Convex Optimization

Autumn 2022

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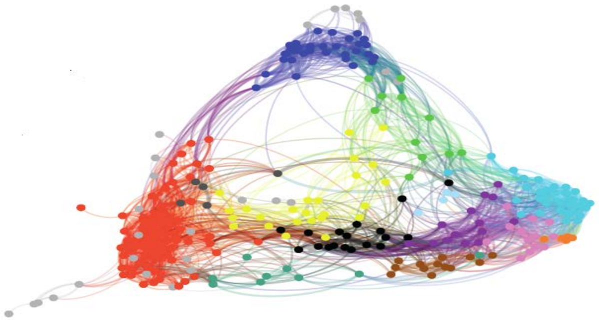
Some slides are made by Prof Yuanming Shi

Outline

$f(x) = x$, $f(x,y) = x \cdot y$

- **Data science models**
  - Linear, bilinear, quadratic, low-rank, and deep models
- **Large-scale optimization**
  - Constrained vs. unconstrained, convex vs. nonconvex, deterministic vs. stochastic, solvability vs. scalability
- **High-dimensional statistics**
  - Convex geometry, local geometry, global geometry
- **Topics and grading**
  - Theoretical foundations, first-order methods, second-order methods, stochastic methods, machine learning approaches, and applications.

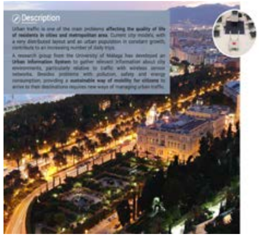
Motivations: The Era of Big Data



Intelligent IoT applications

Autonomous vehicles

Smart home



Smart city

Smart health

Smart agriculture

Smart drones

Financial big data

- **Financial data & AI technologies:** set up analytic models to gain valuable insights for better business decisions

Large Volume

data generating at speed of 1TB/day in NYSE (2013)

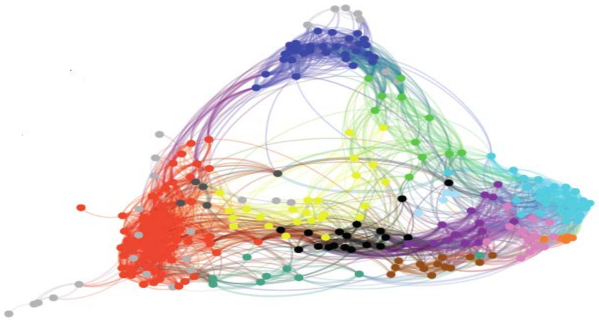
typically 100,000trans/s in High-frequency Trading

High Velocity

Wide Variety

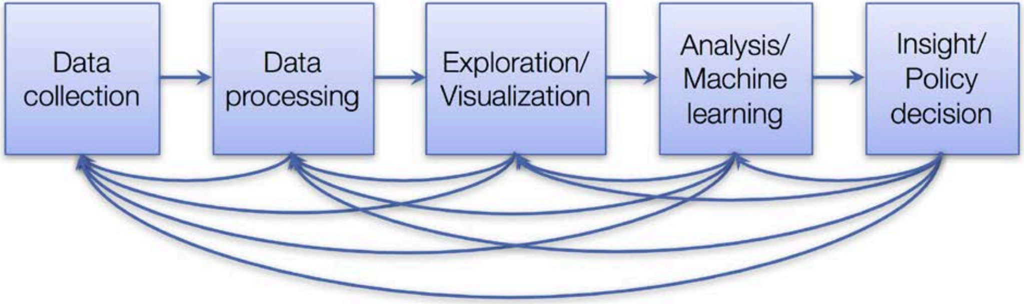
various data sources and types

Vignettes A: Data Science Models



What is data science?

- **Some possible definitions**
  - Data science is the application of ***computational*** and ***statistical*** techniques to address or gain insight into some problem in the **real world**



Challenges

- Retrieve or infer information from high-dimensional/large-scale data

limited processing ability
(computation, storage, ...)

2.5 exabytes of data
are generated every day (2012)

exabyte ⟶ zettabyte ⟶ yottabyte...??

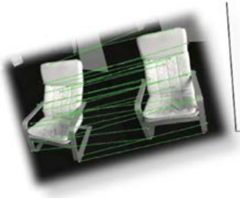
We're interested in the ***information*** rather than the data

Challenges:

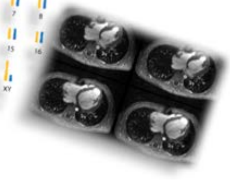
- High computational cost
- Only limited memory is available
- Do NOT want to compromise statistical accuracy

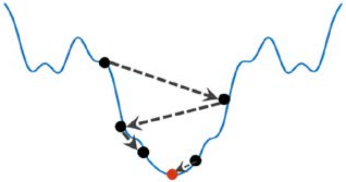
Optimization for data science

- Optimization has transformed algorithm design

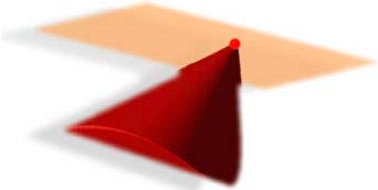


Learning/
Inference





Optimization Problems
(e.g., empirical risk minimization,
maximum likelihood estimation)



(Convex) optimization is *almost* a tool

Optimization problem

- General optimization problem in standard form:

$$\begin{array}{ll} \text{minimize} & f_0(\boldsymbol{x}) \\ \text{subject to} & f_i(\boldsymbol{x}) \leq 0, i = 1, \ldots, m \end{array}$$

