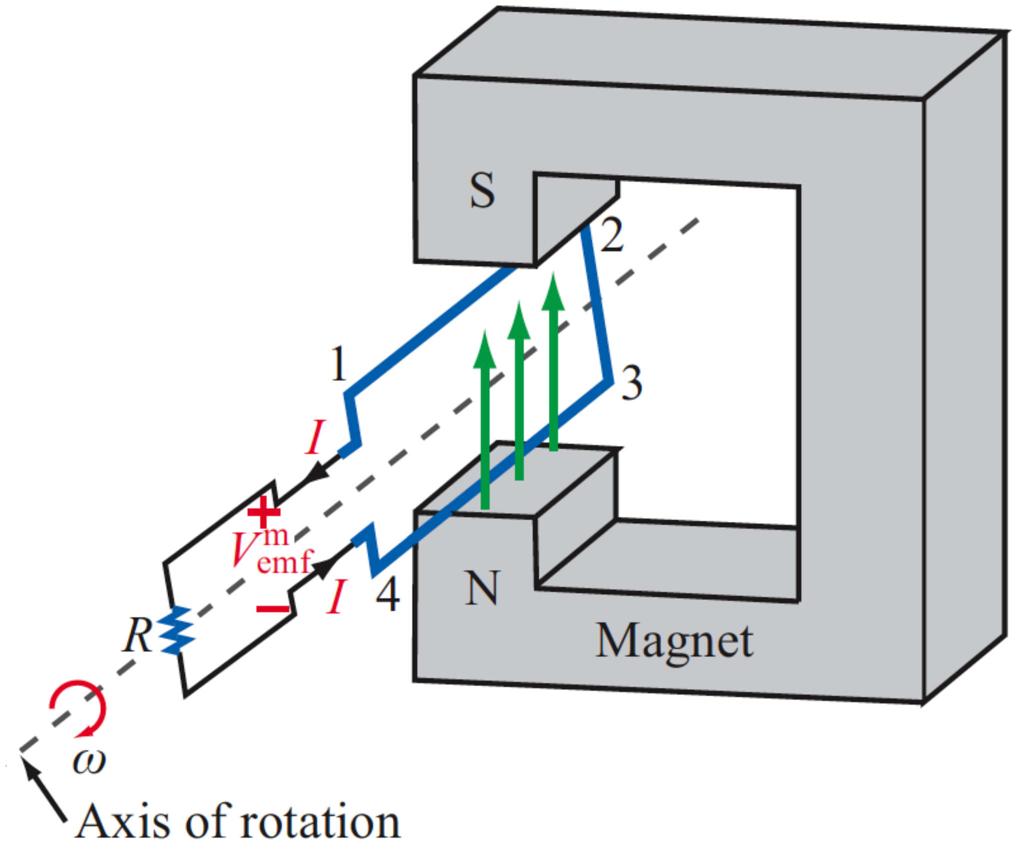


(b) ac generator

6. MAXWELL'S EQUATIONS IN TIME-VARYING FIELDS

7e Applied EM by Ulaby and Ravaioli

Chapter 6 Overview

Chapter Contents

Objectives

Upon learning the material presented in this chapter, you should be able to:

1. Apply Faraday's law to compute the voltage induced by a stationary coil placed in a time-varying magnetic field or moving in a medium containing a magnetic field.
2. Describe the operation of the electromagnetic generator.
3. Calculate the displacement current associated with a time-varying electric field.
4. Calculate the rate at which charge dissipates in a material with known ϵ and σ.

Maxwell's Equations

Table 6-1: Maxwell's equations.

Reference	Differential Form	Integral Form	
Gauss's law	$\nabla \cdot \mathbf{D} = \rho_v$	$\oint_S \mathbf{D} \cdot d\mathbf{s} = Q$	(6.1)
Faraday's law	$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$	$\oint_C \mathbf{E} \cdot d\mathbf{l} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{s}$	(6.2)*
Gauss's law for magnetism	$\nabla \cdot \mathbf{B} = 0$	$\oint_S \mathbf{B} \cdot d\mathbf{s} = 0$	(6.3)
Ampère's law	$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$	$\oint_C \mathbf{H} \cdot d\mathbf{l} = \int_S \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}\right) \cdot d\mathbf{s}$	(6.4)

*For a stationary surface S.

In this chapter, we will examine Faraday's and Ampère's laws

Electromotive Force

Electromotive force (voltage) induced by time-varying magnetic flux:

$$V_{\mathrm{emf}} = -N\,\frac{d\Phi}{dt} = -N\,\frac{d}{dt}\int_S \mathbf{B}\cdot d\mathbf{s} \qquad \text{(V)}$$

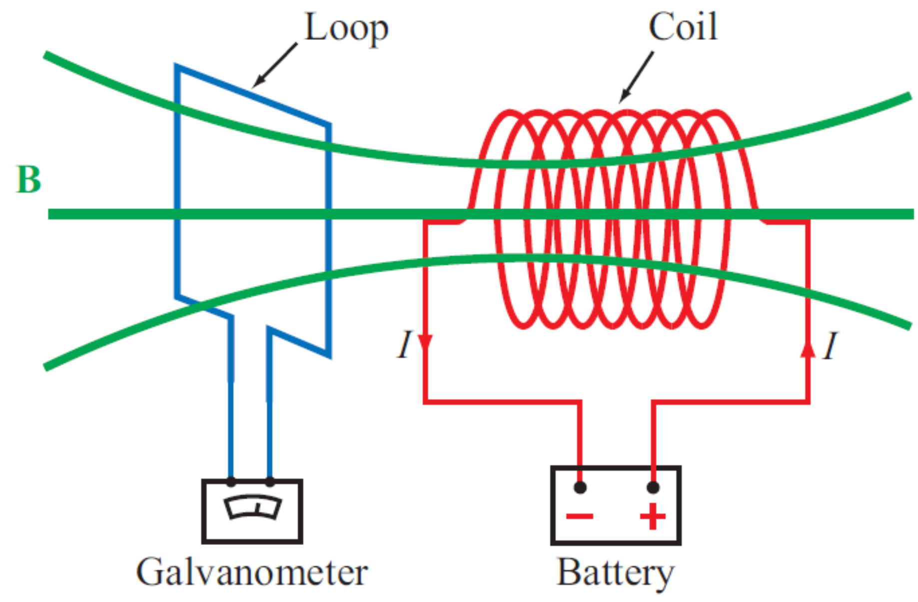


Figure 6-1: The galvanometer (predecessor of the ammeter) shows a deflection whenever the magnetic flux passing through the square loop changes with time.

Magnetic fields can produce an electric current in a closed loop, but only if the magnetic flux linking the surface area of the loop changes with time.The key to the induction process is change.

Three types of EMF

1. A time-varying magnetic field linking a stationary loop; the induced emf is then called the ***transformer emf***, $V_{\mathrm{emf}}^{\mathrm{tr}}$.

2. A moving loop with a time-varying surface area (relative to the normal component of **B**) in a static field **B**; the induced emf is then called the ***motional emf***, $V_{\mathrm{emf}}^{\mathrm{m}}$.

3. A moving loop in a time-varying field **B**.

The total emf is given by

$$V_{\mathrm{emf}} = V_{\mathrm{emf}}^{\mathrm{tr}} + V_{\mathrm{emf}}^{\mathrm{m}}, \qquad (6.7)$$

Stationary Loop in Time-Varying B

It is important to remember that $\mathbf{B}_{\text{ind}}$ serves to oppose the change in $\mathbf{B}(t)$, and not necessarily $\mathbf{B}(t)$ itself.

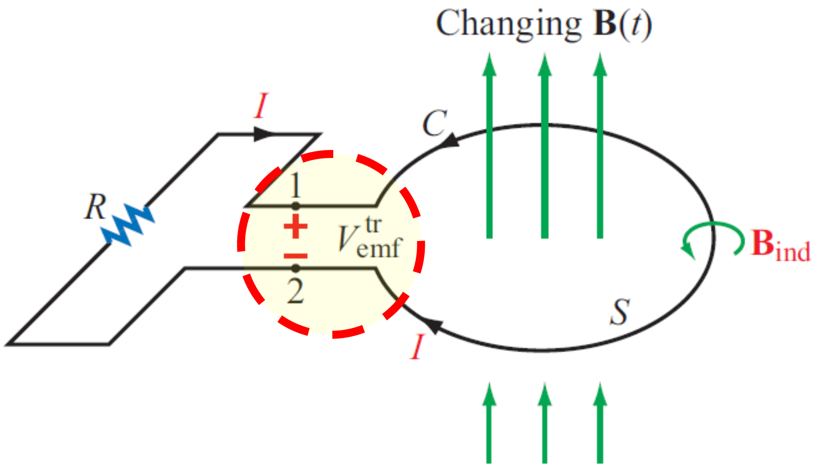
$$V_{\text{emf}}^{\text{tr}} = -N \int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{s} \quad \text{(transformer emf)},$$

The connection between the direction of $d\mathbf{s}$ and the polarity of $V_{\text{emf}}^{\text{tr}}$ is governed by the following right-hand rule: if $d\mathbf{s}$ points along the thumb of the right hand, then the direction of the contour C indicated by the four fingers is such that it always passes across the opening from the positive terminal of $V_{\text{emf}}^{\text{tr}}$ to the negative terminal.

