CS271 Computer Graphics II

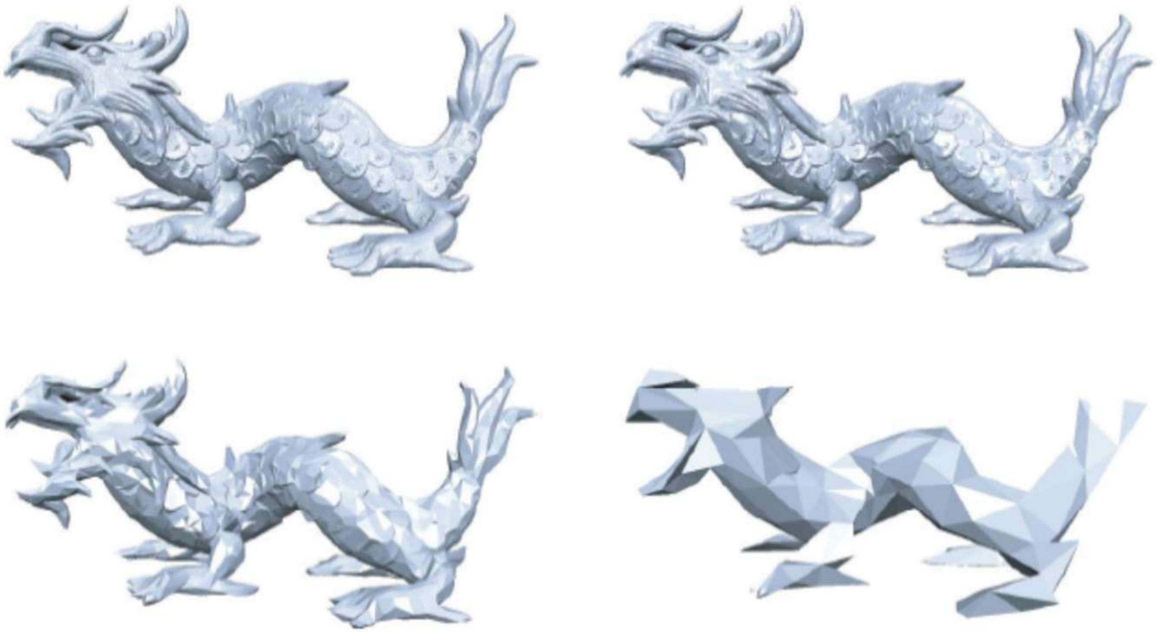
Lecture 6

Mesh Simplification

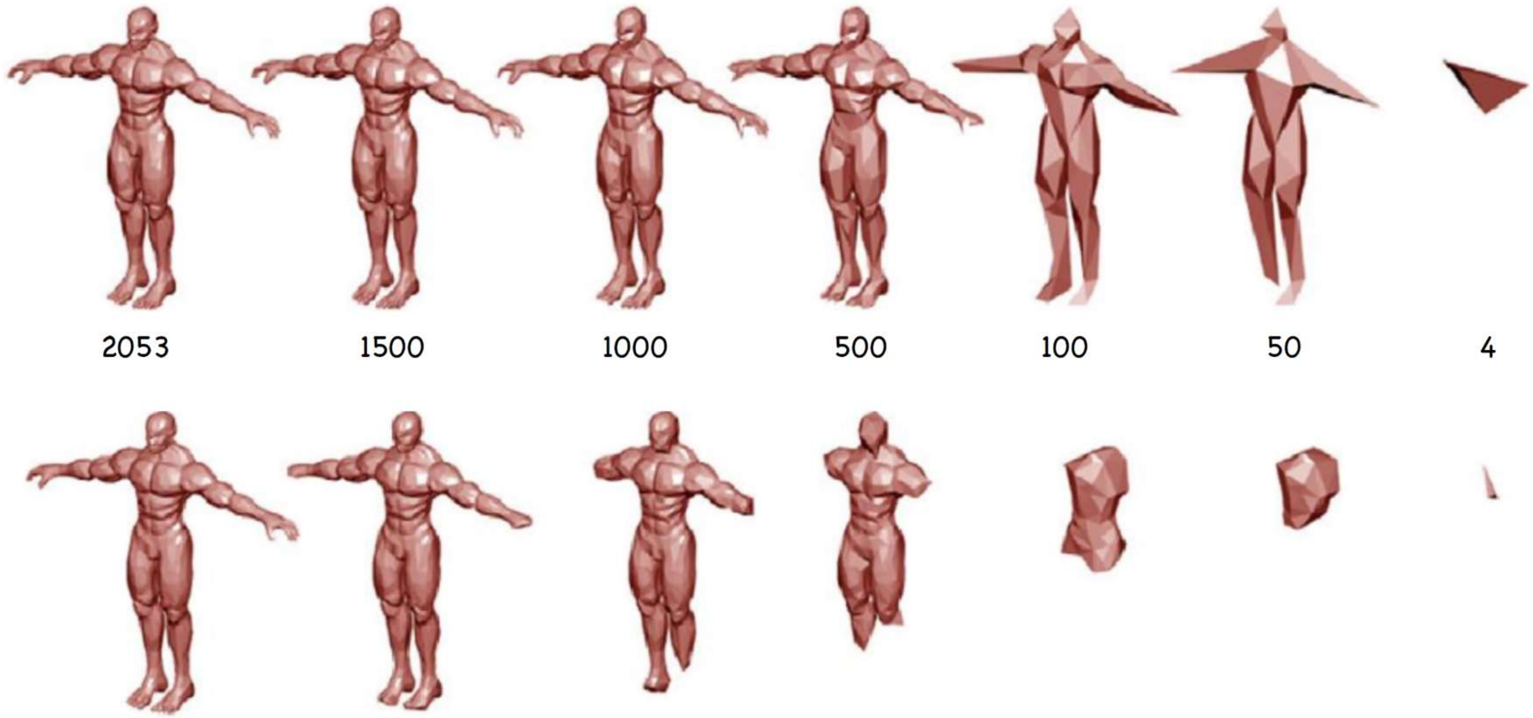
Definition

Simplification, aka decimation, approximation, downsampling

- Transform a given polygonal mesh into another mesh with fewer faces, edges, and vertices
- The simplification or approximation procedure is usually controlled by user-defined quality criteria

