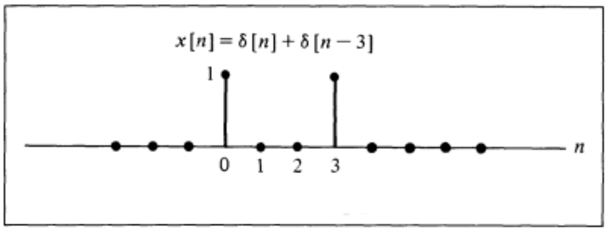
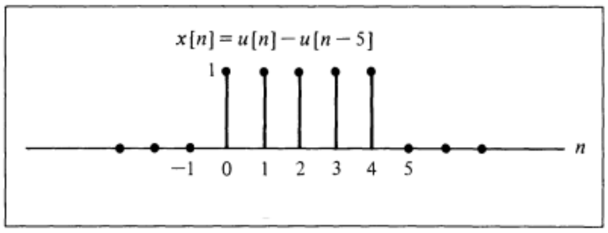
1. [15 points] Sketch each of the following signals.

$$
\begin{aligned}
&\textbf{(a)} x[n] = \delta[n] + \delta[n-3] \\
&\textbf{(b)} x[n] = u[n] - u[n-5] \\
&\textbf{(c)} x[n] = \delta[n] + \frac{1}{2}\delta[n-1] + (\frac{1}{2})^2\delta[n-2] + (\frac{1}{2})^3\delta[n-3] \\
&\textbf{(d)} x(t) = u(t+3) - u(t-3) \\
&\textbf{(e)} x(t) = \delta(t+2) \\
&\textbf{(f)} x(t) = e^{-t}u(t)
\end{aligned}
$$

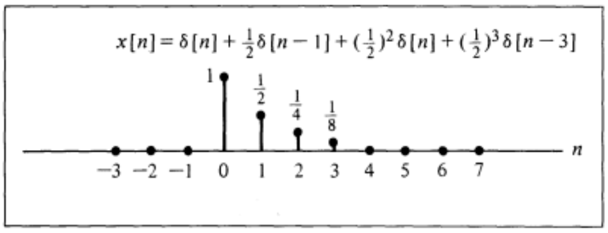
(a)



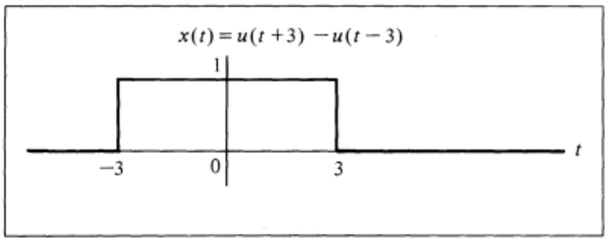
(b)



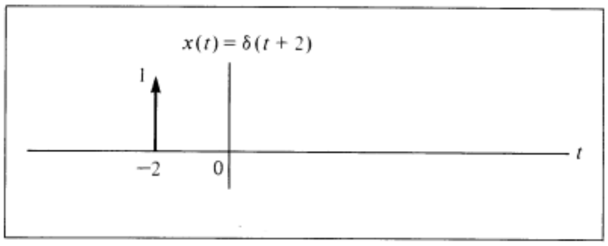
(c)



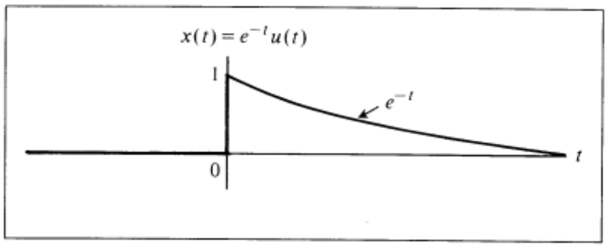
(d)



(e)



(f)



2. [10 points] For x(t) indicated in Figure 1,sketch the following.

$$\textbf{(a)}x(1-t)[u(t+1)-u(t-2)]$$
$$\textbf{(b)}x(1-t)[u(t+1)-u(2-3t)]$$

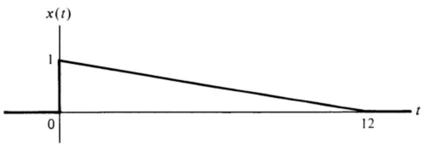


Figure 1: x(t)

Solution:

We are given Figure S3.4-1.

