

Numerical Optimization

Lecture 7: Sensitivity Analysis

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Slides were made by Prof Hao Wang.

本节内容

- 灵敏度分析
 - ◆ 增加单个变量
 - ◆ 增加单个约束（等式、不等式）
 - ◆ 改变右端向量某分量
 - ◆ 改变目标某系数
 - ◆ 改变系数矩阵某分量
 - ◆ 影子价格和既约费用

Command Window

```
>> load sc50b
>> options = optimoptions(@linprog,'Algorithm','dual-simplex','Display','iter');
>> ub = [];
>> [x,fval,exitflag,output,lambda] = linprog(f,A,b,Aeq,beq,lb,ub,options);
|
LP preprocessing removed 2 inequalities, 16 equalities,
16 variables, and 26 non-zero elements.
```

Iter	Time	Fval	Primal Infeas	Dual Infeas
0	0.001	0.000000e+00	0.000000e+00	1.305012e+00
7	0.001	-1.587054e+02	3.760622e+02	0.000000e+00
34	0.001	-7.000000e+01	0.000000e+00	0.000000e+00

```
Optimal solution found.

>> exitflag

exitflag =

    1

>> output

output =

    struct with fields:

        iterations: 34
        constrviolation: 1.7053e-13
        message: 'Optimal solution found.'
        algorithm: 'dual-simplex'
        firstorderopt: 1.2590e-14

>> lambda

lambda =

    struct with fields:

        lower: [48x1 double]
        upper: [48x1 double]
        eqlin: [20x1 double]
        ineqlin: [30x1 double]
```

```
>> lambda

lambda =

    struct with fields:

        lower: [48x1 double]
        upper: [48x1 double]
        eqlin: [20x1 double]
        ineqlin: [30x1 double]

>> lambda.eqlin

ans =

    0.1750
    0.1750
    0.1750
    0.1750
    0.7500
    0.1312
    0.1313
    0.1313
    0.1313
    0.5625
    0.0984
    0.0984
    0.0984
    0.0984
    0.4219
    0.0738
    0.0738
    0.0738
    0.0738
    0.3164
```

Sensitivity Analysis

- ◆ In many real-world problem, the following can occur:
 - The input data is not very accurate
 - We don't know all of the constraints ahead of time
 - We don't know all the variables ahead of time.
- ◆ Because of this, we want to analyze the dependence of the model on the input data, i.e.,
 - the matrix A
 - the right-hand side vector b , and
 - the cost vector c
- ◆ We would also like to know the effect of additional variables and constraints
- ◆ This is done using **sensitivity analysis**

The Fundamental Idea

- ◆ Using the simplex algorithm to solve a standard form problem, we know that if B is an optimal basis, then two conditions are satisfied:
 - $B^{-1}b \geq 0$
 - $c^T - c_B^T B^{-1}A \geq 0$
- ◆ When the problem is changed, we can check to see how **these conditions are affected**.
- ◆ This is the simplest kind of analysis.
- ◆ When using the simplex method, we always have B^{-1} available, so we can easily **recompute appropriate quantities**
- ◆ Where is B^{-1} in the simplex tableau?

Adding a New Variable

- ◆ Suppose we want to consider **adding a new variable** to the problem, e.g., we want to consider a new product to our line
- ◆ We simply compute the reduced cost of the new variable as

$$c_j - c_B^T B^{-1} A_j$$

where A_j is the column corresponding to the new variable in the matrix.

- ◆ If the reduced cost is **nonnegative**, then we should not consider adding the product
- ◆ Otherwise, it is eligible to enter the basis and we can **reoptimize** from the current feasible (yet now non-optimal) basis

Adding a New Inequality Constraint

- ◆ Assume the new constraint is **NOT** satisfied by the current optimal solution
- ◆ Suppose we want to **introduce a new constraint** of the form

$$a_{m+1}^T x \geq b_{m+1}$$

- ◆ The new constraint matrix (in standard form) would look like

$$\begin{bmatrix} A & 0 \\ a_{m+1}^T & -1 \end{bmatrix}$$

- ◆ Hence, the new basis matrix would look like

$$\bar{B} = \begin{bmatrix} B & 0 \\ a^T & -1 \end{bmatrix}$$

- ◆ The new basis inverse would then be

$$\bar{B}^{-1} = \begin{bmatrix} B^{-1} & 0 \\ a^T B^{-1} & -1 \end{bmatrix}$$

Adding a New Inequality Constraint

- ◆ The vector of reduced costs is

$$[c^T \ 0] - [c_B^T \ 0] \begin{bmatrix} B^{-1} & 0 \\ a^T B^{-1} & -1 \end{bmatrix} \begin{bmatrix} A & 0 \\ a_{m+1}^T & -1 \end{bmatrix} = [c^T - c_B^T B^{-1} A, \ 0]$$

and so the reduced costs remain unchanged.

