

CS244: THEORY OF COMPUTATION

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Recap

- ▶ Interactive Proof Systems: provide a way to define a probabilistic analog of the class **NP**
 - ▶ Interactive polynomial Time (IP)
 - ▶ $\mathbf{NP} \subseteq \mathbf{IP}$, $\mathbf{BPP} \subseteq \mathbf{IP}$, $\mathbf{IP} = \mathbf{PSPACE}$
 - ▶ $\mathbf{NONISO} \in \mathbf{coNP} \cap \mathbf{IP}$
- ▶ Parallel Computation
 - ▶ Uniform Boolean Circuits
 - ▶ Simultaneous size-depth circuit complexity $(f(n), g(n))$
 - ▶ \mathbf{NC}^i : polynomial size and $O(\log^i n)$ depth
 - ▶ $\mathbf{NC}^1 \subseteq \mathbf{L} \subseteq \mathbf{NL} \subseteq \mathbf{NC}^2$ and $\mathbf{NC} \subseteq \mathbf{P}$
- ▶ Cryptography
 - ▶ One-way functions: allow us to construct secure private-key cryptosystems
 - ▶ One-way function: PTIME length-preserving function f such that for every PTIME PTM M , every k , and sufficiently large n , if we pick a random w of length n and run M on input $f(w)$,

$$Pr_{M,w}[M(f(w)) = y, \text{ where } f(y) = f(w)] \leq n^{-k}$$

- ▶ Trapdoor functions: allow us to construct public-key cryptosystems
- ▶ Trapdoor-way function: PTIME length-preserving indexing function f that has an auxiliary PTIME PTM G and an auxiliary function $h: \Sigma^* \times \Sigma^* \rightarrow \Sigma^*$ such that \dots
 $Pr_{E,w}[E(i, f_i(w)) = y, \text{ where } f_i(y) = f_i(w)] \leq n^{-k}$ and
 $h(t, f_i(w)) = y, \text{ where } f_i(y) = f_i(w)$

Outline

Visibly pushdown automata (VPA)

Closure properties

Visibly pushdown grammar (VPG)

Logical characterization

Equivalence of NFA and MSO

Equivalence of VPA and MSO_μ

Decision problems

Motivation

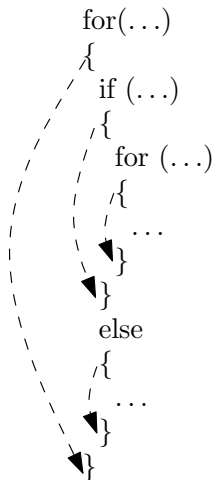
Parenthesises in arithmetic expressions

$$(((5 + x) * y + z) * (u - v)) / w$$

The diagram illustrates the evaluation order of the arithmetic expression $(((5 + x) * y + z) * (u - v)) / w$. Dashed arrows indicate the sequence of operations from left to right, showing how sub-expressions are evaluated and then combined. The first arrow connects the opening parenthesis to the closing parenthesis of the innermost sub-expression $(5 + x)$. The second arrow connects the opening parenthesis to the closing parenthesis of the sub-expression $((5 + x) * y + z)$. The third arrow connects the opening parenthesis to the closing parenthesis of the sub-expression $((5 + x) * y + z) * (u - v)$. The final arrow connects the opening parenthesis to the closing parenthesis of the entire expression $((5 + x) * y + z) * (u - v) / w$.

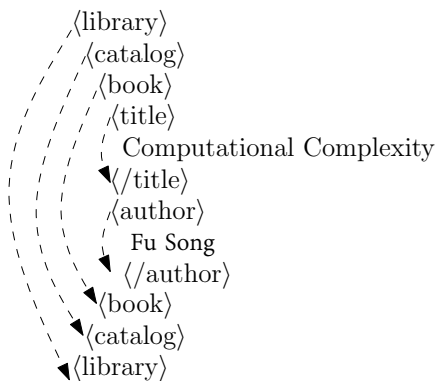
Motivation

Curly brackets in C Programs



Motivation

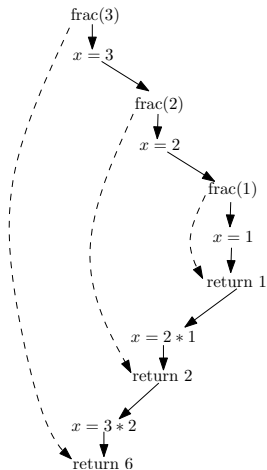
XML documents



Motivation

Recursive function calls and returns

```
frac(int y)
{
  int x = y;
  if (y >= 2)
  {
    x = y * frac(y - 1);
    return x;
  }
  else
  {
    return x;
  }
}
```




Motivation

$$\begin{array}{c} \uparrow \\ (((5 + x) * y + z) * (u - v)) / w \end{array}$$


⊥

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Stack operations determined by the input symbol

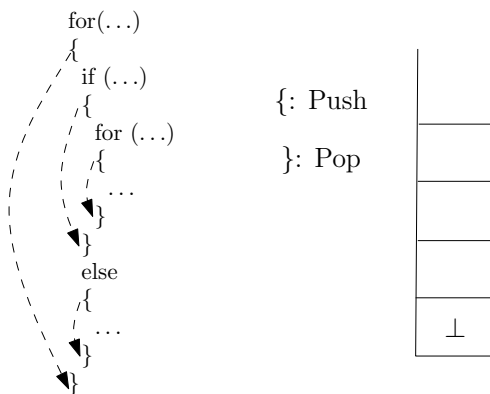
$(((5 + x) * y + z) * (u - v)) / w$

(: Push
): Pop



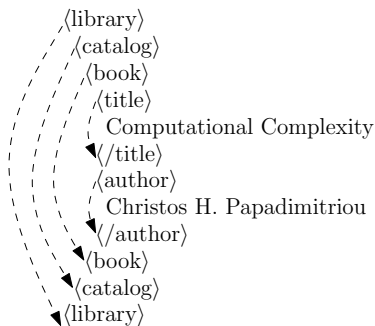
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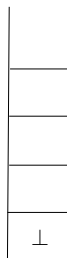
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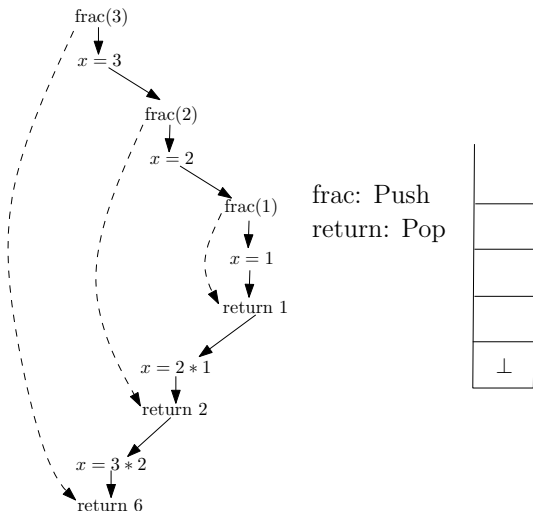
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Motivation

Stack operations determined by the input symbol



Visibly pushdown automata (VPA)

The alphabet Σ is partitioned into $\tilde{\Sigma} = \langle \Sigma_c, \Sigma_r, \Sigma_l \rangle$

- ▶ Σ_c : finite set of **calls**,
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A **(nondeterministic) VPA** \mathcal{A} is a 7-tuple $(Q, \tilde{\Sigma}, \Gamma, \delta, q_0, \perp, F)$, where

- ▶ Q is a finite set of **states**,
- ▶ $\tilde{\Sigma}$ is the **input alphabet**,
- ▶ Γ is the **stack alphabet**,
- ▶ $\delta \subseteq Q \times \Sigma_c \times Q \times (\Gamma \setminus \{\perp\}) \cup Q \times \Sigma_r \times \Gamma \times Q \cup Q \times \Sigma_l \times Q$,
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Remark:

- ▶ No **ε -transitions**,
- ▶ Exactly **one symbol** is pushed in **each** call transition.

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A deterministic VPA is **complete** if “at most” is replaced by “**exactly**”.

