CS244: Theory of Computation

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Fall 2022

Outline

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  - For a maximization problem, a k-optimal approximation algorithm always finds a solution that is at least $\frac{1}{k}$ times the size of the optimal

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VERTEX-COVER is **NP***-complete.*

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- $|Y|$ is equal to or lager than $|H|$, i.e., $2|Y| \geq 2|H| \geq |X|$
- This algorithm is a 2-optimal approximation algorithm

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- The size of a cut is the number of cut edges
- The MAX-CUT problem asks for a largest cut in a graph G.

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- $\sum_{v \in V} |CutEdge(v)| = 2|CutEdge|$, because every cut edge is counted once for each of its two endpoints
- $|CutEdge| = \frac{\sum_{v \in V} |CutEdge(v)|}{2}$
- $\sum_{v \in V} |CutEdge(v)| \geq \sum_{v \in V} |UNCutEdge(v)|$
- $\sum_{v \in V} |CutEdge(v)| + \sum_{v \in V} |UNCutEdge(v)| = 2|E| \Rightarrow$ $\sum_{v \in V} |CutEdge(v)| \geq |E| \Rightarrow |CutEdge| \geq \frac{|E|}{2}$

Outline

Advanced Topics In Complexity Theory

- Approximation Algorithms
- Probabilistic Algorithms
- Alternation
- Interactive Proof Systems
- Parallel Computation
- Cryptography

Probabilistic Algorithm

Nature Perspective

“There are several reasons why probabilistic programming could prove to be revolutionary for machine intelligence and scientific modelling.”[1]

REVIEW

doi:10.1038/nature14541

Probabilistic machine learning and artificial intelligence

Zoubin Ghahramani[1]

[1]Zoubin Ghahramani leads the Cambridge Machine Learning Group, and holds positions at CMU, UCL, and the Alan Turing Institute.

Probabilistic Algorithm

- A probabilistic algorithm is an algorithm designed to use the outcome of a random process, e.g., by "flip a coin"[2]

[2]Probabilistic Programming at RWTH-Aachen University https://moves.rwth-aachen.de/teaching/ws-1819/probabilistic-programming

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- Sometimes, calculating the best choice may require excessive time, and estimating it may introduce a bias that invalidates the result

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Sorting by flipping coins

Quicksort:

```
QS(A) =
  if |A| <= 1 { return A; }
  i := ceil(|A|/2);
  A< := {a in A | a < A[i]};
  A> := {a in A | a > A[i]};
  return QS(A<) ++ A[i] ++ QS(A>)
```

