TA Lecture Special - Midterm Review

November 16, 2023

School of Information Science and Technology,
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Outline

Midterm Review

Exercises in Previous TA Lectures

Summary of Counting

Choose k objects out of n objects, the number of possible ways:

	Order Matters	Order Not Matter
with replacement	n^k	$\binom{n+k-1}{k}$
without replacement	$n(n-1)\cdots(n-k+1)$	$\binom{n}{k}$

Property of Probability

Probability has the following properties, for any events A and B:

1. $P(A^c) = 1 - P(A)$.
2. If $A \subseteq B$, then $P(A) \leq P(B)$.
3. $P(A \bigcup B) = P(A) + P(B) - P(A \cap B)$.

For any events $A_1, \ldots, A_n$:

$$P(\bigcup_{i=1}^{n} A_i) = \sum_{i} P(A_i) - \sum_{i<j} P(A_i \cap A_j) + \sum_{i<j<k} P(A_i \cap A_j \cap A_k)$$

$$+ \ldots + (-1)^{n+1} P(A_1 \cap \cdots \cap A_n).$$

Conditional Probability

Definition

If A and B are events with $P(B) > 0$, then the *conditional probability* of A given B, denoted by $P(A|B)$, is defined as

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

- $P(A)$: *prior probability* of A.
- $P(A|B)$: *posterior probability* of A.

Theorem

For any events $A_1, \ldots, A_n$ with positive probabilities,

$$P(A_1, \ldots, A_n) = P(A_1)P(A_2|A_1)P(A_3|A_1, A_2) \cdots P(A_n|A_1, \ldots, A_{n-1}).$$

Conditional Probability is Also Probability

- Condition on an event E, we update our beliefs to be consistent with this knowledge.
- $P(\cdot|E)$ is also a probability function with sample space S:
  - $0 \leq P(\cdot|E) \leq 1$ with $P(S|E) = 1$ and $P(\emptyset|E) = 0$.
  - If events $A_1, A_2, \ldots$ are disjoint, then $P(\cup_{j=1}^{\infty} A_j|E) = \sum_{j=1}^{\infty} P(A_j|E)$.
  - $P(A^c|E) = 1 - P(A|E)$.
  - Inclusion-exclusion: $P(A \cup B|E) = P(A|E) + P(B|E) - P(A \cap B|E)$.

Key Tool: Bayes Rule and LOTP

Theorem

For any events A and B with positive probabilities,

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}.$$

Theorem

Let $A_1, \ldots, A_n$ be a partition of the sample space S (i.e., the A_i are disjoint events and their union is S), with $P(A_i) > 0$ for all i. Then

$$P(B) = \sum_{i=1}^{n} P(B|A_i)P(A_i).$$

Key Tool: Independence

Definition

Events A and B are *independent* if

$$P(A \cap B) = P(A)P(B).$$

If $P(A) > 0$ and $P(B) > 0$, then this is equivalent to

$$P(A|B) = P(A)$$
$$P(B|A) = P(B)$$

“Independence”

Definition

Events A, B and C are *independent* if all of the following equations hold:

$$\begin{aligned} P(A \cap B) &= P(A)P(B) \\ P(A \cap C) &= P(A)P(C) \\ P(B \cap C) &= P(B)P(C) \\ P(A \cap B \cap C) &= P(A)P(B)P(C). \end{aligned}$$

Definition

Events A and B are said to be *conditionally independent* given E if:

$$P(A \cap B|E) = P(A|E)P(B|E).$$

PMF: Watch the Support

Theorem

Let X be a discrete r.v. with support $x_1, x_2, \ldots$ (assume these values are distinct and, for notational simplicity, that the support is countably infinite; the analogous results hold if the support is finite). The PMF p_X of X must satisfy the following two criteria:

- *Nonnegative:* $p_X(x) > 0$ *if* $x = x_j$ *for some* j, *and* $p_X(x) = 0$ *otherwise;*
- *Sums to 1:* $\sum_{j=1}^{\infty} p_X(x_j) = 1$.

PMF: Watch the Support

Theorem

Let X be a discrete r.v. and $g : \mathbb{R} \to \mathbb{R}$. Then the support of $g(X)$ is the set of all y such that $g(x) = y$ for at least one x in the support of X, and the PMF of $g(X)$ is

$$P(g(X) = y) = \sum_{x:g(x)=y} P(X = x)$$

for all y in the support of $g(X)$.

Memoryless Property

Theorem

Suppose for any positive integer n, discrete random variable X satisfies

$$P(X \geq n + k | X \geq k) = P(X \geq n)$$

for $k = 0, 1, 2, \ldots$, then $X \sim \mathrm{Geom}(p)$.

Expectation

Definition

The *expected value* (also called the *expectation* or *mean*) of a discrete r.v. X whose distinct possible values are $x_1, x_2, \cdots$ is defined by

$$E(X) = \sum_{j=1}^{\infty} x_j P(X = x_j)$$

If the support is finite, then this is replaced by a finite sum. We can also write

$$E(X) = \sum_{x} \underbrace{x}_{\text{value}} \underbrace{P(X = x)}_{\text{PMF at } x}$$

where the sum is over the support of X.

