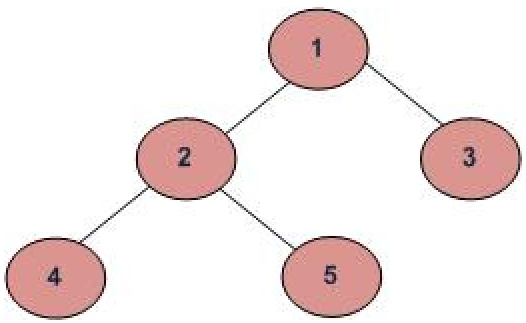
CS101 Data Structures

Binary Trees

Textbook Ch B.5.3, 10.4

Orders of DFS traversal

- Inorder (Left, Root, Right) : 4 2 5 1 3
- Preorder (Root, Left, Right) : 1 2 4 5 3
- Postorder (Left, Right, Root) : 4 5 2 3 1
- Breadth-First or Level Order Traversal: 1 2 3 4 5



Outline

- Binary tree
- Perfect binary tree
- Complete binary tree
- Left-child right-sibling binary tree

Outline

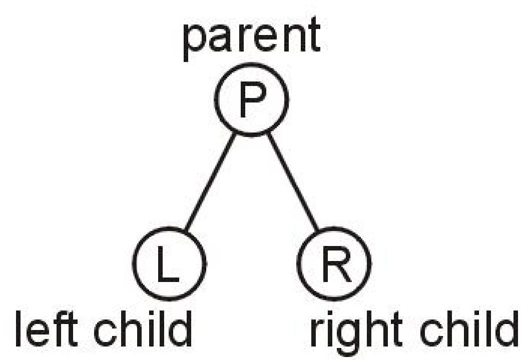
In this talk, we will look at the binary tree data structure:

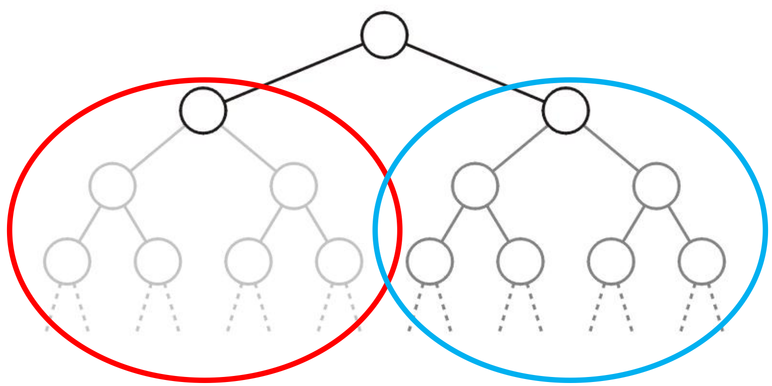
- Definition
- Properties
- Application
  - Expression trees

Definition

A binary tree is a restriction where each node has exactly two children:

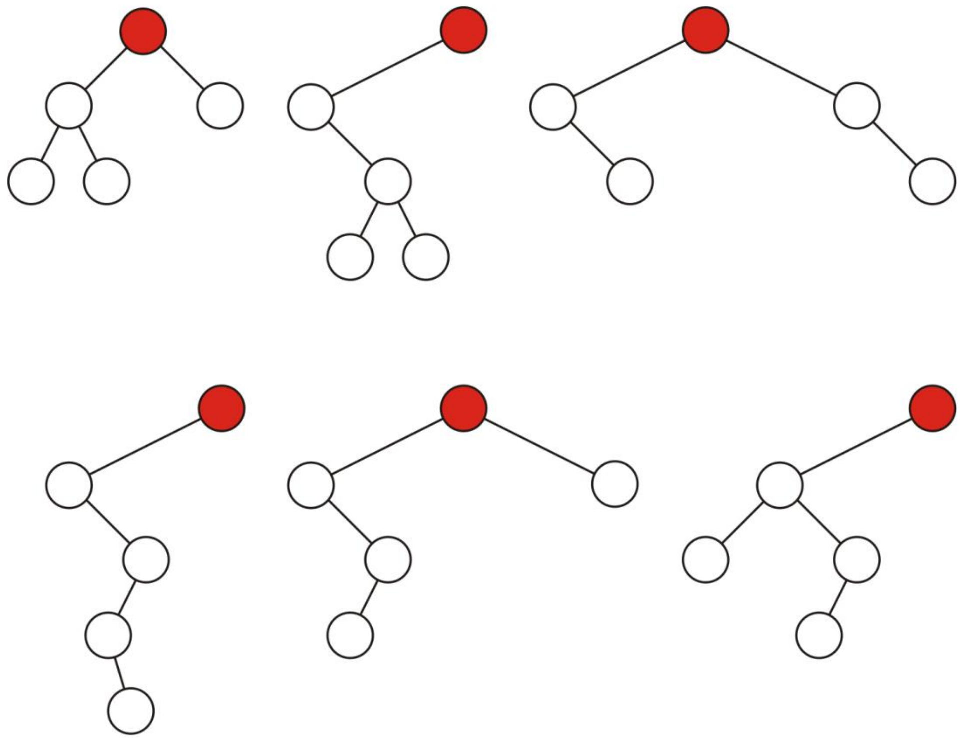
- Each child is either empty or another binary tree
- This restriction allows us to label the children as *left* and *right* subtrees





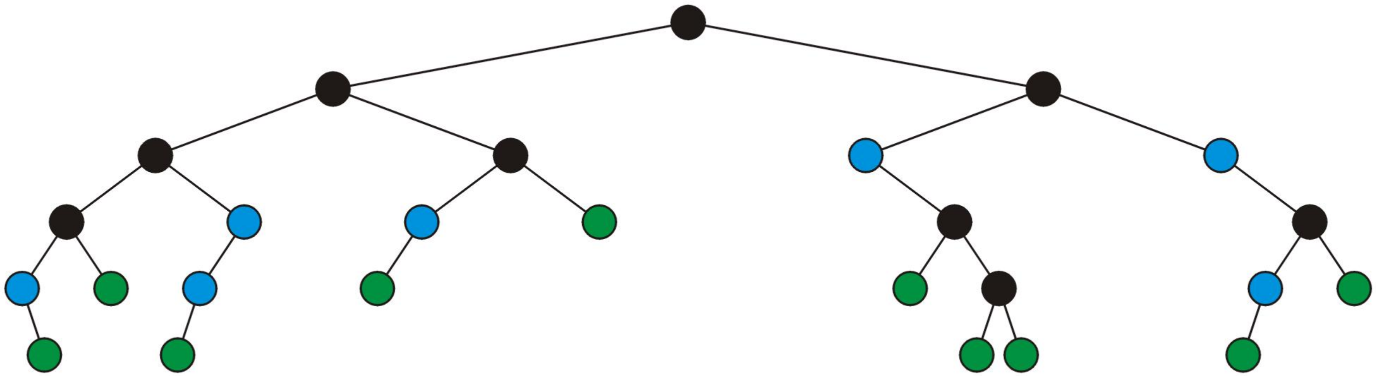
Definition

Some binary trees with five nodes:



Definition

A *full* node is a node where both the left and right sub-trees are non-empty trees

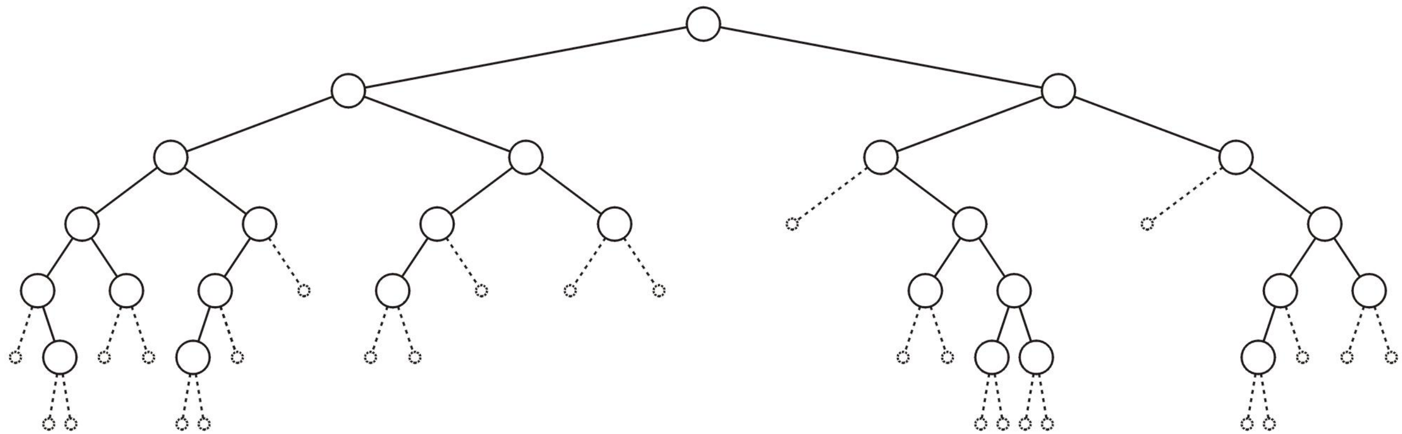


Legend:

full nodes ● neither ● leaf nodes ●

Definition

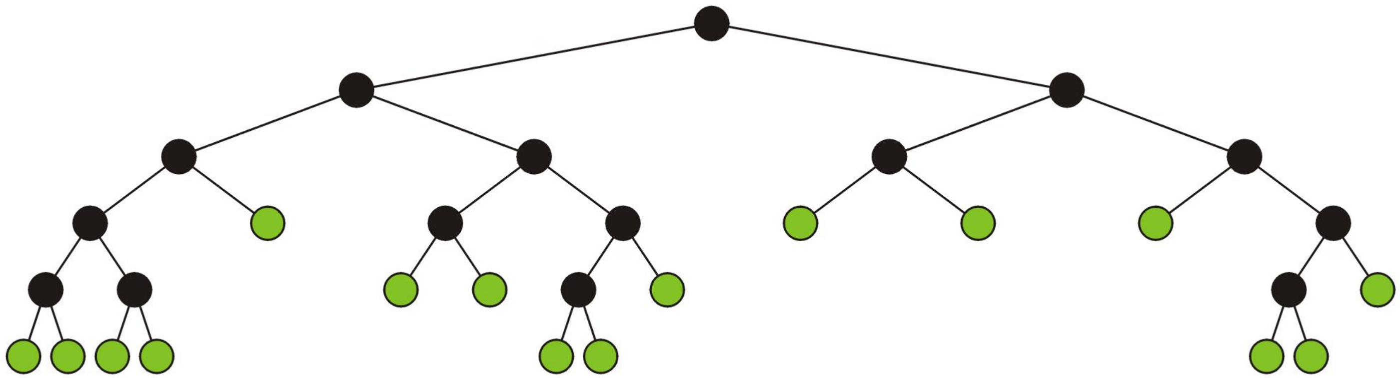
An *empty node* or a *null sub-tree* is any location where a new leaf node could be appended



Definition

A *full binary tree* is where each node is:

- A full node, or
- A leaf node



These have applications in

- Expression trees
- Huffman encoding

Binary Node Class

The binary node class is similar to the single node class:

```
template <typename Type>
class Binary_node {
    protected:
        Type element;
        Binary_node *left_tree;
        Binary_node *right_tree;

    public:
        Binary_node( Type const & );

        Type retrieve() const;
        Binary_node *left() const;
        Binary_node *right() const;

        bool is_leaf() const;
        int size() const;
        int height() const;

}
```

Binary Node Class

We will usually only construct new leaf nodes

```
template <typename Type>
Binary_node<Type>::Binary_node( Type const &obj ):
element( obj ),
left_tree( nullptr ),
right_tree( nullptr ) {
    // Empty constructor
}
```

Binary Node Class

The accessors are similar to that of Single_list

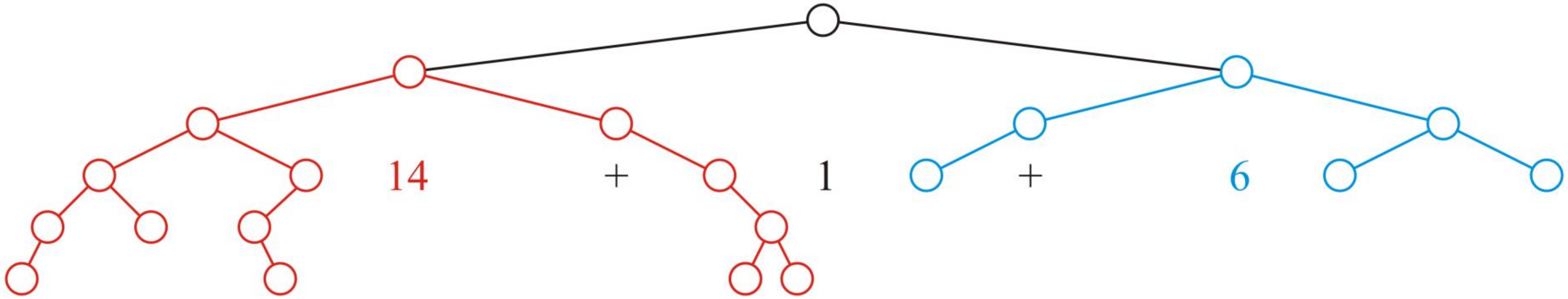
```
template <typename Type>
Type Binary_node<Type>::retrieve() const {
    return element;
}

template <typename Type>
Binary_node<Type> *Binary_node<Type>::left() const {
    return left_tree;
}

template <typename Type>
Binary_node<Type> *Binary_node<Type>::right() const {
    return right_tree;
}
```

Binary Node Class

```
template <typename Type>
bool Binary_node<Type>::is_leaf() const {
    return left() == nullptr && right() == nullptr;
}
```

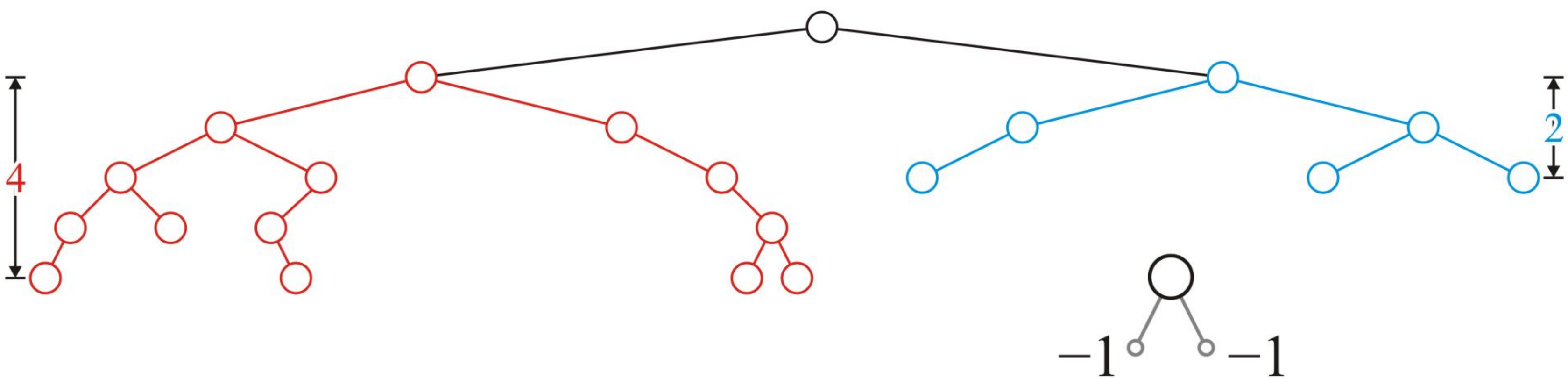
Size

```
template <typename Type>
int Binary_node<Type>::size() const {
    return 1
           + left() == nullptr? 0 : left()->size()
           + right() == nullptr? 0 : right()->size();
}
```



Height

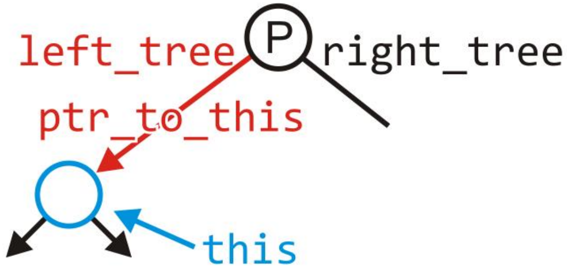
```
template <typename Type>
int Binary_node<Type>::height() const {
    return empty() ? -1 :
   1 + std::max( left()->height(), right()->height() );
}
```



Clear

Removing all the nodes in a tree is similarly recursive:

```
template <typename Type>
void Binary_node<Type>::clear( Binary_node *&ptr_to_this ) {
    if ( empty() ) {
        return;
    }

    left()->clear( left_node );
    right()->clear( right_node );

    delete this;
    ptr_to_this = nullptr;
}
```



Run Times

Recall that with linked lists and arrays, some operations would run in $\Theta(n)$ time

The run times of operations on binary trees, we will see, depends on the height of the tree

We will see that:

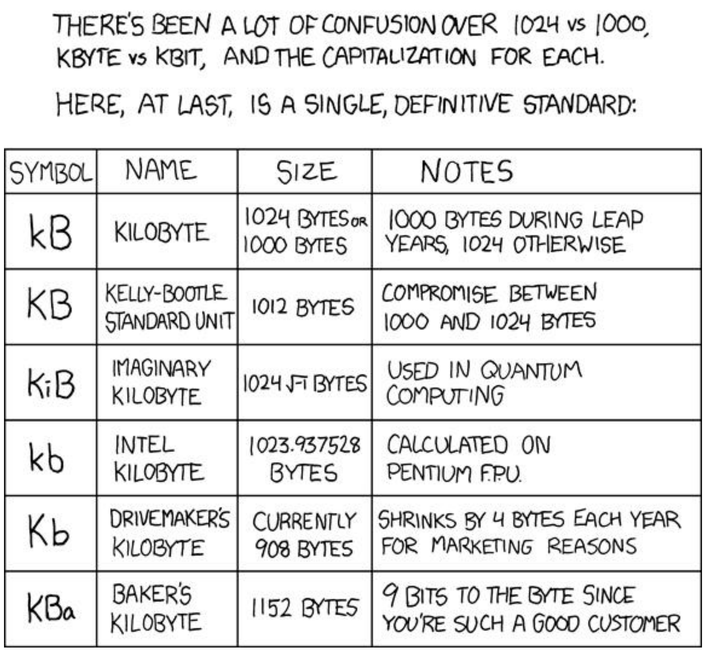
- The worst is clearly $\Theta(n)$
- Under average conditions, the height is $\Theta\left(\sqrt{n}\right)$
- The best case is $\Theta(\ln(n))$

Run Times

If we can achieve and maintain a height $\Theta(\lg(n))$, we will see that many operations can run in $\Theta(\lg(n))$ we

Logarithmic time is not significantly worse than constant time:

$$\lg(\,1000\,) \approx 10 \qquad \text{kB}$$
$$\lg(\,1\,000\,000\,) \approx 20 \qquad \text{MB}$$
$$\lg(\,1\,000\,000\,000\,) \approx 30 \qquad \text{GB}$$
$$\lg(\,1\,000\,000\,000\,000\,) \approx 40 \qquad \text{TB}$$
$$\lg(\,1000^n\,) \approx 10\,n$$


THERE'S BEEN A LOT OF CONFUSION OVER 1024 vs 1000, KBYTE vs KBIT, AND THE CAPITALIZATION FOR EACH.
HERE, AT LAST, IS A SINGLE, DEFINITIVE STANDARD:

SYMBOL	NAME	SIZE	NOTES
kB	KILOBYTE	1024 BYTES OR 1000 BYTES	1000 BYTES DURING LEAP YEARS, 1024 OTHERWISE
KB	KELLY-BOOTLE STANDARD UNIT	1012 BYTES	COMPROMISE BETWEEN 1000 AND 1024 BYTES
KiB	IMAGINARY KILOBYTE	1024 $\sqrt{-1}$ BYTES	USED IN QUANTUM COMPUTING
kb	INTEL KILOBYTE	1023.937528 BYTES	CALCULATED ON PENTIUM F.P.U.
Kb	DRIVEMAKER'S KILOBYTE	CURRENTLY 908 BYTES	SHRINKS BY 4 BYTES EACH YEAR FOR MARKETING REASONS
KBa	BAKER'S KILOBYTE	1152 BYTES	9 BITS TO THE BYTE SINCE YOU'RE SUCH A GOOD CUSTOMER

http://xkcd.com/394/

Application: Ropes

In 1995, Boehm *et al*. introduced the idea of a rope, or a *heavyweight* string

Application: Ropes

Alpha-numeric data is stored using a *string* of characters

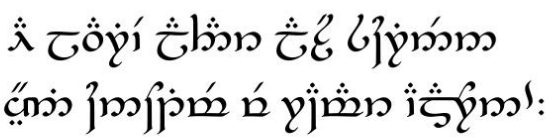
– A character (or char) is a numeric value from 0 to 255 where certain numbers represent certain letters

For example,

‘A’	65	01000001_2
‘B’	66	01000010_2
‘a’	97	01100001_2
‘b’	98	01100010_2
‘ ’	32	00100000_2

Unicode extends character encoding beyond the Latin alphabet

– Still waiting for the Tengwar characters…



J.R.R. Tolkien

Application: Ropes

A C-style string is an array of characters followed by the character with a numeric value of 0

```
char * story = "In a hole there lived a hobbit.";
```

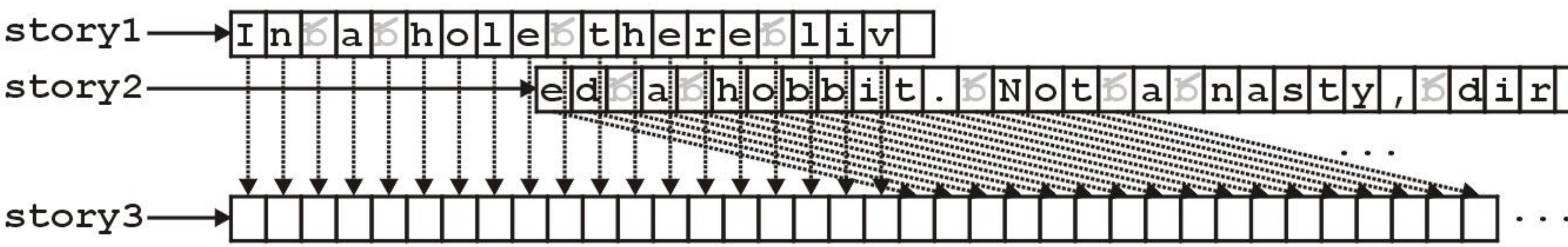
One problem with using arrays is the runtime required to concatenate two strings

Application: Ropes

Concatenating two strings requires the operations of:

- Allocating more memory, and
- Coping both strings $\Theta(n + m)$

```
char * story1 = "In a hole there liv";
char * story2 = "ed a hobbit. Not a nasty, di";
```

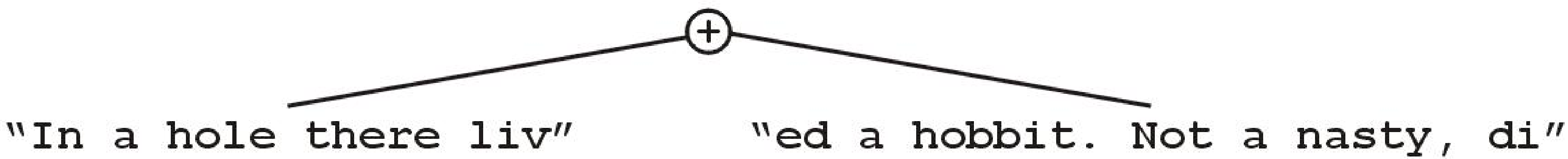


Application: Ropes

The rope data structure:

- Stores strings in the leaves,
- Internal nodes (full) represent the concatenation of the two strings, and
- Represents the string with the right sub-tree concatenated onto the end of the left

The previous concatenation may now occur in $\Theta(1)$ time

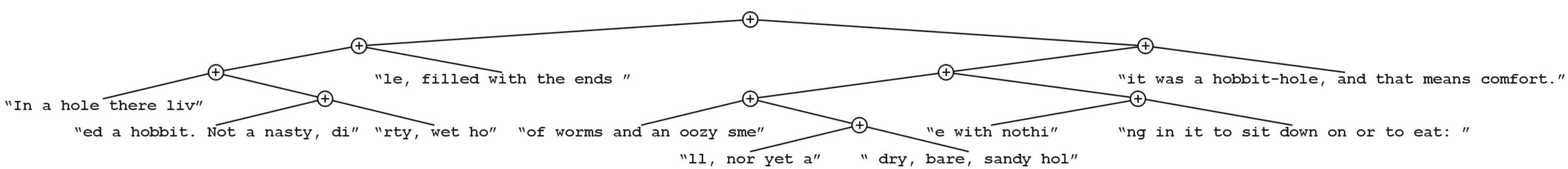


Application: Ropes

The string

"In a hole there lived a hobbit. Not a nasty, dirty, wet hole, filled with the ends of worms and an oozy smell, nor yet a dry, bare, sandy hole with nothing in it to sit down on or to eat: it was a hobbit-hole, and that means comfort."

may be represented using the rope



References: http://en.wikipedia.org/wiki/Rope_(computer_science)
J.R.R. Tolkien, *The Hobbit*

Application: Ropes

Additional information may be useful:

- Recording the number of characters in both the left and right sub-trees

It is also possible to eliminate duplication of common sub-strings

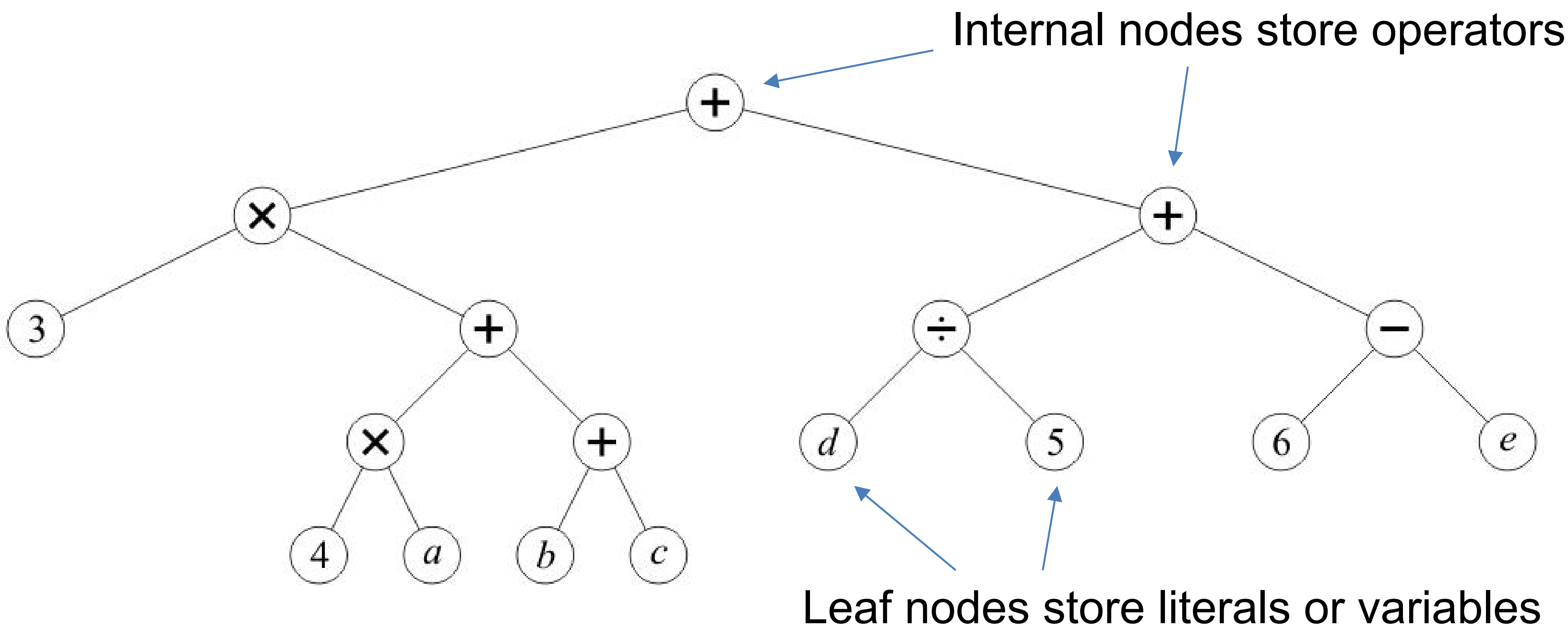
```
"In a hole there lived a hobbit. Not a nasty, dirty, wet hole,
 filled_with_the ends of worms and an oozy smell, nor yet a dry, bare,
 sandy hole_with_nothing in it to sit down on or to eat: it was a
 hobbit-hole, and that means comfort."
```

References: http://en.wikipedia.org/wiki/Rope_(computer_science)
J.R.R. Tolkien, *The Hobbit*

Application: Expression Trees

Any basic mathematical expression containing binary operators may be represented using a (full) binary tree

– For example, $3*(4a + b + c) + d/5 + (6 - e)$



Application: Expression Trees

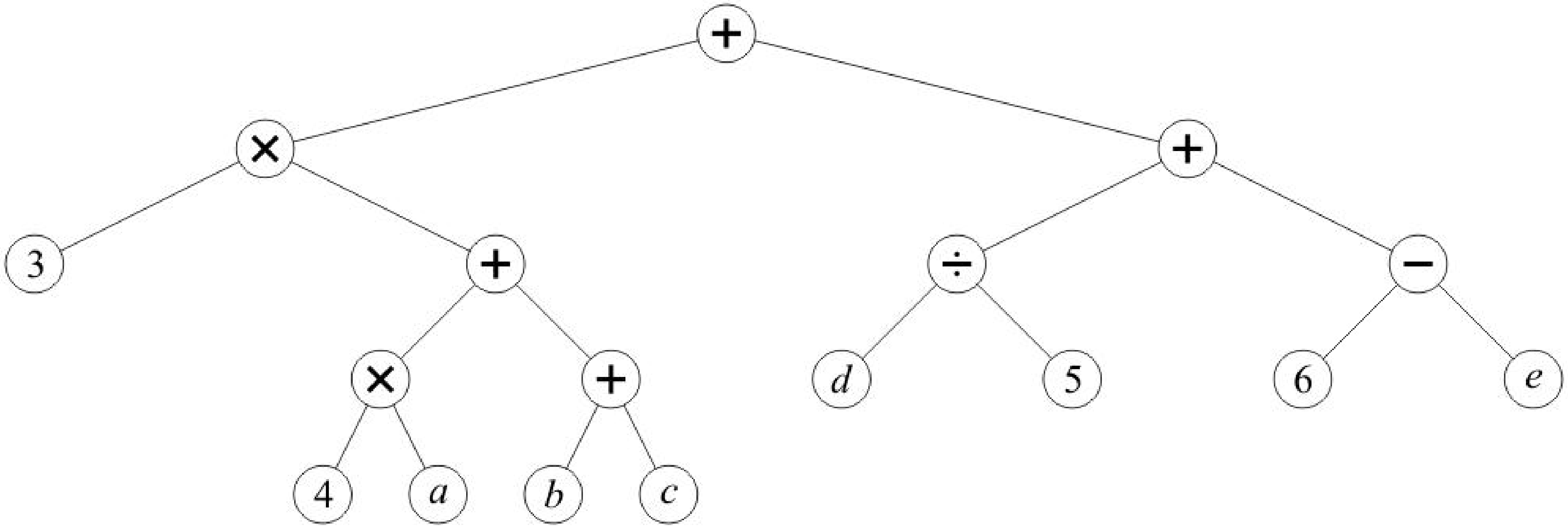
Observations:

- Internal nodes store operators
- Leaf nodes store literals or variables
- No nodes have just one sub tree
- The order is not relevant for
  - Addition and multiplication (commutative)
- Order is relevant for
  - Subtraction and division (non-commutative)
- It is possible to replace non-commutative operators using the unary negation and inversion:

$$(a/b) = a\ b^{-1} \qquad (a - b) = a + (-b)$$

Application: Expression Trees

A post-order depth-first traversal converts such a tree to the reverse-Polish format



$$3 \; 4 \; a \; \times \; b \; c \; + \; + \; \times \; d \; 5 \; \div \; 6 \; e \; - \; + \; +$$

Application: Expression Trees

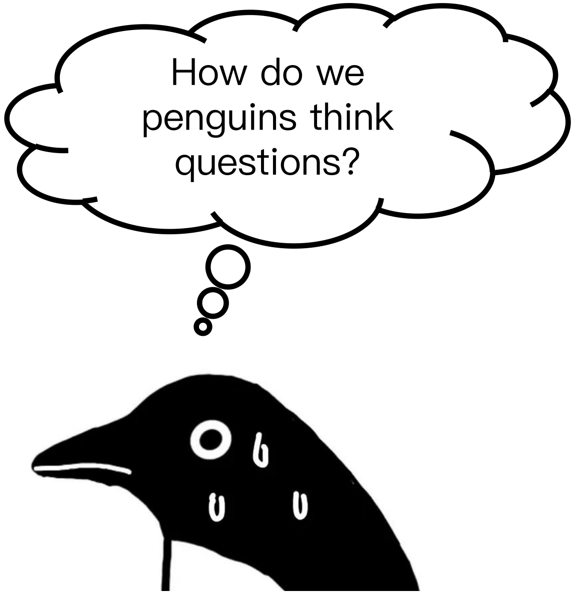
Computers think in post-order:

- Both operands must be loaded into registers
- The operation is then called on those registers

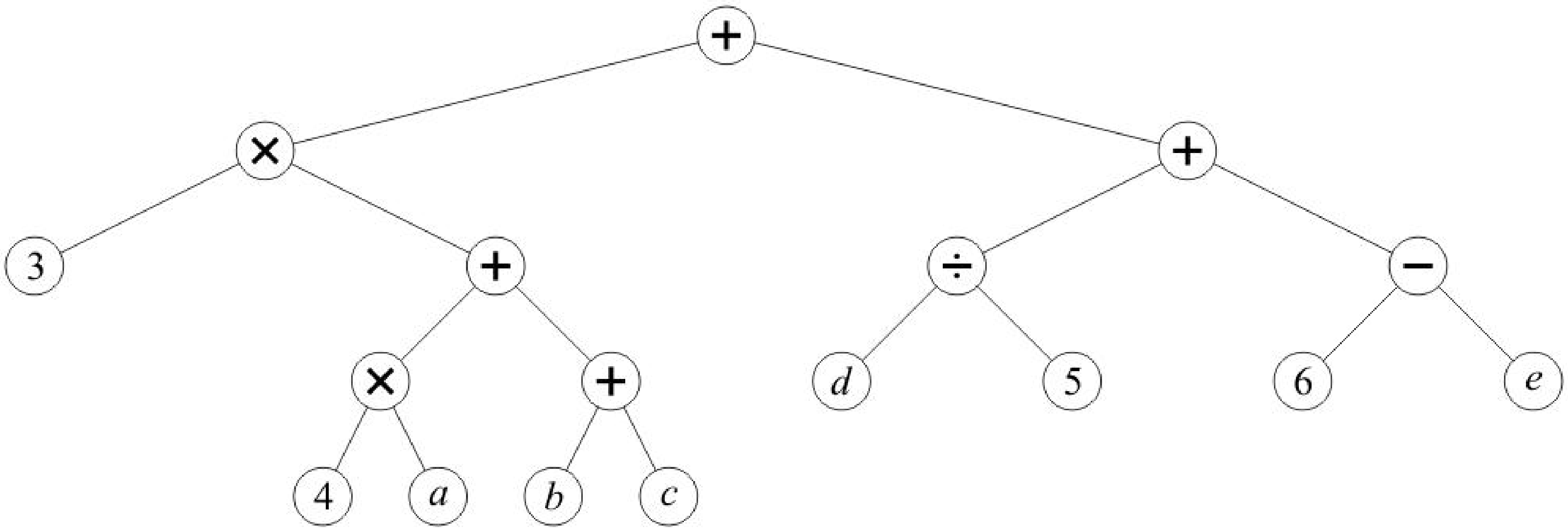
Humans think in in-order:

- First, the left sub-tree is traversed
- Then, the current node is visited
- Finally, the right-sub-tree is traversed

