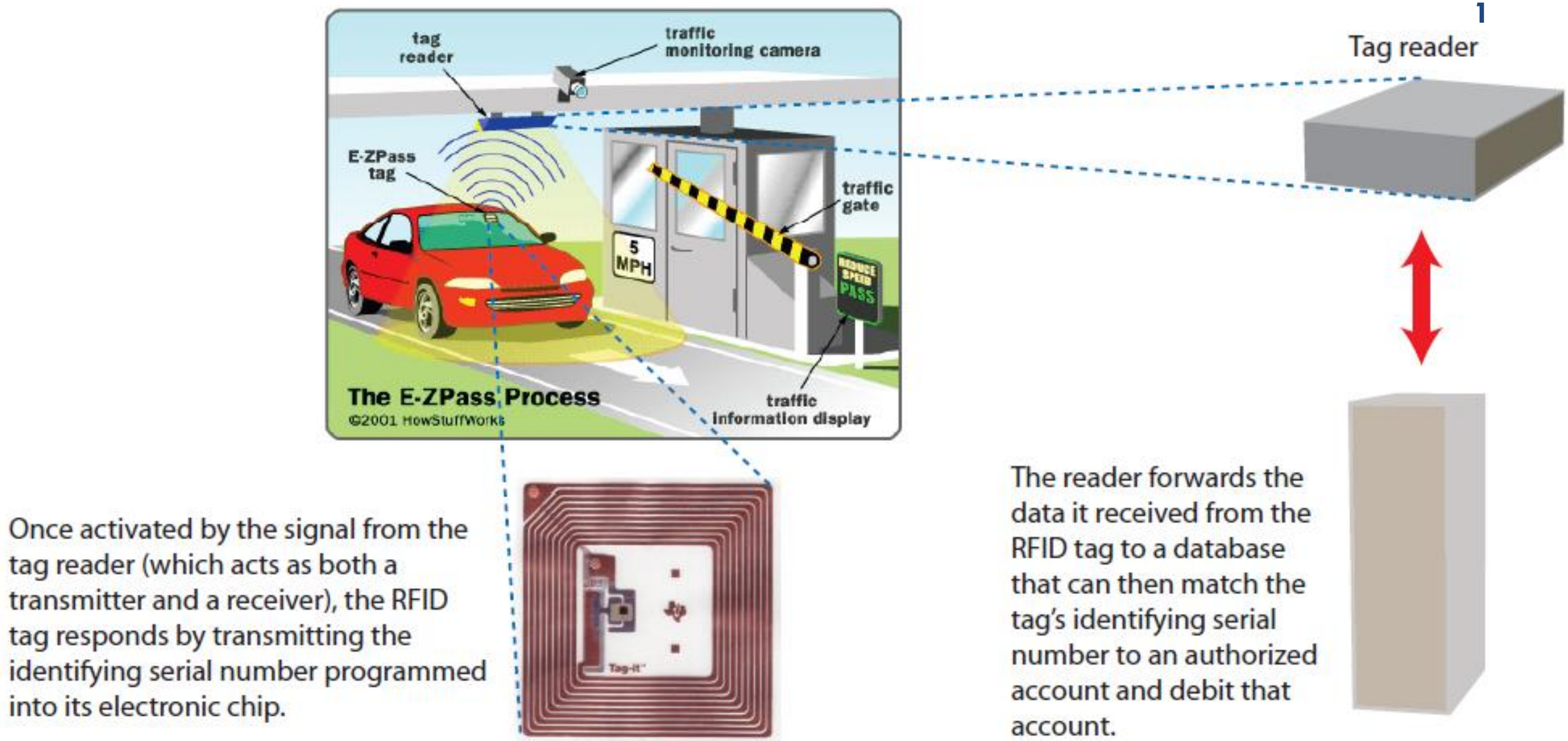


Figure TF13-2: How an RFID system works is illustrated through this EZ-Pass example. (Tag courtesy of Texas Instruments.)

7. PLANE WAVE PROPAGATION

7e Applied EM by Ulaby and Ravaioli

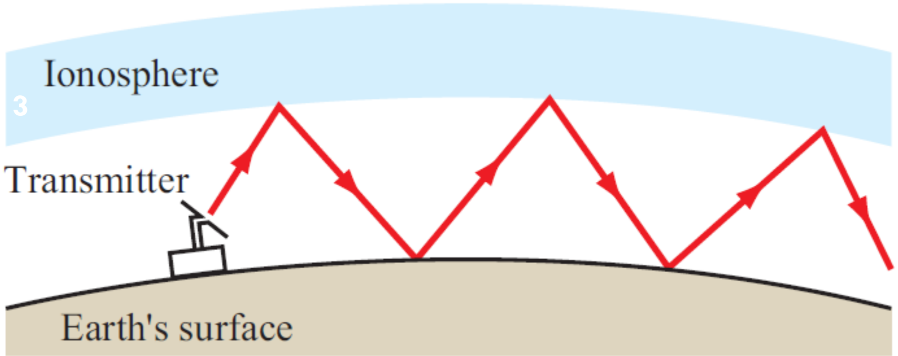
Chapter 7 Overview

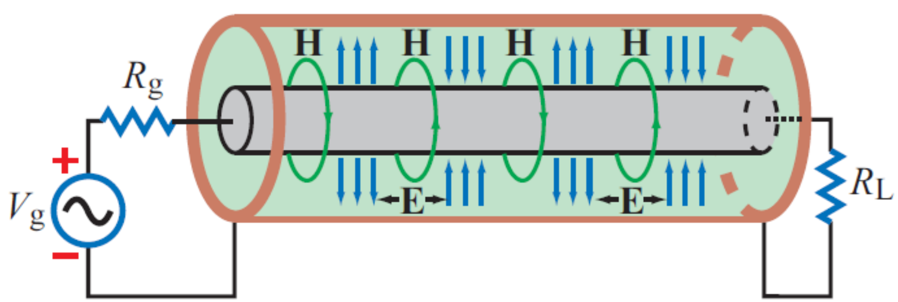
Chapter Contents

Objectives

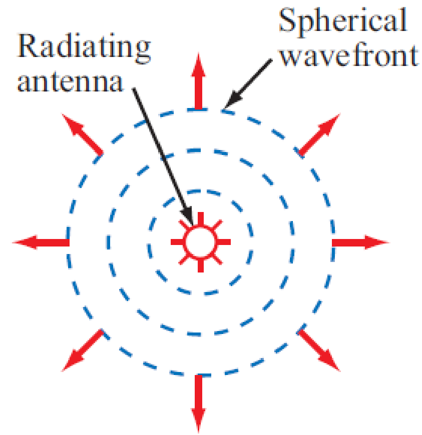
Upon learning the material presented in this chapter, you should be able to:

1. Describe mathematically the electric and magnetic fields of TEM waves.
2. Describe the polarization properties of an EM wave.
3. Relate the propagation parameters of a wave to the constitutive parameters of the medium.
4. Characterize the flow of current in conductors and use it to calculate the resistance of a coaxial cable.
5. Calculate the rate of power carried by an EM wave, in both lossless and lossy media.

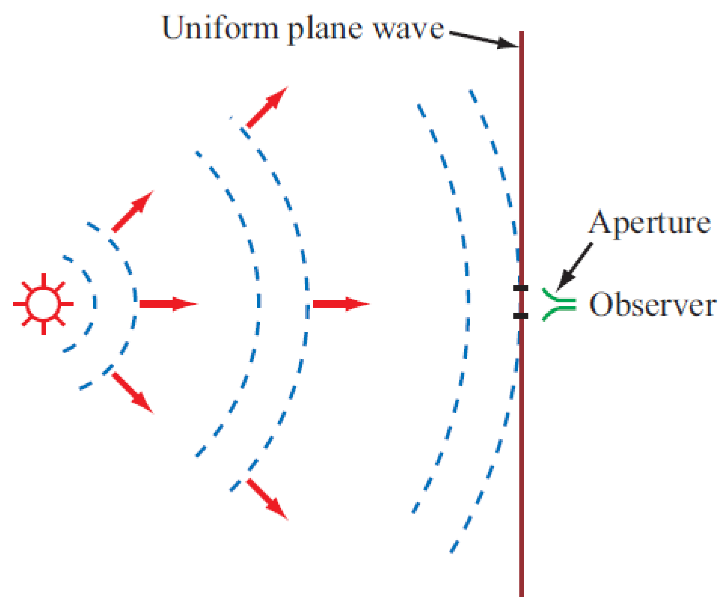




Guided EM Waves



(a) Spherical wave



(b) Plane-wave approximation

Unbounded EM Waves

Time-Harmonic Fields

For sinusoidal time variations: $$\mathbf{E}(x, y, z; t) = \mathfrak{Re}\left[\widetilde{\mathbf{E}}(x, y, z)\, e^{j\omega t}\right]$$

For a linear, isotropic, and homogeneous medium with ε and μ, the Maxwell's equations in phasor form are

$$\nabla \cdot \widetilde{\mathbf{E}} = \tilde{\rho}_{\mathrm{v}}/\varepsilon,$$
$$\nabla \times \widetilde{\mathbf{E}} = -j\omega\mu\widetilde{\mathbf{H}},$$
$$\nabla \cdot \widetilde{\mathbf{H}} = 0,$$
$$\nabla \times \widetilde{\mathbf{H}} = \widetilde{\mathbf{J}} + j\omega\varepsilon\widetilde{\mathbf{E}}.$$

$$\mathbf{D} = \epsilon\mathbf{E} \text{ and } \mathbf{B} = \mu\mathbf{H},$$

Complex Permittivity

$$\nabla \times \widetilde{\mathbf{H}} = \widetilde{\mathbf{J}} + j\omega\varepsilon\widetilde{\mathbf{E}}$$
$$= (\sigma + j\omega\varepsilon)\widetilde{\mathbf{E}} = j\omega\left(\varepsilon - j\frac{\sigma}{\omega}\right)\widetilde{\mathbf{E}}.$$

By defining the ***complex permittivity*** ε_c as

$$\boxed{\varepsilon_c = \varepsilon - j\frac{\sigma}{\omega}\,, \qquad (7.4)}$$

$$\varepsilon_c = \varepsilon - j\frac{\sigma}{\omega} = \varepsilon' - j\varepsilon''$$

Eq. (7.3) can be rewritten as

$$\nabla \times \widetilde{\mathbf{H}} = j\omega\varepsilon_c\widetilde{\mathbf{E}}.$$

$$\varepsilon' = \varepsilon, \tag{7.8a}$$

$$\varepsilon'' = \frac{\sigma}{\omega}\,. \tag{7.8b}$$

For a lossless medium with $\sigma = 0$, it follows that $\varepsilon'' = 0$ and $\varepsilon_c = \varepsilon' = \varepsilon$.

Wave Equations

$$\nabla \cdot \widetilde{\mathbf{E}} = 0, \quad (7.6a)$$

$$\nabla \times \widetilde{\mathbf{E}} = -j\omega\mu\widetilde{\mathbf{H}}, \quad (7.6b)$$

$$\nabla \cdot \widetilde{\mathbf{H}} = 0, \quad (7.6c)$$

$$\nabla \times \widetilde{\mathbf{H}} = j\omega\varepsilon_c\widetilde{\mathbf{E}}. \quad (7.6d)$$

Source free

$$\nabla \times (\nabla \times \widetilde{\mathbf{E}}) = -j\omega\mu(\nabla \times \widetilde{\mathbf{H}}). \quad (7.9)$$

Upon substituting Eq. (7.6d) into Eq. (7.9) we obtain

$$\nabla \times (\nabla \times \widetilde{\mathbf{E}}) = -j\omega\mu(j\omega\varepsilon_c\widetilde{\mathbf{E}}) = \omega^2\mu\varepsilon_c\widetilde{\mathbf{E}}. \quad (7.10)$$

From Eq. (3.113), we know that the curl of the curl of $\widetilde{\mathbf{E}}$ is

$$\nabla \times (\nabla \times \widetilde{\mathbf{E}}) = \nabla(\nabla \cdot \widetilde{\mathbf{E}}) - \nabla^2\widetilde{\mathbf{E}}, \quad (7.11)$$

where $\nabla^2\widetilde{\mathbf{E}}$ is the Laplacian of $\widetilde{\mathbf{E}}$, which in Cartesian coordinates is given by

$$\nabla^2\widetilde{\mathbf{E}} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\widetilde{\mathbf{E}}. \quad (7.12)$$

In view of Eq. (7.6a), the use of Eq. (7.11) in Eq. (7.10) gives

$$\nabla^2\widetilde{\mathbf{E}} + \omega^2\mu\varepsilon_c\widetilde{\mathbf{E}} = 0, \quad (7.13)$$

which is known as the ***homogeneous wave equation for*** $\widetilde{\mathbf{E}}$. By defining the ***propagation constant*** γ as

$$\gamma^2 = -\omega^2\mu\varepsilon_c, \quad (7.14)$$

Eq. (7.13) can be written as

Helmholtz equation

$$\boxed{\nabla^2\widetilde{\mathbf{E}} - \gamma^2\widetilde{\mathbf{E}} = 0. \quad (7.15)}$$

To derive Eq. (7.15), we took the curl of both sides of Eq. (7.6b) and then we used Eq. (7.6d) to eliminate $\widetilde{\mathbf{H}}$ and obtain an equation in $\widetilde{\mathbf{E}}$ only. If we reverse the process, that is, if we start by taking the curl of both sides of Eq. (7.6d) and then use Eq. (7.6b) to eliminate $\widetilde{\mathbf{E}}$, we obtain a wave equation for $\widetilde{\mathbf{H}}$:

$$\boxed{\nabla^2\widetilde{\mathbf{H}} - \gamma^2\widetilde{\mathbf{H}} = 0. \quad (7.16)}$$

Since the wave equations for $\widetilde{\mathbf{E}}$ and $\widetilde{\mathbf{H}}$ are of the same form, so are their solutions.

