SI231b: Matrix Computations

Lecture 4: Basic Concepts (Part 3)

Yue Qiu

qiuyue@shanghaitech.edu.cn

School of Information Science and Technology

ShanghaiTech University

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Basic Concepts: Part 3

- sums of subspaces
- dimension of subspaces, rank
- inner product, orthogonality
- matrix products, computational complexity

Sums of Subspaces

If $\mathcal{X}$ and $\mathcal{Y}$ are subspaces of a vector space $\mathcal{V}$, define the sum of two subspaces by

$$\mathcal{X} + \mathcal{Y} = \{\mathbf{x} + \mathbf{y} | \mathbf{x} \in \mathcal{X} \text{ and } \mathbf{y} \in \mathcal{Y}\}$$

then

- the sum $\mathcal{X} + \mathcal{Y}$ is again a subspace of $\mathcal{V}$
- if $\mathcal{S}_X$, $\mathcal{S}_Y$ spans $\mathcal{X}$ and $\mathcal{Y}$, then $\mathcal{S}_X \cup \mathcal{S}_Y$ spans $\mathcal{X} + \mathcal{Y}$

Examples

- If $\mathcal{X} \subset \mathbb{R}^2$ and $\mathcal{Y} \subset \mathbb{R}^2$ are subspaces defined by two different lines through the origin, then $\mathcal{X} + \mathcal{Y} = \mathbb{R}^2$
- If $\mathcal{X}$ is a subspace represents a plane passing through the origin in $\mathbb{R}^3$ and $\mathcal{Y}$ is a subspace defined by the line through the origin that is perpendicular to $\mathcal{X}$, $\mathcal{X} + \mathcal{Y} = \mathbb{R}^3$

Direct Sum of Subspaces

Let $\mathcal{X}$ and $\mathcal{Y}$ be subspaces of a vector space $\mathcal{V}$, then $\mathcal{V}$ is said to be a direct sum of $\mathcal{X}$ and $\mathcal{Y}$, i.e., $\mathcal{V} = \mathcal{X} \oplus \mathcal{Y}$, if

$$\mathcal{V} = \mathcal{X} + \mathcal{Y} \quad \text{and} \quad \mathcal{X} \cap \mathcal{Y} = \{0\}$$

Equivalently,

Every vector $\mathbf{u}$ from the vector space $\mathcal{V}$ can be uniquely represented by

$$\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2$$

with $\mathbf{u}_1 \in \mathcal{X}$ and $\mathbf{u}_2 \in \mathcal{Y}$. Then we use

$$\mathcal{V} = \mathcal{X} \oplus \mathcal{Y}$$

to represent the direct sum of $\mathcal{X}$ and $\mathcal{Y}$.

Example:

$$\text{span}\{\mathbf{e_1},\ \mathbf{e_2}\} \oplus \text{span}\{\mathbf{e_3}\} = \mathbb{R}^{\mathbf{3}}$$

Dimension of Subspaces

The dimension of a nontrivial subspace $\mathcal{S}$ is defined as the number of elements of a basis for $\mathcal{S}$.

- the dimension of of the trivial subspace $\{\mathbf{0}\}$ is defined as 0.
- $\dim \mathcal{S}$ will be used as the notation for denoting the dimension of $\mathcal{S}$
- physical meaning: effective degrees of freedom of the subspace
- examples:
  - $\dim \mathbb{R}^m = m$
  - if k is the number of maximal linearly independent vectors of $\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$, then $\dim \operatorname{span}\{\mathbf{a}_1, \ldots, \mathbf{a}_n\} = k$.

