Numerical Optimization

Lecture 5:　Duality

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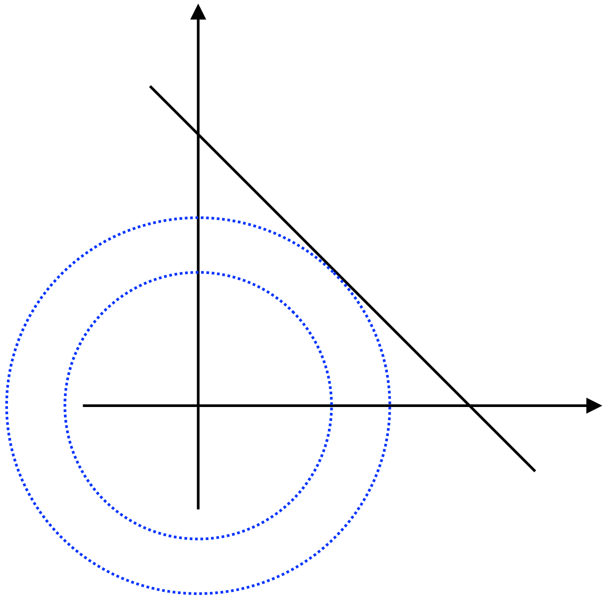
Constrained to unconstrained

Constrained optimization

$$\begin{array}{ll} \text{minimize} & x^2 + y^2 \\ \text{subject to} & x + y = 1 \end{array}$$

Formulate the unconstrained problem

$$L(x, y, p) = x^2 + y^2 + p(1 - x - y)$$



The minimizer is

$$x = y = \frac{p}{2}$$

Lagrange duality：

$$\min_x \quad c^T x \quad \text{s.t. } a_i^T x \le b_i, i = 1,\ldots,m\,.$$

Lagrange函数： $L(x,\lambda) = c^T x + \sum_{i=1}^{m} \lambda_i (a_i^T x - b_i)$

无约束问题： $\min_x L(x,\lambda) = c^T x + \sum_{i=1}^{m} \lambda_i (a_i^T x - b_i)$

- 可以容许约束$a_i^T x \le b_i$可以被违反$a_i^T x > b_i$
- 但约束的违反量$a_i^T x - b_i$要产生代价： $\lambda_i (a_i^T x - b_i)$
- 单位违反量产生的价格$\lambda_i \ge 0$

$$\min_{x} \quad c^T x \quad \text{s.t. } a_i^T x \leq b_i, i = 1,\ldots,m\,.$$

$$f(x) := \max_{\lambda \geq 0} \quad L(x,\lambda) = c^T x + \sum_{i=1}^{m} \lambda_i (a_i^T x - b_i)$$

- 给定x，对偶玩家以λ为变量极大化收入，即通过操控违反行为的定价λ极大化罚款收入$L(x,\lambda)$
- 由于x为参量，最优的λ取决于x：$\lambda = \lambda(x)$

$$g(\lambda) := \min_{x} \quad L(x,\lambda) = c^T x + \sum_{i=1}^{m} \lambda_i (a_i^T x - b_i)$$

- 给定$\lambda \geq 0$，原玩家以x为变量极小化损失，即通过操控x甚至违反约束，来尽量极小化损失$L(x,\lambda)$
- 由于$\lambda \geq 0$为参量，最优的x取决于λ：$x = x(\lambda)$

Primal:
$$\min_x \quad c^T x \quad \text{s.t. } a_i^T x \le b_i, i = 1,\ldots,m.$$

$$f(x) := \max_{\lambda \ge 0} \quad L(x,\lambda) = c^T x + \sum_{i=1}^{m} \lambda_i (a_i^T x - b_i)$$

- 给定x，对偶玩家以λ为变量极大化收入，即通过操控违反行为的定价λ极大化罚款收入$L(x,\lambda)$
- 原玩家再以x为变量来极小化损失（博弈的思想）

$$\min_x f(x) = \min_x \boxed{\max_{\lambda \ge 0} \quad L(x,\lambda) = c^T x + \sum_{i=1}^{m} \lambda_i (a_i^T x - b_i)}$$

Primal objective: $f(x)$

Primal problem: min max

Dual:

$$\min_{x} \quad c^T x \quad \text{s.t. } a_i^T x \le b_i, i = 1,\ldots,m\,.$$

$$g(\lambda) := \min_{x} \quad L(x,\lambda) = c^T x + \sum_{i=1}^{m} \lambda_i (a_i^T x - b_i)$$

- 给定$\lambda \ge 0$，原玩家以x为变量极小化损失，即通过操控x甚至违反约束，来尽量极小化损失$L(x,\lambda)$
- 对偶玩家再以λ为变量来极大化收入（博弈的思想）

$$\max_{\lambda\ge 0} g(\lambda) = \max_{\lambda \ge 0} \boxed{\min_{x} \quad L(x,\lambda) = c^T x + \sum_{i=1}^{m} \lambda_i (a_i^T x - b_i)}$$

Dual objective: $g(\lambda)$

Dual problem: max min

Primal问题(min-max):

$$\min_{x} f(x) = \min_{x} \max_{\lambda} L(x,\lambda) = c^T x + \sum_{i=1}^{m} \lambda_i (a_i^T x - b_i)$$

$$f(x) \begin{cases} = +\infty & \text{if } \exists i, a_i^T x > b_i \\ < +\infty & \text{if } \forall i, a_i^T x \le b_i \end{cases}$$

$\Longrightarrow$ We don't care about this case.

