Online Lecture Notes

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1 Summary of Lecture 4: Nonlinear Differential Equation

The first technique that we learned in Lecture 4 is the separation of variables for ODEs of the form

$$\dot{x}(t) = f_1(x(t))f_2(t) \quad \text{and} \quad x(0) = x_0$$

In this case, we can sometimes solve the nonlinear differential equation explicitly by assuming $f_1(x(t)) \neq 0$ (otherwise our solution is constant) such that

$$\frac{\dot{x}(t)}{f_1(x(t))} = f_2(t) \qquad \Longrightarrow \qquad \int_0^t \frac{\dot{x}(\tau)}{f_1(x(\tau))} = \int_0^t f_2(\tau)\,\mathrm{d}\tau$$

If we are “lucky” we can find explicit expressions for the integrals on the left and the right and eliminate $x(t)$.

1.1 Examples from the Mid-Term Exam 2020:

1.1.1 Example 1

Consider the case

$$\dot{x}(t) = e^t, \quad x(0) = 1$$

then we can directly integrate on both sides

$$\int_0^t \dot{x}(\tau)\,\mathrm{d}\tau = \int_0^t e^\tau\,\mathrm{d}\tau$$

where we find

$$\int_0^t \dot{x}(\tau)\,\mathrm{d}\tau = x(t) - x(0) = x(t) - 1 \qquad \text{and} \qquad \int_0^t e^\tau\,\mathrm{d}\tau = e^t - e^0 = e^t - 1$$

Thus, we have to solve the equation

$$x(t) - 1 = e^t - 1 \qquad \text{and} \qquad x(t) = e^t$$

1.1.2 Example 2

Consider the case

$$\dot{x}(t) = x(t)e^{-t}, \quad x(0) = 1 \ .$$

In this case, separation of variables yields

$$\int_0^t \frac{\dot{x}(\tau)}{x(\tau)} \, \mathrm{d}\tau = \int_0^t e^{-\tau} \, \mathrm{d}\tau \ .$$

We work out the terms separately:

$$\int_0^t \frac{\dot{x}(\tau)}{x(\tau)} \, \mathrm{d}\tau = \log(x(t)) - \log(x(0)) = \log(x(t))$$

and

$$\int_0^t e^{-\tau} \, \mathrm{d}\tau = -e^{-t} + 1 \ .$$

The last step is to solve the equation

$$\log(x(t)) = e^{-t} + 1 \qquad \Longrightarrow \qquad x(t) = e^{1-e^{-t}}$$

1.2 Theorem of Picard Lindelöf

If f is globally Lipschitz continuous, the differential equation

$$\dot{x}(t) = f(x(t)) \quad \text{with} \quad x(0) = x_0$$

has a unique solution. This is also correct in the vector-valued states and right-hand sides, $f : \mathbb{R}^{n_x} \to \mathbb{R}^{n_x}$, where n_x is the number of states. In practice, for differentiable f, we can check this condition by checking whether $\|f'(x)\|$ is globally bounded by a constant $L < \infty$, since for differentiable f, we have

$$\begin{aligned} \|f(y) - f(x)\| &= \left\| \int_0^1 f'(x + s(y-x))(y-x)\mathrm{d}s \right\| \\ &\leq \int_0^1 \|f'(x + s(y-x))\| \|y-x\| \, \mathrm{d}s \\ &\leq L \int_0^1 \|y-x\| \, \mathrm{d}s = L\|y-x\| \ . \end{aligned} \tag{1}$$

Thus, if $\|f'(x)\| \leq L$ holds globally, then f is Lipschitz continuous.