Expectation

The expected value of a sum of r.v.s is the sum of the individual expected values.

Theorem

For any r.v.s X, Y and any constant c,

$$E(X+Y) = E(X) + E(Y)$$

$$E(cX) = cE(X)$$

Theorem

If X is a discrete r.v. and g is a function from $\mathbb{R}$ to $\mathbb{R}$, then

$$E(g(X)) = \sum_x g(x) P(X = x)$$

where the sum is taken over all possible values of X.

Indicator: Fundamental Bridge

Theorem

There is a one-to-one correspondence between events and indicator r.v.s, and the probability of an event A is the expected value of its indicator r.v. I_A:

$$P(A) = E(I_A).$$

Key Tool: Indicator

Let A and B be events. Then the following properties hold.

1. $(I_A)^k = I_A$ for any positive integer k.
2. $I_{A^c} = 1 - I_A$.
3. $I_{A \cap B} = I_A I_B$.
4. $I_{A \cup B} = I_A + I_B - I_A I_B$.

- Given n events $A_1, \ldots, A_n$ and indicators $I_j, j = 1, \ldots, n$.
- $X = \sum_{j=1}^{n} I_j$: the number of events that occur
- $\binom{X}{2} = \sum_{i<j} I_i I_j$: the number of pairs of distinct events that occur
- $E(\binom{X}{2}) = \sum_{i<j} P(A_i \cap A_j)$
  - $E(X^2) = 2\sum_{i<j} P(A_i \cap A_j) + E(X)$.
  - $Var(X) = 2\sum_{i<j} P(A_i \cap A_j) + E(X) - (E(X))^2$.

Variance

- For any r.v. X, $\mathrm{Var}(X) = E\left(X^2\right) - (EX)^2$.
- $Var(X + c) = Var(X)$ for any constant c.
- $Var(cX) = c^2 Var(X)$ for any constant c.
- If X and Y are independent, then $Var(X + Y) = Var(X) + Var(Y)$.
- $Var(X) \geq 0$ with equality if and only if $P(X = a) = 1$ for some constant a.

Key Tool: PGF

Definition

The *probability generating function* (PGF) of a nonnegative integer-valued r.v. X with PMF $p_k = P(X = k)$ is the generating function of the PMF. By LOTUS, this is

$$E\left(t^X\right) = \sum_{k=0}^{\infty} p_k t^k.$$

The PGF converges to a value in $[-1, 1]$ for all t in $[-1, 1]$ since $\sum_{k=0}^{\infty} p_k = 1$ and $\left|p_k t^k\right| \leq p_k$ for $|t| \leq 1$.

Let X be a nonnegative integer-valued r.v. with PMF $p_k = P(X = k)$, and the PGF of X is $g(t) = \sum_{k=0}^{\infty} p_k t^k$, we have

- $E(X) = g'(t)|_{t=1}$
- $E(X(X-1)) = g''(t)|_{t=1}$

PDF

Definition

For a continuous r.v. X with CDF F, the *probability density function* (PDF) of X is the derivation f of the CDF, given by $f(x) = F'(x)$. The *support* of X, and of its distribution, is the set of all x where $f(x) > 0$.

Theorem

The PDF f of a continuous r.v. must satisfy the following two criterias:

- *Nonnegative:* $f(x) \geq 0$;
- *Integrates to 1:* $\int_{-\infty}^{\infty} f(x)dx = 1$.

