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Lecture 3 : Feedback Control System Characteristics

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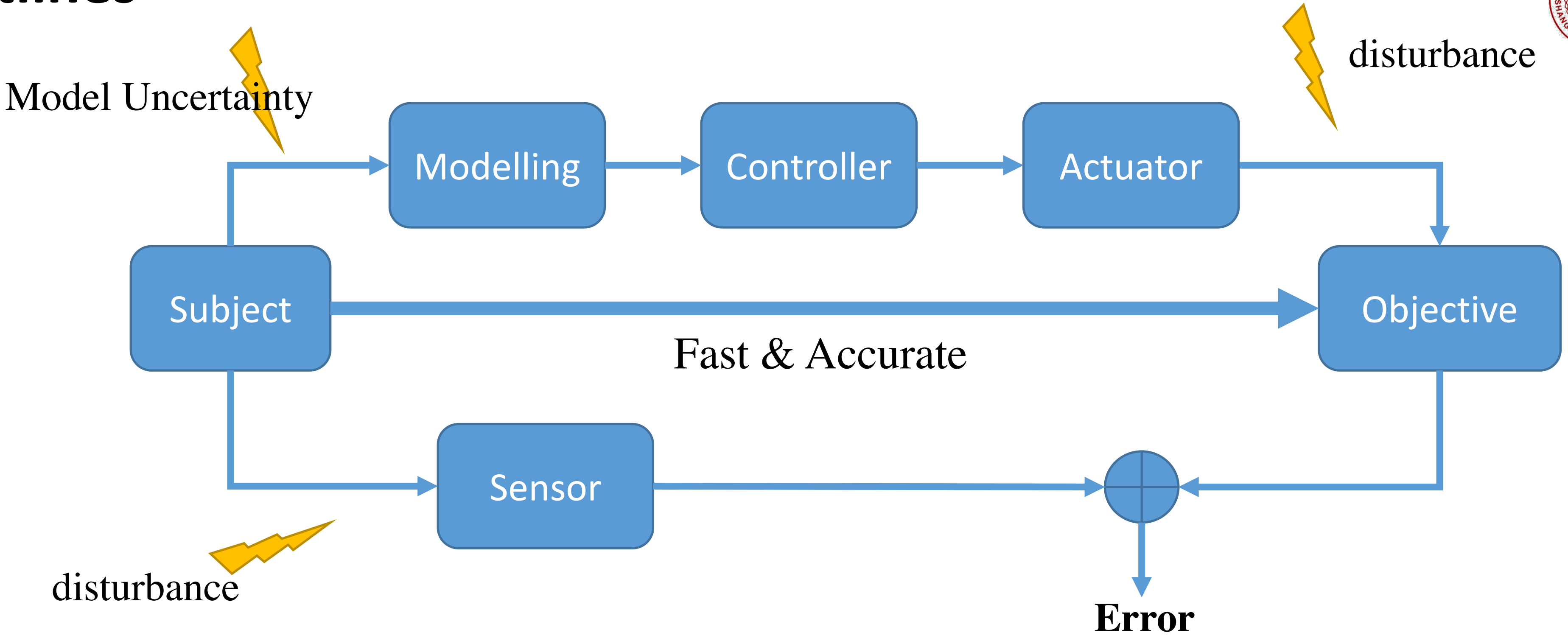
Outlines



In this chapter, we explore

- ❑ the central role of error signals in feedback control
- ❑ The effect of parameter variations, disturbance and measurement noise.
- ❑ the transient response and the steady-state response of a system.
- ❑ have a sense of feedback controller design process.

Open-loop Vs Closed-loop

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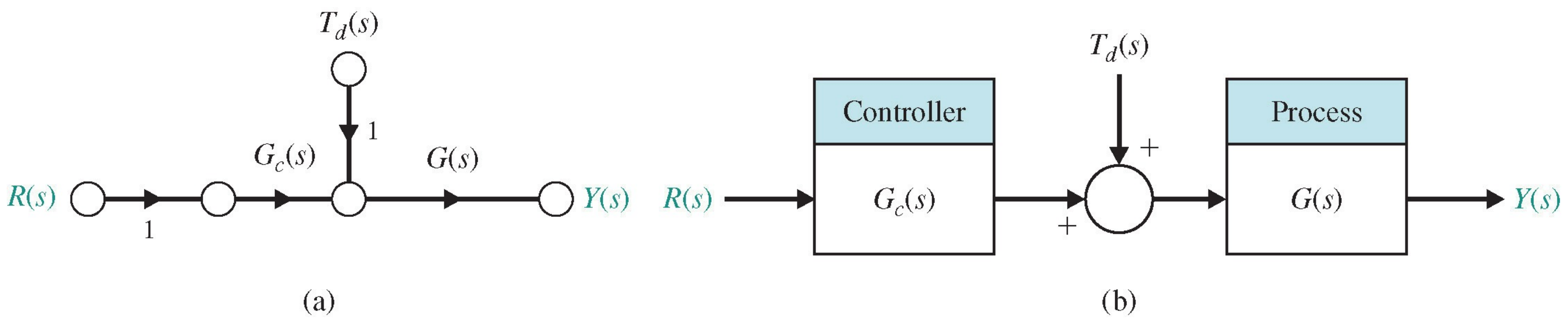
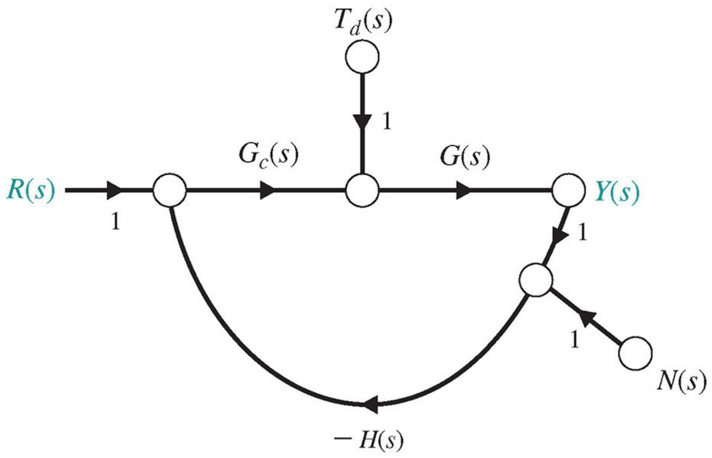
An open-loop system operates without feedback and directly generates the output in response to an input signal.



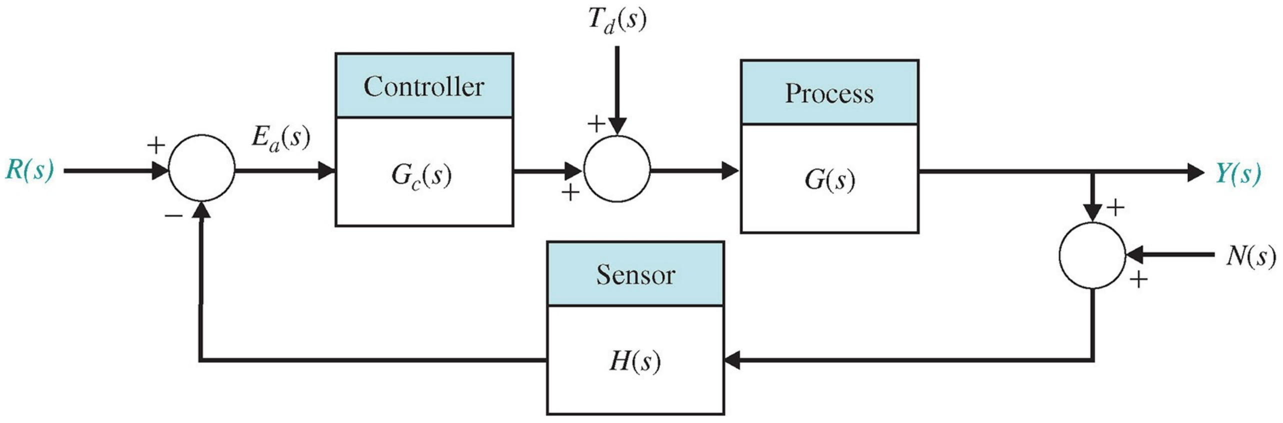
- The disturbance $T_d(s)$ directly influences the output.
- The control system is highly sensitive to disturbances and to variations in parameters of $G(s)$.

