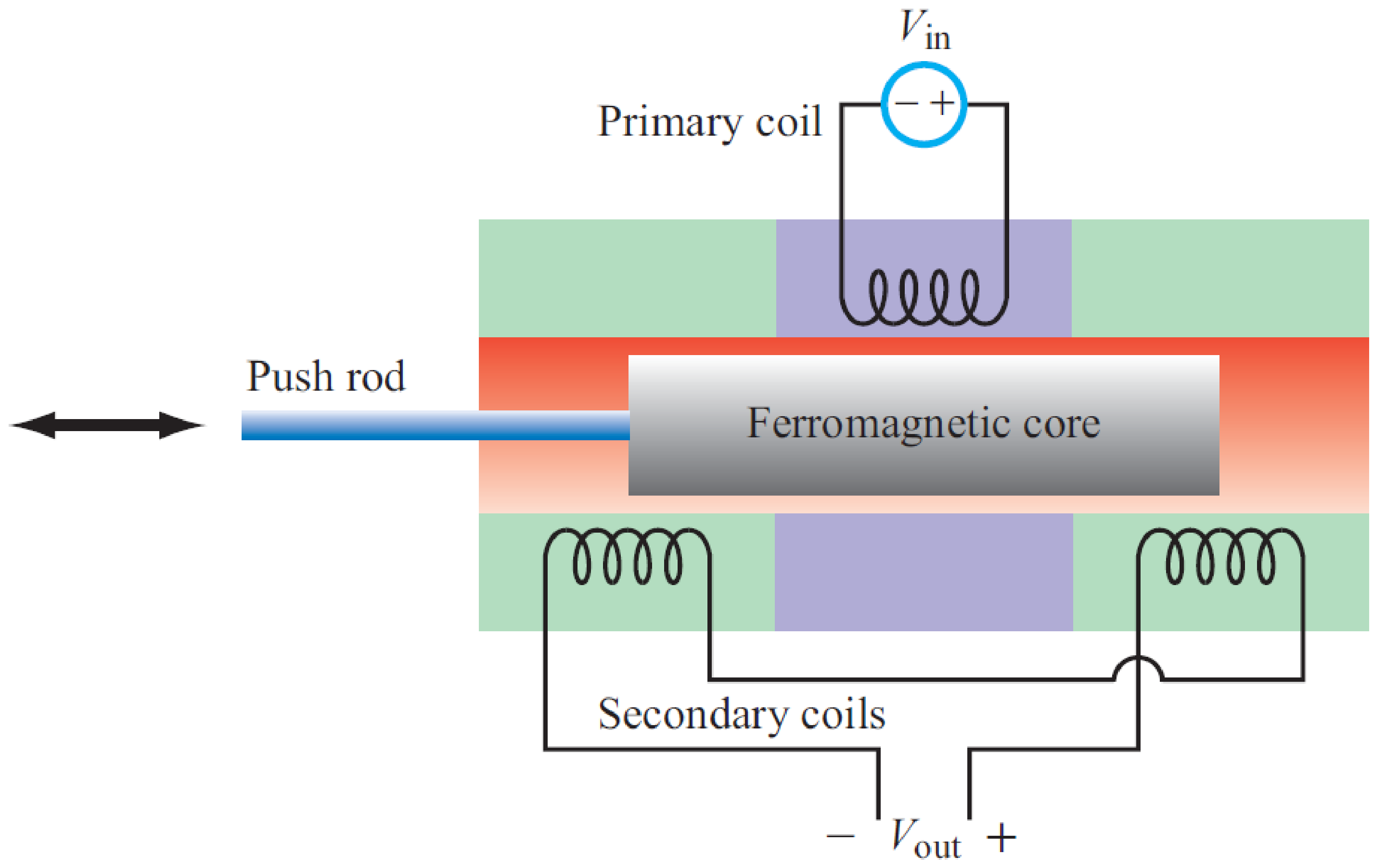


Figure TF11-1: Linear variable differential transformer (LVDT) circuit.

5. MAGNETOSTATICS

7e Applied EM by Ulaby and Ravaioli

Chapter 5 Overview

Chapter Contents

Objectives

Upon learning the material presented in this chapter, you should be able to:

1. Calculate the magnetic force on a current-carrying wire placed in a magnetic field and the torque exerted on a current loop.
2. Apply the Biot–Savart law to calculate the magnetic field due to current distributions.
3. Apply Ampère's law to configurations with appropriate symmetry.
4. Explain magnetic hysteresis in ferromagnetic materials.
5. Calculate the inductance of a solenoid, a coaxial transmission line, or other configurations.
6. Relate the magnetic energy stored in a region to the magnetic field distribution in that region.

Electric vs Magnetic Comparison

Table 5-1: Attributes of electrostatics and magnetostatics.

Attribute	Electrostatics	Magnetostatics
Sources	Stationary charges ρ_v	Steady currents $\mathbf{J}$
Fields and Fluxes	$\mathbf{E}$ and $\mathbf{D}$	$\mathbf{H}$ and $\mathbf{B}$
Constitutive parameter(s)	ε and σ	μ
Governing equations		
• Differential form	$\nabla \cdot \mathbf{D} = \rho_v$ $\nabla \times \mathbf{E} = 0$	$\nabla \cdot \mathbf{B} = 0$ $\nabla \times \mathbf{H} = \mathbf{J}$
• Integral form	$\oint_S \mathbf{D} \cdot d\mathbf{s} = Q$ $\oint_C \mathbf{E} \cdot d\mathbf{l} = 0$	$\oint_S \mathbf{B} \cdot d\mathbf{s} = 0$ $\oint_C \mathbf{H} \cdot d\mathbf{l} = I$
Potential	Scalar V, with $\mathbf{E} = -\nabla V$	Vector $\mathbf{A}$, with $\mathbf{B} = \nabla \times \mathbf{A}$
Energy density	$w_e = \frac{1}{2}\varepsilon E^2$	$w_m = \frac{1}{2}\mu H^2$
Force on charge q	$\mathbf{F}_e = q\mathbf{E}$	$\mathbf{F}_m = q\mathbf{u} \times \mathbf{B}$
Circuit element(s)	C and R	L

Electric & Magnetic Forces

Note the sign of q

- Magnetic force $\quad \mathbf{F}_{\mathrm{m}} = q\mathbf{u} \times \mathbf{B} \qquad \text{(N)}$

Tesla(T)

- Electromagnetic (Lorentz) force

$$\mathbf{F} = \mathbf{F}_{\mathrm{e}} + \mathbf{F}_{\mathrm{m}} = q\mathbf{E} + q\mathbf{u} \times \mathbf{B} = q(\mathbf{E} + \mathbf{u} \times \mathbf{B}).$$

- Three PMW differences between $\mathbf{F}_e$ and $\mathbf{F}_m$

NOT BMW

1. Whereas the electric force is always in the direction of the electric field, the magnetic force is always perpendicular to the magnetic field. **Perpendicular**
2. Whereas the electric force acts on a charged particle whether or not it is moving, the magnetic force acts on it only when it is in motion. **Moving**
3. Whereas the electric force expends energy in displacing a charged particle, the magnetic force does no work when a particle is displaced. **Work**

(because $\mathbf{F}_{\mathrm{m}} \perp \boldsymbol{u}$, *only moving direction changes, not speed*)

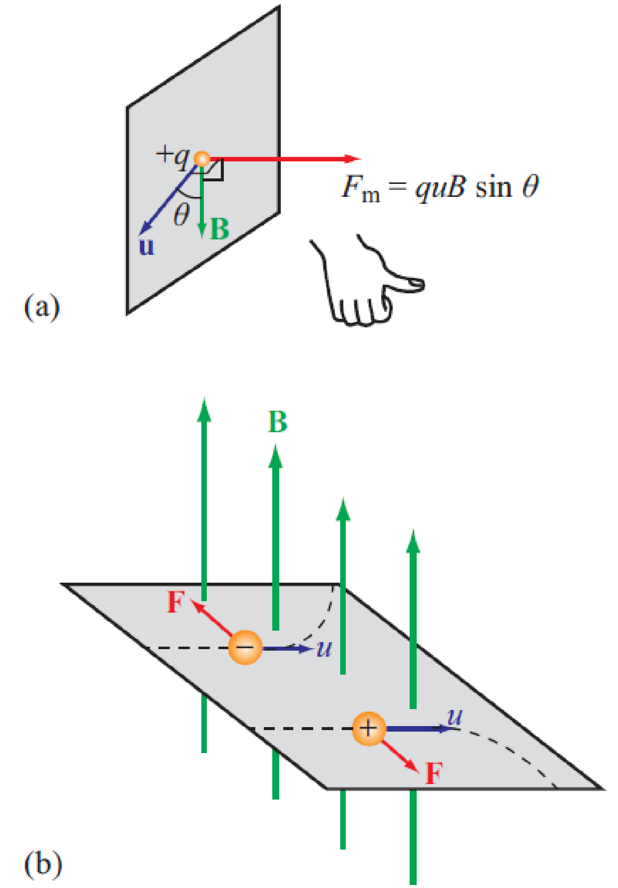


Figure 5-1: The direction of the magnetic force exerted on a charged particle moving in a magnetic field is (a) perpendicular to both **B** and **u** and (b) depends on the charge polarity (positive or negative).

Magnetic Force on a Current Element

$d\mathbf{l}$ is the dispacement vector in the direction of current

- Differential force $d\mathbf{F}_{\mathrm{m}}$ on a differential current $Id\boldsymbol{l}$:

$$d\mathbf{F}_{\mathrm{m}} = \frac{dq}{dt}(\boldsymbol{u}dt) \times \mathbf{B} = I d\boldsymbol{l} \times \mathbf{B} \quad \text{(N)} \qquad (5.9)$$

For a closed circuit of contour C carrying a current I, the total magnetic force is

$$\mathbf{F}_{\mathrm{m}} = I \oint_C d\mathbf{l} \times \mathbf{B} \qquad \text{(N).} \qquad (5.10)$$

If the closed wire in Fig. 5-3(a) (see the right figure) is resides in a uniform **B**, then **B** can be taken outside the integral in Eq. (5.10), in which case

$$\mathbf{F}_{\mathrm{m}} = I \left(\oint_C d\mathbf{l} \right) \times \mathbf{B} = 0. \qquad (5.11)$$

This result, which is a consequence of the fact that the vector sum of the infinitesimal vectors $d\mathbf{l}$ over a closed path equals zero, states that the total magnetic force on any closed current loop in a uniform magnetic field is zero.

