Resistive Circuit $\left\{ \begin{array}{l} \text{VCR} \leftarrow \\ + \\ \text{KCL/KVL} \leftarrow \end{array} \right\}$

Lecture 8

- Sinusoidal Steady-State Analysis

Outline

- Kirchhoff's laws in phasor domain
- Generalized impedance
- General AC phasor analysis
- Phasor diagram

Kirchhoff's Laws in the Phasor Domain

- Let $v_1, v_2, \cdots v_n$ be the voltages around a closed loop. Then according to KVL

$$v_1 + v_2 + \cdots + v_n = 0$$

$$\mathbf{V}_1 + \mathbf{V}_2 + \cdots + \mathbf{V}_n = 0$$

Similarly, KCL holds for phasors:

$$i_1 + i_2 + \cdots + i_n = 0$$

$$\mathbf{I}_1 + \mathbf{I}_2 + \cdots + \mathbf{I}_n = 0,$$

Proof

If

$$v_1 + v_2 + \cdots + v_n = 0$$

where v_i are sinusoidal voltages of the same frequency, then

$$\dot{\mathbf{V}}_1 + \dot{\mathbf{V}}_2 + \cdots + \dot{\mathbf{V}}_n = 0$$

Proof:

$$v_1 + v_2 + \cdots + v_n = 0$$

$$\Updownarrow$$

$$V_{m1}\cos(\omega t + \theta_1) + V_{m2}\cos(\omega t + \theta_2) + \cdots + V_{mn}\cos(\omega t + \theta_n) = 0$$

$$\Updownarrow$$

$$\mathrm{Re}\left(V_{m1}e^{j\theta_1} \cdot e^{j\omega t}\right) + \cdots + \mathrm{Re}\left(V_{mn}e^{j\theta_n} \cdot e^{j\omega t}\right) = 0$$

$$\Updownarrow$$

$$\mathrm{Re}\left((\dot{\mathbf{V}}_1 + \cdots + \dot{\mathbf{V}}_n) \cdot e^{j\omega t}\right) = 0 \quad Where\ \mathbf{V}_k = V_{mk}e^{j\theta_k}$$

$$\Updownarrow$$

$$\dot{\mathbf{V}}_1 + \dot{\mathbf{V}}_2 + \cdots + \dot{\mathbf{V}}_n = 0$$

Outline

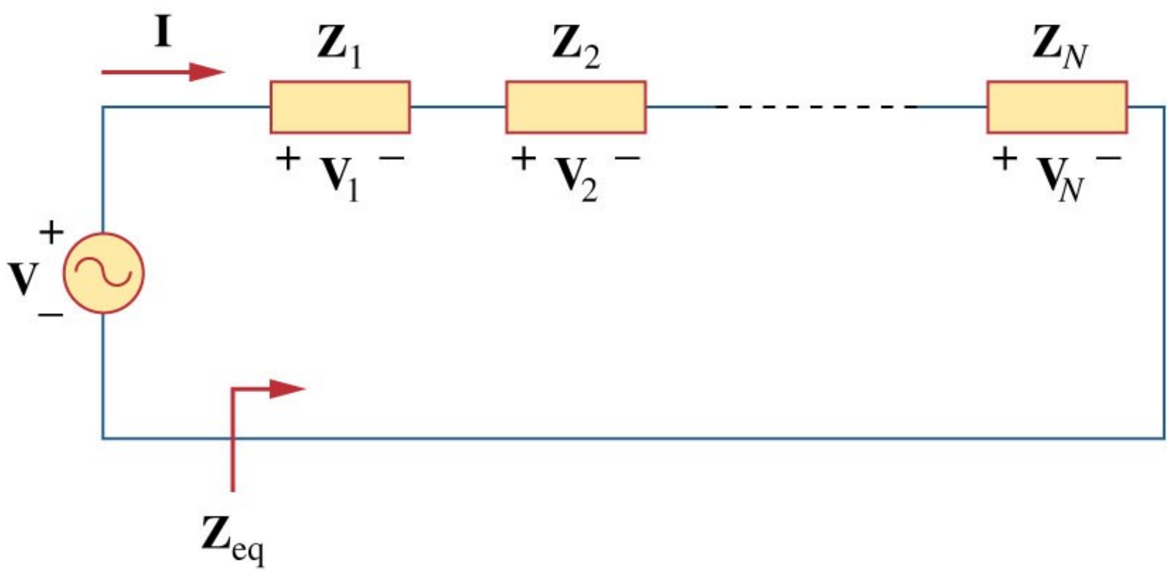
- Kirchhoff's laws in phasor domain
- Generalized impedance
- General AC phasor analysis
- Phasor diagram

Impedance and Admittance

Resistor	$\mathbf{Z} = R$	$\mathbf{Y} = 1/R$
Inductor	$\mathbf{Z} = j\omega L$	$\mathbf{Y} = 1/j\omega L$
Capacitor	$\mathbf{Z} = 1/j\omega C$	$\mathbf{Y} = j\omega C$

Series Impedance

- In phasor domain, combinations of *impedance* will follow the rules for ***resistors:***



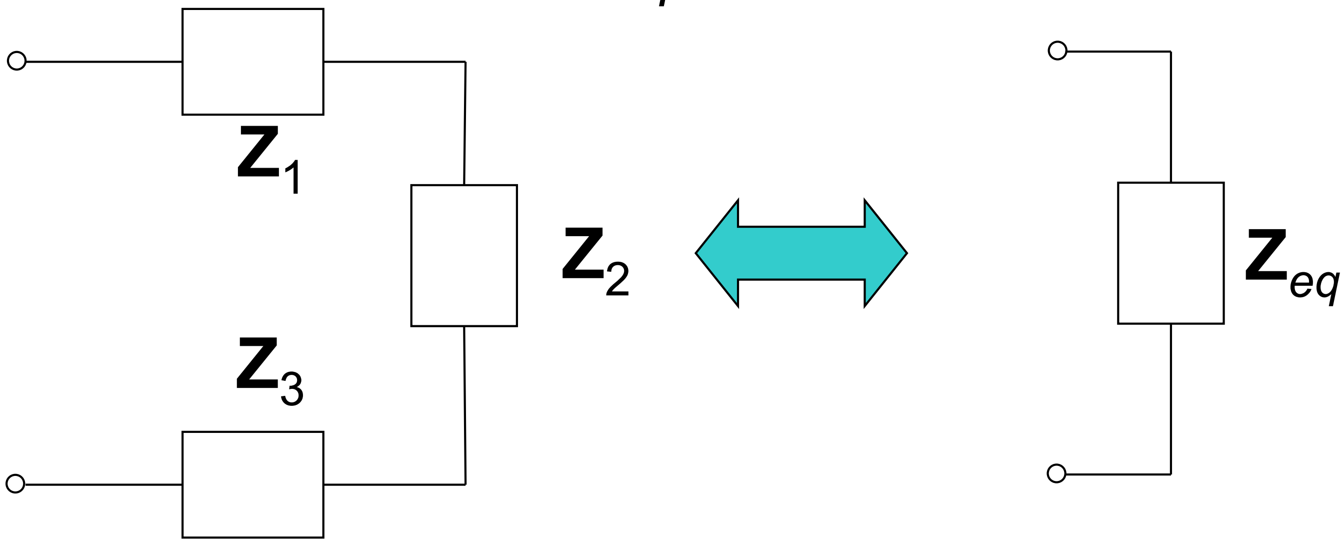
$$Z_{eq} = Z_1 + Z_2 + Z_3 + \cdots$$

$$V_1 = \frac{Z_1}{Z_1 + \cdots + Z_N} V$$

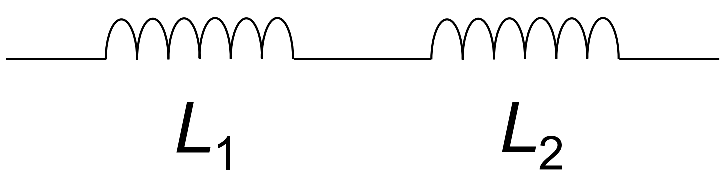
$$V_2 = \frac{Z_2}{Z_1 + \cdots + Z_N} V$$

