SI231 - Matrix Computations, 2022 Fall

Homework Set #4

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Acknowledgements:

1) Deadline: **2022-12-02 23:59:59**
2) **Late Policy details** can be found on piazza.
3) Submit your homework in **Homework 4** on **Gradescope**. Entry Code: **4V2N55**. **Make sure that you have correctly select pages for each problem.** If not, you probably will get 0 point.
4) No handwritten homework is accepted. You need to write LaTeX. (If you have difficulties in using LaTeX, you are allowed to use **MS Word or Pages** for the first and the second homework to accommodate yourself.)
5) Use the given template and give your solution in English. Solution in Chinese is not allowed.
6) Your homework should be uploaded in the PDF format, and the naming format of the file is not specified.

I. Eigenvalue

Problem 1. (8 points + 2 points + 12 points + 8 points) For the matrix below

$$A = \begin{bmatrix} -1 & -3 & -3 \\ 3 & 5 & 3 \\ -1 & -1 & 1 \end{bmatrix}$$

1) Calculate the characteristic polynomial of A ,
2) Find the eigenvalues of A ,
3) Find a basis for each eigenspace of A ,
4) Determine whether or not A is diagonalizable. If A is diagonalizable, then find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$.

Solution:

1) The characteristic polynomial of A is

$$\begin{aligned}
\det(A-\lambda I) &= \begin{vmatrix} -1-\lambda & -3 & -3 \\ 3 & 5-\lambda & 3 \\ -1 & -1 & 1-\lambda \end{vmatrix} = (-1)\begin{vmatrix} 1+\lambda & 3 & 3 \\ 3 & 5-\lambda & 3 \\ -1 & -1 & 1-\lambda \end{vmatrix} \\
&= (-1)\begin{vmatrix} 0 & 2-\lambda & 4-\lambda^2 \\ 0 & 2-\lambda & 6-3\lambda \\ -1 & -1 & 1-\lambda \end{vmatrix} \\
&= (-1)(-1)\begin{vmatrix} 2-\lambda & 4-\lambda^2 \\ 2-\lambda & 6-3\lambda \end{vmatrix} = (2-\lambda)\begin{vmatrix} 1 & 4-\lambda^2 \\ 1 & 6-3\lambda \end{vmatrix} \\
&= (2-\lambda)\left(6-3\lambda-\left(4-\lambda^2\right)\right) = (2-\lambda)\left(\lambda^2-3\lambda+2\right) \\
&= -(\lambda-1)(\lambda-2)^2.
\end{aligned}$$

(8 points)

2) A has eigenvalues 1 and 2 , with algebraic multiplicities 1 and 2 respectively. (2 points)
3) The eigenspace of A associated to the eigenvalue 1 is the null space of the matrix $A - I$. To find a basis for the eigenspace, row reduce this matrix.

$$A - I = \begin{bmatrix} -2 & -3 & -3 \\ 3 & 4 & 3 \\ -1 & -1 & 0 \end{bmatrix} \longrightarrow \cdots \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus, the general solution to the equation $(A-I)\mathbf{x} = \mathbf{0}$ is $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \\ 1 \end{bmatrix} x_3$ where x_3 is arbitrary. Letting

$x_3 = 1$ gives $\mathcal{B}_1 = \left\langle \begin{bmatrix} 3 \\ -3 \\ 1 \end{bmatrix} \right\rangle$ as a basis of the eigenspace associated to the eigenvalue 1. The eigenspace of A associated to the eigenvalue 2 is the null space of the matrix $A - 2I$. To find a basis for the eigenspace, row reduce this matrix.

$$A - 2I = \begin{bmatrix} -3 & -3 & -3 \\ 3 & 3 & 3 \\ -1 & -1 & -1 \end{bmatrix} \longrightarrow \cdots \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus, the general solution to the equation $(A - 2I)\mathbf{x} = \mathbf{0}$ is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} x_3$$

where x_2 and x_3 are arbitrary. Thus $\mathcal{B}_2 = \left\langle \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\rangle$ as a basis of the eigenspace associated to the eigenvalue 2.(12 points)

4) A is diagonalizable since there is a basis of $\mathbb{R}^3$ consisting of eigenvectors of A . Specifically, concatenate $\mathcal{B}_1$ and $\mathcal{B}_2$ to get such a basis

$$\mathcal{B} = \left\langle \begin{bmatrix} 3 \\ -3 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\rangle$$

