Homework 4

Due date: 8th, NOV.

Turn in your homework in class

Rules:

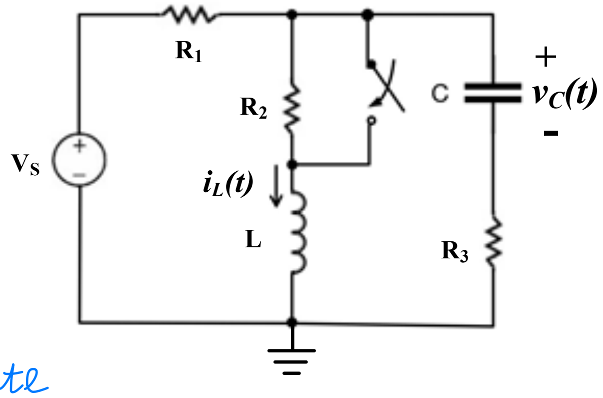
- Work on your own. Discussion is permissible, but extremely similar submissions will be judged as plagiarism.
- Please show all intermediate steps: a correct solution without an explanation will get zero credit.
- Please submit on time. No late submission will be accepted.
- Please prepare your submission in English only. No Chinese submission will be accepted.

0.5' per unit

no more than 1' for unit per question

10'

1. For the circuit below, the switch has been open for a long time. The switch is closed at $t = 0$s immediately. Given **Vs** = 10V, $\mathbf{R_1}$ = 2Ω, $\mathbf{R_2}$ = 3Ω, $\mathbf{R_3}$ = 1Ω, **L** = 4H, **C** = 2F, please find $v_C(0^+)$, $dv_C(0^+)/dt$, $i_L(0^+)$, $di_L(0^+)/dt$. (Hint: remember to assign units for your answers)



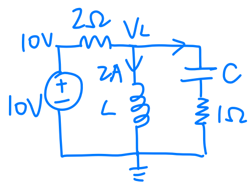
$t=0^-$: Steady state

4'

$V_C(0^+) = V_C(0^-) = 6V$

$i_L(0^+) = i_L(0^-) = 2A$

$t = 0^+$:



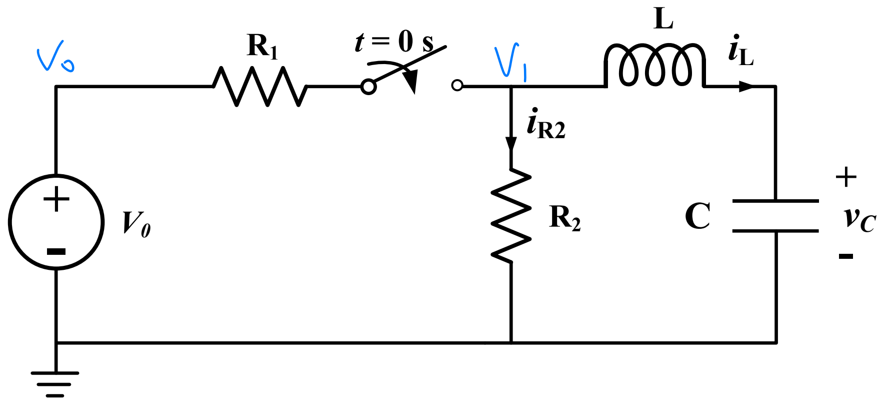
$$\begin{cases} \frac{10-V_L}{2} = 2 + C\frac{dV_C(0^+)}{dt} \\ V_L = L\frac{di(0^+)}{dt} \\ C\frac{dV_C(0^+)}{dt} = \frac{V_L-6}{1} \end{cases}$$

6'

$$\frac{di_L(0^+)}{dt} = 1.5 \ A/s \qquad \frac{dV_C(0^+)}{dt} = 0 \ V/s$$

22'

2. Determine $v_c(t)$, $i_L(t)$, and $i_{R2}(t)$ in the circuit for $t \geq 0$, given that $V_0 = 12V$, $R_1 = 2\ \Omega$, $R_2 = 2\ \Omega$, $L = 0.25$ H and $C = 0.5$ F. Note that the switch has been open for a long time before $t = 0$s