Wave Nature

$$\nabla^2 \widetilde{\mathbf{E}} - \gamma^2 \widetilde{\mathbf{E}} = 0 \qquad \nabla^2 \widetilde{\mathbf{H}} - \gamma^2 \widetilde{\mathbf{H}} = 0 \qquad \gamma^2 = -\omega^2 \mu \epsilon_c,$$

$$\nabla^2 \widetilde{\mathbf{E}} + \omega^2 \mu \epsilon_c \widetilde{\mathbf{E}} = 0,$$

Conventional wave equation
(mechanical displacement *u* of a wave)

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u \qquad \nabla^2 u + \frac{\omega^2}{c^2} u = 0 \qquad \boxed{c = \frac{1}{\sqrt{\mu\varepsilon}}}$$

Propagation Speed

Explanation of Wave Speed

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u \qquad \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial z^2} \qquad \frac{\partial u}{\partial t} = \frac{u(t+\Delta t) - u(t)}{\Delta t}$$

$$\frac{\partial^2 u}{\partial t^2} = \frac{\left.\frac{\partial u}{\partial t}\right|_{t+\Delta t} - \left.\frac{\partial u}{\partial t}\right|_{t}}{\Delta t} = \frac{\frac{u(t+2\Delta t) - u(t+\Delta t)}{\Delta t} - \frac{u(t+\Delta t) - u(t)}{\Delta t}}{\Delta t} = \frac{u(t+2\Delta t) - 2u(t+\Delta t) + u(t)}{\Delta t^2}$$

$$\frac{u(t+2\Delta t) - 2u(t+\Delta t) + u(t)}{\Delta t^2} = c^2 \frac{u(z+2\Delta z) - 2u(z+\Delta z) + u(z)}{\Delta z^2}$$

$$\frac{u(t+\Delta t) - 2u(t) + u(t-\Delta t)}{\Delta t^2} = c^2 \frac{u(z+\Delta z) - 2u(z) + u(z-\Delta z)}{\Delta z^2}$$

$$\frac{u(t+\Delta t, z) - 2u(t,z) + u(t-\Delta t, z)}{\Delta t^2} = c^2 \frac{u(t, z+\Delta z) - 2u(t,z) + u(t, z-\Delta z)}{\Delta z^2}$$

$$\frac{u(t,z)}{\Delta t^2} = c^2 \frac{u(t,z)}{\Delta z^2} \qquad \frac{\Delta z}{\Delta t} = c$$

$$\frac{u(t+\Delta t, z)}{\Delta t^2} = c^2 \frac{u(t, z-\Delta z)}{\Delta z^2} \qquad u(t+\Delta t, z) = u(t, z-\Delta z)$$

$$\frac{u(t-\Delta t, z)}{\Delta t^2} = c^2 \frac{u(t, z+\Delta z)}{\Delta z^2} \qquad u(t-\Delta t, z) = u(t, z+\Delta z)$$

Wave Equation Expansion

$$\nabla^2\widetilde{\mathbf{E}} + \omega^2\mu\epsilon_c\widetilde{\mathbf{E}} = 0,$$

Expand ∇^2 in the wave equation in Cartesian coordinate system

$$\nabla^2\boldsymbol{E} = \hat{x}\nabla^2 E_x + \hat{y}\nabla^2 E_y + \hat{z}\nabla^2 E_z$$

$$\nabla^2 E_x + \omega^2\mu\varepsilon E_x = 0$$

$$\nabla^2 E_y + \omega^2\mu\varepsilon E_y = 0$$

$$\nabla^2 E_z + \omega^2\mu\varepsilon E_z = 0$$

$$\nabla^2 E_x = \frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_x}{\partial y^2} + \frac{\partial^2 E_x}{\partial z^2}$$

$$\nabla^2 E_y = \frac{\partial^2 E_y}{\partial x^2} + \frac{\partial^2 E_y}{\partial y^2} + \frac{\partial^2 E_y}{\partial z^2}$$

$$\nabla^2 E_z = \frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + \frac{\partial^2 E_z}{\partial z^2}$$

Lossless Media

If the medium is ***nonconducting*** **($\sigma = 0$)**, the wave does not suffer any attenuation as it travels and hence the medium is said to be ***lossless.***

$$\gamma^2 = -\omega^2 \mu \varepsilon. \tag{7.17}$$

For lossless media, it is customary to define the ***wavenumber*** k as

$$\boxed{k = \omega\sqrt{\mu\varepsilon}\ . \qquad (7.18)}$$

In view of Eq. (7.17), $\gamma^2 = -k^2$ and Eq. (7.15) becomes

$$\nabla^2 \widetilde{\mathbf{E}} + k^2 \widetilde{\mathbf{E}} = 0. \tag{7.19}$$

Uniform Plane Wave

$$\nabla^2 \widetilde{\mathbf{E}} + \omega^2 \mu \epsilon_c \widetilde{\mathbf{E}} = 0$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + k^2 \right) \widetilde{E}_x = 0, \qquad (7.22)$$

and similar expressions apply to $\widetilde{E}_y$ and $\widetilde{E}_z$.

A ***uniform plane wave*** *is characterized by electric and magnetic fields that have uniform properties at all points across an infinite plane.*

If this happens to be the x–y plane, then $\mathbf{E}$ and $\mathbf{H}$ do not vary with x and y. Hence, $\partial \widetilde{E}_x / \partial x = 0$ and $\partial \widetilde{E}_x / \partial y = 0$, and Eq. (7.22) reduces to

$$\frac{d^2 \widetilde{E}_x}{dz^2} + k^2 \widetilde{E}_x = 0. \qquad (7.23)$$

Transverse Electromagnetic (TEM) Wave

$$\frac{d^2\tilde{E}_y}{dz^2}+k^2\tilde{E}_y=0 \qquad \frac{d^2\tilde{H}_x}{dz^2}+k^2\tilde{H}_x=0 \qquad \frac{d^2\tilde{H}_y}{dz^2}+k^2\tilde{H}_y=0$$

Similar expressions apply to $\tilde{E}_y$, $\tilde{H}_x$, and $\tilde{H}_y$. The remaining components of $\tilde{\mathbf{E}}$ and $\tilde{\mathbf{H}}$ are zero; that is, $\tilde{E}_z = \tilde{H}_z = 0$. To show that $\tilde{E}_z = 0$, let us consider the z component of Eq. (7.6d),

$$\nabla \times \tilde{\mathbf{H}} = j\omega\epsilon_c\tilde{\mathbf{E}}.$$

$$\hat{\mathbf{z}}\left(\frac{\partial \tilde{H}_y}{\partial x} - \frac{\partial \tilde{H}_x}{\partial y}\right) = \hat{\mathbf{z}}\,j\omega\epsilon\tilde{E}_z. \qquad (7.24)$$

Since $\partial\tilde{H}_y/\partial x = \partial\tilde{H}_x/\partial y = 0$, it follows that $\tilde{E}_z = 0$. A similar examination involving Eq. (7.6b) reveals that $\tilde{H}_z = 0$.

Uniform Plane Wave Solution

$$\frac{d^2 \widetilde{E}_x}{dz^2} + k^2 \widetilde{E}_x = 0.$$

-z propagation

General Form of the Solution:

$$\widetilde{E}_x(z) = \widetilde{E}_x^+(z) + \widetilde{E}_x^-(z) = E_{x0}^+ e^{-jkz} + E_{x0}^- e^{jkz}$$

+z propagation

For a wave travelling along +z only:

$$\widetilde{\mathbf{E}}(z) = \hat{\mathbf{x}}\widetilde{E}_x^+(z) = \hat{\mathbf{x}} E_{x0}^+ e^{-jkz}$$

Assume for the time being that $\widetilde{\mathbf{E}}$ only has a component along x (i.e., $\widetilde{E}_y = 0$) and that $\widetilde{E}_x$ is associated with a wave traveling in the $+z$ direction only (i.e., $E_{x0}^- = 0$). Under these conditions,

$$\nabla \times \widetilde{\mathbf{E}} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \widetilde{E}_x^+(z) & 0 & 0 \end{vmatrix}$$

$$= -j\omega\mu(\hat{\mathbf{x}}\widetilde{H}_x + \hat{\mathbf{y}}\widetilde{H}_y + \hat{\mathbf{z}}\widetilde{H}_z).$$

Application of $\nabla \times \widetilde{\mathbf{E}} = -j\omega\mu\widetilde{\mathbf{H}}$ yields:

$$\widetilde{H}_y(z) = \frac{k}{\omega\mu} E_{x0}^+ e^{-jkz} = H_{y0}^+ e^{-jkz}$$

Proof of TEM Nature

Fields do not change with x and y $\Rightarrow$ $\dfrac{d^2\widetilde{E}_x}{dz^2} + k^2\widetilde{E}_x = 0.$

Fields do not change with x and z $\Rightarrow$ $\dfrac{d^2\tilde{E}_x}{dy^2} + k^2\tilde{E}_x = 0$

Fields do not change with y and z $\Rightarrow$ $\dfrac{d^2\tilde{E}_x}{dx^2} + k^2\tilde{E}_x = 0$

$$\tilde{E}_x(x) = E_x^+ e^{-jkx} + E_x^- e^{jkx}$$

$$E_x = H_x = 0$$

$$\nabla \times \widetilde{\mathbf{E}} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \widetilde{E}_x^+(x) & 0 & 0 \end{vmatrix}$$

$$= -j\omega\mu(\hat{\mathbf{x}}\widetilde{H}_x + \hat{\mathbf{y}}\widetilde{H}_y + \hat{\mathbf{z}}\widetilde{H}_z)$$

$\Rightarrow$ $H = 0$

Intrinsic Impedance

$$\eta = \frac{E_x^+}{H_y^+}$$ Intrinsic impedance of the medium or wave impedance

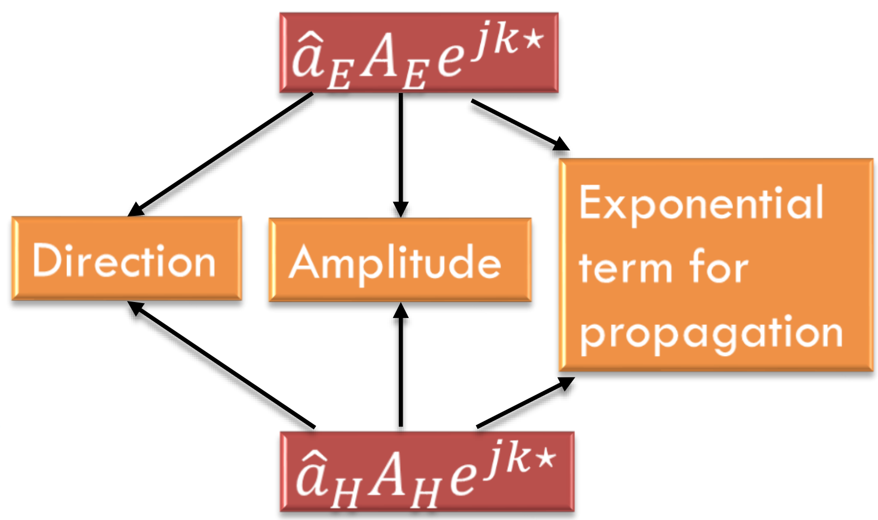
$$\widetilde{H}_y(z) = \frac{k}{\omega\mu} E_{x0}^+ e^{-jkz}$$

$$\boxed{\eta = \frac{\omega\mu}{k} = \frac{\omega\mu}{\omega\sqrt{\mu\varepsilon}} = \sqrt{\frac{\mu}{\varepsilon}} \qquad (\Omega)}$$