Series Impedance

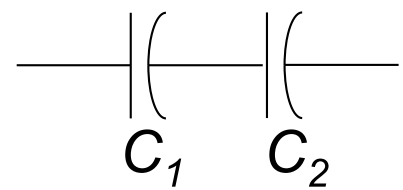
$$\mathbf{Z}_{eq} = \mathbf{Z}_1 + \mathbf{Z}_2 + \mathbf{Z}_3$$



For example:



$$\mathbf{Z}_{eq} = j\omega L_1 + j\omega L_2$$



$$\mathbf{Z}_{eq} = \frac{1}{j\omega C_1} + \frac{1}{j\omega C_2}$$

Impedance combination for RLC Circuit

$R_1 + j\omega L_1$

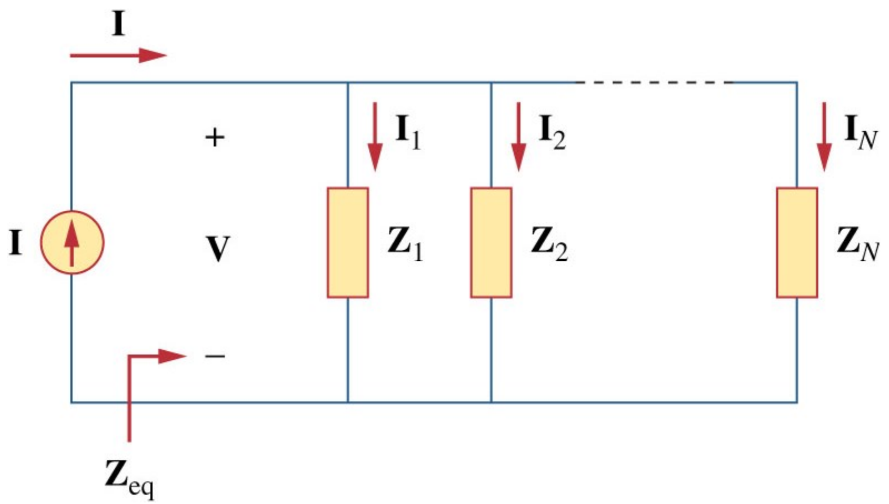
(a) RL

(b) RC

(c) LC

Parallel Combination

- Likewise, elements in parallel will combine in the same fashion as resistors in parallel:



$$\frac{1}{Z_{eq}} = \frac{1}{Z_1} + \frac{1}{Z_2} + \frac{1}{Z_3} + \cdots \frac{1}{Z_N}$$

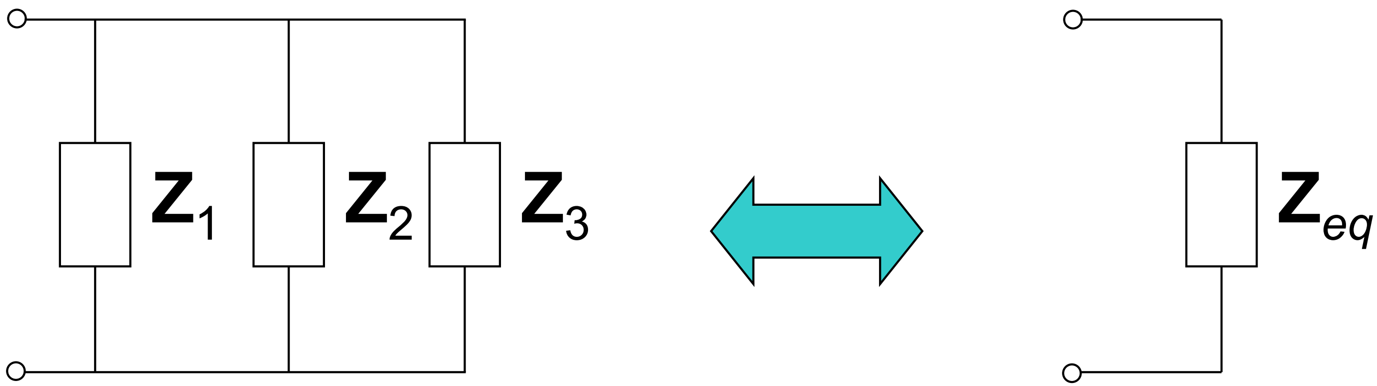
$$Y_{eq} = Y_1 + Y_2 + Y_3 + \cdots Y_N$$

$$I_1 = \frac{Y_1}{Y_1 + \cdots + Y_N} I$$

$$I_2 = \frac{Y_2}{Y_1 + \cdots + Y_N} I$$

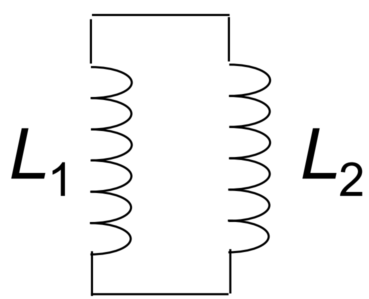
……

Parallel Impedance

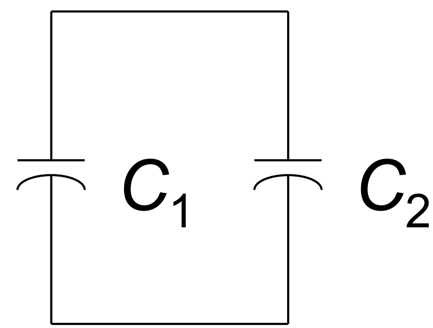


$$1/\mathbf{Z}_{eq} = 1/\mathbf{Z}_1 + 1/\mathbf{Z}_2 + 1/\mathbf{Z}_3$$

For example:



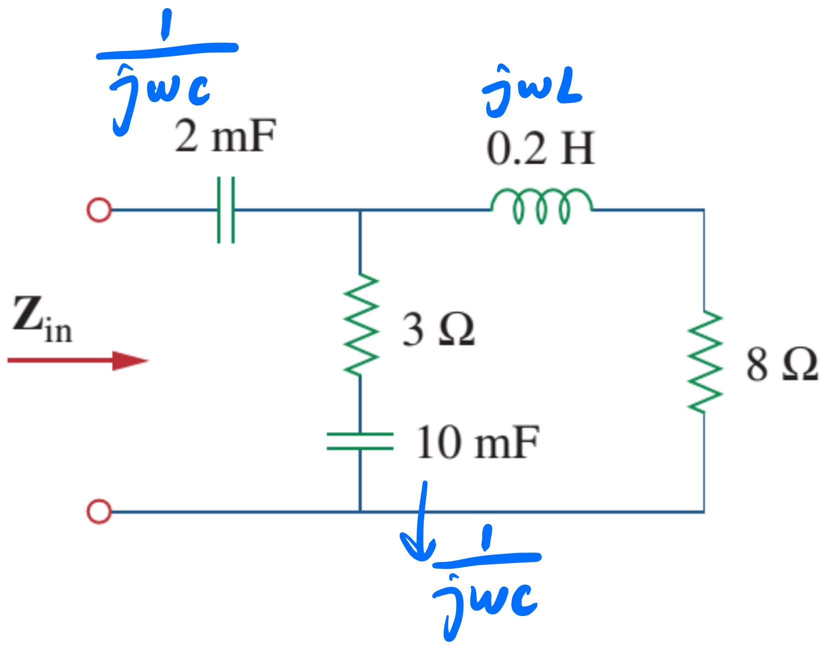
$$\mathbf{Z}_{eq} = j\omega \frac{L_1 L_2}{(L_1 + L_2)}$$

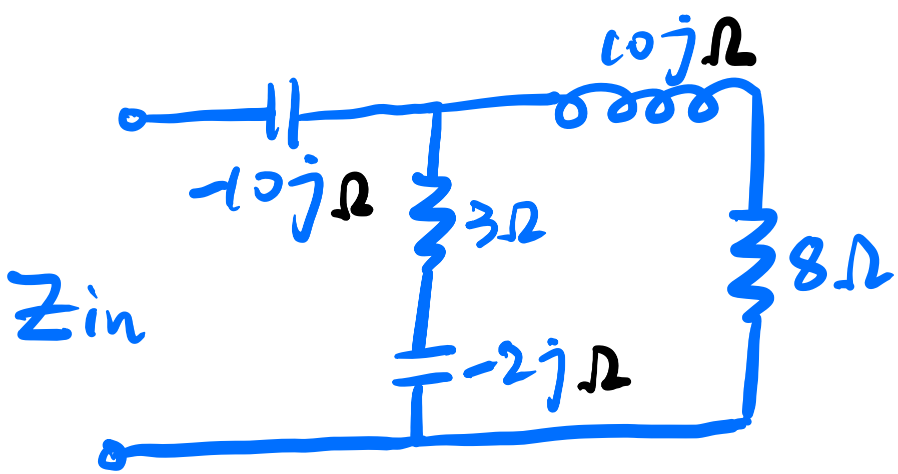


$$\mathbf{Z}_{eq} = \frac{1}{j\omega(C_1 + C_2)}$$

Exercise

- Find the input impedance of the circuit below. $\omega = 50$ rad/s.

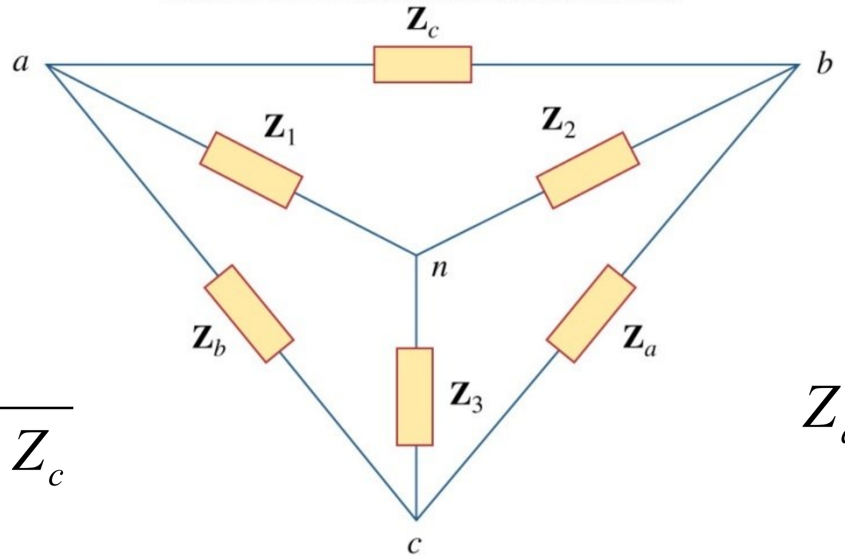




$$Z_{in} = (8+10j) \parallel (3-2j) + (-10j)$$

$$= 3.22 - j11.07$$

Delta-Wye Transformation in Phasor Domain



$$Z_1 = \frac{Z_b Z_c}{Z_a + Z_b + Z_c}$$

$$Z_2 = \frac{Z_c Z_a}{Z_a + Z_b + Z_c}$$

$$Z_3 = \frac{Z_a Z_b}{Z_a + Z_b + Z_c}$$

$$Z_a = \frac{Z_1 Z_2 + Z_2 Z_3 + Z_3 Z_1}{Z_1}$$

$$Z_b = \frac{Z_1 Z_2 + Z_2 Z_3 + Z_3 Z_1}{Z_2}$$

$$Z_c = \frac{Z_1 Z_2 + Z_2 Z_3 + Z_3 Z_1}{Z_3}$$

Impedance and Admittance

Resistor	$\mathbf{Z} = R$	$\mathbf{Y} = 1/R$
Inductor	$\mathbf{Z} = j\omega L$	$\mathbf{Y} = 1/j\omega L$
Capacitor	$\mathbf{Z} = 1/j\omega C$	$\mathbf{Y} = j\omega C$

Impedance is voltage/current

$$\mathbf{Z} = R + jX$$

R = resistance = Re(Z)

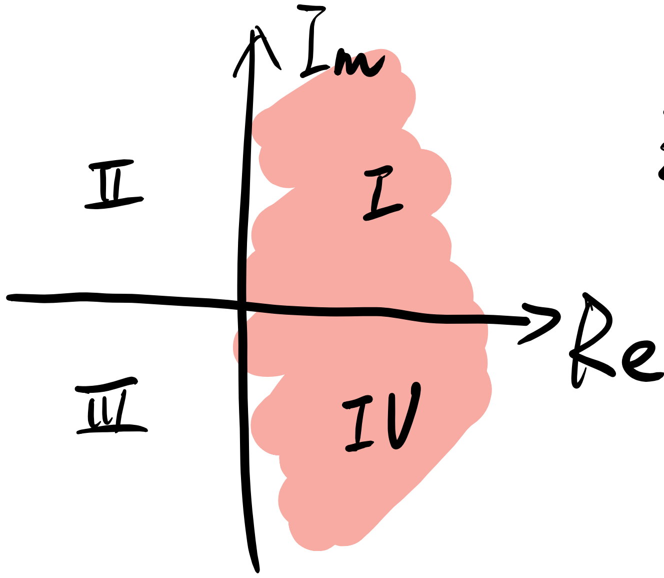
X = reactance = Im(Z)

Admittance is current/voltage

$$\mathbf{Y} = \frac{1}{\mathbf{Z}} = G + jB$$

G = conductance = Re(Y)

B = susceptance = Im(Y)



$$Z = |Z| \angle \theta$$

$$|\theta| \leq 90°$$

$$Y = \frac{1}{Z} = \frac{1}{|Z|} \cdot \angle -\theta$$

Outline

- Kirchhoff's laws in phasor domain
- Generalized impedance
- General AC phasor analysis
  - Nodal/mesh analysis
  - Superposition
  - Source transformation/Thevenin/Norton
- Phasor diagram