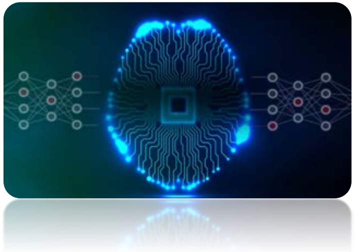
  - $\boldsymbol{x} = (x_1, \ldots, x_n)$: optimization variables
  - $f_0 : \mathbb{R}^n \to \mathbb{R}$: objective function
  - $f_i : \mathbb{R}^n \to \mathbb{R}, i = 1, \ldots, m$: constraint functions

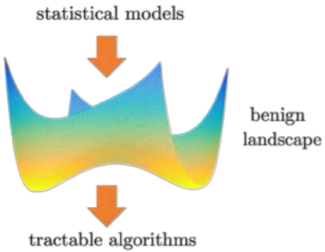
- **Goal:** find optimal solution $\boldsymbol{x}^\star$ minimizing f_0 while satisfying constraints
- **Three basic elements:** 1) variables, 2) constraints, and 3) objective

High-dimensional data analysis

(big) data

Models: (deep) machine learning





Methods:

1. Large-scale optimization
2. High-dimensional statistics
3. Device-edge-cloud computing

Linear model

- Let $\boldsymbol{x}^\natural \in \mathbb{R}^d$ be an unknown structured sparse signal
  - Individual sparsity for compressed sensing
- Let $f : \mathbb{R}^d \to \mathbb{R}$ be a convex function that reflects structure, e.g., ℓ_1-norm
- Let $\boldsymbol{A} \in \mathbb{R}^{m \times d}$ be a measurement operator
- **Observe** $\boldsymbol{z} = \boldsymbol{A}\boldsymbol{x}^\natural$
- Find estimate $\hat{\boldsymbol{x}}$ by solving **convex program**

$$\text{minimize} \quad f(\boldsymbol{x}) \quad \text{subject to} \quad \boldsymbol{A}\boldsymbol{x} = \boldsymbol{z}$$

- **Hope:** $\hat{\boldsymbol{x}} = \boldsymbol{x}^\natural$


MR scanner


MR image

$$\min_{x \in \mathbb{R}^n} \quad |x|_1 \qquad |x|_1 = \sum_{i=1}^{n} |x_i|$$

$$\text{s.t.} \quad Ax = b$$

$$\min_{x,t} \quad \sum_{i=1}^{n} t_i \qquad x \in \mathbb{R}^n$$

$$\text{s.t.} \quad Ax = b$$

$$-t_i \leq x_i \leq t_i$$

$$Ax \leq b$$

$$\downarrow$$

$$\begin{cases} Ax + y = b \\ y \geq 0 \end{cases}$$

Bilinear model

image deblurring

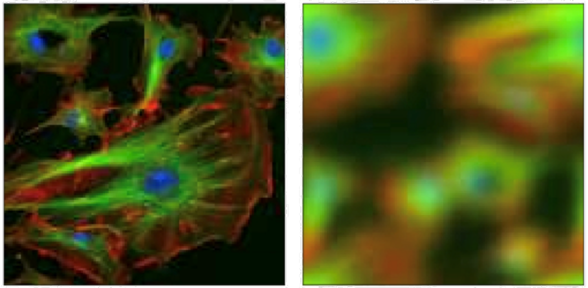


Fig. credit: Romberg

multipath in wireless comm

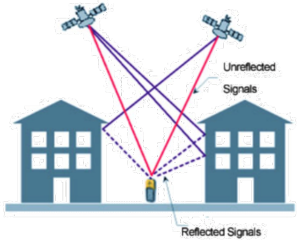


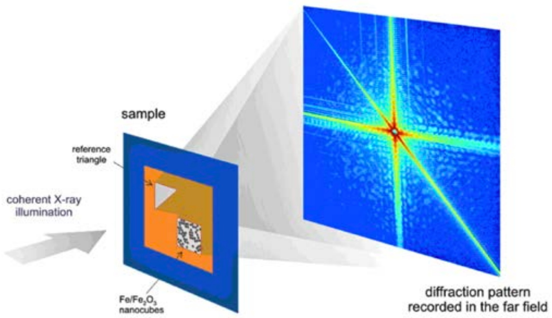
Fig. credit: EngineeringsALL

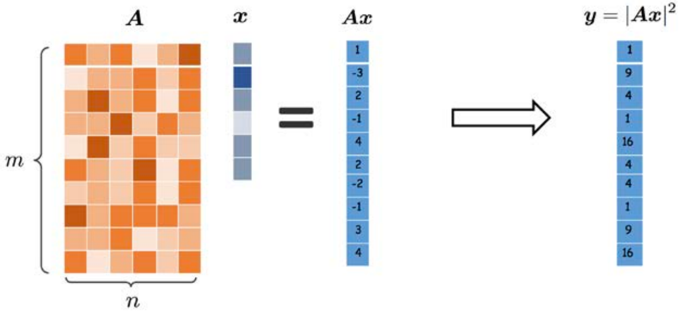
- **Blind deconvolution:** reconstruct two signals from their convolution

$$\text{find} \quad \boldsymbol{x}, \boldsymbol{h} \quad \text{subject to} \quad z_i = \boldsymbol{b}_i^* \boldsymbol{h} \boldsymbol{x}^* \boldsymbol{a}_i, \quad 1 \le i \le m$$