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A run of a VPA \mathcal{A} over a word $w = a_1 \dots a_n$ is

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Remark: Acceptance of VPAs are defined by final states, not by empty stack.

Well-matched words

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Remark. As a result of the acceptance by final states,

*VPAs over $\tilde{\Sigma}$ may accept **non-well-matched** words.*

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A language $L \subseteq \Sigma^*$ is a **visibly pushdown language with respect to $\tilde{\Sigma}$** if
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Example:

*The language $\{a^n b^n \mid n \geq 1\}$ is a VPL
with respect to $\tilde{\Sigma} = \langle \{a\}, \{b\}, \emptyset \rangle$.*

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Visibly pushdown languages (VPL)

A language $L \subseteq \Sigma^*$ is a **visibly pushdown language with respect to $\tilde{\Sigma}$** if
there is a VPA \mathcal{A} over $\tilde{\Sigma}$, satisfying that $\mathcal{L}(\mathcal{A}) = L$.

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Theorem

$VPL \subsetneq CFL$.

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Proposition. For every CFL $L \subseteq \Sigma^*$, there are a VPL $L' \subseteq (\Sigma')^*$ with respect to some $\widetilde{\Sigma'}$ and a homomorphism $h : (\Sigma')^* \rightarrow \Sigma^*$ such that $L = h(L')$.

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Outline

Visibly pushdown automata (VPA)

Closure properties

Visibly pushdown grammar (VPG)

Logical characterization

Equivalence of NFA and MSO

Equivalence of VPA and MSO_μ

Decision problems

Union and intersection

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Note that CFL are not closed under intersection

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- ▶ The construction will have a component S that is a set of “**summary**” edges that keeps track of what state transitions are possible from a push-transition to the corresponding pop-transition.

Determinization

For every VPA \mathcal{A} , can we construct a deterministic VPA \mathcal{A}' such that $\mathcal{L}(\mathcal{A}) = \mathcal{L}(\mathcal{A}')$?

Theorem. For every VPA \mathcal{A} , a deterministic VPA \mathcal{A}' can be constructed such that $\mathcal{L}(\mathcal{A}) = \mathcal{L}(\mathcal{A}')$. Moreover, if \mathcal{A} has n states, we can construct \mathcal{A}' with $O(2^{n^2})$ states and with stack alphabet of size $O(2^{n^2} \cdot |\Sigma_c|)$.

Let $\mathcal{A} = (Q, \tilde{\Sigma}, \Gamma, \delta, q_0, \perp, F)$ be a VPA. we construct

$\mathcal{A}' = (Q', \tilde{\Sigma}, \Gamma', \delta', (\text{Id}_Q, \{q_0\}), \perp, F')$, where $\text{Id}_Q = \{(q, q) \mid q \in Q\}$

- ▶ Do a **subset construction** but **postpone** handling the push-transitions that \mathcal{A} does.
- ▶ Instead, we store the **call actions** and simulate the push-transitions corresponding to them later, namely at the time of the **corresponding pop-transition**.
- ▶ The construction will have a component S that is a set of “**summary**” edges that keeps track of what state transitions are possible from a push-transition to the corresponding pop-transition.
- ▶ Using the **summary** information, the set of reachable states is updated.

Determinization

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 - ▶ $x \in \Sigma^*$ such that every call in x is matched by a return, but there may be unmatched returns

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- ▶ R_1 is the set of states reachable by \mathcal{A} from any initial state on x
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- ▶ R is the set of states reachable by \mathcal{A} from any initial state on xa_1ya_2z

Illustration of the intuition of the proof of the Theorem

In an obviously way, we can define $(q, \alpha) \xrightarrow{w} (q', \alpha')$:

the reachability of the config. (q', α') from (q, α) by reading w .

Observation. Suppose $(q, \alpha) \xrightarrow{w} (q', \alpha')$ and w is well-matched, then $\alpha = \alpha'$.

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Point I.

*A well-matched word w can be seen as a relation $S_w \subseteq Q \times Q$,
without changing the content of the stack.*

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Point II.

Suppose w is well-matched.

- ▶ $S_\varepsilon = \text{Id}_Q$.
- ▶ If $w = aw'$ with $a \in \Sigma_l$, then
$$S_w = \{(q, q') \mid \exists q''. (q, a, q'') \in \delta, (q'', q') \in S_{w'}\}.$$
Similarly for $w = w'a$.
- ▶ If $w = aw'b$ with $a \in \Sigma_c$ and $b \in \Sigma_r$, then
$$S_w = \{(q, q') \mid \exists q_1, q_2, \gamma. (q, a, q_1, \gamma) \in \delta, (q_1, q_2) \in S_{w'}, (q_2, b, \gamma, q') \in \delta\}.$$

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Point III. Get inspirations from the subset construction for NFAs.

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Question:

What info. should be remembered after reading a word w in a NFA?

Answer:

The set of states reachable from q_0 after reading w .

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Question:

*What info. should be remembered
after reading a word w in a nondeterministic VPA?*

Answer:

Let me think for a while ...

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Consider $w_1aw_2aw_3bb$ s.t. $a \in \Sigma_c, b \in \Sigma_r$ and w_1, w_2, w_3 are well-matched.

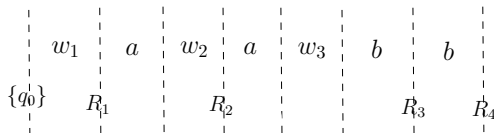


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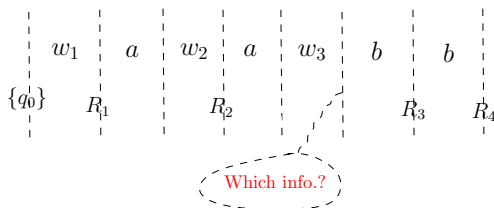


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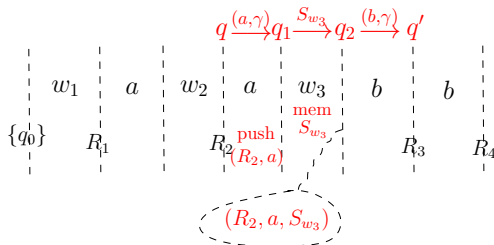


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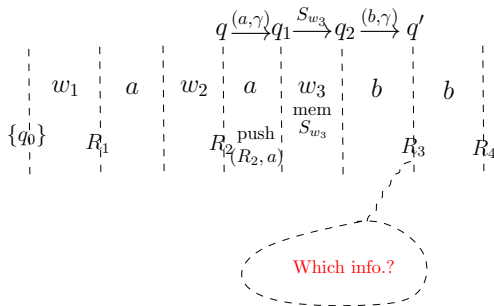


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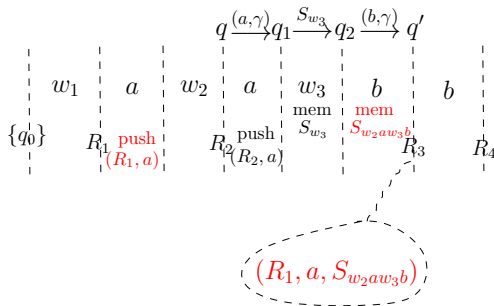


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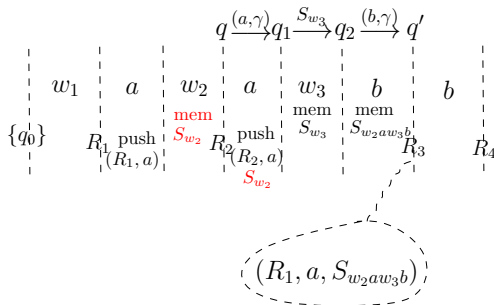
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- ▶ $S' = \text{Id}_Q$ (note $\text{Id}_Q = \{(q, q) \mid q \in Q\}$), the initialization of the summary;
- ▶ $R' = \{q' \mid \exists q \in R, \gamma \in \Gamma : (q, a_3, q', \gamma) \in \delta\}$, all the states reachable by \mathcal{A} after reading a_3

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If $a_3 \in \Sigma_c$, then $((S, R), a_3, (\text{Id}_Q, R'), (S, R, a_3)) \in \delta'$,
where $R' = \{q' \mid \exists q \in R, \gamma \in \Gamma : (q, a_3, q', \gamma) \in \delta\}$.