Summary: This is a plane wave with

$$\widetilde{\mathbf{E}}(z) = \hat{\mathbf{x}}\widetilde{E}_x^+(z) = \hat{\mathbf{x}}E_{x0}^+ e^{-jkz},$$

$$\widetilde{\mathbf{H}}(z) = \hat{\mathbf{y}}\frac{\widetilde{E}_x^+(z)}{\eta} = \hat{\mathbf{y}}\frac{E_{x0}^+}{\eta} e^{-jkz}.$$



Time-Domain Solution

In the general case, E_{x0}^+ is a complex quantity with magnitude $|E_{x0}^+|$ and phase angle ϕ^+. That is,

$$E_{x0}^+ = |E_{x0}^+| e^{j\phi^+}. \tag{7.33}$$

The instantaneous electric and magnetic fields therefore are

$$\begin{aligned} \mathbf{E}(z,t) &= \mathfrak{Re}\left[\widetilde{\mathbf{E}}(z)\, e^{j\omega t}\right] \\ &= \hat{\mathbf{x}} |E_{x0}^+| \cos(\omega t - kz + \phi^+) \quad \text{(V/m)}, \end{aligned} \tag{7.34a}$$

and

$$\begin{aligned} \mathbf{H}(z,t) &= \mathfrak{Re}\left[\widetilde{\mathbf{H}}(z)\, e^{j\omega t}\right] \\ &= \hat{\mathbf{y}} \frac{|E_{x0}^+|}{\eta} \cos(\omega t - kz + \phi^+) \quad \text{(A/m)}. \end{aligned} \tag{7.34b}$$

Wave's Phase Velocity

$$u_p = \frac{\omega}{k} = \frac{\omega}{\omega\sqrt{\mu\varepsilon}} = \frac{1}{\sqrt{\mu\varepsilon}} \qquad \text{(m/s)}, \qquad (7.35)$$

and its wavelength is

$$\lambda = \frac{2\pi}{k} = \frac{u_p}{f} \qquad \text{(m)}. \qquad (7.36)$$

In vacuum, $\varepsilon = \varepsilon_0$ and $\mu = \mu_0$, and the phase velocity u_p and the intrinsic impedance η given by Eq. (7.31) are

$$u_p = c = \frac{1}{\sqrt{\mu_0\varepsilon_0}} = 3 \times 10^8 \qquad \text{(m/s)}, \qquad (7.37)$$

$$\eta = \eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} = 377\ (\Omega) \approx 120\pi \quad (\Omega), \qquad (7.38)$$

Example 7-1: EM Plane Wave in Air

This example is analogous to the "Sound Wave in Water" problem given by Example 1-1.

The electric field of a 1-MHz plane wave traveling in the $+z$-direction in air points along the x-direction. If this field reaches a peak value of 1.2π (mV/m) at $t = 0$ and $z = 50$ m, obtain expressions for $\mathbf{E}(z, t)$ and $\mathbf{H}(z, t)$, and then plot them as a function of z at $t = 0$.

Solution: At $f = 1$ MHz, the wavelength in air is

$$\lambda = \frac{c}{f} = \frac{3 \times 10^8}{1 \times 10^6} = 300 \text{ m},$$

and the corresponding wavenumber is $k = (2\pi/300)$ (rad/m). The general expression for an x-directed electric field traveling in the $+z$-direction is given by Eq. (7.34a) as

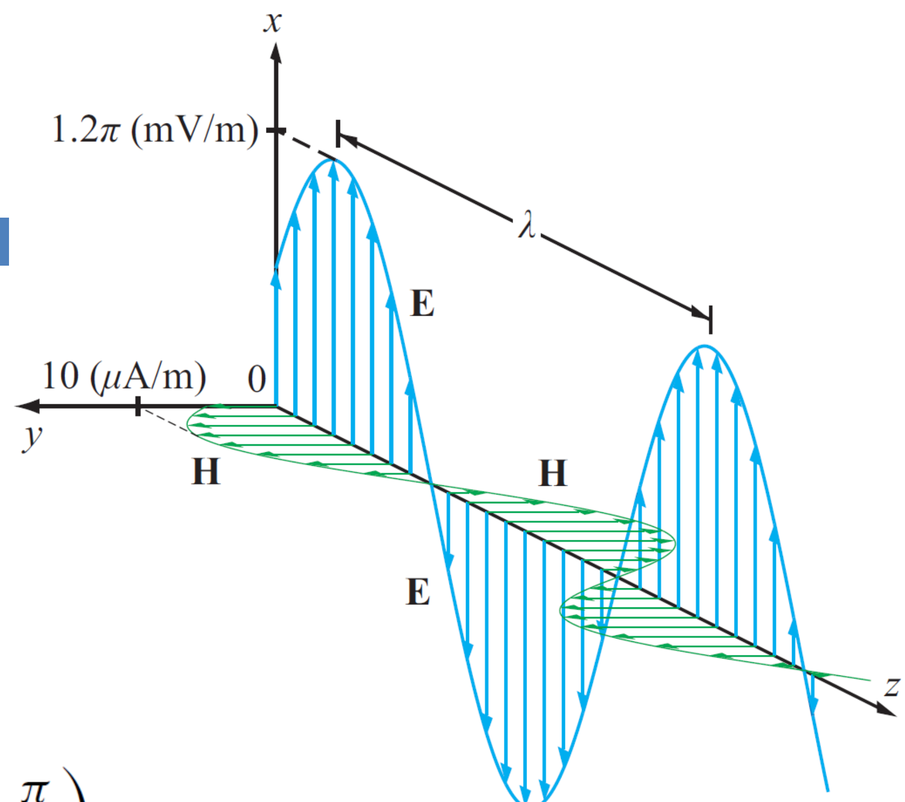
$$\begin{aligned}\mathbf{E}(z, t) &= \hat{\mathbf{x}}|E_{x0}^+| \cos(\omega t - kz + \phi^+)\\ &= \hat{\mathbf{x}}\, 1.2\pi \cos\left(2\pi \times 10^6 t - \frac{2\pi z}{300} + \phi^+\right) \quad \text{(mV/m)}.\end{aligned}$$

The field $\mathbf{E}(z, t)$ is maximum when the argument of the cosine function equals zero or a multiple of 2π. At $t = 0$ and $z = 50$ m, this condition yields

$$-\frac{2\pi \times 50}{300} + \phi^+ = 0 \qquad \text{or} \qquad \phi^+ = \frac{\pi}{3}.$$

Cont.

Example 7-1 cont.



Hence,

$$\mathbf{E}(z,t) = \hat{\mathbf{x}}\, 1.2\pi \cos\left(2\pi \times 10^6 t - \frac{2\pi z}{300} + \frac{\pi}{3}\right) \quad \text{(mV/m)},$$

and from Eq. (7.34b) we have

$$\mathbf{H}(z,t) = \hat{\mathbf{y}}\, \frac{E(z,t)}{\eta_0}$$
$$= \hat{\mathbf{y}}\, 10 \cos\left(2\pi \times 10^6 t - \frac{2\pi z}{300} + \frac{\pi}{3}\right) \quad (\mu\text{A/m}),$$

General Plane Wave

+z propagation e^{-jkz} Wave number vector $\vec{k} = k\hat{z}$ $e^{-jkz} = e^{-j\vec{k}\cdot z\hat{z}}$

General-direction propagation $e^{-j\vec{k}\cdot\vec{r}} = e^{-j\left(k_x x + k_y y + k_z z\right)}$

$$\vec{k} = k_x\hat{x} + k_y\hat{y} + k_z\hat{z} = k\hat{k} \qquad \vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$$

$$\vec{E} = \vec{E}_0 e^{-j\vec{k}\cdot\vec{r}} = (A\hat{x} + B\hat{y} + C\hat{z})e^{-j\vec{k}\cdot\vec{r}}$$

Directional Relation Between **E** and **H**

$$\nabla \times \vec{E} = -j\omega\mu\vec{H}$$

$$\nabla \times (\psi \mathbf{A}) = \nabla\psi \times \mathbf{A} + \psi \nabla \times \mathbf{A}$$

$$\nabla \times \vec{E}_0 = 0$$

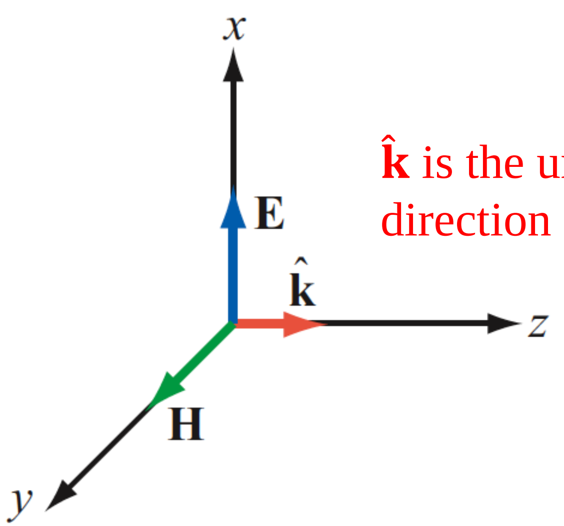
$$\vec{H} = \frac{j}{\omega\mu}\nabla \times \vec{E} = \frac{j}{\omega\mu}\nabla \times \left(\vec{E}_0 e^{-j\vec{k}\cdot\vec{r}}\right)$$

$$= \frac{-j}{\omega\mu}\vec{E}_0 \times \nabla e^{-j\vec{k}\cdot\vec{r}} = \frac{-j}{\omega\mu}\vec{E}_0 \times \left(-j\vec{k}e^{-j\vec{k}\cdot\vec{r}}\right)$$

$$= \frac{\vec{k}}{\omega\mu} \times \vec{E}_0 e^{-j\vec{k}\cdot\vec{r}} = \frac{k}{\omega\mu}\hat{k} \times \vec{E}_0 e^{-j\vec{k}\cdot\vec{r}}$$

$$= \frac{1}{\eta}\hat{k} \times \vec{E}_0 e^{-j\vec{k}\cdot\vec{r}} = \frac{1}{\eta}\hat{k} \times \vec{E}$$

Cont.



$\hat{\mathbf{k}}$ is the unit vector in the direction of propagation

Figure 7-4: A transverse electromagnetic (TEM) wave propagating in the direction $\hat{\mathbf{k}} = \hat{\mathbf{z}}$. For all TEM waves, $\hat{\mathbf{k}}$ is parallel to $\mathbf{E} \times \mathbf{H}$.

For Any TEM Wave

$$\widetilde{\mathbf{H}} = \frac{1}{\eta}\,\hat{\mathbf{k}} \times \widetilde{\mathbf{E}}, \qquad (7.39a)$$

$$\widetilde{\mathbf{E}} = -\eta\,\hat{\mathbf{k}} \times \widetilde{\mathbf{H}}. \qquad (7.39b)$$

$$\hat{k} \times \eta\widetilde{\mathbf{H}} = \hat{k} \times \left(\hat{k} \times \widetilde{\mathbf{E}}\right)$$

$$\eta\hat{k} \times \widetilde{\mathbf{H}} = \left(\hat{k} \cdot \widetilde{\mathbf{E}}\right)\hat{k} - \left(\hat{k} \cdot \hat{k}\right)\widetilde{\mathbf{E}}$$

$$\eta\hat{k} \times \widetilde{\mathbf{H}} = -\widetilde{\mathbf{E}}$$

The following right-hand rule applies: when we rotate the four fingers of the right hand from the direction of **E** *toward that of* **H**, *the thumb points in the direction of wave travel,* $\hat{\mathbf{k}}$.