$\Longrightarrow$ This implies primal feasibility.

Dual问题(max-min):

$$\max_{\lambda \ge 0} g(\lambda) = \max_{\lambda \ge 0} \min_{x} L(x,\lambda) = c^T x + \sum_{i=1}^{m} \lambda_i (a_i^T x - b_i)$$

$$g(\lambda) \begin{cases} = -\infty & \text{if } c + \sum_{i=1}^{m} \lambda_i a_i \ne 0 \\ > -\infty & \text{if } c + \sum_{i=1}^{m} \lambda_i a_i = 0 \end{cases}$$

$\Longrightarrow$ We don't care about this case.

$\Longrightarrow$ This implies dual feasibility.

$$\min\ b^T x \quad \text{s.t. } A^T x \leq c$$

The Lagrangian is

$$L(x, \lambda) = b^T x + \lambda^T (A^T x - c),\ \lambda \geq 0$$

Let's determine λ, using the max-min

The dual objective is

$$g(\lambda) = \min_x L(x, \lambda) = b^T x + \lambda^T (A^T x - c)$$

$$= \min_x\ (b + A\lambda)^T x - \lambda^T c$$

Notice that we want to maximize $g(\lambda)$. So we only care about the case $g(\lambda) > -\infty$, which means $b + A\lambda = 0$

$$\Longrightarrow \quad \max\ -c^T \lambda \quad \text{s.t. } A\lambda = -b,\ \lambda \geq 0$$

$$\min\ c^T x \quad \text{s.t. } Ax = b,\ x \geq 0 \quad \Longrightarrow \quad Ax \geq b,\ Ax \leq b$$

The Lagrangian is

$$L(x,u,w,v) = c^T x + u^T(-Ax + b) + w^T(Ax - b) - v^T x, \quad u,w,v \geq 0$$

$$= c^T x - (u-w)^T(Ax - b) - v^T x$$

$$= c^T x - \lambda^T(Ax - b) - v^T x, \ \text{ with } \lambda \text{ free}, v \geq 0$$

The dual objective is

$$g(\lambda, v) = \min_x L(x,\lambda,v) = \min_x (c - A^T\lambda - v)^T x + \lambda^T b$$

Notice that we want to maximize $g(\lambda, v)$. So we only care about the case $g(\lambda, v) > -\infty$, which means $c - A^T\lambda - v = 0$

$$\Longrightarrow \quad \max\ \lambda^T b \quad \text{s.t. } A^T\lambda \leq c$$

$$\begin{aligned} \text{minimize} \quad & -x_1 - 4x_2 - 3x_3 \\ \text{subject to} \quad & 2x_1 + 2x_2 + x_3 = 4 \\ & x_1 + 2x_2 + 2x_3 \le 6 \\ & x_1 \ge 0, x_2 \ge 0, x_3 \ge 0 \end{aligned}$$

$$\begin{aligned} L(x,\lambda) &= -x_1 - 4x_2 - 3x_3 + \lambda_1(2x_1 + 2x_2 + x_3 - 4) \\ &\quad +\lambda_2(x_1 + 2x_2 + 2x_3 - 6) - \mu_1 x_1 - \mu_2 x_2 - \mu_3 x_3 \\ &= \boxed{(-1 + 2\lambda_1 + \lambda_2 - \mu_1)}x_1 + \boxed{(-4 + 2\lambda_1 + 2\lambda_2 - \mu_2)}x_2 \\ &\quad + \boxed{(-3 + \lambda_1 + 2\lambda_2 - \mu_3)}x_3 \boxed{-4\lambda_1 - 6\lambda_2} \end{aligned}$$

λ_1 is free , $\lambda_2 \ge 0,\ \mu_1 \ge 0,\ \mu_2 \ge 0,\ \mu_3 \ge 0$

再写出对偶问题:

对偶问题(The dual problem)

给定数据$A \in \mathbb{R}^{m\times n}, b \in \mathbb{R}^m, c \in \mathbb{R}^n$

- 原始问题 (primal)：

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array}$$

$x \in \mathbb{R}^n$：原始变量

- 对偶问题 (dual)：

$$\begin{array}{ll} \text{maximize} & b^T \lambda \\ \text{subject to} & A^T \lambda \leq c \end{array}$$

$\lambda \in \mathbb{R}^m$：对偶变量

❖注意分清对偶变量(dual variable)，对偶问题(dual problem)，对偶目标值(dual value)，对偶间隙(duality gap)，对偶理论(duality)

❖对于线性规划，对偶是相互的，即对偶问题的对偶是原始问题

对偶问题：经济解释

配餐问题：确定食品数量，满足营养需求，花费最小?

