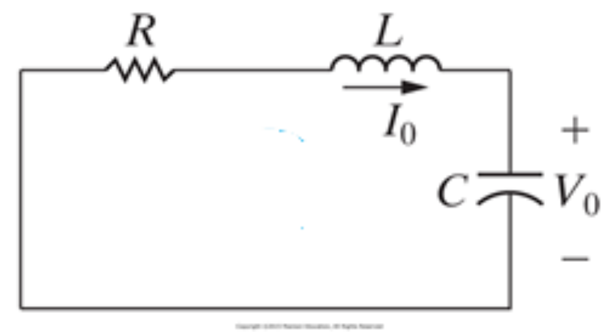
Properties of Series RLC Network - v(t) $V_c(t)$



- Behavior captured by "damping"
  - Gradual loss of the initial stored energy V_0
  - α determines the rate of damping

$$\alpha = \frac{R}{2L} \qquad \omega_0 = \frac{1}{\sqrt{LC}}$$

$\alpha \gtreqless \omega_0$

– $\alpha > \omega_0$, overdamped

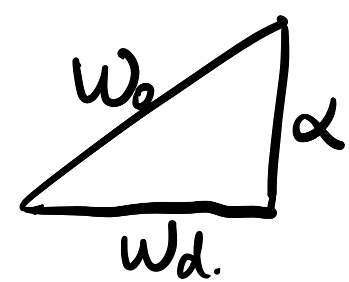
→ $v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$ ① $S_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$

– $\alpha = \omega_0$, critically damped

→ $v(t) = (A_1 t + A_2) e^{-\alpha t}$ ② $S_1 = S_2 = -\alpha \pm \cancel{\sqrt{\alpha^2 - \omega_0^2}} = -\alpha$

