Discrete Mathematics: Lecture 26

Paths and Isomorphism, Counting Paths, Euler Paths and Circuits

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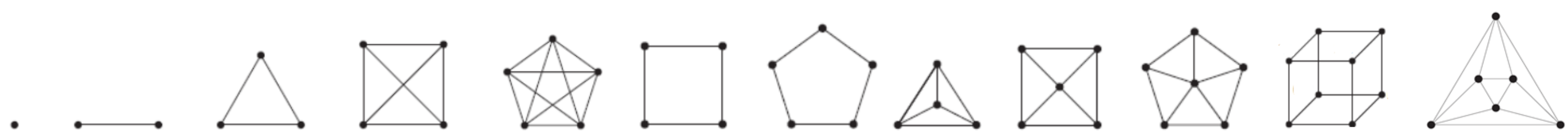
Vertex Connectivity

DEFINITION: A connected undirected graph $G = (V, E)$ is said to be **nonseparable**不可分的 if G has no cut vertex.

DEFINITION: Let $G = (V, E)$ be a connected simple graph.

- **vertex cut**点割集: A subset $V' \subseteq V$ such that $G - V'$ is disconnected
- **vertex connectivity**点连通度 $\kappa(G)$: the minimum number of vertices whose removal disconnect G or results in K_1; equivalently,
  - if G is disconnected, $\kappa(G) = 0$; //additional definition
  - if $G = K_n, \kappa(G) = n - 1$ //K_n has no vertex cut
  - else, $\kappa(G)$ is the minimum size of a vertex cut of G



0 1 2 3 4 2 2 3 3 3 3 4

These graphs are all nonseparable

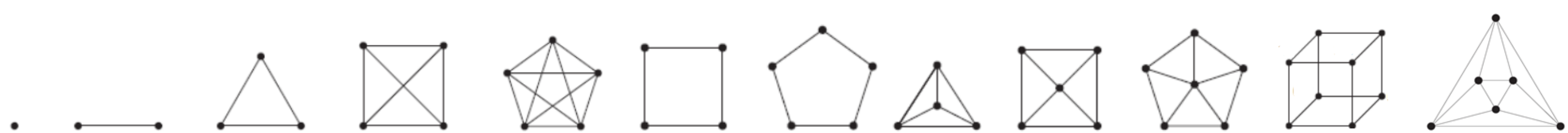
Edge Connectivity

DEFINITION: Let $G = (V, E)$ be a connected simple graph. $E' \subseteq E$ is an **edge cut**边割集 of G if $G - E'$ is disconnected.

DEFINITION: Let $G = (V, E)$ be a simple graph.

The **edge connectivity**边连通度 *($\lambda(G)$)* of G is defined as below:

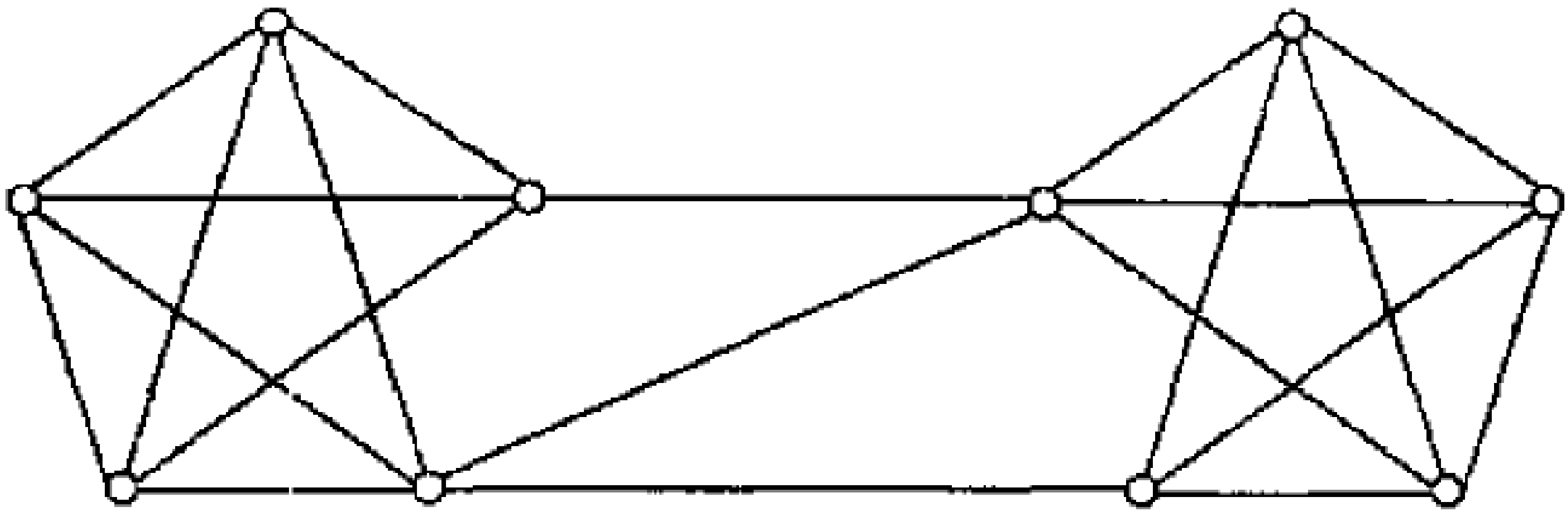
- G disconnected*:* $\lambda(G) = 0$
- G connected*:*
  - $|V| = 1$: $\lambda(G) = 0$
  - $|V| > 1$: $\lambda(G)$ is the minimum size of edge cuts of G.



0 1 2 3 4 2 2 3 3 3 3 4

Connectivity

THEOREM: Let $G = (V, E)$ be a simple graph. Then $\kappa(G) \leq \lambda(G) \leq \delta(G)$, where $\delta(G) = \min_{v \in V} \deg(v)$ is the least degree of $G's$ vertices.



- $\kappa(G) = 2$
- $\lambda(G) = 3$
- $\delta(G) = 4$

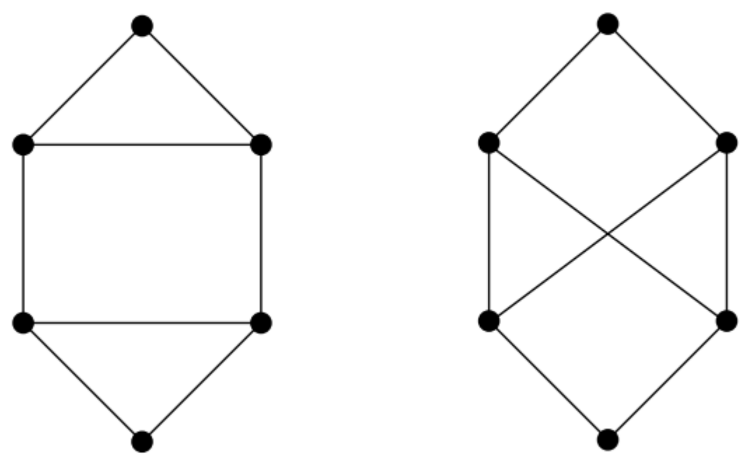
https://cp-algorithms.com/graph/edge_vertex_connectivity.html

http://www.math.caltech.edu/~2014-15/2term/ma006b/05%20connectivity%201.pdf

Paths and Isomorphism

Theorem

The existence of a simple circuit of length k, $k \geq 3$ is an isomorphism invariant for simple graphs.



G_1 G_2

6 vertices, 8 edges

Degree sequence: 3, 3, 3, 3, 2, 2

Paths and Isomorphism*

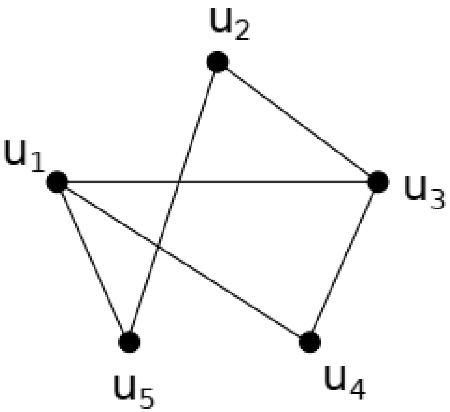
Theorem

The existence of a simple circuit of length k, $k \geq 3$ is an isomorphism invariant for simple graphs.

