EE160 Homework 4

Deadline: 2022-12-24, 23:59:59, Submit your homework on Blackboard
(Hint:You can use MATLAB to help you do the homework.)

1. A block diagram of a turret lathe control system is shown in Figure 1. The parameters in the system are $n = 0.2$, $J = 10^{-3}$ and $b = 2.0 \times 10^{-2}$. It is necessary to attain an accuracy of 4.7×10^{-4} inches. To satisfy this condition, a steady-state position accuracy of 2% is specified for a ramp input. Design a cascade compensator to be inserted before the controller in order to provide a response to a step command with a percent overshoot of $P.O \leq 4.5\%$. A suitable damping ratio for this system is $\zeta \geq 0.7$. The gain of the controller is $K_R = 5$. Design a suitable phase-lag compensator with the following two methods.

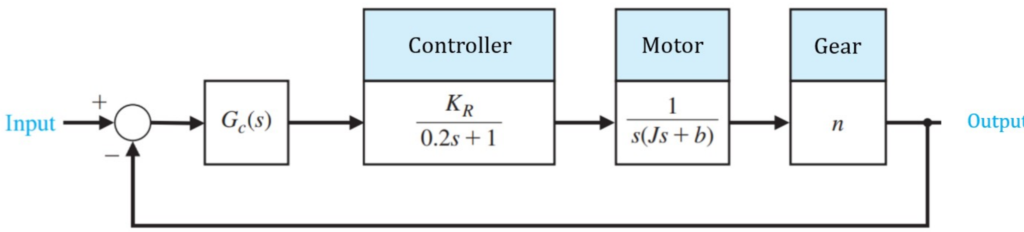


Figure 1: A feedback control system

   (a) Use bode plot.(10')
   (b) Use root locus.(10')

Solution: The transfer function is $G(s) = \frac{5000}{s(s+5)(s+20)}$. To meet the steady-state accuracy. We need $K_v \geq 50$. The uncompensated $K_v = 50$, so the steady-state accuracy can be met. Thus $K = 1$. The uncompensated system has $P.M. = -15°$.

(a)

$$T(s) = \frac{Y(s)}{R(s)} = \frac{5000}{s^3 + 25s^2 + 100s + 5000}$$

We can let

$$\frac{K\alpha(1+\tau s)}{(1+\alpha\tau s)}$$

Since $\zeta = 0.7$, we can know the linear approximation is $\zeta = 0.01 P.M.$. Thus, the phase margin of desired the system is $P.M. = \frac{\eta}{0.01} = 70°$.

$$\tan P.M. = \frac{1-\alpha}{2\sqrt{\alpha}}$$

$$\frac{1-\alpha}{\alpha+1} = \sin P.M. = \sin 70°$$

$$\alpha = 32.16$$

We can choose $\alpha = 50$ to satisfy the loop gain and the phase margin.
We can let the compensator satisfy

