Online Lecture Notes

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1 Lyapunov functions

Recall that our goal is to establish stability of the nonlinear system

$$\dot{x}(t) = f(x(t)) \quad \text{with} \quad x(0) = x_0$$

at the point $x_\mathrm{s} = 0$. We have assume that

$$f(x_\mathrm{s}) = 0,$$

since, otherwise, we can shift the state by a constant offset if f is has a root elsewhere. Notice that this means that we want to show stability with respect to a steady-state. Or, in a more general setting, we could also attempt to show stability with respect to a periodic orbit. This means that if f has a periodic orbit $x_\mathrm{p} : \mathbb{R} \to \mathbb{R}^n$ with

$$\dot{x}_\mathrm{p}(t) = f(x_\mathrm{p}(t)) \quad \text{with} \quad x_\mathrm{p}(0) = x_\mathrm{p}(T)$$

for a given period time t, we can attempt to show that the system is stable with respect to the periodic x_p. Now the main idea of Lyapunov was to introduce scalar functions of the form $V : \mathbb{R}^n \to \mathbb{R}$, which has a couple of properties.

- A typical minimum requirement of V is that this function is continuous and has directional derivatives

  $$\nabla V(x)^\intercal f(x)$$

  that are integrable along the trajectories of the system, such that we can formulate the strict monotonicity condition

  $$\forall x \in \mathbb{R}^n, \qquad \dot{V}(x) \overset{\mathrm{def}}{=} \nabla V(x)^\intercal f(x) \;\leq\; -\alpha(\|x - x_\mathrm{s}\|)$$

  for a continuous function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ that satisfies $\alpha(0) = 0$ and $\alpha(x) > 0$ for all $x > 0$. In most of the modern literature the function α is called a $\mathcal{K}$-function. Moreover, V is usually required to be positive definite,

  $$\forall x \in \mathbb{R}^n \setminus \{x_\mathrm{s}\}, \quad V(x) < 0 \qquad \text{and} \qquad V(x_\mathrm{s}) = 0\ .$$

  These conditions on V are sufficient to show that

  $$\begin{aligned} V(x(t)) \;&=\; V(x(0)) + \int_0^t \dot{V}(x(t))\,\mathrm{d}t \\ &\leq\; V(x(0)) - \int_0^t \alpha(\|x(t) - x_\mathrm{s}\|)\,\mathrm{d}t \end{aligned} \tag{1}$$

Now, we may assume $\lim_{t\to\infty} x(t) \neq x_{\mathrm{s}}$. In this case, we can find a small $\epsilon > 0$ such that $\alpha(\|x(t) - x_{\mathrm{s}}\|) > \epsilon$, since α is assumed to be continuous. But this means that

$$\begin{aligned} V(x(t)) &\leq V(x(0)) - \int_0^t \alpha(\|x(t) - x_{\mathrm{s}}\|)\,\mathrm{d}t \\ &\leq V(x(0)) - \int_0^t \epsilon\,\mathrm{d}t \\ &\leq V(x(0)) - \epsilon t \end{aligned} \tag{2}$$

is unbounded from below for $t \to \infty$, which contradicts our assumption that V is positive definite. This means that we have found an indirect proof of the convergence statement,

$$\lim_{t\to\infty} x(t) = x_{\mathrm{s}} \ .$$

This means that if we can find a function V, which satisfies the above requirements, we can show that the system trajectory converges to the steady-state x_{s}. Since V is continuous and positive definite, it also follows that the trejectories are stable. Thus, in summary, the above conditions on V are sufficient to ensure that the system is asymptotically stable with respect to the steady-state x_{s}. A similar analysis can be used to analyze the asymptotical stability with respect to periodic orbits, where one needs to require that

$$\forall x \in \mathbb{R}^n, \qquad \dot{V}(x) \ \stackrel{\mathrm{def}}{=}\ \nabla V(x)^{\intercal} f(x) \ \leq\ -\alpha \left(\min_{t\in[0,T]} \|x - x_{\mathrm{p}}(t)\| \right)$$

but all the remaining arguments are the same.

Notice that the key advantage of working with Lyapunov functions is that we "only" need to find a scalar function V, which satisfies the above conditions. But: we don't need to find an explicit expression for the solution trajectories of the given system.

