SI231B - Matrix Computations, Spring 2022-23

Homework Set #1

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Acknowledgements:

1) Deadline: **2023-03-13 23:59:59**
2) Please submit your assignments via Blackboard.
3) You can write your homework using latex/word or you can write in handwriting and submit the scanned pdf.

Problem 1. (20 points)

1) Prove that $\mathsf{rank}(\mathbf{A}+\mathbf{B}) \leq \mathsf{rank}(\mathbf{A}) + \mathsf{rank}(\mathbf{B})$. (10 points)
2) Prove that $\mathsf{rank}(\mathbf{AB}) \leq \min\{\mathsf{rank}(\mathbf{A}), \mathsf{rank}(\mathbf{B})\}$ and discuss when the equality holds. (10 points)
   (*Hint*: It suffices to show dim $\mathcal{R}(\mathbf{AB}) \leq$ dim $\mathcal{R}(\mathbf{A})$ and dim $\mathcal{R}(\mathbf{B}^T\mathbf{A}^T) \leq$ dim $\mathcal{R}(\mathbf{B}^T)$.)

Solution:

1)

$$\begin{aligned}\mathsf{rank}(\mathbf{A}) + \mathsf{rank}(\mathbf{B}) &= \mathsf{rank}\left(\begin{bmatrix}\mathbf{A} & \mathbf{0}\\ \mathbf{0} & \mathbf{B}\end{bmatrix}\right) = \mathsf{rank}\left(\begin{bmatrix}\mathbf{A} & \mathbf{A}+\mathbf{B}\\ \mathbf{0} & \mathbf{B}\end{bmatrix}\right)\\ &\geq \mathsf{rank}\left(\begin{bmatrix}\mathbf{A}+\mathbf{B}\\ \mathbf{B}\end{bmatrix}\right) \geq \mathsf{rank}(\mathbf{A}+\mathbf{B}).\end{aligned}$$

2) Since the columns of matrix $\mathbf{AB}$ can be seen as the linear combinations of the columns in $\mathbf{A}$, we can have $\mathsf{rank}(\mathbf{AB}) \leq \mathsf{rank}(\mathbf{A})$. And since the rows of matrix $\mathbf{AB}$ can be seen as the linear combinations of the rows in $\mathbf{B}$, we can have $\mathsf{rank}(\mathbf{AB}) \leq \mathsf{rank}(\mathbf{B})$. Hence $\mathsf{rank}(\mathbf{AB}) \leq \min\{\mathsf{rank}(\mathbf{A}), \mathsf{rank}(\mathbf{B})\}$. Moreover, if $\mathbf{A}$ has full column rank, we have $\mathsf{rank}(\mathbf{AB}) = \mathsf{rank}(\mathbf{B})$ and if $\mathbf{B}$ has full row rank, we have $\mathsf{rank}(\mathbf{AB}) = \mathsf{rank}(\mathbf{A})$. Therefore, we have $\mathsf{rank}(\mathbf{AB}) = \min\{\mathsf{rank}(\mathbf{A}), \mathsf{rank}(\mathbf{B})\}$ if the columns of $\mathbf{A}$ are linearly independent or the rows of $\mathbf{B}$ are linearly independent.

Problem 2. (20 points)

From the slides, we know that the direct sum of two subspaces $\mathcal{S}_1$ and $\mathcal{S}_2$ is defined as $\mathcal{S}_3 = \mathcal{S}_1 \oplus \mathcal{S}_2$ if $\mathcal{S}_1 + \mathcal{S}_2 = \mathcal{S}_3$ and $\mathcal{S}_1 \cap \mathcal{S}_2 = \{\mathbf{0}\}$.

1) Suppose $\mathbf{a}_1 + \mathbf{a}_2 = \mathbf{0}$, $\mathbf{a}_1 \in \mathcal{S}_1$, $\mathbf{a}_2 \in \mathcal{S}_2$. Prove that $\mathcal{S}_3 = \mathcal{S}_1 \oplus \mathcal{S}_2$ iff $\mathbf{a}_1 = \mathbf{a}_2 = \mathbf{0}$. (10 points)
2) Suppose $\mathbf{a} = \mathbf{a}_1 + \mathbf{a}_2$, $\mathbf{a} \in \mathcal{S}_1 + \mathcal{S}_2$, $\mathbf{a}_1 \in \mathcal{S}_1$, $\mathbf{a}_2 \in \mathcal{S}_2$. Prove that $\mathcal{S}_3 = \mathcal{S}_1 \oplus \mathcal{S}_2$ iff $\mathbf{a}_1$ and $\mathbf{a}_2$ are unique. (10 points)

   (*Hint*: You can use the conclusion in 1).)

