Homework 1

Due: Mar. 15th

1. In a patient study for a new test for multiple sclerosis (MS), 32 of the 100 patients studied actually have MS. For the data given below, complete the two by two matrices and construct an ROC. The number of lesions (50, 40, 30, 20 or 10) corresponds to the different threshold values for designating MS as the diagnosis.
   (1) Calculate the prevalence and accuracy for every threshold;
   (2) Plot the ROC curve.

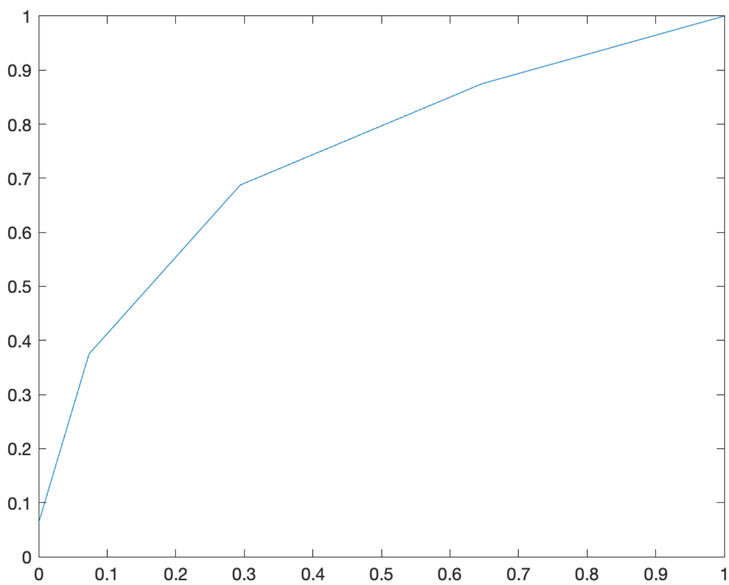
50 lesions		40 lesions		30 lesions		20 lesion		10 lesions	
2	0	12	5	22	20	28	44	32	68

Solution:

50 lesions		40 lesions		30 lesions		20 lesion		10 lesions	
2	0	12	5	22	20	28	44	32	68
30	68	20	63	10	48	4	24	0	0

(1) prevalence = 32 / 100 = 0.32
accuracy: (2+68)/100=0.7; 0.75; 0.7; 0.52; 0.32

(2)



2. Let the original image $f(x,y)=\begin{bmatrix}10 & 8 & 9 & 5 & 8 & 7\\ 8 & 5 & 7 & 0 & 1 & 6\\ 0 & 4 & 6 & 11 & 14 & 8\\ 7 & 6 & 5 & 8 & 5 & 9\\ 3 & 3 & 7 & 12 & 15 & 11\\ 9 & 4 & 17 & 15 & 6 & 14\end{bmatrix}$ $(0 \le x,y \le 5)$, the transformed image is $g\,(x,y)$. If rotate f counter-clockwise 30 degrees around the pixel $f\,(2{,}2)$ to get g, then
   (1) Forward mapping is $g = T \cdot f$, inverse mapping is $f = T^c \cdot g$. Derive the transformation matrix T and T^c
   (2) Calculate the gray value of $g\,(3{,}3)$ using nearest neighbour and bilinear interpolation method

Solution:

(1)

$$T=\begin{bmatrix}1 & 0 & 2\\ 0 & 1 & 2\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\cos 30^\circ & -\sin 30^\circ & 0\\ \sin 30^\circ & \cos 30^\circ & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & -2\\ 0 & 1 & -2\\ 0 & 0 & 1\end{bmatrix}=\begin{bmatrix}\frac{\sqrt{3}}{2} & -\frac{1}{2} & 3-\sqrt{3}\\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 1-\sqrt{3}\\ 0 & 0 & 1\end{bmatrix}$$

$$T^c = T^{-1} = \begin{bmatrix}\frac{\sqrt{3}}{2} & \frac{1}{2} & 1-\sqrt{3}\\ -\frac{1}{2} & \frac{\sqrt{3}}{2} & 3-\sqrt{3}\\ 0 & 0 & 1\end{bmatrix}$$

