically valid**普遍有效 if it is **T** in every interpretation

- $\forall x\ (P(x) \vee \neg P(x))$ is logically valid

DEFINITION: A WFF is **unsatisfiable**不可满足 if it is **F** in every interpretation

- $\exists x\ (P(x) \wedge \neg P(x))$ is unsatisfiable

DEFINITION: A WFF is **satisfiable**可满足 if it is **T** in some interpretation

- $\forall x\ (x^2 > 0)$
  - true when domain= nonzero real numbers

THEOREM: Let A be any WFF. A is logically valid iff $\neg A$ is unsatisfiable.

Rule of Substitution: Let A be a tautology in propositional logic. If we substitute any propositional variable in A with an arbitrary WFF from predicate logic, then we get a logically valid WFF.

- $p \vee \neg p$ is a tautology; hence, $P(x) \vee \neg P(x)$ is logically valid

Review: Logically Equivalent (Proposition)

DEFINITION: Let A and B be WFFs in propositional variables $p_1, \dots, p_n$.

- A and B are **logically equivalent** (等值) if they always have the same truth value for every truth assignment (of $p_1, \dots, p_n$)
  - Notation: $A \equiv B$

THEOREM: $A \equiv B$ if and only if $A \leftrightarrow B$ is a tautology.

- $A \equiv B$
- iff for any truth assignment, A, B take the same truth values
- iff for any truth assignment, $A \leftrightarrow B$ is true
- iff $A \leftrightarrow B$ is a tautology

THEOREM: $A \equiv A$; If $A \equiv B$, then $B \equiv A$; If $A \equiv B, B \equiv C$, then $A \equiv C$

QUESTION: How to prove $A \equiv B$?

Review: Logical Equivalence (Predicate)

DEFINITION: Two WFFs A,B are **logically equivalent**等值 if they always have the same truth value in every interpretation.

- notation: $A \equiv B$; example: $\forall x\, P(x) \wedge \forall x\, Q(x) \equiv \forall x\, (P(x) \wedge Q(x))$

THEOREM: $A \equiv B$ iff $A \leftrightarrow B$ is logically valid.

- $A \equiv B$
- iff A, B have the same truth value in every interpretation I
- iff $A \leftrightarrow B$ is true in every interpretation I
- iff $A \leftrightarrow B$ is logically valid

THEOREM: $A \equiv B$ iff $A \rightarrow B$ and $B \rightarrow A$ are both logically valid.

- $A \leftrightarrow B \equiv (A \rightarrow B) \wedge (B \rightarrow A)$

Review: Tautological Implications (Proposition)

DEFINITION: Let A and B be WFFs in propositional variables $p_1, \ldots, p_n$.

- A **tautologically implies** (重言蕴涵) B if every truth assignment that causes A to be true causes B to be true.
  - Notation: $A \Rightarrow B$, called a **tautological implication**
  - $A^{-1}(\mathbf{T}) \subseteq B^{-1}(\mathbf{T})$; $B^{-1}(\mathbf{F}) \subseteq A^{-1}(\mathbf{F})$

THEOREM: $A \Rightarrow B$ iff $A \rightarrow B$ is a tautology.

- $A \Rightarrow B$ iff $A^{-1}(\mathbf{T}) \subseteq B^{-1}(\mathbf{T})$ iff $A \rightarrow B$ is a tautology

THEOREM: $A \Rightarrow B$ iff $A \wedge \neg B$ is a contradiction.

- $A \rightarrow B \equiv \neg A \vee B \equiv \neg(A \wedge \neg B)$

Proving $\boldsymbol{A} \Rightarrow \boldsymbol{B}$: (1) $A^{-1}(\mathbf{T}) \subseteq B^{-1}(\mathbf{T})$; (2) $B^{-1}(\mathbf{F}) \subseteq A^{-1}(\mathbf{F})$;

(3) $A \rightarrow B$ is a tautology; (4) $A \wedge \neg B$ is a contradiction

Tautological Implication (Predicate)

DEFINITION: Let A and B be WFFs in predicate logic. A **tautologically implies**(重言蕴涵) B if every interpretation that causes A to be true causes B to be true.

