

(b) ac generator

6. MAXWELL'S EQUATIONS IN TIME-VARYING FIELDS

Chapter 6 Overview

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Objectives

Upon learning the material presented in this chapter, you should be able to:

1. Apply Faraday's law to compute the voltage induced by a stationary coil placed in a time-varying magnetic field or moving in a medium containing a magnetic field.
2. Describe the operation of the electromagnetic generator.
3. Calculate the displacement current associated with a time-varying electric field.
4. Calculate the rate at which charge dissipates in a material with known ϵ and σ .

Maxwell's Equations

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Table 6-1: Maxwell's equations.

Reference	Differential Form	Integral Form
Gauss's law	$\nabla \cdot \mathbf{D} = \rho_v$	$\oint_S \mathbf{D} \cdot d\mathbf{s} = Q \quad (6.1)$
Faraday's law	$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$	$\oint_C \mathbf{E} \cdot d\mathbf{l} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{s} \quad (6.2)^*$
Gauss's law for magnetism	$\nabla \cdot \mathbf{B} = 0$	$\oint_S \mathbf{B} \cdot d\mathbf{s} = 0 \quad (6.3)$
Ampère's law	$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$	$\oint_C \mathbf{H} \cdot d\mathbf{l} = \int_S \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \right) \cdot d\mathbf{s} \quad (6.4)$
*For a stationary surface S .		

In this chapter, we will examine Faraday's and Ampère's laws

Electromotive Force

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Electromotive force (voltage) induced by time-varying magnetic flux:

$$V_{\text{emf}} = -N \frac{d\Phi}{dt} = -N \frac{d}{dt} \int_S \mathbf{B} \cdot d\mathbf{s} \quad (\text{V})$$

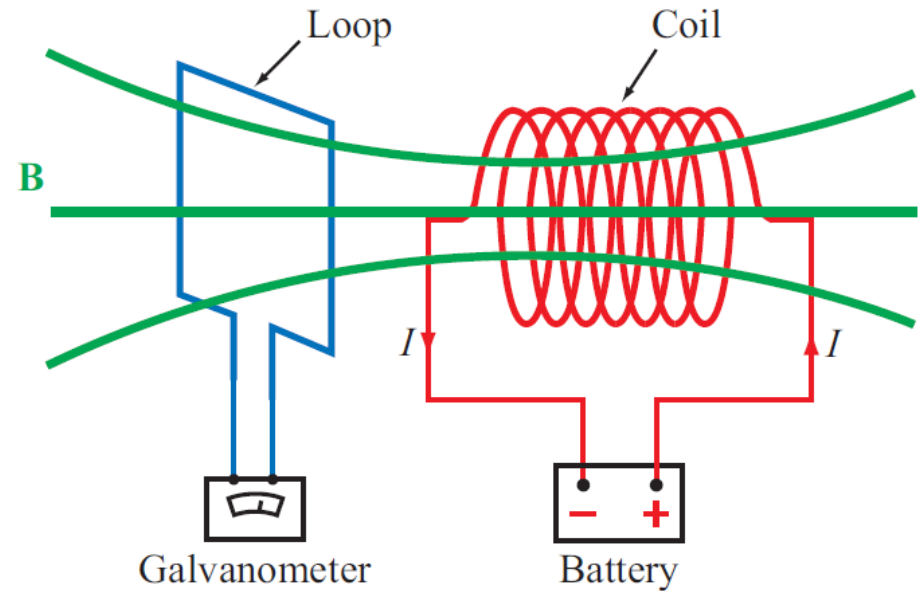


Figure 6-1: The galvanometer (predecessor of the ammeter) shows a deflection whenever the magnetic flux passing through the square loop changes with time.

Magnetic fields can produce an electric current in a closed loop, but only if the magnetic flux linking the surface area of the loop changes with time. The key to the induction process is change.

Three types of EMF

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1. A time-varying magnetic field linking a stationary loop; the induced emf is then called the *transformer emf*, $V_{\text{emf}}^{\text{tr}}$.
2. A moving loop with a time-varying surface area (relative to the normal component of \mathbf{B}) in a static field \mathbf{B} ; the induced emf is then called the *motional emf*, $V_{\text{emf}}^{\text{m}}$.
3. A moving loop in a time-varying field \mathbf{B} .

The total emf is given by

$$V_{\text{emf}} = V_{\text{emf}}^{\text{tr}} + V_{\text{emf}}^{\text{m}}, \quad (6.7)$$

Stationary Loop in Time-Varying \mathbf{B}

It is important to remember that \mathbf{B}_{ind} serves to oppose the change in $\mathbf{B}(t)$, and not necessarily $\mathbf{B}(t)$ itself.

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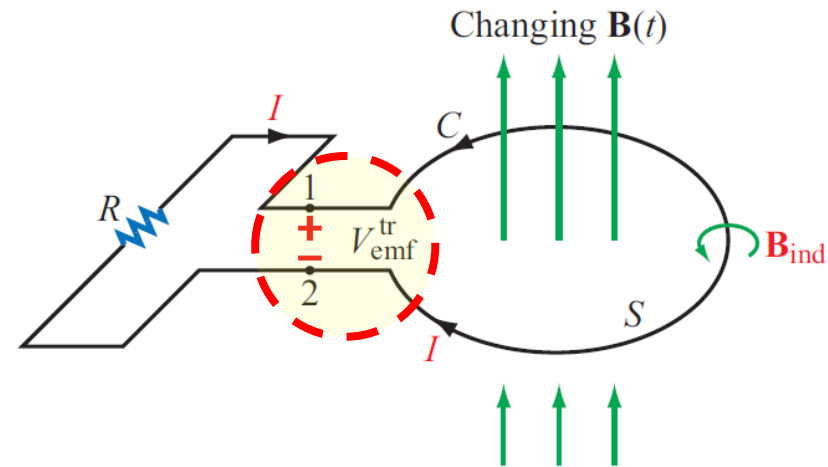
$$V_{\text{emf}}^{\text{tr}} = -N \int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{s} \quad (\text{transformer emf}),$$

The connection between the direction of $d\mathbf{s}$ and the polarity of $V_{\text{emf}}^{\text{tr}}$ is governed by the following right-hand rule: if $d\mathbf{s}$ points along the thumb of the right hand, then the direction of the contour C indicated by the four fingers is such that it always passes across the opening from the positive terminal of $V_{\text{emf}}^{\text{tr}}$ to the negative terminal.

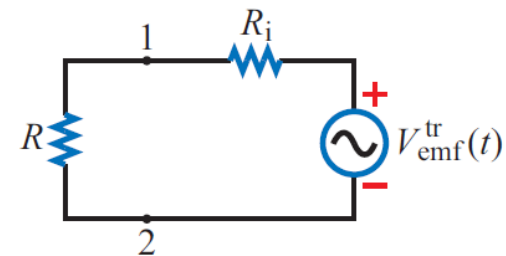
$$I = \frac{V_{\text{emf}}^{\text{tr}}}{R + R_i}. \quad (6.9)$$

For good conductors, R_i usually is very small, and it may be ignored in comparison with practical values of R .

*The polarity of $V_{\text{emf}}^{\text{tr}}$ and hence the direction of I is governed by **Lenz's law**, which states that the current in the loop is always in a direction that opposes the change of magnetic flux $\Phi(t)$ that produced I .*



(a) Loop in a changing \mathbf{B} field



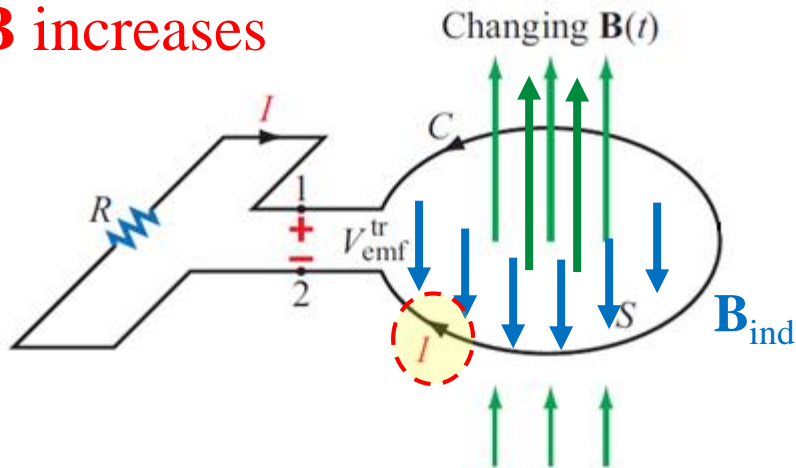
(b) Equivalent circuit

Figure 6-2: (a) Stationary circular loop in a changing magnetic field $\mathbf{B}(t)$, and (b) its equivalent circuit.