参数：n种食品，m种营养成份

c_j—第j种食品的单价

a_{ij}—每单位第j种食品含第i种营养的数量

b_i— 为了健康，每天最少要摄入的第i种营养的数量

变量：x_j—摄入第j种食品的数量

Primal：

$$
\begin{array}{ll}
\text{minimize} & c_1x_1 + c_2x_2 + \cdots + c_nx_n \\
\text{subject to} & a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \geq b_1 \\
& a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \geq b_2 \\
& \vdots \\
& a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \geq b_m \\
& x_1 \geq 0, x_2 \geq 0, \cdots, x_n \geq 0
\end{array}
$$

$$
\begin{array}{ll}
\text{maximize} & \lambda_1 b_1 + \lambda_2 b_2 + \cdots + \lambda_m b_m \\
\text{subject to} & \lambda_1 a_{11} + \cdots + \lambda_m a_{m1} \leq c_1 \\
& \lambda_1 a_{12} + \cdots + \lambda_m a_{m2} \leq c_2 \\
& \vdots \\
& \lambda_1 a_{1n} + \cdots + \lambda_m a_{mn} \leq c_n \\
& \lambda_i \geq 0, i = 1, \cdots, m.
\end{array}
$$

Dual：保健品公司药剂师、开发营养丸、给每个成分定价问题

cheat-sheet

给定数据 A, b, c；记 A 的第 i 行为 a^i，A 的第 j 列为 a_j

- 原始问题(primal)：

$$
\begin{array}{lll}
\textbf{minimize} & c^{\mathrm{T}}x & \\
\textbf{subject to} & a^i x \geq b_i, & i \in M_1 \\
& a^i x \leq b_i, & i \in M_2 \\
& a^i x = b_i, & i \in M_3 \\
& x_j \geq 0, & j \in N_1 \\
& x_j \leq 0, & j \in N_2 \\
& x_j \text{ 无限制} & j \in N_3
\end{array}
$$

- 对偶问题(dual)：

$$
\begin{array}{lll}
\textbf{maximize} & \lambda^{\mathrm{T}}b & \\
\textbf{subject to} & \lambda_i \geq 0, & i \in M_1 \\
& \lambda_i \leq 0, & i \in M_2 \\
& \lambda_i \text{ 无限制}, & i \in M_3 \\
& \lambda^{\mathrm{T}}a_j \leq c_j, & j \in N_1 \\
& \lambda^{\mathrm{T}}a_j \geq c_j, & j \in N_2 \\
& \lambda^{\mathrm{T}}a_j = c_j, & j \in N_3
\end{array}
$$

Duality Scheme

$$\min c^T x$$
$$Ax \left\{ \begin{array}{c} \geq \\ = \\ \leq \end{array} \right\} b$$
$$x \left\{ \begin{array}{c} \geq \\ = \\ \leq \end{array} \right\} 0$$

$$\max b^T y$$
$$A^T y \left\{ \begin{array}{c} \geq \\ = \\ \leq \end{array} \right\} c$$
$$y \left\{ \begin{array}{c} \geq \\ = \\ \leq \end{array} \right\} 0$$

Primal problem	**Dual problem**
minimize	maximize
Constraints	**Variables**
$\sum_{j=1}^{n} a_{ij}x_j \geq b_i$	$y_i \geq 0$
$\sum_{j=1}^{n} a_{ij}x_j = b_i$	y_i is free
$\sum_{j=1}^{n} a_{ij}x_j \leq b_i$	$y_i \leq 0$
Variables	**Constraints**
x_j is free	$a_j^T y = c_j$
$x_j \geq 0$	$a_j^T y \leq c_j$
$x_j \leq 0$	$a_j^T y \geq c_j$
$x_j = 0$	no constraint

对偶问题：例子

$$
\begin{aligned}
\text{minimize} \quad & x_1 + 2x_2 + 3x_3 \\
\text{subject to} \quad & -x_1 + 3x_2 = 5 \\
& 2x_1 - x_2 + 3x_3 \geq 6 \\
& x_3 \leq 4 \\
& x_1 \geq 0 \\
& x_2 \leq 0 \\
& x_3 \text{无限制}
\end{aligned}
$$

$$
\begin{aligned}
\text{maximize} \quad & 5\lambda_1 + 6\lambda_2 + 4\lambda_3 \\
\text{subject to} \quad & \lambda_1 \text{无限制} \\
& \lambda_2 \geq 0 \\
& \lambda_3 \leq 0 \\
& -\lambda_1 + 2\lambda_2 \leq 1 \\
& 3\lambda_1 - \lambda_2 \geq 2 \\
& 3\lambda_2 + \lambda_3 = 3
\end{aligned}
$$

弱对偶(Weak Duality)

$$\begin{aligned} \text{minimize} \quad & c^T x \\ \text{subject to} \quad & Ax = b \\ & x \geq 0 \end{aligned} \quad \longleftrightarrow \quad \begin{aligned} \text{maximize} \quad & b^T \lambda \\ \text{subject to} \quad & \lambda^T A \leq c^T \end{aligned}$$

弱对偶定理. 设$\boldsymbol{x}$和$\boldsymbol{\lambda}$分别是原问题和对偶问题的可行解，则$\boldsymbol{c}^T\boldsymbol{x} \geq \boldsymbol{\lambda}^T\boldsymbol{b}$.

