Problem description: Solve the following initial-value problem

$$f'(x) = y(x, f) \quad , x \in [0, 2] \tag{1}$$

with $f(0) = 1$ and $y(x, f) = x + f$.

Requirements:

i). For the ordinary differential equation problem (1), applying the

1). Fourth-order Adams-Bashforth technique as g_1,

2). Adams-Bashforth Three-Step Explicit Method as g_2,

3). Adams-Moulton Four-Step Implicit Method as g_3,

4). Adams Fourth-Order Predictor-Corrector Method as g_4,

with different step-sizes $h_1 = \frac{1}{5}, h_2 = \frac{1}{10}, h_3 = \frac{1}{20}, h_4 = \frac{1}{40}, h_5 = \frac{1}{80}$.
Note that you should calculate the initial values before you start to apply linear multistep methods.

ii). Plot your approximation results with different step-size and the real function in one figure for different method. Concretely, you should plot five approximation results with given step-size and the real function in each figure. i.e., $g_1(h_1)$, $g_1(h_2)$, $g_1(h_3)$, $g_1(h_4)$, $g_1(h_5)$ and $f(x)$ should be in one figure. The $g(h_j)$ means the approximation function with node points from step-size $h_j, j = 1, 2, 3, 4, 5$.

iii). Calculate the residuals of $f(\cdot)$ and $g_i(\cdot), i = 1, 2, 3, 4$ with different step-size, i.e., error $= |f(x) - g(x)|, g(\cdot) = g_i(\cdot), i = 1, 2, 3, 4$ where x is the node points obtained according to step-size $h_j, j = 1, 2, 3, 4, 5$. Then plot approximation error with different step-size for the four methods.

iv). Define

$$P_h = \log_2 \frac{|\text{error}_{j-1}|_\infty}{|\text{error}_j|_\infty}, \ j = 2, 3, 4, 5,$$

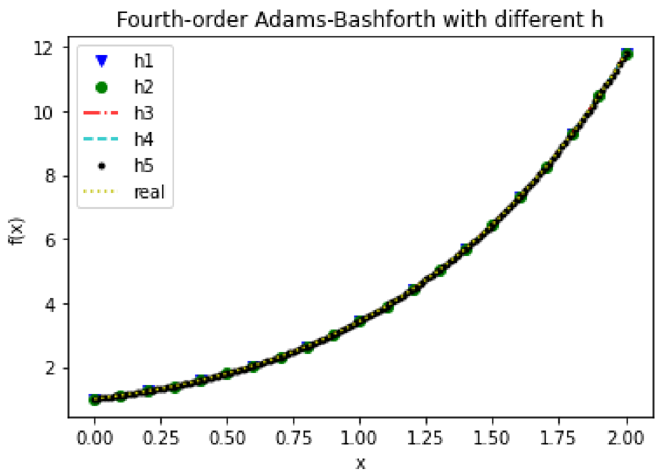
where $\text{error}_j = |f(x_{h_j}) - g(x_{h_j})|$, and $|\cdot|_\infty$ denote the ∞-norm,i.e., $|x|_\infty = \max_i |x_i|, x \in \mathbb{R}^n$. Then apply the definition to calculate the P_h for different method, and complete the following table. For convenience, keeping five decimals for your results when you calculate P_h.

j	$P_h(g_1)$	$P_h(g_2)$	$P_h(g_3)$	$P_h(g_3)$	$P_h(g_4)$	$P_h(g_4)$
2	3.49377	2.66105	4.52256	5.15274	1.65414	-0.11878
3	3.76631	2.84277	4.78202	4.24942	3.29962	3.18844
4	3.88679	2.92368	4.89509	4.71222	3.6946	3.66511
5	3.94419	2.9623	4.94762	4.8673	3.8556	3.84446

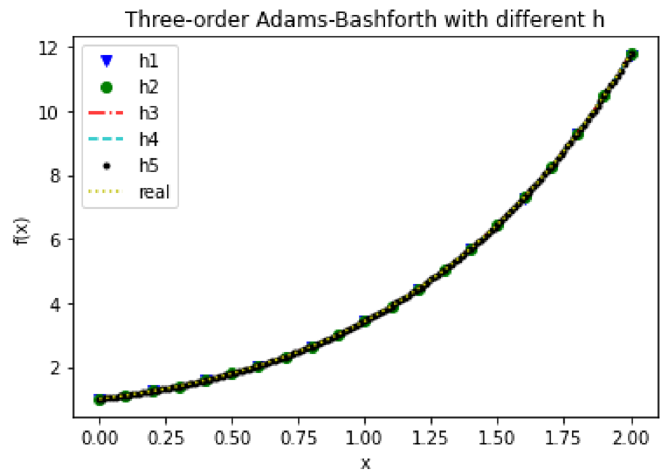
Table 1

Solution: Initialization with RK6 for $g_{1,2,3,4}$.