Quadratic model

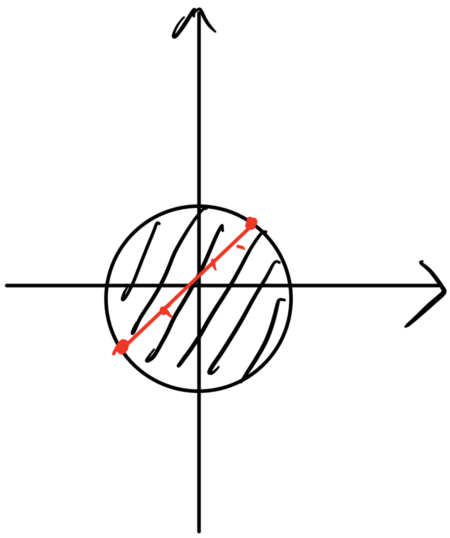
- **Phase retrieval:** recover signal from intensity (missing phase)





- Recover $z^\natural \in \mathbb{R}^n$ from m random quadratic measurements

$$\text{find} \quad z \qquad \text{subject to} \;\; y_r = |\langle a_r, z \rangle|^2, \quad r = 1, 2, \ldots, m$$



$$\left\{ f(x) = t \right.$$ nonconvex

$$f(x) = x^T Q x + p x$$

$x \in \mathbb{R}^n$, $Q \in S_+$

$$f(x) \le t$$ convex

Low-rank model

$$\begin{bmatrix} \checkmark & ? & ? & ? & \checkmark & ? \\ ? & ? & \checkmark & \checkmark & ? & ? \\ \checkmark & ? & ? & \checkmark & ? & ? \\ ? & ? & \checkmark & ? & ? & \checkmark \\ \checkmark & ? & ? & ? & ? & ? \\ ? & \checkmark & ? & ? & \checkmark & ? \\ ? & ? & \checkmark & \checkmark & ? & ? \end{bmatrix}$$

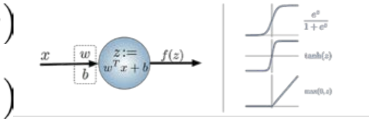
Fig. credit: Candès

- Given partial samples Ω of a low-rank matrix $M^{\natural}$, fill in missing entries

$$\underset{M \in \mathbb{C}^{m \times n}}{\text{minimize}} \quad \text{rank}(M) \quad \text{subject to} \quad Y_{i,k} = M_{i,k}, \quad (i,k) \in \Omega$$

Deep models

- **Data:** n observations $\{\boldsymbol{x}_i, y_i\}_{i=1}^n \in \mathcal{X} \times \mathcal{Y}$
- **Prediction function:** $h(\boldsymbol{x}, \boldsymbol{\theta}) \in \mathbb{R}$ parameterized by $\boldsymbol{\theta} \in \mathbb{R}^d$
  - linear predictions: $h(\boldsymbol{x}, \boldsymbol{\theta}) = \boldsymbol{\theta}^T \Phi(\boldsymbol{x})$ using features $\Phi(\boldsymbol{x})$
  - neural networks: $h(\boldsymbol{x}, \boldsymbol{\theta}) = \boldsymbol{\theta}_m^T \sigma(\boldsymbol{\theta}_{m-1}^T \sigma(\cdots \boldsymbol{\theta}_2^T \sigma(\boldsymbol{\theta}_1^T \boldsymbol{x})))$



- Estimating $\boldsymbol{\theta}$ parameters is an optimization problem (ℓ: loss function)

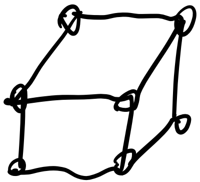
$$\text{minimize} \quad f(\boldsymbol{\theta}) := \frac{1}{n} \sum_{i=1}^{n} \ell(h(\boldsymbol{x}_i, \boldsymbol{\theta}), y_i) \quad \text{subject to} \quad \mathcal{R}(\boldsymbol{\theta}) \leq \tau$$

  - $\mathcal{R}$:regularization function encoding prior information (e.g., sparse) on $\boldsymbol{\theta}$

Key benefits of looking at problems in AI as optimization problems:
separate out the *definition* of the problem from the *method for solving it!*

