EE150: Signals and Systems, Spring 2022
Homework 2

(Due Thursday, Mar. 17 at 11:59pm (CST))

1. [12 points] Determine the continuous-time convolution of x(t) and h(t) for the following three cases:

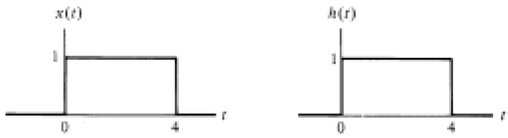


Figure 1.1: (a)

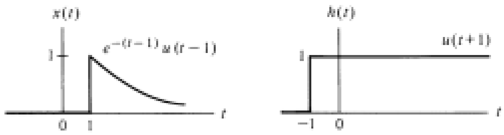


Figure 1.2: (b)

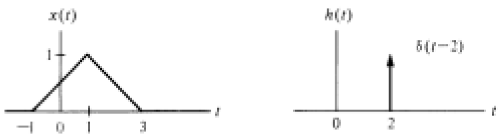


Figure 1.3: (c)

(a) $\begin{cases} t & 0 \le t < 4 \\ 8-t & 4 \le t < 8 \\ 0 & t<0,\ t \ge 8 \end{cases}$

(b) $y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$

$= \int_{-\infty}^{\infty} e^{-(\tau-1)} u(\tau-1)\, u(t-\tau+1)\, d\tau$

$= \begin{cases} \int_{1}^{t+1} e^{-(\tau-1)}\, d\tau & t>0 \\ 0, & t<0 \end{cases}$

$= \begin{cases} 1-e^{-t} & t>0 \\ 0 & t<0 \end{cases}$

(c) $y(t) = x(t-2)$

$= \begin{cases} 0 & t<1,\ t\ge 5 \\ \frac{1}{2}(x-1) & 1 \le t < 3 \\ -\frac{1}{2}(x-5) & 3 \le t < 5 \end{cases}$

2. [12 points] Consider a discrete-time, linear, shift-invariant system that has unit sample response $h[n]$ and input $x[n]$.

(a) Sketch the response of this system if $x[n] = \delta[n - n_0]$ for some $n_0 > 0$, and $h[n] = (\frac{1}{2})^n u[n]$.

(b) Evaluate and sketch the output of the system if $h[n] = (\frac{1}{2})^n u[n]$ and $x[n] = u[n]$.

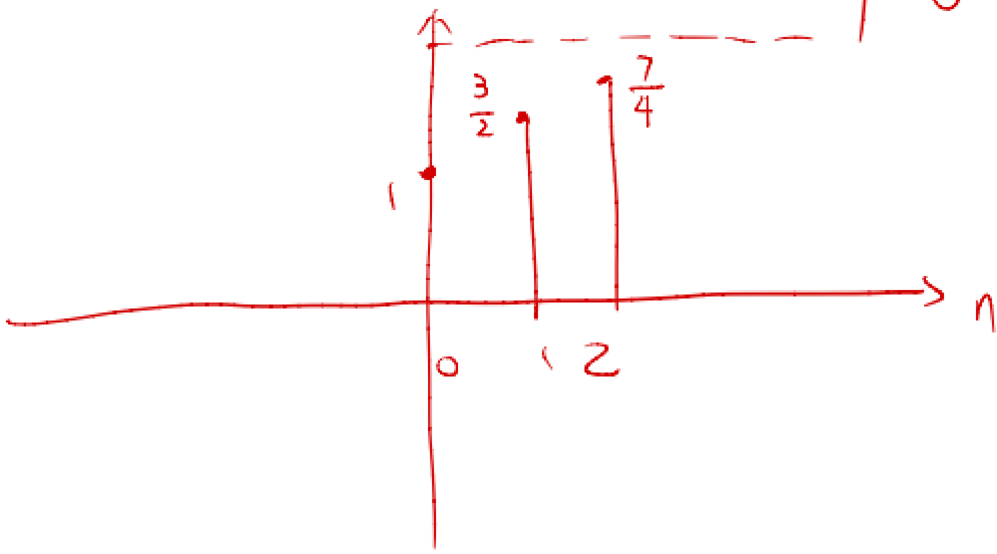
(c) Consider reversing the role of the input and system response in part (b). That is,

$$h[n] = u[n],$$
$$x[n] = (\frac{1}{2})^n u[n]$$

Evaluate the system output $y[n]$ and sketch.