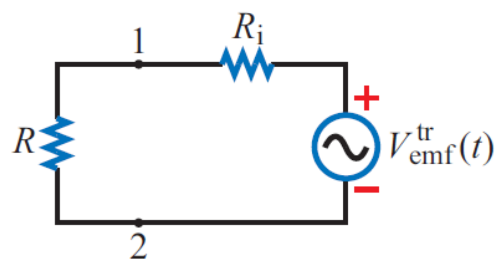
$$I = \frac{V_{\text{emf}}^{\text{tr}}}{R + R_{\text{i}}} . \tag{6.9}$$

For good conductors, R_{i} usually is very small, and it may be ignored in comparison with practical values of R.

*The polarity of $V_{\text{emf}}^{\text{tr}}$ and hence the direction of I is governed by **Lenz's law**, which states that the current in the loop is always in a direction that opposes the change of magnetic flux $\Phi(t)$ that produced I.*



(a) Loop in a changing **B** field

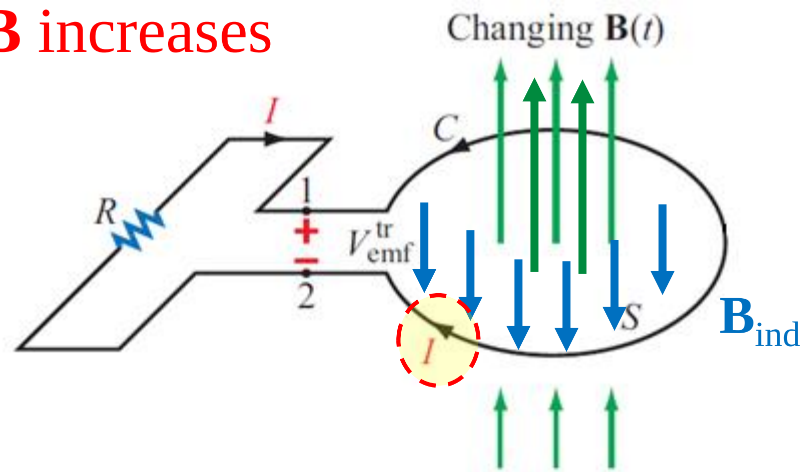


(b) Equivalent circuit

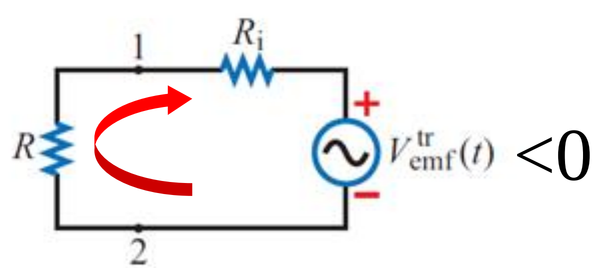
Figure 6-2: (a) Stationary circular loop in a changing magnetic field $\mathbf{B}(t)$, and (b) its equivalent circuit.

Lenz's Law

B increases

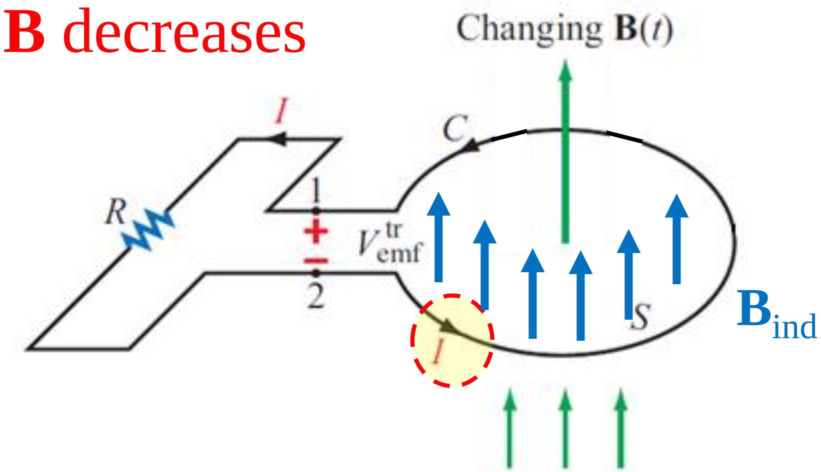


(a) Loop in a changing **B** field

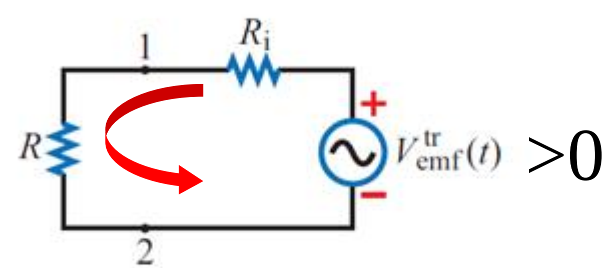


(b) Equivalent circuit

B decreases



(a) Loop in a changing **B** field



(b) Equivalent circuit

$$V_{\text{emf}}^{\text{tr}} = -N \int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{s} \quad \text{(transformer emf)},$$

$\mathbf{B}_{\text{ind}}$ serves to oppose the *change in* $\mathbf{B}(t)$, and not necessarily $\mathbf{B}(t)$ itself

Faraday's Law

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \text{(Faraday's law).}$$

*Establish a link between **B** and **E***

$$V_{\text{emf}}^{\text{tr}} = \oint_C \mathbf{E} \cdot d\mathbf{l}.$$

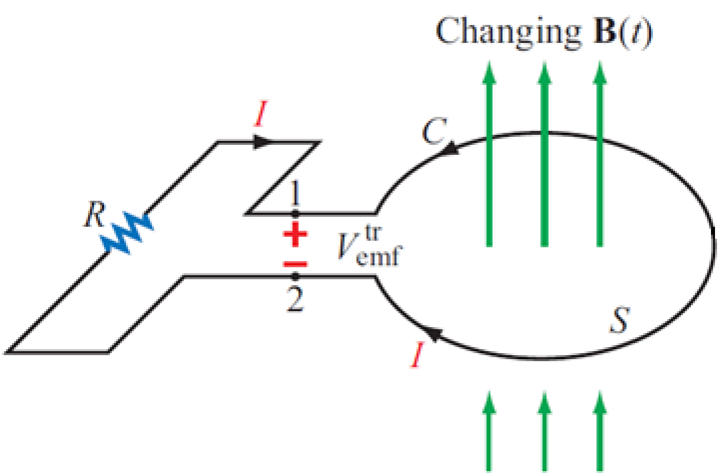
$$V_{\text{emf}}^{\text{tr}} = -N \int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{s}$$

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{s},$$

Integral form of Faraday's law

Stokes's theorem

$$\int_S (\nabla \times \mathbf{E}) \cdot d\mathbf{s} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{s},$$



Example 6-1: Inductor in a Changing Magnetic Field

An inductor is formed by winding N turns of a thin conducting wire into a circular loop of radius a. The inductor loop is in the x–y plane with its center at the origin, and connected to a resistor R, as shown in Fig. 6-3. In the presence of a magnetic field $\mathbf{B} = B_0(\hat{\mathbf{y}}2 + \hat{\mathbf{z}}3)\sin\omega t$, where ω is the angular frequency, find

(a) the magnetic flux linking a single turn of the inductor,

(b) the transformer emf, given that $N = 10$, $B_0 = 0.2$ T, $a = 10$ cm, and $\omega = 10^3$ rad/s,

(c) the polarity of $V_{\text{emf}}^{\text{tr}}$ at $t = 0$, and

(d) the induced current in the circuit for $R = 1$ kΩ (assume the wire resistance to be much smaller than R).

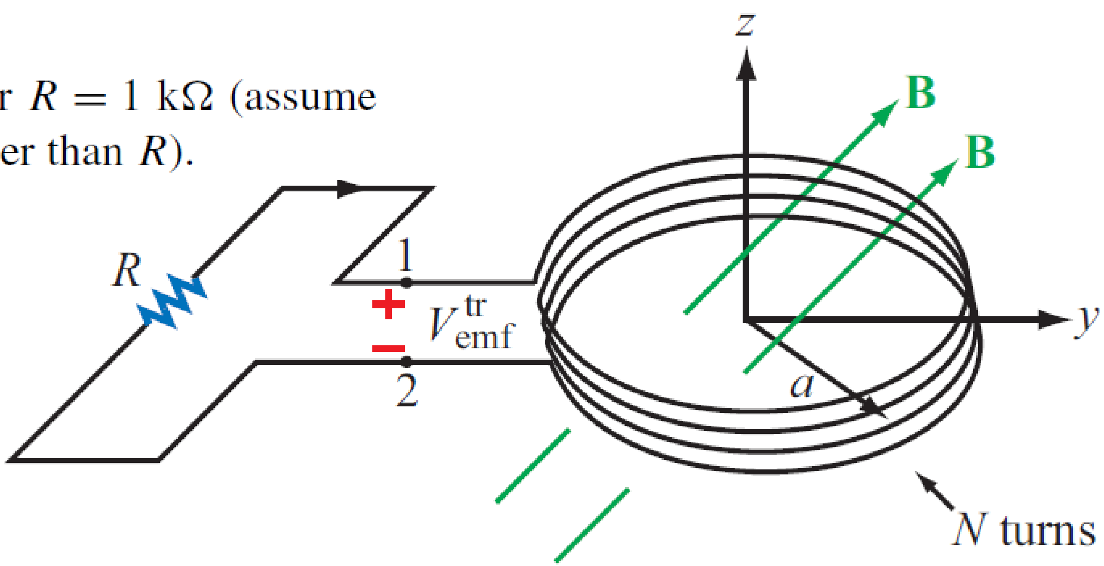


Figure 6-3: Circular loop with N turns in the x–y plane. The magnetic field is $\mathbf{B} = B_0(\hat{\mathbf{y}}2 + \hat{\mathbf{z}}3)\sin\omega t$ (Example 6-1).

cont.

