Online Lecture Notes

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1 Stability Analysis

The basic definition of btability is based on an ϵ-δ-statement. Namely, a system of the form

$$\dot{x}(t) = f(x(t)) \qquad \text{with} \qquad x(0) = x_0$$

is called stable at 0, if there exists for every $\epsilon > 0$ a $\delta > 0$ such that the solution trajectory of the system satisfies $\|x(t)\| \leq \epsilon$ for all initial values x_0 with $\|x_0\| \leq \delta$. Additionally, we call the system asymptotically stable, if it is stable and satisfies

$$\lim_{t \to \infty} x(t) = 0 \; .$$

The stability convergence requirement in the last setence are independent of each other and both necessary for asymptotic stability.

1.1 Stability of LTI systems

The stability of LTI systems of the form

$$\dot{x}(t) = Ax(t) \qquad \text{with} \qquad x(0) = x_0$$

can be analyzed by writing the explicit solution in the form

$$x(t) = e^{At} x_0 \; .$$

Thus, it sufficient to analyze boundedness and convergence of the matrix exponential e^{At}. It turns out that

1. The term $\|e^{At}\|$ remain bounded for all times $t \in \mathbb{R}_+$ if and only if the eigenvalues of A have all non-positive real part and all the dimension of the Jordan block of all purely imaginary eigenvalues is equal to 1. Notice that this is the same as saying that there exists a constant $M < \infty$ such that

   $$\forall t \in \mathbb{R}_+, \qquad \|e^{At}\| \leq M \; .$$

   Moreover, this statement is equivalent to saying that the above linear system is stable (proof: see our slides).

2. The term $\|e^{At}\|$ converges to 0 if and only if all eigenvalues of A have strictly negative real part. (same as the analysis in Lecture 6). This condition is equivalent to saying that $x(t)$ is asymptotically stable.

Notice that for LTI systems, convergence of the solution trajectory,

$$\lim_{t \to \infty} x(t) = 0,$$

for all $x_0 \in \mathbb{R}^n$ implies also that the system is stable (and, thus, asymptotically stable). But this property only holds for linear systems—for nonlinear systems such implications do not hold in general.

1.2 Stability of Periodic LTV System

Recall that a linear time-varying (LTV) system can be written in the form

$$\dot{x}(t) = A(t)x(t) \qquad \text{with} \qquad x(0) = x_0 \ .$$

The solution trajectory can be written in the form

$$x(t) = G(t,0)x_0 \ ,$$

where G denotes the fundamental solution (see Lecture 7). In particular, if A is periodic, such that

$$\forall t \in \mathbb{R}, \qquad A(t+T) = A(t)$$

for a given period time $T > 0$, then we have that

$$\forall t \in [NT, (N+1), T], \qquad x(t) = G(t - NT, 0)G(T,0)^N x_0$$

for $N \in \mathbb{N}$. Since the function

$$G(t - NT, 0)$$

is continuous and bounded on the interval $[NT, (N+1), T]$, it is sufficient to analyze the properties of term

$$G(T,0)^N,$$

recalling that $G(T,0)$ denotes the monodromy matrix of the periodic LTV system. The analysis of this term is analogous to the analysis of the matrix exponential in the context of the above LTI system. In detail, we find that

1. The term $\|G(T,0)\|$ remains bounded for all times $N \in \mathbb{N}$ if and only if the eigenvalues of $G(T,0)$ are all in the closed unit disc and all the dimension of the Jordan block of all eigenvalues on the unit circle are equal to 1. Notice that this is the same as saying that there exists a constant $M < \infty$ such that
$$\forall N \in \mathbb{N}, \qquad \|G(T,0)^N\| \leq M \ .$$
Moreover, this statement is equivalent to saying that the above LTV is stable (proof: see our slides).

2. The term $\|G(T,0)^N\|$ converges to 0 if and only if all eigenvalues of the monodromy matrix $G(T,0)$ are all contained in the open unit disc. This condition is equivalent to saying that $x(t)$ is asymptotically stable.

1.3 Lyapunov Functions

Lyapunov functions are among the most important tools for analyzing the stability properties of linear and nonlinear systems. The theory itself is extremely general. In this lecture, we cover only some of the most basic ideas. Here, first of the all, the key idea is to introduce a function $V : \mathbb{R}^{n_x} \to \mathbb{R}$, which either weakly or strong descends along the trajectories of the system. This means that we analyze the expression

$$V(x(t))$$

as a function of t. This has the advantage that we can analyze a scalar function rather than the vector-valued function $x(t)$, which may be hard to do directly. If the function V is differentiable, it time derivative is given by

$$\dot{V}(x(t)) \stackrel{\text{def}}{=} \frac{\mathrm{d}}{\mathrm{d}t} V(x(t)) \;=\; \nabla V(x(t))\dot{x}(t) \; .$$