6'
$$\begin{cases} i_L = C\dfrac{dV_c}{dt} \\ \dfrac{V_0 - V_1}{2} = i_L + \dfrac{V_1}{2} \Rightarrow V_1 = \dfrac{V_0 - 2i_L}{2} \\ V_1 = V_c + L\dfrac{di_L}{dt} \end{cases}$$

$V_c(0^+) = 0$ V

$\dfrac{dV_c(0^+)}{dt} = 0$ V/s 2'

$$\frac{1}{2}V_0 - i_L = V_c + L\frac{di_L}{dt}$$

$$6 - 0.5\frac{dV_c}{dt} = V_c + 0.25 \times 0.5 \times \frac{d^2V_c}{dt^2}$$

$$6 - 0.5\frac{dV_c}{dt} = V_c + \frac{1}{8}\frac{d^2V_c}{dt^2}$$

2' $$\frac{d^2V_c}{dt^2} + 4\frac{dV_c}{dt} + 8V_c = 48$$

4'
$$\begin{cases} \alpha = 2 < \omega_0 = \sqrt{8} = 2\sqrt{2} \text{ rad/s} \quad \omega_d = 2 \text{ rad/s} \\ V_c(\infty) = 6\text{ V} \end{cases}$$

$$V_c(t) = e^{-2t}(B_1\cos 2t + B_2 \sin 2t) + 6 \quad (V)$$

$$V_c(0) = B_1 + 6 = 0$$

$$\frac{dV_c(0)}{dt} = -2e^{-2t}(B_1\cos 2t + B_2\sin 2t) + (-2B_1\sin 2t + 2B_2\cos 2t)e^{-2t}$$
$$= -2B_1 + 2B_2 = 0$$

2'
$$\begin{cases} B_1 = -6 \\ B_2 = -6 \end{cases}$$

$$V_c(t) = e^{-2t}(-6\cos 2t - 6\sin 2t) + 6 \quad (V)$$

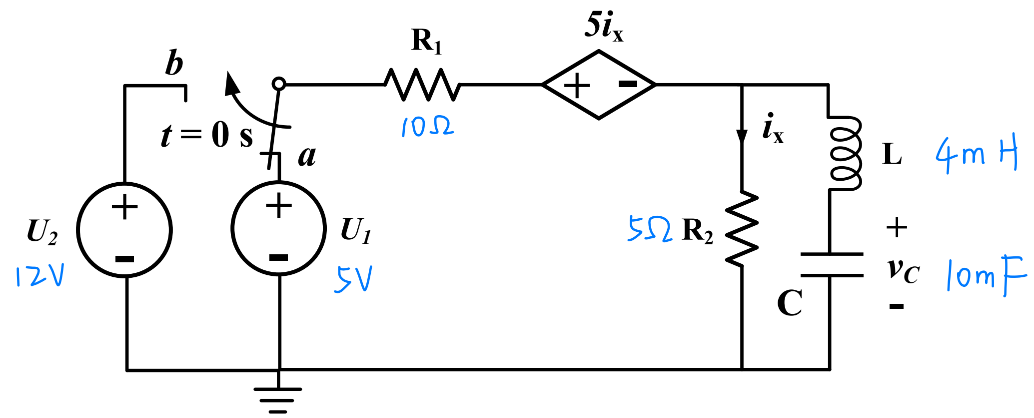
$$i_L(t) = i_c(t) = C\frac{dV_c(t)}{dt} = 6e^{-2t}(\cos 2t + \sin 2t) + 6(\sin 2t - \cos 2t)e^{-2t}$$
$$= 12e^{-2t}\sin 2t \quad (A)$$

$$i_{R_2}(t) = \frac{V_c + L\frac{di_L(t)}{dt}}{2} = -6e^{-2t}\sin 2t + 3 \quad (A)$$

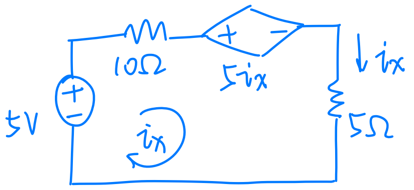
2' × 3

22'

3. In the circuit below, R_1= 10Ω , R_2= 5Ω , L = 4mH , C = 10mF, U_1 =5V, and U_2 =12V. When t = 0s, the switch changes from node a to node b immediately. Assume that the circuit reaches steady state before t = 0. Determine the expression for $v_C(t)$ and i_x(t) when $t \geq$ 0s.