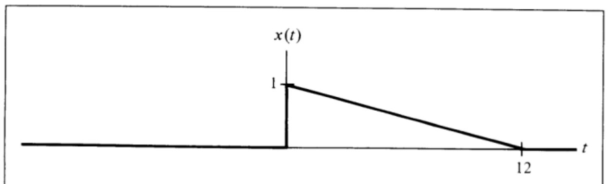


Figure S3.4-1

$x(-t)$ and $x(1-t)$ are as shown in Figures S3.4-2 and S3.4-3.

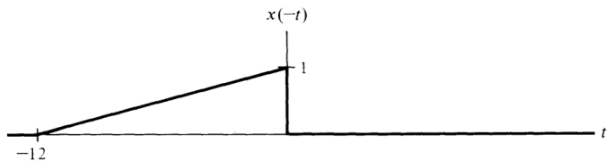


Figure S3.4-2

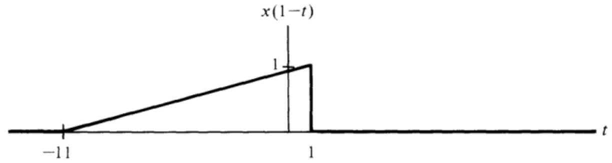


Figure S3.4-3

(a) $u(t+1) - u(t-2)$ is as shown in Figure S3.4-4.

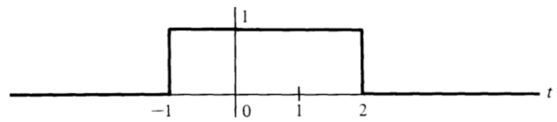


Figure S3.4-4

Hence, $x(1-t)[u(t+1) - u(t-2)]$ looks as in Figure S3.4-5.

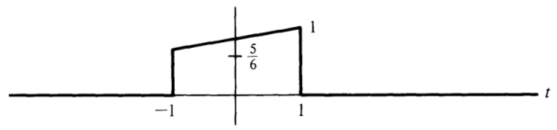


Figure S3.4-5

(b) $-u(2-3t)$ looks as in Figure S3.4-6.

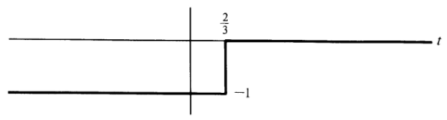


Figure S3.4-6

Hence, $u(t+1) - u(2-3t)$ is given as in Figure S3.4-7.

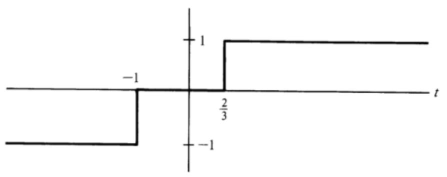


Figure S3.4-7

So $x(1-t)[u(t+1) - u(2-3t)]$ is given as in Figure S3.4-8.

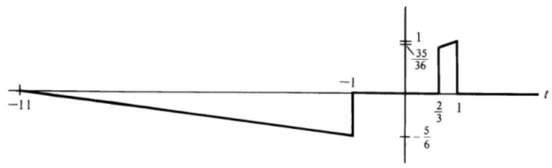


Figure S3.4-8

3. [10 points] Determine whether each of the following signals is periodic.

$$(\mathbf{a}) x(t) = 2e^{j(t+\frac{\pi}{4})}u(t)$$

$$(\mathbf{b}) x[n] = \sum_{k=-\infty}^{\infty} (\delta[n-4k] - \delta[n-1-4k])$$

Solution: (a)

$x_1(t)$ is not periodic because it is zero for $t < 0$ and have non-zero value for $t \geq 0$.

(b)

We can know that

We can let $j = k - 1$

This signal is periodic and the period is 4.

4. [15 points] Consider a discrete-time system with input x[n] and output y[n]

$$y[n] = \sum_{k=n-n_0}^{n+n_0} x[k]$$

where n_0 is a finite positive integer.