Open-loop Vs Closed-loop

A closed-loop system uses a measurement of the output signal and **a comparison** with the desired output to generate **an error signal** that is used by the controller to adjust the actuator.



(a)



(b)

Despite the cost and increased system complexity, CL FB sys has advantages:

a) Decreased sensitivity of the system to variations in the parameters of the process.
b) Improved rejection of the disturbances.
c) Improved measurement noise attenuation.
d) Improved reduction of the steady-state error of the system.
e) Easy control and adjustment of the transient response of the system.

Error Signal Analysis

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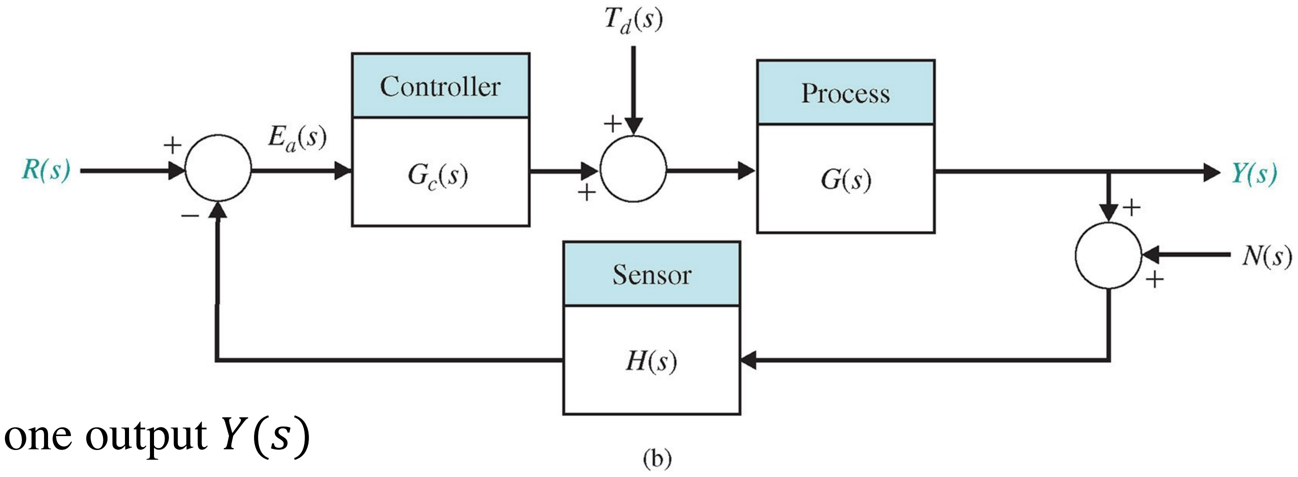
For ease of discussion, we assume

$$H(s) = 1$$

that is, a unity feedback system.



The system has three inputs $R(s), T_d(s), N(s)$ and one output $Y(s)$

$$Y(s) = \frac{G_c(s)G(s)}{1 + G_c(s)G(s)}R(s) + \frac{G(s)}{1 + G_c(s)G(s)}T_d(s) - \frac{G_c(s)G(s)}{1 + G_c(s)G(s)}N(s).$$

Define the tracking error as

$$E(s) = R(s) - Y(s)$$

Then we have

$$E(s) = \frac{1}{1 + G_c(s)G(s)}R(s) - \frac{G(s)}{1 + G_c(s)G(s)}T_d(s) + \frac{G_c(s)G(s)}{1 + G_c(s)G(s)}N(s).$$

Error Signal Analysis

$$E(s) = \frac{1}{1 + G_c(s)G(s)}R(s) - \frac{G(s)}{1 + G_c(s)G(s)}T_d(s) + \frac{G_c(s)G(s)}{1 + G_c(s)G(s)}N(s).$$

we see that (for a given $G(s)$), if we want to minimize the tracking error, we want both

Sensitivity Function

$$S(s) = \frac{1}{1 + G_c(s)G(s)}$$

Loop Gain

$$L(s) = G_c(s)G(s).$$

Complementary sensitivity function

$$C(s) = \frac{G_c(s)G(s)}{1 + G_c(s)G(s)}$$

Characteristic Polynomial

$$F(s) = 1 + L(s).$$

Small and satisfy

$$S(s) + C(s) = 1.$$

hence design compromises must be made. But first we need to understand what it means for a transfer function to be "large" or to be "small."

Sensitivity to G(s) variation

System is always subject to

- a changing environment
- aging
- uncertainty and other factors that affect a control process.

a closed-loop system senses the change in the output due to the process changes and attempts to correct the output.

Letting $T_d(s) = N(s) = 0$, we have

$$E(s) = \frac{1}{1 + G_c(s)G(s)} R(s)$$

However, large loop gain may cause the system response to be highly oscillatory and even unstable.

We can conclude

- if $G_c(s)G(s) \gg 1$ all complex frequencies of interest, error is relatively small
- increasing the magnitude of the loop gain reduces the effect of $G(s)$ on the output
- increasing the magnitude of the loop gain also reduces the effect of the variation of the parameters of the process

Sensitivity to G(s) variation

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- increasing the magnitude of the loop gain also reduces the effect of the variation of the parameters of the process

Suppose the process (or plant) $G(s)$ undergoes a change such that the true plant model is $G(s) + \Delta G(s)$

Utilizing the principle of superposition

$$E(s) + \Delta E(s) = \frac{1}{1 + G_c(s)(G(s) + \Delta G(s))} R(s).$$

Substituting $E(s) = \dfrac{1}{1 + G_c(s)G(s)} R(s)$

$$\Delta E(s) = \frac{-G_c(s)\Delta G(s)}{(1 + G_c(s)G(s) + G_c(s)\Delta G(s))(1 + G_c(s)G(s))} R(s).$$

$$\approx \frac{-G_c(s)\Delta G(s)}{(1 + L(s))^2} R(s) \approx -\frac{1}{L(s)} \frac{\Delta G(s)}{G(s)} R(s)$$

Via approximations: $G_c(s)G(s) \gg G_c(s)\,\Delta G(s)$ and $1 + L(s) \approx L(s)$

larger $L(s)$ implies smaller sensitivity

Sensitivity to G(s) variation

The system sensitivity is defined as
the ratio of the percentage change in the system transfer function to the percentage change of the process transfer function.

$$S = \frac{\Delta T(s)/T(s)}{\Delta G(s)/G(s)} = \frac{\partial T/T}{\partial G/G}$$

- The sensitivity of the **open-loop system** to changes in the plant $\mathrm{T}(s) = G(s)$ is equal to 1.
- The sensitivity of the **closed-loop system**

$$S_G^T = \frac{\partial T}{\partial G} \cdot \frac{G}{T} = \frac{G_c}{(1 + G_c G)^2} \cdot \frac{G}{GG_c/(1 + G_c G)}$$
$$= \frac{1}{1 + G_c(s)G(s)}$$

with $T(s) = \dfrac{G_c(s)G(s)}{1 + G_c(s)G(s)}$.

We find that the sensitivity of the system can be reduced below that of the open loop system by increasing open loop gain $\mathrm{L}(s)$

Sensitivity to G(s) variation

Sometimes, we further seek to determine S_α^T , α is a parameter within the transfer function $G(s)$.

Using the chain rule yields

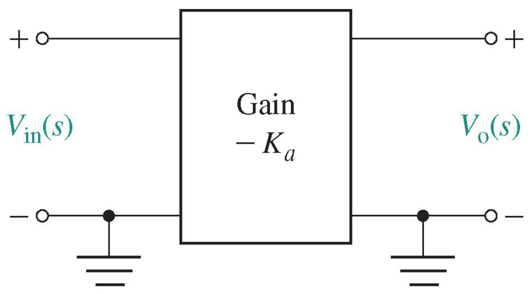
$$S_\alpha^T = S_G^T S_\alpha^G.$$

When $G(s) = \frac{N(s,\alpha)}{D(s,\alpha)}$, then

$$S_\alpha^G = \frac{\partial \ln G}{\partial \ln \alpha} = \frac{\partial \ln N}{\partial \ln \alpha}\bigg|_{\alpha=\alpha_0} - \frac{\partial \ln D}{\partial \ln \alpha}\bigg|_{\alpha=\alpha_0} = S_\alpha^N - S_\alpha^D,$$

with α_0 is the nominal value of the parameter.