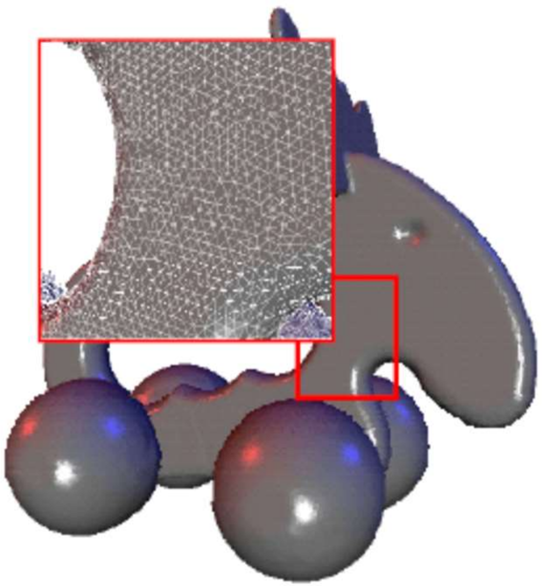


Curvature-preserved vs. Curvature-removed Criteria

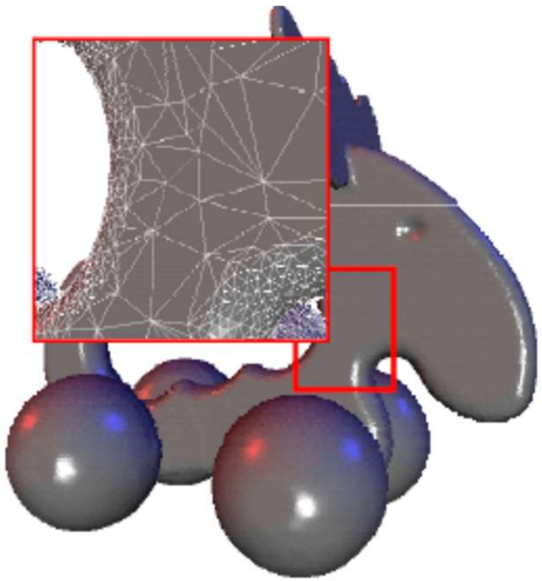


Mesh Simplification Applications

- Oversampled 3D scan data



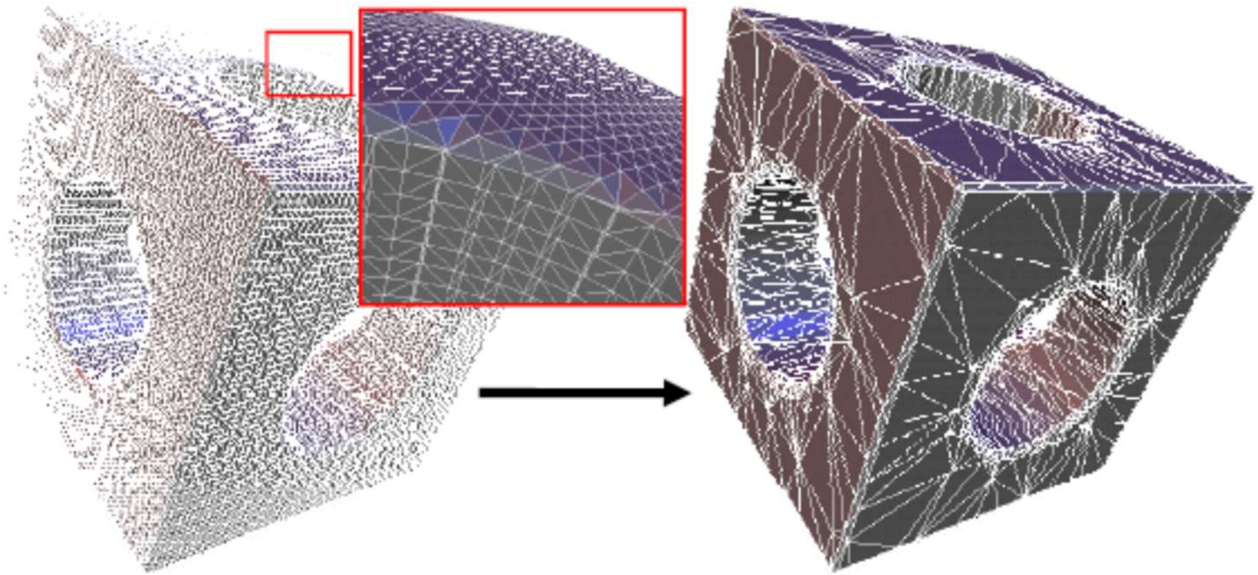
~150k triangles



~80k triangles

Mesh Simplification Applications

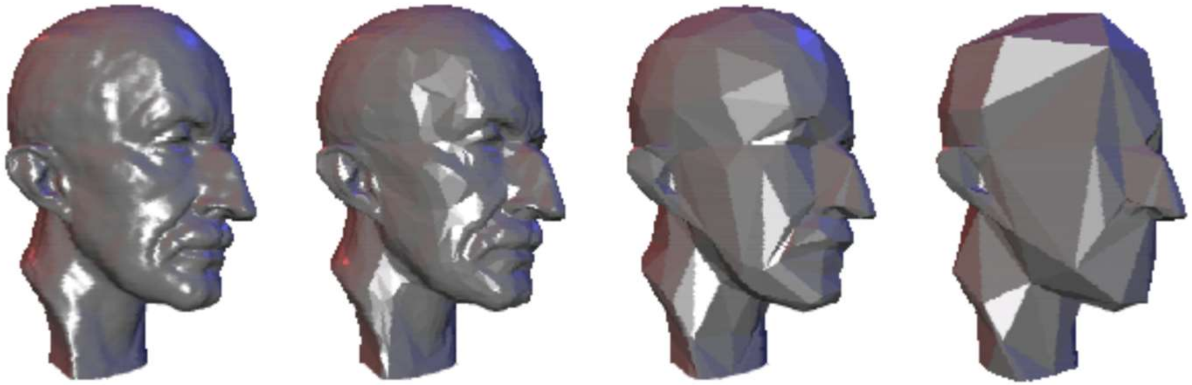
- Over-tessellation: e.g., iso-surface extraction



Mesh Simplification Applications

Multi-resolution hierarchies for

- Efficient geometry processing
- Level-of-detail (LOD) rendering



Mesh Simplification Applications

- Adaption to hardware capabilities

Mesh Simplification

Adjust the complexity of a geometry data set