If we set

$$P = \begin{bmatrix} \frac{3}{\sqrt{19}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{3}{\sqrt{19}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{19}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

then P is invertible and

$$P^{-1}AP = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

(8 points)

II. EIGENSPACE

Problem 2. (20 points)

Let $T : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ be the linear transformation given by $T(A) = A^T$ where A^T is the transpose of A . If $\exists\lambda \in \mathbb{R}, X \in \mathbb{R}^{2\times 2} \Longrightarrow T(X) = \lambda X$,then λ is the eigenvalue of T and X is the eigenvector associated to the eigenvalue λ . Please find the eigenvalues of T and the dimensions of the eigenspaces.

Solution:

This can be done by writing a matrix of A , but it can actually be done directly. Suppose we have

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} = T \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \lambda \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Then, $a = \lambda a, c = \lambda b, b = \lambda c$, and $d = \lambda d$. If a or d is nonzero, these imply immediately that $\lambda = 1$. Otherwise, either c or b is not zero, then either $c = \lambda b = \lambda^2 c$ or $b = \lambda^2 b$ implies that $\lambda = \pm 1$. Thus, the eigenvalues of T are 1 and -1 .

For $\lambda = 1$, we have must have $c = b$ and no other conditions. Thus, the eigenspace for $\lambda = 1$ is

$$\left\{ \begin{pmatrix} a & b \\ b & d \end{pmatrix} : a, b, d \in \mathbb{R} \right\} = \text{Span} \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}$$

and this eigenspace has dimension equal to 3 .

For $\lambda = -1$, we must have $a = 0$ since $a = -a$ and similarly $d = 0$. We also have $b = -c$. Thus, the eigenspace is

$$\left\{ \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix} : b \in \mathbb{R} \right\} = \text{Span} \left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}$$

and this eigenspace has dimension equal to 1.(20 points)

III. Eigenvectors of Symmetric Matrix

Problem 3. (8 points + 12 points + 15 points) Let $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{R}^n$ be two linearly independent vectors, with unit norm ($\|\boldsymbol{\alpha}\|_2 = \|\boldsymbol{\beta}\|_2 = 1$). Define the symmetric matrix $\mathbf{A} = \boldsymbol{\alpha}\boldsymbol{\beta}^T + \boldsymbol{\beta}\boldsymbol{\alpha}^T$.

1) Prove that $\boldsymbol{\alpha} + \boldsymbol{\beta}$ and $\boldsymbol{\alpha} - \boldsymbol{\beta}$ are eigenvectors of $\mathbf{A}$, and determine the corresponding eigenvalues.

   Hint: The notation $c = \boldsymbol{\alpha}^T\boldsymbol{\beta}$ may be useful.
2) Find the nullspace and rank of $\mathbf{A}$.
3) Find an eigenvalue decomposition of $\mathbf{A}$, in terms of $\boldsymbol{\alpha}, \boldsymbol{\beta}$.

Solution:

1) $\mathbf{A}\boldsymbol{\alpha} = c\boldsymbol{\alpha} + \boldsymbol{\beta}$, $\mathbf{A}\boldsymbol{\beta} = \boldsymbol{\alpha} + c\boldsymbol{\beta}$, then $\mathbf{A}(\boldsymbol{\alpha} + \boldsymbol{\beta}) = (c+1)(\boldsymbol{\alpha} + \boldsymbol{\beta})$, $\mathbf{A}(\boldsymbol{\alpha} - \boldsymbol{\beta}) = (c-1)(\boldsymbol{\alpha} - \boldsymbol{\beta})$, where $c = \boldsymbol{\alpha}^T\boldsymbol{\beta}$.

   Therefore, $\boldsymbol{\alpha} + \boldsymbol{\beta}$ and $\boldsymbol{\alpha} - \boldsymbol{\beta}$ are eigenvectors of $\mathbf{A}$, with eigenvalues $c+1, c-1$. (8 points)
2) For any $\mathbf{x} \in \mathcal{N}(\mathbf{A})$, $\mathbf{A}\mathbf{x} = \mathbf{0}$. $\mathbf{A}\mathbf{x} = \boldsymbol{\alpha}\left(\boldsymbol{\beta}^T\mathbf{x}\right) + \boldsymbol{\beta}\left(\boldsymbol{\alpha}^T\mathbf{x}\right) = \mathbf{0}$.