Worst case complexity:
$O(N^2)$ comparisons

Randomised Quicksort:

```
rQS(A) =
  if |A| <= 1 { return A; }
  i := Unif[1...|A|];
  A< := {a in A | a < A[i]};
  A> := {a in A | a > A[i]};
  return rQS(A<) ++ A[i] ++ rQS(A>)
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Worst case complexity:
$O(N \log N)$ expected comparisons

Monte Carlo: Matrix multiplication

Input: three N^2 square matrices A, B, and C
Output: yes, if $A \cdot B = C$; no, otherwise

- until end 1960s: cubic ($= 3$)
- 1969: 2.808
- 1978: 2.796
- 1979: 2.780
- 1981: 2.522
- 1984: 2.496
- 1989: 2.376
- 2014: 2.373
- 2100: $\cdots$

Monte Carlo: Freivald's matrix multiplication

Input: three $\mathcal{O}(N^2)$ square matrices A, B, and C

Output: yes, if $A \times B = C$; no, otherwise

Deterministic: compute $A \times B$ and compare with C

Complexity: in $\mathcal{O}(N^3)$, best known complexity $\mathcal{O}(N^{2.37})$

Randomised:

1. take a random bit-vector $\vec{x}$ of size N
2. compute $A \times (B\,\vec{x}) - C\,\vec{x}$
3. output yes if this yields the null vector; no otherwise
4. repeat these steps k times

Complexity: in $\mathcal{O}(k{\cdot}N^2)$, with false positive with probability $\leqslant 2^{-k}$

Probabilistic Turing Machine

Definition

A probabilistic Turing machine M is a type of nondeterministic Turing machine in which each nondeterministic step is called a coin-flip step and has two legal next moves. We assign a probability to each branch b of M's computation on input w as follows. Define the probability of branch b to be

$$Pr[b] = 2^{-k}$$

where k is the number of coin-flip steps that occur on branch b.

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$$Pr[M \text{ rejects } w] = 1 - Pr[M \text{ accepts } w]$$

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For Probabilistic TM M of a language A and a small probability of error $0 \leq \epsilon < \frac{1}{2}$, for every string w,

- if $w \in A$, then $Pr[M \text{ accepts } w] \geq 1 - \epsilon$ i.e., $Pr[M \text{ rejects } w] \leq \epsilon$
- if $w \notin A$, then $Pr[M \text{ rejects } w] \geq 1 - \epsilon$, i.e., $Pr[M \text{ accepts } w] \leq \epsilon$

Note: $Pr[M \text{ rejects } w] = 1 - Pr[M \text{ accepts } w]$

Consider $\epsilon = \frac{1}{4}$, where $Pr[M \text{ accepts } w] = \frac{1}{2}$, then

neither $w \in A$ nor $w \notin A$

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We may consider error probability bounds that depend on the input length n, e.g., $\epsilon = \frac{1}{2^n}$

Bounded-Error Probabilistic Polynomial-Time (BPP)

Time and space complexity of a probabilistic Turing machine in the same way we do for a nondeterministic Turing machine: by using the worst case computation branch on each input.

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Bounded-Error **P**robabilistic **P**olynomial-Time (**BPP**) is the class of languages that are decided by probabilistic polynomial time Turing machines with an error probability of $\frac{1}{3}$.

Note: $\frac{1}{3}$ can be any bounded error $0 \leq \epsilon < \frac{1}{2}$.

Bounded-Error Probabilistic Polynomial-Time (BPP)

Lemma

Let the bounded error $0 \leq \epsilon < \frac{1}{2}$. Then for any polynomial $p(n)$, a probabilistic polynomial time Turing machine M_1 that operates with error probability ϵ has an equivalent probabilistic polynomial time Turing machine M_2 that operates with an error probability of $\frac{1}{2^{p(n)}}$.

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1. M_2 simulates M_1 by running it a polynomial number of times and taking the majority vote of the outcomes
2. The probability of error decreases exponentially with the number of runs of M_1 made

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M_2 On input x

1. Calculate k (see analysis below)

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3. If most runs of M_1 accept, then accept; otherwise, reject

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- The probability that M_2 obtains S: $P_s \leq (1 - \epsilon_x)^c (\epsilon_x)^w$
- $0 \leq \epsilon_x \leq \epsilon < \frac{1}{2} \Rightarrow (\epsilon - \epsilon_x) \geq (\epsilon - \epsilon_x)(\epsilon + \epsilon_x)$

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Agrawal, Kayal, Saxena

undergraduate students at the time!

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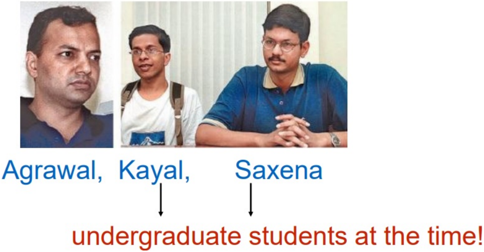