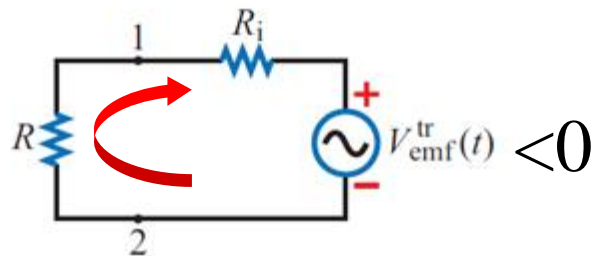
Lenz's Law

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B increases



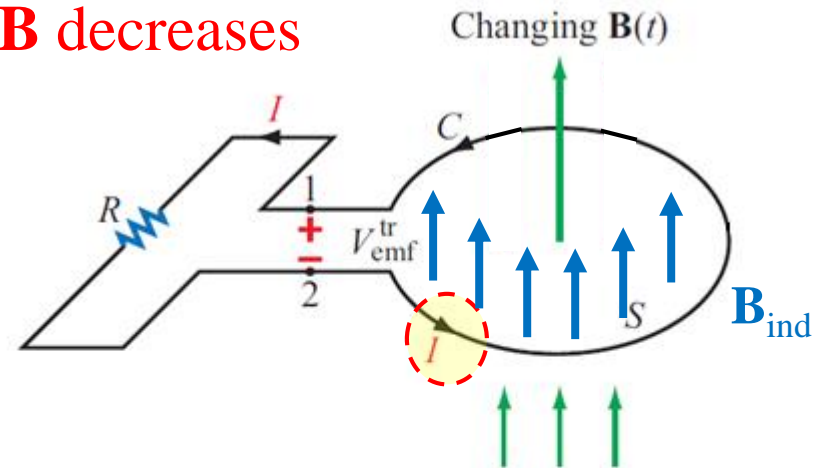
(a) Loop in a changing **B** field



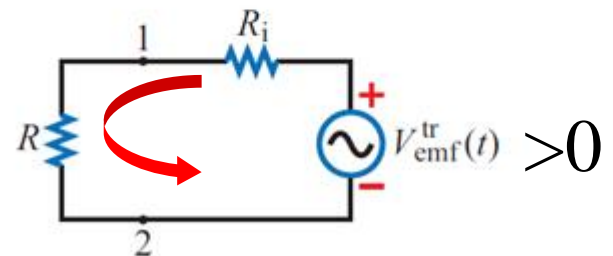
(b) Equivalent circuit

$$V_{\text{emf}}^{\text{tr}} = -N \int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{s} \quad (\text{transformer emf}),$$

B decreases



(a) Loop in a changing **B** field



(b) Equivalent circuit

B_{ind} serves to oppose the *change in* **B(t)**, and not necessarily **B(t)** itself

Faraday's Law

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$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad (\text{Faraday's law}).$$

Establish a link between \mathbf{B} and \mathbf{E}

$$V_{\text{emf}}^{\text{tr}} = \oint_C \mathbf{E} \cdot d\mathbf{l}.$$

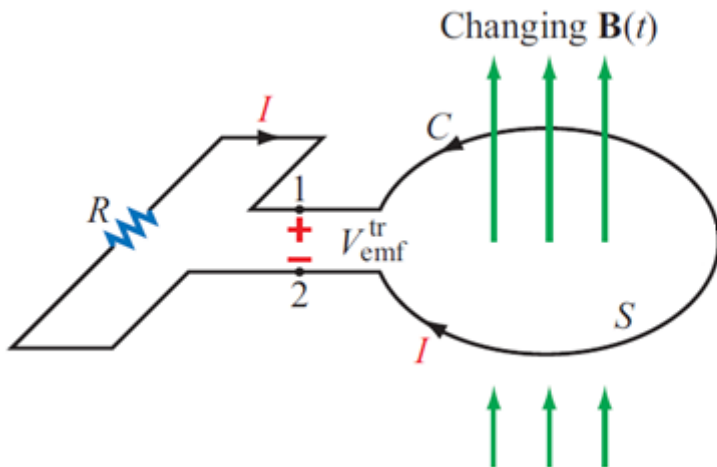
$$V_{\text{emf}}^{\text{tr}} = -N \int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{s}$$

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = - \int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{s},$$

Integral form of Faraday's law

Stokes's theorem

$$\int_S (\nabla \times \mathbf{E}) \cdot d\mathbf{s} = - \int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{s},$$



Example 6-1: Inductor in a Changing Magnetic Field

An inductor is formed by winding N turns of a thin conducting wire into a circular loop of radius a . The inductor loop is in the x - y plane with its center at the origin, and connected to a resistor R , as shown in Fig. 6-3. In the presence of a magnetic field $\mathbf{B} = B_0(\hat{\mathbf{y}}2 + \hat{\mathbf{z}}3) \sin \omega t$, where ω is the angular frequency, find

- (a) the magnetic flux linking a single turn of the inductor,
- (b) the transformer emf, given that $N = 10$, $B_0 = 0.2$ T, $a = 10$ cm, and $\omega = 10^3$ rad/s,
- (c) the polarity of $V_{\text{emf}}^{\text{tr}}$ at $t = 0$, and
- (d) the induced current in the circuit for $R = 1$ k Ω (assume the wire resistance to be much smaller than R).

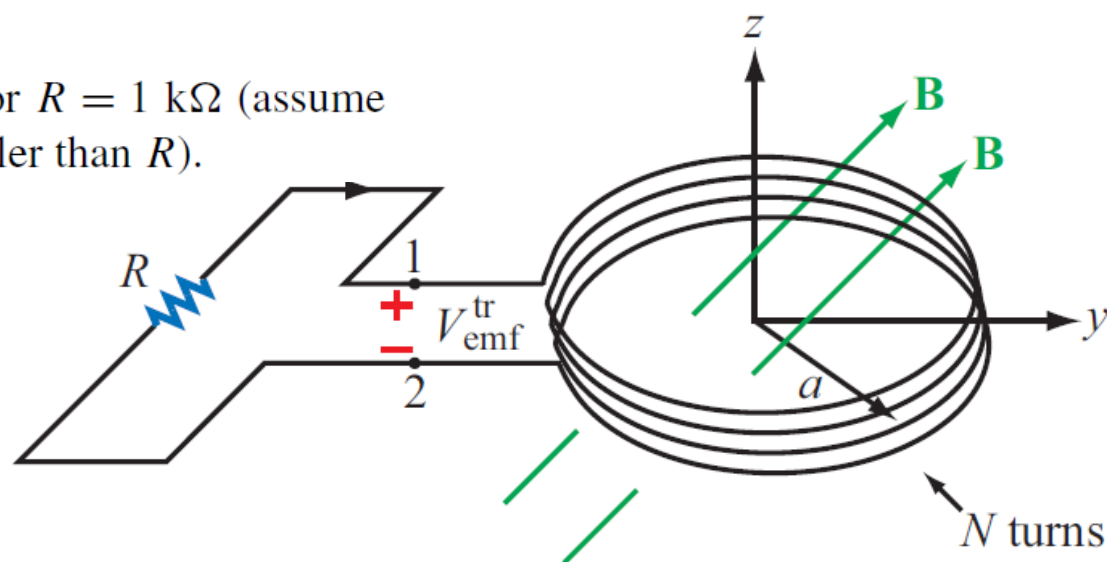


Figure 6-3: Circular loop with N turns in the x - y plane. The magnetic field is $\mathbf{B} = B_0(\hat{\mathbf{y}}2 + \hat{\mathbf{z}}3) \sin \omega t$ (Example 6-1).

cont.

Example 6-1 Solution

Solution: (a) The magnetic flux linking each turn of the inductor is

$$\begin{aligned}\Phi &= \int_S \mathbf{B} \cdot d\mathbf{s} \\ &= \int_S [B_0(\hat{\mathbf{y}}2 + \hat{\mathbf{z}}3) \sin \omega t] \cdot \hat{\mathbf{z}} d\mathbf{s} \\ &= 3\pi a^2 B_0 \sin \omega t.\end{aligned}$$

(b) To find $V_{\text{emf}}^{\text{tr}}$, we can apply Eq. (6.8) or we can apply the general expression given by Eq. (6.6) directly. The latter approach gives

$$\begin{aligned}V_{\text{emf}}^{\text{tr}} &= -N \frac{d\Phi}{dt} \\ &= -\frac{d}{dt}(3\pi N a^2 B_0 \sin \omega t) \\ &= -3\pi N \omega a^2 B_0 \cos \omega t.\end{aligned}$$

For $N = 10$, $a = 0.1$ m, $\omega = 10^3$ rad/s, and $B_0 = 0.2$ T,

$$V_{\text{emf}}^{\text{tr}} = -188.5 \cos 10^3 t \quad (\text{V}).$$

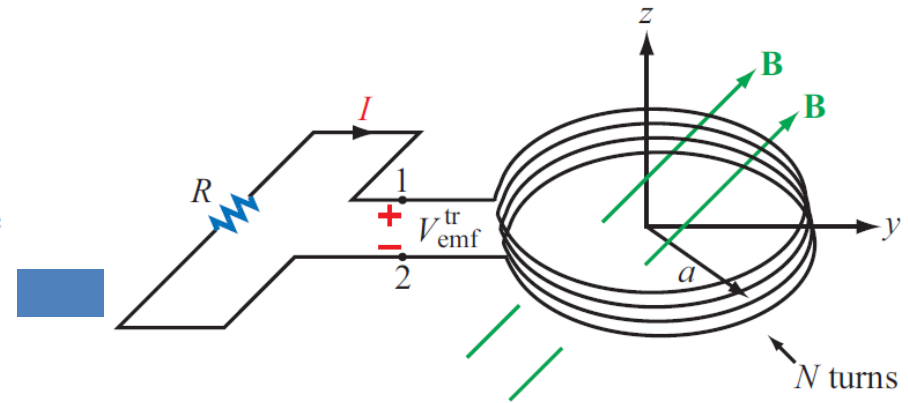


Figure 6-3: Circular loop with N turns in the x - y plane. The magnetic field is $\mathbf{B} = B_0(\hat{\mathbf{y}}2 + \hat{\mathbf{z}}3) \sin \omega t$ (Example 6-1).

