SI231B - Matrix Computations, Spring 2022-23

Homework Set #2

Prof. Ziping Zhao

Acknowledgements:

1) Deadline: **2023-03-26 23:59:59**
2) Please submit your assignments via Blackboard.
3) You can write your homework using latex/word or you can write in handwriting and submit the scanned pdf.

Problem 1. (20 points)

1) Given matrix $\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$, compute its QR decomposition using Gram-Schmidt Orthogonality.
2) Please solve the least square problem via QR decomposition: $\min ||\mathbf{A}\mathbf{x} - \mathbf{b}||_2$, where $\mathbf{b} = [1, -1, 0, 1]^T$.

Problem 2. (20 points)

Consider two full-column rank matrices $\mathbf{A} \in \mathbb{R}^{m \times n_1}$ and $\mathbf{B} \in \mathbb{R}^{m \times n_2}$ with $n_1 < m$ and $n_2 < m$. Suppose $\mathcal{R}(\mathbf{A})^{\perp} \cap \mathcal{R}(\mathbf{B})^{\perp} = \{\mathbf{0}\}$. Find a semi-orthogonal matrix $\mathbf{P}$ based on QR decompositions of $\mathbf{A}$ and $\mathbf{B}$, where the columns of $\mathbf{P}$ form an orthonormal basis for $\mathcal{R}(\mathbf{A}) \cap \mathcal{R}(\mathbf{B})$.

(*Hint:* You may use the orthogonal compliment of $\mathcal{R}(\mathbf{A}) \cap \mathcal{R}(\mathbf{B})$ as $(\mathcal{R}(\mathbf{A}) \cap \mathcal{R}(\mathbf{B}))^{\perp} = \mathcal{R}(\mathbf{A})^{\perp} + \mathcal{R}(\mathbf{B})^{\perp}$.)

Problem 3. (20 points)

For $\mathbf{A} \in \mathbb{R}^{m \times n}$, $\mathbf{b} \in \mathbb{R}^n$, $\mathbf{y} \in \mathbb{R}^m$, and $\lambda \in \mathbb{R}^+$, derive the optimal solution of

$$\underset{\mathbf{x} \in \mathbb{R}^n}{\text{minimize}} \quad \left\|\mathbf{y} - \mathbf{A}\mathbf{x}\right\|_2^2 + \lambda \left\|\mathbf{b} - \mathbf{x}\right\|_2^2.$$

Problem 4. (20 points)

Given

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ -1 & 2 \\ 1 & -1 \end{bmatrix}.$$

Find a point in the column space of $\mathbf{A}$ to make it closest to point $\mathbf{p} = [1, 0, 2]^T$.

Hints: Orthogonal projection of vector $\mathbf{a}$ onto a nonzero vector $\mathbf{b}$ is defined as

$$\text{proj}_{\mathbf{b}}(\mathbf{a}) = \frac{\langle \mathbf{a}, \mathbf{b} \rangle}{\langle \mathbf{b}, \mathbf{b} \rangle} \mathbf{b}$$

where the $\langle \cdot, \cdot \rangle$ is the inner product of vectors.

Problem 5. (20 points)

Given a matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 4 & 4 & -1 \\ 4 & -2 & 2 & 0 \end{bmatrix}$$

1) Use Householder reflection to give the full QR decomposition of $\mathbf{A}^T$, i.e., $\mathbf{A}^T = \mathbf{QR}$ with $\mathbf{Q}$ being a square and orthogonal matrix.
2) Let $\mathbf{b} = [5, 10, 4]^T$, give one possible solution of linear system $\mathbf{Ax} = \mathbf{b}$ via QR decomposition of $\mathbf{A}^T$.
3) Let $\mathbf{c} = [1, 2, 3, 4]^T$ and W be the kernel space of $\mathbf{A}$. Decompose $\mathbf{c}$ with respect to W as $\mathbf{c} = \mathbf{w} + \mathbf{z}$, where $\mathbf{w} \in W, \mathbf{z} \in W^{\perp}$.

Hints: The orthogonal projector onto $\mathcal{R}(\mathbf{A})$ ($\mathbf{A}$ has full column rank) is $\mathbf{AA}^{\dagger}$.