Lecture 9

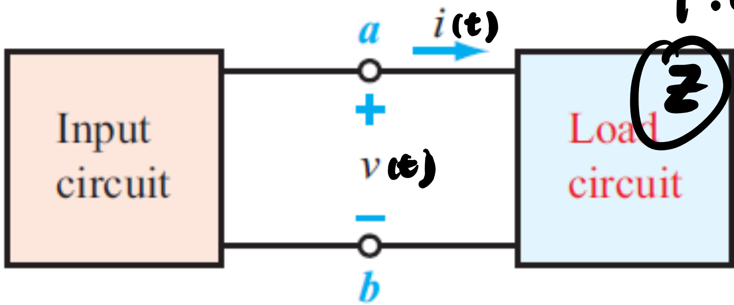
- AC Power Calculation

Outline

- Instantaneous power
- Average power
- Apparent power
- Power factor
- Complex power

AC Power in Time Domain: Instantaneous

Time Domain



P.S.C.

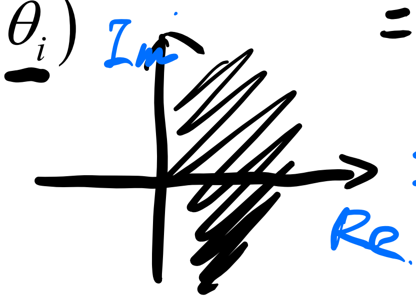
$v(t) \rightarrow \dot{V} = V_m \angle \theta_v$

$i(t) \rightarrow \dot{I} = I_m \angle \theta_i$

$Z = \dot{V}/\dot{I} = \frac{V_m}{I_m} \angle \theta_v - \theta_i$

$= R + jX$

$$v(t) = V_m \cos(\omega t + \theta_v) \quad i(t) = I_m \cos(\omega t + \theta_i)$$



$\Rightarrow |\theta_v - \theta_i| \le 90^\circ$

Instantaneous power:
power at any instant of time.

$$p(t) = v(t)i(t) = V_m I_m \cos(\omega t + \theta_v) \cos(\omega t + \theta_i)$$

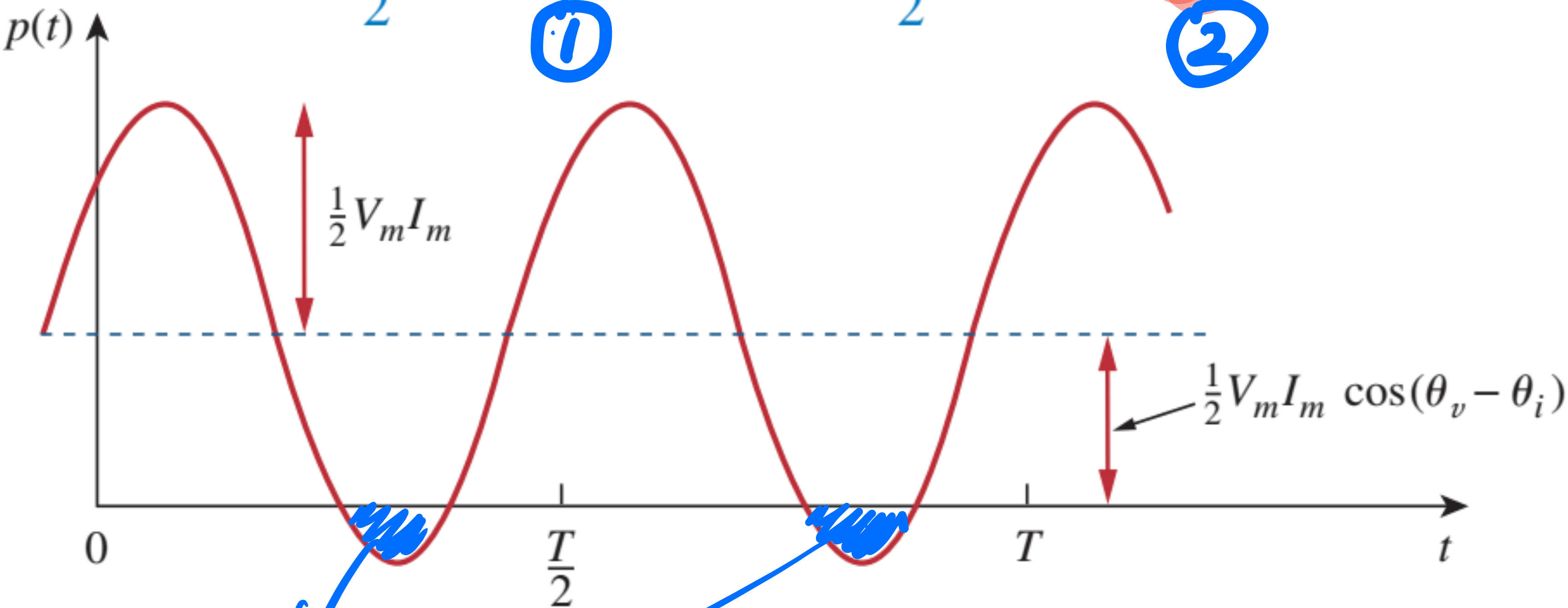
abs.

$$p(t) = \frac{1}{2} V_m I_m \cos(\theta_v - \theta_i) + \frac{1}{2} V_m I_m \cos(2\omega t + \theta_v + \theta_i)$$

AC Power in Time Domain: Instantaneous

"$$p(t) = \frac{1}{2}V_m I_m \cos(\theta_v - \theta_i) + \frac{1}{2}V_m I_m \cos(2\omega t + \theta_v + \theta_i)$$"