(c) At $t = 0$, $d\Phi/dt > 0$ and $V_{\text{emf}}^{\text{tr}} = -188.5$ V. Since the flux is increasing, the current I must be in the direction shown in Fig. 6-3 in order to satisfy Lenz's law. Consequently, terminal 2 is at a higher potential than terminal 1 and

$$\begin{aligned}V_{\text{emf}}^{\text{tr}} &= V_1 - V_2 \\ &= -188.5 \quad (\text{V}).\end{aligned}$$

(d) The current I is given by

$$\begin{aligned}I &= \frac{V_2 - V_1}{R} \\ &= \frac{188.5}{10^3} \cos 10^3 t \\ &= 0.19 \cos 10^3 t \quad (\text{A}).\end{aligned}$$

Example 6-2: Lenz's Law

Determine voltages V_1 and V_2 across the $2\text{-}\Omega$ and $4\text{-}\Omega$ resistors shown in Fig. 6-4. The loop is located in the x - y plane, its area is 4 m^2 , the magnetic flux density is $\mathbf{B} = -\hat{\mathbf{z}}0.3t$ (T), and the internal resistance of the wire may be ignored.

Solution: The flux flowing through the loop is

$$\begin{aligned}\Phi &= \int_S \mathbf{B} \cdot d\mathbf{s} = \int_S (-\hat{\mathbf{z}}0.3t) \cdot \hat{\mathbf{z}} d\mathbf{s} \\ &= -0.3t \times 4 = -1.2t \quad (\text{Wb}),\end{aligned}$$

and the corresponding transformer emf is

$$V_{\text{emf}}^{\text{tr}} = -\frac{d\Phi}{dt} = 1.2 \quad (\text{V}).$$

The total voltage of 1.2 V is distributed across two resistors in series. Consequently,

$$\begin{aligned}I &= \frac{V_{\text{emf}}^{\text{tr}}}{R_1 + R_2} \\ &= \frac{1.2}{2 + 4} = 0.2\text{ A},\end{aligned}$$

and

$$V_1 = IR_1 = 0.2 \times 2 = 0.4\text{ V},$$

$$V_2 = IR_2 = 0.2 \times 4 = 0.8\text{ V}.$$

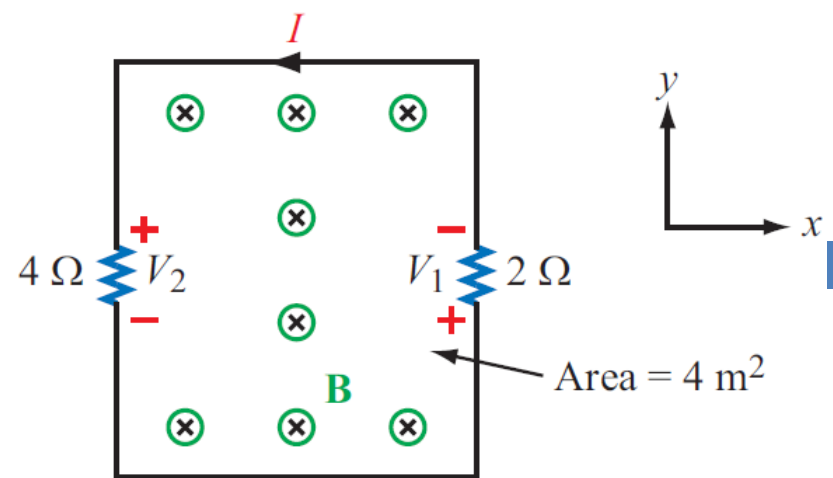


Figure 6-4: Circuit for Example 6-2.

Motional EMF

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Magnetic force on charge q moving with velocity \mathbf{u} in a magnetic field \mathbf{B} :

$$\mathbf{F}_m = q(\mathbf{u} \times \mathbf{B}).$$

This magnetic force is equivalent to the electrical force that would be exerted on the particle by the electric field \mathbf{E}_m given by

$$\mathbf{E}_m = \frac{\mathbf{F}_m}{q} = \mathbf{u} \times \mathbf{B}.$$

This, in turn, induces a voltage difference between ends 1 and 2, with end 2 being at the higher potential. The induced voltage is

$$V_{\text{emf}}^m = V_{12} = \int_2^1 \mathbf{E}_m \cdot d\mathbf{l} = \int_2^1 (\mathbf{u} \times \mathbf{B}) \cdot d\mathbf{l}.$$

$$V_1 - V_2$$

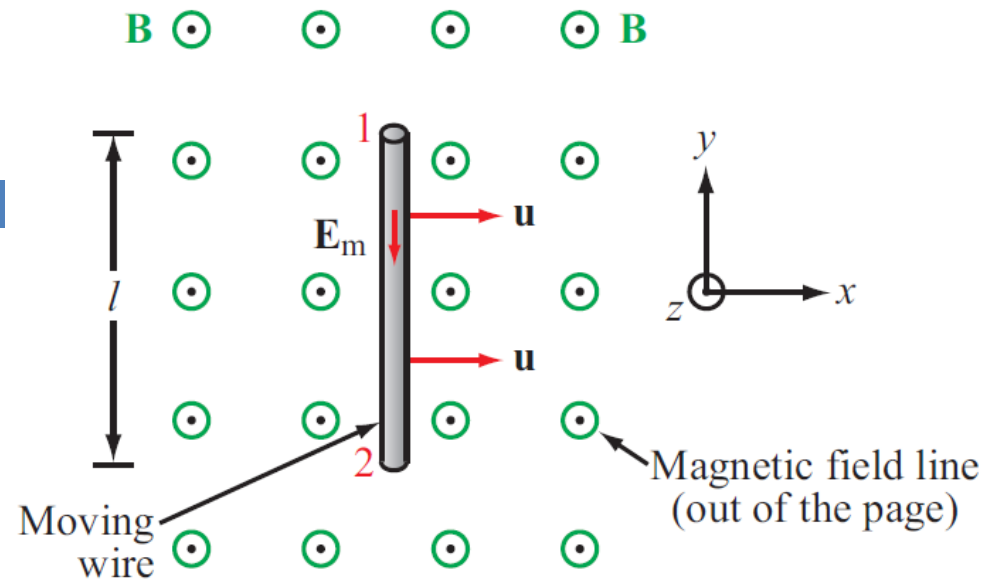


Figure 6-7: Conducting wire moving with velocity \mathbf{u} in a static magnetic field.

For the conducting wire, $\mathbf{u} \times \mathbf{B} = \hat{x}u \times \hat{z}B_0 = -\hat{y}uB_0$ and $d\mathbf{l} = \hat{y} dl$. Hence,

$$V_{\text{emf}}^m = V_{12} = -uB_0l. \quad (6.25)$$

\mathbf{E} points from 1 to 2

Motional EMF

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In general, if any segment of a closed circuit with contour C moves with a velocity \mathbf{u} across a static magnetic field \mathbf{B} , then the induced motional emf is given by

$$V_{\text{emf}}^{\text{m}} = \oint_C (\mathbf{u} \times \mathbf{B}) \cdot d\mathbf{l} \quad (\text{motional emf}). \quad (6.26)$$

Only those segments of the circuit that cross magnetic field lines contribute to $V_{\text{emf}}^{\text{m}}$.

Example 6-3: Sliding Bar

The rectangular loop shown in Fig. 6-8 has a constant width l , but its length x_0 increases with time as a conducting bar slides with uniform velocity \mathbf{u} in a static magnetic field $\mathbf{B} = \hat{\mathbf{z}}B_0x$. Note that \mathbf{B} increases linearly with x . The bar starts from $x = 0$ at $t = 0$. Find the motional emf between terminals 1 and 2 and the current I flowing through the resistor R . Assume that the loop resistance $R_i \ll R$.

