SI231B - Matrix Computations, Spring 2022-23

Homework Set #3

Prof. Ziping Zhao

Acknowledgements:

1) Deadline: **2023-04-08 23:59:59**
2) Please submit your assignments via Blackboard.
3) You can write your homework using latex/word or you can write in handwriting and submit the scanned pdf.

Problem 1. (20 points)

For the matrix below

$$\mathbf{A} = \begin{bmatrix} 5 & -1 & -1 \\ 3 & 1 & -1 \\ 4 & -2 & 1 \end{bmatrix}$$

1) Calculate the characteristic polynomial of $\mathbf{A}$. (5 points)
2) Find the eigenvalues of $\mathbf{A}$. (5 points)
3) Find a basis for each eigenspace of $\mathbf{A}$. (5 points)
4) Determine whether or not $\mathbf{A}$ is diagonalizable. If $\mathbf{A}$ is diagonalizable, then find an invertible matrix $\mathbf{V}$ and a diagonal matrix $\mathbf{\Lambda}$ such that $\mathbf{V}^{-1}\mathbf{A}\mathbf{V} = \mathbf{\Lambda}$. (5 points)

Problem 2. (20 points)

1) Let $\mathbf{A}$ be the adjacency matrix of an undirected graph $G = (V, E)$ and λ_1 the largest eigenvalue of $\mathbf{A}$, and

$$a_{i,j} = \begin{cases} 1, & \text{if vertex } i \text{ and vertex } j \text{ are adjacent;} \\ 0, & \text{otherwise.} \end{cases}$$

   Show that λ_1 is at least the average degree of the vertices in G. (10 points)

2) Let $\mathbf{A}$ be a symmetric matrix with the largest eigenvalue α_1. Let $\mathbf{B}$ be the matrix obtained by removing the last row and column from $\mathbf{A}$. And β_1 is the largest eigenvalue of $\mathbf{B}$. Show that $\alpha_1 \geq \beta_1$. (10 points)

   (*Hint:* You can use the Rayleigh quotient to prove the two problems.)

Problem 3. (20 points)

Let $\mathbf{A}$ be an $n \times n$ square matrix.

1) Suppose that $\mathbf{A}^{-1}$ exists. Prove: if λ is an eigenvalue of $\mathbf{A}$, then $1/\lambda$ is an eigenvalue of $\mathbf{A}^{-1}$. (5 points)
2) Prove that if $\mathbf{A}^2 = \mathbf{I}$, then the eigenvalue of $\mathbf{A}$ must be 1 or -1. (5 points)
3) Suppose that λ_1 and λ_2 are two distinct eigenvalues of $\mathbf{A}$. And suppose that $\mathbf{x}_1$ is an eigenvector of $\mathbf{A}$ under λ_1, and $\mathbf{x}_2$ is an eigenvector of $\mathbf{A}$ under λ_2. Prove that there dose not exists any real number t such that $t\mathbf{x}_1 = \mathbf{x}_2$. (5 points)
4) Suppose that λ_1 and λ_2 are two distinct eigenvalues of $\mathbf{A}$. And suppose that $\mathbf{x}_1$ is an eigenvector of $\mathbf{A}$ under λ_1, and $\mathbf{x}_2$ is an eigenvector of $\mathbf{A}$ under λ_2. Prove that $\mathbf{x}_1 + \mathbf{x}_2$ is not an eigenvector of $\mathbf{A}$. (5 points)

Problem 4. (20 points) For $\mathbf{A} \in \mathbb{R}^{m \times n}$, prove that $\mathbf{A}^T\mathbf{A}$ and $\mathbf{A}^T$ have the same range space.

Problem 5. (20 points) Time-of-arrival (TOA) based source localization is a scenario in which the location of a target sensor is determined based on the TOA measurements of the target sensor collected by many anchors.

Define $\mathbf{x}^* \in \mathbb{R}^n$ be the unknown true position of the target sensor. Define $\{\mathbf{a}_i\}_{i=1}^m \subseteq \mathbb{R}^n$ be the known position of the ith anchor and suppose that the vectors $\{\mathbf{a}_i - \mathbf{a}_1\}_{i=1}^m$ span $\mathbb{R}^n$ ($m \geq n+1$). Then the TOA based *range measurement* between the target and the ith anchor is modeled as

$$r_i = \|\mathbf{x}^* - \mathbf{a}_i\|_2 + \omega_i, \quad i = 1, ..., m,$$

where $\{\omega_i\}_{i=1}^m \subseteq \mathbb{R}$ is the noise.