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- ▶ If a_3 is the current input at configuration $((S, R), (S_2, R_2, a_2)(S_1, R_1, a_1)\perp)$
- ▶ If $a_3 \in \Sigma_r$: \mathcal{A}' goes to $((S', R'), (S_1, R_1, a_1)\perp)$, where S' and R' are defined below.

$$\begin{aligned} & ((S, R), a_3, (S_2, R_2, a_2), (S', R')) \in \delta', \text{ where} \\ S' &= \left\{ (q, q') \mid \begin{array}{l} \exists q_1, q_2, q_3, \gamma \in \Gamma : (q, q_1) \in S_2, (q_2, q_3) \in S, \\ (q_1, a_2, q_2, \gamma) \in \delta, (q_3, a_3, \gamma, q') \in \delta \end{array} \right\}, \\ R' &= \left\{ q' \mid \begin{array}{l} \exists q_1, q_2, q_3, \gamma \in \Gamma : q_1 \in R_2, (q_2, q_3) \in S, \\ (q_1, a_2, q_2, \gamma) \in \delta, (q_3, a_3, \gamma, q') \in \delta \end{array} \right\} \end{aligned}$$

$$(q, q_1) \in S_2: (q, \alpha) \xRightarrow{y}^* (q_1, \alpha)$$

Determinization

- ▶ After reading $w = xa_1ya_2z$, the VPA \mathcal{A}' we construct will reach the configuration

$$\left((S, R), (S_2, R_2, a_2)(S_1, R_1, a_1)\perp \right)$$

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$$(q, q_1) \in S_2: (q, \alpha) \xRightarrow{y}^* (q_1, \alpha) \text{ and } (q_1, a_2, q_2, \gamma) \in \delta: (q_1, \alpha) \xRightarrow{a_2}^* (q_2, \gamma\alpha)$$

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$$\left((S, R), (S_2, R_2, a_2)(S_1, R_1, a_1)\perp \right)$$

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$$\begin{aligned} (q, q_1) \in S_2: (q, \alpha) &\xRightarrow{y}^* (q_1, \alpha) \text{ and } (q_1, a_2, q_2, \gamma) \in \delta: (q_1, \alpha) \xRightarrow{a_2}^* (q_2, \gamma\alpha) \\ (q_2, q_3) \in S: (q_2, \gamma\alpha) &\xRightarrow{z}^* (q_3, \gamma\alpha) \end{aligned}$$

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 - ▶ $S' = \{(q, q') \mid \exists q'' : (q, q'') \in S, (q'', a_3, q') \in \delta\}$, the initialization of the summary;
 - ▶ $R' = \{q' \mid \exists q \in R : (q, a_3, q') \in \delta\}$

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$$(q, \alpha) \xRightarrow{za_3}^* (q', \alpha)$$

Determinization

We construct $\mathcal{A}' = (Q', \tilde{\Sigma}, \Gamma', \delta', (\text{Id}_Q, \{q_0\}), F')$:

Determinization

We construct $\mathcal{A}' = (Q', \tilde{\Sigma}, \Gamma', \delta', (\text{Id}_Q, \{q_0\}), F')$:

- ▶ Q' : (S, R) such that $S \subseteq Q \times Q, R \subseteq Q$,
- ▶ Γ' : letters (S, R, a) such that $S \subseteq Q \times Q, R \subseteq Q, a \in \Sigma_c$,
- ▶ $F' = \{(S, R) \mid R \cap F \neq \emptyset\}$,
- ▶ δ' :

Local if $a \in \Sigma_l$, then $((S, R), a, (S', R')) \in \delta'$, where
 $R' = \{q' \mid \exists q \in R. (q, a, q') \in \delta\}$,
 $S' = \{(q, q') \mid \exists q_1. (q, q_1) \in S, (q_1, a, q') \in \delta\}$.

Call if $a \in \Sigma_c$, then $((S, R), a, (\text{Id}_Q, R'), (S, R, a)) \in \delta'$, where
 $R' = \{q' \mid \exists q \in R, \gamma \in \Gamma. (q, a, q', \gamma) \in \delta\}$.

Return if $a \in \Sigma_r$, then $((S, R), a, (S'', R'', a'), (S', R')) \in \delta'$, where

$$S' = \left\{ (q, q') \mid \begin{array}{c} \exists q_1, q_2, q_3, \gamma \in \Gamma. \\ (q, q_1) \in S'', (q_1, a', q_2, \gamma) \in \delta, (q_2, q_3) \in S, (q_3, a, \gamma, q') \in \delta \end{array} \right\},$$
$$R' = \left\{ q' \mid \begin{array}{c} \exists q_1, q_2, q_3, \gamma \in \Gamma. \\ q_1 \in R'', (q_1, a', q_2, \gamma) \in \delta, (q_2, q_3) \in S, (q_3, a, \gamma, q') \in \delta \end{array} \right\},$$

or $((S, R), a, \perp, (S', R')) \in \delta'$, where

$$S' = \{(q, q') \mid \exists q''. (q, q'') \in S, (q'', a, \perp, q') \in \delta\},$$
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Complementation

Theorem. VPLs with respect to $\tilde{\Sigma}$ are closed under complementation.

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For a VPA \mathcal{A} , first construct a deterministic VPA \mathcal{A}' such that $\mathcal{L}(\mathcal{A}) = \mathcal{L}(\mathcal{A}')$. Then, complement the set of final states.

Outline

Visibly pushdown automata (VPA)

Closure properties

Visibly pushdown grammar (VPG)

Logical characterization

Equivalence of NFA and MSO

Equivalence of VPA and MSO_μ

Decision problems

Visibly pushdown grammar (VPG)

A CFG $G = (\mathcal{N}, \Sigma, \mathcal{P}, S)$ is a VPG over $\tilde{\Sigma}$ if \mathcal{N} can be partitioned into \mathcal{N}_0 and \mathcal{N}_1 , and each rule in \mathcal{P} is of the following forms,

- ▶ $X \rightarrow \varepsilon$,
- ▶ $X \rightarrow aY$ such that if $X \in \mathcal{N}_0$, then $a \in \Sigma_l$, $Y \in \mathcal{N}_0$,
- ▶ $X \rightarrow aYbZ$ such that $a \in \Sigma_c$, $b \in \Sigma_r$, $Y \in \mathcal{N}_0$, and if $X \in \mathcal{N}_0$, then $Z \in \mathcal{N}_0$.

The non-terminals in \mathcal{N}_0 derive **only well-matched words** where there is a one-to-one correspondence between calls and returns.

The non-terminals in \mathcal{N}_1 derive words that can contain **unmatched calls** as well as **unmatched returns**.

Visibly pushdown grammar (VPG)

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- ▶ $X \rightarrow \varepsilon$,
- ▶ $X \rightarrow aY$ such that if $X \in \mathcal{N}_0$, then $a \in \Sigma_l, Y \in \mathcal{N}_0$,
- ▶ $X \rightarrow aYbZ$ such that $a \in \Sigma_c, b \in \Sigma_r, Y \in \mathcal{N}_0$, and if $X \in \mathcal{N}_0$, then $Z \in \mathcal{N}_0$.

The non-terminals in \mathcal{N}_0 derive **only well-matched words** where there is a one-to-one correspondence between calls and returns.

The non-terminals in \mathcal{N}_1 derive words that can contain **unmatched calls** as well as **unmatched returns**.

Example: Let $\tilde{\Sigma} = (\{a\}, \{b\}, \emptyset)$. Then the VPG

$$S \rightarrow aSbC \mid aTbC, T \rightarrow \varepsilon, C \rightarrow \varepsilon,$$

such that $\mathcal{N}_0 = \{S, T, C\}$ defines $\{a^n b^n \mid n \geq 1\}$.

Equivalence of VPA and VPG

Theorem. $\text{VPA} \equiv \text{VPG}$.

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From VPA to VPG.

Let $\mathcal{A} = (Q, \tilde{\Sigma}, \Gamma, \delta, q_0, \perp, F)$ be a VPA.