AC Phasor Analysis General Procedure

Step 1: Adopt cosine reference

$$v_s(t) = 12\sin(\omega t - 45^\circ) = 12\cos(\omega t - 45^\circ - 90^\circ) = 12\cos(\omega t - 135^\circ) \text{ V}.$$

$$\mathbf{V}_s = 12e^{-j135^\circ} \text{ V}.$$

Step 2: Transform circuit to phasor domain

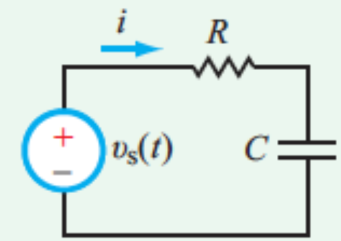
Step 3: Cast KCL and/or KVL equations in phasor domain

$$\mathbf{Z}_R\mathbf{I} + \mathbf{Z}_C\mathbf{I} = \mathbf{V}_s,$$

which is equivalent to

$$\left(R + \frac{1}{j\omega C}\right)\mathbf{I} = 12e^{-j135^\circ}.$$

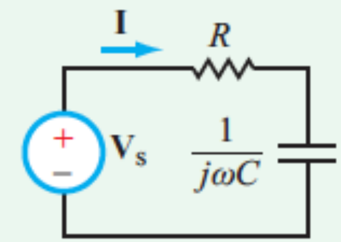
Step 1
Adopt Cosine Reference
(Time Domain)



$v_s(t) = 12\sin(\omega t - 45^\circ)$ (V)

Step 2
Transfer to Phasor Domain

$i \rightarrow \mathbf{I}$
$v \rightarrow \mathbf{V}$
$R \rightarrow \mathbf{Z}_R = R$
$L \rightarrow \mathbf{Z}_L = j\omega L$
$C \rightarrow \mathbf{Z}_C = 1/j\omega C$



$\mathbf{V}_s = 12e^{-j135^\circ}$ (V)

Step 3
Cast Equations in
Phasor Form

$$\mathbf{I}\left(R + \frac{1}{j\omega C}\right) = \mathbf{V}_s$$

AC Phasor Analysis General Procedure

Step 4: Solve for unknown variable

$$\mathbf{I} = \frac{12e^{-j135^\circ}}{R + \frac{1}{j\omega C}} = \frac{j12\omega C e^{-j135^\circ}}{1 + j\omega RC}.$$

Using the specified values, namely $R = \sqrt{3}\ \text{k}\Omega$, $C = 1\ \mu\text{F}$, and $\omega = 10^3$ rad/s,

$$\mathbf{I} = \frac{j12 \times 10^3 \times 10^{-6} e^{-j135^\circ}}{1 + j10^3 \times \sqrt{3} \times 10^3 \times 10^{-6}} = \frac{j12e^{-j135^\circ}}{1 + j\sqrt{3}}\ \text{mA}.$$

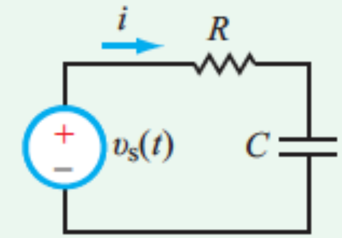
$$\mathbf{I} = \frac{12e^{-j135^\circ} \cdot e^{j90^\circ}}{2e^{j60^\circ}} = 6e^{j(-135^\circ + 90^\circ - 60^\circ)} = 6e^{-j105^\circ}\ \text{mA}.$$

Step 5: Transform solution back to time domain

To return to the time domain, we apply the fundamental relation between a sinusoidal function and its phasor counterpart, namely

$$i(t) = \mathfrak{Re}[\mathbf{I}e^{j\omega t}] = \mathfrak{Re}[6e^{-j105^\circ} e^{j\omega t}] = \underline{6\cos(\omega t - 105^\circ)\ \text{mA}}.$$

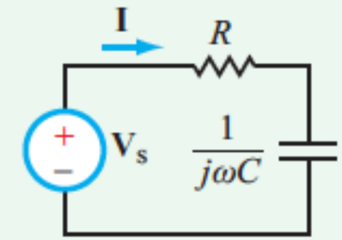
Step 1
Adopt Cosine Reference
(Time Domain)



$v_s(t) = 12\sin(\omega t - 45^\circ)$ (V)

Step 2
Transfer to Phasor Domain

$i \rightarrow \mathbf{I}$
$v \rightarrow \mathbf{V}$
$R \rightarrow \mathbf{Z}_R = R$
$L \rightarrow \mathbf{Z}_L = j\omega L$
$C \rightarrow \mathbf{Z}_C = 1/j\omega C$



$\mathbf{V}_s = 12e^{-j135^\circ}$ (V)

Step 3
Cast Equations in
Phasor Form

$\mathbf{I}\left(R + \frac{1}{j\omega C}\right) = \mathbf{V}_s$

Step 4
Solve for Unknown Variable
(Phasor Domain)

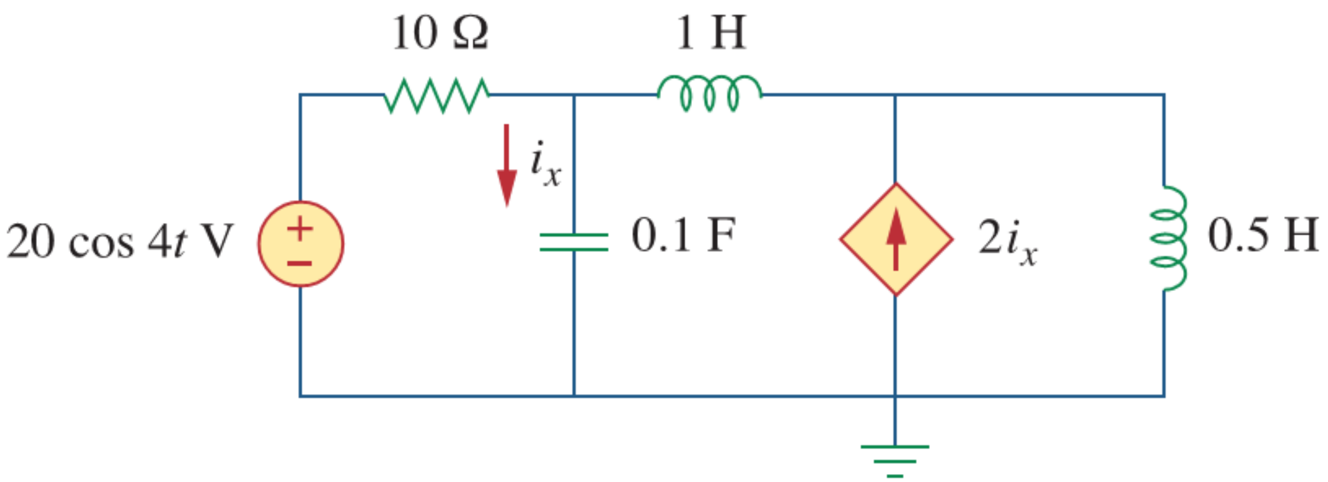
$\mathbf{I} = \frac{\mathbf{V}_s}{R + \frac{1}{j\omega C}}$

