SI152 Numerical Optimization
Quiz 2

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Note:

- Please provide enough calculation process to get full marks.
- Please **write your name** on your answer sheet and **submit your answer sheet only**.

Exercise 1. (Revised simplex) (3+2 pts)

Question 1 (3 pts)

Please solve the following problem via **revised simplex method**.

$$\begin{aligned} \min \quad & 2x_1 + x_2 - x_3 - 3x_4 + x_5 \\ \text{s.t.} \quad & -3x_1 + x_2 + x_3 - x_4 + 2x_5 \le 5, \\ & 2x_1 - x_3 + x_4 - x_5 \le 6, \\ & x_2 + 2x_3 - x_4 + x_5 \le 3, \\ & x_j \ge 0, j = 1, \cdots, 5 \end{aligned}$$

Solution:
Step 0: First introduce the slack variable x_6, x_7, x_8, then reformulate the above problem to standard form, where the coefficient matrix are

$$A = \begin{bmatrix} -3 & 1 & 1 & -1 & 2 & 1 & 0 & 0 \\ 2 & 0 & -1 & 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

$$b = \begin{bmatrix} 5 \\ 6 \\ 3 \end{bmatrix}$$

We also have

$$B^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$B^{-1}b = \begin{bmatrix} 5 \\ 6 \\ 3 \end{bmatrix}$$

Step 1: We have

$$r_N = \begin{bmatrix} 2 \\ 1 \\ -1 \\ -3 \\ 1 \end{bmatrix}$$

So we choose x_4 and construct the following revised simplex table

var	B^{-1}			x_B	x_4
x_6	1	0	0	5	-1
x_7	0	1	0	6	1
x_8	0	0	1	3	-1
λ^T	0	0	0	0	3

.

Complete the pivot

var	B^{-1}			x_B
x_6	1	1	0	11
x_4	0	1	0	6
x_8	0	1	1	9
λ^T	0	-3	0	-18

.

Step 2: We have

$$r_N = \begin{bmatrix} 8 \\ 1 \\ -4 \\ -2 \\ 3 \end{bmatrix}, x_3 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

So we choose x_3 and construct the following revised simplex table

var	B^{-1}			x_B	x_3
x_6	1	1	0	11	0
x_4	0	1	0	6	-1
x_8	0	1	1	9	1
λ^T	0	-3	0	-18	4

.

Complete the pivot

var	B^{-1}			x_B
x_6	1	1	0	11
x_4	0	2	1	15
x_3	0	1	1	9
λ^T	0	-7	-4	-54

.

Step 3: We have

$$r_N = \begin{bmatrix} 16 \\ 5 \\ -2 \\ 7 \\ 4 \end{bmatrix}, x_3 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

So we choose x_5 and construct the following revised simplex table

var	B^{-1}			x_B	x_5
x_6	1	1	0	11	1
x_4	0	2	1	15	-1
x_3	0	1	1	9	0
λ^T	0	-7	-4	-54	2

.

Complete the pivot

var	B^{-1}			x_B
x_6	1	1	0	11
x_4	1	3	1	26
x_3	0	1	1	9
λ^T	-2	-9	-4	-76

.

Step 4: We have

$$r_N = \begin{bmatrix} 14 \\ 7 \\ 2 \\ 9 \\ 4 \end{bmatrix}$$

, so we reach the optimal solution, the optimal solution is

$$(x_1, x_2, x_3, x_4, x_5) = (0, 0, 9, 26, 11).$$

The optimal objective is -76

Question 2 (2 pts)

Suppose the matrix A in standard form have size $n \times m$. Please briefly explain in which case between the relationship of n and m, the revised simplex method will be significantly superior to simplex method.

Solution:
The revised simplex method only save the data in calculation, which reduce the memory cost during computation. When m is much larger than m, the revised simplex method is significantly superior to simplex method.