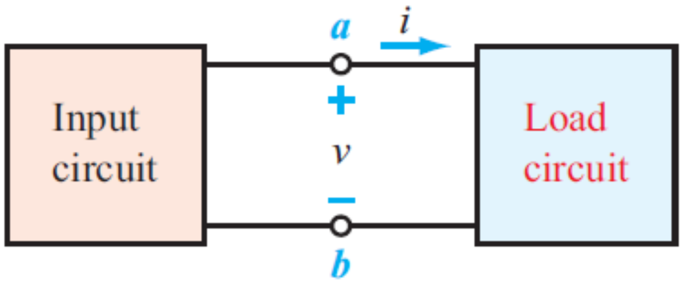
① ②



p(t) < 0

Average Power *P* (*Capitalized*)

$$v(t)=V_m\cos(\omega t+\theta_v) \quad i(t)=I_m\cos(\omega t+\theta_i)$$



$$\boxed{p(t)=\frac{1}{2}V_mI_m\cos(\theta_v-\theta_i)+\frac{1}{2}V_mI_m\cos(2\omega t+\theta_v+\theta_i)}$$

Average (or real) power (unit: watts) W

The **average power,** in watts, is the average of the instantaneous power over one period.

$$P=\frac{1}{T}\int_0^T p(t)\,dt$$

Average Power *P (time domain)*

$$P = \frac{1}{T}\int_0^T \frac{1}{2}V_m I_m \cos(\theta_v - \theta_i)\,dt + \frac{1}{T}\int_0^T \frac{1}{2}V_m I_m \cos(2\omega t + \theta_v + \theta_i)\,dt$$

$$= \frac{1}{2}V_m I_m \cos(\theta_v - \theta_i)\frac{1}{T}\int_0^T dt + \frac{1}{2}V_m I_m \frac{1}{T}\int_0^T \cos(2\omega t + \theta_v + \theta_i)\,dt$$

ω

$$\boxed{P = \frac{1}{2}\underline{V_m}\,\underline{I_m}\cos(\underline{\theta_v} - \underline{\theta_i})}$$

Average Power *P (phasor domain)*

$$v(t) = V_m \cos(\omega t + \theta_v) \quad i(t) = I_m \cos(\omega t + \theta_i)$$

$$\dot{\mathbf{V}} = V_m\angle\theta_v \text{ and } \dot{\mathbf{I}} = I_m\angle\theta_i, \quad , \quad \dot{I}^* = I_m \angle -\theta_i$$

$$\frac{1}{2}\dot{\mathbf{V}}\dot{\mathbf{I}}^* = \frac{1}{2}V_m I_m \angle\theta_v - \theta_i$$

$$= \frac{1}{2}V_m I_m[\cos(\theta_v - \theta_i) + j\sin(\theta_v - \theta_i)]$$

$$\boxed{P = \frac{1}{2}\mathrm{Re}[\dot{\mathbf{V}}\dot{\mathbf{I}}^*] = \frac{1}{2}V_m I_m \cos(\theta_v - \theta_i)}$$

Two special cases for average power *P*

- For a purely resistive load R: $\dot{V} = V_m \angle \theta_v$, $\dot{I} = I_m \angle \theta_i$, $\theta_v - \theta_i = 0$

$\dot{V} = R \cdot \dot{I}$

$$P = \frac{1}{2} V_m I_m = \frac{1}{2} I_m^2 R = \frac{1}{2} |\dot{\mathbf{I}}|^2 R \quad \text{where } |\dot{\mathbf{I}}|^2 = \dot{\mathbf{I}} \times \dot{\mathbf{I}}^*$$

> 0

- For a purely reactive load:

$$P = \frac{1}{2} V_m I_m \cos 90^\circ = 0$$

cap: $\theta_v - \theta_i = -90^\circ$

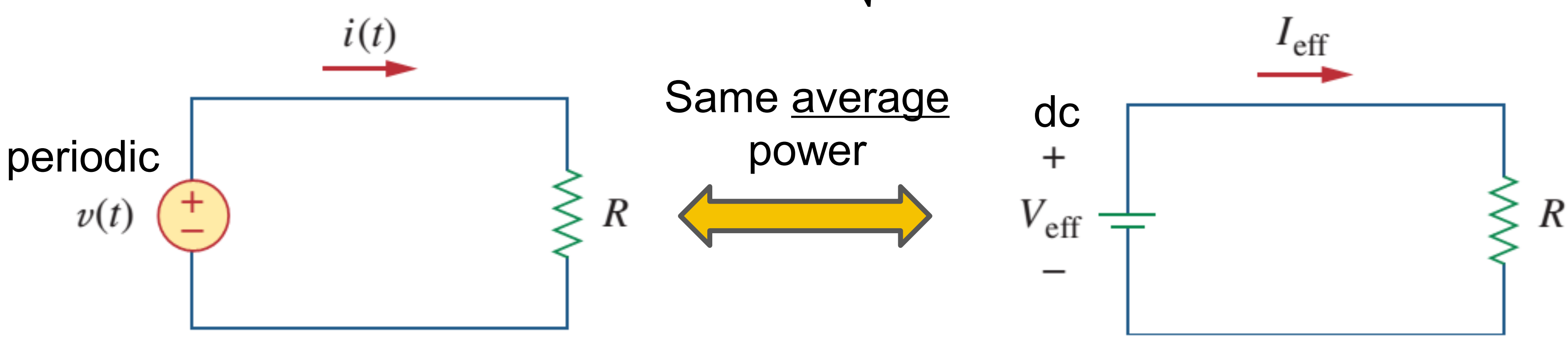
ind: $\theta_v - \theta_i = 90^\circ$

A resistive load (R) absorbs ave. power ~~at all times~~, while a reactive load (L or C) absorbs zero average power.