– $\alpha < \omega_0$, underdamped

→ $v(t) = e^{-\alpha t}(B_1 \cos \omega_d t + B_2 \sin \omega_d t)$ ③

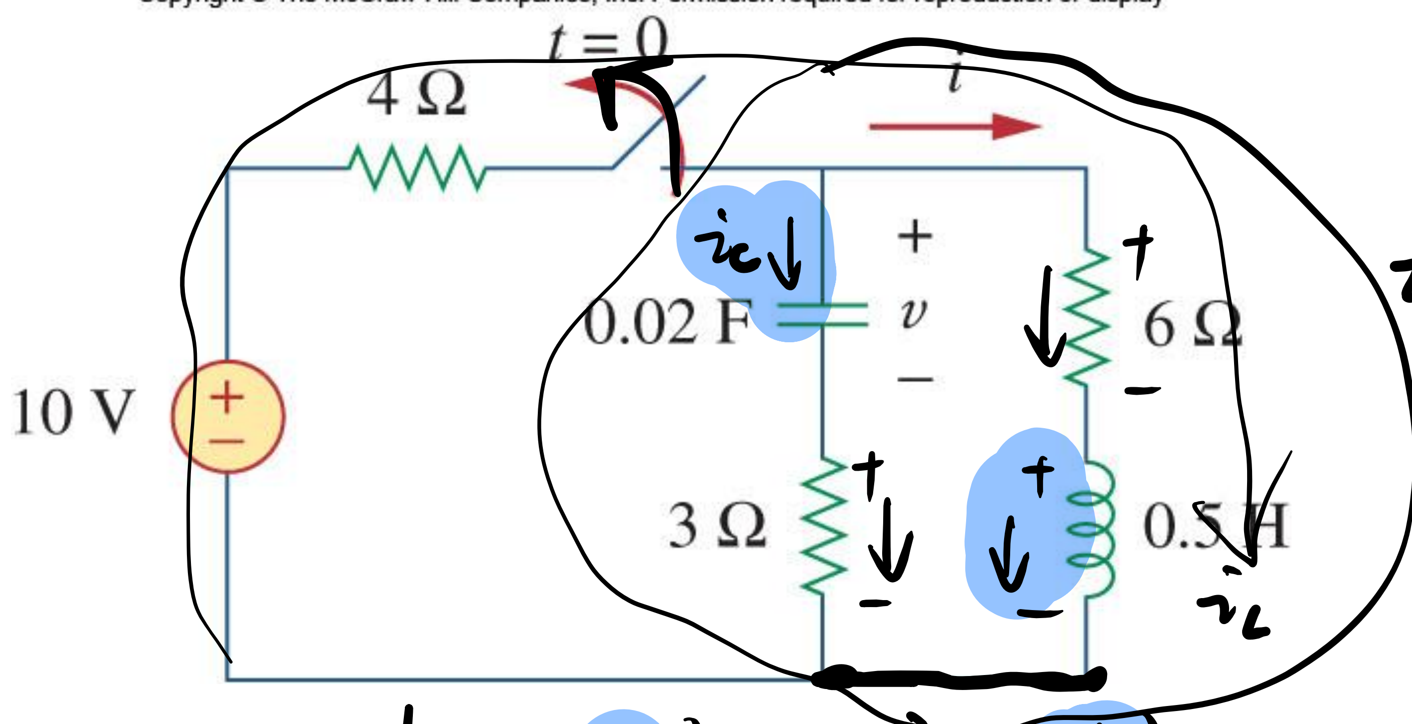


$\omega_d = \sqrt{\omega_0^2 - \alpha^2}$

[Source: Berkeley]

Example

- Find $v(t)$ in the circuit below. Assume the circuit has reached steady state at $t = 0^-$.



$i_c(0^+) + i_L(0^+) = 0$

$$\frac{dV_c(0^+)}{dt} = \frac{i_c(0^+)}{C} = \frac{-i_L(0^+)}{C} = \frac{-i_L(0^-)}{C}$$

$$i_L(0^-) = \frac{10}{4+6} = 1A$$

$$\frac{dV_c(0^+)}{dt} = \frac{-1}{0.02} = 50V/s$$

$t>0$ S-RLC-Network

$R = 9\Omega$, $L = 0.5H$. $C = 0.02F$

$$\left.\begin{aligned} \alpha &= \frac{R}{2L} = \frac{9}{1} = 9 \\ \omega_0 &= \frac{1}{\sqrt{LC}} = 10 \end{aligned}\right\} \quad \alpha < \omega_0 \quad \text{underdamped.}$$

$$V_c(t) = e^{-\alpha t}[B_1 \cos \underline{\omega_d} t + B_2 \sin \underline{\omega_d} t]$$

$$\omega_d = \sqrt{\omega_0^2 - \alpha^2} = 4.36$$

B_1, B_2 ?

$\langle 1 \rangle \quad V_c(0^+) = V_c(0^-) = 6V$

$t<0$

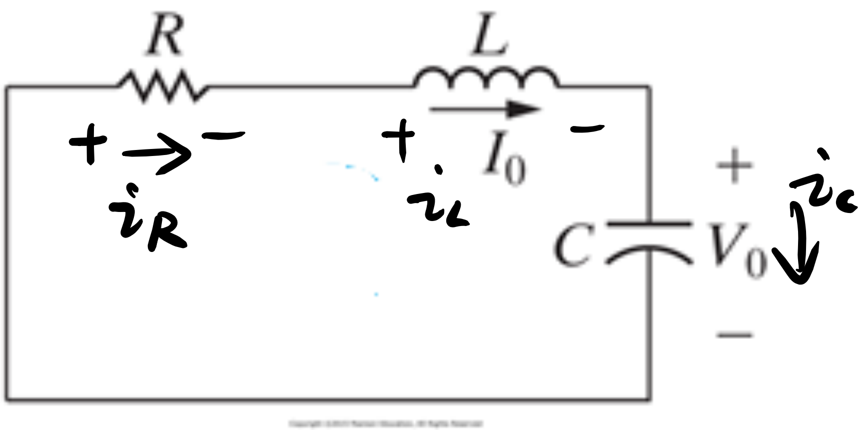
$\langle 2 \rangle \quad \dfrac{dV_c(0^+)}{dt} = -50\,V/s$