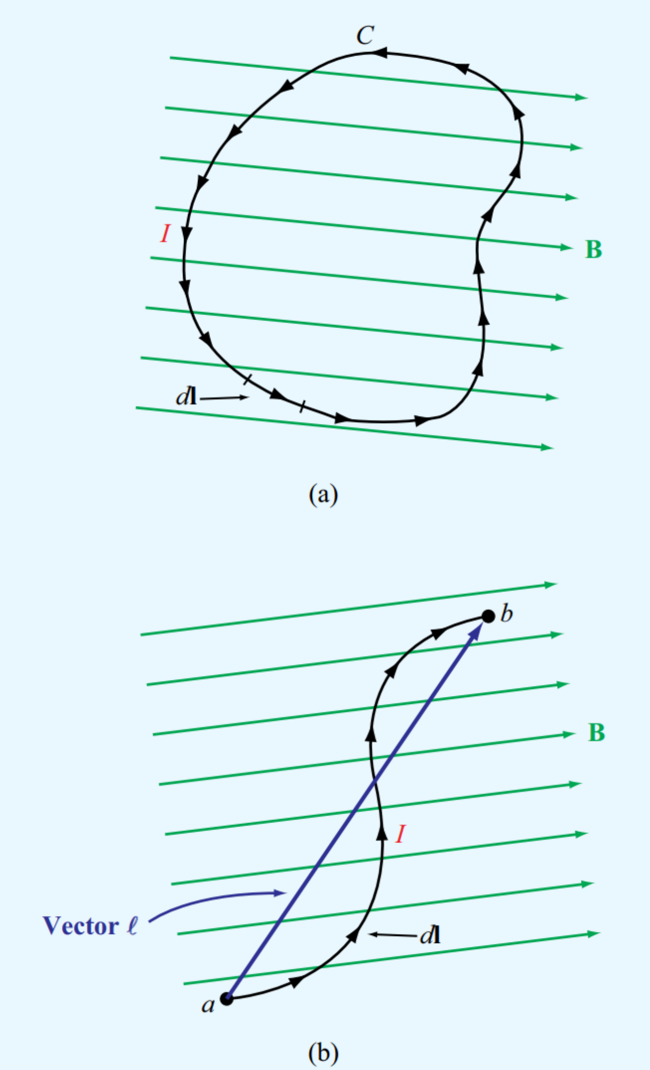


Figure 5-3 In a uniform magnetic field, (a) the net force on a closed current loop is zero because the integral of the displacement vector $d\mathbf{l}$ over a closed contour is zero, and (b) the force on a line segment is proportional to the vector between the end point ($\mathbf{F}_{\mathrm{m}} = I\boldsymbol{\ell} \times \mathbf{B}$).

Torque

$$\mathbf{T} = \mathbf{d} \times \mathbf{F} \qquad (\mathrm{N \cdot m})$$

d = moment arm
F = force
T = torque

d connects the rotation axis and the application point of **F**

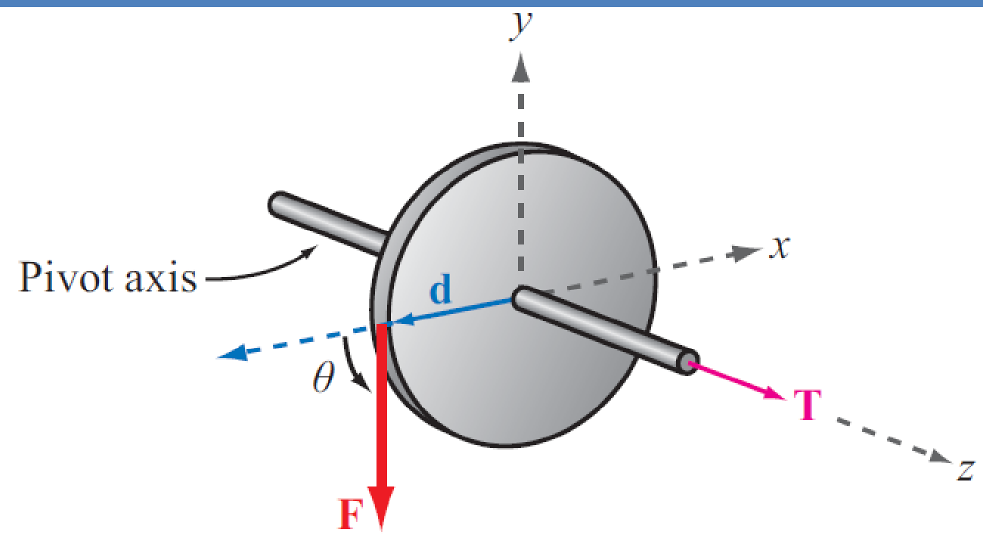


Figure 5-5: The force **F** acting on a circular disk that can pivot along the *z*-axis generates a torque $\mathbf{T} = \mathbf{d} \times \mathbf{F}$ that causes the disk to rotate.

These directions are governed by the following ***right-hand rule:*** *when the thumb of the right hand points along the direction of the torque, the four fingers indicate the direction that the torque tries to rotate the body.*

Magnetic Torque on a Current Loop

- When **B** is in the plane of the loop

a rectangular conducting loop carries a current I.

$$\mathbf{F}_1 = I(-\hat{\mathbf{y}}b) \times (\hat{\mathbf{x}}B_0) = \hat{\mathbf{z}}IbB_0,$$

$$\mathbf{F}_3 = I(\hat{\mathbf{y}}b) \times (\hat{\mathbf{x}}B_0) = -\hat{\mathbf{z}}IbB_0.$$

No forces on arms 2 and 4 (because **I** // **B**)

Magnetic torque:

$$\begin{aligned}\mathbf{T} &= \mathbf{d}_1 \times \mathbf{F}_1 + \mathbf{d}_3 \times \mathbf{F}_3 \\ &= \left(-\hat{\mathbf{x}}\,\frac{a}{2}\right) \times (\hat{\mathbf{z}}IbB_0) + \left(\hat{\mathbf{x}}\,\frac{a}{2}\right) \times (-\hat{\mathbf{z}}IbB_0) \\ &= \hat{\mathbf{y}}IabB_0 = \hat{\mathbf{y}}IAB_0,\end{aligned}$$

Area of the loop=ab

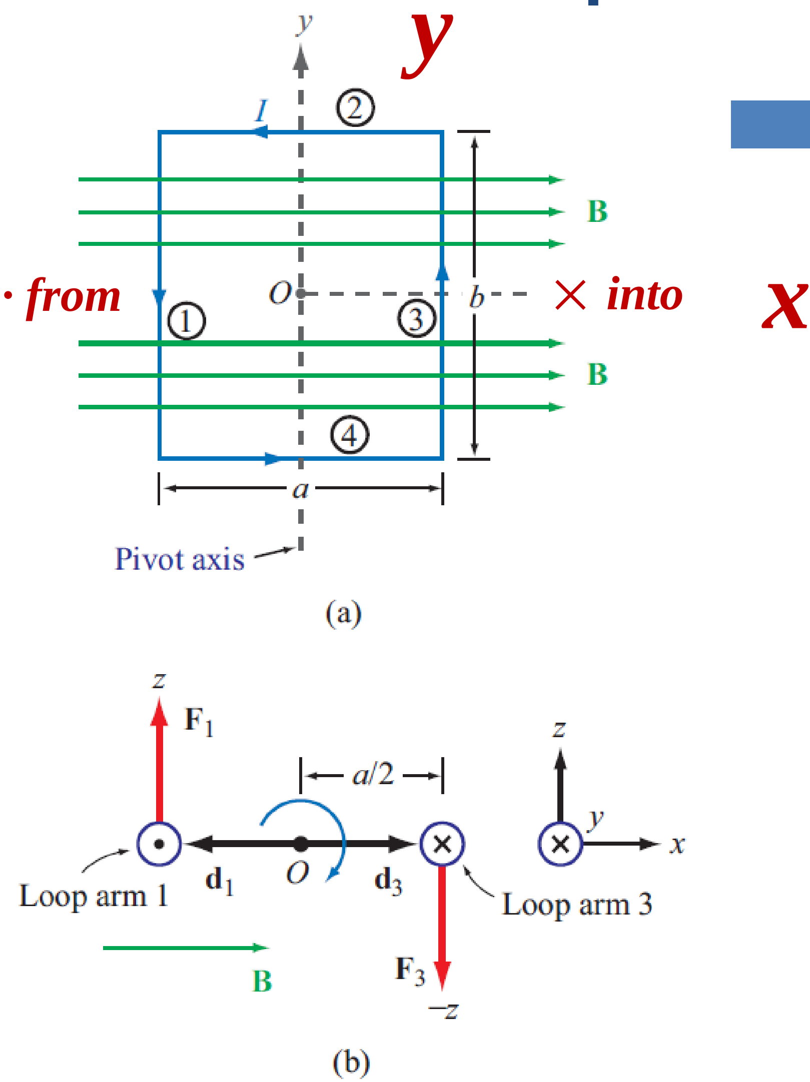


Figure 5-6: Rectangular loop pivoted along the y-axis: (a) front view and (b) bottom view. The combination of forces $\mathbf{F}_1$ and $\mathbf{F}_3$ on the loop generates a torque that tends to rotate the loop in a clockwise direction as shown in (b).

Inclined Loop

- For a loop with *N* turns and whose surface normal is at angle ϑ (relative to the direction of ***B***)

T: Maximum for parallel B ($\theta = 90^\circ$),
Zero for perpendicular B ($\theta = 0$).

$$T = NIAB_0 \sin\theta. \qquad (5.18)$$

The quantity NIA is called the ***magnetic moment*** m of the loop. Now, consider the vector

$$\boxed{\mathbf{m} = \hat{\mathbf{n}}\, NIA = \hat{\mathbf{n}}\, m \qquad (\text{A}\cdot\text{m}^2), \qquad (5.19)}$$

where $\hat{\mathbf{n}}$ is the surface normal of the loop and governed by the following ***right-hand rule: when the four fingers of the right hand advance in the direction of the current I, the direction of the thumb specifies the direction of*** $\hat{\mathbf{n}}$. In terms of $\mathbf{m}$, the torque vector $\mathbf{T}$ can be written as

$$\boxed{\mathbf{T} = \mathbf{m} \times \mathbf{B} \qquad (\text{N}\cdot\text{m}). \qquad (5.20)}$$

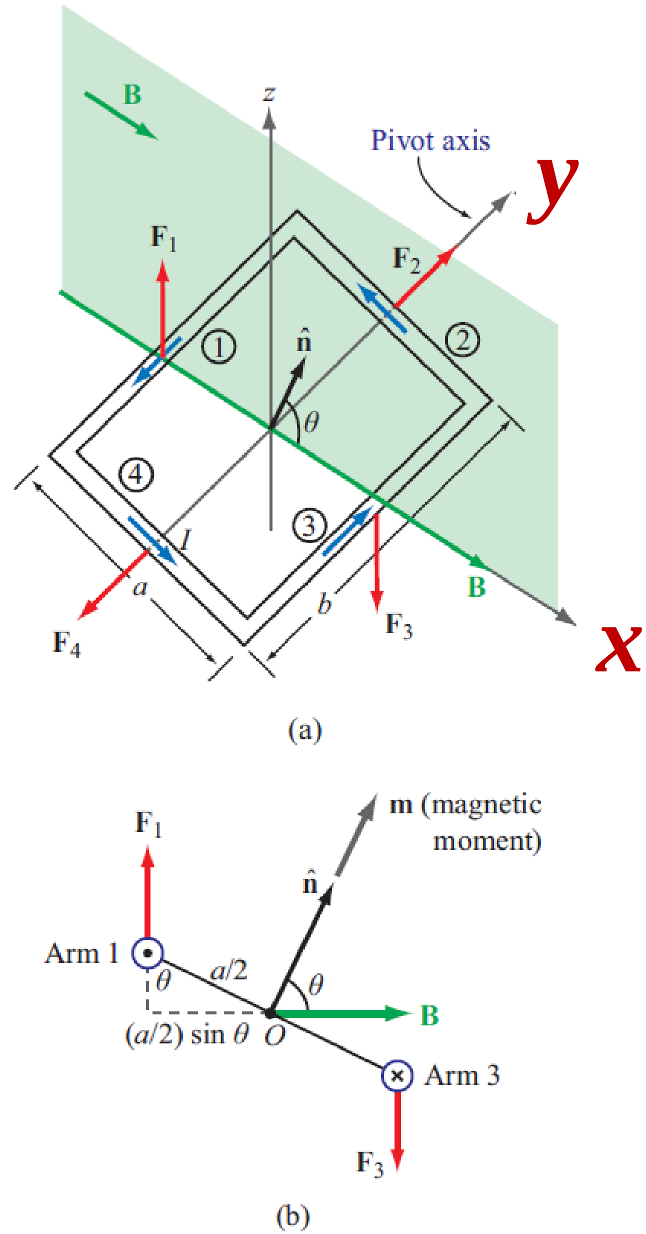


Figure 5-7: Rectangular loop in a uniform magnetic field with flux density **B** whose direction is perpendicular to the rotation axis of the loop, but makes an angle θ with the loop's surface normal $\hat{\mathbf{n}}$.