$$G_c(s) = \frac{bs+1}{50bs+1}$$

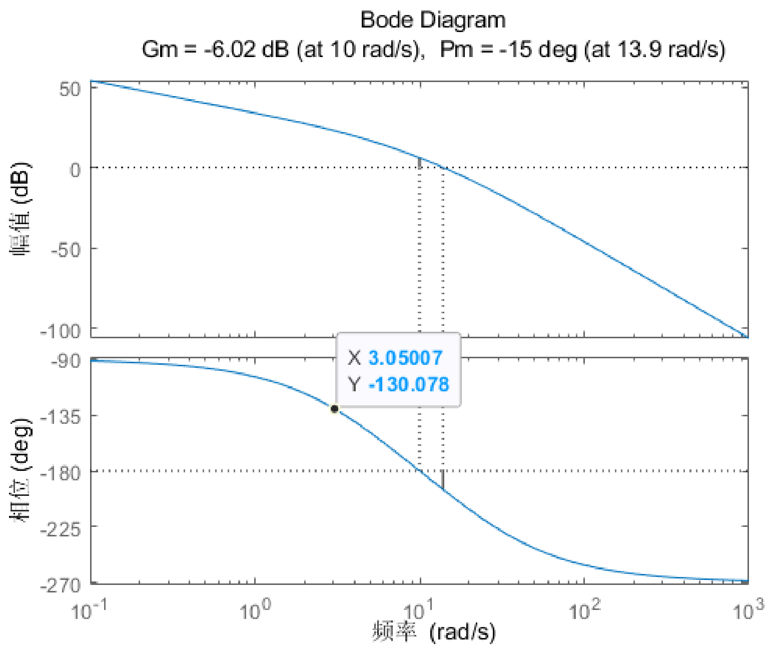


Figure 2: bode plot of the open-loop system

We can know that the phase margin changes with b in the following way:

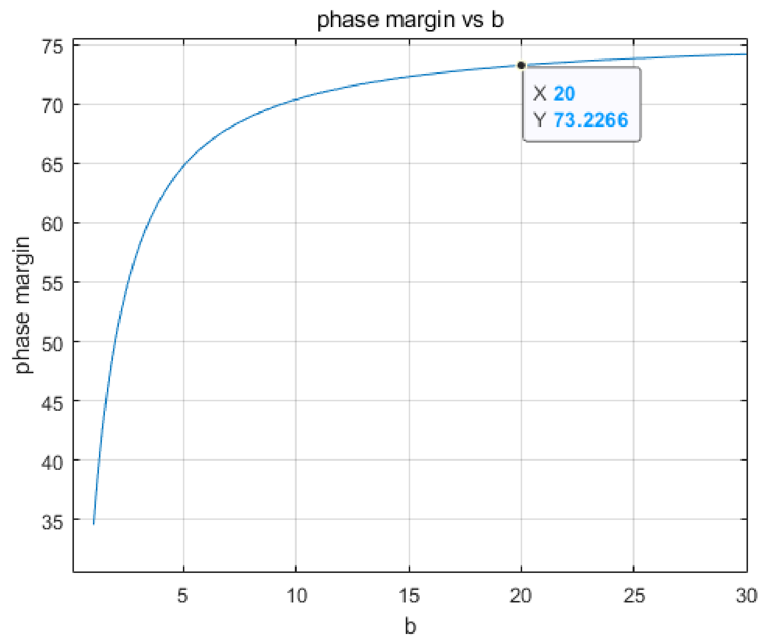


Figure 3: Phase margin versus b

We can choose $b = 15$ The transfer function of the compensator is

$$G_c(s) = \frac{1+20s}{1+1000s}$$

The total loop transfer function is

$$G_c(s)G(s) = \frac{5000(1+20s)}{s(s+5)(s+20)(1+1000s)}$$

The step response is

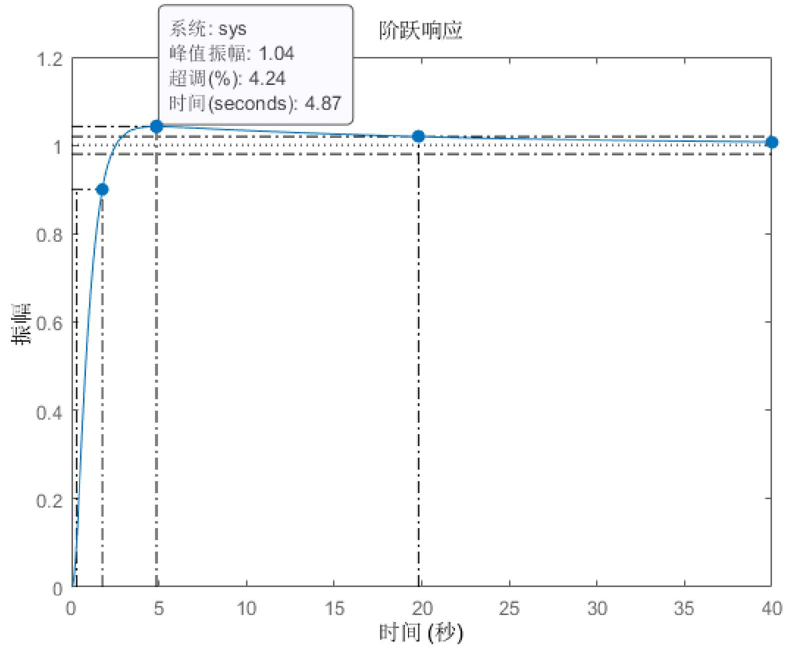


Figure 4: The step response of the closed-loop system

We can see that P.O. and steady-state accuracy requirement is satisfied.