We describe a much simpler probabilistic polynomial time algorithm for primality testing, in $O(n^2)$, with tiny error probability

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- Therefore, the number of witnesses must be at least as large as the number of nonwitnesses in $\mathbb{Z}_p^+$

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- then $b' - 1, b' + 1 \in \mathbb{Z}_p^+$, but cp cannot be expressed by a product of two numbers that are smaller than it is, a contradiction

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Corollary

If p is an odd number and there exists a nontrivial square root (i.e., not ± 1) of 1, modulo p, then p is composite number.

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10. Accept if all the tests have been passed

- Line 8: Miller-Rabin test, $x_{j-1}^2 = x_j = 1$ and $x_{j-1} \notin \{1, -1\}$, then find a nontrivial square root of 1, modulo p, then p is composite
- Line 9: Fermat test, i.e., $a_i^{p-1} \not\equiv_p 1$, implies that p is not prime

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If PRIME rejects p, then p is not prime,

i.e., if p is prime, $Pr[\textit{PRIME accepts } p] = 1$

PRIME on input p:

1. If p is even: accept if $p = 2$, otherwise reject;
2. Select $a_1, \cdots, a_k$ randomly in $\mathbb{Z}_p^+$
3. For each $i = 1$ to k:
4. Let $p - 1 = s \times 2^\ell$ such that s is odd [Note $p - 1$ is even]
5. Let $x_0 = a_i^s \pmod{p}$
6. For each $j = 1$ to ℓ:
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- hence $t^{s\times 2^j} \not\equiv_p \pm 1$,
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    - if $t^{s\times 2^{j+1}} \not\equiv_p 1$, contradicts that j is the largest position having -1

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If p is an odd composite number, the number of nonwitnesses of p in $\mathbb{Z}_p^+$ is at most $\frac{p-1}{2}$.

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PRIME on input p:

1. If p is even: accept if $p = 2$, otherwise reject;
2. Select $a_1, \cdots, a_k$ randomly in $\mathbb{Z}_p^+$
3. For each $i = 1$ to k:
4. Let $p - 1 = s \times 2^\ell$ such that s is odd [Note $p - 1$ is even]
5. Let $x_0 = a_i^s \pmod{p}$
6. For each $j = 1$ to ℓ:
7. $x_j = x_{j-1}^2 \pmod{p}$ $[x_j = a_i^{s \times 2^j} \pmod{p}]$
8. If $x_j == 1 \wedge x_{j-1} \notin \{1, -1\}$, then reject
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Theorem

If p is prime, $Pr[PRIME\ accepts\ p] = 1$.
If p is not prime, $Pr[PRIME\ accepts\ p] \leq 2^{-k}$,
i.e., $Pr[PRIME\ rejects\ p] \geq 1 - 2^{-k}$.

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RP is the class of languages that are decided by probabilistic polynomial time TM where inputs in the language are accepted with a probability of at least $\frac{1}{2}$, and inputs not in the language are rejected with a probability of 1.

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Theorem

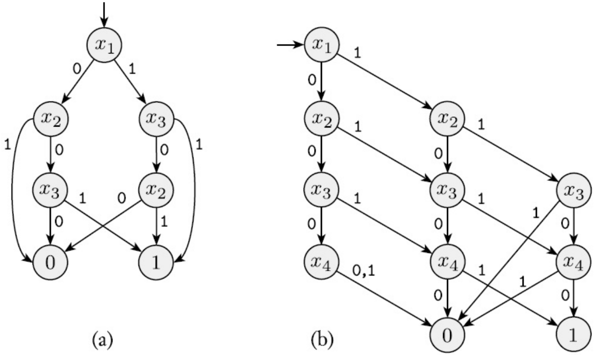
COMPOSITES $\in$ RP *and PRIME* $\in$ *co*RP

Branching Program

Definition

A branching program is a directed acyclic graph where all nodes are labeled by variables, except for two output nodes labeled 0 or 1.
The nodes that are labeled by variables are called query nodes. Every query node has two outgoing edges: one labeled 0 and the other labeled 1. Both output nodes have no outgoing edges. One of the nodes in a branching program is designated the start node.

$$f : \{0,1\}^V \to \{0,1\}$$



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- *co***NP**-hardness. $\overline{SAT} \leq_{\mathrm{P}} EQ_{BP}$. ϕ to P_1 and P_2 denotes the Boolean function that is always 0

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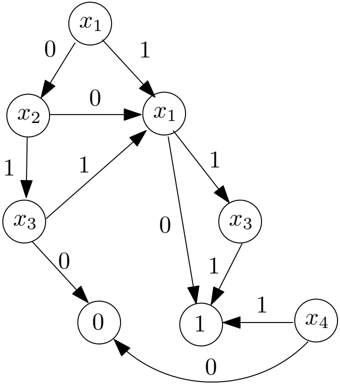
$$(x_1 \vee \neg x_2 \vee x_3) \wedge (\neg x_1 \vee x_3 \vee x_4)$$



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2. This algorithm introduces the technique of assigning non-Boolean values to normally Boolean variables in order to analyze the behavior of some Boolean function of those variables

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Randomly selecting a non-Boolean assignment to the variables, and evaluate P_1 and P_2 in a suitably defined manner. If P_1 and P_2 are not equivalent, the random evaluations will likely be unequal.