Example Feedback amplifier



Open-loop amplifier

$$V_o(s) = -K_a V_{in}(s).$$

The transfer function of the amplifier without feedback

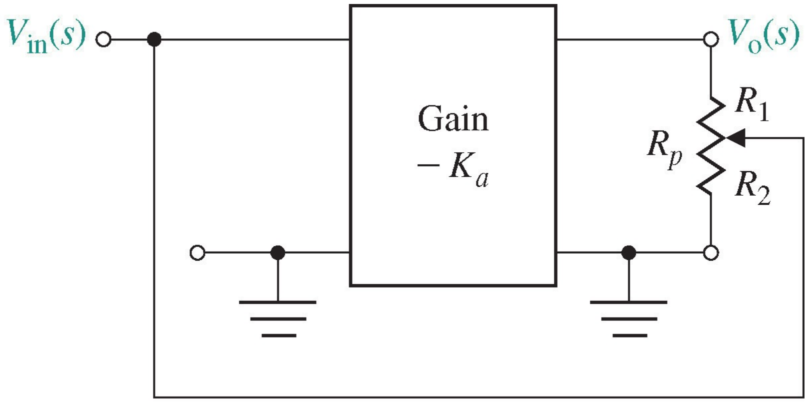
$$T(s) = -K_a,$$

The sensitivity to changes in the amplifier gain

$$S_{K_a}^T = 1$$

Sensitivity to G(s) variation

Now we add feedback using a potentiometer $R_p = R_1 + R_2$.



Amplifier with feedback

The closed-loop transfer function of the feedback amplifier is

$$T(s) = \frac{-K_a}{1 + K_a\beta}$$

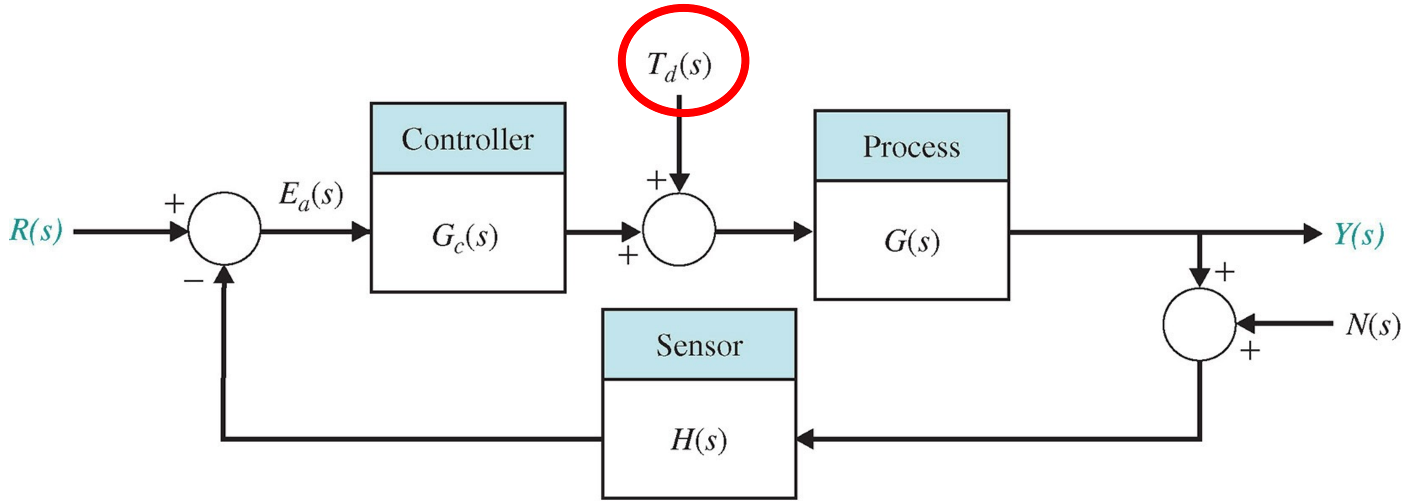
The sensitivity of the closed-loop feedback amplifier is

$$S_{K_a}^{T} = S_G^T S_{K_a}^G = \frac{1}{1 + K_a\beta}$$

Then if β or K_a large, the sensitivity is low.

Disturbance Attenuation

Motivation:
Many control systems are subject to extraneous disturbance signals that cause the system to provide an inaccurate output.



The benefit of feedback systems is that the effect of distortion, noise, and unwanted disturbances can be effectively reduced. When $N(s) = R(s) = 0$,

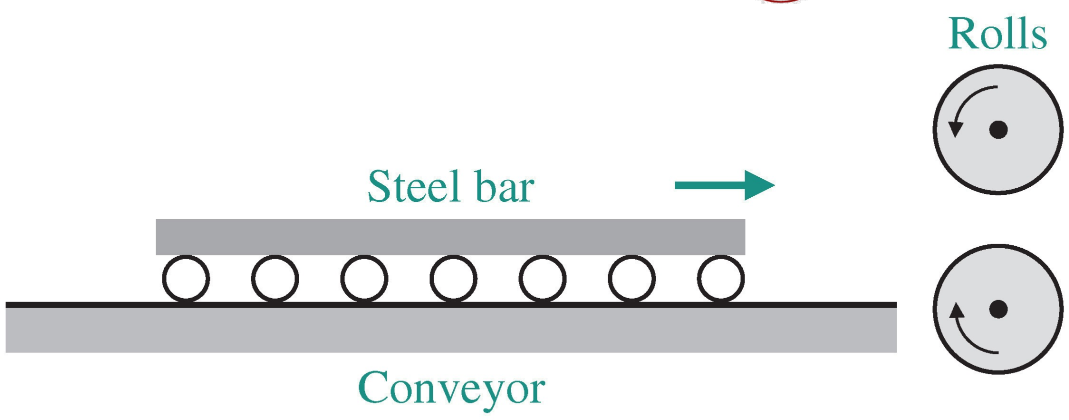
$$E(s) = -S(s)G(s)T_d(s) = -\frac{G(s)}{1 + L(s)}T_d(s)$$