(b) Since we require $\zeta = 0.7$ to meet the $P.O.$ specifications. We can also know that the uncompensated loop transfer function is

$$L(j\omega) = G_c(j\omega)G(j\omega) = \frac{5000K}{j\omega(j\omega+5)(j\omega+20)}$$

The root locus is as follow:

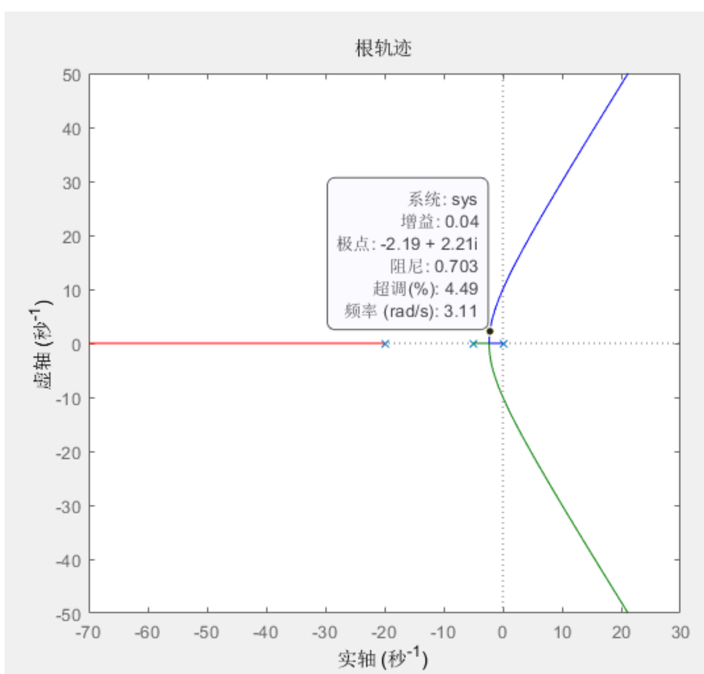


Figure 5: root locus of the system

The roots are $s = -2.19 \pm 2.21j$. The gain $K = 0.04$. Therefore, $K_v = \frac{5000}{20*5} \cdot K = 2$. Thus, the required ratio of the zero to the pole of the compensator is

