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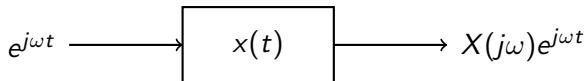000000000

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EE150 Signals and Systems
– Part 5: Discrete-time Fourier Transform (DTFT)

Continuous-time Fourier Transform

Continuous-time Fourier transform of $x(t)$ can be interpreted as

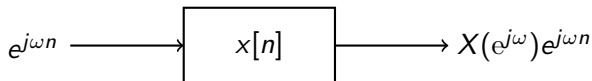$$e^{j\omega t} \longrightarrow \boxed{x(t)} \longrightarrow X(j\omega)e^{j\omega t}$$

LTI system with impulse response $x(t)$

$X(j\omega)$: eigenvalue of $e^{j\omega t}$

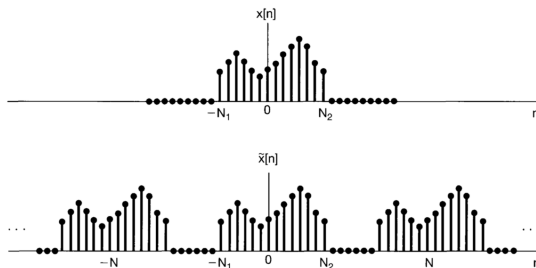$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

Discrete-time Fourier Transform

Discrete-time LTI system with impulse response $x[n]$

$$e^{j\omega n} \longrightarrow \boxed{x[n]} \longrightarrow X(e^{j\omega})e^{j\omega n}$$

$$\begin{aligned}
X(e^{j\omega})e^{j\omega n} &= e^{j\omega n} * x[n] \\
&= \sum_{m=-\infty}^{\infty} x[m]e^{j\omega(n-m)} \\
&= e^{j\omega n} \sum_{m=-\infty}^{\infty} x[m]e^{-j\omega m}
\end{aligned}$$

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Fourier Series to Fourier Transform

$$\tilde{x}[n] = \sum_{k=\langle N \rangle} a_k e^{jk(2\pi/N)n} \qquad a_k = \frac{1}{N} \sum_{n=\langle N \rangle} \tilde{x}[n] e^{-jk(2\pi/N)n}$$

As $x[n] = \tilde{x}[n]$ for $-N_1 \leq n \leq N_2$, we have

$$a_k = \frac{1}{N} \sum_{n=-N_1}^{N_2} x[n] \mathrm{e}^{-jk(2\pi/N)n} = \frac{1}{N} \sum_{n=-\infty}^{\infty} x[n] \mathrm{e}^{-jk(2\pi/N)n}$$

Fourier Series to Fourier Transform cont.

1. Define $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$

2. As $a_k = \frac{1}{N}X(e^{jk\omega_0})$, then

$$\tilde{x}[n] = \sum_{k=<N>} \frac{1}{N}X(e^{jk\omega_0})e^{jk\omega_0 n}$$

$$= \frac{1}{2\pi} \sum_{k=<N>} X(e^{jk\omega_0})e^{jk\omega_0 n}\omega_0$$

3. As $N \to \infty$, we have $\tilde{x}[n] \to x[n]$.

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Discrete-time Fourier Transform
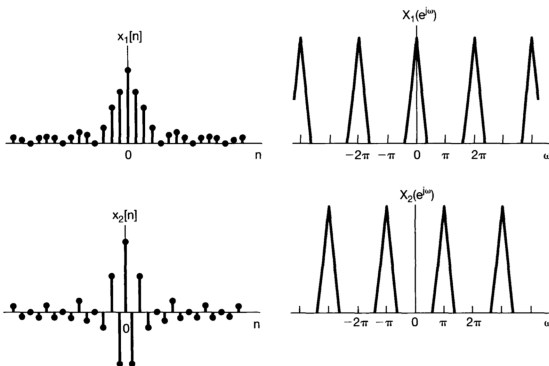
- Forward and inverse transforms

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \qquad \text{(DTFT)}$$

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega \qquad \text{(Inverse DTFT)}$$

- $x[n]$: linear combination of complex exponential infinitesimally close in frequency with amplitudes $X(e^{j\omega})(\mathrm{d}\omega/2\pi)$
- $X(e^{j\omega})$ is the spectrum of $x[n]$

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Discrete-time Fourier Transform

