Discrete Mathematics: Lecture 28

Homeomorphic, Kuratowski's Theorem, Graph Coloring, Tree

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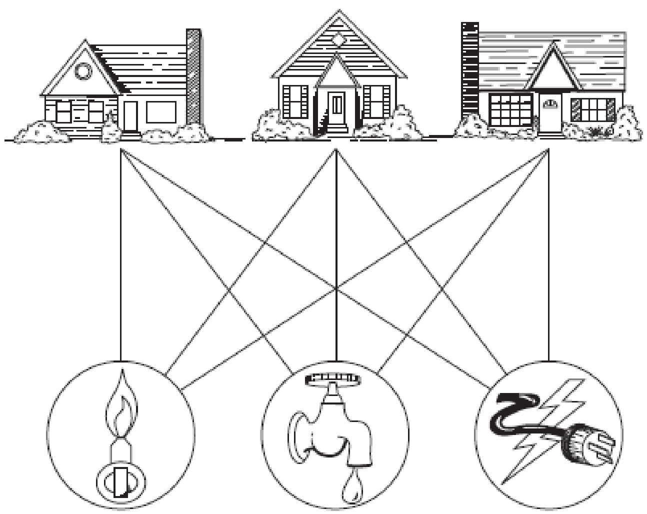
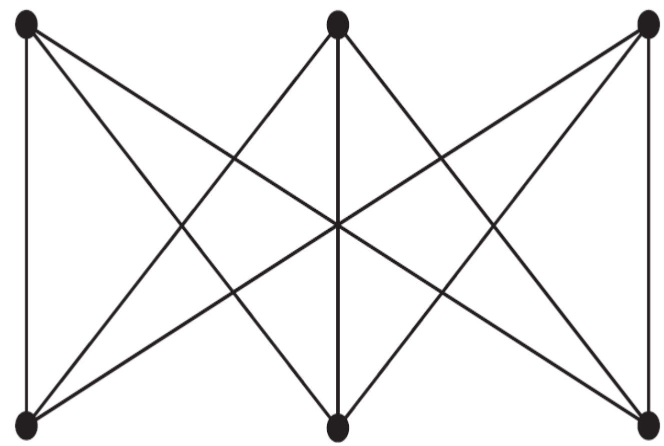
Planar Graph

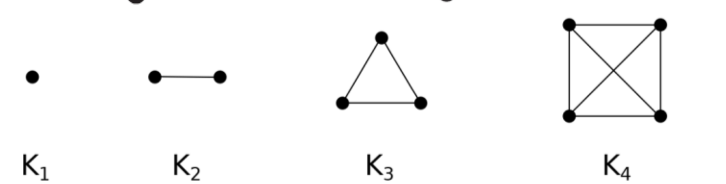
DEFINITION: Let $G = (V, E)$ be an undirected graph. G is called a **planar graph**平面图 if it can be drawn in the plane without any edges crossing.

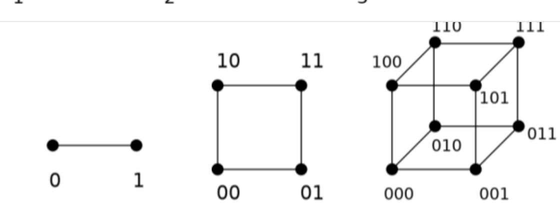
- Crossing of edges: an intersection other than endpoints (vertices)
- **planar representation**平面表示: a drawing w/o edge crossing; **nonplanar**非平面的





- K_1, K_2, K_3, K_4 are planar graphs
- $K_{1,n}, K_{2,n}$ are planar graphs
- $C_n\ (n \geq 3), W_n (n \geq 3)$ are planar graphs
- Q_1, Q_2, Q_3 are planar graphs

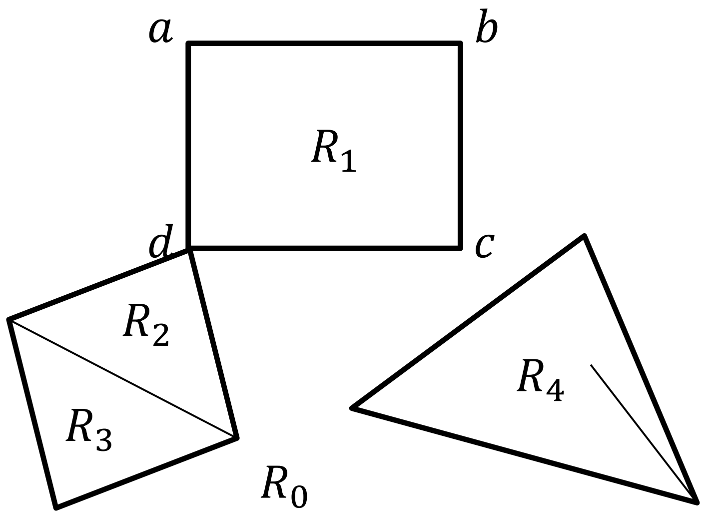




Regions

DEFINITION: Let $G = (V, E)$ be a planar graph. Then the plane is divided into several **regions**面 by the edges of G.