$$\left|\frac{z}{p}\right| = \alpha = \frac{K_{v,comp}}{K_{v,unc}} = 25$$

We can choose $z = 0.004$ and then $p = 0.004/25$. The compensator is

$$G_c(s) = \frac{0.04(250s+1)}{(6250s+1)}$$

The compensated system loop transfer function is

$$L(s) = G_c(s)G(s) = \frac{200(250s+1)}{s(s+5)(s+20)(6250s+1)}$$

The step response of the system is

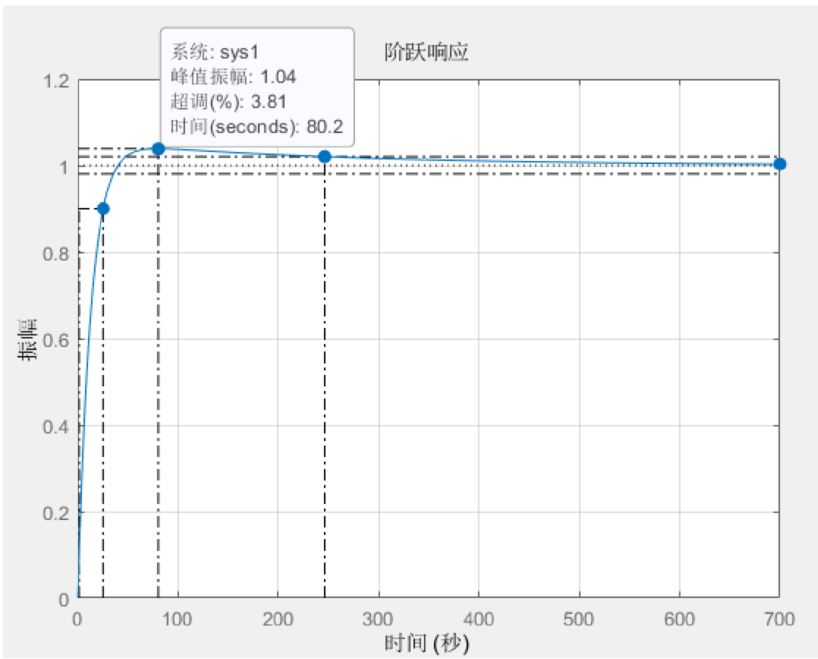


Figure 6: The step response of the closed-loop system

We can see that P.O. and steady-state accuracy requirement is satisfied.

2. Consider a unity feedback system in Figure 7. We want the step response of the system to have a percent overshoot of $P.O. \leq 9\%$ and a settling time (with a 2% criterion) of $T_s \leq 4$ s.

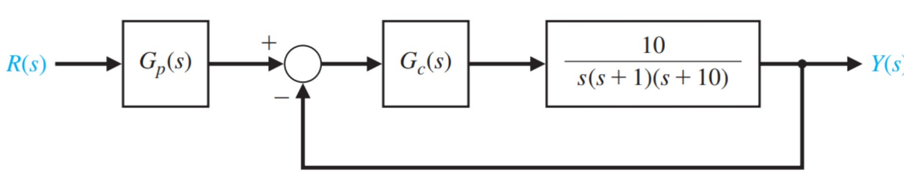


Figure 7: A feedback control system

(a) Design a phase-lead compensator $G_c(s)$ to achieve the dominant roots desired.(15')

(b) If we add a PD controller between the system and the compensator, and change the system to $\frac{1}{s(s+1)(s+2)}$. Design a first-order compensator($\frac{s+z}{s+p}$) and a first-order prefilter($\frac{z}{s+z}$), and determine the coefficients that yield the optimal deadbeat response.(15')

Solution:

(a) From the overshoot specification $P.O. = 9\%$. Thus, $\zeta \geq 0.61$. The plant transfer function is

$$G(s) = \frac{10}{s(s+1)(s+10)}$$

Let $G_p = 1$. We can know the uncompensated system, $K = 20$ is a good choice. $K_v = 20/10 = 2$, $\alpha = 50/2 = 25$. We can choose the compensator

$$G_c(s) = K\frac{s+0.5}{s+12.5}$$

By root locus, we can know that $K = 60$ yield $P.O. \approx 9\%$. The compensated system is $\frac{600(s+0.5)}{s(s+12.5)(s+1)(s+10)}$. The step response is

