Online Lecture Notes

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1 Matrix Exponentials (continued)

The components of the matrix exponential

$$X(t) = e^{tA}$$

could be:

1. Exponential functions. Examples: the scalar case, or A is diagonal.
2. Polynomials. This happens whenever A is nilponent, $A^m = 0$, since

   $$e^{tA} = \sum_{i=0}^{\infty} \frac{1}{i!} t^i A^i = \sum_{i=0}^{m-1} \frac{1}{i!} t^i A^i$$

   is a polynomial function in t.
3. Sine or cosine functions. This happens for example if we consider the harmonic oscillator, $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Or, in more generality, this happens if A is skew-symmetric, see the discussion below.

The goal of this lecture is to show that the above set of functions is complete. Essentially, this means that the components of any matrix eponential can be written as products or linear combinations of exponentials, polynomials, and sine and cosine functions. There are no other possibilities: for instance the components of $X(t)$ cannot possibly contain a "log" function or a "square-root" or other stuff that is not a exponential, polynomial, or sine or cosine function.

1.1 Interlude: Closer look at the harmonic oscillator case

Let us recall once more how to work out the matrix exponential explicitly by using its series expansion for the harmonic oscillator case

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} .$$

There is three steps:

1. We need to work out the matrix power of A. Let us start with

$$A^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad A^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

   Thus, we have

$$A^i = \begin{cases} I & \text{if } i \equiv 0 \bmod 4 \\ A & \text{if } i \equiv 1 \bmod 4 \\ -I & \text{if } i \equiv 2 \bmod 4 \\ -A & \text{if } i \equiv 3 \bmod 4 \end{cases}$$

   for all $i \in \mathbb{N}$. This means that we have found a general expression for A^i.

2. The second step is to use this result and substitute it into the series expansion of the e-function:

$$e^{At} = \sum_{i=0}^{\infty} \frac{1}{i!} t^i A^i = \begin{pmatrix} 1 - \frac{1}{2!}t^2 + \frac{1}{4!}t^4 - \frac{1}{6!}t^6 \pm \ldots & t - \frac{1}{3!}t^3 + \frac{1}{5!}t^5 - \frac{1}{7!}t^7 \pm \ldots \\ -\left(t - \frac{1}{3!}t^3 + \frac{1}{5!}t^5 - \frac{1}{7!}t^7 \pm \ldots\right) & 1 - \frac{1}{2!}t^2 + \frac{1}{4!}t^4 - \frac{1}{6!}t^6 \pm \ldots \end{pmatrix}$$

3. The third is to re-identify the terms in the series expansion of the e-function. In general, this is difficult, but here we are "lucky" in the sense we can see directly that we have the series expansions of the sine and cosine functions in the components:

$$e^{At} = \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix}$$

   Thus, this is an example for a matrix whose exponential functions yields sine and cosine functions.

1.2 A short remark about skew-symmetric systems

The harmonic oscillator is an example for a skew-symmetric system. This means that the matrix A satisfies

$$A^{\intercal} = -A \tag{1}$$

In the above example, we have seen that the corresponding matrix exponential yields a "rotation matrix", since we have

$$\begin{aligned} e^{At}\left[e^{At}\right]^{\intercal} &= \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix}\begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix} && (2) \\ &= \begin{pmatrix} \cos(t)^2 + \sin(t)^2 & 0 \\ 0 & \cos(t)^2 + \sin(t)^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} && (3) \end{aligned}$$

Matrices which have this property are called orthogonal matrices. Geometrically, these matrices describe rotations in the plane or in dimension n this would be combined rotations around all coordinate axis. The same happens for all skew symmetric matrices, since we have

$$e^{At}\left[e^{At}\right]^{\intercal} = e^{At}e^{A^{\intercal}t} \overset{(1)}{=} e^{At}e^{-At} = I\ .$$

This follows from the properties of the matrix exponential! In summary, if A is skew-symmetrix, then e^{At} is orthogonal.

2 Analysis of General Matrix Eponentials using Linear Algebra

If we want to work out general matrix exponentials, it is cumbersome to always write out the series expansion of the e-function. This means that we need to proceed in a more systematic way for analyzing the function $X(t) = e^{At}$. Tools from Linear Algebra can be used to do this in a very systematic way.

2.1 Diagonalizable Matrices

Many matrices $A \in \mathbb{R}^{n\times n}$ can be diagonalized. This means that we can find a potential complex-valued diagonal matrix $D \in \mathbb{C}^{n\times n}$ and a potentially complex-valued invertible matrix $T \in \mathbb{C}^{n\times n}$ such that

$$A = TDT^{-1} \; .$$

Recall from LA: the diagonal entries of the matrix D are called the eigenvalues of the matrix A. Just to see this, we could write the above equation in the form

$$AT = TD$$

We can write out this equation columnwise, finding that

$$AT_j = T_j D_{jj} \; ,$$

where $D_{jj} \in \mathbb{C}$ is the diagonal element of D at index j, while $T_j \in \mathbb{C}^n$ denotes the jth comonent of the matrix A. This means that D_{jj} is an eigenvalue of A and T_j is the corresponding eigenvector. This means that we can find the diagonalization of A by working out the eigenvalues of the matrix A and their corresponding eigenvectors and sort the eigenvectors columnwise into the matrix T and collect the eigenvalues in the matrix D.