Dimension of Subspaces

Properties:

- let $\mathcal{S}_1, \mathcal{S}_2 \subseteq \mathbb{R}^m$ be subspaces. If $\mathcal{S}_1 \subseteq \mathcal{S}_2$, then $\dim \mathcal{S}_1 \leq \dim \mathcal{S}_2$.
- let $\mathcal{S}_1, \mathcal{S}_2 \subseteq \mathbb{R}^m$ be subspaces. If $\mathcal{S}_1 \subseteq \mathcal{S}_2$ and $\dim \mathcal{S}_1 = \dim \mathcal{S}_2$, then $\mathcal{S}_1 = \mathcal{S}_2$.
- let $\mathcal{S} \subseteq \mathbb{R}^m$ be a subspace. Then

$$\dim \mathcal{S} = m \Longleftrightarrow \mathcal{S} = \mathbb{R}^m.$$

- let $\mathcal{S}_1, \mathcal{S}_2 \subseteq \mathbb{R}^m$ be subspaces. We have $\dim(\mathcal{S}_1 + \mathcal{S}_2) \leq \dim \mathcal{S}_1 + \dim \mathcal{S}_2$.
  - as a more advanced result, we also have

$$\dim(\mathcal{S}_1 + \mathcal{S}_2) = \dim \mathcal{S}_1 + \dim \mathcal{S}_2 - \dim(\mathcal{S}_1 \cap \mathcal{S}_2).$$

  - if $\mathcal{S} = \mathcal{S}_1 \oplus \mathcal{S}_2$, then

$$\dim \mathcal{S} = \dim \mathcal{S}_1 + \dim \mathcal{S}_2$$

Range Spaces

1. The range of a matrix $\mathbf{A} \in \mathbb{R}^{m\times n}$ denoted by $\mathcal{R}(\mathbf{A})$, is defined to be the subspace of $\mathbb{R}^m$ generated by the range of $\mathbf{Ax}$

$$\mathcal{R}(\mathbf{A}) = \{\mathbf{y} \in \mathbb{R}^m \mid \mathbf{y} = \mathbf{Ax},\ \mathbf{x} \in \mathbb{R}^n\} \subset \mathbb{R}^m$$

   - also called column space

2. The range of $\mathbf{A}^T$ is the subspace of $\mathbb{R}^n$ defined by

$$\mathcal{R}(\mathbf{A}^T) = \left\{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x} = \mathbf{A}^T\mathbf{y},\ \mathbf{y} \in \mathbb{R}^m\right\} \subset \mathbb{R}^n$$

   - also called row space

3. $\mathcal{R}(\mathbf{A})$ is the set of all "images" of vectors $\mathbf{x} \in \mathbb{R}^n$ under transformation by $\mathbf{A}$, sometimes $\mathcal{R}(\mathbf{A})$ is called the image space of $\mathbf{A}$.

Null Spaces

1. The null space of a matrix $\mathbf{A} \in \mathbb{R}^{m\times n}$ denoted by $\mathcal{N}(\mathbf{A})$, is defined to be the subspace of $\mathbb{R}^n$ with

$$\mathcal{N}(\mathbf{A}) = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{A}\mathbf{x} = \mathbf{0}\} \subset \mathbb{R}^n$$

   - $\mathcal{N}(\mathbf{A})$ is simply the set of all solutions to the homogeneous system $\mathbf{A}\mathbf{x} = \mathbf{0}$.

2. Similarly, the nullspace of $\mathbf{A}^T$, i.e., $\mathcal{N}(\mathbf{A}^T)$

$$\mathcal{N}(\mathbf{A}^T) = \left\{\mathbf{y} \in \mathbb{R}^m \mid \mathbf{A}^T\mathbf{y} = \mathbf{0}\right\} \subset \mathbb{R}^m$$

   - also called left-hand nullspace of $\mathbf{A}$ since it is the set of all solutions to the left-hand homogeneous system $\mathbf{y}^T\mathbf{A} = \mathbf{0}^T$

Rank

The rank of a matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$, denoted by $\text{rank}(\mathbf{A})$, is defined as the number of elements of a maximal linearly independent subset of $\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$.