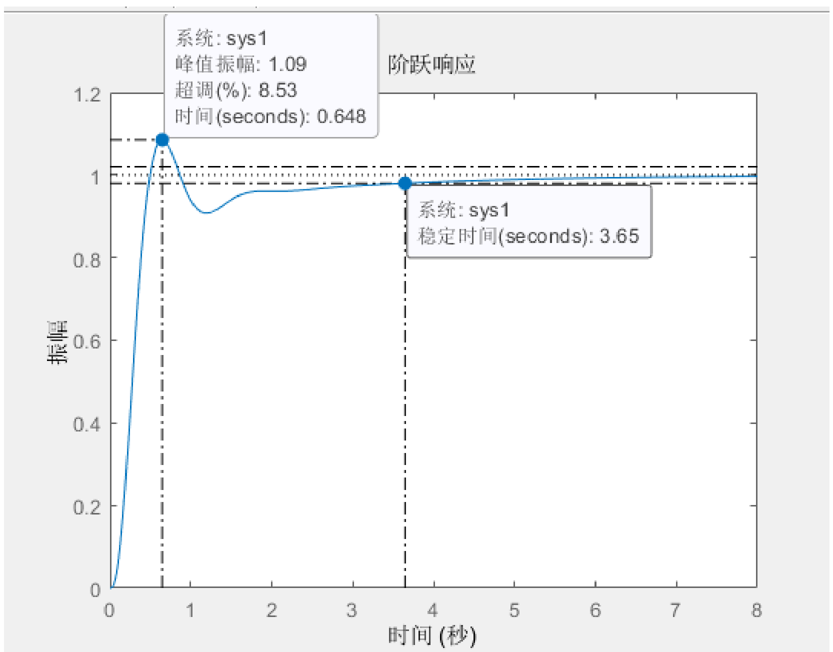


Figure 8: The step response of the closed-loop system

(b) The PD controller is $K_d s + K_p$. In order to simplify our problem, we can choose $\frac{K_d}{K_p} = 1$ The closed-loop transfer function is

$$\begin{aligned} T(s) &= \frac{K_p z}{(s+p)s(s+2) + K_p(s+z)} \\ &= \frac{K_p z}{s^3 + (2+p)s^2 + (K_p + 2p)s + K_p z} \end{aligned}$$

We use Table 10.2 to determine the required coefficients

Table 10.2 Coefficients and Response Measures of a Deadbeat System

System Order	Coefficients α	β	γ	δ	ϵ	Percent Overshoot P.O.	Percent Overshoot P.U.	90% Rise Time T_r	Settling Time T_s
2nd	1.82					0.10%	0.00%	3.47	4.82
3rd	1.90	2.20				1.65%	1.36%	3.48	4.04
4th	2.20	3.50	2.80			0.89%	0.95%	4.16	4.81
5th	2.70	4.90	5.40	3.40		1.29%	0.37%	4.84	5.43
6th	3.15	6.50	8.70	7.55	4.05	1.63%	0.94%	5.49	6.04

Note: All times are normalized.

Figure 9: Table 10.2

$$\alpha = 1.9, \beta = 2.2$$

If we select $T_s = 2s$, then $\omega_n T_s = 4.04$, and thus $\omega_n = 2.02$

$$q(s) = s^3 + \alpha\omega_n s^2 + \beta\omega_n^2 s + \omega_n^3 = s^3 + 3.84s^2 + 8.98s + 8.24$$