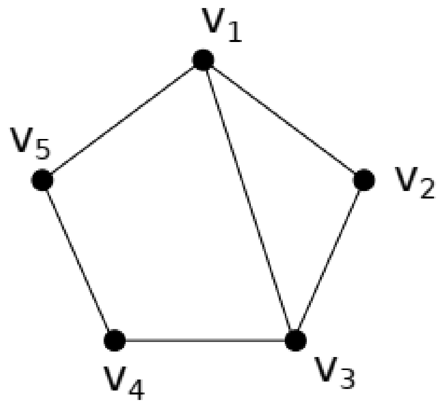
Proof: Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be isomorphic graphs: there is a bijective function $f : V_1 \rightarrow V_2$ respecting adjacency conditions.
Assume G_1 has a simple circuit of length k: $u_0, u_1, \ldots, u_k = u_0$, with $u_i \in V_1$ for $0 \leq i \leq k$. Let's denote $v_i = f(u_i)$, for $0 \leq i \leq k$.

$$(u_i, u_{i+1}) \in E_1 \Rightarrow (f(u_i), f(u_{i+1})) = (v_i, v_{i+1}) \in E_2, \text{ for } 0 \leq i \leq k-1.$$

So $v_0, \ldots, v_k$ is a path of length k in G_2.
It is a circuit because $v_k = f(u_k) = f(u_0) = v_0$.
It is simple: if not, at least one edge is traversed more than once, so it would mean that there exist $0 \leq i \neq j \leq k-1$ such that $(v_i, v_{i+1}) = (v_j, v_{j+1})$. But this implies $(u_i, u_{i+1}) = (u_j, u_{j+1})$ by bijectivity of f. This is impossible because $u_0, u_1, \ldots, u_k$ is simple.



G



H

5 vertices, 6 edges
Degree sequence: 3, 3, 2, 2, 2
1 simple circuit of length 3,
1 simple circuit of length 4,
1 simple circuit of length 5.

Isomorphic graphs ?

If there is an iso $f : V_G \to V_H$, the simple circuit of length 5 u_1, u_4, u_3, u_2, u_5 must be sent to the simple circuit of length 5 in H, respecting the degrees of vertices.
Check that $f(u_1) = v_1, f(u_4) = v_2, f(u_3) = v_3, f(u_2) = v_4, f(u_5) = v_5$ is an isomorphism by writing adjacency matrices.

Counting Paths Between Vertices

Theorem

Let G be a graph with adjacency matrix A with respect to the ordering of vertices $v_1, \ldots, v_n$. The number of different paths of length $r \geq 1$ from v_i to v_j equals the (i, j) entry of the matrix A^r.

Proof: By induction

• $r = 1$: the number of paths of length 1 from v_i to v_j is equal to the (i, j) entry of A by definition of A, as it corresponds to the number of edges from v_i to v_j.

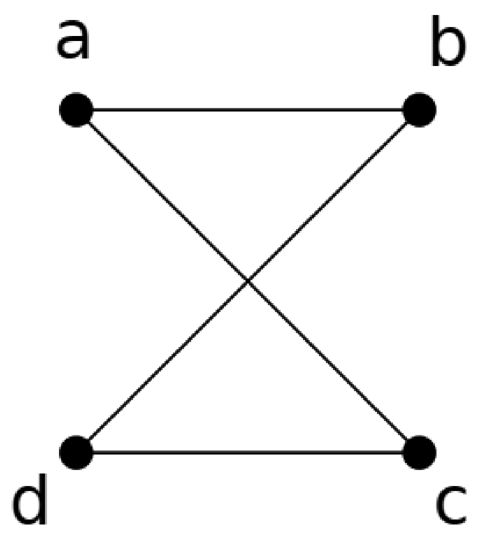
• Assume the (i,j) entry of the matrix A^r is the number of different paths of length r from v_i to v_j.
We can write $A^{r+1} = A^r A$
Let's denote $A^r = (b_{ij})_{1\le i,j\le n}$, and $A = (a_{ij})_{1\le i,j\le n}$. The (i,j) entry of A^{r+1} is given by:

$$\sum_{k=1}^{n} b_{ik}a_{kj} = b_{i1}a_{1j} + b_{i2}a_{2j} + \cdots + b_{in}a_{nj} \tag{1}$$

By hypothesis: b_{ik} equals the number of paths of length r from v_i to v_k.
"Path of length $r+1$ from v_i to v_j = path of length r from v_i to any vertex v_k + an edge from v_k to v_j."
This is equal to the sum (1).

Example

How many paths of length four are there from a to d in the simple graph G



G

with ordering of vertices (a, b, c, d, e):

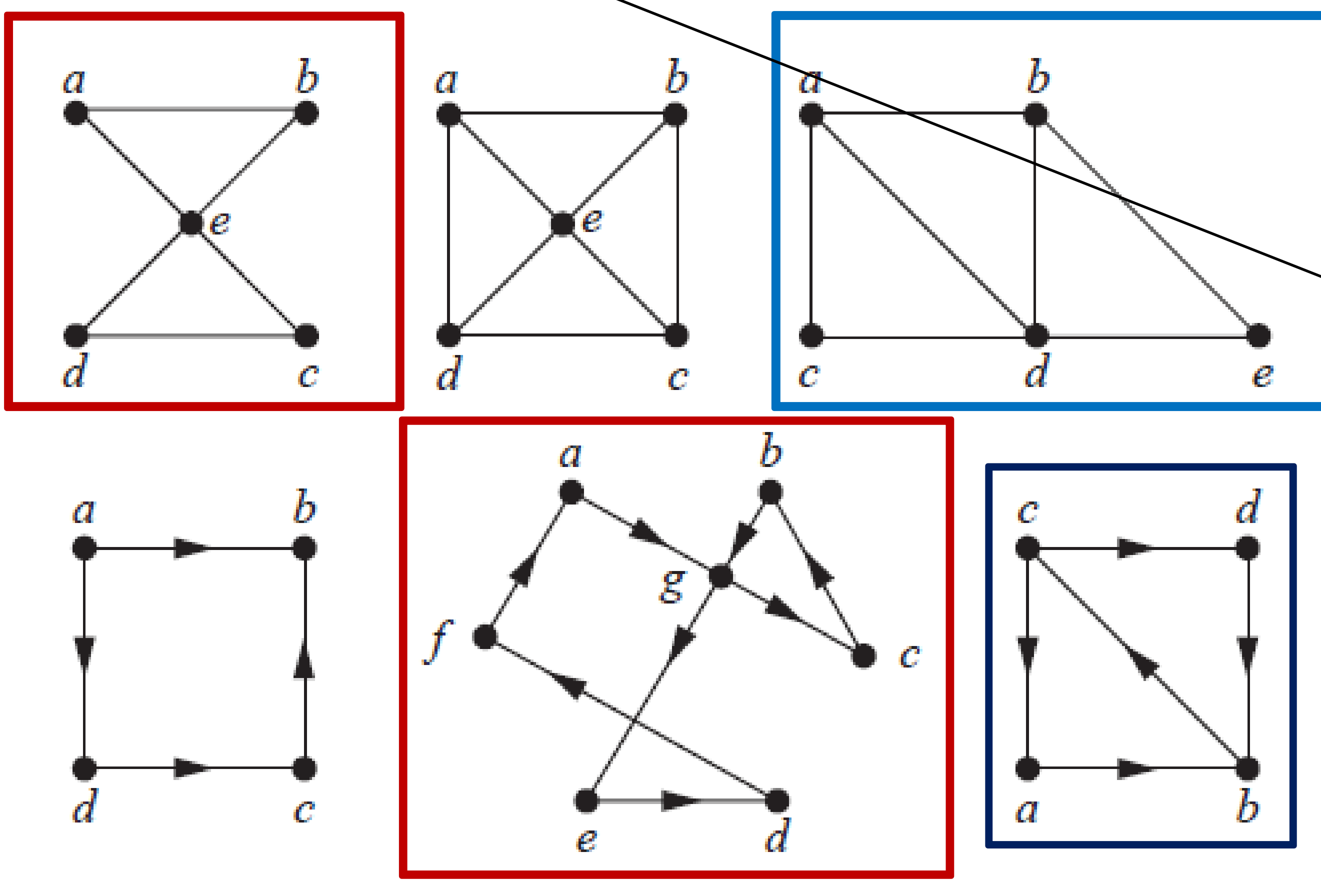
$$A_G = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

$$A_G^2 = \begin{pmatrix} 2 & 0 & 0 & 2 \\ 0 & 2 & 2 & 0 \\ 0 & 2 & 2 & 0 \\ 2 & 0 & 0 & 2 \end{pmatrix} \quad A_G^3 = \begin{pmatrix} 0 & 4 & 4 & 0 \\ 4 & 0 & 0 & 4 \\ 4 & 0 & 0 & 4 \\ 0 & 4 & 4 & 0 \end{pmatrix} \quad A_G^4 = \begin{pmatrix} 8 & 0 & 0 & 8 \\ 0 & 8 & 8 & 0 \\ 0 & 8 & 8 & 0 \\ 8 & 0 & 0 & 8 \end{pmatrix}$$

Euler Paths and Circuits

DEFINITION: Let $G = (V, E)$ be a graph.