Weak Duality: for any primal-dual feasible (x, λ)

$$f(x) = \max_{\lambda \geq 0} L(x, \lambda) \geq L(x, \lambda) \geq \min_x L(x, \lambda) = g(\lambda)$$

推论1. 设$\boldsymbol{x}$和$\boldsymbol{\lambda}$分别是原始问题和对偶问题的可行解，如果$\boldsymbol{c}^T\boldsymbol{x} = \boldsymbol{\lambda}^T\boldsymbol{b}$. 设$\boldsymbol{x}$和$\boldsymbol{\lambda}$分别是原问题和对偶问题的最优解

推论2. 如果原始问题与对偶问题之一无界，则另一个问题没有可行解

强对偶(Strong Duality)

如何由原始问题的解得到对偶问题的解?

定理. 设标准形线性规划问题有最优解 $\boldsymbol{x}^*$，$\boldsymbol{B}$ 是最优基本可行解对应的基，则

$$\boldsymbol{\lambda}^* = (\boldsymbol{c}_B^{\mathrm{T}} \boldsymbol{B}^{-1})^T$$

是其对偶问题的最优解。（如何验证？）

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array} \quad \longleftrightarrow \quad \begin{array}{ll} \text{maximize} & b^T \lambda \\ \text{subject to} & \lambda^T A \leq c^T \end{array}$$

强对偶(Strong Duality)

强对偶定理. 如果原始问题和对偶问题之一有解，则另一个问题也有解，且最优值相等.

原问题 \ 对偶问题	不可行	无(上)界	有最优解
不可行	√	√	×
无(下)界	√	×	×
有最优解	×	×	√

思考题：写出两阶段法中第一阶段的辅助问题的对偶问题？这个对偶问题有最优解吗？为什么？

与单纯形法的关系 $\boldsymbol{\lambda}^* = (\boldsymbol{c}_B^{\mathrm{T}}\boldsymbol{B}^{-1})^{\boldsymbol{T}}$

Where is the reduced cost?

$$\begin{aligned} \text{minimize} \quad & c^T x \\ \text{subject to} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

x是谁的乘子?

$$\begin{aligned} \text{maximize} \quad & b^T \lambda \\ \text{subject to} \quad & \lambda^T A \leq c^T \end{aligned}$$

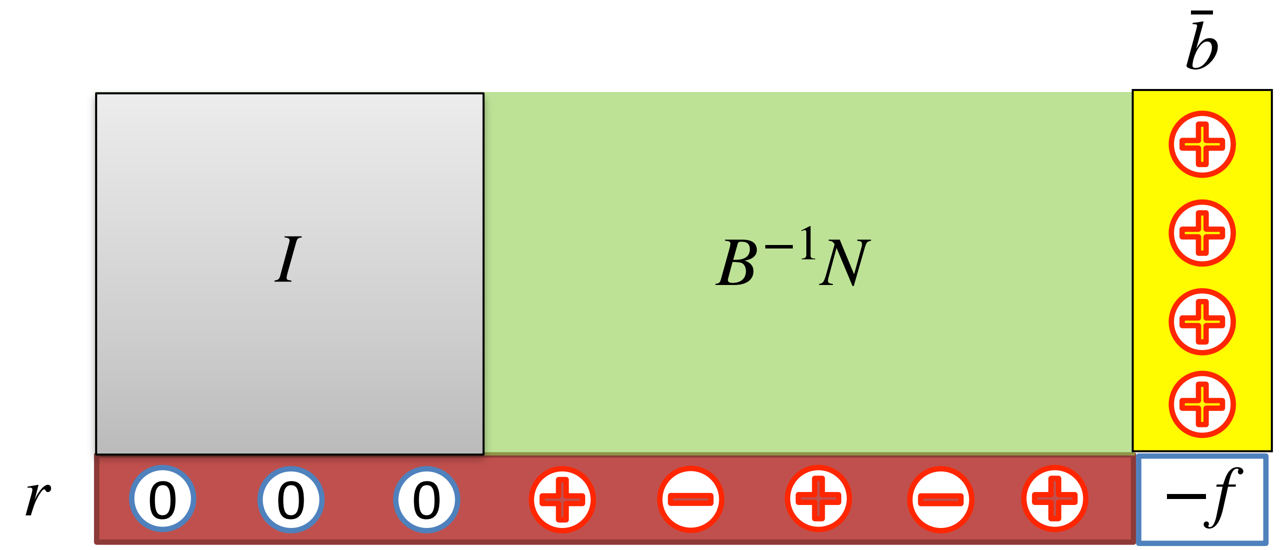
λ, r是谁的乘子?

$$\begin{aligned} r^T &= c^T - c_B^T B^{-1} A \\ &= c^T - \lambda^T A \end{aligned}$$

$$\begin{aligned} \text{maximize} \quad & b^T \lambda \\ \text{subject to} \quad & \lambda^T A + r^T = c^T \\ & r \geq 0 \end{aligned}$$

$r^T \geq 0 \implies$ dual feasibility

与单纯形法的关系



$$
\begin{array}{ll}
\text{minimize} & c^T x \\
\text{subject to} & Ax = b \\
 & x \geq 0
\end{array}
\qquad
\begin{array}{ll}
\text{maximize} & b^T \lambda \\
\text{subject to} & \lambda^T A + r^T = c^T \\
 & r \geq 0
\end{array}
$$