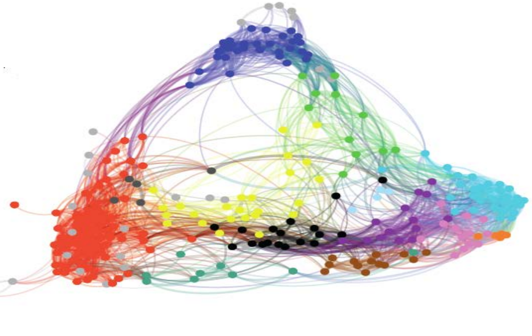
$\mathbb{R}^n$ ∞ ✓

$\{0,1\}^n = 2^n$ ✓



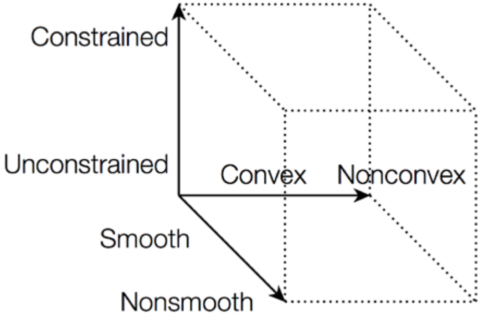
Vignettes B: Large-Scale Optimization

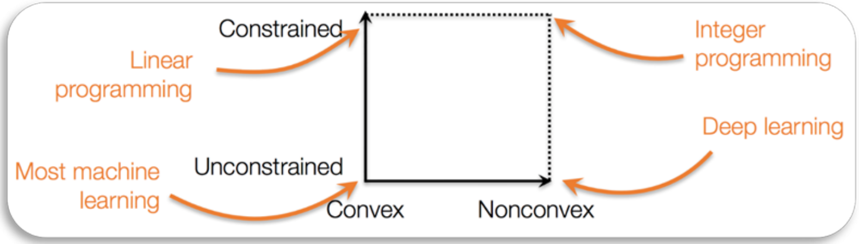
$[0,1]$



Classes of optimization problems

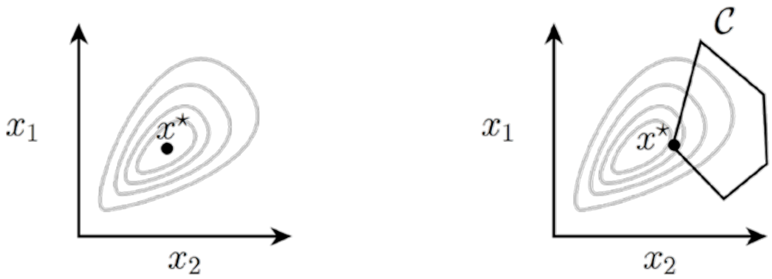
- Types of optimization problems: linear programming, nonlinear programming, integer programming, geometric programming, …





- We focus on three dimensions: ***unconstrained vs. constrained, convex vs. nonconvex, and smooth vs. nonsmooth***

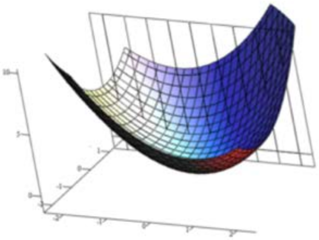
Constrained vs. unconstrained optimization

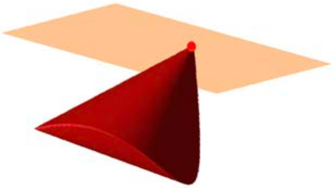


- **Unconstrained optimization:** every point $\boldsymbol{x} \in \mathbb{R}^n$ is feasible, so only focus is on minimizing $f(\boldsymbol{x})$
- **Constrained optimization:** it may be difficult to even *find* a feasible point $\boldsymbol{x} \in \mathcal{C}$

Typically leads to different classes of algorithms

Convex vs. nonconvex optimization





Convex optimization:

1) All local optima are global optima
2) Can be solved in polynomial-time

"... the great watershed in optimization isn't between linearity and nonlinearity, but convexity and nonconvexity"

— R. Rockafellar '1993

$f_1(x)$

$f_2(x)$

Deterministic vs. stochastic optimization

- **Stochastic optimization**

$$\text{minimize} \quad f(\boldsymbol{x}) := \mathbb{E}[F(\boldsymbol{x}, \boldsymbol{\xi})] \quad \text{subject to} \quad \boldsymbol{x} \in \mathcal{X}$$

  - f: loss; $\boldsymbol{x}$: parameters; $\boldsymbol{\xi}$: data samples

- **Example:** supervised machine learning (finite-sum problems)

$$\text{minimize} \quad f(\boldsymbol{x}) := \frac{1}{n}\sum_{i=1}^{n} f_i(\boldsymbol{x}) = \frac{1}{n}\sum_{i=1}^{n} \ell(b_i - \boldsymbol{a}_i^T \boldsymbol{x})$$

  - Data observations: $(\boldsymbol{a}_i, b_i) \in \mathbb{R}^d \times \mathbb{R}$; loss function: $\ell : \mathbb{R}^d \to \mathbb{R}$

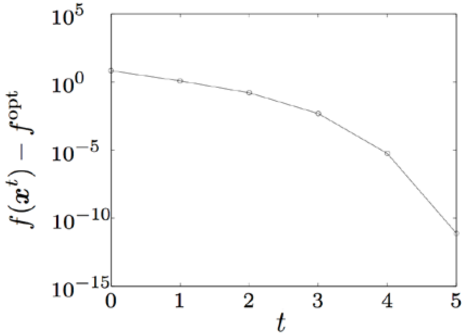
- **Stochastic gradient:** $\boldsymbol{x}_{k+1} = \boldsymbol{x}_k - \alpha_k \nabla f_{i(k)}(\boldsymbol{x}_k)$
  - $i(k) \in \{1, 2, \ldots, n\}$ uniformly at random; unbiased estimate: $\mathbb{E}[\nabla f_{i(k)}] = \nabla f$

Scaling issues: solvability vs. scalability

- Polynomial-time algorithms might be *useless* in large-scale applications
- **Example:** Newton's method

$$\text{minimize}_{\boldsymbol{x} \in \mathbb{R}^n} \quad f(\boldsymbol{x})$$

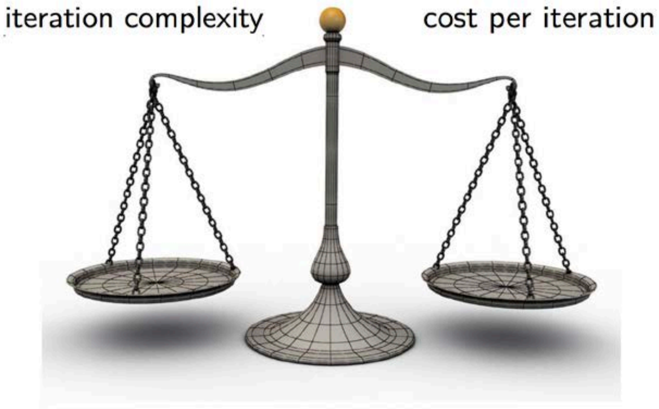
$$\boldsymbol{x}^{t+1} = \boldsymbol{x}^t - (\nabla^2 f(\boldsymbol{x}^t))^{-1} \nabla f(\boldsymbol{x}^t)$$