1. $e^{j\omega n}$ is periodic in terms of ω: $e^{j\omega n} = e^{j(\omega + k2\pi)n}, k \in \mathbb{Z}$
2. Hence, $X(e^{j\omega})$ is periodic (with period 2π)
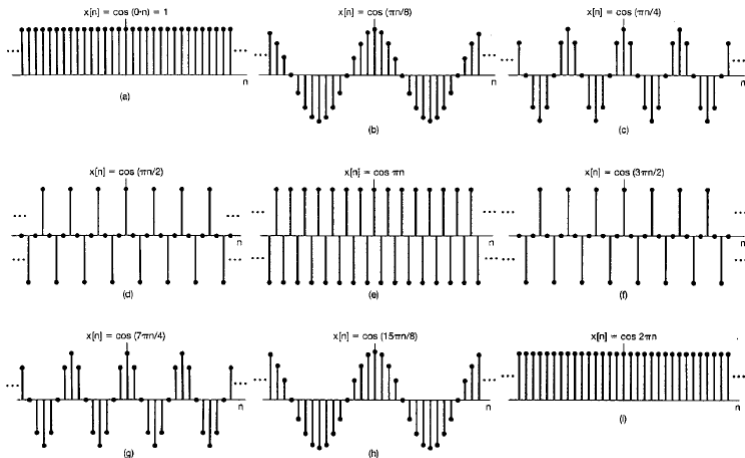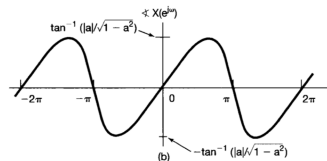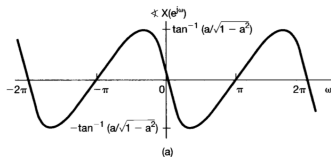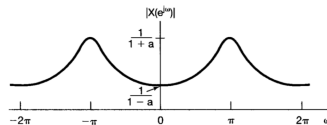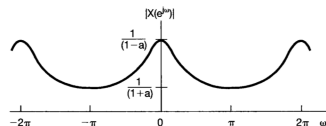
Discrete-time Complex Exponential

Figure 1.27   Discrete-time sinusoidal sequences for several different frequencies.

Example 5.1

Calculate the DTFT of signal $x[n] = a^n u[n]$, $|a| < 1$

Solution:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} a^n u[n] e^{-j\omega n} = \sum_{n=0}^{\infty} (ae^{-j\omega})^n = \frac{1}{1 - ae^{-j\omega}}$$

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Example 5.2

Calculate the DTFT of signal $x[n] = a^{|n|}, |a| < 1$.

Solution:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} a^{|n|}e^{-j\omega n} = \sum_{n=0}^{\infty} a^n e^{-j\omega n} + \sum_{n=-\infty}^{-1} a^{-n}e^{-j\omega n}$$

$$= \frac{1}{1 - ae^{-j\omega}} + \frac{ae^{j\omega}}{1 - ae^{j\omega}} = \frac{1 - a^2}{1 - 2a\cos\omega + a^2}$$

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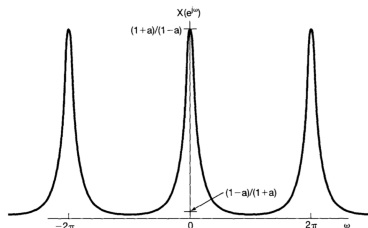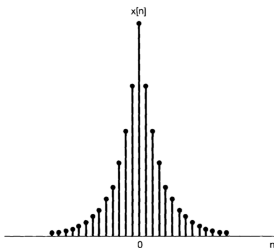
Example 5.3
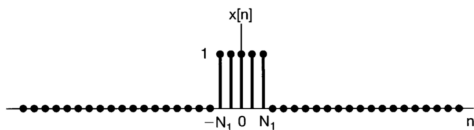
$$x[n] = \begin{cases} 1, & |n| \leq N_1 \\ 0, & |n| > N_1 \end{cases}$$

$$X(e^{j\omega}) = \sum_{n=-N_1}^{N_1} e^{-j\omega n} = \frac{\sin\omega\left(N_1 + \frac{1}{2}\right)}{\sin(\omega/2)}$$

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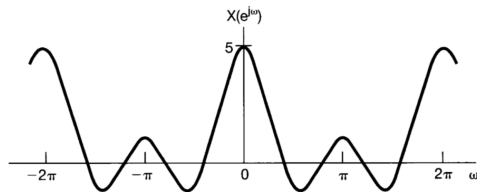
Convergence Issues of DTFT

1. $x[n]$ is absolutely summable

$$\sum_{n=-\infty}^{\infty} |x[n]| < \infty$$

2. OR $x[n]$ has finite energy

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty$$

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Fourier Transform for Periodic Signals

Fourier transform can be applied to periodic signal

Consider $x(t)$ and its FT $X(e^{j\omega})$

Assume $X(e^{j\omega}) = \sum_{l=-\infty}^{\infty} 2\pi\delta(\omega - \omega_0 - 2\pi l)$

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega})e^{j\omega n}\mathrm{d}\omega = \frac{1}{2\pi} \int_{2\pi} \sum_{l=-\infty}^{\infty} 2\pi\delta(\omega - \omega_0 - 2\pi l)e^{j\omega n}\mathrm{d}\omega$$

$$= e^{j(\omega_0 + 2\pi r)n} = e^{j\omega_0 n}$$

Hence,

$$x[n] = \sum_{k=<N>} a_k e^{jk(2\pi/N)n} \xleftrightarrow{\quad FT \quad} X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta\left(\omega - \frac{2\pi k}{N}\right)$$

