EE160 Homework 1 Solution

1. Fig. 1 below shows a classical RLC circuit consisting of a resistor with given resistance $R = 3\Omega$, two inductors with given inductance $L_1 = 5H$ and $L_2 = 4H$, and a capacitor with given capacitance $C = 2F$.

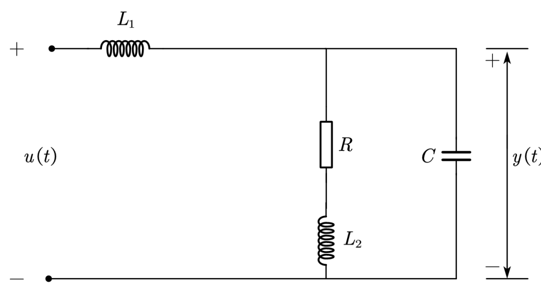
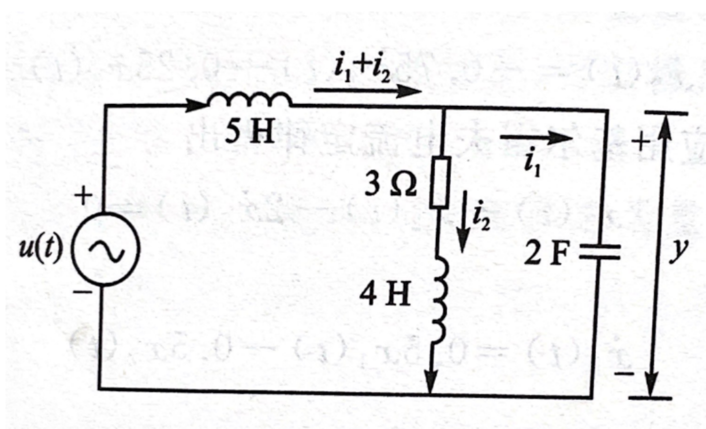


Figure 1: RLC system

   (a) Show that the system output $y(t)$ satisfies a differential equation with respect to the input $u(t)$. **(10')**

   Solution. Consider the circuit network in Fig. 1 and redraw it.



   Let $i_1(t)$ represent the current flowing through the $2F$ capacitor, and use $i_2(t)$ to represent the current flowing through the series branch of the 3Ω resistor and the $4H$ inductor, then we have

$$i_1(t) = 2\dot{y}(t) \tag{1a}$$

$$y(t) = 3i_2(t) + 4\dot{i}_2(t) \tag{1b}$$

   The current flowing through the $5H$ inductor is $i_1(t) + i_2(t)$, so the voltage across the inductor is $5\dot{i}_1(t) + 5\dot{i}_2(t)$. Applying Kirchhoff's voltage law to the outer loop. We can obtain

$$5\dot{i}_1(t) + 5\dot{i}_2(t) + y(t) - u(t) = 0 \tag{2}$$

in order to derive the differential equation linking $u(t)$ and $y(t)$, $i_1(t)$ and $i_2(t)$ must be eliminated from equations $(1)-(2)$. First, substitute equation $(1a)$ into equation (2) to get

$$10\ddot{y}(t) + 5\dot{i}_2(t) + y(t) = u(t) \tag{3}$$

then derivation of it, we can get

$$10y^{(3)}(t) + 5\ddot{i}_2(t) + \dot{y}(t) = \dot{u}(t) \tag{4}$$

multiply the formula (3) by 3 and the formula (4) by 4 to get

$$40y^{(3)}(t) + 5(4\ddot{i}_2(t) + 3\dot{i}_2(t)) + 30\ddot{y}(t) + 4\dot{y}(t) + 3y(t) = 4\dot{u}(t) + 3u(t) \tag{5}$$

substituting into the derivative of $(1b)$, the above equation becomes

$$40y^{(3)}(t) + 30\ddot{y}(t) + 9\dot{y}(t) + 3y(t) = 4\dot{u}(t) + 3u(t) \tag{6}$$

(b) Write the transfer function $\frac{Y(s)}{U(s)}$, with initial conditions $y(0) = 0$, $\dot{y}(0) = 0$, $\ddot{y}(0) = 0$. **(10')**
Solution.