Equivalence of VPA and VPG

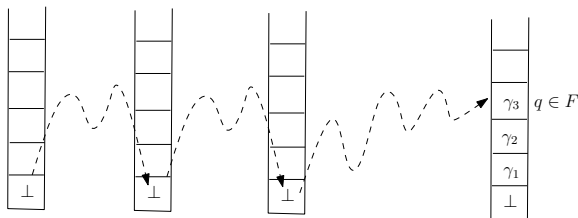
Theorem. $VPA \equiv VPG$.

From VPA to VPG.

Let $\mathcal{A} = (Q, \tilde{\Sigma}, \Gamma, \delta, q_0, \perp, F)$ be a VPA.

The intuition: Utilizing the nonterminals $[q, \gamma, p]$ with the meaning

*the top symbol of the stack is γ ,
and from state q ,
by reading a well-matched word,
state p can be reached.*



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- ▶ $S = (q_0, \perp)$,
- ▶ $\mathcal{N}_0 = \{[q, \gamma, p] \mid q, p \in Q, \gamma \in \Gamma \setminus \{\perp\}\},$

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Construct a VPG $(\mathcal{N}_0, \mathcal{N}_1, \tilde{\Sigma}, \mathcal{P}, S)$ as follows.

- ▶ $S = (q_0, \perp)$,
- ▶ $\mathcal{N}_0 = \{[q, \gamma, p] \mid q, p \in Q, \gamma \in \Gamma \setminus \{\perp\}\}$,
- ▶ $\mathcal{N}_1 = \{(q, \perp) \mid q \in Q\} \cup \{q \mid q \in Q\}$,

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- ▶ $S = (q_0, \perp)$,
- ▶ $\mathcal{N}_0 = \{[q, \gamma, p] \mid q, p \in Q, \gamma \in \Gamma \setminus \{\perp\}\}$,
- ▶ $\mathcal{N}_1 = \{(q, \perp) \mid q \in Q\} \cup \{q \mid q \in Q\}$,
 - ▶ (q, \perp) : the state is q and the stack is **empty**,
 - ▶ q : the state is q and the stack is **nonempty**.

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- ▶ $S = (q_0, \perp)$,
- ▶ $\mathcal{N}_0 = \{[q, \gamma, p] \mid q, p \in Q, \gamma \in \Gamma \setminus \{\perp\}\}$,
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From VPA to VPG.

Let $\mathcal{A} = (Q, \tilde{\Sigma}, \Gamma, \delta, q_0, \perp, F)$ be a VPA.

Construct a VPG $(\mathcal{N}_0, \mathcal{N}_1, \tilde{\Sigma}, \mathcal{P}, S)$ as follows.

- ▶ $S = (q_0, \perp)$,
- ▶ $\mathcal{N}_0 = \{[q, \gamma, p] \mid q, p \in Q, \gamma \in \Gamma \setminus \{\perp\}\}$,
- ▶ $\mathcal{N}_1 = \{(q, \perp) \mid q \in Q\} \cup \{q \mid q \in Q\}$,
 - ▶ (q, \perp) : the state is q and the stack is **empty**,
 - ▶ q : the state is q and the stack is **nonempty**.
- ▶ \mathcal{P} is defined by the following rules,
 - ▶ if $(q, a, q') \in \delta$ s.t. $a \in \Sigma_I$, then
 $(q, \perp) \rightarrow a(q', \perp)$, $q \rightarrow aq'$, $[q, \gamma, p] \rightarrow a[q', \gamma, p]$.

Equivalence of VPA and VPG

From VPA to VPG.

Let $\mathcal{A} = (Q, \tilde{\Sigma}, \Gamma, \delta, q_0, \perp, F)$ be a VPA.

Construct a VPG $(\mathcal{N}_0, \mathcal{N}_1, \tilde{\Sigma}, \mathcal{P}, S)$ as follows.

- ▶ $S = (q_0, \perp)$,
- ▶ $\mathcal{N}_0 = \{[q, \gamma, p] \mid q, p \in Q, \gamma \in \Gamma \setminus \{\perp\}\}$,
- ▶ $\mathcal{N}_1 = \{(q, \perp) \mid q \in Q\} \cup \{q \mid q \in Q\}$,
 - ▶ (q, \perp) : the state is q and the stack is **empty**,
 - ▶ q : the state is q and the stack is **nonempty**.
- ▶ \mathcal{P} is defined by the following rules,
 - ▶ if $(q, a, q') \in \delta$ s.t. $a \in \Sigma_l$, then
 $(q, \perp) \rightarrow a(q', \perp)$, $q \rightarrow aq'$, $[q, \gamma, p] \rightarrow a[q', \gamma, p]$.
 - ▶ if $(q, a, q', \gamma), (p', b, \gamma, p) \in \delta$ s.t. $a \in \Sigma_c, b \in \Sigma_r$, then
 $[q, \gamma_1, r] \rightarrow a[q', \gamma, p']b[p, \gamma_1, r]$, $(q, \perp) \rightarrow a[q', \gamma, p']b(p, \perp)$,
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 $q \rightarrow a[q', \gamma, p']bp$.
 - ▶ if $(q, a, q', \gamma) \in \delta$ s.t. $a \in \Sigma_c$, then
 $(q, \perp) \rightarrow aq'$, $q \rightarrow aq'$, $(q, \perp) \rightarrow a[q', \gamma, p]$, $q \rightarrow a[q', \gamma, p]$.

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From VPA to VPG.

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Construct a VPG $(\mathcal{N}_0, \mathcal{N}_1, \tilde{\Sigma}, \mathcal{P}, S)$ as follows.

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 - ▶ if $(q, a, q') \in \delta$ s.t. $a \in \Sigma_l$, then
 $(q, \perp) \rightarrow a[q', \perp]$, $q \rightarrow aq'$, $[q, \gamma, p] \rightarrow a[q', \gamma, p]$.
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 $[q, \gamma_1, r] \rightarrow a[q', \gamma, p']b[p, \gamma_1, r]$, $(q, \perp) \rightarrow a[q', \gamma, p']b(p, \perp)$,
 $q \rightarrow a[q', \gamma, p']bp$.
 - ▶ if $(q, a, q', \gamma) \in \delta$ s.t. $a \in \Sigma_c$, then
 $(q, \perp) \rightarrow aq'$, $q \rightarrow aq'$, $(q, \perp) \rightarrow a[q', \gamma, p]$, $q \rightarrow a[q', \gamma, p]$.
 - ▶ if $(q, a, \perp, q') \in \delta$ s.t. $a \in \Sigma_r$, then $(q, \perp) \rightarrow a(q', \perp)$.

Equivalence of VPA and VPG

From VPA to VPG.

Let $\mathcal{A} = (Q, \tilde{\Sigma}, \Gamma, \delta, q_0, \perp, F)$ be a VPA.

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 - ▶ if $(q, a, \perp, q') \in \delta$ s.t. $a \in \Sigma_r$, then $(q, \perp) \rightarrow a(q', \perp)$.
 - ▶ $\forall q \in Q. [q, \gamma, q] \rightarrow \varepsilon$,
 - ▶ $\forall q \in F. q \rightarrow \varepsilon, (q, \perp) \rightarrow \varepsilon$.

Equivalence of VPA and VPG: continued

From VPG to VPA.

Let $G = (\mathcal{N}_0, \mathcal{N}_1, \tilde{\Sigma}, \mathcal{P}, S)$ be a VPG.

Construct VPA $\mathcal{A} = (\mathcal{N}, \tilde{\Sigma}, \Sigma_r \times \mathcal{N} \cup \{\perp, \$\}, \delta, S, F)$ as follows.

- ▶ δ is defined by the following rules,
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Equivalence of VPA and VPG: continued

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 - ▶ if $X \rightarrow \varepsilon$ and $X \in \mathcal{N}_0$, then $(X, b, (b, Y), Y) \in \delta$.

Equivalence of VPA and VPG: continued

From VPG to VPA.