- The infinite region is **exterior region**外部面. The others are **interior regions**内部面.
- The **boundary**边界 of a region is a subset of E.
- The **degree**度数 of a region is the number of edges on its boundary.
  - If an edge is shared by R_i, R_j, then it contributes 1 to $\deg(R_i)$, $\deg(R_j)$
  - If an edge is on the boundary of a single region R_i, then it contributes 2 to $\deg(R_i)$



- The plane is divided into 5 regions R_0, R_1, R_2, R_3, R_4
  - R_0 is the exterior region
  - R_1, R_2, R_3, R_4 are interior regions
- The boundary of R_1; $\deg(R_1) = 4$
- There are 4 edges on the boundary of R_4
  - $\deg(R_4) = 1 + 1 + 1 + 2 = 5$ because one of the edges contribute 2 to $\deg(R_4)$
- $\deg(R_0) = 11, \deg(R_1) = 4, \deg(R_2) = 3, \deg(R_3) = 3, \deg(R_4) = 5$

Euler's Formula

THEOREM: Let $G = (V, E)$ be a connected planar simple graph with e edges and v vertices. Let r be the number of regions in a planar representation of G. Then $r = e - v + 2$.

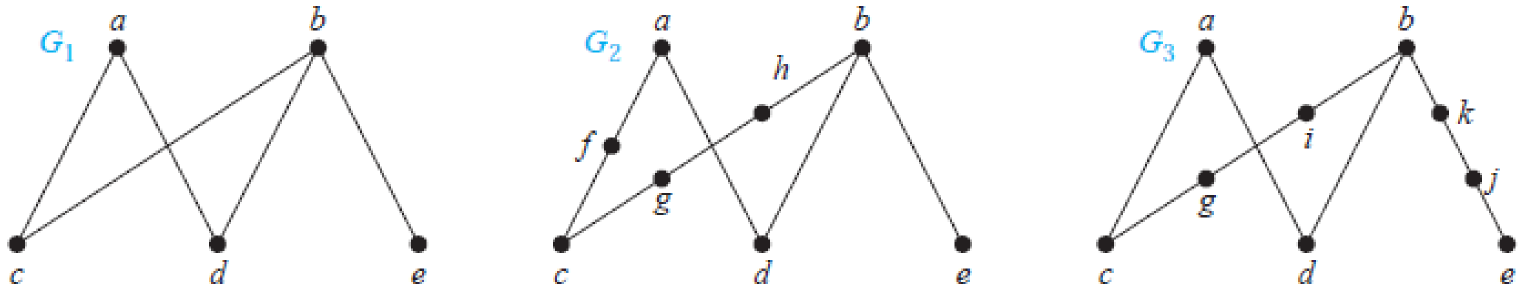
THEOREM: Let G be a planar simple graph with p connected components. Then $|V(G)| - |E(G)| + |R(G)| = p + 1$.

- Let $G_1, G_2, \ldots, G_p$ be the connected components of G.
  - By Euler's formula, $|R(G_i)| = |E(G)_i| - |V(G_i)| + 2$ for all $i \in [p]$
- $|V(G)| = |V(G_1)| + |V(G_2)| + \cdots + |V(G_p)|$
- $|E(G)| = |E(G_1)| + |E(G_2)| + \cdots + |E(G_p)|$
- $|R(G)| = |R(G_1)| + |R(G_2)| + \cdots + \left|R\left(G_p\right)\right| - p + 1$
- $|V(G)| - |E(G)| + |R(G)| = \sum_{i=1}^{p}(|V(G_i)| - |E(G_i)| + |R(G_i)|) - p + 1$
  $= 2p - p + 1 = p + 1$

Homeomorphic

DEFINITION: Let $G = (V, E)$ be a graph and $\{u, v\} \in E$.

- **elementary subdivision**初等细分: $G' = (V \cup \{w\}, E - \{u, v\} + \{u, w\} + \{v, w\})$
- Two graphs are **homeomorphic**同胚的 if they can be obtained from the same graph via elementary subdivisions

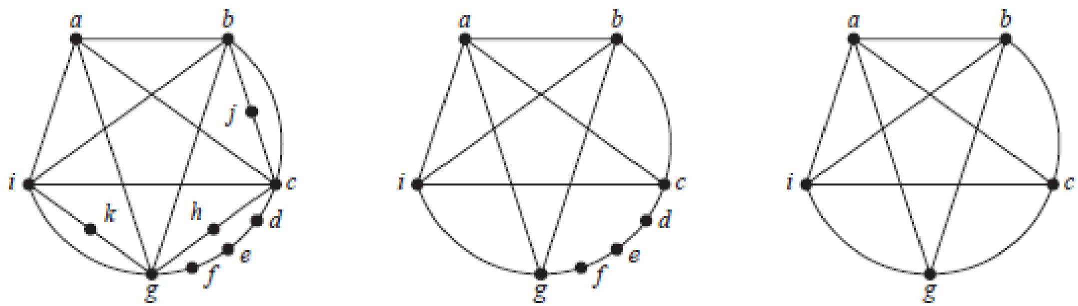


G_2 and G_3 are homeomorphic

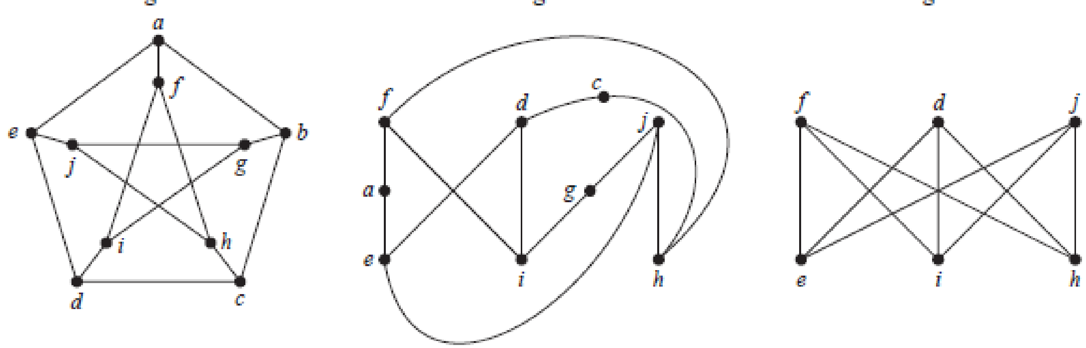
Kuratowski's Theorem

THEOREM: A graph G is nonplanar if and only if it has a subgraph homeomorphic to $K_{3,3}$ or K_5.