- $\text{rank}(\mathbf{A})$ is the maximum number of linearly independent columns of $\mathbf{A}$
- $\dim \mathcal{R}(\mathbf{A}) = \text{rank}(\mathbf{A})$ by definition

Facts:

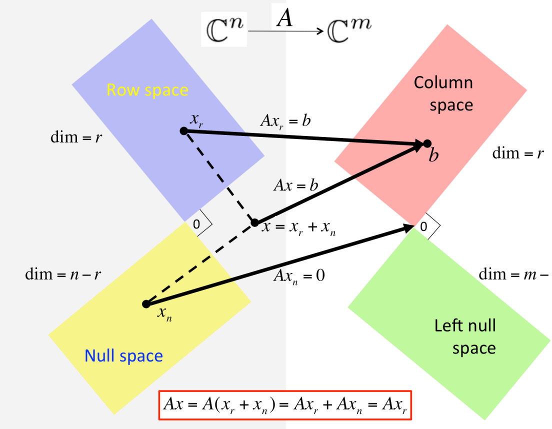
- $\text{rank}(\mathbf{A}) = \text{rank}(\mathbf{A}^T)$, i.e., the rank of $\mathbf{A}$ is also the maximum number of linearly independent rows of $\mathbf{A}$

  Proof?

- $\text{rank}(\mathbf{A} + \mathbf{B}) \le \text{rank}(\mathbf{A}) + \text{rank}(\mathbf{B})$
- $\text{rank}(\mathbf{AB}) \le \min\{\text{rank}(\mathbf{A}), \text{rank}(\mathbf{B})\}$.
  - Equality holds when $\mathbf{A}$ and $\mathbf{B}$ are full rank.

Rank

- $\mathbf{A} \in \mathbb{R}^{m \times n}$ is said to have
  - full column rank if the columns of $\mathbf{A}$ are linearly independent (more precisely, the collection of *all* columns of $\mathbf{A}$ is linearly independent)
    - $\mathbf{A} \in \mathbb{R}^{m \times n}$ being of full column rank $\Longleftrightarrow m \geq n, \text{rank}(\mathbf{A}) = n$
  - full row rank if the rows of $\mathbf{A}$ are linearly independent
    - $\mathbf{A} \in \mathbb{R}^{m \times n}$ being of full row rank $\Longleftrightarrow m \leq n, \text{rank}(\mathbf{A}) = m$
  - full rank if $\text{rank}(\mathbf{A}) = \min\{m, n\}$; i.e., it has either full column rank or full row rank
  - rank deficient if $\text{rank}(\mathbf{A}) < \min\{m, n\}$



- $\mathcal{R}(\mathbf{A}) \perp \mathcal{N}(\mathbf{A}^T)$
- $\mathcal{R}(\mathbf{A}^T) \perp \mathcal{N}(\mathbf{A})$
- Details will follow in the later part

Rank Nulllity Theorem

Theorem

For $\mathbf{A} \in \mathbb{R}^{m \times n}$, we have

$$rank(\mathbf{A}) + \dim \mathcal{N}(\mathbf{A}) = n$$

Can we prove this?

Equivalently, we have

$$\mathcal{R}(\mathbf{A}) \oplus \mathcal{N}(\mathbf{A}^T) = \mathbb{R}^m$$

$$\mathcal{R}(\mathbf{A}^T) \oplus \mathcal{N}(\mathbf{A}) = \mathbb{R}^n$$

Inner Product

The inner product of two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ is defined as

$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^{n} y_i x_i = \mathbf{y}^T \mathbf{x}.$$

- $\mathbf{x}, \mathbf{y}$ are said to be orthogonal to each other if $\langle \mathbf{x}, \mathbf{y} \rangle = 0$
- $\mathbf{x}, \mathbf{y}$ are said to be parallel if $\mathbf{x} = \alpha \mathbf{y}$ for some α

The angle between two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ is defined as

$$\theta = \arccos\left(\frac{\mathbf{y}^T \mathbf{x}}{\|\mathbf{x}\|_2 \|\mathbf{y}\|_2} \right).$$

- $\mathbf{x}, \mathbf{y}$ are orthogonal if $\theta = \pi/2$
- $\mathbf{x}, \mathbf{y}$ are parallel if $\theta = 0$ or $\theta = \pi$

Important Inequalities for Inner Product

Cauchy-Schwartz inequality:

$$|\mathbf{x}^T\mathbf{y}| \leq \|\mathbf{x}\|_2\|\mathbf{y}\|_2.$$

Also, the above equality holds if and only if $\mathbf{x} = \alpha\mathbf{y}$ for some $\alpha \in \mathbb{R}$.