Wave decomposition

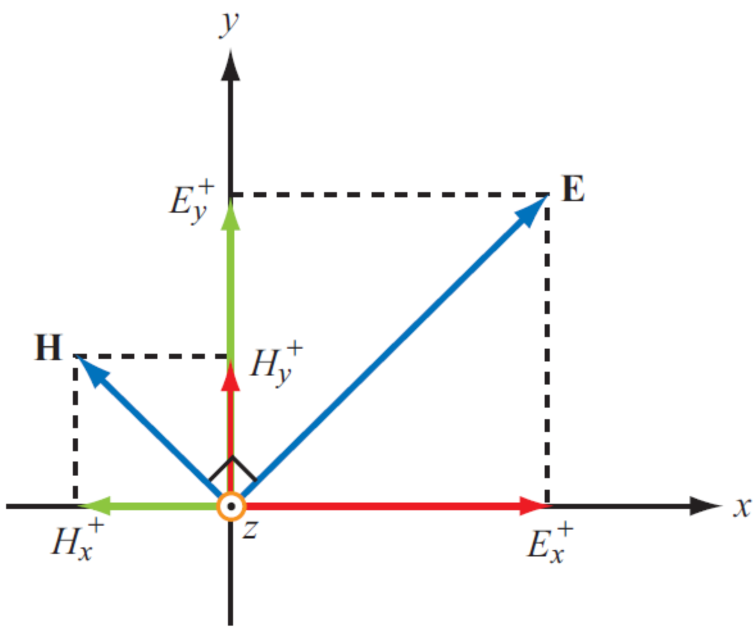


Figure 7-6: The wave (**E**, **H**) is equivalent to the sum of tw waves, one with fields (E_x^+, H_y^+) and another with (E_y^+, H_x^+ with both traveling in the $+z$-direction.

In general, a uniform plane wave traveling in the $+z$-direction may have both x- and y-components, in which case $\widetilde{\mathbf{E}}$ is given by

$$\widetilde{\mathbf{E}} = \hat{\mathbf{x}}\, \widetilde{E}_x^+(z) + \hat{\mathbf{y}}\, \widetilde{E}_y^+(z), \tag{7.43a}$$

and the associated magnetic field is

$$\widetilde{\mathbf{H}} = \hat{\mathbf{x}}\, \widetilde{H}_x^+(z) + \hat{\mathbf{y}}\, \widetilde{H}_y^+(z). \tag{7.43b}$$

Application of Eq. (7.39a) gives

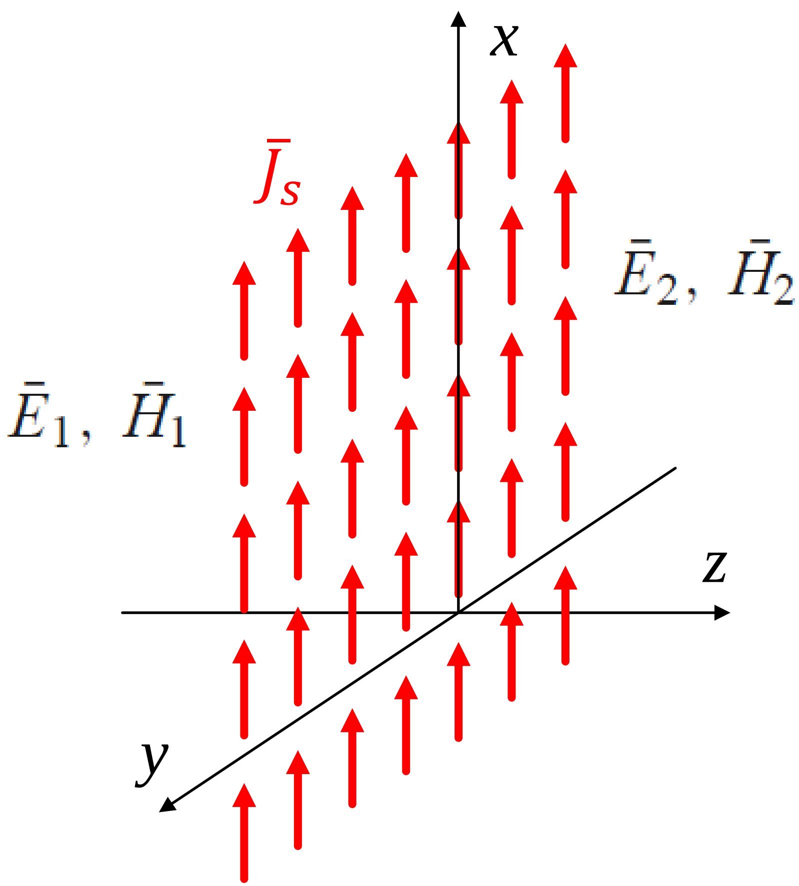
$$\widetilde{\mathbf{H}} = \frac{1}{\eta}\hat{\mathbf{z}} \times \widetilde{\mathbf{E}} = -\hat{\mathbf{x}}\frac{\widetilde{E}_y^+(z)}{\eta} + \hat{\mathbf{y}}\frac{\widetilde{E}_x^+(z)}{\eta}. \tag{7.44}$$

By equating Eq. (7.43b) to Eq. (7.44), we have

$$\widetilde{H}_x^+(z) = -\frac{\widetilde{E}_y^+(z)}{\eta}, \qquad \widetilde{H}_y^+(z) = \frac{\widetilde{E}_x^+(z)}{\eta}. \tag{7.45}$$

Example

- An infinite sheet of surface current can be considered as a source for plane waves. If an electric surface current density $\bar{J}_s = J_0\hat{x}$ exists on the $z = 0$ plane in free-space, find the resulting fields by assuming plane waves on either side of the current sheet and enforcing boundary conditions.



- Since source does not vary with x and y, the field does not either. So the wave only has z variation, propagating from the source to $+z$ and $-z$ directions
- The boundary conditions at $z = 0$ are

$$\hat{n} \times (\bar{E}_2 - \bar{E}_1) = \hat{z} \times (\bar{E}_2 - \bar{E}_1) = 0,$$

$$\hat{n} \times (\bar{H}_2 - \bar{H}_1) = \hat{z} \times (\bar{H}_2 - \bar{H}_1) = J_0\hat{x}$$

- H must be in the y direction and E must be in the x direction

Cont.

- Then the fields can be written in the following form with A and B are arbitrary numbers representing field amplitudes in the $z < 0$ and $z > 0$ region

$$\text{for } z < 0, \qquad \bar{E}_1 = \hat{x} A \eta_0 e^{jk_0 z},$$
$$\bar{H}_1 = -\hat{y} A e^{jk_0 z},$$
$$\text{for } z > 0, \qquad \bar{E}_2 = \hat{x} B \eta_0 e^{-jk_0 z},$$
$$\bar{H}_2 = \hat{y} B e^{-jk_0 z},$$

- With the first boundary condition at $z = 0$ (E_x continuous), we get $A = B$
- Use the second boundary condition at $z = 0$, we get
  $-B - A = J_0$
  So $A = B = -J_0/2$

Wave Polarization

*The **polarization** of a uniform plane wave describes the locus traced by the tip of the* **E** *vector (in the plane orthogonal to the direction of propagation) at a given point in space as a function of time.*

Plane wave propagating along +z :

$$\widetilde{\mathbf{E}}(z) = \hat{\mathbf{x}}\widetilde{E}_x(z) + \hat{\mathbf{y}}\widetilde{E}_y(z),$$

with

$$\widetilde{E}_x(z) = \boxed{E_{x0}}\, e^{-jkz},$$

$$\widetilde{E}_y(z) = \boxed{E_{y0}}\, e^{-jkz},$$

Generally complex numbers

If: $E_{x0} = a_x,$

$E_{y0} = a_y e^{j\delta},$

Phase difference

then

$$\widetilde{\mathbf{E}}(z) = (\hat{\mathbf{x}}a_x + \hat{\mathbf{y}}a_y e^{j\delta})e^{-jkz},$$

$$a_x \geq 0 \qquad a_y \geq 0$$

and the corresponding instantaneous field is

$$\begin{aligned}\mathbf{E}(z,t) &= \mathfrak{Re}\left[\widetilde{\mathbf{E}}(z)\, e^{j\omega t}\right] \\ &= \hat{\mathbf{x}}a_x \cos(\omega t - kz) \\ &\quad + \hat{\mathbf{y}}a_y \cos(\omega t - kz + \delta).\end{aligned}$$

Polarization State

Polarization state describes the trace of **E** as a function of time at a fixed z

Magnitude of E

$$|\mathbf{E}(z,t)| = [E_x^2(z,t) + E_y^2(z,t)]^{1/2}$$
$$= [a_x^2 \cos^2(\omega t - kz) + a_y^2 \cos^2(\omega t - kz + \delta)]^{1/2}$$

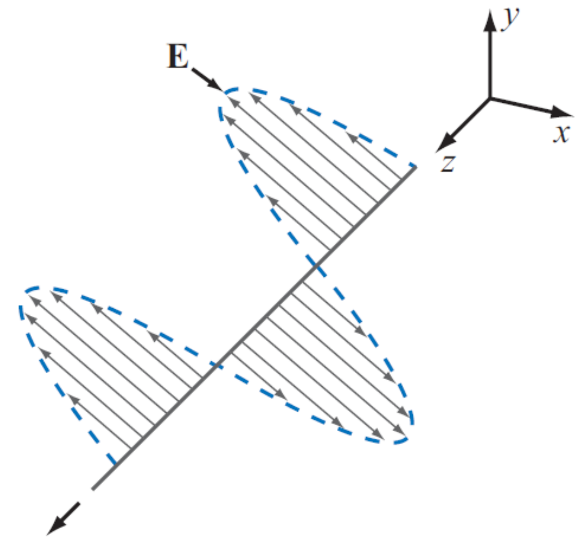
Inclination Angle

$$\psi(z,t) = \tan^{-1}\left(\frac{E_y(z,t)}{E_x(z,t)}\right)$$

Linear Polarization:

$\delta = 0$ or $\delta = \pi$

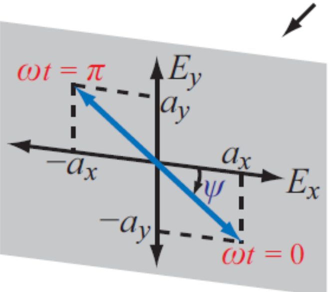
A wave is said to be linearly polarized if for a fixed z, the tip of $\mathbf{E}(z, t)$ traces a straight line segment as a function of time. This happens when $E_x(z, t)$ and $E_y(z, t)$ are ***in-phase*** *(i.e., $\delta = 0$) or* ***out-of-phase*** *($\delta = \pi$).*



Under these conditions Eq. (7.50) simplifies to

$$\mathbf{E}(0, t) = (\hat{\mathbf{x}} a_x + \hat{\mathbf{y}} a_y) \cos(\omega t - kz) \quad \text{(in-phase)}, \qquad (7.53a)$$

$$\mathbf{E}(0, t) = (\hat{\mathbf{x}} a_x - \hat{\mathbf{y}} a_y) \cos(\omega t - kz) \quad \text{(out-of-phase)}. \qquad (7.53b)$$



E traces a line(in blue) as the wave traverses a fixed plane

Let us examine the out-of-phase case. The field's magnitude is

$$|\mathbf{E}(z, t)| = [a_x^2 + a_y^2]^{1/2} |\cos(\omega t - kz)|, \qquad (7.54a)$$

and the inclination angle is

$$\psi = \tan^{-1}\left(\frac{-a_y}{a_x}\right) \quad \text{(out-of-phase)}. \qquad (7.54b)$$

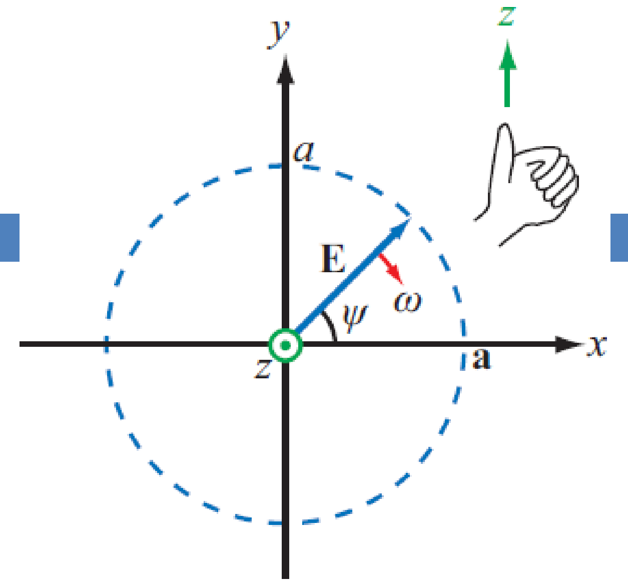
Special cases

If $a_y = 0$, then $\psi = 0°$ or $180°$, and the wave is x-polarized; conversely, if $a_x = 0$, then $\psi = 90°$ or $-90°$, and the wave is y-polarized.