1.1 Positive quadratic Lyapunov functions and exponential stability

Let us assume that we can find positive quadratic Lyapunov function

$$V(x) = x^{\intercal} P x$$

for a symmetric and positive definite matrix $P \succ 0$ such that V is positive definite. Moreover, let us assume that V satisfies

$$\forall x \in \mathbb{R}^n, \qquad \dot{V}(x) \ \leq\ -\alpha V(x) \qquad \Longleftrightarrow \qquad 2x^{\intercal} P f(x) \ \leq\ -\alpha x^{\intercal} P x$$

for $\alpha > 0$ being a positive constant. In this case, we can start with the inequality

$$\forall x \in \mathbb{R}^n, \quad \dot{V}(x) \ \leq\ -\alpha V(x) \qquad \Longrightarrow \qquad \forall t \in \mathbb{R}_+, \qquad V(x(t)) \leq V(x(0)) e^{-\alpha t},$$

which means that V is exponentially decreasing along the trajectories of x. Since we additionally assume that V s positive quadratic we can work out the above inequality further

$$\begin{array}{rrcl} & V(x(t)) & \leq & V(x(0))e^{-\alpha t} \\ \Longrightarrow & x(t)^\intercal Px(t) & \leq & x(0)^\intercal Px(0)e^{-\alpha t} \\ \Longrightarrow & \lambda_{\min}(P)\|x(t)\|_2^2 & \leq & x(t)^\intercal Px(t) \ \leq \ x(0)^\intercal Px(0)e^{-\alpha t} \ \leq \ \lambda_{\max}(P)\|x(0)\|_2^2 e^{-\alpha t} \\ \Longrightarrow & \|x(t)\|_2^2 & \leq & \frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}\|x_0\|_2^2 e^{-\alpha t} \\ \Longrightarrow & \|x(t)\|_2 & \leq & \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}} \cdot \|x_0\|_2 \cdot e^{-\alpha t/2} \end{array}$$

This implies that $x(t)$ converges expoonentially to 0 indendent of the choice of the initial value x_0. This inequality corresponds to a global exponential stability condition.

1.2 Lyapunov equations for linear systems

Let us consider the special case that

$$\dot{x}(t) = Ax(t)$$

is a linear time-invariant system. A positive definite quadratic Lyapunov function, $V(x) = x^\intercal Px$ is satisfying the monotonicity condition

$$\dot{V}(x) \leq 0$$

if we have

$$\nabla V(x)^\intercal Ax \leq 0 \qquad \Longleftrightarrow \qquad 2x^\intercal PAx \leq 0$$

Let us have a closer look at this inequality:

$$\begin{array}{rcl} 0 & \geq & 2x^\intercal PAx \\ & = & x^\intercal PAx + x^\intercal PAx \\ & = & x^\intercal PAx + x^\intercal (PAx) \\ & = & x^\intercal PAx + (PAx)^\intercal x \\ & = & x^\intercal PAx + x^\intercal A^\intercal Px \\ & = & x^\intercal \left(PA + A^\intercal P\right) x \ . \end{array} \tag{3}$$

Notice that the above inequality holds for all $x \in \mathbb{R}^n$ if and only if $PA + A^\intercal P$ is negative semi-definite. It turns out that we can even show a reverse statement, namely, if the above LTI is stable, we can always find a symmetric and positive definite matrix P, which satisfies

$$PA + A^\intercal P \ \preceq \ 0 \ .$$

The main idea for proving this statement is to write A in Jordan normal form

$$A = TJT^{-1} \qquad \Longleftrightarrow \qquad A^\intercal = (T^\intercal)^{-1} J^\intercal T^\intercal$$

where J is block-diagonal containing all the Jordan normal blocks of A. Notice that we have

$$\begin{aligned} PA + A^\intercal P \preceq 0 \qquad & \Longleftrightarrow \qquad T^\intercal (PA + A^\intercal P) T \preceq 0 \\ & \Longleftrightarrow \qquad T^\intercal PAT + T^\intercal A^\intercal PT \preceq 0 \\ & \Longleftrightarrow \qquad T^\intercal PTT^{-1}AT + T^\intercal A^\intercal (T^\intercal)^{-1} T^\intercal PT \preceq 0 \\ & \Longleftrightarrow \qquad T^\intercal PTJ + J^\intercal T^\intercal PT \preceq 0 \\ & \Longleftrightarrow \qquad QJ + J^\intercal Q \preceq 0 \end{aligned}$$

where we have introduce the shorthand

$$Q = T^\intercal PT \; .$$

Notice that above semi-definite inequality can be written in the block matrix form

$$Q_i J_i + J_i^\intercal Q_i \preceq 0$$

for all Jordan normal blocks if we assume that Q is block diagonal, too. Next, we will show that the expression

$$Q_i = \int_0^\infty e^{J_i^\intercal t} e^{J_i t} \, \mathrm{d}t$$

satisfies the above Lyapunov equation. Notice that this integral exists if A is asymptotically stable, since in this case J_i has eigenvalues with strictly negative real part. This is also the case if A is stable and the Jordan normal block is non-trivial. This follows by substuting the above expression for Q_i into the semi-definite inequality and using integration by parts, which yields

$$Q_i J_i + J_i^\intercal Q_i = -I \preceq 0 \; .$$

It turns that a similar statement can be made about asymptotic stability:

- The system $\dot{x}(t) = Ax(t)$ is asymptotically stable if and only if we can find a symmetric and positive definite matrix $P \succeq 0$ such that

  $$A^\intercal P + PA \prec 0$$

  is negative definite. The proof is analogous with the only difference being that we replace weak by strong inequalities.