When $t=0^-$ steady state



$5 = 10i_x + 5i_x + 5i_x$

$i_x = 0.25$ A

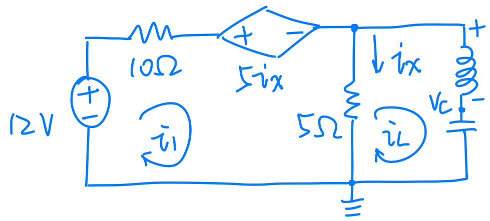
$V_C(0^+) = V_C(0^-) = 5 \times 0.25 = 1.25$ V

$i_L(0^+) = i_L(0^-) = 0$ A

$\frac{dV_C(0^+)}{dt} = 0$ V/s

4'

When $t>0$,



$$\begin{cases} 12 = 10i_1 + 5(i_1 - i_L) + 5(i_1 - i_L) \\ i_L = C\frac{dV_C}{dt} \\ i_1 - i_L = \dfrac{V_C + L\frac{di_L}{dt}}{5} \end{cases}$$

6'

2' $\Rightarrow \frac{d^2V_C}{dt^2} + \frac{5}{2L}\frac{dV_C}{dt} + \frac{1}{LC}V_C = \frac{3}{LC}$

4'

$$\begin{cases} \alpha = \frac{5}{4L} = 312.5 \text{ Np/s} > \omega_0 = \sqrt{\frac{1}{LC}} = \sqrt{25000} \text{ rad/s} \\ S_1 = -312.5 + \sqrt{312.5^2 - 25000} \approx -42.95 \\ S_2 = -312.5 - \sqrt{312.5^2 - 25000} \approx -582.05 \end{cases}$$

$V_C(t) = A_1 e^{S_1 t} + A_2 e^{S_2 t} + V_C(\infty)$

$V_C(0) = A_1 + A_2 + 3 = 1.25$ $\quad \frac{dV_C(0)}{dt} = S_1 A_1 + S_2 A_2 = 0$

2' $\begin{cases} A_1 = -1.8894 \\ A_2 = 0.1394 \end{cases}$

$V_C(t) = -1.89e^{-42.95t} + 0.14e^{-582.05t} + 3$ (V)

$i_L(t) = C\frac{dV_C(t)}{dt} = 10^{-2}(81.15e^{-42.95t} - 81.14e^{-582.05t})$ (A)

$i_x(t) = \dfrac{V_C(t) + L\frac{di_L(t)}{dt}}{5} = -0.41e^{-42.95t} + 0.41e^{-582.05t} + 0.6$ (A)

2' × 2

26'

4. In the circuit shown in Fig.9, the switch was closed at $t = 0$ and re-opened at $t =$ 0.5s. Determine the response $i_L(t)$ for $t \geq 0$, there's no energy stored in the inductor and capacitor.

Assume that $V_S = 18\text{V}$, $R_S = 1\Omega$, $R_1 = 5\Omega$, $R_2 = 2\Omega$, L= 2H and $C_1 = \frac{1}{17}$F.

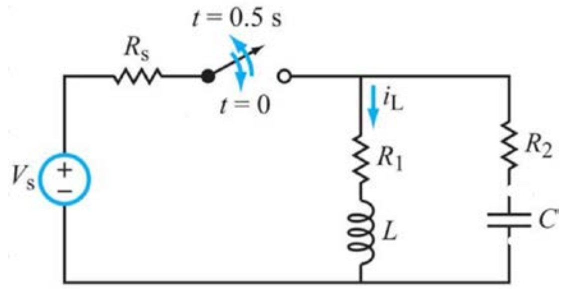


Figure 9

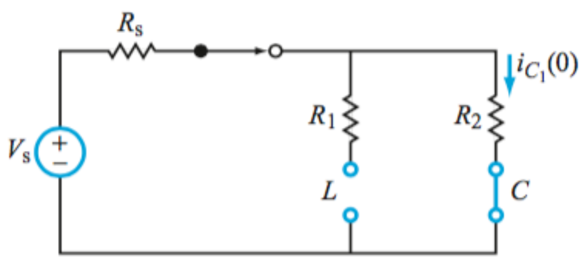
Solution:

Time Segment 1: $0 \leq t \leq 0.5$ **s**

Prior to $t = 0$, the circuit contained no sources. Hence,

2'

$$i_{L_1}(0) = i_{L_1}(0^-) = 0, \qquad \text{[open-circuit equivalent]}$$
$$\upsilon_{C_1}(0) = \upsilon_{C_1}(0^-) = 0. \qquad \text{[short-circuit equivalent]}$$



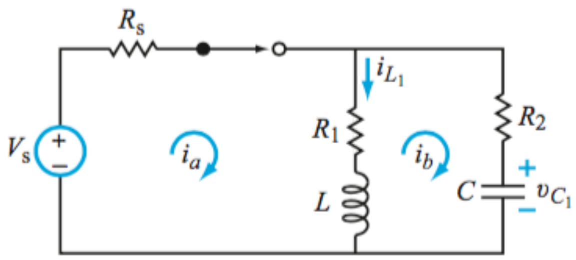
(a) At $t = 0$

At $t = 0$ (Fig. (a)):

$$i_{C_1}(0) = \frac{V_s}{R_s + R_2} = \frac{18}{1+2} = 6 \text{ A}, \tag{1}$$

2'

$$\upsilon'_{C_1}(0) = \frac{i_{C_1}(0)}{C} = \frac{6}{\frac{1}{17}} = 102 \text{ V/s}. \tag{2}$$



(b) At $0 \le t \le 0.5$ s

At $0 \le t \le 0.5$ s (Fig. (b)):

$$-V_s + R_s i_a + i_b R_2 + v_{C_1} = 0, \qquad \text{[outer loop]} \tag{3}$$

$$i_b = C\frac{dv_{C_1}}{dt} = Cv'_{C_1}. \tag{4}$$

Using Eq. (4) in Eq. (3) and solving for i_a gives

$$i_a = \frac{V_s - v_{C_1} - R_2 C v'_{C_1}}{R_s}. \tag{5}$$

The left loop equation is:

4'

$$-V_s + R_s i_a + R_1(i_a - i_b) + L(i'_a - i'_b) = 0. \tag{6}$$

The derivative of Eq. (5) gives

$$i'_a = \frac{-v'_{C_1} - R_2 C v''_{C_1}}{R_s}. \tag{7}$$

Using Eqs. (4), (5), and (7) in (6) gives:

$$-V_s + (V_s - v_{C_1} - R_2 C v'_{C_1}) + \frac{R_1}{R_s}(V_s - v_{C_1} - R_2 C v'_{C_1}) - R_1 C v'_{C_1} + \frac{L}{R_s}[-v'_{C_1} - R_2 C v''_{C_1}] - LC v''_{C_1} = 0. \tag{8}$$

Collecting like terms leads to:

$$v''_{C_1}\left[LC\left(1 + \frac{R_2}{R_s}\right)\right] + v'_{C_1}\left[R_2 C + \frac{R_1 R_2 C}{R_s} + R_1 C + \frac{L}{R_s}\right] + v_{C_1}\left[1 + \frac{R_1}{R_s}\right] = \frac{R_1 V_s}{R_s}. \tag{9}$$

2'

or equivalently

$$v''_{C_1} + a v'_{C_1} + b v'_{C_1} = c, \tag{10}$$

where

$$a = \frac{R_s(R_1 + R_2)C + R_1 R_2 C + L}{(R_s + R_2)LC} = \frac{1(5+2)(\frac{1}{17}) + 5 \times 2 \times (\frac{1}{17}) + 2}{(1+2) \times 2 \times \frac{1}{17}} = 8.5,$$

$$b = \frac{R_s + R_1}{(R_s + R_2)LC} = \frac{1+5}{(1+2) \times 2 \times \frac{1}{17}} = 17,$$

$$c = \frac{R_1 V_s}{(R_s + R_2)LC} = \frac{5 \times 18}{(1+2) \times 2 \times \frac{1}{17}} = 255.$$