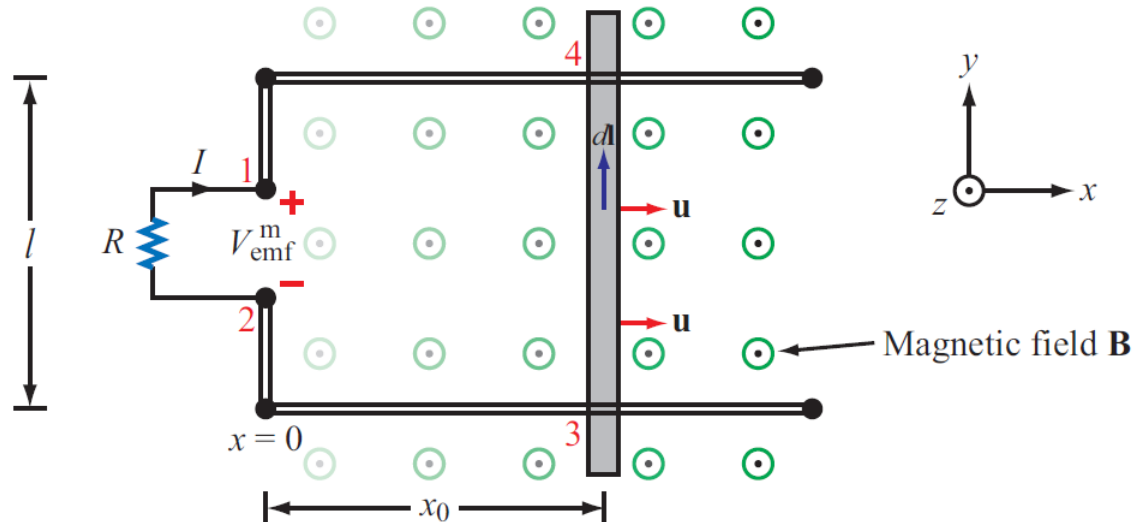
Note that B increases with x

$$\mathbf{B} = \hat{\mathbf{z}}B_0x$$

$$\begin{aligned} V_{\text{emf}}^{\text{m}} &= V_{12} = V_{43} = \int_3^4 (\mathbf{u} \times \mathbf{B}) \cdot d\mathbf{l} \\ &= \int_3^4 (\hat{\mathbf{x}}u \times \hat{\mathbf{z}}B_0x_0) \cdot \hat{\mathbf{y}} dl = -uB_0x_0l. \end{aligned}$$

The length of the loop is related to u by $x_0 = ut$. Hence

$$V_{\text{emf}}^{\text{m}} = -B_0u^2lt \quad (\text{V}).$$



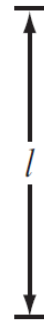
How to define the polarity of V_{emf} in this condition?

Example 6-3: cont.

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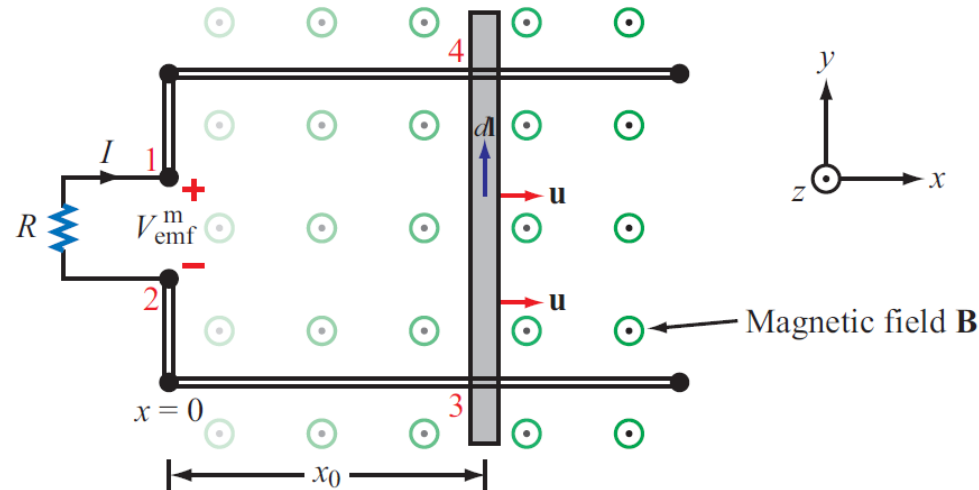
Use Faraday's law

$$\begin{aligned}\Phi &= \int_S \mathbf{B} \cdot d\mathbf{s} = \int_S (\hat{\mathbf{z}} B_0 x) \cdot \hat{\mathbf{z}} dx dy \\ &= B_0 l \int_0^{x_0} x dx = \frac{B_0 l x_0^2}{2}\end{aligned}$$



Note that B increases with x

$$\mathbf{B} = \hat{\mathbf{z}} B_0 x$$



$$V_{\text{emf}} = -\frac{d\Phi}{dt} = -\frac{d}{dt} \left(\frac{B_0 l u^2 t^2}{2} \right) = -B_0 u^2 l t \quad (\text{V}),$$

The same as the result obtained
by the motional emf expression

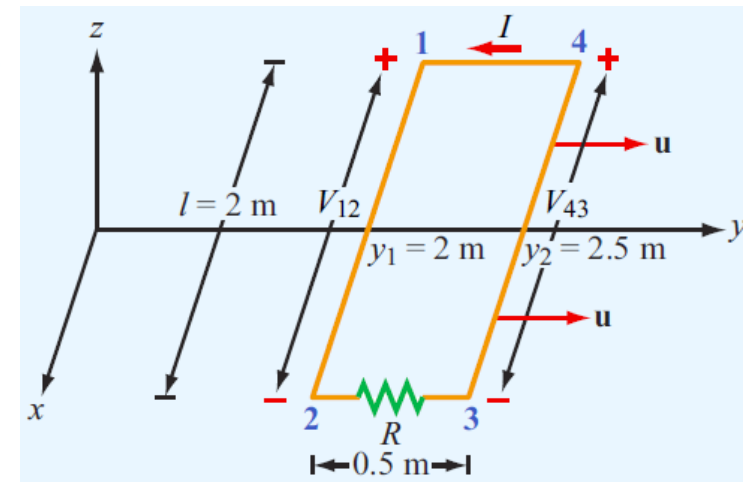
Example 6-4: Moving Loop

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The rectangular loop shown in **Fig. 6-9** is situated in the x - y plane and moves away from the origin with velocity $\mathbf{u} = \hat{\mathbf{y}}5$ (m/s) in a magnetic field given by

$$\mathbf{B}(y) = \hat{\mathbf{z}}0.2e^{-0.1y} \quad (\text{T}).$$

If $R = 5 \, \Omega$, find the current I at the instant that the loop sides are at $y_1 = 2$ m and $y_2 = 2.5$ m. The loop resistance may be ignored.



$$\begin{aligned} V_{12} &= \int_2^1 [\mathbf{u} \times \mathbf{B}(y_1)] \cdot d\mathbf{l} = \int_{l/2}^{-l/2} (\hat{\mathbf{y}}5 \times \hat{\mathbf{z}}0.2e^{-0.2}) \cdot \hat{\mathbf{x}} dx \\ &= -e^{-0.2}l = -2e^{-0.2} = -1.637 \quad (\text{V}). \end{aligned}$$

$$\mathbf{B}(y_1) = \hat{\mathbf{z}}0.2e^{-0.1y_1} = \hat{\mathbf{z}}0.2e^{-0.2} \quad (\text{T}).$$

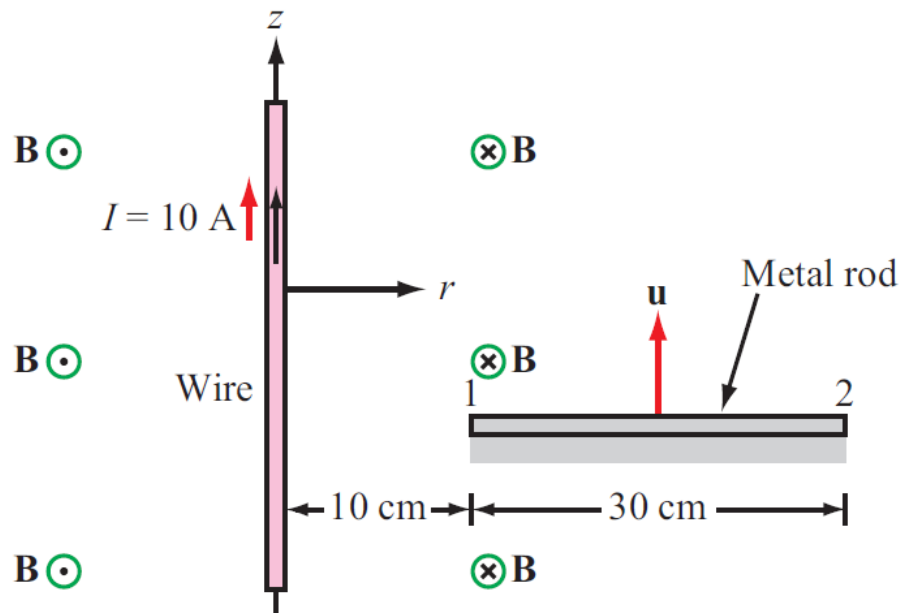
$$V_{43} = -u B(y_2)l = -5 \times 0.2e^{-0.25} \times 2 = -1.558 \quad (\text{V}).$$

$$I = \frac{V_{43} - V_{12}}{R} = \frac{0.079}{5} = 15.8 \text{ (mA)}.$$

Example 6-5: Moving Rod Next to a Wire

The wire shown in Fig. 6-10 carries a current $I = 10$ A. A 30-cm-long metal rod moves with a constant velocity $\mathbf{u} = \hat{\mathbf{z}}5$ m/s. Find V_{12} .

$$\mathbf{B} = \hat{\phi} \frac{\mu_0 I}{2\pi r}$$



$$\begin{aligned} V_{12} &= \int_{40 \text{ cm}}^{10 \text{ cm}} (\mathbf{u} \times \mathbf{B}) \cdot d\mathbf{l} \\ &= \int_{40 \text{ cm}}^{10 \text{ cm}} \left(\hat{\mathbf{z}}5 \times \hat{\phi} \frac{\mu_0 I}{2\pi r} \right) \cdot \hat{\mathbf{r}} dr \\ &= -\frac{5\mu_0 I}{2\pi} \int_{40 \text{ cm}}^{10 \text{ cm}} \frac{dr}{r} \\ &= -\frac{5 \times 4\pi \times 10^{-7} \times 10}{2\pi} \times \ln \left(\frac{10}{40} \right) \\ &= 13.9 \quad (\mu\text{V}). \end{aligned}$$