Example 6-1 Solution

Solution: **(a)** The magnetic flux linking each turn of the inductor is

$$\Phi = \int_S \mathbf{B} \cdot d\mathbf{s}$$
$$= \int_S [B_0(\hat{\mathbf{y}}\, 2 + \hat{\mathbf{z}}\, 3) \sin \omega t] \cdot \hat{\mathbf{z}}\; ds$$
$$= 3\pi a^2 B_0 \sin \omega t.$$

(b) To find $V_{\text{emf}}^{\text{tr}}$, we can apply Eq. (6.8) or we can apply the general expression given by Eq. (6.6) directly. The latter approach gives

$$V_{\text{emf}}^{\text{tr}} = -N \frac{d\Phi}{dt}$$
$$= -\frac{d}{dt}(3\pi N a^2 B_0 \sin \omega t)$$
$$= -3\pi N \omega a^2 B_0 \cos \omega t.$$

For $N = 10$, $a = 0.1$ m, $\omega = 10^3$ rad/s, and $B_0 = 0.2$ T,

$$V_{\text{emf}}^{\text{tr}} = -188.5 \cos 10^3 t \qquad \text{(V)}.$$

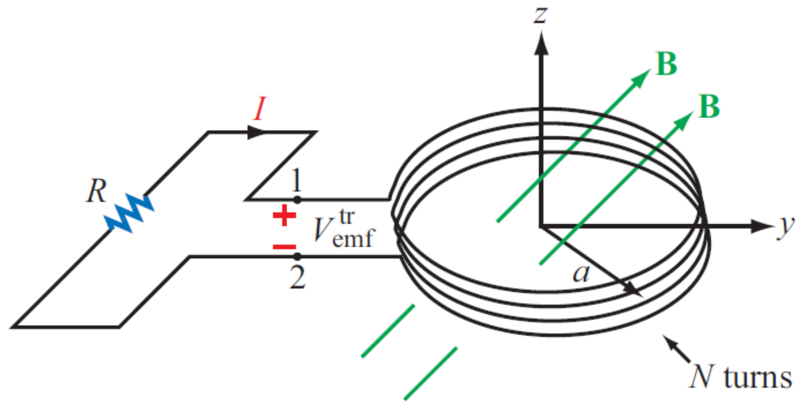


Figure 6-3: Circular loop with N turns in the x–y plane. The magnetic field is $\mathbf{B} = B_0(\hat{\mathbf{y}}2 + \hat{\mathbf{z}}3) \sin \omega t$ (Example 6-1).

(c) At $t = 0$, $d\Phi/dt > 0$ and $V_{\text{emf}}^{\text{tr}} = -188.5$ V. Since the flux is increasing, the current I must be in the direction shown in Fig. 6-3 in order to satisfy Lenz's law. Consequently, terminal 2 is at a higher potential than terminal 1 and

$$V_{\text{emf}}^{\text{tr}} = V_1 - V_2$$
$$= -188.5 \qquad \text{(V)}.$$

(d) The current I is given by

$$I = \frac{V_2 - V_1}{R}$$
$$= \frac{188.5}{10^3} \cos 10^3 t$$
$$= 0.19 \cos 10^3 t \qquad \text{(A)}.$$

Example 6-2: Lenz's Law

Determine voltages V_1 and V_2 across the 2-Ω and 4-Ω resistors shown in Fig. 6-4. The loop is located in the x–y plane, its area is 4 m^2, the magnetic flux density is $\mathbf{B} = -\hat{\mathbf{z}}0.3t$ (T), and the internal resistance of the wire may be ignored.

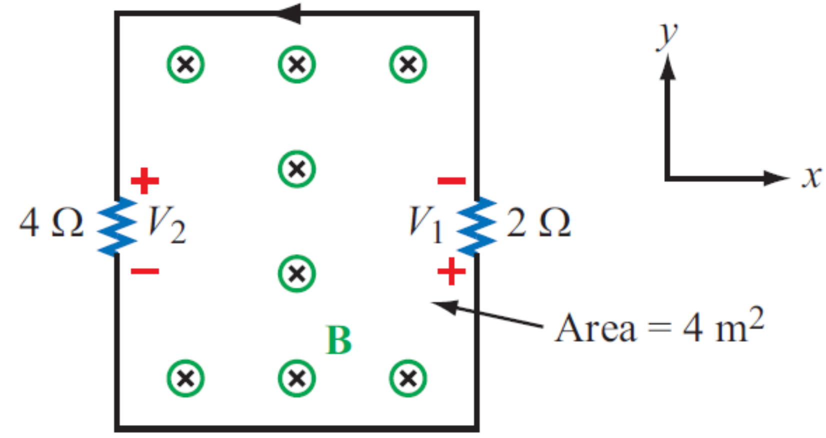


Figure 6-4: Circuit for Example 6-2.

Solution: The flux flowing through the loop is

$$\Phi = \int_S \mathbf{B} \cdot d\mathbf{s} = \int_S (-\hat{\mathbf{z}}0.3t) \cdot \hat{\mathbf{z}}\, ds$$

$$= -0.3t \times 4 = -1.2t \qquad \text{(Wb)},$$

and the corresponding transformer emf is

$$V_{\text{emf}}^{\text{tr}} = -\frac{d\Phi}{dt} = 1.2 \qquad \text{(V)}.$$

The total voltage of 1.2 V is distributed across two resistors in series. Consequently,

$$I = \frac{V_{\text{emf}}^{\text{tr}}}{R_1 + R_2} = \frac{1.2}{2+4} = 0.2 \text{ A},$$

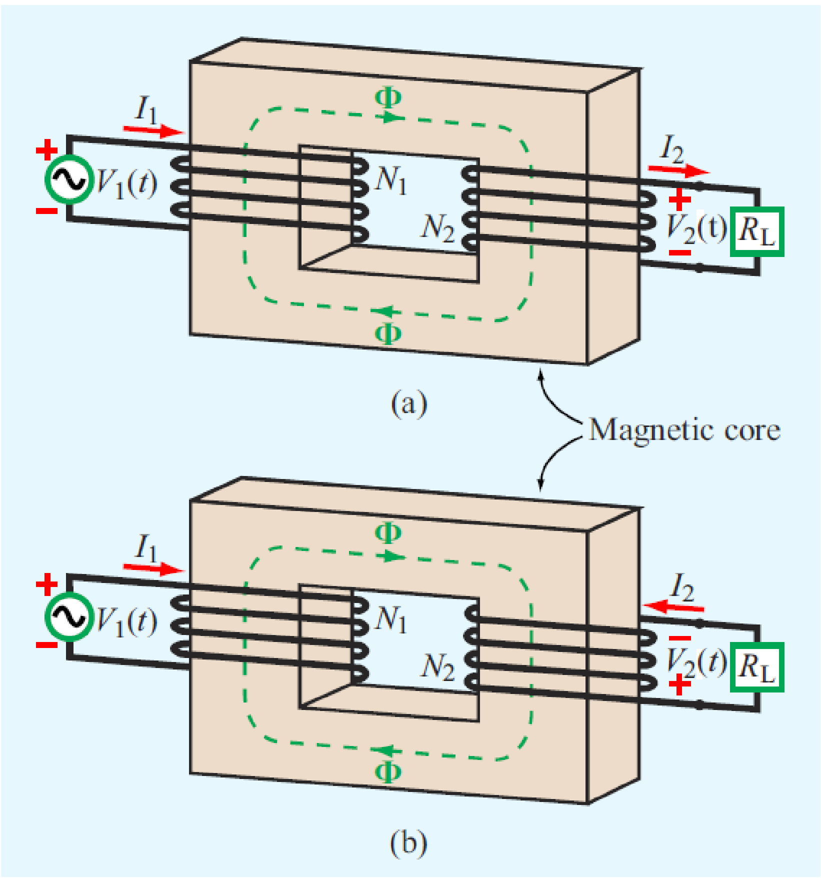
and

$$V_1 = IR_1 = 0.2 \times 2 = 0.4 \text{ V},$$

$$V_2 = IR_2 = 0.2 \times 4 = 0.8 \text{ V}.$$

Ideal transformer

Transformer is used to convert the AC voltage from one voltage to another. An ideal transformer has infinite permeability, and the magnetic flux is confined within the core.