1.3 Numerical Integration

In this lecture we discussed two types of numerical integration schemes, namely, Taylor model based integration and Runge-Kutta integrators. The step of the Taylor model can be found by computing the coefficient functions Φ_i recursively, as

$$\Phi_{i+1}(t,x) = \frac{\partial}{\partial t}\Phi_i(t,x) + \frac{\partial}{\partial x}\Phi_i(t,x)f(t,x) \qquad \text{with} \qquad \Phi_0(t,x) = x$$

which yields the coeficients of the solution trajectory of the ODE

$$\dot{x}(t) = f(t, x(t)),\ x(t_0) = x_0 \qquad \Longrightarrow \qquad x(t) \approx \sum_{i=0}^{s} \Phi_i(t_0, x_0) \frac{(t-t_0)^i}{i!} \ .$$

This can be used to compute approximations of $x(t)$ up to any order. We can also run this in a loop by breaking long horizon into N shorter ones (Taylor model based integration). Runge-Kutta methods, on the other hand, avoid computing derivatives of f by evaluating f at more than one point and matching all consistency conditions. Most important examples are the Euler integetor,

$$y_{i+1} = y_i + hf(t_i, y_i),$$

as well as Heun's method

$$\begin{aligned} k_1 &= f(t_i, y_i) && (2) \\ k_2 &= f(t_i + h, y_i + hk_1) && (3) \\ y_{i+1} &= y_i + h\frac{k_1 + k_2}{2} \ , && (4) \end{aligned}$$

which are consistent up to order 1 or 2, respectively.

1.4 Linear Approximation of Nonlinear ODE

Consider a nonlinear control system with steady-state at $x_\mathrm{s}, u_\mathrm{s}$,

$$\dot{x}(t) = f(x(t), u(t)) \qquad \text{with} \qquad 0 = f(x_\mathrm{s}, u_\mathrm{s}) \ .$$

In this case, we can find an approximation of the nonlinear control system by using a first order Taylor approximation of f instead of f, since

$$f(x,u) \ \approx \ \underbrace{f(x_\mathrm{s}, u_\mathrm{s})}_{=0} + A(x - x_\mathrm{s}) + B(u - u_\mathrm{s})$$

with the first order partial derivatives

$$A = \frac{\partial f}{\partial x}(x_\mathrm{s}, u_\mathrm{s}) \qquad \text{and} \qquad B = \frac{\partial f}{\partial u}(x_\mathrm{s}, u_\mathrm{s}) \ .$$

This leads to a linear control system of the form

$$\dot{z}(t) = Az(t) + Bv(t)$$

with $z(t) = x(t) - x_\mathrm{s}$ and $v(t) = u(t) - u_\mathrm{s}$. This makes the connection between Lecture 4 and Lecture 5 !!!!

1.4.1 Example: Controlled Pendulum

Consider a pendulum with length l and mass m in a gravitation field with gravitational constant g. The intertial of the pendulum is $J = ml^2$ and the gravitational force component depends on the excitation angle θ as $F_g = -mg\sin(\theta)$.

We could consider an additional torque $T = u$ that we can control. In this example, Newton's equation of motions yield the nonlinear system

$$J\ddot{\theta} = T - F_g l = u - mgl\sin(\theta)$$

we can write this in the form

$$\ddot{\theta} = \frac{u}{ml^2} - \frac{g}{l}\sin(\theta)$$

We can write this in the form

$$\dot{x}(t) = f(x(t), u(t)) = \begin{pmatrix} x_2(t) \\ \frac{u}{ml^2} - \frac{g}{l}\sin(x_1(t)) \end{pmatrix}$$

which is a nonlinear control system in standard form with steady state at $(0, 0)$. The corresponding partial derivatives are

$$A = \frac{\partial f(0,0)}{\partial x} = \begin{pmatrix} 0 & 1 \\ -\frac{g}{l} & 0 \end{pmatrix} \quad \text{and} \quad B = \frac{\partial f(0,0)}{\partial u} = \begin{pmatrix} 0 \\ \frac{1}{ml^2} \end{pmatrix}$$

Thus, we find the coefficients of the corresponding linear control system in standard form (see Lecture 6).