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*Adapt \mathcal{A} into $\mathcal{A}' = (\mathcal{N} \times \Gamma, \tilde{\Sigma}, \Gamma, \delta', (S, \perp), \{(X, \gamma) \mid X \rightarrow \varepsilon, \gamma = \$, \perp\})$
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 - ▶ if $X \rightarrow aY$ s.t. $a \in \Sigma_I$, then $\forall \gamma. ((X, \gamma), a, (Y, \gamma)) \in \delta'$,

Equivalence of VPA and VPG: continued

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Equivalence of VPA and VPG: continued

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Equivalence of VPA and VPG: continued

From VPG to VPA.

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Outline

Visibly pushdown automata (VPA)

Closure properties

Visibly pushdown grammar (VPG)

Logical characterization

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Equivalence of VPA and MSO_μ

Decision problems

Outline

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Monadic Second-Order Logic (MSO)

Syntax.

$\varphi := P_\sigma(x) \mid x = y \mid \text{succ}(x, y) \mid X(x) \mid \varphi_1 \vee \varphi_2 \mid \neg \varphi_1 \mid \exists x \varphi_1 \mid \exists X \varphi_1,$

where $\sigma \in \Sigma$, x, y are position (first-order) variables, X is a set (second-order) variables

An MSO **sentence** is a MSO formula without free variables.

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Semantics.

A **structure** \mathcal{S} over Σ is

- ▶ a domain $S = \{1, \dots, n\}$,
- ▶ an interpretation of all the unary predicates $P_\sigma \in \Sigma$ over S , denoted by $(P_\sigma)^\mathcal{S}$.

Monadic Second-Order Logic (MSO)

Syntax.

$\varphi := P_\sigma(x) \mid x = y \mid \text{succ}(x, y) \mid X(x) \mid \varphi_1 \vee \varphi_2 \mid \neg \varphi_1 \mid \exists x \varphi_1 \mid \exists X \varphi_1,$

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A **structure** \mathcal{S} over Σ is

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- ▶ an interpretation of all the unary predicates $P_\sigma \in \Sigma$ over S , denoted by $(P_\sigma)^{\mathcal{S}}$.

Example. Let $\Sigma = \{a, b\}$. Then $\mathcal{S} = (\{1, 2, 3\}, (P_a)^{\mathcal{S}} = \{1\}, (P_b)^{\mathcal{S}} = \{2, 3\})$ is a structure over Σ .

Monadic Second-Order Logic (MSO)

Syntax.

$\varphi := P_\sigma(x) \mid x = y \mid \text{succ}(x, y) \mid X(x) \mid \varphi_1 \vee \varphi_2 \mid \neg \varphi_1 \mid \exists x \varphi_1 \mid \exists X \varphi_1,$

where $\sigma \in \Sigma$, x, y are position (first-order) variables, X is a set (second-order) variables

An MSO **sentence** is a MSO formula without free variables.

Semantics.

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A word $w = a_1 \dots a_n$ can be seen as a structure \mathcal{S}_w over Σ ,

- ▶ the domain of \mathcal{S}_w , denoted by S_w , is $\{1, \dots, n\}$,
- ▶ the interpretation of every $P_\sigma \in \Sigma$ is the set of positions with the letter σ in w .

Monadic Second-Order Logic (MSO)

Syntax.

$\varphi := P_\sigma(x) \mid x = y \mid \text{succ}(x, y) \mid X(x) \mid \varphi_1 \vee \varphi_2 \mid \neg \varphi_1 \mid \exists x \varphi_1 \mid \exists X \varphi_1,$

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An MSO **sentence** is a MSO formula without free variables.

Semantics. Given a MSO formula φ , a **valuation** of $\text{free}(\varphi)$ over a structure \mathcal{S} is a mapping \mathcal{I} such that

- ▶ for every $x \in \text{free}(\varphi)$, $\mathcal{I}(x) \in S$,
- ▶ for every $X \in \text{free}(\varphi)$, $\mathcal{I}(X) \subseteq S$.

Monadic Second-Order Logic (MSO)

Syntax.

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where $\sigma \in \Sigma$, x, y are position (first-order) variables, X is a set (second-order) variables

An MSO **sentence** is a MSO formula without free variables.

Semantics. A MSO formula φ is satisfied over a word $w = a_1 \dots a_n$, with a valuation \mathcal{I} of $\text{free}(\varphi)$ over S_w , denoted by $(w, \mathcal{I}) \models \varphi$, is defined as follows,

- ▶ $(w, \mathcal{I}) \models P_\sigma(x)$ iff $a_{\mathcal{I}(x)} = \sigma$,
- ▶ $(w, \mathcal{I}) \models x = y$ iff $\mathcal{I}(x) = \mathcal{I}(y)$,
- ▶ $(w, \mathcal{I}) \models \text{suc}(x, y)$ iff $\mathcal{I}(x) + 1 = \mathcal{I}(y)$,
- ▶ $(w, \mathcal{I}) \models X(x)$ iff $\mathcal{I}(x) \in \mathcal{I}(X)$,
- ▶ $(w, \mathcal{I}) \models \varphi_1 \vee \varphi_2$ iff $(w, \mathcal{I}) \models \varphi_1$ or $(w, \mathcal{I}) \models \varphi_2$,
- ▶ $(w, \mathcal{I}) \models \neg \varphi_1$ iff not $(w, \mathcal{I}) \models \varphi_1$,
- ▶ $(w, \mathcal{I}) \models \exists x \varphi_1$ iff there is $j \in S_w$ such that $(w, \mathcal{I}[x \rightarrow j]) \models \varphi_1$,
- ▶ $(w, \mathcal{I}) \models \exists X \varphi_1$ iff there is $J \subseteq S_w$ such that $(w, \mathcal{I}[X \rightarrow J]) \models \varphi_1$.

Monadic Second-Order Logic (MSO)

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$\varphi := P_\sigma(x) \mid x = y \mid \text{succ}(x, y) \mid X(x) \mid \varphi_1 \vee \varphi_2 \mid \neg \varphi_1 \mid \exists x \varphi_1 \mid \exists X \varphi_1,$

where $\sigma \in \Sigma$, x, y are position (first-order) variables, X is a set (second-order) variables

An MSO **sentence** is a MSO formula without free variables.

Semantics.

Let φ be a MSO sentence.

The language defined by φ , denoted $\mathcal{L}(\varphi)$: The set of words satisfying φ .

A language $L \subseteq \Sigma^*$ is **MSO-definable**
if

there is a MSO sentence φ such that $\mathcal{L}(\varphi) = L$.

Monadic Second-Order Logic (continued)

Abbreviations.

- ▶ $\varphi_1 \wedge \varphi_2 = \neg(\neg\varphi_1 \vee \neg\varphi_2),$
- ▶ $\varphi_1 \rightarrow \varphi_2 = \neg\varphi_1 \vee \varphi_2,$
- ▶ $\forall x\varphi_1 = \neg\exists x(\neg\varphi_1),$
- ▶ $x < y = \forall X ((X(x) \wedge \overset{\neg x = y \wedge}{\forall z_1 \forall z_2 (X(z_1) \wedge \text{suc}(z_1, z_2) \rightarrow X(z_2))}) \rightarrow X(y)) ,$
- ▶ $\text{first}(x) = \forall y(x = y \vee x < y),$
- ▶ $\text{last}(x) = \forall y(x = y \vee y < x).$

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- ▶ $\varphi_1 \wedge \varphi_2 = \neg(\neg\varphi_1 \vee \neg\varphi_2),$
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Example.

$$\neg\exists x \text{ first}(x), \text{ i.e., } \varepsilon$$

Monadic Second-Order Logic (continued)

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- ▶ $\forall x\varphi_1 = \neg\exists x(\neg\varphi_1),$
- ▶ $x < y = \neg x = y \wedge \forall X ((X(x) \wedge \forall z_1\forall z_2(X(z_1) \wedge \text{succ}(z_1, z_2) \rightarrow X(z_2))) \rightarrow X(y)) ,$
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$$\exists x\exists y(P_a(x) \wedge P_b(y) \wedge x < y),$$

i.e., all the words that has two positions where b occurs later than a

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i.e., all the words that has two positions where b occurs later than a

$$\exists X \left(\begin{array}{c} \exists x(\text{first}(x) \wedge X(x)) \wedge \\ \forall x \forall y \forall z (\text{succ}(x, y) \wedge \text{succ}(y, z) \wedge X(x) \rightarrow X(z)) \\ \wedge \forall x (X(x) \rightarrow P_a(x)) \end{array} \right),$$

i.e., non-empty words and all the even positions have a

NFA \equiv MSO

From NFA to MSO

Let $\mathcal{A} = (Q, \Sigma, \delta, q_0, F)$ be a NFA.