Notice that if we can find such a diagonalization of A, it is easy to work the corresponding matrix exponential:

$$\begin{aligned}
e^{At} &= \sum_{i=0}^{\infty} \frac{1}{i!} A^i t^i = \sum_{i=0}^{\infty} \frac{1}{i!} \left[TDT^{-1} \right]^i t^i && (4) \\
&= \sum_{i=0}^{\infty} \frac{1}{i!} \underbrace{\left[TDT^{-1} \right] \left[TDT^{-1} \right] \cdot \ldots \cdot \left[TDT^{-1} \right]}_{i \text{ times}} t^i && (5) \\
&= \sum_{i=0}^{\infty} \frac{1}{i!} T \underbrace{D \cdot D \cdot \ldots \cdot D}_{i \text{ times}} T^{-1} t^i && (6) \\
&= T \left[\sum_{i=0}^{\infty} \frac{1}{i!} D^i t^i \right] T^{-1} = T e^{Dt} T^{-1} && (7)
\end{aligned}$$

This equation is useful, since it reduces the task of computing e^{At} to computing e^{Dt} for a diagonal matrix D. But we already know that the matrix exponential of a diagonal matrix can be found by taking the matrix exponential of all of its diagonal elements:

$$e^{At} = Te^{Dt}T^{-1} \qquad \text{with} \qquad e^{Dt} = \begin{pmatrix} e^{D_{11}t} & 0 & \dots & 0 \\ 0 & e^{D_{22}t} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & & e^{D_{nn}t} \end{pmatrix} .$$

2.2 Example

Let us come back to our harmonic oscillator example recalling that in this case we have

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} .$$

This is an example for a diagonalizable matrix. Let us work out the eigenvalues and eigenvectors of A!

1. The first step is to work the eigenvalues of the matrix A by solving the characterstic equation

   $$0 = \det(A - \lambda I) = \det\left(\begin{pmatrix} -\lambda & 1 \\ -1 & -\lambda \end{pmatrix}\right) = \lambda^2 + 1 \quad \Longleftrightarrow \quad \lambda^2 = -1$$

   Thus, the eigenvalues of the matrix A are given by $D_{11} = i$ and $D_{22} = -i$ with $i = \sqrt{-1}$.

2. The second is to work out the correponding eigenvectors T_1 and T_2. This means we need to solve the conditions

   $$AT_1 = iT_1 \qquad \text{and} \qquad AT_2 = -iT_2$$

   Let us write out the first equation componentwise

   $$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} T_{11} \\ T_{12} \end{pmatrix} = \begin{pmatrix} iT_{11} \\ iT_{12} \end{pmatrix}$$

   Of course, there is now more than one solution to this equations system (since we can scale T_1), but, of course, the easiest is normalize T_1. This means that we determine the scaling such that

   $$\|T_1\|_2 = \sqrt{|T_{11}|^2 + |T_{12}|^2} = 1 .$$

   This yields

   $$T_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}$$

   Analogously, we can work out the corresponding second eigenvector

   $$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} T_{11} \\ T_{12} \end{pmatrix} = \begin{pmatrix} -iT_{11} \\ -iT_{12} \end{pmatrix} \qquad \Longleftrightarrow \qquad T_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}$$

3. The third is to put everything back together: we have

$$T = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} .$$

   Thus, the matrix exponential of the diagonal part is given by

$$e^{Dt} = \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix} = \begin{pmatrix} \cos(t) + i\sin(t) & 0 \\ 0 & \cos(t) - i\sin(t) \end{pmatrix} .$$

   The complete matrix exponential is then given by

$$\begin{aligned} e^{At} &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \begin{pmatrix} \cos(t) + i\sin(t) & 0 \\ 0 & \cos(t) - i\sin(t) \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} \\ &= \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix} \end{aligned} \tag{8}$$

   Thus, we get the same result as before, but by using LA tools.

2.3 Summary of our results so far

If A is diagonalizable, the matrix exponential $e^{At} = Te^{Dt}T^{-1}$ is a linear combination of the diagonal modes with complex-valued coefficients $D_{jj} = \sigma_j + \omega_j\sqrt{-1}$,

$$e^{D_{jj}t} = e^{(\sigma_j + \omega_j\sqrt{-1})t} = e^{\sigma_j t}e^{i\omega_j t} = e^{\sigma_j t}\left(\cos(\omega_j t) + \sin(\omega_j t)\sqrt{-1}\right) .$$

This means that the real parts of the eigenvalues of A can be interpreted as exponential growth or decay factors while the imaginary parts of A can be interpreted as oscillation frequencies. Thus, if A is diagonalizable, all components of $X(t)$ are linear combinations and products of exponential functions and sine and cosine functions.

Outlook: in the next lecture, we will see that A is not always diagonalizable. In this case, we can also get polynomial contributions.