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2. Assign 1 to the start nodes.

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5. Assign $\sum_{(v',v) \in E_0} (1 - a_j) p(v') + \sum_{(v',v) \in E_1} a_j p(v')$ to v, where a_j denotes the variable of v'
6. Accept if the assigned values of 1-labeled output nodes in P_1 and P_2 are identical; otherwise reject.

Read-once Branching Program

Theorem

EQ_{ROBP} is in **BPP**.

D on input $\langle P_1, P_2 \rangle$, two read-once BP over variables $x_1, \cdots, x_m$:

1. Select elements a_1 through a_m at random from a finite field $\mathcal{F}$ with at least $3m$ elements.
2. Assign 1 to the start nodes.
3. For BP P_i:
4. For each node v of P_i in topological order:
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- For each y_j^i that is 1, it can be replaced by $x_j + (1 - x_j)$, then the product term $y_1^i y_2^i \cdots y_m^i$ is split into two product terms: $(y_1^i y_2^i \cdots y_m^i)[y_j^i \mapsto x_j]$ and $(y_1^i y_2^i \cdots y_m^i)[y_j^i \mapsto (1 - x_j)]$
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Lemma

Let $\mathcal{F}$ be a finite field with $f > 0$ elements, and p be a non-zero polynomial on variables $x_1, \cdots, x_m$, where each variable has degree at most d. If $a_1, \cdots, a_m$ are selected randomly from $\mathcal{F}$, then

$$Pr[p(a_1, \cdots, a_m) = 0] \leq \frac{md}{f}$$

By induction on m:

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  - Then, the probability that $a_1, \cdots, a_m$ is a root of p is at most $\frac{(m-1)d}{f} + \frac{d}{f} = \frac{md}{f}$

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Since $f = 3m$ and $d = 1$, we have:

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Outline

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  - An alternating computation has two designated cases: accepts if any of the children branching accepts, or accepts if all of the children branchings, seen as AND/OR-tree accept.
- Using alternation, we may simplify various proofs in time/space complexity theory and exhibit a surprising connection between the time and space complexity measures

Alternating TM

Definition

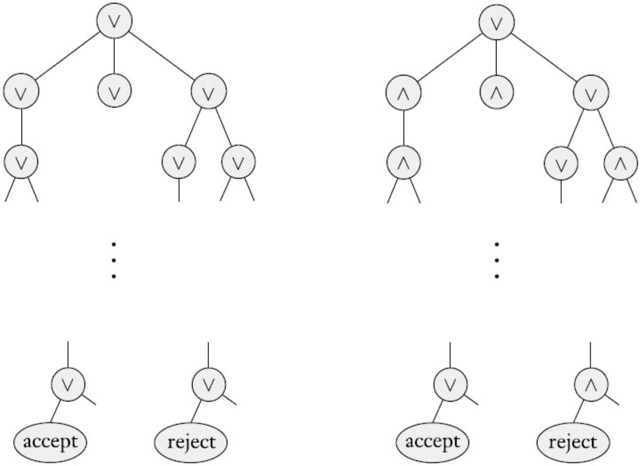
A alternating TM is a 7-tuple, $(Q, \Sigma, \Gamma, \delta, q_0, q_{\text{accept}}, q_{\text{rejcet}})$, where

1. Q is finite set of states such that $Q \setminus \{q_{\text{accept}}, q_{\text{rejcet}}\} = Q_\exists \uplus Q_\forall$
2. Σ is a finite nonempty input alphabet not containing the blank symbol $\sqcup$,
3. Γ is finite nonempty tape alphabet, where $\sqcup \in \Gamma$ and $\Sigma \subseteq \Gamma$,
4. δ is a transition function

$$\delta : Q \times \Gamma \to \mathcal{P}(Q \times \{L, R\}).$$

5. $q_0 \in Q$ is the start state,
6. $q_{\text{accept}} \in Q$ is the accept state, and
7. $q_{\text{reject}} \in Q$ is the reject state, where $q_{\text{reject}} \neq q_{\text{accept}}$.

Alternating TM



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$$\mathbf{AP} = \bigcup_k \mathbf{ATIME}(n^k)$$

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3. Evaluate ϕ and ψ on this assignment

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Two Boolean formulas are equivalent if they have the same set of variables and are true on the same set of assignments to those variables (i.e., they describe the same Boolean function). A Boolean formula is minimal if no shorter Boolean formula is equivalent to it.

$$\text{MIN-FORMULA} = \{\langle \phi \rangle \mid \phi \text{ is a minimal formula}\}$$

Theorem

MIN-FORMULA is in **AP**.

M on input $\langle \phi \rangle$:

1. Universally select all formulas ψ that are shorter than of ϕ
2. Existentially select an assignment to the variables of ϕ
3. Evaluate ϕ and ψ on this assignment
4. If ϕ and ψ evaluate to the same value, reject; otherwise accept

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It is not known whether MIN-FORMULA is in **NP** or in co**NP**

Connection between the time and space complexity

Theorem

$$\forall f(n) \geq n, \textbf{ATIME}(f(n)) \subseteq \text{SPACE}(f(n)) \subseteq \textbf{ATIME}(f^2(n))$$

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Corollary

AL=**P**, **AP**=PSPACE *and* **APSPACE**=EXPTIME

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Open problems: $\mathbf{NP} = \mathbf{AP}$? and $\mathbf{P} = \mathbf{AP}$?

