Numerical Optimization, 2023 Fall
Homework 3

Due 23:59 (CST), Nov. 16, 2023

Problem 1. Prove the dual of the dual of a linear programming (standard form) is itself.[25pts]

For the linear programming (standard form) problem:

$$\begin{aligned} \min_{\boldsymbol{x}} \quad & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{aligned} \tag{1}$$

We can write its dual problem as:

$$\begin{aligned} \max_{\boldsymbol{\lambda}} \quad & \boldsymbol{b}^\top \boldsymbol{\lambda} \\ \text{s.t.} \quad & \boldsymbol{A}^\top \boldsymbol{\lambda} \leq \boldsymbol{c} \end{aligned} \tag{2}$$

which is equivalent to:

$$\begin{aligned} \min_{\boldsymbol{\lambda}} \quad & -\boldsymbol{b}^\top \boldsymbol{\lambda} \\ \text{s.t.} \quad & -\boldsymbol{A}^\top \boldsymbol{\lambda} \geq -\boldsymbol{c} \end{aligned} \tag{3}$$

Then we can write the dual of the dual problem as:

$$\begin{aligned} \max_{\boldsymbol{x}} \quad & -\boldsymbol{c}^\top \boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{aligned} \tag{4}$$

which can be written as:

$$\begin{aligned} \min_{\boldsymbol{x}} \quad & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{aligned} \tag{5}$$

Problem 2. Prove the dual objective increases after a pivot of the dual simplex method.[25pts]

We update the dual variables $\boldsymbol{\lambda}$ to $\hat{\boldsymbol{\lambda}} = \boldsymbol{\lambda} + \frac{r_p}{y_{pq}}\boldsymbol{u}_p$, where $\boldsymbol{u}_p$ is the pivot element and $\boldsymbol{u}_p$ is the p-th row of the inverse of the basis matrix B^{-1}. Then we have the new dual objective:

$$\begin{aligned}\hat{\boldsymbol{\lambda}}^\top \boldsymbol{b} &= (\boldsymbol{\lambda} + \frac{r_p}{y_{pq}}\boldsymbol{u}_p)^\top \\ &= \boldsymbol{\lambda}^T \boldsymbol{b} + \frac{r_p}{y_{pq}}\boldsymbol{u}_p^\top \boldsymbol{b} \\ &= \boldsymbol{\lambda}^T \boldsymbol{b} + \frac{r_p}{y_{pq}}\bar{b}_p\end{aligned} \tag{6}$$

Because $r_p \geq 0, y_{pq} < 0, \bar{b}_p < 0$ in the dual simplex method, we have $\hat{\boldsymbol{\lambda}}^\top \boldsymbol{b} \geq \boldsymbol{\lambda}^\top \boldsymbol{b}$.

Problem 3. Let $L(\boldsymbol{x}, \boldsymbol{\lambda})$ be the Lagrangian of a linear programming problem, and $(\boldsymbol{x}^*, \boldsymbol{\lambda}^*)$ be the optimal primal-dual solution. Prove that

$$L(\boldsymbol{x}, \boldsymbol{\lambda}^*) \geq L(\boldsymbol{x}^*, \boldsymbol{\lambda}^*) \geq L(\boldsymbol{x}^*, \boldsymbol{\lambda}),$$

for any primal feasible $\boldsymbol{x}$ and dual feasible $\boldsymbol{\lambda}$.[25pts]

Suppose that the primal problem is

$$\begin{aligned} \min_{\boldsymbol{x}} \quad & \boldsymbol{c}^T\boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}, \end{aligned} \tag{1}$$

then the Lagrangian of the primal problem is

$$L(\boldsymbol{x}, \boldsymbol{\lambda}) = \boldsymbol{c}^T\boldsymbol{x} + \boldsymbol{\lambda}^T(\boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}) = -\boldsymbol{b}^T\boldsymbol{\lambda} + (\boldsymbol{A}^T\boldsymbol{\lambda} + \boldsymbol{c})^T\boldsymbol{x},$$

and the dual problem is

$$\begin{aligned} \max_{\boldsymbol{\lambda}} \quad & -\boldsymbol{b}^T\boldsymbol{\lambda} \\ \text{s.t.} \quad & \boldsymbol{A}^T\boldsymbol{\lambda} + \boldsymbol{c} = \boldsymbol{0} \\ & \boldsymbol{\lambda} \geq \boldsymbol{0}. \end{aligned} \tag{2}$$

