

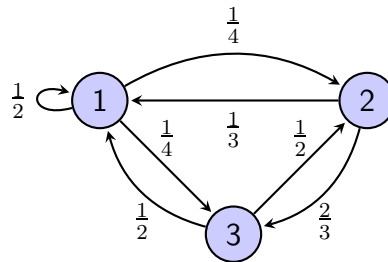
## Homework 12

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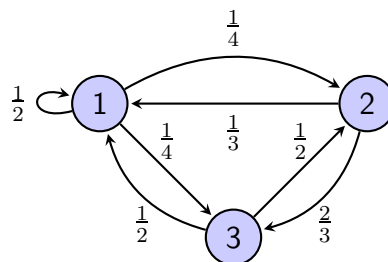
Due: 2024/1/7 10:59pm

1. Given a coin with the probability  $p$  of landing heads.  $p$  is unknown and we need to estimate its value through data. In our data collection model, we have  $n$  independent tosses, result of each toss is either Head or Tail. Let  $X$  denote the number of heads in the total  $n$  tosses. Now we conduct experiments to collect data and find  $X = k$ . Then we need to find  $\hat{p}$ , the estimation of  $p$ .
  - (a) Assume  $p$  is an unknown constant. Find  $\hat{p}$  through the MLE (Maximum Likelihood Estimation) rule.
  - (b) Assume  $p$  is a random variable with a prior distribution  $p \sim \text{Beta}(a, b)$ , where  $a$  and  $b$  are known constants. Find  $\hat{p}$  through the MAP (Maximum a Posterior Probability) rule.
  - (c) Assume  $p$  is a random variable with a prior distribution  $p \sim \text{Beta}(a, b)$ , where  $a$  and  $b$  are known constants. Find  $\hat{p}$  through the MMSE (Minimal Mean Squared Error) rule.
2. Let  $X$  be the height of a randomly chosen adult man, and  $Y$  be his father's height, where  $X$  and  $Y$  have been standardized to have mean 0 and standard deviation 1. Suppose that  $(X, Y)$  is Bivariate Normal, with  $X, Y \sim \mathcal{N}(0, 1)$  and  $\text{Corr}(X, Y) = \rho$ .
  - (a) Let  $y = ax + b$  be the equation of the best line for predicting  $Y$  from  $X$  (in the sense of minimizing the mean squared error), *e.g.*, if we were to observe  $X = 1.3$  then we would predict that  $Y$  is  $1.3a + b$ . Now suppose that we want to use  $Y$  to predict  $X$ , rather than using  $X$  to predict  $Y$ . Give and explain an intuitive guess for what the slope is of the best line for predicting  $X$  from  $Y$ .
  - (b) Find a constant  $c$  (in terms of  $\rho$ ) and an r.v.  $V$  such that  $Y = cX + V$ , with  $V$  independent of  $X$ .  
Hint: Start by finding  $c$  such that  $\text{Cov}(X, Y - cX) = 0$ .
  - (c) Find a constant  $d$  (in terms of  $\rho$ ) and an r.v.  $W$  such that  $X = dY + W$ , with  $W$  independent of  $Y$ .
  - (d) Find  $E(Y|X)$  and  $E(X|Y)$ .
  - (e) Reconcile (a) and (d), giving a clear and correct intuitive explanation.

3. Two chess players, Vishy and Magnus, play a series of games. Given  $p$ , the game results are i.i.d. with probability  $p$  of Vishy winning, and probability  $q = 1 - p$  of Magnus winning (assume that each game ends in a win for one of the two players). But  $p$  is unknown, so we will treat it as an r.v. To reflect our uncertainty about  $p$ , we use the prior  $p \sim \text{Beta}(a, b)$ , where  $a$  and  $b$  are known positive integers and  $a \geq 2$ .
- Find the expected number of games needed in order for Vishy to win a game (including the win). Simplify fully; your final answer should not use factorials or  $\Gamma$ .
  - Explain in terms of independence vs. conditional independence the direction of the inequality between the answer to (a) and  $1 + E(G)$  for  $G \sim \text{Geom}(\frac{a}{a+b})$ .
  - Find the conditional distribution of  $p$  given that Vishy wins exactly 7 out of the first 10 games.
4. Given a Markov chain with state-transition diagram shown as follows:



- Is this chain irreducible?
  - Is this chain aperiodic?
  - Find the stationary distribution of this chain.
  - Is this chain reversible?
5. Given a Markov chain with state-transition diagram shown as follows:



- Find  $P(X_3 = 3 | X_2 = 2)$  and  $P(X_4 = 1 | X_3 = 2)$ .
- If  $P(X_0 = 2) = \frac{2}{5}$ , find  $P(X_0 = 2, X_1 = 3, X_2 = 1)$ .

- (c) Find  $P(X_2 = 1|X_0 = 2)$ ,  $P(X_2 = 2|X_0 = 2)$ , and  $P(X_2 = 3|X_0 = 2)$ .
- (d) Find  $E(X_2|X_0 = 2)$ .

6. **(Optional Challenging Problem)** A fair coin is flipped repeatedly. We use H to denote “Head appeared” and T to denote the “Tail appeared”.

- (a) What is the expected number of flips until the pattern HTHT is observed?
- (b) What is the expected number of flips until the pattern THTT is observed?
- (c) What is the probability that pattern HTHT is observed earlier than THTT?