Online Lecture Notes

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1 Announcements

It seems that we to keep on running the lecture online for a long time. The rough time plan is

1. Today we complete Lecture 6 + send out Homework 6. This complete all the material that you need for the mid-term exam.

2. Next week we will recall all the previous material and have time for questions. We will spend a bit more time on Lecture 4 and 5 during the wrap up.

3. The tentative date for the mid-term exam is May 10. Start around 8am. The exam will be a take-home exam—we will send you via email / skype / tencent. You will have 24h to complete all the problems. The difficult level will be very similar to the exam from previous years. The content that will be covered is Lecture 1-6 and Homework 1-6.

2 Proportional Differential (PD) Control

In order to understand why one might want to introduce a differential control gain in addition to a proportional control gain, it is important to understand the limitations of propportional goal.

2.1 An Example where Proportional Control Fails

Consider a a "simple" car with mass m, which can be modeled by Newton's equation of motion,

$$m\ddot{p}(t) = u(t) \; .$$

Here, p denotes the position of the car, m the mass of the car. We can control the force $u(t)$ at time t. In order to reformulate the above differential equation as a linear control system in standard form, we introduce the stacked state

$$x(t) = \begin{pmatrix} p(t) \\ v(t) \end{pmatrix} ,$$

where $v(t) = \dot{p}(t)$ denotes the velocity of the car at time t. Our differential equation now has the form

$$\dot{x}(t) = Ax(t) + Bu(t) \qquad \text{with} \quad A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \qquad B = \begin{pmatrix} 0 \\ \frac{1}{m} \end{pmatrix} \; .$$

Let's say our goal is to stabilize the car at the position $p_{\mathrm{ref}} = 0$, where we can measure only the position $p(t)$,

$$y(t) = p(t) = Cx(t) \quad \text{and} \quad C = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

If we introduce a proportional controller of the form $u(t) = Ky(t)$, the our closed-loop system gain has the form

$$A_{\mathrm{cl}} = A + BKC = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{1}{m} \end{pmatrix} K \begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \frac{K}{m} & 0 \end{pmatrix}$$

Recall that this closed-loop system is asymptotically stable if the eigenvalues of A_{cl} have strictly negative real part. Let us work the eigenvalues explicitly:

$$0 = \det(A_{\mathrm{cl}} - \lambda I) = \det\left(\begin{pmatrix} -\lambda & 1 \\ \frac{K}{m} & -\lambda \end{pmatrix} \right) = \lambda^2 - \frac{K}{m}$$

This means that the eigenvalues have the form

$$\lambda_1 = \sqrt{\frac{K}{m}} \qquad \text{and} \qquad \lambda_2 = -\sqrt{\frac{K}{m}}$$

If we choose $K \geq 0$, then $\lambda_1 \geq 0$ is not strictly negative. If we choose $K < 0$, then both eigenvalues are imaginary, which means that our closed-loop system will oscillate with frequency $\omega = \sqrt{-\frac{K}{m}}$, but the states are not converging to zero (harmonic oscillator!).

Summary: this example shows that in general a proportional controller alone is not sufficient to asymptotically stabilize linear control systems. In this example, the closed-loop system is "osciallating". Such oscillation can eventually be damped out by introducing a differential gain. Essentially, this means that we would to feedback the "velocity" of the system, too.

2.2 Proportional Differential Control

The general form of a proprtional-differential controller is given by

$$u(t) = u_{\mathrm{ref}} + K(y(t) - y_{\mathrm{ref}}) + K_{\mathrm{D}}\dot{y}(t)$$

This means that we have the following relations:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + b && (1) \\ y(t) &= Cx(t) + d && (2) \\ u(t) &= u_{\mathrm{ref}} + K(y(t) - y_{\mathrm{ref}}) + K_{\mathrm{D}}\dot{y}(t) && (3) \\ 0 &= Ax_{\mathrm{ref}} + Bu_{\mathrm{ref}} + b && (4) \\ y_{\mathrm{ref}} &= Cx_{\mathrm{ref}} + d && (5) \end{aligned}$$

