SI231b: Matrix Computations

Lecture 2: Basic Concepts (Part 1)

邱越

qiuyue@shanghaitech.edu.cn

School of Information Science and Technology
ShanghaiTech University

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Basic Concepts: Part 1

- notation and conventions
- vector spaces
- vector norms and matrix norms
- subspaces

Notation and Conventions

$\mathbb{R}$	the set of real numbers, or real space
$\mathbb{C}$	the set of complex numbers, or complex space
$\mathbb{R}^n$	n-dimensional real space
$\mathbb{C}^n$	n-dimensional complex space
$\mathbb{R}^{m\times n}$	set of all $m \times n$ real-valued matrices
$\mathbb{C}^{m\times n}$	set of all $m \times n$ complex-valued matrices
$\mathbf{x}$	column vector
$x_i, [\mathbf{x}]_i$	ith entry of $\mathbf{x}$
$\mathbf{A}$	matrix
$a_{ij}, [\mathbf{A}]_{ij}$	(i, j)th entry of $\mathbf{A}$

Notation and Conventions

- vector: $\mathbf{x} \in \mathbb{R}^n$ means that $\mathbf{x}$ is a real-valued n-dimensional column vector, i.e.,

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \qquad x_i \in \mathbb{R} \text{ for all } i.$$

  Similarly, $\mathbf{x} \in \mathbb{C}^n$ means that $\mathbf{x}$ is a complex-valued n-dimensional column vector.

- transpose: let $\mathbf{x} \in \mathbb{R}^n$. The notation $\mathbf{x}^T$ means that

$$\mathbf{x}^T = \begin{bmatrix} x_1, & x_2, & \ldots, & x_n \end{bmatrix}.$$

- Hermitian transpose: let $\mathbf{x} \in \mathbb{C}^n$. The notation $\mathbf{x}^H$ means that

$$\mathbf{x}^H = \begin{bmatrix} x_1^*, & x_2^*, & \ldots, & x_n^* \end{bmatrix},$$

  where the superscript $*$ denotes the complex conjugate.

Notation and Conventions

- matrix: $\mathbf{A} \in \mathbb{R}^{m\times n}$ means that $\mathbf{A}$ is real-valued $m \times n$ matrix

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}, \qquad a_{ij} \in \mathbb{R} \text{ for all } i, j.$$

  Similarly, $\mathbf{A} \in \mathbb{C}^{m\times n}$ means that $\mathbf{A}$ is a complex-valued $m \times n$ matrix.

- unless specified, we denote the ith column of a matrix $\mathbf{A} \in \mathbb{R}^{m\times n}$ as $\mathbf{a}_i \in \mathbb{R}^m$, i.e.,

$$\mathbf{A} = \begin{bmatrix} \mathbf{a}_1, & \mathbf{a}_2, & \dots, & \mathbf{a}_n \end{bmatrix}.$$

  The same notation applies to $\mathbf{A} \in \mathbb{C}^{m\times n}$.

- **transpose:** let $\mathbf{A} \in \mathbb{R}^{m \times n}$. The notation $\mathbf{A}^T$ means that

$$\mathbf{A}^T = \begin{bmatrix} a_{11} & a_{21} & \dots & a_{m1} \\ a_{12} & a_{22} & \dots & a_{m2} \\ \vdots & & & \vdots \\ a_{1n} & a_{m2} & \dots & a_{mn} \end{bmatrix} \in \mathbb{R}^{n \times m}.$$

  - or, we have $\mathbf{B} = \mathbf{A}^T \iff b_{ij} = a_{ji}$ for all i, j.
  - properties:
    - $(\mathbf{AB})^T = \mathbf{B}^T \mathbf{A}^T$
    - $(\mathbf{A}^T)^T = \mathbf{A}$
    - $(\mathbf{A} + \mathbf{B})^T = \mathbf{A}^T + \mathbf{B}^T$

Notation and Conventions

- Hermitian transpose: let $\mathbf{A} \in \mathbb{C}^{m \times n}$. The notation $\mathbf{A}^H$ means that

$$\mathbf{A}^H = \begin{bmatrix} a_{11}^* & a_{21}^* & \dots & a_{m1}^* \\ a_{12}^* & a_{22}^* & \dots & a_{m2}^* \\ \vdots & & & \vdots \\ a_{1n}^* & a_{m2}^* & \dots & a_{mn}^* \end{bmatrix} \in \mathbb{C}^{n \times m}.$$

