Rotary Inverted Pendulum Lab

1 Specifications

- The project should contain at most three students as a group.
- The amount of contribution/work of each student involved should be clearly stated.

2 Background

The rotary inverted pendulum (RIP) is a highly non-linear, dynamically unstable and multivariable system which is widely used in many real-world applications such as maintaining equilibrium state for two legged robots, stabilization of cabin in a ship, altitude control of rockets and satellite, missile stabilization in upward direction, sky scrapping buildings etc.

3 Modeling of the Pendulum

A schematic representation of Rotary Inverted Pendulum is shown in Figure 1.

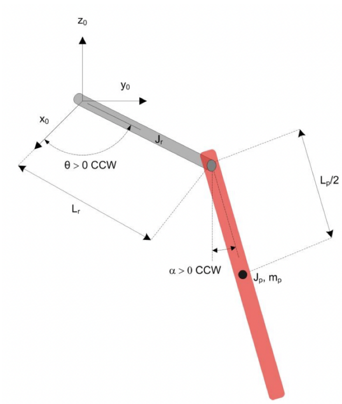

Figure 1: Rotary Inverted Pendulum

The angle of the rotary arm θ, the angle of the pendulum α, and the motor applied torque τ defined as positive when rotated in the counter-clockwise (CCW) direction. The applied torque at the base of the rotary arm generated by the servomotor is described by the equation

$$\tau = \frac{k_t \left(V_m - k_m \dot{\theta} \right)}{R_m} \tag{1}$$

where V_m is the input voltage.

The equations of motion of the system are given as

$$\begin{aligned}
&\left(m_pL_r^2 + \frac{1}{4}m_pL_p^2 - \frac{1}{4}m_pL_p^2\cos^2(\alpha) + J_r\right)\ddot{\theta} - \left(\frac{1}{2}m_pL_pL_r\cos(\alpha)\right)\ddot{\alpha} \\
&\quad + \left(\frac{1}{2}m_pL_p^2\sin(\alpha)\cos(\alpha)\right)\dot{\theta}\dot{\alpha} + \left(\frac{1}{2}m_pL_pL_r\sin(\alpha)\right)\dot{\alpha}^2 = \tau - D_r\dot{\theta} \\
&\frac{1}{2}m_pL_pL_r\cos(\alpha)\ddot{\theta} + \left(J_p + \frac{1}{2}m_pL_p^2\right)\ddot{\alpha} - \frac{1}{4}m_pL_p^2\cos(\alpha)\cos(\alpha)\dot{\theta}^2 \\
&\quad + \frac{1}{2}m_pL_pg\sin(\alpha) = -D_p\dot{\alpha}
\end{aligned} \tag{2}$$

To apply linear control methods, we need to linearize the equations of motion. This can be done using the Taylor expansion approximation about a specific equilibrium point, assuming that we only allow small deviation of the states from this equilibrium point.

- For equilibrium point $[\theta, \alpha, \dot{\theta}, \dot{\alpha}] = [0, 0, 0, 0]$, the resulting linearized equation of motion becomes

$$\begin{aligned}
\left(m_pL_r^2 + J_r\right)\ddot{\theta} - \frac{1}{2}m_pL_pL_r\ddot{\alpha} &= \tau - D_r\dot{\theta} \\
\frac{1}{2}m_pL_pL_r\ddot{\theta} + \left(J_p + \frac{1}{4}m_pL_p^2\right)\ddot{\alpha} + \frac{1}{2}m_pL_pg\alpha &= -D_p\dot{\alpha}
\end{aligned} \tag{3}$$

- For equilibrium point $[\theta, \alpha, \dot{\theta}, \dot{\alpha}] = [0, \pi, 0, 0]$, the resulting linearized equation of motion becomes

$$\begin{aligned}
\left(m_pL_r^2 + J_r\right)\ddot{\theta} + \frac{1}{2}m_pL_pL_r\ddot{\alpha} &= \tau - D_r\dot{\theta} \\
-\frac{1}{2}m_pL_pL_r\ddot{\theta} + \left(J_p + \frac{1}{4}m_pL_p^2\right)\ddot{\alpha} - \frac{1}{2}m_pL_pg\alpha &= -D_p\dot{\alpha}
\end{aligned} \tag{4}$$

Letting $z = [\theta, \alpha]^\top$, linearized equation of motion becomes $M\ddot{z} + G\dot{z} + Kz = FV_m$.