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M_d on input w:

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3. The depth of M_a's computation tree is at most $O(f(n))$, and during each recursion M_d only need to store the nondeterministic choice instead of the configuration
4. The total space is $O(f(n))$, but the total time is $O(2^{f(n)})$

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CANYIELD on input c_1, c_2, and t:

1. If $t = 1$, then test whether $c_1 = c_2$ or whether c_1 yields c_2 in one step according to the rules of M_d. Accept if either test succeeds; reject if both fail.

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3. Existentially to guess a configuration c_m of M_d [$O(f(n))$ time]

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M_a on input w:

1. CANYIELD$(c_{start}, c_{accept}, 2^{df(n)})$, where d is selected such that M has no more than $2^{df(n)}$ configurations within its space bound

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M_a uses at most $O(f(n)) \times \log 2^{df(n)} = O(f^2(n))$ time, space is $O(f(n))$

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3. Repeatedly scans all the nodes in the graph until no update:
    - Mask a node if it is universal configuration and all its successors are masked

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- The number of nodes is at most $2^{O(f(n))}$, hence the size of the graph
- Thus, the graph can be constructed in $2^{O(f(n))}$ time

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- Thus, the graph can be constructed in $2^{O(f(n))}$ time
- It scans at most $2^{O(f(n))}$ times and each scan is in $2^{O(f(n))}$ time
- Total: $2^{O(f(n))}$ time, and also space

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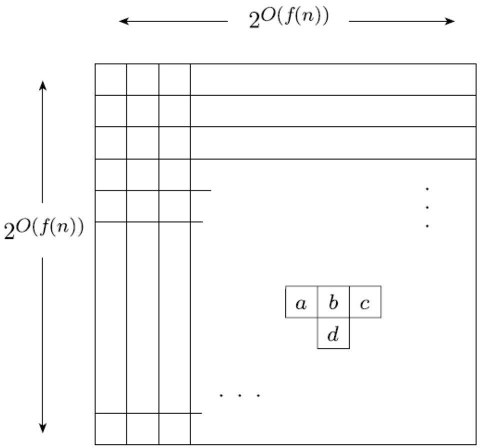

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Let M_d be a $2^{k \cdot f(n)}$ time deterministic TM, we construct a $O(f(n))$ space alternating TM M_a that simulates M_d as follows:
Assume that M_d moves its head to the left-hand end of the tape and write $\sqcup$ on the tape on acceptance

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CellCheck on i, j, s:

1. If $i = 0$, then accept if $s = w_j$, otherwise reject.

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CellCheck on i, j, s:

1. If $i = 0$, then accept if $s = w_j$, otherwise reject.
2. Otherwise, existentially guesses three symbols s_1, s_2, s_2

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CellCheck on i, j, s:

1. If $i = 0$, then accept if $s = w_j$, otherwise reject.
2. Otherwise, existentially guesses three symbols s_1, s_2, s_2
3. Verify whether $cell[i-1, j-1, s_1], cell[i-1, j, s_2], cell[i-1, j+1, s_3]$ can yield $cell[i, j, s]$ in M_d's transition function

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4. If yes, then universally run $CellCheck(i-1, j-1, s_1)$, $CellCheck(i-1, j, s_2)$ and $CellCheck(i-1, j+1, s_3)$

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Assume that M_d moves its head to the left-hand end of the tape and write $\sqcup$ on the tape on acceptance

CellCheck on i, j, s:

1. If $i = 0$, then accept if $s = w_j$, otherwise reject.
2. Otherwise, existentially guesses three symbols s_1, s_2, s_2
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M_d on input w:

1. Universally $CellCheck(2^{k \cdot f(n)}, 0, q_{accept})$, $CellCheck(2^{k \cdot f(n)}, 1, \sqcup)$, $\cdots$ $CellCheck(2^{k \cdot f(n)}, 2^{k \cdot f(n)}, \sqcup)$

Connection between the time and space complexity

$$\forall f(n) \geq \log n, \mathbf{ASPACE}(f(n)) \supseteq \text{TIME}(2^{O(f(n))})$$

Let M_d be a $2^{k \cdot f(n)}$ time deterministic TM, we construct a $O(f(n))$ space alternating TM M_a that simulates M_d as follows:
Assume that M_d moves its head to the left-hand end of the tape and write ␣ on the tape on acceptance

CellCheck on i, j, s:

1. If $i = 0$, then accept if $s = w_j$, otherwise reject.
2. Otherwise, existentially guesses three symbols s_1, s_2, s_2
3. Verify whether $cell[i-1, j-1, s_1], cell[i-1, j, s_2], cell[i-1, j+1, s_3]$ can yield $cell[i, j, s]$ in M_d's transition function
4. If yes, then universally run $CellCheck(i-1, j-1, s_1)$, $CellCheck(i-1, j, s_2)$ and $CellCheck(i-1, j+1, s_3)$

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We only need to store i, j, s which only need space $O(f(n))$ using binary representation, time is still $O(2^{f(n)})$