Effective Value (RMS)

- For any periodic function $x(t)$ in general, its rms value is

$$\frac{1}{T}\int_0^T \frac{v(t)}{R}\cdot v(t)\,dt \qquad X_{\text{eff}} = X_{\text{rms}} = \sqrt{\frac{1}{T}\int_0^T x^2(t)\,dt} \qquad \frac{1}{T}\int_0^T \frac{V_{\text{eff}}}{R}\cdot V_{\text{eff}}\,dt$$



$$\Rightarrow \quad V_{\text{eff}} = \sqrt{\frac{1}{T}\int_0^T v^2(t)\,dt}$$

Similarly: $$I_{\text{eff}} = \sqrt{\frac{1}{T}\int_0^T i^2(t)\,dt}$$

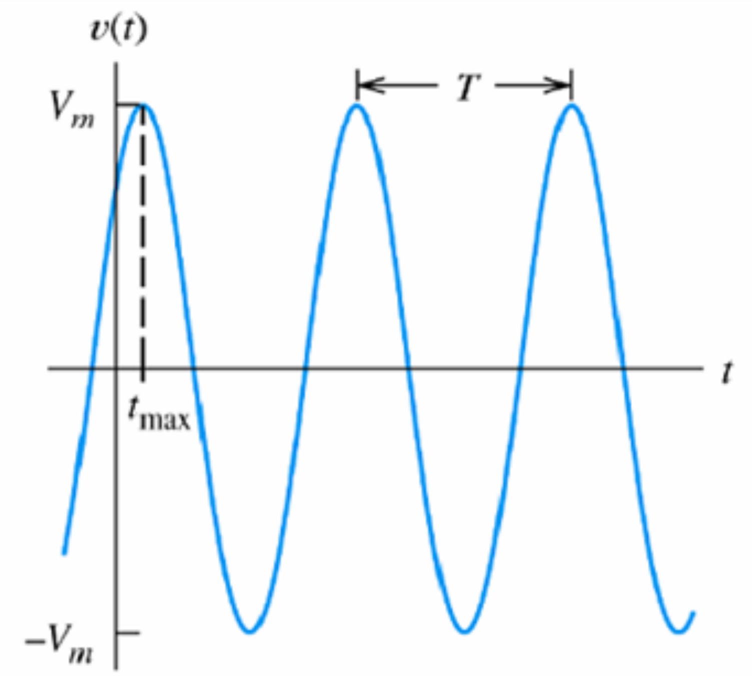
RMS of a sinusoidal signal

- The RMS value of $v(t) = V_m \cos(\omega t + \phi)$ is

$$V_{\text{rms}} = \sqrt{\frac{1}{T}\int_{t_0}^{t_0+T} v^2(t)dt}$$

$$= \sqrt{\frac{1}{T}\int_{t_0}^{t_0+T} V_m^2 \cdot \underbrace{\cos^2(\omega t + \phi)}_{\frac{1+\cos(2\omega t+2\phi)}{2}}dt}$$

$$= \frac{V_m}{\sqrt{2}}$$



Average Power

$$P = \frac{1}{2}V_m I_m \cos(\theta_v - \theta_i) = \frac{V_m}{\sqrt{2}}\frac{I_m}{\sqrt{2}}\cos(\theta_v - \theta_i)$$

$$= \dot{V}_{rms} \cdot \dot{I}_{rms}^{*} \qquad = V_{\text{rms}} I_{\text{rms}} \cos(\theta_v - \theta_i)$$

V_s : $v(t) = V_m \cos(\omega t + \phi)$ $\quad 5\cos(2t + 30°)$

Mag : $\dot{V}_m = V_m \cdot \angle\phi \longrightarrow \dot{V}_m = 5\angle 30°$

RMS : $\dot{V}_{rms} = V_{rms} \cdot \angle\phi \quad \dot{V}_{rms} = \frac{5}{\sqrt{2}}\angle 30°$

$= \frac{V_m}{\sqrt{2}}\angle\phi$

Outline

- Instantaneous power
- Average power
- Apparent power
- Power factor
- Complex power

Apparent Power

$$P = \frac{1}{2} V_m I_m \cos(\theta_v - \theta_i) = \frac{V_m}{\sqrt{2}} \frac{I_m}{\sqrt{2}} \cos(\theta_v - \theta_i)$$

$$= \dot{V}_{rms} \cdot \dot{I}^{*}_{rms} \quad = \boxed{V_{\text{rms}} I_{\text{rms}}} \cos(\theta_v - \theta_i)$$

$$\boxed{S \ or \ Sa = V_{rms} I_{rms}}$$

Unit: volt-amp (VA)

It seems <u>apparent</u> that the power should be the voltage-current product, *by analogy with dc resistive circuits.*

Power Factor

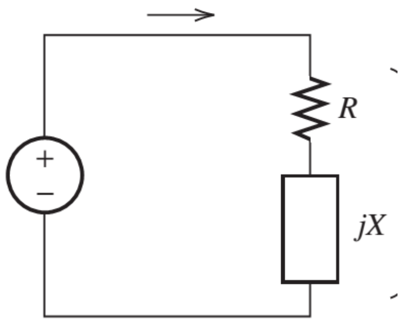
$$P = \frac{1}{2} V_m I_m \cos(\theta_v - \theta_i) = \frac{V_m}{\sqrt{2}} \frac{I_m}{\sqrt{2}} \cos(\theta_v - \theta_i)$$
$$= V_{\text{rms}} I_{\text{rms}} \boxed{\cos(\theta_v - \theta_i)}$$

- The power factor

$$\boxed{\text{pf} = \frac{P}{S_{a}} = \cos(\theta_v - \theta_i)}$$

  - $(\theta_v - \theta_i)$ is called <u>power factor angle</u>.
    - >0 means a *lagging* pf (current lags voltage)
    - <0 means a *leading* pf (current leads voltage)
  - pf ranges from 0 to 1.

Power Factor-2



Power factor leading and lagging relationships for a load $\mathbf{Z} = R + jX$.