Solution:

1) $\Rightarrow$

   Suppose $\mathbf{a}_1 + \mathbf{a}_2 = \mathbf{0}$, then $\mathbf{a}_1 = -\mathbf{a}_2 \in \mathcal{S}_1 \cap \mathcal{S}_2$.

   Since $\mathcal{S}_3 = \mathcal{S}_1 \oplus \mathcal{S}_2$, we have $\mathcal{S}_1 \cap \mathcal{S}_2 = \{\mathbf{0}\}$.

   This implies $\mathbf{a}_1 = \mathbf{a}_2 = \mathbf{0}$.

   $\Leftarrow$

   For any $\mathbf{a} \in \mathcal{S}_1 \cap \mathcal{S}_2$, $\mathbf{0} = \mathbf{a} + (-\mathbf{a})$, $\mathbf{a} \in \mathcal{S}_1$, $-\mathbf{a} \in \mathcal{S}_2$.

   Thus $\mathbf{a} = -\mathbf{a} = \mathbf{0}$.

2) $\Rightarrow$

   Suppose $\mathbf{a} \in \mathcal{S}_1 + \mathcal{S}_2$, $\mathbf{a} = \mathbf{a}_1 + \mathbf{a}_2 = \mathbf{b}_1 + \mathbf{b}_2$, where $\mathbf{a}_i, \mathbf{b}_i \in \mathcal{S}_i$, $i = 1, 2$.

   We have $(\mathbf{a}_1 - \mathbf{b}_1) + (\mathbf{a}_2 - \mathbf{b}_2) = \mathbf{0}$.

   Since $\mathcal{S}_3 = \mathcal{S}_1 \oplus \mathcal{S}_2$, from 1) we can conclude that $\mathbf{a}_1 - \mathbf{b}_1 = \mathbf{0}$ and $\mathbf{a}_2 - \mathbf{b}_2 = \mathbf{0}$, which implies $\mathbf{a}_1$ and $\mathbf{a}_2$ are unique.

   $\Leftarrow$

   Suppose $\mathbf{0} = \mathbf{a}_1 + \mathbf{a}_2$, where $\mathbf{a}_1 \in \mathcal{S}_1$, $\mathbf{a}_2 \in \mathcal{S}_2$.

   If $\mathbf{a}_1$ and $\mathbf{a}_2$ are unique, then $\mathbf{a}_1 = \mathbf{a}_2 = \mathbf{0}$.

   From 1) we can conclude that $\mathcal{S}_3 = \mathcal{S}_1 \oplus \mathcal{S}_2$.

Problem 3. (20 points)

Given a matrix as follows:

$$\mathbf{A} = \begin{bmatrix} -1 & 3 & 3 & 2 \\ -1 & 4 & 7 & 10 \\ 2 & -4 & -1 & 5 \end{bmatrix}.$$

derive its LU decomposition.

Solution:

$$\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -2 & 2 & 1 \end{bmatrix}$$

$$\mathbf{U} = \begin{bmatrix} -1 & 3 & 3 & 2 \\ 0 & 1 & 4 & 8 \\ 0 & 0 & -3 & -7 \end{bmatrix}$$

Problem 4. (20 points)

Given a matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$, suppose that the LDM (LDU) decomposition of $\mathbf{A}$ exists, prove that

1) The LDM (LDU) decomposition of $\mathbf{A}$ is *uniquely* determined; (10 points)
2) If $\mathbf{A}$ is a symmetric matrix, then its LDM (LDU) decomposition must be $\mathbf{A} = \mathbf{L}\mathbf{D}\mathbf{L}^T$, which is called LDL (LDL^T) decomposition in this case. (10 points)

(*Hints*: The existence of the LDM (LDU) decomposition implies the non-singularity of the matrix.)

Solution:

1) We could assume that A has two LDM decompositions, which is $\mathbf{A} = \mathbf{L}_1\mathbf{D}_1\mathbf{M}_1^T = \mathbf{L}_2\mathbf{D}_2\mathbf{M}_2^T$, our goal is to prove that $\mathbf{L}_1 = \mathbf{L}_2$, $\mathbf{D}_1 = \mathbf{D}_2$ and $\mathbf{M}_1 = \mathbf{M}_2$.