Then, we determine that

$$p = 1.84, z = 1.55, K_d = 5.30, K_p = 5.30$$

We can see the step response of the system is

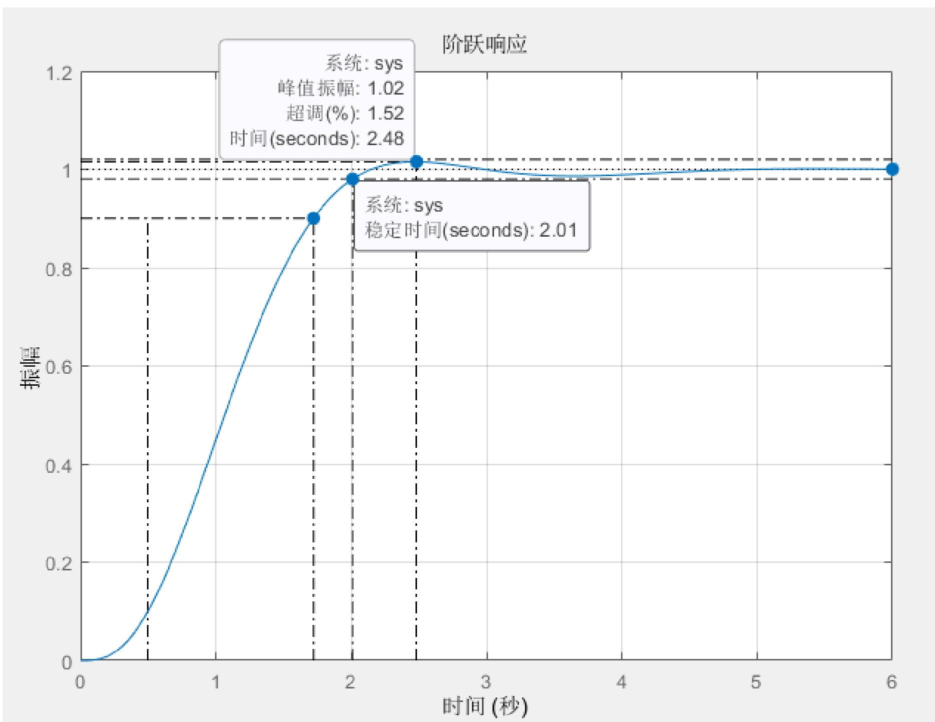


Figure 10: The step response of the closed-loop system

3. Prove the following propositions:

(a) If two state-space models share the same controllable canonical form, the two models are consistent in controllability. (5')

(b) Consider a general system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \tag{1}$$

Define the dual system of (1)

$$\begin{aligned} \dot{x} &= A^\top x + C^\top u \\ y &= B^\top x \end{aligned} \tag{2}$$

Show that system (1) is observable if and only if system (2) is controllable. (5')

Solution:

(a)

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}$$

z = P x, P is invertable.

$$\begin{aligned} \dot{z} &= PAP^{-1}z + PBu \\ y &= CP^{-1}x \end{aligned}$$

$$R_1 = \begin{bmatrix} PB & PAP^{-1}PB & PA^2P^{-1}PB & \cdots & PA^{n-1}P^{-1}PB \end{bmatrix} = P\begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix}$$
$$R_0 = \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix}$$

rank(R_0)=rank(R_1)

4.

$$R_o = \begin{bmatrix} C & CA & CA^2 & \cdots & CA^{n-1} \end{bmatrix}$$
$$R_c = \begin{bmatrix} C^\top & A^\top C^\top & (A^2)^\top C^\top & \cdots & (A^{n-1})^\top C^\top \end{bmatrix}$$
$$R_o = R_c^\top$$

rank(R_o)=rank(R_c)

5. Consider the third-order system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -8 & -3 & -1 \end{bmatrix} \quad B = \begin{bmatrix} -1 \\ 2 \\ -6 \end{bmatrix} \quad C = \begin{bmatrix} 2 & 8 & 10 \end{bmatrix}$$

(a) Sketch a block diagram model of the system. (5')

(b) Write the transfer function of the system $G(s) = Y(s)/U(s)$. (5')

(c) Check the controllability and observability of the system. (5')

(d) Design a full-state observer for the system with an expected settling time of less than 1 second. (5')

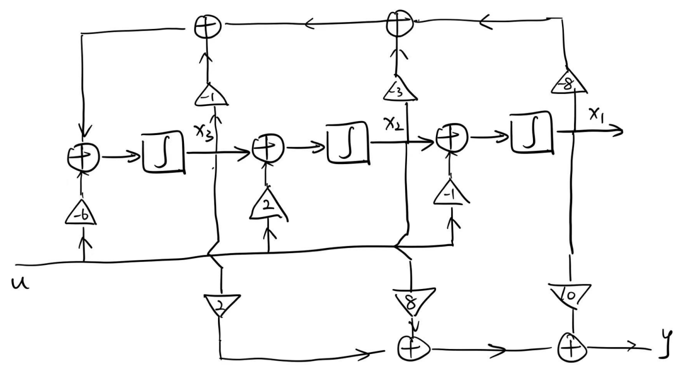
(e) Suppose system state $x(t)$ is available. Design a full-state feedback controller for the system. The desired poles of the closed-loop system are $\begin{bmatrix} -4+j3 & -4-j3 & -8 \end{bmatrix}$. (7')

(f) Prove that if (A, B) are controllable, (A, C) are observable, the closed-loop system with full-state observer-based feedback controller is stable. Verify the proposition with the control scheme designed above. (8')

(g) Consider a piece-wise constant reference signal $r(t)$. Design a compensator such that the tracking error $y(t) - r(t)$ asymptotically converges to zero. (5')

