Numerical Optimization, 2021 Fall
Homework 3 Solution

1 Extreme Points

1. Prove that there is a one-to-one correspondence between the extreme points of the following two sets. [12pts]

$$\begin{aligned} S_1 &= \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}\}, \\ S_2 &= \{(\boldsymbol{x}, \boldsymbol{y}) \in \mathbb{R}^n \times \mathbb{R}^m \mid \boldsymbol{A}\boldsymbol{x} + \boldsymbol{y} = \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}, \boldsymbol{y} \geq \boldsymbol{0}\}. \end{aligned} \tag{1}$$

Please note that the "one-to-one correspondence" here means "given any extreme point of a set, we can construct an extreme point of the other set".

Consider an extreme point of set S_1 and suppose there are k active constraints satisfying $x_i = 0,\ i = 1, \ldots, k$, then there are $n - k$ $(\leq m)$ active constraints satisfying $\boldsymbol{A}_j^T \boldsymbol{x} = b_j,\ j = 1, \ldots, n - k$. For set S_2, let $y_i = 0,\ i = 1, \ldots, n - k$ and $y_i = b_i - \boldsymbol{A}_i^T \boldsymbol{x} \geq 0$ for the remaining. Now, the constraint $\boldsymbol{A}\boldsymbol{x} + \boldsymbol{y} = \boldsymbol{b}$ is satisfied, which provides m active constraints, and there are k active constraints in $\boldsymbol{x} \geq \boldsymbol{0}$ as well as $n - k$ active constraints in $\boldsymbol{y} \geq \boldsymbol{0}$. Thus, there are $m + k + (n - k) = m + n$ active constraints in total, which constructs an extreme point of set S_2.

On the other hand, for an extreme point of set S_2, there are k active constraints satisfying $x_i = 0,\ i = 1, \ldots, k$ and there are $n - k$ active constraints satisfying $y_j = 0,\ j = 1, \ldots, n - k$. Hence, there are n active constraints satisfied in S_1, which makes up an extreme point of set S_1.

2. Given set $P = \{\boldsymbol{x} \in \mathbb{R}^2 \mid 0 \leq x_1 \leq 1\}$ illustrated by Figure 1, answer the following questions

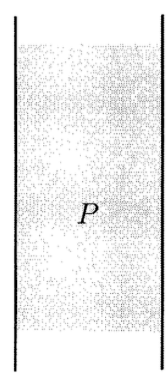

图 1: Set P

(1) Does set P have extreme point(s)? [3pts]

No. P contains a line so it doesn't have extreme point.

(2) What is the standard form of P? [5pts]

We can write $\boldsymbol{x}$ as $(x_1, x_2)^T$, where x_2 is a free variable. Thus, we can replace x_2 with two positive variables x_2^+ and x_2^-, and introduce a slack variable s to the inequality containing x_1. The standard form of P is

$$P = \{\boldsymbol{x} = (x_1, x_2^+, x_2^-, s) \in \mathbb{R}^4 \mid x_1 + s = 1, x_1, x_2^+, x_2^-, s \geq 0\} \tag{2}$$

(3) Does the standard form of P have extreme point(s)? If so, give an example and explain why it is an extreme point. [5pts]

Yes, one example would be $\bar{\boldsymbol{x}} = (1, 0, 0, 0)^T$. Explanation:

(a) All equalities (Only $x_1 + s = 1$ in this problem) are active;

(b) Out of the active constraints ($x_1 + s = 1$, $x_2^+ = 0$, $x_2^- = 0$, $s = 0$), four of them are linearly independent;

(c) All constraints are satisfied.

Therefore, $\bar{\boldsymbol{x}}$ is a basic feasible solution, which means it is an extreme point as well.

2 Pivot

Show that if the vectors $\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_m$ are a basis in E^m, the vectors $\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_{p-1}, \boldsymbol{a}_q, \boldsymbol{a}_{p+1}, \ldots, \boldsymbol{a}_m$ also are a basis if and only if $\bar{a}_{pq} \neq 0$, where $\bar{a}_{pq}$ is defined by the tableau shown in Table 1. [10pts]

x_1	x_2	x_3	$\cdots$	x_m	x_{m+1}	x_{m+2}	$\cdots$	x_n	
1	0	0	$\cdots$	0	$\bar{a}_{1(m+1)}$	$\bar{a}_{1(m+2)}$	$\cdots$	$\bar{a}_{1n}$	$\bar{a}_{10}$
0	1	0	$\cdots$	0	$\bar{a}_{2(m+1)}$	$\bar{a}_{2(m+2)}$	$\cdots$	$\bar{a}_{2n}$	$\bar{a}_{20}$
0	0	1	$\cdots$	0	$\bar{a}_{3(m+1)}$	$\bar{a}_{3(m+2)}$	$\cdots$	$\bar{a}_{3n}$	$\bar{a}_{30}$
.	.	.		.	.	.		.	.
.	.	.		.	.	.		.	.
.	.	.		.	.	.		.	.
0	0	0	$\cdots$	1	$\bar{a}_{m(m+1)}$	$\bar{a}_{m(m+2)}$	$\cdots$	$\bar{a}_{mn}$	$\bar{a}_{m0}$