$\Rightarrow B_1 = 6 \qquad B_2 = 0.82$

$V_c(t) = e^{-9t}[6\cos 4.36t + 0.82\sin 4.36t]$

$t>0$

Source-Free Series RLC Circuit



$$\Longrightarrow i_L(t)$$

$$L\frac{di_L(t)}{dt} = V_L(t)$$

$$i_L(t) = i_R = i_c$$

$$i_R \cdot R = i_L(t)\cdot R = V_R(t)$$

$$\frac{1}{C}\int_{-\infty}^{t} i_L(t)\,dt = \frac{1}{C}\int_{-\infty}^{t} i_c(t)\,dt = V_c(t)$$

$$V_L + V_R + V_c = 0$$

$$\left[L \cdot \frac{di_L(t)}{dt} + i_L(t) \cdot R + \frac{1}{C}\int_{-\infty}^{t} i_L(t) \cdot dt \right]' = 0$$

$$\frac{d^2 i_L(t)}{dt^2} + \frac{R}{L} \cdot \frac{di_L(t)}{dt} + \frac{1}{LC} \cdot i_L(t) = 0$$

$$\alpha = \frac{R}{2L} \qquad \omega_0 = \frac{1}{\sqrt{LC}}$$

$$\alpha \begin{matrix} > \\ = \\ < \end{matrix} \omega_0$$

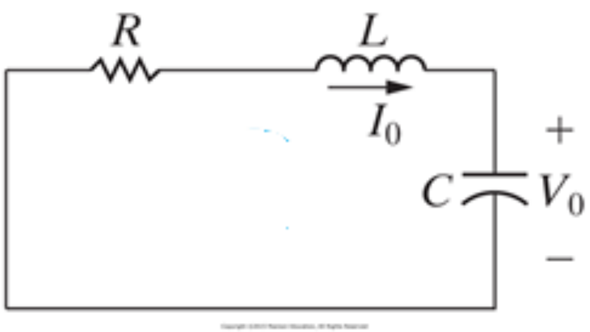
$$\frac{d^2 V_c}{dt^2} + \frac{R}{L} \cdot \frac{dV_c}{dt} + \frac{1}{LC} V_c = 0$$

Properties of Series RLC Network - *i(t)*

- Behavior captured by damping
  - Gradual loss of the initial stored energy
  - α determines the rate of damping

$$\alpha = \frac{R}{2L} \qquad \omega_0 = \frac{1}{\sqrt{LC}}$$

- $\alpha > \omega_0$, overdamped
  $$i(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$
- $\alpha = \omega_0$, critically damped
  $$i(t) = (A_1 t + A_2) e^{-\alpha t}$$
- $\alpha < \omega_0$, underdamped
  $$i(t) = e^{-\alpha t}(B_1 \cos \omega_d t + B_2 \sin \omega_d t)$$