Proof:
Since $(\boldsymbol{x}^*, \boldsymbol{\lambda}^*)$ is the optimal primal-dual solution, we have

$$\boldsymbol{A}\boldsymbol{x}^{\boldsymbol{*}} \leq \boldsymbol{b}$$

$$\boldsymbol{A}^T\boldsymbol{\lambda}^{\boldsymbol{*}} + \boldsymbol{c} = \boldsymbol{0}.$$

According to the strong duality,

$$\boldsymbol{c}^T\boldsymbol{x}^* = -\boldsymbol{\lambda}^{*T}\boldsymbol{b}.$$

Consider

$$\begin{aligned} L(\boldsymbol{x}, \boldsymbol{\lambda}^{\boldsymbol{*}}) &= -\boldsymbol{b}^T\boldsymbol{\lambda}^{\boldsymbol{*}} + (\boldsymbol{A}^T\boldsymbol{\lambda}^{\boldsymbol{*}} + \boldsymbol{c})^T\boldsymbol{x} = -\boldsymbol{b}^T\boldsymbol{\lambda}^{\boldsymbol{*}} \\ L(\boldsymbol{x}^*, \boldsymbol{\lambda}^{\boldsymbol{*}}) &= -\boldsymbol{b}^T\boldsymbol{\lambda}^{\boldsymbol{*}} + (\boldsymbol{A}^T\boldsymbol{\lambda}^{\boldsymbol{*}} + \boldsymbol{c})^T\boldsymbol{x}^{\boldsymbol{*}} = -\boldsymbol{b}^T\boldsymbol{\lambda}^{\boldsymbol{*}} \\ L(\boldsymbol{x}^*, \boldsymbol{\lambda}) &= \boldsymbol{c}^T\boldsymbol{x}^{\boldsymbol{*}} + \boldsymbol{\lambda}^T(\boldsymbol{A}\boldsymbol{x}^{\boldsymbol{*}} - \boldsymbol{b}) \leq \boldsymbol{c}^T\boldsymbol{x}^{\boldsymbol{*}} = -\boldsymbol{b}^T\boldsymbol{\lambda}^{\boldsymbol{*}}, \end{aligned}$$

we have

$$L(\boldsymbol{x}, \boldsymbol{\lambda}^*) \geq L(\boldsymbol{x}^*, \boldsymbol{\lambda}^*) \geq L(\boldsymbol{x}^*, \boldsymbol{\lambda}).$$

Problem 4. Construct a linear programming problem for which both the primal and the dual problem has no feasible solution.[25pts]

Construct a linear programming problem

$$
\begin{aligned}
\min_{x_1,x_2} \quad & x_1 - 2x_2 \\
\text{s.t.} \quad & x_1 - x_2 \leq 1 \\
& x_1 - x_2 \geq 2 \\
& x_1, x_2 \geq 0.
\end{aligned}
\tag{3}
$$

It is impossible to satisfy $x_1 - x_2 \leq 1$ and $x_1 - x_2 \geq 2$ at the same time, the primal problem has no feasible solution.
The dual problem is

$$
\begin{aligned}
\max_{\lambda_1,\lambda_2} \quad & \lambda_1 + 2\lambda_2 \\
\text{s.t.} \quad & \lambda_1 + \lambda_2 \leq 1 \\
& -\lambda_1 - \lambda_2 \leq -2 \\
& \lambda_1 \leq 0, \lambda_2 \geq 0.
\end{aligned}
\tag{4}
$$

The second constrain $-\lambda_1 - \lambda_2 \leq -2$ can be written as $\lambda_1 + \lambda_2 \geq 2$.
It is impossible to satisfy $\lambda_1 + \lambda_2 \leq 1$ and $\lambda_1 + \lambda_2 \geq 2$ at the same time, the dual problem has no feasible solution.