Online Lecture Notes

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1 Multivariate Linear Input-Output System

In this lecture we consider systems of the form

$$\dot{x}(t) = Ax(t) + Bu(t) + b \tag{1}$$
$$y(t) = Cx(t) + d \tag{2}$$

with input (control) $u(t)$ and output (sensor data) $y(t)$.

1.1 Steady-State Relations

At the steady-state the equation

$$0 = Ax_{\mathrm{ref}} + Bu_{\mathrm{ref}} + b \tag{3}$$
$$y_{\mathrm{ref}} = Cx_{\mathrm{ref}} + d \tag{4}$$

If A is invertible, we have

$$x_{\mathrm{ref}} = -A^{-1}(Bu_{\mathrm{ref}} + b),$$

which can be substituted into the other equation in order to find

$$y_{\mathrm{ref}} = Cx_{\mathrm{ref}} + d = -CA^{-1}(Bu_{\mathrm{ref}} + b) + d = -[CA^{-1}B]u_{\mathrm{ref}} + [d - CA^{-1}b]$$

In general, we have $n_y > n_u$, which means that we might not be able to bring the system to any output. However, for the special case that $n_y = n_u$ and $CA^{-1}B$ invertible, we can, at least in principle, bring the system to any given set point y_{ref} by choosing the appropriate u_{ref}, given by

$$u_{\mathrm{ref}} = [CA^{-1}B]^{-1}[d - CA^{-1}b - y_{\mathrm{ref}}] \ .$$

1.2 Proportional Controllers

The main idea of proportional control is to introduce a feedback law of the form

$$\forall y \in \mathbb{R}^{n_y}, \qquad \mu(y) = u_{\mathrm{ref}} + K(y - y_{\mathrm{ref}}) \ .$$

Notice that in contrast to the actual control input function u the function μ is a function of the current output data y rather than of time t. This means that we feedback the control signal

$$u(t) = u_{\mathrm{ref}} + K(y(t) - y_{\mathrm{ref}})$$

This means that we have the following relations:

$$
\begin{aligned}
0 &= Ax_{\text{ref}} + Bu_{\text{ref}} + b && (5)\\
y_{\text{ref}} &= Cx_{\text{ref}} + d && (6)\\
\dot{x}(t) &= Ax(t) + Bu(t) + b && (7)\\
y(t) &= Cx(t) + d && (8)\\
u(t) &= u_{\text{ref}} + K(y(t) - y_{\text{ref}}) && (9)
\end{aligned}
$$

Thus, we have in total one differential equation and 4 algebraic relation that we need to substitute if we want to work out the closed-loop trajectory. Let us substitute these equations step-by-step:

$$
\begin{aligned}
\dot{x}(t) &\overset{(7)}{=} Ax(t) + Bu(t) + b && (10)\\
&\overset{(9)}{=} Ax(t) + B\left[u_{\text{ref}} + K(y(t) - y_{\text{ref}})\right] + b && (11)\\
&\overset{(8)}{=} Ax(t) + B\left[u_{\text{ref}} + K(Cx(t) + d - y_{\text{ref}})\right] + b && (12)\\
&\overset{(6)}{=} Ax(t) + B\left[u_{\text{ref}} + KC(x(t) - x_{\text{ref}})\right] + b && (13)\\
&\overset{(5)}{=} A(x(t) - x_{\text{ref}}) + BKC(x(t) - x_{\text{ref}}) && (14)\\
&= (A + BKC)(x(t) - x_{\text{ref}}) && (15)
\end{aligned}
$$