EXAMPLE: The following graph is nonplanar.



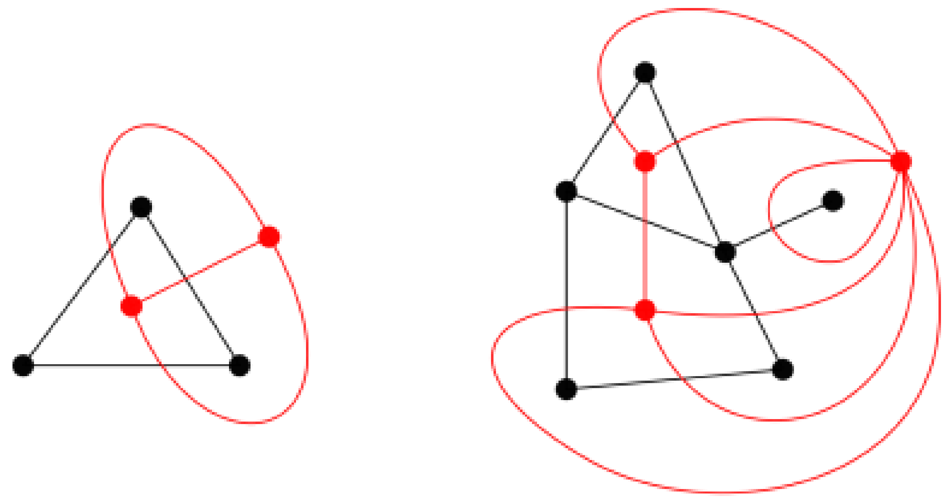
There is a subgraph homoemorphic to K_5



There is a subgraph homoemorphic to $K_{3,3}$

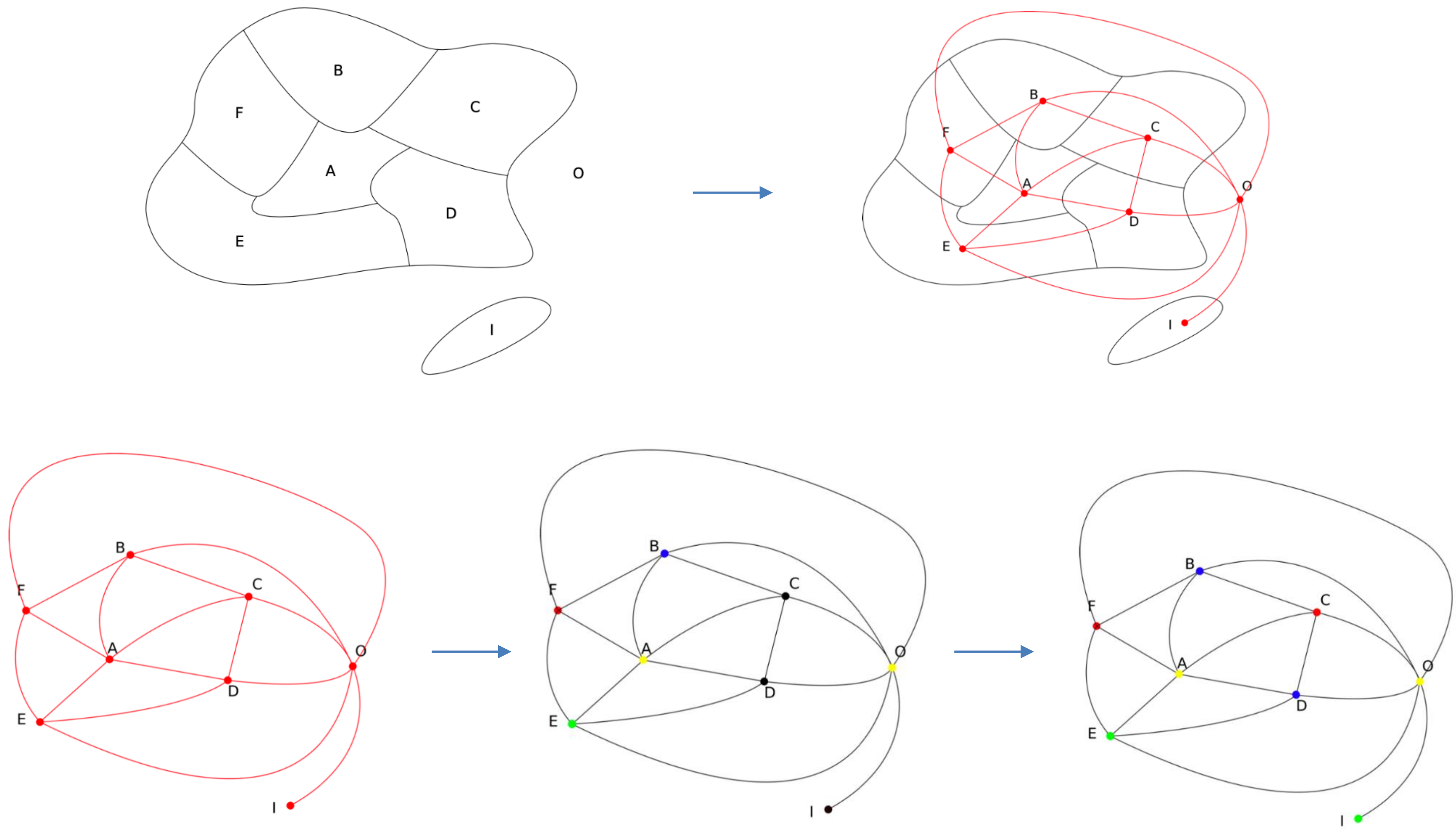
Dual Graph

Let G be a planar graph and assume we take a planar representation of G that we denote also G. The **dual of** G is the graph G^* that has a vertex for each face of G and an edge connecting two vertices if the corresponding faces in G have a common edge in their boundary.



Remark: The dual of a planar simple graph is not necessarily simple.

Coloring a Map

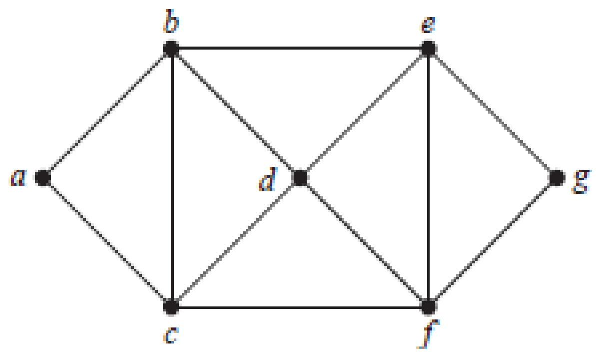