Understanding Proofs

Target	Proof	Other side
$\mathbf{ATIME}(f(n)) \subseteq \mathrm{SPACE}(f(n))$	post-order tree travseral tree depth$=O(f(n))$	$\mathrm{TIME}(2^{f(n)})$
$\mathrm{SPACE}(f(n)) \subseteq \mathbf{ATIME}(f^2(n))$	CANYIELD	$\mathbf{ASPACE}(f(n))$
$\mathbf{ASPACE}(f(n)) \subseteq \mathrm{TIME}(2^{O(f(n))})$	graph travseral graph size$=O(2^{f(n)})$	$\mathrm{SPACE}(2^{f(n)})$
$\mathrm{TIME}(2^{O(f(n))}) \subseteq \mathbf{ASPACE}(f(n))$	*CellCheck*	$\mathbf{ATIME}(2^{f(n)})$

Polynomial Hierarchy

Definition

For every $i \geq 1$ a language L is Σ_i^p if there is a polynomial time TM M and a polynomial time computable function q such that

$$\begin{aligned} x \in L \quad \iff \quad & \exists u_1 \in \{0,1\}^{q(|x|)} \forall u_2 \in \{0,1\}^{q(|x|)} \cdots \\ & Q_i u_i \in \{0,1\}^{q(|x|)} M(x, u_1, \cdots, u_i) = accept \end{aligned}$$

where $Q_i = \forall$ if i is even, otherwise $Q_i = \exists$.

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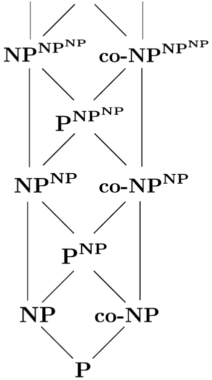

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Similar to the case $i = 1$, do it by yourself.

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where $Q_i \in \{\forall, \exists\}$ and ϕ is an unquantified boolean formula

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$$Q_1 x_1 Q_2 x_2 \cdots Q_n x_n \phi(x_1, x_2, \cdots, x_n)$$

where $Q_i \in \{\forall, \exists\}$ and ϕ is an unquantified boolean formula

$$TQBF = \left\{ \langle \varphi \rangle \;\middle|\; \varphi \text{ is a true fully quantifier Boolean formula} \right\}.$$

Theorem

TQBF is PSPACE-*complete.*

Examples continued

- Define

$$\Sigma_i\mathrm{SAT} = \{\exists x_1 \forall x_2 \cdots Q_i x_i \phi(x_1, x_2, \cdots, x_i) = \mathit{true}\}$$

where $Q_i = \forall$ if i is even, otherwise $\exists$, each x_i is a vector of Boolean variables

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Theorem

$\Sigma_i\mathrm{SAT}$ *is* Σ_i^p*-complete.*

Examples continued

Theorem (Meyer and Stockmeyer, 1972)

Σ_iSAT is Σ_i^p-complete.

- Σ_iSAT is in Σ_i^p. Consider $\exists x_1 \forall x_2 \cdots Q_i x_i \phi(x_1, x_2, \cdots, x_i)$.

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$$\begin{aligned} \phi \in \Sigma_i\text{SAT} \quad \Longleftrightarrow \quad & \exists x_1' \in \{0,1\}^{q(|\phi|)} \forall x_2' \in \{0,1\}^{q(|\phi|)} \dots \\ & Q_i X_i' \in \{0,1\}^{q(|\phi|)} M(\phi(x_1', x_2', \cdots, x_i')) \end{aligned}$$

where $q(|\phi|) = \max_{i=1}^{i} |x_i|$

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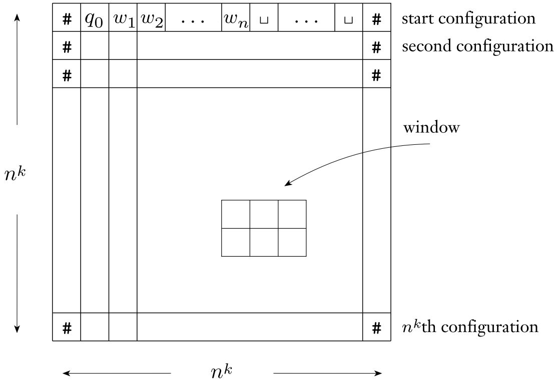
- Σ_iSAT is Σ_i^p-hard.

Recall: SAT is **NP**-hard

Let N be an NTM that decides a language A in time n^k for some $k \in \mathbb{N}$.

A tableau for N on w is an $n^k \times n^k$ table whose rows are the configurations of the branch of the computation of N on input w.



$$\varphi_{\text{cell}} \wedge \varphi_{\text{start}} \wedge \varphi_{\text{move}} \wedge \varphi_{\text{accept}}.$$

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Outline

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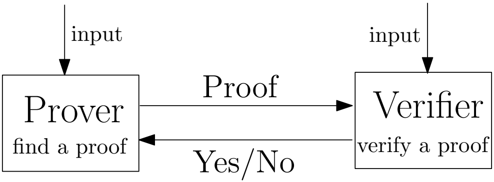
SAT problem



Interactive Proof Systems

- Verifier is a function $V : \Sigma^* \times \Sigma^* \times \Sigma^* \to \Sigma^* \cup \{accept, reject\}$
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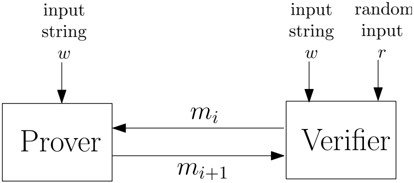
Interactive Proof Systems



$\forall i \geq 1,\ i$ is odd

$P(w, m_1 \sharp \cdots \sharp m_i) = m_{i+1}$

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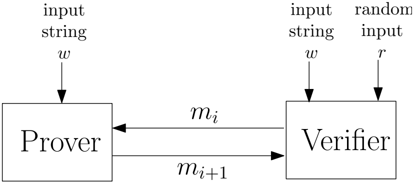

$\forall i \geq 1,\ i$ is odd $\quad P(w, m_1 \sharp \cdots \sharp m_i) = m_{i+1}$

$\forall i \geq 0,\ i$ is even $\quad V(w, r, m_1 \sharp \cdots \sharp m_i) = m_{i+1}$

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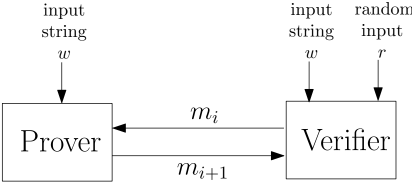

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$$\Pr[V \leftrightarrow P \text{ accepts } w] = \Pr[(V \leftrightarrow P)(w, r) = \text{accepts}]$$

Interactive Polynomial Time

Definition

A language A is in interactive polynomial Time (IP) if some polynomial time computable function V exists such that for some (arbitrary) function P and for every (arbitrary) function $\widetilde{P}$ and for every string w with length n