Biot-Savart Law

$I\mathrm{d}\boldsymbol{l}$ is known as "current element"
$\hat{\mathbf{R}}$ is from *the current element* to point P

- This law relates the magnetic field **H** at any point in space to the current *I* that generates **H**
- For most materials the flux and field are linearly related by $\mathbf{B} = \mu\mathbf{H}$
- Differential magnetic field *dH* generated by steady current *I* through differential length vector **dl**

$$d\mathbf{H} = \frac{I}{4\pi}\,\frac{d\mathbf{l} \times \hat{\mathbf{R}}}{R^2}$$

(A/m)
ampere·m/m²

- ➢ *Magnitude:* varies as R^{-2}
- ➢ *Direction*: orthogonal to ($I d\mathbf{l} \times \mathbf{R}$)

- Total **B** due to the current

where l is the line path along which I exists.

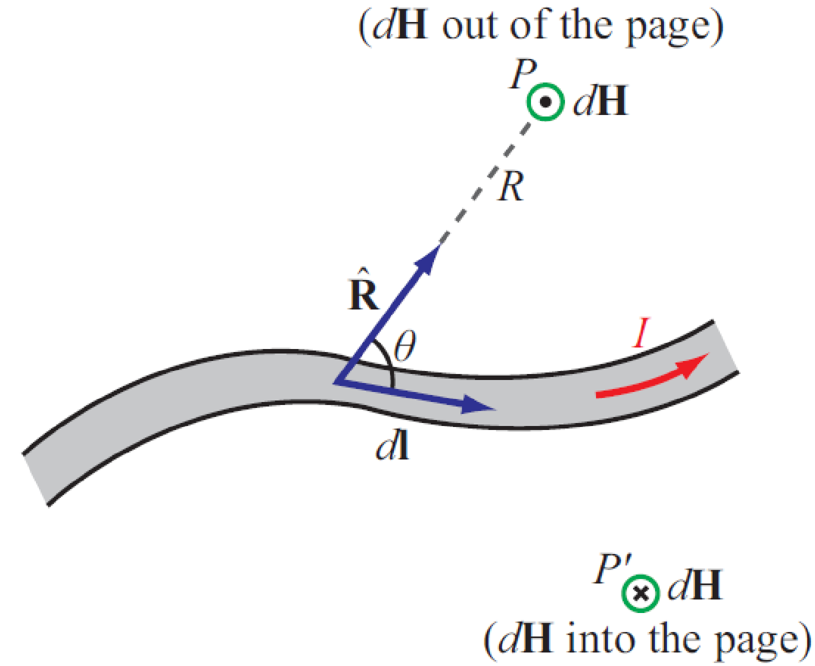


Figure 5-8: Magnetic field $d\mathbf{H}$ generated by a current element $I\,d\mathbf{l}$. The direction of the field induced at point P is opposite to that induced at point P'.

$$\mathbf{H} = \frac{I}{4\pi}\int_l \frac{d\mathbf{l} \times \hat{\mathbf{R}}}{R^2} \quad \text{(A/m)}, \qquad (5.22)$$

Magnetic Field due to Distributed Current Densities

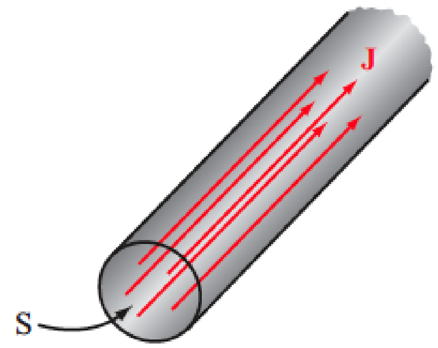
- Currents (in A) $\quad I \qquad I = \int_l J_s \, dl \qquad I = \int_S \mathbf{J} \cdot d\mathbf{s}$
- Current Elements (in A·m) $\quad I \, d\mathbf{l} \leftrightarrow \mathbf{J}_s \, ds \leftrightarrow \mathbf{J} \, d\mathcal{V}$

  (A)·m $\qquad$ (A/m) ·m^2 $\qquad$ (A/m^2) ·m^3

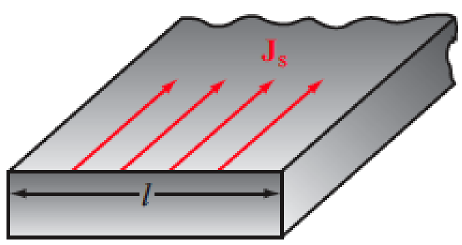
$$\mathbf{H} = \frac{I}{4\pi} \int_l \frac{d\mathbf{l} \times \hat{\mathbf{R}}}{R^2} \quad \text{(A/m)}, \qquad (5.22)$$

$$\mathbf{H} = \frac{1}{4\pi} \int_S \frac{\mathbf{J}_s \times \hat{\mathbf{R}}}{R^2} \, ds \quad \text{(surface current)},$$

$$\mathbf{H} = \frac{1}{4\pi} \int_{\mathcal{V}} \frac{\mathbf{J} \times \hat{\mathbf{R}}}{R^2} \, d\mathcal{V} \quad \text{(volume current)}.$$



(a) Volume current density **J** in A/m^2



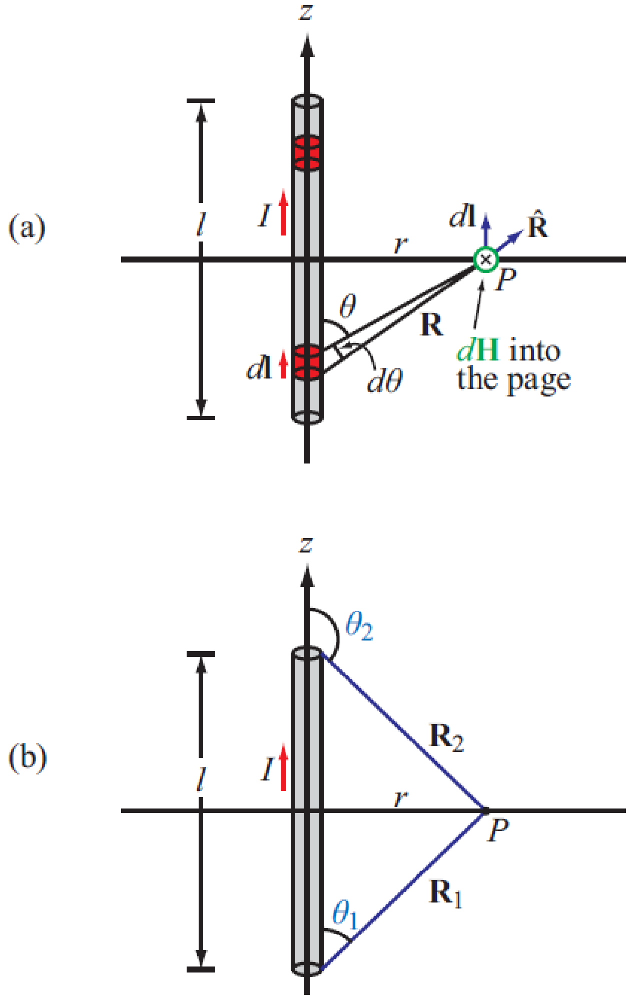
(b) Surface current density $\mathbf{J}_s$ in A/m

Step 1: determine the current element $Id\boldsymbol{l}$, $\mathbf{J}_s$ds, or $\mathbf{J}dV$
Step 2: determine the unit distance vector $\hat{R}$
Step 3: do the integral

Figure 5-9: (a) The total current crossing the cross section S of the cylinder is $I = \int_S \mathbf{J} \cdot d\mathbf{s}$. (b) The total current flowing across the surface of the conductor is $I = \int_l J_s \, dl$.

Example 5-2: Magnetic Field of Linear Conductor

Step 1: determine the current element $I d\mathbf{l}$, $\mathbf{J}_s ds$, or $\mathbf{J} dV$
Step 2: determine the unit distance vector $\hat{\boldsymbol{R}}$
Step 3: express $I d\boldsymbol{l} \times \boldsymbol{R}$ in coordinates, then do the integral



Solution: From Fig. 5-10, the differential length vector $d\mathbf{l} = \hat{\mathbf{z}}\, dz$. Hence, $d\mathbf{l} \times \hat{\mathbf{R}} = dz\, (\hat{\mathbf{z}} \times \hat{\mathbf{R}}) = \hat{\boldsymbol{\phi}} \sin\theta\, dz$, where $\hat{\boldsymbol{\phi}}$ is the azimuth direction and θ is the angle between $d\mathbf{l}$ and $\hat{\mathbf{R}}$. Application of Eq. (5.22) gives

$$\mathbf{H} = \frac{I}{4\pi} \int_{z=-l/2}^{z=l/2} \frac{d\mathbf{l} \times \hat{\mathbf{R}}}{R^2} = \hat{\boldsymbol{\phi}} \frac{I}{4\pi} \int_{-l/2}^{l/2} \frac{\sin\theta}{R^2}\, dz. \qquad (5.25)$$

Both R and θ are dependent on the integration variable z, but the radial distance r is not. For convenience, we will convert the integration variable from z to θ by using the transformations

$$R = r \csc\theta, \qquad (5.26a)$$

$$z = -r \cot\theta, \qquad (5.26b)$$

$$dz = r \csc^2\theta\, d\theta. \qquad (5.26c)$$

Figure 5-10: Linear conductor of length l carrying a current I. (a) The field $d\mathbf{H}$ at point P due to incremental current element $d\mathbf{l}$. (b) Limiting angles θ_1 and θ_2, each measured between vector $I\, d\mathbf{l}$ and the vector connecting the end of the conductor associated with that angle to point P (Example 5-2).

