Discrete Mathematics: Lecture 21

predicate logic, WFFs, from NL to WFFs, logic equivalence, tautological implication

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Well-Formed Formulas

DEFINIITON: well-formed formulas合式公式/formulas

1) propositional constants, propositional variables, and propositional functions without connectives are WFFs
2) If A is a WFF, then $\neg A$ is also a WFF
3) If A, B are WFFs and there is no individual variable x which is bound in one of A, B but free in the other, then $(A \land B), (A \lor B), (A \rightarrow B), (A \leftrightarrow B)$ are WFFs.
4) If A is a WFF with a free individual variable x, then $\forall x\ A, \exists x\ A$ are WFFs.
5) WFFs can be constructed with 1)-4).
   - Example: $\forall x\ F(x) \lor G(x), \forall x P(y)$ are not WFFs
   - Example: $\exists x\ (A(x) \rightarrow \forall y\ B(x, y))$ is a WFF

Precedence: $\forall, \exists$ have higher precedence than $\neg, \land, \lor, \rightarrow, \leftrightarrow$

- $\forall x\ P(x) \rightarrow Q(y)$ means $\left(\forall x P(x)\right) \rightarrow Q(y)$, not $\forall x(P(x) \rightarrow Q(y))$

From Natural Language to WFFs

The Method of Translation:

- Introduce symbols to represent propositional constants, propositional variables, individual constants, individual variables, predicate constants, predicate variables, functions of individuals
- Construct WFFs with 1)-4) such that WFFs reflect the real meaning of the natural language

EXAMPLE: All irrational numbers are real numbers.

- Every irrational number is a real number.
- For every x, if x is an irrational number, then x is a real number.
  - $I(x) =$"x is an irrational number"
  - $R(x) =$"x is a real number"
    - Translation: $\forall x\ (I(x) \rightarrow R(x))$

From Natural Language to WFFs

EXAMPLE: Some real numbers are irrational numbers.

- There is a real number which is also an irrational number.
- There is an x such that x is a real number and also an irrational number.
    - $I(x) =$"x is an irrational number"
    - $R(x) =$"x is a real number"
        - Translation: $\exists x\ (R(x) \land I(x))$

EXAMPLE: There is a symbol that can not be understood by any person's brain.

- There is a symbol such that any person's brain can not understand it.
- There is an x such that x is a symbol and any person's brain can not understand x.
    - $S(x)$: "x is a symbol"
    - Translation: $\exists x\ (S(x) \land (\cdots))$

From Natural Language to WFFs

EXAMPLE: There is a symbol that can not be understood by any person's brain.

- Any person's brain can not understand x.
- For any y, if y is a person, then y's brain cannot understand x.
  - $P(y)$: "y is a person"
  - Translation: $\forall y\ (P(y) \rightarrow (\cdots))$
- y's brain cannot understand x
  - $U(z, x)$: "z can understand x"
  - $b(y)$ =the brain of y
    - Translation: $\neg U(b(y), x)$
- Translation: $\exists x\ \Big(S(x) \wedge \forall y\ \big(P(y) \rightarrow \neg U(b(y), x)\big)\Big)$

Interpretation

DEFINITION: an **interpretation**解释 requires one to (remove all uncertainty)

- assign a concrete proposition to every **proposition variable**
- assign a concrete predicate to every **predicate variable**
- restrict the domain of every **bound individual variable**
- assign a concrete individual to every **free individual variable**
- choose a concrete **function**, if there is any

EXAMPLE: $\exists x P(x) \rightarrow q$

- Domain of x={Alice, Bob, Eve}
- $P(x)$ =“x gets A+”
- q =“I get A+”
- If at least one of Alice, Bob, and Eve gets A+, then I get A+.

Types of WFFs

DEFINITION: A WFF is **logically valid**普遍有效 if it is **T** in every interpretation

- $\forall x\,(P(x) \vee \neg P(x))$ is logically valid

DEFINITION: A WFF is **unsatisfiable**不可满足 if it is **F** in every interpretation

- $\exists x\,(P(x) \wedge \neg P(x))$ is unsatisfiable

DEFINITION: A WFF is **satisfiable**可满足 if it is **T** in some interpretation

- $\forall x\,(x^2 > 0)$
  - true when domain= nonzero real numbers

THEOREM: Let A be any WFF. A is logically valid iff $\neg A$ is unsatisfiable.