表 1: Tableau

This proof is equivalent to proving the fact that the linear independence condition holds if and only if $\bar{a}_{pq} \neq 0$.

1. If $\bar{a}_{pq} \neq 0$, the vectors $\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_{p-1}, \boldsymbol{a}_q, \boldsymbol{a}_{p+1}, \ldots, \boldsymbol{a}_m$ are linearly independent:
   - The proof is provided on the lecture slides (Lecture 3, Page 19)
2. If the vectors $\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_{p-1}, \boldsymbol{a}_q, \boldsymbol{a}_{p+1}, \ldots, \boldsymbol{a}_m$ are linearly independent, $\bar{a}_{pq} \neq 0$:
   - This can be proved by proving its contrapositive, i.e. if $\bar{a}_{pq} = 0$, the vectors $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_q, \ldots, \boldsymbol{a}_m$ are linearly dependent. The proof can be derived from the following equation

$$
\begin{aligned}
\boldsymbol{a}_q = \boldsymbol{B}\boldsymbol{y}_q &= \sum_{i=1}^{m} \bar{a}_{iq}\boldsymbol{a}_i \\
&= \bar{a}_{1q}\boldsymbol{a}_1 + \cdots + \bar{a}_{(p-1)q}\boldsymbol{a}_{p-1} + \boxed{0 \cdot \boldsymbol{a}_q} + \bar{a}_{(p+1)q}\boldsymbol{a}_{p+1} + \cdots + \bar{a}_{mq}\boldsymbol{a}_m
\end{aligned} \tag{3}
$$

3 Reduced Cost

If $r_j > 0$ for every j corresponding to a variable x_j that is not basic, show that the corresponding basic feasible solution is the unique optimal solution. [10pts]

We first show that the BFS is optimal and then show its uniqueness.

1. Because $r_j \geq 0$ for every j corresponding to a nonbasic variable, the corresponding basic feasible solution is an optimal solution by the optimal criterion listed on lecture slides (Lecture 3, Page 11).
2. Since $r_j > 0$, change the basis will strictly increase the objective function value. Therefore, the optimal solution is unique.

4 Two-Phase Simplex

Use the two-phase simplex procedure to solve the following problem [10pts]

$$
\begin{aligned}
\min \quad & -3x_1 + x_2 + 3x_3 - x_4 \\
\text{s.t.} \quad & x_1 + 2x_2 - x_3 + x_4 = 0 \\
& 2x_1 - 2x_2 + 3x_3 + 3x_4 = 9 \\
& x_1 - x_2 + 2x_3 - x_4 = 6 \\
& x_i \geq 0, \quad i = 1, 2, 3, 4.
\end{aligned} \tag{4}
$$

For **phase I**, we add artificial variables x_5, x_6, x_7 and change the objective function to $x_5 + x_6 + x_7$. The resulting tableau is

$$
\begin{array}{ccccccc|c}
1 & 2 & -1 & 1 & 1 & 0 & 0 & 0 \\
2 & -2 & 3 & 3 & 0 & 1 & 0 & 9 \\
1 & -1 & 2 & -1 & 0 & 0 & 1 & 6 \\
\hline
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0
\end{array}
$$

This does not represent a basic feasible solution yet because we need zeros under the basic columns (we need to three 1's in the bottom row to be 0). Using elementary row operations, we can eliminate the unwanted 1's. Notice that this is identical to pivoting on the 1's in the artificial columns. The resulting basic feasible tableau is:

$$
\begin{array}{ccccccc|c}
\boxed{1} & 2 & -1 & 1 & 1 & 0 & 0 & 0 \\
2 & -2 & 3 & 3 & 0 & 1 & 0 & 9 \\
1 & -1 & 2 & -1 & 0 & 0 & 1 & 6 \\
\hline
-4 & 1 & -4 & -3 & 0 & 0 & 0 & -15
\end{array}
$$