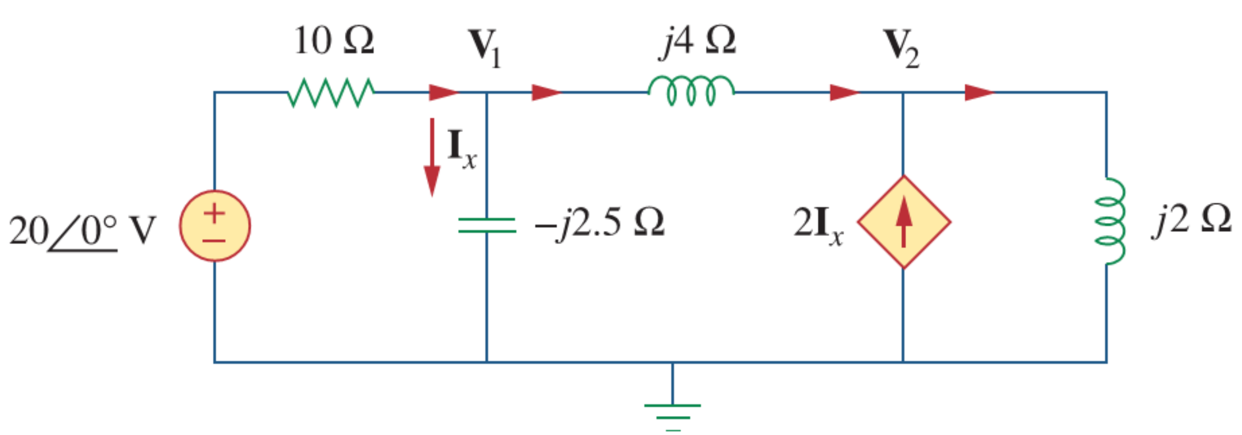
Step 5
Transform Solution
Back to Time Domain

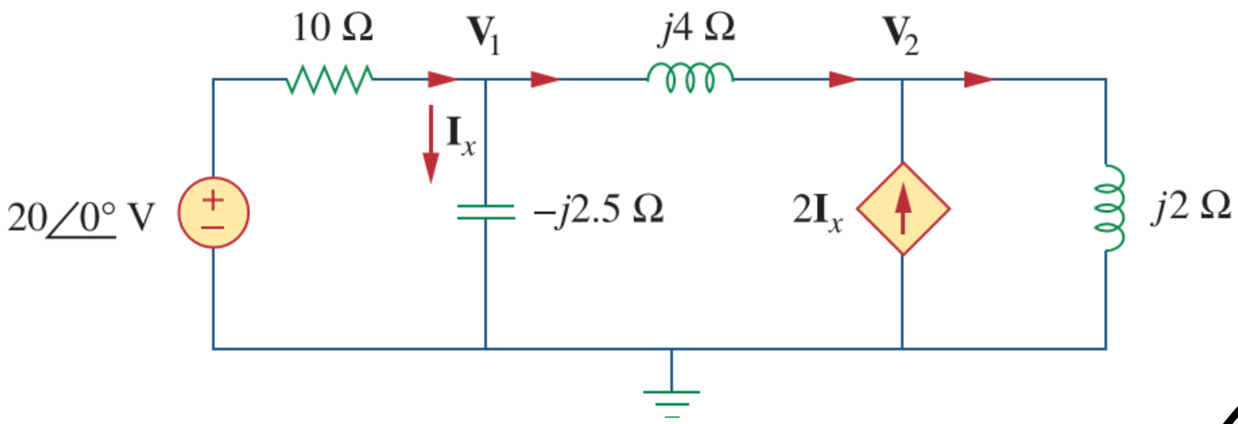
$i(t) = \mathfrak{Re}[\mathbf{I}e^{j\omega t}]$
$= 6\cos(\omega t - 105^\circ)$
(mA)

Nodal Analysis

- Example---Find i_x





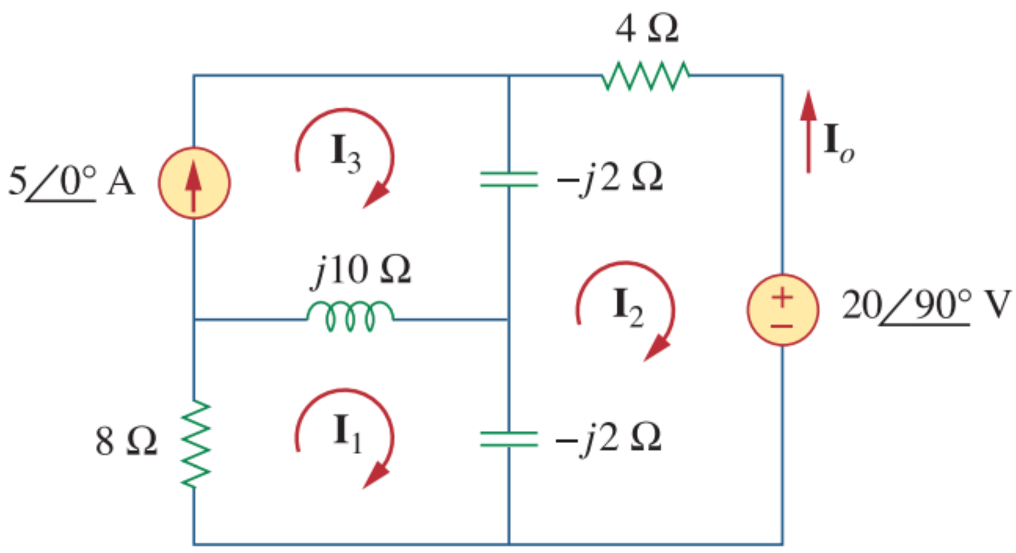


$$\frac{20 - \mathbf{V}_1}{10} = \frac{\mathbf{V}_1}{-j2.5} + \frac{\mathbf{V}_1 - \mathbf{V}_2}{j4}$$

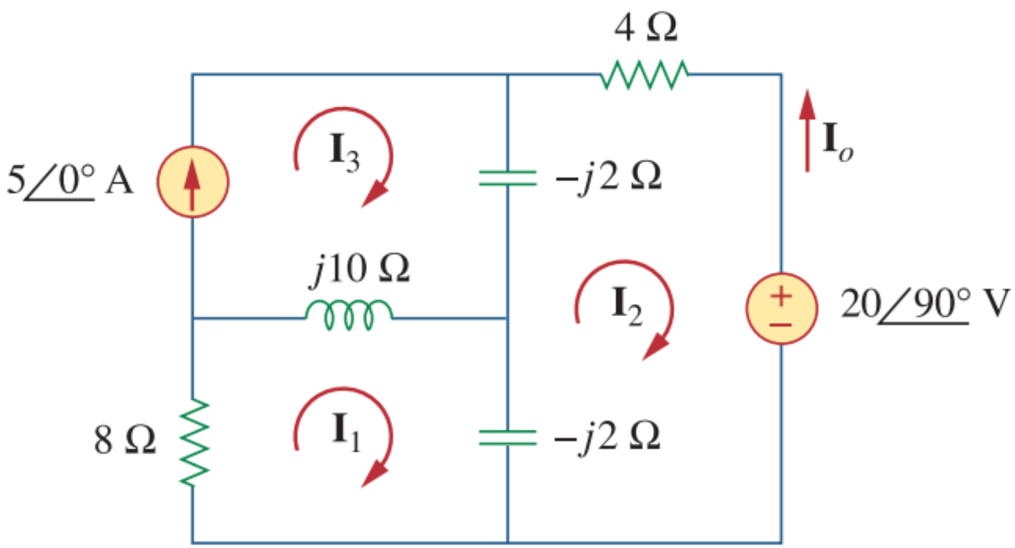
$$2\mathbf{I}_x + \frac{\mathbf{V}_1 - \mathbf{V}_2}{j4} = \frac{\mathbf{V}_2}{j2}$$

$$I_x = \frac{V_1}{-2.5j}$$

Mesh Analysis



Mesh Analysis



Applying KVL to mesh 1, we obtain

$$(8 + j10 - j2)\mathbf{I}_1 - (-j2)\mathbf{I}_2 - j10\mathbf{I}_3 = 0 \qquad \textbf{(10.3.1)}$$