(a) Is this system linear?

(b) Is this system time-invariant?

(c) If $x[n]$ is known to be bounded by a finite integer B (i.e., $|x[n]| <$ B for all n), it can be shown that $y[n]$ is bounded by a finite number C. We conclude that the given system is stable. Express C in terms of B and n_0.

(a) Suppose that $x_1[n] \to y_1[n]$. $x_2[n] \to y_2[n]$

Let $x_3[n] = a x_1[n] + b x_2[n]$. $a, b \in R$

$x_3[n] \to y_3[n] = \sum_{k=n-n_0}^{n+n_0} x_3[k] = \sum_{k=n-n_0}^{n+n_0} a x_1[k] + b x_2[k] = a y_1[n] + b y_2[n]$

Therefore. this system is linear.

(b) $x_1[n] \to y_1[n] = \sum_{k=n-n_0}^{n+n_0} x_1[k]$, $y_1[n-n'] = \sum_{k=n-n_0-n'}^{n+n_0-n'} x_1[k]$

Suppose that $x_2[n] = x_1[n-n']$

$x_2[n] \to y_2[n] = \sum_{k=n-n_0}^{n+n_0} x_1[k-n'] = \sum_{k'=n-n_0-n'}^{n+n_0-n'} x_1[k'] = y_1[n-n']$

Therefore. this system is time-invariant.

(c) $|y[n]| = \left|\sum_{k=n-n_0}^{n+n_0} x[k]\right| = \sum_{k=n-n_0}^{n+n_0} |x[k]| < \sum_{k=n-n_0}^{n+n_0} B = (2n_0+1)B$.

Therefore, $C \leq (2n_0+1)B$

5. [10 points] Consider the following systems

$$H : y(t) = \int_{-\infty}^{t} x(\tau)d\tau$$
$$G : y(t) = x(2t),$$

where the input is x(t) and the output is y(t).

(a) What is H^{-1}? What is G^{-1}?

(b) Consider the system in Figure 2. Find the inverse F^{-1} and draw it in block diagram form in terms of H^{-1} and G^{-1}.

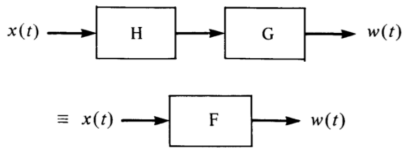


Figure 2: System of 3.(b)

(a) H^{-1}: $y(t) = \frac{dx(t)}{dt}$

G^{-1}: $y(t) = x(\frac{t}{2})$

(b) F: $w(t) = \int_{-\infty}^{2t} x(\tau)\,d\tau$

$$\frac{dw(t)}{dt} = 2x(2t)$$

let $k = 2t$ $\quad \frac{dw(\frac{k}{2})}{d\frac{k}{2}} = 2x(k)$

$$x(k) = \frac{dw(\frac{k}{2})}{dk}$$

Therefore: F^{-1}: $w(t) = \frac{dx(\frac{t}{2})}{dt}$

$w(t) \rightarrow F^{-1} \rightarrow x(t)$

$\equiv$ $w(t) \rightarrow G^{-1} \rightarrow H^{-1} \rightarrow x(t)$

6. [15 points] Determine whether or not each of the following discrete-time signals is periodic.If the signal is periodic, determine its fundamental period.

$$
\begin{aligned}
&\textbf{(a)} x[n] = sin(\frac{6\pi}{7}n+1) \\
&\textbf{(b)} x[n] = cos(\frac{\pi}{8}n^2) \\
&\textbf{(c)} x[n] = 2cos(\frac{\pi}{4}n) + sin(\frac{\pi}{8}n) - 2cos(\frac{\pi}{2}n+\frac{\pi}{6})
\end{aligned}
$$

Solution:

(a) Periodic, period = 7.

(b) Periodic, period=8.

(c) Periodic. $T_1 = 8$, $T_2 = 16$, $T_3 = 4$. The period is $T = 16$.

7. [10 points]

(a) Consider a system with input x(t) and with output y(t) given by

$$y(t) = \sum_{n=-\infty}^{+\infty} x(t)\delta(t-nT)$$

(i) Is this system linear?