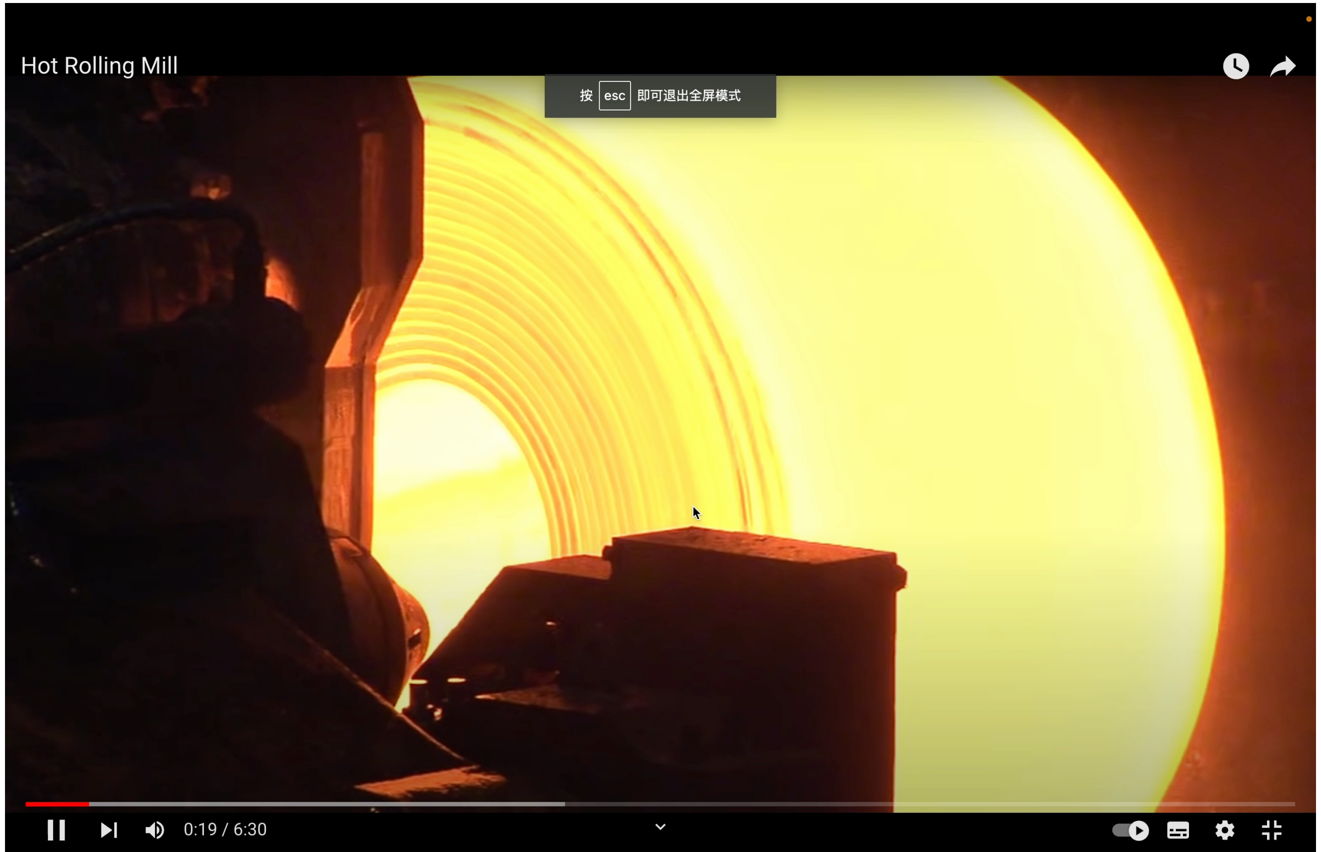
- For a fixed $G(s)$, as the loop gain $L(s)$ increases, the effect of $T_d(s)$ on the tracking error decreases
- Equivalently, we wish to design the controller $G_c(s)$ so that the sensitivity function $S(s)$ is small

Disturbance Attenuation

Example Steel rolling mill

When the bar engages in the rolls, the load on the rolls increases immediately to a large value. This loading effect can be approximated by a step change of disturbance torque.





Disturbance Attenuation

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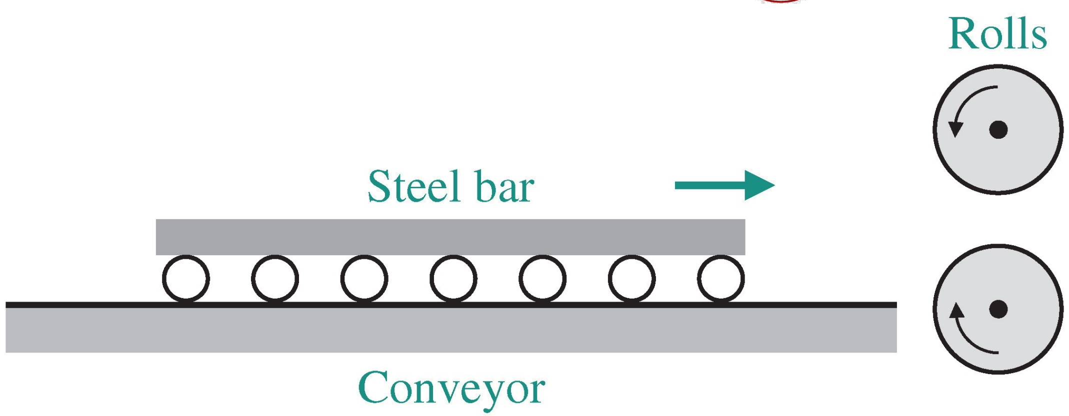
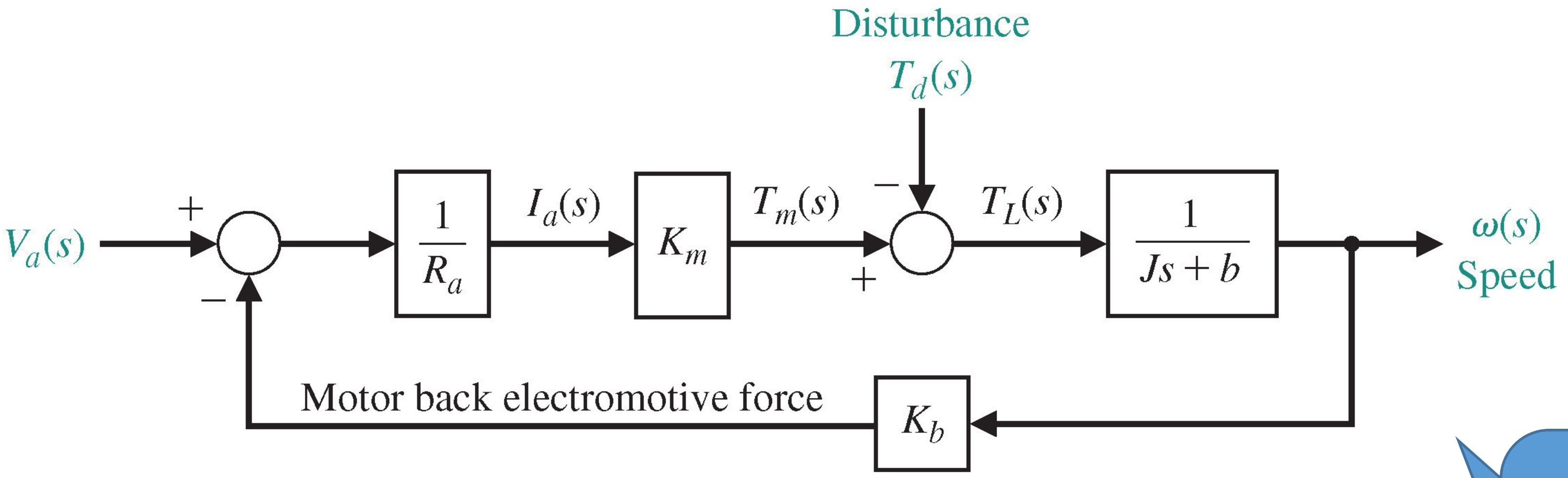




an armature-controlled DC motor with a load torque disturbance

This is not a closed-loop system!!!

