Numerical analysis(SI211) Fall 2021-22 Homework 1

Prof. Jiahua Jiang

Name: xxx Student No.: xxx E-mail: xxx@shanghaitech.edu.cn

Acknowledgements:

1. Deadline: **2021-10-29 11:59:00**, no late submission is allowed.
2. No handwritten homework is accepted. You should submit your homework in Blackboard with PDF format, we recommend you use LaTeX. (If you have difficulty in using LaTeX,you are allowed to use Word for the first homework to accommodate yourself.)
3. Giving your solution in English, solution in Chinese is not allowed.
4. Make sure that your codes can run and are consistent with your solutions, you can use any programming language.
5. Your PDF should be named as "your student id+HW1.pdf", package all your code into "your_student_id+_Code1.zip" and upload.Don't put your PDF in your code file.
6. Plagiarism is not allowed. Those plagiarized solutions and codes will get 0 point.

---

1. **Taylor's Theorem**(20 points.) Consider the function

$$f(x) = \frac{\sin(x) - x}{x^3}, \tag{1}$$

(a) Find the third Taylor polynomials $P_3(x)$ and truncation error for the function.

(b) About $x_0 = 10^{-5}$, approximate $f'(0.1)$ using $P_3'(0.1)$ and compare the numerical error with truncation error.

Solution: Since $f \in C^\infty(\mathbb{R})$, Taylor's Theorem can be applied for any $n \geq 0$.

$$f'(x) = \frac{\cos(x) - 1}{x^3} - \frac{3(-x + \sin(x))}{x^4},$$

$$f^{(2)}(x) = -\frac{1}{x^3}(\sin(x) + \frac{6(\cos(x) - 1)}{x} + \frac{12(x - \sin(x))}{x^2}),$$

$$f^{(3)}(x) = \frac{1}{x^3}(-\cos(x) + \frac{9\sin(x)}{x} + \frac{36(\cos(x) - 1)}{x^2} + \frac{60(x - \sin(x))}{x^3}),$$

$$f^{(4)}(x) = \frac{1}{x^3}(\sin(x) + \frac{12\cos(x)}{x} - \frac{72\sin(x)}{x^2} - \frac{240(\cos(x) - 1)}{x^3} - \frac{360(x - \sin(x))}{x^4})$$

(a) For $n = 3$ and a fixed point x_0, we have

$$P_3(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f^{(2)}(x_0)}{2!}(x - x_0)^2 + \frac{f^{(3)}(x_0)}{3!}(x - x_0)^3 \tag{2}$$

The truncation error associated with $P_3(x)$ is given

$$R_3(x) = \frac{f^{(4)}(\xi(x))}{4!}(x - x_0)^4.$$

(b) For $n = 2$ and a fixed point x_0, Taylor expansion for $f'(x)$ is

$$f'(x) = f'(x_0) + f^{(2)}(x_0)(x - x_0) + \frac{f^{(3)}(x_0)}{2!}(x - x_0)^2 + \frac{f^{(4)}(\theta(x))}{3!}(x - x_0)^3. \tag{3}$$

Also, we can obtain

$$P_3'(x) = f'(x_0) + f^{(2)}(x_0)(x - x_0) + \frac{f^{(3)}(x_0)}{2!}(x - x_0)^2. \tag{4}$$

Upon comparing (3) and (4), we find the Taylor polynomials of (3) is the same as (4), which means that the truncation error of $P_3'(x)$ approximating $f'(x)$ is the truncation error of (3).