- $w \in A$ implies that $\Pr[V \leftrightarrow P \text{ accepts } w] \geq \frac{2}{3}$
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Theorem

NP $\subseteq$ IP *and* **BPP** $\subseteq$ IP.

Graph isomorphism

Definition

Two graphs are isomorphic if they are same up-to node renaming

$$\mathrm{ISO} = \{\langle G, H\rangle \mid G \text{ and } H \text{ are isomorphic graphs}\}$$

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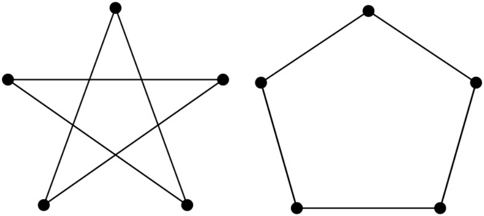

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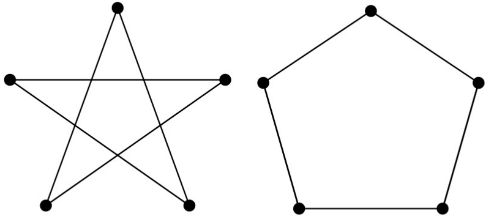

Theorem

$ISO \in \mathbf{NP}$.

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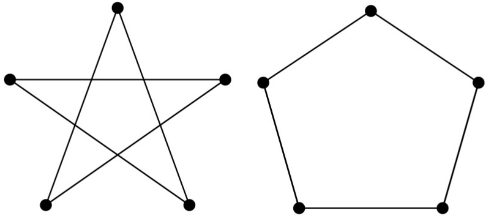

Theorem

ISO $\in$ **NP**.

Open problem: ISO is **NP**-complete or ISO $\in$ **P**?

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- If $\langle G_1, G_2\rangle \in \text{ISO}$, then no Prover even with unlimited computational power can identify whether H came from G_1 or G_2, it has 50-50 chance

Graph non-isomorphism

$$\text{NONISO} = \{\langle G, H \rangle \mid G \text{ and } H \text{ are not isomorphic graphs}\}$$

Theorem

$NONISO \in co\mathbf{NP} \cap \mathrm{IP}$

1. Verifier randomly selects either G_1 or G_2 and then randomly reorders its nodes to obtain a graph H
2. The Verifier sends H to the Prover
3. The Prover must respond by declaring whether G_1 or G_2 was the source of H
4. If Prover's response is correct, then accept; otherwise reject

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- The Verifier can repeat the above protocol in order to get the desired error probability

Theorem
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- For any language in PSPACE, a Prover can convince a probabilistic polynomial time Verifier about the membership of a string in the language, even though a conventional proof of membership might be exponentially long
- Proof: read the textbook.

Outline

Parallel Computer

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  - explore the possibility that parallelism may not be suitable for certain other problem

Boolean Circuits

Definition

A Boolean circuit is a collection of gates and inputs connected by wires. Cycles aren't permitted. Gates take three forms: AND gates, OR gates, and NOT gates

- The size of a circuit C is the number of gates that it contains
- The size complexity of a circuit family $(C_0, C_1, C_2, \cdots)$ is the function $f : \mathbb{N} \to \mathbb{N}$, where $f(n)$ is the size of C_n
- The depth of a circuit is the length (number of wires) of the longest path from an input variable to the output gate. Depth minimal circuits and circuit families, and the depth complexity of circuit families are similar

Definition

The circuit complexity of a language is the size complexity of a minimal circuit family for that language.
The circuit depth complexity of a language is defined similarly, using depth instead of size.

Uniform Boolean Circuits

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A family of circuits $(C_0, C_1, C_2, \dots)$ is uniform if some log space transducer T outputs $\langle C_n \rangle$ when T's input is 1^n.

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- A uniform family of circuits is a model of a parallel computer, where each gate to be an individual processor
- A language has simultaneous size-depth circuit complexity at most $(f(n), g(n))$ if a uniform circuit family exists for that language with size complexity $f(n)$ and depth complexity $g(n)$

Example

The m-input parity function $\texttt{parity}_m : \{0,1\}^m \to \{0,1\}$ outputs 1 if an odd number of 1's appear in the input variables.

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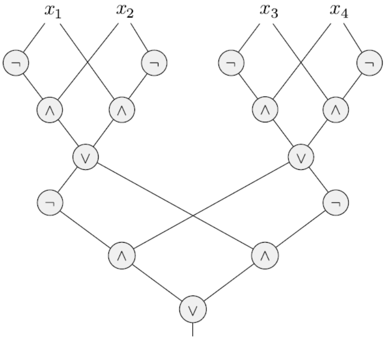

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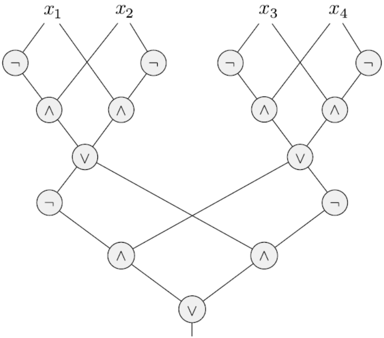

The simultaneous size-depth circuit complexity: $(O(n), O(\log n))$, as each $\oplus$ costs 5 gates

Boolean Matrix Multiplication

- The input of Boolean matrix multiplication has $2m^2 = n$ variables representing two $m \times m$ matrices $A = \{a_{ik}\}_{1 \leq i,k \leq m}$ and $B = \{b_{ik}\}_{1 \leq i,k \leq m}$

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The simultaneous size-depth circuit complexity:
$(O(m^3), O(\log m)) = (O(n^{1.5}), O(\log n))$