CDF

Theorem

Let X be a continuous r.v. with PDF f. Then the CDF of X is given by

$$F(x) = \int_{-\infty}^{x} f(t)dt.$$

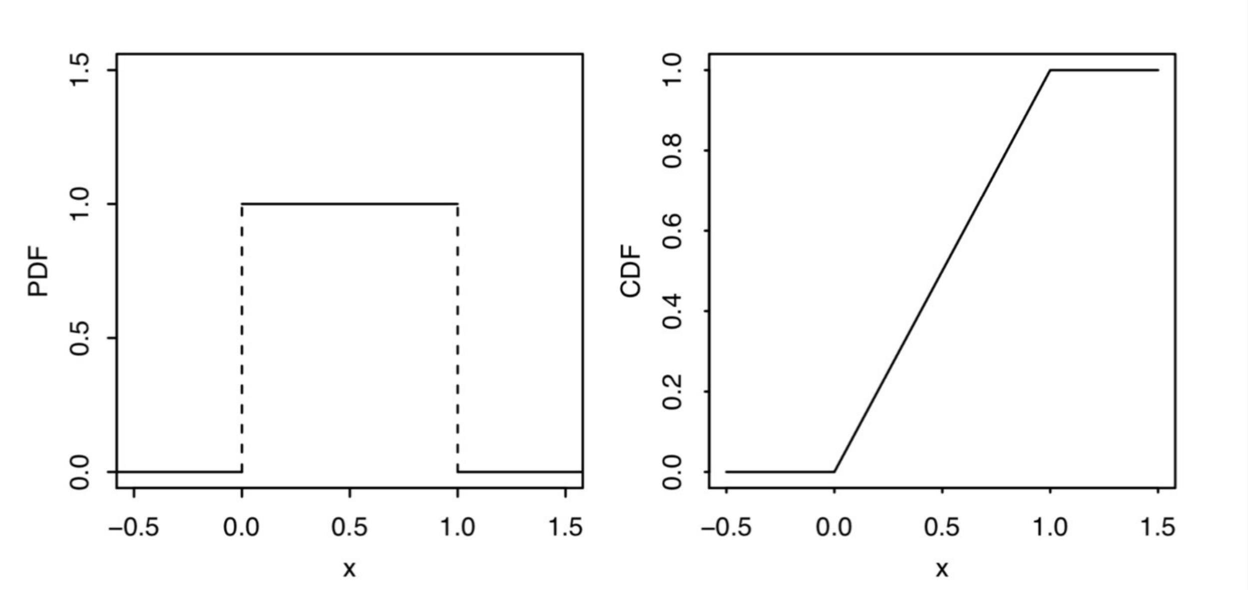
Uniform Distribution

Definition

A continuous r.v. U is said to have the *Uniform Distribution* on the interval (a, b) if its PDF is

$$f(x) = \begin{cases} \dfrac{1}{b-a} & \text{if} a < x < b, \\ 0 & \text{otherwise}. \end{cases}$$

We denote this by $U \sim \mathrm{Unif(a, b)}$.



Exponential Distribution

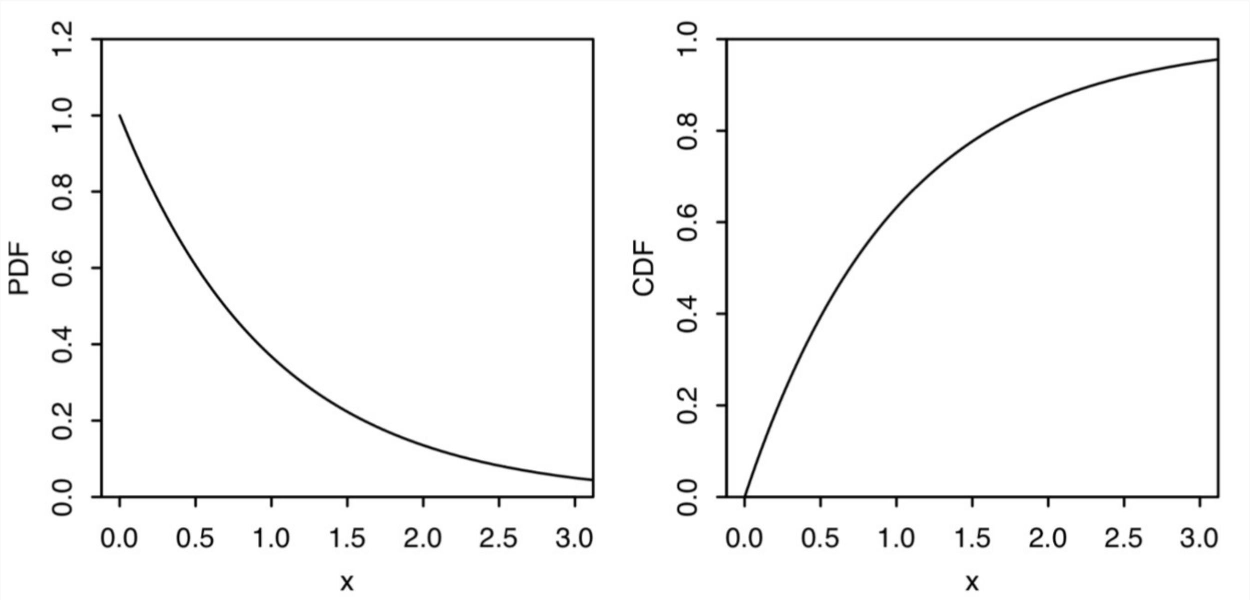
Definition

A continuous r.v. X is said to have the *Exponential Distribution* with parameter λ if its PDF is

$$f(x) = \lambda e^{-\lambda x}, \ x > 0.$$

We denote this by $X \sim \mathrm{Expo}(\lambda)$. The corresponding CDF is

$$F(X) = 1 - e^{-\lambda x}, x > 0.$$



Standard Normal Distribution

Definition

A continuous r.v. Z is said to have the *standard Normal distribution* if its PDF φ is given by

$$\varphi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}, -\infty < z < \infty.$$

We write this as $Z \sim \mathcal{N}(0, 1)$ since, as we will show, Z has mean 0 and variance 1.
The standard Normal CDF Φ is the accumulated area under the PDF:

$$\Phi(z) = \int_{-\infty}^{z} \varphi(t) dt = \int_{-\infty}^{z} \varphi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dt.$$

Normal Distribution

Definition

If $Z \sim \mathcal{N}(0,1)$, then

$$X = \mu + \sigma Z$$

is said to have the *Normal distribution* with mean μ and variance σ^2. We denoote this by $X \sim \mathcal{N}(\mu, \sigma^2)$.