Using the simplex method, we have

$$
\begin{array}{ccccccc|c}
1 & 2 & -1 & 1 & 1 & 0 & 0 & 0 \\
0 & -6 & \boxed{5} & 1 & -2 & 1 & 0 & 9 \\
0 & -3 & 3 & -2 & -1 & 0 & 1 & 6 \\
\hline
0 & 9 & -8 & 1 & 4 & 0 & 0 & -15
\end{array}
$$

$$\downarrow$$

$$
\begin{array}{ccccccc|c}
1 & 4/5 & 0 & 6/5 & 3/5 & 1/5 & 0 & 9/5 \\
0 & -6/5 & 1 & 1/5 & -2/5 & 1/5 & 0 & 9/5 \\
0 & \boxed{3/5} & 0 & -13/5 & 1/5 & -3/5 & 1 & 3/5 \\
\hline
0 & -3/5 & 0 & 13/5 & 4/5 & 8/5 & 0 & -3/5
\end{array}
$$

$$\downarrow$$

$$
\begin{array}{ccccccc|c}
1 & 0 & 0 & 14/3 & 1/3 & 1 & -4/3 & 1 \\
0 & 0 & 1 & -5 & 0 & -1 & 2 & 3 \\
0 & 1 & 0 & -13/3 & 1/3 & -1 & 5/3 & 1 \\
\hline
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0
\end{array}
$$

which shows the final phase I tableau.

. . . .

For **phase II**, we replace the last row with the true objective function. Zero out the elements under the basic variables (same as pivoting on elements $(1,1)$; $(3,2)$ and $(2,3)$). The resulting tableau is (without artificial variables)

$$
\begin{array}{cccc|c}
1 & 0 & 0 & 14/3 & 1 \\
0 & 0 & 1 & -5 & 3 \\
0 & 1 & 0 & -13/3 & 1 \\
\hline
-3 & 1 & 3 & -1 & 0
\end{array}
$$

$$\downarrow$$

$$
\begin{array}{cccc|c}
1 & 0 & 0 & 14/3 & 1 \\
0 & 0 & 1 & -5 & 3 \\
0 & 1 & 0 & -13/3 & 1 \\
\hline
0 & 0 & 0 & 97/3 & -7
\end{array}
$$

Since the last row is nonnegative, we are done. The solution is $(x_1, x_2, x_3, x_4) = (1, 1, 3, 0)$. Value of objective function is $-(-7) = 7$.

5 Big-M Method

It is possible to combine the two phases of the two-phase method into a single procedure by the *big-M method*. Given the linear program in standard form

$$\begin{aligned} \min \quad & \boldsymbol{c}^T\boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}, \end{aligned} \tag{5}$$

one forms the approximating problem

$$\begin{aligned} \min \quad & \boldsymbol{c}^T\boldsymbol{x} + M\sum_{i=1}^{m} y_i \\ \text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} + \boldsymbol{y} = \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0}, \boldsymbol{y} \geq \boldsymbol{0}. \end{aligned} \tag{6}$$

In this problem, $\boldsymbol{y} = (y_1, y_2, \ldots, y_m)^T$ is a vector of artificial variables and M is a large constant. The term $M\sum_{i=1}^{m} y_i$ serves as a penalty term for nonzero y_i's.
If this problem is solved by the simplex method, show the following:

(1) If an optimal solution is found with $\boldsymbol{y} = \boldsymbol{0}$, then the corresponding $\boldsymbol{x}$ is an optimal basic feasible solution to the original problem. [5pts]
Let $(\boldsymbol{x}, \boldsymbol{0})$ be an optimal solution to the approximating problem. First note that $\boldsymbol{x}$ is a basic feasible solution to the original problem. Now suppose $\boldsymbol{x}$ is not an optimal solution to the original problem, then there exists $\boldsymbol{x}' \geq \boldsymbol{0}$ such that $\boldsymbol{A}\boldsymbol{x}' = \boldsymbol{b}$ and $\boldsymbol{c}^T\boldsymbol{x}' < \boldsymbol{c}^T\boldsymbol{x}$. Clearly, $(\boldsymbol{x}', \boldsymbol{0})$ is a feasible solution to the approximating problem and $\boldsymbol{c}^T\boldsymbol{x}' + 0M < \boldsymbol{c}^T\boldsymbol{x} + 0M$, which contradicts to the assumption that $(\boldsymbol{x}, \boldsymbol{0})$ is an optimal solution to the approximating problem.

