EE160 Homework 1 Solution

1. (3 points) *Scalar linear differential equations*

   For scalar linear differential equation

$$\dot{x}(t) = ax(t) + b \quad \text{with} \quad a \neq 0,$$

   the general solution is

$$x(t) = e^{at}(x_0 - x_s) + x_s \quad \text{where steady state } x_s = -\frac{b}{a}$$

   (a) $a = -2, b = 4 \quad \Rightarrow x(t) = -2e^{-2t} + 2$

   (b) First shift t and set $y(t) = x(t+1)$ then $y(t)$ satisfies

$$\dot{y}(t) = y(t) - 2 \quad \text{with} \quad y(0) = 1$$

   which yields $y(t) = -e^t + 2$ and $x(t) = y(t-1) = -e^{t-1} + 2$

   (c) Reverse t and set $z(t) = x(-t)$, then $z(t)$ satisfies

$$\dot{z}(t) = z(t) - 2 \quad \text{with} \quad z(1) = 1$$

   which is the same as question (b), hence $z(t) = -e^{t-1} + 2$ and $x(t) = z(-t) = -e^{-t-1} + 2$.

2. (3 points) *Functional equation of the exponential function*

   (a) The equation has explicit solution

$$x(t) = e^{at}.$$

   Recall the property of exponential function and

$$x(t_1 + t_2) = e^{a(t_1+t_2)} = e^{at_1}e^{at_2} = x(t_1)x(t_2).$$

   (b) **Method 1:** First we show that for any $t \in \mathbb{R}$, $x(t) > 0$. $x(t) = x(\frac{t}{2})^2 \geq 0$ and if there exist some t_0 such that $x(t_0) = 0$ then $x(0) = x(-t_0)x(t_0) = 0$ which is in contradiction of the condition that $x(0) = 1$. Then we set

$$y(t) = \ln(x(t)), \tag{1}$$

   function $y(t)$ is well-defined on $\mathbb{R}$ and satisfies

$$\begin{aligned} y(t_1 + t_2) &= \ln(x(t_1 + t_2)) = \ln(x(t_1)x(t_2)) = \ln(x(t_1)) + \ln(x(t_2)) \\ &= y(t_1) + y(t_2) \end{aligned}$$

   then take the derivative of t_1 on both side and consider t_2 as a constant,

$$\dot{y}(t_1 + t_2) = \dot{y}(t_1) \quad \text{for any } t_2 \in \mathbb{R}$$

   which means $\dot{y}(t) = c$ is a constant function. From (1), we know that

$$\dot{y}(t) = \frac{\dot{x}(t)}{x(t)} = c \quad \Rightarrow \quad \dot{x}(t) = cx(t)$$

then we finish the proof.

Method 2: Given the condition, obviously for any $t \in \mathbb{R}$ and small $\Delta t \in R$

$$x(t + \Delta t) = x(t)x(\Delta t)$$

hence according to the definition, the derivative of $x(t)$

$$\begin{aligned}
\dot{x}(t) &= \lim_{\Delta t \to 0} \frac{x(t + \Delta t) - x(t)}{\Delta t} \\
&= \lim_{\Delta t \to 0} \frac{x(\Delta t) - 1}{\Delta t} \cdot x(t) = \lim_{\Delta t \to 0} \frac{x(\Delta t) - x(0)}{\Delta t} \cdot x(t) \\
&= \dot{x}(0)x(t)
\end{aligned}$$

the last equality holds because $x(t)$ is differentiable at 0.

3. (4 points) *Heat transfer through a wall.*

   (a) Total energy is invariant,

$$C_1 T_1(t) + C_2 T_2(t) = C_1 T_1(0) + C_2 T_2(0) \quad \Rightarrow \quad T_2(t) = \frac{C_1}{C_2}\left(T_1(0) - T_1(t)\right) + T_2(0) \tag{2}$$

   then the derivative of $T_1(t)$,

$$\begin{aligned}
\dot{T}_1(t) &= \frac{\dot{E}_1(t)}{C_1} = \frac{k}{C_1}\left(\frac{C_1}{C_2}\left(T_1(0) - T_1(t)\right) + T_2(0) - T_1(t)\right) \\
&= -\left(\frac{k}{C_1} + \frac{k}{C_2}\right) \cdot T_1(t) + k \cdot \left(\frac{T_1(0)}{C_1} + \frac{T_2(0)}{C_2}\right)
\end{aligned} \tag{3}$$

   (b) The steady-state T_s of above linear differential equation (3) can be calculated,

$$T_s = \frac{k \cdot \left(\frac{T_1(0)}{C_1} + \frac{T_2(0)}{C_2}\right)}{\left(\frac{k}{C_1} + \frac{k}{C_2}\right)} = \frac{C_2 T_1(0) + C_1 T_2(0)}{C_1 + C_2}$$

and the solution of (3) is

$$T_1(t) = \exp\left[-\left(\frac{k}{C_1} + \frac{k}{C_2}\right) \cdot t\right]\left(T_1(0) - T_s\right) + T_s.$$

Since k, C_1, C_2 are all positive, $T_1(t)$ will converge to T_s as $t \to \infty$. Substituting $T_1(t)$ with its limit T_s in (2), we get

$$\lim_{t \to \infty} T_2(t) = \frac{C_1}{C_2}\left(T_1(0) - T_s\right) + T_2(0) = T_s.$$