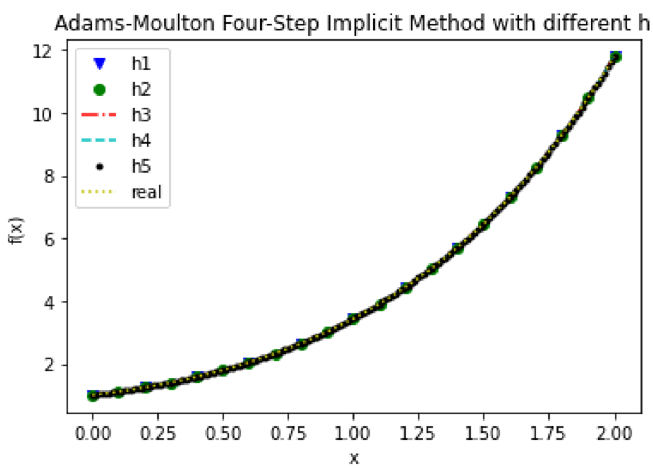
ii) The approximation for the problem (1) with different method as follows:



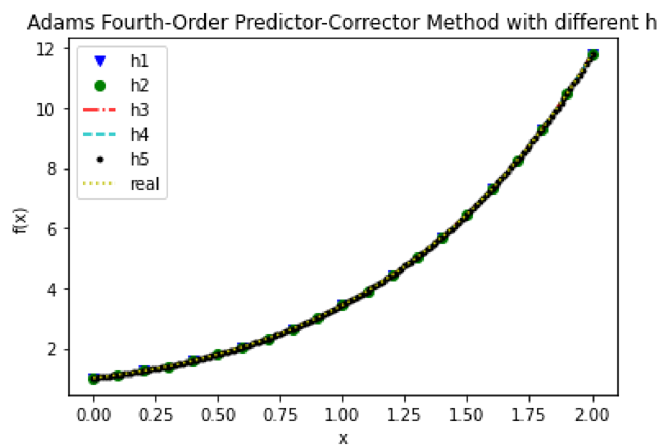
(a) Fourth-order Adams-Bashforth



(b) Adams-Bashforth Three-Step Explicit Method



(c) Adams-Moulton Four-Step Implicit Method



(d) Adams Fourth-Order Predictor-Corrector Method

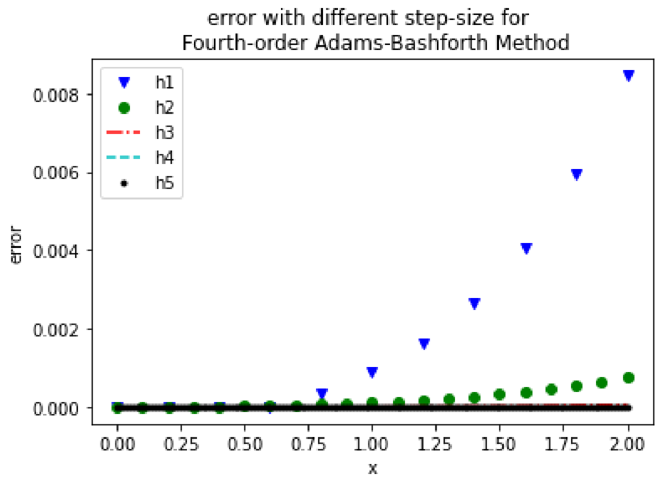
Figure 1: Approximation (1) With Different Method.

While we can calculate the "real" solution for problem (1) as

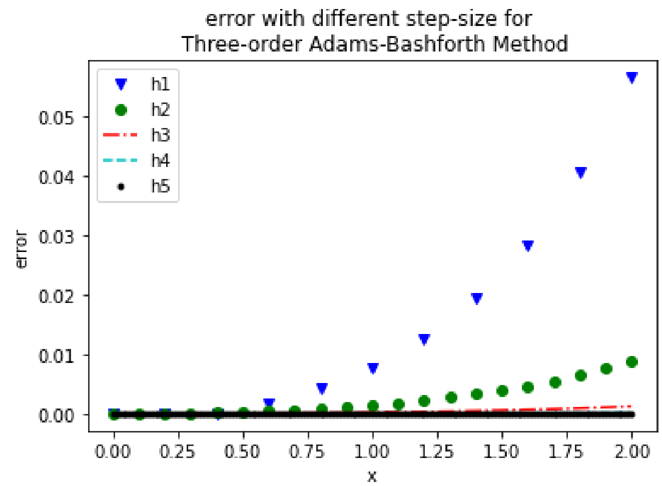
$$f(x) = 2e^x - x - 1,$$

the Figure 1 shows that all the method in the requirements have the good approximation for the ODE (1).