- Since many decimation schemes work iteratively, i.e., they decimate a mesh by removing one vertex at a time, they usually can be inverted
- Hierarchical method

Problem Statement

- Given:M = (V, F)
- Find: M' = (V', F') such that
- |V'| = n < |V| and || M – M'|| is minimal, or
- ||M-M'|| < ε and |V'| is minimal

Respect additional fairness criteria

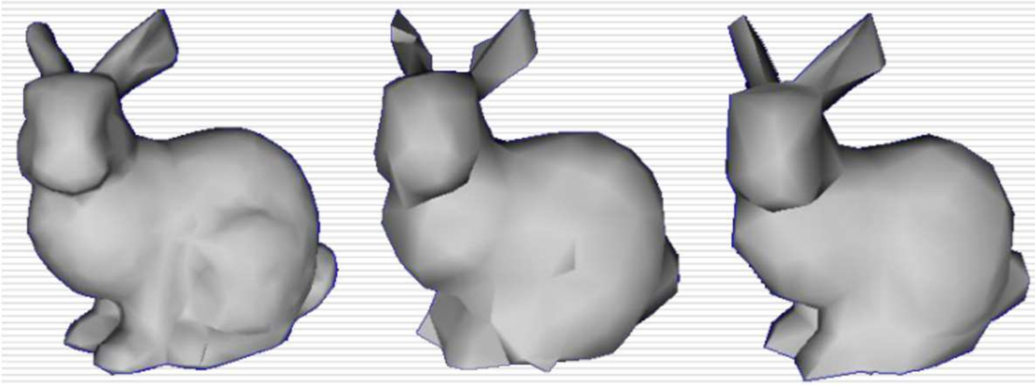
- Normal deviation, triangle shape, scalar attributes, etc.

Mesh Simplification

Start with the original fine mesh

- **Simply progressively**
  E.g., collapse edges, vertex clustering

- **Aim to keep original appearance**
  Normal deviation, triangle shape, scalar attributes
  Error control

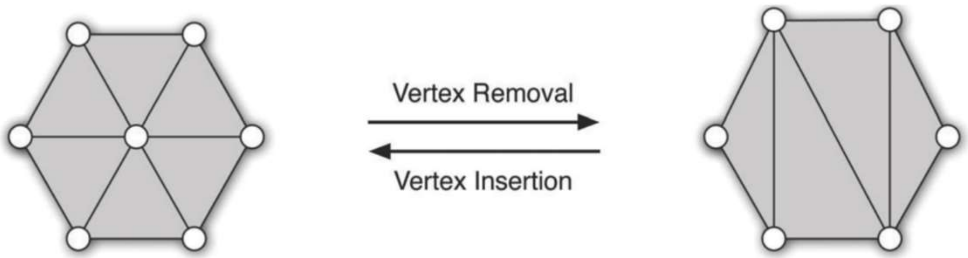


2000 faces	488 faces (local error)	488 faces (global error)