$$G(s) = \frac{Y(s)}{U(s)} = \frac{4s+3}{40s^3 + 30s^2 + 9s + 3} \tag{7}$$

2. Find the inverse Laplace transform $x(t)$ of the following functions **(10')**

(a) $X(s) = \frac{e^{-s}}{s-1}$

(b) $X(s) = \frac{1}{s(s+2)^3(s+3)}$

(c) $X(s) = \frac{s+1}{s(s^2+2s+2)}$

Solution.
(a)

$$x(t) = e^{t-1}$$

(b)

$$X(s) = \frac{-1}{2(s+2)^3} + \frac{1}{4(s+2)^2} - \frac{3}{8(s+2)} + \frac{1}{24s} + \frac{1}{3(s+3)}$$

then

$$x(t) = \frac{-t^2}{4}e^{-2t} + \frac{t}{4}e^{-2t} - \frac{3}{8}e^{-2t} + \frac{1}{3}e^{-3t} + \frac{1}{24}$$

(c)

$$X(s) = \frac{1}{2s} - \frac{\frac{1}{2}s}{s^2+2s+2} = \frac{1}{2s} - \frac{1}{2}\frac{s+1}{(s+1)^2+1} + \frac{1}{2}\frac{1}{(s+1)^2+1}$$

then

$$x(t) = \frac{1}{2} + \frac{1}{2}e^{-t}(sint - cost)$$

3. A feedback control system has the structure shown in Fig. 2, determine the closed-loop transfer function $\frac{Y(s)}{R(s)}$ by block diagram simplification. **(10')**

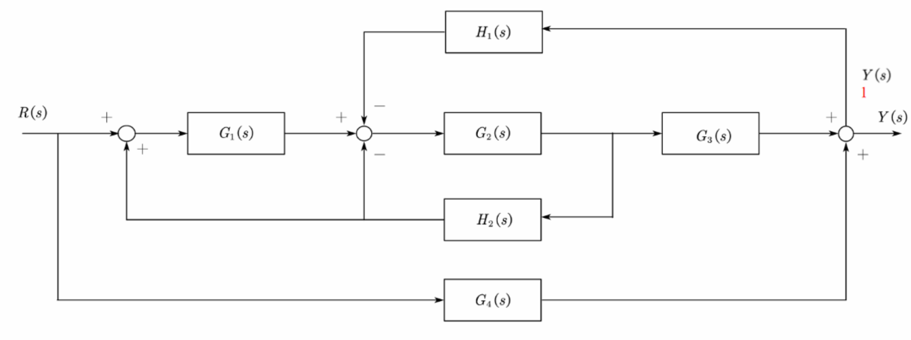
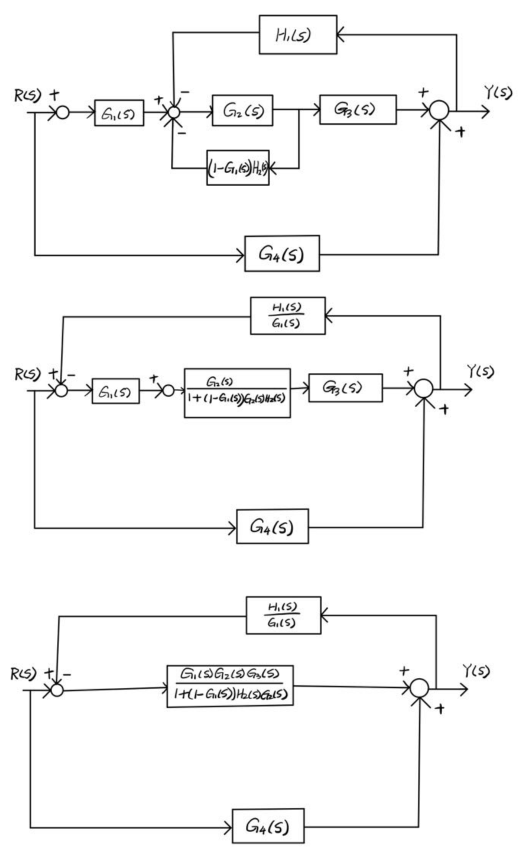


Figure 2: Block diagram

Solution.
Note: In this problem, the signal at segment 1 in the figure is also $Y(s)$ (where the drawing is slightly irregular, the signal should branch on the line, not branch on the summation module). If you think that segment 1 is before the summation module, we also think that you are right, see Solution 2.
Solution 1.