From NFA to MSO

Let $\mathcal{A} = (Q, \Sigma, \delta, q_0, F)$ be a NFA. Let $Q = \{q_0, q_1, \dots, q_n\}$. Construct the MSO formula φ as follows,

$$\exists X_{q_0} \dots X_{q_n} (\varphi_{\text{unique}} \wedge \varphi_{\text{init}} \wedge \varphi_{\text{trans}} \wedge \varphi_{\text{final}}),$$

where

- ▶ X_q stands for the positions where the run is in state q ,
- ▶ $\varphi_{\text{unique}} = \bigwedge_{q \neq q'} \forall x \neg (X_q(x) \wedge X_{q'}(x))$
- ▶ $\varphi_{\text{init}} = \exists x (\text{first}(x) \wedge \bigvee_{(q_0, a, q) \in \delta} (P_a(x) \wedge X_q(x)))$,
- ▶ $\varphi_{\text{trans}} = \forall x \forall y (\text{suc}(x, y) \rightarrow \bigvee_{(q, a, q') \in \delta} X_q(x) \wedge P_a(y) \wedge X_{q'}(y))$,
- ▶ $\varphi_{\text{final}} = \exists x (\text{last}(x) \wedge \bigvee_{q \in F} X_q(x))$.

Then $\mathcal{L}(\varphi) = \mathcal{L}(\mathcal{A})$ if $q_0 \notin F$ and $\mathcal{L}(\varphi \vee \forall x (\neg \text{first}(x))) = \mathcal{L}(\mathcal{A})$ if $q_0 \in F$.

From MSO to NFA.**A normal form for MSO formulas**

New modalities,

$$X \subseteq Y, \text{ Singleton}(X), \text{ suc}(X, Y).$$

Then a MSO formula φ can be transformed into a normal form φ' by the following rules,

- ▶ if $\varphi = P_\sigma(x)$, then $\varphi' = \text{Singleton}(X) \wedge X \subseteq P_\sigma$,
- ▶ if $\varphi = x = y$, then $\varphi' = \text{Singleton}(X) \wedge \text{Singleton}(Y) \wedge X \subseteq Y \wedge Y \subseteq X$,
- ▶ if $\varphi = \text{suc}(x, y)$, then $\varphi' = \text{suc}(X, Y)$,
- ▶ if $\varphi = Z(x)$, then $\varphi' = \text{Singleton}(X) \wedge X \subseteq Z$,
- ▶ if $\varphi = \varphi_1 \vee \varphi_2$, then $\varphi' = \varphi'_1 \vee \varphi'_2$,
- ▶ if $\varphi = \neg\varphi_1$, then $\varphi' = \neg\varphi'_1$,
- ▶ if $\varphi = \exists x\varphi_1$, then $\varphi' = \exists X(\text{Singleton}(X) \wedge \varphi'_1)$,
- ▶ if $\varphi = \exists X\varphi_1$, then $\varphi' = \exists X\varphi'_1$.

From MSO to NFA.

$$\varphi := X \subseteq P_\sigma \mid P_\sigma \subseteq X \mid X \subseteq Y \mid \text{Singleton}(X) \mid \text{succ}(X, Y) \mid \varphi_1 \vee \varphi_2 \mid \neg \varphi_1 \mid \exists X \varphi_1.$$

Let $\varphi(X_1, \dots, X_k)$ be a MSO formula in the prefix normal form.

We construct a NFA $\mathcal{A} = (Q, \Sigma \times \{0, 1\}^k, \delta, q_0, F)$ as follows.

Consider $\varphi(X_1, \dots, X_k)$ with X_1, \dots, X_k free variables, $(a, b_1 \dots b_k)$ at position p such that $a \in \Sigma$ and $b_i \in \{0, 1\}$ denotes that

- ▶ a is the symbol at position p ,
- ▶ $b_i = 1$ denotes that $p \in X_i$
- ▶ $b_i = 0$ denotes that $p \notin X_i$

From MSO to NFA.

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input	a_1	a_2	a_3	a_4	a_5	$a_6 \dots$
X_1	1	0	1	0	1	$0 \dots$
X_2	0	0	1	1	0	$1 \dots$

X_1 : even positions

X_2 : prime positions

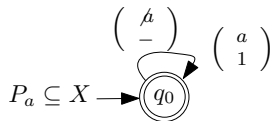
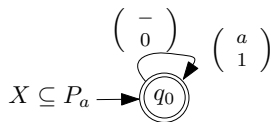
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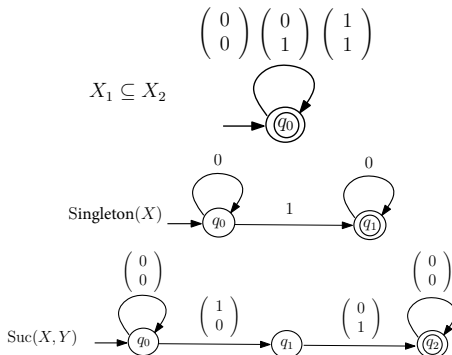
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Let $\varphi(X_1, \dots, X_k)$ be a MSO formula in the normal form.

We construct a NFA $\mathcal{A} = (Q, \Sigma \times \{0, 1\}^k, \delta, q_0, F)$ as follows.

- ▶ $\varphi = \varphi_1 \vee \varphi_2$
NFAs are closed under union,
- ▶ $\varphi = \neg \varphi_1$
NFAs are closed under complementation,
- ▶ $\varphi = \exists X_1 \varphi_1$
NFAs are closed under projection (a special case of homomorphisms),
e.g. $(b_1, \dots, b_k) \rightarrow (b_2, \dots, b_k).$

Then $\mathcal{L}(\varphi) = \mathcal{L}(\mathcal{A})$

Outline

Visibly pushdown automata (VPA)

Closure properties

Visibly pushdown grammar (VPG)

Logical characterization

Equivalence of NFA and MSO

Equivalence of VPA and MSO_μ

Decision problems

Fix $\tilde{\Sigma}$.

Given a word $w = a_1 \dots a_n \in \Sigma^*$, a binary relation $\mu(x, y)$ can be defined such that

$\mu(i, j)$ iff a_i is a call and a_j is a matching return.

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Example. In the word “(()) ((”, $\mu(1, 4)$, $\mu(2, 3)$, $\mu(5, 6)$ hold.

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Syntax of MSO_μ over $\tilde{\Sigma}$.

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where $\sigma \in \Sigma$.

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where $\sigma \in \Sigma$.

Semantics of MSO_μ over $\tilde{\Sigma}$.

► $(w, \mathcal{I}) \models \mu(x, y)$ iff $\mu(\mathcal{I}(x), \mathcal{I}(y))$ holds on w .

Fix $\tilde{\Sigma}$.

Given a word $w = a_1 \dots a_n \in \Sigma^*$, a binary relation $\mu(x, y)$ can be defined such that

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► $(w, \mathcal{I}) \models \mu(x, y)$ iff $\mu(\mathcal{I}(x), \mathcal{I}(y))$ holds on w .

Example. Let $\tilde{\Sigma} = (\{a\}, \{b\}, \{c\})$

$$\forall x (P_a(x) \rightarrow \exists y \exists z (P_b(y) \wedge P_c(z) \wedge x < z \wedge z < y \wedge \mu(x, y)))$$

From VPA to MSO_μ

Let $\mathcal{A} = (Q, \tilde{\Sigma}, \Gamma, \delta, q_0, \perp, F)$ be a VPA, $Q = \{q_0, \dots, q_n\}$, $\Gamma = \{\gamma_1, \dots, \gamma_k\}$.