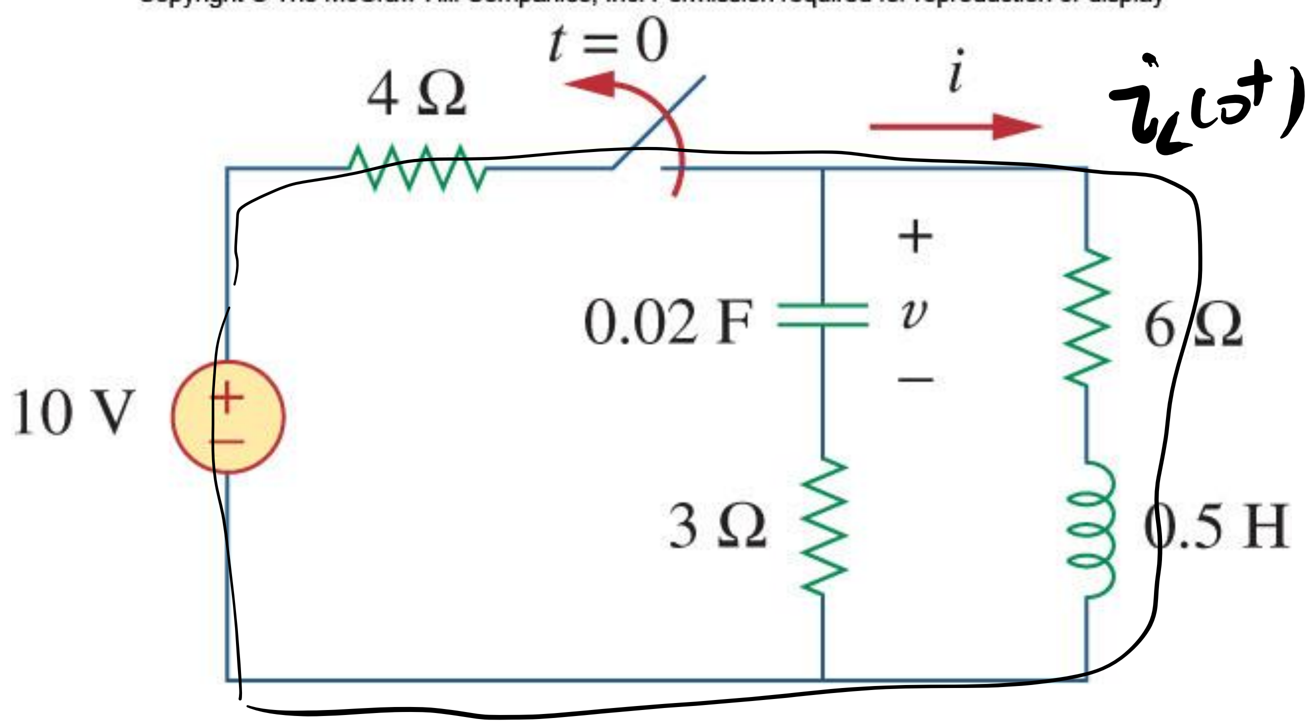


$i_L(0^+)$ ✓

$\frac{d i_L(0^+)}{dt}$ ✓ $\Rightarrow A_1, A_2$

Example

- Find $\boldsymbol{i(t)}$ in the circuit below. Assume the circuit has reached steady state at $t = 0^{-}$.



$i_L(0^+) = i_L(0^-) = 1A$

$\alpha = \frac{R}{2L} = 9$, $\omega_0 = \frac{1}{\sqrt{LC}} = 10$

$\alpha < \omega_0$ underdamped

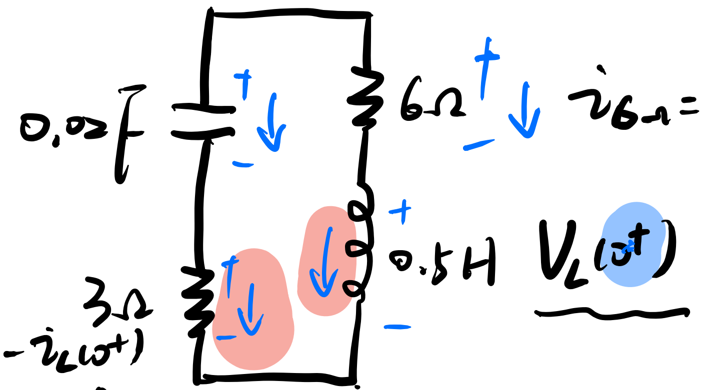
- $\alpha < \omega_0$, underdamped

$\Longrightarrow$ $i(t) = e^{-\alpha t}(B_1 \cos \omega_d t + B_2 \sin \omega_d t)$