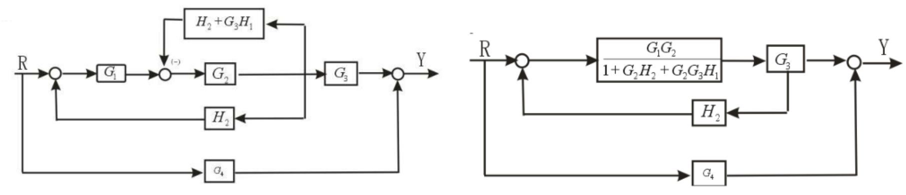


Transfer function

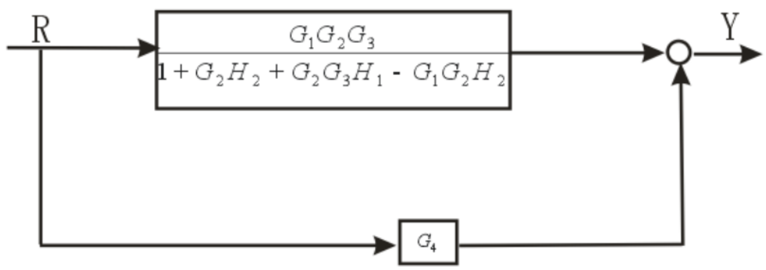
$$(Y(s) - G_4(s)R(s))\frac{1 + (1 - G_1(s))H_2(s)G_2(s)}{G_1(s)G_2(s)G_3(s)} + \frac{H_1(s)Y(s)}{G_1(s)} = R(s) \tag{8}$$

$$\frac{Y(s)}{R(s)} = \frac{G_1(s)G_2(s)G_3(s) + G_4(s) + G_4(s)G_2(s)H_2(s) - G_4(s)G_1(s)G_2(s)H_2(s)}{1 + G_2H_2 + G_2G_3H_1 - G_1G_2H_2} \tag{9}$$

Solution 2:



(a) (b)



(C)

Transfer function

$$\frac{Y(s)}{R(s)} = G_4 + \frac{G_1G_2G_3}{1 + G_2H_2 + G_2G_3H_1 - G_1G_2H_2}$$