Circular Polarization

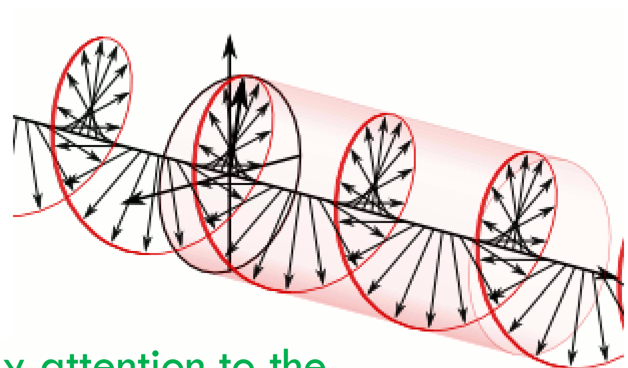
Polarization Handedness

Polarization handedness is defined in terms of the rotation of **E** *as a function of time in a fixed plane orthogonal to the direction of propagation, which is opposite of the direction of rotation of* **E** *as a function of distance at a fixed point in time.*

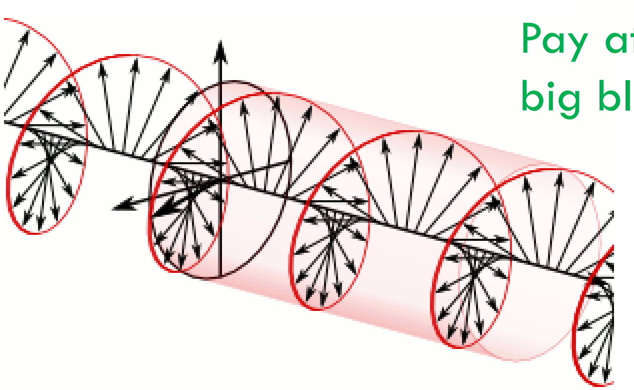


(a) LHC polarization

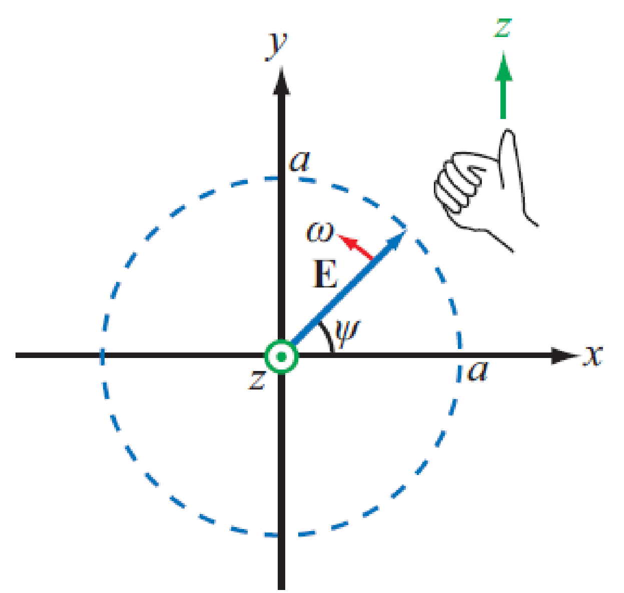
Left-hand



Pay attention to the big black arrow



Right-hand



(b) RHC polarization

LH Circular Polarization:

$$a_x = a_y = a \text{ and } \delta = \pi/2$$

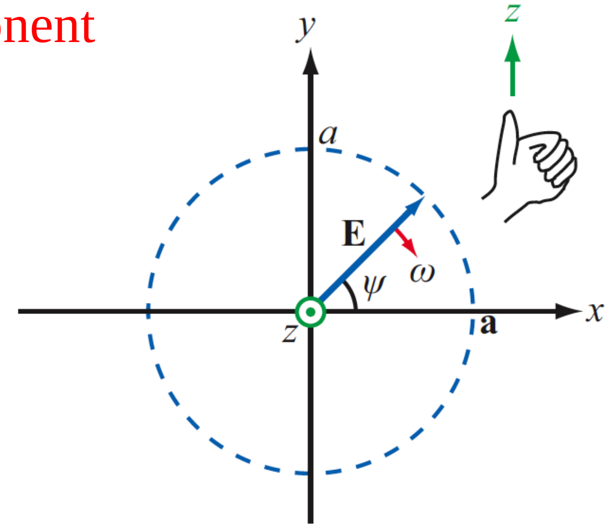
y component is 90° ahead of *x* component

(a) Left-Hand Circular (LHC) Polarization

For $a_x = a_y = a$ and $\delta = \pi/2$, Eqs. (7.49) and (7.50) become

$$\widetilde{\mathbf{E}}(z) = (\hat{\mathbf{x}}a + \hat{\mathbf{y}}ae^{j\pi/2})e^{-jkz} = a(\hat{\mathbf{x}} + j\hat{\mathbf{y}})e^{-jkz},$$

$$\mathbf{E}(z,t) = \mathfrak{Re}\left[\widetilde{\mathbf{E}}(z)\, e^{j\omega t}\right] = \hat{\mathbf{x}}a\cos(\omega t - kz) + \hat{\mathbf{y}}a\cos(\omega t - kz + \pi/2) = \hat{\mathbf{x}}a\cos(\omega t - kz) - \hat{\mathbf{y}}a\sin(\omega t - kz).$$



(a) LHC polarization

phase of *x* component needs to take T/4 to become the current phase of *y* component

The corresponding field magnitude and inclination angle are

$$|\mathbf{E}(z,t)| = \left[E_x^2(z,t) + E_y^2(z,t)\right]^{1/2} = [a^2\cos^2(\omega t - kz) + a^2\sin^2(\omega t - kz)]^{1/2} = \boxed{a},$$

Constant magnitude

$$\psi(z,t) = \tan^{-1}\left[\frac{E_y(z,t)}{E_x(z,t)}\right] = \tan^{-1}\left[\frac{-a\sin(\omega t - kz)}{a\cos(\omega t - kz)}\right] = \boxed{-(\omega t - kz)}.$$

RH Circular Polarization:

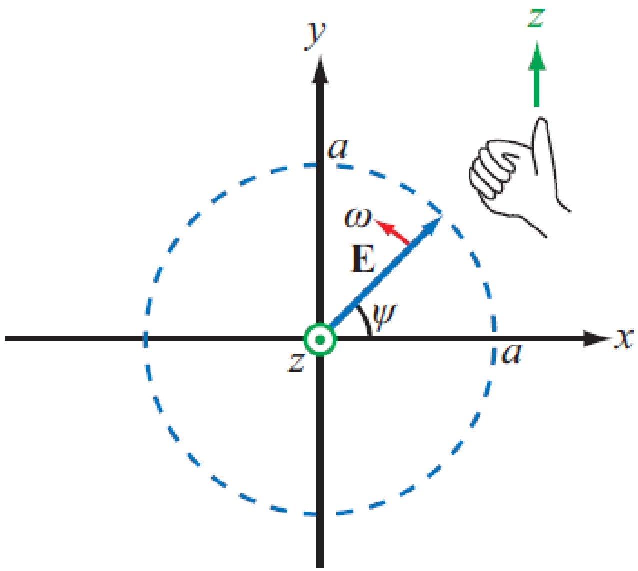
$a_x = a_y = a$ and $\delta = -\pi/2$

x component is 90° ahead of y component

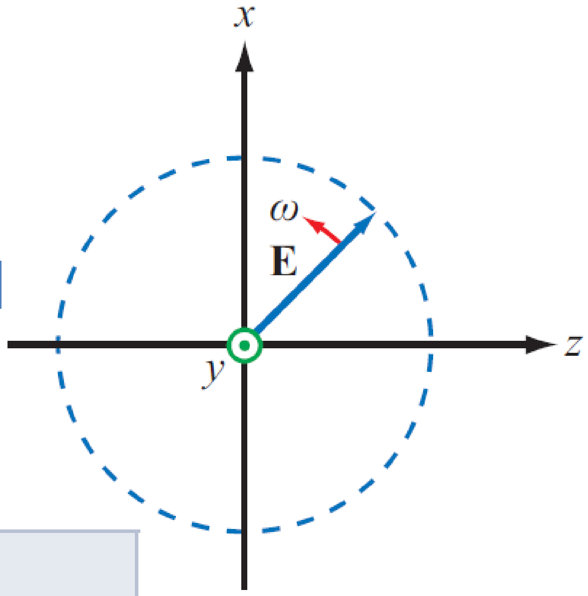
(b) Right-Hand Circular (RHC) Polarization

For $a_x = a_y = a$ and $\delta = -\pi/2$, we have

$$\boxed{|\mathbf{E}(z,t)| = a, \qquad \psi = (\omega t - kz).}$$



(b) RHC polarization



Example 7-2: RHC Polarized Wave

An RHC polarized plane wave with electric field magnitude of 3 (mV/m) is traveling in the $+y$-direction in a dielectric medium with $\varepsilon = 4\varepsilon_0$, $\mu = \mu_0$, and $\sigma = 0$. If the frequency is 100 MHz, obtain expressions for $\mathbf{E}(y, t)$ and $\mathbf{H}(y, t)$.

Solution: Since the wave is traveling in the $+y$-direction, its field must have components along the x- and z-directions. The rotation of $\mathbf{E}(y, t)$ is depicted in Fig. 7-10, where $\hat{\mathbf{y}}$ is out of the page. By comparison with the RHC polarized wave shown in Fig. 7-8(b), we assign the z-component of $\widetilde{\mathbf{E}}(y)$ a phase angle of zero and the x-component a phase shift of $\delta = -\pi/2$.

Cont.

Example 7-2 cont.

Wave with electric field magnitude of 3 (mV/m) traveling in the +*y*-*direction*

With $\omega = 2\pi f = 2\pi \times 10^8$ (rad/s), the wavenumber k is

$$k = \frac{\omega\sqrt{\varepsilon_r}}{c} = \frac{2\pi \times 10^8\sqrt{4}}{3 \times 10^8} = \frac{4}{3}\pi \qquad \text{(rad/m)},$$

$$\begin{aligned}\widetilde{\mathbf{E}}(y) &= \hat{\mathbf{x}}\widetilde{E}_x + \hat{\mathbf{z}}\widetilde{E}_z \\ &= \hat{\mathbf{x}}ae^{-j\pi/2}e^{-jky} + \hat{\mathbf{z}}ae^{-jky} \\ &= (-\hat{\mathbf{x}}j + \hat{\mathbf{z}})3e^{-jky} \qquad \text{(mV/m)},\end{aligned}$$

and application of (7.39a) gives

$$\begin{aligned}\widetilde{\mathbf{H}}(y) &= \frac{1}{\eta}\,\hat{\mathbf{y}} \times \widetilde{\mathbf{E}}(y) \\ &= \frac{1}{\eta}\,\hat{\mathbf{y}} \times (-\hat{\mathbf{x}}j + \hat{\mathbf{z}})3e^{-jky} \\ &= \frac{3}{\eta}(\hat{\mathbf{z}}j + \hat{\mathbf{x}})e^{-jky} \qquad \text{(mA/m)}.\end{aligned}$$

Cont.