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Fourier Transform for Periodic Signals cont.

$$x[n] = a_0 + a_1 e^{j(2\pi/N)n} + a_2 e^{j2(2\pi/N)n} + \cdots + a_{N-1} e^{j(N-1)(2\pi/N)n}$$

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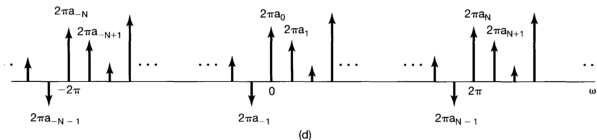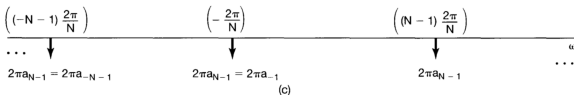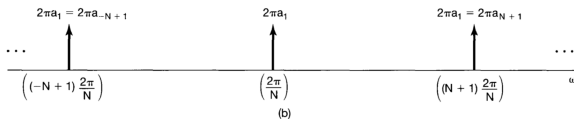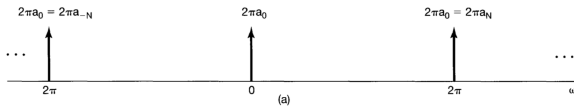
Example 5.5

1. Consider the periodic signal
   $x[n] = \cos \omega_0 n = \frac{1}{2} e^{j\omega_0 n} + \frac{1}{2} e^{-j\omega_0 n}$ with $\omega_0 = \frac{2\pi}{5}$

2. Hence, $X(e^{j\omega}) =$
   $\sum_{l=-\infty}^{\infty} \pi\delta\left(\omega - \frac{2\pi}{5} - 2\pi l\right) + \sum_{l=-\infty}^{\infty} \pi\delta\left(\omega + \frac{2\pi}{5} - 2\pi l\right)$

3. It equals $X(e^{j\omega}) = \pi\delta\left(\omega - \frac{2\pi}{5}\right) + \pi\delta\left(\omega + \frac{2\pi}{5}\right), -\pi \leq \omega \leq \pi$

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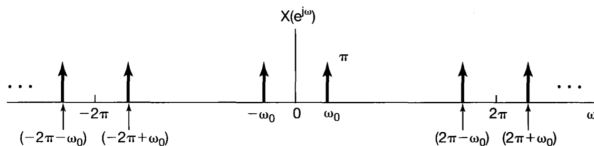
Properties of DTFT

$$x[n] \xleftarrow{\quad FT \quad} X(e^{j\omega})$$

1. Periodicity: DTFT is always periodic in ω with period 2π, i.e., $X(e^{j(\omega+2\pi)}) = X(e^{j\omega})$.

2. Linearity: $ax_1[n] + bx_2[n] \xleftarrow{\quad FT \quad} aX_1(e^{j\omega}) + bX_2(e^{j\omega})$

3. Time shifting: $x[n - n_0] \xleftarrow{\quad FT \quad} e^{-j\omega n_0} X(e^{j\omega})$

4. Frequency shifting: $e^{j\omega_0 n} x[n] \xleftarrow{\quad FT \quad} X(e^{j(\omega-\omega_0)})$

5. Conjugation and conjugate symmetry:
   $x^*[n] \xleftarrow{\quad FT \quad} X^*(e^{-j\omega})$
   If $x[n]$ is real valued, then its transform $X(e^{j\omega})$ is conjugate symmetric, i.e., $X(e^{j\omega}) = X^*(e^{-j\omega})$

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Properties of DTFT cont.

6. Differencing

$$x[n] - x[n-1] \xleftrightarrow{\quad FT \quad} \left(1 - \mathrm{e}^{-j\omega}\right) X(\mathrm{e}^{j\omega})$$

Proof.

$$x[n] - x[n-1] = x[n] * [\delta[n] - \delta[n-1]]$$
$$\delta[n-k] \xleftrightarrow{\quad FT \quad} e^{-j\omega k}$$

□

7. Accumulation: $y[n] = \sum_{m=-\infty}^{n} x[m]$

$$\sum_{m=-\infty}^{n} x[m] \xleftrightarrow{\quad FT \quad} \frac{1}{1 - \mathrm{e}^{-j\omega}} X(\mathrm{e}^{j\omega}) + \pi X(\mathrm{e}^{j0}) \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k)$$

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Example 5.8

Derive the Fourier transform of unit step $x[n] = u[n]$

Solution: $g[n] = \delta[n] \xleftarrow{\quad FT \quad} G(e^{j\omega}) = 1$

Unit step function is the running sum of the unit impulse

$$x[n] = \sum_{m=-\infty}^{n} g[m]$$

Taking the Fourier transform of both sides and using accumulation yield

$$X(e^{j\omega}) = \frac{1}{1 - e^{-j\omega}} G(e^{j\omega}) + \pi G(e^{j0}) \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k)$$

$$= \frac{1}{1 - e^{-j\omega}} + \pi \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k)$$

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Properties of DTFT cont.