In this homework, three x_0 are accepted, that are $\pi/12, 0, 10^{-5}$. Given $f'(0.1) = 0.0016658731813441463$,

- If $x_0 = \pi/12$, $P_3'(0.1) = 0.00166254639521111$, numerical error is $3.326786e-06$, truncation error is bounded by 3.354722e-6. Thus numerical error is less than truncation error.
- If $x_0 = 0$, firstly obtain Taylor expansion at $x_0 = 0$,

$$\sin(x) = x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{\cos(\xi(x))}{5040}x^7$$

  At x_0,

$$f(x) = \frac{1}{6} + \frac{x^2}{120} - \frac{\cos(\xi(x))}{5040}x^4$$

  , $P_3(x) = \frac{1}{6} + \frac{x^2}{120}$, $P_3'(x) = \frac{x}{60}$. $P_3'(0.1) = 0.00166667$
  Numerical error is $7.934853225204704e-07$.

$$f'(x) = \frac{x}{60} + \frac{1}{7!}(-4\cos(\xi(x)) + \xi(x)\sin(\xi(x)))x^3$$

  The truncation error is bounded by $7.93650793624781e-7$.
- If $x_0 = 10^{-5}$, the procedure is the same as $\pi/12$. However this value may meet the round off error.

2. **Lagrange Interpolating Polynomials**(20 points.)For the given 3 points:

x	0.32	0.34	0.36
$f(x) = \sin(x)$	0.314567	0.333487	0.352274

(a) Using the first-order Lagrange Interpolating Polynomials to estimate the value of $f(x)$ at the point $x = 0.3367$, indicates the nodes you selected and give the error estimate.

(b) Using the second-order Lagrange Interpolating Polynomials to estimate the value of $f(x)$ at the point $x = 0.3367$, and give the error estimate.

Solution:

(a) From the Theorem 3.29(P110) we know that to obtain the first-order Lagrange Polynomial we need at most two different points. Note that $x = 0.3367$ is between $x_0 = 0.32$ and $x_1 = 0.34$, from the Theorem 3.1(P106) and Theorem 3.3(P112) , we know that if we want to a more accurately approximate we had better to choose x_0 and x_1. Thus we have

$$\begin{aligned}\sin 0.3367 &\approx L_1(0.3367) \\ &= y_0 + \tfrac{y_1-y_0}{x_1-x_0}(0.3367 - x_0) \\ &= 0.330365,\end{aligned} \tag{5}$$

truncation error:

$$R_n(x) = f^{(n+1)}(\xi(x)) \prod_{i=0}^{n}(x - x_i),$$

with $\xi(x) \in [\min\{x_i\}, \max\{x_i\}], i = 0, \cdots, n$.

$$\begin{aligned}
|R_1| &\leq \max_{\xi(x)\in[x_0,x_1]} \tfrac{|f''(\xi(x))|}{2} |(x_0 - x)(x_1 - x)| \\
&\leq \max_{\xi(x)\in[x_0,x_1]} \tfrac{|f''(\xi(x))|}{2} \cdot \max_{x\in[x_0,x_1]} |(x_0 - x)(x_1 - x)| \\
&\leq 0.92 \times 10^{-5}
\end{aligned} \tag{6}$$

[Warning: In this homework, you can choose x_0 and x_2 or choose x_1 and x_2 to approximate the value at $x = 0.3367$, but the latter option is forbidden in engineering.]

(b) The second-order Lagrange Interpolating Polynomials:

$$\begin{aligned}
P_2(x) &= \tfrac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} y_0 + \tfrac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} y_1 + \tfrac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} y_2 \\
&= 0.330283
\end{aligned} \tag{7}$$

truncation error:

$$\begin{aligned}
|R_2| &\leq \max_{\xi(x)\in[x_0,x_2]} \tfrac{|f^{(3)}(\xi(x))|}{6} |(x_0 - x)(x_1 - x)(x_2 - x)| \\
&\leq \max_{\xi(x)\in[x_0,x_2]} \tfrac{|f^{(3)}(\xi(x))|}{6} \cdot \max_{x\in[x_0,x_2]} |(x_0 - x)(x_1 - x)(x_2 - x)| \\
&\leq 2.0316 \times 10^{-7}.
\end{aligned} \tag{8}$$

3. **Coding of Hermite Interpolation**(20 points.) Given the points:

x	1/3	1	5/3
$f(x) = e^x \sin(x)$	0.4566	2.2874	5.2702
$f'(x) = e^x (\sin(x) + \cos(x))$	1.7754	3.7560	4.7634

Table 1: Nodes For Hermite Interpolation Polynomial

(a) Determining the Hermite Interpolation Polynomial $H(x)$ with the highest order with nodes from Table(1), your function should output the coefficients of $H(x)$ with coefficients are sorted in order of order from largest to smallest.