The matrix $A_{\text{cl}} = A + BKC$ is called the closed-loop system gain. The explicit solution for the closed-loop trajectory is given by

$$
x(t) = e^{A_{\text{cl}}t}[x(0) - x_{\text{ref}}] + x_{\text{ref}} \ .
$$

1.3 Properties of the closed-loop trajectory in dependence on K

Since the closed gain matrix $A_{\text{cl}} = A + BKC$ depends on K, we can analyze the closed-loop trajectory in dependence on K. In general, this is not entirely trivial, since the explicit solution depends on the matrix exponential

$$
e^{A_{\text{cl}}t} = e^{(A+BKC)t} \ .
$$

We know from Lecture 5 that this matrix exponential depends on the eigenvalues of the matrix $A + BKC$. In particular, we have

$$
\lim_{t\to\infty} e^{(A+BKC)t} = 0
$$

if all eigenvalues of $A+BKC$ have strictly negative real part. This follows from the fact that the Jordan normal decomposition of $A + BKC$ has the form

$$
A + BKC = T(D + N)T^{-1} \qquad \overset{\text{Lecture 5}}{\Longrightarrow} \qquad e^{(A+BKC)t} = Te^{Dt}e^{Nt}T^{-1},
$$

where e^{Nt} is a polynomial in t, which is for $t \to \infty$ overpowered by the exponential diagonal function e^{Dt} as long as all the diagonal elements of D (= to the eigenvalues of $A+BKC$) have strictly negative real part. Thus, if we manage to choose K in such a way that the eigenvalues of $A + BKC$ have strictly negative real part, then we have

$$\lim_{t\to\infty} x(t) \;=\; \lim_{t\to\infty} e^{(A+BKC)t}(x(0) - x_{\mathrm{ref}}) + x_{\mathrm{ref}} \;=\; x_{\mathrm{ref}};$$

that is, the closed trajectory converges to the reference point as desired.

1.4 Example

Let us consider the specific example

$$A = \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix} \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad C = (\ 1\ 0\ )\ .$$

This corresponds to a system with 2 states, 1 control input, and 1 measurement. The corresponding closed-loop system matrix is given by

$$A_{\mathrm{cl}} = A + BKC = A = \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} K \begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1+K & -2 \end{pmatrix}\ .$$

In order to analyze under which conditions the scalar proportional control gain $K \in \mathbb{R}^{1\times 1}$ stabilizes the system, we need to work out the eigenvalues of the matrix A_{cl}. For this aim, we first work out the roots of the characteristic polynomial

$$0 \;=\; \det(A_{\mathrm{cl}} - \lambda I) \;=\; \det\left(\begin{pmatrix} 1-\lambda & 1 \\ 1+K & -2-\lambda \end{pmatrix} \right) \tag{16}$$

$$\;=\; (1-\lambda)(-2-\lambda) - (1+K) \tag{17}$$

$$\;=\; \lambda^2 + \lambda - K - 3 \tag{18}$$

Thus, the eigenvalues are given by

$$\lambda_{1,2} = -\frac{1}{2} \pm \sqrt{\frac{1}{4} + 3 + K}\ .$$

Clearly, we have $\mathrm{Re}(\lambda_1) \geq \mathrm{Re}(\lambda_2)$. Thus, if we want to achieve that both eigenvalues have strictly negative real part, it is sufficient to ensure that $\mathrm{Re}(\lambda_1) < 0$. There are three cases

- Case 1: $K \geq -3$. In this case, we have $\mathrm{Re}(\lambda_1) = \lambda_1 \geq 0$, which means that the closed-loop system matrix is not asymptotically stabilizing.
- Case 2: $-3 - \frac{1}{4} \leq K < -3$. In this case both eigenvalues are real and we have
  $$\lambda_2 \leq -\frac{1}{2} \leq \lambda_1 < 0\ .$$
  In this case, the closed-loop system is asymptotically stable and $x(t)$ converges to its reference without any oscillations, since both eigenvalues are real.

- Case 3: $K < -3 - \frac{1}{4}$. In this case we have

$$\mathrm{Re}(\lambda_1) = \mathrm{Re}(\lambda_2) = -\frac{1}{2} \qquad \text{and} \qquad \omega = \mathrm{Im}(\lambda_1) = -\mathrm{Im}(\lambda_2) > 0 \; .$$

  This means that the closed-loop system is also asymptotically stabilizing, but it oscillates with frequency ω before converging to the reference state.