   According to the problem, the existence of the LDM decomposition implies that $\mathbf{A}$ is nonsingular, i.e. $\det(\mathbf{A}) \neq 0$. Since $\det(\mathbf{A}) = \det(\mathbf{L}) \cdot \det(\mathbf{D}) \cdot \det(\mathbf{M}^T)$, we geet that $\mathbf{L}_1, \mathbf{D}_1, \mathbf{M}_1, \mathbf{L}_2, \mathbf{D}_2$ and $\mathbf{M}_2$ are all nonsingular.

   Since $\mathbf{L}_1\mathbf{D}_1\mathbf{M}_1^T = \mathbf{L}_2\mathbf{D}_2\mathbf{M}_2^T$, we have

$$\mathbf{L}_2^{-1}\mathbf{L}_1 = \mathbf{D}_2\mathbf{M}_2^T(\mathbf{M}_1^T)^{-1}\mathbf{D}_1^{-1}$$

   As we can see, the left hand side $\mathbf{L}_2^{-1}\mathbf{L}_1$ is a lower triangular matrix, and the r.h.s $\mathbf{D}_2\mathbf{M}_2^T(\mathbf{M}_1^T)^{-1}\mathbf{D}_1^{-1}$ is an upper triangular matrix. Thus, they are all diagonal matrices.

   Because of the diagonal entries of $\mathbf{L}_2^{-1}\mathbf{L}_1$ equal to one, $\mathbf{L}_2^{-1}\mathbf{L}_1 = \mathbf{I}$, and then $\mathbf{L}_1 = \mathbf{L}_2$. Similarly, we could derive that $\mathbf{M}_1 = \mathbf{M}_2$. After that, we could get $\mathbf{D}_1 = \mathbf{L}_1^{-1}\mathbf{A}(\mathbf{M}_1^T)^{-1} = \mathbf{L}_2^{-1}\mathbf{A}(\mathbf{M}_2^T)^{-1} = \mathbf{D}_2$. QED.

2) Since $\mathbf{A} = \mathbf{L}\mathbf{D}\mathbf{M}^T$ and $\mathbf{A}$ is symmetric, we have $\mathbf{L}\mathbf{D}\mathbf{M}^T = \mathbf{A} = \mathbf{A}^T = \mathbf{M}\mathbf{D}\mathbf{L}^T$. From the proof in 1), we learn that $\mathbf{L}$ and $\mathbf{M}^T$ are both invertible. Then we can derive that

$$\mathbf{D}\mathbf{M}^T(\mathbf{L}^T)^{-1} = \mathbf{L}^{-1}\mathbf{M}\mathbf{D}$$

   As we know, $\mathbf{M}^T, (\mathbf{L}^T)^T$ are all upper triangular, $\mathbf{L}^{-1}, \mathbf{M}$ are both lower triangular. Then, the left hand side is upper triangular, and right hand size is lower triangular. We can find that both left and right hand side are all diagonal matrices. Besides, $\mathbf{D}$ is diagonal matrix, so $\mathbf{L}^{-1}\mathbf{M}$ is also diagonal matrix. Moreover, the diagonal entries of $\mathbf{L}^{-1}\mathbf{M}$ must be one, so we could get $\mathbf{L}^{-1}\mathbf{M} = \mathbf{I}$, i.e. $\mathbf{L} = \mathbf{M}$.

Problem 5. (20 points)

Consider matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$ in the following form,

$$\mathbf{A} = \begin{bmatrix} b_1 & c_1 & 0 & 0 & 0 & 0 \\ a_2 & b_2 & c_2 & 0 & 0 & 0 \\ 0 & a_3 & b_3 & c_3 & 0 & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & a_{n-1} & b_{n-1} & c_{n-1} \\ 0 & 0 & \cdots & 0 & a_n & b_n \end{bmatrix},$$

where a_j, b_j, and c_j are non-zero entries. The matrix in such form is known as a **Tridiagonal Matrix** in the sense that it contains three diagonals.

1) LU decomposition is particularly efficient in the case of tridiagonal matrices. Find the LU decomposition of $\mathbf{A}$ (derivation is expected) and try to complete the Algorithm 1. (15 points)

Algorithm 1: LU decomposition for tridiagonal matrices

Input : Tridiagonal matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$.

Output: LU decomposition of $\mathbf{A}$.