<u>**Rule of Substitution**</u>: Let A be a tautology in propositional logic. If we substitute any propositional variable in A with an arbitrary WFF from predicate logic, then we get a logically valid WFF.

- $p \vee \neg p$ is a tautology; hence, $P(x) \vee \neg P(x)$ is logically valid

Logical Equivalence

DEFINITION: Two WFFs A,B are **logically equivalent**等值 if they always have the same truth value in every interpretation.

- notation: $A \equiv B$; example: $\forall x\, P(x) \wedge \forall x\, Q(x) \equiv \forall x\, (P(x) \wedge Q(x))$

THEOREM: $A \equiv B$ iff $A \leftrightarrow B$ is logically valid.

- $A \equiv B$
- iff A, B have the same truth value in every interpretation I
- iff $A \leftrightarrow B$ is true in every interpretation I
- iff $A \leftrightarrow B$ is logically valid

THEOREM: $A \equiv B$ iff $A \rightarrow B$ and $B \rightarrow A$ are both logically valid.

- $A \leftrightarrow B \equiv (A \rightarrow B) \wedge (B \rightarrow A)$

Rule of Substitution

METHOD*:* Applying the rule of substitution to the logical equivalences in propositional logic, we get logical equivalences in predicate logic.

$P \vee Q \equiv Q \vee P$	$A(x) \vee B(y) \equiv B(y) \vee A(x)$
$(P \wedge Q) \wedge R \equiv P \wedge (Q \wedge R)$	$\big(A(x) \wedge B(y)\big) \wedge c \equiv A(x) \wedge (B(y) \wedge c)$
$P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$	$A(x) \wedge (B(y) \vee c) \equiv \big(A(x) \wedge B(y)\big) \vee (A(x) \wedge c)$
$P \wedge (P \vee Q) \equiv P$	$A(x) \wedge \big(A(x) \vee B(y)\big) \equiv A(x)$
$\neg(P \wedge Q) \equiv \neg P \vee \neg Q$	$\neg\big(A(x) \wedge B(y)\big) \equiv \neg A(x) \vee \neg B(y)$
$P \rightarrow Q \equiv \neg P \vee Q$	$A(x) \rightarrow (\forall y\, B(y)) \equiv \neg A(x) \vee \big(\forall y\, B(y)\big)$
$P \leftrightarrow Q \equiv (P \rightarrow Q) \wedge (Q \rightarrow P)$	$A(x) \leftrightarrow c \equiv (A(x) \rightarrow c) \wedge (c \rightarrow A(x))$

De Morgan's Laws for Quantifiers

THEOREM: $\neg\forall x\ P(x) \equiv \exists x\ \neg P(x)$

- Show that $\neg\forall x\ P(x) \rightarrow \exists x\ \neg P(x)$ is logically valid
  - Suppose that $\neg\forall x\ P(x)$ is $\mathbf{T}$ in an interpretation I
  - $\forall x\ P(x)$ is $\mathbf{F}$ in I
  - There is an x_0 such that $P(x_0)$ is $\mathbf{F}$ in I
  - There is an x_0 such that $\neg P(x_0)$ is $\mathbf{T}$ in I
  - $\exists x\ \neg P(x)$ is $\mathbf{T}$ in I
- Show that $\exists x\ \neg P(x) \rightarrow \neg\forall x\ P(x)$ is logically valid
  - Suppose that $\exists x \neg P(x)$ is $\mathbf{T}$ in an interpretation I
  - There is an x_0 such that $\neg P(x_0)$ is $\mathbf{T}$ in I
  - There is an x_0 such that $P(x_0)$ is $\mathbf{F}$ in I
  - $\forall x P(x)$ is $\mathbf{F}$ in I
  - $\neg\forall x P(x)$ is $\mathbf{T}$ in I

THEOREM: $\neg\exists x\ P(x) \equiv \forall x\ \neg P(x)$.