Complementary condition
互补

Primal problem

$$\min c^T x$$
$$Ax = b$$
$$x \geq 0$$

Dual problem

$$\max \ \lambda^T y$$
$$A^T \lambda \leq c$$

We need to solve:

$$Ax = b$$
$$x \geq 0$$

$$A^T \lambda + r = c$$
$$r \geq 0$$

$$\boxed{b^T \lambda = c^T x}$$

The last condition $b^T \lambda = c^T x$ can be written as

$$0 = c^T x - b^T \lambda = c^T x - (Ax)^T \lambda = (c - A^T \lambda)^T x = r^T x$$

Because $r \geq 0$ and $x \geq 0$ the condition $\boxed{r^T x = 0}$ implies

$$\boxed{r_i x_i = 0 \qquad \forall i}$$

Complementarity:

$$\begin{aligned} \text{minimize} \quad & c^T x \\ \text{subject to} \quad & Ax = b \longleftarrow \lambda \\ & x \geq 0 \longleftarrow r \end{aligned}$$

互补：$r^T x = 0$

- 互补条件指的是对于**最优解处inactive的约束，其乘子为0**：

$$x_i^* > 0 \implies r_i^* = 0$$

- 对于一般的问题：$\min_x \quad c^T x \quad$ **s.t.** $a_i^T x \leq b_i, i = 1,\ldots,m$.

$$a_i^T x^* - b_i < 0 \implies \lambda_i^* = 0,$$

即 $(a_i^T x^* - b_i)\lambda_i^* = 0$

- 对于已经严格遵循约束的解，处罚定价应为0

$$L(x, \lambda) = c^T x + \sum_{i=1}^{m} \lambda_i (a_i^T x - b_i)$$

$$\left.\begin{array}{l}\text{Primal feasible}\\ \text{Dual feasible}\\ \text{Duality Gap} = \text{Primal} - \text{Dual} = 0\end{array}\right\} \Longleftrightarrow \text{Primal-dual optimal}$$

$$\left.\begin{array}{l}\text{Primal feasible}\\ \text{Dual feasible}\\ \text{Complementarity}\end{array}\right\} \Longleftrightarrow \text{Primal-dual optimal}$$

例子：对偶问题与单纯形法的关系

考虑问题

$$\begin{aligned} \text{minimize} \quad & -x_1 - 4x_2 - 3x_3 \\ \text{subject to} \quad & 2x_1 + 2x_2 + x_3 \le 4 \\ & x_1 + 2x_2 + 2x_3 \le 6 \\ & x_1 \ge 0, x_2 \ge 0, x_3 \ge 0 \end{aligned}$$

对偶问题

$$\begin{aligned} \text{maximize} \quad & 4\lambda_1 + 6\lambda_2 \\ \text{subject to} \quad & 2\lambda_1 + \lambda_2 \le -1 \\ & 2\lambda_1 + 2\lambda_2 \le -4 \\ & \lambda_1 + 2\lambda_2 \le -3 \\ & \lambda_1 \le 0, \lambda_2 \le 0 \end{aligned}$$

引入松弛变量→标准形→利用单纯形法求解

	x_1	x_2	x_3	x_4	x_5	$B^{-1}b$
	2	$\boxed{2}$	1	1	0	4
	1	2	2	0	1	6
r^T	-1	-4	-3	0	0	0

	x_1	x_2	x_3	x_4	x_5	$B^{-1}b$
	1	1	$\frac{1}{2}$	$\frac{1}{2}$	0	2
	-1	0	$\boxed{1}$	-1	1	2
r^T	3	0	-1	2	0	8

	x_1	x_2	x_3	x_4	x_5	$B^{-1}b$
	$\frac{3}{2}$	1	0	1	$-\frac{1}{2}$	1
	-1	0	1	-1	1	2
r^T	2	0	0	1	1	10

why?

原问题
最优解

$$x_1^* = 0$$
$$x_2^* = 1$$
$$x_3^* = 2$$

对偶问题
最优解

$$\lambda_1^* = -1$$
$$\lambda_2^* = -1$$

对偶单纯形法：对偶可行基本解

$$\begin{array}{ll} \text{maximize} & b^T\lambda \\ \text{subject to} & \lambda^T A \le c^T \end{array}$$

如果知道哪些是active constraints，问题该如何求解?

定义 假设$\boldsymbol{x}_B = \boldsymbol{B}^{-1}\boldsymbol{b}$是$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$的基本解（未必可行）. 如果 $\boldsymbol{\lambda}^T = \boldsymbol{c}_B^T\boldsymbol{B}^{-1}$是对偶问题的可行解，即$(\boldsymbol{r}^T =)\ \boldsymbol{c}^T - \boldsymbol{\lambda}^T\boldsymbol{A} \ge \boldsymbol{0}$，则称 $\boldsymbol{x}$ 是标准形问题的对偶可行基本解.