- Attains ϵ accuracy within $\mathcal{O}(\log \log \frac{1}{\epsilon})$ iterations; requires $\nabla^2 f(\boldsymbol{x}) \in \mathbb{R}^{n \times n}$
- A single iteration may last forever; prohibitive storage requirement

Iteration complexity vs. per-iteration cost

computational cost = iteration complexity (#iterations) x cost per iteration



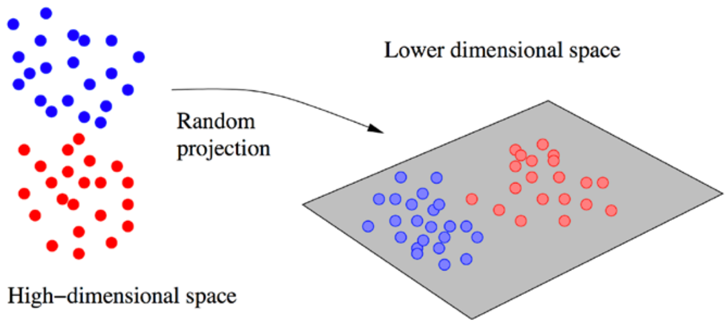
Large-scale problems call for methods with *cheap iterations*

First-order methods

- **First-order methods:** methods that exploit only information on function values and (sub)gradients without using Hessian information
  - cheap iterations
  - low memory requirements

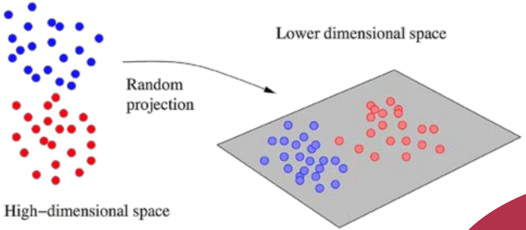
Randomized and approximation methods

- **Optimization for high-dimensional data analysis:** polynomial-time algorithms often not fast enough: further *approximations* are essential

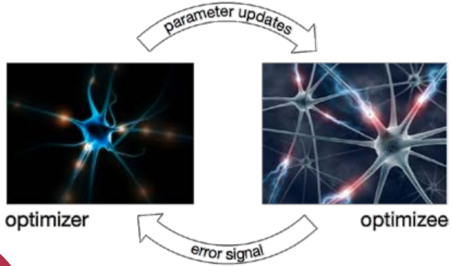


- **Randomized and stochastic methods:** project data into subspace, and solve reduced dimension problem

Advanced large-scale optimization

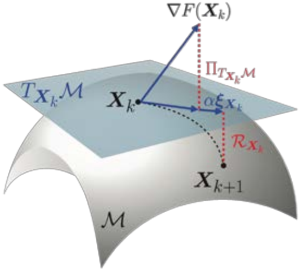


randomized methods

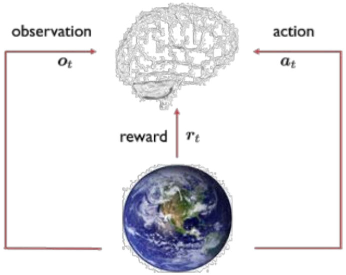


learning to optimize

Goal: scalable, real-time, parallel, distributed, automatic, etc.

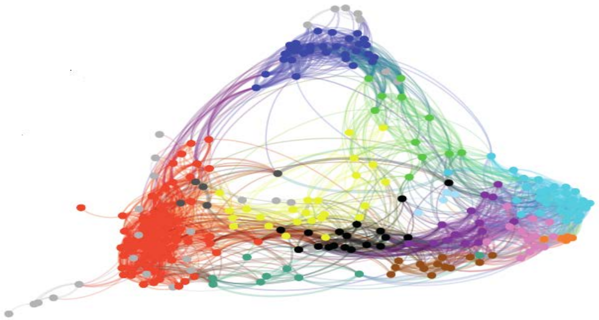


nonconvex optimization on manifold



deep reinforcement learning

Vignettes C: High-Dimensional Statistics

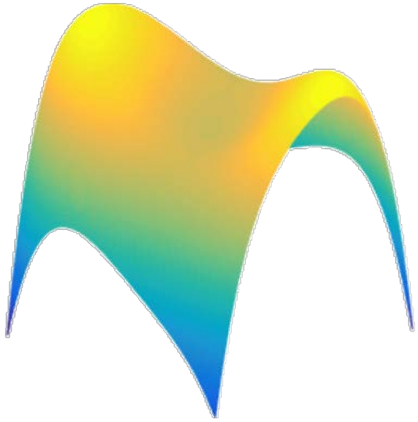


Nonconvex problems are everywhere

- Empirical risk minimization is usually nonconvex

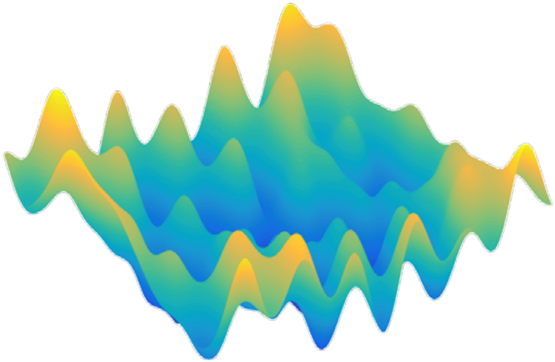
$$\underset{\boldsymbol{x}}{\text{minimize}} \quad f(\boldsymbol{x};\boldsymbol{\theta})$$