Symbol	Description	Units	Values
k_t	Torque constant	$N \cdot m/A$	0.042
k_m	Motor back-emf constant	$V \cdot s/rad$	0.042
R_m	Terminal resistance	Ω	8.4
L_r	Length of the rotary arm	m	0.085
J_r	Rotary arm Inertia	$kg \cdot m^2$	2.3117×10^{-4}
D_r	Vicious damping coefficient	$N \cdot m \cdot s/rad$	7.5×10^{-4}
L_p	Length of the pendulum	m	0.129
m_p	Mass of pendulum	kg	0.024
J_p	Pendulum Inertia	$kg \cdot m^2$	$m_p \cdot L_p^2/12$
D_p	Pendulum damping coefficient	$N \cdot m \cdot s/rad$	2.5×10^{-4}
g	Gravity constant	m/s^2	9.8

Question 1 *Find the M, G, K and F matrices corresponding to the linearized equation of motion at the equilibrium point $[\theta, \alpha, \dot{\theta}, \dot{\alpha}] = [0, 0, 0, 0]$.*

Question 2 *Find the M, G, K and F matrices corresponding to the linearized equation of motion at the equilibrium point $[\theta, \alpha, \dot{\theta}, \dot{\alpha}] = [0, \pi, 0, 0]$.*

4 State Space Equation of the Pendulum

Using the linearized equation of motion from the previous part, let $x = [z^\top, \dot{z}^\top]^\top$, then create a state space model of the RIP system in Matlab with

$$\begin{aligned}
\dot{x} &= Ax + Bu = \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}G \end{bmatrix} x + \begin{bmatrix} 0 \\ M^{-1}F \end{bmatrix} V_m \\
y &= Cx + Du
\end{aligned} \tag{5}$$

where $y = [\theta, \alpha]^\top$.

- Define the state space matrices A, B, C, D in Matlab.
- Use the command `ss(A,B,C,D)` to create the state space model.

Question 3 *Show your state space model at* $[\theta, \alpha, \dot{\theta}, \dot{\alpha}] = [0, 0, 0, 0]$*, find the pole locations of the system. Determine whether the system is stable and explain why.*

Question 4 *Show your state space model at* $[\theta, \alpha, \dot{\theta}, \dot{\alpha}] = [0, \pi, 0, 0]$*, find the pole locations of the system. Determine whether the system is stable and explain why.*

5 Comparison of State Space and Physical Model

Next we validate the linearized mathematical model of the pendulum. Using the given physical RIP model (`Inverted Pendulum System` shown in Figure 2) to compare the measured response of the physical RIP model to the linearized state space model at $[\theta, \alpha, \dot{\theta}, \dot{\alpha}] = [0, 0, 0, 0]$.

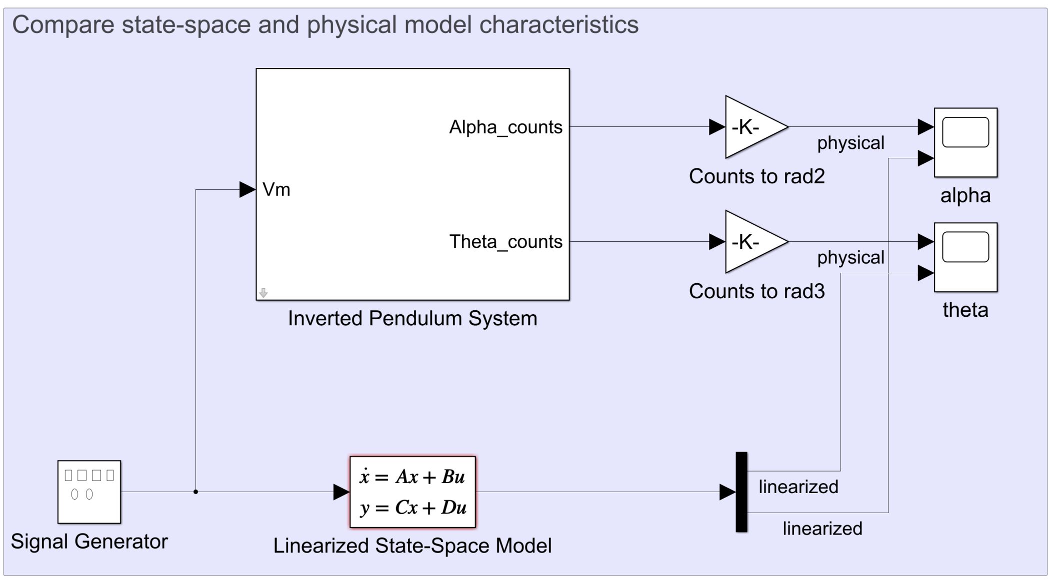

Figure 2: RIP model

- Set Signal Generator to a **square** form with **1 amplitude** and **1 Hz frequency**.
- Set Linearized State Space Model to the linearized model at $[\theta, \alpha, \dot{\theta}, \dot{\alpha}] = [0, 0, 0, 0]$.
- Run the model for 10 seconds.

Question 5 *Compare the outputs of the RIP model to the linearized state space model. Submit figures of angle responses of the rotary arm and pendulum (θ and α), respectively. Does the model accurately predict the system behavior? Explain the reason of any discrepancies.*

6 Design of Pendulum Control Law

The state space equation of the inverted pendulum are given in Part 4 as

$$P(s): \quad \begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned} \tag{6}$$

for rotary arm angle θ, inverted pendulum angle α, input $u = V_m$, where $x = [\theta, \alpha, \dot{\theta}, \dot{\alpha}]^\top$ and $y = [\theta, \alpha]^\top$. The above state space equation has 2 outputs and 1 input, thus it is a single-input multi-output (SIMO) system.