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The NC^i computable or NC computable problems may be considered to be highly parallelizable with a moderate number of processors

Boolean Matrix Multiplication$\in \mathrm{NC}^1$

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$\mathsf{NC}^1 \subseteq \mathbf{L}$, *i.e., problems that are solvable in logarithmic depth are also solvable in logarithmic space.*

Let A be a language in NC^1.

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The depth of C_n is $O(\log n)$, therefore recursion depth is $O(\log n)$, hence space of M

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- C_n has polynomial size and $O(\log^2 n)$ depth, see Theorem 8.25 (*PATH* is **NL**-complete) and Theorem 9.30 (TIME$(t(n)) \Rightarrow$ circuit complexity $O(t^2(n))$).

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A polynomial time algorithm can run the log space transducer to generate circuit C_n and simulate it on an input of length n

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$$\text{CIRCUIT-VALUE} = \{\langle C, x\rangle \mid C \text{ is a Boolean circuit and } C(x) = 1\}$$

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  - The reduction can be carried out in log space because the circuit it produces has a simple and repetitive structure.

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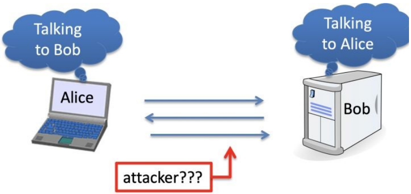
  - NP-completeness concerns worst-case complexity, but, we need to measure average-case complexity rather than worst-case complexity, e.g., integer factorization

The One Time Pad & Perfect Security



- Three spaces (K, M, C) and two algorithms (E, D), $E : K \times M \to C$ is enc. alg., $D : K \times C \to M$ is dec. alg.
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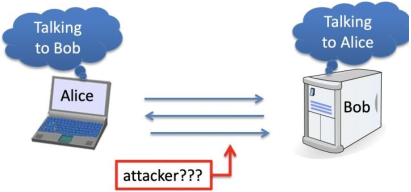

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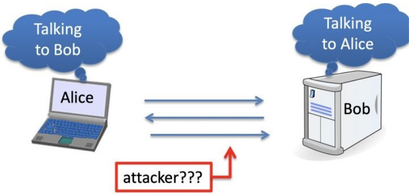
The One Time Pad & Perfect Security



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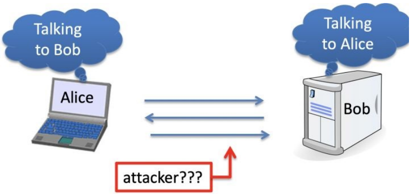

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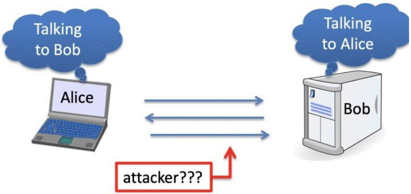

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- 2nd place Bell Labs Prize, InstaHide: Instance-hiding Schemes for Private Distributed Learning (Yangsibo Huang, Zhao Song, Kai Li, Sanjeev Arora, ICML 2020)

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Note that M may sometimes fail to accept on input w:

$$\sum_{x \in \Sigma^*} Pr[M(w) = x] \leq 1$$

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For one-way permutations, any probabilistic polynomial time algorithm has only a small probability of inverting f; that is, it is unlikely to compute w from $f(w)$

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For one-way functions, any probabilistic polynomial time algorithm is unlikely to be able to find any y that maps to $f(w)$.

One-way function: example

The multiplication function mult is a candidate for a one-way function.

Definition

- Let $\Sigma = \{0, 1\}$, for any $w \in \Sigma^*$, let $\mathrm{mult}(w)$ be the string representing the product of the first and second halves of w, i.e.,

$$\mathrm{mult}(w) = u \cdot v$$

  where $w = uv$ such that $|u| = |v|$ if $|w|$ is even, and $|u| = |v| + 1$ if $|w|$ odd.

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Despite a great deal of research into the integer factorization problem, no probabilistic polynomial time algorithm is known that can invert mult , even on a polynomial fraction of inputs

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We don't know whether the existence of a one-way function alone is enough to allow the construction of a public-key cryptosystem

Trapdoor function

- A family of functions $\{f_i\}_{i \in \Sigma^*}$ can be represented by the single function $f : \Sigma^* \times \Sigma^* \to \Sigma^*$ such that for every $i \in \Sigma^*$ and $w \in \Sigma^*$: $f(i, w) = f_i(w)$

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- f is length-preserving if for each $i \in \Sigma^*$, the functions f_i is length preserving.

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- The trapdoor function f: $f_{N,e}(w) = w^e \pmod{N}$
- The inverting function h: $h(d, x) = x^d \pmod{N}$

$$
\begin{aligned}
h(d, f_{N,e}(w)) &= (w^e \pmod{N})^d \pmod{N} \\
&= w^{de} \pmod{N} = w
\end{aligned}
$$