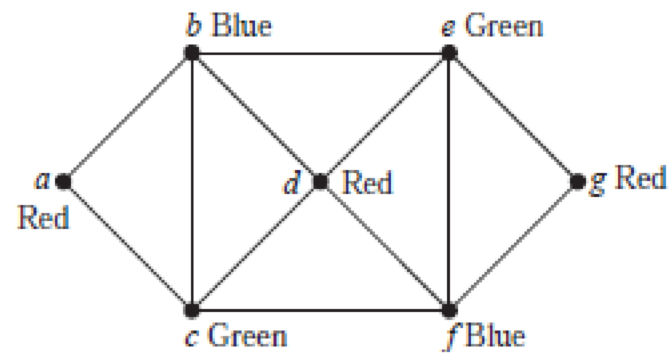
Coloring regions of the map ⇔ Coloring vertices of the dual graph

Graph Coloring

DEFINITION: Let $G = (V, E)$ be a simple graph. A ***k*-coloring**$_{k\text{-着色}}$ of G is a map $f: V \to [k]$ such that $f(u) \neq f(v)$ whenever $\{u, v\} \in E$.

- **chromatic number ($\chi(G)$)**$_{色数}$**:** the least k s.t. G has a k-coloring.





$$\chi(G) = 3$$

The chromatic number is at least 3 because *a; b; c* is a circuit of length 3

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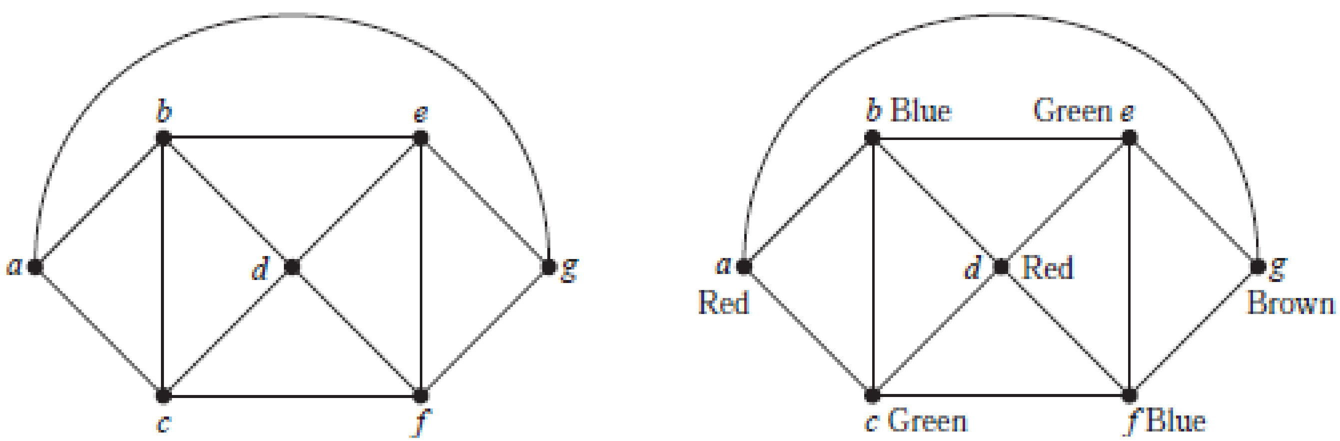
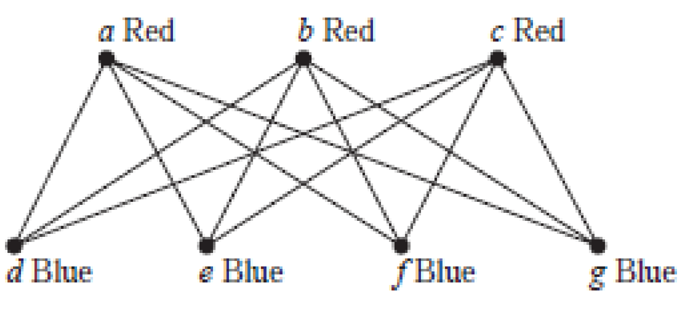