Load Type	$\phi_z = \boxed{(\theta_v - \theta_i)}$	I-V Relationship	*pf*
Purely Resistive ($X = 0$)	$\phi_z = 0$	**I** in-phase with **V**	1
Inductive ($X > 0$)	$0 < \phi_z \leq 90°$	**I** lags **V**	lagging
Purely Inductive ($X > 0$ and $R = 0$)	$\phi_z = 90°$	**I** lags **V** by 90°	lagging
Capacitive ($X < 0$)	$-90° \leq \phi_z < 0$	**I** leads **V**	leading
Purely Capacitive ($X < 0$ and $R = 0$)	$\phi_z = -90°$	**I** leads **V** by 90°	leading

Power Factor-3

$$P = \frac{1}{2} V_m I_m \cos(\theta_v - \theta_i) = \frac{V_m}{\sqrt{2}} \frac{I_m}{\sqrt{2}} \cos(\theta_v - \theta_i)$$
$$= V_{\text{rms}} I_{\text{rms}} \boxed{\cos(\theta_v - \theta_i)}$$

- The power factor

$$\boxed{\text{pf} = \frac{P}{S_a} = \cos(\theta_v - \theta_i)}$$

  - $(\theta_v - \theta_i)$ is called power factor angle.
  - $(\theta_v - \theta_i)$ is equal to the angle of the load impedance

$$\mathbf{Z} = \frac{\dot{\mathbf{V}}}{\dot{\mathbf{I}}} = \frac{V_m \angle \theta_v}{I_m \angle \theta_i} = \frac{V_m}{I_m} \angle \theta_v - \theta_i$$

Also $$\mathbf{Z} = \frac{\dot{\mathbf{V}}}{\dot{\mathbf{I}}} = \frac{\dot{\mathbf{V}}_{\text{rms}}}{\dot{\mathbf{I}}_{\text{rms}}} = \frac{V_{\text{rms}}}{I_{\text{rms}}} \angle \theta_v - \theta_i$$

Outline

- Instantaneous power
- Average power
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- Power factor
- **Complex power**

Complex Power

$$v(t) = V_m \cos(\omega t + \theta_v) \Longrightarrow \mathbf{V} = V_m \angle \theta_v$$

$$i(t) = I_m \cos(\omega t + \theta_i) \Longrightarrow \mathbf{I} = I_m \angle \theta_i$$

We observe that:

$$\frac{1}{2}\mathbf{V}\mathbf{I}^* = \frac{1}{2} V_m I_m \angle(\theta_v - \theta_i)$$

$$= \frac{1}{2} V_m I_m \cos(\theta_v - \theta_i) + j\frac{1}{2} V_m I_m \sin(\theta_v - \theta_i)$$

- Define a ***single*** **power metric**

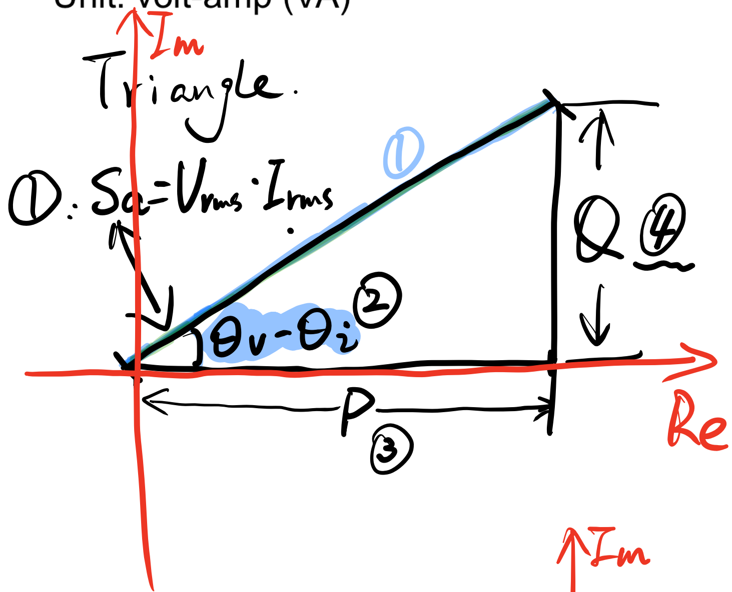
$$\boxed{\tilde{\mathbf{S}} = \frac{1}{2}\dot{\mathbf{V}}_m\dot{\mathbf{I}}_m^* = \dot{\mathbf{V}}_{\mathrm{rms}}\dot{\mathbf{I}}_{\mathrm{rms}}^* = V_{rms} I_{rms} \angle(\theta_v - \theta_i) = P + jQ}$$

Unit: volt-amp (VA)

Complex power is important in power analysis because it contains *all* the information pertaining to the power absorbed by a given load.

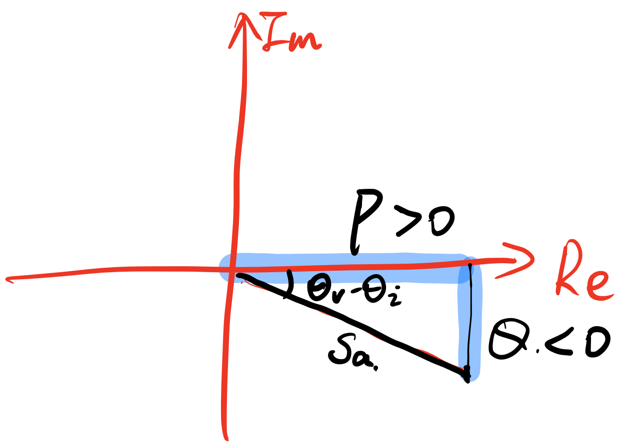
$$\boxed{\tilde{S} = \frac{1}{2}\dot{V}_m\dot{I}_m^* = \dot{V}_{rms}\dot{I}^*_{rms} = \overset{1}{V_{rms}I_{rms}}\angle\overset{2}{(\theta_v - \theta_i)} = \overset{3}{P} + j\overset{4}{Q}}$$

Unit: volt-amp (VA)