(ii) Is this system time-invariant?

For each part, if your answer is yes, show your reason, else produce a counterexample.

(b) Suppose that the input to this system is $x(t) = \cos 2t$. Sketch and label carefully the output y(t) for each of the following values of T: T=1, $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{12}$. Make sure that all your sketches should have the same horizontal and vertical scales.

a)

i) $Z(t) = a\,x_1(t) + b\,x_2(t)$

$$\sum_{n=-\infty}^{\infty} Z(t)\,\delta(t-nT)$$

$$= \sum_{n=-\infty}^{\infty} \left(a x_1(t) + b x_2(t)\right) \delta(t-nT)$$

$$= a\sum_{n=-\infty}^{\infty} x_1(t)\,\delta(t-nT) + b\sum_{n=-\infty}^{\infty} x_2(t)\,\delta(t-nT)$$

$\Rightarrow$ This system is linear ✓

ii) let $b(t) = a(t+m)$

$$\sum_{n=-\infty}^{\infty} b(t)\,\delta(t-nT) = \sum_{n=-\infty}^{\infty} a(t+m)\,\delta(t-nT)$$

$$= \begin{cases} a(t+m) & \text{if } t = nT, \\ 0 & \text{otherwise} \end{cases}$$

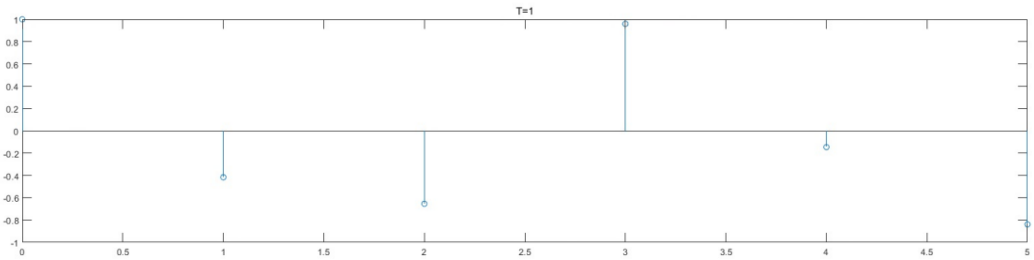
$$y(t+m) = \sum_{n=-\infty}^{\infty} a(t+m)\,\delta(t+m-nT)$$

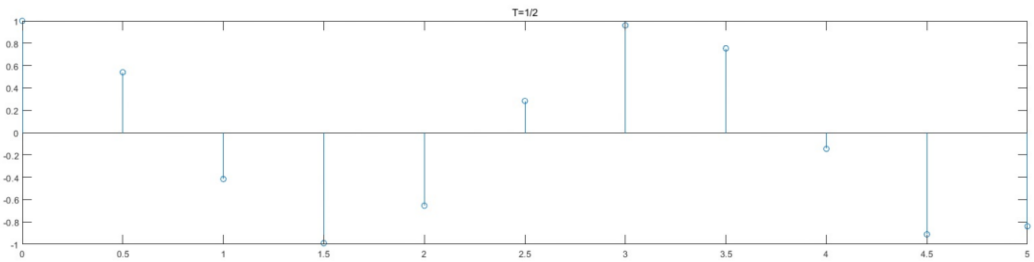
$$= \begin{cases} a(t+m) & \text{if } t+m = nT \\ 0 & \text{otherwise} \end{cases}$$