From VPA to MSO_μ

Let $\mathcal{A} = (Q, \tilde{\Sigma}, \Gamma, \delta, q_0, \perp, F)$ be a VPA, $Q = \{q_0, \dots, q_n\}$, $\Gamma = \{\gamma_1, \dots, \gamma_k\}$. Define $\varphi := \exists X_{q_0} \dots X_{q_n} P_{\gamma_1} \dots P_{\gamma_k} (\varphi_{\text{unique}} \wedge \varphi_{\text{init}} \wedge \varphi_{\text{trans}} \wedge \varphi_{\text{final}})$ as follows,

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- ▶ X_q stands for the positions where the run is in state q
- ▶ P_γ stands for the positions where r is pushed/popped

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- ▶ X_q stands for the positions where the run is in state q
- ▶ P_γ stands for the positions where r is pushed/popped
- ▶ $\varphi_{\text{unique}} = \bigwedge_{q \neq q'} \forall x \neg (X_q(x) \wedge X_{q'}(x)) \wedge \bigwedge_{\gamma \neq \gamma'} \forall x \neg (P_\gamma(x) \wedge P_{\gamma'}(x))$

From VPA to MSO_μ

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- ▶ X_q stands for the positions where the run is in state q
- ▶ P_γ stands for the positions where r is pushed/popped
- ▶ $\varphi_{unique} = \bigwedge_{q \neq q'} \forall x \neg (X_q(x) \wedge X_{q'}(x)) \wedge \bigwedge_{\gamma \neq \gamma'} \forall x \neg (P_\gamma(x) \wedge P_{\gamma'}(x))$

$$\text{▶ } \varphi_{init} = \exists x \left(\text{first}(x) \wedge \left(\begin{array}{c} \bigvee_{(q_0, a, q) \in \delta} (P_a(x) \wedge X_q(x)) \bigvee \\ \bigvee_{(q_0, a, q, \gamma) \in \delta} (P_a(x) \wedge X_q(x) \wedge P_\gamma(x)) \bigvee \\ \bigvee_{(q_0, a, \perp, q) \in \delta} (P_a(x) \wedge X_q(x) \wedge P_\perp(x)) \end{array} \right) \right),$$

From VPA to MSO_μ

Let $\mathcal{A} = (Q, \tilde{\Sigma}, \Gamma, \delta, q_0, \perp, F)$ be a VPA, $Q = \{q_0, \dots, q_n\}$, $\Gamma = \{\gamma_1, \dots, \gamma_k\}$.

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- $\varphi_{\text{trans}} = \forall x \forall y (\text{succ}(x, y) \rightarrow \psi_{\text{call}} \vee \psi_{\text{return}} \vee \psi_{\text{local}}),$

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- ▶ $\varphi_{\text{trans}} = \forall x \forall y (\text{succ}(x, y) \rightarrow \psi_{\text{call}} \vee \psi_{\text{return}} \vee \psi_{\text{local}})$, where
 - ▶ $\psi_{\text{call}} = \bigvee_{(q, a, q', \gamma) \in \delta} (X_q(x) \wedge P_a(y) \wedge X_{q'}(y) \wedge P_\gamma(y))$,

From VPA to MSO_μ

Let $\mathcal{A} = (Q, \tilde{\Sigma}, \Gamma, \delta, q_0, \perp, F)$ be a VPA, $Q = \{q_0, \dots, q_n\}$, $\Gamma = \{\gamma_1, \dots, \gamma_k\}$.

Define $\varphi := \exists X_{q_0} \dots X_{q_n} P_{\gamma_1} \dots P_{\gamma_k} (\varphi_{\text{unique}} \wedge \varphi_{\text{init}} \wedge \varphi_{\text{trans}} \wedge \varphi_{\text{final}})$ as follows,

- ▶ $\varphi_{\text{trans}} = \forall x \forall y (\text{succ}(x, y) \rightarrow \psi_{\text{call}} \vee \psi_{\text{return}} \vee \psi_{\text{local}})$, where
 - ▶ $\psi_{\text{call}} = \bigvee_{(q, a, q', \gamma) \in \delta} (X_q(x) \wedge P_a(y) \wedge X_{q'}(y) \wedge P_\gamma(y))$,
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► $\varphi_{\text{final}} = \exists x \left(\text{last}(x) \wedge \bigvee_{q \in F} X_q(x) \right).$

From VPA to MSO_μ

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Then $\mathcal{L}(\varphi) = \mathcal{L}(\mathcal{A})$ if $q_0 \notin F$ and $\mathcal{L}(\varphi \vee \forall x (\neg \text{first}(x))) = \mathcal{L}(\mathcal{A})$ if $q_0 \in F$.

VPA \equiv MSO_μ : continued

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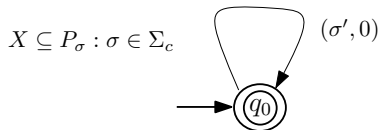
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where \downarrow =push, \uparrow =pop. If push, $\sigma' \in \Sigma_c$; If pop, $\sigma' \in \Sigma_r$; otherwise $\sigma' \in \Sigma_l$; If σ'' in stack, $\sigma'' \in \Sigma_c$.

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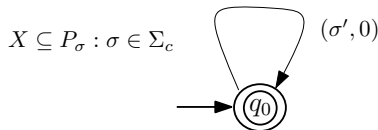
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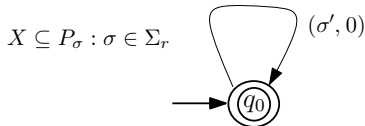
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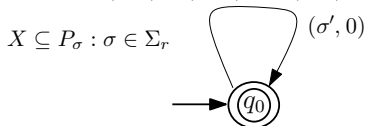
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VPA \equiv MSO_μ : continued

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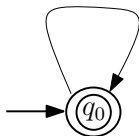
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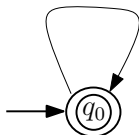
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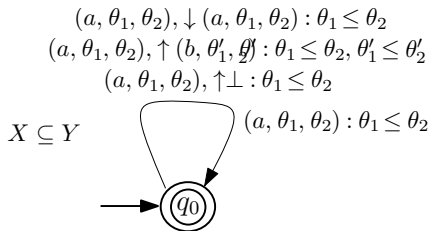
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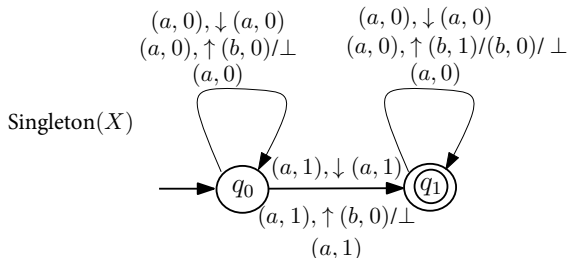
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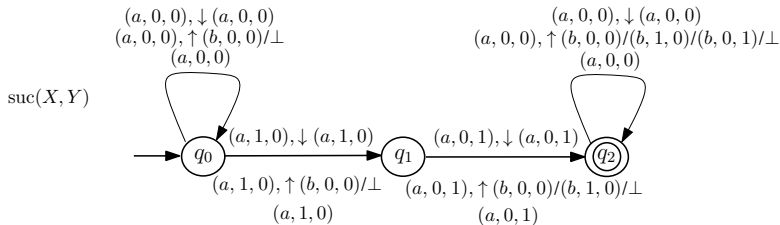
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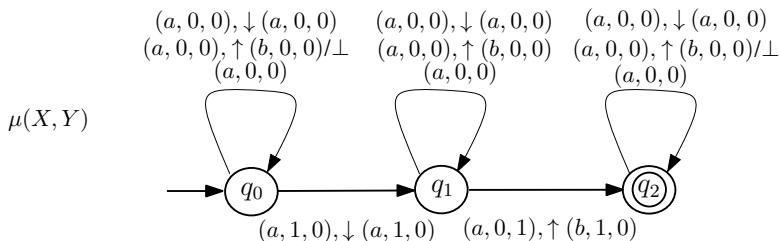
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Satisfiability of MSO and MSO_μ

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- ▶ **Upper Bound** [Büchi, Elgot, Trakhtenbrot, 1957-8 (independently)]:
Nonelementary Growth $2^{\dots^{2^n}}$, (tower of height $O(n)$), due to complementation
- ▶ **Lower Bound** [Stockmeyer, 1974]: Satisfiability of FO over finite words is **nonelementary** (no bounded height tower).