Example 7-2 cont.

$$\eta = \frac{\eta_0}{\sqrt{\varepsilon_r}} \simeq \frac{120\pi}{\sqrt{4}} = 60\pi \qquad (\Omega).$$

The instantaneous fields $\mathbf{E}(y,t)$ and $\mathbf{H}(y,t)$ are

$$\begin{aligned}\mathbf{E}(y,t) &= \mathfrak{Re}\left[\widetilde{\mathbf{E}}(y)\, e^{j\omega t}\right] \\ &= \mathfrak{Re}\left[(-\hat{\mathbf{x}}j + \hat{\mathbf{z}})3e^{-jky}e^{j\omega t}\right] \\ &= 3[\hat{\mathbf{x}}\sin(\omega t - ky) + \hat{\mathbf{z}}\cos(\omega t - ky)] \qquad (\text{mV/m})\end{aligned}$$

and

$$\begin{aligned}\mathbf{H}(y,t) &= \mathfrak{Re}\left[\widetilde{\mathbf{H}}(y)\, e^{j\omega t}\right] \\ &= \mathfrak{Re}\left[\frac{3}{\eta}(\hat{\mathbf{z}}j + \hat{\mathbf{x}})e^{-jky}e^{j\omega t}\right] \\ &= \frac{1}{20\pi}[\hat{\mathbf{x}}\cos(\omega t - ky) - \hat{\mathbf{z}}\sin(\omega t - ky)] \ (\text{mA/m}).\end{aligned}$$

Elliptical Polarization: General Case

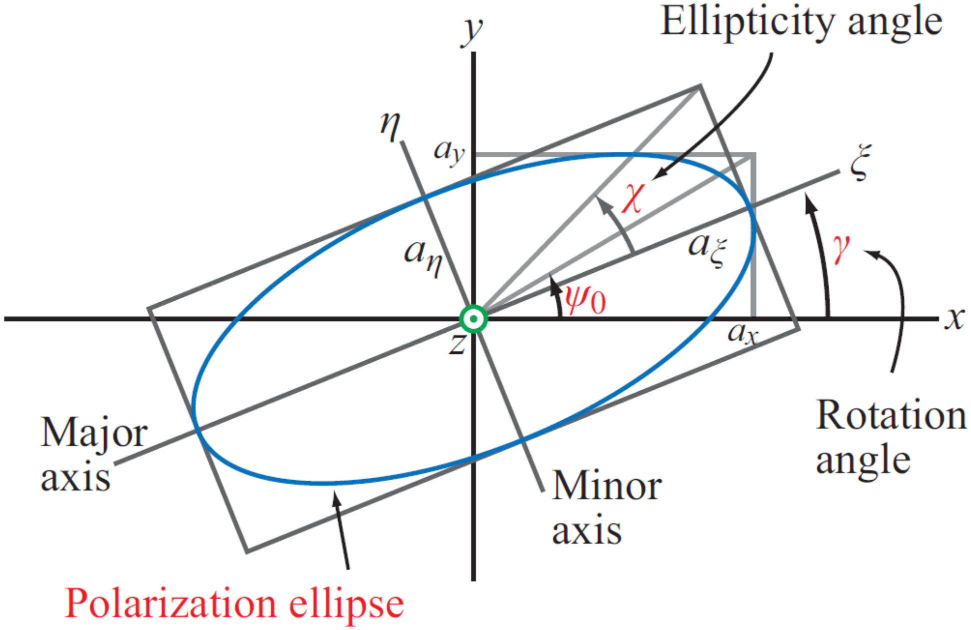
Linear and circular polarizations are special cases of elliptical polarization

$$\tan 2\gamma = (\tan 2\psi_0)\cos\delta \quad (-\pi/2 \le \gamma \le \pi/2),$$
$$\sin 2\chi = (\sin 2\psi_0)\sin\delta \quad (-\pi/4 \le \chi \le \pi/4),$$

where ψ_0 is an ***auxiliary angle*** defined by

$$\tan\psi_0 = \frac{a_y}{a_x} \qquad \left(0 \le \psi_0 \le \frac{\pi}{2}\right).$$

$$\gamma > 0 \text{ if } \cos\delta > 0,$$
$$\gamma < 0 \text{ if } \cos\delta < 0.$$



Positive values of χ, corresponding to $\sin\delta > 0$, are associated with left-handed rotation, and negative values of χ, corresponding to $\sin\delta < 0$, are associated with right-handed rotation.

Example 7-3: Polarization State

Determine the polarization state of a plane wave with electric field

$$\mathbf{E}(z,t) = \hat{\mathbf{x}}\,3\cos(\omega t - kz + 30^\circ) - \hat{\mathbf{y}}\,4\sin(\omega t - kz + 45^\circ) \qquad \text{(mV/m)}.$$

Solution: We begin by converting the second term to a cosine reference,

$$\begin{aligned}\mathbf{E} &= \hat{\mathbf{x}}\,3\cos(\omega t - kz + 30^\circ) \\ &\quad - \hat{\mathbf{y}}\,4\cos(\omega t - kz + 45^\circ - 90^\circ) \\ &= \hat{\mathbf{x}}\,3\cos(\omega t - kz + 30^\circ) - \hat{\mathbf{y}}\,4\cos(\omega t - kz - 45^\circ).\end{aligned}$$

The corresponding field phasor $\widetilde{\mathbf{E}}(z)$ is

$$\begin{aligned}\widetilde{\mathbf{E}}(z) &= \hat{\mathbf{x}}\,3e^{-jkz}e^{j30^\circ} - \hat{\mathbf{y}}\,4e^{-jkz}e^{-j45^\circ} \\ &= \hat{\mathbf{x}}\,3e^{-jkz}e^{j30^\circ} + \hat{\mathbf{y}}\,4e^{-jkz}e^{-j45^\circ}e^{j180^\circ} \\ &= \hat{\mathbf{x}}\,3e^{-jkz}e^{j30^\circ} + \hat{\mathbf{y}}\,4e^{-jkz}e^{j135^\circ},\end{aligned}$$

Cont.

Example 7-3 cont.

$$\psi_0 = \tan^{-1}\left(\frac{a_y}{a_x}\right) = \tan^{-1}\left(\frac{4}{3}\right) = 53.1^\circ.$$

From Eq. (7.59a),

$$\tan 2\gamma = (\tan 2\psi_0)\cos\delta = \tan 106.2^\circ \cos 105^\circ = 0.89,$$

which gives two solutions for γ, namely $\gamma = 20.8^\circ$ and $\gamma = -69.2^\circ$. Since $\cos\delta < 0$, the correct value of γ is -69.2°. From Eq. (7.59b),

$$\sin 2\chi = (\sin 2\psi_0)\sin\delta = \sin 106.2^\circ \sin 105^\circ = 0.93 \quad \text{or} \quad \chi = 34.0^\circ.$$

The magnitude of χ indicates that the wave is elliptically polarized and its positive polarity specifies its rotation as left handed.

Lossy Media

For a uniform plane wave with electric field $\widetilde{\mathbf{E}} = \hat{\mathbf{x}}\, \widetilde{E}_x(z)$ traveling along the z-direction, the wave equation given by Eq. (7.61) reduces to

$$\frac{d^2\, \widetilde{E}_x(z)}{dz^2} - \gamma^2\, \widetilde{E}_x(z) = 0. \qquad (7.67)$$

with

$$\gamma^2 = -\omega^2 \mu \varepsilon_c = -\omega^2 \mu (\varepsilon' - j\varepsilon''), \qquad (7.62)$$

where $\varepsilon' = \varepsilon$ and $\varepsilon'' = \sigma/\omega$. Since γ is complex, we express it as

$$\gamma = \alpha + j\beta, \qquad (7.63)$$

where α is the medium's ***attenuation constant*** and β its ***phase constant***. By replacing γ with $(\alpha + j\beta)$ in Eq. (7.62), we obtain

$$\begin{aligned}(\alpha + j\beta)^2 &= (\alpha^2 - \beta^2) + j2\alpha\beta \\ &= -\omega^2 \mu \varepsilon' + j\omega^2 \mu \varepsilon''. \qquad (7.64)\end{aligned}$$

Lossless

$$\gamma = jk = j\beta$$

Cont.

Lossy Media

The rules of complex algebra require the real and imaginary parts on one side of an equation to equal the real and imaginary parts on the other side. Hence,

$$\alpha^2 - \beta^2 = -\omega^2 \mu \varepsilon', \qquad (7.65a)$$

$$2\alpha\beta = \omega^2 \mu \varepsilon''. \qquad (7.65b)$$

Solving these two equations for α and β gives

$$\alpha = \omega \left\{ \frac{\mu\varepsilon'}{2} \left[\sqrt{1 + \left(\frac{\varepsilon''}{\varepsilon'}\right)^2} - 1 \right] \right\}^{1/2} \quad \text{(Np/m)}, \qquad (7.66a)$$

$$\beta = \omega \left\{ \frac{\mu\varepsilon'}{2} \left[\sqrt{1 + \left(\frac{\varepsilon''}{\varepsilon'}\right)^2} + 1 \right] \right\}^{1/2} \quad \text{(rad/m)}. \qquad (7.66b)$$

Both are positive

Check when ε” = 0

Attenuation

E and H fields:

+z propagation

−z propagation

$$e^{\gamma z} = e^{\alpha z} e^{j\beta z}$$

$$\widetilde{\mathbf{E}}(z) = \hat{\mathbf{x}}\widetilde{E}_x(z) = \hat{\mathbf{x}}E_{x0}e^{-\gamma z} = \hat{\mathbf{x}}E_{x0}e^{-\alpha z}e^{-j\beta z}. \quad (7.68)$$

The associated magnetic field $\widetilde{\mathbf{H}}$ can be determined by applying Eq. (7.2b): $\nabla \times \widetilde{\mathbf{E}} = -j\omega\mu\widetilde{\mathbf{H}}$, or using Eq. (7.39a): $\widetilde{\mathbf{H}} = (\hat{\mathbf{k}} \times \widetilde{\mathbf{E}})/\eta_c$, where η_c is the ***intrinsic impedance of the lossy medium***. Both approaches give

$$\widetilde{\mathbf{H}}(z) = \hat{\mathbf{y}}\,\widetilde{H}_y(z) = \hat{\mathbf{y}}\,\frac{\widetilde{E}_x(z)}{\eta_c} = \hat{\mathbf{y}}\,\frac{E_{x0}}{\eta_c}e^{-\alpha z}e^{-j\beta z}, \quad (7.69)$$

where

$$\boxed{\eta_c = \sqrt{\frac{\mu}{\varepsilon_c}} = \sqrt{\frac{\mu}{\varepsilon'}}\left(1 - j\frac{\varepsilon''}{\varepsilon'}\right)^{-1/2} \quad (\Omega). \quad (7.70)}$$

Cont.

Attenuation

Magnitude of *E*

$$|\widetilde{E}_x(z)| = |E_{x0}e^{-\alpha z}e^{-j\beta z}| = |E_{x0}|e^{-\alpha z}$$

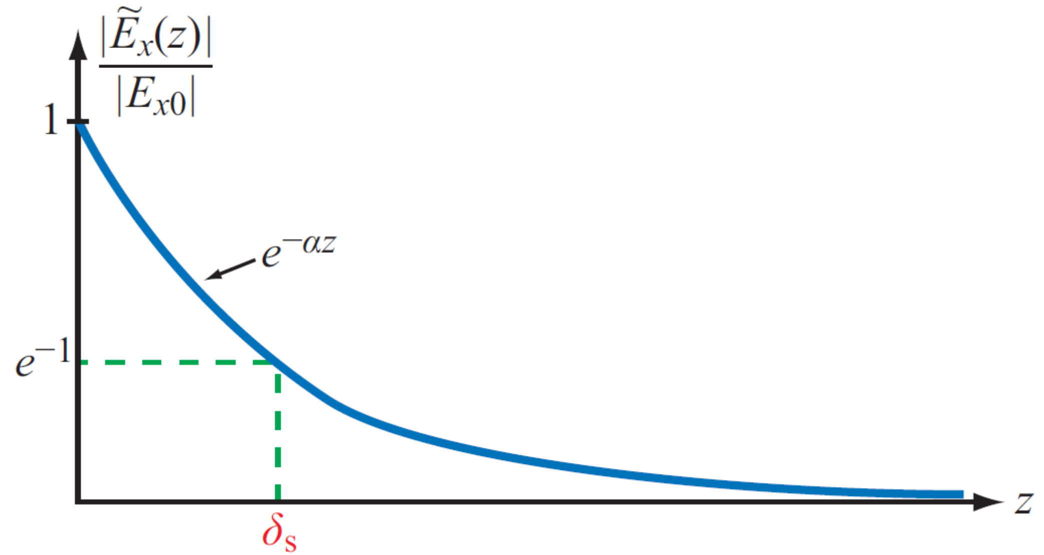


Figure 7-13: Attenuation of the magnitude of $\widetilde{E}_x(z)$ with distance z. The skin depth δ_s is the value of z at which $|\widetilde{E}_x(z)|/|E_{x0}| = e^{-1}$, or $z = \delta_s = 1/\alpha$.

Skin depth

$$\boxed{\delta_s = \frac{1}{\alpha} \qquad \text{(m)}, \qquad (7.72)}$$

the wave magnitude decreases by a factor of $e^{-1} \approx 0.37$ (Fig. 7-13). At depth $z = 3\delta_s$, the field magnitude is less than 5% of its initial value, and at $z = 5\delta_s$, it is less than 1%.

