Discrete Mathematics: Lecture 29

Tree, Tree Traversals, Spanning Trees, DFS, BFS

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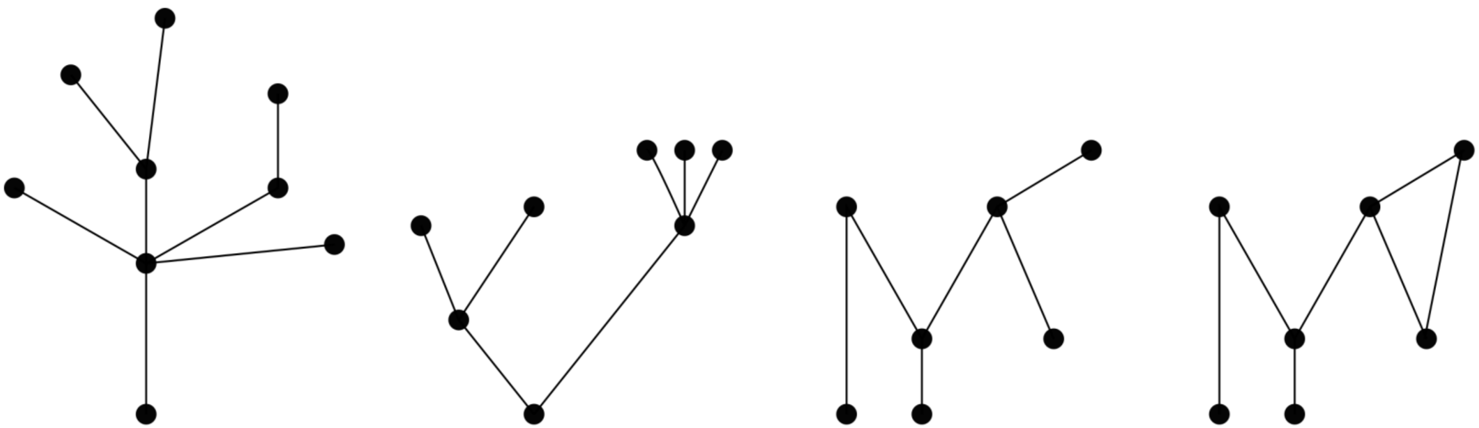
Spring Semester, 2022

Notes by Prof. Liangfeng Zhang

Tree

Definition

- A **tree** is a connected undirected graph with no simple circuits.
- A **forest** is an graph such that each of its connected components is a tree.



G H I K

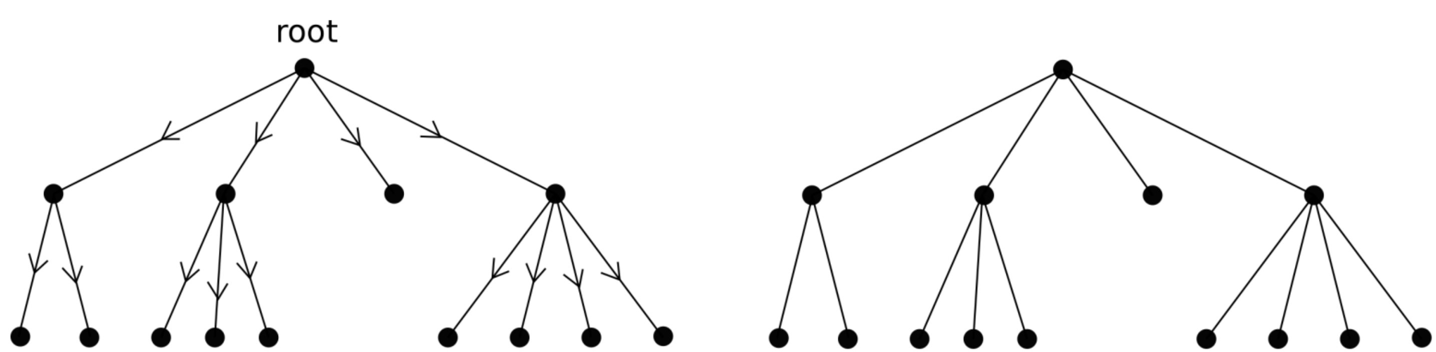
G, H, I are trees, but K is not a tree.

Rooted Tree

Definition

A **rooted tree** is a tree in which one vertex has been designated as the root and every edge is directed away from the root.

Remarks: • A rooted tree is a directed graph.
• We usually draw a rooted tree with its root at the top of the graph.
• We usually omit the arrows on the edges to indicate the direction because it is uniquely determined by the choice of the root.
• Any non rooted tree can be changed to a rooted tree by choosing a vertex for the root.