Let $R(s) = V_a(s) = 0$

$$E(s) = -\omega(s) = \frac{1}{Js + b + K_m K_b / R_a} T_d(s).$$

Disturbance Attenuation

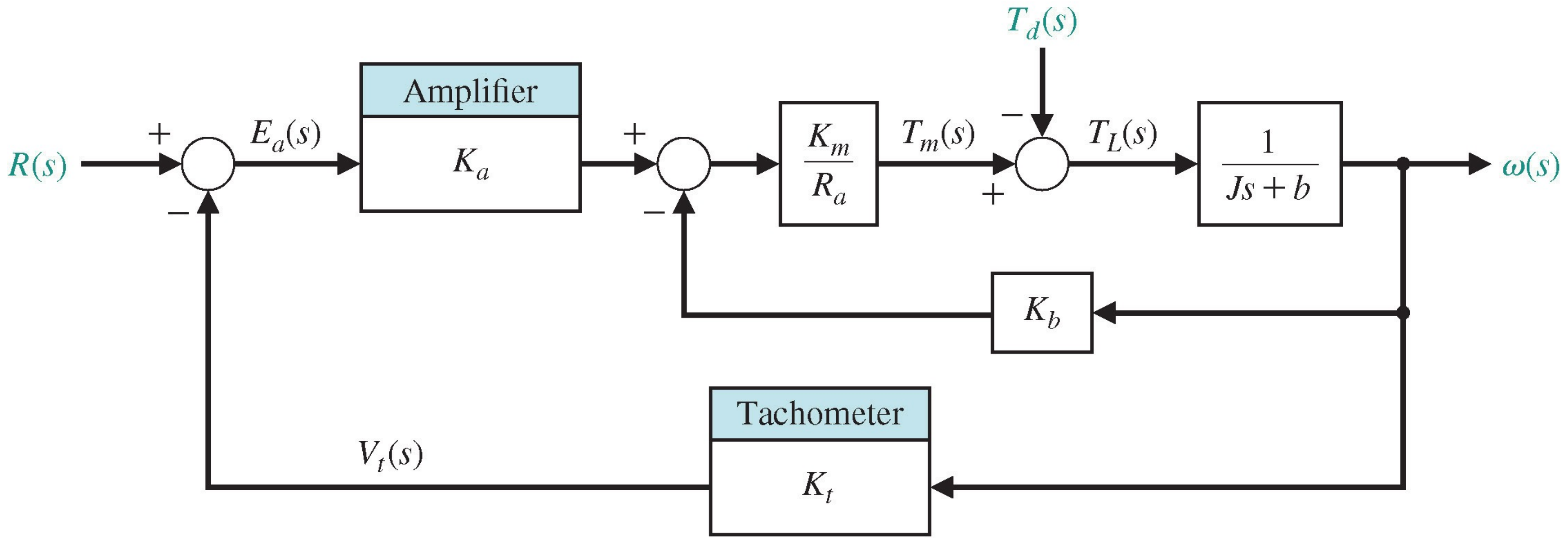
$$E(s) = -\omega(s) = \frac{1}{Js + b + K_m K_b / R_a} T_d(s).$$

$$T_d(s) = D/s$$

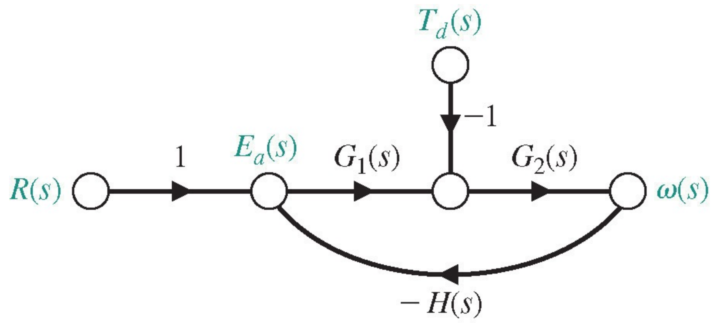
The steady-state error is found by using the final-value theorem

$$\lim_{t \to \infty} E(t) = \lim_{s \to 0} sE(s) = \lim_{s \to 0} s \frac{1}{Js + b + K_m K_b / R_a} \left(\frac{D}{s} \right) = \frac{D}{b + K_m K_b / R_a} = -\omega_0(\infty).$$

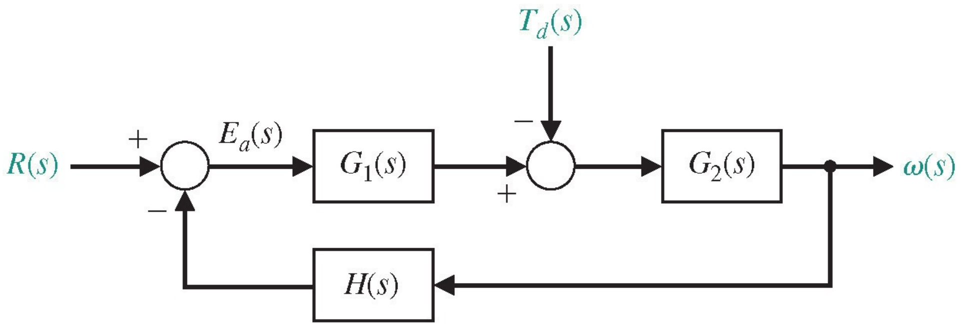
Adding a speed tachometer to form a closed-loop system



Disturbance Attenuation



(a)



(b)

where $G_1(s) = K_a K_m / R_a$, $G_2(s) = 1/(Js + b)$, and $H(s) = K_t + K_b/K_a$.

Now, the error of the closed-loop system becomes

$$E(s) = -\omega(s) = \frac{G_2(s)}{1 + G_1(s)G_2(s)H(s)} T_d(s) \approx \frac{1}{G_1(s)H(s)} T_d(s).$$

Then if $G_1(s)H(s)$ is made sufficiently large, the effect of the disturbance can be decreased

$$G_1(s)H(s) = \frac{K_a K_m}{R_a}\left(K_t + \frac{K_b}{K_a}\right) \approx \frac{K_a K_m K_t}{R_a}$$

since $K_a \gg K_b$. Thus, we seek a large amplifier gain, K_a, while minimizing R_a.

Disturbance Attenuation

Recall

$$\omega(s) = \frac{-1}{Js + b + (K_m/R_a)(K_t K_a + K_b)} T_d(s)$$

The steady-state output is obtained by utilizing the final-value theorem

$$\lim_{t \to \infty} \omega(t) = \lim_{s \to 0} (s\omega(s)) = \frac{-1}{b + (K_m/R_a)(K_t K_a + K_b)} D$$

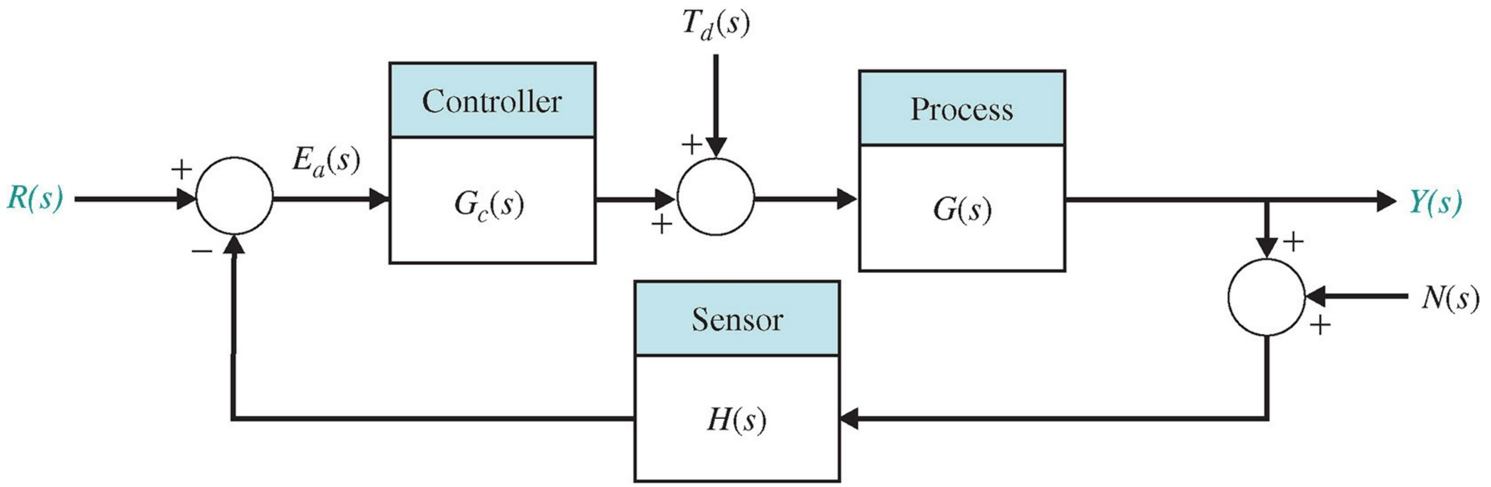
when the amplifier gain K_a is sufficiently large, we have

$$\omega(\infty) \approx \frac{-R_a}{K_a K_m K_t} D = \omega_c(\infty).$$

The ratio of closed-loop to open-loop steady-state speed output due to an undesired disturbance is

$$\frac{\omega_c(\infty)}{\omega_0(\infty)} = \frac{R_a b + K_m K_b}{K_a K_m K_t}$$