Theorem

Let $X \sim \mathcal{N}(\mu, \sigma^2)$. Then the CDF of X is

$$F(x) = \Phi\left(\frac{x-\mu}{\sigma}\right),$$

and the PDF of X is

$$f(x) = \varphi\left(\frac{x-\mu}{\sigma}\right)\frac{1}{\sigma}$$

Central Limit Theorem

Theorem

As $n \to \infty$,

$$\sqrt{n}\left(\frac{\bar{X}_n - \mu}{\sigma}\right) \to \mathcal{N}(0, 1) \text{ in distribution.}$$

In words, the CDF of the left-hand side approaches the CDF of the standard Normal distribution.

MGF

Definition

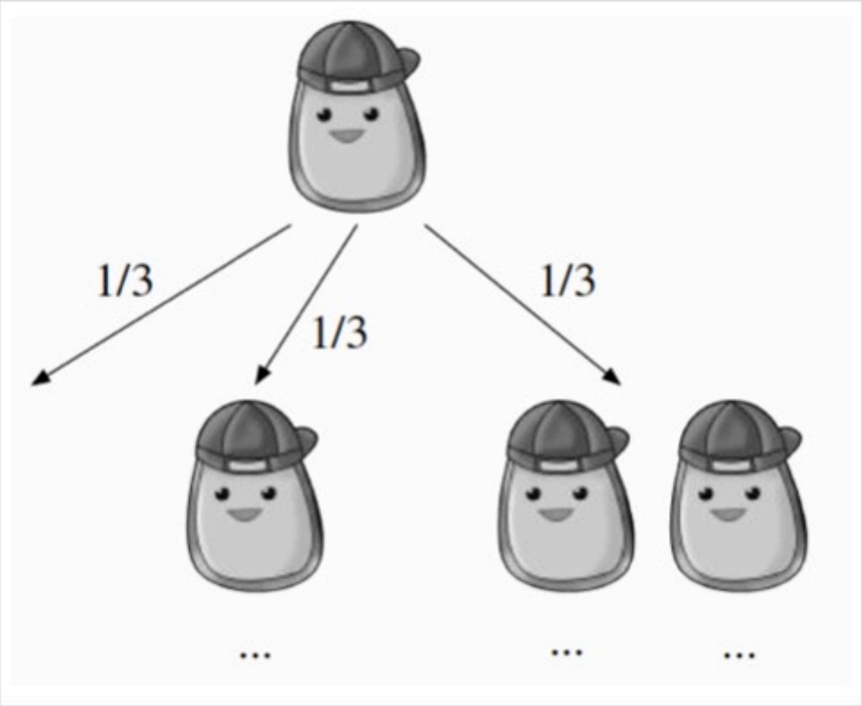
The *moment generating function* (MGF) of an rr.v. X is $M(t) = E(e^{tX})$, as a function of t, if this is finite on some open interval $(-a, a)$ containing 0. Otherwise we say the MGF of X does not exist

Probabilistic Model: Birthday Problem

There are k people in a room. Assume each person's birthday is equally likely to be any of the 365 days of the year (we exclude February 29), and that people's birthdays are independent (we assume there are no twins in the room). What is the probability that two or more people in the group have the same birthday?

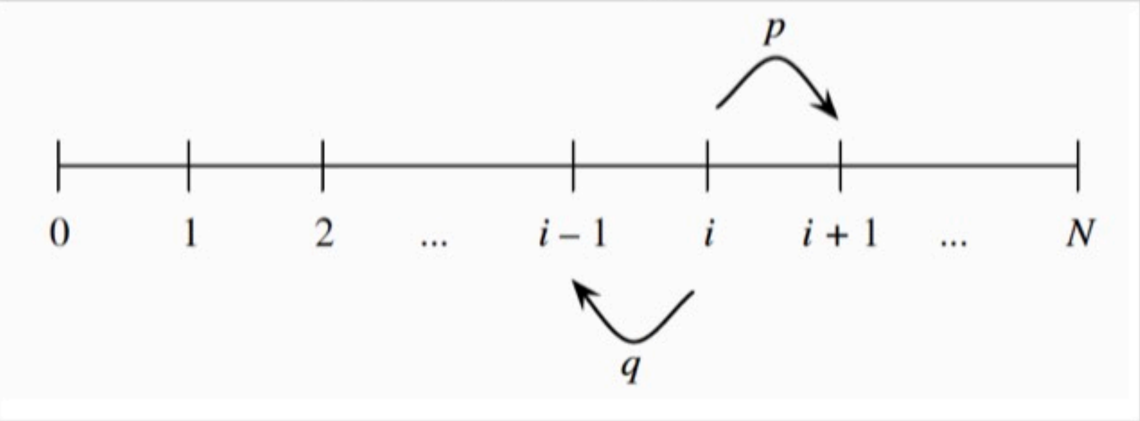
Probabilistic Model: Branching Process

A single amoeba, Bobo, lives in a pond. After one minute Bobo will either die, split into two amoebas, or stay the same, with equal probability, and in subsequent minutes all living amoebas will behave the same way, independently. What is the probability that the amoeba population will eventually die out?