De Morgan's Laws for Quantifiers

EXAMPLE: $R(x)$:"x is a real number"; $Q(x)$:"x is a rational number"

- $\neg\forall x\ (R(x) \rightarrow Q(x))$
  - Not all real numbers are rational numbers
- Negation: $\exists x\ \neg\big(R(x) \rightarrow Q(x)\big) \equiv \exists x\ (R(x) \wedge \neg Q(x))$
  - There is a real number which is not rational

EXAMPLE: Let the domain be the set of all real numbers. Let $Q(x)$:"x is a rational number" and $I(x)$:"x is an irrational number"

- $\neg\exists x\ (Q(x) \wedge I\ (x))$
  - No real number is both rational and irrational.
- Negation: $\forall x\ \neg\big(Q(x) \wedge I(x)\big) \equiv \forall x\ (\neg Q(x) \vee \neg I(x))$
  - Any real number is either not rational or not irrational.

Distributive Laws for Quantifiers

THEOREM: $\forall x\, (P(x) \wedge Q(x)) \equiv \forall x\, P(x) \wedge \forall x\, Q(x)$

- <u>Show that $\forall x\, (P(x) \wedge Q(x)) \rightarrow \forall x\, P(x) \wedge \forall x\, Q(x)$ is logically valid</u>
  - Suppose that $\forall x\, (P(x) \wedge Q(x))$ is $\mathbf{T}$ in an interpretation I
  - $P(x) \wedge Q(x)$ is $\mathbf{T}$ for every x in I
  - $P(x)$ is $\mathbf{T}$ for every x in I and $Q(x)$ is $\mathbf{T}$ for every x in I
  - $\forall x\, P(x)$ is $\mathbf{T}$ in I and $\forall x\, Q(x)$ is $\mathbf{T}$ in I
  - $\forall x\, P(x) \wedge \forall x\, Q(x)$ is $\mathbf{T}$ in I
- <u>Show that $\forall x\, P(x) \wedge \forall x\, Q(x) \rightarrow \forall x\, (P(x) \wedge Q(x))$ is logically valid.</u>
  - Suppose that $\forall x\, P(x) \wedge \forall x\, Q(x)$ is $\mathbf{T}$ in an interpretation I
  - $\forall x\, P(x)$ is $\mathbf{T}$ in I and $\forall x\, Q(x)$ is $\mathbf{T}$ in I
  - $P(x)$ is $\mathbf{T}$ for every x in I and $Q(x)$ is $\mathbf{T}$ for every x in I
  - $P(x) \wedge Q(x)$ is $\mathbf{T}$ for every x in I
  - $\forall x\, (P(x) \wedge Q(x))$ is $\mathbf{T}$ in I

THEOREM: $\exists x\, (P(x) \vee Q(x)) \equiv \exists x\, P(x) \vee \exists x\, Q(x)$.

Tautological Implication

DEFINITION: Let A and B be WFFs in predicate logic. A **tautologically implies**(重言蕴涵) B if every interpretation that causes A to be true causes B to be true.

- notation: $A \Rightarrow B$, called a **tautological implication**(重言蕴涵)

THEOREM: $A \Rightarrow B$ iff $A \rightarrow B$ is logically valid.

- $A \Rightarrow B$
- iff every interpretation that causes A to be true causes B to be true
- iff there is no interpretation such that $(A, B) = (\mathbf{T}, \mathbf{F})$
- Iff $A \rightarrow B$ is true in every interpretation
- iff $A \rightarrow B$ is logically valid

THEOREM: $A \Rightarrow B$ iff $A \wedge \neg B$ is unsatisfiable.

- $A \rightarrow B \equiv \neg A \vee B \equiv \neg(A \wedge \neg B)$

Rule of Substitution

Name	Tautological Implication	NO.
Conjunction(合取)	$(P) \wedge (Q) \Rightarrow P \wedge Q$	1
Simplification(化简)	$P \wedge Q \Rightarrow P$	2
Addition(附加)	$P \Rightarrow P \vee Q$	3
Modus ponens(假言推理)	$P \wedge (P \rightarrow Q) \Rightarrow Q$	4
Modus tollens(拒取)	$\neg Q \wedge (P \rightarrow Q) \Rightarrow \neg P$	5
Disjunctive syllogism(析取三段论)	$\neg P \wedge (P \vee Q) \Rightarrow Q$	6
Hypothetical syllogism(假言三段论)	$(P \rightarrow Q) \wedge (Q \rightarrow R) \Rightarrow (P \rightarrow R)$	7
Resolution (归结)	$(P \vee Q) \wedge (\neg P \vee R) \Rightarrow Q \vee R$	8

EXAMPLE: $P \wedge (P \rightarrow Q) \Rightarrow Q$ is a TI in propositional logic.

