SI231 - Matrix Computations, 2022 Fall

Homework Set #1

Prof. Yue Qiu

Acknowledgements:

1) Deadline: **2022-10-08 10:59:59**
2) **Late Policy details** can be found on piazza.
3) Submit your homework in **Homework 1** on **Gradscope**. Entry Code: **4V2N55**. Make sure that you have correctly select pages for each problem. If not, you probably will get 0 point.
4) No handwritten homework is accepted. You need to write LaTeX. (If you have difficulties in using LaTeX, you are allowed to use **MS Word or Pages** for the first and the second homework to accommodate yourself.)
5) Use the given template and give your solution in English. Solution in Chinese is not allowed.
6) Your homework should be uploaded in the PDF format, and the naming format of the file is not specified.

I. Subspace

Problem 1. (Yuhuang Meng, 5 points $\times$ 3) Let $\mathbf{V}$ be the space of all $n \times n$ matrices over $\mathbb{R}$. Which of following sets of matrices $\mathbf{A}$ in $\mathbf{V}$ are subspaces of $\mathbf{V}$?

1) all invertible $\mathbf{A}$.
2) all $\mathbf{A}$ such that $\mathbf{AB} = \mathbf{BA}$, where $\mathbf{B}$ is some fixed matrix in $\mathbf{V}$.
3) all $\mathbf{A}$ such that $\mathbf{A}^2 = \mathbf{A}$.

Solution:

1) This is not a subspace. For instance, $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$. Both $\mathbf{A}$ and $\mathbf{B}$ are invertible, but $\mathbf{A} + \mathbf{B} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is not invertible. The subset is not closed w.r.t. matrix addition. Therefore it could not be a subspace. (5 points)
2) This is a subspace. Suppose $\mathbf{A}_1$ and $\mathbf{A}_2$ satisfy $\mathbf{A}_1\mathbf{B} = \mathbf{B}\mathbf{A}_1$ and $\mathbf{A}_2\mathbf{B} = \mathbf{B}\mathbf{A}_2$. Let $\alpha_1, \alpha_2 \in \mathbb{R}$,
$$(\alpha_1\mathbf{A}_1 + \alpha_2\mathbf{A}_2)\mathbf{B} = \alpha_1\mathbf{A}_1\mathbf{B} + \alpha_2\mathbf{A}_2\mathbf{B} = \mathbf{B}(\alpha_1\mathbf{A}_1) + \mathbf{B}(\alpha_2\mathbf{A}_2) = \mathbf{B}(\alpha_1\mathbf{A}_1 + \alpha_2\mathbf{A}_2)$$
Hence $\alpha_1\mathbf{A}_1 + \alpha_2\mathbf{A}_2$ is in this subset. (5 points)
3) This is not a subspace. Consider the case $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, where $\mathbf{A}^2 = \mathbf{A}$, $\mathbf{B}^2 = \mathbf{B}$. However, $(\mathbf{A} + \mathbf{B})^2 \neq \mathbf{A} + \mathbf{B}$. This subset is not closed under addition. (5 points)

II. Four Fundamental Subspaces

Problem 2. (Yuhuang Meng, 5 points $\times$ 3) Consider two matrices $\mathbf{A} \in \mathbb{R}^{m\times n}$, $\mathbf{B} \in \mathbb{R}^{n\times p}$,

1) what is the relationship between $\mathcal{N}(\mathbf{B})$ and $\mathcal{N}(\mathbf{AB})$? Are they necessarily equal? If yes, prove your statement, otherwise, give a counterexample.
2) if $\mathcal{N}(\mathbf{A}) = \{\mathbf{0}\}$ and $\mathcal{N}(\mathbf{B}) = \{\mathbf{0}\}$, please find $\mathcal{N}(\mathbf{AB})$.
3) if the columns of $\mathbf{A}$ and $\mathbf{B}$ are linearly independent, are the columns of $\mathbf{AB}$ linearly independent as well?

Solution:

1) $\mathcal{N}(\mathbf{B}) \subseteq \mathcal{N}(\mathbf{AB})$

   $\forall \mathbf{x} \in \mathcal{N}(\mathbf{B})$, $\mathbf{Bx} = \mathbf{0}$, then we have $(\mathbf{AB})\mathbf{x} = \mathbf{A}(\mathbf{Bx}) = \mathbf{A0} = \mathbf{0}$, *i.e.*, $\mathbf{x} \in \mathcal{N}(\mathbf{AB})$. (2 points)

   They are not necessarily equal. (1 points) There is a counterexample,

$$\mathbf{A} = \mathbf{B} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \mathbf{AB} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

   We have $\mathcal{N}(\mathbf{B}) = \text{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$, $\mathcal{N}(\mathbf{AB}) = \mathbb{R}^2$. In this example, $\mathcal{N}(\mathbf{B})$ is a strict subset of $\mathcal{N}(\mathbf{AB})$. (2 points)