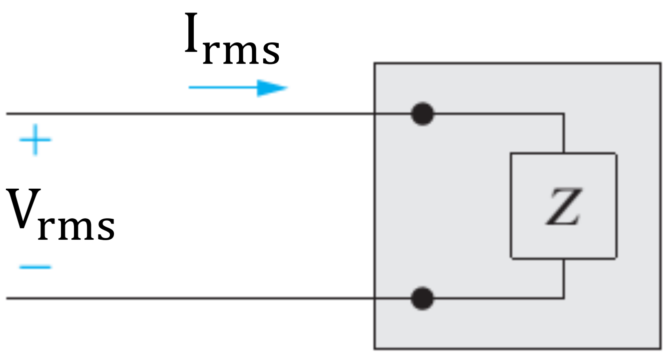


P ≥ 0 ✓
≤ 0
Both

Q ≥ 0
≤ 0
Both ✓



Another Way to Calculate Complex Power using impedance



$$\dot{\mathbf{V}}_{\mathrm{rms}} = \dot{\mathbf{I}}_{\mathrm{rms}} Z$$

$$\tilde{\mathbf{S}} = \dot{\mathbf{V}}_{\mathrm{rms}} \dot{\mathbf{I}}^*_{\mathrm{rms}} = \dot{\mathbf{V}}_{\mathrm{rms}} \left(\frac{\dot{\mathbf{V}}_{\mathrm{rms}}}{Z}\right)^* = \frac{|\dot{\mathbf{V}}_{\mathrm{rms}}|^2}{Z^*}$$

$$\begin{aligned}\tilde{\mathbf{S}} &= \dot{\mathbf{V}}_{\mathrm{rms}} \dot{\mathbf{I}}^*_{\mathrm{rms}} \\ &= \dot{\mathbf{I}}_{\mathrm{rms}} Z \dot{\mathbf{I}}^*_{\mathrm{rms}} \\ &= |\dot{\mathbf{I}}_{\mathrm{rms}}|^2 Z \\ &= |\mathbf{I}_{\mathrm{rms}}|^2 (R + jX) \\ &= |\mathbf{I}_{\mathrm{rms}}|^2 R + j|\mathbf{I}_{\mathrm{rms}}|^2 X \\ &= I_{rms}^2 R + j I_{rms}^2 X\end{aligned}$$

$$\boxed{\tilde{\mathbf{S}} = I^2_{\mathrm{rms}} \mathbf{Z} = \frac{V^2_{\mathrm{rms}}}{\mathbf{Z}^*} = \dot{\mathbf{V}}_{\mathrm{rms}} \dot{\mathbf{I}}^*_{\mathrm{rms}}}$$

$$\boxed{\begin{aligned} P &= \mathrm{Re}(\mathbf{S}) = I^2_{\mathrm{rms}} R \\ Q &= \mathrm{Im}(\mathbf{S}) = I^2_{\mathrm{rms}} X \end{aligned}}$$

Power Triangle

$$P = \mathrm{Re}(\mathbf{S}) = I_{\mathrm{rms}}^2 R$$
$$Q = \mathrm{Im}(\mathbf{S}) = I_{\mathrm{rms}}^2 X$$

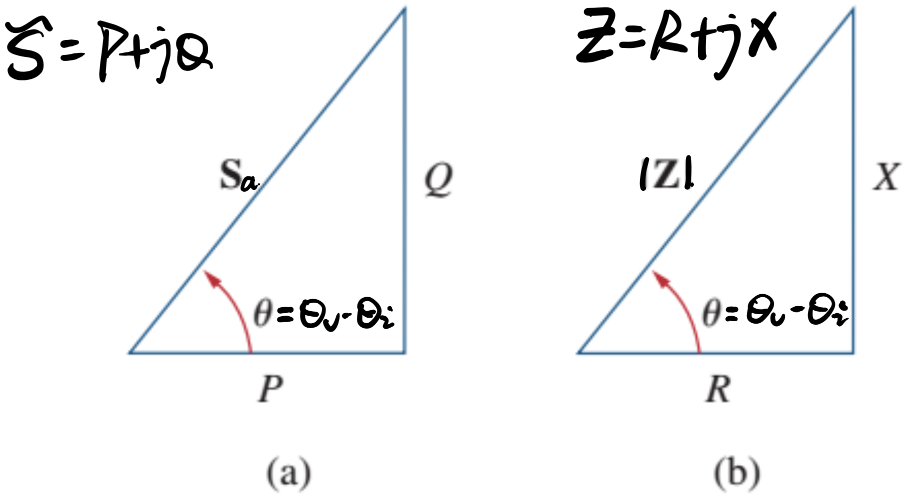


Figure 11.21
(a) Power triangle, (b) impedance triangle.

	Quantity	Units
$\tilde{S}$	Complex power	volt-amps (VA)
P	Average power	watts (W)
Q	Reactive power	var 乏

$$\mathbf{S} = \frac{1}{2}\mathbf{V}\mathbf{I}^* = \mathbf{V}_{\mathrm{rms}}\mathbf{I}^*_{\mathrm{rms}} = V_{rms}I_{rms}\angle(\theta_v - \theta_i) = P + jQ$$

$$P = \mathrm{Re}(\mathbf{S}) = I^2_{\mathrm{rms}}R$$

$$Q = \mathrm{Im}(\mathbf{S}) = I^2_{\mathrm{rms}}X$$

- Average (or real) power

$$P = \mathrm{Re}[\frac{1}{2}\mathbf{V}\mathbf{I}^*]$$

Unit: W

- Reactive power

$$Q = \mathrm{Im}[\frac{1}{2}\mathbf{V}\mathbf{I}^*]$$

Unit: Volt Amperes Reactive (VAR)

- Apparent power

$$S_a = |\mathbf{S}| = \frac{1}{2}V_m I_m = V_{rms}I_{rms}$$

$$= \sqrt{P^2 + Q^2}$$

Unit: volt-amp (VA)