Cont.

Example 5-2: Magnetic Field of Linear Conductor (cont.)

$$\mathbf{H} = \hat{\boldsymbol{\phi}} \frac{I}{4\pi} \int_{\theta_1}^{\theta_2} \frac{\sin\theta \; r \csc^2\theta \; d\theta}{r^2 \csc^2\theta}$$

$$= \hat{\boldsymbol{\phi}} \frac{I}{4\pi r} \int_{\theta_1}^{\theta_2} \sin\theta \; d\theta$$

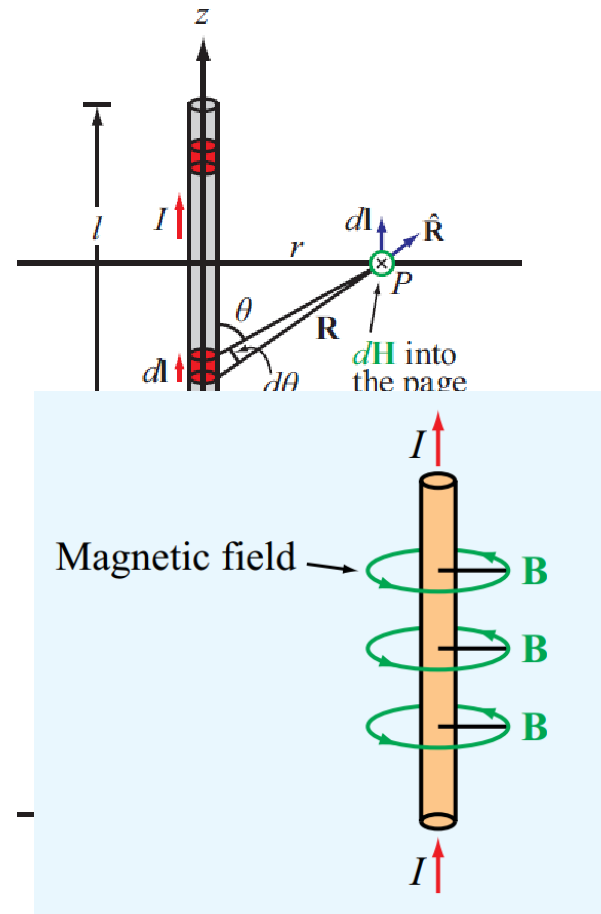
$$= \hat{\boldsymbol{\phi}} \frac{I}{4\pi r} (\cos\theta_1 - \cos\theta_2),$$

$$\cos\theta_1 = \frac{l/2}{\sqrt{r^2 + (l/2)^2}} ,$$

$$\cos\theta_2 = -\cos\theta_1 = \frac{-l/2}{\sqrt{r^2 + (l/2)^2}} .$$

(a)

(b)



Hence,

$$\mathbf{B} = \mu_0 \mathbf{H} = \hat{\boldsymbol{\phi}} \frac{\mu_0 I l}{2\pi r \sqrt{4r^2 + l^2}} \quad \text{(T)}. \qquad (5.29)$$

For an infinitely long wire with $l \gg r$, Eq. (5.29) reduces to

$$\mathbf{B} = \hat{\boldsymbol{\phi}} \frac{\mu_0 I}{2\pi r} \qquad \textbf{(infinitely long wire)}. \qquad (5.30)$$

▶ This is a very important and useful expression to keep in mind. It states that in the neighborhood of a linear conductor carrying a current I, the induced magnetic field forms concentric circles around the wire (**Fig. 5-11**), and its intensity is directly proportional to I and inversely proportional to the distance r. ◀

Example 5-3: Magnetic Field of a Loop

A circular loop of radius a carries a steady current *I*. Determine the magnetic field **H** at a point on the axis of the loop, *i.e.* at (0, 0, *z*).

- Magnitude of $d\mathbf{H}$ due to current element $I d\mathbf{l}$ is

$$dH = \frac{I}{4\pi R^2} |d\mathbf{l} \times \hat{\mathbf{R}}| = \frac{I\, dl}{4\pi (a^2 + z^2)}$$

- $d\mathbf{H}$ is in the r–z plane, thereby components dH_r and dH_z

(1) dH_z due to d**l** and dl' add with each other
(2) dH_r due to d**l** and dl' cancel with each other

- Hence only H_z exists for a pair of elements $I d\mathbf{l}$ and $I d\mathbf{l}'$, which is

$$d\mathbf{H} = \hat{\mathbf{z}}\, dH_z = \hat{\mathbf{z}}\, dH \cos\theta = \hat{\mathbf{z}} \frac{I \cos\theta}{4\pi (a^2 + z^2)}\, dl$$

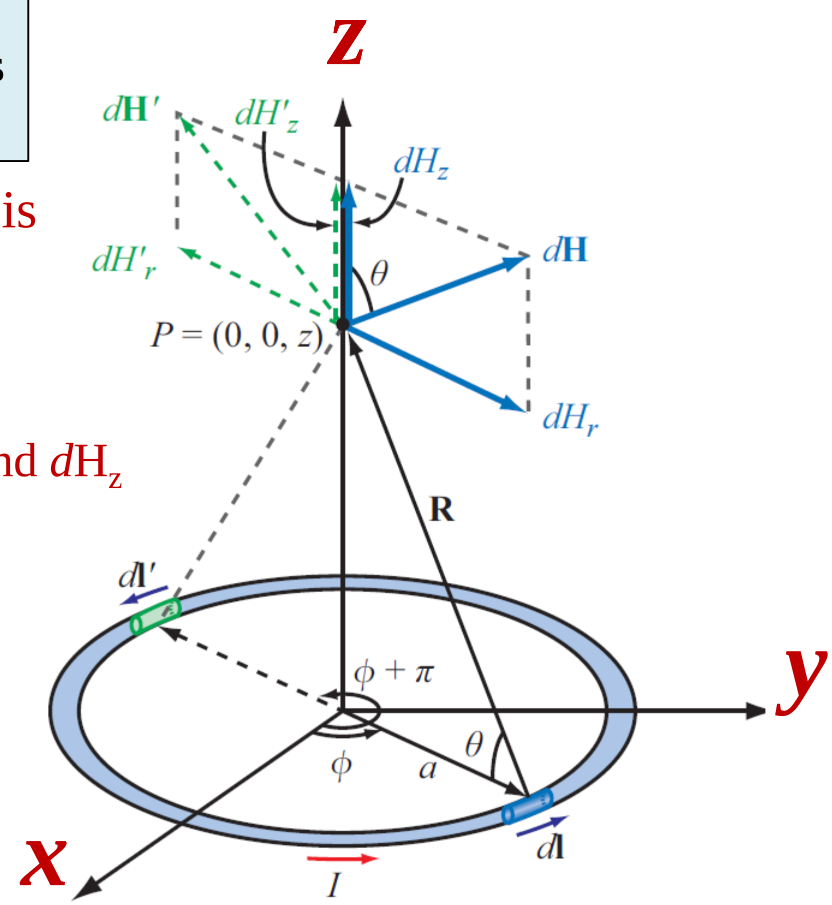


Figure 5-12: Circular loop carrying a current *I* (Example 5-3).

Cont.

Example 5-3: Magnetic Field of a Loop (cont.)

- Total H-field due to the entire current loop

$$\mathbf{H} = \hat{\mathbf{z}} \frac{I \cos\theta}{4\pi(a^2 + z^2)} \oint dl = \hat{\mathbf{z}} \frac{I \cos\theta}{4\pi(a^2 + z^2)} (2\pi a). \quad (5.33)$$

Upon using the relation $\cos\theta = a/(a^2 + z^2)^{1/2}$, we obtain

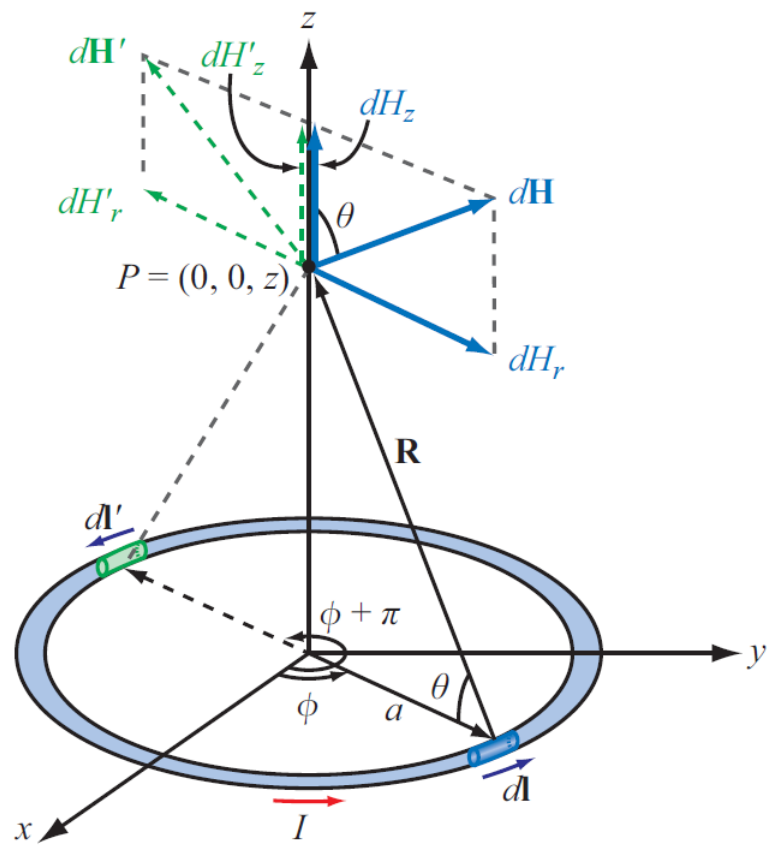
$$\boxed{\mathbf{H} = \hat{\mathbf{z}} \frac{I a^2}{2(a^2 + z^2)^{3/2}} \quad \text{(A/m)}. \quad (5.34)}$$

At the center of the loop ($z = 0$), Eq. (5.34) reduces to

$$\boxed{\mathbf{H} = \hat{\mathbf{z}} \frac{I}{2a} \quad (\text{at } z = 0),} \quad (5.35)$$

and at points very far away from the loop such that $z^2 \gg a^2$, Eq. (5.34) simplifies to

$$\mathbf{H} = \hat{\mathbf{z}} \frac{I a^2}{2|z|^3} \quad (\text{at } |z| \gg a). \quad (5.36)$$



'igure 5-12: Circular loop carrying a current I (Example 5-3).

The axial H-field decay so fast with $|z|^3$ that your handphone has to be placed very close to the charging pad!