  - or, we have $\mathbf{B} = \mathbf{A}^H \Longleftrightarrow b_{ij} = a_{ji}^*$ for all $i,\ j$.
  - properties (the same as transpose):
    - $(\mathbf{A}\mathbf{B})^H = \mathbf{B}^H \mathbf{A}^H$
    - $(\mathbf{A}^H)^H = \mathbf{A}$
    - $(\mathbf{A} + \mathbf{B})^H = \mathbf{A}^H + \mathbf{B}^H$

Notation and Conventions

- trace: let $\mathbf{A} \in \mathbb{R}^{n \times n}$, the trace of $\mathbf{A}$ is defined by

$$\operatorname{tr}(\mathbf{A}) = \sum_{i=1}^{n} a_{ii}.$$

  - properties:
    - $\operatorname{tr}\left(\mathbf{A}^T\right) = \operatorname{tr}(\mathbf{A})$
    - $\operatorname{tr}(\mathbf{A} + \mathbf{B}) = \operatorname{tr}(\mathbf{A}) + \operatorname{tr}(\mathbf{B})$
    - $\operatorname{tr}(\mathbf{AB}) = \operatorname{tr}(\mathbf{BA})$ for $\mathbf{A}, \mathbf{B}$ of appropriate sizes

- matrix power: let $\mathbf{A} \in \mathbb{R}^{n \times n}$, then the notation $\mathbf{A}^2$ means $\mathbf{A}^2 = \mathbf{AA}$, and $\mathbf{A}^k$ means

$$\mathbf{A}^k = \underbrace{\mathbf{AA} \cdots \mathbf{A}}_{k \ \mathbf{A}\text{'s}}.$$

- all-ones vector: we use the notation

$$\mathbf{1} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$$

to denote a vector of all 1's.

- zero vectors or matrices: we use the notation $\mathbf{0}$ to denote either a vector of all zeros, or a matrix of all zeros.

- unit vectors: unit vectors are vectors that have only one nonzero element and the nonzero element is 1. We use the notation

$$\mathbf{e}_i = \begin{bmatrix} 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \end{bmatrix}^T$$

to denote a unit vector with the nonzero element at the ith entry.

- identity matrix:

$$\mathbf{I} = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix},$$

where, as a convention, the empty entries are assumed to be zero.

- diagonal matrices: we use the notation

$$\mathrm{diag}(a_1, \ldots, a_n) = \begin{bmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{bmatrix}$$

to denote a diagonal matrix with diagonals $a_1, \ldots, a_n$. We also use the shorthand notation $\mathrm{diag}(\mathbf{a}) = \mathrm{diag}(a_1, \ldots, a_n)$.

Notation and Conventions

- The matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$ is said to be
  - square if $m = n$;
  - 'tall' if $m > n$;
  - 'fat' if $m < n$.
- The matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$ is said to be
  - upper triangular if $a_{ij} = 0$ for all $i > j$;
  - strictly upper triangular if $a_{ij} = 0$ for all $i \geq j$;
  - lower triangular if $a_{ij} = 0$ for all $i < j$;
  - strictly lower triangular if $a_{ij} = 0$ for all $i \leq j$.

Examples:

$$\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 2 & 0 \\ \frac{1}{8} & 3 & 2 \end{bmatrix}, \qquad \mathbf{U} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}.$$

Vector Space

A vector space involves four things

- two sets $\mathcal{V}$ and $\mathcal{F}$
- two algebraic operations, i.e., vector addition and scalar multiplication;

where

- $\mathcal{V}$ is a nonempty set of objects called **vectors**;
- $\mathcal{F}$ is a scalar field, for this course, either $\mathbb{R}$ or $\mathbb{C}$;
- vector addition, denoted by $\mathbf{x} + \mathbf{y}$ for $\mathbf{x},\ \mathbf{y} \in \mathcal{V}$;
- scalar multiplication, denoted by $\alpha\mathbf{x}$ for $\alpha \in \mathcal{F}$ and $\mathbf{x} \in \mathcal{V}$.