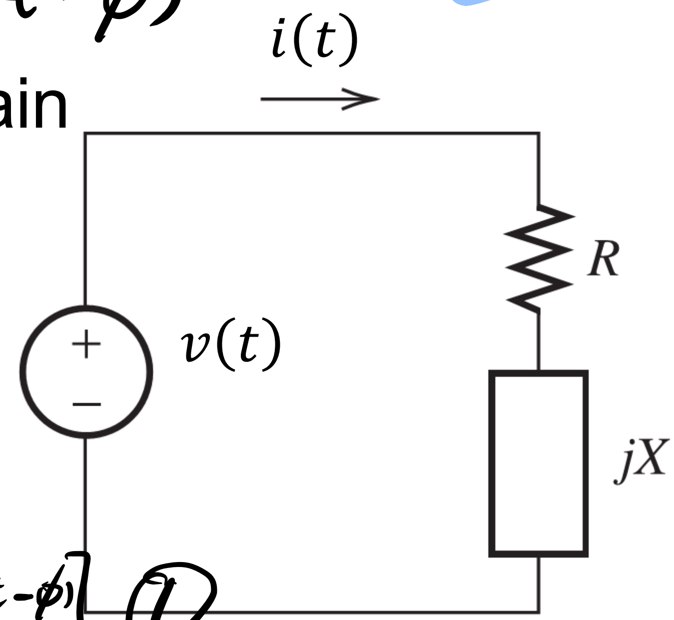
$$\text{Complex Power} = \tilde{\mathbf{S}} = P + jQ = \mathbf{V}_{\text{rms}}(\mathbf{I}_{\text{rms}})^*$$
$$= |\mathbf{V}_{\text{rms}}|\,|\mathbf{I}_{\text{rms}}|\underline{/\theta_v - \theta_i}$$
$$\text{Apparent Power} = S_a = |\tilde{\mathbf{S}}| = |\mathbf{V}_{\text{rms}}|\,|\mathbf{I}_{\text{rms}}| = \sqrt{P^2 + Q^2}$$
$$\text{Real Power} = P = \text{Re}(\tilde{\mathbf{S}}) = S_a\cos(\theta_v - \theta_i)$$
$$\text{Reactive Power} = Q = \text{Im}(\tilde{\mathbf{S}}) = S_a\sin(\theta_v - \theta_i)$$
$$\text{Power Factor} = \frac{P}{S_a} = \cos(\theta_v - \theta_i)$$

Reactive Power Q

$$\begin{cases} v(t) = V_m \cos(\omega t) \\ i(t) = I_m \cos(\omega t - \phi) \end{cases}$$

$$\theta_v - \theta_i = \phi$$

Let us look at Instantaneous power again



$$p(t) = v(t)i(t)$$
$$p(t) = pR(t) + p_X(t)$$
$$p_R(t) =$$
$$p_X(t) =$$

$$P(t) = V(t) \cdot i(t) = \frac{V_m I_m}{2}\left[\cos\phi + \cos(2\omega t - \phi)\right] \quad ①$$

$$P_R(t) = i^2(t) \cdot R = I_m^2 \cos^2(\omega t - \phi) \cdot R$$

$$R + jX = \frac{\dot{V}_m}{\dot{I}_m} = \frac{V_m \angle 0^\circ}{I_m \angle -\phi} = \frac{V_m}{I_m}\angle\phi$$

$$\boxed{R = \frac{V_m}{I_m} \cdot \cos\phi}$$

$$\begin{aligned} P(t) &= V \cdot i \\ &= (V_R + V_X) \cdot i \\ &= V_R \cdot i + V_X \cdot i \\ &= P_R(t) + P_X(t) \end{aligned}$$

$$P_R(t) = I_m^2 \cos^2(\omega t - \phi) \cdot \frac{V_m}{I_m} \cdot \cos\phi$$

$$= \frac{V_m I_m}{2} \cos\phi \left[1 + \cos(2\omega t - 2\phi) \right] \quad ②$$

$$P_x(t) = ① - ②$$

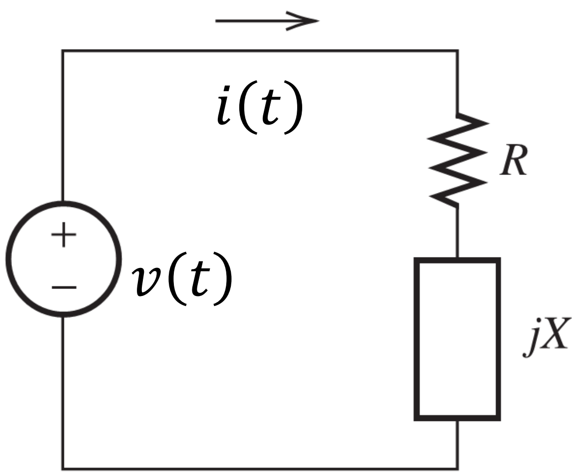
$$= \underline{\frac{V_m I_m}{2} \sin\phi} \cos(2\omega t - 2\phi + 90^\circ) \quad ③$$

$$\downarrow$$

$$\frac{V_m I_m}{2} \sin(\theta_v - \theta_i)$$

Reactive Power *Q*: Peak Exchanged Power

- Definition: The peak instantaneous power associated with the energy storage elements contained in a general load.



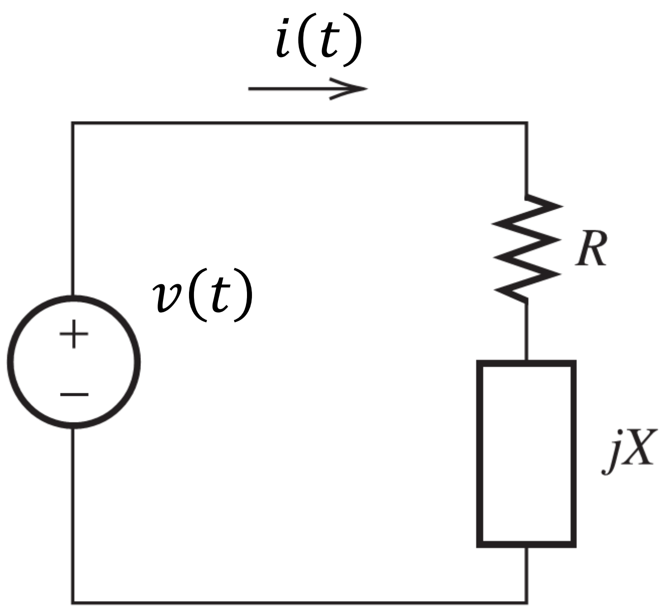
$$Q = \frac{1}{2} V_m I_m \sin(\theta_v - \theta_i) = V_{rms} I_{rms} \sin(\theta_v - \theta_i)$$

$$Q = \begin{cases} 0 & \text{for resistive } (\theta_v - \theta_i = 0^\circ) \\ \frac{1}{2} V_m I_m & \text{for inductive } (\theta_v - \theta_i = 90^\circ) \\ -\frac{1}{2} V_m I_m & \text{for capacitive } (\theta_v - \theta_i = -90^\circ) \end{cases}$$

- Reactive power is still of concern to power-system engineers
  - Transmission lines/transformers/fuses et al. must be capable of withstanding the current associated with reactive power.