8. Time reversal: $x[-n] \xleftrightarrow{\quad FT \quad} X(\mathrm{e}^{-j\omega})$

9. Differentiation in frequency: $nx[n] \xleftrightarrow{\quad FT \quad} j\dfrac{\mathrm{d}X(\mathrm{e}^{j\omega})}{\mathrm{d}\omega}$

10. Parseval's Relation: $\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi} \int_{2\pi} |X(\mathrm{e}^{j\omega})|^2 \mathrm{d}\omega$

11. Time expansion: $x_{(k)}[n] \xleftrightarrow{\quad FT \quad} X(\mathrm{e}^{jk\omega})$
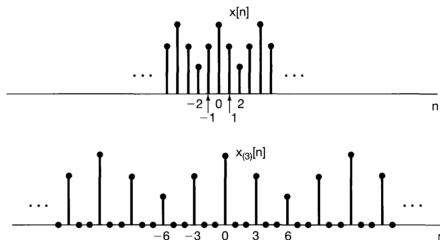
$$x_{(k)}[n] = \begin{cases} x[n/k], & \text{if } n \text{ is a multiple of } k \\ 0, & \text{if } n \text{ is not a multiple of } k \end{cases}$$

$$X_{(k)}(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x_{(k)}[n]e^{-j\omega n} = \sum_{r=-\infty}^{+\infty} x_{(k)}[rk]e^{-j\omega rk}$$

since $x_{(k)}[rk] = x[r]$

$$X_{(k)}(e^{j\omega}) = \sum_{r=-\infty}^{+\infty} x[r]e^{-j(k\omega)r} = X(e^{jk\omega})$$

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Time Expansion
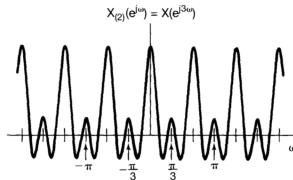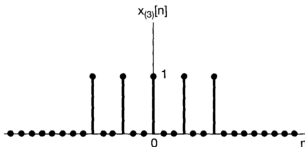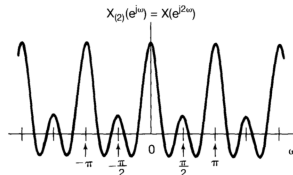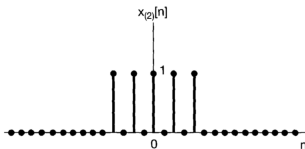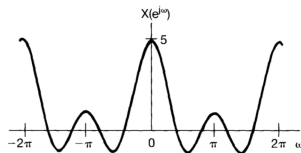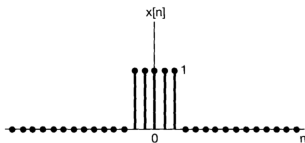
Example 5.9
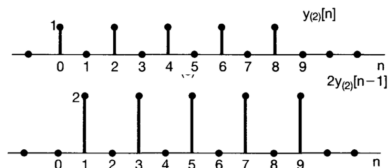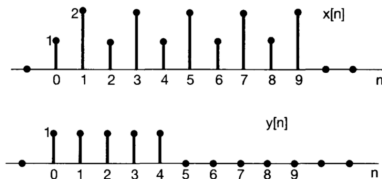
$$x[n] = y_{(2)}[n] + 2y_{(2)}[n-1],$$

$$y_{(2)}[n] = \begin{cases} y[n/2], & \text{if } n \text{ is even} \\ 0, & \text{if } n \text{ is odd} \end{cases}$$

$$Y(e^{j\omega}) = e^{-j2\omega}\frac{\sin(5\omega/2)}{\sin(\omega/2)}$$

$$y_{(2)}[n] \overset{\mathcal{F}}{\longleftrightarrow} e^{-j4\omega}\frac{\sin(5\omega)}{\sin(\omega)}$$

$$2y_{(2)}[n-1] \overset{\mathcal{F}}{\longleftrightarrow} 2e^{-j5\omega}\frac{\sin(5\omega)}{\sin(\omega)}$$

$$X(e^{j\omega}) = e^{-j4\omega}(1 + 2e^{-j\omega})\left(\frac{\sin(5\omega)}{\sin(\omega)}\right)$$

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Convolution Property

- $y[n] = h[n] * x[n]$

$$Y(j\omega) = \sum_n y[n]e^{-j\omega n}$$

$$= \sum_n \sum_m x[m]h[n-m]e^{-j\omega n}$$

$$= \sum_m \left(\sum_n h[n-m]e^{-j\omega(n-m)} \right) x[m]e^{-j\omega m}$$

$$= \sum_m H(j\omega)x[m]e^{-j\omega m}$$

$$= H(j\omega)X(j\omega)$$

- Frequency response $H(e^{j\omega})$ captures the change in complex amplitude of the Fourier Transform of the input at each frequency ω