$\omega_d = \sqrt{\omega_0^2 - \alpha^2} = 4.36$

B_1 B_2 $i_L(0^+) = i_L(0^-) = ?$ 1A

$? \quad \frac{d i_L(0^+)}{dt} = \frac{V_L(0^+)}{L} = \frac{-3}{0.5} = -6A/s$



$V_C(0^+) + V_{3\Omega} = V_{6\Omega} + V_L(0^+)$

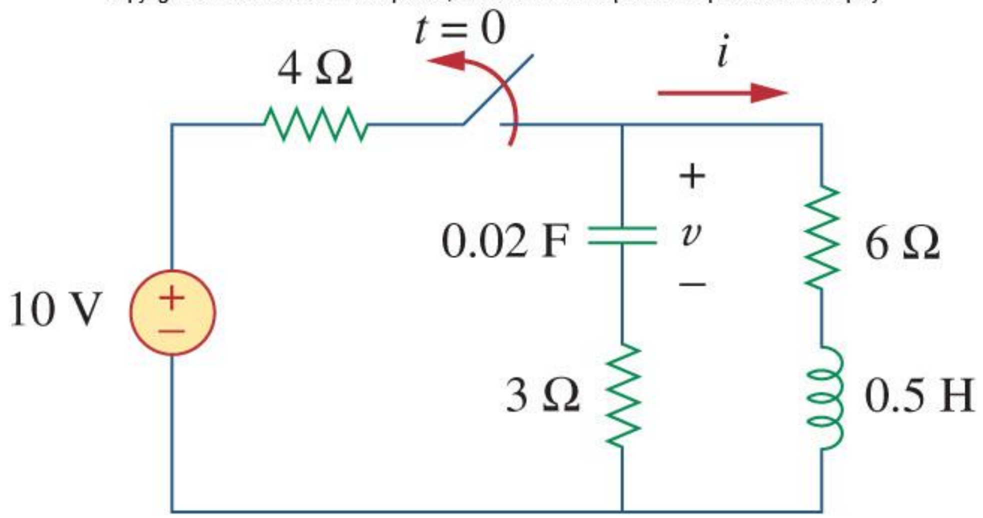
$V_L(0^+) = V_C(0^+) + V_{3\Omega}^{(0^+)} - V_{6\Omega}(0^+)$

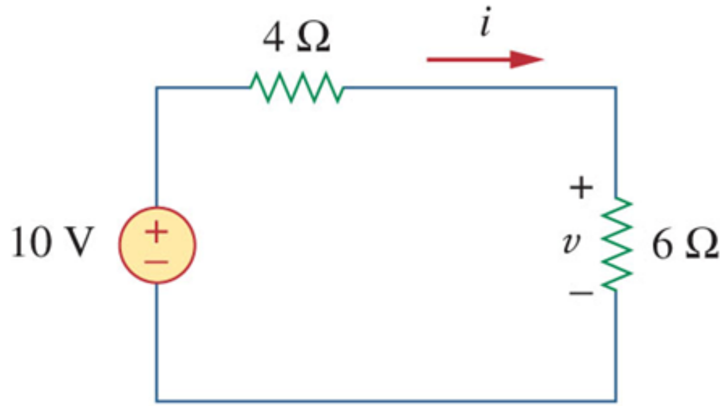
$= V_C(0^-) + i_{3\Omega} \times 3\Omega - i_{6\Omega} \times 6\Omega$

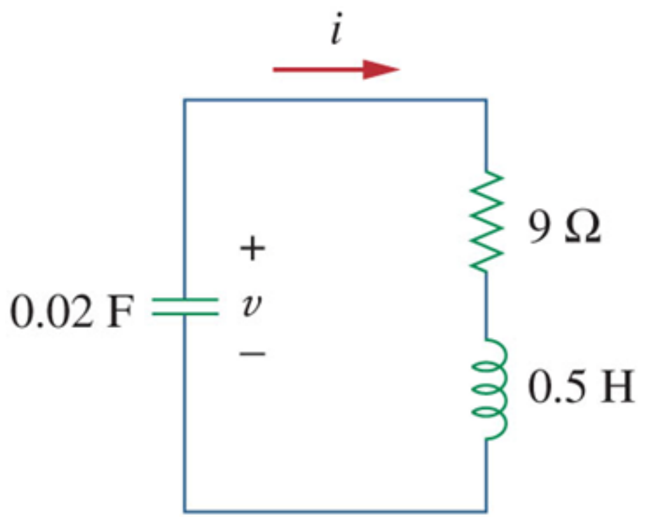
$= 6V + (-1) \times 3 - 1 \times 6 = -3V$

Example

- Find $i(t)$ in the circuit below. Assume the circuit has reached steady state at $t = 0^-$.