Properties of Tree

Tree = connected with no simple circuit (definition)

(1) connected
(2) no simple circuit
(3) $(n-1)$ edges (n=nb of vertices)

Previous theorem: $(1) + (2) \Rightarrow (3)$

We also have: $(1) + (3) \Rightarrow (2)$
$(2) + (3) \Rightarrow (1)$

Example: For what value of m, n the complete bipartite graph $K_{m,n}$ is a tree?

$K_{m,n}$ is connected, has $m + n$ vertices and $m \times n$ edges.
It is a tree if:

$$m \times n = m + n - 1 \Longleftrightarrow (n-1)m = n - 1$$

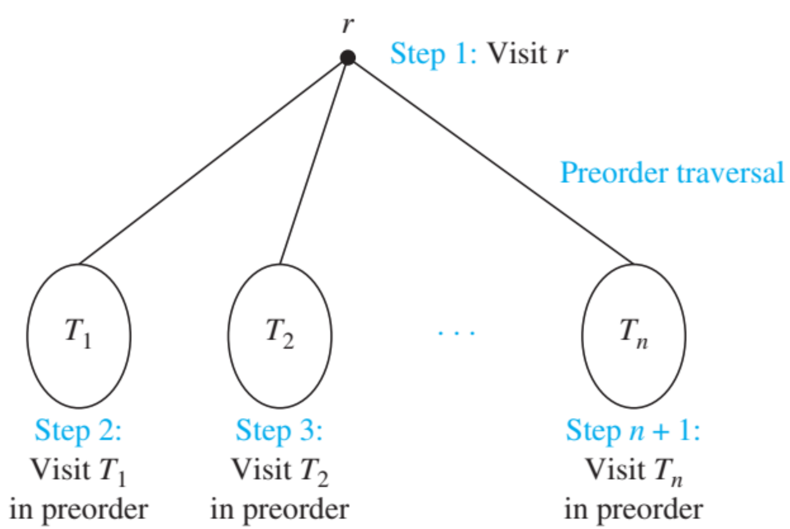
If $n \neq 1$: $m = 1$
If $n = 1$: $m \in \mathbb{N}^*$

Tree Traversals

Preorder traversal algorithm

Recursive definition: Let T be a rooted tree with root r

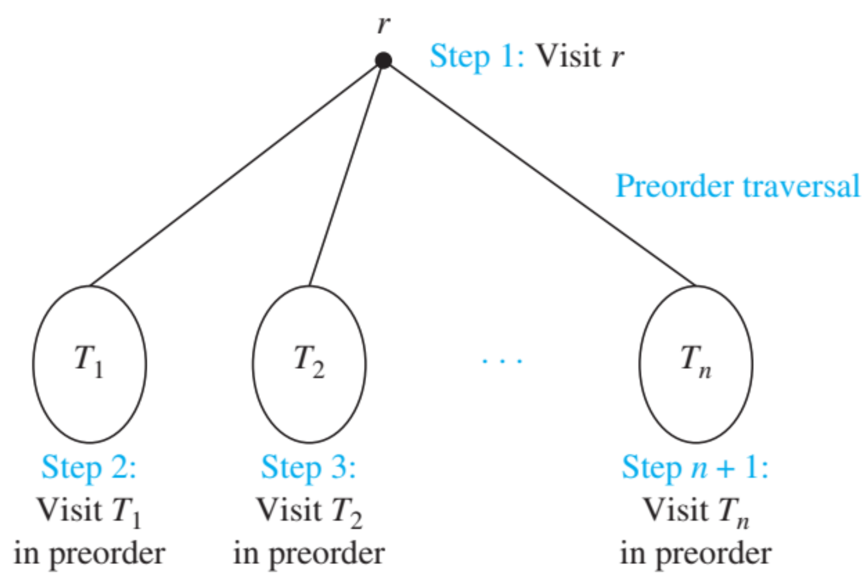
- if T consists only on r: r is the preorder traversal of T.
- otherwise, denote by $T_1, \ldots, T_n$ the subtrees rooted at the children of r, from left to right.
  The preorder traversal of T begins by visiting r, then traverses T_1 in preorder, then T_2 in preorder,..., and finally T_n in preorder.