Solution:

(a) No standard answer. Full score if satisfy the requirements of the block diagram



(b)

$$G(s) = -\frac{46s^2 + 10s + 110}{s^3 + s^2 + 3s + 8}$$

(c)

$$R_c = \begin{bmatrix} 1 & -1 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

full-rank

$$R_o = \begin{bmatrix} -46 & 36 & -8 \\ -10 & 28 & 260 \\ -110 & 368 & -288 \end{bmatrix}$$

full-rank

(d)

$$\text{Re}\{\text{eig}(A - LC)\} < -1$$

(e)

$$K = \begin{bmatrix} 15 & 86 & 192 \end{bmatrix}$$

(f)

$$\begin{bmatrix} A + BK & -BK \\ 0 & A + LC \end{bmatrix}$$

verify:

$$\text{Re}\left\{\text{eig}\left(\begin{bmatrix} A & BK \\ -LC & A + LC + BK \end{bmatrix}\right)\right\} < 0$$

(g) introduce integral term to overcome steady state error. E.g.:

$$e(t) = y(t) - r(t)$$

$$u(t) = -K_i \int_0^t e(\tau) d\tau - K\hat{x}(t)$$

$K_i > 0$

6. Consider a LRC circuit with input voltage $v_i(t)$ and output voltage $v_o(t)$. The system model is given as

$$\frac{V_o(s)}{V_i(s)} = \frac{1}{LCs^2 + RCs + 1}$$

where $L = 0.1$, $R = 0.5$, $C = 20$.

(a) Write the above system into the state space model

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t)$$

where $x(t) = [v_o(t)\ \dot{v}_o(t)]^\top$ (3')

(b) Suppose input $u(t)$ is unknown. Design a system state observer $\hat{x}(t)$ such that the amplitude of the frequency response of observation error $e(t) = C\hat{x}(t) - y(t)$ is smaller than -15 dB. (7')

(c) Design the infinite LQR controller law $u(t) = -Kx(t)$ that minimizes the infinite horizon cost

$$\int_0^\infty 3v_o^2(t) + \dot{v}_o^2 + v_i^2(t)dt$$

Write the corresponding optimal control problem in standard form, indicate Q matrix and R matrix, then solve the control gain K explicitly. (5')

(d) Compare the performance and control effort of (c) and $K = [-0.5 \quad -2]$ using MATLAB. The initial state is given as $x_0 = [3, 2]^\top$, simulation time $T = 100s$. (5')

Solution:

(a)

$$A = \begin{bmatrix} 0 & 1 \\ -0.5 & -5 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0.5 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

(b)

$$G(s) = \frac{E(s)}{U(s)} = -C(sI - A - LC)^{-1}B = \frac{1}{l_1 + 5l_2 - 10s + l_2 s - 2s^2 - 1}$$

$$G(jw) = \frac{1}{(l_1 + 5l_2 - 1 + 2w^2) + (l_2 - 10)jw}$$

$$p(w^2) = (l_1 + 5l_2 - 1 + 2w^2)^2 + (l_2 - 10)^2 w^2 = 0$$

$p(w^2)$ is a quadratic polynomial of w^2, substitute L, then find the min value of $p(w^2)$ and $max_w|G(jw)| = 1/p(w^2) < 15dB$

(c)

$$Q = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \quad R = 1 \quad K = [0.407\ 0.618]$$