(2) If for every $M > 0$ an optimal solution is found with $\boldsymbol{y} \neq \boldsymbol{0}$, then the original problem is infeasible. [10pts]
There is a finite number of basic feasible solutions that contain some of the y_j variables. For every such basic feasible solution, the objective function grows linearly in M (since the part of the function dependent on $\boldsymbol{x}$ is independent of M). For basic feasible solutions that do not contain any y_j variables, the objective function is independent of M. Hence, if for every $M > 0$, an optimal solution is found with $\boldsymbol{y} \neq \boldsymbol{0}$, then the optimal solution grows linearly with M. In particular, by taking the minimum over all basic feasible solutions, we can state that it grows as $\alpha M + \beta$ for some $\alpha > 0$. This directly implies that the original problem is infeasible. Indeed, if it were feasible, the corresponding value of the objective function would provide an upper bound on the optimum. However, the optimum is at least $\alpha M + \beta$ for all M, a contradiction.

(3) If for every $M > 0$ the approximating problem is unbounded, then the original problem is either unbounded or infeasible. [10pts]
The approximating problem is unbounded for all $M > 0$, so for all $M > 0$ there exist $\boldsymbol{u}$, $\boldsymbol{v}$ such that

$$\begin{aligned} \boldsymbol{c}^T\boldsymbol{u} + M\boldsymbol{e}^T\boldsymbol{v} < 0 \\ \boldsymbol{A}\boldsymbol{u} + \boldsymbol{v} = \boldsymbol{0} \\ \boldsymbol{u}, \boldsymbol{v} \geq \boldsymbol{0} \end{aligned} \tag{7}$$

Since it is a homogeneous problem we do not change feasibility by adding constraints $\boldsymbol{u} \leq \boldsymbol{e}$, $\boldsymbol{v} \leq \boldsymbol{e}$ (this can be done by scaling variables) and consider the problem

$$
\begin{aligned}
\min \quad & \boldsymbol{c}^T\boldsymbol{u} + M\boldsymbol{e}^T\boldsymbol{v} \\
\text{s.t.} \quad & \boldsymbol{A}\boldsymbol{u} + \boldsymbol{v} = \boldsymbol{0} \\
& \boldsymbol{u} \geq \boldsymbol{0}, \boldsymbol{v} \geq \boldsymbol{0} \\
& \boldsymbol{u} \leq \boldsymbol{e}, \boldsymbol{v} \leq \boldsymbol{e}
\end{aligned} \tag{8}
$$

This is a bounded problem since the variables are constrained to the unit cube. Hence, there is a finite optimum attained at a basic feasible solution. Note that objective function values for all basic feasible solutions containing nonzero $\boldsymbol{v}$ are positive for sufficiently large M (since there is a finite number of them and the coefficients before M are all positive). Hence, if the approximating problem is unbounded for all $M > 0$, the latter problem must have a solution with $\boldsymbol{v} = \boldsymbol{0}$, which we denote by $(\boldsymbol{u}^*, \boldsymbol{0})$. If the original problem is feasible, consider a feasible point $\boldsymbol{x}^*$. One sees that $\boldsymbol{c}^T(\boldsymbol{x}^* + t\boldsymbol{u}^*) \to -\infty$ as $t \to \infty$. Hence, the problem is unbounded. We have shown that the original problem is either infeasible or unbounded.

For another reference, see page 60 and 61 of the textbook 最优化理论与算法.

(4) Suppose now that the original problem has a finite optimal value $V(\infty)$. Let $V(M)$ be the optimal value of the approximating problem. Show that $V(M) \leq V(\infty)$. [10pts]

Since the set of feasible solutions for the original problem is a subset of the set of feasible solutions of the approximating problem, and the objective function of the approximating problem, when restricted to the set of feasible solutions of the original problem, is the same as that for the original problem, we have this relation between the minimum: $V(M) \leq V(\infty)$.

(5) Show that for $M_1 \leq M_2$ we have $V(M_1) \leq V(M_2)$. [5pts]

It is enough to verify that for every basic feasible solution $(\boldsymbol{x}, \boldsymbol{y})$, $\boldsymbol{c}^T\boldsymbol{x} + M_1\boldsymbol{e}^T\boldsymbol{y} \leq \boldsymbol{c}^T\boldsymbol{x} + M_2\boldsymbol{e}^T\boldsymbol{y}$, which is obvious (the decisive fact here is that the sets of basic feasible solutions are the same for both problems).

(6) Show that there is a value M_0 such that for $M \geq M_0$, $V(M) = V(\infty)$, and hence conclude that the big-M method will produce the right solution for large enough values of M. [10pts]

This is the same as the proof of (2). Indeed, consider basic feasible solutions that contain y_j in the basis. As discussed above, the minimal value of the objective function over these points grows as $\alpha M + \beta$ for some $\alpha > 0$. Hence, for large enough M, any solution that contains $\boldsymbol{y}$ in the basis has a larger objective function value than the optimum.