► The directions of the currents flowing in the two coils, I_1 and I_2, are defined such that, when I_1 and I_2 are both positive, the flux generated by I_2 is opposite to that generated by I_1. The transformer gets its name from the fact that it transforms currents, voltages, and impedances between its primary and secondary circuits, and vice versa. ◄

$$V_1 = -N_1 \frac{d\Phi}{dt}. \quad V_2 = -N_2 \frac{d\Phi}{dt}. \quad \frac{V_1}{V_2} = \frac{N_1}{N_2}.$$

$$P_1 = P_2. \quad \frac{I_1}{I_2} = \frac{N_2}{N_1}.$$

$$R_{in} = \frac{V_2}{I_2}\left(\frac{N_1}{N_2}\right)^2 = \left(\frac{N_1}{N_2}\right)^2 R_L.$$

Motional EMF

Magnetic force on charge *q* moving with velocity **u** in a magnetic field **B**:

$$\mathbf{F}_{\mathrm{m}} = q(\mathbf{u} \times \mathbf{B}).$$

This magnetic force is equivalent to the electrical force that would be exerted on the particle by the (motional) electric field Em given by

$$\mathbf{E}_{\mathrm{m}} = \frac{\mathbf{F}_{\mathrm{m}}}{q} = \mathbf{u} \times \mathbf{B}.$$

This, in turn, induces a voltage difference between ends 1 and 2, with end 2 being at the higher potential. The induced voltage is

$$\underbrace{V_{\mathrm{emf}}^{\mathrm{m}} = V_{12}}_{V_1 - V_2} = \int_2^1 \mathbf{E}_{\mathrm{m}} \cdot d\mathbf{l} = \int_2^1 (\mathbf{u} \times \mathbf{B}) \cdot d\mathbf{l}.$$

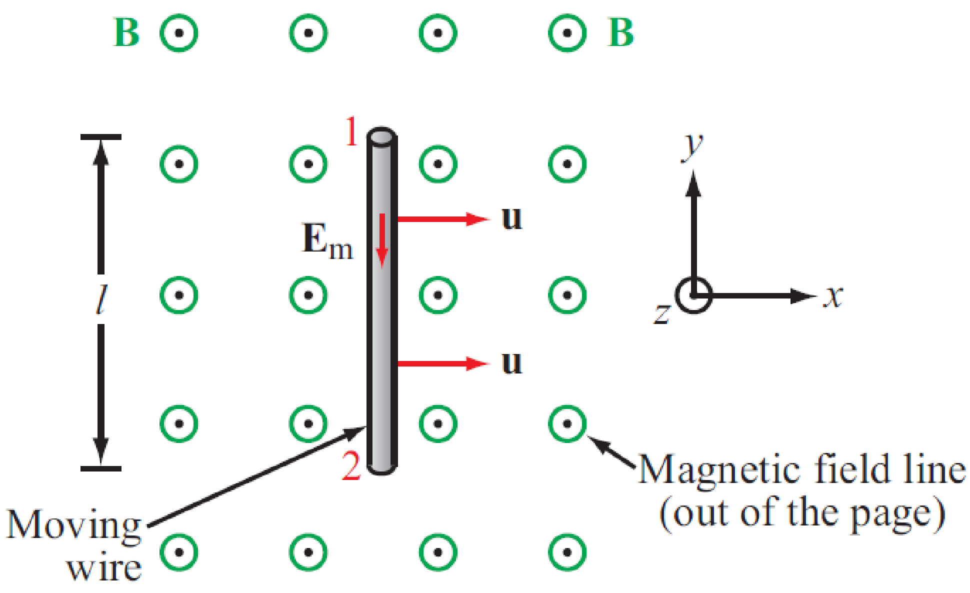


Figure 6-7: Conducting wire moving with velocity **u** in a static magnetic field.

For the conducting wire, $\mathbf{u} \times \mathbf{B} = \hat{\mathbf{x}}u \times \hat{\mathbf{z}}B_0 = -\hat{\mathbf{y}}uB_0$ and $d\mathbf{l} = \hat{\mathbf{y}}\, dl$. Hence,

$$V_{\mathrm{emf}}^{\mathrm{m}} = V_{12} = -uB_0 l. \tag{6.25}$$

E points from 1 to 2

Motional EMF

In general, if any segment of a closed circuit with contour C moves with a velocity $\mathbf{u}$ across a static magnetic field $\mathbf{B}$, then the induced motional emf is given by

$$V_{\mathrm{emf}}^{\mathrm{m}} = \oint_C (\mathbf{u} \times \mathbf{B}) \cdot d\mathbf{l} \qquad \text{(motional emf).} \qquad (6.26)$$

Only those segments of the circuit that cross magnetic field lines contribute to $V_{\mathrm{emf}}^{\mathrm{m}}$.

Example 6-3: Sliding Bar

The rectangular loop shown in Fig. 6-8 has a constant width l, but its length x_0 increases with time as a conducting bar slides with uniform velocity $\mathbf{u}$ in a static magnetic field $\mathbf{B} = \hat{\mathbf{z}} B_0 x$. Note that $\mathbf{B}$ increases linearly with x. The bar starts from $x = 0$ at $t = 0$. Find the motional emf between terminals 1 and 2 and the current I flowing through the resistor R. Assume that the loop resistance $R_i \ll R$.

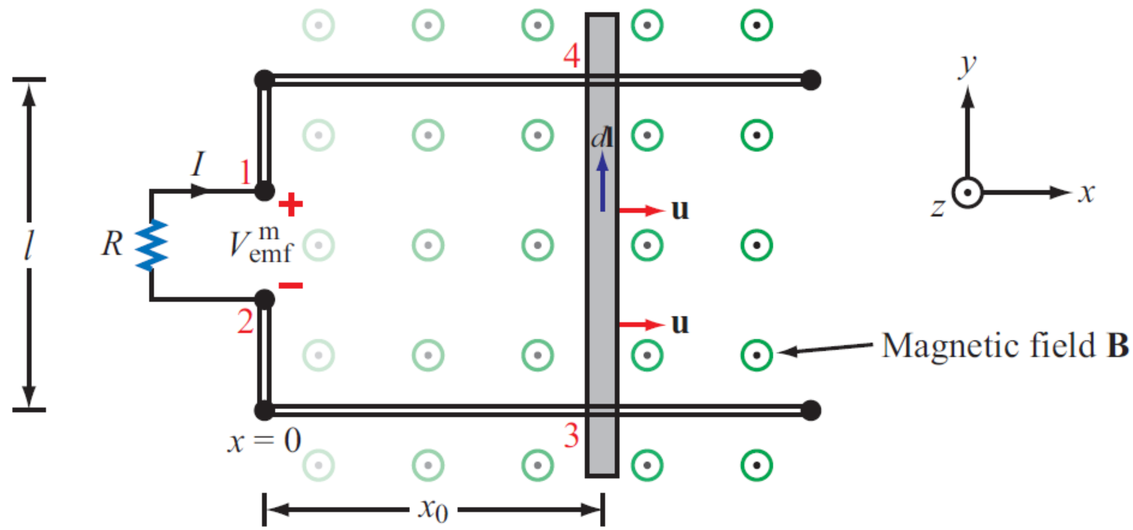
Note that B increases with x

$$\mathbf{B} = \hat{\mathbf{z}} B_0 x$$

$$V^{\mathrm{m}}_{\mathrm{emf}} = V_{12} = V_{43} = \int_3^4 (\mathbf{u} \times \mathbf{B}) \cdot d\mathbf{l}$$

$$V_4 - V_3$$

$$= \int_3^4 (\hat{\mathbf{x}} u \times \hat{\mathbf{z}} B_0 x_0) \cdot \hat{\mathbf{y}}\, dl = -u B_0 x_0 l.$$



The length of the loop is related to u by $x_0 = ut$. Hence

$$V^{\mathrm{m}}_{\mathrm{emf}} = -B_0 u^2 l t \qquad \text{(V)}.$$

How to define the polarity of V_{emf} in this condition?

Example 6-3: cont.

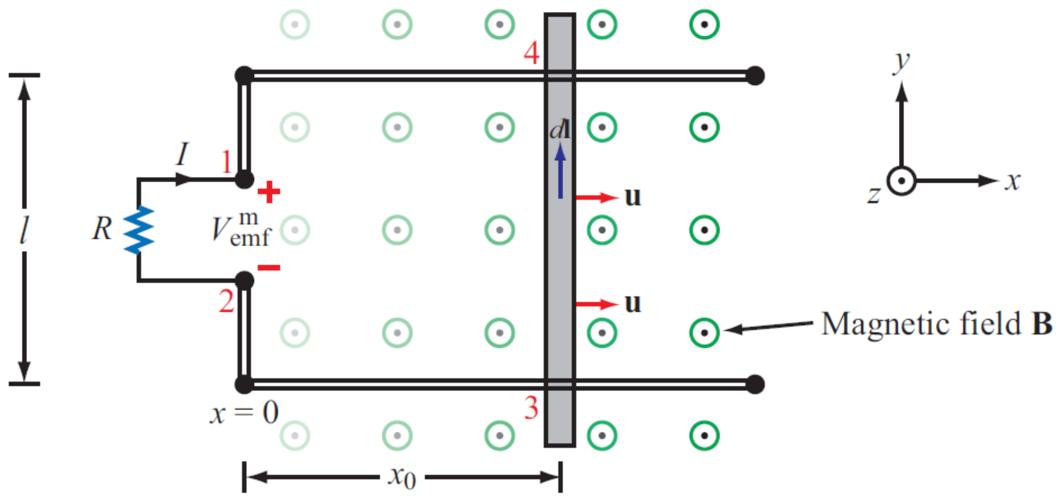
Use Faraday's law

Note that B increases with x

$$\mathbf{B} = \hat{\mathbf{z}} B_0 x$$

$$\Phi = \int_S \mathbf{B} \cdot d\mathbf{s} = \int_S (\hat{\mathbf{z}} B_0 x) \cdot \hat{\mathbf{z}}\, dx\, dy$$

$$= B_0 l \int_0^{x_0} x\, dx = \frac{B_0 l x_0^2}{2}$$



$$V_{\mathrm{emf}} = -\frac{d\Phi}{dt} = -\frac{d}{dt}\left(\frac{B_0 l u^2 t^2}{2}\right) = -B_0 u^2 l t \qquad \text{(V)},$$