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Example 5.11

1. Consider an LTI system with impulse response $h[n] = \delta[n - n_0]$
2. Frequency response is

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \delta[n - n_0]e^{-j\omega n} = e^{-j\omega n_0}$$

3. For any input $x[n]$ with Fourier transform $X(e^{j\omega})$, the Fourier transform of the output is $Y(e^{j\omega}) = e^{-j\omega n_0}X(e^{j\omega})$

4. Frequency response $H(e^{j\omega}) = e^{-j\omega n_0}$ has unity magnitude at all frequencies and a phase characteristic $-\omega n_0$ that is linear with frequency

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Example 5.12

1. Consider the discrete-time ideal lowpass filter $H(e^{j\omega})$
2. Impulse response and frequency response of an LTI system are a Fourier transform pair

$$h[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} H(e^{j\omega}) e^{j\omega n} \mathrm{d}\omega = \frac{1}{2\pi} \int_{-\omega_0}^{\omega_0} e^{j\omega n} \mathrm{d}\omega = \frac{\sin \omega_0 n}{\pi n}$$

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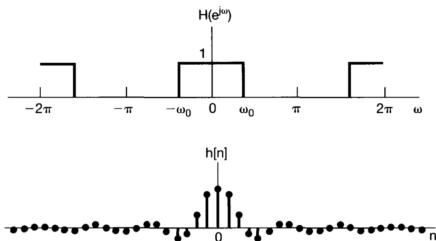
Example 5.13

Consider an LTI system with impulse response $h[n] = \alpha^n u[n]$ with $|\alpha| < 1$, and suppose that the input to this system is $x[n] = \beta^n u[n]$ with $|\beta| < 1$. Evaluate the output $y[n]$.

1. Time-domain convolution: $y[n] = x[n] * h[n]$
2. Convolution property:

$$Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega}) = \frac{1}{(1 - \alpha e^{-jw})(1 - \beta e^{-j\omega})}$$

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Multiplication Property

- $y[n] = x[n]h[n]$

$$y[n] = \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega \int_{-\pi}^{\pi} H(e^{j\rho})e^{j\rho n}d\rho$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega})H(e^{j(\nu-\omega)})d\omega \right) e^{j\nu n}d\nu$$

- So

$$Y(e^{j\nu}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega})H(e^{j(\nu-\omega)})d\omega$$

(periodic convolution)

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Example 5.15

Calculate the Fourier transform $X(e^{j\omega})$ of signal $x[n] = x_1[n]x_2[n]$, where $x_1[n] = \frac{\sin(3\pi n/4)}{\pi n}$ and $x_2[n] = \frac{\sin(\pi n/2)}{\pi n}$

Solution: $X(e^{j\omega}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X_1(e^{j\theta}) X_2(e^{j(\omega-\theta)}) d\theta$
Define $\hat{X}_1(e^{j\omega})$ equals $X_1(e^{j\omega})$ for $-\pi < \omega \leq \pi$, and equals 0 otherwise. Hence,

$$X(e^{j\omega}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{X}_1(e^{j\theta}) X_2(e^{j(\omega-\theta)}) d\theta$$
$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{X}_1(e^{j\theta}) X_2(e^{j(\omega-\theta)}) d\theta$$