Magnetic Dipole

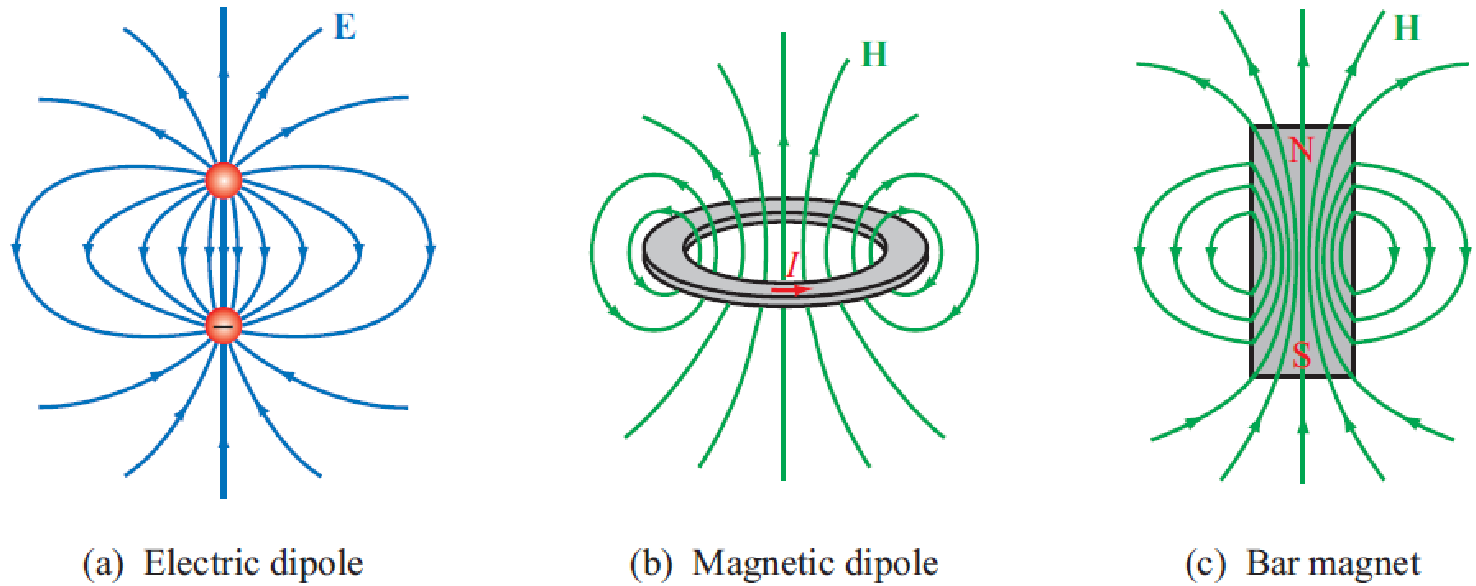


Figure 5-13: Patterns of (a) the electric field of an electric dipole, (b) the magnetic field of a magnetic dipole, and (c) the magnetic field of a bar magnet. Far away from the sources, the field patterns are similar in all three cases.

- Definition: a small current loop, regardless of its shape
- How small: the dimension of the loop is much smaller than the distance where you want to evaluate its field.
- Why dipole: because a loop exhibits a magnetic field pattern similar to the electric field of an electric dipole, it is called a *magnetic dipole*

Forces on Parallel Conductors

- Two wires closely placed with each other

$$\mathbf{B}_1 = -\hat{\mathbf{x}} \frac{\mu_0 I_1}{2\pi d}. \quad (5.39)$$

The force $\mathbf{F}_2$ exerted on a length l of wire I_2 due to its presence in field $\mathbf{B}_1$ may be obtained by applying Eq. (5.12):

$$\mathbf{F}_2 = \underbrace{I_2 l \hat{\mathbf{z}}}_{\text{Current element}} \times \underbrace{\mathbf{B}_1}_{\text{Field}} = I_2 l \hat{\mathbf{z}} \times (-\hat{\mathbf{x}}) \frac{\mu_0 I_1}{2\pi d}$$

$$= -\hat{\mathbf{y}} \frac{\mu_0 I_1 I_2 l}{2\pi d}, \quad (5.40)$$

and the corresponding force per unit length is

$$\mathbf{F}'_2 = \frac{\mathbf{F}_2}{l} = -\hat{\mathbf{y}} \frac{\mu_0 I_1 I_2}{2\pi d}. \quad (5.41)$$

A similar analysis performed for the force per unit length exerted on the wire carrying I_1 leads to

$$\mathbf{F}'_1 = \hat{\mathbf{y}} \frac{\mu_0 I_1 I_2}{2\pi d}. \quad (5.42)$$

- Separation d, *infinitely long*
- Currents I_1 and I_2 along z
- $\mathbf{B}_1$ at I_2 by $\hat{\mathbf{z}}I_1 l$, $\mathbf{B}_2$ at I_1 by $\hat{\mathbf{z}}I_2 l$.

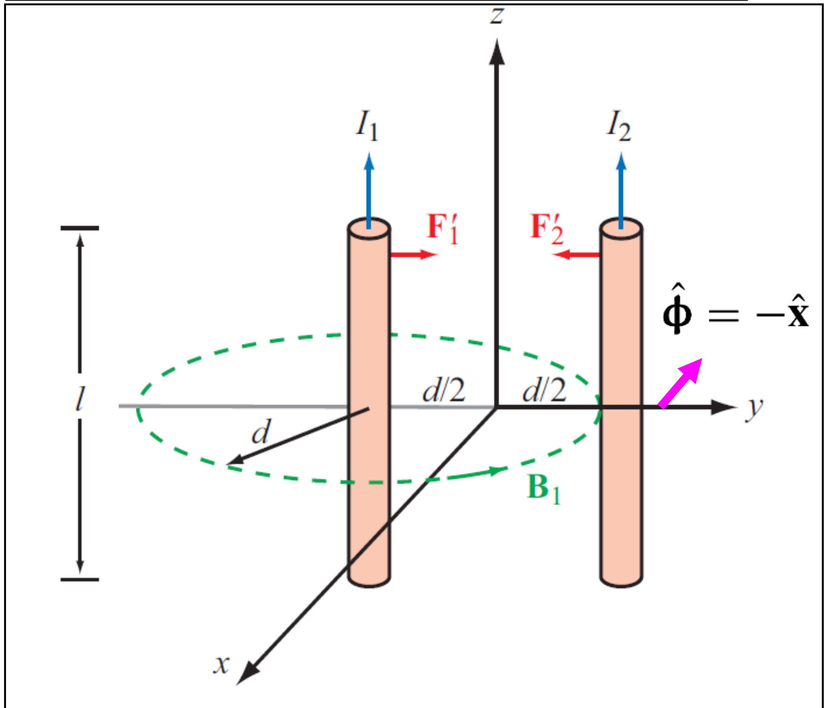


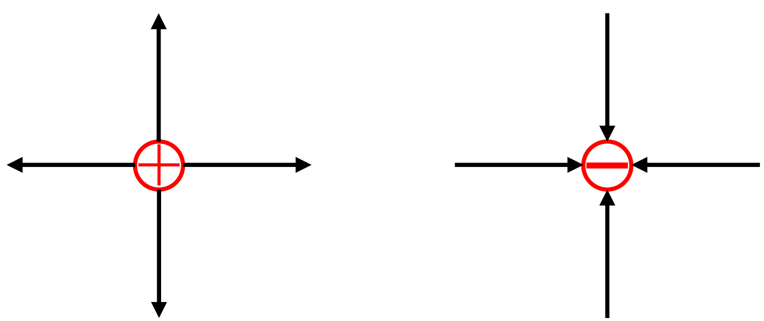
Figure 5-14: Magnetic forces on parallel current-carrying conductors.

Parallel wires attract if their currents are in the same direction, and repel if currents are in opposite directions

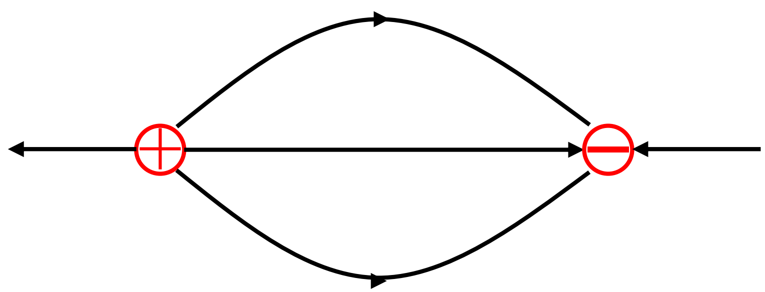
Gauss's Law for Magnetism

Gauss's Law for Electricity

$$\nabla \cdot \mathbf{D} = \rho_V \quad \longleftrightarrow \quad \oint_S \mathbf{D} \cdot d\mathbf{s} = Q.$$



Electrostatics

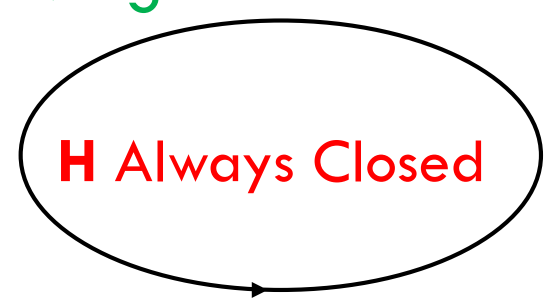


Gauss's Law for Magnetism

$$\nabla \cdot \mathbf{B} = 0 \quad \longleftrightarrow \quad \oint_S \mathbf{B} \cdot d\mathbf{s} = 0.$$

Zero at the right-hand side means that magnetic monopole (charge) does not exist in nature, but magnetic dipole exists.

Magnetostatics



Ampère's Law

Ampère's circuital law states that the line integral of **H** around a closed path is equal to the current traversing the surface bounded by that path.

$$\nabla \times \mathbf{E} = 0 \quad \longleftrightarrow \quad \oint_C \mathbf{E} \cdot d\boldsymbol{\ell} = 0.$$

Conservative

$$\nabla \times \mathbf{H} = \mathbf{J} \quad \longleftrightarrow \quad \oint_C \mathbf{H} \cdot d\boldsymbol{\ell} = I$$

Not Conservative unless $I = 0$

The direction of the path C is taken so that **<u>I and H satisfy the right-hand rule</u>**. That is, if the direction of I is aligned with the direction of the thumb of the right hand, then the direction of the contour C should be chosen along that of the other four fingers.