*This distance δ_s, called the **skin depth** of the medium, characterizes how deep an electromagnetic wave can penetrate into a conducting medium.*

Phase Velocity and Wavelength

Lossless

$$u_p = \frac{\omega}{k} = \frac{\omega}{\omega\sqrt{\mu\varepsilon}} = \frac{1}{\sqrt{\mu\varepsilon}} \quad \text{(m/s)},$$

$$\lambda = \frac{2\pi}{k} = \frac{u_p}{f} \quad \text{(m)}.$$

Lossy $\quad$ k is complex

$$u_p = \frac{\omega}{\beta} \qquad \lambda = \frac{2\pi}{\beta}$$

Low and High Frequency Approximations

Table 7-1: Expressions for α, β, η_c, u_p, and λ for various types of media.

	Any Medium	Lossless Medium ($\sigma = 0$)	Low-loss Medium ($\varepsilon''/\varepsilon' \ll 1$)	Good Conductor ($\varepsilon''/\varepsilon' \gg 1$)	Units
$\alpha =$	$\omega\left[\frac{\mu\varepsilon'}{2}\left[\sqrt{1+\left(\frac{\varepsilon''}{\varepsilon'}\right)^2}-1\right]\right]^{1/2}$	0	$\frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}}$	$\sqrt{\pi f \mu\sigma}$	(Np/m)
$\beta =$	$\omega\left[\frac{\mu\varepsilon'}{2}\left[\sqrt{1+\left(\frac{\varepsilon''}{\varepsilon'}\right)^2}+1\right]\right]^{1/2}$	$\omega\sqrt{\mu\varepsilon}$	$\omega\sqrt{\mu\varepsilon}$	$\sqrt{\pi f \mu\sigma}$	(rad/m)
$\eta_c =$	$\sqrt{\frac{\mu}{\varepsilon'}}\left(1-j\frac{\varepsilon''}{\varepsilon'}\right)^{-1/2}$	$\sqrt{\frac{\mu}{\varepsilon}}$	$\sqrt{\frac{\mu}{\varepsilon}}$	$\boxed{(1+j)\frac{\alpha}{\sigma}}$	(Ω)
$u_p =$	ω/β	$1/\sqrt{\mu\varepsilon}$	$1/\sqrt{\mu\varepsilon}$	$\sqrt{4\pi f/\mu\sigma}$	(m/s)
$\lambda =$	$2\pi/\beta = u_p/f$	u_p/f	u_p/f	u_p/f	(m)

Notes: $\varepsilon' = \varepsilon$; $\varepsilon'' = \sigma/\omega$; in free space, $\varepsilon = \varepsilon_0$, $\mu = \mu_0$; in practice, a material is considered a low-loss medium if $\varepsilon''/\varepsilon' = \sigma/\omega\varepsilon <$ 0.01 and a good conducting medium if $\varepsilon''/\varepsilon' >$ 100.

Example 7-4: Plane Wave in Seawater

A uniform plane wave is traveling in seawater. Assume that the x–y plane resides just below the sea surface and the wave travels in the $+z$-direction into the water. The constitutive parameters of seawater are $\varepsilon_r = 80$, $\mu_r = 1$, and $\sigma = 4$ S/m. If the magnetic field at $z = 0$ is $\mathbf{H}(0, t) = \hat{\mathbf{y}}\, 100 \cos(2\pi \times 10^3 t + 15^\circ)$ (mA/m),

(a) obtain expressions for $\mathbf{E}(z, t)$ and $\mathbf{H}(z, t)$, and

(b) determine the depth at which the magnitude of $\mathbf{E}$ is 1% of its value at $z = 0$.

Solution: (a) Since $\mathbf{H}$ is along $\hat{\mathbf{y}}$ and the propagation direction is $\hat{\mathbf{z}}$, $\mathbf{E}$ must be along $\hat{\mathbf{x}}$. Hence, the general expressions for the phasor fields are

$$\widetilde{\mathbf{E}}(z) = \hat{\mathbf{x}} E_{x0} e^{-\alpha z} e^{-j\beta z}, \qquad (7.78a)$$

$$\widetilde{\mathbf{H}}(z) = \hat{\mathbf{y}} \frac{E_{x0}}{\eta_c} e^{-\alpha z} e^{-j\beta z}. \qquad (7.78b)$$

Cont.

Example 7-4: Plane Wave in Seawater

$$\frac{\varepsilon''}{\varepsilon'} = \frac{\sigma}{\omega\varepsilon} = \frac{\sigma}{\omega\varepsilon_r\varepsilon_0} = \frac{4}{2\pi \times 10^3 \times 80 \times (10^{-9}/36\pi)} = 9 \times 10^5.$$

This qualifies seawater as a good conductor at 1 kHz and allows us to use the good-conductor expressions given in Table 7-1:

$$\alpha = \sqrt{\pi f \mu\sigma} = \sqrt{\pi \times 10^3 \times 4\pi \times 10^{-7} \times 4} = 0.126 \quad \text{(Np/m)}, \tag{7.79a}$$

$$\beta = \alpha = 0.126 \quad \text{(rad/m)}, \tag{7.79b}$$

$$\eta_c = (1 + j)\frac{\alpha}{\sigma} = (\sqrt{2}\, e^{j\pi/4})\frac{0.126}{4} = 0.044 e^{j\pi/4} \quad (\Omega). \tag{7.79c}$$

Cont.

Example 7-4: Plane Wave in Seawater

$$\mathbf{E}(z,t) = \mathfrak{Re}\left[\hat{\mathbf{x}}|E_{x0}|e^{j\phi_0}e^{-\alpha z}e^{-j\beta z}e^{j\omega t}\right]$$

$$= \hat{\mathbf{x}}|E_{x0}|e^{-0.126z}\cos(2\pi \times 10^3 t - 0.126z + \phi_0)$$

$$\text{(V/m)}, \qquad (7.80a)$$

$$\mathbf{H}(z,t) = \mathfrak{Re}\left[\hat{\mathbf{y}}\,\frac{|E_{x0}|e^{j\phi_0}}{0.044e^{j\pi/4}}e^{-\alpha z}e^{-j\beta z}e^{j\omega t}\right]$$

$$= \hat{\mathbf{y}}22.5|E_{x0}|e^{-0.126z}\cos(2\pi \times 10^3 t$$

$$- 0.126z + \phi_0 \boxed{- 45^\circ)} \qquad \text{(A/m)}. \qquad (7.80b)$$

Cont.

Example 7-4: Plane Wave in Seawater

At $z = 0$,

$$\mathbf{H}(0, t) = \hat{\mathbf{y}}\, 22.5 |E_{x0}| \cos(2\pi \times 10^3 t + \phi_0 - 45^\circ) \quad \text{(A/m)}. \tag{7.81}$$

By comparing Eq. (7.81) with the expression given in the problem statement,

$$\mathbf{H}(0, t) = \hat{\mathbf{y}}\, 100 \cos(2\pi \times 10^3 t + 15^\circ) \qquad \text{(mA/m)},$$

we deduce that

$$22.5 |E_{x0}| = 100 \times 10^{-3}$$

or

$$|E_{x0}| = 4.44 \qquad \text{(mV/m)},$$

and

$$\phi_0 - 45^\circ = 15^\circ \qquad \text{or} \qquad \phi_0 = 60^\circ.$$

Cont.

Example 7-4: Plane Wave in Seawater

Hence, the final expressions for $\mathbf{E}(z, t)$ and $\mathbf{H}(z, t)$ are

$$\mathbf{E}(z, t) = \hat{\mathbf{x}}\, 4.44 e^{-0.126z} \cos(2\pi \times 10^3 t - 0.126z + 60^\circ) \quad \text{(mV/m)}, \tag{7.82a}$$

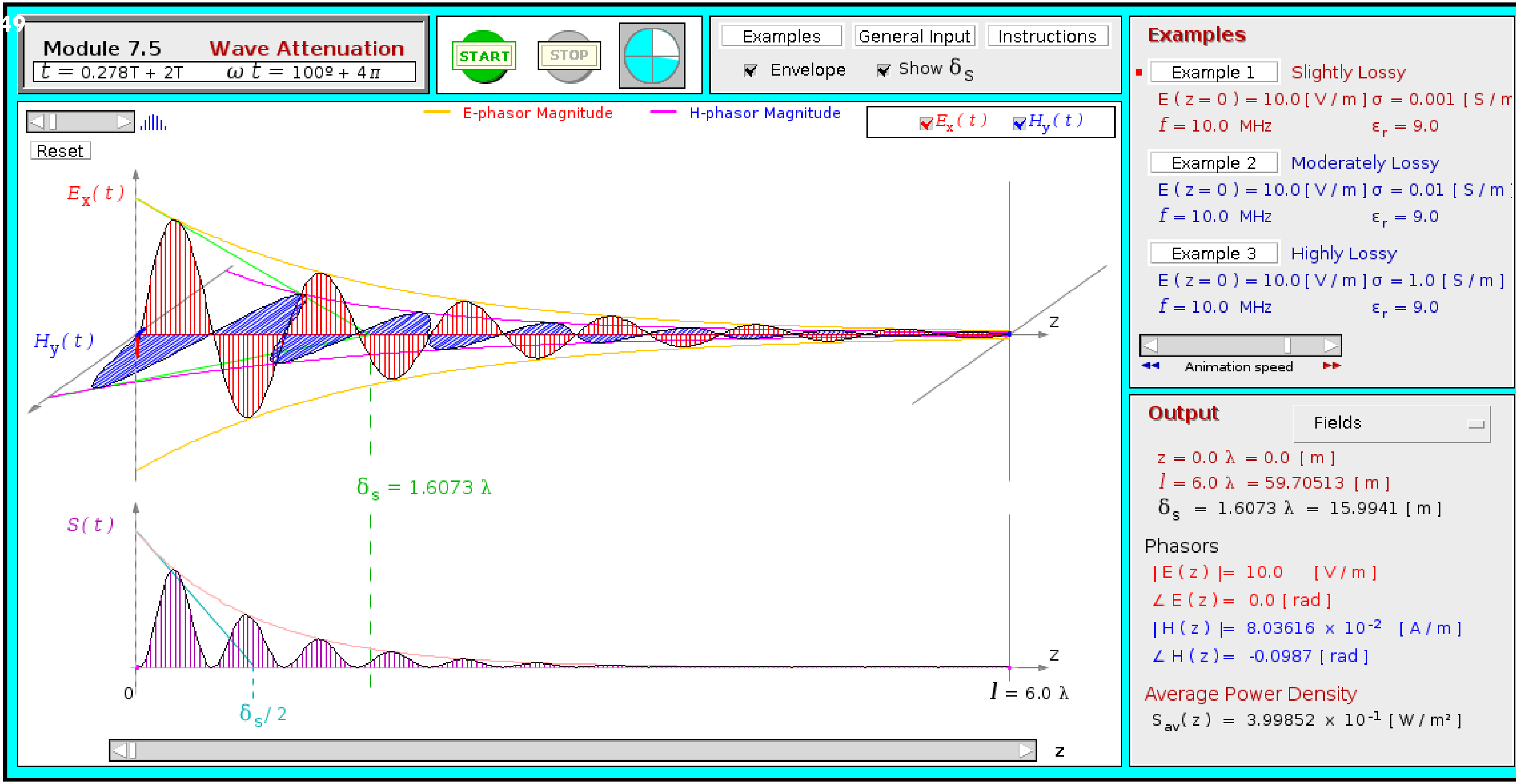
$$\mathbf{H}(z, t) = \hat{\mathbf{y}}\, 100 e^{-0.126z} \cos(2\pi \times 10^3 t - 0.126z + 15^\circ) \quad \text{(mA/m)}. \tag{7.82b}$$

(b) The depth at which the amplitude of $\mathbf{E}$ has decreased to 1% of its initial value at $z = 0$ is obtained from

$$0.01 = e^{-0.126z}$$

or

$$\boxed{z = \frac{\ln(0.01)}{-0.126} = 36.55 \text{ m} \approx 37 \text{ m}.}$$


Module 7.5 Wave Attenuation
t = 0.278T + 2T ω t = 100º + 4π
START
STOP
Examples
General Input
Instructions
Envelope
Show δs
E-phasor Magnitude
H-phasor Magnitude
Ex(t)
Hy(t)
Reset
Ex(t)
Hy(t)
z
δs = 1.6073 λ
S(t)
z
0
δs / 2
l = 6.0 λ
z
Examples
Example 1 Slightly Lossy
E (z = 0) = 10.0 [V / m] σ = 0.001 [S / m
f = 10.0 MHz εr = 9.0
Example 2 Moderately Lossy
E (z = 0) = 10.0 [V / m] σ = 0.01 [S / m]
f = 10.0 MHz εr = 9.0
Example 3 Highly Lossy
E (z = 0) = 10.0 [V / m] σ = 1.0 [S / m]
f = 10.0 MHz εr = 9.0
Animation speed
Output
Fields
z = 0.0 λ = 0.0 [m]
l = 6.0 λ = 59.70513 [m]
δs = 1.6073 λ = 15.9941 [m]
Phasors
| E (z) |= 10.0 [V / m]
∠ E (z) = 0.0 [rad]
| H (z) |= 8.03616 x 10^-2 [A / m]
∠ H (z) = -0.0987 [rad]
Average Power Density
Sav(z) = 3.99852 x 10^-1 [W / m²]