- notation: $A \Rightarrow B$, called a **tautological implication**(重言蕴涵)

THEOREM: $A \Rightarrow B$ iff $A \to B$ is logically valid.

- $A \Rightarrow B$
- iff every interpretation that causes A to be true causes B to be true
- iff there is no interpretation such that $(A, B) = (\mathbf{T}, \mathbf{F})$
- Iff $A \to B$ is true in every interpretation
- iff $A \to B$ is logically valid

THEOREM: $A \Rightarrow B$ iff $A \wedge \neg B$ is unsatisfiable.

- $A \to B \equiv \neg A \vee B \equiv \neg(A \wedge \neg B)$

Rule of Substitution

Name	Tautological Implication	NO.
Conjunction(合取)	$(P) \wedge (Q) \Rightarrow P \wedge Q$	1
Simplification(化简)	$P \wedge Q \Rightarrow P$	2
Addition(附加)	$P \Rightarrow P \vee Q$	3
Modus ponens(假言推理)	$P \wedge (P \rightarrow Q) \Rightarrow Q$	4
Modus tollens(拒取)	$\neg Q \wedge (P \rightarrow Q) \Rightarrow \neg P$	5
Disjunctive syllogism(析取三段论)	$\neg P \wedge (P \vee Q) \Rightarrow Q$	6
Hypothetical syllogism(假言三段论)	$(P \rightarrow Q) \wedge (Q \rightarrow R) \Rightarrow (P \rightarrow R)$	7
Resolution (归结)	$(P \vee Q) \wedge (\neg P \vee R) \Rightarrow Q \vee R$	8

EXAMPLE: $P \wedge (P \rightarrow Q) \Rightarrow Q$ is a TI in propositional logic.

- $A(x) \wedge \big(A(x) \rightarrow B(y)\big) \Rightarrow B(y)$ must be a TI in predicate logic.
  - Rule of substitution: let $P = A(x)$ and $Q = B(y)$

Tautological Implications

- $\forall x P(x) \vee \forall x\, Q(x) \Rightarrow \forall x\, (P(x) \vee Q(x))$
- $\exists x \big(P(x) \wedge Q(x)\big) \Rightarrow \exists x\, P(x) \wedge \exists x Q(x)$
- $\forall x\, \big(P(x) \to Q(x)\big) \Rightarrow \forall x P(x) \to \forall x\, Q(x)$
- $\forall x\, \big(P(x) \to Q(x)\big) \Rightarrow \exists x\, P(x) \to \exists x\, Q(x)$
- $\forall x\, \big(P(x) \leftrightarrow Q(x)\big) \Rightarrow \forall x\, P(x) \leftrightarrow \forall x\, Q(x)$
- $\forall x\, \big(P(x) \leftrightarrow Q(x)\big) \Rightarrow \exists x\, P(x) \leftrightarrow \exists x\, Q(x)$
- $\forall x\, \big(P(x) \to Q(x)\big) \wedge \forall x\, \big(Q(x) \to R(x)\big) \Rightarrow \forall x\, (P(x) \to R(x))$
- $\forall x\, \big(P(x) \to Q(x)\big) \wedge P(a) \Rightarrow Q(a)$

Examples

EXAMPLE: $\underline{\forall x\left(P(x) \rightarrow Q(x)\right) \wedge P(a)} \Rightarrow \underline{Q(a)}$

- Suppose that the left hand side is true in an interpretation I (domain=D)
  - $\forall x\left(P(x) \rightarrow Q(x)\right)$ is $\mathbf{T}$ and $P(a)$ is $\mathbf{T}$
    - $P(a) \rightarrow Q(a)$ is $\mathbf{T}$ and $P(a)$ is $\mathbf{T}$
      - $Q(a)$ is $\mathbf{T}$ in I.

EXAMPLE: Tautological implication in the following proof?