Exercise 2. (complementary slackness) (2 pts)

Consider the standard form of an arbitrary optimization problem:

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \quad & f_0(\boldsymbol{x}) \\ \text{s.t.} \quad & f_i(\boldsymbol{x}) \leq 0, \quad i = 1, \cdots, m, \\ & h_i(\boldsymbol{x}) = 0, \quad i = 1, \cdots, p. \end{aligned}$$

Suppose the strong duality holds for the primal problem and the dual problem, please prove that the complementary slackness holds, i.e.,

$$\sum_{i=1}^{m} \lambda_i^\star f_i(\boldsymbol{x}^\star) = 0$$

where $\boldsymbol{x}^\star$ is the optimal solution of the primal problem and $(\boldsymbol{\lambda}^\star, \boldsymbol{\nu}^\star)$ is the optimal solution of the dual problem.

(Note: When strong duality holds, $f_0(\boldsymbol{x}^\star) = g(\boldsymbol{\lambda}^\star, \boldsymbol{\nu}^\star)$.)

Solution:
When strong duality holds,

$$\begin{aligned} f_0(\boldsymbol{x}^\star) &= g(\boldsymbol{\lambda}^\star, \boldsymbol{\nu}^\star) \\ &= \inf_{\boldsymbol{x}} (f_0(\boldsymbol{x}) + \sum_{i=1}^{m} \lambda_i^\star f_i(\boldsymbol{x}) + \sum_{i=1}^{p} \nu_i^\star h_i(\boldsymbol{x})) \\ &\leq f_0(\boldsymbol{x}^\star) + \sum_{i=1}^{m} \lambda_i^\star f_i(\boldsymbol{x}^\star) + \sum_{i=1}^{p} \nu_i^\star h_i(\boldsymbol{x}^\star) \\ &\leq f_0(\boldsymbol{x}^\star) \end{aligned}$$

So the inequality above means equality, i.e.,

$$f_0(\boldsymbol{x}^\star) + \sum_{i=1}^{m} \lambda_i^\star f_i(\boldsymbol{x}^\star) + \sum_{i=1}^{p} \nu_i^\star h_i(\boldsymbol{x}^\star) = f_0(\boldsymbol{x}^\star)$$

where $\sum_{i=1}^{p} \nu_i^\star h_i(\boldsymbol{x}^\star) = 0$, so

$$\sum_{i=1}^{m} \lambda_i^\star f_i(\boldsymbol{x}^\star) = 0$$

Exercise 3. (Dual problem) (3 pts)

Consider the following maximum entropy problem.

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \quad & \sum_{i=1}^{n} x_i \log x_i \\ \text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} \preceq \boldsymbol{b} \\ & \boldsymbol{1}^T \boldsymbol{x} = 1 \end{aligned}$$

where $A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m$. Please provide the dual problem of the above problem.

Solution:
First, calculate the Lagrangian

$$\mathcal{L}(\boldsymbol{x}, \boldsymbol{\lambda}, \nu) = \sum_{i=1}^{n} x_i \log x_i + \boldsymbol{\lambda}^T (\boldsymbol{A}\boldsymbol{x} - b) + \nu(\boldsymbol{1}^T \boldsymbol{x} - 1).$$

So we have

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial x_i} &= (\log x_i + 1) + (a_i^T \boldsymbol{\lambda}) + \nu = 0, \\ x_i &= e^{(-1 - a_i^T \boldsymbol{\lambda} - \nu)}.\end{aligned}$$

So

$$\begin{aligned}g(\boldsymbol{\lambda}, \nu) &= \sum_{i=1}^{n} (-1 - a_i^T \boldsymbol{\lambda} - \nu) e^{(-1 - a_i^T \boldsymbol{\lambda} - \nu)} + a_i^T \boldsymbol{\lambda} e^{(-1 - a_i^T \boldsymbol{\lambda} - \nu)} + \nu e^{(-1 - a_i^T \boldsymbol{\lambda} - \nu)} - \boldsymbol{\lambda}^T b - \nu \\ &= -\boldsymbol{\lambda}^T b - \nu - \sum_{i=1}^{n} e^{(-1 - a_i^T \boldsymbol{\lambda} - \nu)}.\end{aligned}$$

The dual problem is

$$\begin{aligned}\min \qquad & -\boldsymbol{\lambda}^T b - \nu - \sum_{i=1}^{n} e^{(-1 - a_i^T \boldsymbol{\lambda} - \nu)} \\ \text{s.t.} \qquad & \boldsymbol{\lambda} \succeq 0\end{aligned}$$