Online Lecture Notes

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1 Summary of the main idea of Lecture

The main point of Lecture 3 was to say that changing from an open-loop control input $u(t)$ to a closed-loop control input,

$$u(t) = \mu(x(t)),$$

in dependence on x, has many practical advantages. In the easiest, for a scalar linear system,

$$\dot{x}(t) = ax(t) + bu(t)$$

and a linear feedback law $\mu(x) = kx$, the closed-loop system has the form

$$\dot{x}(t) = (a + bk)x(t) \ .$$

This system is stable for $a + bk < 0$. If we find a k such that $a + bk < 0$, then there exist an open neighborhood of a and b such that $(a + \delta a) + (b + \delta)k < 0$ for small data perturbations δa and δb in the open neighborhood of a and b. This is the same as saying the system is "robustly stable" with respect to small perturbations of a and b. Similarly, if there are small errors in the initial value, these errors are damped out exponentially in time, since $a + bk$ is strictly negative. Also, we discussed in the previous lecture how to analyze systems of the form

$$\dot{x}(t) = ax(t) + bkx(t) + cw(t)$$

with external disturbance signal $w(t)$. Also there we have seen that a bounded-input-bounded-output lemma holds as long as $a + bk < 0$. Since we can choose k, this can always be achieved (as long as $b \neq 0$).

2 Multivariate Linear Time-Invariant Control Systems

In Lecture 6, we are analyzing linear input output systems of the form

$$\dot{x}(t) = Ax(t) + Bu(t) + b \quad (1)$$
$$y(t) = Cx(t) + d \quad (2)$$

Here, $y(t) \in \mathbb{R}^{n_y}$ is called the output function. In practice, this is the function that we can measure, while $x(t)$ itself may be unknown to us. For instance, if we have, say $n_x = 5$ states, but we can only measure the first and the third state, $x_1(t)$ and $x_3(t)$, we could model this situation by setting

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} .$$

Now, there is different ways of how to control this system. In the easiest case, we can design open-loop control input for $u(t)$. This means, we only need to solve the first differential equation in order to find $x(t)$. And, after this, in the second step, we can substitute $x(t)$ into the equation $y(t) = Cx(t) + d$ in order to predict our measurements $y(t)$ in dependence on the time t. Or, in another setting, we could also implement a closed-loop feedback control law of the form $u(t) = \mu(y(t))$, but in this case, the differential equation for x and the algebraic equation for y are coupled. This would lead to a closed-loop system of the form

$$\dot{x}(t) = Ax(t) + B\mu(y(t)) + b \quad (3)$$
$$y(t) = Cx(t) + d . \quad (4)$$

As in our previous analysis $x(0) = x_0$ denotes the initial state, which may or may not be given.

2.1 Open-Loop Control

In the open-loop case it is sufficient to first analyze the differential equation

$$\dot{x}(t) = Ax(t) + Bu(t) \quad \text{with} \quad x(0) = x_0 .$$

Here, we assume that $b = 0$ (which we can achieve by removing offsets–just shift the states or controls). The basic idea to "guess" an explicit solution for this open-loop controlled system in depedence on t is to generalize the result of Lecture 2 by passing from the exponential term "e^{at} to the matrix exponential "e^{At}", where the matrix exponential is defined as in Lecture 5. This means: our conjecture is that

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)\,\mathrm{d}\tau .$$

It is easy to prove that this expression solves the open-loop differential equation. Namely, x satisfies the differential equation

$$
\begin{aligned}
\dot{x}(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\left[e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)\,\mathrm{d}\tau\right] \\
&= Ae^{At}x_0 + e^{A(t-t)}Bu(t) + \int_0^t Ae^{A(t-\tau)}Bu(\tau)\,\mathrm{d}\tau \\
&= Ae^{At}x_0 + Bu(t) + A\int_0^t e^{A(t-\tau)}Bu(\tau)\,\mathrm{d}\tau \\
&= A\left[e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)\,\mathrm{d}\tau\right] + Bu(t) \\
&= Ax(t) + Bu(t) \qquad (5)
\end{aligned}
$$

as well as the initial value condition

$$
x(0) = e^0 x_0 + \int_0^0 \ldots\,\mathrm{d}\tau = x_0 \ .
$$

This completes out proof for the open-loop case. Notice that this means that all the results from Lecture 2 generalize trivially by replacing exponential with matrix exponentials. In particular, the linear superposition holds: if

$$
u(t) = \sum_{i=0}^{n} v_i \varphi_i(t)
$$

for scalar basis functions, but vector valued coefficients $v_i \in \mathbb{R}^{n_u}$, we have

$$
u(t) = \sum_{i=0}^{n} v_i \varphi_i(t) \qquad \Longrightarrow \qquad x(t) = \sum_{i=0}^{n} \Phi_i(t) v_i
$$

for $x(0) = 0$ and repsonse functions

$$
\Phi_i(t) = \int_0^t e^{A(t-\tau)}B\varphi_i(\tau)\,\mathrm{d}\tau \ \in \ \mathbb{R}^{n_x \times n_u} \ .
$$

Here, we can choose all kinds of basis function φ_i, such as, piecewise constant functions, polynomials, trigonemtric basis functions (discrete Fourier transform), and so on (see Lecture 2).

2.2 Closed-Loop Control

Similarly, if $C = I$ (such that we can all states we can generalize the results from Lecture 3. In the easiest case, we can introduce a linear feedback law

$$
u(t) = Kx(t)
$$

for a feedback gain matrix $K \in \mathbb{R}^{n_u \times n_x}$. This means that the corresponding closed-loop system has the form

$$
\dot{x}(t) = Ax(t) + BKx(t) \quad \text{with} \quad x(0) = x_0 \ ,
$$

which can also be written in the form

$$\dot{x}(t) = (A + BK)x(t) \quad \text{with} \quad x(0) = x_0 \ ,$$

which has the unique solution

$$x(t) = e^{(A+BK)t}x_0 \ .$$

The analysis of the matrix exponential in dependence on the feedback gain K is, however, more difficult than in the scalar case. In order to stabilize, we want to choose K such that all eigenvalues of the matrix $A + BK$ have strictly negative real part, such that

$$\lim_{t\to\infty} e^{(A+BK)t} = 0 \ .$$