初始对偶可行基本解 → 新的对偶可行基本解 →

→ ……迭代…… → “原始可行＋对偶可行”的基本解！

基本解(互补性)＋可行＋对偶可行＝最优解

对偶单纯形法:

设对偶可行基本解 $\boldsymbol{x}$ 对应的基$B = [a_1, \ldots, a_p, \ldots, a_m]$

不妨设$\bar{b}_p < 0$; 此外还假设$\boldsymbol{\lambda}^T = \boldsymbol{c}_B^T \boldsymbol{B}^{-1}$非退化，即

$$
\begin{aligned}
(r_j =)c_j - \lambda^{\mathrm{T}} a_j &= 0, \quad j = 1, 2, \cdots, m \\
(r_j =)c_j - \lambda^{\mathrm{T}} a_j &> 0, \quad j = m+1, \cdots, n
\end{aligned}
$$

回忆修正单纯形法

令$\hat{\lambda}^T = \lambda^T + \frac{r_p}{y_{pq}} u_p$, 其中$u_p$是$B^{-1}$的第$p$行

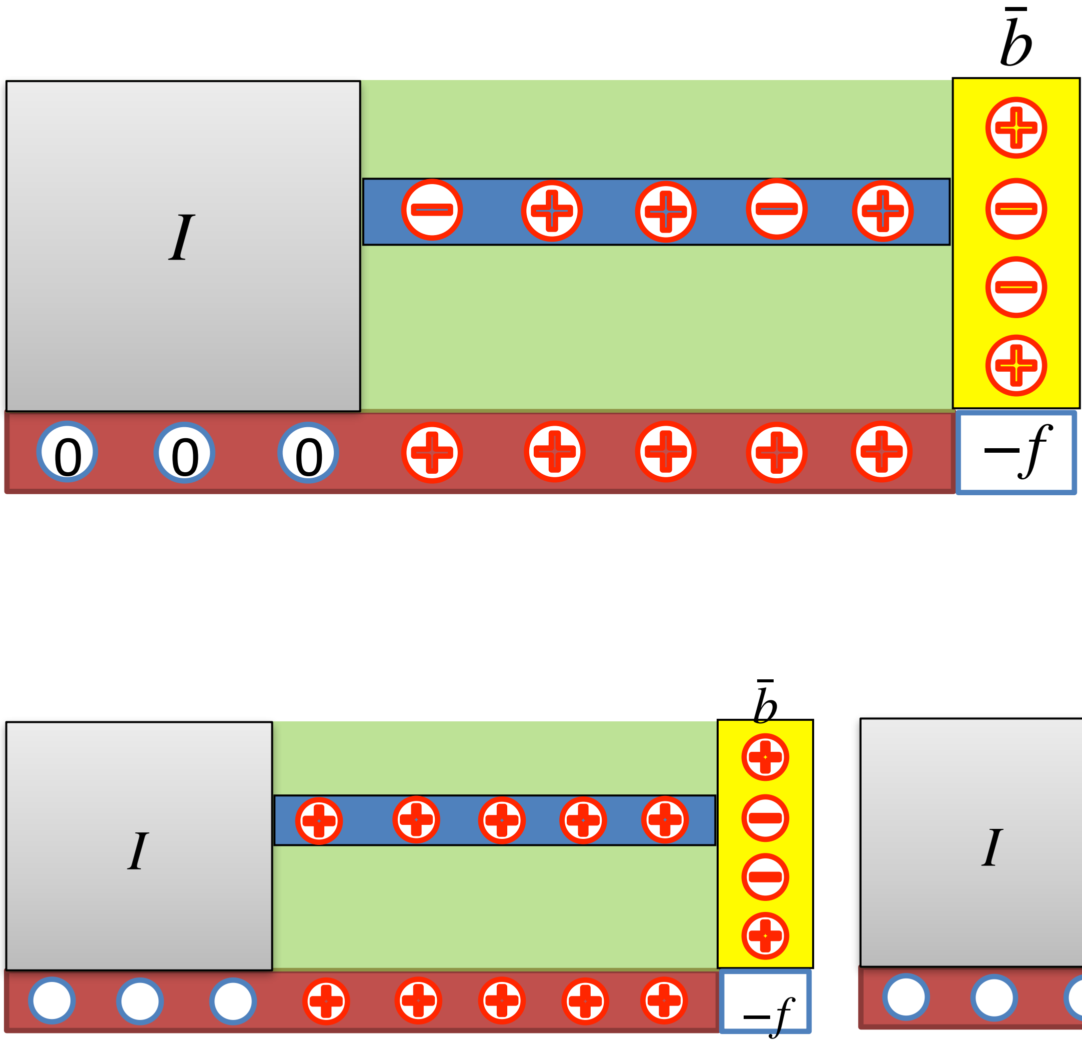
于是有$\hat{\lambda}^T b = \lambda^T b - \epsilon \bar{b}_p$

对偶目标函数值

目的：找新的$\hat{\lambda}$使m个等式中的某个与其他$n-m$个不等式 中的某个角色互换(即$\hat{\lambda}$是对偶问题的与λ相邻的极点)，同时使对偶问题的目标函数值增大！

与单纯形法的关系

$$r_j' = r_j + \frac{r_q}{y_{pq}} y_{pj} \geq 0$$



对偶单纯形法：计算步骤

步0 给定对偶可行基本解对应的单纯形表.