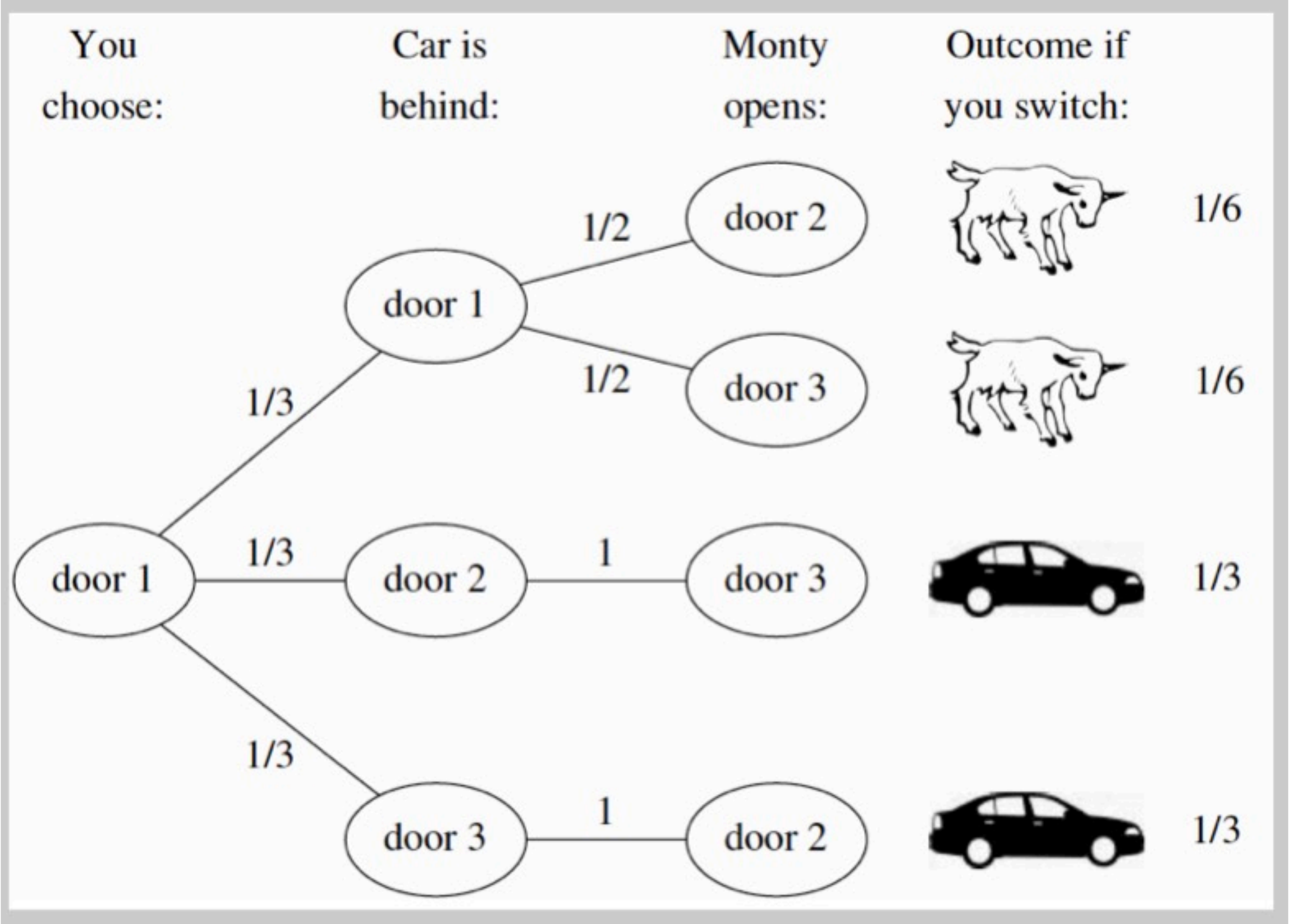


Probabilistic Model: Gambler's Ruin

Two gamblers, A and B, make a sequence of dollar 1 bets. In each bet, gambler A has probability p of winning, and gambler B has probability $q = 1 - p$ of winning. Gambler A starts with i dollars and gambler B starts with $N - i$ dollars; the total wealth between the two remains constant since every time A loses a dollar, the dollar goes to B, and vice versa. The game ends when either A or B is ruined (run out of money). What is the probability that A wins the game (walking away with all the money)?