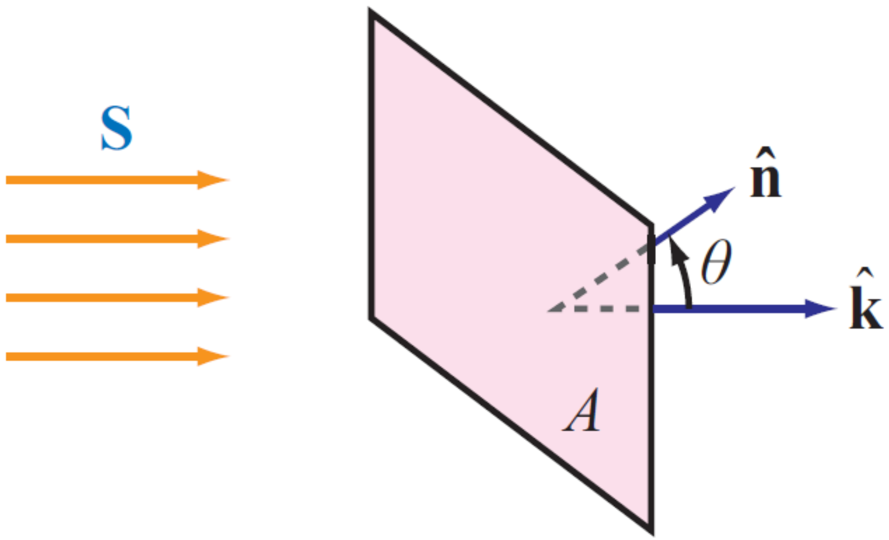
Power Density

Poynting vector:

$$\mathbf{S} = \mathbf{E} \times \mathbf{H} \qquad (\text{W/m}^2).$$

Total power intercepted by A:

$$P = \int_A \mathbf{S} \cdot \hat{\mathbf{n}}\, dA \qquad (\text{W}).$$



Time-average power density:

$$\mathbf{S}_{\text{av}} = \frac{1}{2} \mathfrak{Re}\left[\widetilde{\mathbf{E}} \times \widetilde{\mathbf{H}}^*\right] \qquad (\text{W/m}^2).$$

Plane Wave in Lossless Medium

For a plane wave with **E** field :

$$\widetilde{\mathbf{E}}(z) = \hat{\mathbf{x}}\,\widetilde{E}_x(z) + \hat{\mathbf{y}}\,\widetilde{E}_y(z)$$
$$= (\hat{\mathbf{x}}\,E_{x0} + \hat{\mathbf{y}}\,E_{y0})e^{-jkz},$$

the time-average power density carried by the wave is:

$$\mathbf{S}_{\text{av}} = \hat{\mathbf{z}}\,\frac{1}{2\eta}(|E_{x0}|^2 + |E_{y0}|^2)$$
$$= \hat{\mathbf{z}}\,\frac{|\widetilde{\mathbf{E}}|^2}{2\eta} \qquad (\text{W/m}^2),$$

Plane Wave in Lossy Medium

For a plane wave travelling in a lossy medium:

$$\widetilde{\mathbf{E}}(z) = \hat{\mathbf{x}}\, \widetilde{E}_x(z) + \hat{\mathbf{y}}\, \widetilde{E}_y(z)$$
$$= (\hat{\mathbf{x}}\, E_{x0} + \hat{\mathbf{y}}\, E_{y0}) e^{-\alpha z} e^{-j\beta z},$$

$$\widetilde{\mathbf{H}}(z) = \frac{1}{\eta_c}(-\hat{\mathbf{x}}\, E_{y0} + \hat{\mathbf{y}}\, E_{x0}) e^{-\alpha z} e^{-j\beta z},$$

the power density is :

$$\mathbf{S}_{av}(z) = \frac{1}{2} \mathfrak{Re}\left[\widetilde{\mathbf{E}} \times \widetilde{\mathbf{H}}^*\right]$$
$$= \frac{\hat{\mathbf{z}}(|E_{x0}|^2 + |E_{y0}|^2)}{2} e^{-2\alpha z}\, \mathfrak{Re}\left(\frac{1}{\eta_c^*}\right).$$

By expressing η_c in polar form as

$$\eta_c = |\eta_c| e^{j\theta_\eta},$$

$$\mathbf{S}_{av}(z) = \hat{\mathbf{z}}\, \frac{|\widetilde{E}(0)|^2}{2|\eta_c|}\, e^{-2\alpha z} \cos\theta_\eta \quad (\mathrm{W/m^2})$$

Whereas the fields $\widetilde{\mathbf{E}}(z)$ and $\widetilde{\mathbf{H}}(z)$ decay with z as $e^{-\alpha z}$, the power density $\mathbf{S}_{av}$ decreases as $e^{-2\alpha z}$.

Example 7-6: Power Received by a Submarine Antenna

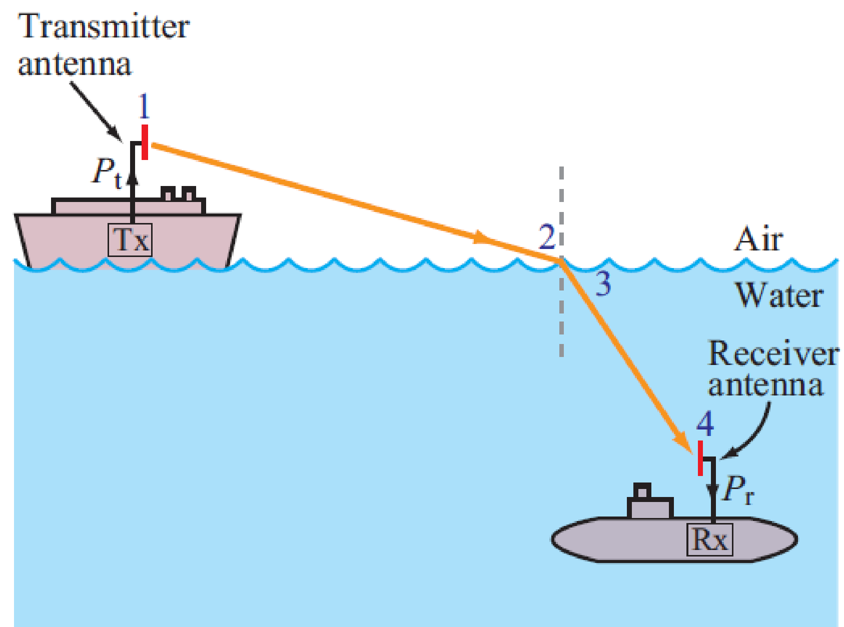
A submarine at a depth of 200 m below the sea surface uses a wire antenna to receive signal transmissions at 1 kHz. Determine the power density incident upon the submarine antenna due to the EM wave of Example 7-4.

Solution: From Example 7-4, $|\tilde{E}(0)| = |E_{x0}| = 4.44$ (mV/m), $\alpha = 0.126$ (Np/m), and $\eta_c = 0.044\angle 45°$ (Ω). Application of Eq. (7.109) gives

$$\begin{aligned}
\mathbf{S}_{\text{av}}(z) &= \hat{\mathbf{z}}\,\frac{|E_0|^2}{2|\eta_c|}\,e^{-2\alpha z}\cos\theta_\eta \\
&= \hat{\mathbf{z}}\,\frac{(4.44\times 10^{-3})^2}{2\times 0.044}\,e^{-0.252z}\cos 45° \\
&= \hat{\mathbf{z}}\,0.16e^{-0.252z} \qquad (\text{mW/m}^2).
\end{aligned}$$

At $z = 200$ m, the incident power density is

$$\begin{aligned}
\mathbf{S}_{\text{av}} &= \hat{\mathbf{z}}\,(0.16\times 10^{-3}e^{-0.252\times 200}) \\
&= 2.1\times 10^{-26} \qquad (\text{W/m}^2).
\end{aligned}$$



Power Density of Different Polarization

Linear Polarization

$$\vec{E} = \hat{x}E_0 e^{-jkz} \qquad \vec{H} = \hat{y}\frac{E_0}{\eta_0}e^{-jkz}$$

$$\vec{S}_{av} = \frac{1}{2}\mathrm{Re}\left[\vec{E}\times\vec{H}^*\right] = \boxed{\hat{z}\frac{E_0^2}{2\eta_0}}$$

Circular Polarization

$$\vec{E} = E_0(\hat{x}+j\hat{y})e^{-jkz} \qquad \vec{H} = \frac{E_0}{\eta_0}(\hat{y}-j\hat{x})e^{-jkz}$$

$$\vec{S}_{av} = \frac{1}{2}\mathrm{Re}\left[\vec{E}\times\vec{H}^*\right] = \frac{E_0^2}{2\eta_0}\mathrm{Re}\left[(\hat{x}+j\hat{y})e^{-jkz}\times(\hat{y}+j\hat{x})e^{jkz}\right]$$

Total fields

$$= \frac{E_0^2}{2\eta_0}\mathrm{Re}[(\hat{x}+j\hat{y})\times(\hat{y}+j\hat{x})] = \frac{E_0^2}{2\eta_0}\mathrm{Re}[\hat{z}+\hat{z}] = \boxed{\hat{z}\frac{E_0^2}{\eta_0}}$$

Summary

Chapter 7 Relationships

Complex Permittivity

$$\varepsilon_c = \varepsilon' - j\varepsilon''$$

$$\varepsilon' = \varepsilon$$

$$\varepsilon'' = \frac{\sigma}{\omega}$$

Lossless Medium

$$k = \omega\sqrt{\mu\varepsilon}$$

$$\eta = \sqrt{\frac{\mu}{\varepsilon}} \qquad (\Omega)$$

$$u_p = \frac{\omega}{k} = \frac{1}{\sqrt{\mu\varepsilon}} \qquad \text{(m/s)}$$

$$\lambda = \frac{2\pi}{k} = \frac{u_p}{f} \qquad \text{(m)}$$

Wave Polarization

$$\widetilde{\mathbf{H}} = \frac{1}{\eta}\,\hat{\mathbf{k}} \times \widetilde{\mathbf{E}}$$

$$\widetilde{\mathbf{E}} = -\eta\,\hat{\mathbf{k}} \times \widetilde{\mathbf{H}}$$

Maxwell's Equations for Time-Harmonic Fields

$$\nabla \cdot \widetilde{\mathbf{E}} = 0$$

$$\nabla \times \widetilde{\mathbf{E}} = -j\omega\mu\widetilde{\mathbf{H}}$$

$$\nabla \cdot \widetilde{\mathbf{H}} = 0$$

$$\nabla \times \widetilde{\mathbf{H}} = j\omega\varepsilon_c\widetilde{\mathbf{E}}$$

Lossy Medium

$$\alpha = \omega\left\{\frac{\mu\varepsilon'}{2}\left[\sqrt{1 + \left(\frac{\varepsilon''}{\varepsilon'}\right)^2} - 1\right]\right\}^{1/2} \qquad \text{(Np/m)}$$

$$\beta = \omega\left\{\frac{\mu\varepsilon'}{2}\left[\sqrt{1 + \left(\frac{\varepsilon''}{\varepsilon'}\right)^2} + 1\right]\right\}^{1/2} \qquad \text{(rad/m)}$$

$$\eta_c = \sqrt{\frac{\mu}{\varepsilon_c}} = \sqrt{\frac{\mu}{\varepsilon'}}\left(1 - j\frac{\varepsilon''}{\varepsilon'}\right)^{-1/2} \qquad (\Omega)$$

$$\delta_s = \frac{1}{\alpha} \qquad \text{(m)}$$

Power Density

$$\mathbf{S}_{av} = \tfrac{1}{2}\,\mathfrak{Re}\left[\widetilde{\mathbf{E}} \times \widetilde{\mathbf{H}}^*\right] \qquad (\text{W/m}^2)$$