In order to find the differential equation for the closed-loop system state, we need to substitute the above relations. This yields

$$\begin{aligned} \dot{x}(t) &\overset{(1)}{=} Ax(t) + Bu(t) + b \\ &\overset{(3)}{=} Ax(t) + B\left[u_{\mathrm{ref}} + K(y(t) - y_{\mathrm{ref}}) + K_{\mathrm{D}}\dot{y}(t)\right] + b \\ &\overset{(2)}{=} Ax(t) + B\left[u_{\mathrm{ref}} + K(Cx(t) + d - y_{\mathrm{ref}}) + K_{\mathrm{D}}C\dot{x}(t)\right] + b\,. && (6) \end{aligned}$$

Here, we see that the behavior of this equation is different from the proportional controller, since we get an implicit equation for $\dot{x}(t)$. Thus, if we want to continue, we need to eliminate $\dot{x}(t)$ first. Let us put all the corresponding term that depend on $\dot{x}(t)$ to the left:

$$
\begin{aligned}
[I - BK_{\mathrm{D}}C]\dot{x}(t) &\overset{(6)}{=} Ax(t) + B\left[u_{\mathrm{ref}} + K(Cx(t) + d - y_{\mathrm{ref}})\right] + b \\
&\overset{(5)}{=} Ax(t) + B\left[u_{\mathrm{ref}} + KC(x(t) - x_{\mathrm{ref}})\right] + b \\
&\overset{(4)}{=} A\left(x(t) - x_{\mathrm{ref}}\right) + BKC(x(t) - x_{\mathrm{ref}}) \\
&= (A + BKC)(x(t) - x_{\mathrm{ref}}) \qquad (7)
\end{aligned}
$$

If the matrix $[I - BK_{\mathrm{D}}C]$ is invertible, the closed loop system takes the form

$$\dot{x}(t) = [I - BK_{\mathrm{D}}C]^{-1}(A + BKC)(x(t) - x_{\mathrm{ref}}) \; .$$

This means that our closed-loop system matrix is given by

$$A_{\mathrm{cl}} = [I - BK_{\mathrm{D}}C]^{-1}(A + BKC) \; .$$

Interesting exercise: show that this gives the same result as the similar expression for the lecture slides. Also notice that the example from the lecture slides will be discussed in Homework 6.

2.3 Proportional Integral (PI) Control

An integral control gain can be used to compensate offset errors, which only contribute on the "long run" under the integral. This means that we consider closed-loop control laws of the form

$$u(t) = Ky(t) + K_{\mathrm{I}} \int_0^t y(\tau)\,\mathrm{d}\tau$$

assuming for simplicity that $u_{\mathrm{ref}} = 0$, $y_{\mathrm{ref}} = 0$, and $d = 0$. The closed-loop system then takes the form

$$\dot{x}(t) = Ax(t) + B\left(KCx(t) + K_{\mathrm{I}} \int_0^t Cx(\tau)\,\mathrm{d}\tau\right)$$

Now, the main idea is to introduce the auxiliary state

$$z(t) = \begin{pmatrix} x(t) \\ \int_0^t x(\tau)\,\mathrm{d}\tau \end{pmatrix}$$

such that

$$\dot{z}_1(t) = Az_1(t) + BKCz_1(t) + BK_{\mathrm{I}}Cz_2(t) \quad \text{and} \qquad \dot{z}_2(t) = x_1(t) = z_1(t)$$

Thus we have

$$\dot{z}(t) = \underbrace{\begin{pmatrix} A + BKC & BK_{\mathrm{I}}C \\ I & 0 \end{pmatrix}}_{=A_{\mathrm{cl}}} z(t)$$

The closed loop system is asymptotically stable if all eigenvalues of A_{cl} have strictly negative real part.