Online Lecture Notes

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1 Announcements

This lecture will be online until further notice. The new lecture is, as today from 9:00 to 10:20. The plan for the coming lectures is to repeat the material from Lecture 1 to 5 and also complete Lecture before the mid-term exam.

1.1 Mid-Term Exam

It seems that we have to run the mid-term exam, as there are no changes to the situation. The tentative plan would be to schedule the mid-term exam on

- Mid-Term Exam: April 28, 2022.

The idea is to basically convert the mid-term exam into "24-hour-take-home exam"—so, basically, this will be a homework, but you only have one day to submit the solutions by email. The exam will start on April 28 at 9:00 am by us sending you the exam questions by skype/tencent/email. There will be no lecture on April 28, but we use the lecture directly to start working on the problems. This means that the submission deadline for sending us answers to the mid-term exam questions would be on April 29, at 8:59. The material will be about Lecture 1–6. This will be a very similar to a homework—the grading will be exactly as planned.

2 Summary of important results from Lecture 2

Recall that Lecture 2 was about scalar linear open-loop control systems of the form

$$\dot{x}(t) = Ax(t) + Bu(t) \ .$$

The goal is to analyze this system. Here, the initial state x_0 might or might be given depending on the context. In Lecture 2, the coefficients $A = a$ and $B = b$ are both scalar,

$$\dot{x}(t) = ax(t) + bu(t) \ ,$$

while Lecture 6 will revisit the same problem for the multivariate case. In this scalar case, we learned in Lecture 2 that

$$x(t) = e^{at}x(0) + \int_0^t e^{a(t-\tau)} bu(\tau)\,\mathrm{d}\tau$$

is the unique explicit solution for x in dependence on $x(0)$ for a given control input function u. One of the key points that we made in Lecture 2 is that the function x depends linearly on u. This means that the so-called linear superposition principle holds: if u is given in parametric form

$$u(t) = \sum_{i=0}^{n} v_i \varphi(t)$$

with coefficients $v_0, v_1, \ldots, v_n$ and basis functions $\varphi_0, \varphi_1, \ldots, \varphi_n$, then the corresponding output

$$\begin{aligned}
x(t) &= e^{at}x(0) + \int_0^t e^{a(t-\tau)} bu(\tau)\,\mathrm{d}\tau && (1)\\
&= e^{at}x(0) + \int_0^t e^{a(t-\tau)} b \left[\sum_{i=0}^{n} v_i \varphi(\tau) \right] \mathrm{d}\tau && (2)\\
&= e^{at}x(0) + \sum_{i=0}^{n} v_i \underbrace{b \int_0^t e^{a(t-\tau)} \varphi(\tau)\,\mathrm{d}\tau}_{=\Phi_i(t)} && (3)\\
&= e^{at}x(0) + \sum_{i=0}^{n} v_i \Phi_i(t) && (4)
\end{aligned}$$

If $x(0) = 0$, this becomes even easier to remember, since

$$u(t) = \sum_{i=0}^{n} v_i \varphi(t) \qquad \Longrightarrow \qquad x(t) = \sum_{i=0}^{n} v_i \Phi_i(t) \ .$$

This is called the linear superposition principle. If you understood the derivation up to this point, you have basically understood all the theory material of Lecture 2. The rest of Lecture was merely about how to apply this formula in different context, for different basis functions φ_i. We essentially discussed:

1. Piecewise constant inputs $u(t)$, for instance by setting

   $$\varphi_i(t) = \begin{cases} 1 & \text{if } t \in [t_i, t_{i+1}] \\ 0 & \text{otherwise,} \end{cases}$$

   where $[t_i, t_{i+1}]$ denotes the i-th interval on which u is constant.

2. The second example is that $u(t)$ could be a superposition of trigonometric functions

$$u(t) = \sum_{k=0}^{n} \left[v_{2k+1} \cos(k\omega t) + v_{2k} \sin(k\omega t) \right]$$

where $\omega > 0$ is a given frequency. This expression can be interpreted as a (finite) discrete Fourier approximation of a periodic input signal. In this case our basis functions

$$\forall k \in \mathbb{N}, \qquad \varphi_{2k+1}(t) = \cos(k\omega t) \quad \text{and} \quad \varphi_{2k}(t) = \sin(k\omega t)\ .$$

In principle, we could even do this for $n = \infty$ as long as the corresponding limits exist.

The corresponding explicit formulas for the functions

$$\Phi_i(t) = b \int_0^t e^{a(t-\tau)} \varphi(\tau) \, \mathrm{d}\tau$$

have been worked out in Lecture 2. The basic tip is really: the details of working out the integrals may be cumbersome, but this is straightforward in princple as long as we can find explicit expressions for these integrals. The basic theory is really simple, but most of the work is actually to work the above integral explicitly (this may be the case for some of our exercise / exam questions). For instance, if you want to work with this out for the discrete Fourier transform, it is helpful to use complex numbers,

$$\cos(k\omega t) = \mathrm{Re}(e^{ik\omega t}) \qquad \text{and} \qquad \sin(k\omega t) = \mathrm{Im}(e^{ik\omega t}),$$

with $i = \sqrt{-1}$.