2) Suppose that $\mathbf{x} \in \mathcal{N}(\mathbf{AB})$, *i.e.*, $\mathbf{ABx} = \mathbf{0}$. This implies that $\mathbf{Bx} \in \mathcal{N}(\mathbf{A})$. Therefore, $\mathbf{Bx} = \mathbf{0}$, this implies that $\mathbf{x} \in \mathcal{N}(\mathbf{B})$. Thus we have $\mathbf{x} = \mathbf{0}$. From this we conclude that $\mathcal{N}(\mathbf{AB}) = \{\mathbf{0}\}$. (5 points)
3) A matrix has linearly independent columns if and only if its nullspace is trivial. Since the columns of $\mathbf{A}$ and $\mathbf{B}$ are linearly independent, we have $\mathcal{N}(\mathbf{A}) = \{\mathbf{0}\}, \mathcal{N}(\mathbf{B}) = \{\mathbf{0}\}$.

   According to the above conclusion, $\mathcal{N}(\mathbf{AB}) = \{\mathbf{0}\}$, which implies that the columns of $\mathbf{AB}$ are linearly independent as well. (5 points)

Problem 3. (Bin Li,15 points)

1) Let $n > 0$ and let A be an $n \times n$ matrix. For all $t \geq 0$, let $\mathcal{N}_t$ be the nullspace of A^t,where by convention $A^0 = 1_{n\times n}$ (identity matrix).Prove that:
   (a) $\mathcal{N}_t \subseteq \mathcal{N}_{t+1}$ for all t.
   (b) The dimension of $\mathcal{N}_t$(the nullity of A^t) is eventually constant, that is there is a number d such that $dim(\boldsymbol{\mathcal{N}}_t) = d$ for all sufficiently large t.
   (c) If T is the least t such that $dim(\mathcal{N}_t) = d$,then $T \leq d$.

Solution:

1) (a) If $v \in \boldsymbol{\mathcal{N}}_t$ then $A^t v = 0$,so $A^{t+1}v = A(A^t v) = A0 = 0 \Longrightarrow v \in \boldsymbol{\mathcal{N}}_{t+1}$,then $\mathcal{N}_t \subseteq \mathcal{N}_{t+1}$ for all t.(3 points)
   (b) $dim(\mathcal{N}_t)$ is an integer,is increasing and bounded above by n,so is eventually constant.(3 points)
   (c) $\mathcal{N}_t \neq \mathcal{N}_{t+1}$ if and only if $dim(\mathcal{N}_t) < dim(\mathcal{N}_{t+1})$.(2 points) If $\mathcal{N}_t = \mathcal{N}_{t+1}$ then we note that

$$v \in \mathcal{N}_{t+2} \Longrightarrow Av \in \mathcal{N}_{t+1} \Longrightarrow Av \in \mathcal{N}_t \Longrightarrow v \in \mathcal{N}_{t+1}$$

   so that $\mathcal{N}_{t+1} = \mathcal{N}_{t+2}$. (3 points)

   It follows that as function of t the number $dim(\mathcal{N}_t)$ is strictly increasing for an initial segment of $\mathbb{N}$,and then becomes constant. Since the eventual value is d,clearly $T \leq d$.(4 points)

Problem 4. (Jianguo Huang.15 points $\times$ 1) In $\mathbb{R}^4$, $V_1 = span < \alpha_1, \alpha_2, \alpha_3 >$, $V_2 = span < \beta_1, \beta_2 >$,where

$$\alpha_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \quad \alpha_2 = \begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \alpha_3 = \begin{bmatrix} 0 \\ 3 \\ 2 \\ 1 \end{bmatrix}, \quad \beta_1 = \begin{bmatrix} 2 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \quad \beta_2 = \begin{bmatrix} 1 \\ -1 \\ 3 \\ 7 \end{bmatrix},$$

then, please find a set of bases and the number of dimension of the subspace $V_1 + V_2$ and the subspace $V_1 \bigcap V_2$.

Solution: $V_1 + V_2 = span < \alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2 >$. So Let

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 0 & 2 & 1 \\ 2 & 1 & 3 & -1 & -1 \\ 1 & 1 & 2 & 0 & 3 \\ 0 & 1 & 1 & 1 & 7 \end{bmatrix}$$

by the elementary row operation for A, we can get a simple matrix, shown following

$$\begin{bmatrix} 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{1}$$

From matrix 1, $< \alpha_1, \alpha_2, \beta_1 >$ is the basis of $V_1 + V_2$ (4 points)and $dim(V_1 + V_2) = 3$(3 points). Simultaneously,from matrix 1, we can get that $\{\alpha_1, \alpha_2\}$ are the basis of V_1 and $\{\beta_1, \beta_2\}$ are the basis of V_2. Then,we have that $dim(V_1) = 2$ and $dim(V_2) = 2$. then ,