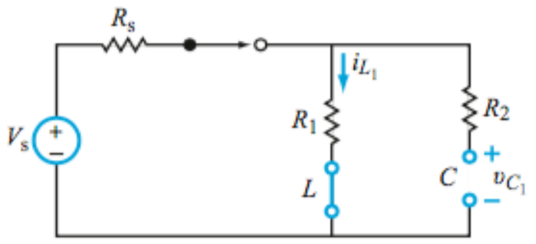
$$\alpha = \frac{a}{2} = 4.25 \text{ Np/s},$$

$$\omega_0 = \sqrt{b} = \sqrt{17} = 4.12 \text{ rad/s},$$

4'

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2} = -4.25 + \sqrt{4.25^2 - 17} = -3.22 \text{ Np/s},$$

$$s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2} = -4.25 - \sqrt{4.25^2 - 17} = -5.28 \text{ Np/s}.$$



(c) At $t = \infty$ (had the switch remained closed)

Had the switch remained closed, at $t = \infty$, the circuit becomes as shown in Fig. (c), in which case

1'

$$v_{C_1}(\infty) = i_{L_1} R_1 = \frac{V_s R_1}{R_s + R_1} = \frac{18 \times 5}{1+5} = 15 \text{ V}.$$

From Table 6-2,

2'

$$A_1 = \frac{v'_{C_1}(0) - s_2[v_{C_1}(0) - v_{C_1}(\infty)]}{s_1 - s_2} = \frac{102 + 5.28[0-15]}{-3.22 + 5.28} = 11.05 \text{ V},$$

$$A_2 = -\left[\frac{v'_{C_1}(0) - s_1[v_{C_1}(0) - v_{C_1}(\infty)]}{s_1 - s_2}\right] = \frac{102 + 3.22[0-15]}{-3.22 + 5.28} = -26.05 \text{ V}.$$

Hence, $v_C(t)$ is given by

$$\begin{aligned} v_{C_1}(t) &= v_{C_1}(\infty) + A_1 e^{s_1 t} + A_2 e^{s_2 t} \\ &= 15 + 11.05e^{-3.22t} - 26.05e^{-5.28t} \qquad \text{(V)}, \qquad \text{for } 0 \le t \le 0.5 \text{ s}. \end{aligned} \tag{11}$$

From Fig. (b), the current $i_{L_1}(t)$ is given by

$$i_{L_1}(t) = i_a - i_b. \tag{12}$$

Using Eqs. (4) and (5) in Eq. (12) gives:

$$\begin{aligned} i_{L_1}(t) &= \frac{V_s}{R_s} - \frac{v_{C_1}}{R_s} - \frac{R_2 C}{R_s} v'_{C_1} - C v'_{C_1} \\ &= \frac{V_s}{R_s} - \frac{v_{C_1}}{R_s} - C\left(1 + \frac{R_2}{R_s}\right) v'_{C_1}. \end{aligned} \tag{13}$$

From Eq. (11),

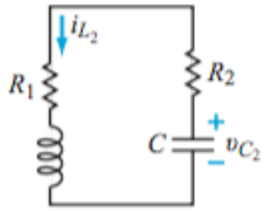
$$\begin{aligned} v'_{C_1}(t) &= -3.22 \times 11.05e^{-3.22t} + 5.28 \times 26.05e^{-5.28t} \\ &= -35.59e^{-3.22t} + 137.59e^{-5.28t} \qquad \text{(V/s)}. \end{aligned} \tag{14}$$

Using Eqs. (11) and (14) in Eq. (13), and then simplifying terms, leads to

2'

$$i_{L_1}(t) = [3 - 4.77e^{-3.22t} + 1.77e^{-5.28t}] \quad \text{(A)}, \qquad \text{for } 0 \le t \le 0.5 \text{ s}. \qquad (15)$$