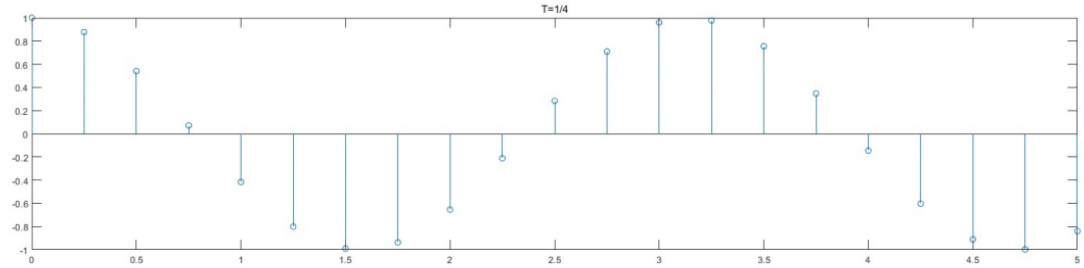
$\Rightarrow$ not time-invariant

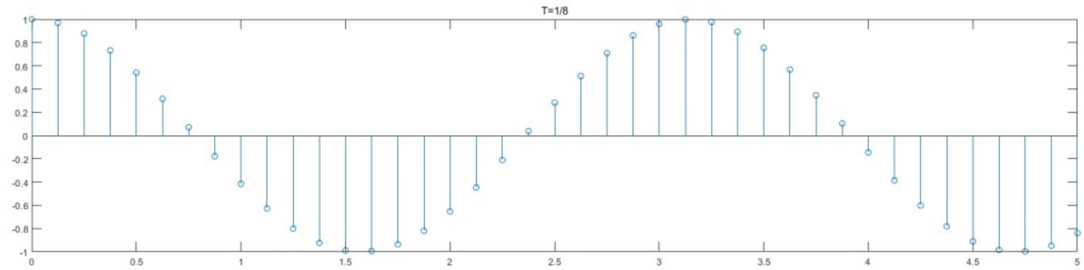
Example : $a(t) = \sin t \qquad m = \frac{\pi}{2} \qquad a(t+\frac{\pi}{2}) = \cos t$

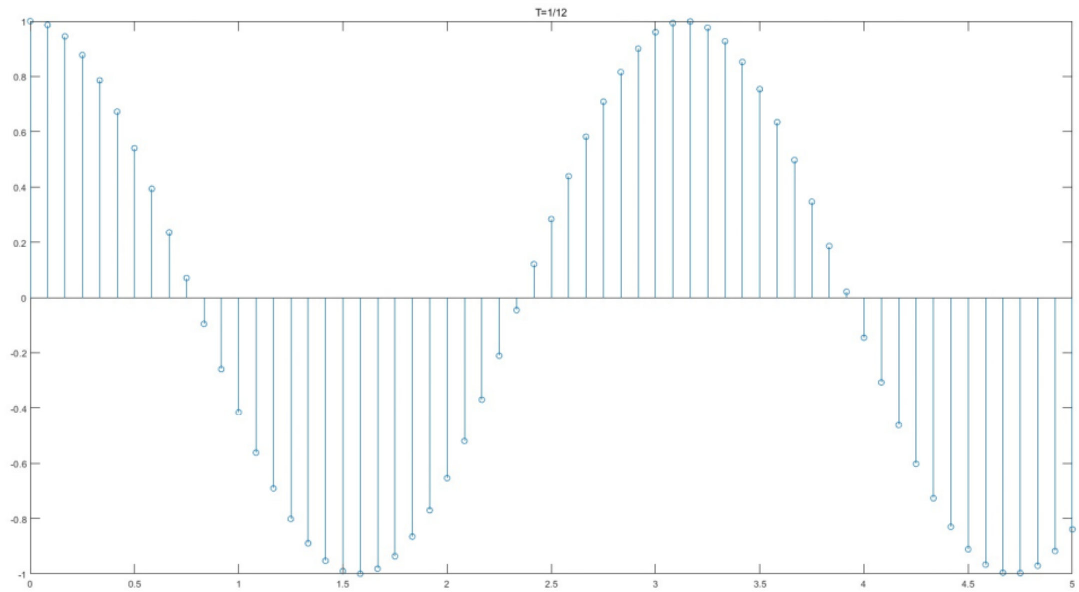
b) $y(t) = \begin{cases} \cos 2t & t = nT \\ 0 & \text{otherwise} \end{cases}$











8. [15 points] In this chapter, we introduced a number of general properties of systems. In particular, a system may or may not be
(1) Memoryless
(2) Time invariant
(3) Linear
(4) Causal
(5) Stable
Determine which of these properties hold and which do not hold for each of the following continuous-time systems. Justify your answers. In each example, $y(t)$ denotes the system output and $x(t)$ is the system input.