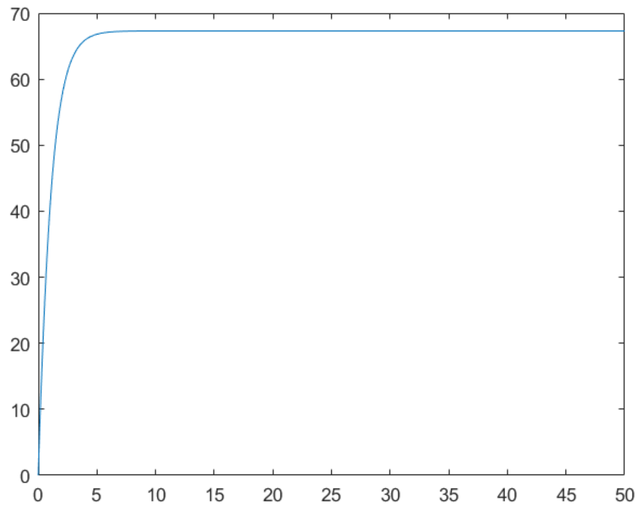


Figure 11: K cost

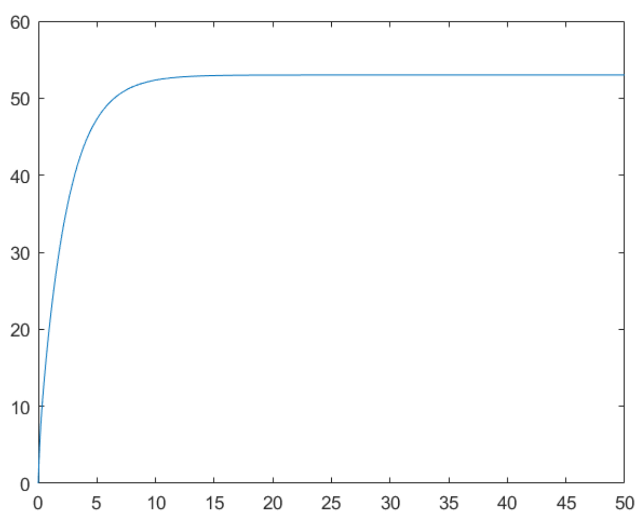


Figure 12: LQR cost

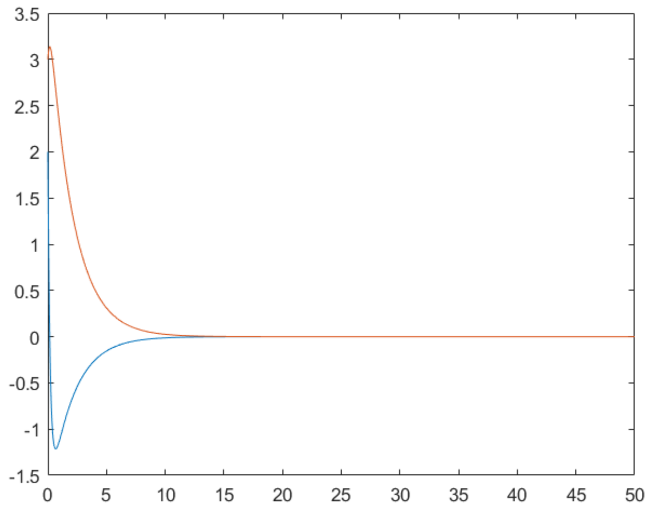


Figure 13: K state

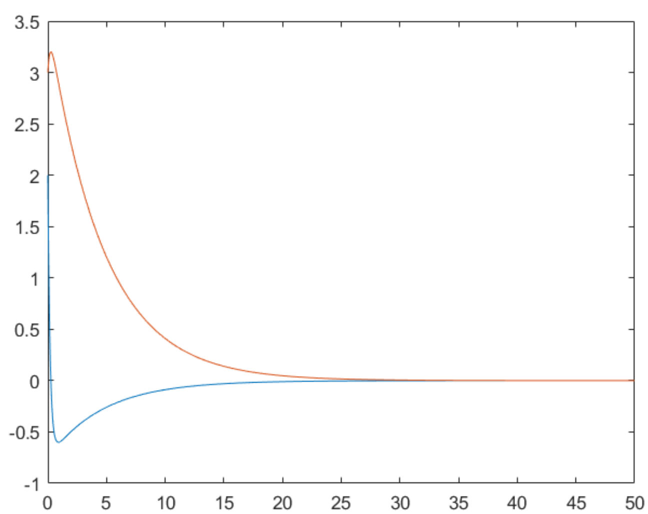


Figure 14: LQR state