$$dim(V_1 \bigcap V_2) = dim(V_1) + dim(V_2) - dim(V_1 + V_2) = 1(4points)$$

Since $dim(V_1 \bigcap V_2) = 1$, finding a basis vector means that finding a nonzero vector $(x_1, x_2, x_3, -1)^T \in V_1 \bigcap V_2$. For $\{\alpha_1, \alpha_2, \beta_1\}$ is the basis of $V_1 + V_2$ and $\beta_2 \in V_2$, there exist a set of nonzero numbers x_1, x_2, x_3 satisfying

$$\beta_2 = x_1\alpha_1 + x_2\alpha_2 + x_3\beta_1$$

by the matrix 1, we can get the solution $(x_1, x_2, x_3) = (-1, 4, 3)$. then, $-\alpha_1 + 4\alpha_2 = -3\beta_1 + \beta_2 \in V_1 \bigcap V_2$. So, we can get that $-\alpha_1 + 4\alpha_2 = (-5, 2, 3, 4)^T$. Finally, the base of $V_1 \bigcap V_2$ is $(-5, 2, 3, 4)^T$.(4 points)

III. Subspace and matrix norm

Problem 5. (Bin Li, 4 points $\times$ 5 + 5 points)

1) Determine whether or not each of the following is a subspace of $\mathbb{R}^2$.Justify your answer.
   (a) $X_1 = \{(x,y) \in \mathbb{R}^2 |\ x+y=0\}$
   (b) $X_2 = \{(x,y) \in \mathbb{R}^2 |\ x-1=0\}$
   (c) $X_3 = \{(x,y) \in \mathbb{R}^2 |\ xy=0\}$
   (d) $X_4 = \{(1,0),(0,1)\}$
   (e) $X_5 = \text{span}\,\{(1,0),(0,1)\}$
2) Is $\|A\|_{\text{max}} = \underset{1\le i,j\le n}{max} |a_{i,j}|$ a matrix norm ? If yes, prove your answer. If no, give a counterexample.

Solution:

1) (a) Yes,X_1 is a subspace.(1 points) Given any $(x,y),(x',y') \in X_1$ and $c \in \mathbb{R}$,we must check that $(cx+x', cy+y') \in X_1$.Indeed,$(cx+x')+(cy+y') = c(x+y)+(x'+y') = c\cdot 0+0 = 0$.(3 points)
   (b) No,X_2 is not a subspace.(1 points)It does not contain $(0,0)$.(It also fails to be closed under addition or scalar multiplication.)(3 points)
   (c) No,X_3 is not a subspace.(1 points)It is not closed under addition:$(1,0) \in X_3$ and $(0,1) \in X_3$,but their sum $(1,1)$ is not in X_3.(3 points)
   (d) No,X_4 is not a subspace.(1 points)It does not contain the zero vector.(It also fails to be closed under addition or scalar multiplication.)(3 points)
   (e) Yes,X_5 is a subspace.(1 points)The span of any set of vectors is always a subspace.(3 points)
2) No,(1 points)take for instance

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \text{ and } B = A^T = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$$

   wherece

$$AB = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}.$$

   Therefore $\|AB\|_{max} = 2$ and $\|A\|_{max}\|B\|_{max} = 1\cdot 1 = 1$,and certainly 2 isn't less than or equal to 1,so $\|\cdot\|_{max}$ isn't submultiplicative and hence isn't a matrix norm.(4 points)

Problem 6. (Jianguo Huang.5 points $\times$ 3)

1) show $||AB||_p \leq ||A||_p||B||_p$ where $1 < p < \infty$. (Hint: It just only be proved by **definition of matrix norm**, i.e. $||A||_p = \max_x \frac{||Ax||_p}{||x||_p} = \max_{||x||_p=1} ||Ax||_p$).
2) let λ is the eigenvalue of matrix A. show $|\lambda| \leq ||A||$ for any matrix norm.
3) $A \in \mathbb{R}^{n\times n}$. if $A^TA = I$, show that $||A||_F = \sqrt{n}$.

Solution:

1) since $||A||_p = \max_x \frac{||Ax||_p}{||x||_p}$,

$$||Ax||_p = ||A\frac{x}{||x||_p}||_p||x||_p \leq \max_x \frac{||Ax||_p}{||x||_p}||x||_P = ||A||_p||x||_p(2points)$$

.

$$||AB||_p = \max_{||x||_p=1} ||ABx||_p \leq \max_{||x||_p=1} ||A||_p||Bx||_p = ||A||_p \max_{||x||_p=1} ||Bx||_p = ||A||_p||B||_p.(3points)$$

2) x is a eigenvector of λ, $\lambda x = Ax$.

$$|\lambda|||x|| = ||\lambda x|| = ||Ax|| \leq ||A||||x||$$

. Finally, $|\lambda| \leq ||A||$

3)

$$||A||_F = \sqrt{tr(A^TA)} = \sqrt{tr(I)} = \sqrt{n}(5points)$$