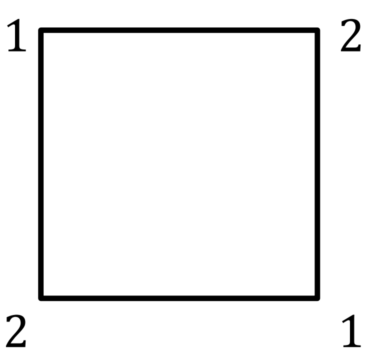
$\chi(G) = 4$

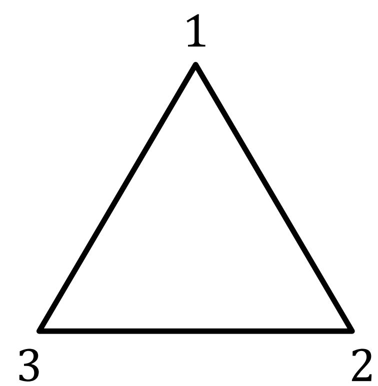
Graph Coloring

THEOREM: Let $G = (V, E)$ be a simple graph.

- $1 \leq \chi(G) \leq |V|$
- $\chi(G) = 1$ iff $E = \emptyset$
- $\chi(G) = 2$ iff G is bipartite and $|E| \geq 1$.
- $\chi(K_n) = n$ for every integer $n \geq 1$.
    - $\chi(G) \geq n$ if G has a subgraph isomorphic to K_n
- $\chi(C_n) = 2$ if $2|n$; $\chi(C_n) = 3$ if $2|(n-1)$; $(n \geq 3)$
- $\chi(G) \leq \Delta(G) + 1$, where $\Delta(G) = \max\{\deg(v): v \in V\}$.

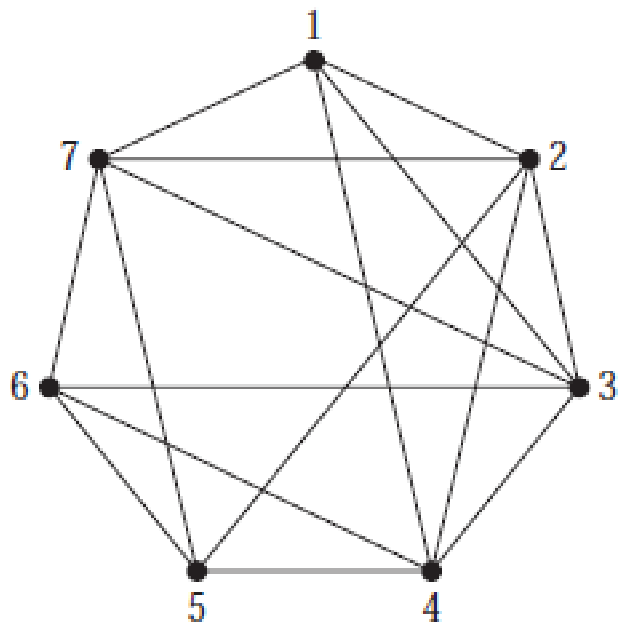


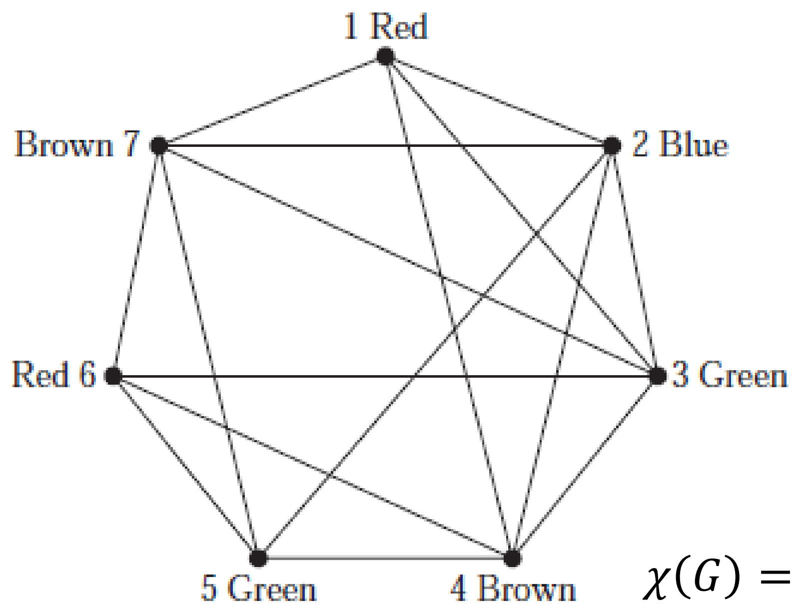




Application

PROBLEM: How can the final exams at a university be scheduled so that no student has two exams at the same time?