Summary



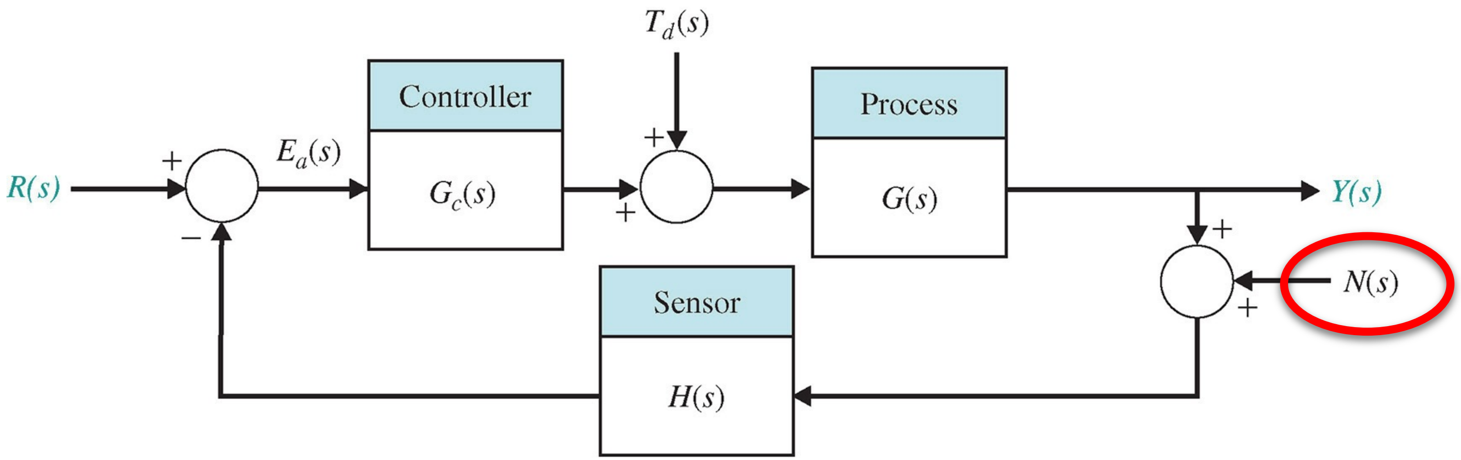
- the sensitivity of the system w.r.t process variations can be reduced below that of the open loop system by increasing open loop gain $L(s)$

$$S_G^T = \frac{1}{1 + G_c(s)G(s)}$$

- For a fixed $G(s)$, as the loop gain $L(s)$ increases, the effect of $T_d(s)$ on the tracking error decreases

$$E(s) = -S(s)G(s)T_d(s) = -\frac{G(s)}{1 + L(s)}T_d(s)$$

Measurement Noise Attenuation



When $R(s) = T_d(s) = 0$, it follows

$$E(s) = C(s)N(s) = \frac{L(s)}{1 + L(s)}N(s).$$

- As the loop gain $L(s)$ decreases, the effect of $N(s)$ on the tracking error decreases.
- The complementary sensitivity function $C(s)$ is small when the loop gain $L(s)$ is small.

For effective measurement noise attenuation, we need **a small** loop gain over the frequencies associated with the expected noise signals.

Trade off in Controller Design

$$E(s) = \frac{1}{1 + G_c(s)G(s)} R(s) - \frac{G(s)}{1 + G_c(s)G(s)} T_d(s) + \frac{G_c(s)G(s)}{1 + G_c(s)G(s)} N(s).$$

$$S(s) + C(s) = 1.$$

It is evident that, for a given $G(s)$ to reduce the tracking error $E(s)$, we desire $L(s) = G_c(s)G(s).$

- $L(s)$ to be large to reduces the sensitivity of the system to parameter variations
- $L(s)$ to be large over the range of frequencies that characterize the input disturbances $T_d(s)$.
- $L(s)$ to be small over the range of frequencies that characterize the measurement noise $N(s)$.

the trade-off in the control design process is evident.

Fortunately, the apparent conflict can be somehow addressed since

- low frequencies generally associated with input disturbances
- high frequencies generally associated with measurement noise

But not always true!

Control of transient response

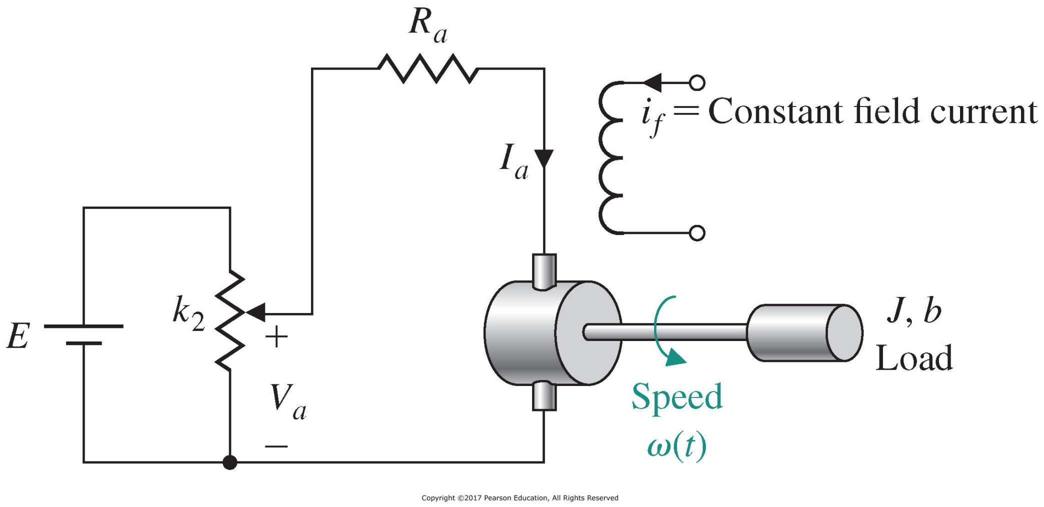
The transient response is the response of a system as a function of time before steady-state

Example Speed control system

$$\frac{\omega(s)}{V_a(s)} = G(s) = \frac{K_1}{\tau_1 s + 1},$$

where

$$K_1 = \frac{K_m}{R_a b + K_b K_m} \quad \text{and} \quad \tau_1 = \frac{R_a J}{R_a b + K_b K_m}.$$



In the case of a steel mill, the inertia of the rolls is quite large, and a large armature-controlled motor is required.

If the steel rolls are subjected to a step command

$$R(s) = V_a(s) = \frac{k_2 E}{S}$$

The output response of the open-loop control

$$\omega(s) = K_a G(s) R(s) \xrightarrow{\mathcal{L}^{-1}} \omega(t) = K_a K_1 (k_2 E)(1 - e^{-t/\tau_1})$$

Control of transient response

The transient response is the response of a system as a function of time before steady-state

Example Speed control system

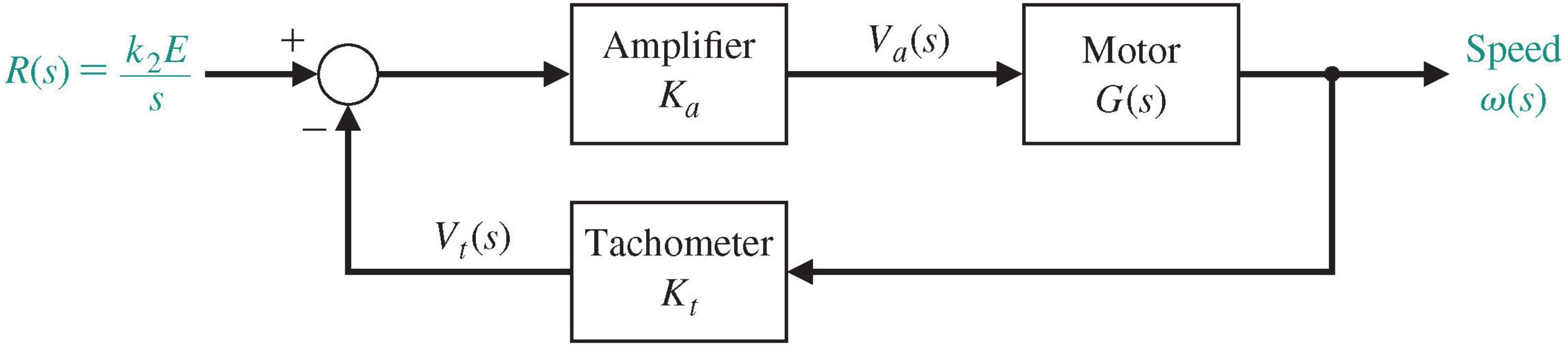
$$\omega(t) = K_a K_1 (k_2 E)(1 - e^{-t/\tau_1})$$

time constant $\tau_1 = \dfrac{R_a J}{R_a b + K_b K_m}$.

