Numerical Optimization

Lecture 19: Quadratic Programming

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1. Quadratic Programming
（二次规划）

Unconstrained quadratic optimization

Consider the quadratic optimization problem

$$\min_{x} \ f(x) = g^T x + \tfrac{1}{2} x^T H x,$$

where H is symmetric. Necessary optimality conditions are

$$\nabla f(x) = g + Hx = 0.$$

If $H \succeq 0$, then f is convex, in which case the above conditions are also sufficient.

Thus, we have the following cases based on g and H:

- If $H \succeq 0$, then any solution to $g + Hx = 0$ is optimal.
- If $H \succeq 0$ and $g + Hx = 0$ has no solution, or if

  $$d^T H d < 0 \ \text{ for some } \ d \neq 0,$$

  then the problem is unbounded.

投资组合优化(Portfolio Optimization)

投资组合$\{1,\ldots,n\}$, 可能受益为r_i（random variable）

对受益与风险的这种进行建模

$$\mu_i = E[r_i],\ \sigma_i^2 = E[(r_i-\mu_i)^2],\ \rho_{ij} = \frac{E[(r_i-\mu_i)(r_j-\mu_j)]}{\sigma_i\sigma_j}$$

假定所有资金均为投资，不允许卖空

投资组合(portfolio): 设对第i项投资的资金投放比例为x_i

$$\sum_{i=1}^{n} x_i = 1,\quad x \geq 0$$

投资组合的收益(return): $R = \sum_{i=1}^{n} x_i r_i$

投资组合的风险(risk):

$$E[(R-E[R])^2] = \sum_{i=1}^{n}\sum_{j=1}^{n} x_i x_j \sigma_i \sigma_j \rho_{ij} = x^T G x,$$ G协方差矩阵

投资组合优化(Portfolio Optimization)

投资组合的风险和收益的折衷

$$\min_x \quad -\mu^T x + \alpha x^T G x \qquad \textbf{s.t.} \quad \sum_{i=1}^n x_i = 1, \quad x \geq 0$$

$$\min_x \quad x^T G x \qquad \textbf{s.t.} \quad \sum_{i=1}^n x_i = 1, \quad x \geq 0, \quad \mu^T x \geq \bar{r}$$

$$\min_x \quad -\mu^T x \qquad \textbf{s.t.} \quad \sum_{i=1}^n x_i = 1, \quad x \geq 0, \quad x^T G x \leq \bar{\sigma}$$

Many other variants!

Equality constrained quadratic optimization

Consider the equality constrained quadratic optimization problem

$$\min_x \ f(x) = g^T x + \tfrac{1}{2} x^T H x$$
$$\text{s.t. } c(x) = Ax + b = 0.$$

The Lagrangian is

$$L(x, \lambda) = g^T x + \tfrac{1}{2} x^T H x + \lambda^T (Ax - b),$$

so necessary optimality conditions are

$$\begin{bmatrix} \nabla_x L(x,\lambda) \\ c(x) \end{bmatrix} = \begin{bmatrix} g + Hx + A^T \lambda \\ Ax + b \end{bmatrix} = 0 \implies \begin{bmatrix} H & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix} = - \begin{bmatrix} g \\ b \end{bmatrix}.$$

- The critical cone at (x, λ) is $\{d : \nabla c(x)^T d = 0\} = \{d : Ad = 0\}$.
- Thus, the necessary conditions above are sufficient as long as

$$d^T \nabla^2_{xx} L(x, \lambda) d = d^T H d \geq 0 \ \text{ for all } d \text{ such that } Ad = 0.$$

Algorithms

Algorithms for convex quadratic optimization include:

- Active-set methods
- Interior-point methods

If a problem is nonconvex, then the choice of algorithm depends on the type of solution required. If a local solution is sufficient, then the above methods apply.

- The problem of determining whether a feasible point to a nonconvex quadratic optimization problem is a minimizer (local and/or global) is NP-hard.

We overview an active-set method for convex quadratic optimization.

Optimal active-set

If an optimal active-set $\mathcal{A}_*$ (i.e., a set of inequalities satisfied as equalities at a solution) is known in advance, then a solution x_* can be found as a solution to

$$\begin{aligned} \min_x \; & g^T x + \tfrac{1}{2} x^T H x \\ \text{s.t. } & A^i x + b^i = 0, \; i \in \mathcal{E} \cup \mathcal{A}_*, \end{aligned}$$

i.e., a solution to

$$\begin{aligned} g + Hx + \sum_{i \in \mathcal{E} \cup \mathcal{A}_*} A^{i^T} \lambda^i &= 0 \\ A^i x + b^i &= 0, \; i \in \mathcal{E} \cup \mathcal{A}_*. \end{aligned}$$

Active-set iteration

Suppose we have an iterate x_k and a guess $\mathcal{A}_k$ of an optimal active set.