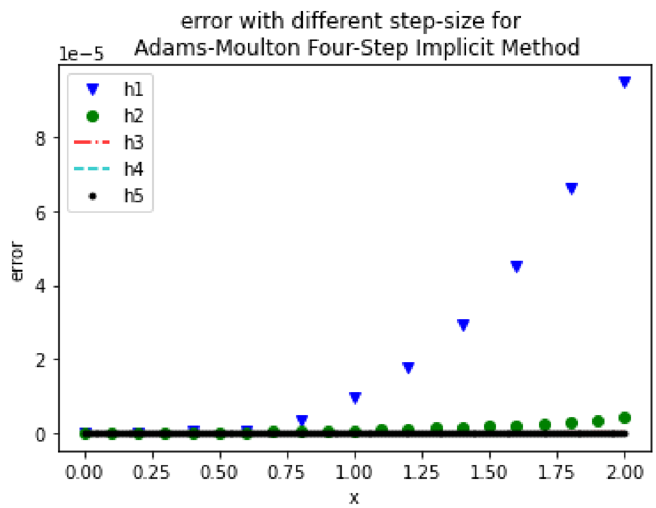
iii) The local truncation error for different method as follows:



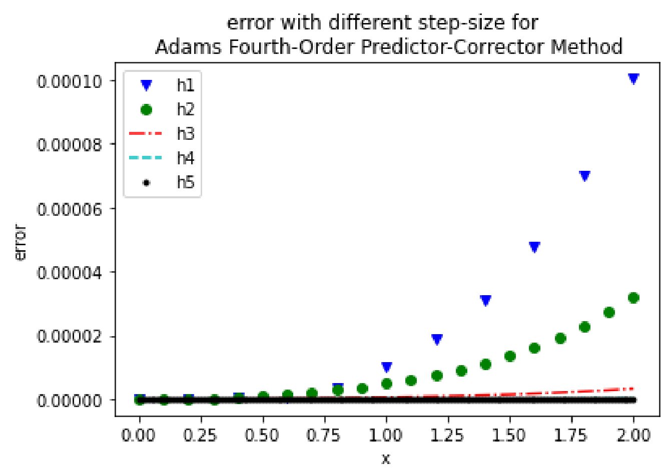
(a) Fourth-order Adams-Bashforth



(b) Adams-Bashforth Three-Step Explicit Method



(c) Adams-Moulton Four-Step Implicit Method



(d) Adams Fourth-Order Predictor-Corrector Method

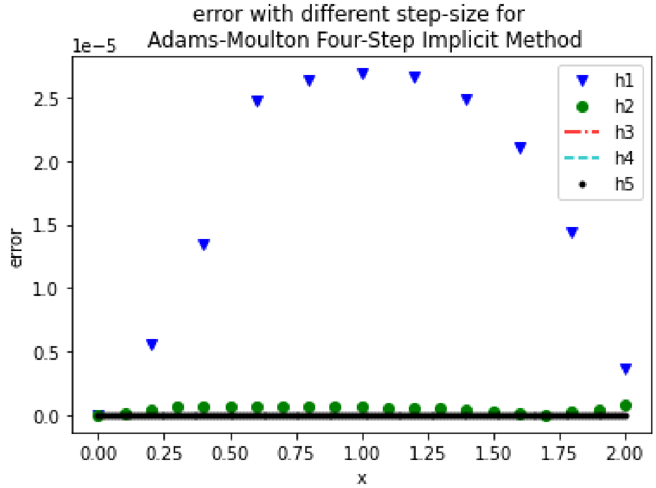
Figure 2: Local Truncation With Different Method For Solving Problem (1).

The Figure 2 shows that with the step-size becoming smaller all the method in the requirements make the smaller actual error.

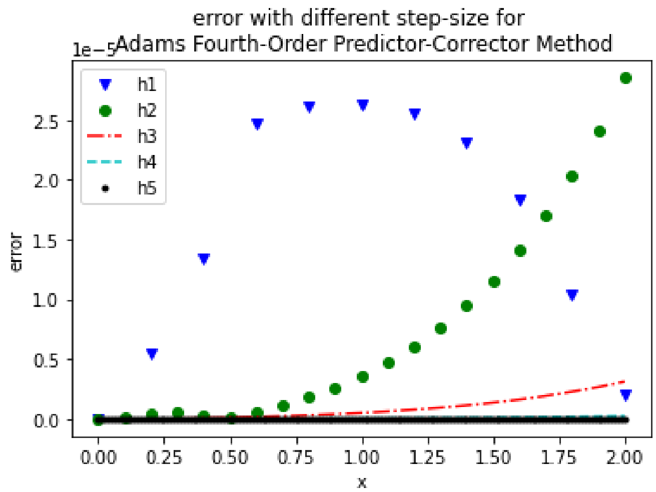
As for the theoretical error estimation, Adams-Moulton Four-Step Implicit Method has the local truncation error with the highest order,i.e., is $O(h^5)$,so we need Initialize g_3 with RK5 for, or other methods with higher order. The Adams-Bashforth Three-Step Explicit Method has has the local truncation error with the lowest order,i.e., is $O(h^3)$.