Summary of the direction of $d\mathbf{l}$

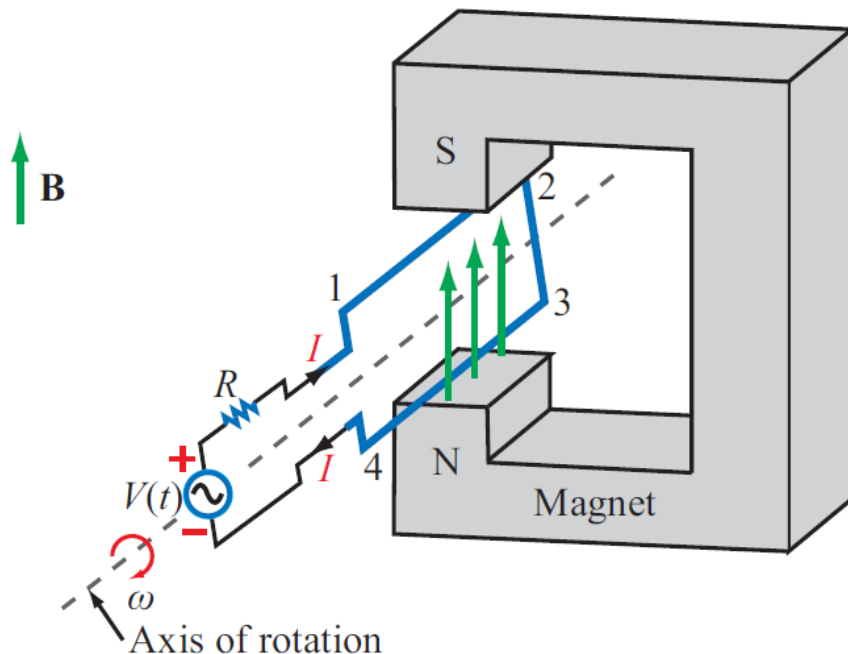
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$$V_{12} = \int_2^1 [\mathbf{u} \times \mathbf{B}(y_1)] \cdot d\mathbf{l} = \int_{l/2}^{-l/2} (\hat{\mathbf{y}}5 \times \hat{\mathbf{z}}0.2e^{-0.2}) \cdot \hat{\mathbf{x}} dx \quad \text{From example 6-4}$$

- If points 1 and 2 are used (first result) to do the integration, actual direction of $d\mathbf{l}$ should be used and it can be negative
- If real values of points 1 and 2 are used (first result), $d\mathbf{l}$ has to be always positive

EM Motor / Generator Reciprocity

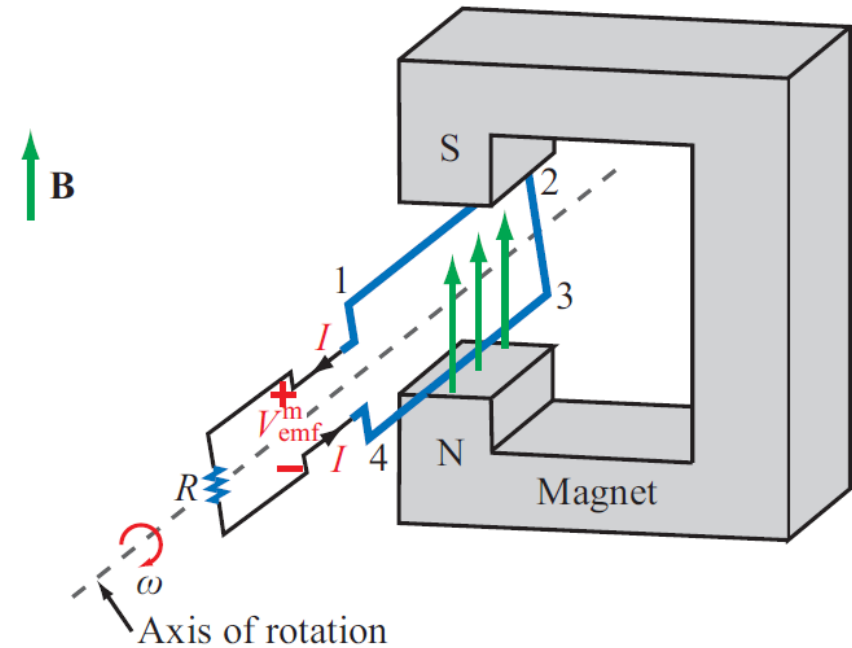
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(a) ac motor

Motor: Electrical to
mechanical energy conversion

Wires with current are subject to force
exerted by magnetic field



(b) ac generator

Generator: Mechanical to
electrical energy conversion

Time varying magnetic flux induces
current in circuit

EM Generator EMF

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As the loop rotates with an angular velocity ω about its own axis, segment 1–2 moves with velocity \mathbf{u} given by

$$\mathbf{u} = \hat{\mathbf{n}}\omega \frac{w}{2}$$

Also: $\hat{\mathbf{n}} \times \hat{\mathbf{z}} = \hat{\mathbf{x}} \sin \alpha.$

Segment 3–4 moves with velocity $-\mathbf{u}$. Hence:

$$\begin{aligned} V_{\text{emf}}^{\text{m}} = V_{14} &= \int_2^1 (\mathbf{u} \times \mathbf{B}) \cdot d\mathbf{l} + \int_4^3 (\mathbf{u} \times \mathbf{B}) \cdot d\mathbf{l} \\ &= \int_{-l/2}^{l/2} \left[\left(\hat{\mathbf{n}}\omega \frac{w}{2} \right) \times \hat{\mathbf{z}}B_0 \right] \cdot \hat{\mathbf{x}} dx \\ &\quad + \int_{l/2}^{-l/2} \left[\left(-\hat{\mathbf{n}}\omega \frac{w}{2} \right) \times \hat{\mathbf{z}}B_0 \right] \cdot \hat{\mathbf{x}} dx. \end{aligned}$$

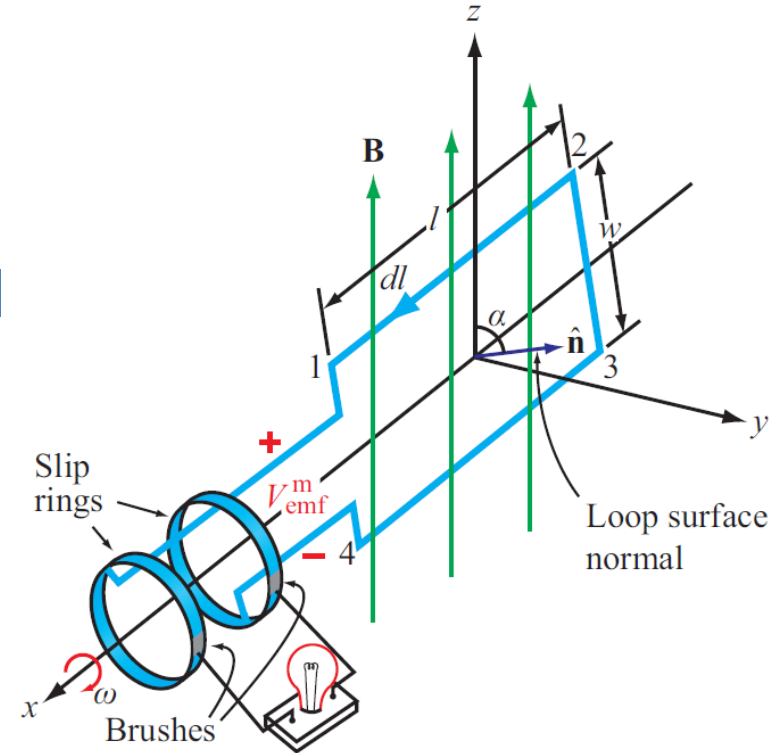


Figure 6-12: A loop rotating in a magnetic field induces an emf.

$$V_{\text{emf}}^{\text{m}} = wl\omega B_0 \sin \alpha = A\omega B_0 \sin \alpha,$$

$$\alpha = \omega t + C_0,$$

$$V_{\text{emf}}^{\text{m}} = A\omega B_0 \sin(\omega t + C_0) \quad (\text{V}).$$

Another Way

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$$\begin{aligned}\Phi &= \int_S \mathbf{B} \cdot d\mathbf{s} = \int_S \hat{\mathbf{z}} B_0 \cdot \hat{\mathbf{n}} \, ds \\ &= B_0 A \cos \alpha \\ &= B_0 A \cos(\omega t + C_0)\end{aligned}$$

$$\begin{aligned}V_{\text{emf}} &= -\frac{d\Phi}{dt} = -\frac{d}{dt}[B_0 A \cos(\omega t + C_0)] \\ &= A\omega B_0 \sin(\omega t + C_0),\end{aligned}$$

Moving Conductor in Time-Varying Magnetic Field

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$$\begin{aligned} V_{\text{emf}} &= V_{\text{emf}}^{\text{tr}} + V_{\text{emf}}^{\text{m}} \\ &= \oint_C \mathbf{E} \cdot d\mathbf{l} \\ &= - \int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{s} + \oint_C (\mathbf{u} \times \mathbf{B}) \cdot d\mathbf{l}. \end{aligned}$$