步1 若对每个 $\boldsymbol{i}$ 都有$\bar{b}_i \geq 0$，停；当前**DFBS**是最优的.

步2 选取 $\boldsymbol{p}$ 满足$\bar{b}_p < 0$，这时，第 $\boldsymbol{p}$ 个基变量出基.

步3 若$(y_{p1}, y_{p2}, \ldots, y_{pn}) \geq 0$，问题无可行解；否则，选 $\boldsymbol{q}$ 满足

$$\hat{\epsilon} = \frac{r_q}{-y_{pq}} = \min\left\{\frac{r_j}{-y_{pj}} : y_{pj} < 0, j = 1, \cdots, n\right\}$$

步4 以 y_{pq} 为转轴元进行转轴，更新单纯形表，返步**1.**

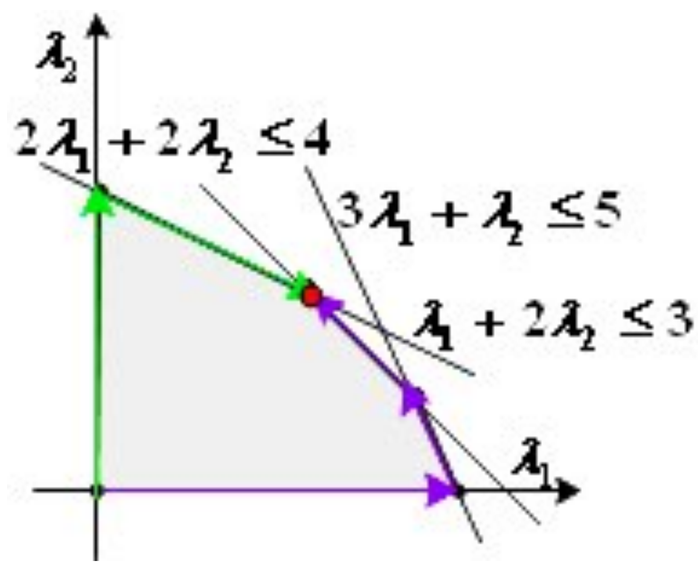
对偶单纯形法：例子

$$\begin{array}{ll} \text{minimize} & 3x_1 + 4x_2 + 5x_3 \\ \text{subject to} & x_1 + 2x_2 + 3x_3 \geq 5 \\ & 2x_1 + 2x_2 + x_3 \geq 6 \\ & x_1 \geq 0, x_2 \geq 0, x_3 \geq 0 \end{array}$$

1) 写出对偶问题并用图解法求解。
2) 用对偶单纯形法求解所给问题。

解. 对偶问题为

$$\begin{array}{ll} \text{maximize} & 5\lambda_1 + 6\lambda_2 \\ \text{subject to} & \lambda_1 + 2\lambda_2 \leq 3 \\ & 2\lambda_1 + 2\lambda_2 \leq 4 \\ & 3\lambda_1 + \lambda_2 \leq 5 \\ & \lambda_1 \geq 0, \lambda_2 \geq 0 \end{array}$$



对偶单纯形法：例子

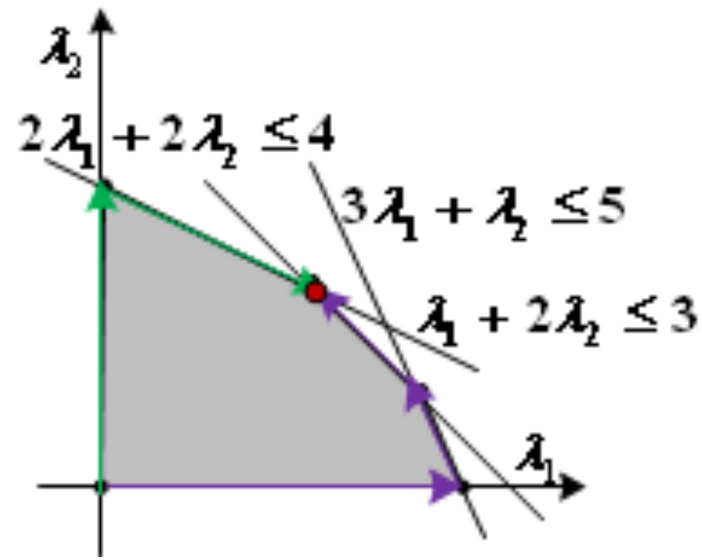
$$\begin{array}{ll} \text{minimize} & 3x_1 + 4x_2 + 5x_3 \\ \text{subject to} & x_1 + 2x_2 + 3x_3 \geq 5 \\ & 2x_1 + 2x_2 + x_3 \geq 6 \\ & x_1 \geq 0, x_2 \geq 0, x_3 \geq 0 \end{array}$$