For mesh 2,

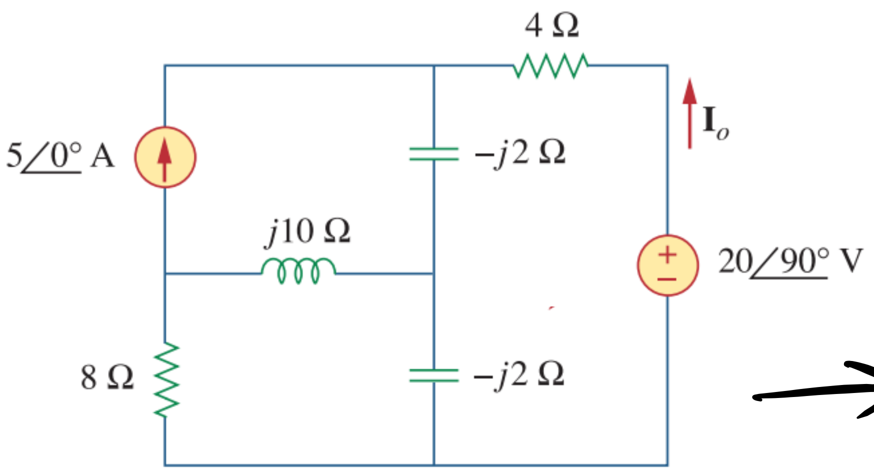
$$(4 - j2 - j2)\mathbf{I}_2 - (-j2)\mathbf{I}_1 - (-j2)\mathbf{I}_3 + 20\underline{/90^\circ} = 0 \qquad \textbf{(10.3.2)}$$

For mesh 3, $\mathbf{I}_3 = 5$. Substituting this in Eqs. (10.3.1) and (10.3.2), we get

$$(8 + j8)\mathbf{I}_1 + j2\mathbf{I}_2 = j50 \qquad \textbf{(10.3.3)}$$

$$j2\mathbf{I}_1 + (4 - j4)\mathbf{I}_2 = -j20 - j10 \qquad \textbf{(10.3.4)}$$

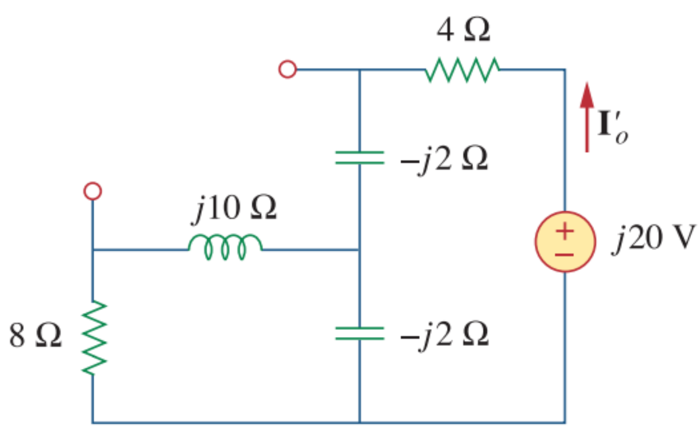
Superposition-Example



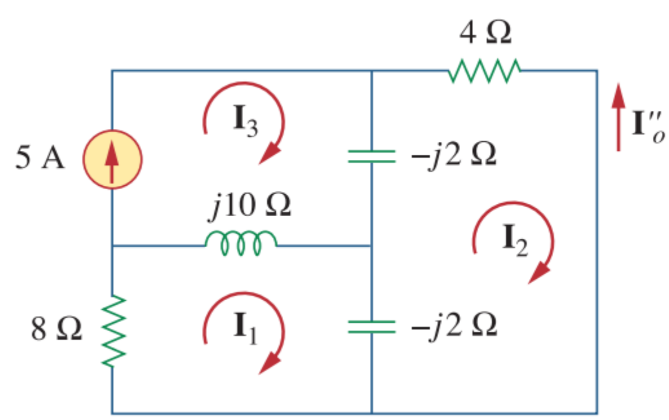
the same freq. ω

$\rightarrow I_o = I_o' + I_o''$

$\hookrightarrow i_o(t)$ ----

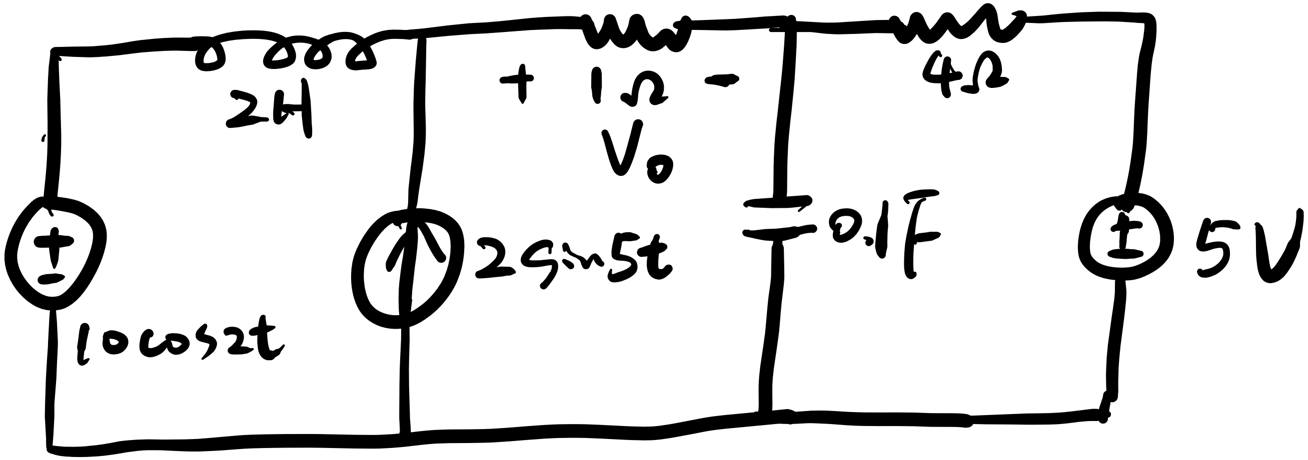


(a)



(b)