- **Euler Path**欧拉路径: a simple path that traverses every edge of G.
- **Euler Circuit**欧拉回路: a simple circuit that traverses every edge of G.



Remark: When G has multiple edges, these edges will be given different names and considered as different. This is implicit in the textbook.

Euler Circuits

THEOREM: Let $G = (V, E)$ be a connected multigraph of order ≥ 2.

Then G has an Euler circuit iff $2|\deg(x)$ for every $x \in V$.

- $\Rightarrow$: Let P: $\{x_0, x_1\}, \ldots, \{x_{i-1}, x_i\}, \ldots, \{x_{n-1}, x_n\}$ be an Euler circuit, $x_0 = x_n$
  - Every occurrence of x_i in P contributes 2 to $\deg(x_i)$
    - Every vertex x_i has an even degree
- $\Leftarrow$: Let P: $\{x_0, x_1\}, \ldots, \{x_{n-1}, x_n\}$ be a longest simple path in G.
  - Let $H = G[P]$, the subgraph of G induced by all edges in P
    - If $x_n \neq x_0$, then $\deg_H(x_n)$ is odd and so P cannot be longest.
      - $x_n = x_0$, P is a simple circuit, and $2|\deg_H(x_i)$ for all i.
    - If $\exists i \in \{0,1, \ldots, n-1\}$ such that $\deg_H(x_i) < \deg_G(x_i)$,
      - then $\exists y \in V$ such that $\{x_i, y\} \notin P$
        - $y, x_i, x_{i+1}, \ldots, x_n, x_1, \ldots, x_{i-1}, x_i$ is longer than P
    - Hence, $\deg_H(x_i) = \deg_G(x_i)$ for all $i \in \{0,1, \ldots, n-1\}$.
      - $V = \{x_0, x_1, \ldots, x_{n-1}\}$ and $H = G$.
        - P is an Euler circuit.

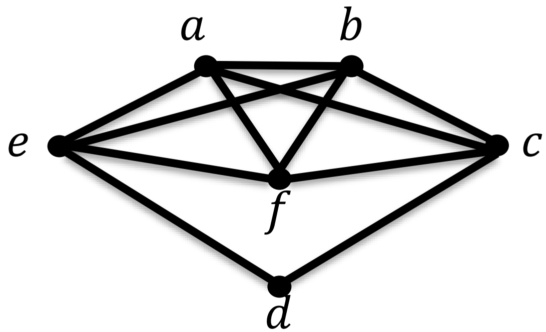
Remark: H contains all vertices of G. Otherwise, P can be extended.

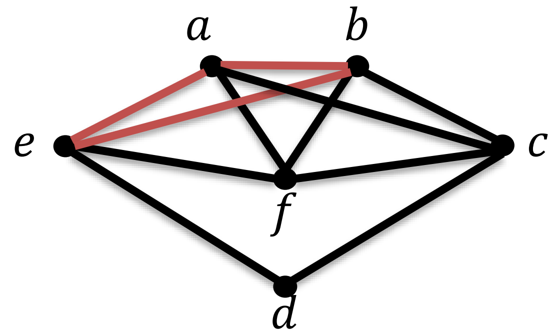
Construction

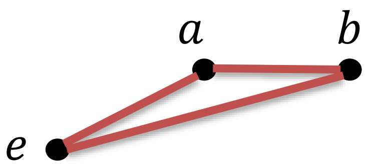
ALGORITHM (Hierholzer):

- **Input**: $G = (V, E)$, a connected multigraph, $2|\deg(x), \forall x \in V$
- **Output**: an Euler circuit
  - $\mathbf{circuit} := $ a circuit in G
  - $H := G - \mathbf{circuit} - $ isolated vertices
  - while H has edges do
    - $\mathbf{subcircuit} := $ a circuit in H that intersects $\mathbf{circuit}$
    - $H := H - \mathbf{subcircuit} - $ isolated vertices
    - $\mathbf{circuit} := \mathbf{circuit} \cup \mathbf{subcircuit}$
  - return $\mathbf{circuit}$

