SI211 Homework 7

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1. *Newton's Method.* Consider the function $f : \mathbb{R} \mapsto \mathbb{R}$
$$f(x) = x^3 - 5x^2 + 7x - 3\,.$$
Use Newton's method to find the zeros of f from two different initial guess: $x_0 \in \{0, 4\}$. Write out the first 5 iterations.

2. *Nonlinear Least Squares.* Consider the nonlinear function
$$f(x,y) = \frac{1}{2}(x-3)^2 + \frac{1}{2}\left(5(y-x^2)\right)^2 + \frac{1}{2}y^2$$
   (a) Write out the gradient vector and the Hessian matrix.
   (b) Rewrite the function as $f(x,y) = \frac{1}{2}\|R(x,y)\|$, $R\colon \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}^3$.
   (c) Derive the Gauss-Newton Hessian approximation. Compare the approximation to the exact Hessian. Under which condition will these two coincide?
   (d) Write a computer program to minimize $f(x)$, first using Newton's method, and then use Gauss-Newton method. The termination condition is $\|\nabla f\|_2 \leq 10^{-3}$. Plot:
      i. the solution trajectories in the 2-D plane.
      ii. the difference of exact Hessian and its Gauss-Newton approximation versus iteration number.

3. *Elimination of Variables.* Consider the given equality constraints
$$\begin{aligned} 8x_1 - 6x_2 + x_3 + 9x_4 + 4x_5 &= 6 \\ 3x_1 - 2x_2 - x_4 + 6x_5 + 4x_6 &= -4 \end{aligned}$$
   (a) Write out A and b needed to represent the constraints as $Ax = b$.
   (b) Eliminate x_3 and x_6 explicitly using the notation $y = (x_3, x_6)^\mathsf{T}$ and $z = (x_1, x_2, x_4, x_5)^\mathsf{T}$ (same notation as on Slide 10-11). Hint: what permutation matrix P can you use to reach $AP(P^\mathsf{T}x) = b$?
   (c) Write out $\tilde{F}(z)$ as on Slide 10-11. Is $\min_z \tilde{F}(z)$ still a constrained optimization problem?

4. *Equality-Constrained Optmization.* Consider the optimization problem
$$\min f(x_1, x_2) = 2(x_1^2 + x_2^2 - 1) - x_1 \qquad \text{subject to } x_1^2 + x_2^2 - 1 = 0\,.$$
Please write a computer program according to Slide 10-50. You may use a fixed step size of $\alpha = 0.9$.