This is called an *in-order* traversal

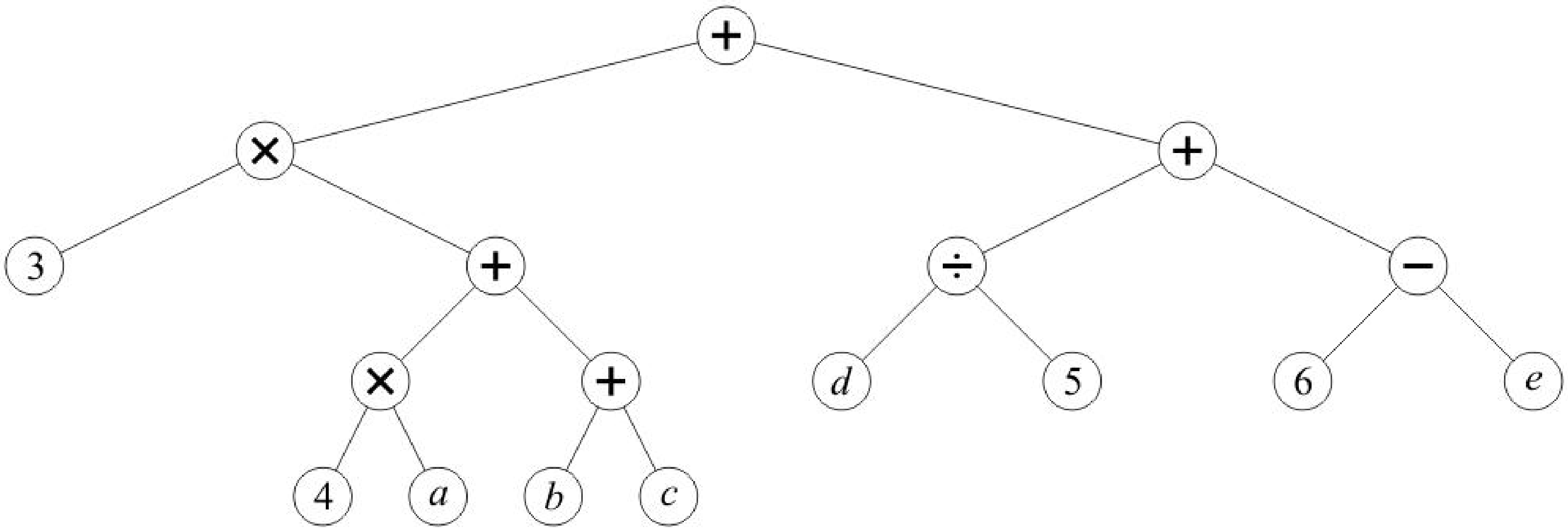


In-order Traversal



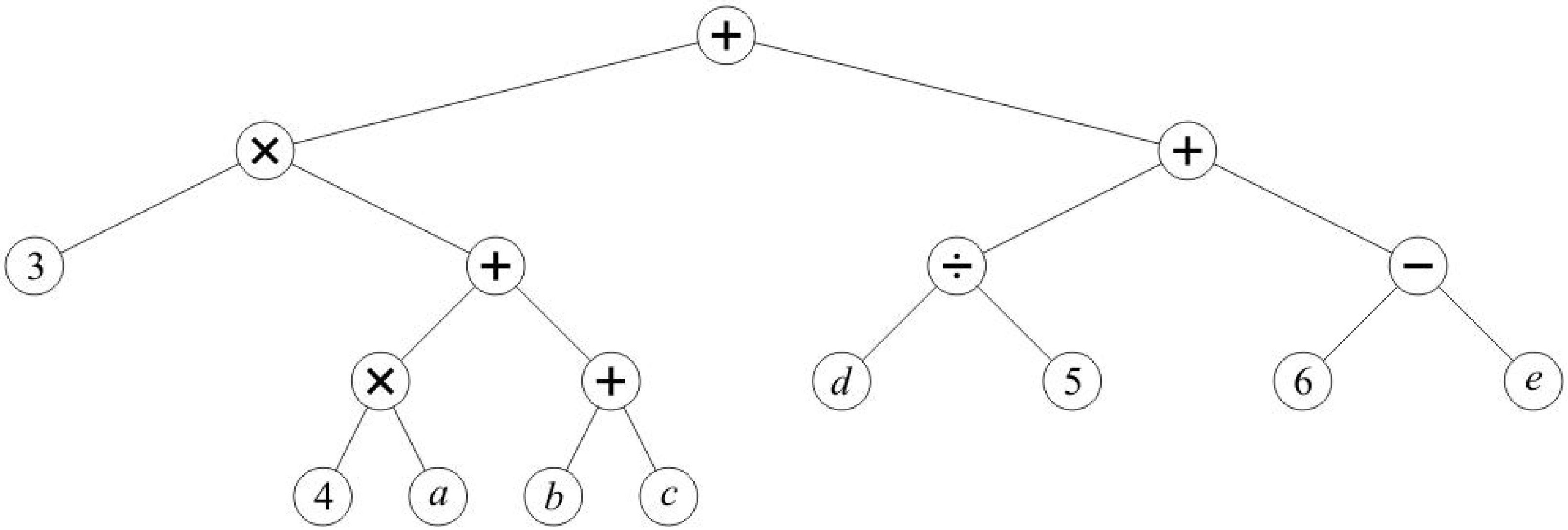
3

In-order Traversal



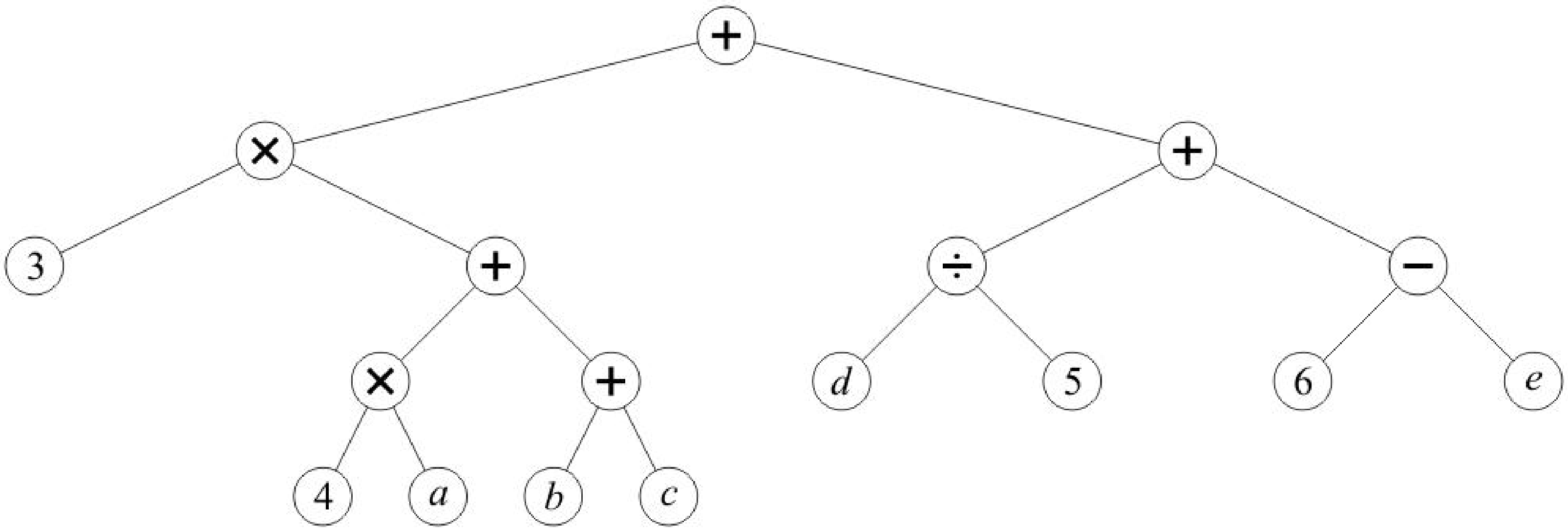
$3 \times$

In-order Traversal



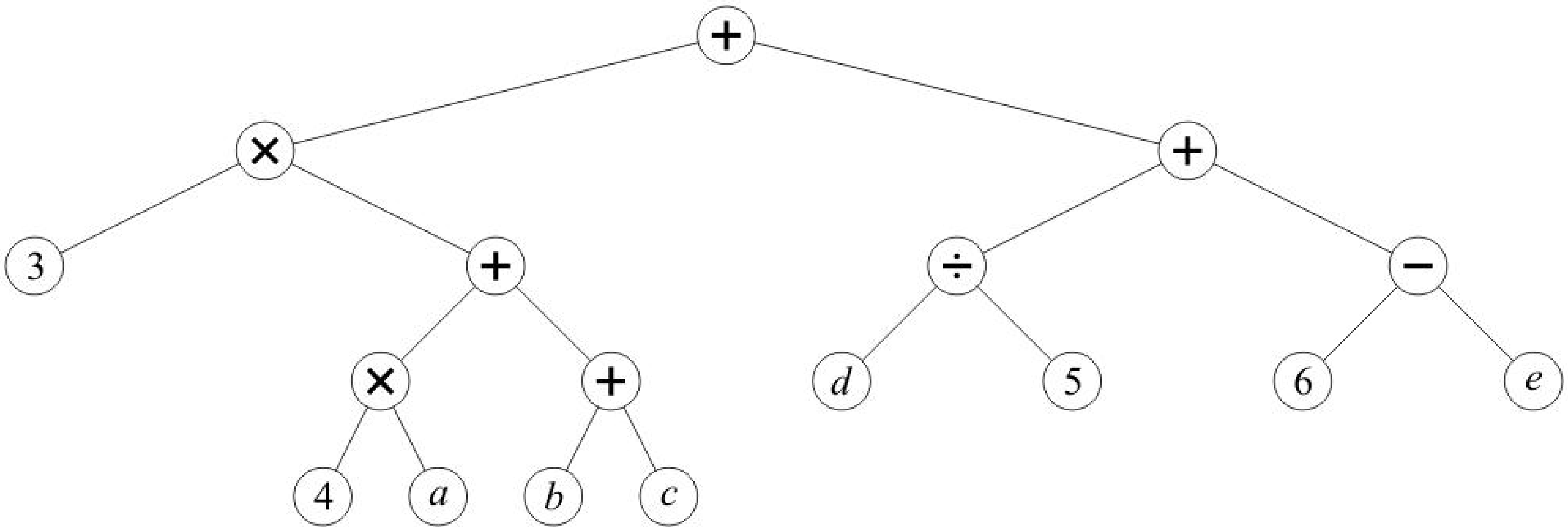
3×4

In-order Traversal



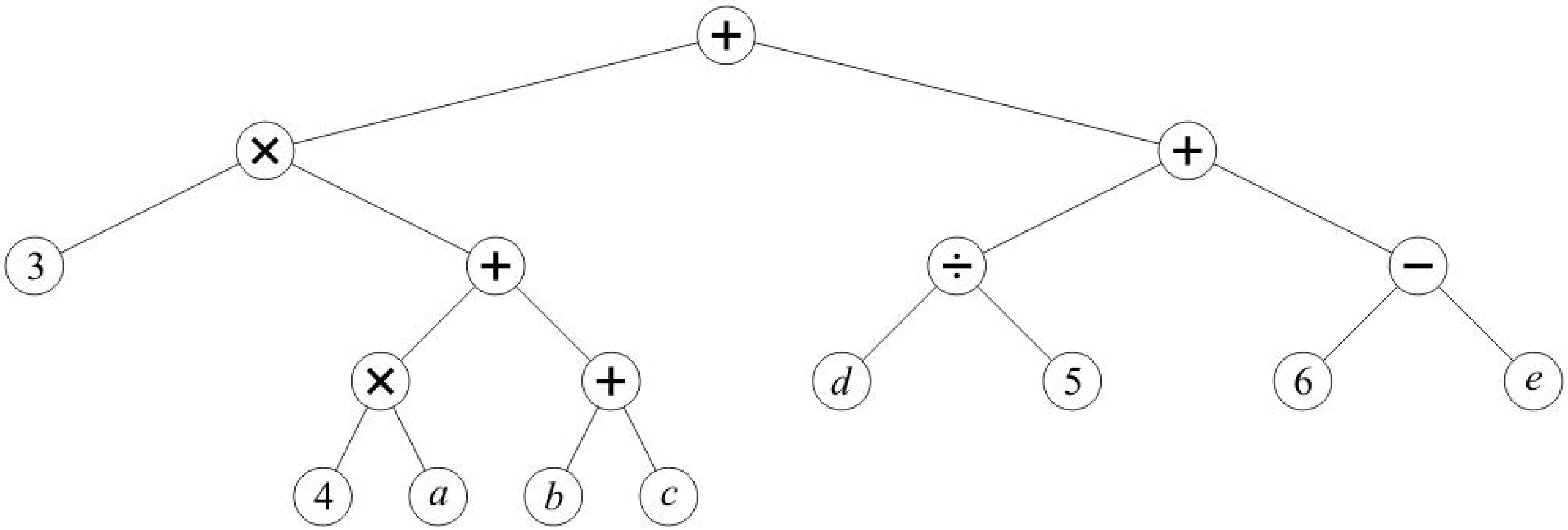
$3 \times 4 \times$

In-order Traversal



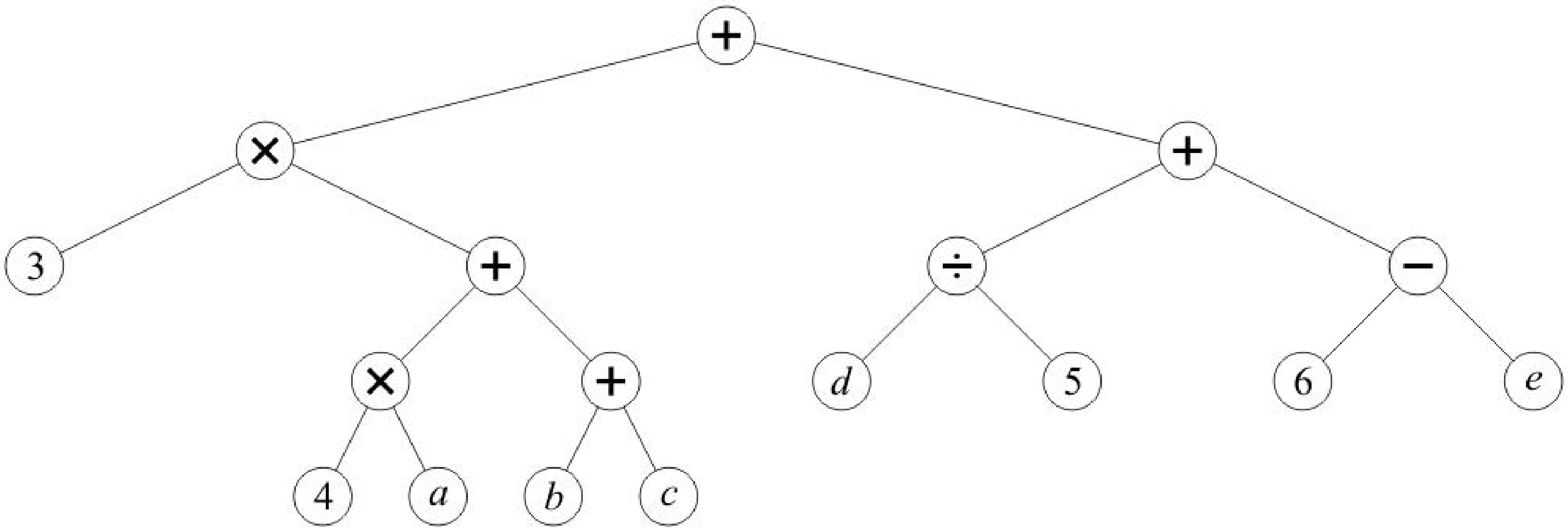
$3 \times 4 \times a$

In-order Traversal



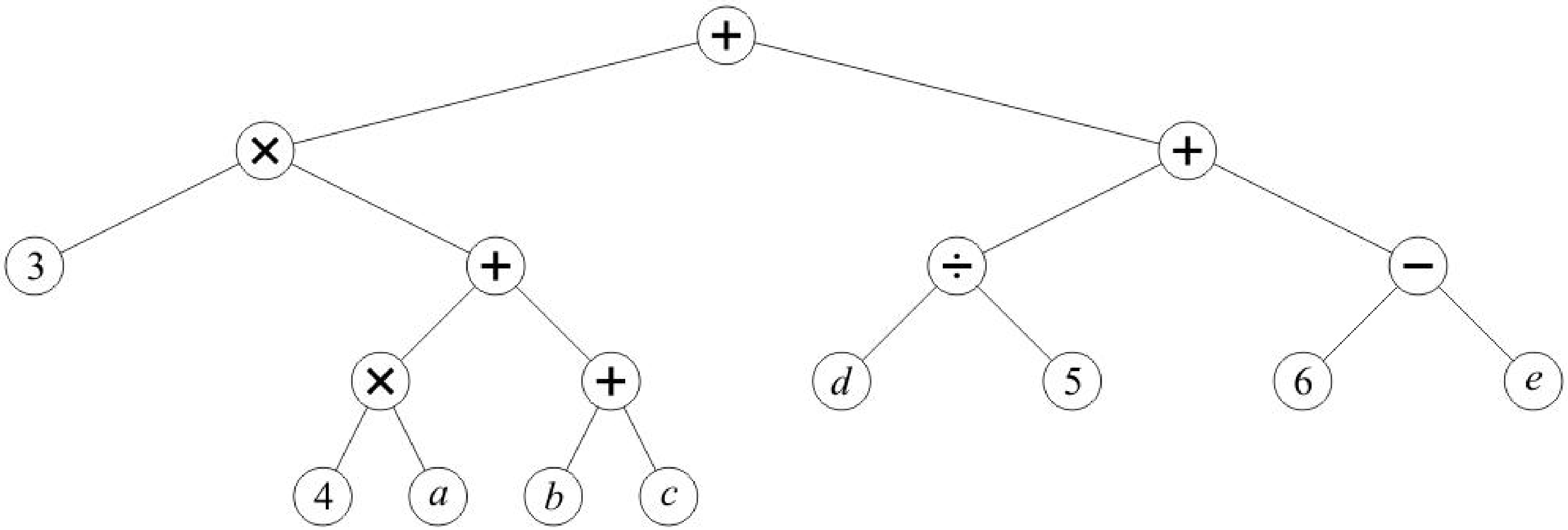
$3 \times 4 \times a +$

In-order Traversal



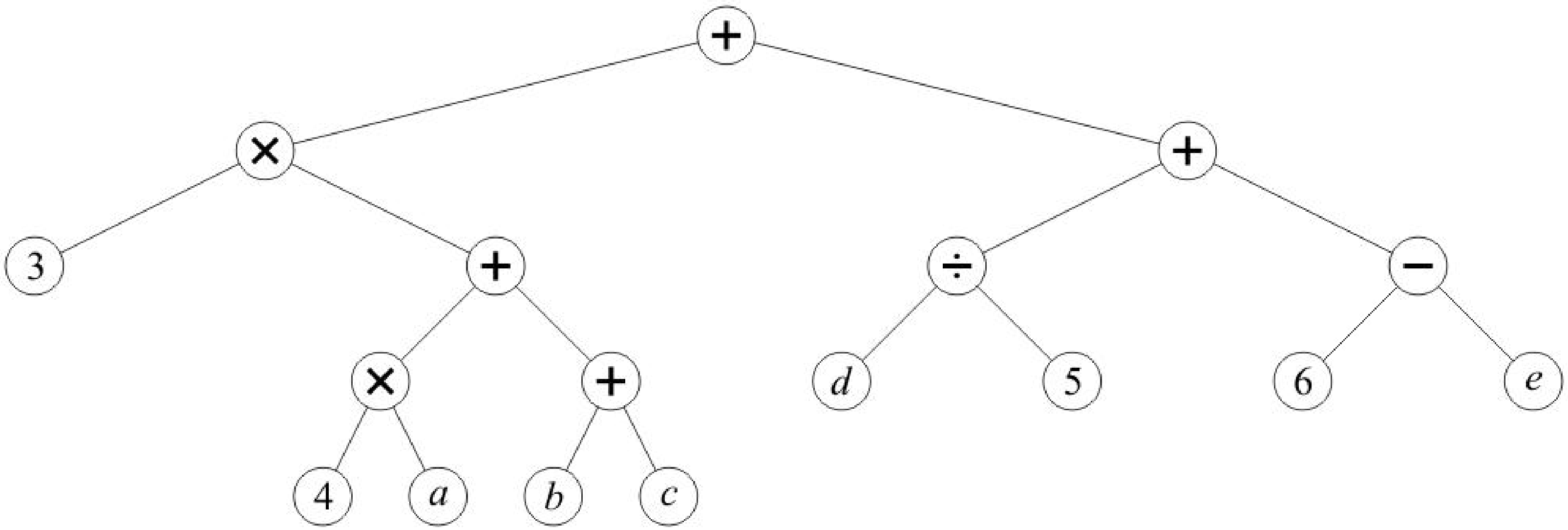
$3 \times 4 \times a + b$

In-order Traversal



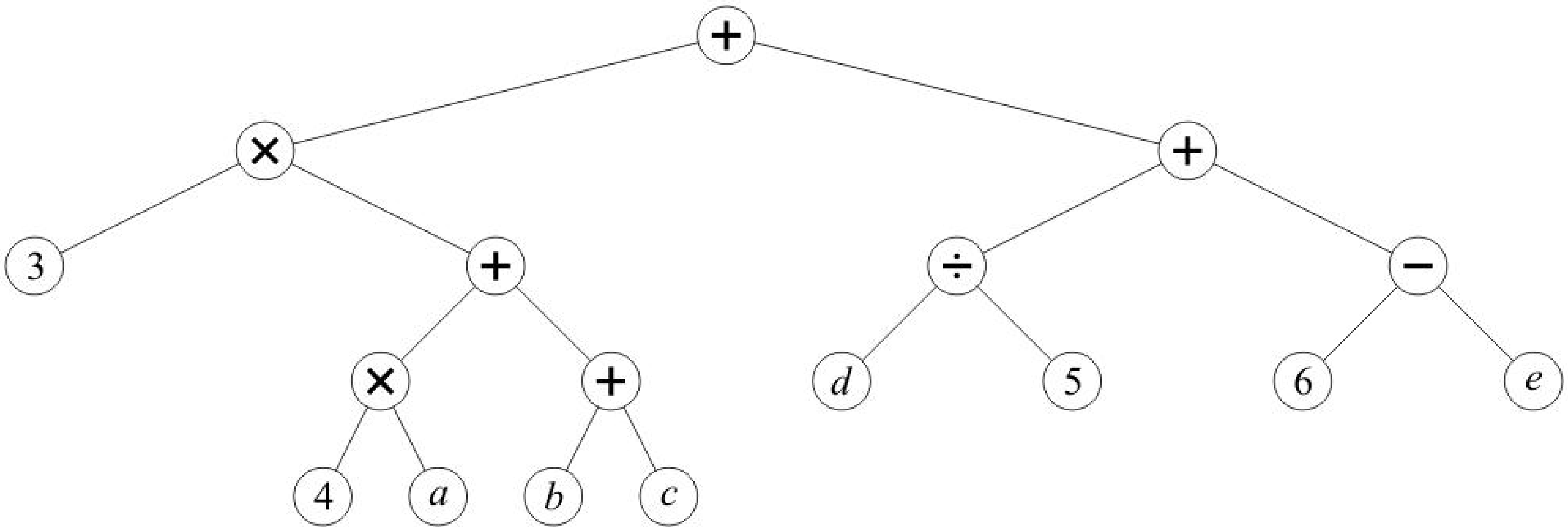
$3 \times 4 \times a + b +$

In-order Traversal



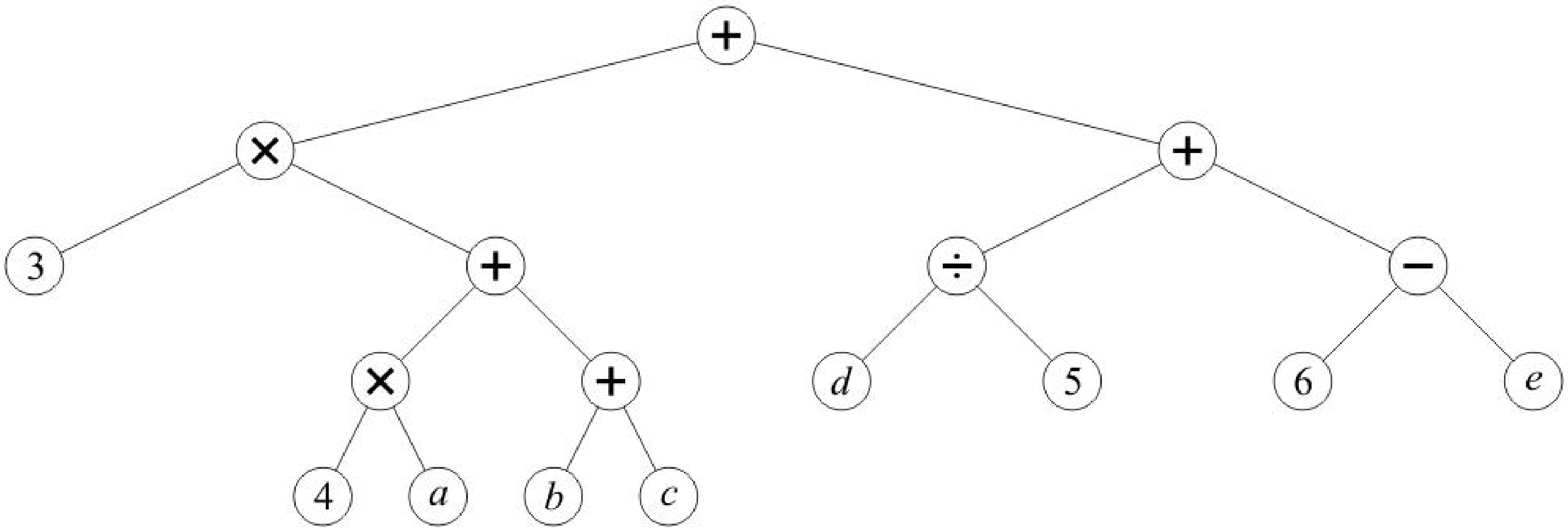
$3 \times 4 \times a + b + c$

In-order Traversal



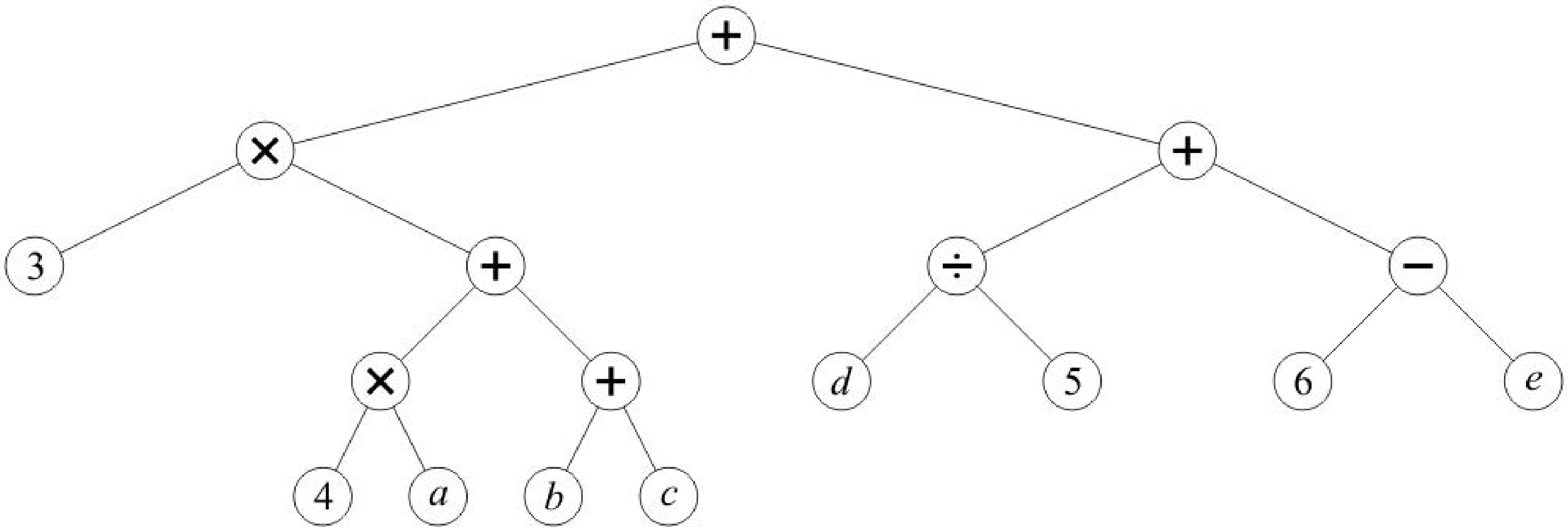
$3 \times 4 \times a + b + c +$

In-order Traversal



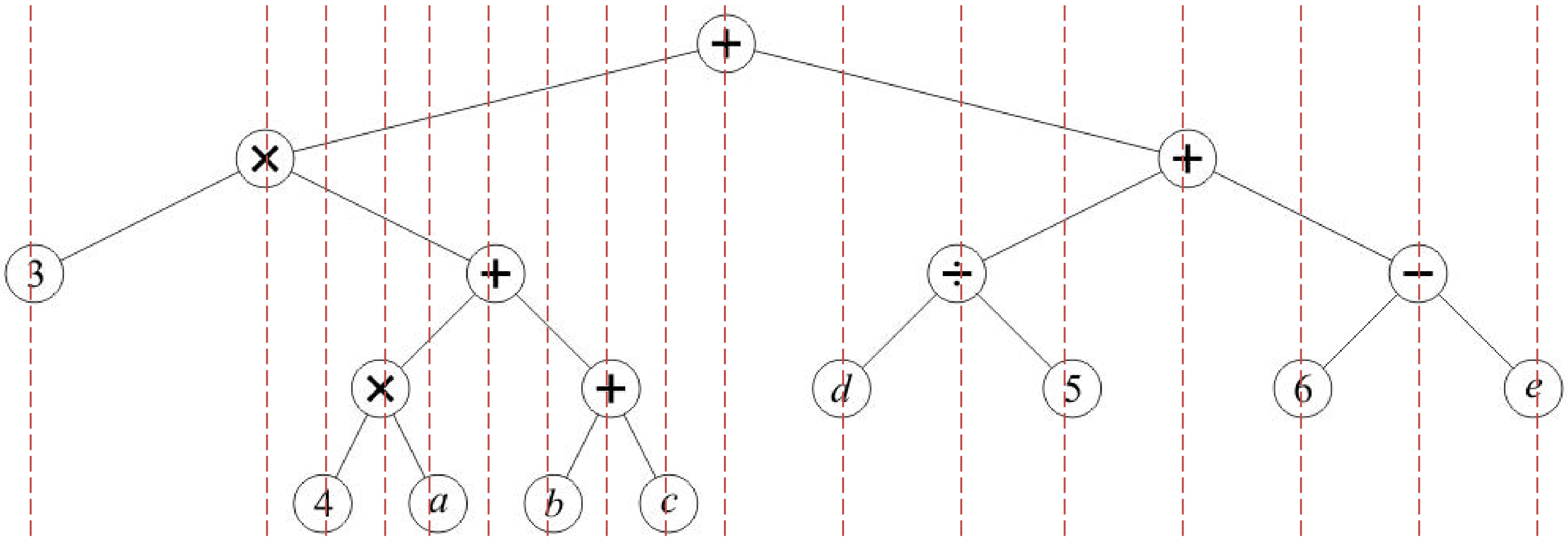
$$3 \times 4 \times a + b + c + d \div 5 + 6 - e$$

In-order Traversal



$$3 \times (4 \times a + (b + c)) + (d \div 5 + (6 - e))$$

In-order Traversal



$$3 \times 4 \times a + b + c + d \div 5 + 6 - e$$

Summary

In this talk, we introduced binary trees

- Each node has two distinct and identifiable sub-trees
- Either sub-tree may optionally be empty
- The sub-trees are ordered relative to the other

We looked at:

- Properties
- Applications

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Sincerely,