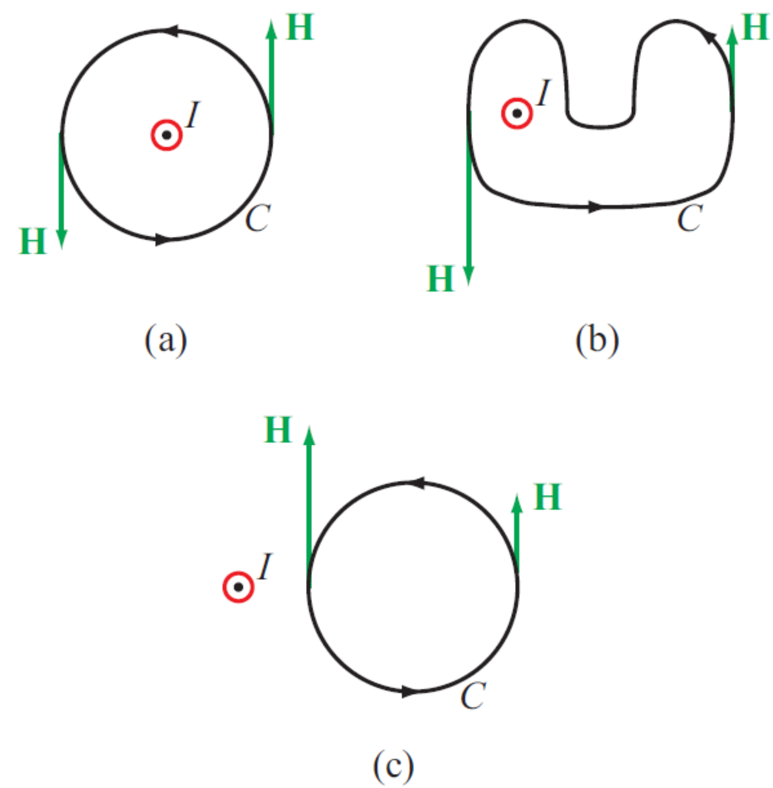


Figure 5-16: Ampère's law states that the line integral of **H** around a closed contour C is equal to the current traversing the surface bounded by the contour. This is true for contours (a) and (b), but the line integral of **H** is zero for the contour in (c) because the current I (denoted by the symbol $\odot$) is not enclosed by the contour C.

Internal Magnetic Field of a Long Wire

Current I along infinitely long wire. Find **H** at r from the wire for (a) $r \leq a$ (inside the wire) and (b) $r \geq a$ (outside the wire)

(a) $r \leq a$

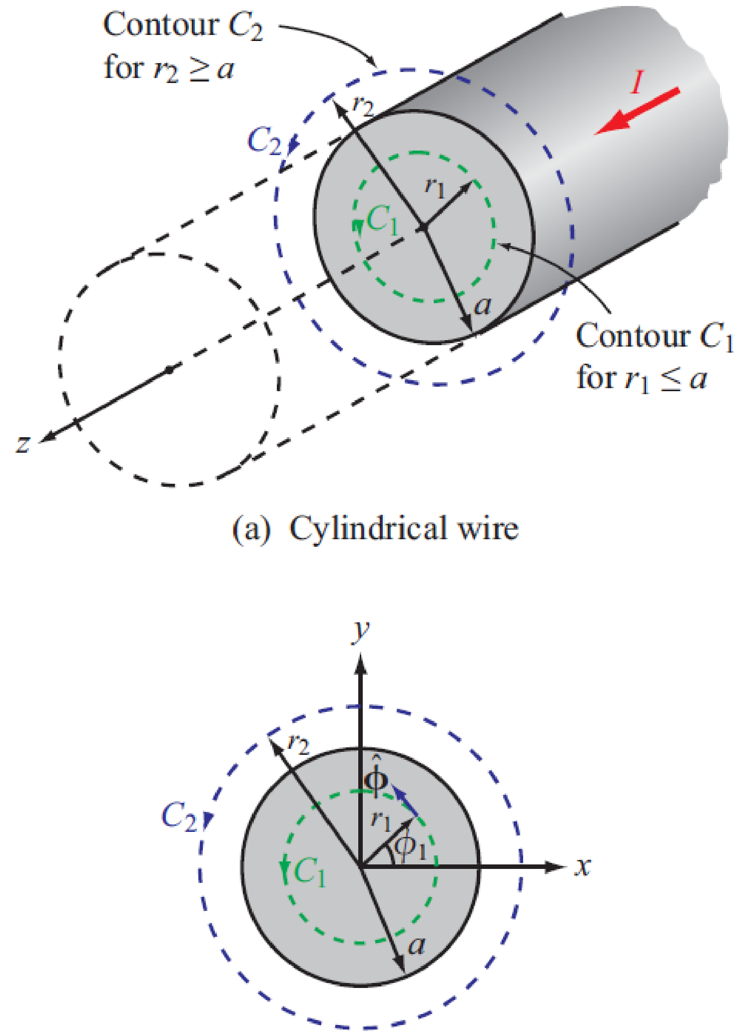
$$\oint_{C_1} \mathbf{H}_1 \cdot d\mathbf{l}_1 = I_1, \qquad \oint_{C_1} \mathbf{H}_1 \cdot d\mathbf{l}_1 = \int_0^{2\pi} H_1(\hat{\boldsymbol{\phi}} \cdot \hat{\boldsymbol{\phi}}) r_1 \, d\phi = 2\pi r_1 H_1$$

The current I_1 flowing through the area enclosed by C_1 is equal to the total current I multiplied by the ratio of the area enclosed by C_1 to the total cross-sectional area of the wire:

$$I_1 = \left(\frac{\pi r_1^2}{\pi a^2}\right) I = \left(\frac{r_1}{a}\right)^2 I.$$

Equating both sides of Eq. (5.48) and then solving for $\mathbf{H}_1$ yields

$$\boxed{\mathbf{H}_1 = \hat{\boldsymbol{\phi}} H_1 = \hat{\boldsymbol{\phi}} \frac{r_1}{2\pi a^2} I \qquad (\text{for } r_1 \leq a).} \qquad (5.49\text{a})$$



(a) Cylindrical wire

(b) Wire cross section

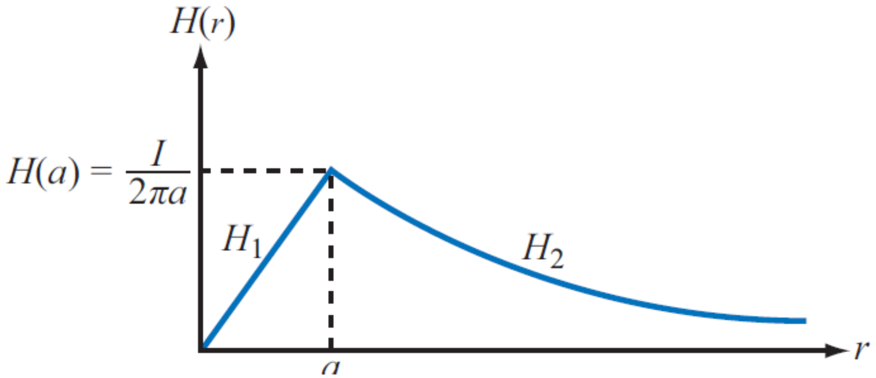
Cont.

External Magnetic Field of Long Conductor

(b) $r \geq a$

(b) For $r = r_2 \geq a$, we choose path C_2, which encloses all the current I. Hence, $\mathbf{H}_2 = \hat{\boldsymbol{\phi}}\, H_2$, $d\boldsymbol{\ell}_2 = \hat{\boldsymbol{\phi}}\, r_2\, d\phi$, and

$$\boxed{\oint_{C_2} \mathbf{H}_2 \cdot d\mathbf{l}_2 = 2\pi r_2 H_2 = I,}$$



which yields

$$\mathbf{H}_2 = \hat{\boldsymbol{\phi}} H_2 = \hat{\boldsymbol{\phi}}\, \frac{I}{2\pi r_2} \qquad (\text{for } r_2 \geq a). \qquad (5.49\text{b})$$

Magnetic Field of Toroid

Applying Ampere's law over contour C:

$$\oint_C \mathbf{H} \cdot d\boldsymbol{\ell} = I$$

Ampere's law states that the line integral of **H** around a closed contour C *is equal to the current traversing the* surface bounded by the contour.

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = \int_0^{2\pi} (-\hat{\boldsymbol{\phi}} H) \cdot \hat{\boldsymbol{\phi}} r \; d\phi = -2\pi r H = -NI.$$

Hence, $H = NI/(2\pi r)$ and

$$\mathbf{H} = -\hat{\boldsymbol{\phi}} H = -\hat{\boldsymbol{\phi}} \frac{NI}{2\pi r} \qquad \boxed{(\text{for } a < r < b).}$$

The magnetic field outside the toroid is zero. Why?

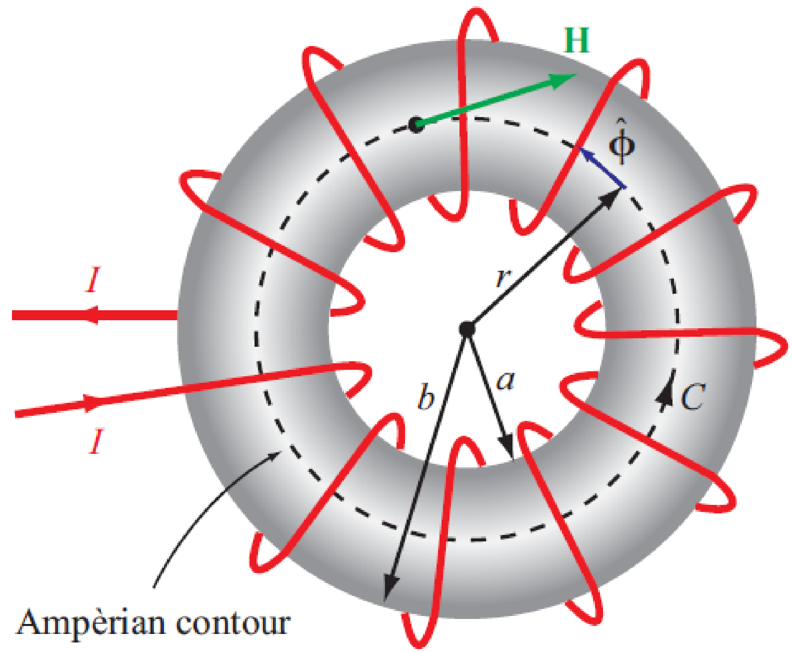
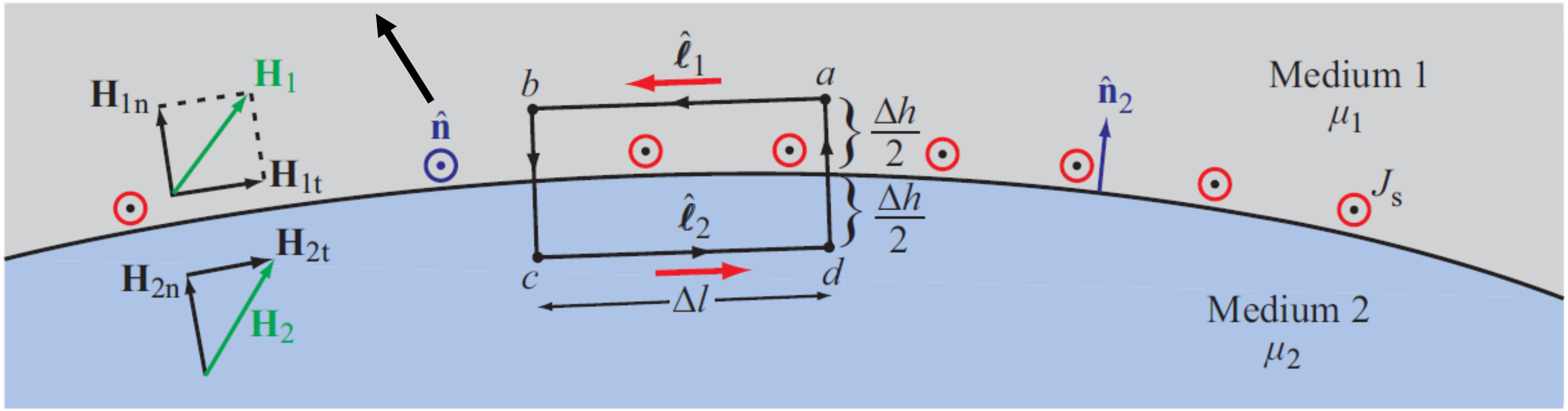


Figure 5-18: Toroidal coil with inner radius a and outer radius b. The wire loops usually are much more closely spaced than shown in the figure (Example 5-5).