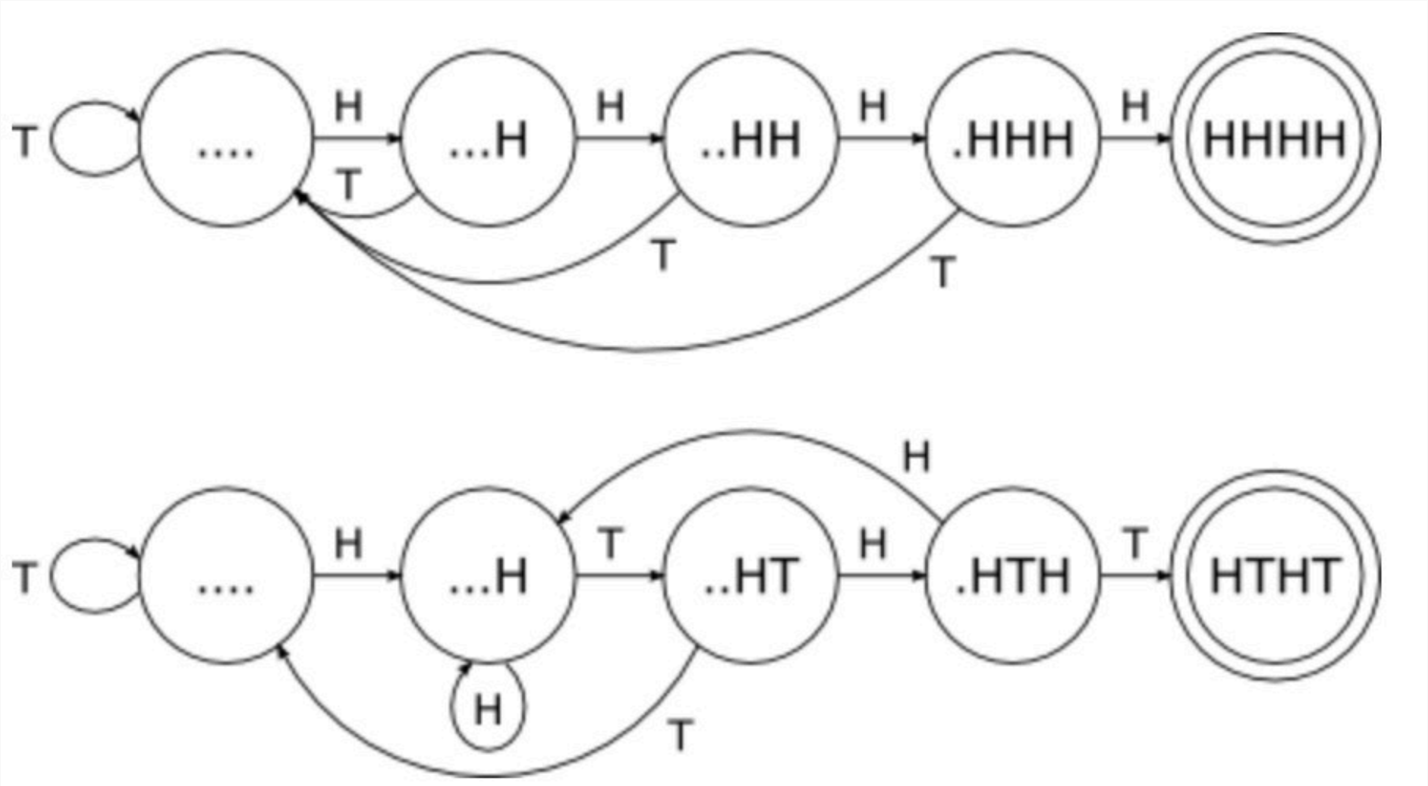
Probabilistic Model: Monty Hall



Probabilistic Model: Coupon Collector

Suppose there are n types of toys, which you are collecting one by one, with the goal of getting a complete set. When collecting toys, the toy types are random (as is sometimes the case, for example, with toys included in cereal boxes or included with kids' meals from a fast food restaurant). Assume that each time you collect a toy, it is equally likely to be any of the n types. Let N denote the number of toys needed until you have a complete set. Find $E(N)$ and $Var(N)$.

Probabilistic Model: Pattern Matching



Outline

Midterm Review

Stat110.net

Exercises in Previous TA Lectures

Problem 1

$$\frac{C_n^k}{n^k}$$

There are n balls in a jar, labeled with the numbers $1, 2, \cdots, n$. A total of k balls are drawn, one by one *with replacement*, to obtain a sequence of numbers.

(a) What is the probability that the sequence obtained is strictly increasing?

(b) What is the probability that the sequence obtained is increasing? (Note: In this book, "increasing" means "nondecreasing".)

$$\frac{C_{n+k-1}^k}{n^k}$$

Problem 1

(a) The total possible orderings of k numbers from n is n^k. For any selection of k distinct integers from n possible integers, we can order them in exactly one way to ensure strictly increasing order. Thus the total possible strict sequence is C_n^k. Thus, $P = \frac{C_n^k}{n^k}$.

(b) The number of increasing sequences of k numbers from n is equal to choosing k items from n without order with replacement, i.e. C_{n+k-1}^k. Thus, $P = \frac{C_{n+k-1}^k}{n^k}$.

Problem 2

There are three boxes:

a A box containing two gold coins;

b A box containing two silver coins;

c A box containing one gold coin and a silver coin.

After choosing a box randomly and withdrawing one coin randomly, if that happens to be a gold coin, find the probability of the next coin drawn from the same box also being a gold coin.

Problem 2

Let $G1, G2$ be the event of first and the second coin drawn is a gold coin. Let A, B, C be the event of the box chosen is a, b, c respectively.