4. Derive the transfer function $\frac{C(s)}{R(s)}$ of the signal flow graph in Fig. 3. **(10')**

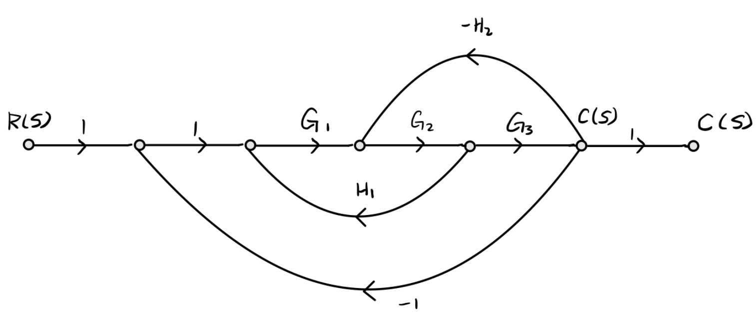
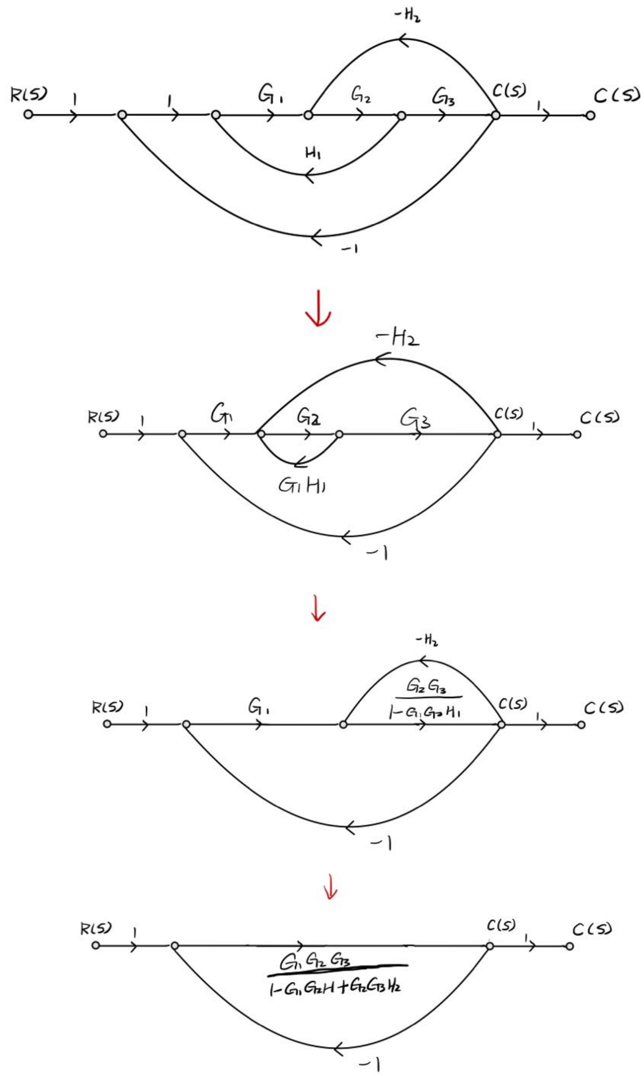


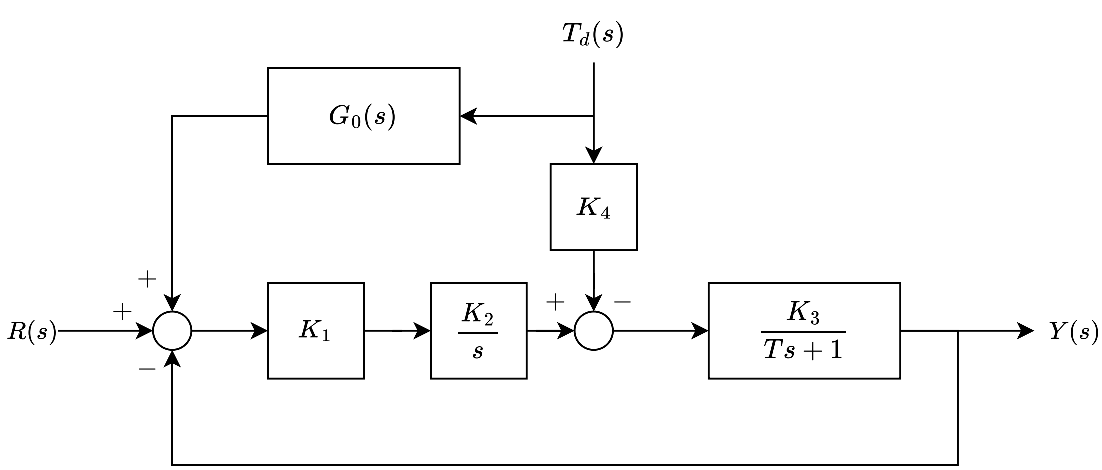
Figure 3: Signal flow graph

Solution.