- All rational numbers are real numbers $\boxed{\forall x\left(P(x) \rightarrow Q(x)\right)}$
- 1/3 is a rational number $\boxed{P(1/3)}$
- 1/3 is a real number $\boxed{Q(1/3)}$
  - $P(x)$ =“x is a rational number”
  - $Q(x)$ =“x is a real number”
  - rule of inference: $\forall x\left(P(x) \rightarrow Q(x)\right) \wedge P(1/3) \Rightarrow Q(1/3)$

Examples

EXAMPLE: $\underline{\forall x\,(P(x) \to Q(x)) \wedge \forall x\,(Q(x) \to R(x))} \Rightarrow \underline{\forall x\,(P(x) \to R(x))}$

- Suppose that the left hand side is $\mathbf{T}$ in an interpretation I (domain=D)
  - $\forall x\,(P(x) \to Q(x))$ is $\mathbf{T}$ and $\forall x\,(Q(x) \to R(x))$ is $\mathbf{T}$
  - $P(x) \to Q(x)$ is $\mathbf{T}$ for all $x \in D$ and $Q(x) \to R(x)$ is $\mathbf{T}$ for all $x \in D$
    - $P(x) \to R(x)$ is $\mathbf{T}$ for all $x \in D$
    - $\forall x\,(P(x) \to R(x))$ is $\mathbf{T}$ in I.

EXAMPLE: Tautological implication in the following proof?

- All integers are rational numbers. $\boxed{\forall x(P(x) \to Q(x))}$
- All rational numbers are real numbers. $\boxed{\forall x(Q(x) \to R(x))}$
- All integers are real numbers. $\boxed{\forall x(P(x) \to R(x))}$
  - $P(x) =$"x is an integer"
  - $Q(x) =$"x is a rational number"
  - $R(x) =$"x is a real number"
  - rule of inference: $\forall x\,(P(x) \to Q(x)) \wedge \forall x\,(Q(x) \to R(x)) \Rightarrow \forall x\,(P(x) \to R(x))$

Building Arguments

QUESTION: Given the premises $P_1, \ldots, P_n$, show a conclusion Q, that is, show that $P_1 \wedge \cdots \wedge P_n \Rightarrow Q$.

Name	Operations
Premise	Introduce the <u>given formulas</u> $P_1, \ldots, P_n$ in the process of constructing proofs.
Conclusion	Quote the <u>intermediate formula</u> that have been deducted.
Rule of replacement	Replace a formula with a <u>logically equivalent</u> formula.
Rules of Inference	Deduct a new formula with a <u>tautological implication</u>.
Rule of substitution	Deduct a formula from a <u>tautology</u>.

Rules of Inference for ∀, ∃

Name	Rules of Inference	NO.
Universal Instantiation 全称量词消去	$\forall\, x\, P(x) \Rightarrow P(a)$ a is any individual in the domain of x	1
Universal Generalization 全称量词引入	$P(a) \Rightarrow \forall x\, P(x)$ a takes any individual in the domain of x	2
Existential Instantiation 存在量词消去	$\exists\, x\, P(x) \Rightarrow P(a)$ a is a specific individual in the domain of x	3
Existential Generalization 存在量词引入	$P(a) \Rightarrow \exists x\, P(x)$ a is a specific individual in the domain of x	4

Building Arguments

EXAMPLE: Show that the following premises 1, 2 lead to conclusion 3.

1. "A student in this class has not read the book," $\exists x\big(C(x) \land \neg B(x)\big)$
2. "Everyone in this class passed the exam," $\forall x(C(x) \rightarrow P(x))$
3. "Someone who passed the exam has not read the book." $\exists x(P(x) \land \neg B(x))$

- **Translate the premises and the conclusion into formulas.**
  - $C(x)$: "x is in the class"; $B(x)$: "x has read the book"; $P(x)$: "x passed the exam"
- **?** $\exists x\big(C(x) \land \neg B(x)\big) \land \forall x(C(x) \rightarrow P(x)) \Rightarrow \exists x(P(x) \land \neg B(x))$

(1)	$\exists x\big(C(x) \land \neg B(x)\big)$	Premise
(2)	$C(a) \land \neg B(a)$	Existential instantiation from (1)
(3)	$C(a)$	Simplification from (2)
(4)	$\forall x(C(x) \rightarrow P(x))$	Premise
(5)	$C(a) \rightarrow P(a)$	Universal instantiation from (4)
(6)	$P(a)$	Modus ponens from (3) and (5)
(7)	$\neg B(a)$	Simplification from (2)
(8)	$P(a) \land \neg B(a)$	Conjunction from (6) and (7)
(9)	$\exists x(P(x) \land \neg B(x))$	Existential generalization from (8)