1 *Complete the algorithm here...*

2) Consider symmetric tridiagonal matrices

$$\mathbf{A} = \begin{bmatrix} a & a & 0 \\ a & a+b & b \\ 0 & b & b+c \end{bmatrix},$$

and give the LU decompositions of matrix $\mathbf{A}$. (5 points)

Solution:

1) Following the standard procedure of LU decomposition, we have that

$$\mathbf{M}_1 = \begin{bmatrix} 1 & & & \\ -\frac{a_2}{b_1} & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \end{bmatrix}, \quad \mathbf{M}_1\mathbf{A} = \begin{bmatrix} b_1 & c_1 & & & & \\ 0 & b_2^{(1)} & c_2 & & & \\ 0 & a_3 & b_3 & c_3 & & \\ 0 & 0 & \ddots & \ddots & \ddots & \\ \vdots & \vdots & \ddots & a_{n-1} & b_{n-1} & c_{n-1} \\ 0 & 0 & \cdots & 0 & a_n & b_n \end{bmatrix}$$

where $b_2^{(1)} = b_2 - a_2 c_1 / b_1$, and then

$$\mathbf{M}_2 = \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & -\frac{a_3}{b_2^{(1)}} & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \end{bmatrix}, \quad \mathbf{M}_1\mathbf{A} = \begin{bmatrix} b_1 & c_1 & & & & \\ 0 & b_2^{(1)} & c_2 & & & \\ 0 & 0 & b_3^{(2)} & c_3 & & \\ 0 & 0 & \ddots & \ddots & \ddots & \\ \vdots & \vdots & \ddots & a_{n-1} & b_{n-1} & c_{n-1} \\ 0 & 0 & \cdots & 0 & a_n & b_n \end{bmatrix}$$

where $b_3^{(2)} = b_3 - a_3 c_2 / b_2^{(1)}$. After $n-1$ steps, we have

$$\mathbf{A} = \begin{bmatrix} 1 & & & & \\ -\frac{a_2}{b_1} & 1 & & & \\ & -\frac{a_3}{b_2^{(1)}} & 1 & & \\ & & \ddots & \ddots & \\ & & & -\frac{a_n}{b_{n-1}^{(n-2)}} & 1 \end{bmatrix} \cdot \begin{bmatrix} b_1 & c_1 & & & & \\ 0 & b_2^{(1)} & c_2 & & & \\ 0 & 0 & b_3^{(2)} & c_3 & & \\ 0 & 0 & \ddots & \ddots & \ddots & \\ \vdots & \vdots & \ddots & & b_{n-1}^{(n-2)} & c_{n-1} \\ 0 & 0 & \cdots & 0 & 0 & b_n^{(n-1)} \end{bmatrix} \tag{1}$$

$$= \begin{bmatrix} 1 & & & & \\ \alpha_2 & 1 & & & \\ & \alpha_3 & 1 & & \\ & & \ddots & \ddots & \\ & & & \alpha_n & 1 \end{bmatrix} \cdot \begin{bmatrix} \beta_1 & c_1 & & & & \\ 0 & \beta_2 & c_2 & & & \\ 0 & 0 & \beta_3 & c_3 & & \\ 0 & 0 & \ddots & \ddots & \ddots & \\ \vdots & \vdots & \ddots & & \beta_{n-1} & c_{n-1} \\ 0 & 0 & \cdots & 0 & 0 & \beta_n \end{bmatrix} \tag{2}$$

Thus, the iteritive form is

$$\begin{aligned} &\beta_1 = b_1, \\ &\alpha_k = \frac{a_k}{\beta_{k-1}},\ k = 2, cdots, n \\ &\beta_k = b_k - \alpha_k c_{k-1},\ k = 2, \ldots, n \end{aligned}$$

The algorithm is as Algorithm 2 .

2) Follow the algorithm in (1), the LU decomposition could be directly obtained as

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} a & a & 0 \\ 0 & b & b \\ 0 & 0 & c \end{bmatrix}$$

Algorithm 2: LU decomposition for tridiagonal matrices

Input : Tridiagonal matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$.

Output: LU decomposition of $\mathbf{A}$.

```
Initialize β_1 = b_1
for k = 2 to n do
    if β_{k-1} == 0 then
        Stop the program.
    end
    α_k = α_k / β_{k-1}
    β_k = b_k − α_k c_{k-1}
end
return L, U as form (2)
```