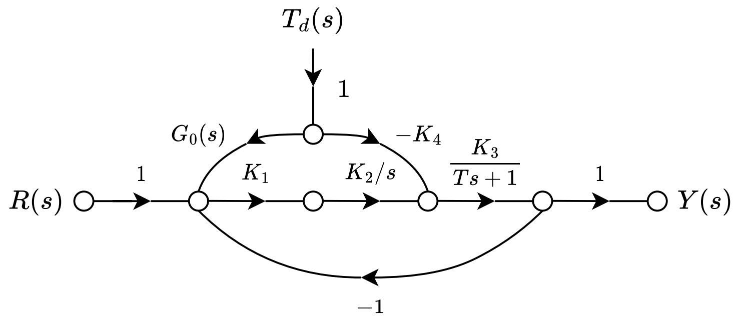
Transfer function: $\frac{C(s)}{R(s)} = \frac{G_1G_2G_3}{1+G_1G_2G_3-G_1G_2H_1+G_2G_3H_2}$.

5. Consider the following system block diagram



(a) Convert the block diagram to the signal flow graph. **(10')**

Solution. The corresponding signal flow graph (10')



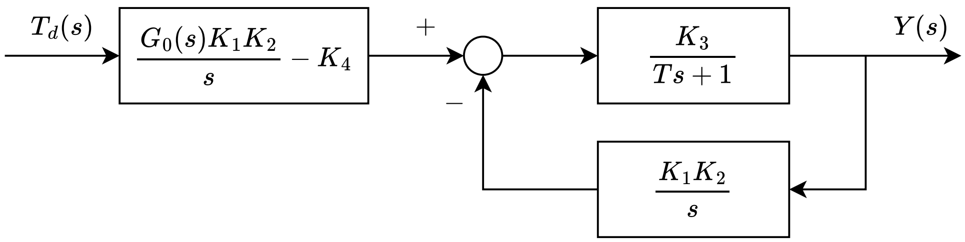
Note: if you write $-K_4$ as K_4 or forget the negative feedback, you will lose 1 point. Also, you should mark $R(s)$ and $Y(s)$.

(b) Determine the corresponding transfer function for $\frac{Y(s)}{R(s)}$ and $\frac{Y(s)}{T_d(s)}$. **(10')**

Solution. Note, T is a constant and s is a complex variable, it is not T_s. First, let $T_d(s) = 0$, the open-loop transfer function is $\frac{K_1K_2K_3}{s(Ts+1)}$, with the unity negative feedback

$$\frac{Y(s)}{R(s)} = \frac{\frac{K_1K_2K_3}{s(Ts+1)}}{1 + \frac{K_1K_2K_3}{s(Ts+1)}} = \frac{K_1K_2K_3}{Ts^2 + s + K_1K_2K_3}. \quad (5')$$

Second, let $R(s) = 0$, and first simplify the block diagram as



Then we can derive the transfer function between $Y(s)$ and $T_d(s)$.

$$\begin{aligned}\frac{Y(s)}{T_d(s)} &= \left[\frac{G_0(s)K_1K_2}{s} - K_4\right] \cdot \frac{\frac{K_3}{Ts+1}}{1 + \frac{K_3}{Ts+1} \cdot \frac{K_1K_2}{s}} \\ &= \left[\frac{G_0(s)K_1K_2}{s} - K_4\right] \cdot \frac{K_3 s}{Ts^2 + s + K_1K_2K_3} \quad (5') \\ &= \frac{K_3(G_0(s)K_1K_2 - K_4 s)}{Ts^2 + s + K_1K_2K_3}\end{aligned}$$

Note: if you did not simplify your result, you would lose 1 point.

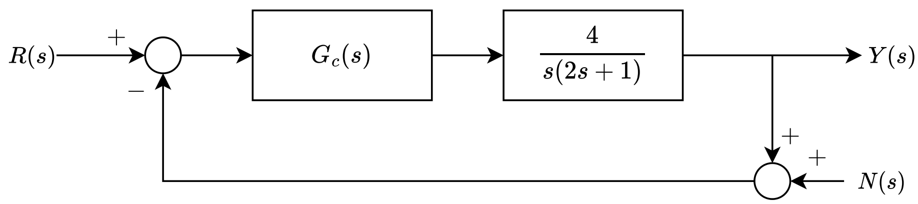
(c) To eliminate the impact of $T_d(s)$ on $Y(s)$, what should $G_0(s)$ be? **(10')**

Solution. To cancel the influence of $T_d(s)$, we can set the transfer function of $\frac{Y(s)}{T_d(s)}$ to zero, i.e.

$$\frac{Y(s)}{T_d(s)} = 0 \Rightarrow G_0(s)K_1K_2 - K_4 s = 0 \Rightarrow G_0(s) = \frac{K_4 s}{K_1K_2} \quad (10')$$

Note: if you get a wrong $G_0(s)$, you will lose 4 points.