 Equivalent

$$V_{\text{emf}} = -\frac{d\Phi}{dt} = -\frac{d}{dt} \int_S \mathbf{B} \cdot d\mathbf{s} \quad (\text{total emf}).$$

Example 6-6

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$$\mathbf{B} = \hat{\mathbf{z}}B_0 \cos \omega t$$

$$\Phi = B_0 A \cos \omega t \quad \text{From p. 20}$$

$$\Phi = B_0 A \cos^2 \omega t$$

$$V_{\text{emf}} = -\frac{\partial \Phi}{\partial t}$$

$$= -\frac{\partial}{\partial t} (B_0 A \cos^2 \omega t)$$

$$= 2B_0 A \omega \cos \omega t \sin \omega t = B_0 A \omega \sin 2\omega t.$$

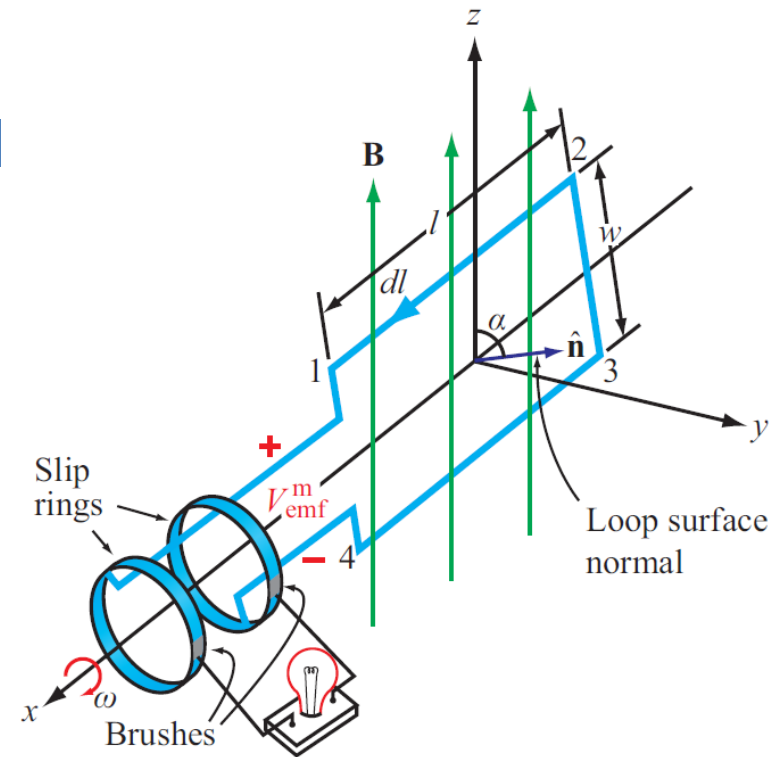


Figure 6-12: A loop rotating in a magnetic field induces an emf.

Charge Current Continuity Equation

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Current I out of a volume is equal to rate of decrease of charge Q contained in that volume:

$$I = -\frac{dQ}{dt} = -\frac{d}{dt} \int_V \rho_v dV$$

$$\oint_S \mathbf{J} \cdot d\mathbf{s} = -\frac{d}{dt} \int_V \rho_v dV$$

$$\oint_S \mathbf{J} \cdot d\mathbf{s} = \int_V \nabla \cdot \mathbf{J} dV = -\frac{d}{dt} \int_V \rho_v dV$$

Used Divergence Theorem

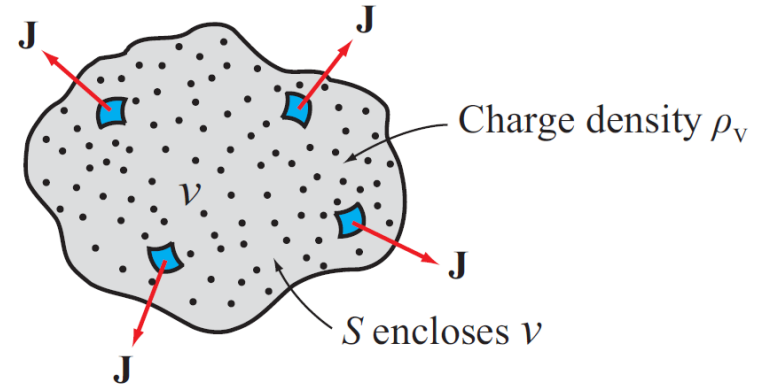


Figure 6-14: The total current flowing out of a volume V is equal to the flux of the current density \mathbf{J} through the surface S , which in turn is equal to the rate of decrease of the charge enclosed in V .

Volume current density

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho_v}{\partial t}, \quad (6.54)$$

which is known as the *charge-current continuity relation*, or simply the *charge continuity equation*.

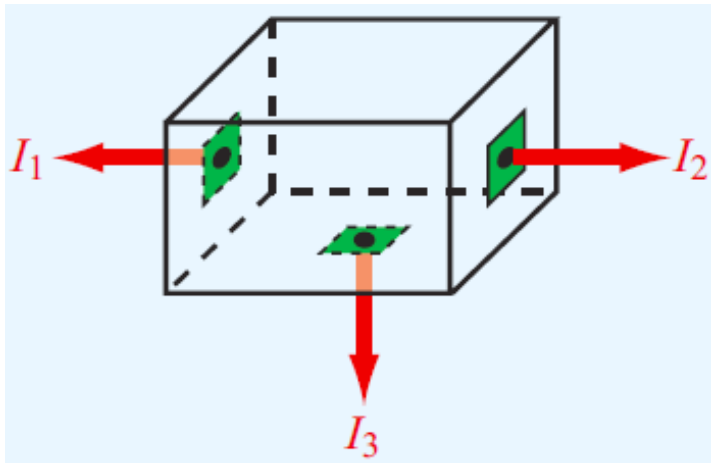
Charge Current Continuity Equation

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$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho_v}{\partial t}, \quad (6.54)$$

$$\nabla \cdot \mathbf{J} = 0,$$

Net current flowing out of ΔV is zero



$$\oint_S \mathbf{J} \cdot d\mathbf{s} = -\frac{d}{dt} \int_V \rho_v dV$$

$$\oint_S \mathbf{J} \cdot d\mathbf{s} = 0 \quad (\text{Kirchhoff's current law}).$$

$$\sum_i I_i = 0 \quad (\text{Kirchhoff's current law}),$$

Ampère's Law for Dynamic Fields

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$$\nabla \times \mathbf{H} = \mathbf{J}$$

$$\nabla \cdot (\nabla \times \mathbf{H}) = \nabla \cdot \mathbf{J}$$

$$0 = \nabla \cdot \mathbf{J} \quad \text{True for static fields, but} \\ \text{not for dynamic fields}$$

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho_v}{\partial t}, \quad (6.54)$$

$$\nabla \cdot (\nabla \times \mathbf{H}) - \frac{\partial \rho_v}{\partial t} = \nabla \cdot \mathbf{J}$$

$$\nabla \cdot \mathbf{D} = \rho_v$$

$$\nabla \cdot (\nabla \times \mathbf{H}) - \frac{\partial (\nabla \cdot \mathbf{D})}{\partial t} = \nabla \cdot \mathbf{J}$$

Need to add a term for dynamic field

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \quad (\text{Ampère's law}).$$



Added by Maxwell

Displacement Current

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$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \quad (\text{Ampère's law}). \quad (6.41)$$

Integrating both sides of Eq. (6.41) over an arbitrary open surface S with contour C , we have

$$\int_S (\nabla \times \mathbf{H}) \cdot d\mathbf{s} = \int_S \mathbf{J} \cdot d\mathbf{s} + \int_S \frac{\partial \mathbf{D}}{\partial t} \cdot d\mathbf{s}. \quad (6.42)$$

This term is
conduction
current I_C

This term must
represent a
current

Application of Stokes's theorem gives:

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = I_C + \int_S \frac{\partial \mathbf{D}}{\partial t} \cdot d\mathbf{s} \quad (\text{Ampère's law})$$

Cont.

Displacement Current

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$$\oint_C \mathbf{H} \cdot d\mathbf{l} = I_c + \int_S \frac{\partial \mathbf{D}}{\partial t} \cdot d\mathbf{s} \quad (\text{Ampère's law})$$

Define the displacement current as:

$$I_d = \int_S \mathbf{J}_d \cdot d\mathbf{s} = \int_S \frac{\partial \mathbf{D}}{\partial t} \cdot d\mathbf{s}, \quad (6.44)$$

where $\mathbf{J}_d = \partial \mathbf{D} / \partial t$ represents a *displacement current density*.
In view of Eq. (6.44),

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = I_c + I_d = I, \quad (6.45)$$

The displacement current does not involve real charges; it is an equivalent current that depends on $\partial \mathbf{D} / \partial t$

Capacitor Circuit

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Given: Wires are perfect conductors and capacitor insulator material is perfect dielectric.

