EE160 Homework 8 Solution

1. *(6 points) Infinite Horizon Optimal Control.*

   Solution:

   (a) Solving the algebraic Riccati equation, we'll get

   $$0 = 4P_\infty + 2 - \frac{P_\infty^2}{3} \quad \Rightarrow \quad P_\infty = 6 + \sqrt{42} > 0$$

   control gain

   $$K_\infty = -\left(2 + \sqrt{\frac{14}{3}}\right).$$

   Then closed-loop system is given by

   $$\dot{x} = a_{\mathrm{cl}}\, x, \quad a_{\mathrm{cl}} = -\sqrt{\frac{14}{3}} \quad \Rightarrow \quad x(t) = e^{-\sqrt{\frac{14}{3}}t}$$

   where the feedback control law

   $$u = K_\infty x = -\left(2 + \sqrt{\frac{14}{3}}\right) e^{-\sqrt{\frac{14}{3}}t}$$

   (b) The steady state of the control system must satisfy

   $$x_s + u_s + 1 = 0$$

   however that could make the cost function diverge to infinity, which means this optimal control problem has no solution.

   (c) Refer to solution of final exam 2019 please!

2. *(4 points) Finite Horizon Optimal Control.*

   Solution: The Riccati equations are given by

   $$\begin{aligned} \dot{P}(t) &= -2P(t) + 1 - P(t)^2 \\ P(10) &= 5 \end{aligned}$$

   solve this ODE with the boundary condition first

   $$\begin{aligned} & \dot{P}(t) = -2P(t) + 1 - P(t)^2 \\ \Rightarrow \quad & \frac{\dot{P}(t)}{(P(t) + (-1+\sqrt{2}))(P(t) + (-1-\sqrt{2}))} = -1 \\ \Rightarrow \quad & \frac{\dot{P}(t)}{P(t) + (-1+\sqrt{2})} - \frac{\dot{P}(t)}{P(t) + (-1-\sqrt{2})} = 2\sqrt{2} \\ \Rightarrow \quad & \log\left(\left|\frac{P(10) + (-1+\sqrt{2})}{P(10) + (-1-\sqrt{2})}\right|\right) - \log\left(\left|\frac{P(t) + (-1+\sqrt{2})}{P(t) + (-1-\sqrt{2})}\right|\right) = 2\sqrt{2}(10 - t) \end{aligned}$$

substituting $P(10) = 5$, we'll have

$$P(t) = \frac{\left(5\sqrt{2} - 4\right) e^{2\sqrt{2}t} + e^{20\sqrt{2}} \left(5\sqrt{2} + 4\right)}{\left(\sqrt{2} + 6\right) e^{2\sqrt{2}t} + e^{20\sqrt{2}} \left(\sqrt{2} - 6\right)}, \quad K(t) = -P(t)$$

To get the explicit expression of $x(t)$, let's calculate the ODE of the control system,

$$\begin{aligned}
& \dot{x}(t) = -x(t) + u(t) = -(1 + P(t))x(t) \\
\Rightarrow \quad & \frac{\dot{x}(t)}{x(t)} = -(1 + P(t)) \\
\Rightarrow \quad & \log(x(t)) = -t + \int_0^t P(\tau) \mathrm{d}\tau \\
\Rightarrow \quad & x(t) = \frac{\left(e^{20\sqrt{2}} \left(6\sqrt{2} - 19\right) + 17\right) e^{\sqrt{2}t}}{e^{20\sqrt{2}} \left(6\sqrt{2} - 19\right) + 17 e^{2\sqrt{2}t}} \quad \text{and} \quad u(t) = -P(t)x(t)
\end{aligned}$$