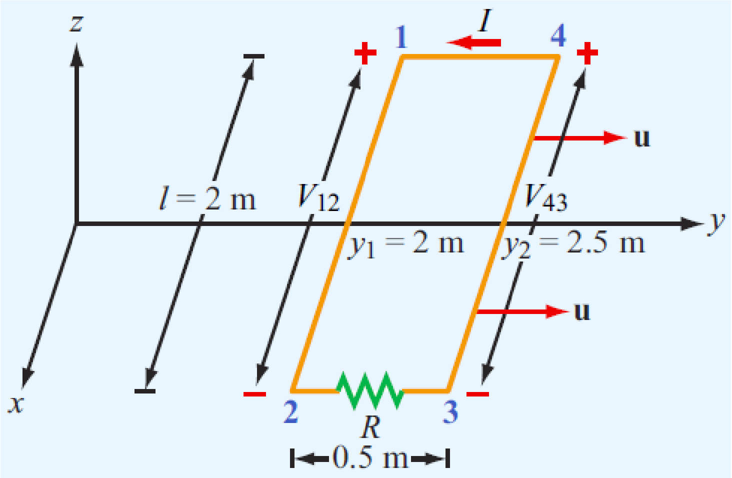
The same as the result obtained by the motional emf expression

Example 6-4: Moving Loop

The rectangular loop shown in Fig. 6-9 is situated in the x–y plane and moves away from the origin with velocity $\mathbf{u} = \hat{\mathbf{y}}5$ (m/s) in a magnetic field given by

$$\mathbf{B}(y) = \hat{\mathbf{z}}\,0.2e^{-0.1y} \qquad \text{(T)}.$$

If $R = 5\ \Omega$, find the current I at the instant that the loop sides are at $y_1 = 2$ m and $y_2 = 2.5$ m. The loop resistance may be ignored.



$$V_{12} = \int_2^1 [\mathbf{u} \times \mathbf{B}(y_1)] \cdot d\mathbf{l} = \int_{l/2}^{-l/2} (\hat{\mathbf{y}}5 \times \hat{\mathbf{z}}0.2e^{-0.2}) \cdot \hat{\mathbf{x}}\, dx$$

$$= -e^{-0.2}l = -2e^{-0.2} = -1.637 \qquad \text{(V)}.$$

$$\mathbf{B}(y_1) = \hat{\mathbf{z}}\,0.2e^{-0.1y_1} = \hat{\mathbf{z}}\,0.2e^{-0.2} \qquad \text{(T)}.$$

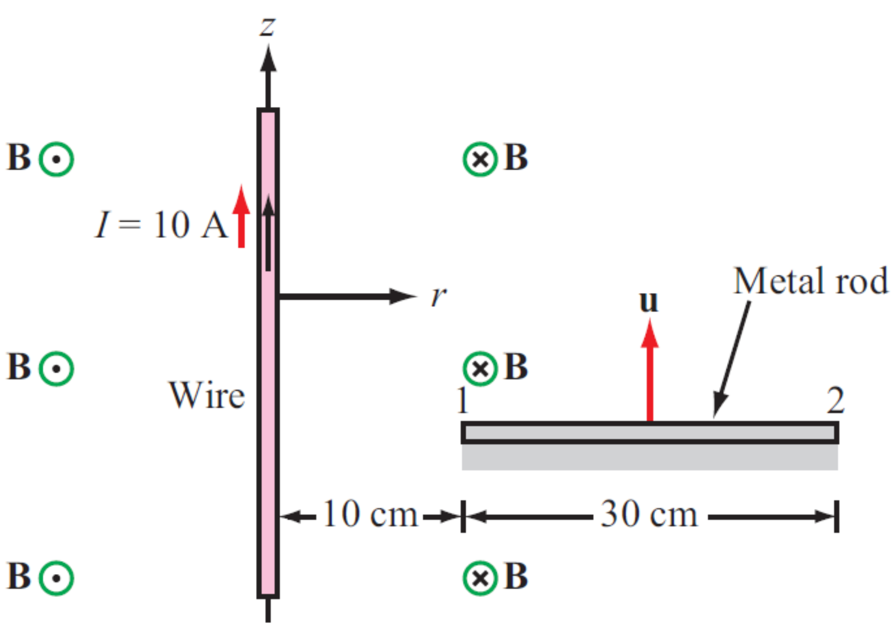
$$V_{43} = -u\,B(y_2)\,l = -5 \times 0.2e^{-0.25} \times 2 = -1.558 \qquad \text{(V)}.$$

$$I = \frac{V_{43} - V_{12}}{R} = \frac{0.079}{5} = 15.8 \text{ (mA)}.$$

Example 6-5: Moving Rod Next to a Wire

The wire shown in Fig. 6-10 carries a current $I = 10$ A. A 30-cm-long metal rod moves with a constant velocity $\mathbf{u} = \hat{\mathbf{z}}5$ m/s. Find V_{12}.

$$\mathbf{B} = \hat{\boldsymbol{\phi}} \frac{\mu_0 I}{2\pi r}$$



$$V_{12} = \int_{40\text{ cm}}^{10\text{ cm}} (\mathbf{u} \times \mathbf{B}) \cdot d\mathbf{l}$$

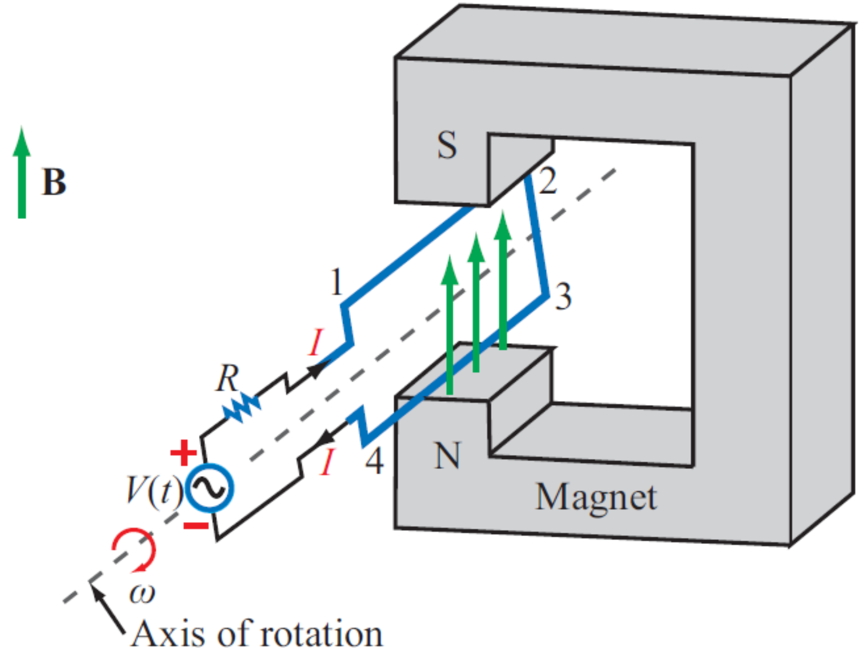
$$= \int_{40\text{ cm}}^{10\text{ cm}} \left(\hat{\mathbf{z}}5 \times \hat{\boldsymbol{\phi}} \frac{\mu_0 I}{2\pi r} \right) \cdot \hat{\mathbf{r}}\, dr$$

$$= -\frac{5\mu_0 I}{2\pi} \int_{40\text{ cm}}^{10\text{ cm}} \frac{dr}{r}$$

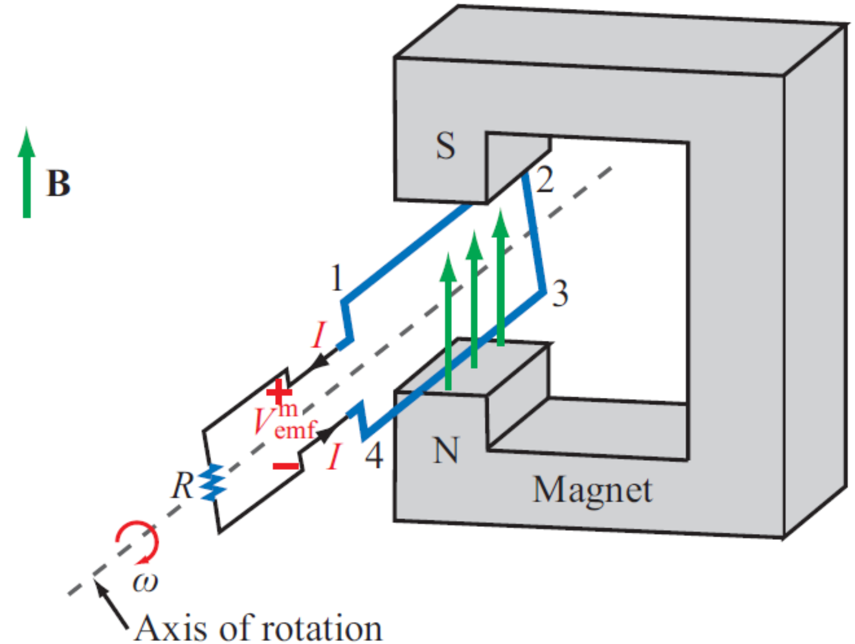
$$= -\frac{5 \times 4\pi \times 10^{-7} \times 10}{2\pi} \times \ln\left(\frac{10}{40}\right)$$