Superposition-Example 2



$$V_o = V_o' + V_o'' + V_o'''$$

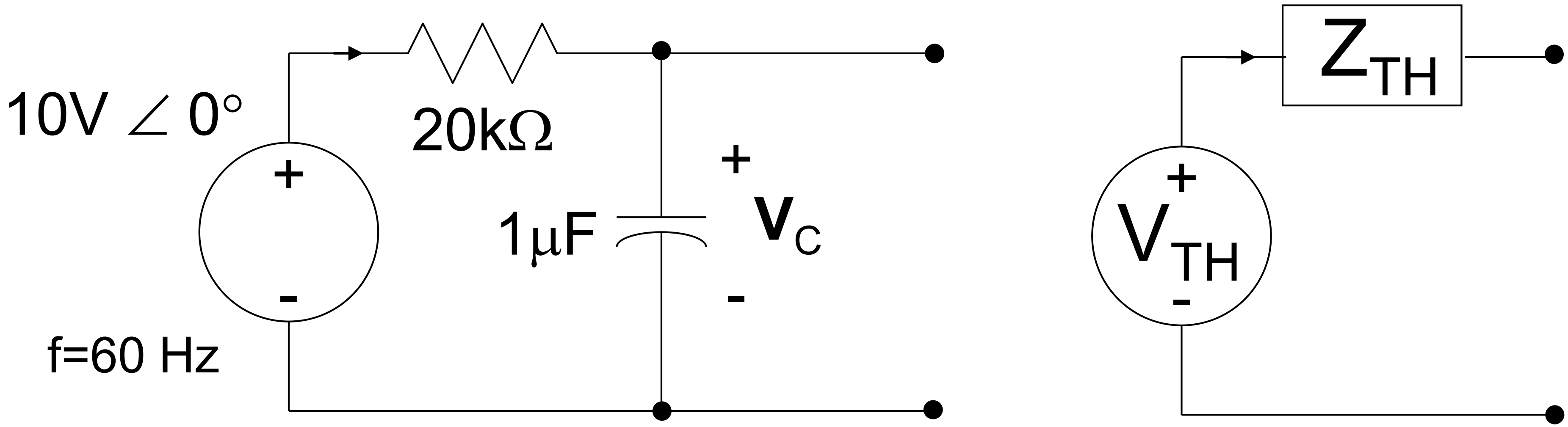
D.C.

$$-1V + \text{Phasor}'' + \text{Phasor}'''$$

$$\downarrow \qquad \downarrow \omega \qquad \downarrow \omega$$

$$V_o(t) = \underline{-1V} + \underline{\cdots\cos(5t - \cdots)} + \underline{\cos(2t - \cdots)}$$

Thevenin Equivalent

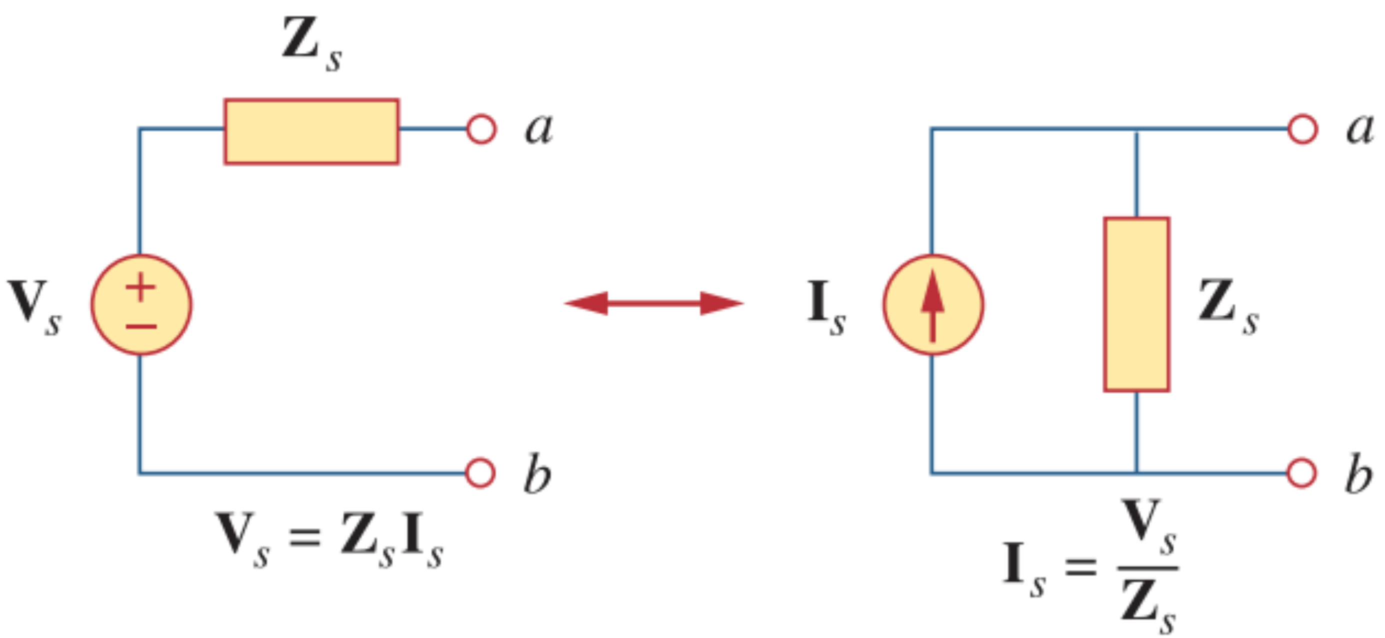


$\mathbf{Z}_R$ = R= 20kΩ = 20kΩ ∠ 0°
$\mathbf{Z}_C$ = 1/*j* (2πf x 1μF) = 2.65kΩ ∠ -90°

$$\mathbf{V}_{TH} = \mathbf{V}_{OC} = 10\text{V} \angle 0^\circ \left(\frac{2.65\text{k}\Omega\angle -90^\circ}{2.65\text{k}\Omega\angle -90^\circ + 20\text{k}\Omega\angle 0^\circ} \right) = 1.31\angle -82.4$$

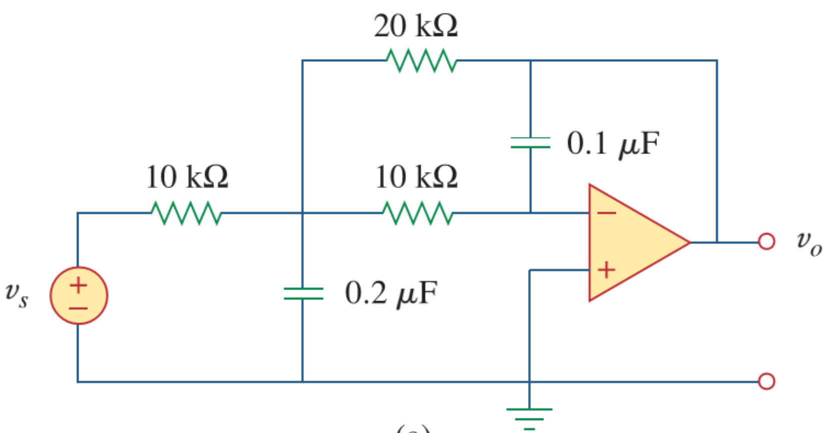
$$\mathbf{Z}_{TH} = \mathbf{Z}_R \,||\, \mathbf{Z}_\text{C} = \left(\frac{20\text{k}\Omega\angle 0^\circ \cdot 2.65\text{k}\Omega\angle -90^\circ}{2.65\text{k}\Omega\angle -90^\circ + 20\text{k}\Omega\angle 0^\circ} \right) = 2.62\angle -82.4$$

Source transformation/Norton



$$\mathbf{V}_s = \mathbf{Z}_s\mathbf{I}_s \quad \Leftrightarrow \quad \mathbf{I}_s = \frac{\mathbf{V}_s}{\mathbf{Z}_s}$$

AC Op Amp Circuits



(a)

Question 1: Are op amps used in ac circuits?