- Proof: suppose $\mathbf{y} \neq \mathbf{0}$; the case of $\mathbf{y} = \mathbf{0}$ is trivial. For any $\alpha \in \mathbb{R}$,

$$0 \leq \|\mathbf{x} - \alpha\mathbf{y}\|_2^2 = (\mathbf{x} - \alpha\mathbf{y})^T(\mathbf{x} - \alpha\mathbf{y}) = \|\mathbf{x}\|_2^2 - 2\alpha\mathbf{x}^T\mathbf{y} + \alpha^2\|\mathbf{y}\|_2^2. \qquad (*)$$

Also, the equality above holds if and only if $\mathbf{x} = \beta\mathbf{y}$ for some β. Let

$$f(\alpha) = \|\mathbf{x}\|_2^2 - 2\alpha\mathbf{x}^T\mathbf{y} + \alpha^2\|\mathbf{y}\|_2^2.$$

The function f is minimized when $\alpha = (\mathbf{x}^T\mathbf{y})/\|\mathbf{y}\|_2^2$. Plugging this α back to $(*)$ leads to the desired result.

Hölder inequality:

$$|\mathbf{x}^T\mathbf{y}| \leq \|\mathbf{x}\|_p \|\mathbf{y}\|_q,$$

for any p, q such that $1/p + 1/q = 1,\ p \geq 1$.

- examples:
  - $(p, q) = (2, 2)$: Cauchy-Schwartz inequality
  - $(p, q) = (1, \infty)$: $|\mathbf{x}^T\mathbf{y}| \leq \|\mathbf{x}\|_1 \|\mathbf{y}\|_\infty$.

    This can be easily verified to be true:

    $$|\mathbf{x}^T\mathbf{y}| \leq \sum_{i=1}^n |x_i y_i| \leq \max_j |y_j| \left(\sum_{i=1}^n |x_i|\right) = \|\mathbf{x}\|_1 \|\mathbf{y}\|_\infty.$$

Invertible Matrices

Facts (for a nonsingular $\mathbf{A}$):

- $\mathbf{A}^{-1}$ always exists and is unique (or there are no two inverses of $\mathbf{A}$)
- $\mathbf{A}^{-1}$ is nonsingular
- $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$
- $(\mathbf{A}^{-1})^{-1} = \mathbf{A}$
- $(\mathbf{A}\mathbf{B})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$, where $\mathbf{A}, \mathbf{B}$ are square and nonsingular
- $(\mathbf{A}^{T})^{-1} = (\mathbf{A}^{-1})^{T}$
  - as a shorthand notation, we will denote $\mathbf{A}^{-T} = (\mathbf{A}^{T})^{-1}$

Let $\mathbf{A} \in \mathbb{R}^{m\times k}$, $\mathbf{B} \in \mathbb{R}^{k\times n}$, and consider

$$\mathbf{C} = \mathbf{AB}.$$

- column representation:

$$\mathbf{c}_i = \mathbf{A}\mathbf{b}_i, \quad i = 1, \dots, n$$

- inner-product representation: redefine $\mathbf{a}_i \in \mathbb{R}^k$ as the ith row of $\mathbf{A}$.

$$\mathbf{AB} = \begin{bmatrix} \mathbf{a}_1^T \\ \vdots \\ \mathbf{a}_m^T \end{bmatrix} \begin{bmatrix} \mathbf{b}_1 & \cdots & \mathbf{b}_n \end{bmatrix} = \begin{bmatrix} \mathbf{a}_1^T\mathbf{b}_1 & \cdots & \mathbf{a}_1^T\mathbf{b}_n \\ \vdots & & \vdots \\ \mathbf{a}_m^T\mathbf{b}_1 & \cdots & \mathbf{b}_m^T\mathbf{b}_n \end{bmatrix}$$