- $A(x) \wedge \left(A(x) \rightarrow B(y)\right) \Rightarrow B(y)$ must be a TI in predicate logic.
  - Rule of substitution: let $P = A(x)$ and $Q = B(y)$

Tautological Implications

- $\forall x P(x) \lor \forall x\, Q(x) \Rightarrow \forall x\, (P(x) \lor Q(x))$
- $\exists x\big(P(x) \land Q(x)\big) \Rightarrow \exists x\, P(x) \land \exists x Q(x)$
- $\forall x\, \big(P(x) \to Q(x)\big) \Rightarrow \forall x P(x) \to \forall x\, Q(x)$
- $\forall x\, \big(P(x) \to Q(x)\big) \Rightarrow \exists x\, P(x) \to \exists x\, Q(x)$
- $\forall x\, \big(P(x) \leftrightarrow Q(x)\big) \Rightarrow \forall x\, P(x) \leftrightarrow \forall x\, Q(x)$
- $\forall x\, \big(P(x) \leftrightarrow Q(x)\big) \Rightarrow \exists x\, P(x) \leftrightarrow \exists x\, Q(x)$
- $\forall x\, \big(P(x) \to Q(x)\big) \land \forall x\, \big(Q(x) \to R(x)\big) \Rightarrow \forall x\, (P(x) \to R(x))$
- $\forall x\, \big(P(x) \to Q(x)\big) \land P(a) \Rightarrow Q(a)$

Examples

EXAMPLE: $\underline{\forall x \left(P(x) \to Q(x)\right) \land P(a)} \Rightarrow \underline{Q(a)}$

- Suppose that the left hand side is true in an interpretation I (domain=D)
  - $\forall x \left(P(x) \to Q(x)\right)$ is $\mathbf{T}$ and $P(a)$ is $\mathbf{T}$
    - $P(a) \to Q(a)$ is $\mathbf{T}$ and $P(a)$ is $\mathbf{T}$
      - $Q(a)$ is $\mathbf{T}$ in I.

EXAMPLE: Tautological implication in the following proof?

- All rational numbers are real numbers $\boxed{\forall x\left(P(x) \to Q(x)\right)}$
- 1/3 is a rational number $\boxed{P(1/3)}$
- 1/3 is a real number $\boxed{Q(1/3)}$
  - $P(x) =$"x is a rational number"
  - $Q(x) =$"x is a real number"
  - rule of inference: $\forall x \left(P(x) \to Q(x)\right) \land P(1/3) \Rightarrow Q(1/3)$

Examples

EXAMPLE: $\underline{\forall x\left(P(x) \rightarrow Q(x)\right) \wedge \forall x\left(Q(x) \rightarrow R(x)\right)} \Rightarrow \underline{\forall x\,(P(x) \rightarrow R(x))}$

- Suppose that the left hand side is $\mathbf{T}$ in an interpretation I (domain=D)
  - $\forall x\left(P(x) \rightarrow Q(x)\right)$ is $\mathbf{T}$ and $\forall x\left(Q(x) \rightarrow R(x)\right)$ is $\mathbf{T}$
  - $P(x) \rightarrow Q(x)$ is $\mathbf{T}$ for all $x \in D$ and $Q(x) \rightarrow R(x)$ is $\mathbf{T}$ for all $x \in D$
    - $P(x) \rightarrow R(x)$ is $\mathbf{T}$ for all $x \in D$
    - $\forall x\,(P(x) \rightarrow R(x))$ is $\mathbf{T}$ in I.

EXAMPLE: Tautological implication in the following proof?

- All integers are rational numbers. $\boxed{\forall x\left(P(x) \rightarrow Q(x)\right)}$
- All rational numbers are real numbers. $\boxed{\forall x\left(Q(x) \rightarrow R(x)\right)}$
- All integers are real numbers. $\boxed{\forall x\left(P(x) \rightarrow R(x)\right)}$
  - $P(x)$ =“x is an integer”
  - $Q(x)$ =“x is a rational number”
  - $R(x)$ =“x is a real number”
  - rule of inference: $\forall x\left(P(x) \rightarrow Q(x)\right) \wedge \forall x\left(Q(x) \rightarrow R(x)\right) \Rightarrow \forall x\,(P(x) \rightarrow R(x))$