So, the probability of the next coin drawn from the same box also being a gold coin can be expressed as below:

$$\begin{aligned}
P(G_2|G_1) =& P(G_2|A, G_1)P(A|G_1) + P(G_2|B, G_1)P(B|G_1) \\
&+ P(G_2|C, G_1)P(C|G_1) \\
=& 1 \times P(A|G_1) + 0 + 0 = \frac{P(G_1|A)P(A)}{P(G_1)} \\
=& \frac{P(G_1|A)P(A)}{P(G_1|A)P(A) + P(G_1|B)P(B) + P(G_1|C)P(C)} \\
=& \frac{1 \times \frac{1}{3}}{1 \times \frac{1}{3} + 0 + \frac{1}{2} \times \frac{1}{3}} = \frac{2}{3}
\end{aligned}$$

Problem 3

$$P_{ij} = P_{i-1,j} \frac{j}{n} + P_{i-1,j-1} \frac{n-(j-1)}{n}$$

There are n types of toys, which you are collecting one by one. Each time you buy a toy, it is randomly determined which type it has, with equal probabilities. Let $p_{i,j}$ be the probability that just after you have bought your ith toy, you have exactly j toy types in your collection, for $i \geq 1$ and $0 \leq j \leq n$.

(a) Find a recursive equation expressing $p_{i,j}$ in terms of $p_{i-1,j}$ and $p_{i-1,j-1}$, for $i \geq 2$ and $1 \leq j \leq n$.

(b) Describe how the recursion from (a) can be used to calculate $p_{i,j}$.

Problem 3

(a) There are two ways to have exactly j toy types just after buying your ith toy: either you have exactly $j-1$ toy types just after buying your $i-1st$ toy and then the ith toy you buy is of a type you don't already have, or you already have exactly j toy types just after buying your $i-1st$ toy and then the ith toy you buy is of a type you do already have. Conditioning on how many toy types you have just after buying your $i-1st$ toy,

$$p_{ij} = p_{i-1,j-1}\frac{n-j+1}{n} + p_{i-1,j}\frac{j}{n}$$

for $i \geq 2$ and $1 \leq j \leq n$, with $j \leq i$ and

$$p_{i,j} = 0, \forall j > i.$$

Problem 3

(b) First note that $p_{11} = 1$, and $p_{ij} = 0$ for $j = 0$ or $j > i$. Now suppose that we have computed $p_{i-1,1}, p_{i-1,2}, \ldots, p_{i-1,i-1}$ for some $i \geq 2$. Then we can compute $p_{i,1}, p_{i,2}, \ldots, p_{i,i}$ using the recursion from (a). We can then compute $p_{i+1,1}, p_{i+1,2}, \ldots, p_{i+1,i+1}$ using the recursion from (a), and so on. By induction, it follows that we can obtain any desired p_{ij} recursively by this method.

Problem 4

Consider the following 7-door version of the Monty Hall problem. There are 7 doors, behind one of which there is a car (which you want), and behind the rest of which there are goats (which you don't want). Initially, all possibilities are equally likely for where the car is. You choose a door. Monty Hall then opens 3 goat doors, and offers you the option of switching to any of the remaining 3 doors.

Assume that Monty Hall knows which door has the car, will always open 3 goat doors and offer the option of switching, and that Monty chooses with equal probabilities from all his choices of which goat doors to open. Should you switch? What is your probability of success if you switch?

Problem 4

Assume the doors are labeled such that you choose Door 1 (to simplify notation), and suppose first that you follow the "stick to your original choice" strategy. Let S be the event of success in getting the car, and let C_j be the event that the car is behind Door j. Conditioning on which door has the car, we have

$$P(S) = \frac{1}{7}. = P(C_1) \qquad P(S|M_{ijk})$$

Let M_{ijk} be the event that Monty opens Doors i, j, k. Then

$$P(S) = \sum_{i,j,k} P(S|M_{i,j,k})P(M_{i,j,k}), 2 < j < k \leq 7 \quad P(M_{ijk})$$

By symmetry, this gives $P(S|M_{i,j,k}) = P(S) = \frac{1}{7}$. Thus the conditional probability that the car is behind 1 of the remaining 3 doors is 6/7, which gives 2/7 for each. So you should switch, thus making your probability of success 2/7 rather than 1/7

Problem 5

Let X and Y be independent geometric random variables, where X has parameter p and Y has parameter q.

(a) What is the probability that $X = Y$?

(b) What is $E[\max(X, Y)]$?

(c) What is $P(\min(X, Y) = k)$?

(d) What is $E[X|X \leq Y]$?

Problem 5

(a)

$$\mathbb{P}[X = Y] = \sum_x (1-p)^x p (1-q)^x q$$

$$= \sum_x [(1-p)(1-q)]^x pq$$

$$= \frac{pq}{1-(1-p)(1-q)}$$

$$= \frac{pq}{p+q-pq}$$

X Y

$0 < ? < 1,$

$$\sum_{n=0}^{+\infty} t^n = \frac{1}{1-t}$$

(b) Because $\max(X, Y) = X + Y - \min(X, Y)$ so $\mathbb{E}[\max(X, Y)] = \mathbb{E}[X] + \mathbb{E}[Y] - \mathbb{E}[\min(X, Y)]$. From below, in part (c), we know that $\min(X, Y)$ is a geometric random variable mean $p + q - pq$. Therefore, $\mathbb{E}[\min(X, Y)] = \frac{1+pq-p-q}{p+q-pq}$, and we get

$$\mathbb{E}[\max(X, Y)] = \frac{1-p}{p} + \frac{1-q}{q} - \frac{1+pq-p-q}{p+q-pq}$$

Problem 5

(c) We split this event into two disjoint events.