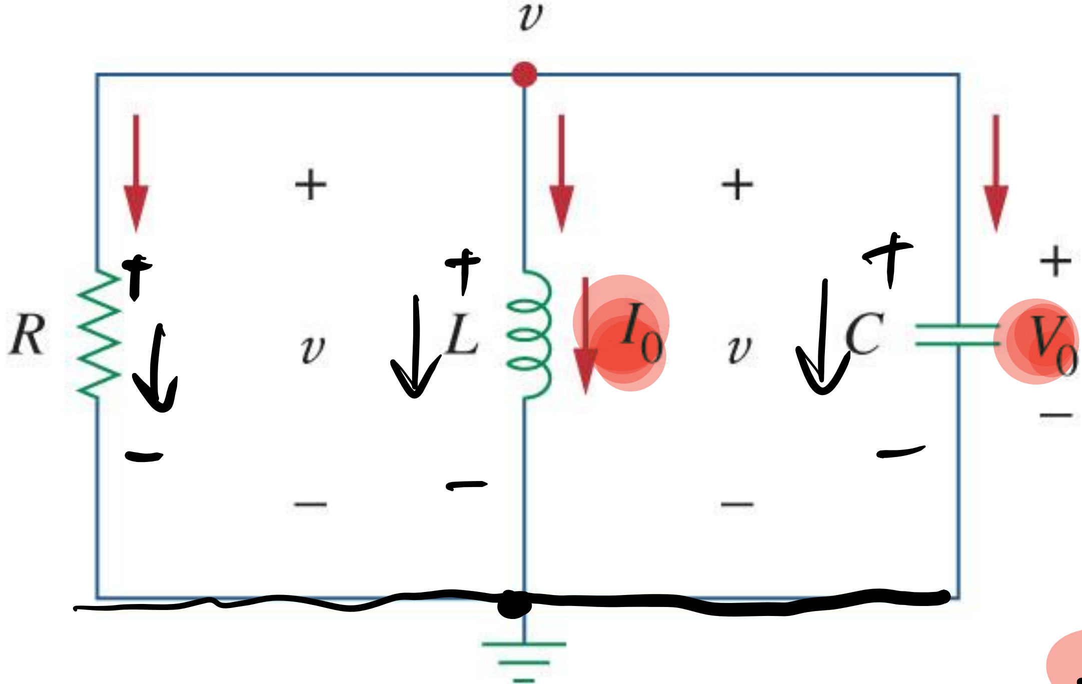




$$\alpha = \frac{R}{2L} = 9 \qquad \omega_0 = \frac{1}{\sqrt{LC}} = 10$$

$$s_{1,2} = -\alpha \mp \sqrt{\alpha^2 - \omega_0^2} = -9 \mp j4.359$$

Source-Free Parallel RLC Network



$V_c(t)$?

$$i_c(t) = C \cdot \frac{dV_c(t)}{dt}$$

$$i_R(t) = \frac{V_R(t)}{R} = \frac{V_c(t)}{R}$$

$$i_L(t) = \frac{1}{L}\int_{-\infty}^{t} V_L(t)\,dt$$

$$= \frac{1}{L}\int_{-\infty}^{t} V_c(t)\,dt$$

KCL:

$$\left[C \cdot \frac{dV_c}{dt} + \frac{V_c}{R} + \frac{1}{L}\int_{-\infty}^{t} V_c \, dt \right]' = 0$$

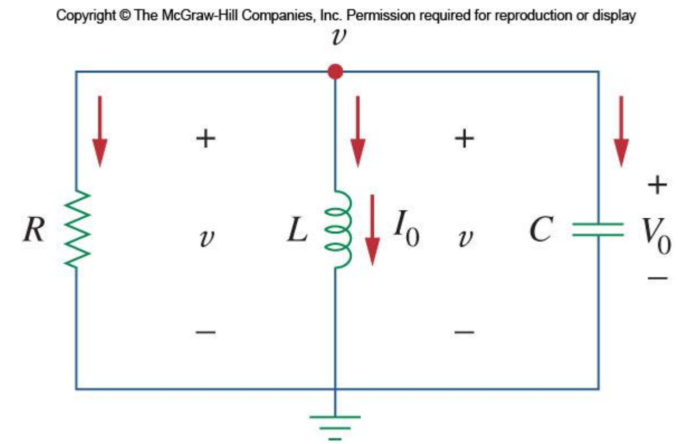
$$\frac{d^2 V_c}{dt^2} + \frac{1}{RC}\frac{dV_c}{dt} + \frac{1}{LC} V_c = 0$$

$$\alpha = \frac{1}{2RC} \quad , \quad \omega_0 = \frac{1}{\sqrt{LC}}$$

$$\alpha \begin{matrix} > \\ = \\ < \end{matrix} \omega_0$$

Source-Free Parallel RLC Network - *v(t)*

$$\frac{d^2v}{dt^2}+\frac{1}{RC}\frac{dv}{dt}+\frac{1}{LC}v=0$$



- The characteristic equation is:

$$s^2+\frac{1}{RC}s+\frac{1}{LC}=0$$

From this, we can find the roots of the characteristic equation to be:

$$s_{1,2}=-\alpha\pm\sqrt{\alpha^2-\omega_0^2}$$

$$\alpha=\frac{1}{2RC}\qquad \omega_0=\frac{1}{\sqrt{LC}}$$

As in last time, there are three scenarios to consider.