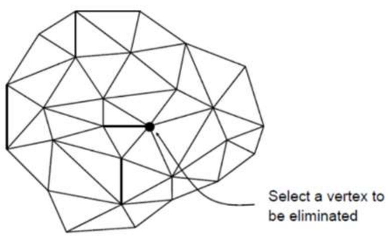
Local Operations

Local Simplification Operator

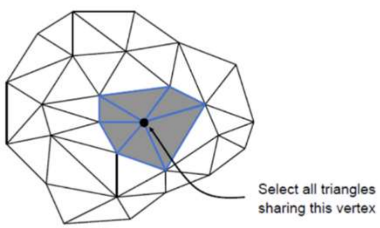
- Vertex Removal



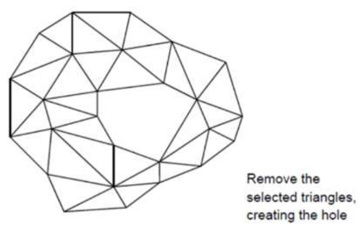
- Deletion & Insertion
- Reversible



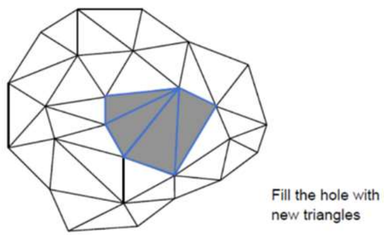
Select a vertex to be eliminated



Select all triangles sharing the vertex



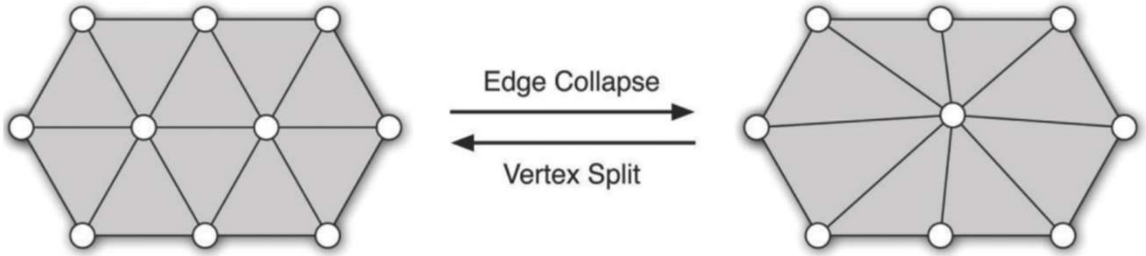
Remove the selected triangles, creating the hole



Fill the hole with new triangles

Local Simplification Operator

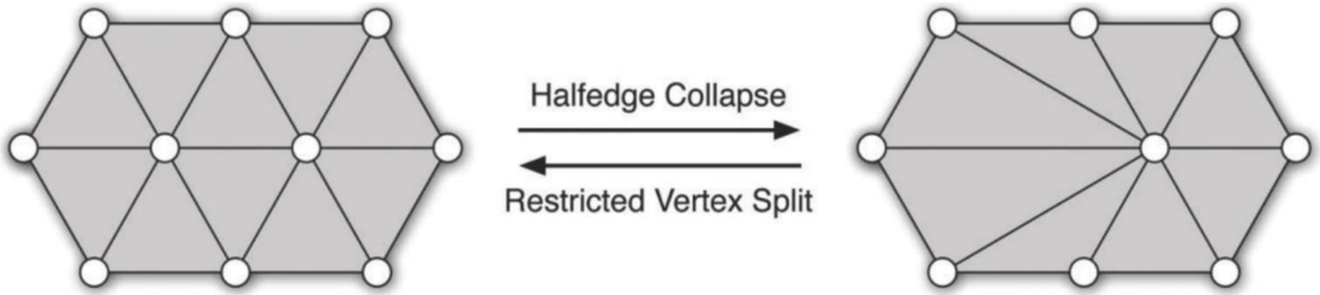
- **Edge Collapse**



- Merge two adjacent vertices
- Simple to implement
- Well-suited for implementing geomorphing