Outline

Visibly pushdown automata (VPA)

Closure properties

Visibly pushdown grammar (VPG)

Logical characterization

Equivalence of NFA and MSO

Equivalence of VPA and MSO_μ

Decision problems

Nonemptiness

Theorem. The nonemptiness of VPA can be solved in $O(n^3)$ time.

A VPA can be transformed into an equivalent VPG in $O(n^3)$ time.

The emptiness of a CFG can be solved in linear time.

Language inclusion and universality problems

Theorem. The language inclusion problem and universality problem of VPA is EXPTIME-complete.

Upper bound.

Given two VPAs \mathcal{A}_1 and \mathcal{A}_2 ,

- ▶ determinize \mathcal{A}_2 into \mathcal{A}'_2 ,
- ▶ complement \mathcal{A}'_2 into \mathcal{B} ,
- ▶ test whether $\mathcal{L}(\mathcal{A}_1) \cap \mathcal{L}(\mathcal{B}) = \emptyset$.

Language inclusion and universality problems

Theorem. The language inclusion problem and universality problem of VPA is EXPTIME-complete.

Upper bound.

Given two VPAs \mathcal{A}_1 and \mathcal{A}_2 ,

- ▶ determinize \mathcal{A}_2 into \mathcal{A}'_2 ,
- ▶ complement \mathcal{A}'_2 into \mathcal{B} ,
- ▶ test whether $\mathcal{L}(\mathcal{A}_1) \cap \mathcal{L}(\mathcal{B}) = \emptyset$.

$$\mathcal{L}(\mathcal{A}_1) \subseteq \mathcal{L}(\mathcal{A}_2) \iff \mathcal{L}(\mathcal{A}_1) \cap \mathcal{L}(\mathcal{B}) = \emptyset$$

$$\mathcal{L}(\mathcal{A}_2) = \tilde{\Sigma}^* \iff \mathcal{L}(\mathcal{B}) = \emptyset$$

The determinization procedure can be fulfilled in EXPTIME.

Language inclusion and universality problems

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The determinization procedure can be fulfilled in EXPTIME. To show EXPTIME-hardness, we show that universality of VPA is EXPTIME-hard.

$$\mathcal{L}(\mathcal{A}) = \tilde{\Sigma}^* \iff \mathcal{L}(\mathcal{A}) \subseteq \mathcal{L}(\mathcal{A}')$$

where $\mathcal{L}(\mathcal{A}') = \tilde{\Sigma}^*$

Language inclusion and universality problems

Theorem. The language inclusion of VPA is EXPTIME-complete.

Lower bound.

The universality of VPA is EXPTIME-hard.

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Result from complexity theory: **APSPACE** = EXPTIME.

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An **alternating** TM (ATM) is a TM $M = (Q_{\exists}, Q_{\forall}, \Sigma, \Gamma, \delta, q_0, B, F)$ such that

- ▶ the state set is divided into two disjoint subsets, Q_{\exists} (“existential” state), Q_{\forall} (“universal” state),
- ▶ for every $q \in Q$ and $a \in \Gamma$, $|\delta(q, a)| = 2$.

A **run** of an ATM M over an input $w \in \Sigma^*$ is a **configuration tree** s.t.

- ▶ the root of the tree is the initial configuration,
- ▶ we assume that for every node (configuration) $\alpha q \beta$ in the tree, if $q \in Q_{\exists}$, then
 $\alpha q \beta$ has one of its successor config. as its **unique** child in the tree,
- ▶ for every node (configuration) $\alpha q \beta$ in the tree, if $q \in Q_{\forall}$, then
 the **two** successor config. of $\alpha q \beta$ are both its children in the tree.

APSPACE: The class of languages accepted by ATMs using polynomial space.

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Reduction from

the membership problem of alternating TMs using polynomial space.

Let $M = (Q_{\exists}, Q_{\forall}, \Sigma, \Gamma, \delta, q_0, B, F)$ be an ATM using linear space, say cn .

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- We will construct a VPA \mathcal{B} that accepts all the **non-accepting computation histories** of M over an input w , then

$$\mathcal{L}(\mathcal{B}) = \tilde{\Sigma}^* \iff M \text{ does not accept } w$$

- To construct \mathcal{B} , we first construct a deterministic VPA \mathcal{A} that accepts **all the accepting computation histories** of M over an input w , then **complement** \mathcal{A}

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Use C_x 's (where $x \in \{0, 1\}^*$) to denote the nodes of t ,

e.g. the root is C_ε , while the left child of the root is C_0 , and so on.

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Encode t by a word θ which is generated by a DFS traversal of t .

Language inclusion and universality problems

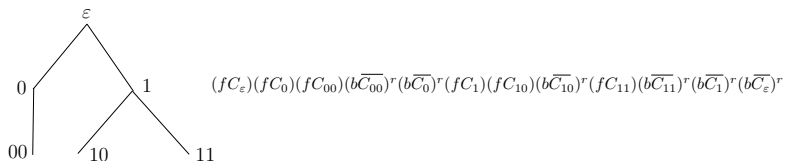
Theorem. The language inclusion of VPA is EXPTIME-complete.

Lower bound.

The universality of VPA is EXPTIME-hard.

Initially set $\theta = \varepsilon$.

1. The traversal starts from the root C_ε .
2. When a node C_x is visited for the **first** time, then $\theta = \theta(fC_x)$,
3. When a node C_x is visited again by **backtracking from its right-child**, then $\theta = \theta(b\overline{C_x})^r$.
4. Each leaf is an accepting configuration.



Language inclusion and universality problems

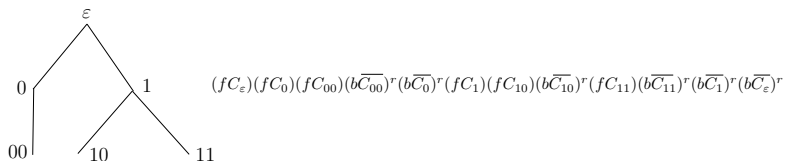
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Let $\Gamma' = \Gamma \cup Q \cup \overline{\Gamma} \cup \overline{Q} \cup \{f, b\}$, $\tilde{\Gamma}' = \langle \Gamma \cup Q \cup \{f\}, \overline{\Gamma} \cup \overline{Q} \cup \{b\} \rangle$.

Language inclusion and universality problems

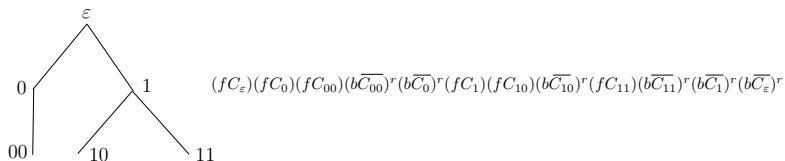
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The format of a successful computation θ , e.g. *well-matched call-returns*.

Language inclusion and universality problems

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Lower bound.

The universality of VPA is EXPTIME-hard.

The set of unsuccessful computations of M

can be accepted by a nondeterministic VPA \mathcal{B} of polynomial size.

M does not accept w iff $\mathcal{L}(\mathcal{B}) = (\Gamma')^$.*

Equivalence problem

Theorem. The equivalence of VPA is EXPTIME-complete.

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Upper bound.

Invoke two times of inclusion testing.

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$$\mathcal{L}(\mathcal{A}) = \tilde{\Sigma}^* \iff \mathcal{L}(\mathcal{A}) = \mathcal{L}(\mathcal{A}')$$

where $\mathcal{L}(\mathcal{A}') = \tilde{\Sigma}^*$

Summary

Closure Properties

	Union	Intersection	Complement	Concatenation	Kleene-*
Regular	YES	YES	YES	YES	YES
CFL	YES	NO	NO	YES	YES
DCFL	NO	NO	YES	NO	NO
VPL	YES	YES	YES	YES	YES

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Decision problems

	Emptiness	Universality/Equivalence	Inclusion
NFA	NL	PSPACE	PSPACE
PDA	P	Undecidable	Undecidable
DPDA	P	Decidable	Undecidable
VPA	P	EXPTIME	EXPTIME