$$\textbf{(a)} y(t) = cos(3t)x(t)$$

$$\textbf{(b)} y(t) = \begin{cases} 0, & x(t) < 0 \\ x(t) + x(t-2), & x(t) \geq 0 \end{cases}$$

$y(t) = [\cos(3t)]x(t)$

Memoryless:

y only depends on x(t), which is the present value,so memoryless

Time invariant: Consider, a shift t_0 in the output $y(t)$ i.e.,

$$\begin{aligned} y(t-t_0) &= [\cos(3(t-t_0))]x(t-t_0) \\ y(t-t_0) &= [\cos(3t-3t_0)]x(t-t_0) \end{aligned}$$

If the input is shifted to t_0 and then passed through the system, then the output is,

$$x(t-t_0) \longrightarrow \quad y(t) = [\cos(3t)]x(t-t_0)$$

From the above equation, it is clear that a shift of t_0 in the input doesn't have a corresponding shift in the output, which implies that the system is $\mathbf{Time\text{-}variant}$.

Linear: Consider,

$$\begin{aligned} y_1(t) &= [\cos(3t)]x_1(t) \\ y_2(t) &= [\cos(3t)]x_2(t) \end{aligned}$$

Now let us consider a third input $x_3(t)$ such that $x_3(t)$ is a linear combination of $x_1(t)$ and $x_2(t)$. i.e.,

$$x_3(t) = ax_1(t) + bx_2(t)$$

Therefore, the output $y_3(t)$ is given as,

$$\begin{aligned} y_3(t) &= [\cos(3t)]x_3(t) \\ y_3(t) &= [\cos(3t)]\{ax_1(t) + bx_2(t)\} \\ y_3(t) &= a[\cos(3t)]x_1(t) + b[\cos(3t)]x_2(t) \\ y_3(t) &= ay_1(t) + by_2(t) \end{aligned}$$

From the above expression, it is clear that the system satisfies both additivity and homogeneity properties. Therefore, the system is $\mathbf{Linear}$.

Casual:

y only depends on x(t), which is the present value,so casual

Stable: Let us consider, $|x(t)| < B$ for all t, then,

$(B \neq +\infty)$

$$\begin{aligned} |y(t)| &= |\cos(3t)||x(t)| \leq 1 \cdot x(t) \\ |y(t)| &\leq |x(t)| \leq B \end{aligned}$$

So it's stable

$$y(t) = \begin{cases} 0, & x(t) < 0 \\ x(t) + x(t-2), & x(t) \geq 0 \end{cases}$$

Memoryless:

y(t) depends on x(t) in time t-2, so not memoryless

Time invariant:

Consider, a shift t_0 in the output $y(t)$ i.e.,

$$y(t-t_0) = \begin{cases} 0, & t < t_0 \\ x(t-t_0) + x(t-t_0-2), & t \geq t_0 \end{cases}$$

If the input is shifted to t_0 and then passed through the system, then the output is,

$$x(t-t_0) \longrightarrow \quad y(t) = \begin{cases} 0, & t < 0 \\ x(t-t_0) + x(t-t_0-2), & t \geq 0 \end{cases}$$

So that $x(t-t_0) \to y(t-t_0)$, it's time-invariant

Linear: Consider $x_1(t) \to y_1(t)$, $x_2(t) \to y_2(t)$

and $x_1(t) = -1$ for all t, $x_2(t) = 1$ for all t

So that $y_1(t) = 0$ $y_2(t) = 2$

taking $a = b = 1$, so $x_3(t) = x_1(t) + x_2(t) = 0$

$y_3(t) = 0 \neq y_1(t) + y_2(t) = 2$

Not linear

Casual:

y(t) depends on x(t) in time t-2 and t, which is now and past, so casual.

Stable:

Let us consider, $|x(t)| < \infty$, for all t, then,

$$\begin{aligned} |y(t)| &\leq \begin{cases} 0, & |x(t)| < 0 \\ |x(t)| + |x(t-2)|, & |x(t)| \geq 0 \end{cases} && (\because |a+b| \leq |a| + |b|) \\ \implies |y(t)| &< \infty && (\because |x(t)| < \infty) \end{aligned}$$

From the above expression, it is clear that the system is $\textbf{Stable}$.