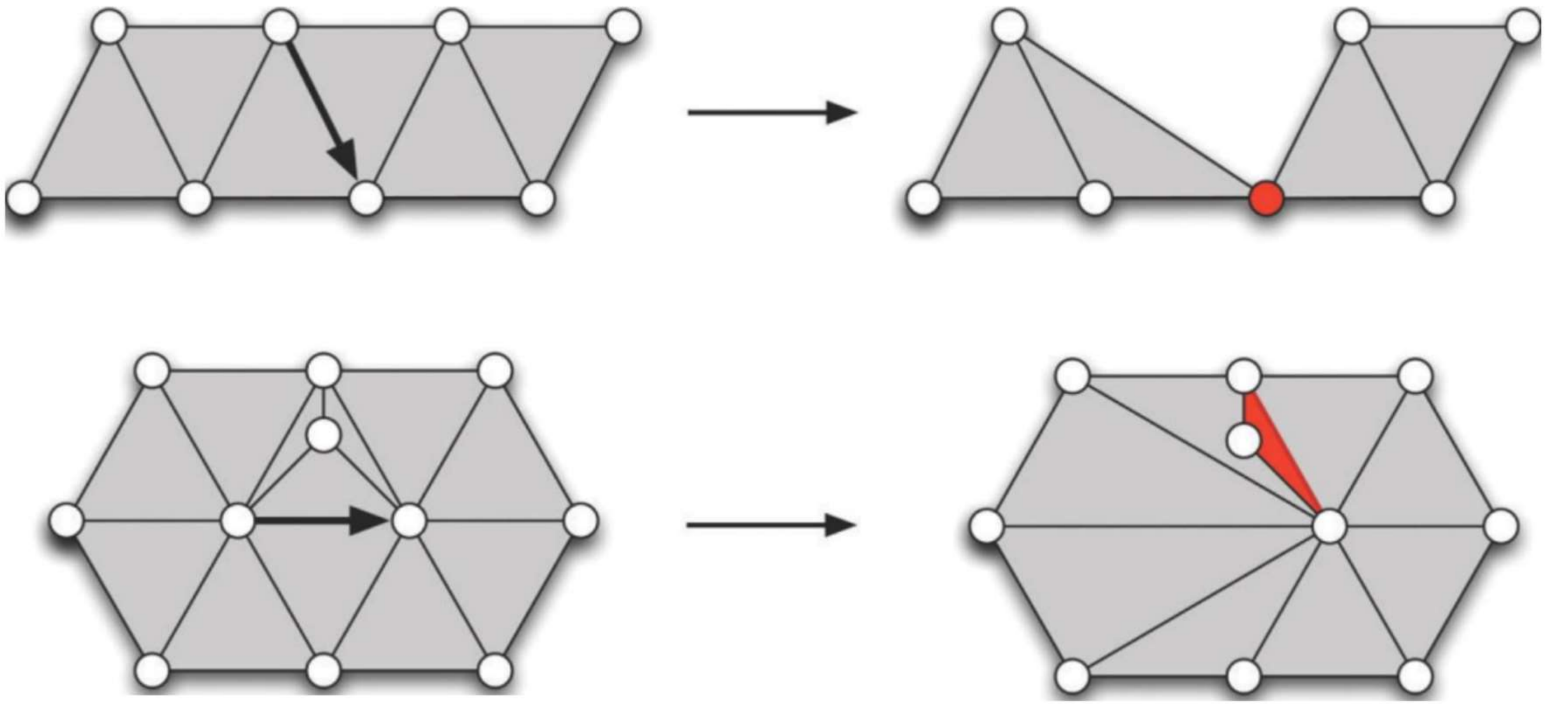
Local Simplification Operator

- **Half-edge Collapse**



- **Collapse edge into one end point**
  Special case of vertex removal
  Special case of edge collapse
- After collapse: n(E) - 3, n(V) – 1, n(F) – 2
- According to Euler Formula: unchanged
- Half-edge collapsing would not change the genus of a mesh
- Should determine whether collapse is ok (may introduce non-manifold structure)

Topologically Illegal (half-)edge Collapses



Simplification via Edge Collapse

One popular scheme: iteratively collapse edges

Greedy algorithm from a general overview:

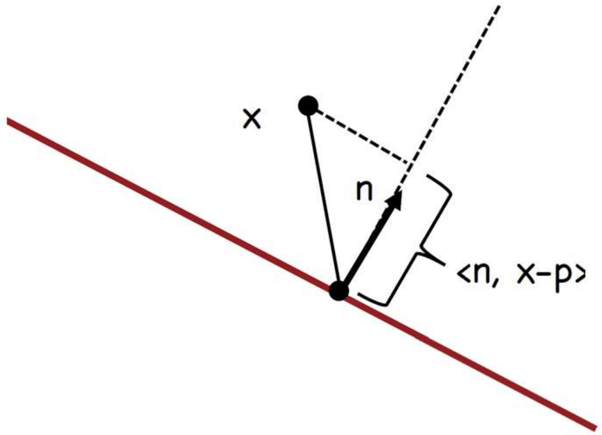
- Assign each edge a cost
- Collapse edge with least cost
- Repeat until target number of elements is reached
- Particularly effective cost function: **Quadric Error Metric**

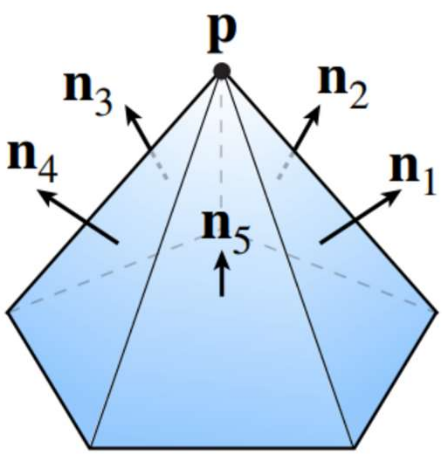
Quadric Error Metric

Quadric Error Metric (QEM)

Approximate distance to a collection of triangles

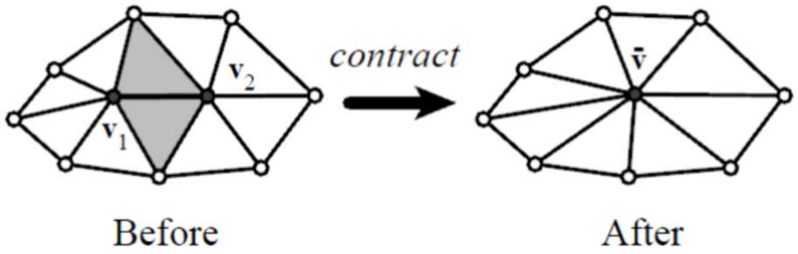
- Q: Distance to plane w/ normal n passing through point p?
- A: dist(x) = <n, x> - <n, p> = <n, x-p>
- Quadric error is then sum of squared point-to-plane distances





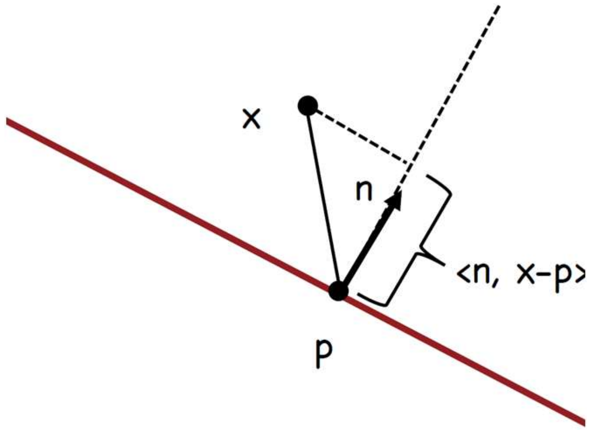
$$Q(x) := \sum_{i=1}^{k} < n_i, x - p >^2$$

$$Q^e = Q_1^v + Q_2^v$$



Quadric Error – Homogeneous Coordinates

Suppose in coordinates we have
- A query point **x = (x, y, z)**
- A normal **n = (a, b, c)**
- An offset **d := <n, p>**