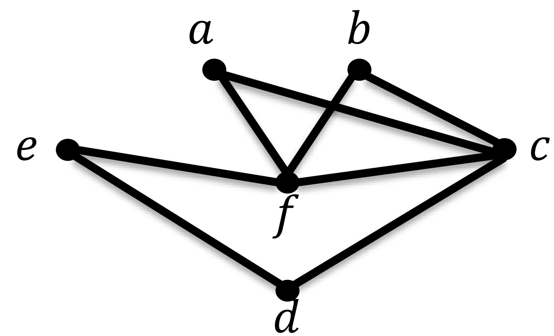
Example





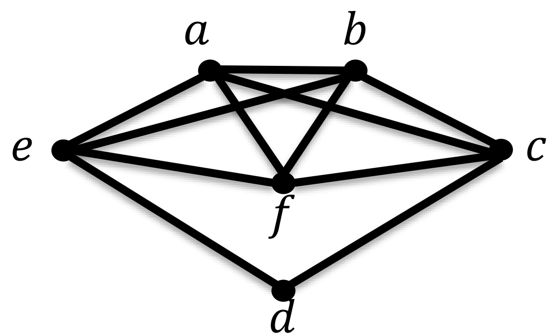


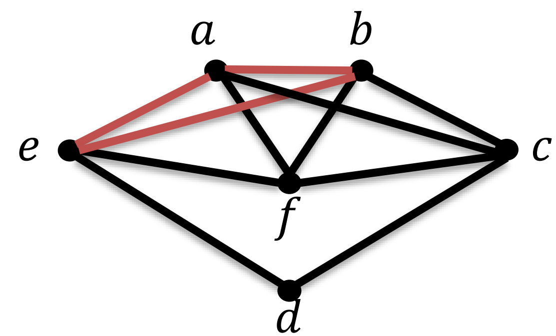
circuit $= a, b, e, a$

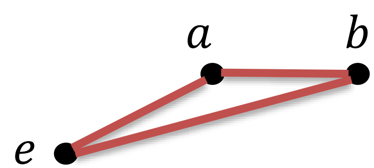


H

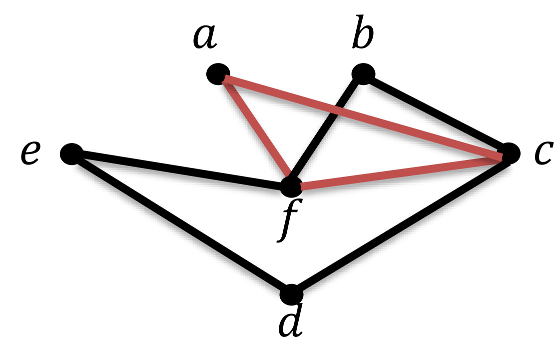
Example







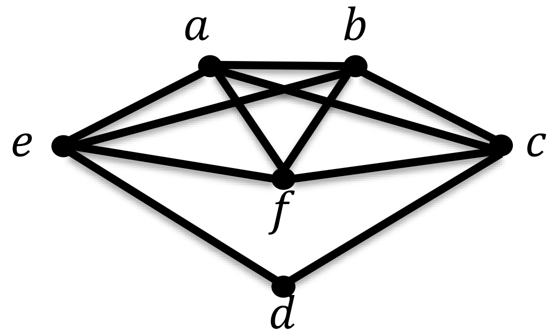
circuit $= a, b, e, a$

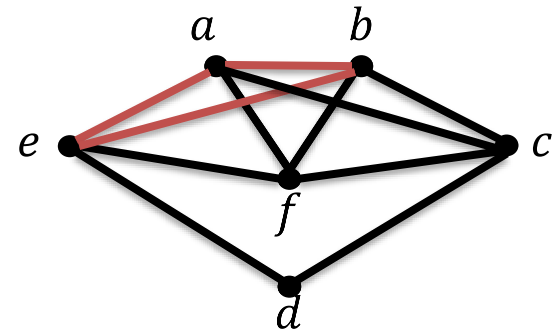


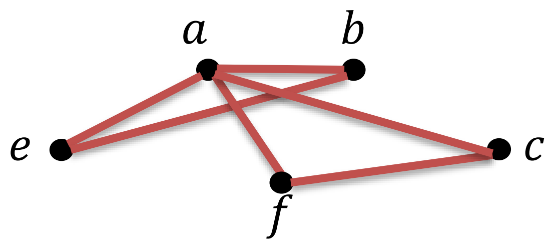
H

subcircuit $= a, c, f, a$

Example

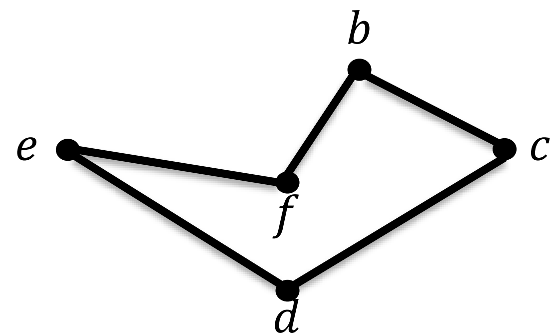






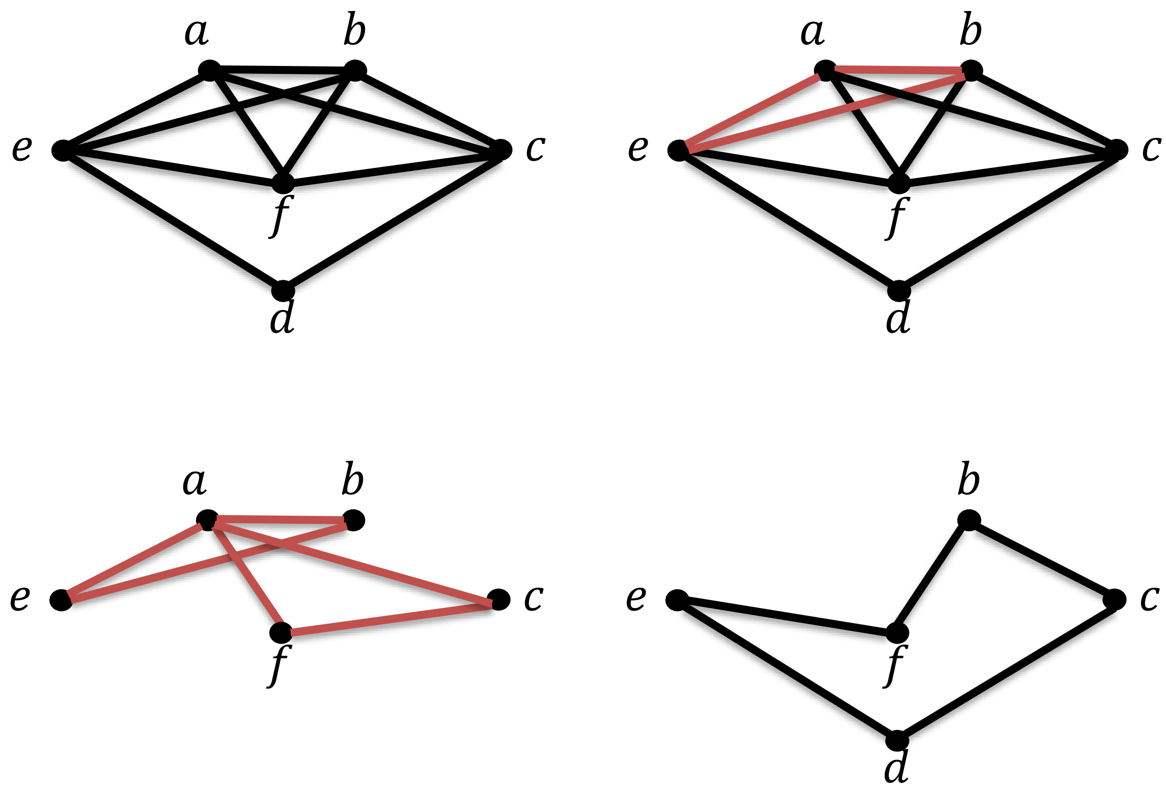
circuit $= a, b, e, a$

subcircuit $= a, c, f, a$



H

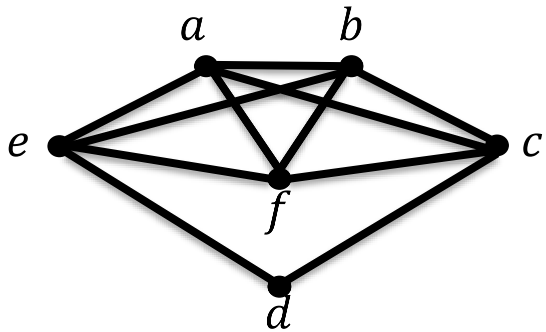
Example

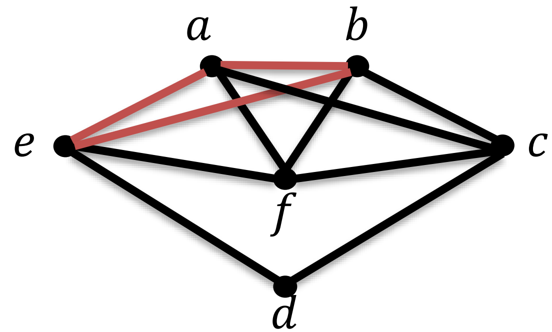


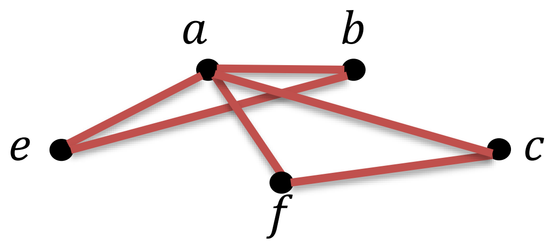
circuit $= a, b, e, a, c, f, a$

H

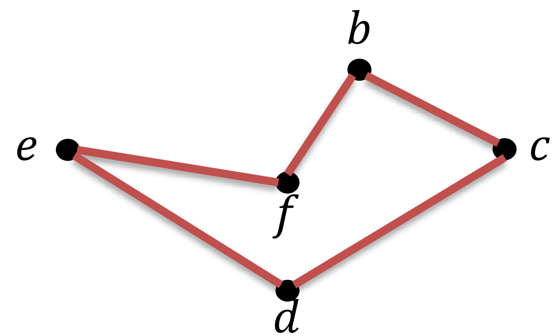
Example







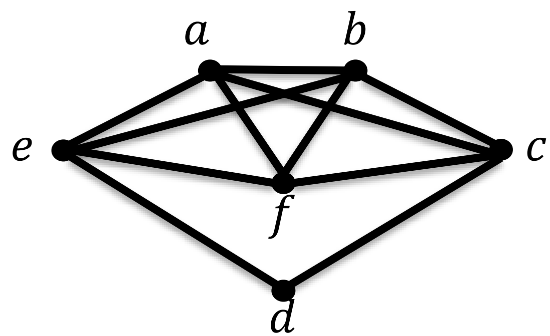
circuit $= a, b, e, a, c, f, a$

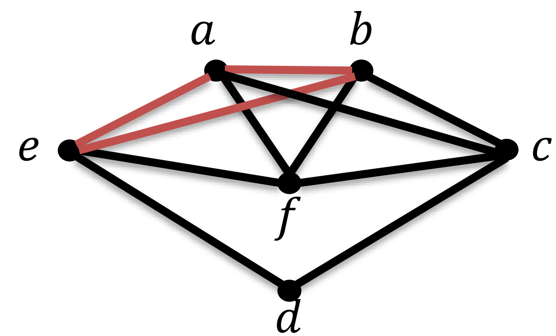


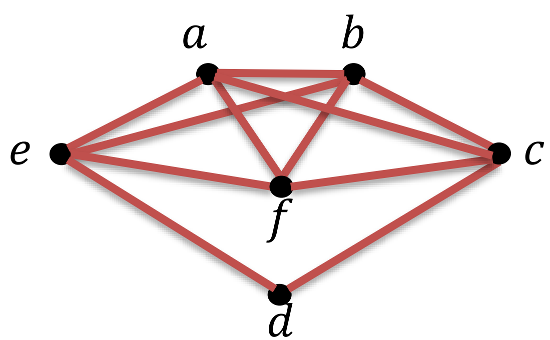
H

subcircuit $= c, d, e, f, b, c$

Example





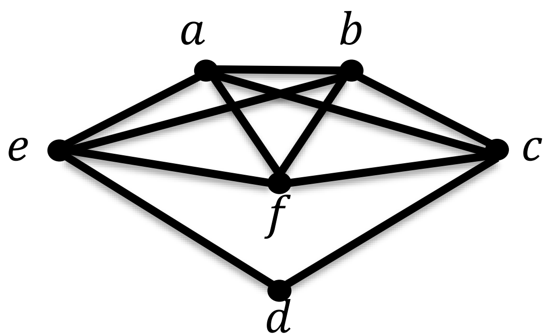


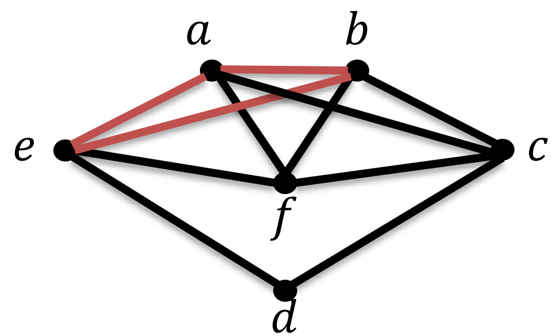
circuit $= a, b, e, a, c, f, a$

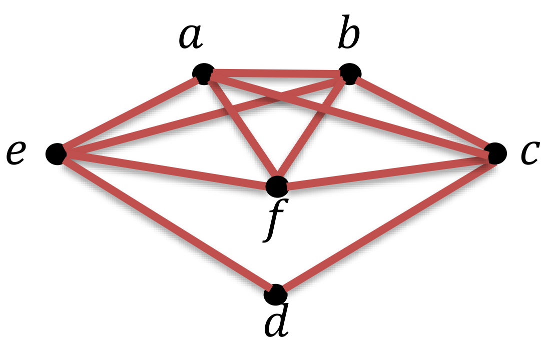
H

subcircuit $= c, d, e, f, b, c$

Example







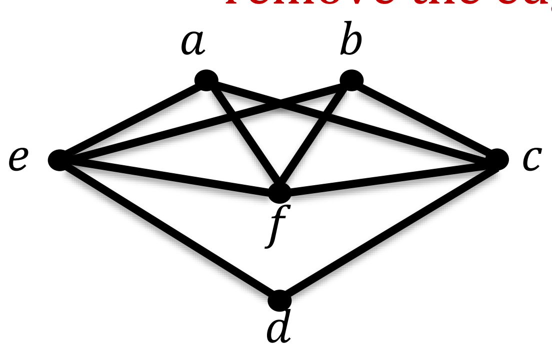
circuit $= a, b, e, a, c, d, e, f, b, c, f, a$

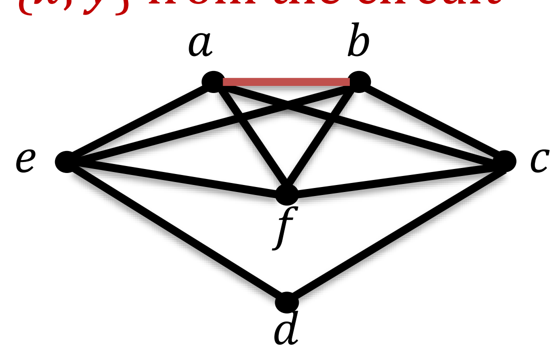
Euler Paths