- ◆ Hence, we have a dual feasible basis and we apply **dual simplex**.
- ◆ The tableau can be computed as

$$\bar{B}^{-1} \begin{bmatrix} A & 0 \\ a_{m+1}^T & -1 \end{bmatrix} = \begin{bmatrix} B^{-1} A & 0 \\ a^T B^{-1} A - a_{m+1}^T & 1 \end{bmatrix}$$

- ◆ Note that $B^{-1}A$ is available from the original tableau

Adding a New Equality Constraint

- ◆ Assume the new constraint is **NOT** satisfied by the current optimal solution
- ◆ We **introduce an artificial variable** x_{n+1} , as in the two-phase method, and consider the LP (assuming $a_{m+1}^T x^* > b_{m+1}$)

$$\begin{aligned} \min \quad & c^T x + Mx_{n+1} \\ \text{s.t.} \quad & Ax = b \\ & a_{m+1}^T x - x_{n+1} = b_{m+1} \\ & x \geq 0, x_{n+1} \geq 0 \end{aligned}$$

- ◆ We can obtain a primal feasible basis by making the new variable basic.
- ◆ The new tableau can be computed as before.
- ◆ If the new problem is feasible and M is large enough, then the solution will have $x_{n+1} = 0$
- ◆ The values of the remaining variables will yield an optimal solution to the original problem with the additional constraint

Changes to the Right-hand Side

- ◆ Suppose we change b_i to $b_i + \delta$
- ◆ The values of the basic variables change from $B^{-1}b$ to $B^{-1}(b + \delta e^i)$, where e^i is the i th unit vector

- ◆ The feasibility condition is then

$$B^{-1}(b + \delta e^i) \geq 0$$

- ◆ If g is the i th column of B^{-1} , then the feasibility condition becomes

$$x_B + \delta g \geq 0$$

- ◆ This is equivalent to

$$\max_{\{j|g_j>0\}} \left(-\frac{x_{B(j)}}{g_j} \right) \leq \delta \leq \min_{\{j|g_j<0\}} \left(-\frac{x_{B(j)}}{g_j} \right)$$

- ◆ If δ is outside the allowable range, we can **reoptimize using dual simplex**

Changes in the Cost Vector

- ◆ Suppose we **change some cost coefficient** from c_j to $c_j + \delta$
- ◆ If c_j is the cost coefficient of a nonbasic variable, then we need recalculate its reduced cost $c_i - c_B B^{-1} a_j + \delta$
- ◆ The reduced cost itself increases by δ and the current solution remains optimal as long as $\delta \geq -r_j$
- ◆ If c_l is the cost coefficient of the l th basic variable, then c_B becomes $c_B + \delta e_l$ and the new optimality conditions are

$$(c_B + \delta e_l)^T B^{-1} A \leq c^T$$

- ◆ This is equivalent to

$$\delta q \leq r$$

where q is the l th row of $B^{-1}A$, which is available in the simplex tableau

Changes in a Nonbasic Column of A

◆ Suppose we **change some entry** a_{ij} from the constraint matrix to $a_{ij} + \delta$

◆ If column j is nonbasic, then B does not change and we only need to check the reduced cost of column j

◆ The new reduced cost is

$$c_j - c_B^T B^{-1} (A_j + \delta e^i)$$

◆ This means the current solution remains optimal if

$$r_j - \delta p_i \geq 0$$

◆ Otherwise, we **reoptimize with primal simplex**

Changes in a Basic Column of A

- ◆ This case is more complicated; the **tragedy case**
- ◆ $B + \delta e^i$ may not be a basis anymore;
- ◆ $B + \delta e^i$ may even not invertible
- ◆ $B + \delta e^i$ is invertible, but $(B + \delta e^i)^{-1}$ way different from B^{-1}
- ◆ Generally we have to reoptimize, may starting from the beginning

Sensitivity and Shadow Price

$$\begin{aligned} z(b) = \quad & \text{minimize} \quad c^T x \\ & \text{subject to} \quad Ax = b \\ & \quad \quad \quad x \geq 0 \end{aligned}$$

- ◆ Suppose the optimal basis matrix is B with $B^{-1}b > 0$ (nondengenerate)
 $\lambda^T = c_B^T B^{-1}$

- ◆ Question: if b changes a little bit, how would the optimal value change?

$$b \leftarrow b + \delta \implies z(\delta) - z = ?$$

- ◆ Let $\frac{\partial}{\partial b_j} z(b)$ denote the partial derivative of $z(b)$ w.r.t. b_j

$$\frac{\partial}{\partial b_j} z(b) = \lambda_j$$

- ◆ We call λ_j is the marginal price, or shadow price, associated with b_j
- ◆ What is the best δ ?

Sensitivity and Reduced Cost

- ◆ Suppose we **change some cost coefficient** from c_j to $c_j + \delta$
- ◆ If c_j is the cost coefficient of a nonbasic variable, then we need recalculate its reduced cost $c_i - c_B B^{-1} a_j + \delta$
- ◆ The reduced cost itself increases by δ and the current solution remains optimal as long as $\delta \geq -r_j$
- ◆ If you reduce c_j more than r_j , the current basis must change, and x_j may enter the basis