Test Paper of "SI231b: Matrix Computations"

Total points: 110

Your grade: min{your real grade, 100}

Problem 1 (15 points). Subspace problems: let $\mathcal{V} \subset \mathbb{R}^m$ be a subspace of dimension n, and let $v_1, v_2, \cdots, v_n$ be its basis. Define

$$\mathcal{T} = \{x | x = Ty,\ y \in \mathcal{V}\},$$

where $T \in \mathbb{R}^{m\times m}$ is nonsingular. Show that $\mathcal{T}$ is also a subspace and give its basis.

Problem 2 (20 points). LU factorization problems: for $A \in \mathbb{R}^{n\times n}$, suppose its LU factorization exists and its lower-triangular factor L has unit diagonal entries, i.e., $l_{ii} = 1$.

(1) (5 points). Show the uniqueness of this LU factorization;

(2) (5 points). Given $A = \begin{bmatrix} -3 & 1 & -2 \\ 1 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix}$, compute its LU factorization;

(3) (5 points). Solve the sequence of linear systems $Ax_i = b_i$ for $i = 1, 2, 3$, where $b_1 = \begin{bmatrix}1 & 0 & 0\end{bmatrix}^T$, $b_2 = \begin{bmatrix}0 & 1 & 0\end{bmatrix}^T$, $b_3 = \begin{bmatrix}0 & 0 & 1\end{bmatrix}^T$.

(4) (5 points). Compute the matrix condition norm $\kappa(A) = \|A\|\|A^{-1}\|$ using induced 1-norm and infinity norm.

Problem 3 (25 points). QR factorization problems: given a matrix

$$A = \begin{bmatrix} 1 & 3 \\ 1 & 3 \\ 1 & -1 \\ -1 & 1 \end{bmatrix},$$

(1) (5 points). Compute one orthonormal basis set for $\mathcal{R}(A)$;

(2) (5 points). Give the (reduced) QR factorization of A;

(3) (5 points). For $b = \begin{bmatrix} 6 & 6 & 8 & 8 \end{bmatrix}^T$, determine the optimal solution for $\min_x \|b - Ax\|_2$;

(4) (10 points). Give the orthogonal projector matrix that projects onto $\mathcal{N}(A^T)$.

Probme 4 (25 points). Eigenvalue problems:

(1) (5 points). For real symmetric matrix $A \in \mathbb{R}^{n\times n}$, prove that eigenvectors corresponding to distinct eigenvalues are orthogonal;

(2) (5 points). Given $A \in \mathbb{R}^{n\times n}$, and its eigen-pair $(\lambda,\ v)$. For any $d \in \mathbb{R}^n$, determine the eigen-pair of the matrix $A + vd^T$;

(3) (10 points). Given

$$A = \begin{bmatrix} \frac{1}{3} & \frac{1}{6} \\ -\frac{1}{3} & \frac{1}{3} \end{bmatrix},$$

compute $\lim_{k\to\infty} A^k$;

(4) (5 points). To compute the eigenvalues using the QR iteration, explain the benefit of Hessenberg reduction.

Problem 5 (15 points). Singular value decomposition problems:

(1) (5 points). For $A \in \mathbb{R}^{m\times n}$, if $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n$ are its nonzero singular values, show that

$$\min_{x\in\mathbb{R}^n,\ \|x\|_2=1} \|Ax\|_2 = \sigma_n;$$

(2) (10 points). For a real matrix $A \in \mathbb{R}^{m\times m}$, denote its singular value decomposition by $A = U\Sigma V^T$, determine the singular value decomposition of the matrix

$$\begin{bmatrix} 0 & A^T \\ A & 0 \end{bmatrix}.$$

Bonus (10 points). Show that for a real matrix $A \in \mathbb{R}^{n\times n}$, if $A = -A^T$, then $I - A$ is nonsingular and the matrix $(I - A)^{-1}(I + A)$ is orthogonal. This is known as the *Cayley transform* of A.