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Properties of DTFT

TABLE 5.1 PROPERTIES OF THE DISCRETE-TIME FOURIER TRANSFORM

Section	Property	Aperiodic Signal	Fourier Transform
		$x[n]$	$X(e^{j\omega})$ periodic with
		$y[n]$	$Y(e^{j\omega})$ period 2π
5.3.2	Linearity	$ax[n] + by[n]$	$aX(e^{j\omega}) + bY(e^{j\omega})$
5.3.3	Time Shifting	$x[n - n_0]$	$e^{-j\omega n_0}X(e^{j\omega})$
5.3.3	Frequency Shifting	$e^{j\omega_0 n}x[n]$	$X(e^{j(\omega - \omega_0)})$
5.3.4	Conjugation	$x^*[n]$	$X^*(e^{-j\omega})$
5.3.6	Time Reversal	$x[-n]$	$X(e^{-j\omega})$
5.3.7	Time Expansion	$x_{(k)}[n] = \begin{cases} x[n/k], & \text{if } n = \text{multiple of } k \\ 0, & \text{if } n \neq \text{multiple of } k \end{cases}$	$X(e^{jk\omega})$
5.4	Convolution	$x[n] * y[n]$	$X(e^{j\omega})Y(e^{j\omega})$
5.5	Multiplication	$x[n]y[n]$	$\dfrac{1}{2\pi}\displaystyle\int_{2\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$
5.3.5	Differencing in Time	$x[n] - x[n-1]$	$(1 - e^{-j\omega})X(e^{j\omega})$
5.3.5	Accumulation	$\displaystyle\sum_{k=-\infty}^{n} x[k]$	$\dfrac{1}{1 - e^{-j\omega}}X(e^{j\omega})$ $+ \pi X(e^{j0})\displaystyle\sum_{k=-\infty}^{+\infty} \delta(\omega - 2\pi k)$
5.3.8	Differentiation in Frequency	$nx[n]$	$j\dfrac{dX(e^{j\omega})}{d\omega}$
5.3.4	Conjugate Symmetry for Real Signals	$x[n]$ real	$\begin{cases} X(e^{j\omega}) = X^*(e^{-j\omega}) \\ \mathfrak{Re}\{X(e^{j\omega})\} = \mathfrak{Re}\{X(e^{-j\omega})\} \\ \mathfrak{Im}\{X(e^{j\omega})\} = -\mathfrak{Im}\{X(e^{-j\omega})\} \\ |X(e^{j\omega})| = |X(e^{-j\omega})| \\ \sphericalangle X(e^{j\omega}) = -\sphericalangle X(e^{-j\omega}) \end{cases}$
5.3.4	Symmetry for Real, Even Signals	$x[n]$ real an even	$X(e^{j\omega})$ real and even
5.3.4	Symmetry for Real, Odd Signals	$x[n]$ real and odd	$X(e^{j\omega})$ purely imaginary and odd
5.3.4	Even-odd Decomposition of Real Signals	$x_e[n] = \mathcal{E}v\{x[n]\}$ $[x[n]$ real$]$	$\mathfrak{Re}\{X(e^{j\omega})\}$
		$x_o[n] = \mathcal{O}d\{x[n]\}$ $[x[n]$ real$]$	$j\mathfrak{Im}\{X(e^{j\omega})\}$
5.3.9	Parseval's Relation for Aperiodic Signals	$\displaystyle\sum_{n=-\infty}^{+\infty} |x[n]|^2 = \dfrac{1}{2\pi}\displaystyle\int_{2\pi} |X(e^{j\omega})|^2 d\omega$	

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Basic DTFT Pairs

Signal	Fourier Transform	Fourier Series Coefficients (if periodic)
$\sum_{k=\langle N \rangle} a_k e^{jk(2\pi/N)n}$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta\left(\omega - \dfrac{2\pi k}{N}\right)$	a_k
$e^{j\omega_0 n}$	$2\pi \sum_{l=-\infty}^{+\infty} \delta(\omega - \omega_0 - 2\pi l)$	(a) $\omega_0 = \dfrac{2\pi m}{N}$ $a_k = \begin{cases} 1, & k = m, m \pm N, m \pm 2N, \dots \\ 0, & \text{otherwise} \end{cases}$ (b) $\dfrac{\omega_0}{2\pi}$ irrational $\Rightarrow$ The signal is aperiodic
$\cos \omega_0 n$	$\pi \sum_{l=-\infty}^{+\infty} \{\delta(\omega - \omega_0 - 2\pi l) + \delta(\omega + \omega_0 - 2\pi l)\}$	(a) $\omega_0 = \dfrac{2\pi m}{N}$ $a_k = \begin{cases} \frac{1}{2}, & k = \pm m, \pm m \pm N, \pm m \pm 2N, \dots \\ 0, & \text{otherwise} \end{cases}$ (b) $\dfrac{\omega_0}{2\pi}$ irrational $\Rightarrow$ The signal is aperiodic
$\sin \omega_0 n$	$\dfrac{\pi}{j} \sum_{l=-\infty}^{+\infty} \{\delta(\omega - \omega_0 - 2\pi l) - \delta(\omega + \omega_0 - 2\pi l)\}$	(a) $\omega_0 = \dfrac{2\pi r}{N}$ $a_k = \begin{cases} \frac{1}{2j}, & k = r, r \pm N, r \pm 2N, \dots \\ -\frac{1}{2j}, & k = -r, -r \pm N, -r \pm 2N, \dots \\ 0, & \text{otherwise} \end{cases}$ (b) $\dfrac{\omega_0}{2\pi}$ irrational $\Rightarrow$ The signal is aperiodic
$x[n] = 1$	$2\pi \sum_{l=-\infty}^{+\infty} \delta(\omega - 2\pi l)$	$a_k = \begin{cases} 1, & k = 0, \pm N, \pm 2N, \dots \\ 0, & \text{otherwise} \end{cases}$
Periodic square wave $x[n] = \begin{cases} 1, & |n| \le N_1 \\ 0, & N_1 < |n| \le N/2 \end{cases}$ and $x[n + N] = x[n]$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta\left(\omega - \dfrac{2\pi k}{N}\right)$	$a_k = \dfrac{\sin[(2\pi k/N)(N_1 + \frac{1}{2})]}{N \sin[2\pi k/2N]}, \ k \ne 0, \pm N, \pm 2N, \dots$ $a_k = \dfrac{2N_1 + 1}{N}, \ k = 0, \pm N, \pm 2N, \dots$
$\sum_{k=-\infty}^{+\infty} \delta[n - kN]$	$\dfrac{2\pi}{N} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \dfrac{2\pi k}{N}\right)$	$a_k = \dfrac{1}{N}$ for all k

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Basic DTFT Pairs cont.