2.1 Control systems with additional disturbance term

Notice that the above control system model is, in practice, often augmented by an additional input w, which is chosen by “nature”. This means that we can choose the input $u(t)$ but someone else (often “nature”) chooses the input $w(t)$. An example would be: we are controlling a airplane, which means that there are several input u that we can choose (for position of ailerons, elevators, thrusters, ...), but the external distubance $w(t)$, for instance a wind turbulence, cannot be chosen by us. Very often, in practice, we don’t know what w is. In many cases it is sufficient to model such situations by using linear model. Here, we discuss the scalar case

$$\dot{x}(t) = ax(t) + bu(t) + cw(t)\ .$$

In order to analyze such system, we can, at least in principle use the same technique as we discussed in Lecture 2, since here both u and w enter linearly. This means that the superposition principle still holds,

$$x(t) = e^{at}x(0) + b \int_0^t e^{a(t-\tau)} u(\tau) \, \mathrm{d}\tau + c \int_0^t e^{a(t-\tau)} w(\tau) \, \mathrm{d}\tau$$

So, essentially this means that $u(t)$ does not affect the disturbance term

$$c \int_0^t e^{a(t-\tau)} w(\tau) \, \mathrm{d}\tau$$

as this term only depends on $w(t)$—but we cannot choose w. So, basically, in the open-loop case, there is nothing that we can do about this term—it will always enter additively. In the open-loop case, this only works well if $a < 0$ and $w(t) \leq \overline{w}$ is bounded, since we then have

$$\begin{aligned}
\left| c \int_0^t e^{a(t-\tau)} w(\tau)\,\mathrm{d}\tau \right| &\leq c\overline{w} \left| \int_0^t e^{a(t-\tau)}\,\mathrm{d}\tau \right| \leq c\overline{w}e^{at} \left| \left| -\frac{1}{a} e^{-a\tau} \right|_0^t \right| \\
&= c\overline{w}e^{at} \left| \left[-\frac{1}{a} e^{-at} + \frac{1}{a} \right] \right| = c\overline{w} \left| \left[-\frac{1}{a} + \frac{e^{at}}{a} \right] \right| \\
&\leq \left| \frac{c}{a} \right| \overline{w} \qquad (5)
\end{aligned}$$

This means that if $a < 0$ is negative, the influence of the bounded uncertainty remains bounded. There is actually a name for this result: it's called the bounded-input-bounded-output lemma. This states that if $a < 0$ and w bounded, then x is bounded. However, if $a < 0$ is close to zero, this term could still be very large. The amplification factor depends on the ratio $\left|\frac{c}{a}\right|$.

3 Summary of important results from Lecture 3

The main idea of Lecture 3 was to introduce closed-loop systems. There is not much new math in this lecture, but this is really all about developing an understanding of the difference between open-loop and closed-loop controlled systems. The main difference is that

- In the open-loop case we choose the input $u(t)$ in advance without measuring $x(t)$ and apply it "blindly" to the system. This means that we cannot react to any events or external distubances that effect out system behavior. Whenever you can meausre $x(t)$ online, one would try to close to loop and not use open-loop controls anymore.

- The alternative is to control the system in "closed-loop" mode. This means that we first measure $x(t)$ at time t and then choose $u(t)$ in dependence on $x(t)$. Mathematically, this means that now $u(t)$ depends on $x(t)$. The corresponding relation is called the feedback law

  $$u(t) = \mu(x(t)) \; .$$

  This means that we do not choose the function u, but we are choosing the function μ. This can help us to compensate disturbances online.

There are many ways to choose the feedback law μ. However, in the easiest case we can use an affine function of the form

$$\mu(x) = k\left[x - x_{\mathrm{s}}\right] + u_{\mathrm{s}} \; ,$$

where the pair $(x_{\mathrm{s}}, u_{\mathrm{s}})$ denotes a steady-state (tracking point), such that

$$a x_{\mathrm{s}} + b u_{\mathrm{s}} = 0 \; .$$

The coefficient k is called the proportional control gain. For our mathematical analysis we may assume $x_{\mathrm{s}} = 0$ and $u_{\mathrm{s}} = 0$, since we can always correct the linear differential by introducing constant offsets (see Lecture 1 or 2). In this case, the closed-loop system takes the form

$$\dot{x}(t) = ax(t) + b\mu(t) + cw(t) \ .$$

If $\mu(x) = kx$, this simplifies to

$$\dot{x}(t) = ax(t) + bkx(t) + cw(t) = (a + bk)x(t) + cw(t) \ .$$

This means that our closed-loop system response is given by

$$x(t) = e^{(a+bk)t}x(0) + \int_0^t e^{(a+bk)(t-\tau)}cw(\tau)\,\mathrm{d}\tau \ .$$

The bounded-input-bounded-output lemma yields

$$|x(t)| \leq |e^{(a+bk)t}x(0)| + \left|\frac{c}{a+bk}\right| \overline{w}$$

for bounded uncertainty inputs $w(t) \leq \overline{w}$ and $a + bk < 0$. In the scalar case, if $b \neq 0$, this can always be achieved by choosing an appropriate feedback gain k, such that $a + bk < 0$.