Compute d_k as the solution to the subproblem

$$\begin{aligned} \min_{d} \; & g^T(x_k + d) + \tfrac{1}{2}(x_k + d)^T H (x_k + d) \\ \text{s.t. } & A^i(x_k + d) + b^i = 0, \;\; i \in \mathcal{E} \cup \mathcal{A}_k. \end{aligned}$$

- If $x_k + d_k$ is feasible, then set $x_{k+1} \leftarrow x_k + d_k$ and let $\mathcal{A}_{k+1} \leftarrow \mathcal{A}_k$.
- Else, set $x_{k+1} \leftarrow x_k + \alpha_k d_k$, where α_k is the largest value such that x_{k+1} satisfies all constraints. Let $\mathcal{A}_{k+1}$ be the set of constraints active at x_{k+1}.

Continue this process until $d_k = 0$ for some k...

Optimality check

Eventually, we obtain a solution (x_k, λ_k) of the KKT conditions

$$
\begin{aligned}
g + Hx + \sum_{i \in \mathcal{E} \cup \mathcal{A}_k} {A^i}^T \lambda^i &= 0 \\
A^i x + b^i &= 0, \; i \in \mathcal{E} \cup \mathcal{A}_k
\end{aligned}
$$

(with $\lambda^i = 0$ for $i \in \mathcal{I} \backslash \mathcal{A}_k$). The KKT conditions for the quadratic problem

$$
\begin{aligned}
g + Hx + {A^{\mathcal{E}}}^T \lambda^{\mathcal{E}} + {A^{\mathcal{I}}}^T \lambda^{\mathcal{I}} &= 0 \\
A^{\mathcal{E}} x + b^{\mathcal{E}} &= 0 \\
A^{\mathcal{I}} x + b^{\mathcal{I}} &\leq 0 \\
\lambda^{\mathcal{I}} &\geq 0 \\
\lambda^{\mathcal{I}} \cdot (A^{\mathcal{I}} x + b^{\mathcal{I}}) &= 0
\end{aligned}
$$

will be satisfied as long as x_k is feasible and $\lambda_k^i \geq 0, \; i \in \mathcal{A}_k$.

Finding an improving direction

Suppose a solution x_k to

$$
\begin{aligned}
\min_x \; & g^T x + \tfrac{1}{2} x^T H x \\
\text{s.t. } & A^i x + b^i = 0, \; i \in \mathcal{E} \cup \mathcal{A}_k
\end{aligned}
$$

is feasible, but not optimal for the linear optimization problem.

- Consider $j \in \mathcal{A}_k$ such that $\lambda_k^j < 0$.
- An improving direction is obtained by considering the problem

$$
\begin{aligned}
\min_d \; & g^T (x_k + d) + \tfrac{1}{2} (x_k + d)^T H (x_k + d) \\
\text{s.t. } & A^i (x_k + d) + b^i = 0, \; i \in \mathcal{E} \cup (\mathcal{A}_k \backslash j),
\end{aligned}
$$

  or, equivalently,

$$
\begin{aligned}
\min_d \; & (g + H x_k)^T d + \tfrac{1}{2} d^T H d \\
\text{s.t. } & A^i d = 0, \; i \in \mathcal{E} \cup (\mathcal{A}_k \backslash j).
\end{aligned}
$$

- If this problem is unbounded or has a solution $d \neq 0$, then such a d with

$$
g^T d < 0 \quad \text{and} \quad A^i d = 0, \; i \in \mathcal{E} \cup (\mathcal{A}_k \backslash j),
$$

  is an improving direction from x_k.

Improving directions and feasibility

Do we know if such a direction will maintain feasibility for all constraints?

- If we are not at a degenerate point in that

$$A^i x + b^i < 0, \ i \in \mathcal{I} \backslash \mathcal{A}_k,$$

  then we maintain feasibility for these constraints for any small displacement.
- If we are at a degenerate point, then we need to worry...
- Finally, what about for the constraint that we are removing from the active set? We do indeed remain feasible; cf. Theorem 16.5 in N&W.