Example

- Find the average power and reactive power absorbed by an impedance $Z = 30 - j70\Omega$, when a voltage $\dot{\mathbf{V}}_{\mathbf{m}} = 120\angle 0^{\circ}$ is applied across it.



Example

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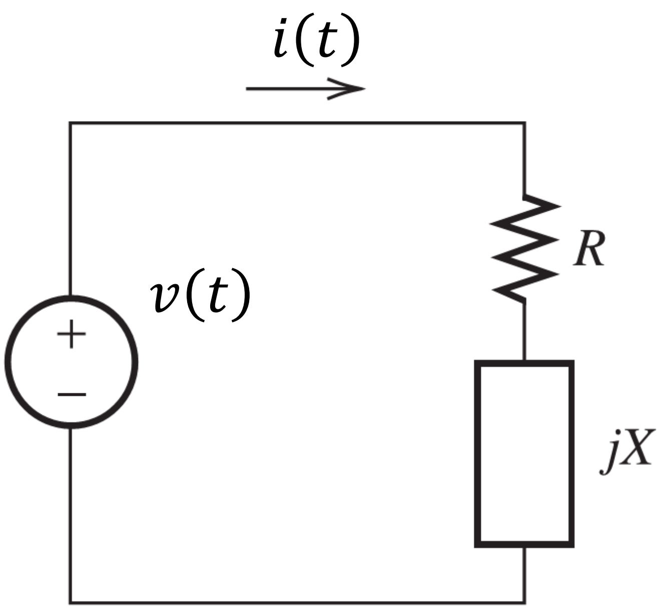


$$\mathbf{I_m} = \frac{\mathbf{V_m}}{Z} = \frac{120\angle 0^{\circ}}{30 - j70} = \frac{120\angle 0^{\circ}}{76.16\angle - 66.8^{\circ}}$$

$$= 1.576\angle 66.8^{\circ}\ \text{A}$$

$$P = \frac{1}{2} V_m I_m \cos(\theta_v - \theta_i) = 37.24\text{W}$$

$$Q = \frac{1}{2} V_m I_m \sin(\theta_v - \theta_i) = -86.91\text{VAR}$$

Exercise

- The voltage across a load is $v(t) = 60\cos(\omega t - 10°)\mathrm{V}$, and the current through the load is $i(t) = 1.5\cos(\omega t + 50°)$. Find
  - The complex and apparent powers.
  - The real and reactive powers.
  - The power factor and the load impedance.

Exercise

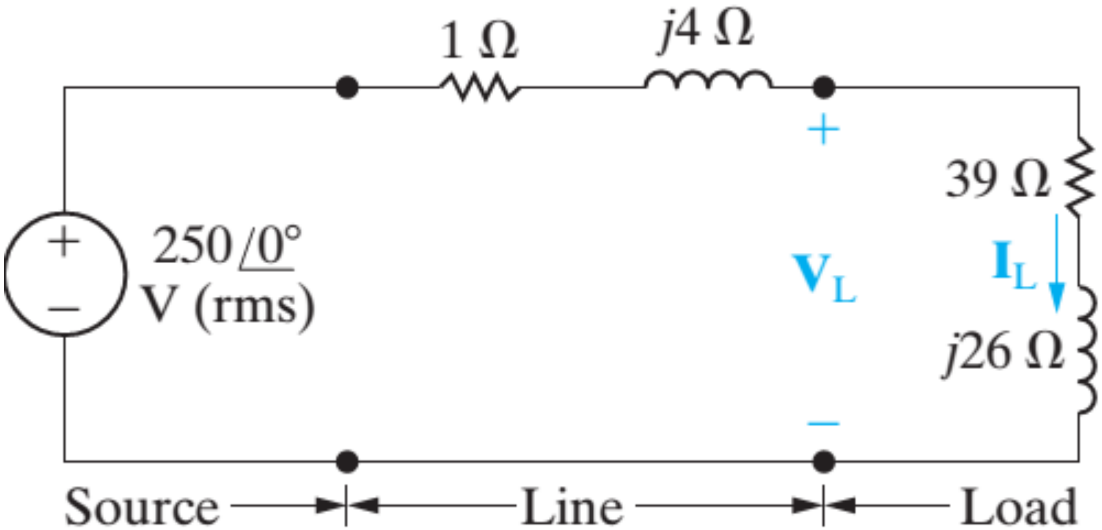
- The voltage across a load is $v(t) = 60\cos(\omega t - 10°)\mathrm{V}$, and the current through the load is $i(t) = 1.5\cos(\omega t + 50°)$. Find
  - The complex and apparent powers.
  - The real and reactive powers.
  - The power factor and the load impedance.

$$\mathbf{S} = \mathbf{V}_{\text{rms}}\mathbf{I}^*_{\text{rms}} = 45\underline{/-60°}\ \text{VA}$$

