Chap 9 — 6

向量场的微商

9.6.1 向量场

场 空间区域中每一点都对应一个量. 若对应的量是向量(数量), 则称之为**向量场(数量场)**.

如空间向量场

$$\boldsymbol{v}(x,y,z)=P(x,y,z)\boldsymbol{i}+Q(x,y,z)\boldsymbol{j}+R(x,y,z)\boldsymbol{k}$$

$(x, y, z) \in D \subset \mathbf{R}^3$. 平面向量场

$$\boldsymbol{v}(x,y)=P(x,y)\boldsymbol{i}+Q(x,y)\boldsymbol{j},\quad (x,y)\in D\subset\mathbb{R}^2$$

位置向量 $\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}.$

定义 设有向量场$\boldsymbol{v}(x, y, z)$, 它对x的**偏微商**定义为

$$\frac{\partial \boldsymbol{v}}{\partial x}=\lim_{\Delta x\to 0}\frac{\boldsymbol{v}(x+\Delta x, y, z)-\boldsymbol{v}(x, y, z)}{\Delta x}$$

想一想 对y, z的偏微商? 显见

$$\frac{\partial \boldsymbol{v}}{\partial x}=\frac{\partial P(x,y,z)}{\partial x}\boldsymbol{i}+\frac{\partial Q(x,y,z)}{\partial x}\boldsymbol{j}+\frac{\partial R(x,y,z)}{\partial x}\boldsymbol{k}$$

光滑向量场 P, Q, R具有连续偏导数.

例 设 $\boldsymbol{v}(x,y,z)=\sin xyz\boldsymbol{i}+x^2\boldsymbol{j}+\mathrm{e}^{x+2y+z}\boldsymbol{k}$, 求 $\frac{\partial \boldsymbol{v}}{\partial x},\frac{\partial \boldsymbol{v}}{\partial y},\frac{\partial \boldsymbol{v}}{\partial z}$

9.6.2 梯度、散度与旋度

设有数量场$\varphi(x, y, z)$, 根据微分及点乘公式, 有

$$\mathrm{d}\varphi = \frac{\partial \varphi}{\partial x}\mathrm{d}x + \frac{\partial \varphi}{\partial y}\mathrm{d}y + \frac{\partial \varphi}{\partial z}\mathrm{d}z$$

$$= \left(\frac{\partial \varphi}{\partial x}\boldsymbol{i} + \frac{\partial \varphi}{\partial y}\boldsymbol{j} + \frac{\partial \varphi}{\partial z}\boldsymbol{k} \right) \cdot (\mathrm{d}x\boldsymbol{i} + \mathrm{d}y\boldsymbol{j} + \mathrm{d}z\boldsymbol{k})$$

$$= \mathbf{grad}\,\varphi \cdot \mathrm{d}\boldsymbol{r}$$

- **函数微分是梯度与位置向量微分之点乘.**

➢ Hamilton算子

$$\nabla = \frac{\partial}{\partial x}\boldsymbol{i} + \frac{\partial}{\partial y}\boldsymbol{j} + \frac{\partial}{\partial z}\boldsymbol{k}$$

➢ ∇算子兼具**偏微商**和**向量**属性, 它‘数乘’函数得

$$\nabla\varphi = \mathbf{grad}\,\varphi = \frac{\partial\varphi}{\partial x}\boldsymbol{i} + \frac{\partial\varphi}{\partial y}\boldsymbol{j} + \frac{\partial\varphi}{\partial z}\boldsymbol{k}$$

➢ 数量场的**梯度是向量场**

定义 设有向量场$\boldsymbol{v}(x, y, z) = P\boldsymbol{i} + Q\boldsymbol{j} + R\boldsymbol{k}$.

1° $\boldsymbol{v}$ 的**散度** $\operatorname{div}\boldsymbol{v} \overset{\text{def}}{=} \nabla\cdot\boldsymbol{v} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$

➢ 向量场的**散度是数量场**

2° $\boldsymbol{v}$ 的**旋度**

$$\mathbf{rot}\,\boldsymbol{v} \overset{\text{def}}{=} \nabla \times \boldsymbol{v} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\boldsymbol{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\boldsymbol{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\boldsymbol{k}$$

$$= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

➢ 向量场的**旋度是向量场**

命题(运算性质) 设φ, ψ是数量场, $\boldsymbol{a}$, $\boldsymbol{b}$是向量场, 则有

$$\nabla f(\varphi)=f'(\varphi)\nabla\varphi$$

$$\nabla(\varphi+\psi)=\nabla\varphi+\nabla\psi$$

$$\nabla\cdot(\boldsymbol{a}+\boldsymbol{b})=\nabla\cdot\boldsymbol{a}+\nabla\cdot\boldsymbol{b}$$

$$\nabla\times(\boldsymbol{a}+\boldsymbol{b})=\nabla\times\boldsymbol{a}+\nabla\times\boldsymbol{b}$$

$$\nabla(\varphi\psi)=\psi\nabla\varphi+\varphi\nabla\psi$$

$$\nabla\cdot(\varphi\boldsymbol{a})=\nabla\varphi\cdot\boldsymbol{a}+\varphi\nabla\cdot\boldsymbol{a}$$

$$\nabla\times(\varphi\boldsymbol{a})=\nabla\varphi\times\boldsymbol{a}+\varphi\nabla\times\boldsymbol{a}$$

$$\nabla\cdot(\boldsymbol{a}\times\boldsymbol{b})=\nabla\times\boldsymbol{a}\cdot\boldsymbol{b}-\nabla\times\boldsymbol{b}\cdot\boldsymbol{a}$$

此外, 还有

$$\boxed{\begin{gathered}\mathbf{rot}\ \mathbf{grad}\,\varphi=\nabla\times\nabla\varphi=\mathbf{0}\\[2ex] \mathrm{div}\,\mathbf{rot}\,\boldsymbol{a}=\nabla\cdot\nabla\times\boldsymbol{a}=0\end{gathered}}$$

➢ Laplace算子

$$\Delta = \nabla \cdot \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

它作用函数φ得到

$$\Delta \varphi = \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2}$$

例 设 $f(x, y, z) = g(r), r = \sqrt{x^2 + y^2 + z^2}$

1) 求Δf；　**2)** 若$\Delta f = 0$, 求f.