For Surface S_1 :

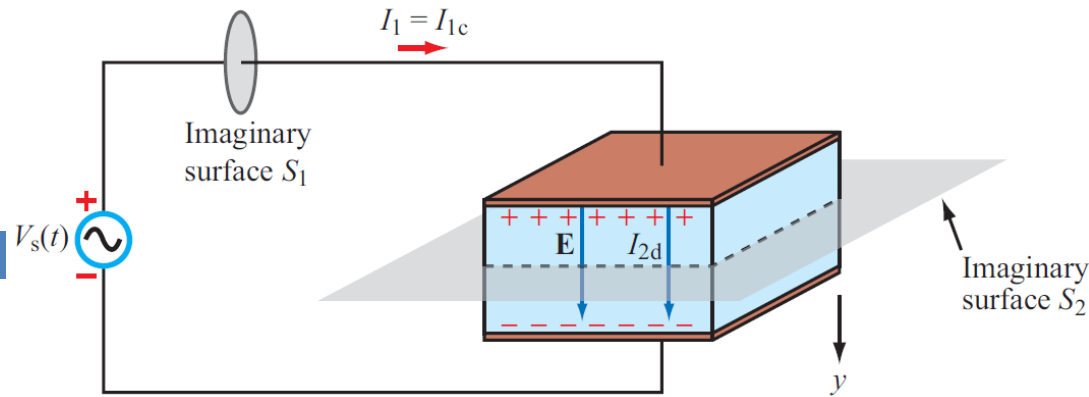
$$I_1 = I_{1c} + I_{1d}$$

$$I_{1c} = C \frac{dV_C}{dt} = C \frac{d}{dt} (V_0 \cos \omega t) = -C V_0 \omega \sin \omega t$$

$$I_{1d} = 0 \quad (\mathbf{D} = 0 \text{ in perfect conductor})$$

$I_1 = I_2$ ensures the continuity of total current flow through the circuit

Although the displacement current does not transport free charges, it behaves like a real current



For Surface S_2 :

$$I_2 = I_{2c} + I_{2d}$$

$$I_{2c} = 0 \text{ (perfect dielectric } \sigma = 0 \text{)}$$

$$\mathbf{E} = \hat{\mathbf{y}} \frac{V_C}{d} = \hat{\mathbf{y}} \frac{V_0}{d} \cos \omega t$$

$$I_{2d} = \int_S \frac{\partial \mathbf{D}}{\partial t} \cdot d\mathbf{s}$$

$$= \int_A \left[\frac{\partial}{\partial t} \left(\hat{\mathbf{y}} \frac{\epsilon V_0}{d} \cos \omega t \right) \right] \cdot (\hat{\mathbf{y}} d\mathbf{s})$$

$$= -\frac{\epsilon A}{d} V_0 \omega \sin \omega t = -C V_0 \omega \sin \omega t$$

Conclusion: $I_1 = I_2$

$$C = \epsilon A / d$$

Example 6-7: Displacement Current Density

The conduction current flowing through a wire with conductivity $\sigma = 2 \times 10^7$ S/m and relative permittivity $\epsilon_r = 1$ is given by $I_c = 2 \sin \omega t$ (mA). If $\omega = 10^9$ rad/s, find the displacement current.

Solution: The conduction current $I_c = JA = \sigma EA$, where A is the cross section of the wire. Hence,

$$\begin{aligned} E &= \frac{I_c}{\sigma A} = \frac{2 \times 10^{-3} \sin \omega t}{2 \times 10^7 A} \\ &= \frac{1 \times 10^{-10}}{A} \sin \omega t \quad (\text{V/m}). \end{aligned}$$

Application of Eq. (6.44), with $D = \epsilon E$, leads to

$$\begin{aligned} I_d &= J_d A \\ &= \epsilon A \frac{\partial E}{\partial t} \\ &= \epsilon A \frac{\partial}{\partial t} \left(\frac{1 \times 10^{-10}}{A} \sin \omega t \right) \\ &= \epsilon \omega \times 10^{-10} \cos \omega t = 0.885 \times 10^{-12} \cos \omega t \quad (\text{A}), \end{aligned}$$

where we used $\omega = 10^9$ rad/s and $\epsilon = \epsilon_0 = 8.85 \times 10^{-12}$ F/m. Note that I_c and I_d are in phase quadrature (90° phase shift between them). Also, I_d is about nine orders of magnitude smaller than I_c , which is why the displacement current usually is ignored in good conductors.

Boundary Conditions

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Table 6-2: Boundary conditions for the electric and magnetic fields.

Field Components	General Form	Medium 1 Dielectric	Medium 2 Dielectric	Medium 1 Dielectric	Medium 2 Conductor
Tangential E	$\hat{n}_2 \times (\mathbf{E}_1 - \mathbf{E}_2) = 0$	$E_{1t} = E_{2t}$		$E_{1t} = E_{2t} = 0$	
Normal D	$\hat{n}_2 \cdot (\mathbf{D}_1 - \mathbf{D}_2) = \rho_s$	$D_{1n} - D_{2n} = \rho_s$		$D_{1n} = \rho_s$	$D_{2n} = 0$
Tangential H	$\hat{n}_2 \times (\mathbf{H}_1 - \mathbf{H}_2) = \mathbf{J}_s$	$H_{1t} = H_{2t}$		$H_{1t} = J_s$	$H_{2t} = 0$
Normal B	$\hat{n}_2 \cdot (\mathbf{B}_1 - \mathbf{B}_2) = 0$	$B_{1n} = B_{2n}$		$B_{1n} = B_{2n} = 0$	
Notes: (1) ρ_s is the surface charge density at the boundary; (2) \mathbf{J}_s is the surface current density at the boundary; (3) normal components of all fields are along \hat{n}_2 , the outward unit vector of medium 2; (4) $E_{1t} = E_{2t}$ implies that the tangential components are equal in magnitude and parallel in direction; (5) direction of \mathbf{J}_s is orthogonal to $(\mathbf{H}_1 - \mathbf{H}_2)$.					

Charge Dissipation

Question 1: What happens if you place a certain amount of free charge inside of a material?

Answer: The charge will move to the surface of the material, thereby returning its interior to a neutral state.

Question 2: How fast will this happen?

Answer: It depends on the material; in a good conductor, the charge dissipates in less than a femtosecond, whereas in a good dielectric, the process may take several hours.

Derivation of charge density equation:

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho_v}{\partial t} . \quad (6.58)$$

In a conductor, the point form of Ohm's law, given by Eq. (4.63), states that $\mathbf{J} = \sigma \mathbf{E}$. Hence,

$$\sigma \nabla \cdot \mathbf{E} = -\frac{\partial \rho_v}{\partial t} . \quad (6.59)$$

Next, we use Eq. (6.1), $\nabla \cdot \mathbf{E} = \rho_v / \epsilon$, to obtain the partial differential equation

$$\frac{\partial \rho_v}{\partial t} + \frac{\sigma}{\epsilon} \rho_v = 0. \quad (6.60)$$

Cont.

Solution of Charge Dissipation Equation

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$$\frac{\partial \rho_v}{\partial t} + \frac{\sigma}{\varepsilon} \rho_v = 0.$$

Given that $\rho_v = \rho_{v0}$ at $t = 0$, the solution of Eq. (6.60) is

$$\rho_v(t) = \rho_{v0} e^{-(\sigma/\varepsilon)t} = \rho_{v0} e^{-t/\tau_r} \quad (\text{C/m}^3), \quad (6)$$

where $\tau_r = \varepsilon/\sigma$ is called the *relaxation time constant*.

For copper: $\tau_r = 1.53 \times 10^{-19} \text{ s}$

For mica: $\tau_r = 5.31 \times 10^4 \text{ s} = 15 \text{ hours}$

EM Potentials

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Static condition

$$V(\mathbf{R}) = \frac{1}{4\pi\epsilon} \int_{V'} \frac{\rho_v(\mathbf{R}_i)}{R'} dV'$$

Dynamic condition

$$V(\mathbf{R}, t) = \frac{1}{4\pi\epsilon} \int_{V'} \frac{\rho_v(\mathbf{R}_i, t)}{R'} dV'$$

Dynamic condition with propagation delay: Similarly, for the magnetic vector potential:

$$V(\mathbf{R}, t) = \frac{1}{4\pi\epsilon} \int_{V'} \frac{\rho_v(\mathbf{R}_i, t - R'/u_p)}{R'} dV' \quad (\text{V}),$$

$$\mathbf{A}(\mathbf{R}, t) = \frac{\mu}{4\pi} \int_{V'} \frac{\mathbf{J}(\mathbf{R}_i, t - R'/u_p)}{R'} dV' \quad (\text{Wb/m}).$$

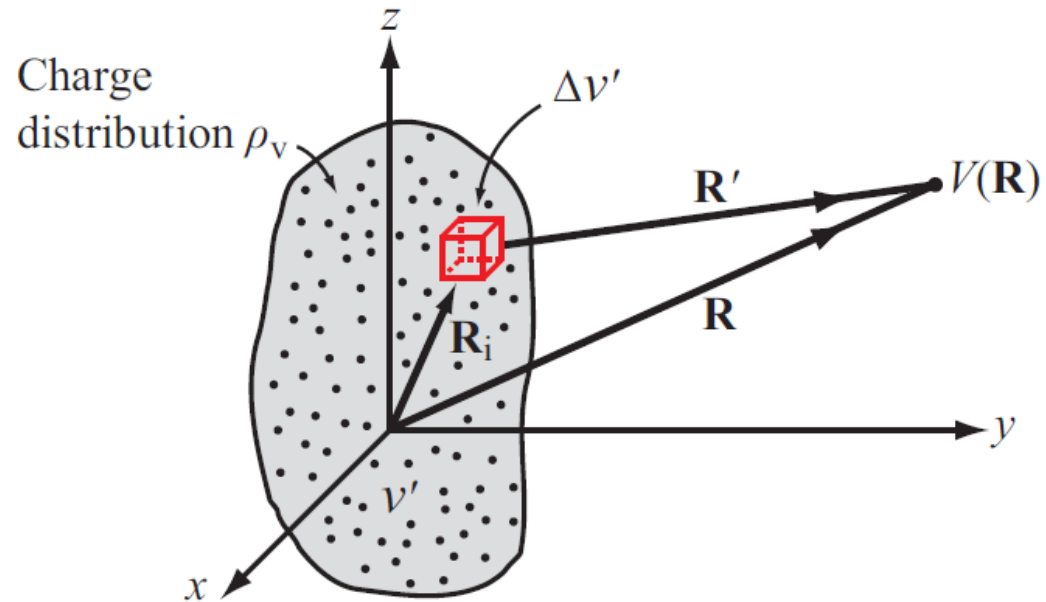


Figure 6-16: Electric potential $V(\mathbf{R})$ due to a charge distribution ρ_v over a volume V' .