Tree Traversals

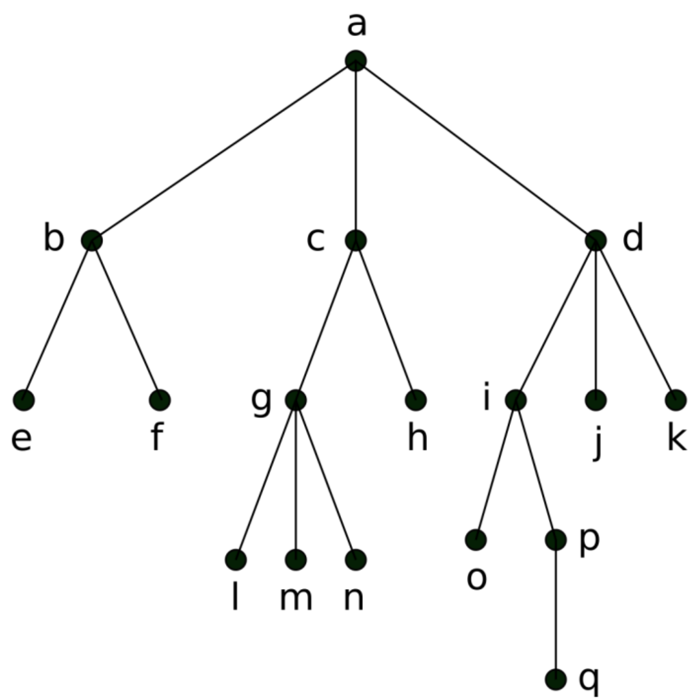
Recursive algorithm:

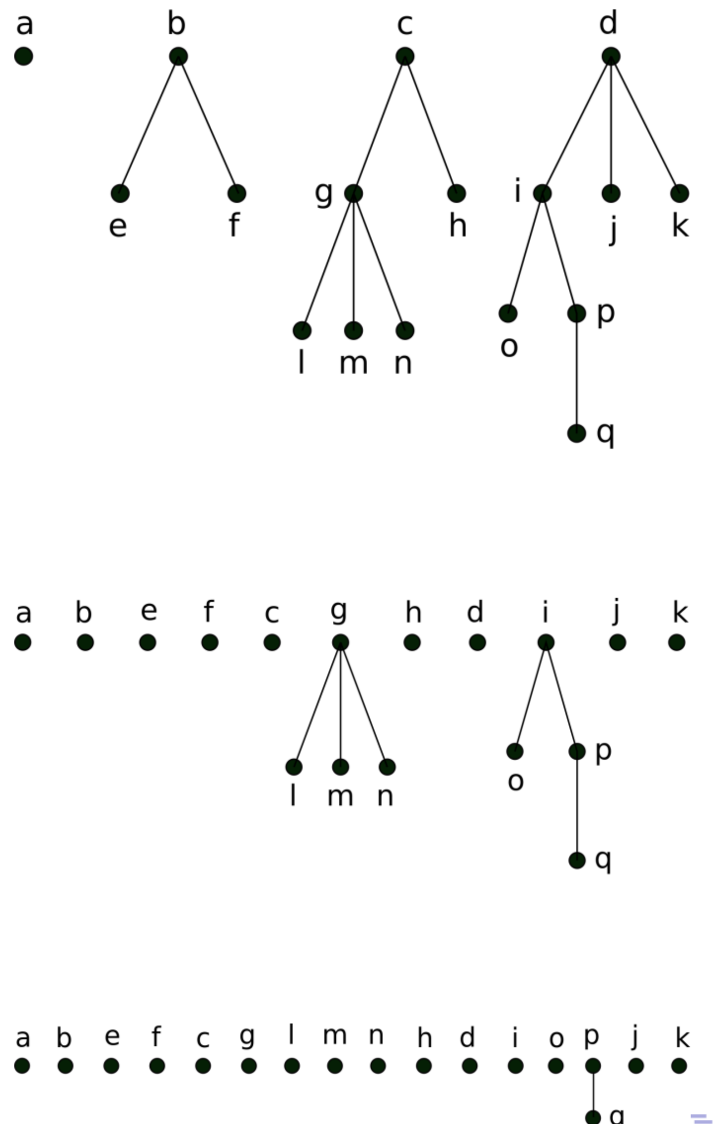
preorder(T: ordered rooted tree)
r:=root of T
list r (add r in the preorder list of the vertices of T)
for each child c of r from left to right
    $T(c)$:= subtree of T with c as its root
    preorder(T(c))



