EE160 Homework 6 Solution

1. *(6 points) Proportional-Differential (PD) Control.*

Solution: Rewrite the control system in standard linear form of

$$\dot{x} = Ax + Bu \quad \text{with output} \quad y = Cx$$

where matrices

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 0 \end{pmatrix} .$$

Now let's have a look of the behaviors of different controller implemented on this system.

(a) Proportional control. With input $u = Ky$, we get a linear closed-loop system

$$\dot{x} = A_{\text{cl}}x \quad \text{with} \quad A_{\text{cl}} = A + BKC = \begin{pmatrix} 0 & 1 \\ -1+K & 0 \end{pmatrix} .$$

Then we know the eigenvalue of A_{cl} are always mirrored, $\lambda_{1,2} = \pm\sqrt{K-1}$, that means we cannot make the system asymptotically stable by tuning proportional control gain K.

(b) Proportional-differential control. With control input in form of (??)

$$u = K_P y(t) + K_D \dot{y}(t) = K_P Cx + K_D C(Ax + Bu) \Longrightarrow u = (K_P C + K_D CA)x$$

the closed-loop system is obtained by

$$\dot{x} = A_{\text{cl}}x \quad \text{with} \quad A_{\text{cl}} = A + BKC + BK_D CA = \begin{pmatrix} 0 & 1 \\ -1+K & K_D \end{pmatrix}$$

so we have to choose control K, K_D satisfying

$$K < 1 \quad \text{and} \quad K_D < 0$$

to make the real part of eigenvalues of A_{cl} are both negative, or to say make the system asymptotically stable, for example $K = \frac{1}{2}, K_D = -1$.

2. *(4 points) Proportional-Integral (PI) Control.*

Solution: Steady state $x_{\text{ref}} = 1$ and $u_{\text{ref}} = -1$, thus we design PI controller in form of

$$u(t) = -1 + K(x(t) - 1) + K_I \int_0^t (x(\tau) - 1)\mathrm{d}\tau .$$

Introduce the auxiliary state

$$z(t) = \begin{pmatrix} x(t) - 1 \\ \int_0^t (x-1)\mathrm{d}\tau \end{pmatrix}$$

and the dynamics of it is given by

$$\dot{z}(t) = \begin{pmatrix} \dot{x}(t) \\ x(t) - 1 \end{pmatrix} = \begin{pmatrix} (1+K)(z_1 - 1) + K_I z_2 \\ z_1 \end{pmatrix} = A_{\text{cl}}\, z ,$$

$$\text{and} \quad A_{\text{cl}} = \begin{pmatrix} 1+K & K_I \\ 1 & 0 \end{pmatrix} \quad \text{with eigenvalues} \quad \lambda_{1,2} = \frac{1+K}{2} \pm \sqrt{\left(\frac{1+K}{2}\right)^2 + K_I}$$

so we can choose $K = -2, K_D = -1$ to stabilize the system and let x converge to 1.