The larger the faster

However, because τ_1 is dominated by the load inertia, J it may not be possible to achieve much alteration of the transient response.

Using a tachometer to generate a voltage proportional to the speed

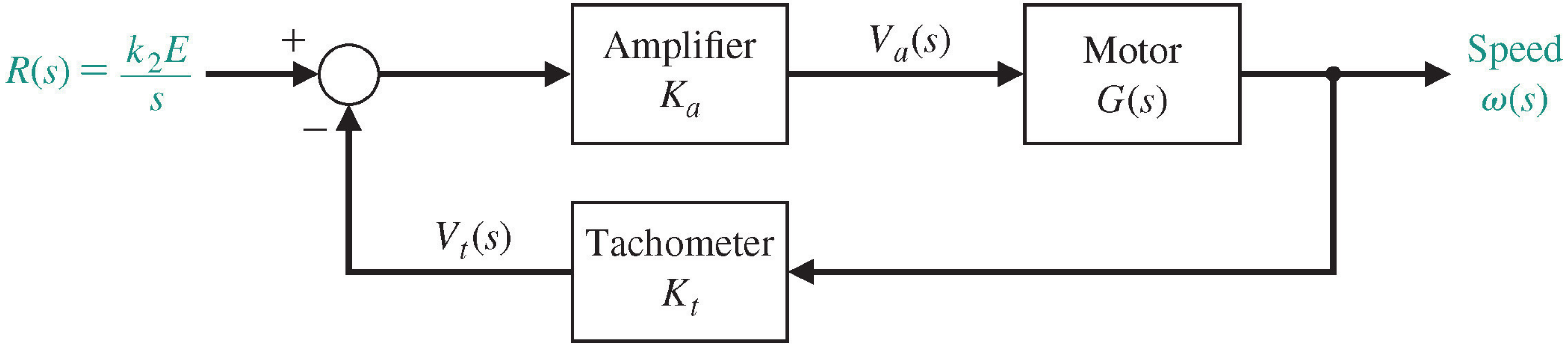


Closed-loop speed control system

Control of transient response

The transient response is the response of a system as a function of time before steady-state

Example Speed control system



The closed-loop transfer function

$$\frac{\omega(s)}{R(s)} = \frac{K_a G(s)}{1 + K_a K_t G(s)} = \frac{K_a K_1/\tau_1}{s + (1 + K_a K_t K_1)/\tau_1}.$$

The transient response to a step change

$$\omega(t) = \frac{K_a K_1}{1 + K_a K_t K_1}(k_2 E)(1 - e^{-pt}),$$

where $p = (1 + K_a K_t K_1)/\tau_1$.

Control of transient response

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The transient response is the response of a system as a function of time before steady-state

Example Speed control system

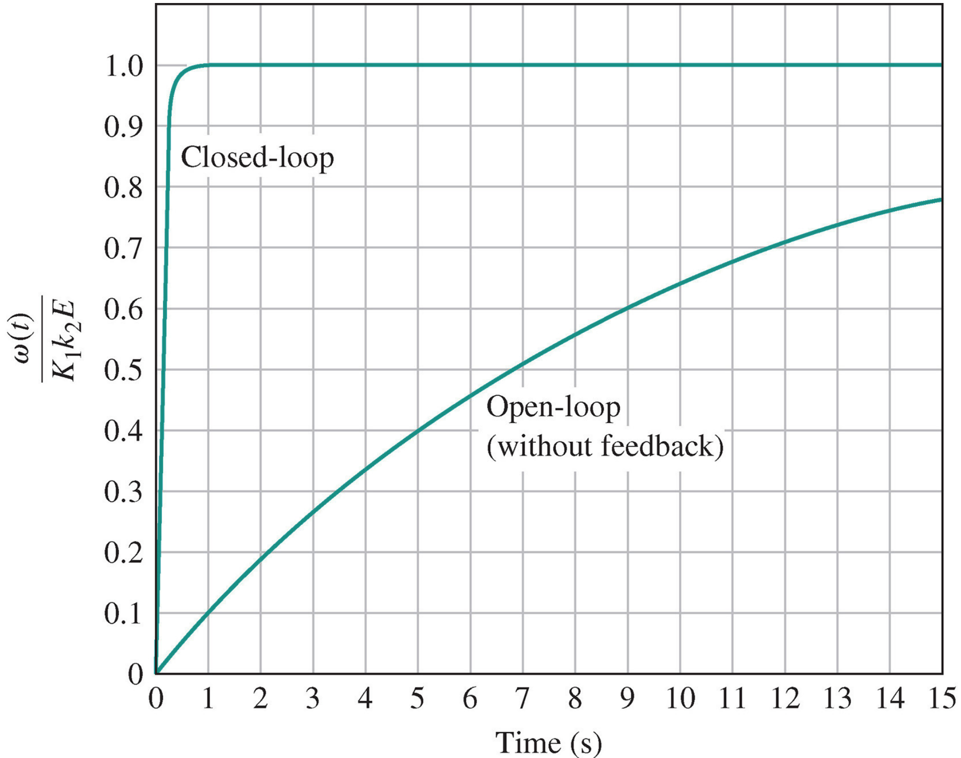
Open-loop v.s Closed-loop

$$p = (1 + K_a K_t K_1)/\tau_1.$$

$$\omega(t) = K_a K_1 (k_2 E)(1 - e^{-t/\tau_1})$$

$$\omega(t) = \frac{K_a K_1}{1 + K_a K_t K_1}(k_2 E)(1 - e^{-pt}),$$

The closed-loop transient response to a step change is $K_a K_t K_1$ times faster than the open loop system

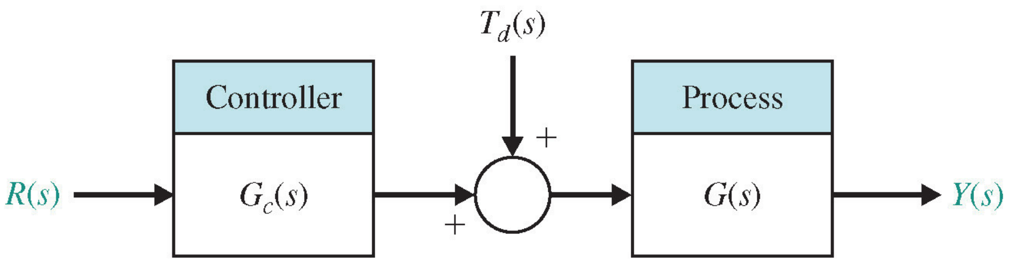


The response of the open-loop and closed-loop speed control system when $\tau = 10$ and $K_1 K_a K_t = 100$. The time to reach 98% of the **final value** for the open-loop and closed-loop system is 40 seconds and 0.4 seconds, respectively

Steady-State Error

The steady-state error is the error after the transient response has decayed, leaving only the continuous response.