  - low-rank matrix completion
  - blind deconvolution/demixing
  - dictionary learning
  - phase retrieval
  - mixture models
  - deep learning
  - …



Nonconvex optimization may be super scary

- **Challenges:** saddle points, local optima, bumps,…



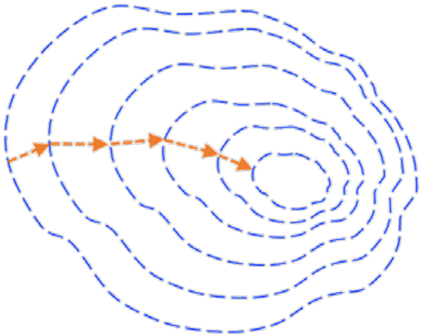


Fig. credit: Chen

- **Fact:** they are usually solved on a daily basis via simple algorithms like (stochastic) gradient descent

Statistical models come to rescue

- **Blessings:** when data are generated by certain statistical models, problems are often much nicer than worst-case instances

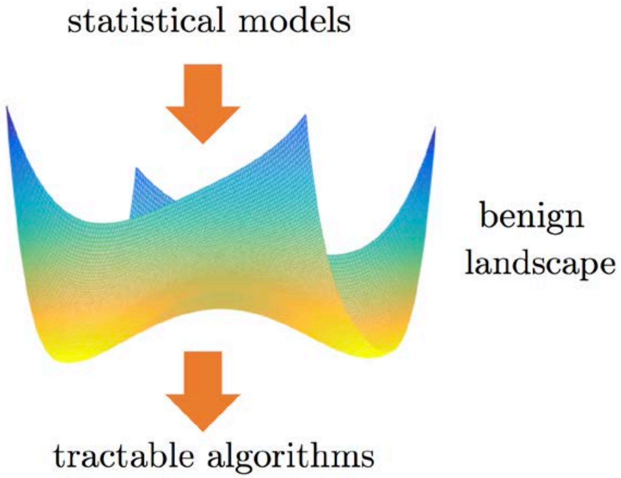
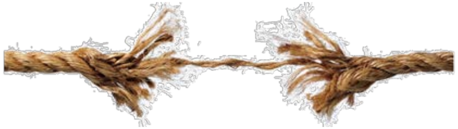


Fig. credit: Chen

Global geometry

- **Proposal:** separation of landscape analysis and generic algorithm design

landscape analysis (statistics)



generic algorithms (optimization)

all local minima are global minima

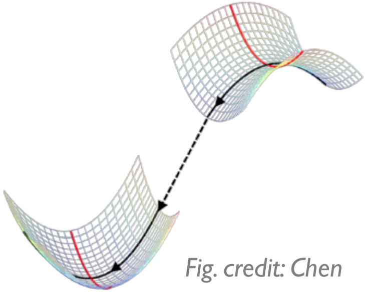


Fig. credit: Chen

all the saddle points can be escaped

- dictionary learning (Sun et al. '15)
- phase retrieval (Sun et al. '16)
- matrix completion (Ge et al. '16)
- synchronization (Bandeira et al. '16)
- inverting deep neural nets (Hand et al. '17)
- ...

- gradient descent (Lee et al. '16)
- trust region method (Sun et al. '16)
- perturbed GD (Jin et al. '17)
- cubic regularization (Agarwal et al. '17)
- Natasha (Allen-Zhu '17)
- ...

Local geometry

- **Initialize** within local basin sufficiently close to ground-truth (i.e., strongly convex, no saddle points/ local minima)
- **Iterative refinement** via some iterative optimization algorithms

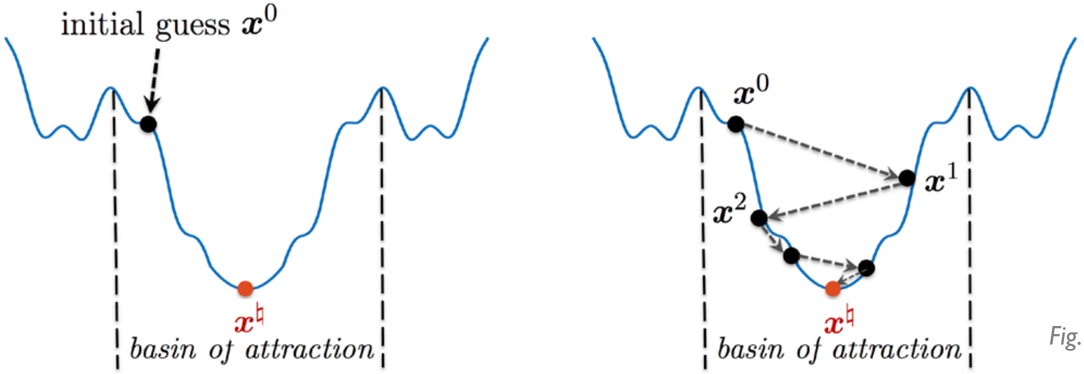
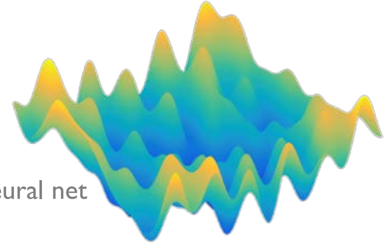


