

$$1. (a) \frac{du(t)}{dt} = \delta(t) \quad 1'$$

$$\text{so } \frac{d}{dt} \{u(-2-t) + u(t-2)\} = -\delta(t+2) + \delta(t-2) \quad 2'$$

$$\begin{aligned} X(j\omega) &= \int_{-\infty}^{\infty} (\delta(t-2) - \delta(t+2)) e^{-j\omega t} dt \\ &= e^{-2j\omega} - e^{2j\omega} = -2j \sin 2\omega \end{aligned} \quad 4'$$

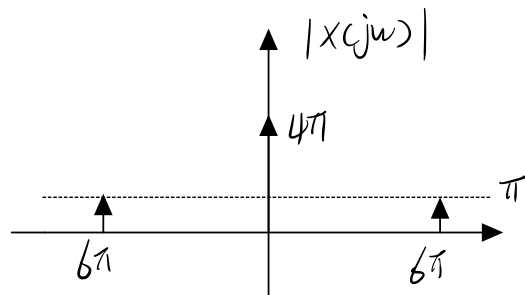
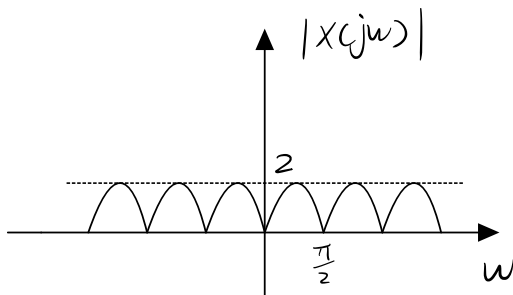
$$(b) \quad \omega_0 = 6\pi \quad 1'$$

Consider the Fourier series:

$$\begin{aligned} x(t) &= 2 + \frac{1}{2} e^{j(6\pi t + \frac{\pi}{8})} + \frac{1}{2} e^{-j(6\pi t + \frac{\pi}{8})} \\ &= 2 + \frac{1}{2} e^{j\frac{\pi}{8}} e^{j6\pi t} + \frac{1}{2} e^{-j\frac{\pi}{8}} e^{-j6\pi t} \end{aligned} \quad 2'$$

where $a_0 = 2$, $a_1 = \frac{1}{2} e^{j\frac{\pi}{8}}$, $a_{-1} = \frac{1}{2} e^{-j\frac{\pi}{8}}$, $a_k = 0$ for other

$$\text{so } X(j\omega) = 4\pi \delta(\omega) + \pi e^{j\frac{\pi}{8}} \delta(\omega - 6\pi) + \pi e^{-j\frac{\pi}{8}} \delta(\omega + 6\pi) \quad 4'$$



$$2. (a) x(t) \xrightarrow{F} X(j\omega) \Rightarrow x(3t) \xrightarrow{F} \frac{1}{3} X(j\frac{\omega}{3})$$

$$\Rightarrow x(3t-2) \xrightarrow{F} e^{-j\omega \frac{2}{3}} \frac{1}{3} X(j\frac{\omega}{3}) \Rightarrow x^*(3t-6) \xrightarrow{F} e^{2j\omega \frac{1}{3}} X^*(-j\frac{\omega}{3})$$

6'

For invertible system: $H_1(j\omega) H_2(j\omega) = 1$

so the Fourier transform of inverse is:

$$3e^{2j\omega} / X^*(-j\frac{\omega}{3})$$

4'

$$(b) x(t) \xrightarrow{F} X(j\omega) \Rightarrow \frac{dx(t)}{dt} \xrightarrow{F} j\omega X(j\omega)$$

$$\Rightarrow \frac{d^2x(t)}{dt^2} \xrightarrow{F} -\omega^2 X(j\omega)$$

5'

$$\Rightarrow \frac{d^2x(t-1)}{dt^2} \xrightarrow{F} -e^{-j\omega} \omega^2 X(j\omega)$$

5'

$$3. (a) x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega \Rightarrow X(j\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

3'

$$\Rightarrow 2\pi X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt, \text{ so } x(t) \xrightarrow{F} 2\pi X(-j\omega)$$

4'

$$(b) \begin{array}{c} \text{[Diagram: A frequency spectrum plot with three rectangular pulses. The first pulse is at } \omega = -\frac{\omega_0}{2} \text{ with height 1. The second pulse is at } \omega = 0 \text{ with height 1. The third pulse is at } \omega = \frac{\omega_0}{2} \text{ with height 1. The x-axis is labeled } \omega \text{ and has ticks at } -\frac{\omega_0}{2}, -\omega_1, \omega_1, \frac{\omega_0}{2}. \end{array} \xrightarrow{F} \sum_{k=-\infty}^{\infty} \frac{2\sin(k\omega_0 T_1)}{k} \delta(\omega - k\omega_0)$$