$$= 13.9 \qquad (\mu\text{V}).$$

EM Motor / Generator Reciprocity



(a) ac motor



(b) ac generator

Motor: Electrical to mechanical energy conversion

Wires with current are subject to force exerted by magnetic field

Generator: Mechanical to electrical energy conversion

Time varying magnetic flux induces current in circuit

EM Generator EMF

As the loop rotates with an angular velocity *ω about its own* axis, segment 1–2 moves with velocity **u** given by

$$\mathbf{u} = \hat{\mathbf{n}}\omega \frac{w}{2}$$

Also: $\hat{\mathbf{n}} \times \hat{\mathbf{z}} = \hat{\mathbf{x}} \sin\alpha.$

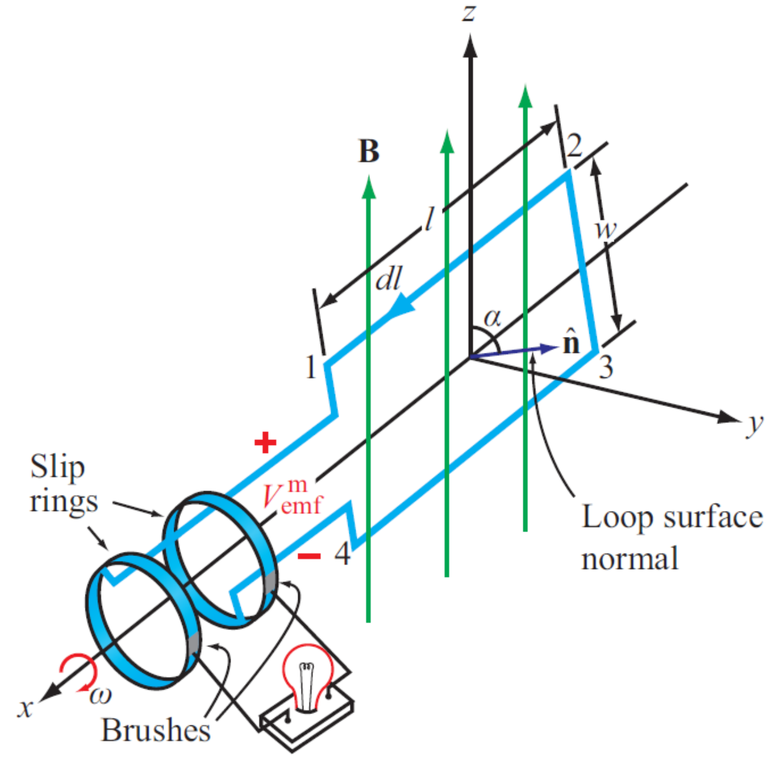


Figure 6-12: A loop rotating in a magnetic field induces an emf.

Segment 3-4 moves with velocity **–u**. Hence:

$$V^{\rm m}_{\rm emf} = V_{14} = \int_2^1 (\mathbf{u}\times\mathbf{B})\cdot d\mathbf{l} + \int_4^3 (\mathbf{u}\times\mathbf{B})\cdot d\mathbf{l}$$

$$= \int_{-l/2}^{l/2} \left[\left(\hat{\mathbf{n}}\omega\frac{w}{2}\right)\times\hat{\mathbf{z}}B_0\right]\cdot\hat{\mathbf{x}}\,dx$$

$$+ \int_{l/2}^{-l/2} \left[\left(-\hat{\mathbf{n}}\omega\frac{w}{2}\right)\times\hat{\mathbf{z}}B_0\right]\cdot\hat{\mathbf{x}}\,dx.$$

$$V^{\rm m}_{\rm emf} = wl\omega B_0\sin\alpha = A\omega B_0\sin\alpha,$$

$$\alpha = \omega t + C_0,$$

$$\boxed{V^{\rm m}_{\rm emf} = A\omega B_0\sin(\omega t + C_0) \qquad \text{(V).}}$$

Another Way

$$\Phi = \int_S \mathbf{B} \cdot d\mathbf{s} = \int_S \hat{\mathbf{z}} B_0 \cdot \hat{\mathbf{n}}\, ds$$

$$= B_0 A \cos\alpha$$

$$= B_0 A \cos(\omega t + C_0)$$

$$V_{\text{emf}} = -\frac{d\Phi}{dt} = -\frac{d}{dt}[B_0 A \cos(\omega t + C_0)]$$

$$= A\omega B_0 \sin(\omega t + C_0),$$

Moving Conductor in Time-Varying Magnetic Field

$$
\begin{aligned}
V_{\text{emf}} &= V_{\text{emf}}^{\text{tr}} + V_{\text{emf}}^{\text{m}} \\
&= \oint_C \mathbf{E} \cdot d\mathbf{l} \\
&= -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{s} + \oint_C (\mathbf{u} \times \mathbf{B}) \cdot d\mathbf{l}.
\end{aligned}
$$

Equivalent

$$
V_{\text{emf}} = -\frac{d\Phi}{dt} = -\frac{d}{dt} \int_S \mathbf{B} \cdot d\mathbf{s} \quad \text{(total emf).}
$$

Example 6-6

$$\mathbf{B} = \hat{\mathbf{z}} B_0 \cos \omega t$$

$$\Phi = B_0 A \cos \omega t$$ From p. 20

$$\Phi = B_0 A \cos^2 \omega t$$

$$V_{\text{emf}} = -\frac{\partial \Phi}{\partial t}$$

$$= -\frac{\partial}{\partial t} (B_0 A \cos^2 \omega t)$$

$$= 2 B_0 A \omega \cos \omega t \sin \omega t = B_0 A \omega \sin 2\omega t.$$

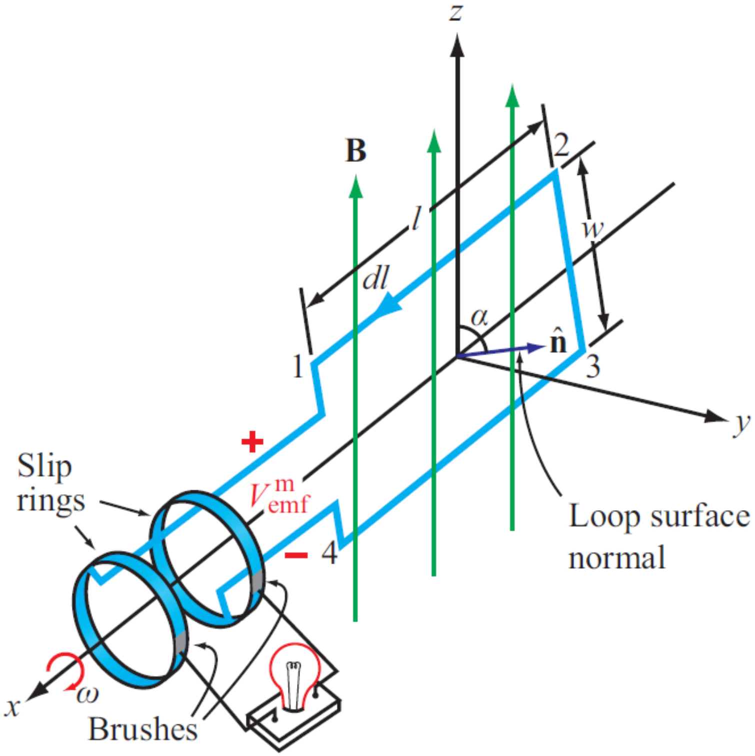


Figure 6-12: A loop rotating in a magnetic field induces an emf.

Displacement Current

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \qquad \text{(Ampère's law).} \qquad (6.41)$$

Integrating both sides of Eq. (6.41) over an arbitrary open surface S with contour C, we have

$$\int_S (\nabla \times \mathbf{H}) \cdot d\mathbf{s} = \int_S \mathbf{J} \cdot d\mathbf{s} + \int_S \frac{\partial \mathbf{D}}{\partial t} \cdot d\mathbf{s}. \qquad (6.42)$$

This term is conduction current I_C

This term must represent a current

Application of Stokes's theorem gives:

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = I_c + \int_S \frac{\partial \mathbf{D}}{\partial t} \cdot d\mathbf{s} \qquad \text{(Ampère's law)}$$

Cont.

Displacement Current

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = I_c + \int_S \frac{\partial \mathbf{D}}{\partial t} \cdot d\mathbf{s} \quad \text{(Ampère's law)}$$

Conduction current density

$$J_c = \sigma E$$

Define the displacement current as:

$$I_d = \int_S \mathbf{J}_d \cdot d\mathbf{s} = \int_S \frac{\partial \mathbf{D}}{\partial t} \cdot d\mathbf{s}, \qquad (6.44)$$

Displacement current density

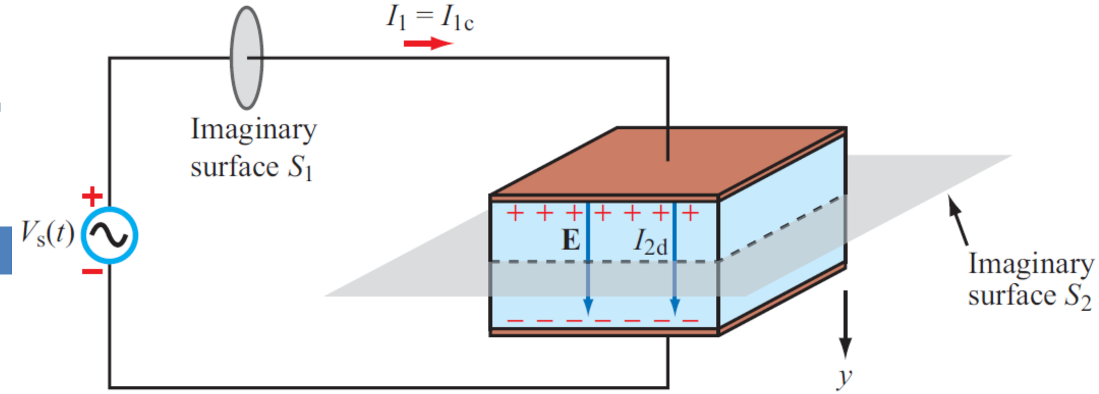
$$J_d = \varepsilon \frac{\partial E}{\partial t}$$

where $\mathbf{J}_d = \partial\mathbf{D}/\partial t$ represents a ***displacement current density***. In view of Eq. (6.44),

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = I_c + I_d = I, \qquad (6.45)$$

The displacement current does not involve real charges; it is an equivalent current that depends on $\partial\mathbf{D}/\partial t$

Capacitor Circuit



Given: Wires are perfect conductors and capacitor insulator material is perfect dielectric.