   Since $\boldsymbol{\beta}^T\mathbf{x}$ and $\boldsymbol{\alpha}^T\mathbf{x}$ are scalars, we can rewrite above equation as: $\mathbf{A}\mathbf{x} = \left(\boldsymbol{\beta}^T\mathbf{x}\right)\boldsymbol{\alpha} + \left(\boldsymbol{\alpha}^T\boldsymbol{x}\right)\boldsymbol{\beta} = \mathbf{0}$.

   We have $\boldsymbol{\alpha}, \boldsymbol{\beta}$ are linearly independent, then $\boldsymbol{\beta}^T\mathbf{x} = \boldsymbol{\alpha}^T\boldsymbol{x} = 0$. So $\mathcal{N}(\mathbf{A})$ is the set of vectors orthogonal to $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, $\mathcal{N}(\mathbf{A}) = \text{span}\{\boldsymbol{\alpha}, \boldsymbol{\beta}\}^{\perp}$. (6 points)

   According to the fact of $\mathbf{A}$ is symmetric, $\mathcal{R}(\mathbf{A}) = \mathcal{R}\left(\mathbf{A}^T\right) = \mathcal{N}(\mathbf{A})^{\perp} = \text{span}\{\boldsymbol{\alpha}, \boldsymbol{\beta}\}$.

   Therefore, $\text{rank}(\mathbf{A}) = 2$. (6 points)
3) First, we need to prove that $\lambda_{1,2} = c \pm 1 \neq 0$. Since $\boldsymbol{\alpha}, \boldsymbol{\beta}$ are two linearly independent vectors and $c = \boldsymbol{\alpha}^T\boldsymbol{\beta} =< \boldsymbol{\alpha}, \boldsymbol{\beta} >$, $|c| \leqslant \|\boldsymbol{\alpha}\|_2\|\boldsymbol{\beta}\|_2 = 1$, while equality holds only when $\boldsymbol{\alpha} = \pm\boldsymbol{\beta}$. Therefore, $|c| < 1$, $\lambda_{1,2} \neq 0$. Thus, we have found two linearly independent eigenvectors $\boldsymbol{\xi}_1 = \boldsymbol{\alpha} + \boldsymbol{\beta}$ and $\boldsymbol{\xi}_2 = \boldsymbol{\alpha} - \boldsymbol{\beta}$ that do not belong to the nullspace of $\mathbf{A}$. (6 points)

   Then, the eigenvalue decomposition of $\mathbf{A}$ is $\mathbf{A} = (c+1)\boldsymbol{\nu}_1\boldsymbol{\nu}_1^T + (c-1)\boldsymbol{\nu}_2\boldsymbol{\nu}_2^T$,

   where $\boldsymbol{\nu}_1 = \boldsymbol{\xi}_1/\|\boldsymbol{\xi}_1\|_2 = (\boldsymbol{\alpha} + \boldsymbol{\beta})/\sqrt{2+2c}$, $\boldsymbol{\nu}_2 = \boldsymbol{\xi}_2/\|\boldsymbol{\xi}_2\|_2 = (\boldsymbol{\alpha} - \boldsymbol{\beta})/\sqrt{2-2c}$. (6 points)

   Hence the eigenvalue decomposition of $\mathbf{A}$ becomes $\mathbf{A} = \frac{1}{2}\left((\boldsymbol{\alpha} + \boldsymbol{\beta})(\boldsymbol{\alpha} + \boldsymbol{\beta})^T - (\boldsymbol{\alpha} - \boldsymbol{\beta})(\boldsymbol{\alpha} - \boldsymbol{\beta})^T\right)$.(3 points)

IV. Diagonalization

Problem 4. (13 points + 12 points) Let A be a real symmetric $n \times n$ matrix with 0 as a simple eigenvalue (that is, the algebraic multiplicity of the eigenvalue 0 is 1), and given a vector $\mathbf{v} \in \mathbb{R}^n$.

1) Prove that for sufficiently small positive real ϵ (Equivalently, ϵ is smaller than all the absolute value of nonzero eigenvalues) , the equation A $\mathbf{x} + \epsilon\mathbf{x} = \mathbf{v}$ has a unique solution $\mathbf{x} = \mathbf{x}(\epsilon) \in \mathbb{R}^n$
2) Evaluate $\lim_{\epsilon\to 0^+} \epsilon\mathbf{x}(\epsilon)$ in terms of $\mathbf{v}$, the eigenvectors of A , and the inner product $\langle , \rangle$ on $\mathbb{R}^n$.