4'

$$\text{so } x(t) \xrightarrow{F} \begin{array}{c} \text{[Diagram: A frequency spectrum plot with three rectangular pulses. The first pulse is at } \omega = -\frac{\omega_0}{2} \text{ with height } \pi. \text{ The second pulse is at } \omega = 0 \text{ with height } \pi. \text{ The third pulse is at } \omega = \frac{\omega_0}{2} \text{ with height } \pi. \text{ The x-axis is labeled } \omega \text{ and has ticks at } -\frac{\omega_0}{2}, -\omega_1, \omega_1, \frac{\omega_0}{2}. \end{array}$$

4'

$$4. (a) \frac{Y(j\omega)}{X(j\omega)} = H(j\omega) \quad 1'$$

$$\text{so } (j\omega)^2 Y(j\omega) + 5j\omega Y(j\omega) + 6Y(j\omega) = j\omega X(j\omega) + 4X(j\omega) \quad 2'$$

$$\Rightarrow \frac{d^2 y(t)}{dt^2} + 5 \frac{dy(t)}{dt} + 6y(t) = \frac{dx(t)}{dt} + 4x(t) \quad 3'$$

$$(b) \quad H(j\omega) = \frac{j\omega + 4}{6 - \omega^2 + 5j\omega} \quad 2'$$

$$= \frac{2}{2 + j\omega} - \frac{1}{3 + j\omega} \quad 2'$$

$$\text{Hence, } h(t) = 2e^{-2t}u(t) - e^{-3t}u(t) \quad 3'$$

$$(c) \quad X(j\omega) = \frac{1}{4 + j\omega} - \frac{1}{(4 + j\omega)^2} \quad 3'$$

$$\text{so } Y(j\omega) = \frac{1}{2} \frac{1}{2 + j\omega} - \frac{1}{2} \frac{1}{4 + j\omega} \quad 2'$$

$$\text{so } y(t) = \frac{1}{2} e^{-2t}u(t) - \frac{1}{2} e^{-4t}u(t) \quad 2'$$

$$5. (a) \quad k_p e(t) + k_i \int_{-\infty}^t e(\tau) d\tau + k_d \frac{de(t)}{dt} = y(t) \quad 3'$$

$$\text{so } k_p \frac{d(x(t) - y(t))}{dt} + k_i (x(t) - y(t)) + k_d \frac{d^2(x(t) - y(t))}{dt^2} = y'(t)$$

$$\Rightarrow k_d \frac{d^2 x(t)}{dt^2} + k_p \frac{dx(t)}{dt} + k_i x(t) = k_d \frac{d^2 y(t)}{dt^2} + (k_p + 1) \frac{dy(t)}{dt} + k_i y(t) \quad 2'$$

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{k_d(j\omega)^2 + k_p j\omega + k_i}{k_d(j\omega)^2 + (k_p + 1)j\omega + k_i} = \frac{-k_d \omega^2 + k_p j\omega + k_i}{-k_d \omega^2 + (k_p + 1)j\omega + k_i} \quad 3'$$

(b) For PID controller:

Assume that the output, input is $y_o(t)$, $x_o(t)$

$$y_o(t) = k_p x_o(t) + k_i \int_{-\infty}^t x_o(\tau) d\tau + k_d x_o'(t)$$

$$\Rightarrow H_{pid}(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{k_p + k_i \frac{1}{j\omega} + k_d j\omega}{1} = \frac{k_p j\omega + k_i - k_d \omega^2}{j\omega} \quad 3'$$

$$\text{let } k_i = 0 \quad H_{pd}(j\omega) = k_p + k_d j\omega \quad 3'$$

$$\text{let } k_d = 0 \quad H_{pi}(j\omega) = \frac{k_p j\omega + k_i}{j\omega} \quad 3'$$

$$(c) \quad H_{pi}(j\omega) H_{pd}(j\omega) = (k_{p1} + \frac{k_i}{j\omega}) (k_{p2} + k_d j\omega) \quad 6'$$

$$= 12 + \frac{5}{j\omega} + 4j\omega$$

so it's equivalent to the pid controller with $2'$

$k_p = 12$, $k_i = 5$, $k_d = 4$. proved.