THEOREM: Let $G = (V, E)$ be a connected multigraph of order ≥ 2. Then G has an Euler path (not Euler circuit) iff G has exactly 2 vertices of odd degree.

ALGORITHM:

- **Input**: $G = (V, E)$, a connected multigraph, $x, y \in V$ have odd degrees
- **Output**: an Euler path
  - $H \coloneqq G + \{x, y\}$
  - $\text{find an Euler circuit using Hierholzer's algorithm}$
  - $\text{remove the edge } \{x, y\} \text{ from the circuit}$





$a, c, d, e, f, b, a, e, b, c, f, a$

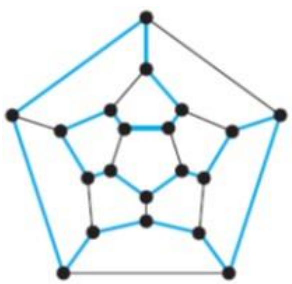
$a, c, d, e, f, b, a, e, b, c, f, a$

$a, e, b, c, f, a, c, d, e, f, b$

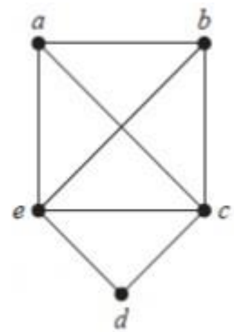
Hamilton Paths and Circuits

DEFINITION: Let $G = (V, E)$ be a graph.

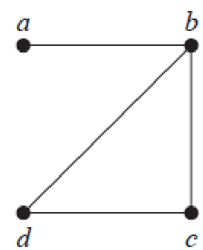
- **Hamilton Path**: A simple path that passes through every vertex exactly once.
- **Hamilton Circuit:** A simple circuit that passes through every vertex exactly once.



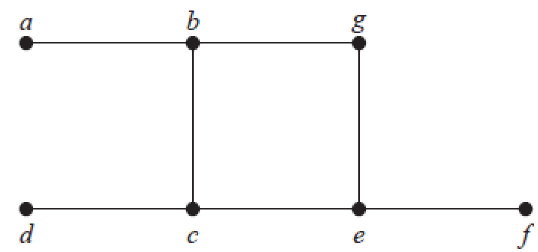
√ Hamilton path
√ Hamilton circuit



√ Hamilton path
√ Hamilton circuit



√ Hamilton path
×Hamilton circuit



× Hamilton path
× Hamilton circuit

Hamilton Circuits

Determine if there is a Hamilton circuit in a given graph G?

- This problem is NP-Complete. //that means very difficult

Necessary conditions on Hamilton circuit.

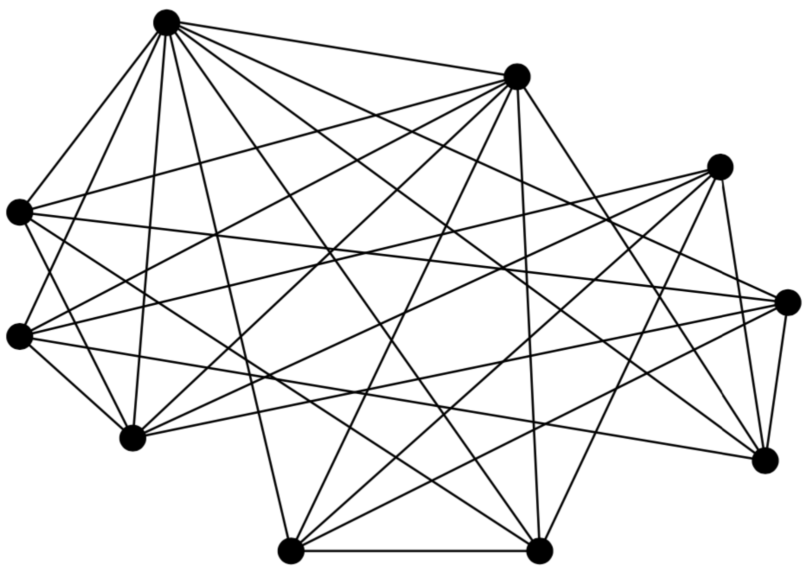
- If G has a vertex of degree 1, then G cannot have a Hamilton circuit.
- If G has a vertex of degree 2, then a Hamilton circuit of G traverses both edges.