Axiomatic Definition of a Vector Space

The set $\mathcal{V}$ is called a vector space over $\mathcal{F}$ when the vector addition and scalar multiplication operations satisfy the following properties

(A1) $\mathbf{x} + \mathbf{y} \in \mathcal{V}$ for all $x, \ y \in \mathcal{V}$, closure for vector addition

(A2) $(\mathbf{x} + \mathbf{y}) + \mathbf{z} = \mathbf{x} + (\mathbf{y} + \mathbf{z})$ for every $\mathbf{x}, \ \mathbf{y}, \ \mathbf{z} \in \mathcal{V}$

(A3) $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$ for every $\mathbf{x}, \ \mathbf{y} \in \mathcal{V}$

(A4) There is an element $\mathbf{0} \in \mathcal{V}$ such that $\mathbf{x} + \mathbf{0} = \mathbf{x}$ for every $\mathbf{x} \in \mathcal{V}$

(A5) $\forall \mathbf{x} \in \mathcal{V}$, $\exists (-\mathbf{x}) \in \mathcal{V}$ such that $\mathbf{x} + (-\mathbf{x}) = \mathbf{0}$

(M1) $\alpha \mathbf{x} \in \mathcal{V}$ for $\forall \alpha \in \mathcal{F}$ and $\mathbf{x} \in \mathcal{V}$, closure for scalar multiplication

(M2) $(\alpha\beta)\mathbf{x} = \alpha(\beta\mathbf{x})$ for $\forall \ \alpha, \ \beta \in \mathcal{F}$ and every $\mathbf{x} \in \mathcal{V}$

(M3) $\alpha(\mathbf{x} + \mathbf{y}) = \alpha\mathbf{x} + \alpha\mathbf{y}$ for every $\alpha \in \mathcal{F}$ and $\forall \ \mathbf{x}, \ \mathbf{y} \in \mathcal{V}$

(M4) $(\alpha + \beta)\mathbf{x} = \alpha\mathbf{x} + \beta\mathbf{x}$ for $\forall \alpha, \ \beta \in \mathcal{F}$ and every $\mathbf{x} \in \mathcal{V}$

(M5) $1\mathbf{x} = \mathbf{x}$ for every $\mathbf{x} \in \mathcal{V}$

Examples of Vector Space

- The set $\mathbb{R}^{m\times n}$ of $m \times n$ real matrices is a vector space over $\mathbb{R}$;
- The set $\mathbb{C}^{m\times n}$ of $m \times n$ complex matrices is a vector space over $\mathbb{C}$;
- The real coordinate spaces

$$\mathbb{R}^n = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \ x_i \in \mathbb{R} \right\},$$

will be the main focus of our course.

Vector Norms

Given an n-dimensional vector $\mathbf{x} \in \mathbb{R}^n$, the non-negative norm $\| \cdot \| : \mathbb{R}^n \to \mathbb{R}$ is defined such that

1. $\|\mathbf{x}\| > 0$ when $\mathbf{x} \neq \mathbf{0}$ and $\|\mathbf{x}\| = 0 \iff \mathbf{x} = \mathbf{0}$
2. $\|k\mathbf{x}\| = |k|\|\mathbf{x}\|$, $\forall k \in \mathbb{R}$
3. $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$, $\forall\ \mathbf{x},\ \mathbf{y} \in \mathbb{R}^n$

- used to measure the length of a vector;
- also used to measure the distance of two vectors, specifically, via $\|\mathbf{x} - \mathbf{y}\|$ where $\mathbf{x},\ \mathbf{y}$ are two vectors.

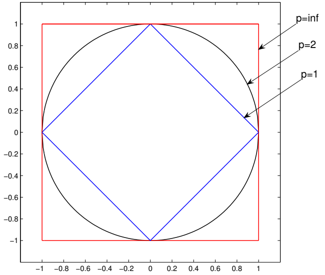
Examples of norm:

- 2-norm or Euclidean norm: $\|\mathbf{x}\|_2 = \sqrt{\sum_{i=1}^{n} |x_i|^2} = (\mathbf{x}^T\mathbf{x})^{1/2}$
- 1-norm, taxicab norm or Manhattan norm: $\|\mathbf{x}\|_1 = \sum_{i=1}^{n} |x_i|$
- ∞-norm: $\|\mathbf{x}\|_\infty = \max_{i=1,\ldots,n} |x_i|$
- p-norm, $p \geq 1$: $\|\mathbf{x}\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}}$