引入盈余变量；并给等式两边同乘−1；得初始表格/第一张单纯形表

	x_1	x_2	x_3	x_4	x_5	$B^{-1}b$
	-1	-2	-3	1	0	-5
	$\boxed{-2}$	-2	-1	0	1	-6
r^T	3	4	5	0	0	0

	x_1	x_2	x_3	x_4	x_5	$B^{-1}b$
	0	$\boxed{-1}$	$-\frac{5}{2}$	1	$-\frac{1}{2}$	-2
	1	1	$\frac{1}{2}$	0	$-\frac{1}{2}$	3
r^T	0	1	$\frac{7}{2}$	0	$\frac{3}{2}$	-9

	x_1	x_2	x_3	x_4	x_5	$B^{-1}b$
	0	1	$\frac{5}{2}$	-1	$\frac{1}{2}$	2
	1	0	-2	1	-1	1
r^T	0	0	1	1	1	-11



最优解：$x^* = (1,2,0)^T$

单纯形乘子的迭代为$(0,0) \to (0,\frac{3}{2}) \to (1,1)$

若第一步让x_4出基，单纯形乘子的迭代为

$$(0,0) \to (\frac{5}{3},0) \to (\frac{3}{2},\frac{1}{2}) \to (1,1)$$

对偶单纯形法：收敛性

定理. 如果标准形线性规划问题的任意的对偶可行基本解所对应的非基变量的相对费用系数大于零，则对偶单纯形法在有限步内终止.

- 如果线性规划问题可以用对偶单纯形法求解，则计算结果只能是不可行或者有解！
- 如果线性规划问题可以用(原)单纯形法求解，则计算结果只能是无界或有解
- 两阶段法可以求解任一线性规划问题；
  - 第**I**阶段的结果为可行或者不可行两种；
  - 对于可行的，在第**II**阶段可得问题无界或有解！

对偶单纯形法：启动

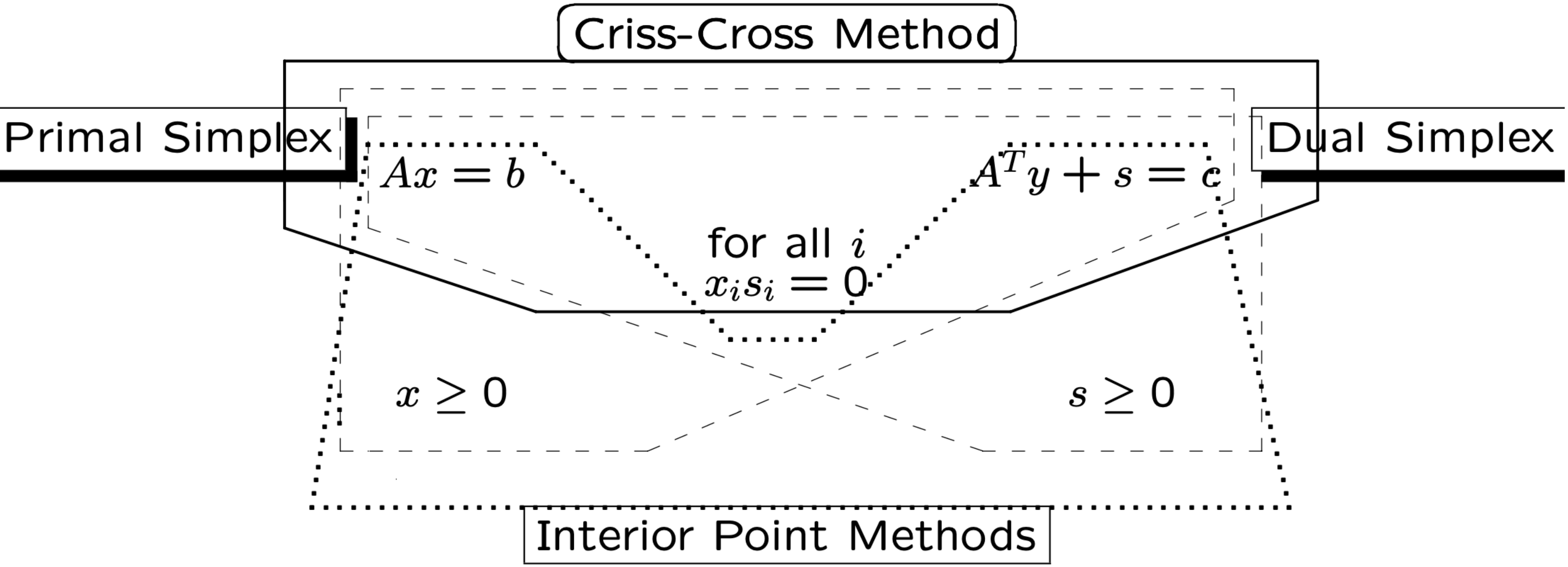
◎ 典型情况(有显然的对偶可行基本解)

- ⊙ “不等式约束” + “$\boldsymbol{x} \geq \mathbf{0}$” + “$\boldsymbol{c} \geq \mathbf{0}$” + “$\mathbf{min}$”

本节的例题

◎ 一般情况 4.2.3节[1]

	单纯形法			对偶单纯形法		
	原始可行	对偶可行	互补性	原始可行	对偶可行	互补性
初始	√	×	√	×	√	√
迭代中	√	×	√	×	√	√
最优终止	√	√	√	√	√	√



All algorithms for LP keep a part of the optimality criteria valid while working towards the others.