SI231b - Matrix Computations, 2020-21 Spring

Homework Set #5

Prof. Ziping Zhao

Notice:

1) Deadline: **2021-05-16 23:59:59**
2) Submit your homework in pdf format to Email: zhangzp1@shanghaitech.edu.cn.
3) You can write your homework using LaTeX/Word, or you can write in handwriting and submit the scanned pdf.

Problem 1. (20 points) Prove that a matrix $\mathbf{A} \in \mathbb{S}^n$ is PSD if and only if it can be factorized as $\mathbf{A} = \mathbf{B}^T\mathbf{B}$ for some $\mathbf{B} \in \mathbb{R}^{m\times n}$. Besides, prove that $\mathbf{A}$ is PD if and only if $\mathbf{B}$ is nonsingular.

Problem 2. (20 points) For $\mathbf{A}, \mathbf{B} \in \mathbb{S}^n$, prove that $\mathsf{tr}(\mathbf{AB}) \geq 0$ holds for any $\mathbf{A} \succeq \mathbf{0}$ if and only if $\mathbf{B} \succeq \mathbf{0}$.

Problem 3. (20 points) For $\mathbf{A}, \mathbf{B} \succeq \mathbf{0}$, prove that $\det(\mathbf{A}+\mathbf{B}) \geq \det(\mathbf{A}) + \det(\mathbf{B})$.

Problem 4. (20 points) Let $\mathbf{X} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^T & \mathbf{C} \end{bmatrix}$ where $\mathbf{A} \in \mathbb{S}^m$, $\mathbf{B} \in \mathbb{R}^{m\times n}$, and $\mathbf{C} \in \mathbb{S}^n$, prove that

- if $\mathbf{C}$ is invertible, then $\mathbf{X}$ is PD if nd only if $\mathbf{C} \succ \mathbf{0}$ and $\mathbf{A} - \mathbf{B}\mathbf{C}^{-1}\mathbf{B}^T \succ \mathbf{0}$;
- if $\mathbf{C} \succ \mathbf{0}$, then $\mathbf{X}$ is PSD if and only if $\mathbf{A} - \mathbf{B}\mathbf{C}^{-1}\mathbf{B}^T \succeq \mathbf{0}$;
- if $\mathbf{C} \succ \mathbf{0}$, then for $\mathbf{b} \in \mathbb{R}^n$, $1 - \mathbf{b}\mathbf{C}^{-1}\mathbf{b}^T \geq 0$ if and only if $\mathbf{C} - \mathbf{b}\mathbf{b}^T \succeq \mathbf{0}$.

Problem 5. (20 points) For $\mathbf{A} \in \mathbb{R}^{m\times n}$ and $\mathbf{M} \in \mathbb{R}^{p\times n}$, find the optimal solution to the following problem:

$$\underset{\mathbf{B}\in\mathbb{R}^{m\times p}}{\mathsf{minimize}} \quad \left\|\mathbf{A} - \mathbf{B}\mathbf{M}\right\|_F^2, \qquad \mathsf{subject\ to} \quad \mathbf{B}^T\mathbf{B} = \mathbf{I}.$$