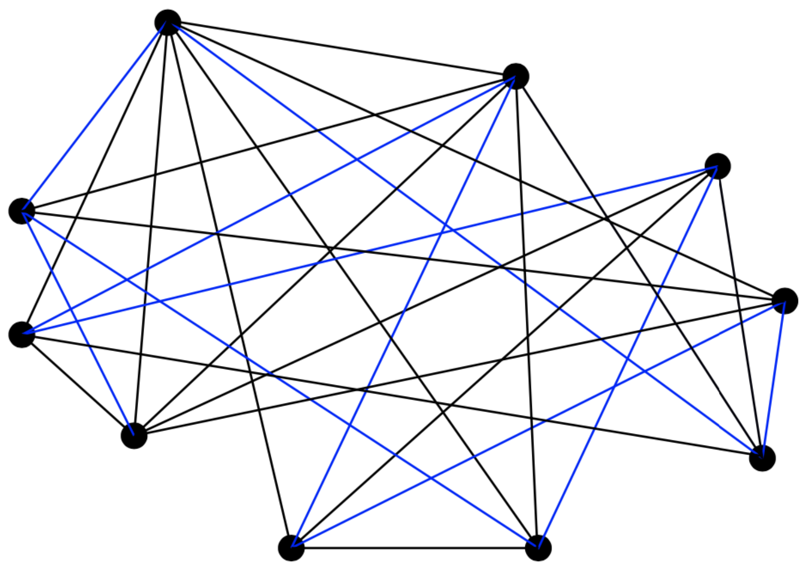
Sufficient conditions on Hamilton circuit.

- **Ore's Theorem:** Let $G = (V, E)$ be a simple graph of order $n \geq 3$. If $\deg(u) + \deg(v) \geq n$ for all $\{u, v\} \notin E$, then G has a Hamilton circuit.
- **Dirac's Theorem:** Let $G = (V, E)$ be a simple graph of order $n \geq 3$. If $\deg(u) \geq n/2$ for every $u \in V$, then G has a Hamilton circuit.
  - This is a corollary of Ore's Theorem
    - $\forall u \in V,\ \deg(u) \geq n/2 \Rightarrow \forall u, v \in V, \deg(u) + \deg(v) \geq n$

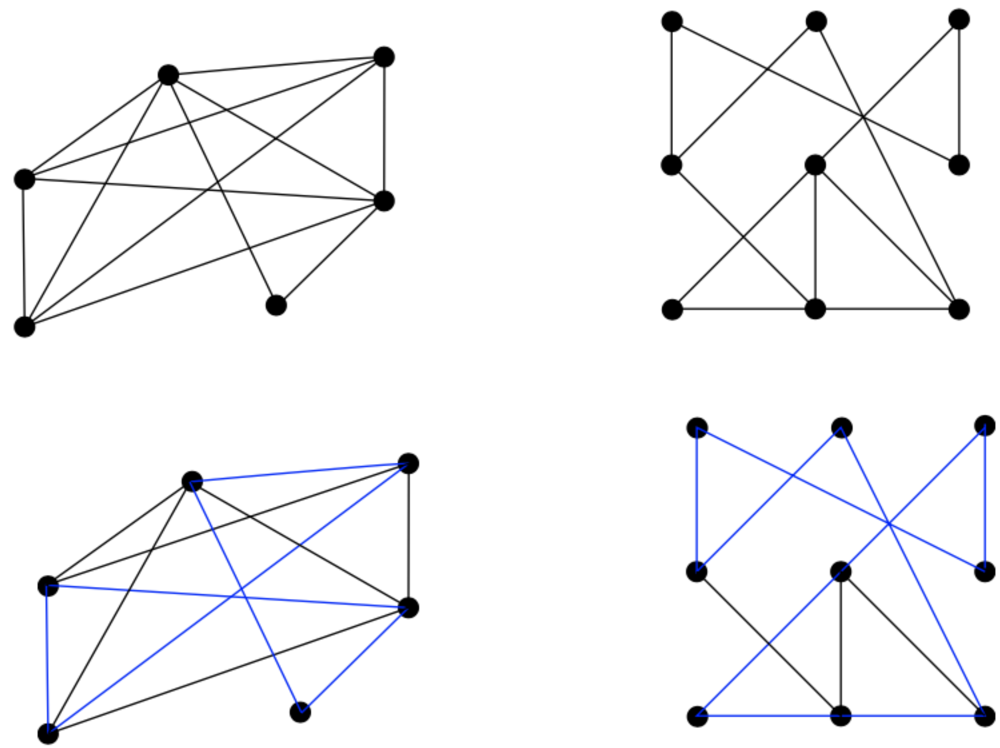
Hamilton Circuits

- Examples (sufficient condition)





Hamilton Circuits

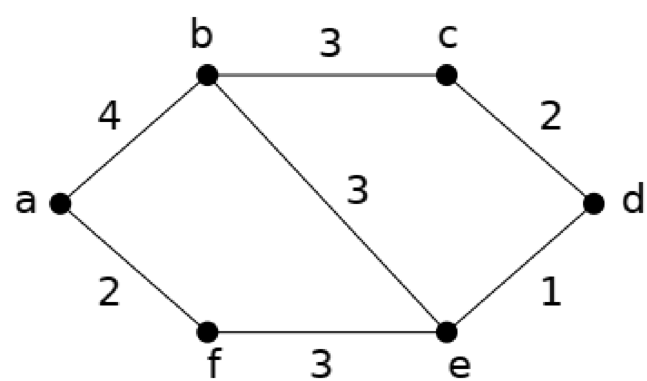


Remark: Dirac's and Ore's Theorems do not give a necessary condition for the existence of a Hamilton circuit!

Shortest Path Problem

Definition

A **weighted graph** is a graph $G = (V, E)$ such that each edge is assigned with a strictly positive number.
The **length** of a path in weighted graph is the sum of the weights of the edges of this path.

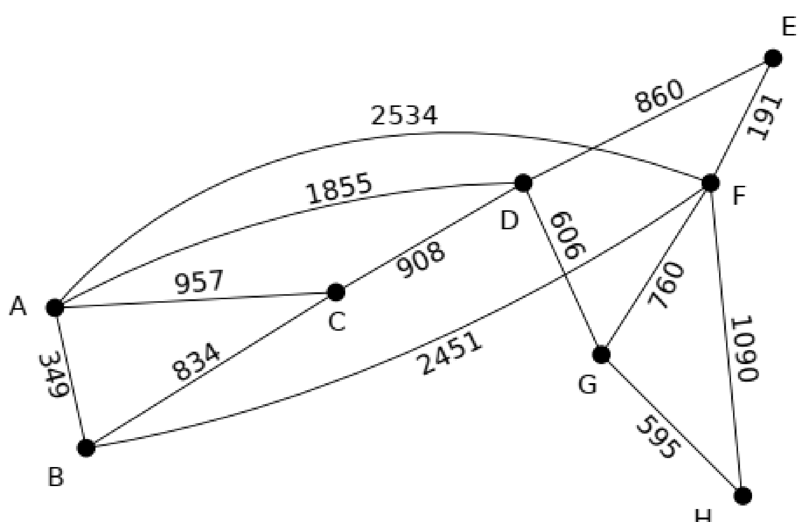


a, b, c is a path of length 7 and b, e, d, c is a path of length 6

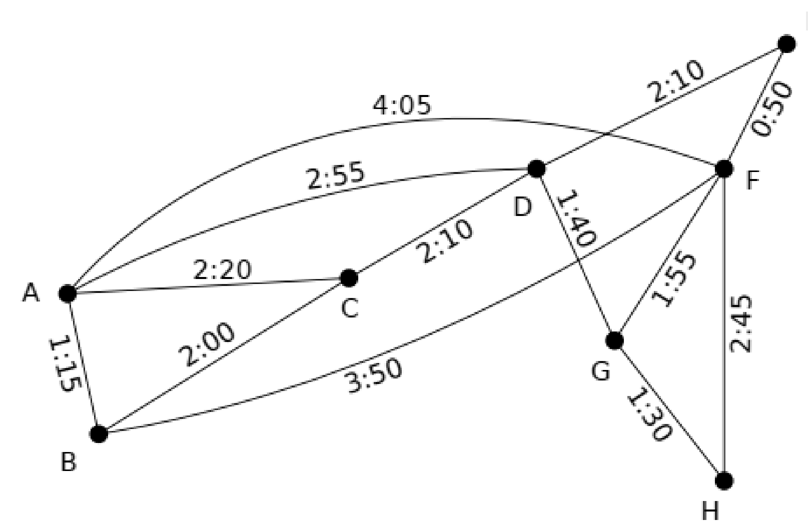
Remark: Observe that in a non-weighted graph the length of a path is the number of edges in the path!