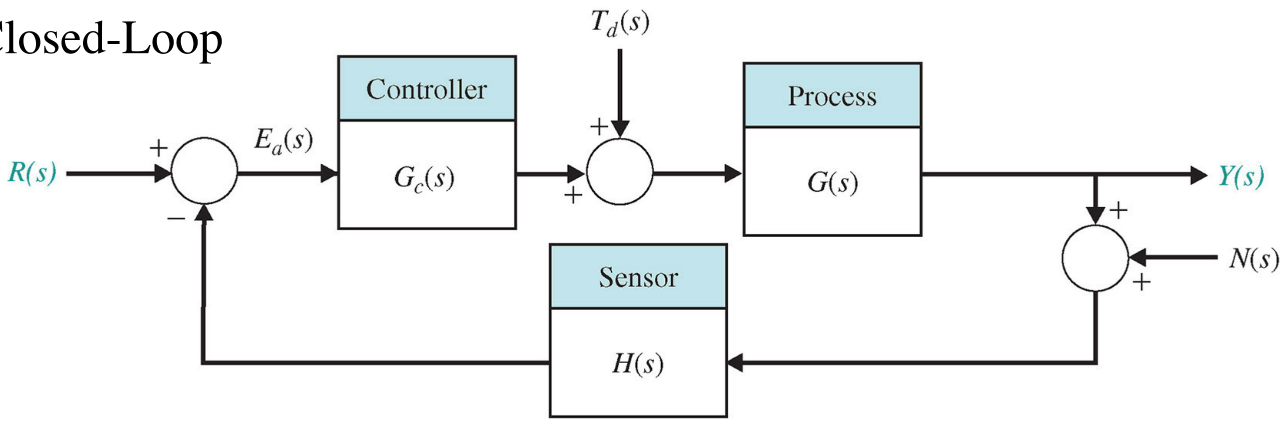
Open-Loop



When $T_d(s) = 0$

$$E_0(s) = R(s) - Y(s) = (1 - G_c(s)G(s))R(s)$$

Closed-Loop



When $T_d(s) = N(s) = 0, H(s) = 1$

$$E_c(s) = \frac{1}{1 + G_c(s)G(s)} R(s).$$

To calculate the steady-state error, we use the final-value theorem

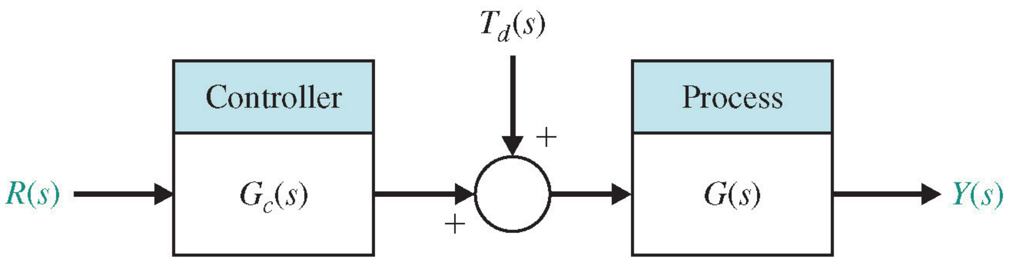
$$\lim_{t \to \infty} e(t) = \lim_{s \to 0} sE(s).$$

w.r.t a unit step input.

Steady-State Error

The steady-state error is the error after the transient response has decayed, leaving only the continuous response.

Open-Loop



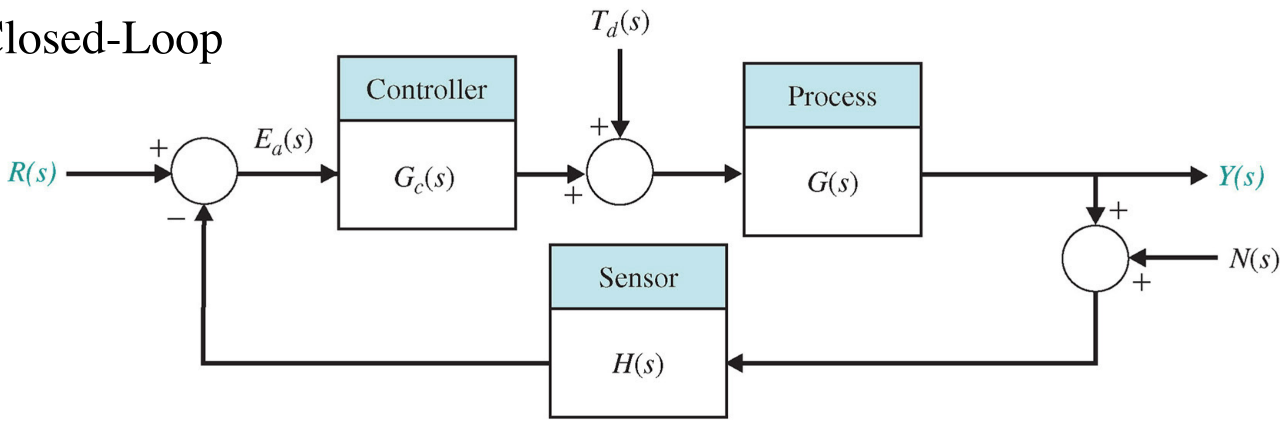
When $T_d(s) = 0$

$$E_0(s) = R(s) - Y(s) = (1 - G_c(s)G(s))R(s)$$

$$e_o(\infty) = \lim_{s\to 0} s(1 - G_c(s)G(s))\left(\frac{1}{s}\right) = \lim_{s\to 0} (1 - G_c(s)G(s))$$
$$= 1 - G_c(0)G(0).$$

- $G(0)$called the DC gain and is normally greater than one. the open-loop control system will usually have a steady-state error of significant magnitude.

Closed-Loop



When $T_d(s) = N(s) = 0, H(s) = 1$

$$E_c(s) = \frac{1}{1 + G_c(s)G(s)}R(s).$$

$$e_c(\infty) = \lim_{s\to 0} s\left(\frac{1}{1 + G_c(s)G(s)}\right)\left(\frac{1}{s}\right) = \frac{1}{1 + G_c(0)G(0)}.$$

- the closed-loop system with a reasonably large DC loop gain $L(0) = G_c(0)G(0)$ will have a small steady-state error.

Steady-State Error

The steady-state error is the error after the transient response has decayed, leaving only the continuous response.

Open-Loop

$$e_o(\infty) = \lim_{s\to 0} s(1 - G_c(s)G(s))\left(\frac{1}{s}\right) = \lim_{s\to 0}(1 - G_c(s)G(s))$$
$$= 1 - G_c(0)G(0).$$

Closed-Loop

$$e_c(\infty) = \lim_{s\to 0} s\left(\frac{1}{1 + G_c(s)G(s)}\right)\left(\frac{1}{s}\right) = \frac{1}{1 + G_c(0)G(0)}.$$

Let $G_c(0)G(0) = 1$?

Theoretically, YES. In practice, NO. $G(s)$ will change due to environmental changes and that the DC gain of the system will no longer be equal to 1.

Steady-State Error

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Consider a unity feedback system with a process transfer function and controller

$$G(s) = \frac{K}{\tau s + 1} \quad \text{and} \quad G_c(s) = \frac{K_a}{\tau_1 s + 1},$$

The desired input variable $R(s) = 1/s$

Open-Loop

$$e_0(\infty) = 1 - G_c(0)G(0) = 1 - KK_a$$

we need calibrate the system so that $KK_a = 1$ and the steady-state error is zero.

Closed-Loop

$$e_c(\infty) = 1 - \frac{KK_a}{1 + KK_a} = \frac{1}{1 + KK_a}.$$

If $KK_a = 100$, $e_c(\infty) = 1/101$

If the process gain drifts or changes by 10 percent

$$\Delta e_o(\infty) = 0.1$$

$$\frac{|\Delta e_o(\infty)|}{|r(t)|} = \frac{0.10}{1},$$

$$\Delta e_c(\infty) = \frac{1}{101} - \frac{1}{91}$$

$$\frac{\Delta e_c(\infty)}{|r(t)|} = 0.0011,$$

the closed-loop relative change is two orders of magnitude lower than that of the open-loop system.

Cost of feedback

Using feedback control allows us to

- Reduce system sensitivity
- Reduce the effect of disturbance
- Improve transient behavior
- Reduce steady state error

Using feedback control also has cost

- an increased number of components and complexity in the system
- loss of gain
- introduction of the possibility of instability

Key words list:
Closed-loop
Tracking error
Sensitivity
Complementary sensitivity
Loop gain
Transient Response
Steady state error

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THANKS!

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