Discrete Mathematics: Lecture 16

proposition, truth value, propositional constant/variable, negation, truth table, conjunction, disjunction, implication, bi-implication, formula

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Overview

- *Combinatorics: complexity analysis, etc*
- *Number theory: cryptography*
- Logic: software engineering, artificial intelligence, database theory, programming language, etc
- Graph theory: software engineering, theoretical computer science
- ...

Textbook: Discrete Mathematics and Its Applications (7th edition)
Kenneth H. Rosen, William C Brown Pub, 2011.

Mathematical Logic

Logic: the study of reasoning, the basis of all mathematical reasoning.

Mathematical logic: the mathematical study of reasoning and the study of mathematical reasoning //foundation of mathematics

- Leibniz: introduced the idea of mathematical logic in "Dissertation on the Art of Combinations" in 1666
- Universal system of reasoning: reasoning based on symbols+calculations
- **Contributors:** Boole, De Morgan, Frege, Peano, Russell, Hilbert, Gödel,...
- **Areas:** (1) set theory, (2) proof theory, (3) recursion theory, (4) model theory, and their foundation (5) propositional logic and predicate logic

Our focus: propositional logic and predicate logic, (naive) set theory

Proposition

Definition: A **proposition** is a declarative sentence(that is, a sentence that declares a fact) that is either true or false.

- Lower-case letters represent propositions: $p, q, r, \ldots$
- **Truth value**: The truth value of p is true (**T**) if p is a true proposition. The truth value of p is false (**F**) if p is a false proposition.

Example:

- Washington, D.C, is the capital of the United States of America.
  (T)
- $1 + 1 = 3$
  (F)
- $(x^2)' = 2x$
  (T)

Proposition

Example:

- Every even integer $n > 2$ is the sum of two primes.
  - Proposition?:
    Yes!
  - Goldbach's conjecture
  - A proposition whose truth value is not known now
- What time is it?
  - Proposition?:
    No!. It's not declarative.
- Do not smoke!
  - Proposition?:
    No!. It's not declarative.
- $x + 1 = 2$.
  - Proposition?:
    No!. It's neither true nor false.

Proposition

Simple Proposition:cannot be broken into 2 or more propositions

- $\sqrt{2}$ is irrational.

Compound Proposition: not simple

- 2 is rational and $\sqrt{2}$ is irrational.

Propositional Constant: a concrete proposition (truth value fixed)

- Every even integer $n > 2$ is the sum of two primes.

Propositional variables:a variable that represents any proposition

- Lower-case letters denote proposition variables: $p, q, r, s, \ldots$
- Truth value is not determined until it is assigned a concrete proposition

Propositional Logic: the area of logic that deals with propositions

Negation: $\neg$

Definition: Let p be any proposition.

- The **negation** of p is the statement "It is not the case that p"
- Notation: $\neg p$; read as "not p"
- **True table:**

p	$\neg p$
T	**F**
F	**T**

Negation: $\neg$

Example:

- $p =$ "Snow is black "
  - $\neg p =$ "It is not the case that snow is black."
  - $\neg p =$ "Snow is not black."
  - $\neg p \neq$ "Snow is white."

- $p =$ "Amy's smartphone has at least 32 GB of memory. "
  - $\neg p =$ "It is not the case that Amy's smartphone has at least 32 GB of memory. "
  - $\neg p =$ "Amy's smartphone does not have at least 32 GB."
  - $\neg p =$ "Amy's smartphone has less than 32 GB."

Conjunction: $\wedge$

Definition: Let p, q be any propositions.

- The **conjunction** of p and q is the statement "p and q"
- Notation: $p \wedge q$; read as "p and q"
- True table:

p	q	$p \wedge q$
T	**T**	**T**
T	**F**	**F**
F	**T**	**F**
F	**F**	**F**

Example:

- $p =$ "2<3"; $q =$ "$2^2 < 3^3$"
  - $p \wedge q =$ "$2 < 3$ and $2^2 < 3^3$."
    (T)
- $p =$ "Dog can fly"; $q =$ "Eagle can fly"
  - $p \wedge q =$ "Dog can fly and Eagle can fly."
    (F)

Disjunction: $\vee$

Definition: Let p, q be any propositions.

- The **disjunction** of p and q is the statement "p or q"
- Notation: $p \vee q$; read as "p or q"
- True table:

p	q	$p \vee q$
T	**T**	**T**
T	**F**	**T**
F	**T**	**T**
F	**F**	**F**

Example:

- $p =$ "2>3"; $q =$ "$2^2 > 3^3$"
  - $p \vee q =$ "$2 > 3$ or $2^2 > 3^3$."
    (F)
- $p =$ "Dog can fly"; $q =$ "Eagle can fly"
  - $p \vee q =$ "Dog can fly or Eagle can fly."
    (T)

Implication: $\rightarrow$

Definition: Let p, q be any propositions.