While the Adams Fourth-Order Predictor-Corrector Method need to predict and correct, it may cause the order for Adams Fourth-Order Predictor-Corrector Method is not approx to 4 if we make a "bad" approximation in the prediction step. To improve the accuracy, we can do more times correct-step.

Here is a typical mistake is we use RK4 for g_3, then you will get the P_h in Table 1, and the actual error as follows.

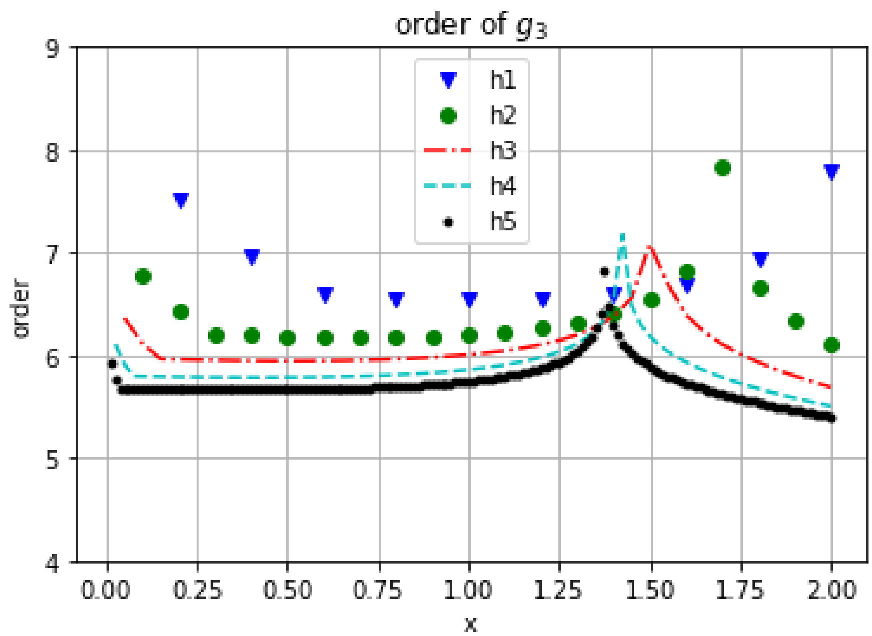


It is wright to use RK4 for g_4, then yow will get the P_h in Table 1 and the actual error as

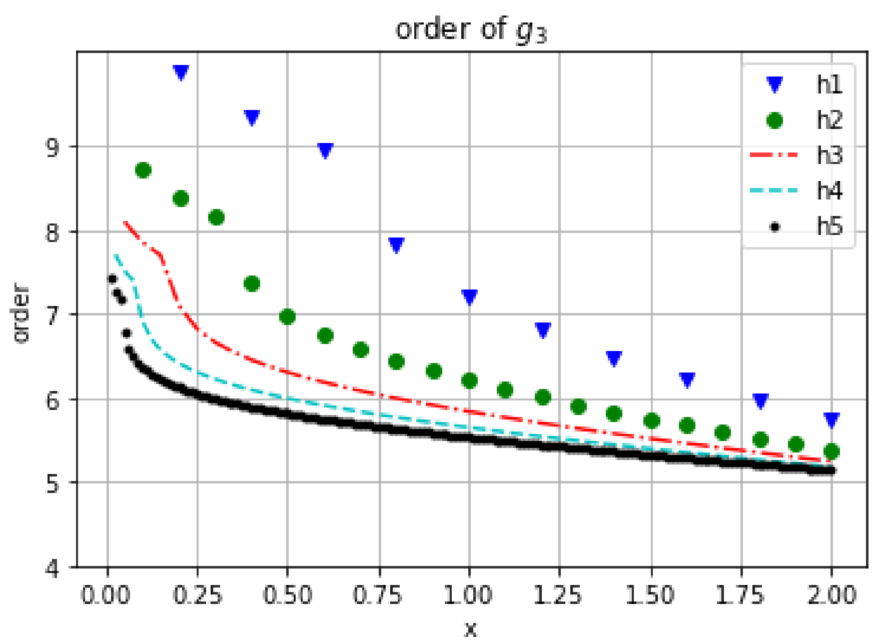


iv) For one method, the number in Table 1 should "limit" to the order for the theoretical error.

Appendix:
Here are two figures to show the asymptotic stability of the "order"



(a) g3 with RK4, non-convergence



(b) g3 with RK6,convergence to the order

Figure 3: The local asymptotic stability