In **homogeneous coordinates**, let
- **u := (x, y, z, 1)**
- **v := (a, b, c, d)**

- Signed distance to plane is then just **<u, v> = ax + by + cz + d**
- Squared distance is **<u, v>2 =** $\mathbf{u^T(vv^T)u =: u^TKu}$
- Matrix $\mathbf{K = vv^T}$ encodes squared distance to plane

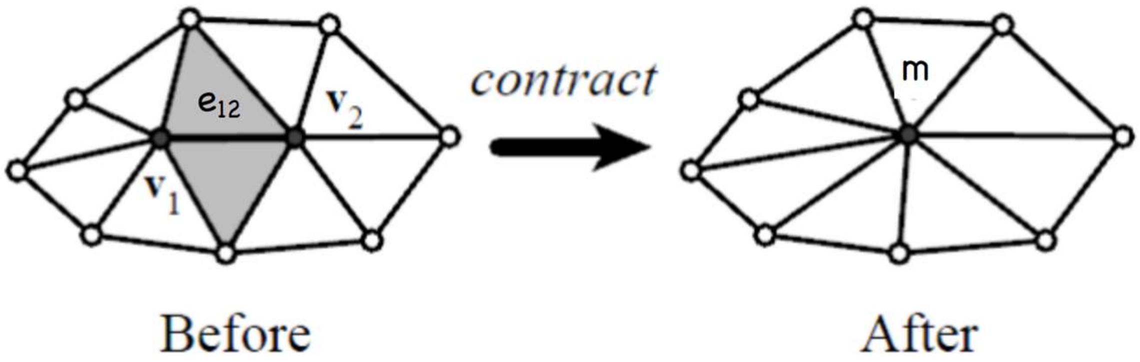
$$K = \begin{bmatrix} a^2 & ab & ac & ad \\ ab & b^2 & bc & bd \\ ac & bc & c^2 & cd \\ ad & bd & cd & d^2 \end{bmatrix}$$

Key idea: sum of matrices K distance to union of planes

$\mathbf{u^TK_1u + u^TK_2u = u^T(K_1+K_2)u}$

Quadric Error of Edge Collapse

- How much does it cost to collapse an edge e12?
  Idea: compute midpoint m, measure error $Q(m) = m^T(K_1+K_2)m$
- Error becomes "score" for e_{12}, determining priority

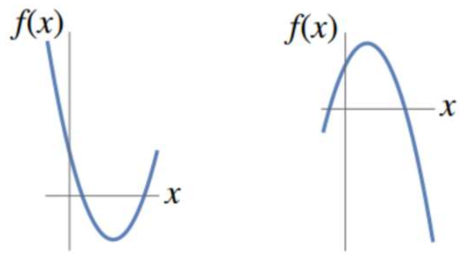


- Better idea: find point x that minimize error!
- But how to minimize quadric error?

Revisit: Minimizing a Quadratic Function

Suppose you have a function $f(x) = ax^2 + bx + c$

- Q: What does the graph of this function look like?
- Q: How do we find the minimum?



- A: Find where the function looks "flat" if we zoom in really close
- i.e., find point x where 1st derivative vanishes:

$$f'(x) = 0$$
$$2ax + b = 0$$
$$x = -b/2a$$

Minimizing Quadratic Polynomial

Not much harder to minimize a quadratic polynomial in **n** variables

- Can always write in terms of a symmetric matrix A
- E.g., in 2D: $\mathbf{f(x, y) = ax^2 + bxy + cy^2 + dx + ey + g}$

$$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} \quad A = \begin{bmatrix} a & b/2 \\ b/2 & c \end{bmatrix} \quad \mathbf{u} = \begin{bmatrix} d \\ e \end{bmatrix}$$

$$\mathbf{f(x, y) = xTAx + uTx + g}$$

(will have the same form for any **n**)

- **Q: How do we find a critical point (min/max/saddle)?**
- **A: Set derivative to zero!**

$$2A\mathbf{x} + \mathbf{u} = 0$$

$$\mathbf{x = -1/2\ A^{-1}u}$$

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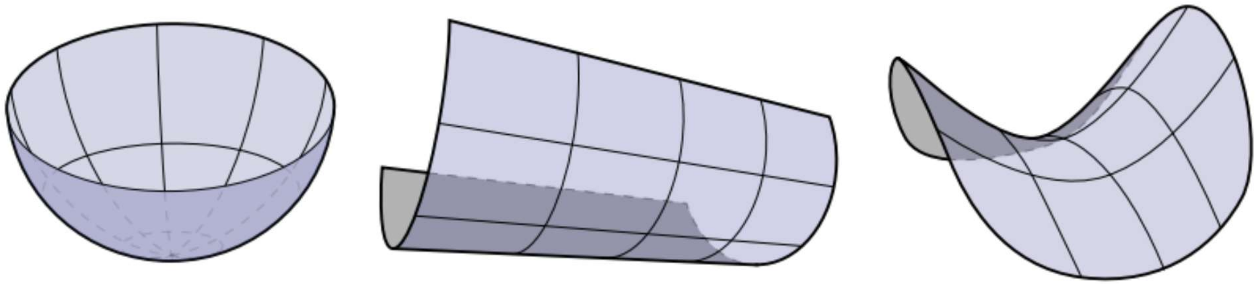
Positive Definite Quadratic Form

Just like our 1D parabola, critical point is not always a min!