Shortest Path Problem

Examples



distance

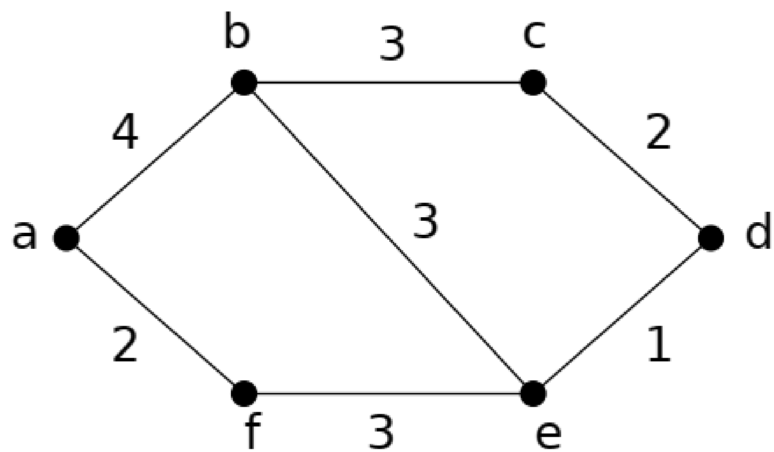


time

What is the shortest path in air distance between cities A and E?
What combination of flights has the smallest total flight time?

Shortest Path Problem

Question: Find the shortest path from a to d.

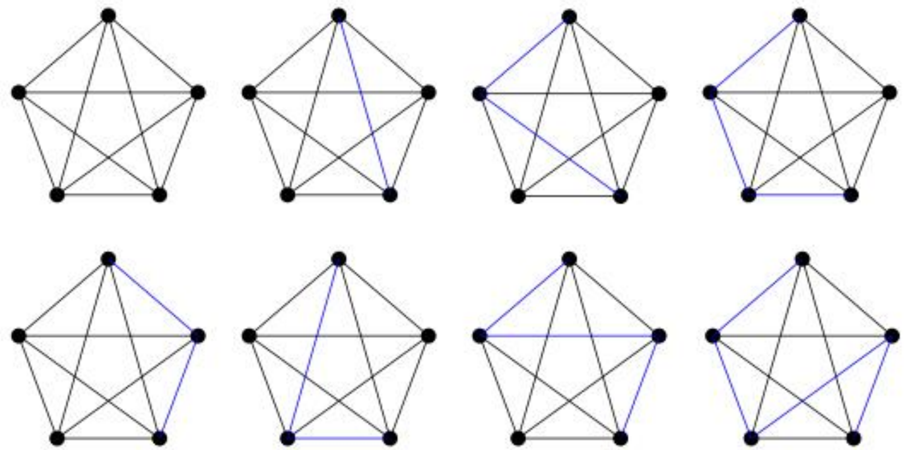


Method: Find the closest vertex to a, then the second closest, the third closest... until we reach d.
$\Rightarrow$ Dijkstra's algorithm

Shortest Path Problem

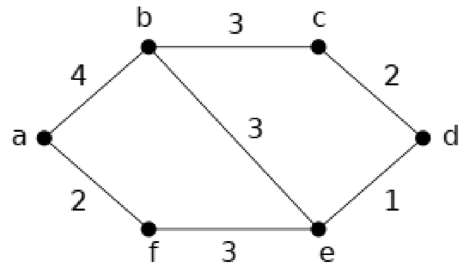
Remarks:

- Of course in the example above, we could have looked at all the paths between a and d and compute their length, but too complicated if the graph has a lot of edges.



- Advantage of Dijkstra's algorithm: we can compute the length of a shortest path from one vertex to all other vertices of the graph.

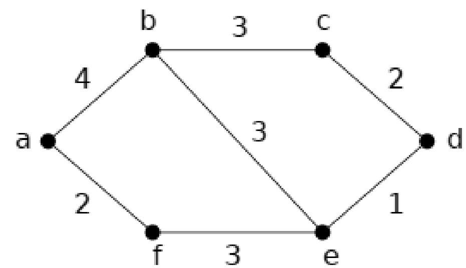
Djikstra's Algorithm



1. Find the closest vertex to $a \rightsquigarrow$ analyse all the edges starting from a:
   a, b of length 4
   a, f of length 2
   $\Rightarrow$ f is the closest vertex to a. The shortest path from a to f has length 2.

2. Find the second closest vertex to $a \rightsquigarrow$ shortest paths from a to a vertex in $\{a, f\}$ followed by an edge from a vertex in $\{a, f\}$ to a vertex not in this set:
   a, b of length 4
   a, f, e of length 5
   $\Rightarrow$ b is the second closest vertex to a. The shortest path from a to b has length 4.

Djikstra's Algorithm



3. Find the third closest vertex to $a \rightsquigarrow$ shortest path from a to a vertex in $\{a, f, b\}$ followed by an edge from a vertex in $\{a, f, b\}$ to a vertex not in this set:
   a, b, c of length 7
   a, b, e of length 7
   a, f, e of length 5
   $\Rightarrow$ e is the third closest vertex to a. The shortest path from a to e has length 5.

4. Find the fourth closest vertex to $a \rightsquigarrow$ shortest path from a to a vertex in $\{a, f, b, e\}$ followed by an edge from a vertex in $\{a, f, b, e\}$ to a vertex not in this set:
   a, b, c of length 7
   a, f, e, d of length 6
   $\Rightarrow$ d is the fourth closest vertex to a. The shortest path from a to d has length 6.

Djikstra's Algorithm

Goal: find the length of a shortest path from a to z with a series of iterations.

- A distinguished set of vertices is constructed by adding one vertex at each iteration.
- A labeling procedure is carried out at each iteration: a vertex w is labeled with the length of a shortest path from a to w that contains only vertices in the distinguished set.
- The vertex added to the distinguished set is one with minimal label among those vertices not already in the set.

Notations: $S_k :=$ distinguished set after k iterations, $L_k(v) :=$ length of a shortest path from a to v containing only vertices in S_k ("label" of v).

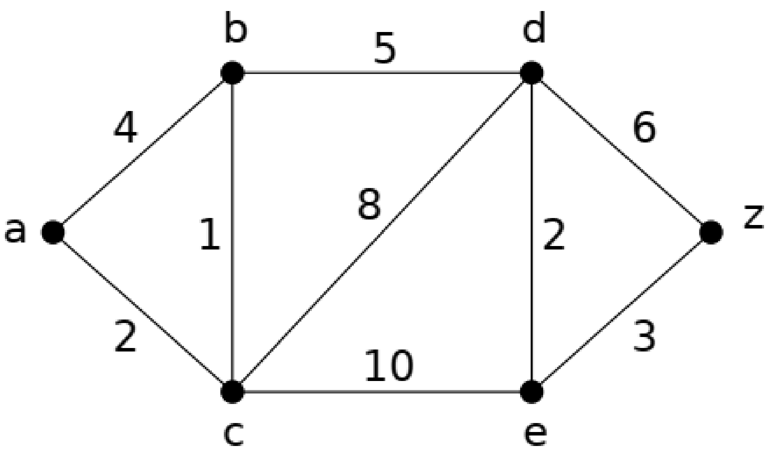
Initialization: $L_0(a) = 0$,
$L_0(v) = \infty$ for every vertex $v \neq a$,
$S_0 = \emptyset$.

***k*th iteration:**

- S_k is formed from S_{k-1} by adding a vertex u not in S_{k-1} with smallest label,
- Update the labels of all vertices not in S_k so that $L_k(v)$ is the length of a shortest path from a to v containing only vertices in S_k, i.e.