Time Harmonic Potentials

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If charges and currents vary sinusoidally with time:

$$\rho_v(\mathbf{R}_i, t) = \rho_v(\mathbf{R}_i) \cos(\omega t + \phi)$$

we can use phasor notation:

$$\rho_v(\mathbf{R}_i, t) = \Re \left[\tilde{\rho}_v(\mathbf{R}_i) e^{j\omega t} \right],$$

with

$$\tilde{\rho}_v(\mathbf{R}_i) = \rho_v(\mathbf{R}_i) e^{j\phi}.$$

Expressions for potentials become:

$$\tilde{V}(\mathbf{R}) = \frac{1}{4\pi\epsilon} \int_{V'} \frac{\tilde{\rho}_v(\mathbf{R}_i) e^{-jkR'}}{R'} dV' \quad (\text{V}).$$

$$\tilde{\mathbf{A}}(\mathbf{R}) = \frac{\mu}{4\pi} \int_{V'} \frac{\tilde{\mathbf{J}}(\mathbf{R}_i) e^{-jkR'}}{R'} dV',$$

Also: $\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}$ (dynamic case).

$$\tilde{\mathbf{H}} = \frac{1}{\mu} \nabla \times \tilde{\mathbf{A}}.$$

Maxwell's equations become:

$$\begin{aligned} \nabla \times \tilde{\mathbf{E}} &= -j\omega\mu\tilde{\mathbf{H}} \\ \text{or} \quad \tilde{\mathbf{H}} &= -\frac{1}{j\omega\mu} \nabla \times \tilde{\mathbf{E}}. \end{aligned}$$

$$\nabla \times \tilde{\mathbf{H}} = j\omega\epsilon\tilde{\mathbf{E}} \quad \text{or} \quad \tilde{\mathbf{E}} = \frac{1}{j\omega\epsilon} \nabla \times \tilde{\mathbf{H}}.$$

$$k = \frac{\omega}{u_p}$$

Example 6-8: Relating \mathbf{E} to \mathbf{H}

In a nonconducting medium with $\varepsilon = 16\varepsilon_0$ and $\mu = \mu_0$, the electric field intensity of an electromagnetic wave is

$$\mathbf{E}(z, t) = \hat{\mathbf{x}} 10 \sin(10^{10}t - kz) \quad (\text{V/m}). \quad (6.88)$$

Determine the associated magnetic field intensity \mathbf{H} and find the value of k .

Solution: We begin by finding the phasor $\tilde{\mathbf{E}}(z)$ of $\mathbf{E}(z, t)$. Since $\mathbf{E}(z, t)$ is given as a sine function and phasors are defined in this book with reference to the cosine function, we rewrite Eq. (6.88) as

$$\begin{aligned} \mathbf{E}(z, t) &= \hat{\mathbf{x}} 10 \cos(10^{10}t - kz - \pi/2) \quad (\text{V/m}) \\ &= \Re \left[\tilde{\mathbf{E}}(z) e^{j\omega t} \right], \end{aligned} \quad (6.89)$$

with $\omega = 10^{10}$ (rad/s) and

$$\tilde{\mathbf{E}}(z) = \hat{\mathbf{x}} 10 e^{-jkz} e^{-j\pi/2} = -\hat{\mathbf{x}} j 10 e^{-jkz}. \quad (6.90)$$

Cont.

To find both $\tilde{\mathbf{H}}(z)$ and k , we will perform a “circle”: we will use the given expression for $\tilde{\mathbf{E}}(z)$ in Faraday’s law to find $\tilde{\mathbf{H}}(z)$; then we will use $\tilde{\mathbf{H}}(z)$ in Ampère’s law to find $\tilde{\mathbf{E}}(z)$, which we will then compare with the original expression for $\tilde{\mathbf{E}}(z)$; and the comparison will yield the value of k . Application of Eq. (6.87) gives

$$\nabla \times \tilde{\mathbf{E}} = -j\omega\mu\tilde{\mathbf{H}}$$

$$\begin{aligned}\tilde{\mathbf{H}}(z) &= -\frac{1}{j\omega\mu} \nabla \times \tilde{\mathbf{E}} \\ &= -\frac{1}{j\omega\mu} \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -j10e^{-jkz} & 0 & 0 \end{vmatrix} \\ &= -\frac{1}{j\omega\mu} \left[\hat{\mathbf{y}} \frac{\partial}{\partial z} (-j10e^{-jkz}) \right] \\ &= -\hat{\mathbf{y}} j \frac{10k}{\omega\mu} e^{-jkz}. \end{aligned} \tag{6.91}$$

Example 6-8 cont.

So far, we have used Eq. (6.90) for $\tilde{\mathbf{E}}(z)$ to find $\tilde{\mathbf{H}}(z)$, but k remains unknown. To find k , we use $\tilde{\mathbf{H}}(z)$ in Eq. (6.86) to find $\tilde{\mathbf{E}}(z)$:

$$\begin{aligned}\tilde{\mathbf{E}}(z) &= \frac{1}{j\omega\epsilon} \nabla \times \tilde{\mathbf{H}} \\ &= \frac{1}{j\omega\epsilon} \left[-\hat{\mathbf{x}} \frac{\partial}{\partial z} \left(-j \frac{10k}{\omega\mu} e^{-jkz} \right) \right] \\ &= -\hat{\mathbf{x}} j \frac{10k^2}{\omega^2\mu\epsilon} e^{-jkz}.\end{aligned}\tag{6.92}$$

$$\nabla \times \tilde{\mathbf{H}} = j\omega\epsilon\tilde{\mathbf{E}}$$

Equating Eqs. (6.90) and (6.92) leads to

$$k^2 = \omega^2\mu\epsilon,$$

$$\tilde{\mathbf{E}}(z) = \hat{\mathbf{x}} 10e^{-jkz} e^{-j\pi/2} = -\hat{\mathbf{x}} j 10e^{-jkz}\tag{6.90}$$

or

$$\begin{aligned}k &= \omega\sqrt{\mu\epsilon} \\ &= 4\omega\sqrt{\mu_0\epsilon_0} \\ &= \frac{4\omega}{c} = \frac{4 \times 10^{10}}{3 \times 10^8} = 133 \quad (\text{rad/m}).\end{aligned}\tag{6.93}$$

Cont.

Example 6-8 cont.

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With k known, the instantaneous magnetic field intensity is then given by

$$\begin{aligned}\mathbf{H}(z, t) &= \Re \left[\tilde{\mathbf{H}}(z) e^{j\omega t} \right] \\ &= \Re \left[-\hat{\mathbf{y}} j \frac{10k}{\omega\mu} e^{-jkz} e^{j\omega t} \right] \\ &= \hat{\mathbf{y}} 0.11 \sin(10^{10}t - 133z) \quad (\text{A/m}). \quad (6.94)\end{aligned}$$

We note that k has the same expression as the phase constant of a lossless transmission line [Eq. (2.49)].

Summary

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Chapter 6 Relationships

Faraday's Law

$$V_{\text{emf}} = -\frac{d\Phi}{dt} = -\frac{d}{dt} \int_S \mathbf{B} \cdot d\mathbf{s} = V_{\text{emf}}^{\text{tr}} + V_{\text{emf}}^{\text{m}}$$

Transformer

$$V_{\text{emf}}^{\text{tr}} = -N \int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{s} \quad (N \text{ loops})$$

Motional

$$V_{\text{emf}}^{\text{m}} = \oint_C (\mathbf{u} \times \mathbf{B}) \cdot d\mathbf{l}$$

Charge-Current Continuity

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho_v}{\partial t}$$

EM Potentials

$$\begin{aligned} \mathbf{E} &= -\nabla V - \frac{\partial \mathbf{A}}{\partial t} \\ \mathbf{B} &= \nabla \times \mathbf{A} \end{aligned}$$

Current Density

$$\text{Conduction} \quad \mathbf{J}_c = \sigma \mathbf{E}$$

$$\text{Displacement} \quad \mathbf{J}_d = \frac{\partial \mathbf{D}}{\partial t}$$

Conductor Charge Dissipation

$$\rho_v(t) = \rho_{v0} e^{-(\sigma/\epsilon)t} = \rho_{v0} e^{-t/\tau_r}$$