- The **conditional statement** $p \rightarrow q$ is the proposition "if p, then q."
  - p: hypothesis; q: conclusion; read as "p implies q", or "if p, then q"
- True table:

p	q	$p \rightarrow q$
T	**T**	**T**
T	**F**	**F**
F	**T**	**T**
F	**F**	**T**

Example:

- $p =$ "you get 100 on the final"; $q =$ "you will receive $A+$"
  - $p \rightarrow q =$ "If you get 100 on the final, you will receive $A+$." **(T)**
  - It is false when you get 100 on the final but don't receive $A+$, which is "when p is true but q is false."**(F)**

Bi-Implication: $\leftrightarrow$

Definition: Let p, q be any propositions.

- The **biconditional statement** $p \leftrightarrow q$ is the proposition "p if and only if q."
  - read as"p if and only if q"
- True table:

p	q	$p \leftrightarrow q$
T	**T**	**T**
T	**F**	**F**
F	**T**	**F**
F	**F**	**T**

Example:

- $p =$ "you can take the flight"; $q =$ "you buy a ticket"
  - $p \leftrightarrow q =$ "You can take the flight if and only if you buy a ticket."
  - False when $(p, q) =$(**T**, **F**) or (**F**, **T**)

Well-Formed Formulas

Definition: recursive definition of **well-formed formulas (WFFs)**

1. propositional constants (**T**, **F**) and propositional variables are WFFs
2. If A is a WFF, then $\neg A$ is a WFF.
3. If A, B are WFFs, then $(A \wedge B), (A \vee B), (A \rightarrow B), (A \leftrightarrow B)$ are WFFs
4. WFFs are results of finitely many applications of $1, 2, 3$.

Remark: well-formed formulas = propositional formulas = formulas

Use tree structure to check

- $\neg(p \wedge q) \rightarrow (r \wedge s)$
  (T)
- $(p \wedge q)\neg r$
  (F)
- $m \leftrightarrow ((p \wedge q) \rightarrow (\neg r \wedge s))$
  (T)

Summary

Proposition: a <u>declarative</u> sentence that is <u>either true or false</u>.

- simple, compound, propositional constant/variable

Logical Connectives: $\neg$ (unary), $\wedge, \vee, \rightarrow, \leftrightarrow$ (binary)

- Truth table
- Example 14 (Textbook Page 11)

Well-Formed Formulas: formulas

- propositional constant, variables
- $\neg A, (A \wedge B), (A \vee B), (A \leftarrow B), (A \leftrightarrow B)$
- Finite

Precedence of Logical Operators

Precedence (priority): $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$

- formulas inside () are computed firstly
- different connectives: $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$ (Decreasing Precedence)
- same connectives: from left to the right

- Example 1: $\neg p \wedge q$: $(\neg p) \wedge q$
- Example 2:$\neg(p \wedge q)$: First (.), then $\neg$.
- Example 3: $p \vee q \wedge r$: $p \vee (q \wedge r)$
- Example 4:

$$\underline{\underline{\underline{(p \rightarrow q)} \wedge \underline{(q \rightarrow r)}} \leftrightarrow \underline{(p \rightarrow r)}}$$

From Natural Language to WFFs

The Method of Translation:

- Introduce symbols to represent simple propositions
- Connect the symbols with logical connectives to obtain WFFs

Example:

- "It is not the case that snow is black."
  - p: "Snow is black"
  - Translation: $\neg p$
  - **Remark:** it is better to choose the simple proposition to be affirmative sentence.
- "π and e are both irrational"
  - p : "π is irrational"; q : "e. is irrational"
  - Translation: $p \wedge q$
- "If π is irrational, then 2π is irrational"
  - p : π is irrational; q : 2π is irrationals
  - Translation: $p \to q$

Example:

- "$e^{\pi} > \pi^{e}$ if and only if $\pi > e \ln \pi$"
  - $p : e^{\pi} > \pi^{e}$; $q : \pi > e \ln \pi$
  - Translation: $p \leftrightarrow q$
- **" $(\sqrt{2})^{\sqrt{2}}$ is rational or irrational."** (ambiguity in natural language)
  - $p =$ "$(\sqrt{2})^{\sqrt{2}}$ is rational "; $q =$ "$(\sqrt{2})^{\sqrt{2}}$ is irrational"
  - Explanation 1: $(\sqrt{2})^{\sqrt{2}}$ cannot be neither rational nor irrational.
  - Emphasis: $(\sqrt{2})^{\sqrt{2}}$ is a real number, only two possibility
  - Translation 1: $p \vee q$ (by default, this is the translation of "or")

  - Explanation 2: $(\sqrt{2})^{\sqrt{2}}$ cannot be both rational or irrational
  - It is obvious that $(\sqrt{2})^{\sqrt{2}}$ is real number. Emphasis: not both
  - Translation 2: $(p \wedge \neg q) \vee (\neg p \wedge q)$ (not both)
- The specific translations remove the ambiguity.