Fig. credit: Chen

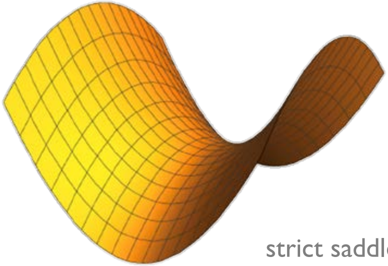
Optimization meets statistics



1-layer neural net

big data & deep learning

statistical models

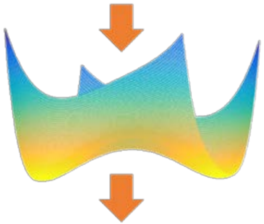


strict saddle points

nonconvex optimization may be super scary

benign geometry: no spurious local optima

statistical models

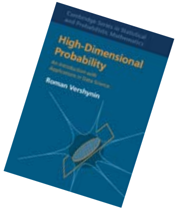


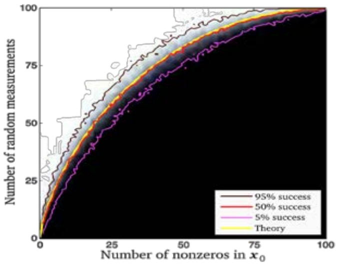
benign
landscape

tractable algorithms

framework: high-dimensional data analysis

high-dimensional
probability & statistics



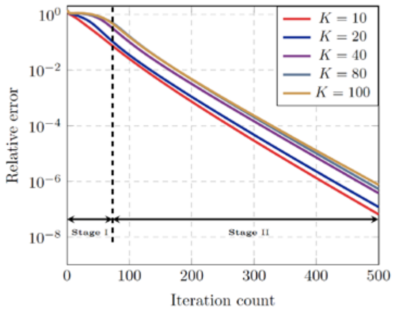


Goals: data sizes, expressivity, information propagations, etc.

Case study: bilinear model

- Demixing from bilinear measurements

$$\begin{aligned} &\text{find} \quad \{\boldsymbol{x}_i\}, \{\boldsymbol{h}_i\} \\ &\text{subject to} \quad z_j = \sum_{i=1}^{s} \boldsymbol{b}_j^* \boldsymbol{h}_i \boldsymbol{x}_i^* \boldsymbol{a}_{ij} \end{aligned}$$



- **Applications**

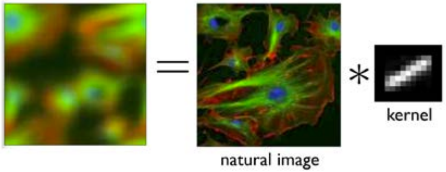
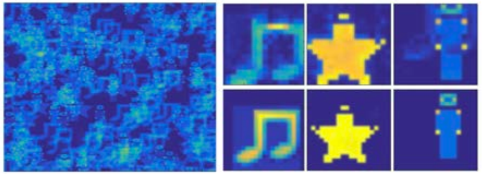
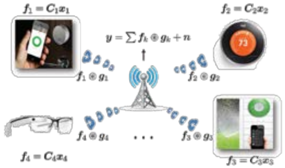


image deblurring



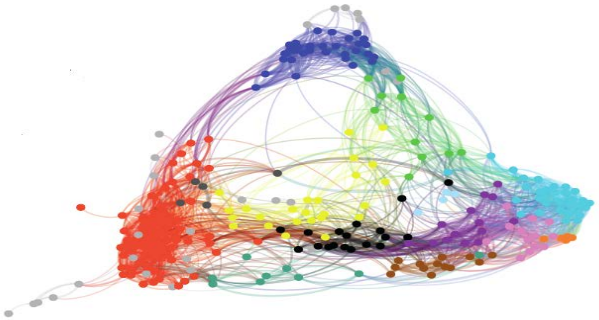
convolutional dictionary learning



low-latency communication

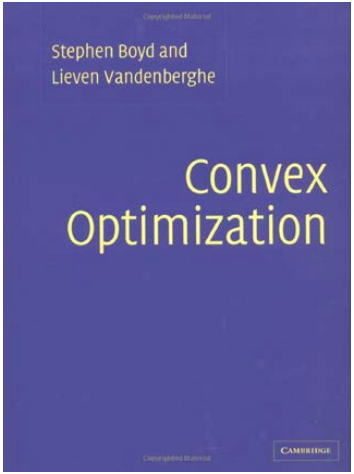
[Ref] J. Dong and Y. Shi, "Nonconvex demixing from bilinear measurements," *IEEE Trans. Signal Process.*, vol. 66, no. 19, pp. 5152-5166, Oct., 2018.

Topics and Grading



Theoretical foundations

- **Main topics:** convex sets, convex functions, convex problems, Lagrange duality and KKT conditions, disciplined convex programming

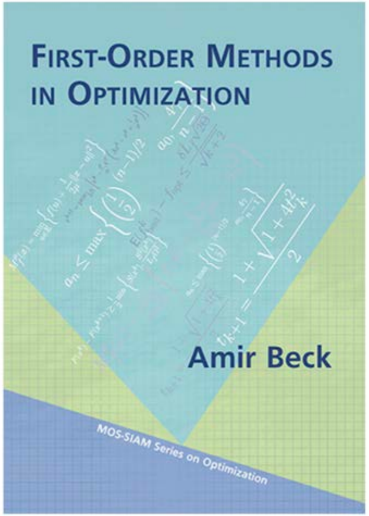


CVX RESEARCH

Convex Optimization, by S. Boyd and L. Vandenberghe, Cambridge University Press, 2003.