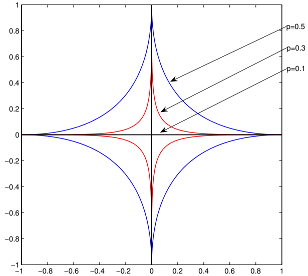
ℓ_p Functions

Let

$$f_p(\mathbf{x}) = \left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}}, \qquad p > 0.$$



(a) Contour plot of $f_p(\mathbf{x}) = 1,\ p \geq 1$



(b) Contour plot of $f_p(\mathbf{x}) = 1,\ p < 1$

- f_p is *not* a norm for $0 < p < 1$, triangular inequality does not hold
- ℓ_1 norm (1-norm) is widely used to get sparse results (cf. Figure(a))

Matrix Norms

Similar as the vector norm, a matrix norm is a function $\|\cdot\|: \mathbb{R}^{m\times n} \to \mathbb{R}$ that satisfies

1. $\|\mathbf{A}\| > 0$ and $\|\mathbf{A}\| = 0 \iff \mathbf{A} = \mathbf{0}$
2. $\|k\mathbf{A}\| = |k|\|\mathbf{A}\|, \forall k \in \mathbb{R}$
3. $\|\mathbf{A}+\mathbf{B}\| \leq \|\mathbf{A}\| + \|\mathbf{B}\|, \forall\ \mathbf{A},\ \mathbf{B} \in \mathbb{R}^{m\times n}$
4. $\|\mathbf{AB}\| \leq \|\mathbf{A}\|\|\mathbf{B}\|$ for all conformable matrices

Naturally, the Frobenius norm given by

$$\begin{aligned}
\|A\|_F^2 &= \sum_{i=1}^{m}\sum_{j=1}^{n}|a_{ij}|^2 \\
&= \sum_{i=1}^{m}\|\mathbf{A}_{i*}\|_2^2 \\
&= \sum_{j=1}^{n}\|\mathbf{A}_{*j}\|_2^2 \\
&= \mathbf{tr}(\mathbf{A}^T\mathbf{A})
\end{aligned}$$

satisfies the above definition. Are there more?

Induced by the aforementioned vector p-norm,

$$\|\mathbf{A}\|_p = \max_{\mathbf{x}\neq\mathbf{0}} \frac{\|\mathbf{A}\mathbf{x}\|_p}{\|\mathbf{x}\|_p} = \max_{\|\mathbf{x}\|_p=1} \|\mathbf{A}\mathbf{x}\|_p$$

- $p = 1$

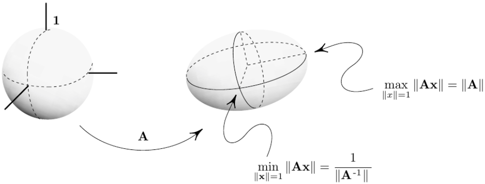
$$\|\mathbf{A}\|_1 = \max_{\|\mathbf{x}\|_1=1} \|\mathbf{A}\mathbf{x}\|_1 = \max_j \sum_{i=1}^{m} |a_{ij}|$$

$$= \text{the largest absolute column sum.}$$

How to prove?

- $p = 2$

$$\|\mathbf{A}\|_2 = \max_{\|\mathbf{x}\|_2=1} \|\mathbf{A}\mathbf{x}\|_2$$



  - also called spectral norm
  - details will be discussed later in this course

- $p = \infty$

$$\|\mathbf{A}\|_\infty = \max_{\|\mathbf{x}\|_\infty=1} \|\mathbf{A}\mathbf{x}\|_\infty = \max_i \sum_{j=1}^{n} |a_{ij}|$$

$$= \text{the largest absolute row sum.}$$

Hands-on exercise

Let $\mathcal{S}$ be a nonempty subset of a vector space $\mathcal{V}$ over $\mathcal{F}$ (i.e., $\mathcal{S} \subset \mathcal{V}$). If $\mathcal{S}$

- is also a vector space over $\mathcal{F}$,
- uses the same addition and scalar multiplication operations

then $\mathcal{S}$ is said to be a subspace of $\mathcal{V}$

Simplified definition

A nonempty subset $\mathcal{S}$ of a vector space $\mathcal{V}$ is a subspace of $\mathcal{V}$ if and only if