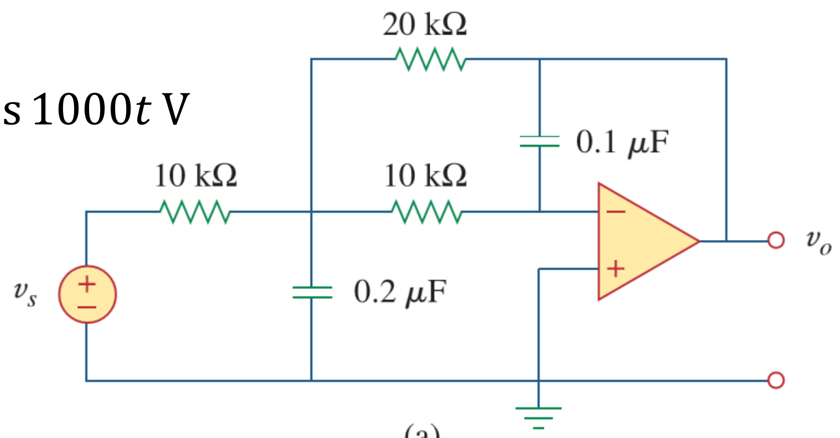
Answer 1: Yes.

Question 2: Is the ideal op-amp model applicable to ac circuits?

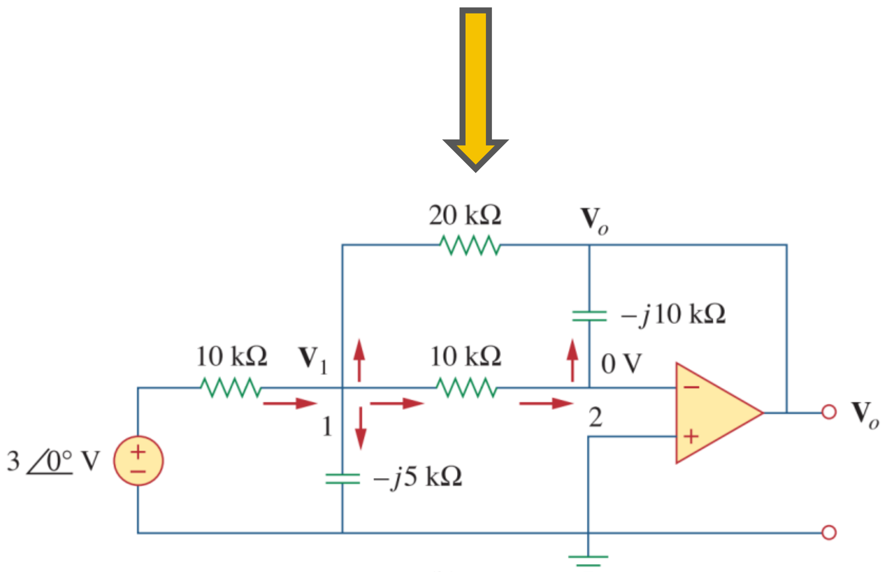
Answer 2: The ideal op-amp model is based on the assumption that the open-loop gain A is very large ($> 10^4$), which is true at dc and low frequencies, but not necessarily so at high frequencies. The range of frequencies over which A is large depends on the specific op-amp design. .

Example –find v_o

$v_s = 3\cos 1000t$ V



(a)

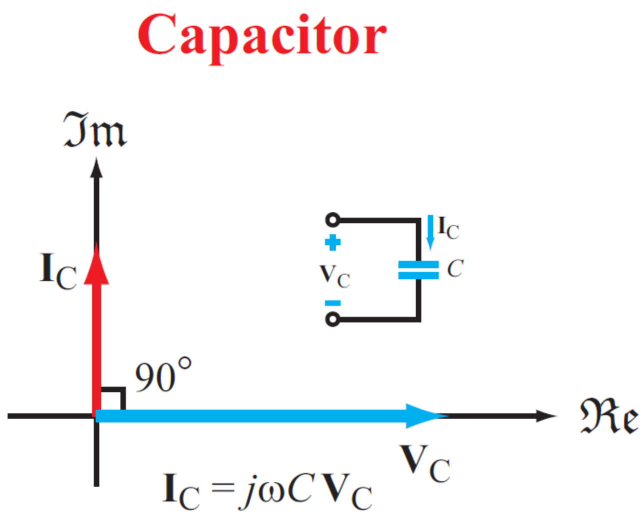


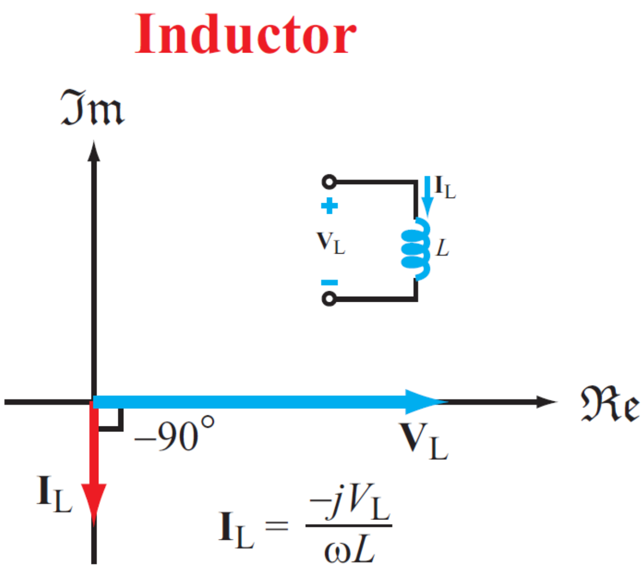
(b)

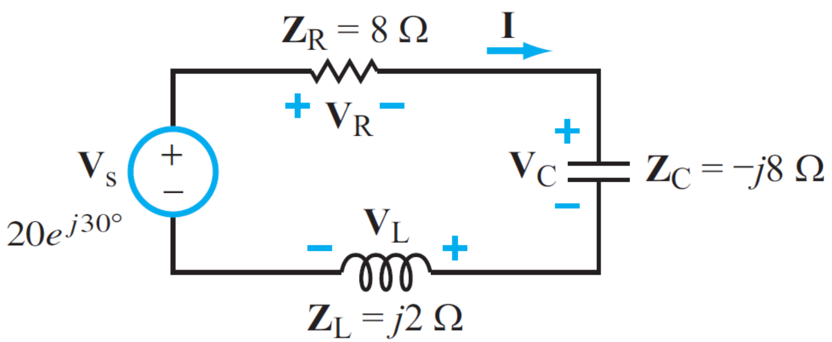
Outline

- Kirchhoff's laws in phasor domain
- Generalized impedance
- General AC phasor analysis
- Phasor diagram

Phasor Diagrams







$$\mathbf{I} = \frac{\mathbf{V}_s}{R + j\omega L - \dfrac{j}{\omega C}} = \frac{20e^{j30^\circ}}{8 + j2 - j8} = \frac{20e^{j30^\circ}}{8 - j6} = \frac{20e^{j30^\circ}}{10e^{-j36.87^\circ}} = 2e^{j66.87^\circ}\ \mathrm{A}$$