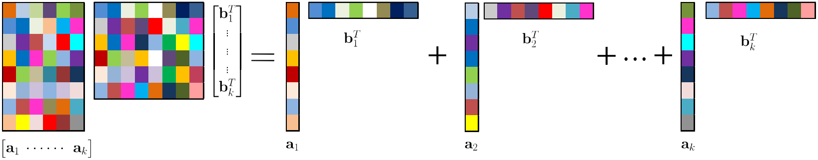
Thus,

$$c_{ij} = \mathbf{a}_i^T\mathbf{b}_j, \quad \text{for any } i, j.$$

Matrix Product Representations

- outer-product representation: redefine $\mathbf{b}_i \in \mathbb{R}^k$ as the ith row of $\mathbf{B}$. Thus,

$$\mathbf{C} = \mathbf{AB} = \sum_{i=1}^{k} \mathbf{a}_i \mathbf{b}_i^T$$



- The matrix of the form $\mathbf{X} = \mathbf{a}\mathbf{b}^T$ for some $\mathbf{a}, \mathbf{b}$ is called a rank-one outer product.
  - It can be verified that $\text{rank}(\mathbf{X}) \leq 1$, and $\text{rank}(\mathbf{X}) = 1$ if $\mathbf{a} \neq \mathbf{0}, \mathbf{b} \neq \mathbf{0}$.
- the outer-product representation

$$\mathbf{C} = \sum_{i=1}^{k} \mathbf{a}_i \mathbf{b}_i^T$$

is a sum of k rank-one outer products.

- does it mean that $\text{rank}(\mathbf{C}) = k$?
  - $\text{rank}(\mathbf{C}) \leq \sum_{i=1}^{k} \text{rank}(\mathbf{a}_i \mathbf{b}_i^T) \leq k$ is true [1]
  - but the above equality is generally not attained; e.g., $k = 2$, $\mathbf{a}_1 = \mathbf{a}_2$, $\mathbf{b}_1 = -\mathbf{b}_2$ leads to $\mathbf{C} = \mathbf{0}$
  - $\text{rank}(\mathbf{C}) = k$ only when $\mathbf{A}$ and $\mathbf{B}$ are full rank (take home exam)

[1] use the rank inequality $\text{rank}(\mathbf{A} + \mathbf{B}) \leq \text{rank}(\mathbf{A}) + \text{rank}(\mathbf{B})$.

Block Matrix Manipulations

Sometimes it may be useful to manipulate matrices in a block form.

- let $\mathbf{A} \in \mathbb{R}^{m \times n}$, $\mathbf{x} \in \mathbb{R}^n$. By partitioning

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_2 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix}$$

where $\mathbf{A}_1 \in \mathbb{R}^{m \times n_1}, \mathbf{A}_2 \in \mathbb{R}^{m \times n_2}, \mathbf{x}_1 \in \mathbb{R}^{n_1}, \mathbf{x}_2 \in \mathbb{R}^{n_2}$, with $n_1 + n_2 = n$, we can write

$$\mathbf{A}\mathbf{x} = \mathbf{A}_1\mathbf{x}_1 + \mathbf{A}_2\mathbf{x}_2$$

- similarly, by partitioning

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix},$$

we can write

$$\mathbf{A}\mathbf{x} = \begin{bmatrix} \mathbf{A}_{11}\mathbf{x}_1 + \mathbf{A}_{12}\mathbf{x}_2 \\ \mathbf{A}_{21}\mathbf{x}_1 + \mathbf{A}_{22}\mathbf{x}_2 \end{bmatrix}$$

Block Matrix Manipulations

- consider $\mathbf{AB}$. By an appropriate partitioning,

$$\mathbf{AB} = \begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_2 \end{bmatrix} \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{B}_2 \end{bmatrix} = \mathbf{A}_1\mathbf{B}_1 + \mathbf{A}_2\mathbf{B}_2$$

- similarly, by an appropriate partitioning,

$$\mathbf{AB} = \begin{bmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \end{bmatrix} \begin{bmatrix} \mathbf{B}_1 & \mathbf{B}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{A}_1\mathbf{B}_1 & \mathbf{A}_1\mathbf{B}_2 \\ \mathbf{A}_2\mathbf{B}_1 & \mathbf{A}_2\mathbf{B}_2 \end{bmatrix}$$