or $$T^c=\begin{bmatrix}1 & 0 & 2\\ 0 & 1 & 2\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\cos(-30^\circ) & -\sin(-30^\circ) & 0\\ \sin(-30^\circ) & \cos(-30^\circ) & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & -2\\ 0 & 1 & -2\\ 0 & 0 & 1\end{bmatrix}=\begin{bmatrix}\frac{\sqrt{3}}{2} & \frac{1}{2} & 1-\sqrt{3}\\ -\frac{1}{2} & \frac{\sqrt{3}}{2} & 3-\sqrt{3}\\ 0 & 0 & 1\end{bmatrix}$$

(2)

$$\begin{bmatrix}x_f\\ y_f\\ 1\end{bmatrix}=T^c\begin{bmatrix}x_g\\ y_g\\ 1\end{bmatrix}=\begin{bmatrix}\frac{\sqrt{3}}{2} & \frac{1}{2} & 1-\sqrt{3}\\ -\frac{1}{2} & \frac{\sqrt{3}}{2} & 3-\sqrt{3}\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}3\\ 3\\ 1\end{bmatrix}=\begin{bmatrix}\frac{5+\sqrt{3}}{2}\\ \frac{3+\sqrt{3}}{2}\\ 1\end{bmatrix}\approx\begin{bmatrix}3.4\\ 2.4\\ 1\end{bmatrix}$$

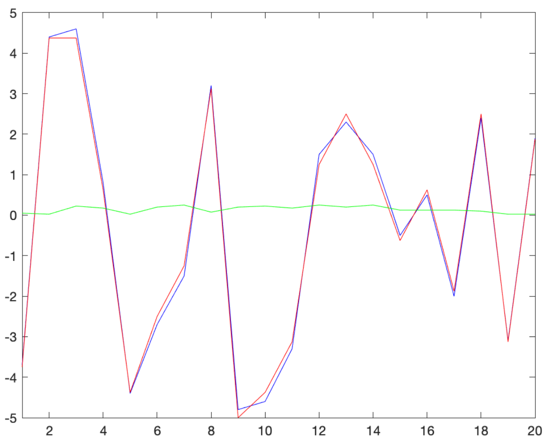
Then $g\,(3{,}3) = f\,(3.4{,}2.4)$, and use the bilinear interpolation between (4,2), (4,3), (3,2), (3,3) four points, the result $f\,(3.4{,}2.4) = 7.0592 \approx 7$.

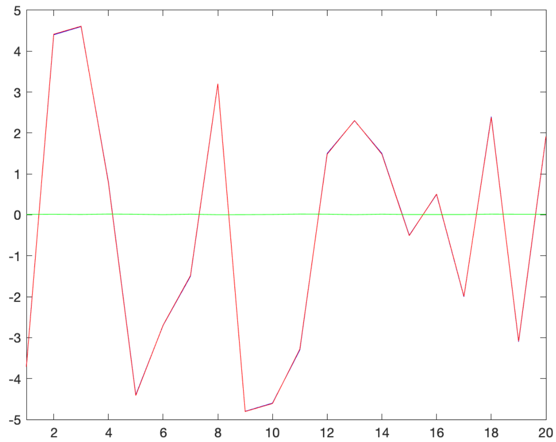
Use the nearest neighbor interpolation, $g\,(3{,}3) = f\,(3{,}2) = 5$.

3. (1) A signal is sampled every 1ms for 20ms, with the following actual values of the analogue voltage at successive sampling times. Plot the values of the voltage recorded by a 5 volt, 4-bit ADC assuming that the noise level is much lower than the signal and so can be neglected. On the same graph, plot the quantization error. Signal(volts) = -3.7, +4.4, +4.6, +0.8, -4.4, -2.7, -1.5, +3.2, -4.8, -4.6, -3.3, +1.5, +2.3, +1.5, -0.5, +0.5, -2, +2.4, -3.1, +1.9.
(2) Using the same signal as in (1), plot the values of the voltage and the quantization error recorded by a 5 volt, 8-bit ADC.

