EE160 Homework 5 Solution

1. *(4 points) Properties of Matrix Exponentials.*

Solution:

(a) Diagonalizable (general) matrix A can be written in form of

$$A = TDT^{-1} \quad (A = T(D+N)T^{-1})$$

where T is invertiable, D is diagonal (and N is nilpotent). Matrix exponential

$$e^A = Te^D T^{-1} \quad \Rightarrow \quad \det(e^A) = \det(T)\det(e^D)\det(T^{-1}) \tag{1}$$

$$e^A = Te^D e^N T^{-1} \quad \Rightarrow \quad \det(e^A) = \det(T)\det(e^D)\det(e^N)\det(T^{-1}) \tag{2}$$

and e^D is also a diagonal matrix with diagonal element

$$(e^D)_{kk} = e^{D_{kk}}$$

then determinant

$$\det(e^D) = \prod_{k=1}^{n} e^{D_{kk}} = e^{\left(\sum_{k=1}^{n} D_{kk}\right)} = e^{\mathrm{Tr(D)}} \ ,$$

and

$$\mathrm{Tr}(A) = \mathrm{Tr}(TDT^{-1}) = \mathrm{Tr}(DTT^{-1}) = \mathrm{Tr}(D) \ ,$$

here we switch D and T, this operation preserves the trace of matrix product. Nilpotent matrix N in Jordan form is upper triangular with all zero diagonal elements and it's matrix exponential is also upper triangular but with diagonal elements all equals 1 (we can see this easily from exponential's finite expansion terms) thus $\det(e^N) = 1$.

(b) Let function $X(t) = e^{At}$ and it satisfies

$$\dot{X}(t) = AX(t) \quad \Longrightarrow \quad \frac{\mathrm{d}^m}{\mathrm{d}t^m} X(t) = A^m X(t), \ \forall m \in \mathbb{N}_+ \ .$$

Assume that order of polynomial function $X(t)$ is n, then $(n+1)$-th derivative of $X(t)$ must be zero, which means

$$A^m X(t) = 0 \ ,$$

because when $t = 0$ this equality still holds for arbitrary vector $X(0)$, the only possibility is $A^m = 0$ and A is nilpotent.

2. *(2 points) Explicit solution of linear time-invariant differential equations.*

Solution: Let $x = [x_1, x_2]$ be the stacked variable, then the differential equations system can be written in a compact form

$$\begin{cases} \dot{x}(t) = Ax(t) \\ x(0) = [0 \ 1]^\top \end{cases} \quad \text{with} \quad A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and solution is given by

$$x(t) = e^{At} x(0) \ .$$

Recall that we've given a very detailed discussion for the reverse situation [seen in example 2.2 from Lecture notes March22] that is

$$e^{-At} = e^{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} t} = \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix}$$

and use the results that orthogonality of e^{-At} and invertiability of matrix exponential, we get

$$e^{At} = (e^{-At})^{-1} = (e^{-At})^\top = \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix}$$

then the solution

$$x(t) = e^{At}x(0) = \begin{pmatrix} -\sin(t) \\ \cos(t) \end{pmatrix}$$

3. *(4 points) High order time-invariant differential equation.*

Solution: To eliminate high order derivative term, we introduce the following variables

$$\begin{aligned} y_1(t) &= x(t) \\ y_2(t) &= \dot{x}(t) \\ y_3(t) &= \ddot{x}(t) \end{aligned}$$

then an equivalent first order differential equations system can be written by

$$\begin{cases} \dot{y}_1(t) = y_2(t) \\ \dot{y}_2(t) = y_3(t) \\ \dot{y}_3(t) = y_1(t) \end{cases} \quad \text{or in standard form} \quad \dot{y} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} y = Ay$$

stacked variable $y = [y_1 \ y_2 \ y_3]^\top$. Since no initial value is provided, solutions of the above linear differential equations is in a general form of

$$y(t) = e^{At}v \ ,$$

where v is an arbitrary 3-dim vector. To get matrix exponential e^{At}, let's first check if A is diagonalizable,

$$\det(\lambda I - A) = \det\left(\begin{pmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ -1 & 0 & \lambda \end{pmatrix}\right) = \lambda^3 - 1$$

with three distinctive roots $\lambda_1 = 1$ and $\lambda_2 = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$, $\lambda_3 = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$, so A is diagonalizable and $A = TDT^{-1}$ with

$$T = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & \frac{-1-\sqrt{3}i}{2} & \frac{-1+\sqrt{3}i}{2} & 1 & \lambda_2 & \lambda_3 \\ 1 & \frac{-1+\sqrt{3}i}{2} & \frac{-1-\sqrt{3}i}{2} & 1 & \lambda_3 & \lambda_2 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \quad \text{and} \quad T^{-1} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ \lambda_3 & \lambda_2 & 1 \\ \lambda_2 & \lambda_3 & 1 \end{pmatrix}$$

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \quad \text{and} \quad e^{Dt} = \begin{pmatrix} e^t & 0 & 0 \\ 0 & e^{\lambda_2 t} & 0 \\ 0 & 0 & e^{\lambda_3 t} \end{pmatrix}$$

Notice that $\lambda_2, \lambda_3 = \frac{-1\pm\sqrt{3}i}{2}$ which are roots of $\lambda^3 = 1$, the following equations can help simplify the result

$$\lambda_2^2 = \lambda_3, \ \lambda_3^2 = \lambda_2, \ \lambda_2\lambda_3 = 1, \ \lambda_2 + \lambda_2 = -1 \ .$$

Now matrix exponential can be computed explicitly

$$e^{At} = Te^{Dt}T^{-1} = \frac{1}{3}\begin{pmatrix} e^t + 2e^{\frac{t}{2}}c & e^t + e^{\frac{t}{2}}(c+\sqrt{3}s) & e^t + e^{\frac{t}{2}}(c-\sqrt{3}s) \\ e^{\frac{t}{2}}(c-\sqrt{3}s) & e^t + 2e^{\frac{t}{2}}c & e^t + e^{\frac{t}{2}}(c+\sqrt{3}s) \\ e^t + e^{\frac{t}{2}}(c+\sqrt{3}s) & e^{\frac{t}{2}}(c-\sqrt{3}s) & e^t + 2e^{\frac{t}{2}}c \end{pmatrix} \,,$$

with shorthands

$$c = \cos\left(\frac{\sqrt{3}}{2}t\right) \quad \text{and} \quad s = \sin\left(\frac{\sqrt{3}}{2}t\right) \,.$$

We can see that arbitrary vector v gives arbitrary linear combination of these three terms

$$e^t,\ e^{-\frac{t}{2}}\cos\left(\frac{\sqrt{3}}{2}t\right) \text{ and } e^{-\frac{t}{2}}\sin\left(\frac{\sqrt{3}}{2}t\right)$$

so all the solutions $x(t)$ are in form of

$$x(t) = \omega_1 e^t + \omega_2 e^{-\frac{t}{2}}\cos\left(\frac{\sqrt{3}}{2}t\right) + \omega_3 e^{-\frac{t}{2}}\sin\left(\frac{\sqrt{3}}{2}t\right) \,,$$

where $\omega_1, \omega_2, \omega_3$ are arbitrary constants.