- Q: In 2D, 3D, nD, when do we get a minimum?
- A: When matrix A is positive-definite:

$$\mathbf{X^T A X > 0 \quad \forall X}$$

- **1D:** Must have xax = $ax^2 > 0$, i.e., a is positive!
- **2D:** Graph of function looks like a “bowl”:



positive definite   positive semidefinite   indefinite

Minimizing Quadric Error

Find "best" point for edge collapse by minimizing quadratic form

$$\min_{\mathbf{u}\in\mathbb{R}^4} \mathbf{u}^T K \mathbf{u}$$

- Already know 4th (homogeneous) coordinate for a point is 1
- So, break up our quadratic function into two pieces:

$$\begin{bmatrix} \mathbf{x}^\mathsf{T} & 1 \end{bmatrix} \begin{bmatrix} B & \mathbf{w} \\ \mathbf{w}^\mathsf{T} & d^2 \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ 1 \end{bmatrix}$$
$$= \mathbf{x}^\mathsf{T} B \mathbf{x} + 2\mathbf{w}^\mathsf{T}\mathbf{x} + d^2$$

- Now we have a quadratic polynomial in the unknown position $\mathbf{x} \in \mathbb{R}^3$
- Can minimize as before:

$$2B\mathbf{x} + 2\mathbf{w} = 0 \quad \Longleftrightarrow \quad \mathbf{x} = -B^{-1}\mathbf{w}$$

QEM Simplification: Final Algorithm

- **Input: a mesh**
- **Output: a simplified mesh**

- **Initialization:**

  Compute **K** for each triangle (squared distance to plane)
  Set $\mathbf{K}_i$ at each vertex to sum of **Ks** from incident triangles
  For each edge $\mathbf{e}_{ij}$:
  Set $\mathbf{K}_{ij} = \mathbf{K}_i + \mathbf{K}_j$
  Find point x minimizing error, set cost to $\mathbf{K}_{ij}(\mathbf{x})$

- **Until we reach target number of triangles:**

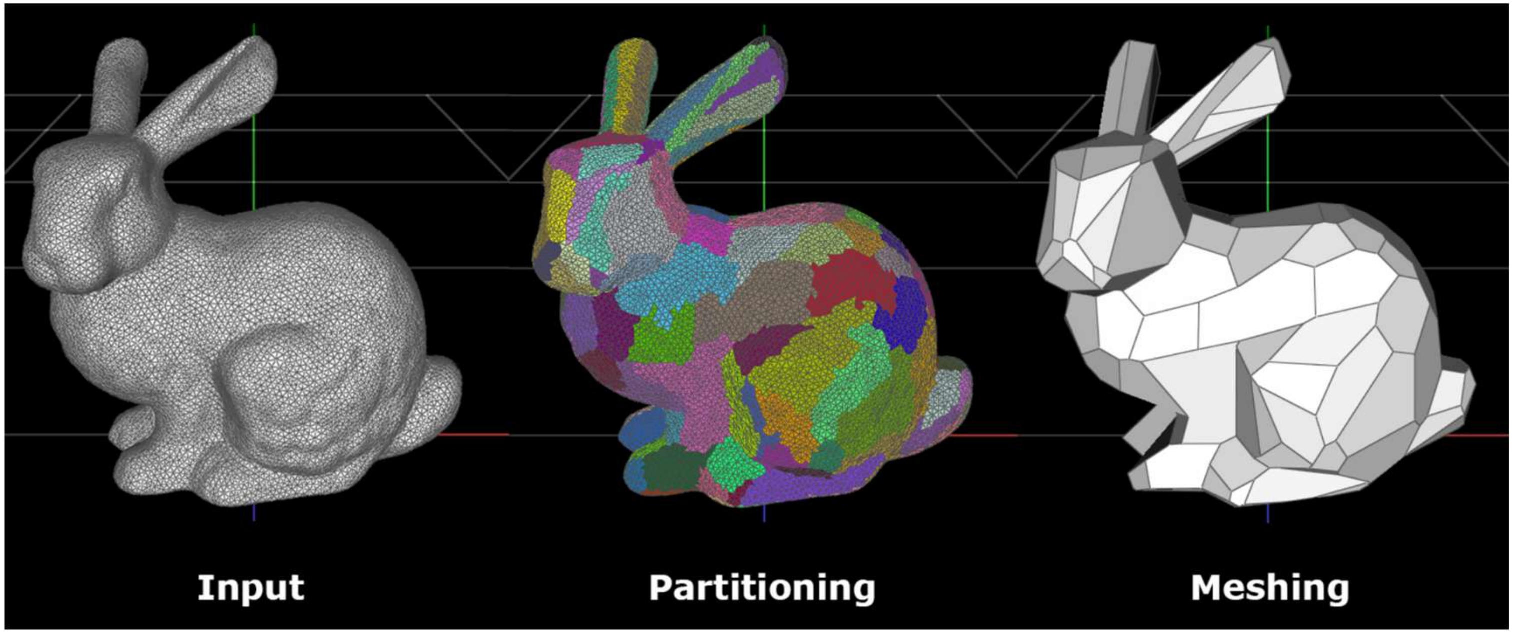
  Collapse edge eij with smallest cost to optimal pont **x**
  Set quadric at new vertex to $\mathbf{K}_{ij}$
  Update cost of edges touching new vertex