Douglas Wilhelm Harder, MMath

dwharder@alumni.uwaterloo.ca

Outline

- Binary tree
- Perfect binary tree
- Complete binary tree
- Left-child right-sibling binary tree

Outline

Introducing perfect binary trees

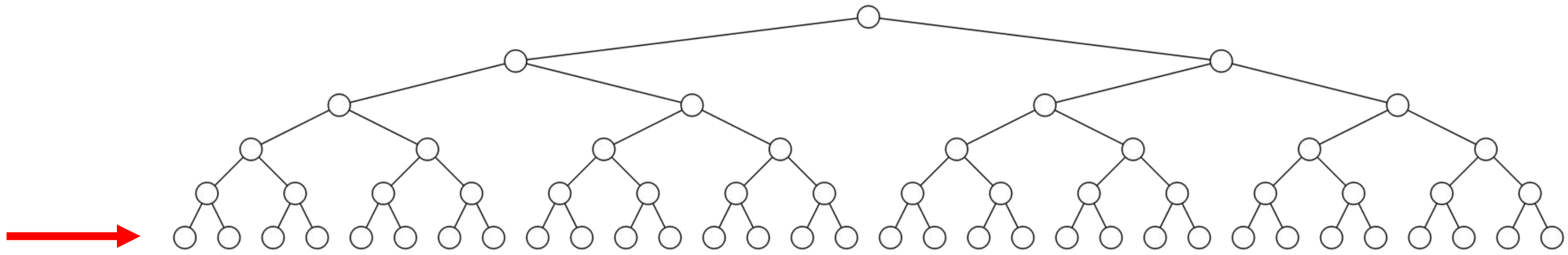
- Definitions and examples
- Number of nodes: $2^{h+1} - 1$
- Logarithmic height
- Number of leaf nodes: 2^h
- Applications

Definition

Standard definition:

– A perfect binary tree of height h is a binary tree where

- All leaf nodes have the same depth h
- All other nodes are full



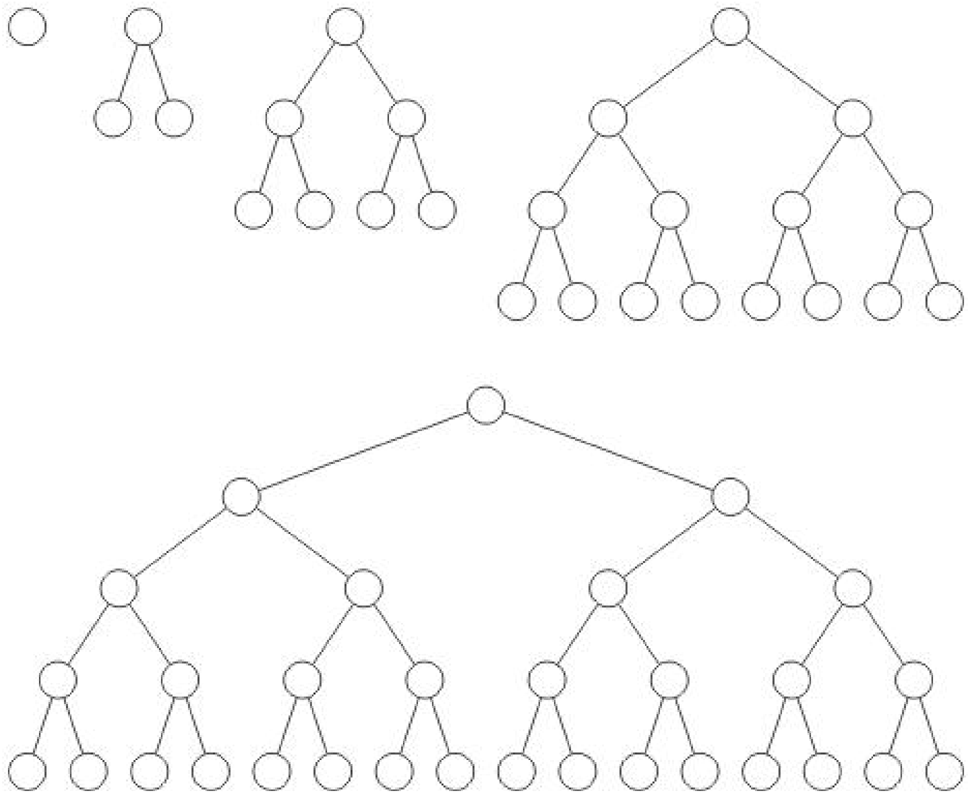
Definition

Recursive definition:

- A binary tree of height $h = 0$ is perfect
- A binary tree with height $h > 0$ is a perfect if both sub-trees are prefect binary trees of height $h - 1$

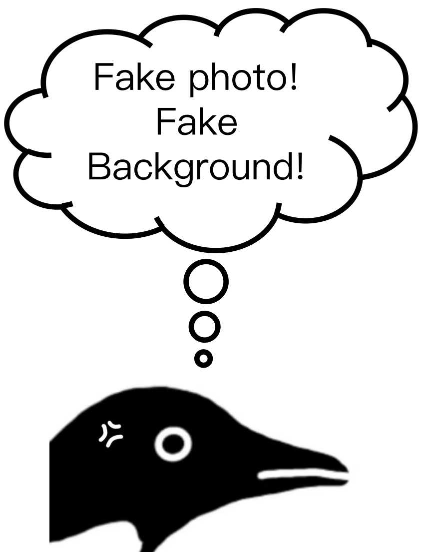
Examples

Perfect binary trees of height $h = 0, 1, 2, 3$ and 4



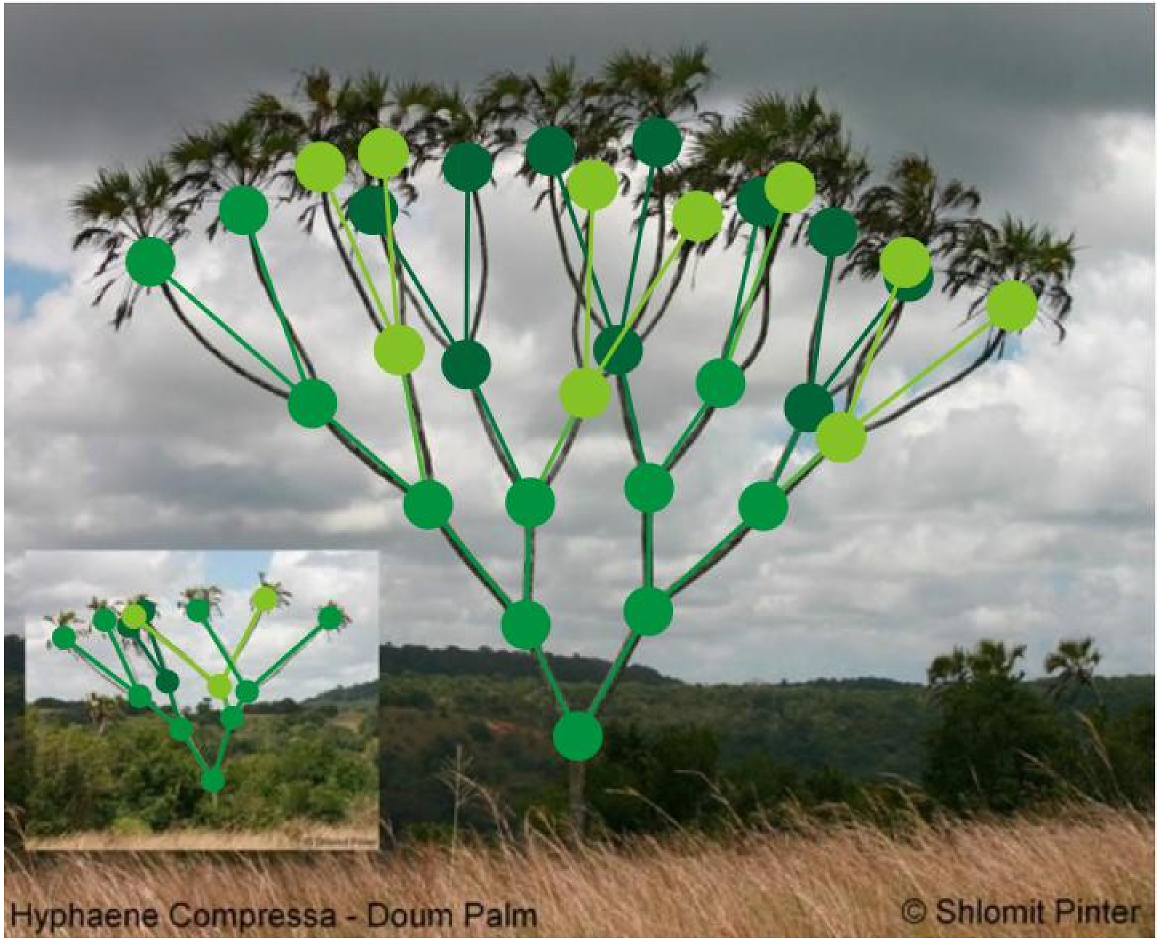
Examples

Perfect binary trees of height $h = 3$ and $h = 4$



Examples

Perfect binary trees of height $h = 3$ and $h = 4$



Theorems

Four theorems of perfect binary trees:

- A perfect binary tree of height h has $2^{h+1} - 1$ nodes
- The height is $\Theta(\ln(n))$
- There are 2^h leaf nodes
- The average depth of a node is $\Theta(\ln(n))$

These theorems will allow us to determine the optimal run-time properties of operations on binary trees

$2^{h+1} - 1$ Nodes

Theorem

A perfect binary tree of height h has $2^{h+1} - 1$ nodes

Proof:

We will use mathematical induction

$2^{h+1} - 1$ Nodes

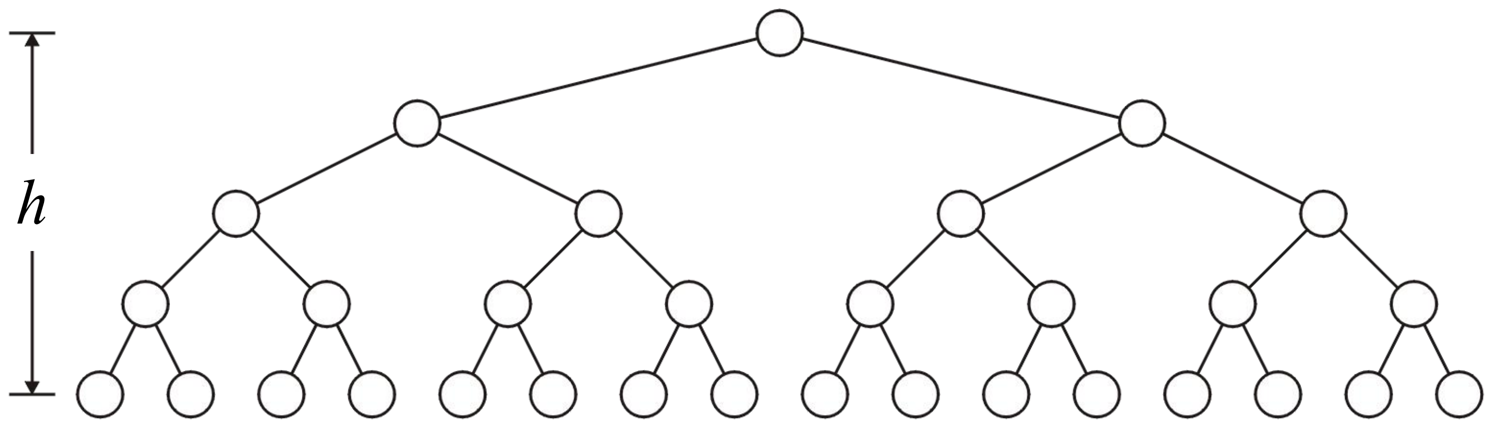
The base case:

- When $h = 0$ we have a single node $n = 1$
- The formula is correct: $2^{0+1} - 1 = 1$

$2^{h+1} - 1$ Nodes

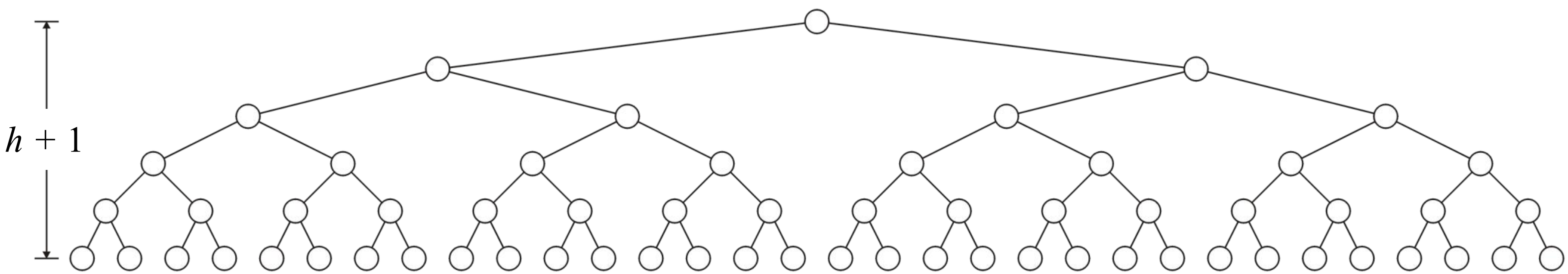
The inductive step:

– Assume that a tree of height h has $n = 2^{h+1} - 1$ nodes



$2^{h+1} - 1$ Nodes

We must show that a tree of height $h + 1$ has
$n = 2^{(h+1)+1} - 1 = 2^{h+2} - 1$ nodes

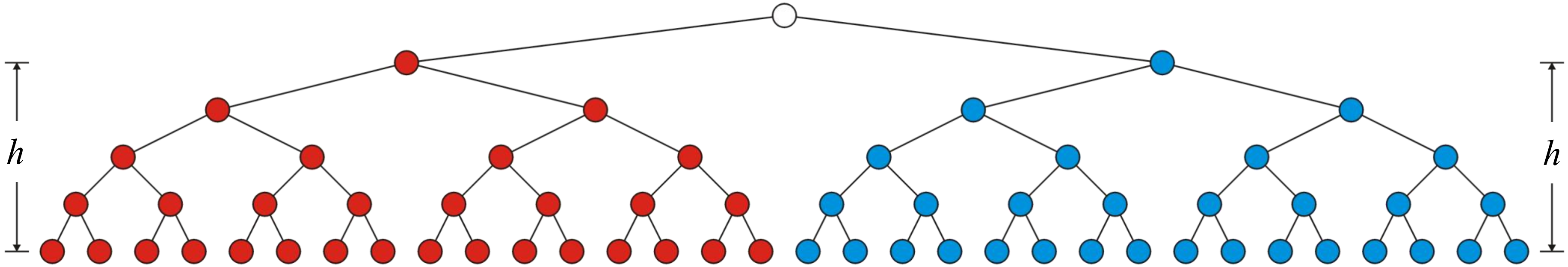


$2^{h+1} - 1$ Nodes

Using the recursive definition, both sub-trees are perfect trees of height h

- By assumption, each sub-tree has $2^{h+1} - 1$ nodes
- Therefore, the total number of nodes is

$$(2^{h+1} - 1) + 1 + (2^{h+1} - 1) = 2^{h+2} - 1$$



$2^{h+1} - 1$ Nodes

Consequently

The statement is true for $h = 0$ and the truth of the statement for an arbitrary h implies the truth of the statement for $h + 1$.

Therefore, by the process of mathematical induction, the statement is true for all $h \geq 0$

Logarithmic Height

Theorem

A perfect binary tree with n nodes has height $\lg(n + 1) - 1$

Proof

Solving $n = 2^{h+1} - 1$ for h:

$$n + 1 = 2^{h+1}$$

$$\lg(n + 1) = h + 1$$

$$h = \lg(n + 1) - 1$$

Logarithmic Height

Lemma

$\lg(n + 1) - 1 = \Theta(\ln(n))$

Proof

$$\lim_{n\to\infty}\frac{\lg(n+1)-1}{\ln(n)} = \lim_{n\to\infty}\frac{\dfrac{1}{(n+1)\ln(2)}}{\dfrac{1}{n}} = \lim_{n\to\infty}\frac{n}{(n+1)\ln(2)} = \lim_{n\to\infty}\frac{1}{\ln(2)} = \frac{1}{\ln(2)}$$

2^h Leaf Nodes

Theorem

A perfect binary tree with height h has 2^h leaf nodes

Proof (by induction):

When $h = 0$, there is $2^0 = 1$ leaf node.

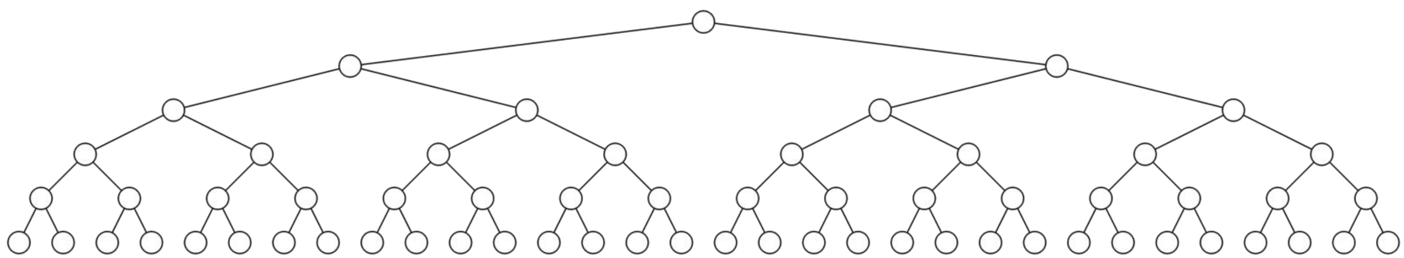
Assume that a perfect binary tree of height h has 2^h leaf nodes and observe that both sub-trees of a perfect binary tree of height $h + 1$ have 2^h leaf nodes.

Consequence: Over half of the nodes are leaf nodes:

$$\frac{2^h}{2^{h+1}-1} > \frac{1}{2}$$

The Average Depth of a Node

The average depth of a node in a perfect binary tree is



Depth	Count
0	1
1	2
2	4
3	8
4	16
5	32

Sum of the depths

$$\frac{\sum_{k=0}^{h} k2^{k}}{2^{h+1}-1} = \frac{h2^{h+1}-2^{h+1}+2}{2^{h+1}-1} = \frac{h(2^{h+1}-1)-(2^{h+1}-1)+1+h}{2^{h+1}-1}$$

$$= h-1+\frac{h+1}{2^{h+1}-1} \approx h-1 = \because\left(\ln(n)\right)$$

Number of nodes

Applications

Perfect binary trees are considered to be the *ideal* case

– The height and average depth are both $\Theta(\ln(n))$

We will attempt to find trees which are as close as possible to perfect binary trees

Summary

We have defined perfect binary trees and discussed:

- The number of nodes: $n = 2^{h+1} - 1$
- The height: $\lg(n + 1) - 1$
- The number of leaves: 2^h
- Half the nodes are leaves
  - Average depth is $\Theta(\ln(n))$
- It is an ideal case

Outline

- Binary tree
- Perfect binary tree
- Complete binary tree
- Left-child right-sibling binary tree

Outline

Introducing complete binary trees

- Background
- Definitions
- Examples
- Logarithmic height
- Array storage

Background

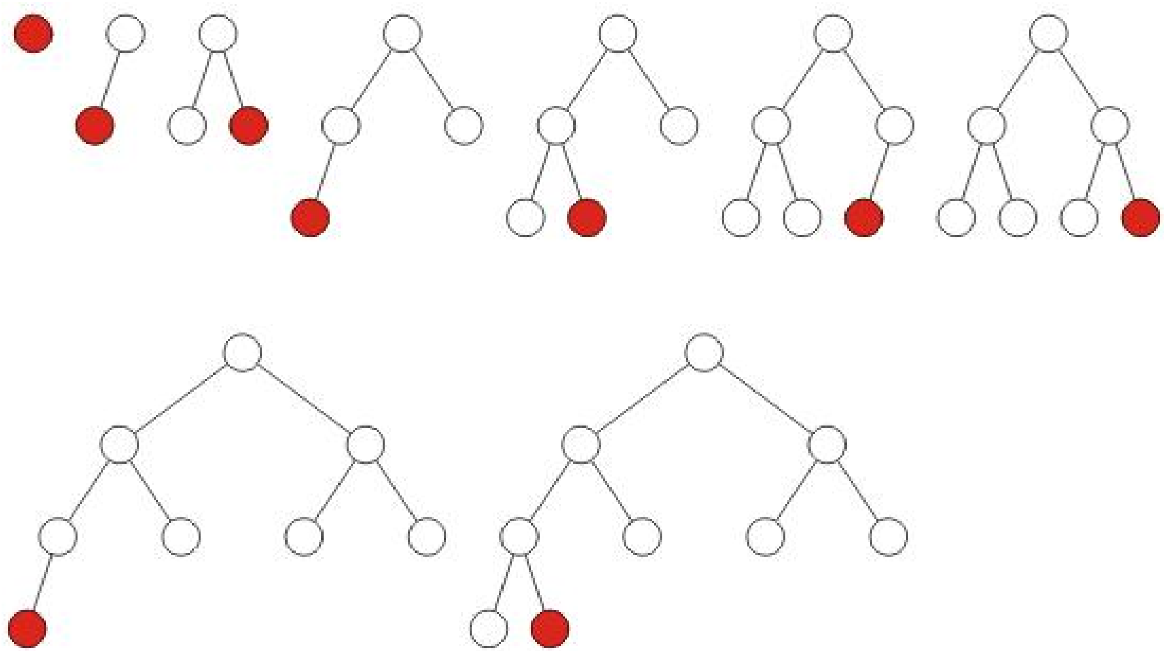
We require binary trees which are

- Similar to perfect binary trees, but
- Defined for any number of nodes

Definition

A complete binary tree filled at each depth from left to right

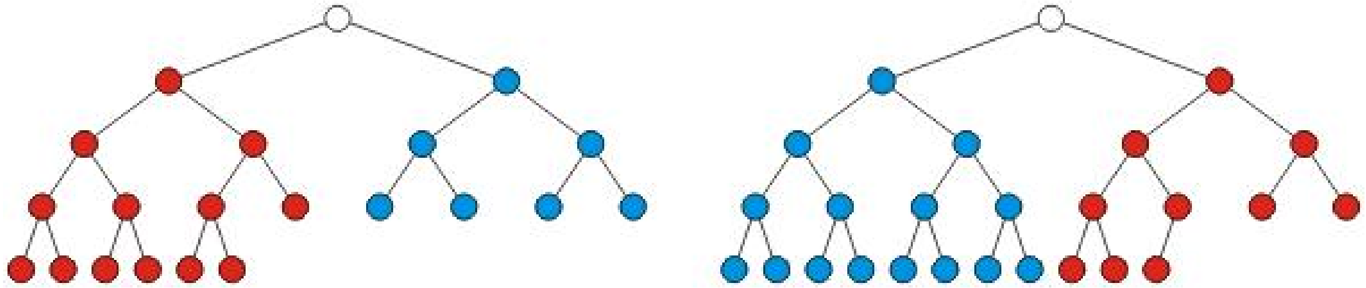
- Identical order to that of a breadth-first traversal



Recursive Definition

Recursive definition: a binary tree with a single node is a complete binary tree of height $h = 0$ and a complete binary tree of height h is a tree where either:

- The left sub-tree is a **complete tree** of height $h - 1$ and the right sub-tree is a **perfect tree** of height $h - 2$, or
- The left sub-tree is **perfect tree** with height $h - 1$ and the right sub-tree is **complete tree** with height $h - 1$



Height

Theorem

The height of a complete binary tree with n nodes is $h = \lg(n)$

Proof:

- Base case:
  - When $n = 1$ then $\lg(1) = 0$ and a tree with one node is a complete tree with height $h = 0$
- Inductive step:
  - Assume that a complete tree with n nodes has height $\lg(n)$
  - Must show that $\lg(n + 1)$ gives the height of a complete tree with $n + 1$ nodes
  - Two cases:
    - If the tree with n nodes is perfect, and
    - If the tree with n nodes is complete but not perfect

Height

Case 1 (the tree with n nodes is perfect):

- If it is a perfect tree then
  - It had $n = 2^{h+1} - 1$ nodes
  - Adding one more node must increase the height
- So, the tree with $n+1$ nodes has height $h+1$ and we have:

$$\lfloor \lg(n+1) \rfloor = \lfloor \lg(2^{h+1} - 1 + 1) \rfloor = \lfloor \lg(2^{h+1}) \rfloor = h+1$$

Height

Case 2 (the tree with n nodes is complete but not perfect):

– If it is not a perfect tree then

$$2^h \le n < 2^{h+1} - 1$$

$$2^h + 1 \le n + 1 < 2^{h+1}$$

$$h < \lg\left(2^h + 1\right) \le \lg\left(n + 1\right) < \lg\left(2^{h+1}\right) = h + 1$$

$$h \le \left\lfloor \lg\left(2^h + 1\right) \right\rfloor \le \left\lfloor \lg\left(n + 1\right) \right\rfloor < h + 1$$

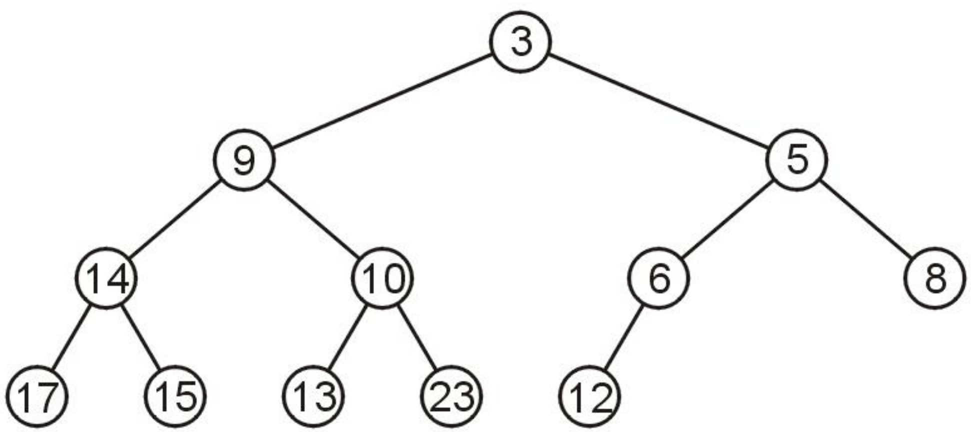
– So, the tree with $n+1$ nodes has height h and we have $\lg(n + 1) = h$

By mathematical induction, the statement must be true for all $n \ge 1$

Array storage

We are able to store a complete tree as an array

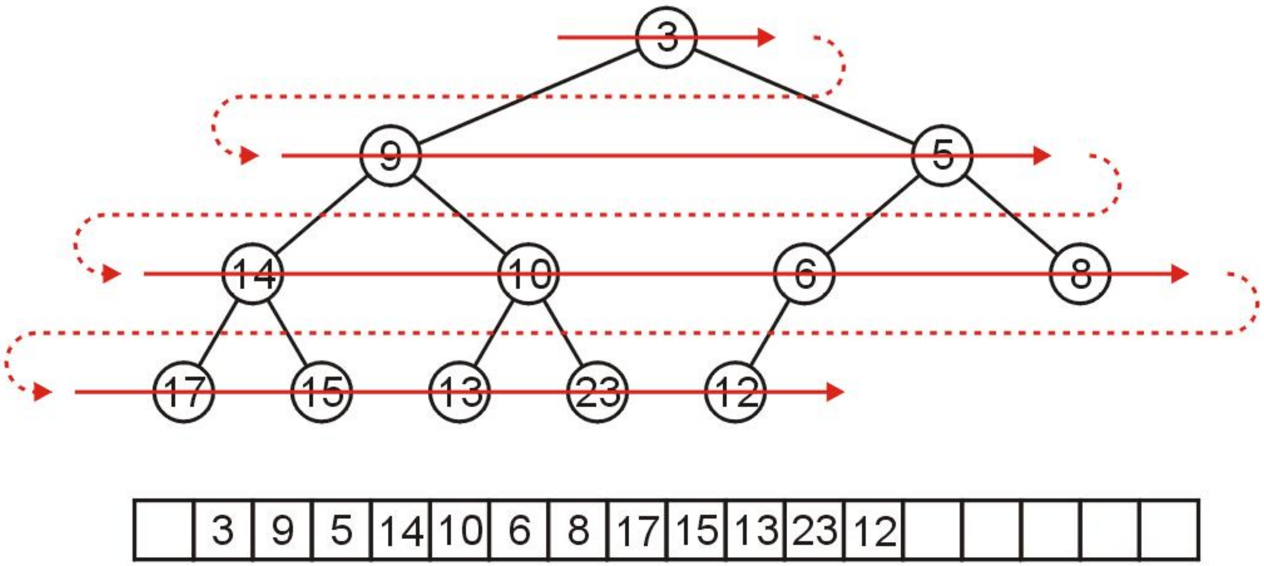
– Traverse the tree in breadth-first order, placing the entries into the array