Tree Traversals

Preorder traversal algorithm



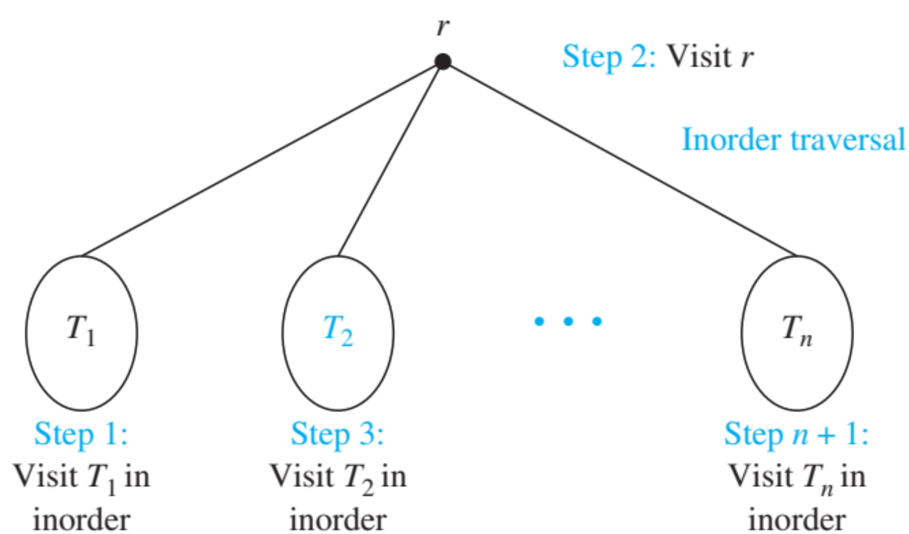


Tree Traversals

Inorder traversal algorithm

Recursive definition: Let T be a rooted tree with root r

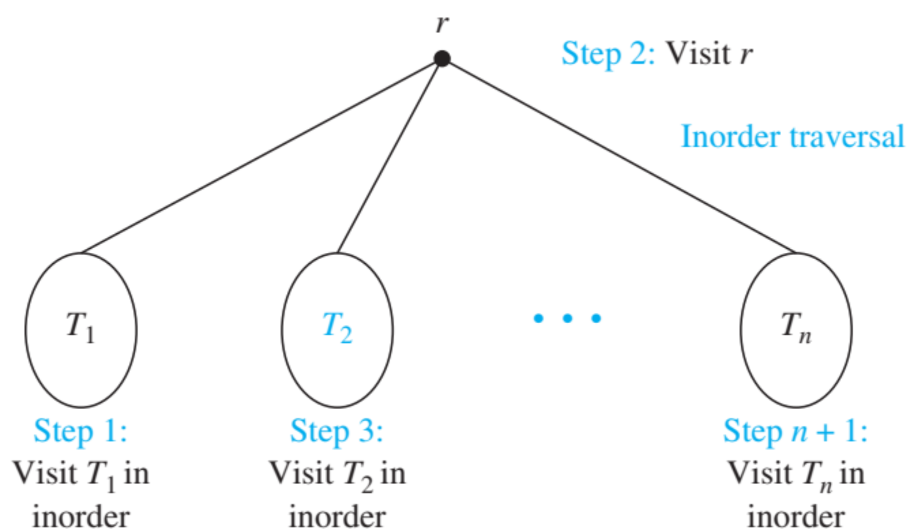
- if T consists only on r: r is the inorder traversal of T.
- otherwise, denote by $T_1, \dots, T_n$ the subtrees rooted at the children of r, from left to right.
  The inorder traversal of T begins by traversing T_1 in inorder, then visiting r, then traversing T_2 in inorder, then T_3 in inorder,..., and finally T_n in inorder.



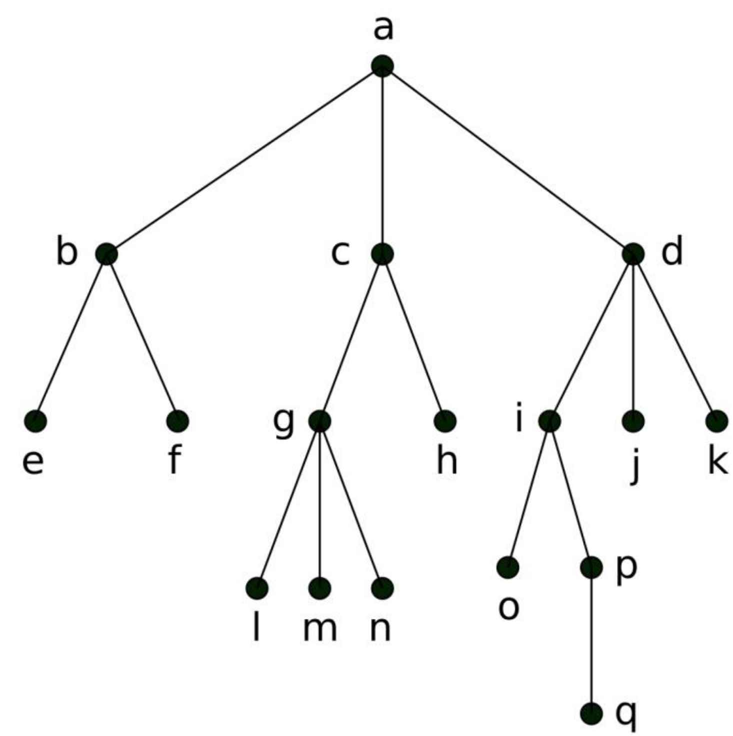
Tree Traversals

Recursive algorithm:

inorder(T: ordered rooted tree)
r:=root of T
if r is a leaf **then** list r
else l := first child of r from left to right
    $T(l)$:= subtree of T with l as its root
    inorder(T(l))
    list r
    for each child c of r from left to right except l
        $T(c)$:= subtree of T with c as its root
        inorder(T(c))