First-order methods

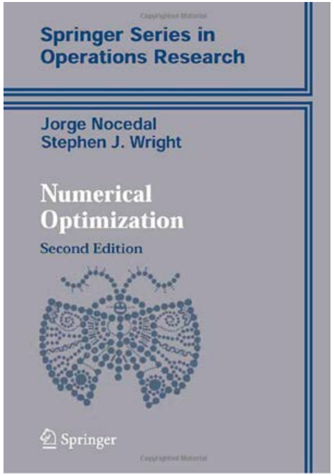
- **Main topics:** gradient methods, subgradient methods, proximal methods



First-order Methods in Optimization, by A. Beck, MOS-SIAM Series on Optimization, 2017.

Second-order methods

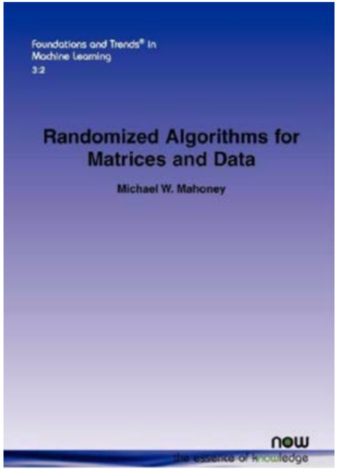
- **Main topics:** Newton method, interior-point methods, quasi-Newton methods



Numerical Optimization, by J. Nocedal and S. Wright, Springer-Verlag, 2006.

Stochastic and randomized methods

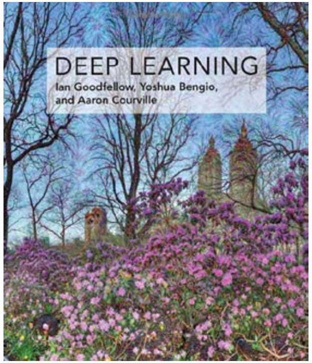
- **Main topics:** stochastic gradient methods, stochastic Newton methods, randomized sketching methods, randomized linear algebra



Lecture Notes on Randomized Linear Algebra, by Mahoney, Michael, 2016.

Machine learning for optimization

- **Main topics:** sparse optimization (deep neural networks), mixed integer nonlinear programming (imitation learning), nonconvex optimization (statistical learning), multiple objective optimization (active learning)



Deep Learning, by I. Goodfellow, Y. Bengio and A. Courville, MIT Press, 2016.

Applications

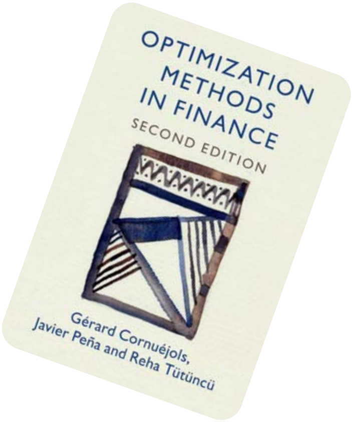
- **Machine learning:** Optimization Layers, Learning to Optimize, Federated learning, etc.

- **Smart grids:** Sparse and low rank optimization (Optimal power flow), mixed integer linear/nonlinear programming (Electric vehicle charging/discharging, Demand response, PMU placement)

- **Finance Engineering:** portfolio optimization, factor models, time series modeling, robust portfolio optimization, risk-parity portfolio, index tracking, pairs trading

Applications in financial engineering

- **Main topics:** portfolio optimization, factor models, time series modeling, robust portfolio optimization, risk-parity portfolio, index tracking, pairs trading



Optimization Methods in Finance, by G. Cornuejols, J. Pena, and R. Tutuncu, Cambridge University Press, 2018.

Prerequisites

- **Warning:** there will be quite a few THEOREMS and PROOFS ...

- Basic linear algebra
- Basic probability
- A programming language (e.g. Matlab, Python, ...)

Somewhat surprisingly, most proofs rely only on basic linear algebra and elementary recursive formula

Grading

- **Homework:** 3 homework assignments
- Quiz： Random in class
- **Course project:**
  - either individually or in groups of two/three
  - list of topics; final report & slides
- **Final Exam:**

$$\text{Grade} = 0.2\text{H} + 0.1\text{Q} + 0.3\text{P} + 0.4\text{E}$$

H: homework; Q: quiz; P: course project; E: final exam

Course information

- **Instructor:** Ye Shi (http://faculty.sist.shanghaitech.edu.cn/faculty/shiye)
  - Email: shiye@shanghaitech.edu.cn
  - Office location: Room 1A-404A, SIST Building
  - Office hours: Wednesday 15:00-16:00 (or by appointments)
- **TAs:**
  - Haixiang Sun (sunhx@shanghaitech.edu.cn)
  - Pengchao Tian (tianpch@shanghaitech.edu.cn)
  - Hongxia Li (lihx2@shanghaitech.edu.cn)
  - Shutong Ding (dingsht@shanghaitech.edu.cn)
  - Office location: Room 1A-405, SIST Building
  - Office hours: Thursday 19:00-20:00 (or by appointments)

Course information

- Use WeChat as the main mode of electronic communication; please post (and answer) questions there!

- Course website: BlackBoard

https://elearning.shanghaitech.edu.cn:8443/webapps/blackboard/execute/modulepage/view?course_id=_3088_1&cmp_tab_id=_7476_1&editMode=true&mode=cpview

Thanks