Time Segment 2: $t > 0.5$ s



(d) After $t = 0.5$ s

After re-opening the switch, the circuit becomes a series RLC circuit as shown in Fig. (d). Since the circuit no longer contains sources,

$$i_{L_2}(\infty) = 0,$$
$$v_{C_2}(\infty) = 0.$$

From Table 6-1, the damping factors are:

2'

$$\alpha = \frac{R}{2L} = \frac{R_1 + R_2}{2L} = \frac{5+2}{2 \times 2} = \frac{7}{4} = 1.75 \text{ Np/s},$$
$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{2 \times \frac{1}{17}}} = 2.92 \text{ rad/s}.$$

Since $\alpha < \omega_0$, the response will be underdamped:

$$i_{L_2}(t) = [D_1 \cos \omega_d(t - 0.5) + D_2 \sin \omega_d(t - 0.5)]e^{-\alpha(t-0.5)}, \qquad (16)$$

where

1'

$$\omega_d = \sqrt{\omega_0^2 - \alpha^2} = 2.33 \text{ rad/s},$$

and the expression in Eq. (16) was shifted in time by 0.5 s. At $t = 0.5$ s, we require that:

$$i_{L_1}(0.5) = i_{L_2}(0.5), \qquad (17a)$$
$$v_{C_1}(0.5) = v_{C_2}(0.5). \qquad (17b)$$

Equating the expressions given by Eqs. (15) and (16) at $t = 0.5$ s gives:

$$3 - 4.78e^{-3.22 \times 0.5} + 1.78e^{-5.28 \times 0.5} = D_1,$$

1'

which gives

$$D_1 = 2.17 \text{ V}. \qquad (18)$$

From the circuit in Fig. (d),

$$v_{C_2}(t) = (R_1 + R_2)i_{L_2} + Li'_{L_2}$$
$$= 7[D_1 \cos \omega_d(t - 0.5) + D_2 \sin \omega_d(t - 0.5)]e^{-\alpha(t-0.5)}$$

$$+2[-\omega_d D_1 \sin\omega_d(t-0.5)+\omega_d D_2 \cos\omega_d(t-0.5)$$
$$-\alpha D_1 \cos\omega_d(t-0.5)-\alpha D_2 \sin\omega_d(t-0.5)]e^{-\alpha(t-0.5)}. \qquad (19)$$

At $t = 0.5$ s, Eqs. (11) and (19) give:

$$v_{C_1}(0.5) = 15 + 11.07e^{-3.22\times0.5} - 26.07e^{-5.28\times0.5} = 15.35 \text{ V}, \qquad (20a)$$
$$v_{C_2}(0.5) = 7D_1 + 2\omega_d D_2 - 2\alpha D_1$$
$$= 7\times2.17 + 2\times2.33D_2 - 2\times1.75\times2.17 = 7.6 + 4.66D_2. \qquad (20b)$$

1'

Equating Eq. (20a) to Eq. (20b) leads to

$$D_2 = 1.66 \text{ V}.$$

Hence,

2'

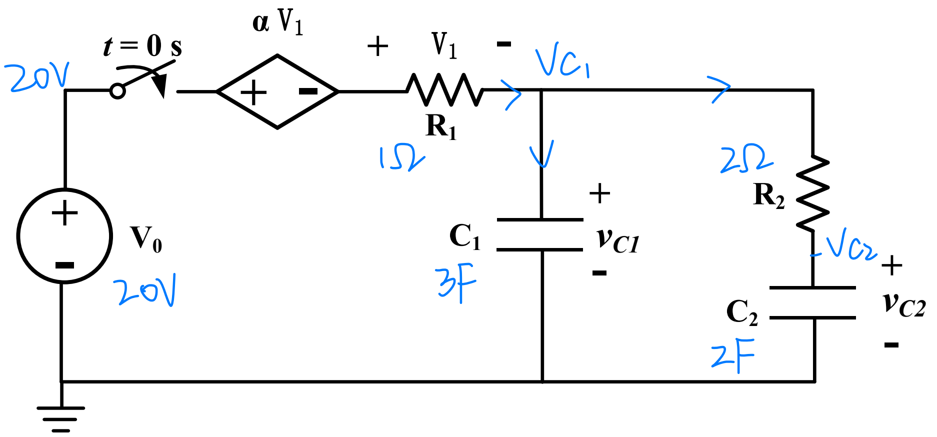
$$i_{L_2}(t) = [2.17\cos 2.33(t-0.5) + 1.66\sin 2.33(t-0.5)]e^{-1.75(t-0.5)} \quad \text{(A)}, \quad \text{for } t \ge 0.5 \text{ s}. \qquad (21)$$