(a) $y[n] = \sum_{m=-\infty}^{\infty} \delta[m-n_0]\, h[n-m] = h[n-n_0]$

(b) $y[n] = \sum_{m=-\infty}^{\infty} (\frac{1}{2})^m u[m]\, u[n-m] = \begin{cases} 2-(\frac{1}{2})^n & n \geq 0 \\ 0 & n < 0 \end{cases}$



(c) Some as (b)

3. [5 points] Compute the convolution $y[n] = x[n] * h[n]$ when

$$x[n] = \alpha^n u[n], 0 < \alpha < 1,$$
$$h[n] = \beta^n u[n], 0 < \beta < 1,$$

Assume that α and β are not equal.

$$\begin{aligned}
y[n] &= x[n] * h[n] \\
&= \sum_{m=-\infty}^{\infty} x[n - m]h[m] \\
&= \sum_{m=-\infty}^{\infty} \alpha^{n-m}\, u\,[n - m]\beta^m u[m] \\
&= \sum_{m=0}^{n} \alpha^{n-m}\beta^m, \qquad n > 0,
\end{aligned}$$

$$\begin{aligned}
y[n] &= \alpha^n \sum_{m=0}^{n} \left(\frac{\beta}{\alpha}\right)^m = \alpha^n \left[\frac{1 - (\beta/\alpha)^{n+1}}{1 - (\beta/\alpha)}\right] \\
&= \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}, \qquad n \geq 0, \\
y[n] &= 0, \qquad n < 0
\end{aligned}$$

4. [16 points] Determine if each of the following statements is true in general. Provide proofs for those that you think are true and counterexamples for those that you think are false.

(a) $x[n] * \{h[n]g[n]\} = \{x[n] * h[n]\}g[n]$

(b) If $y(t) = x(t) * h(t)$, then $y(2t) = 2x(2t) * h(2t)$.

(c) If $x(t)$ and $h(t)$ are odd signals, then $y(t) = x(t) * h(t)$ is an even signal.

(d) If $y(t) = x(t) * h(t)$, then $Ev\{y(t)\} = x(t) * Ev\{h(t)\} + Ev\{x(t)\} * h(t)$.

(Hint: It's taught that for an arbitrary signal $x(t)$, we can have $x(t) = g(t) + h(t)$ where $g(t)$ is an odd signal and $h(t)$ is an even signal, then $Ev\{x(t)\} = h(t)$.)

a) False. let $g[n] = \delta[n]$

$\Rightarrow x[n] * \{h[n]g[n]\}$

$= x[n] * h[0]$

$\{x[n] * h[n]\} g[n]$

$= (x * h)[0]$

different !

b) $y(2t) = \int_{-\infty}^{\infty} x(\tau) h(2t-\tau) d\tau$

let $\tau' = \frac{\tau}{2} \Rightarrow d\tau = 2d\tau'$

$\Rightarrow y(2t) = \int_{-\infty}^{\infty} x(2\tau') h(2t-2\tau') \cdot 2d\tau'$

$= 2x(2t) * h(2t)$ True

c) $y(t) = x(t) * h(t)$

$$y(-t) = x(-t) * h(-t) = \int_{-\infty}^{\infty} x(-\tau)\, h(-t+\tau)\, d\tau$$

odd odd

$$= \int_{-\infty}^{\infty} (-x(\tau)) \cdot (-h(t-\tau))\, d\tau$$

$$= \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

$$= y(t) \quad \Rightarrow \text{True}$$

d) $y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$

$$\text{Even}\{y(t)\} = \frac{1}{2}\left[y(t) + y(-t) \right]$$

$$= \frac{1}{2}\left[\int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau + \int_{-\infty}^{\infty} x(-\tau) h(-t-\tau) d\tau \right]$$

$$= \frac{1}{2}\left[\int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau - \int_{-\infty}^{\infty} x(\tau) h(-t+\tau) d\tau \right]$$

$$= \frac{1}{2}\left[\int_{-\infty}^{\infty} x(\tau) \left[h(t-\tau) - h(-t+\tau) \right] d\tau \right]$$

$$= x(t) * \text{odd}\{h(t)\}$$

$$x(t) * Ev\{h(t)\} + Ev\{x(t)\} * h(t)$$

$$= \frac{1}{2} x(t) * [h(t) + h(-t)] + \frac{1}{2} [x(t) + x(-t)] * h(t)$$

$$= \frac{1}{2} x(t) * h(t) + \frac{1}{2} x(t) * h(-t) + \frac{1}{2} x(t) * h(t)$$

$$+ \frac{1}{2} x(-t) * h(t)$$

$$= x(t) * h(t) + \frac{1}{2} x(t) * h(-t) + \frac{1}{2} x(-t) * h(t)$$

Example: $x(t) = \delta(t-1)$

$h(t) = \delta(t+1)$

$y(t) = \delta(t)$.

False.

5. [9 points] Let $x(t) = u(t-3) - u(t-5)$ and $h(t) = e^{-3t}u(t)$

(a) Compute $y(t) = x(t) * h(t)$.

(b) Compute $g(t) = \frac{dx(t)}{dt} * h(t)$.