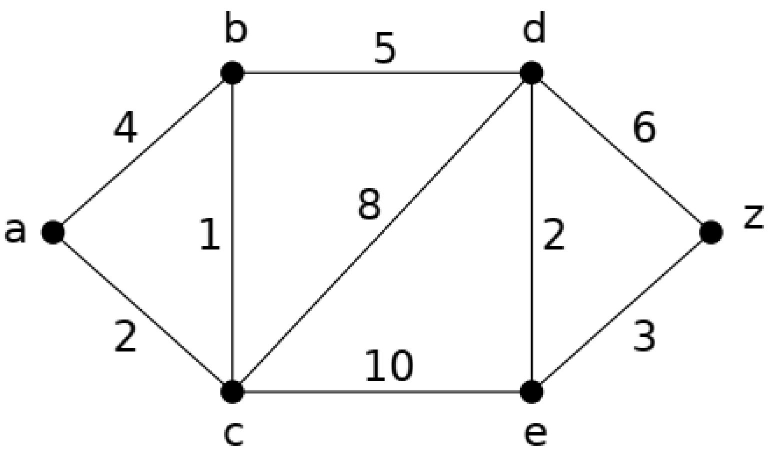
$L_k(v) = min\{L_{k-1}(v), L_{k-1}(u) + w(u, v)\}$ (with $w(u, v)$ length of the edge (u, v))

Djikstra's Algorithm



- **k=0 (initialization):** $S_0 = \emptyset$, $L_0(a) = 0$, $L_0(b) = L_0(c) = L_0(d) = L_0(e) = L_0(z) = \infty$

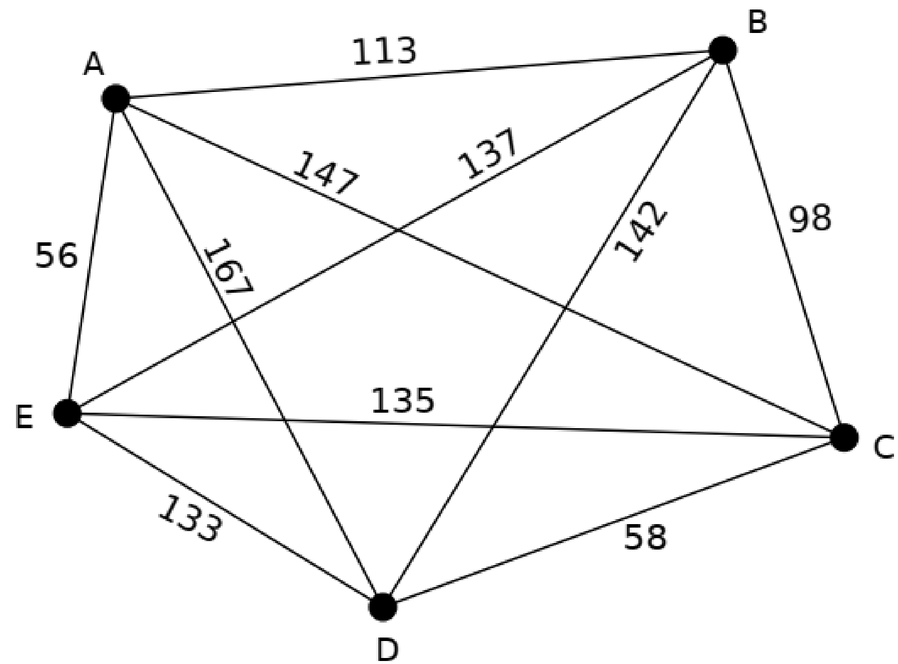
Djikstra's Algorithm



- **k=0 (initialization):** $S_0 = \emptyset$, $L_0(a) = 0$, $L_0(b) = L_0(c) = L_0(d) = L_0(e) = L_0(z) = \infty$

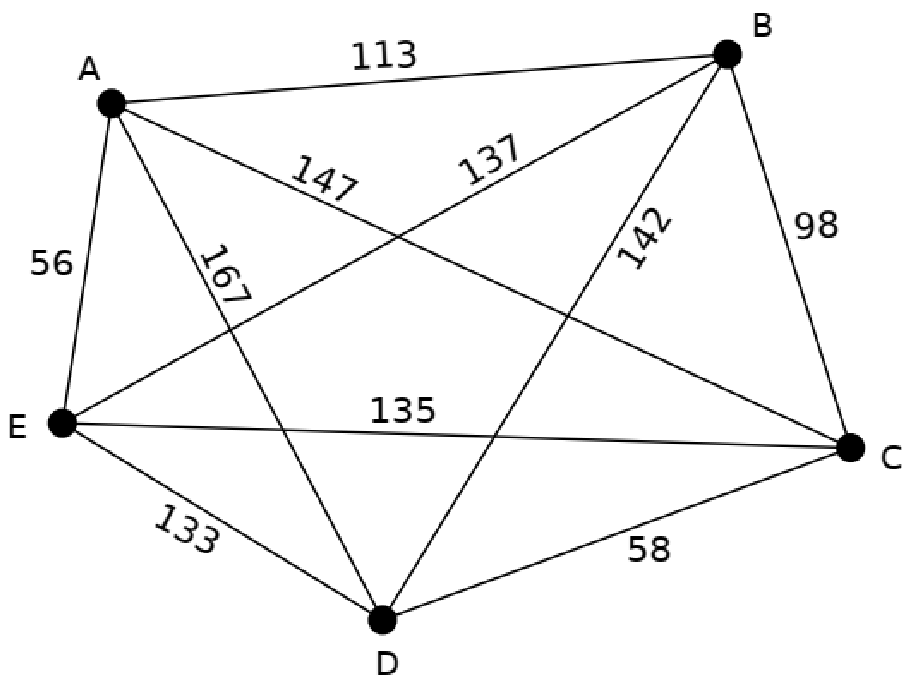
- **k=1:** $u := a \rightsquigarrow S_1 = \{a\}$,
  $L_0(a) + w(a,b) = 4 < L_0(b) \rightsquigarrow L_1(b) = 4$
  $L_0(a) + w(a,c) = 2 < L_0(c) \rightsquigarrow L_1(c) = 2$
- **k=2:** $u := c \rightsquigarrow S_1 = \{a, c\}$,
  $L_1(c) + w(c,b) = 3 < L_1(b) \rightsquigarrow L_2(b) = 3$
  $L_1(c) + w(c,d) = 10 < L_1(d) \rightsquigarrow L_2(d) = 10$
  $L_1(c) + w(c,e) = 12 < L_1(e) \rightsquigarrow L_2(e) = 12$
- **k=3:** $u := b \rightsquigarrow S_1 = \{a, c, b\}$,
  $L_2(b) + w(b,d) = 8 < L_2(d) \rightsquigarrow L_3(d) = 8$
- **k=4:** $u := d \rightsquigarrow S_1 = \{a, c, b, d\}$,
  $L_3(d) + w(d,e) = 10 < L_3(e) \rightsquigarrow L_4(e) = 10$
  $L_3(d) + w(d,z) = 14 < L_3(z) \rightsquigarrow L_4(z) = 14$
- **k=5:** $u := e \rightsquigarrow S_1 = \{a, c, b, d, e\}$,
  $L_4(e) + w(e,z) = 13 < L_4(z) \rightsquigarrow L_5(z) = 13$
- **k=6:** $u := z \rightsquigarrow S_1 = \{a, c, b, d, z\}$,
- **return:** $L(z) = 13$

Traveling Salesperson Problem



Traveling salesperson problem: a traveling salesperson wants to visit each of the cities once and return to his starting point. In which order should he visit these cities to travel the minimum total distance?
$\Rightarrow$ **Hamiltonian circuit with minimum total weight in the complete graph.**

Traveling Salesperson Problem



Route	Tot. dist.
A, B, C, D, E, A	610
A, B, C, E, D, A	516
A, B, E, D, C, A	588
A, B, E, C, D, A	458
A, B, D, E, C, A	540
A, B, D, C, E, A	504
A, D, B, C, E, A	598
A, D, B, E, C, A	576
A, D, E, B, C, A	682
A, D, C, B, E, A	646
A, C, D, B, E, A	670
A, C, B, D, E, A	728

Traveling salesperson problem: a traveling salesperson wants to visit each of the cities once and return to his starting point. In which order should he visit these cities to travel the minimum total distance?
$\Rightarrow$ **Hamiltonian circuit with minimum total weight in the complete graph.**