The expressions given by Eqs. (15) and (21) constitute the complete solution.

$$\begin{cases} i_{L_1}(t) = [3 - 4.77e^{-3.22t} + 1.77e^{-5.28t}] \quad \text{(A)}, \quad \text{for } 0 \le t \le 0.5 \text{ s}. \\ i_{L_2}(t) = [2.17\cos 2.33(t-0.5) + 1.66\sin 2.33(t-0.5)]e^{-1.75(t-0.5)} \quad \text{(A)}, \quad \text{for } t \ge 0.5 \text{ s}. \end{cases}$$

20'

5. For the following circuit, $R_1 = 1\Omega$, $R_2 = 2\Omega$, $C_1 = 3F$, $C_2 = 2F$, $V_0 = 20V$, and the coefficient $\alpha = 1$. The switch closes at $t = 0s$ immediately. Please find the voltage on the capacitors $v_{C1}(t)$ and $v_{C2}(t)$ for $t > 0s$, respectively. Note that the switch has been open for a long time before $t = 0s$.



4'

$V_{c1}(0^-) = V_{c1}(0^+) = 0\ V$ $\quad$ $V_{c2}(0^-) = V_{c2}(0^+) = 0V$

$\frac{dV_{c2}(0^+)}{dt} = 0\ V/s$ $\quad$ $\frac{dV_{c1}(0^+)}{dt} = \frac{10}{3}\ V/s$

6'

$$\begin{cases} 20 - V_1 - V_1 = V_{c1} \\ \frac{V_1}{1} = 3\frac{dV_{c1}}{dt} + 2\frac{dV_{c2}}{dt} \\ \frac{V_{c1} - V_{c2}}{2} = 2\frac{dV_{c2}}{dt} \end{cases}$$

2'

$$\frac{d^2V_{c2}}{dt^2} + \frac{7}{12}\frac{dV_{c2}}{dt} + \frac{1}{24}V_{c2} = \frac{5}{6}$$

$$\alpha = \frac{7}{24} > \omega_0 = \sqrt{\frac{1}{24}}\ rad/s$$

4'

$$S_1 = -\frac{7}{24} + \sqrt{\left(\frac{7}{24}\right)^2 - \frac{1}{24}} = -0.5$$

$$S_2 = -\frac{7}{24} - \sqrt{\left(\frac{7}{24}\right)^2 - \frac{1}{24}} = -\frac{1}{12} \approx -0.083$$

$$V_{c2}(t) = A_1 e^{-0.5t} + A_2 e^{-\frac{1}{12}t} + 20\ (V)$$

$V_{c2}(0) = A_1 + A_2 + 20 = 0$

$\frac{dV_{c2}(0)}{dt} = -0.5A_1 - \frac{1}{12}A_2 = 0$

$$\begin{cases} A_1 = 4 \\ A_2 = -24 \end{cases}$$

2'

$t > 0s$:

1'

$$V_{c2}(t) = 4e^{-0.5t} - 24e^{-\frac{1}{12}t} + 20\ (V)$$

$$V_{c1}(t) = V_{c2} + 4\frac{dV_{c2}}{dt}$$

$$= 4e^{-0.5t} - 24e^{-\frac{1}{12}t} + 20 + 4\left(-2e^{-0.5t} + 2e^{-\frac{1}{12}t}\right)$$

1'

$$= -4e^{-0.5t} - 16e^{-\frac{1}{12}t} + 20\ (V)$$