Tree Traversals



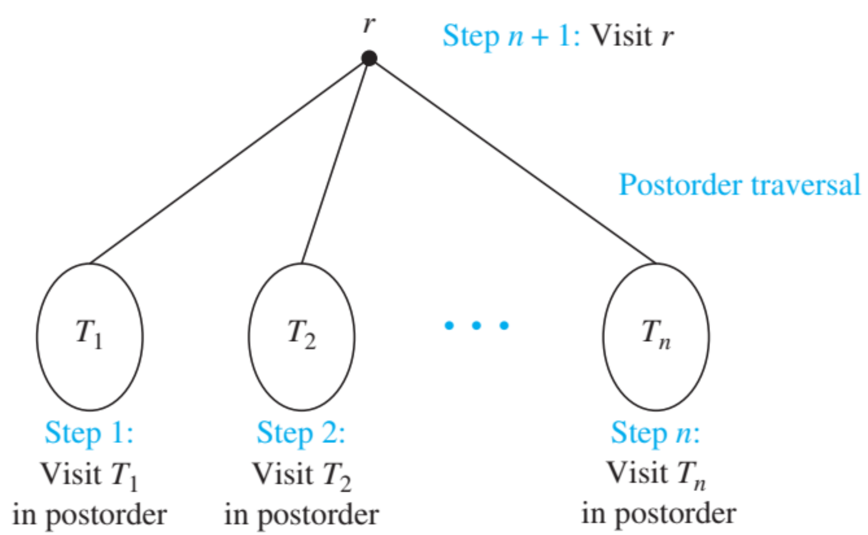
Inorder traversal: e, b, f, a, l, g, m, n, c, h, o, i, q, p, d, j, k

Tree Traversals

Postorder traversal algorithm

Recursive definition: Let T be a rooted tree with root r

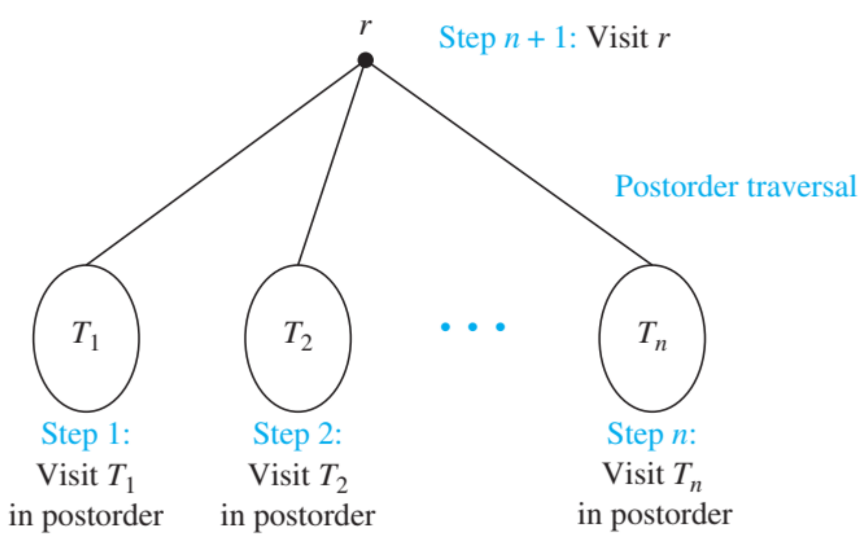
- if T consists only on r: r is the postorder traversal of T.
- otherwise, denote by $T_1, \ldots, T_n$ the subtrees rooted at the children of r, from left to right.
  The postorder traversal of T begins by traversing T_1 in postorder, then T_2 in postorder,..., then T_n in postorder, and ends by visiting the root r.