This means that if $x(t)$ is the solution of a differential equation of the form

$$\dot{x}(t) = f(x(t)) \qquad \text{with} \qquad x(0) = x_0,$$

then we have

$$\dot{V}(x(t)) \;=\; \nabla V(x(t))^{\intercal} f(x(t)) \; .$$

Our main goal is now to construct a function V in such a way that we have

$$\dot{V}(x(t)) = \nabla V(x(t))^{\intercal} f(x(t)) \leq 0 \; ,$$

such that V is at least weakly descending along the trajectories of f. If possible, we would like to additionally introduce the requirement that V is strongly descending along the trajectories of f, which is equivalent to requiring that

$$\dot{V}(x(t)) < 0 \quad \text{whenever} \quad f(x(t)) \neq 0 \; .$$

In practice, we often want to analyze the system properties close to a steady-state. In this case, we may assume that the steady $x_\mathrm{s} = 0$ is at 0. This means that we assume that

$$f(x_\mathrm{s}) = f(0) = 0 \; .$$

In this case, the following definitions make sense:

1. The function V is called positive definite if $V(x) \geq 0$ for all $x \in \mathbb{R}^{n_x}$ and we have $V(x) = 0$ if and only if $x = 0$.

2. The function V is called monotonically decreasing along the vector field f, if we have
$$\nabla V(x)^{\intercal} f(x) \leq 0$$
for all $x \in \mathbb{R}^n$.

3. The function V is called stricly monotonically decreasing along the vector field f, if we have
$$\nabla V(x)^{\intercal} f(x) < 0$$
for all $x \in \mathbb{R}^n \setminus \{0\}$.

Notice that all these requirements of V have to be checked globally for all x, but they can be checked without having an explicit expression for the solution trajectories of the system. This particularly practical for nonlinear systems, since for nonlinear system we usually don't have an explicit expression for the solution trajectories of the system. This means that the only difficulty is to actually find a function V, which satisfies the above properties.

1.4 Invariant Sets

A set $S \subseteq \mathbb{R}^{n_x}$ is called an invariant set of the differential equation

$$\dot{x}(t) = f(x(t)) \quad \text{with} \quad x(0) = x_0$$

if the solution trajectories satisfy

$$\forall t \in \mathbb{R}, \qquad x(t) \in S$$

for all initial values $x_0 \in S$. Notice that if we have a monotonically decreasing Lyapunov function V, which means that

$$\forall x \in \mathbb{R}^{n_x}, \qquad \nabla V(x)^\intercal f(x) \leq 0, \tag{1}$$

then all sub-level sets of V are invariant sets. This follows from the fundamental of calculus,

$$\begin{aligned} V(x(t)) &= V(x(0)) + \int_0^t \dot{V}(x(t))\,\mathrm{d}t \\ &= V(x(0)) + \int_0^t \underbrace{\nabla V(x(t))^\intercal f(x(t))}_{\leq 0}\,\mathrm{d}t \\ &\overset{(1)}{\leq} V(x(0)) \end{aligned} \tag{2}$$

This means that if we introduce the sublevel set

$$S = \{\ x \in \mathbb{R}^{n_x} \ \mid V(x) \leq \alpha\}\ ,$$

then $x_0 \in S$ implies $x(t) \in S$ for all $t \in \mathbb{R}$. This follows from

$$x_0 \in S \quad \Longrightarrow \quad V(x_0) \leq \alpha \quad \overset{(2)}{\Longrightarrow} \quad V(x(t)) \leq V(x_0) \leq \alpha \quad \Longrightarrow \quad x(t) \in S\ .$$

1.5 Conditions for asymptotic convergence

The main idea for analyzing convergence of nonlinear systems of the form

$$\dot{x}(t) = f(x(t)) \quad \text{with} \quad x(0) = x_0$$

is to construct a Lyapunov function V, which has the following properties:

1. V is continuously differentiable,
2. V strictly monotonically decreasing, $\dot{V}(x) < 0$ for $x \neq 0$,

3. V is positive definite, $V(x) \geq 0$ and $V(x) = 0 \;\Leftrightarrow\; x = 0$, and

4. V is radially unbounded, $\lim_{\|x\| \to \infty} V(x) = \infty$.

If these four properties are satisfied, we can prove that the solution trajectory of the nonlinear system satisfies

$$\lim_{t \to \infty} x(t) = 0 \; .$$

The proof of this statement is indirect. We first use that V is monotonically decreasing and bounded from below, which implies that the limit

$$V_\infty = \lim_{t \to \infty} V(x(t))$$

exists. Assume that $V_\infty > 0$. The function V is positive definite and satisfies

$$V(x(t)) \geq V_\infty > 0 \; .$$

This means that we can find a constant $\epsilon > 0$ such that

$$\dot{V}(x(t)) \leq -\epsilon,$$

since V is strictly monotonously decreasing and $\dot{V}$ is continuous. Thus, we have

$$\begin{aligned} V(x(t)) &= V(x(0)) + \int_0^t \dot{V}(x(t)) \, \mathrm{d}t \\ &\leq V(x(0)) - \epsilon t \end{aligned} \tag{3}$$

But this leads to a contradiction, since

$$\lim_{t \to \infty} V(x(t)) \leq -\infty$$

is impossible, since V is positive definite. Consequently, we have $V_\infty = 0$, which implies that

$$\lim_{t \to \infty} V(x(t)) = 0 \qquad \Longrightarrow \qquad \lim_{t \to \infty} x(t) = 0$$

since V is continuous and positive definite.