Array storage

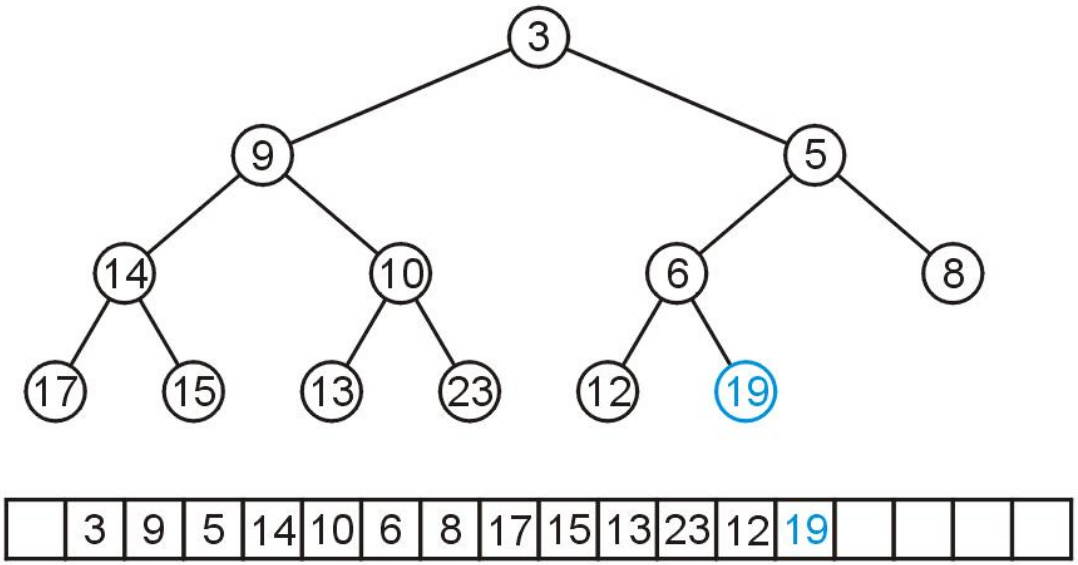
We are able to store a complete tree as an array

- Traverse the tree in breadth-first order, placing the entries into the array



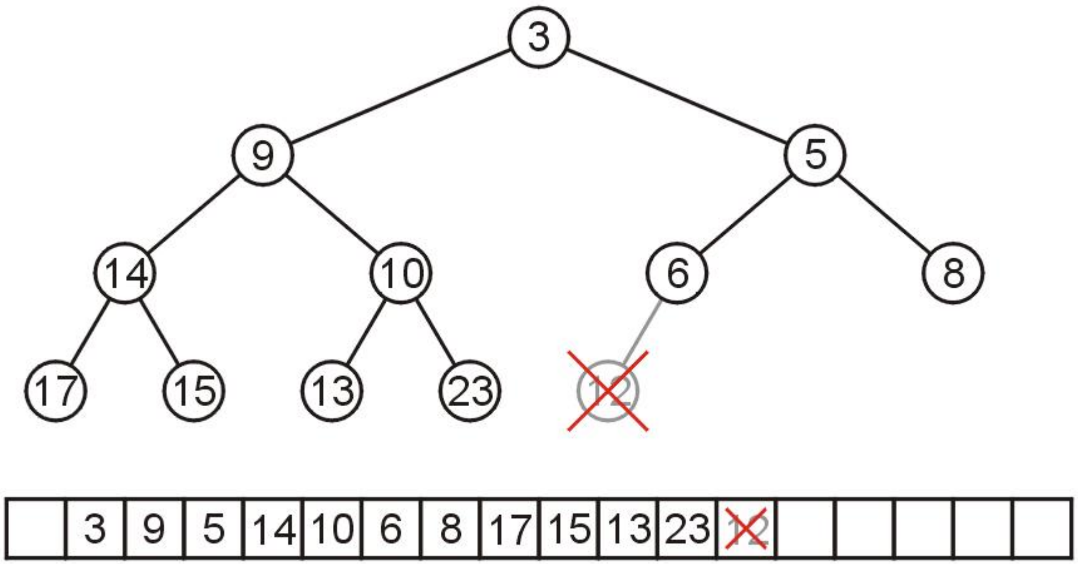
Array storage

To insert another node while maintaining the complete-binary-tree structure, we must insert into the next array location



Array storage

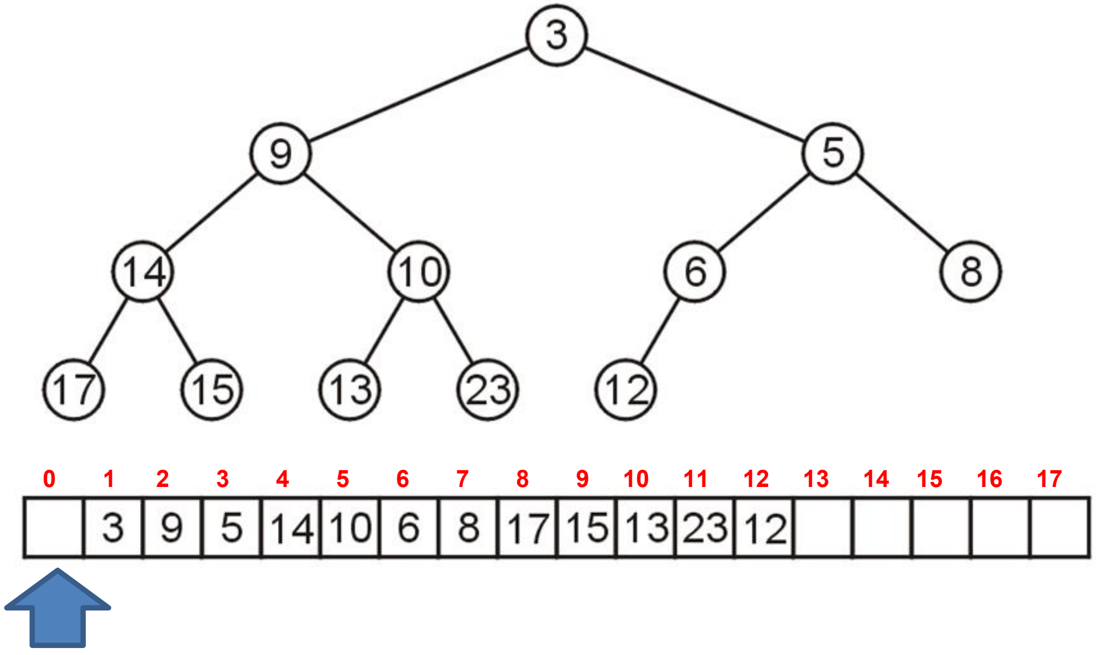
To remove a node while keeping the complete-tree structure, we must remove the last element in the array



Array storage

Leaving the first entry blank yields a bonus:

- The children of the node with index k are in $2k$ and $2k + 1$
- The parent of node with index k is in $k \div 2$

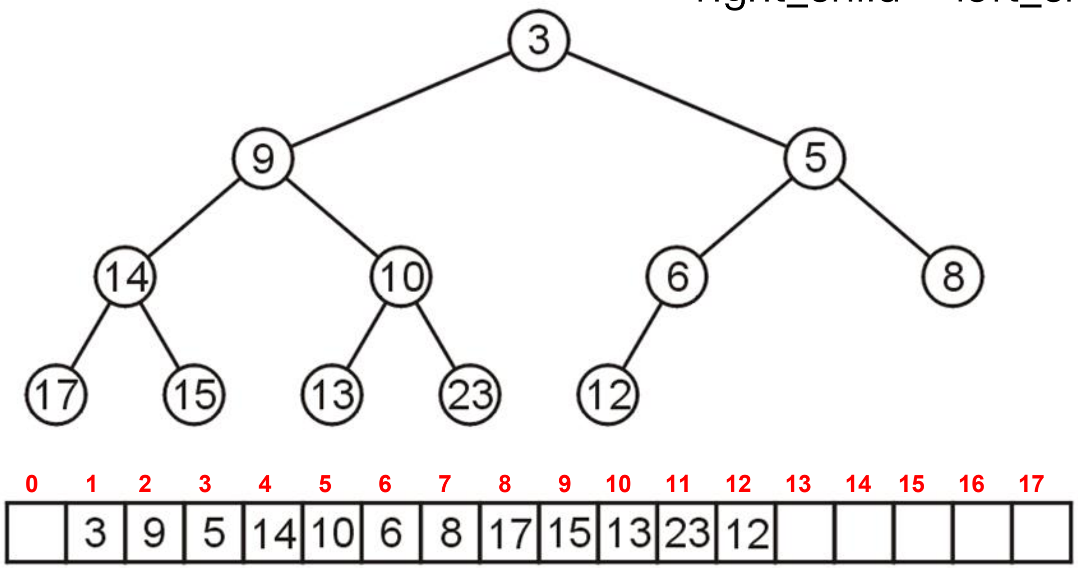


Array storage

Leaving the first entry blank yields a bonus:

– In C++, this simplifies the calculations:

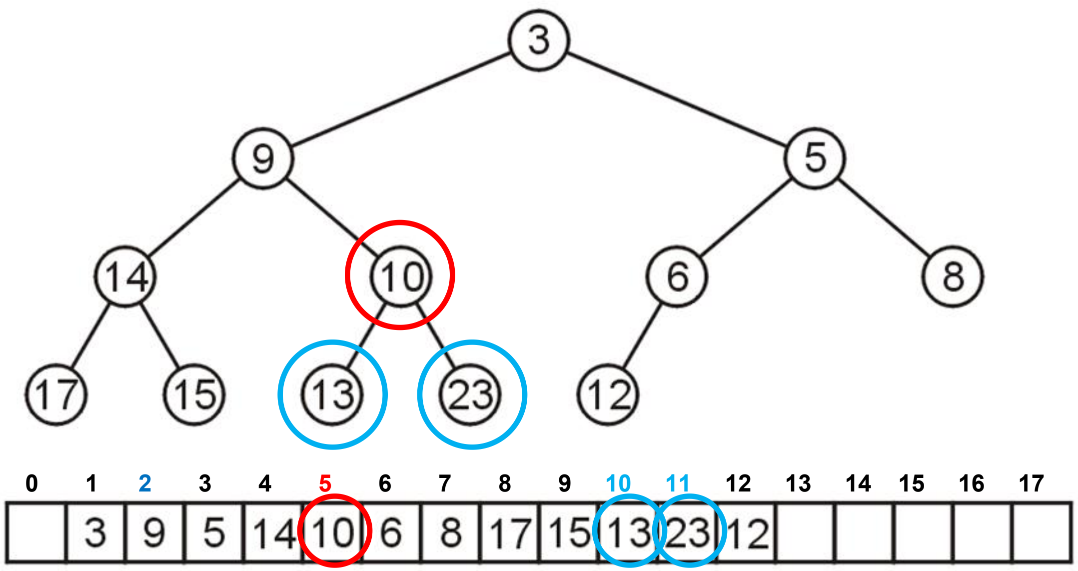
```
parent = k >> 1;
left_child = k << 1;
right_child = left_child | 1;
```



Array storage

For example, node 10 has index 5:

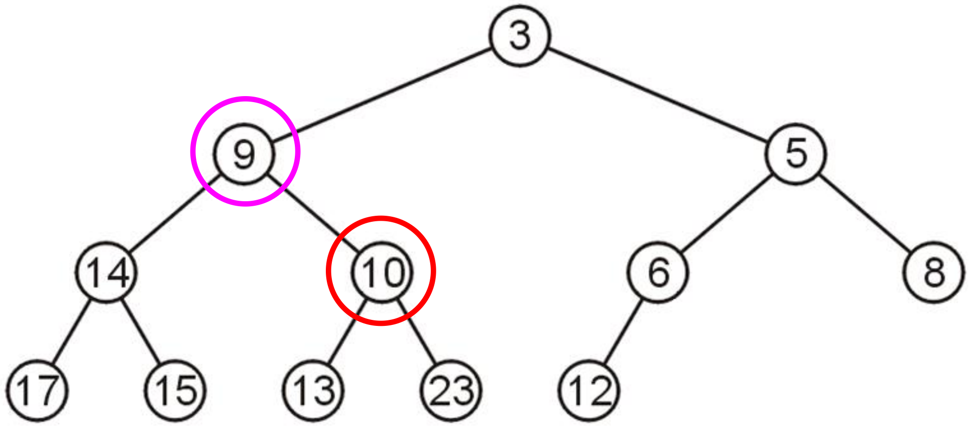
– Its children 13 and 23 have indices 10 and 11, respectively



Array storage

For example, node 10 has index 5:

- Its children 13 and 23 have indices 10 and 11, respectively
- Its parent is node 9 with index 5/2 = **2**



0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
	3	9	5	14	10	6	8	17	15	13	23	12					

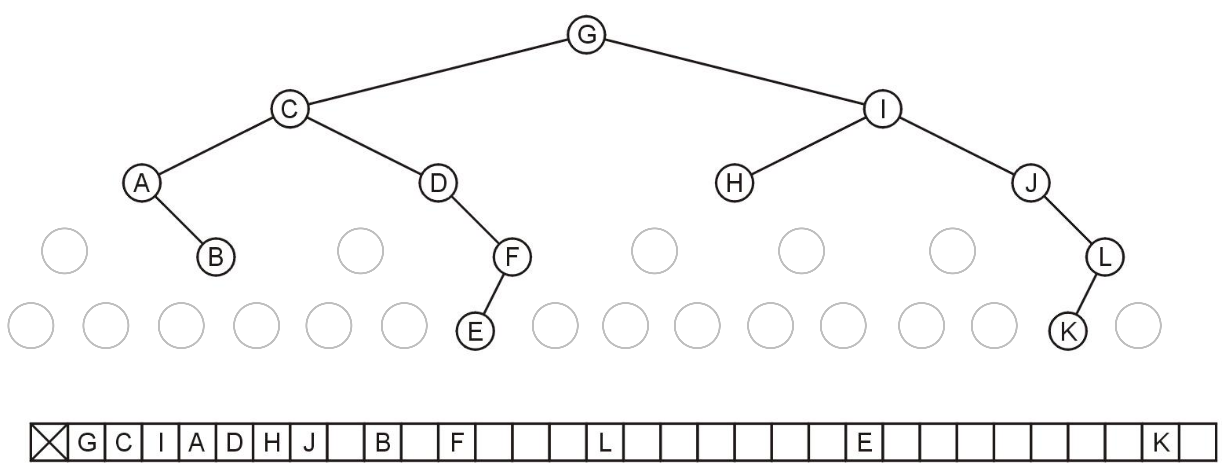
Array storage

Question: why not store any binary tree as an array in this way?

- There is a significant potential for a lot of wasted memory

Consider this tree with 12 nodes would require an array of size 32

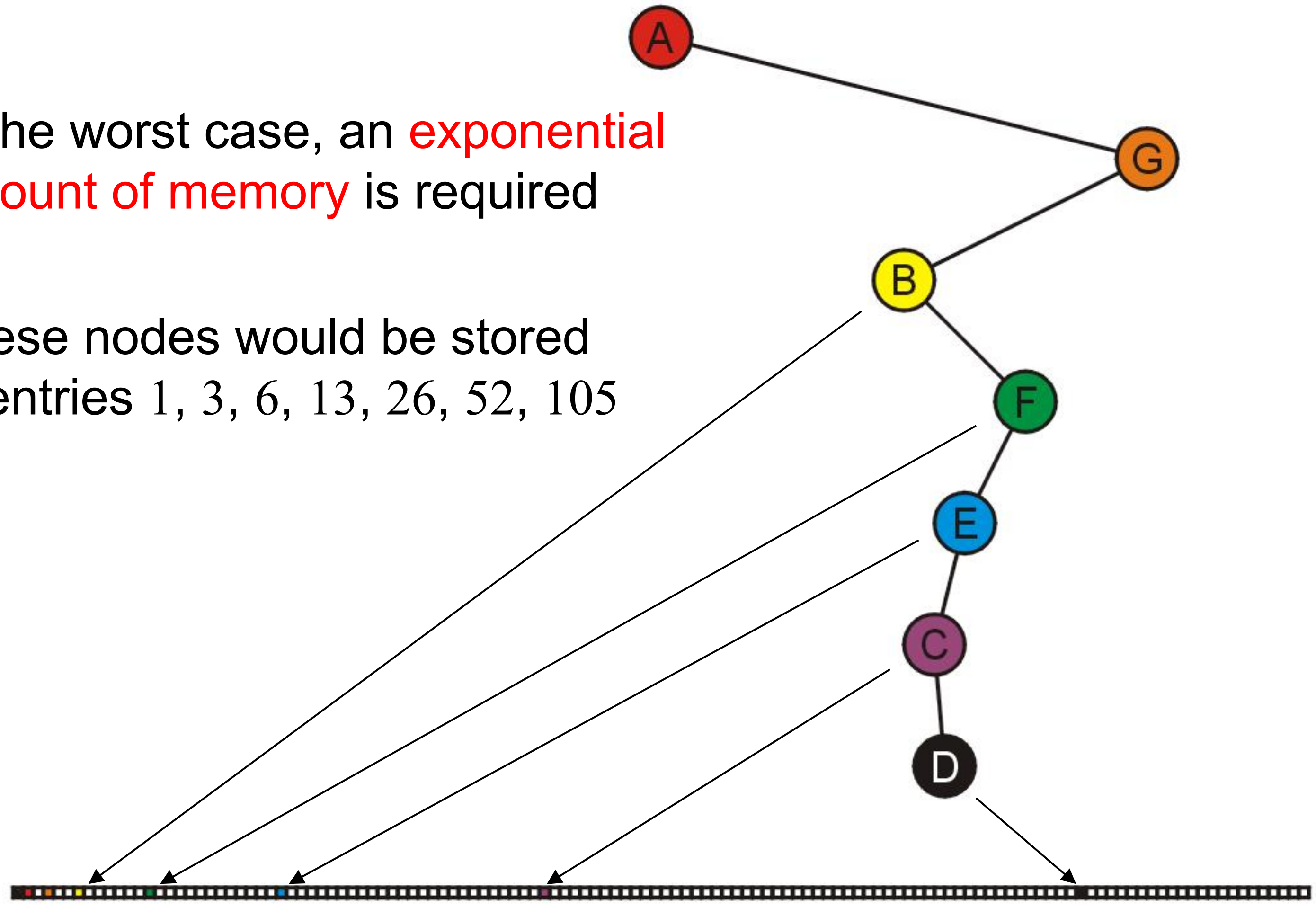
- Adding a child to node K doubles the required memory



Array storage

In the worst case, an exponential amount of memory is required

These nodes would be stored in entries 1, 3, 6, 13, 26, 52, 105



Summary

In this topic, we have covered the concept of a complete binary tree:

- A useful relaxation of the concept of a perfect binary tree
- It has a compact array representation

Outline

- Binary tree
- Perfect binary tree
- Complete binary tree
- Left-child right-sibling binary tree

Background

Our simple tree data structure is node-based where children are stored as a linked list

- Is it possible to store a general tree as a binary tree?