Solution:

1) Recall that the eigenvalues of a real symmetric matrices are all real numbers and it is diagonalizable by an orthogonal matrix. Note that the equation $A\mathbf{x} + \epsilon\mathbf{X} = \mathbf{v}$ can be written as

$$(A + \epsilon I)\mathbf{x} = \mathbf{v}, \tag{*}$$

where I is the $n \times n$ identity matrix. Thus to show that the equation $(*)$ has a unique solution, it suffices to show that the matrix $A + \epsilon I$ is invertible. Since A is diagonalizable, there exists an invertible matrix S such that

$$S^{-1}AS = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix},$$

where λ_i are eigenvalues of A .

Since the algebraic multiplicity of 0 is 1 , without loss of generality, we may assume that $\lambda_1 = 0$ and λ_i, $i > 1$ are nonzero.

Then we have

$$S^{-1}(A + \epsilon I)S = S^{-1}AS + \epsilon I = \begin{bmatrix} \epsilon & 0 & \cdots & 0 \\ 0 & \epsilon + \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \epsilon + \lambda_n \end{bmatrix}.$$

If $\epsilon > 0$ is smaller than the lengths of $|\lambda_i|\,, i > 1$, then none of the diagonal entries $\epsilon + \lambda_i$ are zero. Hence we have

$$\begin{aligned} \det(A + \epsilon I) &= \det(S)^{-1}\det(A + \epsilon I)\det(S) \\ &= \det\left(S^{-1}(A + \epsilon I)S\right) \\ &= \epsilon\,(\epsilon + \lambda_2)\cdots(\epsilon + \lambda_n) \neq 0. \end{aligned}$$

Since $\det(A + \epsilon I) \neq 0$, it yields that A is invertible, hence the equation * has a unique solution

$$\mathbf{x}(\epsilon) = (A + \epsilon I)^{-1}\mathbf{v}.$$

Remark: This result is in general true for any square matrix. Instead of using the diagonalization, we can use the triangulation of a matrix.

2) Let $A = V\Lambda V^T$ and v_i is the $i-$th column vector of V.

Then we compute

$$\begin{aligned}
A\mathbf{x}(\epsilon) &= A(A+\epsilon I)^{-1}\mathbf{v} \\
&= V\Lambda V^T V(\Lambda+\epsilon I)^{-1}V^T v \\
&= V\Lambda(\Lambda+\epsilon I)^{-1}V^T v \\
&= \sum_{i=1} c_i \frac{\lambda_i}{\lambda_i+\epsilon} v_i
\end{aligned}$$

with $c_i =< v_i, v >$.

Therefore we have

$$\begin{aligned}
\lim_{\epsilon\to 0^+} \epsilon\mathbf{x}(\epsilon) &= \lim_{\epsilon\to 0^+} (\mathbf{v} - A\mathbf{x}(\epsilon)) \\
&= \mathbf{v} - \lim_{\epsilon\to 0^+} (A\mathbf{x}(\epsilon)) \\
&= \sum_{i=1}^{n} c_i\mathbf{v}_i - \lim_{\epsilon\to 0^+} \left(\sum_{i=2}^{n} c_i \frac{\lambda_i}{\lambda_i+\epsilon}\mathbf{v}_i \right) \\
&= \sum_{i=1} c_i\mathbf{v}_i - \sum_{i=2}^{n} c_i\mathbf{v}_i \\
&= c_1\mathbf{v}_1
\end{aligned}$$

Using the orthonormality of the basis $\{\mathbf{v}_i\}$, we have

$$\langle \mathbf{v}, \mathbf{v}_1 \rangle = \sum_{i=1}^{n} \langle c_i\mathbf{v}_i, \mathbf{v}_1 \rangle = c_1.$$

Hence the required expression is

$$\lim_{\epsilon\to 0^+} \epsilon\mathbf{X}(\epsilon) = \langle \mathbf{v}, \mathbf{v}_1 \rangle \mathbf{v}_1,$$

where $\mathbf{v}_1$ is the unit eigenvector corresponding to the eigenvalue 0 .