2023Fall Probability & Statistics for EECS 2023/12/04

Homework 8

Professor: Ziyu Shao & Dingzhu Wen *Due: 2023/12/10 10:59pm*

1. Let X and Y be two continuous random variables with joint PDF

$$f_{X,Y}(x,y) = \begin{cases} cx^2y & \text{if } 0 \le y \le x \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

   (a) Find the value of constant c.

   (b) Find the conditional probability $P(Y \le \frac{X}{4} \big| Y \le \frac{X}{2})$.

2. Let X and Y be two integer random variables with joint PMF

$$P_{X,Y}(x,y) = \begin{cases} \frac{1}{6 \cdot 2^{\min(x,y)}} & \text{if } x, y \ge 0, |x-y| \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

   (a) Find the marginal distributions of X and Y.

   (b) Are X and Y independent?

   (c) Find $P(X = Y)$.

3. Let X, Y, Z be r.v.s such that $X \sim \mathcal{N}(0,1)$ and conditional on $X = x, Y$ and Z are i.i.d. $\mathcal{N}(x, 1)$.

   (a) Find the joint PDF of X, Y, Z.

   (b) By definition, Y and Z are conditionally independent given X. Discuss intuitively whether or not Y and Z are also unconditionally independent.

   (c) Find the joint PDF of Y and Z. You can leave your answer as an integral, though the integral can be done with some algebra (such as completing the square) and facts about the Normal distribution.

4. Let X and Y be i.i.d. $\mathcal{N}(0,1)$, and let S be a random sign (1 or -1, with equal probabilities) independent of (X, Y).

   (a) Determine whether or not $(X, Y, X + Y)$ is Multivariate Normal.

   (b) Determine whether or not $(X, Y, SX + SY)$ is Multivariate Normal.

   (c) Determine whether or not (SX, SY) is Multivariate Normal.

5. Let Z_1, Z_2 be two *i.i.d.* random variables satisfying standard normal distributions, *i.e.*, $Z_1, Z_2 \sim \mathcal{N}(0, 1)$. Define

$$X = \sigma_X Z_1 + \mu_X;$$
$$Y = \sigma_Y \left(\rho Z_1 + \sqrt{1-\rho^2} Z_2\right) + \mu_Y,$$

where $\sigma_X > 0$, $\sigma_Y > 0$, $-1 < \rho < 1$.

   (a) Show that X and Y are bivariate normal.
   (b) Find the correlation coefficient between X and Y, *i.e.*, $\mathrm{Corr}(X, Y)$.
   (c) Find the joint PDF of X and Y.

6. (Optional Challenging Problem) Given a random vector $\mathbf{X} = (X_1, \ldots, X_n)^T$, which satisfies a multivariate normal(Gaussian) distribution, i.e., $\mathbf{X} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, where $\boldsymbol{\mu}$ is the mean vector and $\boldsymbol{\Sigma}$ is the covariance matrix. When $\boldsymbol{\Sigma}$ is positive definite, the probability density function is given by

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{1}{(2\pi)^{\frac{n}{2}}\sqrt{\det(\boldsymbol{\Sigma})}} \exp\left(-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})\right).$$

Now we divide $\mathbf{X}(\mathbf{x})$ into two parts:

$$\boldsymbol{X} = \begin{bmatrix} \boldsymbol{X}_A \\ \boldsymbol{X}_B \end{bmatrix}, \quad \boldsymbol{x} = \begin{bmatrix} \boldsymbol{x}_A \\ \boldsymbol{x}_B \end{bmatrix}.$$

and split $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ accordingly:

$$\boldsymbol{\mu} = \begin{bmatrix} \boldsymbol{\mu}_A \\ \boldsymbol{\mu}_B \end{bmatrix}, \quad \boldsymbol{\Sigma} = \begin{bmatrix} \boldsymbol{\Sigma}_{AA} & \boldsymbol{\Sigma}_{AB} \\ \boldsymbol{\Sigma}_{BA} & \boldsymbol{\Sigma}_{BB} \end{bmatrix}.$$

Show the following results:

   (a) Marginal distribution of X_A and X_B are still normal, i.e., $\mathbf{X}_A \sim \mathcal{N}(\boldsymbol{\mu}_A, \boldsymbol{\Sigma}_{AA})$, $\mathbf{X}_B \sim \mathcal{N}(\boldsymbol{\mu}_B, \boldsymbol{\Sigma}_{BB})$.
   (b) $\boldsymbol{\Sigma}_{AB} = 0$ if and only if X_A and X_B are independent.
   (c) Given X_B, the conditional distribution of X_A is still normal, i.e.

$$\mathbf{X}_A | \mathbf{X}_B \sim \mathcal{N}(\boldsymbol{\mu}_{A|B}, \boldsymbol{\Sigma}_{A|B}),$$

   where

$$\boldsymbol{\mu}_{A|B} = \boldsymbol{\mu}_A + \boldsymbol{\Sigma}_{AB}\boldsymbol{\Sigma}_{BB}^{-1}(\mathbf{X}_B - \boldsymbol{\mu}_B)$$
$$\boldsymbol{\Sigma}_{A|B} = \boldsymbol{\Sigma}_{AA} - \boldsymbol{\Sigma}_{AB}\boldsymbol{\Sigma}_{BB}^{-1}\boldsymbol{\Sigma}_{BA}$$

   (d) If $\mathbf{X} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ and $\mathbf{X}' \sim \mathcal{N}(\boldsymbol{\mu}', \boldsymbol{\Sigma}')$ are independent, then

$$\mathbf{X} + \mathbf{X}' \sim \mathcal{N}(\boldsymbol{\mu} + \boldsymbol{\mu}', \boldsymbol{\Sigma} + \boldsymbol{\Sigma}').$$