(c) How is $g(t)$ related to $y(t)$?

(a)
$$y(t) = [u(t-3) - u(t-5)] * h(t)$$
$$= u(t-3) * h(t) - u(t-5) * h(t)$$
$$= \int_{-\infty}^{\infty} u(t-3)\, e^{-3(t-\tau)} u(t-\tau)\, d\tau - \int_{-\infty}^{\infty} u(t-5)\, e^{-3(t-\tau)} u(t-\tau)\, d\tau$$
$$= \int_3^t e^{-3(t-\tau)}\, d\tau - \int_5^t e^{-3(t-\tau)}\, d\tau$$
$$= \begin{cases} 0 & t < 3 \\ \frac{1}{3} - \frac{1}{3} e^{-3(t-3)} & 3 \le t < 5 \\ -\frac{1}{3} e^{-3(t-3)} + \frac{1}{3} e^{-3(t-5)} & 5 \le t \end{cases}$$

(b)
$$g(t) = \frac{dx(t)}{dt} * h(t)$$
$$= \delta(t-3) * e^{-3t}u(t) - \delta(t-5) * e^{-3t}u(t)$$
$$= e^{-3(t-3)} u(t-3) - e^{-3(t-5)} u(t-5)$$
$$= \begin{cases} 0 & t < 3 \\ e^{-3(t-3)} & 3 \le t < 5 \\ e^{-3(t-3)} - e^{-3(t-5)} & 5 \le t \end{cases}$$

(c)
$$g(t) = \frac{dy(t)}{dt}$$

6. [10 points] Consider the cascade interconnection of three causal LTI systems, illustrated in Figure 6.1. The impulse response $h_2[n]$ is

$$h_2[n] = u[n] - u[n-2],$$

and the overall impulse response is as shown in Figure 6.2.

Figure 6.1: LTI systems

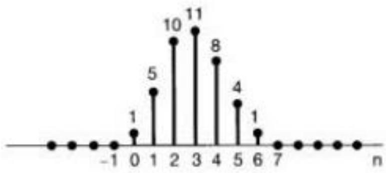


Figure 6.2: overall impulse response

(a) Find the impulse response $h_1[n]$.

(b) Find the response of the overall system to the input

$$x[n] = \delta[n] - \delta[n-1].$$

(a) $h_2[n] = u[n] - u[n-2] = \delta[n] + \delta[n-1]$

So $h_2[n] * h_2[n] = \delta[n] + 2\delta[n-1] + \delta[n-2]$

Assume h[n] is the overall response,

$$h[n] = h_1[n] * (\delta[n] + 2\delta[n-1] + \delta[n-2]) = h_1[n] + 2h_1[n-1] + h_1[n-2]$$

So $\begin{cases} h[0] = h_1[0] = 1 \\ h[1] = h_1[1] + 2h_1[0] = 5 \\ h[2] = h_1[2] + 2h_1[1] + h_1[0] = 10 \\ h[3] = h_1[3] + 2h_1[2] + h_1[1] = 11 \\ h[4] = h_1[4] + 2h_1[3] + h_1[2] = 8 \\ h[5] = h_1[5] + 2h_1[4] + h_1[3] = 4 \\ h[6] = h_1[6] + 2h_1[5] + h_1[4] = 1 \end{cases}$,solve and get $\begin{cases} h_1[0] = 1 \\ h_1[1] = 3 \\ h_1[2] = 3 \\ h_1[3] = 2 \\ h_1[4] = 1 \\ h_1[5] = 0 \\ h_1[6] = 0 \end{cases}$

So $h_1[n] = \delta[n] + 3\delta[n-1] + 3\delta[n-2] + 2\delta[n-3] + \delta[n-4]$

(b) $y[n] = (\delta[n] - \delta[n-1]) * h[n] = h[n] - h[n-1]$

$$\begin{cases} y[0] = h[0] - h[-1] = 1 \\ y[1] = h[1] - h[0] = 4 \\ y[2] = h[2] - h[1] = 5 \\ y[3] = h[3] - h[2] = 1 \\ y[4] = h[4] - h[3] = -3 \\ y[5] = h[5] - h[4] = -4 \\ y[6] = h[6] - h[5] = -3 \\ y[7] = h[7] - h[6] = -1 \end{cases}, \text{ otherwise } 0$$

7. [16 points] Consider the evaluation of

$$y[n] = x_1[n] * x_2[n] * x_3[n],$$

where $x_1[n] = (0.5)^n u[n]$, $x_2[n] = u[n+3]$, and $x_3[n] = \delta[n] - \delta[n-1]$.

(a) Evaluate the convolution $x_1[n] * x_2[n]$.

(b) Convolve the result of part (a) with $x_3[n]$ in order to evaluate $y[n]$.