Full Resolution | 60,000 triangles | 1000 triangles

Variational Shape Approximation

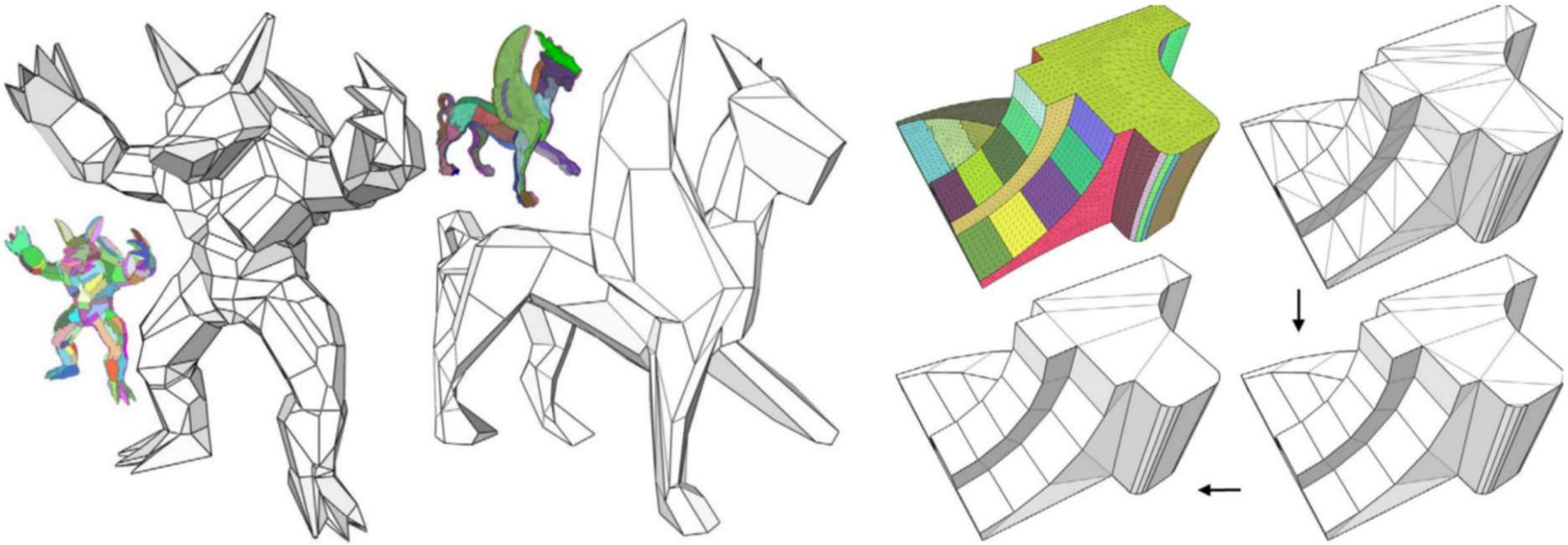
Variational Shape Approximation (VSA)

VSA is highly sensitive to features and symmetries and produces anisotropic meshes of high approximation quality



Variational Shape Approximation (VSA)

- The input shape is approximated by a set of proxies
- A plane in space through the point $\mathbf{x}_i$ with normal direction $\mathbf{n}_i$



Region Representation

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- $\mathbf{M}$: a triangle mesh
- $\mathbf{R} = \{\mathbf{R}_1, \ldots, \mathbf{R}_k\}$: a partition of $\mathbf{M}$ into $\mathbf{k}$ regions $R_1 \cup \ldots \cup R_k = M$
- Proxies: $\mathbf{P} = \{\mathbf{P}_1, \ldots, \mathbf{P}_k\}$, $\mathbf{P}_i = (\mathbf{x}_i, \mathbf{n}_i)$

Distance metrics between $\mathbf{R}_i$ and $\mathbf{P}_i$

- The squared orthogonal distance of x from the plane $\mathbf{P}_i$

$$L^2(R_i, P_i) = \int_{x \in R_i} (n_i^T x - n_i x_i)^2 dA$$

- A measure of the normal field:

$$L^{2,1}(R_i, P_i) = \int_{x \in R_i} ||n(x) - n_i||^2 dA$$

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Goal of VSA

Given a number $\mathbf{k}$ and an error metric $\mathbf{E(L_2 \text{ or } L_{2,1})}$, find a set $\mathbf{R = \{R_1, \ldots, R_k\}}$ of regions and a set $\mathbf{P = \{P_1, \ldots, P_k\}}$ of proxies such that the global distortion

$$E(R, P) = \sum_{i=1}^{k} E(R_i, P_i)$$

is minimized

Lloyd's Clustering Algorithm

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- The algorithm iteratively alternates between a **geometry partitioning** phase and a **proxy fitting** phase
- **Geometry partitioning phase**
  - A set of regions that best fit a **given** set of proxies
  - Modifies the set R of regions to achieve a lower approximation error while keeping the proxies P fixed
- **Proxy fitting phase**
  - The partitioning is kept **fixed,** and the proxies are adjusted to minimize approximation error
  - L^2 metric: the best proxy is the least-squares fitting plane
  - $L^{2,1}$ metric: the proxy normal n_i is just the area-weighted average of the triangle normals
- **Initialization**
  - Randomly picking k triangles as R
  - The planes of k triangles are used to initialize P

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More Paper

- Liu Y J, Xu C X, Fan D, et al. **Efficient construction and simplification of Delaunay meshes**[J]. ACM Transactions on Graphics (TOG), 2015, 34(6): 1-13.

- Yi R, Liu Y J, He Y**. Delaunay mesh simplification with differential evolution**[J]. ACM Transactions on Graphics (TOG), 2018, 37(6): 1-12.

- Liang Y, He F, Zeng X. **3D mesh simplification with feature preservation based on whale optimization algorithm and differential evolution**[J]. Integrated Computer-Aided Engineering, 2020, 27(4): 417-435.