Summary of active set method

Let $\mathcal{A}_k$ be a guess of the optimal active set corresponding to a feasible x_k.

for $k = 0, 1, 2, \ldots$

1. Solve the active-set (equality constrained) QOP to obtain (d_k, λ_k)
2. If $d_k = 0$ and $\lambda_k^{\mathcal{I}} \geq 0$, then stop; x_k is optimal
3. If $d_k \neq 0$ and $x_k + d_k$ is feasible, then set

$$x_{k+1} \leftarrow x_k + d_k \quad \text{and} \quad \mathcal{A}_{k+1} \leftarrow \mathcal{A}_k$$

   and go to step 1
4. If $d_k \neq 0$ and $x_k + d_k$ is infeasible (for any constraint), then set α_k as the largest value such that $x_{k+1} \leftarrow x_k + \alpha_k d_k$ is feasible, set $\mathcal{A}_k$ as the active set at x_{k+1}, and go to step 1
5. Choose any j such that $\lambda_k^j < 0$ and set $\mathcal{A}_{k+1} \leftarrow \mathcal{A}_k \backslash j$
6. Return to step 1

$$\min_x \; q(x) = (x_1 - 1)^2 + (x_2 - 2.5)^2$$

$$\begin{aligned} \text{subject to} \quad x_1 - 2x_2 + 2 &\geq 0, \\ -x_1 - 2x_2 + 6 &\geq 0, \\ -x_1 + 2x_2 + 2 &\geq 0, \\ x_1 &\geq 0, \\ x_2 &\geq 0. \end{aligned}$$

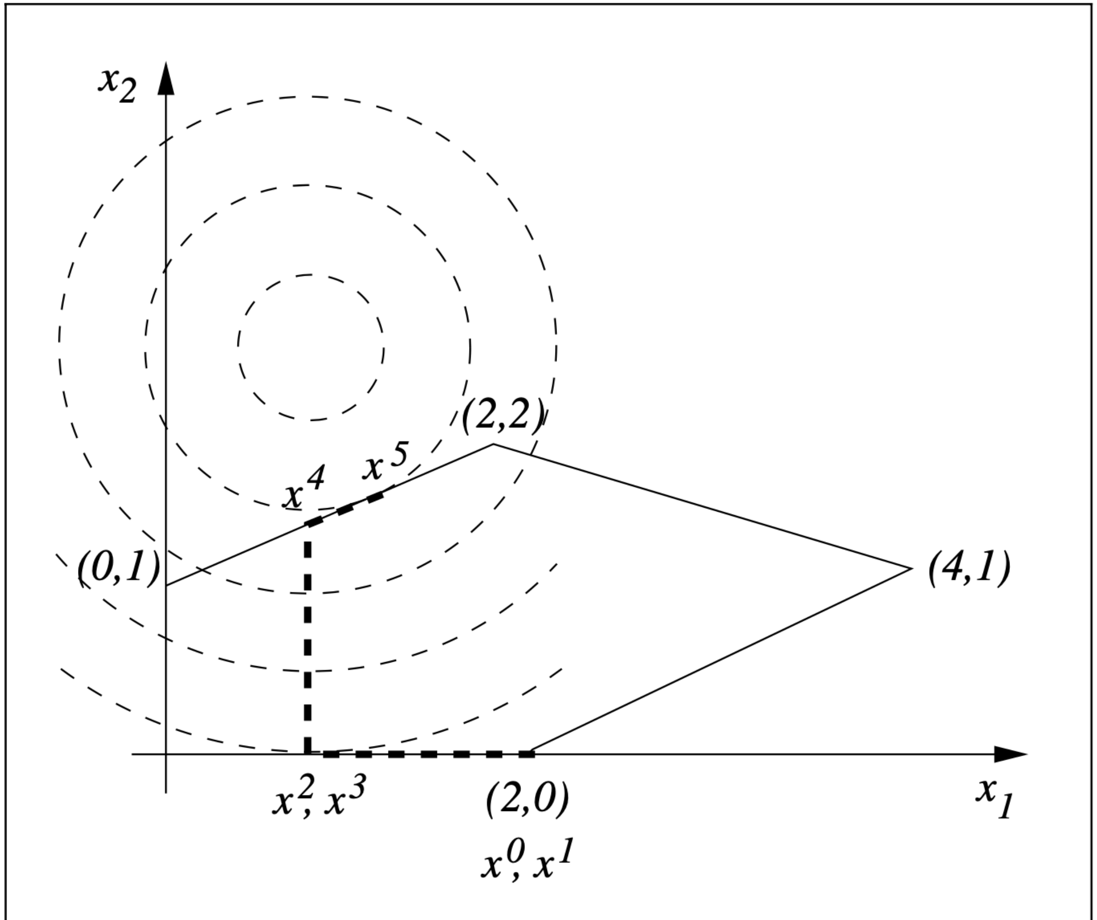


Figure 16.3 Iterates of the active-set method.