$\chi(G) = 4$

- There are 7 different courses, they are vertices of a graph.
- Two courses are adjacent if there is a student registered both courses.
- Choose time slots for the courses such that no two adjacent courses take place at the same time.
  - $\chi(G)$ time slots is needed.

$1 \leq \chi(G) \leq 7$

$1 \leq \chi(G) \leq \Delta(G) + 1 = 6$

$\chi(G) \geq 4$: G has a subgraph isomorphic to K_4

4-coloring Theorem

Theorem (Four coloring Theorem)

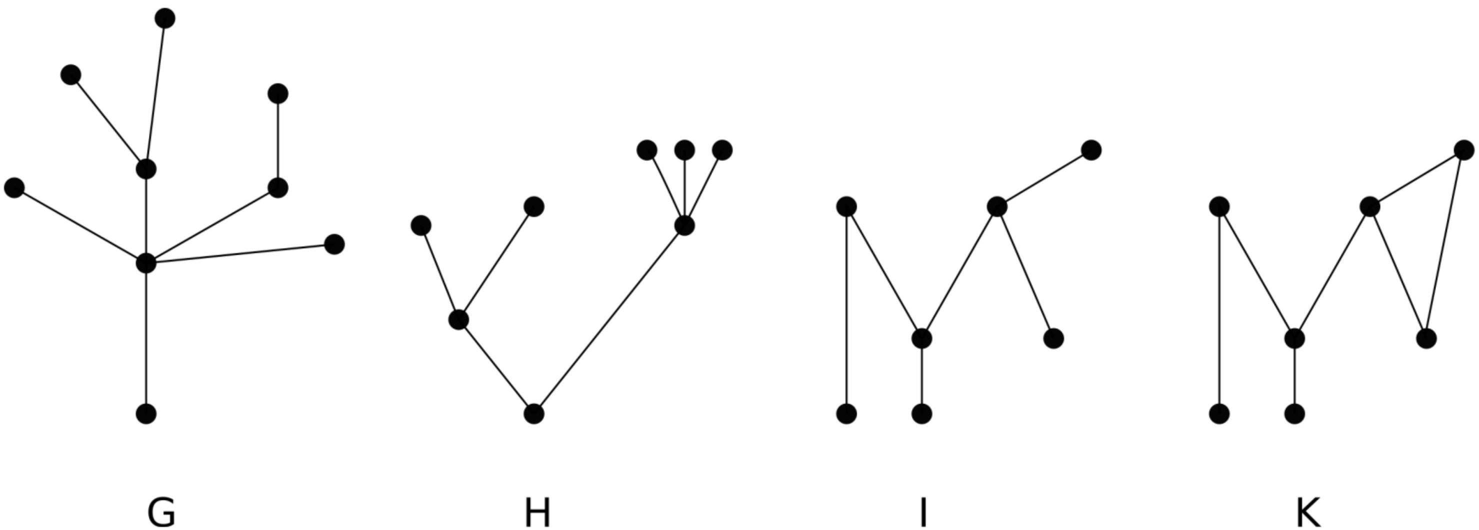
The chromatic number of a simple planar graph is no greater than 4.

Remarks: The proof of the 4-coloring Theorem depends on a computer. The two previous theorems are true for planar graphs only. A non planar graph can have an arbitrarily large chromatic number.

Tree

Definition

- A **tree** is a connected undirected graph with no simple circuits.
- A **forest** is an graph such that each of its connected components is a tree.



G, H, I are trees, but K is not a tree.

Characterization of Tree

Theorem

An undirected graph is a tree if and only if there is a unique simple path between any two of its vertices.

Proof: $(\Rightarrow)$ Assume T is a tree and let u and v be two vertices. T is connected so there is a *simple path* P_1 from u to v. Assume there is a second simple path P_2 from u to v.

Claim: There is a simple circuit in T.
Let $u = x_0, x_1, \ldots, x_n = v$ denote the vertices of P_1 and $u = y_0, y_1, \ldots,$ $y_m = v$ the vertices of P_2.
P_1 and P_2 start at u but are not equal so must diverge at some point.

• If they diverge after one of them has ended, then the remaining part of the other path is a circuit from v to v.

Characterization of Tree

- Otherwise, we can assume

$$x_0 = y_0, x_1 = y_1, \ldots, x_i = y_i$$

and $x_{i+1} \neq y_{i+1}$.
We follow then $y_{i+1}, y_{i+2}, \ldots$ until we reach a vertex of P_1.
Then go back to x_i following P_1 forwards or backwards.

This gives a circuit which is simple because P_1 and P_2 are, and we stop using edges of P_2 as soon as we hit P_1.

($\Leftarrow$) Assume there is a unique simple path between any two vertices of the graph T. Then:

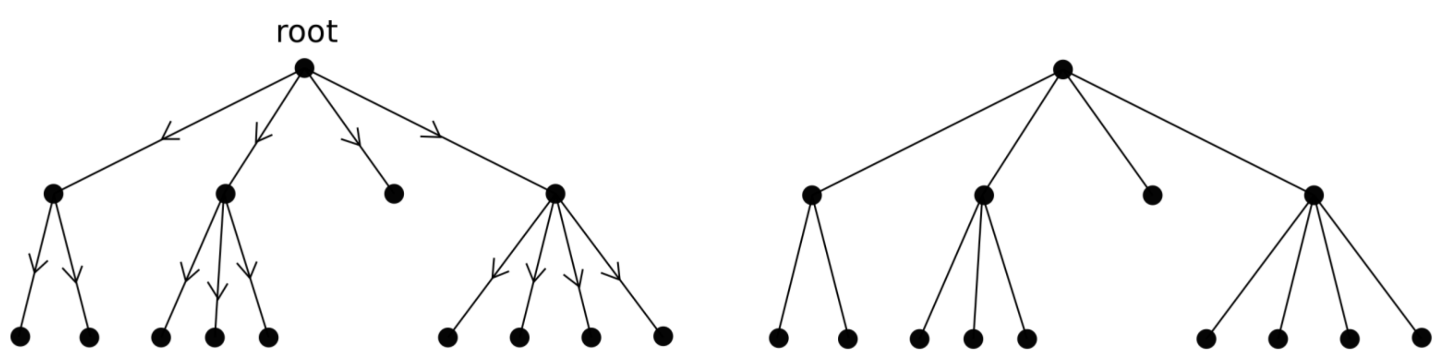
- T is connected (by definition)
- if T has a simple circuit containing the vertices x and y $\leadsto$ two simple paths between x and y. $\square$

Rooted Tree

Definition

A **rooted tree** is a tree in which one vertex has been designated as the root and every edge is directed away from the root.

Remarks: • A rooted tree is a directed graph.
• We usually draw a rooted tree with its root at the top of the graph.
• We usually omit the arrows on the edges to indicate the direction because it is uniquely determined by the choice of the root.
• Any non rooted tree can be changed to a rooted tree by choosing a vertex for the root.

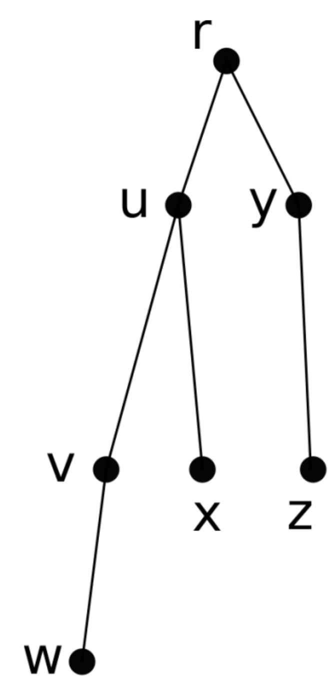


Rooted Tree

Definition

Let T be a rooted tree and v a vertex which is not the root. We call

- **parent** of v the *unique* vertex u such that there is an edge from u to v,
- **child** of v a vertex w such that there is an edge from v to w,
- **siblings** vertices with the same parent,
- **ancestors** of v all vertices in the path from the root to v,
- **descendants** of v all vertices that have v as an ancestor,
- **leaf** a vertex which has no children,
- **internal vertex** a vertex that has children,
- **subtree with** v **at its root** the subgraph of T consisting of v and its descendants and the edges incident to them.

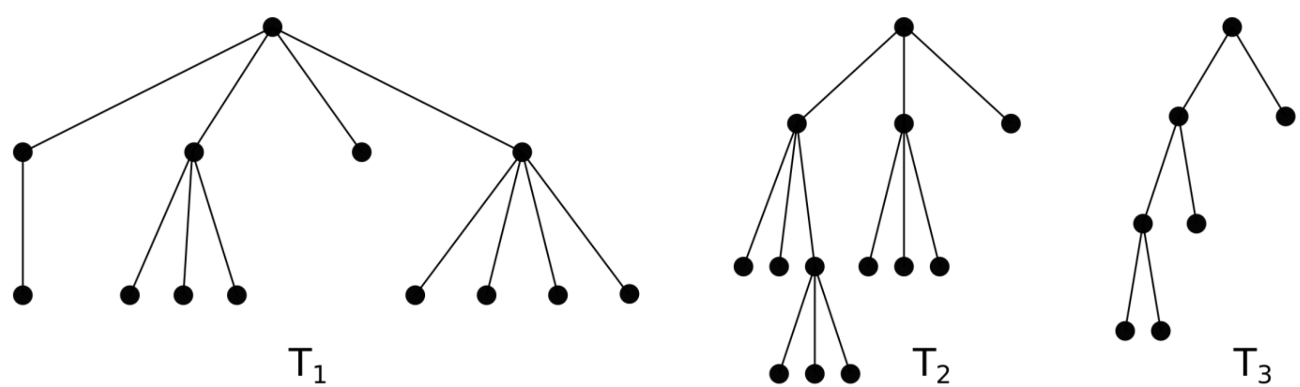


- r is the root
- v is child of u and parent of w
- v and x are siblings

Rooted Tree

Definition

• A rooted tree is called an m-**ary tree** if every internal vertex has no more than m children.
• A rooted tree is called a **full m-ary tree** if every internal vertex has exactly m children.
• An m-ary tree with $m = 2$ is called a **binary tree**. In this case if an internal vertex has two children, they are called **left child** and **right child**. The subtree rooted at the left (resp. right) child of a vertex is called the **left (resp. right) subtree** of this vertex.



T_1 is a 4-ary tree, T_2 a full 3-ary tree, T_3 a full binary tree.

Properties of Tree

Theorem

A tree with n vertices has $n - 1$ edges.

Properties of Tree

Theorem

A tree with n vertices has $n-1$ edges.

Proof: By induction on the number of vertices.

- $n = 1$: A tree with one vertex has no edge.
- $k \rightsquigarrow k+1$: Assume every tree with k vertices has $k-1$ edges.

Let T be a tree with $k+1$ vertices, and v a leaf (which exists because the tree has a finite number of vertices).