The Idea

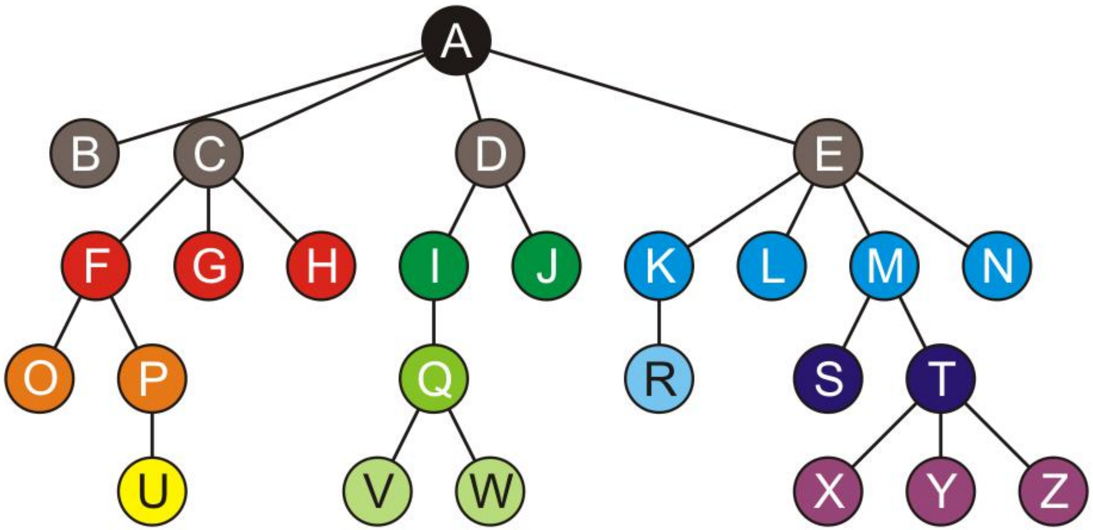
Consider the following:

- The first child of each node is its left sub-tree
- The next sibling of each node is in its right sub-tree

This is called a left-child—right-sibling binary tree

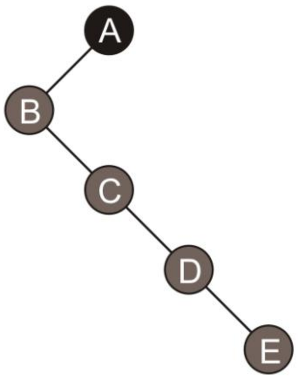
Example

Consider this general tree



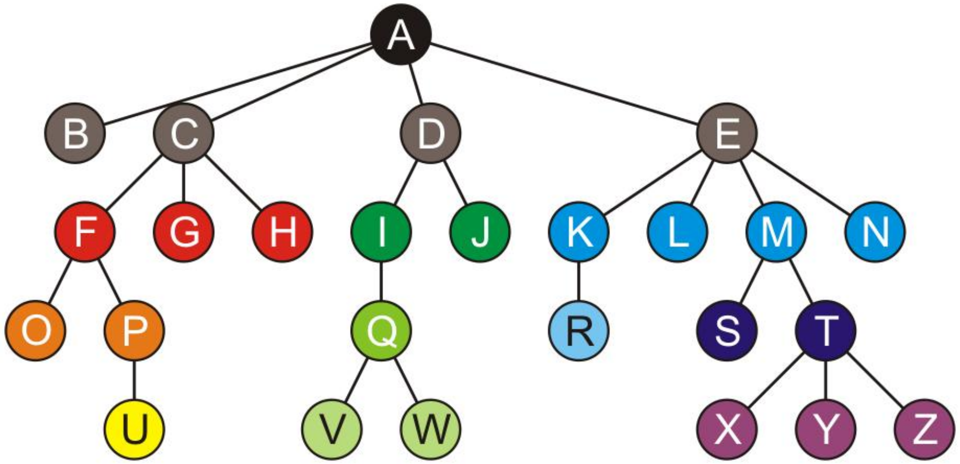
Example

B, the first child of A, is the left child of A



For the three siblings C, D, E:

- C is the right sub-tree of B
- D is the right sub-tree of C
- E is the right sub-tree of D



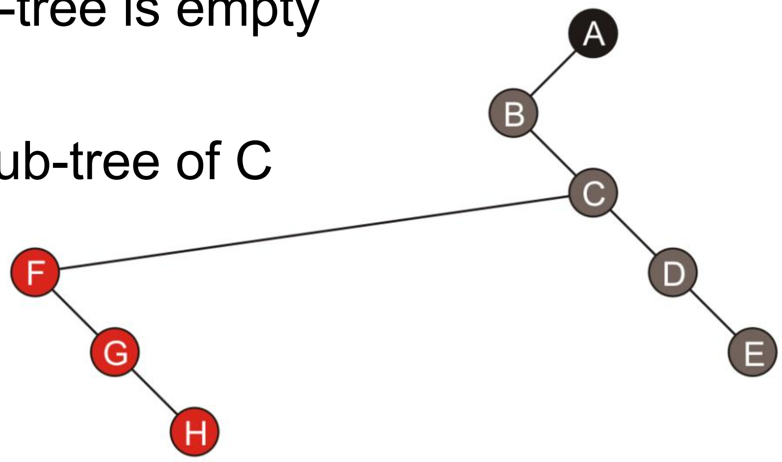
Example

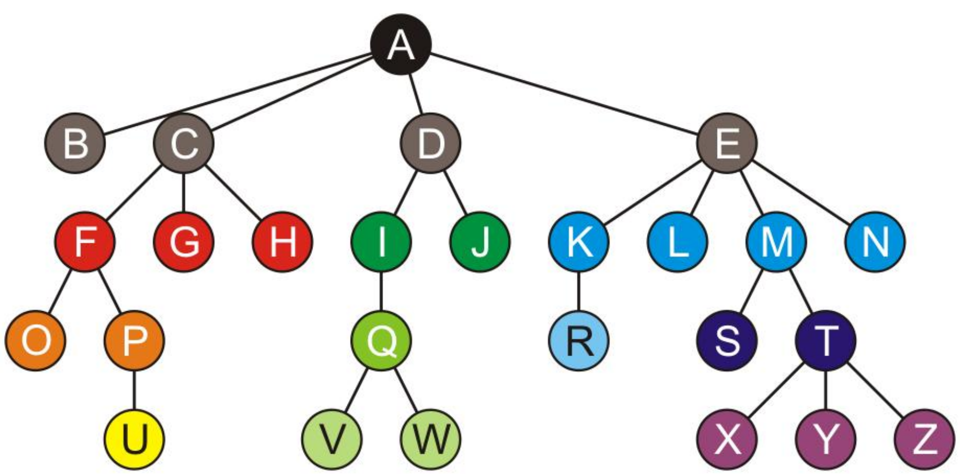
B has no children, so it's left sub-tree is empty

F, the first child of C, is the left sub-tree of C

For the next two siblings:

- G is the right sub-tree of F
- H is the right sub-tree of G

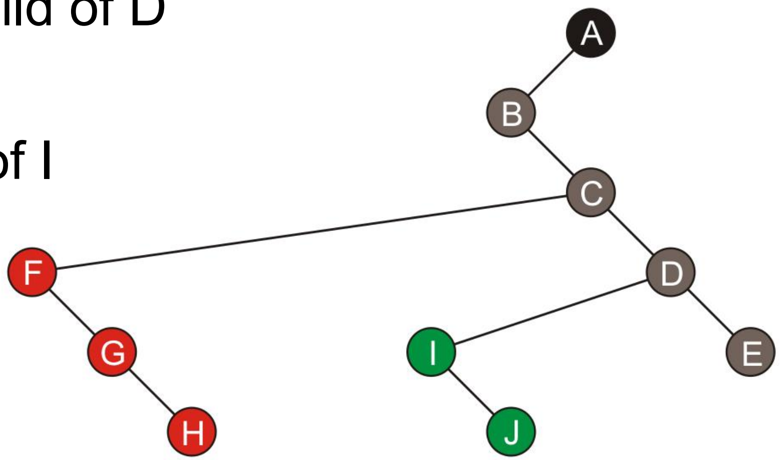


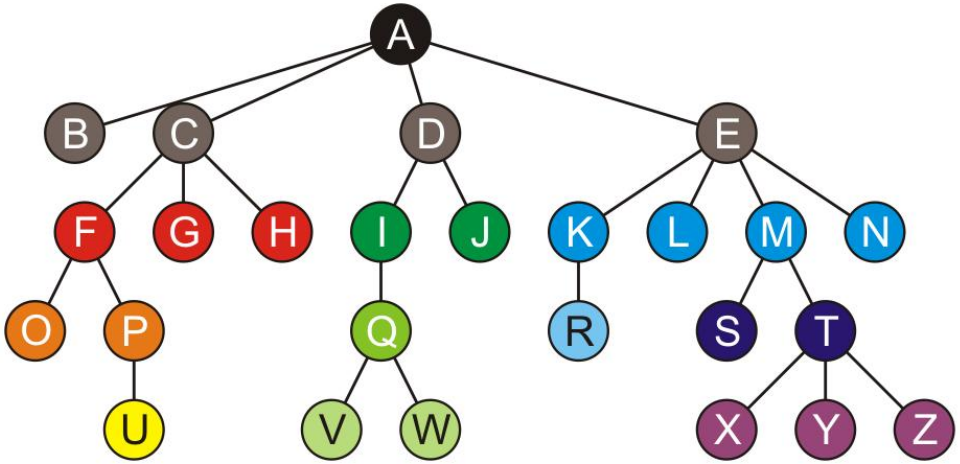


Example

I, the first child of D, is the left child of D

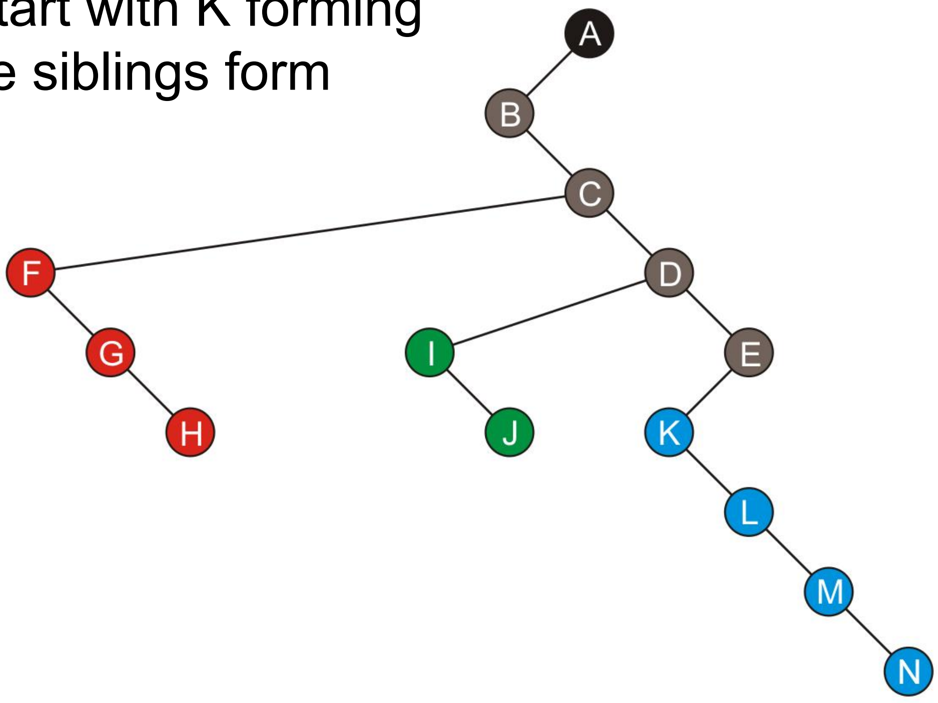
Its sibling J is the right sub-tree of I

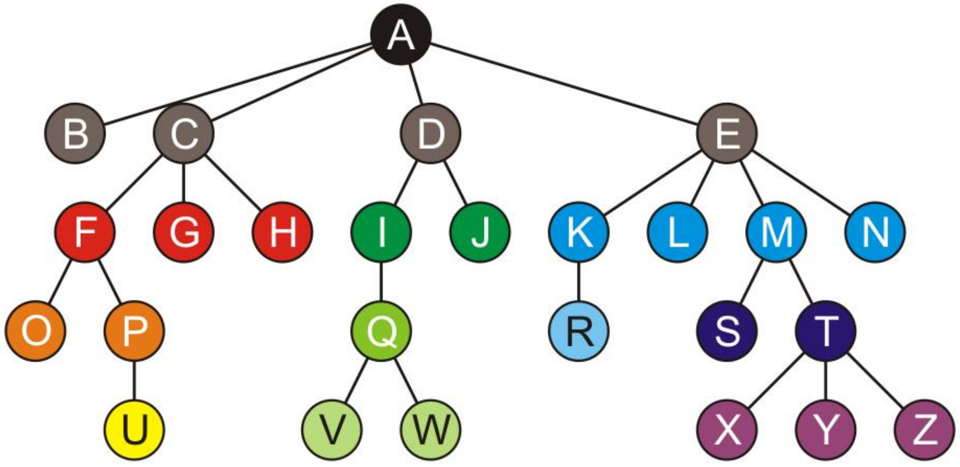




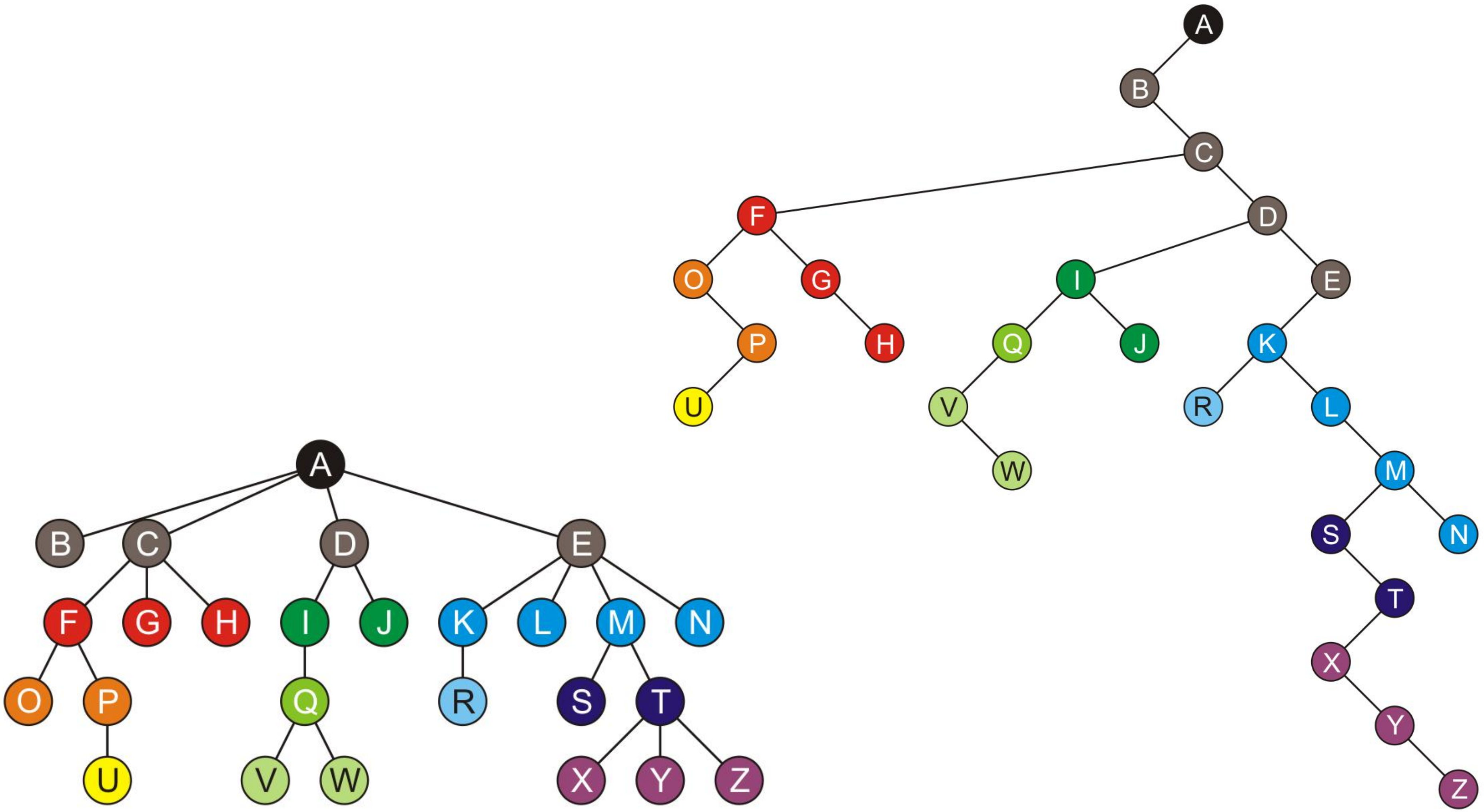
Example

Similarly, the four children of E start with K forming the left sub-tree of E and its three siblings form a chain along the right sub-trees