6. Consider the following block diagram



(a) When $r(t) = t$, $n(t) = 1(t)$, $G_c(s) = 1$, calculate the steady-state error of the system. **(10')**
Solution. First, let $N(s) = 0$ and $R(s) = \frac{1}{s^2}$,

$$E_R(s) = S(s) \cdot R(s) = \frac{1}{1 + \frac{4}{s(2s+1)}} \cdot R(s) = \frac{s(2s+1)}{2s^2+s+4} \cdot \frac{1}{s^2} = \frac{2s+1}{s(2s^2+s+4)} \quad (2')$$

Then based on the final value principle

$$e_R(\infty) = \lim_{s\to 0} sE_R(s) = \lim_{s\to 0} s \cdot \frac{2s+1}{s(2s^2+s+4)} = \frac{1}{4} \quad (2')$$

Second, let $R(s) = 0$ and $N(s) = \frac{1}{s}$,

$$E_N(s) = C(s) \cdot N(s) = \frac{\frac{4}{s(2s+1)}}{1 + \frac{4}{s(2s+1)}} \cdot N(s) = \frac{4}{2s^2+s+4} \cdot \frac{1}{s} = \frac{4}{s(2s^2+s+4)} \quad (2')$$

$$e_N(\infty) = \lim_{s\to 0} sE_N(s) = \lim_{s\to 0} s \cdot \frac{4}{s(2s^2+s+4)} = 1 \quad (2')$$

Thus

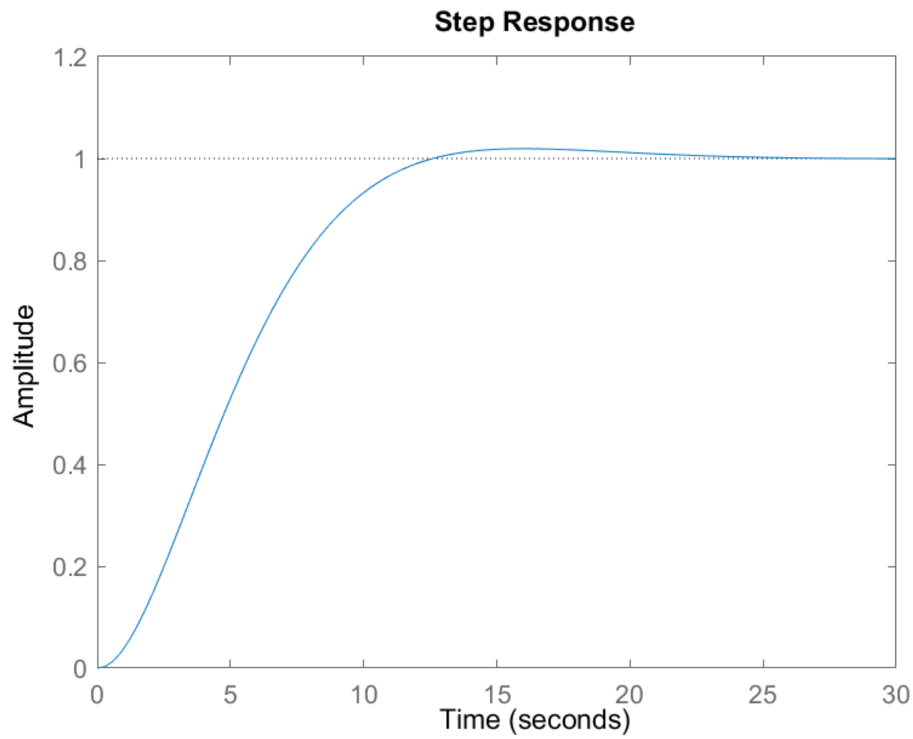
$$e(\infty) = e_R(\infty) + e_N(\infty) = \frac{5}{4} \quad (2')$$

Note: if $e(\infty)$ and $E(s)$ are wrong, you will lose 2+2=4 points. Also, if you write $E(s) = Y(s) - R(s)$, you will lose 5 points.