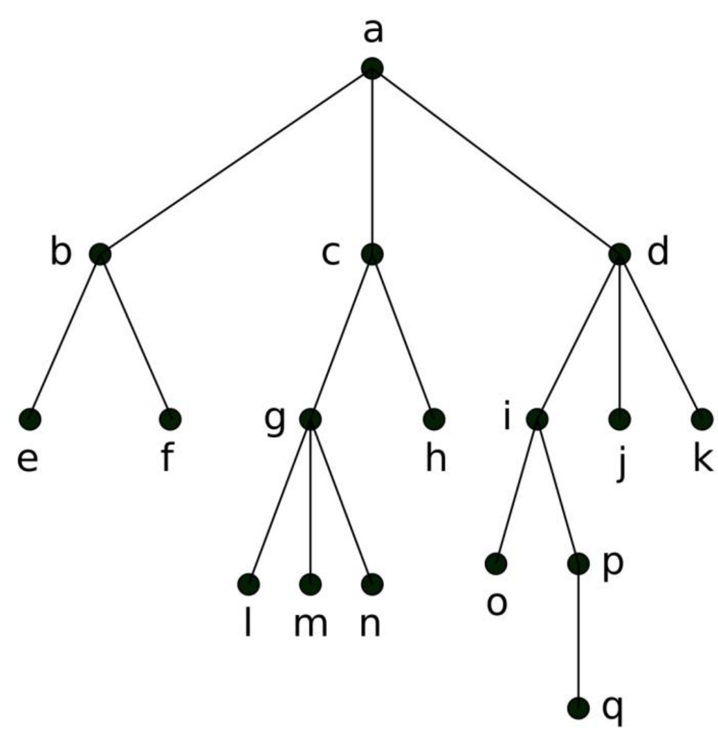
Tree Traversals

Recursive algorithm:

postorder(T: ordered rooted tree)
r:=root of T
for each child c of r from left to right
    $T(c)$:= subtree of T with c as its root
    postorder(T(c))
list r



Tree Traversals



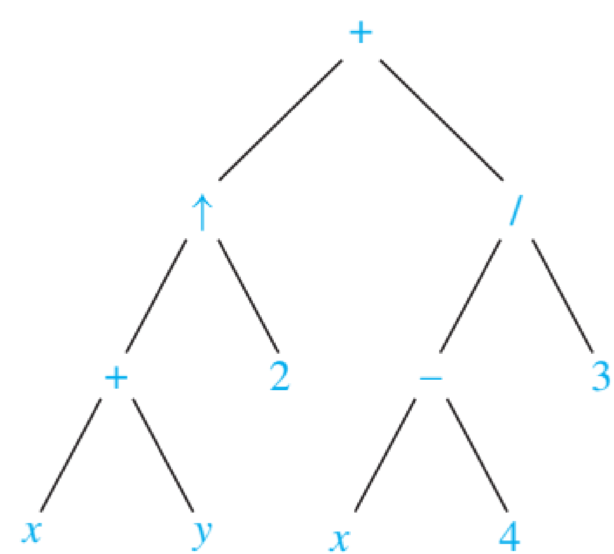
Postorder traversal: e, f, b, l, m, n, g, h, c, o, q, p, i, j, k, d, a

Infix, Prefix, Postfix Notation

Goal: Using ordered rooted trees to represent arithmetic expressions or compound propositions.

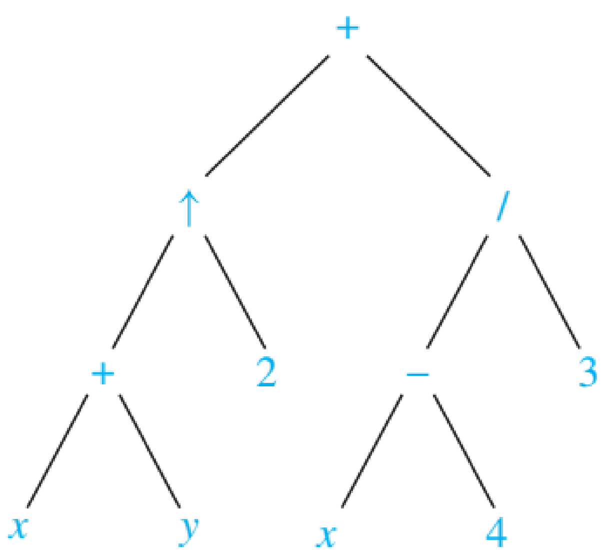
- leaves: numbers or variables,
- internal vertices: operations, where each operation operates on its left and right subtrees in that order (or its only subtree if it is a unary operation).

$$((x + y) \uparrow 2) + ((x - 4)/3)$$



Infix, Prefix, Postfix Notation

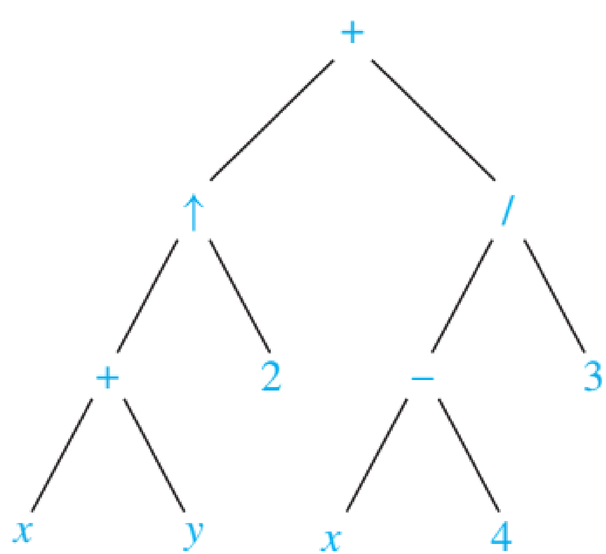
$\Rightarrow$ An inorder traversal of a binary tree representing an expression produces the original expression with the elements and operations in the same order as they originally appear, except for unary operation.
But: inorder traversals give ambiguous expressions $\Rightarrow$ need to include parentheses $\Rightarrow$ **infix form** (fully parenthesized)



$$((x + y) \uparrow 2) + ((x - 4)/3)$$

Infix, Prefix, Postfix Notation

The **prefix form (Polish notation)** of an expression is obtained by traversing its corresponding rooted tree in preorder.
An expression in prefix form (where each operation has a specified number of operands) is unambiguous.