(c) Evaluate the convolution $x_2[n] * x_3[n]$.

(d) Convolve the result of part (c) with $x_1[n]$ in order to evaluate $y[n]$.

Solution:

(a) We can know that

$$y_1[n] = x_1[n] * x_2[n] = \sum_{k=-\infty}^{\infty} x_1[k]x_2[n-k]$$

$$= \sum_{k=0}^{\infty} (0.5)^k u[n+3-k]$$

$$= \sum_{k=0}^{n+3} (0.5)^k$$

This evaluates to

$$y_1[n] = x_1[n] * x_2[n] = 2\{1 - \left(\frac{1}{2}\right)^{n+4}\}\mathrm{u}[n+3]$$

Or

$$y_1[n] = \begin{cases} 2\{1 - \left(\frac{1}{2}\right)^{n+4}\}, n \geq -3 \\ 0, \qquad\qquad otherwise \end{cases}$$

(b) Now,

$$\mathrm{y}[n] = x_3[n] * y_1[n] = y_1[n] - y_1[n-1].$$

Therefore,

$$\mathrm{y}[n] = 2\{1 - \left(\frac{1}{2}\right)^{n+4}\}\mathrm{u}[n+3] - 2\{1 - \left(\frac{1}{2}\right)^{n+3}\}\mathrm{u}[n+2]$$

$$= \left(\frac{1}{2}\right)^{n+3} u[n+3]$$

Or

$$y[n]=\begin{cases}\left(\frac{1}{2}\right)^{n+3} & ,n\geq -2\\ 1, & n=-3\\ 0, & otherwise\end{cases}$$

Therefore, $y[n]=\left(\frac{1}{2}\right)^{n+3}u[n+3]$.

(c) We have

$$y_2[n]=x_2[n]*x_3[n]=u[n+3]-u[n+2]=\delta[n+3]$$

(d) From the result of part(c), we get

$$y[n]=y_2[n]*x_1[n]=x_1[n+3]=\left(\frac{1}{2}\right)^{n+3}u[n+3]$$

8. [10 points]

(a) Consider an LTI system with input and output related through the equation

$$y(t)=\int_{-\infty}^{t}e^{-(t-\tau)}x(\tau-2)d\tau.$$

What is the impulse response $h(t)$ for this system?

(b) Determine the response of the system when the input $x(t)$ is as shown in Figure 8.

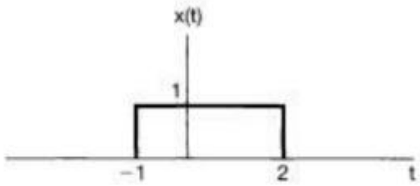


Figure 8: The Figure of $x(t)$

Solution:

(a) Note that

$$y(t)=\int_{-\infty}^{t}e^{-(t-\tau)}x(\tau-2)d\tau=\int_{-\infty}^{t-2}e^{-\left(t-2-\tau'\right)}x\left(\tau'\right)d\tau'$$

Therefore,

$$h(t)=e^{-(t-2)}u(t-2)$$

(b) We have

$$x(t)=u(t+1)-u(t-2)$$

$$y(t)=\int_{-\infty}^{\infty}h(\tau)x(t-\tau)d\tau$$

$$=\int_{2}^{\infty}e^{-(\tau-2)}[u(t-\tau+1)-u(t-\tau-2)]d\tau$$

$$=\int_{2}^{t+1}e^{-(\tau-2)}\,d\tau-\int_{2}^{t-2}e^{-(\tau-2)}d\tau$$

We can know that:

$$y(t) = \begin{cases} 0, & t < 1 \\ 1 - e^{-(t-1)}, & 1 \le t < 4 \\ e^{-(t-4)} - e^{-(t-1)}, & t \ge 4 \end{cases}$$

We can know that

$$y(t) = u(t-1)(1 - e^{-t+1}) - u(t-4)(1 - e^{-t+4})$$

9. [10 points] Suppose that the signal

$$x(t) = u(t + 0.5) - u(t - 0.5)$$

is convolved with the signal

$$h(t) = e^{i\omega_0 t}.$$

(a) Determine a value of ω_0 which ensures that

$$y(0) = 0,$$

where $y(t) = x(t) * h(t)$.

(b) Is your answer to the previous part unique?

(a)

$$y(t) = x(t) * h(t) = \int_{-0.5}^{0.5} e^{i\omega_0 (t-\tau)} d\tau$$

$$y(0) = \int_{-0.5}^{0.5} e^{-i\omega_0 \tau} d\tau = \frac{2}{\omega_0} \sin\left(\frac{\omega_0}{2}\right) = 0$$

So, $\omega_0 = 2\pi$

(b) not unique

$\omega_0 = 2k\pi, \ k \in \mathbb{Z}, \ k \neq 0$