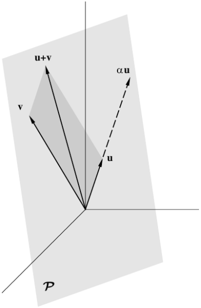
1. $\mathbf{x},\ \mathbf{y} \in \mathcal{S} \Longrightarrow \mathbf{x} +\ \mathbf{y} \in \mathcal{S}$
2. $\mathbf{x} \in \mathcal{S} \Longrightarrow \alpha\mathbf{x} \in \mathcal{S}$ for $\forall \alpha \in \mathcal{F}$

- trivial subspace

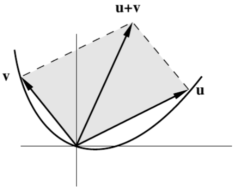
  Given a vector space $\mathcal{V}$, the set $\mathcal{Z} = \{\mathbf{0}\}$ and $\mathcal{V}$ are trivial subspaces.

- nontrivial subspace

  contains at least one nonzero vector



Subspace in $\mathbb{R}^3$



subspace or not?

The curve does not represent a subspace, why?

- no closure for addition

Note: any subspace must contain the trivial subspace $\mathcal{Z} = \{\mathbf{0}\}$, i.e., the zero vector $\{\mathbf{0}\}$

Subspaces and Spanning Sets

Obvious interpretation of subspaces in the visual spaces $\mathbb{R}^2$ and $\mathbb{R}^3$

- flat surfaces passing through the origin

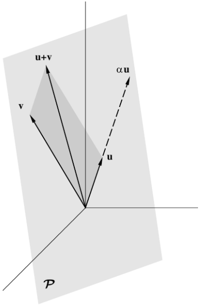
Spanning sets

- for a set of vectors $\mathcal{S} = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_r\}$, the subspaces

$$\mathit{span}(\mathcal{S}) = \{\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \cdots + \alpha_r \mathbf{v}_r\}$$

  generated by forming all linear combinations of vectors from $\mathcal{S}$ is called the space spanned by $\mathcal{S}$

- If $\mathcal{V}$ is a vector space such that $\mathcal{V} = \mathit{span}(\mathcal{S})$, we say $\mathcal{S}$ is a spanning set for $\mathcal{V}$.



- $\mathcal{S} = \{\mathbf{u}, \mathbf{v}\}$ is a spanning set of the indicated plan in the left figure
- The unit vectors

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

spans $\mathbb{R}^3$

If $\mathcal{X}$ and $\mathcal{Y}$ are subspaces of a vector space $\mathcal{V}$,

- $\mathcal{X} \cap \mathcal{Y}$ is also a subspace
- $\mathcal{X} \cup \mathcal{Y}$ need not be a subspace

If $\mathcal{X}$ and $\mathcal{Y}$ are subspaces of a vector space $\mathcal{V}$, define the sum of two subspaces by

$$\mathcal{X} + \mathcal{Y} = \{\mathbf{x} + \mathbf{y} | \mathbf{x} \in \mathcal{X} \text{ and } \mathbf{y} \in \mathcal{Y}\}$$

then

- the sum $\mathcal{X} + \mathcal{Y}$ is again a subspace of $\mathcal{V}$
- if $\mathcal{S}_X$, $\mathcal{S}_Y$ spans $\mathcal{X}$ and $\mathcal{Y}$, then $\mathcal{S}_X \cup \mathcal{S}_Y$ spans $\mathcal{X} + \mathcal{Y}$

Examples

- If $\mathcal{X} \subset \mathbb{R}^2$ and $\mathcal{Y} \subset \mathbb{R}^2$ are subspaces defined by two different lines through the origin, then $\mathcal{X} + \mathcal{Y} = \mathbb{R}^2$
- If $\mathcal{X}$ is a subspace represents a plane passing through the origin in $\mathbb{R}^3$ and $\mathcal{Y}$ is a subspace defined by the line through the origin that is perpendicular to $\mathcal{X}$, $\mathcal{X} + \mathcal{Y} = \mathbb{R}^3$

Readings

- Gene H. Golub and Charles F. Van Loan. *Matrix Computations*, Johns Hopkins University Press, 2013.

  Chapter 1, 2.1 – 2.3