- we showcase two-block partitioning only, but the same manipulations apply to multi-block partitioning like

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \cdots & \mathbf{A}_{1q} \\ \vdots & & \vdots \\ \mathbf{A}_{p1} & \cdots & \mathbf{A}_{pq} \end{bmatrix}$$

Extension to $\mathbb{C}^n$

- all the concepts described above apply to the complex case
- we only need to replace every "$\mathbb{R}$" with "$\mathbb{C}$", and every "T" with "H"; e.g.,
  - $$\mathrm{span}\{\mathbf{a}_1, \ldots, \mathbf{a}_n\} = \{\mathbf{y} \in \mathbb{C}^m \mid \mathbf{y} = \textstyle\sum_{i=1}^n \alpha_i \mathbf{a}_i,\ \boldsymbol{\alpha} \in \mathbb{C}^n\},$$
  - $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{y}^H \mathbf{x}$;
  - $\|\mathbf{x}\|_2 = \sqrt{\mathbf{x}^H \mathbf{x}}$, and so forth.

- the concepts also apply to the matrix case
  - e.g., we may write

$$\mathrm{span}\{\mathbf{A}_1,\ldots,\mathbf{A}_k\} = \{\mathbf{Y}\in\mathbb{R}^{m\times n} \mid \mathbf{Y} = \textstyle\sum_{i=1}^k \alpha_i\mathbf{A}_i,\ \boldsymbol{\alpha}\in\mathbb{R}^k\}.$$

  - sometimes it is more convenient to *vectorize* $\mathbf{X}$ as a vector $\mathbf{x}\in\mathbb{R}^{mn}$, and use the same treatment as in the $\mathbb{R}^n$ case
  - inner product for $\mathbb{R}^{m\times n}$:

$$\langle \mathbf{X},\mathbf{Y}\rangle = \textstyle\sum_{i=1}^m \sum_{j=1}^n x_{ij}y_{ij} = \mathrm{tr}\big(\mathbf{Y}^T\mathbf{X}\big),$$

  - the matrix version of the Euclidean norm is called the Frobenius norm:

$$\|\mathbf{X}\|_F = \sqrt{\textstyle\sum_{i=1}^m \sum_{j=1}^n |x_{ij}|^2} = \sqrt{\mathrm{tr}(\mathbf{X}^T\mathbf{X})}$$

- extension to $\mathbb{C}^{m\times n}$ is just as straightforward as in that to $\mathbb{C}^n$

- every vector/matrix operation such as $\mathbf{x} + \mathbf{y}$, $\mathbf{y}^T\mathbf{x}$, $\mathbf{A}\mathbf{x}$, ... incurs computational costs, and they cost more as the vector and matrix sizes get bigger
- we typically look at floating point arithmetic operations, such as add, subtract, multiply, and divide

- flops: one flop means one floating point arithmetic operation.
- flops count of some standard vector/matrix operations: $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, $\mathbf{A} \in \mathbb{R}^{m \times n}$, $\mathbf{B} \in \mathbb{R}^{n \times p}$,
  - $\mathbf{x} + \mathbf{y}$: n adds, so n flops
  - $\mathbf{y}^T\mathbf{x}$: n multiplies and $n-1$ adds, so $2n-1$ flops
  - $\mathbf{A}\mathbf{x}$: m inner products, so $m(2n-1)$ flops
  - $\mathbf{A}\mathbf{B}$: do "$\mathbf{A}\mathbf{x}$" above p times, so $pm(2n-1)$ flops

