Numerical Optimization: Final Exam

ID: __________________ Name:__________________

December 30, 2021

Problem 1: (16 points) Consider the equality constrained problem

$$\begin{aligned} \min_{x} \quad & f(x) \\ \text{s.t.} \quad & c_i(x) = 0, \quad i = 1, ..., m, \end{aligned}$$

where $f, c_i, i = 1, ..., m$ are smooth functions. Show that in this case the IPM (interior-point method) search step is also a SQP (sequential quadratic programming) step.

Problem 2: (8 points) Fill out the following form. Put a "✓" to indicate that this case could happen. Put a "×" to indicate that this case cannot happen. (see the example in the bottom right cell).

Primal \ Dual	Infeasible	Unbounded (from above)	Optimal solution exists
Infeasible			
Unbounded (from below)			
Optimal solution exists			✓

Problem 3: (20 points = 4×5) For the standard form

$$\min_{x} \quad c^T x \quad \text{s.t. } Ax = b, x \geq 0.$$

(1) Write down the KKT system of this problem.

(2) Write down the conditions in the above KKT system, for which the iterates of the primal simplex method must satisfy.

(3) Write down the conditions in the above KKT system, for which the iterates of the dual simplex method must satisfy.

(4) Write down the conditions in the above KKT system, for which the iterates of the interior point method must satisfy.

(5) Show that the strong duality holds for any KKT point for the LP.

Problem 4: (17points = 7+10) Suppose we are solving the unconstrained optimization problem

$$\min_x \quad f(x),$$

with f being smooth. We use local model

$$m_k(d) = f(x_k) + \nabla f(x_k)^T d + \frac{1}{2} d^T H_k d$$

to approximate $f(x)$ at x_k.

(1) Suppose you were using a multiple of identity matrix αI to approximate the inverse of the Hessian matrix, which may not satisfy the secant equation. Find the α as the least-squares solution of the secant equation.

(2) Suppose we are using Quasi-Newton iteration

$$x^{k+1} \leftarrow x_k - {H_k}^{-1} \nabla f(x_k).$$

If f is strongly convex, then the curvature condition

$$s_k^T y_k > 0$$

holds at each iteration if $x_{k+1} - x_k \neq 0$, where $s_k = x_{k+1} - x_k$ and $y_k = \nabla f(x_{k+1}) - \nabla f(x_k)$.

Problem 5: (25 points = 5+5+15) Suppose we are solving the unconstrained optimization problem

$$\min_{x} \quad f(x),$$

with f being smooth and (locally) L-Lipschitz differentialble on the level set

$$\mathcal{L} := \{x \mid f(x) \leq f(x_0)\}\,.$$

(1) Show that for a given descent direction d_k, moving with sufficiently small stepsize can cause decrease in the objective.

(2) Write down the Armijo backtracking condition for a given descent direction (define the parameters you need)

(3) Find the maximum number of (objective) function evaluations you need for this linear search.

Problem 6: (14 points) Consider the integer linear programming

$$\begin{aligned}\min_{x} \quad & c^T x \\ \text{s.t.} \quad & Ax = b, \\ & Dx \leq e \\ & x \geq 0 \qquad x \text{ is integer}\end{aligned}$$

The Lagrangian relaxation problem is

$$\begin{aligned}\min_{x} \quad & c^T x + \lambda^T (Ax - b) \\ \text{s.t.} \quad & Dx \leq e \\ & x \geq 0 \qquad\qquad\qquad x \text{ is integer}\end{aligned}$$

Show that the linear programming relaxation solution has objective value smaller or equal to the Lagrangian relaxation solution.