EE160 Introduction to Control: Homework 4

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Deadline: April 1, 2022

1. *Explicit solution of nonlinear differential equations.* Solve the following differential equations explicitly by using the method of separation of variables,

   (a) $\dot{x}(t) = \sin(t)x(t) \quad \text{with} \quad x(0) = 1$

   (b) $\dot{x}(t) = 1 - x(t)^2 \quad \text{with} \quad x(0) = 2$

   (c) $\dot{x}(t) = -\frac{1}{x}e^{x(t)^2} \quad \text{with} \quad x(0) = 1$

2. *Picard Iteration.* Find an explicit form of the Picard iterates for the differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 2t(1+x) \quad \text{with} \quad x(0) = 0$$

   and compute the limit function.

3. *Implementation of the classical Runge-Kutta integrator.* Implement the integration method from Algorithm 1.2 in the lecture notes by using the classical Runge-Kutta coefficients with

$$\begin{aligned} k_1 &= f(t_i, y_i) \\ k_2 &= f\left(t_i + \frac{h}{2}, y_i + \frac{h}{2}k_1\right) \\ k_3 &= f\left(t_i + \frac{h}{2}, y_i + \frac{h}{2}k_2\right) \\ k_4 &= f(t_i + h, y_i + hk_3) \\ y_{i+1} &= y_i + h\left(\frac{1}{6}k_1 + \frac{1}{3}k_2 + \frac{1}{3}k_3 + \frac{1}{6}k_4\right) \end{aligned}$$

   Test you implementation by solving the following differential equations numerically:

   (a) $\dot{x}(t) = t^2 \quad \text{with} \quad x(0) = 1.$ Set $[0, T] = [0, 1]$ and $N = 100.$

   (b) $\dot{x}(t) = -tx(t) \quad \text{with} \quad x(0) = 1.$ Set $[0, T] = [0, 3]$ and $N = 100.$

   (c) $\dot{x}(t) = \sin(x(t)) \quad \text{with} \quad x(0) = 1.$ Set $[0, T] = [0, 10]$ and $N = 100.$

   Plot the points (t_i, y_i) for $i \in \{0, 1, \ldots, 100\}$. Also compare the numerical results with the exact results, if you can solve the differential equation explicitly.