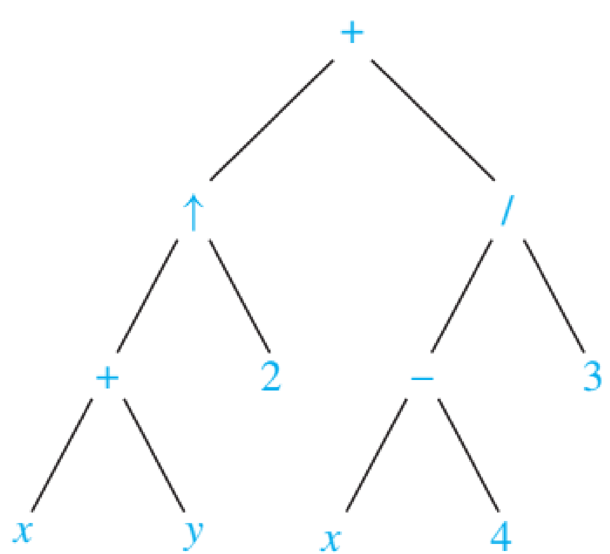


$$+ \uparrow + x \; y \; 2 \; / - x \; 4 \; 3$$

- Evaluate an expression in prefix form by working from right to left.
- When we encounter an operator, we perform the corresponding operation with the two operands immediately to the right of this operand.

Infix, Prefix, Postfix Notation

The **postfix form (reverse Polish notation)** of an expression is obtained by traversing its corresponding rooted tree in postorder. An expression in postfix form (where each operation has a specified number of operands) is unambiguous.



$$x \ y + 2 \uparrow x \ 4 - 3 \ / +$$

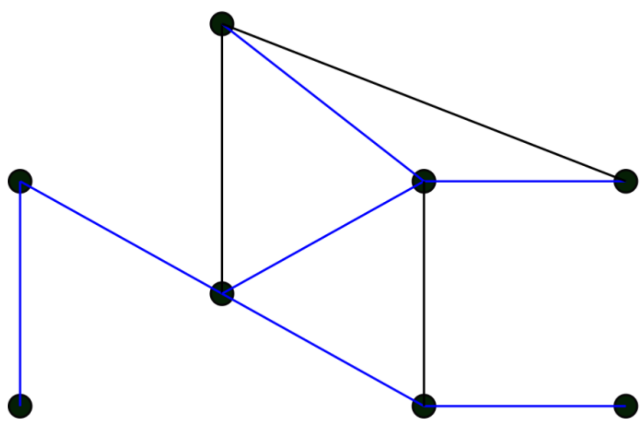
- Work from left to right, carrying out operations whenever an operator follows two operands.
- After an operation is carried out, the result of this operation becomes a new operand.

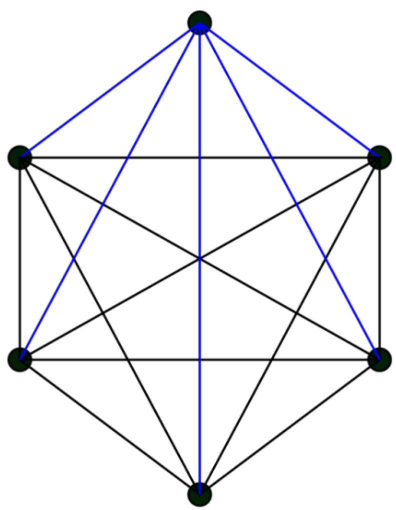
Spanning Trees

Definition

Let G be a simple graph. A **spanning tree** of G is a subgraph of G that is a tree containing every vertex of G.

