Numerical Optimization Midterm Exam

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November 5, 2020

1 ($8 = 4 + 4$ points) Suppose you were restoring a signal from a linear observation operator, i.e., $\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{\theta}$, where $\boldsymbol{A} \in \mathbb{R}^{m \times n}$, $\boldsymbol{y} \in \mathbb{R}^m$, and $\boldsymbol{\theta} \in \mathbb{R}^m$ is the random noise.

(1) Construct a linear programming model to find the signal that can best fit the linear model $\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x}$.

(2) Suppose you know that the signal does not have large differences between adjacent elements. Therefore, you also want to add the "total difference of the signal" $\sum_{i=1}^{n-1} |x_{i+1} - x_i|$ to the objective.

Transform this problem to a linear programming model.

2 (8 = 3 + 2 + 3 points) Does set $P = \{\boldsymbol{x} \in \mathbb{R}^2 \mid -1 \leq x_1 \leq 1\}$ has an extreme point? Write down its standard form. Does this standard form has an extreme point? If it does, then write down an extreme point, and explain why it is an extreme point.

3 ($4 = 8 \times 0.5$ points) Fill out the following form. Put a "✓" to indicate that this case could happen. Put a "✗" to indicate this case cannot happen. (See the example in the bottom right cell)

Dual Problem / Primal Problem	Infeasible	Unbounded (from above)	Has optimal solution
Infeasible			
Unbounded (from below)			
Has optimal solutin			

4 $(6 = 2 + 2 + 2$ points) For the standard form

$$\min_{\boldsymbol{x} \in \mathbb{R}^n} \boldsymbol{c}^\top \boldsymbol{x} \quad \text{s.t.} \quad \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}$$

and consider the following 5 conditions.

$$\begin{array}{lcr} Ax = b & & x \geq 0 \\ & x^T s = 0 & \\ A^T y + s = c & & s \geq 0 \end{array}$$

(1) Draw circles on the conditions that the iterates of the primal simplex must satisfy.

(2) Draw squares on the conditions that the iterates of the dual simplex must satisfy.

(3) Draw lines under the conditions that the iterates of the interior point method must satisfy.

5 ($12 = 2 + 5 \times 2$ points) For the standard form of linear programming and its dual problem.

(1) Write down what is <u>weak duality</u> and prove.

(2) Write down what is <u>strongly duality</u> and prove.

6 $(35 = 5 \times 7$ points) Consider the linear optimization problem

$$\begin{aligned} \min \quad & -4x_1 - x_2 - 3x_3 \\ \text{s.t.} \quad & 2x_1 + 2x_2 + x_3 = 4 \\ & x_1 + 2x_2 = 2 \\ & x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0 \end{aligned} \tag{0.8}$$

(1) Write down its first-phase problem, and use it to find a basic feasible solution.

(2) Turn to the second phase and use primal-simplex method to find the optimal solution.

(3) Define the dual variables and write down the Lagrangian function.

(4) Write down the dual problem. Point out which constraints in the dual problem are active and inactive at the optimal dual solution.

(5) What is $\boldsymbol{B}^{-1}$ in your final tableau?

(6) If the right-hand-side 2 is changed to $2 + \delta$, determine an interval for δ, so that the optimal basis are still the same.

(7) If the second constraint is removed from this problem, is the optimal solution you have found still optimal? If not, find the optimal solution in this new case.

7 ($15 = 4+4+3+4$ points) A large manufacturing company produces liquid nitrogen in 5 plants spread out in Jiangsu Province. Each plant has monthly production capacity.

Plant i	1	2	3	4	5
Capacity p_i	120	95	150	120	140

It has 7 retailers in the same area. Each retailer has a monthly demand to be satisfied.

Retailer j	1	2	3	4	5	6	7
Demand d_j	55	72	80	110	85	30	78

Transportation between any plant i and any retailer j has a cost of c_{ij} dollars per volume unit of nitrogen.

(1) Build a model to decide each retailer should be served by which plant. Your objective is to minimize the total transportation cost.

(2) Since the monthly rental fee is rising up for each plant. The company is considering to shut down some of the plant while still keep all the retailers' demands are satisfied. Suppose the rental fee for each plant is now r_i, build a model to minimize the total cost.

(3) Suppose you want to use the Lagrangian relaxation method. Which constraint(s) do you want to relax? Give your reason(s) for your choice.

(4) Show that for any multipliers in your relaxation, the optimal value of your relaxed problem is always a lower bound for the optimal value of the original problem.

8 (12 = 5 + 2 + 5 points) Suppose in the optimality condition for standard form you relax the complementarity condition to have $x_i s_i = \tau > 0$. Suppose your interior point method has the iteration:

$$(\boldsymbol{x}^{k+1}, \boldsymbol{y}^{k+1}, \boldsymbol{s}^{k+1}) \leftarrow (\boldsymbol{x}^k, \boldsymbol{y}^k, \boldsymbol{s}^k) + (\Delta\boldsymbol{x}, \Delta\boldsymbol{y}, \Delta\boldsymbol{s}).$$

(1) For any $(\boldsymbol{x}^k, \boldsymbol{y}^k, \boldsymbol{s}^k)$ with $\boldsymbol{x}^k \geq \mathbf{0}$, $\boldsymbol{s}^k \geq \mathbf{0}$, derive the system of linear equations that is used for computing $(\Delta\boldsymbol{x}, \Delta\boldsymbol{y}, \Delta\boldsymbol{s})$.

(2) For any $(\boldsymbol{x}^k, \boldsymbol{y}^k, \boldsymbol{s}^k)$ with $\boldsymbol{x}^k \geq \mathbf{0}$, $\boldsymbol{s}^k \geq \mathbf{0}$, show the linear system matrix you have in (1) is nonsigular.

(3) Suppose for any symmetric matrix $\boldsymbol{Q} \in \mathbb{R}^{n\times n}$, you have a very eficient subroutine to compute its Jordan Canonical form decomposition: $\boldsymbol{Q} = \boldsymbol{P}\boldsymbol{\Lambda}\boldsymbol{P}^\top$, where $\boldsymbol{P}^\top = \boldsymbol{P}^{-1}$ and $\boldsymbol{\Lambda}$ is a diagonal matrix consisting of the eigenvalues of $\boldsymbol{Q}$. Now use this decomposition subroutine to design an eficient algorithm for solving the linear equations you have in (1).