Solution:

Voltage	4 bits	8 bits	result-4bit	result-8bit
-3.7	-5	-5	-3.75	-3.7109375
4.4	-4.375	-4.9609375	4.375	4.4140625
4.6	-3.75	-4.921875	4.375	4.609375
0.8	-3.125	-4.8828125	0.625	0.78125
-4.4	-2.5	-4.84375	-4.375	-4.4140625
-2.7	-1.875	-4.8046875	-2.5	-2.6953125
-1.5	-1.25	-4.765625	-1.25	-1.484375
3.2	-0.625	-4.7265625	3.125	3.203125
-4.8	0	-4.6875	-5	-4.8046875
-4.6	0.625	-4.6484375	-4.375	-4.609375
-3.3	1.25	-4.609375	-3.125	-3.28125
1.5	1.875	-4.5703125	1.25	1.484375
2.3	2.5	-4.53125	2.5	2.3046875
1.5	3.125	-4.4921875	1.25	1.484375
-0.5	3.75	-4.453125	-0.625	-0.5078125
0.5	4.375	-4.4140625	0.625	0.5078125
-2	5	-4.375	-1.875	-1.9921875
2.4		-4.3359375	2.5	2.3828125
-3.1		-4.296875	-3.125	-3.0859375
1.9		……	1.875	1.9140625





4. If $\mathrm{LSF}(y) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(y-y_0)^2}{2\sigma^2}\right)$, derive $\mathrm{FWHM} = \left(2\sqrt{2\ln 2}\right)\sigma \cong 2.36\sigma$. (Slide 02, Page 11)

Solution:

$$LSF_{max} = \frac{1}{\sqrt{2\pi\sigma^2}}$$

$$\frac{1}{2} LSF_{max} = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-(y-y_0)^2}{2\sigma^2}\right)$$

$$\frac{1}{2} = \exp\left(\frac{-(y-y_0)^2}{2\sigma^2}\right)$$

$$2\sigma^2 \ln 2 = (y-y_0)^2$$

$$\therefore y_1 = \sqrt{2\sigma^2 \ln 2} + y_0 \,, \quad y_2 = -\sqrt{2\sigma^2 \ln 2} + y_0$$

$$FWHM = y_1 - y_2 = 2\sqrt{2\ln 2}\,\sigma$$

5. Prove: Taking N averages when obtaining an image, the signal-to-noise ratio (SNR) is $sqrt\ (N)$ times the SNR of a single image.

Solution:

$$I = \bar{I} + Noise$$

$$I = \frac{1}{N}\sum_{n=1}^{N}(\bar{I} + Noise_n) = \bar{I} + \frac{1}{N}\sum_{n=1}^{N} Noise_n$$

Thus the signal does not change after N averages.

If original image noise variance is σ^2 and $\sigma^2 = [I - \bar{I}]^2$, The noise variance of image after average is $\mathrm{\sigma_A^2}$ and

$$\mathrm{\sigma_A^2} = \left[\frac{1}{\mathrm{N}}\sum_{\mathrm{N}} \mathrm{I_i} - \bar{\mathrm{I}}\right]^2 = \frac{1}{\mathrm{N}^2}\left[\sum_{\mathrm{N}} \mathrm{I_i} - \mathrm{N}\bar{\mathrm{I}}\right]^2 = \frac{1}{\mathrm{N}^2}\left[\sum_{\mathrm{N}} (\mathrm{I_i} - \bar{\mathrm{I}})\right]^2$$

$$= \frac{1}{\mathrm{N}^2}\sum_{\mathrm{N}} (\mathrm{I_i} - \bar{\mathrm{I}})^2 + \frac{1}{\mathrm{N}^2}\mathrm{var}(\mathrm{I_i} - \bar{\mathrm{I}})$$

where $var(I_i - \bar{I})$ is the cross-correlation function and equal to 0 when $\mathrm{I_i}$ is random number.

$$\sigma_A^2 = \frac{1}{N^2}\sum_{N}(I_i - \bar{I})^2 = \frac{1}{N^2}\cdot N \cdot \sigma^2 = \frac{\sigma^2}{N}$$

The noise variance is reduced to 1 / N times the original, so the SNR is $\mathrm{sqrt\ (N)}$ times the original.