1) Suppose there is no noise, i.e, $\omega_i = 0$ for $i = 1, ..., m$. Prove that $\mathbf{x}^*$ can be recovered based on the following *linear least squares* problem:

$$\mathbf{x}_{\text{LS}} = \arg \min_{\mathbf{x} \in \mathbb{R}^n} \|\mathbf{A}\mathbf{x} - \mathbf{b}\|_2^2,$$

i.e., $\mathbf{x}_{\text{LS}} = \mathbf{x}^*$ in the noiseless case, where

$$\mathbf{A} = \begin{bmatrix} (\mathbf{a}_2 - \mathbf{a}_1)^T \\ \vdots \\ (\mathbf{a}_m - \mathbf{a}_{m-1})^T \end{bmatrix}, \quad \mathbf{b} = \frac{1}{2} \begin{bmatrix} \|\mathbf{a}_2\|_2^2 - \|\mathbf{a}_1\|_2^2 + r_1^2 - r_2^2 \\ \vdots \\ \|\mathbf{a}_m\|_2^2 - \|\mathbf{a}_{m-1}\|_2^2 + r_{m-1}^2 - r_m^2 \end{bmatrix}.$$

Besides, derive the solution of the estimator $\mathbf{x}_{\text{LS}}$. (10 points)

2) Suppose the noise ω_i satisfies $\omega_i \ll \|\mathbf{x}^* - \mathbf{a}_i\|_2$ for $i = 1, ..., m$. We can estimate $\mathbf{x}^*$ based on the following *nonlinear least squares* problem:

$$\hat{\mathbf{x}} = \arg \min_{\mathbf{x} \in \mathbb{R}^n} \sum_{i=1}^{m} \left(\|\mathbf{x} - \mathbf{a}_i\|_2 - r_i\right)^2.$$

Define $\boldsymbol{\omega} = [\omega_1, ..., \omega_m]^T$ and suppose $\|\boldsymbol{\omega}\|_2 \leq c\sqrt{m}\sigma$ where $c, \sigma > 0$ are some constants. Prove that the following result holds:

$$\|\hat{\mathbf{x}} - \mathbf{x}^*\|_2 \leq K_1\sqrt{m}\sigma + K_2 m\sigma^2,$$

for some constants $K_1, K_2 > 0$ which are determined by $\{\mathbf{a}_i\}_{i=1}^m$, c, and $\mathbf{x}^*$. (10 points)

Hint for 2):

- *You can find the upper bound of* $\|\mathbf{x}_{LS} - \mathbf{x}^*\|_2$ *and* $\|\mathbf{x}_{LS} - \hat{\mathbf{x}}\|_2$ *and then combine them to get the result.*
- *Based on 1), you may need to define* $r_i^* = \|\mathbf{x}^* - \mathbf{a}_i\|_2$ *(resp.* $\hat{r}_i = \|\hat{\mathbf{x}} - \mathbf{a}_i\|_2$*) and let* $\mathbf{b}^*$ *(resp.* $\hat{\mathbf{b}}$*) be the vector obtained by replacing* r_i *with* r_i^* *(resp.* $\hat{r}_i$*) in* $\mathbf{b}$*, and then* $\mathbf{x}^*$ *(resp.* $\hat{\mathbf{x}}$*) satisfies* $\mathbf{x}^* = \arg\min_{\mathbf{x} \in \mathbb{R}^n} \|\mathbf{A}\mathbf{x} - \mathbf{b}^*\|_2^2$ *(resp.* $\hat{\mathbf{x}} = \arg\min_{\mathbf{x} \in \mathbb{R}^n} \left\|\mathbf{A}\mathbf{x} - \hat{\mathbf{b}}\right\|_2^2$*).*
- *You may need the relation* $f(\hat{\mathbf{x}}) = \sum_{i=1}^m \left(\|\hat{\mathbf{x}} - \mathbf{a}_i\|_2 - r_i\right)^2 \leq f(\mathbf{x}^*)$.