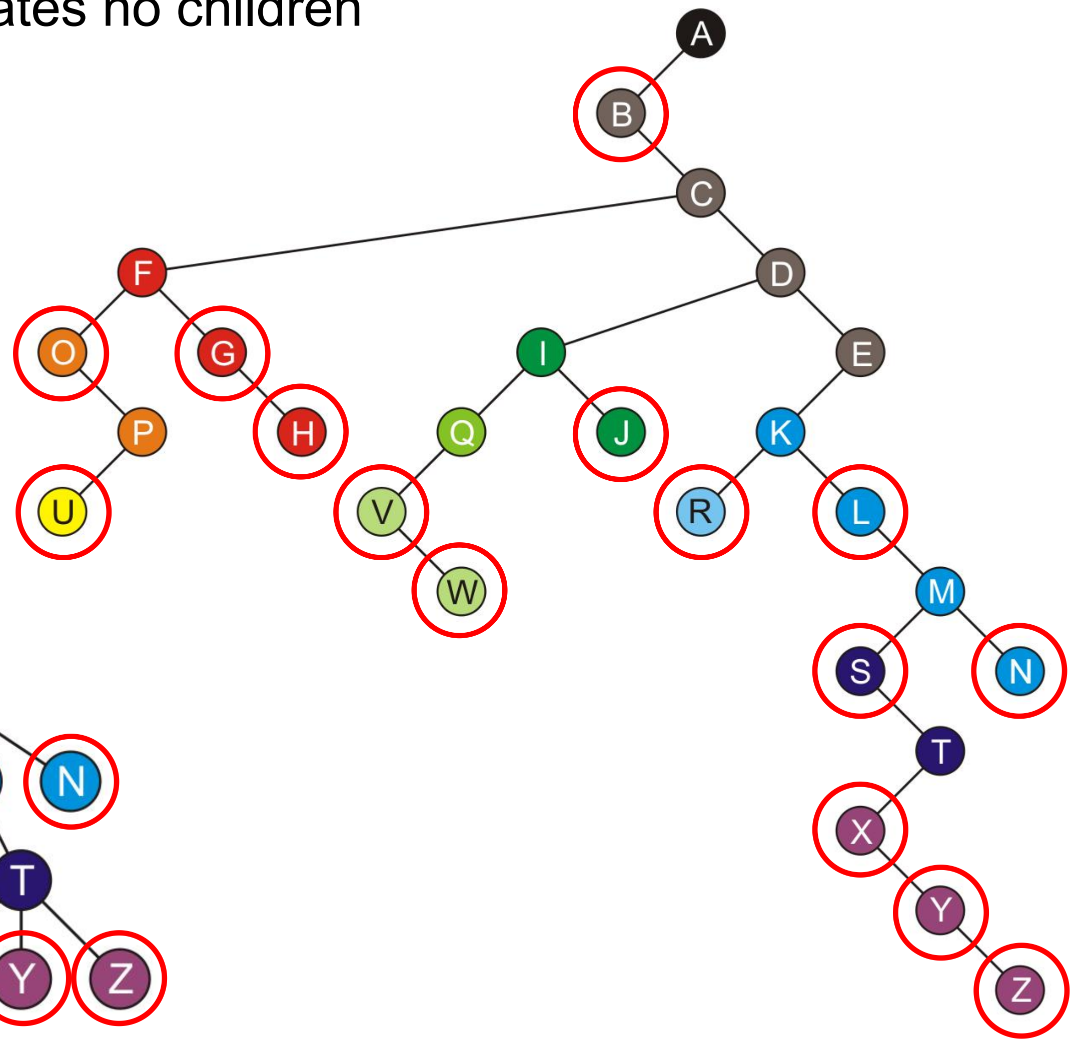


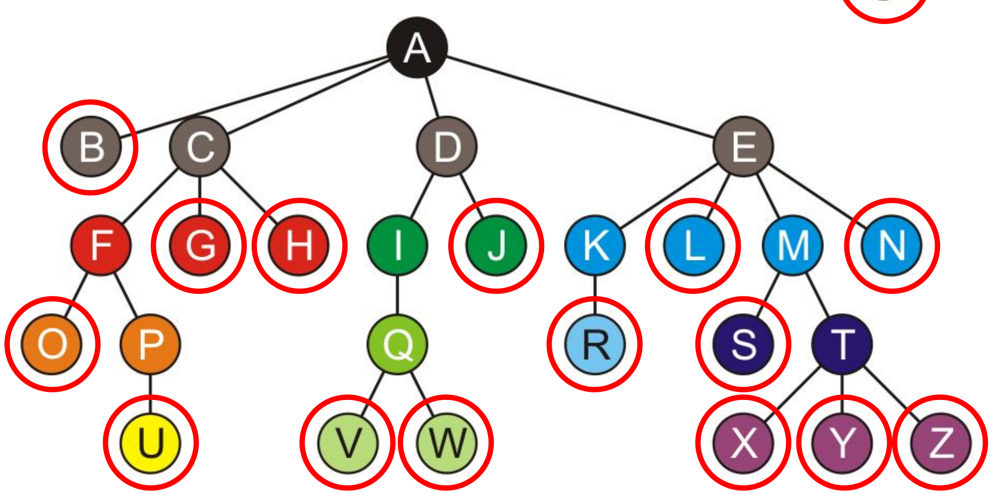
Example


A
B C D E
F G H I J K L M N
O P Q R S T
U V W X Y Z
A
B
C
F D
O G I E
P H Q J K
U V R L
W M
S N
T
X
Y
Z

Example

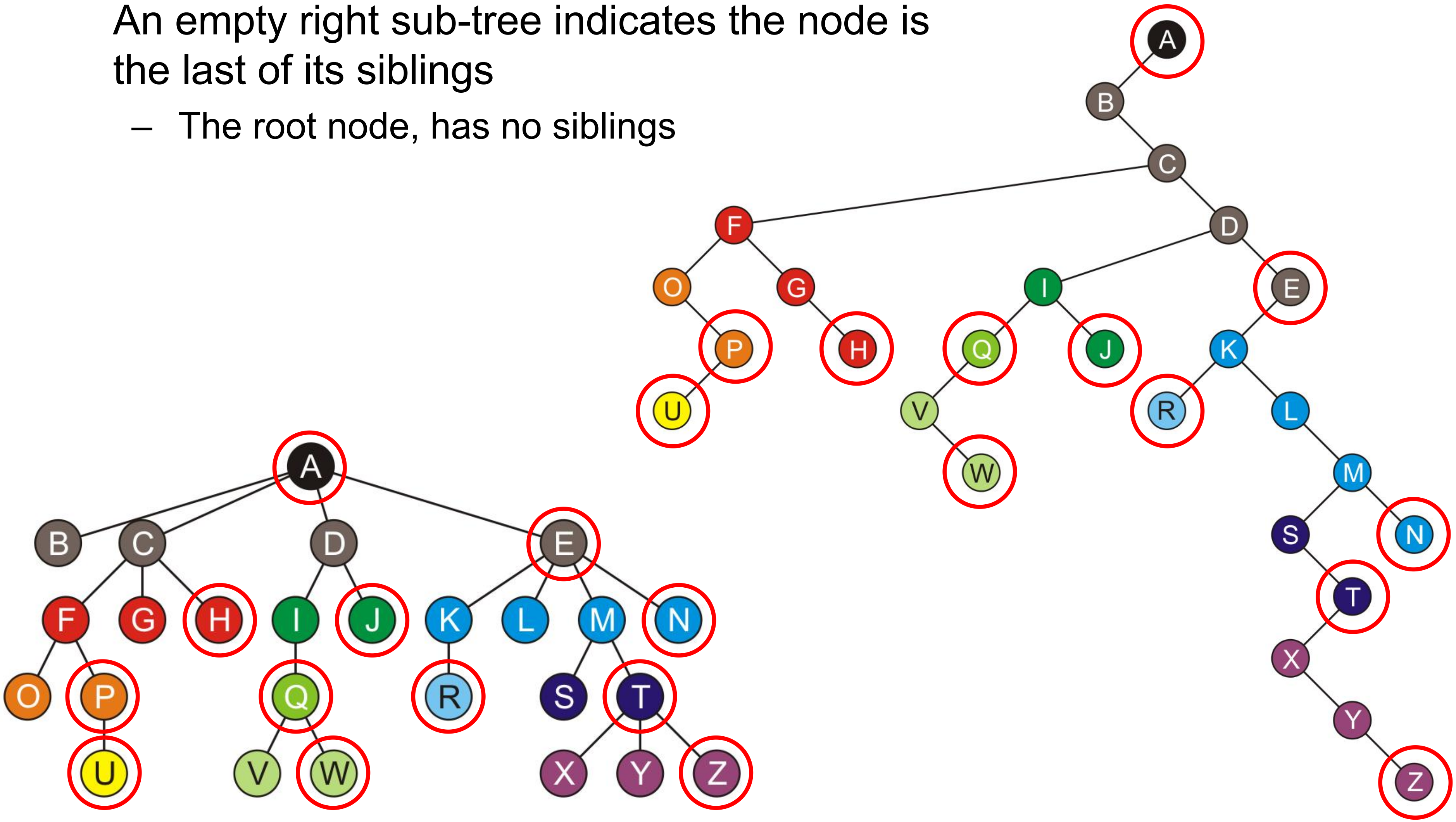
An empty left sub-tree indicates no children





Example

- An empty right sub-tree indicates the node is the last of its siblings
  – The root node, has no siblings



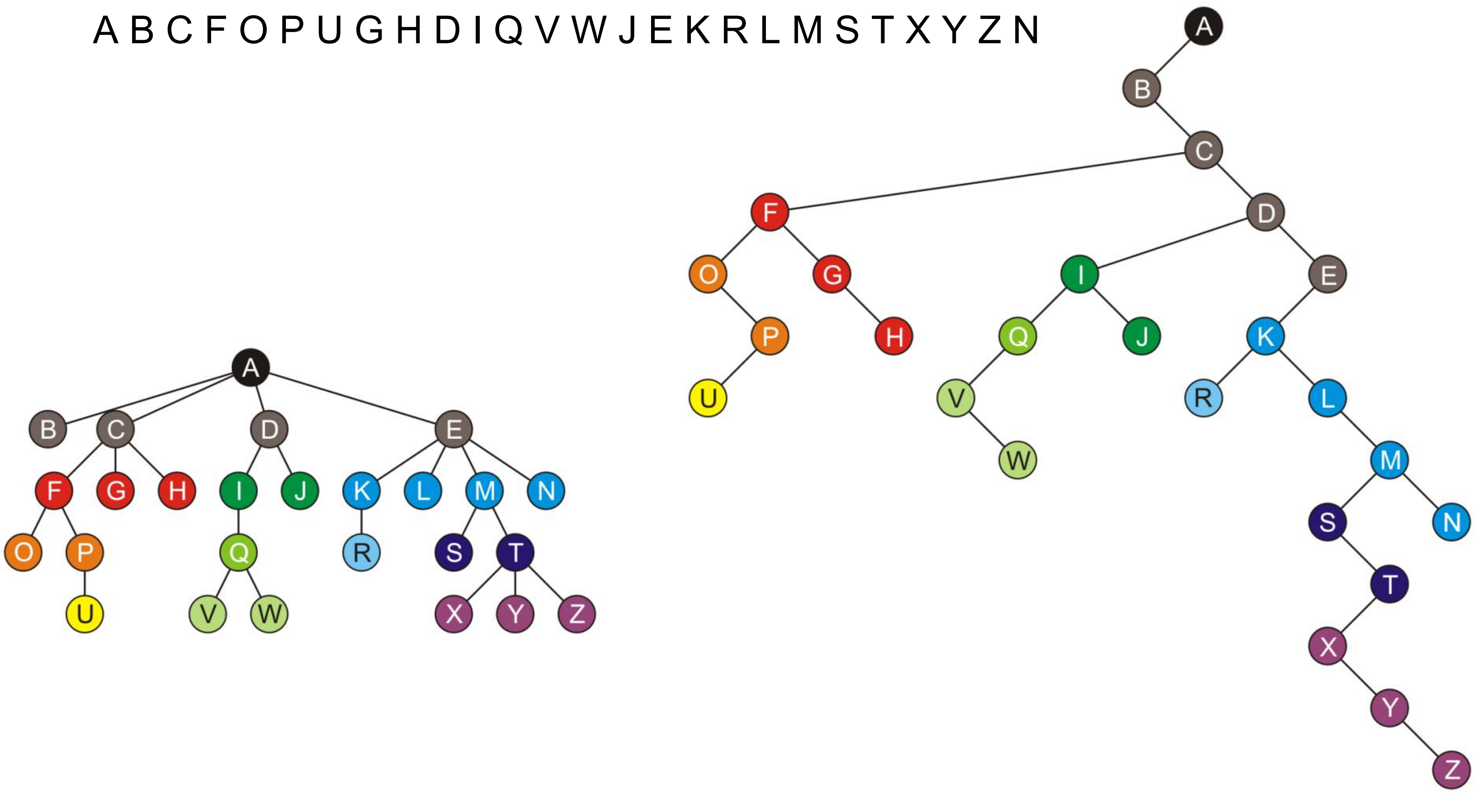
Transformation

The transformation of a general tree into a left-child right-sibling binary tree has been called the Knuth transform

Traversals

A pre-order traversal of the original tree is identical to the pre-order traversal of the Knuth transform

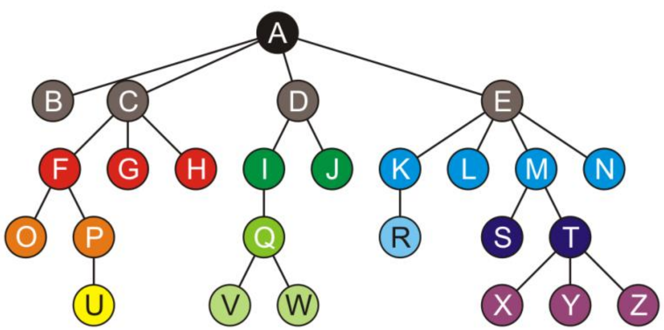
A B C F O P U G H D I Q V W J E K R L M S T X Y Z N

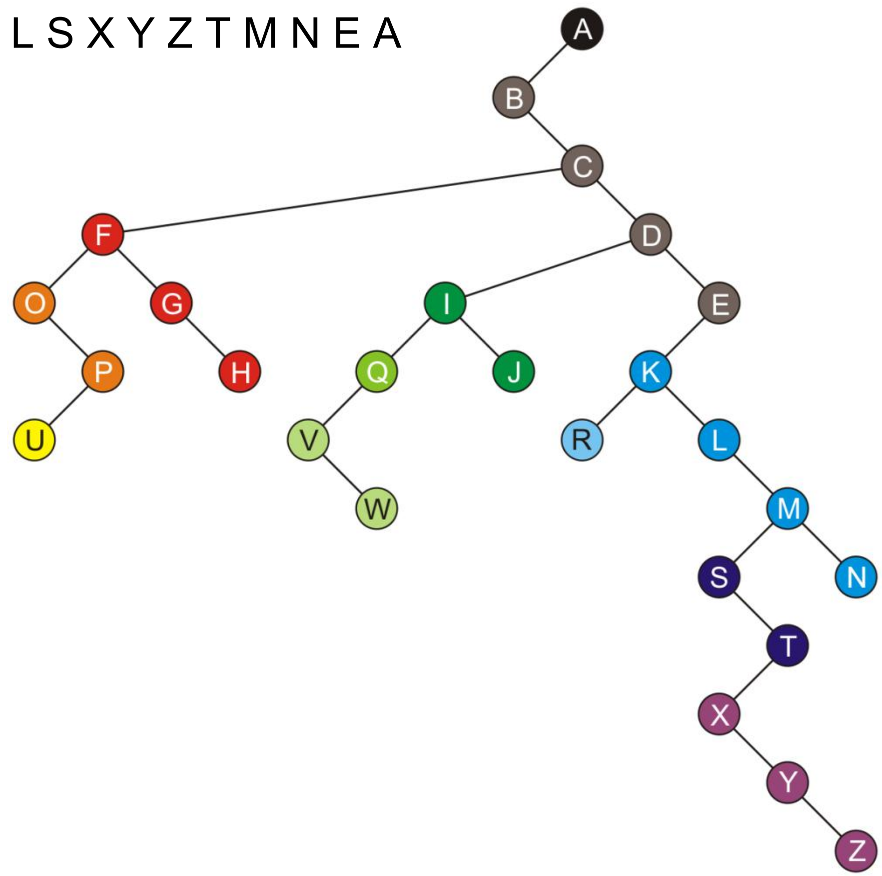


Traversals

A post-order traversal of the original tree is identical to the in-order traversal of the Knuth transform

B O U P F G H C V W Q I J D R K L S X Y Z T M N E A

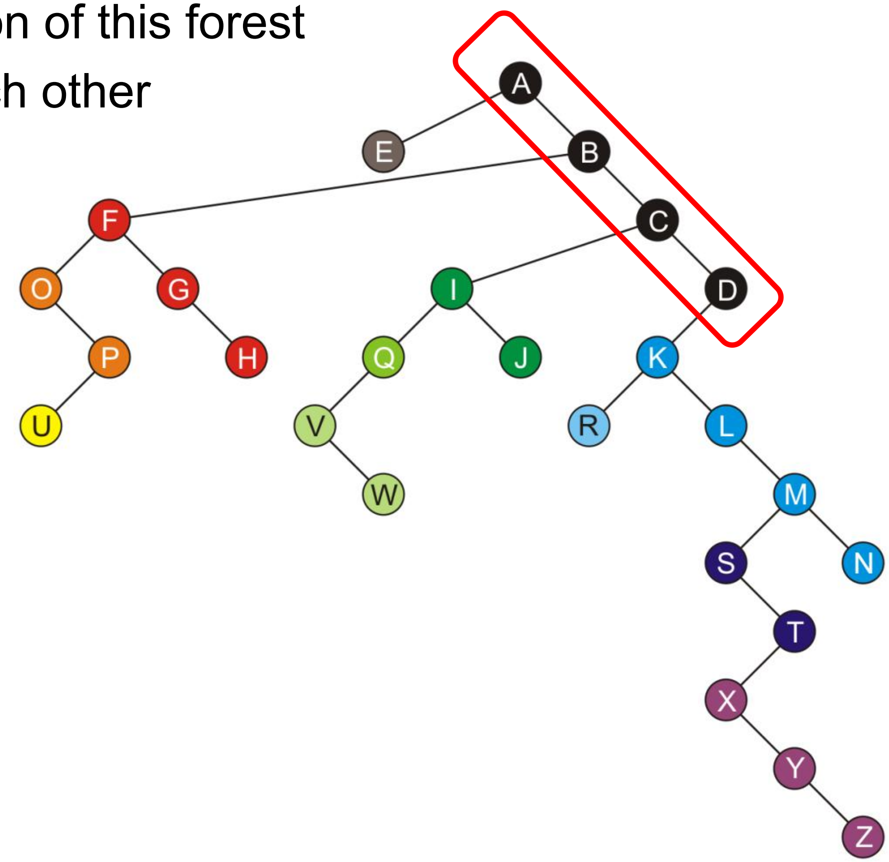


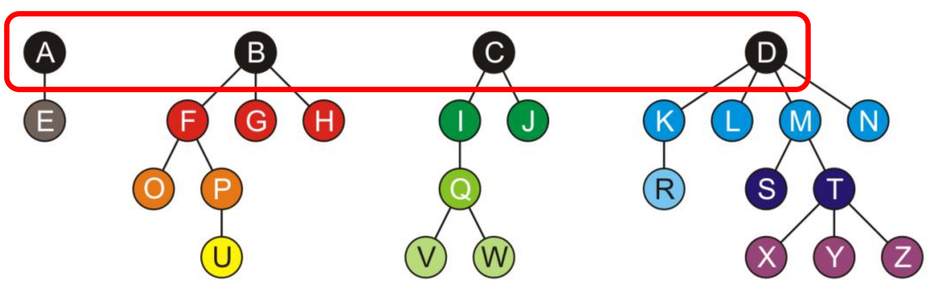


Forests

A forest can be stored in this representation as follows:

- Choose one of the roots of the trees as the root of the binary tree
- Let each subsequent root of a tree be a right child of the previous root
- This is the binary-tree representation of this forest
- Think of the roots as siblings of each other





Implementation

The class is similar to that of a binary tree

```
template <typename Type>
class LCRS_tree {
    private:
        Type element;
        LCRS_tree *first_child_tree;
        LCRS_tree *next_sibling_tree;

    public:
        LCRS_tree();
        LCRS_tree *first_child();
        LCRS_tree *next_sibling();
        // ...
};
```

Implementation

The implementation of various functions now differs

```
template <typename Type>
int LCRS_tree<Type>::degree() const {
    int count = 0;

    for (
        LCRS_tree<Type> *ptr = first_child();
        ptr != nullptr;
        ptr = ptr->next_sibling()
    ) {
        ++count;
    }

    return count;
}
```

Implementation

The implementation of various functions now differs

```
template <typename Type>
bool LCRS_tree<Type>::is_leaf() const {
    return ( first_child() == nullptr );
}
```

Implementation

The implementation of various functions now differs

```
template <typename Type>
LCRS_tree<Type> *LCRS_tree<Type>::child( int n ) const {
    if ( n < 0 || n >= degree() ) {
        return nullptr;
    }

    LCRS_tree<Type> *ptr = first_child();

    for ( int i = 0; i < n; ++i ) {
        ptr = ptr->next_sibling();
    }

    return ptr;
}
```

Implementation

The implementation of various functions now differs

```
template <typename Type>
void LCRS_tree<Type>::append( Type const &obj ) {
    if ( first_child() == nullptr ) {
        first_child_tree = new LCRS_tree<Type>( obj );
    } else {
        LCRS_tree<Type> *ptr = first_child();

        while ( ptr->next_sibling() != nullptr ) {
            ptr = ptr->next_sibling();
        }

        ptr->next_sibling_tree = new LCRS_tree<Type>( obj );
    }
}
```

Implementation

The implementation of various functions now differs

– The size doesn't care that this is a general tree…

```
template <typename Type>
int LCRS_tree<Type>::size() const {
    return 1
      + (  first_child() == nullptr ? 0 :  first_child()->size() )
      + ( next_sibling() == nullptr ? 0 : next_sibling()->size() );
}
```

Implementation

The implementation of various functions now differs

- The height member function is closer to the original implementation

```
template <typename Type>
int LCRS_tree<Type>::height() const {
    int h = 0;

    for (
        LCRS_tree<Type> *ptr = first_child();
        ptr != nullptr;
        ptr = ptr->next_sibling()
    ) {
        h = std::max( h, 1 + ptr->height() );
    }

    return h;
}
```

Summary

This topic has covered a binary representation of general trees

- The first child is the left sub-tree of a node
- Subsequent siblings of that child form a chain down the right sub-trees
- An empty left sub-tree indicates no children
- An empty right sub-tree indicates no other siblings

References

[1] Cormen, Leiserson, Rivest and Stein, *Introduction to Algorithms*, 2nd Ed., MIT Press, 2001, §19.1, pp.457-9.

[2] Weiss, *Data Structures and Algorithm Analysis in C*++, *3rd Ed.*, Addison Wesley, §6.8.1, p.240.

Summary

- Binary tree
  - Each node has two children
  - In-order traversal
- Perfect binary tree
  - Number of nodes, height, number of leaf nodes, average depth
- Complete binary tree
  - Height, array storage
- Left-child right-sibling binary tree