For Surface S_1:

$$I_1 = I_{1c} + I_{1d}$$

$$I_{1c} = C\frac{dV_C}{dt} = C\frac{d}{dt}(V_0 \cos\omega t) = -CV_0\omega\sin\omega t$$

$$I_{1d} = 0$$ (**D** = 0 in perfect conductor)

$I_1 = I_2$ ensures the continuity of total current flow through the circuit

Although the displacement current does not transport free charges, it behaves like a real current

For Surface S_2:

$$I_2 = I_{2c} + I_{2d}$$

$I_{2c} = 0$ (perfect dielectric $\sigma = 0$)

$$\mathbf{E} = \hat{\mathbf{y}}\frac{V_c}{d} = \hat{\mathbf{y}}\frac{V_0}{d}\cos\omega t$$

$$I_{2d} = \int_S \frac{\partial \mathbf{D}}{\partial t}\cdot d\mathbf{s}$$

$$= \int_A \left[\frac{\partial}{\partial t}\left(\hat{\mathbf{y}}\frac{\varepsilon V_0}{d}\cos\omega t\right)\right]\cdot(\hat{\mathbf{y}}\, ds)$$

$$= -\frac{\varepsilon A}{d}V_0\omega\sin\omega t = -CV_0\omega\sin\omega t$$

Conclusion: $I_1 = I_2$

$$C = \epsilon A/d$$

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Example 6-7: Displacement Current Density

The conduction current flowing through a wire with conductivity $\sigma = 2 \times 10^7$ S/m and relative permittivity $\varepsilon_r = 1$ is given by $I_c = 2 \sin \omega t$ (mA). If $\omega = 10^9$ rad/s, find the displacement current.

Solution: The conduction current $I_c = JA = \sigma EA$, where A is the cross section of the wire. Hence,

$$E = \frac{I_c}{\sigma A} = \frac{2 \times 10^{-3} \sin \omega t}{2 \times 10^7 A}$$
$$= \frac{1 \times 10^{-10}}{A} \sin \omega t \qquad \text{(V/m)}.$$

Application of Eq. (6.44), with $D = \varepsilon E$, leads to

$$\begin{aligned} I_d &= J_d A \\ &= \varepsilon A \frac{\partial E}{\partial t} \\ &= \varepsilon A \frac{\partial}{\partial t} \left(\frac{1 \times 10^{-10}}{A} \sin \omega t \right) \\ &= \varepsilon \omega \times 10^{-10} \cos \omega t = 0.885 \times 10^{-12} \cos \omega t \quad \text{(A)}, \end{aligned}$$

where we used $\omega = 10^9$ rad/s and $\varepsilon = \varepsilon_0 = 8.85 \times 10^{-12}$ F/m. Note that I_c and I_d are in phase quadrature (90° phase shift between them). Also, I_d is about nine orders of magnitude smaller than I_c, which is why the displacement current usually is ignored in good conductors.

Boundary Conditions

Table 6-2: Boundary conditions for the electric and magnetic fields.

Field Components	General Form	Medium 1 Dielectric	Medium 2 Dielectric	Medium 1 Dielectric	Medium 2 Conductor
Tangential E	$\hat{\mathbf{n}}_2 \times (\mathbf{E}_1 - \mathbf{E}_2) = 0$	$E_{1t} = E_{2t}$		$E_{1t} = E_{2t} = 0$	
Normal D	$\hat{\mathbf{n}}_2 \cdot (\mathbf{D}_1 - \mathbf{D}_2) = \rho_s$	$D_{1n} - D_{2n} = \rho_s$		$D_{1n} = \rho_s$	$D_{2n} = 0$
Tangential H	$\hat{\mathbf{n}}_2 \times (\mathbf{H}_1 - \mathbf{H}_2) = \mathbf{J}_s$	$H_{1t} = H_{2t}$		$H_{1t} = J_s$	$H_{2t} = 0$
Normal B	$\hat{\mathbf{n}}_2 \cdot (\mathbf{B}_1 - \mathbf{B}_2) = 0$	$B_{1n} = B_{2n}$		$B_{1n} = B_{2n} = 0$	

Notes: (1) ρ_s is the surface charge density at the boundary; (2) $\mathbf{J}_s$ is the surface current density at the boundary; (3) normal components of all fields are along $\hat{\mathbf{n}}_2$, the outward unit vector of medium 2; (4) $E_{1t} = E_{2t}$ implies that the tangential components are equal in magnitude and parallel in direction; (5) direction of $\mathbf{J}_s$ is orthogonal to $(\mathbf{H}_1 - \mathbf{H}_2)$.

Charge Current Continuity Equation

Current I out of a volume is equal to rate of decrease of charge Q contained in that volume:

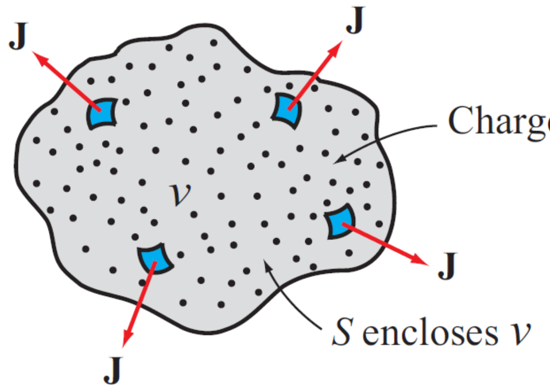


Figure 6-14: The total current flowing out of a volume $\mathcal{V}$ is equal to the flux of the current density **J** through the surface S, which in turn is equal to the rate of decrease of the charge enclosed in $\mathcal{V}$.

$$I = -\frac{dQ}{dt} = -\frac{d}{dt}\int_{\mathcal{V}} \rho_v \, d\mathcal{V}$$

$$\oint_S \mathbf{J} \cdot d\mathbf{s} = -\frac{d}{dt}\int_{\mathcal{V}} \rho_v \, d\mathcal{V}$$

$$\oint_S \mathbf{J} \cdot d\mathbf{s} = \int_{\mathcal{V}} \nabla \cdot \mathbf{J} \, d\mathcal{V} = -\frac{d}{dt}\int_{\mathcal{V}} \rho_v \, d\mathcal{V}$$

Used Divergence Theorem

Volume current density

$$\boxed{\nabla \cdot \mathbf{J} = -\frac{\partial \rho_v}{\partial t}, \qquad (6.54)}$$

which is known as the ***charge-current continuity relation***, or simply the ***charge continuity equation***.

Charge Current Continuity Equation

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho_v}{\partial t}, \qquad (6.54)$$

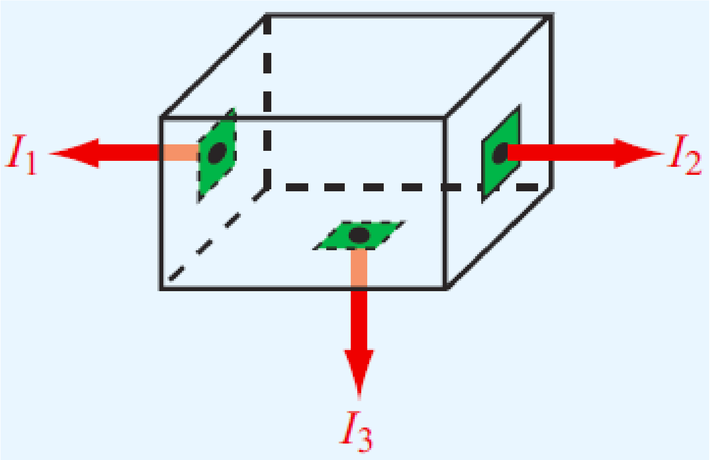
$$\oint_S \mathbf{J} \cdot d\mathbf{s} = -\frac{d}{dt} \int_V \rho_v \, dV$$

$$\nabla \cdot \mathbf{J} = 0,$$

$$\oint_S \mathbf{J} \cdot d\mathbf{s} = 0 \quad \text{(Kirchhoff's current law).}$$

Net current flowing out of ΔV is zero

$$\sum_i I_i = 0 \quad \text{(Kirchhoff's current law),}$$



Ampère's Law for Dynamic Fields

In case the charge density does not change with time

$$\nabla \times \mathbf{H} = \mathbf{J}$$

$$0 = \nabla \cdot \boldsymbol{J}$$

$$\nabla \cdot (\nabla \times \boldsymbol{H}) = \nabla \cdot \boldsymbol{J} = \mathbf{0}$$

In case the charge density changes with time

$$\boxed{\nabla \cdot \mathbf{J} = -\frac{\partial \rho_{\mathrm{v}}}{\partial t}, \qquad (6.54)}$$

$$\nabla \cdot (\nabla \times \boldsymbol{H}) - \frac{\partial \rho_v}{\partial t} = \nabla \cdot \boldsymbol{J}$$

$$\nabla \cdot \mathbf{D} = \rho_{\mathrm{v}}$$

$$\nabla \cdot (\nabla \times \boldsymbol{H}) - \frac{\partial (\nabla \cdot \boldsymbol{D})}{\partial t} = \nabla \cdot \boldsymbol{J}$$

Need to add a term for dynamic field

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \qquad \text{(Ampère's law).}$$

Added by Maxwell

Charge Dissipation

Question 1: What happens if you place a certain amount of free charge inside of a material?
Answer: The charge will move to the surface of the material, thereby returning its interior to a neutral state.