Three Damping Cases - *v(t)*

$$\frac{d^2v}{dt^2}+\frac{1}{RC}\frac{dv}{dt}+\frac{1}{LC}v=0$$

$$s_{1,2}=-\alpha\pm\sqrt{\alpha^2-\omega_0^2}$$

$$\alpha=\frac{1}{2RC}\qquad \omega_0=\frac{1}{\sqrt{LC}}$$

- For the overdamped case, the roots are real and negative,

$\alpha > \omega_0$ $\quad v(t)=A_1e^{s_1t}+A_2e^{s_2t}$

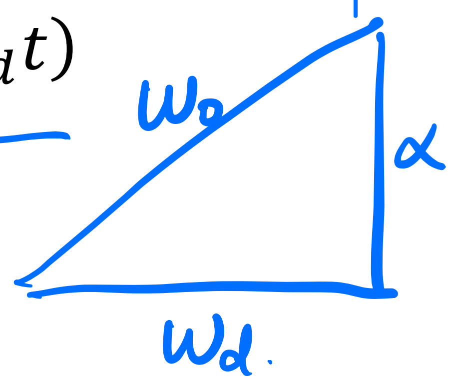
- For critically damped, the roots are real and equal

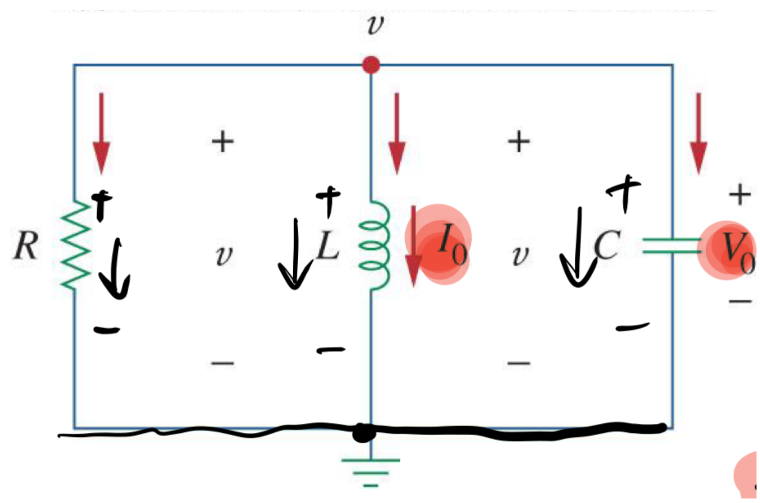
$\alpha = \omega_0$ $\quad v(t)=(A_1t+A_2)e^{-\alpha t}$

- In the underdamped case, the roots are complex

$\alpha < \omega_0$ $\quad v(t)=e^{-\alpha t}(B_1\cos\omega_d t+B_2\sin\omega_d t)$

$$\omega_d=\sqrt{\omega_0^2-\alpha^2}$$





$i_L(t)$?

$$L\frac{d\,i_L(t)}{dt} = V_L(t)$$

$$V_L(t) = V_R(t) = V_C(t)$$

$$i_R(t) = \frac{V_R(t)}{R} = \frac{V_L(t)}{R} = \frac{L}{R}\cdot\frac{d\,i_L(t)}{dt}$$

$$i_C(t) = C\cdot\frac{dV_C(t)}{dt} = LC\cdot\frac{d^2 i_L(t)}{dt^2}$$

$$i_L(t) = i_L(t)$$

KCL: $i_R + i_C + i_L = 0$

$$\frac{d^2 i_L}{dt^2} + \frac{1}{RC}\frac{d i_L}{dt} + \frac{1}{LC} i_L = 0 \qquad (i_L)$$

$$\alpha = \frac{1}{2RC} \qquad \omega_0 = \frac{1}{\sqrt{LC}}$$

$\times 2$

$$\alpha \qquad \omega_0$$

$$\frac{d^2V_c}{dt^2} + \frac{1}{RC}\frac{dV_c}{dt} + \frac{1}{LC}V_c = 0 \qquad (V_c)$$