$$\begin{aligned}
\mathbb{P}[\min(X, Y) = k] &= \mathbb{P}[X = k, Y \geq k] + \mathbb{P}[X > k, Y = k] \\
&= \mathbb{P}[X = k]\mathbb{P}[Y \geq k] + \mathbb{P}[X > k]\mathbb{P}[Y = k]
\end{aligned}$$

because

$\mathbb{P}[X > k] = 1 - \mathbb{P}[x \leq k] = 1 - \sum_{x=0}^{k}(1-p)^x p = (1-p)^k(1-p).$

Finally, we get

$$\begin{aligned}
\mathbb{P}[\min(X, Y) = k] &= (1-p)^k p(1-q)^k + (1-p)^k(1-p)(1-q)^k q \\
&= [(1-p)(1-q)]^k(p + (1-p)q) \\
&= [(1-p)(1-q)]^k(p + q - pq)
\end{aligned}$$

Problem 5

(d)

$$\mathbb{E}[X \mid X \leq Y] = \sum_{x \geq 0} x \mathbb{P}[X = x \mid x \leq Y]$$

$$= \sum_{x \geq 0} x \frac{\mathbb{P}[X = x, x \leq Y]}{\mathbb{P}[X \leq Y]}$$

$P(Y \geq z)$

First, let's consider the denominator.

$$\mathbb{P}[X \leq Y] = \sum_{z \geq 0} \mathbb{P}[X = z, z \leq Y]$$

$X = 0, 1, 2$

$$= \sum_{z} \mathbb{P}[X = z]\mathbb{P}[Y \geq z]$$

$$= \sum_{z} (1-p)^z p (1-q)^z$$

$$= p \sum_{z} [(1 - p - q + pq)]^z = \frac{p}{p + q - pq}$$

Problem 5

Now we can compute the whole equation.

$$\begin{aligned}
\mathbb{E}[X \mid X \leq Y] &= \frac{p+q-pq}{p} \sum_{x \geq 0} x\mathbb{P}[X = x]\mathbb{P}[x \leq Y] \\
&= \frac{p+q-pq}{p} \sum_{x} x(1-p)^x p(1-q)^x \\
&= (p+q-pq) \sum_{x} x(1+pq-p-q)^x
\end{aligned}$$

This is equal to the expectation of a geometric random variable with mean $p+q-pq$. Therefore

$$\mathbb{E}[X \mid X \leq Y] = \frac{1+pq-p-q}{p+q-pq}$$

Problem 6

Suppose there are n types of toys, which you are collecting one by one. Each time you collect a toy, it is equally likely to be any of the n types. What is the expected number of distinct toy types that you have after you have collected t toys? (Assume that you will definitely collect t toys, whether or not you obtain a complete set before then.)

t

Problem 6

Let I_j be the indicator of having the *jth* toy type in your collection after having collected t toys. By symmetry, linearity, and the fundamental bridge, the desired expectation is:

$$n(1 - (\frac{n-1}{n})^t)$$

Problem 7

$\text{H} = p.$

A coin with probability p of Heads is flipped n times. The sequence of outcomes can be divided into runs (blocks of H's or blocks of T's), *e.g.*, HHHTTHTTTH becomes $\boxed{\text{HHH}}\boxed{\text{TT}}\boxed{\text{H}}\boxed{\text{TTT}}\boxed{\text{H}}$, which has 5 runs. Find the expected number of runs.

Problem 7

$\sum \left(I_{n-1} = 0 \quad I_n = 1 \right)$

X [H] H H T n. TTT HH

(1-p) p

Let I_j be the indicator for the event that position j starts a new run, for $1 \le j \le n$. Then $I_1 = 1$ always holds. For $2 \le j \le n$, $I_j = 1$ if and only if the jth toss differs from the $(j-1)st$ toss. So for $2 \le j \le n$,

H (p T (1-p))

$$E(I_j) = P((j-1)st\ toss\ H\ and\ jth\ toss\ T, or\ vice\ versa) = 2p(1-p)$$

Hence, the expected number of runs is $1 + 2(n-1)p(1-p)$.

Problem 8

: CDF. $P(X \le x)$

$= P(U_1 \le x, U_2 \le x, \ldots, U_n \le x)$

Let $U_1, \ldots, U_n$ be i.i.d. Unif(0, 1) and $X = \max(U_1, \ldots, U_n)$.
What is the PDF of X? What is $E(X)$?

$= P(U_1 \le x) \cdot P(U_2 \le x) \cdots P(U_n \le x)$

Problem 8

$X \leq x \iff U_1 \leq x \cap \cdots \cap U_n \leq x.$

$$\begin{aligned} P(X \leq x) =& P(U_1 \leq x, \ldots, U_n \leq x) \\ =& \underbrace{P(U_1 \leq x)}_{x} \cdots P(U_n \leq x) \\ =& x^n. \end{aligned}$$

Thus, $f(x) = nx^{n-1} \quad (0 < x < 1).$

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$$\begin{aligned} E[X] = \int_0^1 xf(x)\,\mathrm{d}x =& \int_0^1 x \cdot nx^{n-1}\,\mathrm{d}x \\ =& n\int_0^1 x^n\,\mathrm{d}x \\ =& \frac{n}{n+1}. \end{aligned}$$