Example:





Spanning Trees

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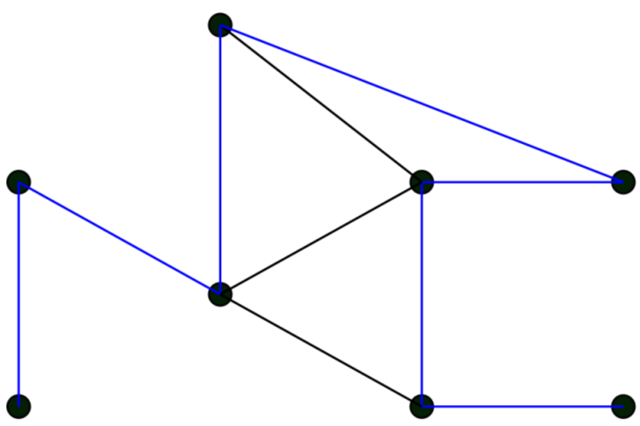
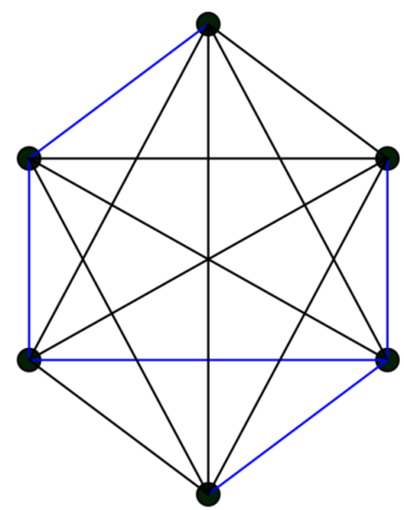




Spanning Trees

Theorem

A simple graph is connected if and only if it has a spanning tree.

Proof:

"$\Leftarrow$" Assume G is a simple graph admitting a spanning tree T:

- T subgraph of G containing all vertices of G,
- by definition of tree, their is a path between any two vertices of T

So their is a path between any two vertices of G.

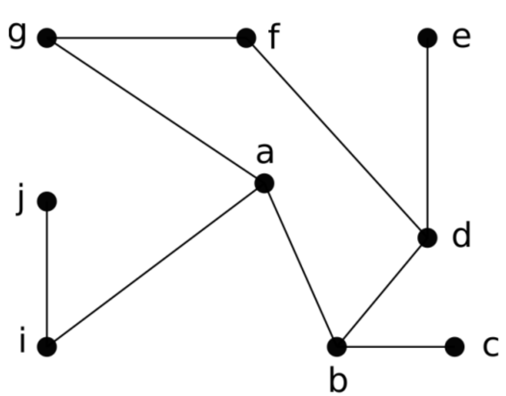
"$\Rightarrow$" Assume G is a simple connected graph.
If it is not a tree, it contains a circuit. Denote G' the subgraph of G obtained by removing one edge of the circuit with endpoints u and v.
There is still a path from u to $v \Rightarrow G'$ is connected.
If G' is not a tree, it contains a circuit, and again take a subgraph removing one edge of the circuit.
Repeat this process until there is no more circuit.
The graph obtained is connected and has no circuit, it is a spanning tree.

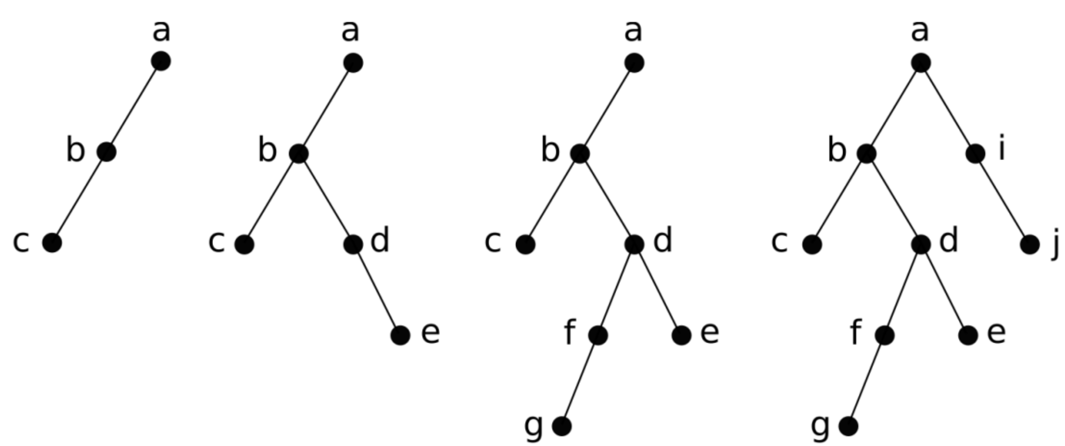
Depth-first Search

Recursive algorithm

DFS(G: connected graph with vertices $v_1, v_2, \ldots, v_n$)
$T :=$ tree consisting only of the vertex v_1
visit(v_1)

visit(v: vertex of G)
for each vertex w adjacent to v and not yet in T
    add vertex w and edge (v, w) to T
    visit(w)





Breadth-first Search

Algorithm

BFS(G: connected graph with vertices $v_1, v_2, \ldots, v_n$)
T:= tree consisting only of vertex v_1
L:= empty list
put v_1 in the list L of unprocessed vertices
while L is not empty
    remove the first vertex v from L
    for each neighbour w of v
        if w is not in L and not in T **then**
            add w to the end of the list L
            add w and the edge (v, w) to T