$$\text{pf} = 0.5\ (\text{leading})$$

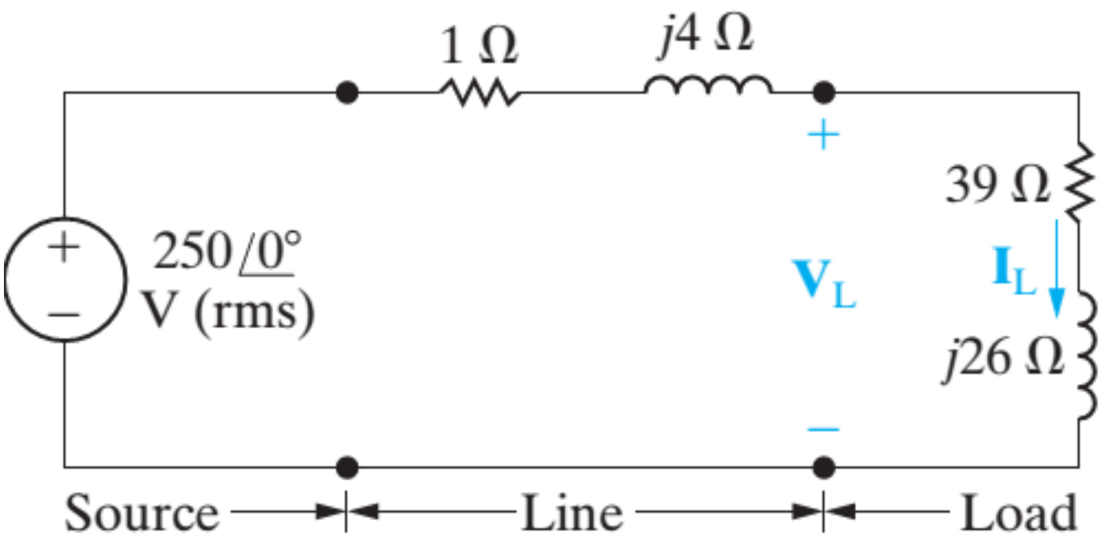
$$\mathbf{Z} = \frac{\mathbf{V}}{\mathbf{I}} = 40\underline{/-60°}\ \Omega$$

Example



- Find $\mathbf{V}_L$ and $\mathbf{I}_L$.
- Find the average and reactive power
  - Delivered to the load
  - Delivered to the line
  - Supplied by the source

Example



- Find $\mathbf{V}_L$ and $\mathbf{I}_L$.
- Find the average and reactive power
  - Delivered to the load
  - Delivered to the line
  - Supplied by the source

$$\mathbf{I}_L = \frac{250\angle 0^\circ}{40+j30} = 4 - j3$$
$$= 5\angle -36.87^\circ \text{ (rms)}$$

$$\mathbf{V}_L = \mathbf{I}_L(39 + j26)$$
$$= 234 - j13$$
$$= 234.36\angle -3.18^\circ$$

Load:

$$\mathbf{V}_L\mathbf{I}_L^* = 975 + j650 \text{ VA}$$

Line:

$$P = (5)^2(1) = 25 \text{ W}$$

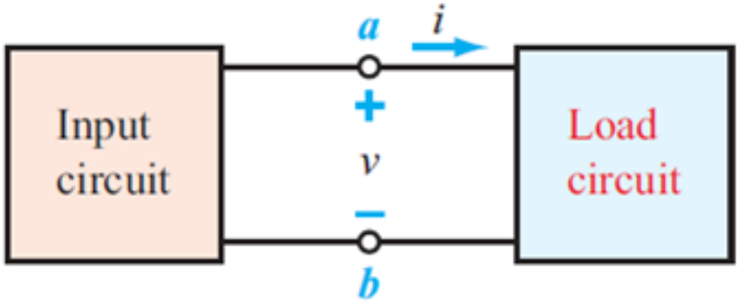
$$Q = (5)^2(4) = 100 \text{ VAR}$$

Source:

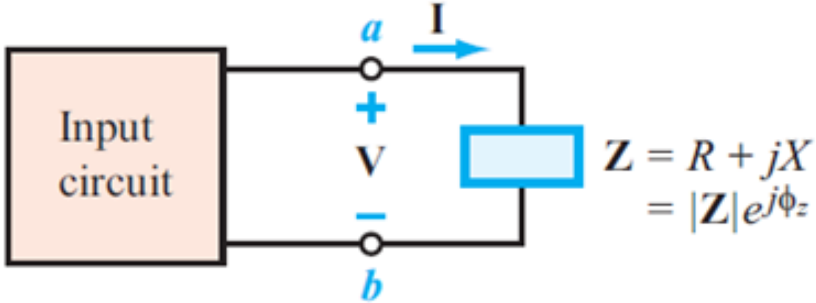
$$250\angle 0^\circ \mathbf{I}_L^* = 1000 + j750 \text{ VA}$$

Complex Power

Time Domain



Phasor Domain



Time Domain		Phasor Domain
$v(t) = V_m \cos(\omega t + \phi_v)$	⟷	$\mathbf{V} = V_m e^{j\phi_v}$
$i(t) = I_m \cos(\omega t + \phi_i)$	⟷	$\mathbf{I} = I_m e^{j\phi_i}$
$V_{rms} = V_m/\sqrt{2}$	⟷	$\mathbf{V}_{rms} = V_{rms} e^{j\phi_v}$
$I_{rms} = I_m/\sqrt{2}$	⟷	$\mathbf{I}_{rms} = I_{rms} e^{j\phi_i}$

Complex Power

$$\mathbf{S} = \frac{1}{2}\mathbf{V}\mathbf{I}^* = \mathbf{V}_{rms}\mathbf{I}^*_{rms} = P + jQ$$

Real Average Power

$$\begin{aligned} P &= \mathfrak{Re}\,[\mathbf{S}] \\ &= V_{rms} I_{rms} \cos(\phi_v - \phi_i) \\ &= I^2_{rms} R \end{aligned}$$

Reactive Power

$$\begin{aligned} Q &= \mathfrak{Im}\,[\mathbf{S}] \\ &= V_{rms} I_{rms} \sin(\phi_v - \phi_i) \\ &= I^2_{rms} X \end{aligned}$$

Apparent Power

$$\begin{aligned} S = |\mathbf{S}| &= \sqrt{P^2 + Q^2} \\ &= V_{rms} I_{rms} \\ &= I^2_{rms} |\mathbf{Z}| \end{aligned}$$

$$\mathbf{S} = S e^{j\phi_s}$$

$$\phi_s = \phi_v - \phi_i = \phi_z$$

Power Factor

$$\begin{aligned} pf &= \frac{P}{S} \\ &= \cos(\phi_v - \phi_i) \\ &= \cos\phi_z \end{aligned}$$

- Maximum average power transfer (不做要求)