Our main focus is to control the pendulum angle α at the straight up position $[\theta, \alpha, \dot{\theta}, \dot{\alpha}] = [0, \pi, 0, 0]$. In order to simplify the controller design problem, we will convert the SIMO system into a single-input single-output (SISO) system by closing the rotary arm position feedback, thus your new SISO plant is $P_\alpha(s)$, which is shown in Figure 3. The rotary arm position controller $C_\theta(s)$ is given as

$$C_\theta(s) = \frac{K_p + K_d s}{Ts + 1} \tag{7}$$

where $K_p = 2$, $K_d = 2$, $T = 0.02$.

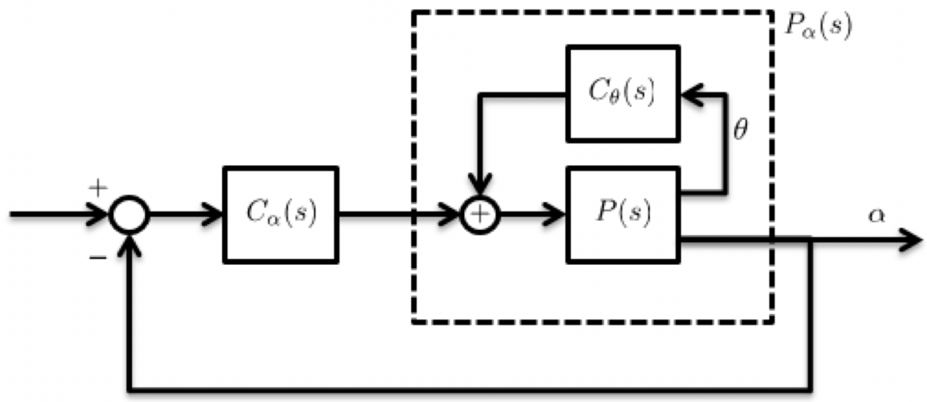

Figure 3: SISO system block diagram

Question 6 *Convert the state space model to the transfer function form, show all transfer functions at $[\theta, \alpha, \dot{\theta}, \dot{\alpha}] = [0, \pi, 0, 0]$. Then derive the form of $P_\alpha(s)$ using $C_\theta(s)$ in (7).*

Question 7 *Use $P_\alpha(s)$ as your new system model, find a control law $C_\alpha(s)$ that keeps the inverted pendulum at the straight up position $[\theta, \alpha, \dot{\theta}, \dot{\alpha}] = [0, \pi, 0, 0]$. Submit in your solutions*

- *the form of $C_\alpha(s)$*
- *the root locus plot*
- *closed-loop transfer function and closed-loop poles*
- *unit step response for 10 seconds of the closed-loop system*

Question 8 *Obtain the Nyquist diagram and determine the stability of the feedback system. To show the relative stability of your system, calculate the gain margin and phase margin. Then show the gain and phase margin on the Bode plots.*

7 Implementation

The controller derived in Part 6 will be tested on the pendulum through the provided simulation model.

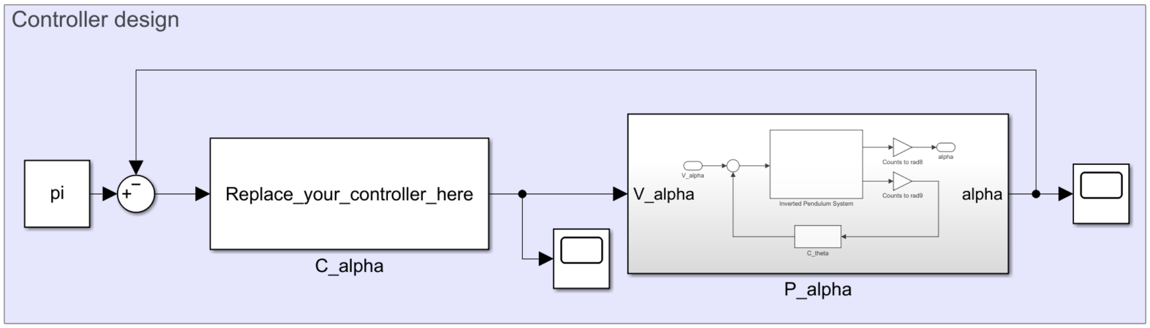

Figure 4: RIP Control

- Implement your control law $C_\alpha(s)$ into the Simulink model. Set the initial pendulum angle to straight up position.
- Run the model for 10 seconds. Add a small deviation to the straight up position, set it as the initial angle and repeat the simulation.

Question 9 *Does the pendulum behave in agreement to your analysis during the controller design process? If not, revise your controller in Part 6, and try again. Include the figures of the pendulum angle* α *and voltage* V_α *(your new control input, shown in Figure 4) in your report.*

8 Hardware Testing

The controller derived in Part 6 will be tested on the pendulum through the hardware platform.

- Implement your control law $C_\alpha(s)$ on the hardware platform. Set the initial pendulum angle to straight up position.
- Add a small deviation to the straight up position, set it as the initial angle, repeat the process and record a short video.

Question 10 *Does the pendulum behave in agreement to your analysis during the controller design process? If not, try to explain why.*