Magnetic Boundary Conditions

Norm of the loop



$$\oint_S \mathbf{D} \cdot d\mathbf{s} = Q \rightarrow \boxed{D_{1n} - D_{2n} = \rho_s.} \qquad (5.78)$$

By analogy, application of Gauss's law for magnetism, as expressed by Eq. (5.44), leads to the conclusion that

$$\oint_S \mathbf{B} \cdot d\mathbf{s} = 0 \rightarrow \boxed{B_{1n} = B_{2n}.} \qquad (5.79)$$

Thus the normal component of **B** *is continuous across the boundary between two adjacent media.*

Tips:
1. the total ***charge density*** enclosed by a box equals to the total flux going outward
- D_n jumps by ρ_s
  (**surface** electric charge density)
- B_n jumps by 0
  (surface magnetic charge density)

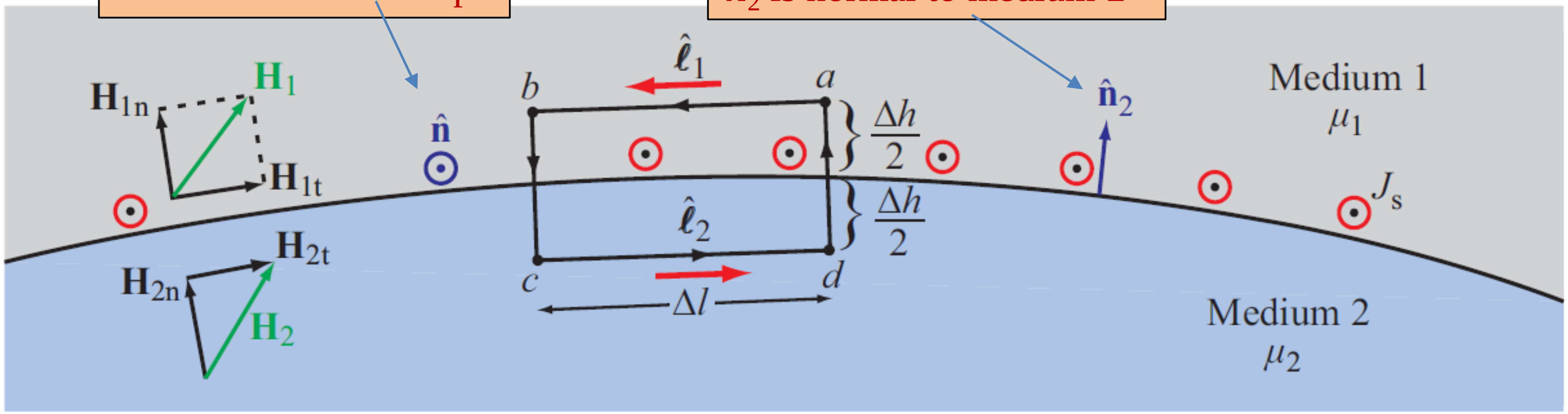
2. the second boundary condition is for B_n, not for H_n. To solve the latter, **B**=μ**H** is needed, where $\mu_1 \neq \mu_2$ will lead to $H_{1n} \neq H_{2n}$

Boundary Conditions

Tips:
the total ***current density*** enclosed by a ***loop*** equals to the circulation
- E_t jumps by ***0*** (magnetic current density?)
- H_t jumps by J_s (electric current density)

n is normal to the loop

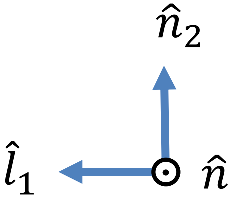
n$_2$ is normal to medium 2



Use Ampère's Law and let $\Delta h \to 0$ (hence a thin line current of length Δl)

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = \int_a^b \mathbf{H}_1 \cdot \hat{\boldsymbol{\ell}}_1 \, d\ell + \int_c^d \mathbf{H}_2 \cdot \hat{\boldsymbol{\ell}}_2 \, d\ell = I, \qquad (5.81)$$

$$\Rightarrow (\mathbf{H}_1 - \mathbf{H}_2) \cdot \hat{\boldsymbol{\ell}}_1 \, \Delta l = \mathbf{J}_s \cdot \hat{\mathbf{n}} \, \Delta l. \qquad (5.82)$$



$$\left\{ \begin{array}{l} \hat{\boldsymbol{\ell}}_1 = \hat{\mathbf{n}} \times \hat{\mathbf{n}}_2 \\ \mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) \end{array} \right.$$

$$\Rightarrow \hat{\mathbf{n}} \cdot [\hat{\mathbf{n}}_2 \times (\mathbf{H}_1 - \mathbf{H}_2)] = \mathbf{J}_s \cdot \hat{\mathbf{n}}$$

valid for arbitrary $\hat{n}$ (loop normal)

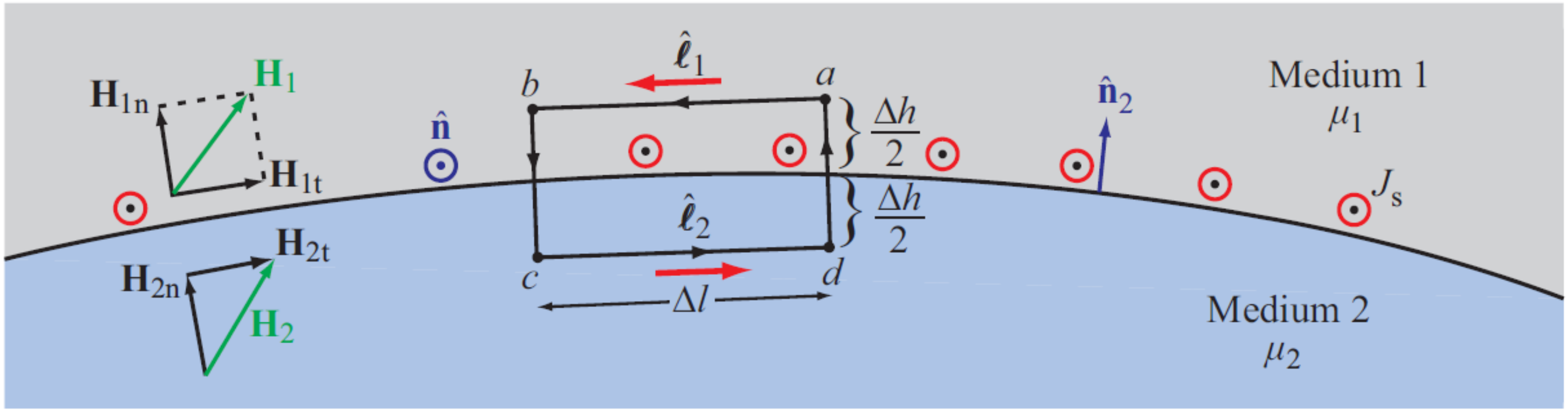
$$\hat{\mathbf{n}}_2 \times (\mathbf{H}_1 - \mathbf{H}_2) = \mathbf{J}_s.$$

$\hat{n} \times \vec{\mathrm{H}}$ is always on the surface

Surface currents can exist only on the surfaces of perfect conductors and superconductors. Hence, *at the interface between media **with finite conductivities***, $\mathbf{J}_s = 0$ and

$$H_{1t} = H_{2t}$$

Boundary Condition for Finite σ



$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} = \sigma \vec{E} + \frac{\partial \vec{D}}{\partial t}$$

$$\vec{J}_s = \int_{h_1}^{h_2} \vec{J}_v dh \approx \frac{\Delta h}{2} \vec{J}_v$$

$$\oint_C \vec{H} \cdot d\vec{l} = \sigma \oiint_S \vec{E} \cdot d\vec{s} + \frac{\partial}{\partial t} \oiint_S \vec{D} \cdot d\vec{s}$$

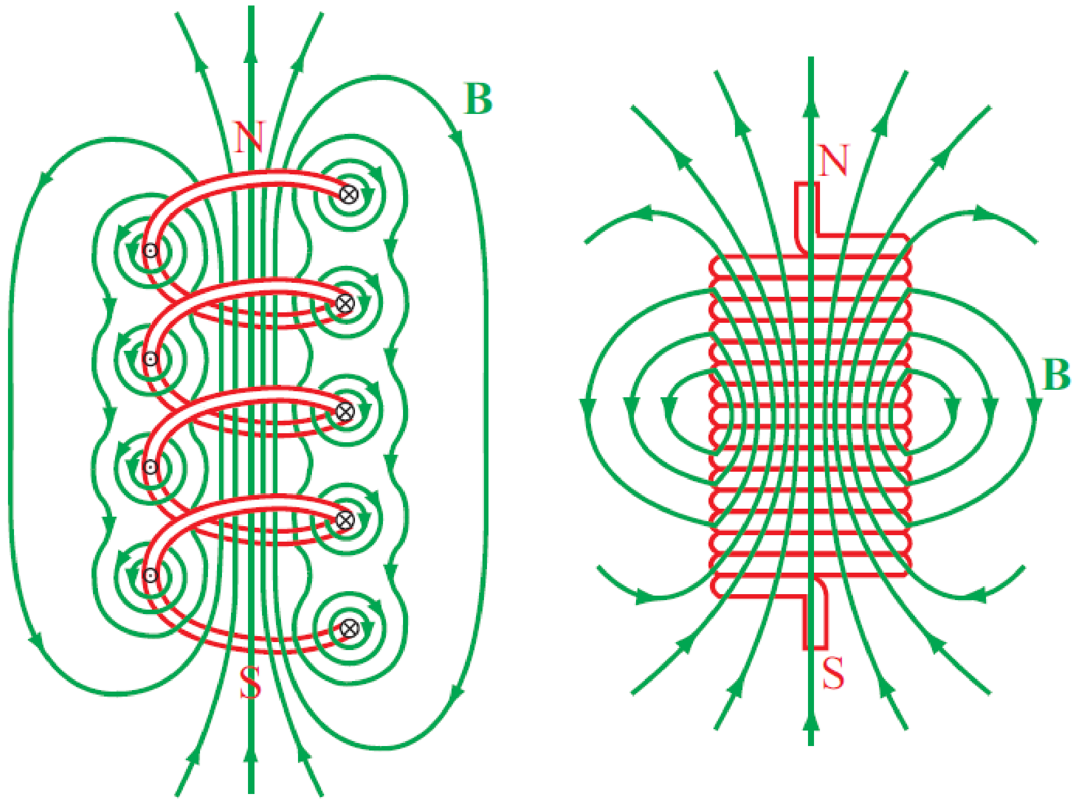
$$\Delta h \to 0$$

$$\vec{J}_s = 0$$

$$\Delta h \to 0 \qquad \oint_C \vec{H} \cdot d\vec{l} = 0$$

Continuous

Solenoid



(a) Loosely wound solenoid

(b) Tightly wound solenoid

Inside the solenoid:

$$\mathbf{B} \simeq \hat{\mathbf{z}}\mu n I = \frac{\hat{\mathbf{z}}\mu N I}{l} \qquad \text{(long solenoid with } l/a \gg 1\text{)}$$