Signal	Fourier Transform	Fourier Series Coefficients (if periodic)
$a^n u[n], \quad |a| < 1$	$\dfrac{1}{1 - ae^{-j\omega}}$	—
$x[n] \begin{cases} 1, & |n| \le N_1 \\ 0, & |n| > N_1 \end{cases}$	$\dfrac{\sin[\omega(N_1 + \frac{1}{2})]}{\sin(\omega/2)}$	—
$\dfrac{\sin Wn}{\pi n} = \dfrac{W}{\pi} \operatorname{sinc}\left(\dfrac{Wn}{\pi}\right)$ $0 < W < \pi$	$X(\omega) = \begin{cases} 1, & 0 \le |\omega| \le W \\ 0, & W < |\omega| \le \pi \end{cases}$ $X(\omega)$ periodic with period 2π	—
$\delta[n]$	1	—
$u[n]$	$\dfrac{1}{1 - e^{-j\omega}} + \displaystyle\sum_{k=-\infty}^{+\infty} \pi\delta(\omega - 2\pi k)$	—
$\delta[n - n_0]$	$e^{-j\omega n_0}$	—
$(n + 1)a^n u[n], \quad |a| < 1$	$\dfrac{1}{(1 - ae^{-j\omega})^2}$	—
$\dfrac{(n + r - 1)!}{n!(r - 1)!} a^n u[n], \quad |a| < 1$	$\dfrac{1}{(1 - ae^{-j\omega})^r}$	—

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Duality in Discrete Time Fourier Series

1. $f[k]$ is the Fourier series coefficient of signal $g[n]$

$$g[n] \xleftrightarrow{FS} f[k], \quad f[k] = \frac{1}{N} \sum_{n=<N>} g[n]\mathrm{e}^{-jk(2\pi/N)n}$$

2. $f[k]$ is a periodic sequence with period N. Its Fourier series expansion is

$$f[n] = \sum_{k=<N>} \frac{1}{N} g[-k]\mathrm{e}^{jk(2\pi/N)n}$$

3. Hence,

$$f[n] \xleftrightarrow{FS} \frac{1}{N} g[-k]$$

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Example 5.16

Consider the following periodic signal with a period of $N = 9$

$$x[n] = \begin{cases} \frac{1}{9}\frac{\sin(5\pi n/9)}{\sin(\pi n/9)}, & n \neq \text{multiple of 9} \\ \frac{5}{9}, & n = \text{multiple of 9} \end{cases}$$

From previous example, we know that

$$g[n] = \begin{cases} 1, & |n| \leq 2 \\ 0, & 2 < |n| \leq 4 \end{cases} \xleftrightarrow{\ FS\ } b_k = \begin{cases} \frac{1}{9}\frac{\sin(5\pi k/9)}{\sin(\pi k/9)}, & k \neq \text{multiple of 9} \\ \frac{5}{9}, & k = \text{multiple of 9} \end{cases}$$

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Discrete-time FT and FS

- Let $x(t)$ be periodic with period 2π ($\omega_0 = 1$)
- FS and DTFT

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jkt} \qquad X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

$$a_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} x(t) e^{-jkt} dt \qquad x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$

- Really one and the same (time-frequency exchange)

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Example 5.17

1. Determine the Fourier transform of sequence $x[n] = \frac{\sin(\pi n/2)}{\pi n}$

2. From Example 3.5, we know that $g(t) = 1$, if $|t| \leq T_1$, and $g(t) = 0$, if $T_1 \leq |t| \leq \pi$. And $g(t)$ is a periodic square with period 2π. Its Fourier coefficients are $a_k = \frac{\sin(kT_1)}{k\pi}$

3. If $T_1 = \frac{\pi}{2}$, then $a_k = x[k]$. Hence,

$$\frac{\sin(\pi k/2)}{\pi k} = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(t) \mathrm{e}^{-jkt} \mathrm{d}t = \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} \mathrm{e}^{-jkt} \mathrm{d}t$$
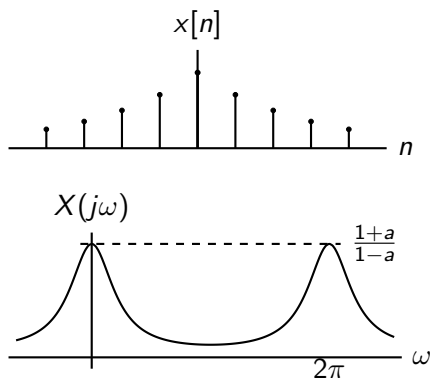
4.

