Inverse Transforms

In principle, we can recover f from F via

$$f(t)=\frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty}F(S)e^{st}ds$$

Surprisingly, this formula isn't really useful!

What is more common/useful as follows:

$$F(s)=\frac{P(s)}{Q(s)}=\frac{b_m s^m+b_{m-1}s^{m-1}+\ldots+b_1 s+b_0}{a_n s^n+a_{n-1}s^{n-1}+\ldots+a_1 s+a_0}$$

Generally

$$F(s)=\frac{P(s)}{Q(s)}=\frac{b_m s^m+b_{m-1}s^{m-1}+\ldots+b_1 s+b_0}{a_n s^n+a_{n-1}s^{n-1}+\ldots+a_1 s+a_0}$$

a_i and b_i are real constants, and the exponents m,n are positive integers

- If m<n, proper rational function
- If m>n, improper rational function

Partial Fraction Expansion with Real Distinct Roots

- Let F(s) be proper rational function, then

$$F(s)=\frac{P(s)}{Q(s)}=\frac{b_m s^m+b_{m-1}s^{m-1}+\ldots+b_1 s+b_0}{a_n s^n+a_{n-1}s^{n-1}+\ldots+a_1 s+a_0}$$

Case I: If the roots are real, $p_i \neq p_j \ for\ \forall i \neq j$

$$F(s)=\frac{P(s)}{Q(s)}=\frac{K_1}{s-p_1}+\frac{K_2}{s-p_2}+\cdots+\frac{K_n}{s-p_n}=\sum_{j=1}^{n}\frac{K_j}{s-p_j}$$

p_j(j=1， 2， …， n) are ***the roots*** of equation $Q(s)$=0

K_j(j=1， 2， …， n) are unknown constants

Partial Fraction Expansion with Real Distinct Roots

$$F(s)=\frac{P(s)}{Q(s)}=\frac{K_1}{s-p_1}+\frac{K_2}{s-p_2}+\cdots+\frac{K_n}{s-p_n}=\sum_{j=1}^{n}\frac{K_j}{s-p_j}$$

Case I:

If the roots are real, $p_i \neq p_j \;\; for\ \forall i \neq j$

$$K_j=\lim_{s\to p_j}(s-p_j)F(s)=(s-p_j)F(s)\Big|_{s=p_j}$$

Exercise

$$F(s)=\frac{s^2+3s+5}{s^3+6s^2+11s+6}$$

$$F(s)=\frac{s^2+3s+5}{(s+1)(s+2)(s+3)}=\frac{K_1}{s+1}+\frac{K_2}{s+2}+\frac{K_3}{s+3}$$

Partial Fraction Expansion with Multiple Roots

- Case II:
- If Q(s) has multiple roots

$$F(s)=\boxed{\frac{K_{11}}{s-p_1}+\frac{K_{12}}{(s-p_1)^2}+\cdots+\frac{K_{1r}}{(s-p_1)^r}}+\boxed{\frac{K_{r+1}}{s-p_{r+1}}\cdots+\frac{K_n}{s-p_n}}$$

$$K_{1r}=(s-p_1)^r F(s)\Big|_{s=p_1}$$

$$K_{1(r-1)}=\frac{d}{ds}[(s-p_1)^r F(s)]_{s=p_1}$$

$$K_{1(r-2)}=\frac{1}{2!}\frac{d^2}{ds^2}[(s-p_1)^r F(s)]_{s=p_1}$$

$$\vdots$$

$$K_{11}=\frac{1}{(r-1)!}\frac{d^{r-1}}{ds^{r-1}}[(s-p_1)^r F(s)]_{s=p_1}$$

Exercise

$$F(s) = \frac{10s^2 + 4}{s(s+1)(s+2)^2}$$

$$F(s) = \frac{K_{11}}{s} + \frac{K_{21}}{s+1} + \frac{K_{31}}{s+2} + \frac{K_{32}}{(s+2)^2}$$

$$\mathrm{f(t)} = \left[1 - 14e^{-t} + (13 + 22t)e^{-2t}\right]u(t)$$

Partial Fraction Expansion with Complex Roots

Case III：
If F(s) has a pole of p_1 expressed by a complex number, then it must have a complex root P_2 as a conjugate of P_1

$$p_1 = \alpha + j\omega \quad p_2 = p_1^* = \alpha - j\omega$$

$$F(s) = \frac{K_1}{s-(\alpha + j\omega)} + \frac{K_2}{s-(\alpha - j\omega)}$$

$$K_1 = [s-(\alpha + j\omega)]F(s)\,|_{s=\alpha + j\omega}$$

$$K_2 = [s-(\alpha - j\omega)]F(s)\,|_{s=\alpha - j\omega} \qquad K_2 = K_1^* = |K_1|e^{-j\varphi_K}$$

$$
\begin{aligned}
f(t) = K_1 e^{(\alpha + j\omega)t} + K_2 e^{(\alpha - j\omega)t} &= |K_1| e^{\alpha t} [e^{j(\omega t + \varphi_K)} + e^{-j(\omega t + \varphi_K)}] \\
&= 2|K_1| e^{\alpha t} \cos(\omega t + \varphi_K)
\end{aligned}
$$

Partial Fraction Expansion with Complex Roots

- Example：
$$F(s)=\frac{s^2+3s+7}{(s^2+4s+13)(s+1)}$$

$$p_1=-2+j3，\quad p_2=-2-j3，\quad p_3=-1$$

$$F(s)=\frac{K_1}{s-(-2+\boxed{j3})}+\frac{K_1^*}{s-(-2-j3)}+\frac{K_3}{s+1}$$

$$K_3=\left.\frac{s^2+3s+7}{s^2+4s+13}\right|_{s=-1}=0.5$$

EXAMPLE：

$$F(s) = \frac{2s^3 + 33s^2 + 93s + 54}{s(s + 1)(s^2 + 5s + 6)}.$$

$$F(s) = \frac{14s^2 + 56s + 152}{(s + 6)(s^2 + 4s + 20)}.$$