Question 2: How fast will this happen?
Answer: It depends on the material; in a good conductor, the charge dissipates in less than a femtosecond, whereas in a good dielectric, the process may take several hours.

Derivation of charge density equation:

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho_{\mathrm{v}}}{\partial t} \ . \qquad (6.58)$$

In a conductor, the point form of Ohm's law, given by Eq. (4.63), states that $\mathbf{J} = \sigma \mathbf{E}$. Hence,

$$\sigma \nabla \cdot \mathbf{E} = -\frac{\partial \rho_{\mathrm{v}}}{\partial t} \ . \qquad (6.59)$$

Next, we use Eq. (6.1), $\nabla \cdot \mathbf{E} = \rho_{\mathrm{v}}/\varepsilon$, to obtain the partial differential equation

$$\frac{\partial \rho_{\mathrm{v}}}{\partial t} + \frac{\sigma}{\varepsilon} \rho_{\mathrm{v}} = 0. \qquad (6.60)$$

Cont.

Solution of Charge Dissipation Equation

$$\frac{\partial \rho_v}{\partial t} + \frac{\sigma}{\varepsilon}\rho_v = 0$$

Given that $\rho_v = \rho_{vo}$ at $t = 0$, the solution of Eq. (6.60) is

$$\rho_v(t) = \rho_{vo} e^{-(\sigma/\varepsilon)t} = \rho_{vo} e^{-t/\tau_r} \qquad (\text{C/m}^3), \qquad (6$$

where $\tau_r = \varepsilon/\sigma$ is called the ***relaxation time constant***.

For copper: $\tau_r = 1.53 \times 10^{-19}$ s

For mica: $\tau_r = 5.31 \times 10^4$ s = 15 hours

Example 6-8: Relating $\mathbf{E}$ to $\mathbf{H}$

In a nonconducting medium with $\varepsilon = 16\varepsilon_0$ and $\mu = \mu_0$, the electric field intensity of an electromagnetic wave is

$$\mathbf{E}(z,t) = \hat{\mathbf{x}}\, 10 \sin(10^{10} t - kz) \qquad \text{(V/m)}. \qquad (6.88)$$

Determine the associated magnetic field intensity **H** and find the value of k.

Solution: We begin by finding the phasor $\widetilde{\mathbf{E}}(z)$ of $\mathbf{E}(z,t)$. Since $\mathbf{E}(z,t)$ is given as a sine function and phasors are defined in this book with reference to the cosine function, we rewrite Eq. (6.88) as

$$\begin{aligned}\mathbf{E}(z,t) &= \hat{\mathbf{x}}\, 10 \cos(10^{10} t - kz - \pi/2) \qquad \text{(V/m)} \\ &= \mathfrak{Re}\left[\widetilde{\mathbf{E}}(z)\, e^{j\omega t}\right],\end{aligned} \qquad (6.89)$$

with $\omega = 10^{10}$ (rad/s) and

$$\widetilde{\mathbf{E}}(z) = \hat{\mathbf{x}}\, 10 e^{-jkz} e^{-j\pi/2} = -\hat{\mathbf{x}}\, j 10 e^{-jkz}. \qquad (6.90)$$

Cont.

Example 6-8: Relating E to H

To find both $\widetilde{\mathbf{H}}(z)$ and k, we will perform a "circle": we will use the given expression for $\widetilde{\mathbf{E}}(z)$ in Faraday's law to find $\widetilde{\mathbf{H}}(z)$; then we will use $\widetilde{\mathbf{H}}(z)$ in Ampère's law to find $\widetilde{\mathbf{E}}(z)$, which we will then compare with the original expression for $\widetilde{\mathbf{E}}(z)$; and the comparison will yield the value of k. Application of Eq. (6.87) gives

$$\nabla \times \widetilde{\mathbf{E}} = -j\omega\mu\widetilde{\mathbf{H}}$$

$$\begin{aligned}
\widetilde{\mathbf{H}}(z) &= -\frac{1}{j\omega\mu} \nabla \times \widetilde{\mathbf{E}} \\
&= -\frac{1}{j\omega\mu} \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -j10e^{-jkz} & 0 & 0 \end{vmatrix} \\
&= -\frac{1}{j\omega\mu} \left[\hat{\mathbf{y}} \, \frac{\partial}{\partial z}(-j10e^{-jkz}) \right] \\
&= -\hat{\mathbf{y}} j \, \frac{10k}{\omega\mu} e^{-jkz}.
\end{aligned} \qquad (6.91)$$

Cont.

Example 6-8 cont.

So far, we have used Eq. (6.90) for $\widetilde{\mathbf{E}}(z)$ to find $\widetilde{\mathbf{H}}(z)$, but k remains unknown. To find k, we use $\widetilde{\mathbf{H}}(z)$ in Eq. (6.86) to find $\widetilde{\mathbf{E}}(z)$:

$$\nabla \times \widetilde{\mathbf{H}} = j\omega\varepsilon\widetilde{\mathbf{E}}$$

$$\begin{aligned}\widetilde{\mathbf{E}}(z) &= \frac{1}{j\omega\varepsilon}\nabla \times \widetilde{\mathbf{H}} \\ &= \frac{1}{j\omega\varepsilon}\left[-\hat{\mathbf{x}}\frac{\partial}{\partial z}\left(-j\frac{10k}{\omega\mu}e^{-jkz}\right)\right] \\ &= -\hat{\mathbf{x}}j\frac{10k^2}{\omega^2\mu\varepsilon}e^{-jkz}. \end{aligned} \qquad (6.92)$$

Equating Eqs. (6.90) and (6.92) leads to

$$k^2 = \omega^2\mu\varepsilon,$$

$$\widetilde{\mathbf{E}}(z) = \hat{\mathbf{x}}\,10e^{-jkz}e^{-j\pi/2} = -\hat{\mathbf{x}}j10e^{-jkz} \qquad (6.90)$$

or

$$\begin{aligned} k &= \omega\sqrt{\mu\varepsilon} \\ &= 4\omega\sqrt{\mu_0\varepsilon_0} \\ &= \frac{4\omega}{c} = \frac{4\times10^{10}}{3\times10^8} = 133 \qquad \text{(rad/m)}. \end{aligned} \qquad (6.93)$$

Cont.

Example 6-8 cont.

With k known, the instantaneous magnetic field intensity is then given by

$$\mathbf{H}(z,t) = \mathfrak{Re}\left[\widetilde{\mathbf{H}}(z)\; e^{j\omega t}\right]$$
$$= \mathfrak{Re}\left[-\hat{\mathbf{y}}\,j\;\frac{10k}{\omega\mu}e^{-jkz}e^{j\omega t}\right]$$
$$= \hat{\mathbf{y}}\,0.11\sin(10^{10}t - 133z) \qquad \text{(A/m)}. \qquad (6.94)$$

We note that k has the same expression as the phase constant of a lossless transmission line [Eq. (2.49)].

Summary

Chapter 6 Relationships

Faraday's Law

$$V_{\text{emf}} = -\frac{d\Phi}{dt} = -\frac{d}{dt}\int_S \mathbf{B}\cdot d\mathbf{s} = V_{\text{emf}}^{\text{tr}} + V_{\text{emf}}^{\text{m}}$$

Transformer

$$V_{\text{emf}}^{\text{tr}} = -N\int_S \frac{\partial \mathbf{B}}{\partial t}\cdot d\mathbf{s} \qquad (N \text{ loops})$$

Motional

$$V_{\text{emf}}^{\text{m}} = \oint_C (\mathbf{u}\times\mathbf{B})\cdot d\mathbf{l}$$

Charge-Current Continuity

$$\nabla\cdot\mathbf{J} = -\frac{\partial \rho_{\text{v}}}{\partial t}$$

EM Potentials

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}$$

$$\mathbf{B} = \nabla\times\mathbf{A}$$

Current Density

Conduction $\mathbf{J}_{\text{c}} = \sigma\,\mathbf{E}$

Displacement $\mathbf{J}_{\text{d}} = \dfrac{\partial \mathbf{D}}{\partial t}$

Conductor Charge Dissipation

$$\rho_{\text{v}}(t) = \rho_{\text{vo}} e^{-(\sigma/\varepsilon)t} = \rho_{\text{vo}} e^{-t/\tau_{\text{r}}}$$

Homework due XXX