- we are often interested in the *order* of the complexity
- big $\mathcal{O}$ notation: given two functions $f(n), g(n)$, the notation

$$f(n) = \mathcal{O}(g(n))$$

  means that there exists a constant $C > 0$ and n_0 such that $|f(n)| \leq C|g(n)|$ for all $n \geq n_0$.
- big $\mathcal{O}$ complexities of standard vector/matrix operations:
  - $\mathbf{x} + \mathbf{y}$: $\mathcal{O}(n)$ flops
  - $\mathbf{y}^T\mathbf{x}$: $\mathcal{O}(n)$ flops
  - $\mathbf{Ax}$: $\mathcal{O}(mn)$ flops
  - $\mathbf{AB}$: $\mathcal{O}(mnp)$ flops

Complexities of Matrix Computations

- Discussion: flop counts do not always translate into the actual efficiency of the execution of an algorithm
- things like pipelining, FPGA, parallel computing (multiple GPUs, multiple servers, cloud computing), etc., can make the story different.
- flop counts also ignore memory usage and other overheads...
- that said, we need at least a crude measure of the computational cost of an algorithm, and counting the flops serves that purpose.

How to Save Computations

- computational complexities depend much on how we design and write an algorithm
- generally, it is about
  - top-down, analysis-guided, designs:
    - seen in class, research papers
    - looks elegant
- facts are
  - usually *not* taught much in class
  - not commonplace in papers
  - subtly depends on your problem at hand
  - a bunch of small differences can make a big difference, say in actual running time
- here we give several, but by no means all, tips for saving computations

- apply matrix operations wisely
- Example: try this on Matlab

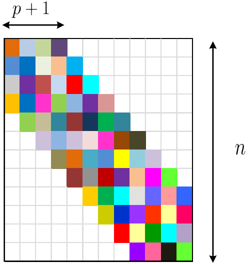
```
>> A=randn(5000,2);
>> B=randn(2,10000);
>> C=randn(10000,10000);
>>
>> tic; D= A*B*C; toc
Elapsed time is 1.334183 seconds.
>> tic; D= (A*B)*C; toc     % ask Matlab to do AB first
Elapsed time is 1.205725 seconds.
>> tic; D= A*(B*C); toc     % ask Matlab to do BC first
Elapsed time is 0.067979 seconds.
```

- let us analyze the complexities in the last example
  - $\mathbf{A} \in \mathbb{R}^{m \times n}, \mathbf{B} \in \mathbb{R}^{n \times p}, \mathbf{C} \in \mathbb{R}^{p \times p}$, with $n \ll \min\{m, p\}$.
  - We want to compute $\mathbf{D} = \mathbf{ABC}$.
  - if we compute $\mathbf{AB}$ first, and then $\mathbf{D} = (\mathbf{AB})\mathbf{C}$, the flop count will be
$$\mathcal{O}(mnp) + \mathcal{O}(mp^2) = \mathcal{O}(m(n+p)p) \approx \mathcal{O}(mp^2)$$
  - if we compute $\mathbf{BC}$ first, and then $\mathbf{D} = \mathbf{A}(\mathbf{BC})$, the flop count will be
$$\mathcal{O}(np^2) + \mathcal{O}(mnp) = \mathcal{O}((m+p)np).$$
  - the 2nd option is preferable if n is much smaller than m, p

- use structures, if available
- example: let $\mathbf{A} \in \mathbb{R}^{n\times n}$ and suppose that

$$a_{ij} = 0 \quad \text{for all } i, j \text{ such that } |i-j| > p,$$

for some integer $p > 0$.



  - such a structured $\mathbf{A}$ is called banded matrix
  - if we don't use structures, computing $\mathbf{Ax}$ requires $\mathcal{O}(n^2)$
  - if we use the banded + sparsity[1] structures, we can compute $\mathbf{Ax}$ with $\mathcal{O}(pn)$
  - different problems may have different fancy/advanced structures to be exploited

[1] a vector or matrix is said to be sparse if it contains many zeros

Readings for lecture 2 and 3

- Carl D. Meyer. *Matrix Analysis and Applied Linear Algebra*, SIAM, 2005.

  Chapter 3.1 – 3.7, 4.1 – 4.5, 5.1 – 5.4

- Gene H. Golub and Charles F. Van Loan. *Matrix Computations*, Johns Hopkins University Press, 2013.

  Chapter 1, 2.1 – 2.3