Inductance

Magnetic Flux

$$\Phi = \int_S \mathbf{B} \cdot d\mathbf{s} \qquad \text{(Wb)}.$$

Flux Linkage

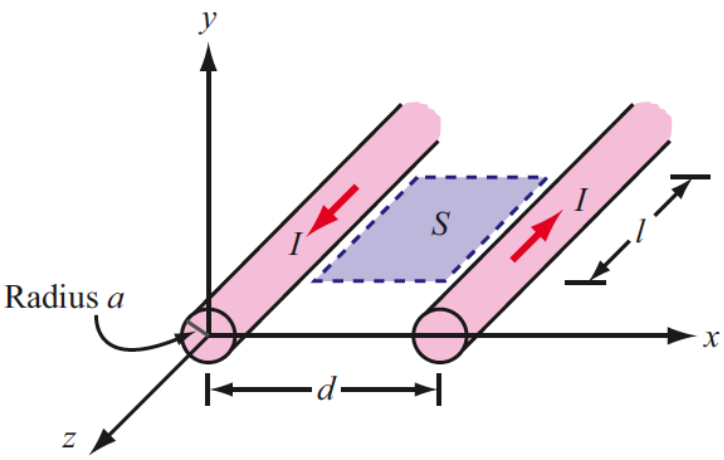
$$\Lambda = N\Phi = \mu \frac{N^2}{l} I S \qquad \text{(Wb)}$$

Inductance

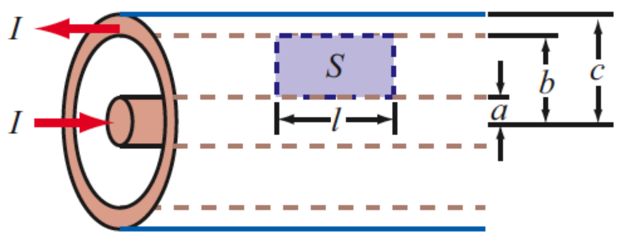
$$L = \frac{\Lambda}{I} = \frac{\Phi}{I} = \frac{1}{I} \int_S \mathbf{B} \cdot d\mathbf{s}. \qquad (5.96)$$

Solenoid

$$L = \mu \frac{N^2}{l} S \qquad \text{(solenoid)}, \qquad (5.95)$$



(a) Parallel-wire transmission line



(b) Coaxial transmission line

Figure 5-27: To compute the inductance per unit length of a two-conductor transmission line, we need to determine the magnetic flux through the area S between the conductors.

Example 5-7: Inductance of Coaxial Cable

The magnetic field in the region S between the two conductors is approximately

$$\mathbf{B} = \hat{\boldsymbol{\phi}}\,\frac{\mu I}{2\pi r}$$

Total magnetic flux through S:

$$\Phi = l \int_a^b B\,dr = l \int_a^b \frac{\mu I}{2\pi r}\,dr = \frac{\mu I l}{2\pi} \ln\left(\frac{b}{a}\right)$$

Inductance per unit length:

$$L' = \frac{L}{l} = \frac{\Phi}{l I} = \frac{\mu}{2\pi} \ln\left(\frac{b}{a}\right).$$

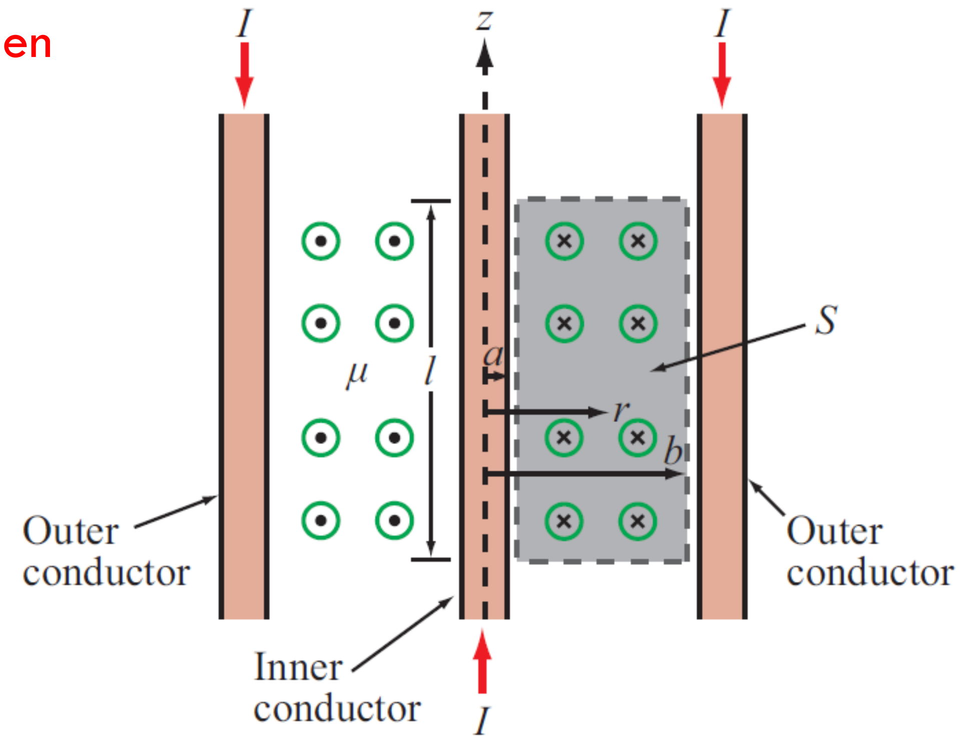


Figure 5-28: Cross-sectional view of coaxial transmission line (Example 5-7).

Magnetic Energy

Example 5-8: Magnetic Energy in a Coaxial Cable

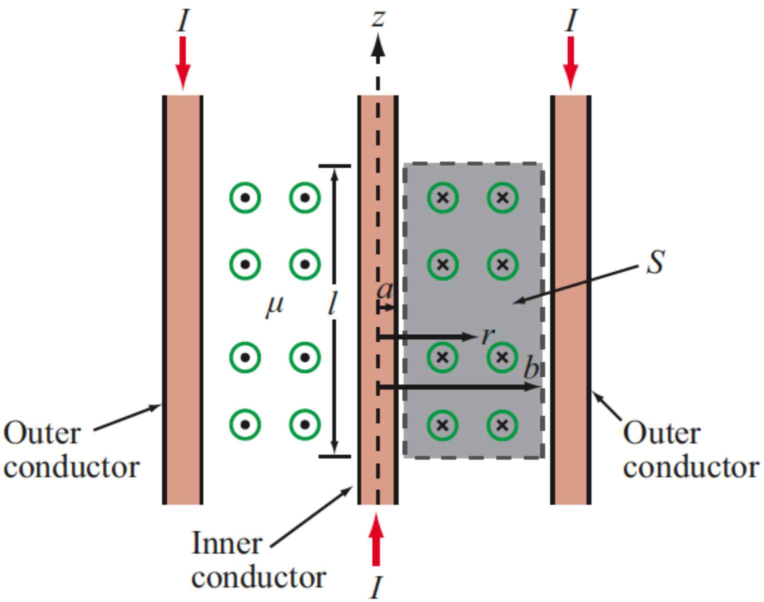
Magnetic field in the insulating material is

$$H = \frac{B}{\mu} = \frac{I}{2\pi r}$$

The magnetic energy stored in the coaxial cable is

$$W_m = \frac{1}{2} \int_{\mathcal{V}} \mu H^2 \, d\mathcal{V} = \frac{\mu I^2}{8\pi^2} \int_{\mathcal{V}} \frac{1}{r^2} \, d\mathcal{V}$$

$$\boxed{w_m = \frac{W_m}{\mathcal{V}} = \frac{1}{2} \mu H^2 \qquad (\text{J/m}^3).}$$



$$W_m = \frac{\mu I^2}{8\pi^2} \int_a^b \frac{1}{r^2} \cdot 2\pi r l \, dr$$

$$= \frac{\mu I^2 l}{4\pi} \ln\left(\frac{b}{a}\right)$$

$$= \frac{1}{2} L I^2 \qquad (\text{J}),$$

Summary

Chapter 5 Relationships

Maxwell's Magnetostatics Equations

Gauss's Law for Magnetism

$$\nabla \cdot \mathbf{B} = 0 \quad \leftrightarrow \quad \oint_S \mathbf{B} \cdot d\mathbf{s} = 0$$

Ampère's Law

$$\nabla \times \mathbf{H} = \mathbf{J} \quad \leftrightarrow \quad \oint_C \mathbf{H} \cdot d\boldsymbol{\ell} = I$$

Lorentz Force on Charge q

$$\mathbf{F} = q(\mathbf{E} + \mathbf{u} \times \mathbf{B})$$

Magnetic Force on Wire

$$\mathbf{F}_\mathrm{m} = I \oint_C d\mathbf{l} \times \mathbf{B} \qquad \text{(N)}$$

Magnetic Torque on Loop

$$\mathbf{T} = \mathbf{m} \times \mathbf{B} \qquad (\text{N}\cdot\text{m})$$

$$\mathbf{m} = \hat{\mathbf{n}}\, NIA \qquad (\text{A}\cdot\text{m}^2)$$

Biot–Savart Law

$$\mathbf{H} = \frac{I}{4\pi} \int_l \frac{d\mathbf{l} \times \hat{\mathbf{R}}}{R^2} \qquad \text{(A/m)}$$

Magnetic Field

Infinitely Long Wire $\quad \mathbf{B} = \hat{\boldsymbol{\phi}} \dfrac{\mu_0 I}{2\pi r} \quad (\text{Wb/m}^2)$

Circular Loop $\quad \mathbf{H} = \hat{\mathbf{z}} \dfrac{I a^2}{2(a^2 + z^2)^{3/2}} \quad \text{(A/m)}$

Solenoid $\quad \mathbf{B} \simeq \hat{\mathbf{z}}\, \mu n I = \dfrac{\hat{\mathbf{z}}\, \mu N I}{l} \quad (\text{Wb/m}^2)$

Vector Magnetic Potential

$$\mathbf{B} = \nabla \times \mathbf{A} \qquad (\text{Wb/m}^2)$$

Vector Poisson's Equation

$$\nabla^2 \mathbf{A} = -\mu \mathbf{J}$$

Inductance

$$L = \frac{\Lambda}{I} = \frac{\Phi}{I} = \frac{1}{I} \int_S \mathbf{B} \cdot d\mathbf{s} \qquad \text{(H)}$$

Magnetic Energy Density

$$w_\mathrm{m} = \frac{1}{2} \mu H^2 \qquad (\text{J/m}^3)$$