(b) When $r(t) = 1(t)$, $n(t) = 0$, plot the step response up to the end time ***30s*** of the closed-loop system with $G_c(s) = \frac{1}{s+20}$ and $\frac{10}{s+20}$, respectively. Give the corresponding ***closed-loop system*** transfer function under each controller.
(Note: You can use MATLAB, Python, etc. Hint: use *tf* function to create the open-loop system, then use *feedback* function to generate the closed-loop system, and use *step* function to get the step response). **(10')**
Solution. When $G_c(s) = \frac{1}{s+20}$, the closed-loop system transfer function is

$$\Phi(s) = \frac{4}{2s^3 + 41s^2 + 20s + 4} \quad (2')$$

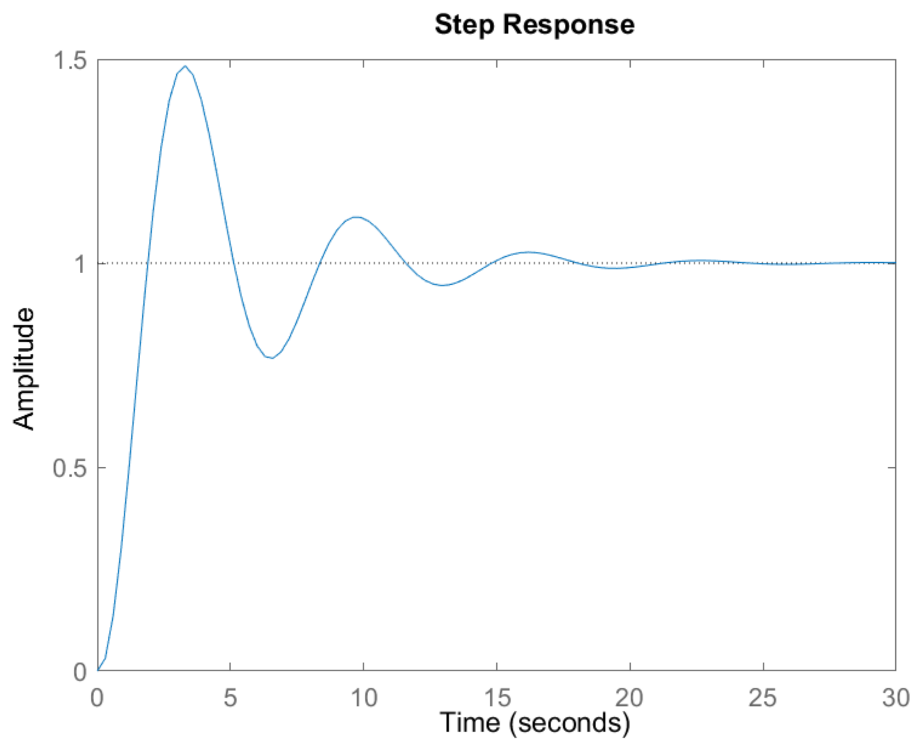
The step response is (3')



When $G_c(s) = \frac{10}{s+20}$, the closed-loop system transfer function is

$$\Phi(s) = \frac{40}{2s^3 + 41s^2 + 20s + 40} \quad (2')$$

The step response is (3')



Note: if you did not simplify your result, you would lose 1 point. If you give the transfer function and step response of the open-loop system, you will lose all points.

MATLAB code:

```
Gc1 = 1 * tf(1, [1 20]) % controller
Gc2 = 10 * tf(1, [1 20]) % controller
sys = tf(4,[2 1 0]) % open-loop system
sys_c1 = feedback(Gc1 * sys, 1, -1) % closed-loop system
step(sys_c1, 30) % step response
sys_c2 = feedback(Gc2 * sys, 1, -1) % closed-loop system
step(sys_c2, 30) % step response
```