Let T' be the tree obtained from T by removing v (and the edge incident to it). T' is a connected tree with k vertices $\Rightarrow$ it has $k-1$ edges by induction hypothesis.

$\Rightarrow$ T has $k+1$ vertices and k edges.

Properties of Tree

Tree = connected with no simple circuit (definition)

(1) connected
(2) no simple circuit
(3) $(n-1)$ edges (n=nb of vertices)

Previous theorem: (1) + (2) $\Rightarrow$ (3)

We also have: (1) + (3) $\Rightarrow$ (2)
(2) + (3) $\Rightarrow$ (1)

Example: For what value of m, n the complete bipartite graph $K_{m,n}$ is a tree?

$K_{m,n}$ is connected, has $m+n$ vertices and $m \times n$ edges.
It is a tree if:

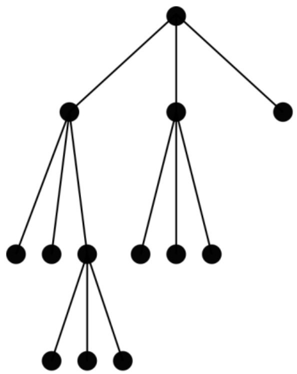
$$m \times n = m + n - 1 \Longleftrightarrow (n-1)m = n - 1$$

If $n \neq 1$: $m = 1$
If $n = 1$: $m \in \mathbb{N}^*$

Properties of Tree

Theorem

A full m-ary tree with i internal vertices contains $n = mi + 1$ vertices.



Proof: Each vertex (except the root) is the child of an internal vertex.
There are i internal vertices, each with m children
$\Rightarrow mi$ vertices + root $= mi + 1$ vertices

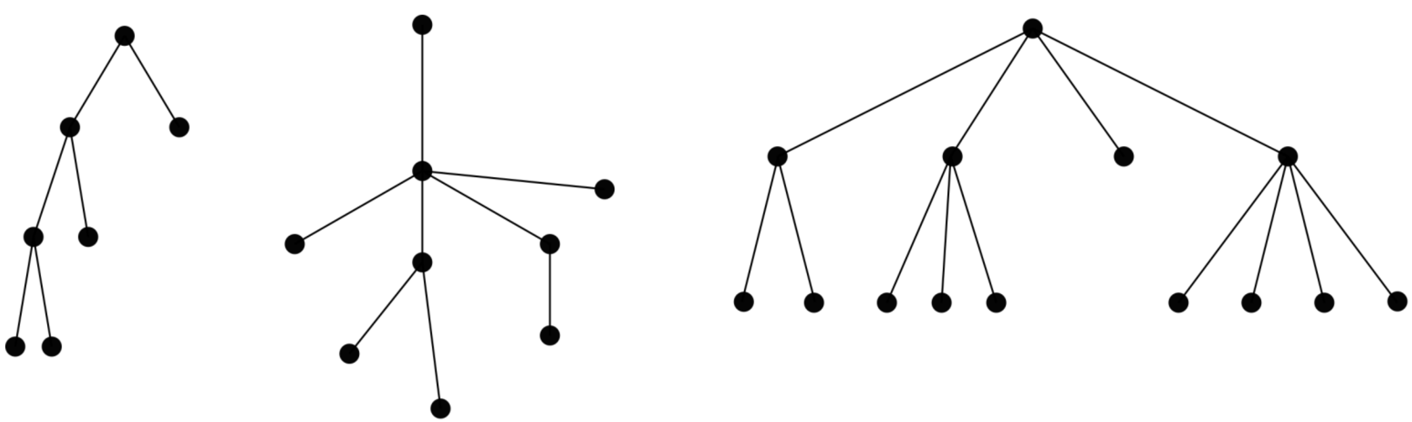
A full m-ary tree with

1. *n vertices has $i = (n-1)/m$ internal vertices and $\ell = \big((m-1)n+1\big)/m$ leaves,*
2. *i internal vertices has $n = mi + 1$ vertices and $\ell = (m-1)i + 1$ leaves,*
3. *ℓ leaves has $n = (m\ell - 1)/(m-1)$ vertices and $i = (\ell - 1)/(m-1)$ internal vertices.*

Balanced m-ary Tree

Definition

- The **level** of a vertex v in a rooted tree is the length of the unique path from the root to this vertex.
- The **height** of a rooted tree is the maximum of the levels of its vertices.
- A rooted m-ary tree of height h is **balanced** if all leaves are at levels h or $h-1$.



Balanced m-ary Tree

Theorem

There are at most m^h leaves in an m-ary tree of height h.

Proof: Induction again!

Corollary

If an m-ary tree of height h has l leaves, then $h \geq \lceil \log_m l \rceil$. If moreover the m-ary tree is full and balanced, then $h = \lceil \log_m l \rceil$.

Balanced m-ary Tree*

Theorem

There are at most m^h leaves in an m-ary tree of height h.

Proof: Induction again!

• An m-ary tree of height 1 consists of a root and its children (at most m) that are leaves. So the tree has at most $m^1 = m$ leaves.

• Assume all m-ary tree of height less or equal to h have at most m^h leaves.
Let T be an m-ary tree of height $h+1$ and denote r its root.
Consider the subtrees rooted at the children of r. Each of them is an m-ary tree of height less or equal to h, so by inductive hypothesis they have at most m^h leaves.
There are at most m of such trees because r has at most m children.
So in total T has at most $m \times m^h$ leaves.