$$\frac{\sin(\pi n/2)}{\pi n} = \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} \mathrm{e}^{jn\omega} \mathrm{d}\omega$$

$$\frac{\sin(\pi n/2)}{\pi n} = \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} \mathrm{e}^{jn\omega} \mathrm{d}\omega \rightarrow X(\mathrm{e}^{j\omega}) = \begin{cases} 1, & |\omega| \leq \frac{\pi}{2} \\ 0, & \frac{\pi}{2} < |\omega| \leq \pi \end{cases}$$

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Discrete-time FT and FS

Example (1). (Aperiodic)

$$x[n] = a^{|n|}, \quad 0 < a < 1$$

$$X(j\omega) = \sum_n a^{|n|} e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} a^n e^{-j\omega n} + \sum_{n=1}^{\infty} a^n e^{j\omega n}$$

$$= \frac{1}{1 - ae^{-j\omega}} + \frac{ae^{j\omega}}{1 - ae^{j\omega}}$$

$$= \frac{1 - a^2}{1 - a \cdot 2\cos\omega + a^2}$$

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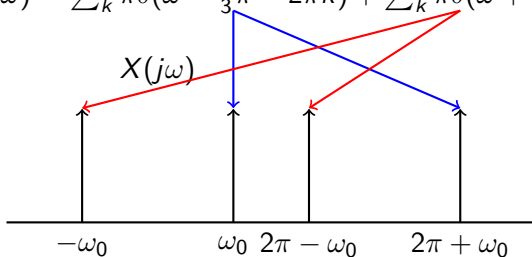
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Discrete-time FT and FS

Example (2). (Periodic)

$$x[n] = \cos(\omega_0 n) = \frac{1}{2}e^{j\omega_0 n} + \frac{1}{2}e^{-j\omega_0 n} \quad \text{with } \omega_0 = \frac{2\pi}{3}$$

From table: $e^{j\omega_0 n} \longleftrightarrow \sum\limits_{k=-\infty}^{\infty} 2\pi\delta(\omega - \omega_0 - 2\pi k)$

$$X(j\omega) = \sum_k \pi\delta(\omega - \tfrac{2}{3}\pi - 2\pi k) + \sum_k \pi\delta(\omega + \tfrac{2}{3}\pi - 2\pi k)$$

Summaries of Fourier Series and Fourier Transform

		Continuous time		Discrete time	
		Time domain	**Frequency domain**	**Time domain**	**Frequency domain**
Fourier Series		$x(t) =$ $\sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$	$a_k =$ $\frac{1}{T_0} \int_{T_0} x(t) e^{-jk\omega_0 t}$	$x[n] =$ $\sum_{k=\langle N \rangle} a_k e^{jk(2\pi/N)n}$	$a_k =$ $\frac{1}{N} \sum_{k=\langle N \rangle} x[n] e^{-jk(2\pi/N)n}$
		continuous time periodic in time	discrete frequency aperiodic in frequency	discrete time periodic in time	discrete frequency periodic in frequency
Fourier Transform		$x(t) =$ $\frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) e^{j\omega t} d\omega$	$X(j\omega) =$ $\int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt$	$x[n] =$ $\frac{1}{2\pi} \int_{2\pi} X(e^{j\omega}) e^{j\omega n}$	$X(e^{j\omega}) =$ $\sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n}$
		continuous time aperiodic in time	continuous frequency aperiodic in frequency	discrete time aperiodic in time	continuous frequency periodic in frequency

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Discrete Fourier Transform (DFT)

Let $x[n]$ be a finite-length sequence of length N.
Suppose $x[n] = 0$ for $n \notin [0 : N-1]$.

The DFT of $x[n]$, denoted as $X[k]$, is defined as

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn} \qquad k = 0, 1, \ldots, N-1$$

$$\text{where} \quad W_N = e^{-j(2\pi/N)}$$

The inverse DFT (IDFT) of $X[k]$ is given by

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] W_N^{-kn} \qquad n = 0, 1, \ldots, N-1$$

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Discrete Fourier Transform (DFT)

$$\begin{bmatrix} X[0] \\ X[1] \\ \vdots \\ X[N-1] \end{bmatrix} = \begin{bmatrix} W_N^{0 \cdot 1} & W_N^{0 \cdot 2} & \cdots & W_N^{0 \cdot N} \\ W_N^{1 \cdot 1} & W_N^{1 \cdot 2} & \cdots & W_N^{1 \cdot N} \\ \vdots & \vdots & \ddots & \vdots \\ W_N^{(N-1) \cdot 1} & W_N^{(N-1) \cdot 2} & \cdots & W_N^{(N-1) \cdot N} \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[N-1] \end{bmatrix}$$

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Important Features of DFT

- one-to-one correspondence between $x[n]$ and $X[k]$.
- an extreme fast algorithm for its calculation, called Fast Fourier Transform (FFT)
- DFT is closely related to discrete Fourier series and Fourier Transform.
- DFT is an appropriate representation for digital computer realization as it is discrete and of finite length in both time and frequency domain.