(b) Plot your Hermite Interpolation Polynomial $H(x)$ and the real function $f(x)$ within the interval [0,2], then compare the difference between $H(x)$ and $f(x)$.

Solution:

(a) The Hermite Interpolation Polynomial is

$$H(x) = -0.1245x^5 + 0.2278x^4 + 0.0614x^3 + 1.1657x^2 + 0.9517x + 0.0053,$$

or

$$\begin{aligned} H(x) = \; & -0.1245(x-x_1)^2(x-x_2)^2(x-x_3) - 0.3119(x-x_1)^2(x-x_2)^2 \\ & +0.0878(x-x_1)^2(x-x_2) + 1.4562(x-x_1)^2 + 1.7754(x-x_1) + 0.4566. \end{aligned}$$

(b) Truncation error:

$$\begin{aligned} |R_5| \leq \; & \max_{x\in[0,2]} \frac{|f^{(6)}(x)|}{6!}|(x-x_0)^2(x-x_1)^2(x-x_2)^2| \\ = \; & \max_{x\in[0,2]} \frac{8|e^x \cos(x)|}{6!}|(x-x_0)^2(x-x_1)^2(x-x_2)^2| \\ \leq \; & 1.0545 \times 10^{-2}. \end{aligned} \tag{9}$$

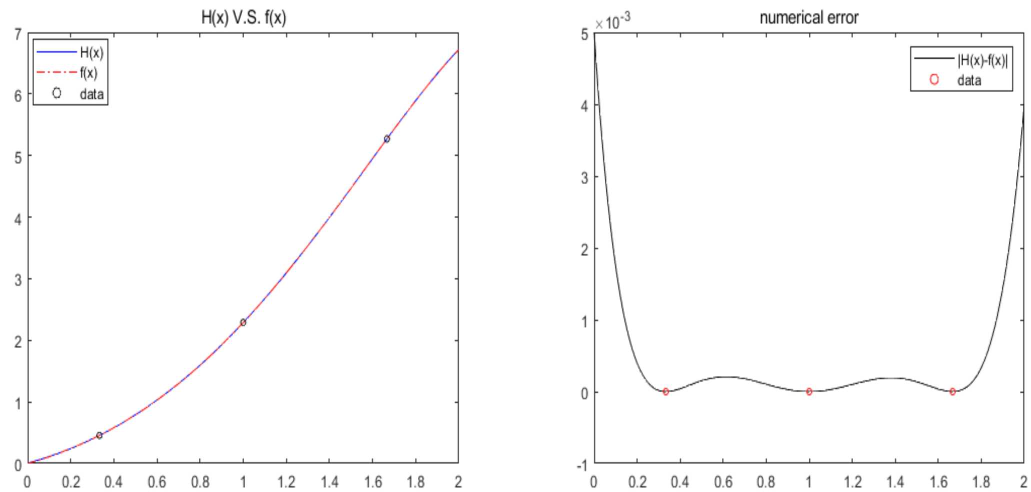

Figure 1: Caption

The sub-figure from the right side shows that the numerical error is small when the x is around the given points, and the numerical error is smoothing when x within the interval $[1/3, 5/3]$, but it increases rapidly when x away the interval $[1/3, 5/3]$. It is obviously that the upper bounded of the numerical error is $|R_N| = 4.9949 \times 10^{-3}$, and the numerical error is far less than truncation error.