$$\boldsymbol{x}^{(0)} = (2, 0)^{\mathrm{T}}$$

$$\mathcal{A} = \{3, 5\}$$

2. Other “easy” constraints

梯度投影法(Gradient Projection Method)

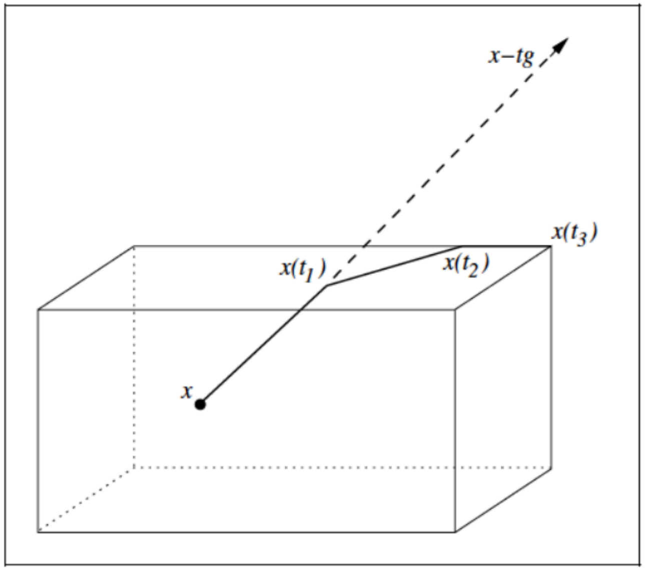
- The problem:

$$\min_x \quad q(x) = \frac{1}{2}x^T G x + x^T c$$
$$\text{subject to} \quad l \le x \le u,$$

- Like in the trust-region case, we look for a Cauchy point, based on a projection on the feasible set.
- G does not have to be psd
- The projection operator:

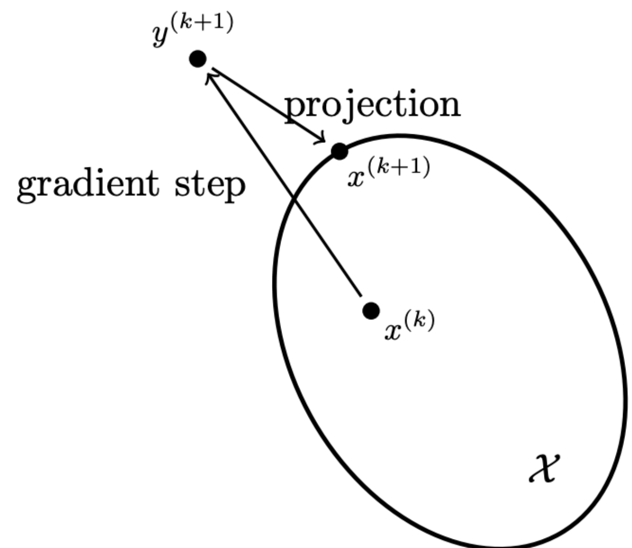
$$P(x, l, u)_i = \begin{cases} l_i & \text{if} \quad x_i < l_i, \\ x_i & \text{if} \quad x_i \in [l_i, u_i], \\ u_i & \text{if} \quad x_i > u_i. \end{cases}$$

- Create a piecewise linear path which is feasible (as opposed to the linear one in the unconstrained case) by projection of gradient.



Projected Gradient Descent

Idea: make sure that points are feasible by projecting onto $\mathcal{X}$



<ins>Algorithm:</ins>

- $y^{(k+1)} = x^{(k)} - t^{(k)} g^{(k)}$
  where $g^{(k)} \in \partial f(x^{(k)})$
- $x^{(k+1)} = \Pi_{\mathcal{X}}(y^{(k+1)})$

The projection operator $\Pi_{\mathcal{X}}$ onto $\mathcal{X}$:

$$\Pi_{\mathcal{X}}(x) = \min_{z \in \mathcal{X}} \|x - z\|$$

Notice: subgradient instead of gradient (even for differentiable functions)

条件梯度法(Conditional Gradient Method) Frank-Wolfe Algorithm

A projection-free algorithm!
Introduced for QP by Marguerite **Frank** and Philip **Wolfe** (1956)

<u>Algorithm</u>

- Initialize: $x^{(0)} \in \mathcal{X}$
- $s^{(k)} = \underset{s \in \mathcal{X}}{\operatorname{argmin}} \langle \nabla f(x^{(k)}), s \rangle$
- $x^{(k+1)} = x^{(k)} + t^{(k)}(s^{(k)} - x^{(k)})$