Three Damping Cases - *i(t)*

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

$$\alpha = \frac{1}{2RC} \qquad \omega_0 = \frac{1}{\sqrt{LC}}$$

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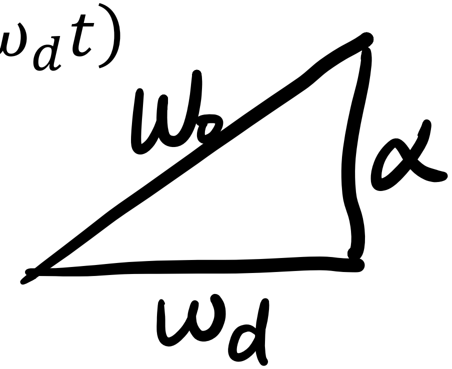
- In the underdamped case, the roots are complex

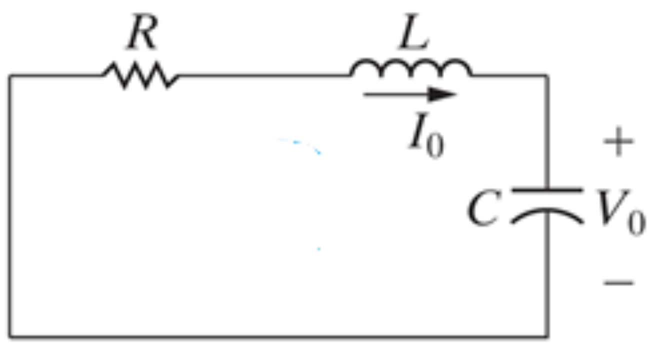
$\alpha < \omega_0$

$$i(t) = e^{-\alpha t}(B_1 \cos \omega_d t + B_2 \sin \omega_d t)$$



Series vs. Parallel (Source-Free RLC Circuit)

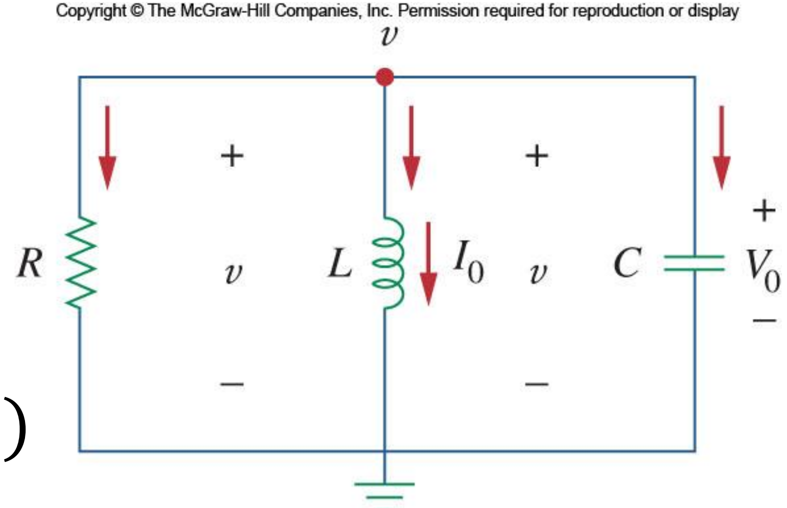
- Series $\alpha = \frac{R}{2L}$ $\omega_0 = \frac{1}{\sqrt{LC}}$



$$v_c(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$
$$v_c(t) = (A_1 t + A_2) e^{-\alpha t}$$
$$v_c(t) = e^{-\alpha t}(B_1 \cos \omega_d t + B_2 \sin \omega_d t)$$

OR $i_L(t)$: 3 cases

- Parallel $\alpha = \frac{1}{2RC}$ $\omega_0 = \frac{1}{\sqrt{LC}}$



$$v_c(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$
$$v_c(t) = (A_1 t + A_2) e^{-\alpha t}$$
$$v_c(t) = e^{-\alpha t}(B_1 \cos \omega_d t + B_2 \sin \omega_d t)$$

OR $i_L(t)$: 3 cases