SI251 - Convex Optimization, Spring 2022
Final Exam

Note: We are interested in the reasoning underlying the solution, as opposed to simply the answer. Thus, solutions with the correct answer but without adequate explanation will not receive full credit; on the other hand, partial solutions with explanation will receive partial credit. Your use of resources should be limited to printed lecture slides, lecture notes, homework, homework solutions, general resources, class reading and textbooks, and other related textbooks on optimization. You should not discuss the final exam problems with anyone. Detected violations of this policy will be processed according to ShanghaiTech's code of academic integrity.

I. Basic Knowledge

1. Consider the set $S = \{\boldsymbol{x} \in \mathbb{R}^m \mid |p(\boldsymbol{x};t)| \leq 1 \text{ for } |t| \leq \pi/3\}$ where $p(\boldsymbol{x};t) = x_1 \cos t + x_2 \cos 2t + \ldots + x_m \cos mt$. Determine whether the set is convex, concave, or neither. Give necessary explanations. (5 points)
2. Consider the logarithmic determinant function: $f(\boldsymbol{X}) = \operatorname{logdet}(\boldsymbol{X})$ in the domain $\mathbb{S}^n = \{\boldsymbol{X} \in \mathbb{R}^{n\times n} \mid \boldsymbol{X} \succ 0\}$. Determine whether the function is convex, concave, or neither. Give necessary explanations. (5 points)
3. Find all stationary points of $f(x,y) = (6x - x^2)(4y - y^2)$ on $\mathbb{R}^2$. For each stationary point, determine if it is a local minimum, local maximum or saddle point. Justify your answer. (5 points)
4. Write down the KKT condition for the following conic optimization problem. (5 points)

$$
\begin{aligned}
\underset{\boldsymbol{x},\boldsymbol{s}}{\text{minimize}} \quad & \boldsymbol{c}^\mathsf{T}\boldsymbol{x} \\
\text{subject to} \quad & \boldsymbol{A}\boldsymbol{x} + \boldsymbol{s} = \boldsymbol{b} \\
& (\boldsymbol{x}, \boldsymbol{s}) \in \mathbb{R}^n \times \mathcal{K},
\end{aligned} \tag{1}
$$

where $\boldsymbol{c}, \boldsymbol{x} \in \mathbb{R}^n$, $\boldsymbol{A} \in \mathbb{R}^{m\times n}$, $\boldsymbol{s}, \boldsymbol{b} \in \mathbb{R}^m$, and $\mathcal{K}$ is a non-empty closed convex cone.

II. Convex Problem

Determine whether these following problems are convex or not respectively.

– If yes, equivalently reformulate the original problem into a standard convex optimization form, i.e., Linear Programming (LP), Second-Order Cone Programming (SOCP) or Semidefinite Programming (SDP).

– If no, relax the original problem to convex problem at first, and then equivalently reformulate the relaxed problem into a standard convex optimization form.

Hint: Standard convex optimization form of Linear Programming (LP):

$$
\begin{aligned}
\underset{\boldsymbol{x}}{\text{minimize}} \quad & \boldsymbol{c}^\mathsf{T}\boldsymbol{x} + d \\
\text{subject to} \quad & \boldsymbol{G}\boldsymbol{x} \leq \boldsymbol{h} \\
& \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}.
\end{aligned} \tag{2}
$$

Standard convex optimization form of Second-Order Cone Programming (SOCP):

$$
\begin{aligned}
\underset{\boldsymbol{x}}{\text{minimize}} \quad & \boldsymbol{f}^\mathsf{T}\boldsymbol{x} \\
\text{subject to} \quad & \|\boldsymbol{A}_i\boldsymbol{x} + \boldsymbol{b}_i\| \leq \boldsymbol{c}_i^\mathsf{T}\boldsymbol{x} + d_i, \ i = 1, \ldots m \\
& \boldsymbol{F}\boldsymbol{x} = \boldsymbol{g}.
\end{aligned} \tag{3}
$$

Standard convex optimization form of Semidefinite Programming (SDP):

$$\begin{aligned} \underset{\boldsymbol{x}}{\text{minimize}} \quad & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{subject to} \quad & x_1 \boldsymbol{F}_1 + x_2 \boldsymbol{F}_2 + \ldots + x_n \boldsymbol{F}_n \preceq \boldsymbol{G} \\ & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}. \end{aligned} \tag{4}$$

1. Consider the following optimization problem

$$\begin{aligned} \underset{r, \boldsymbol{x}_c}{\text{maximize}} \quad & r \\ \text{subjet to} \quad & \boldsymbol{x} \in P \ \ \forall \boldsymbol{x} = \boldsymbol{x}_c + \boldsymbol{u} \ \ \text{with} \ \ \|\boldsymbol{u}\| \leq r, \end{aligned} \tag{5}$$

   where $r \in \mathbb{R}$, $\boldsymbol{x}, \boldsymbol{x}_c, \boldsymbol{u} \in \mathbb{R}^d$ and $P = \{\boldsymbol{x} | \boldsymbol{a}_i^\top \boldsymbol{x} \leq b_i, i = 1, ..., m\}$ with $\boldsymbol{a}_i \in \mathbb{R}^d$ and $b_i \in \mathbb{R}$. (10 points)

2. Consider the following optimization problem

$$\begin{aligned} \underset{\boldsymbol{x}}{\text{minimize}} \quad & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{subject to} \quad & \boldsymbol{a}_i^\top \boldsymbol{x} \leq b_i \quad \forall \boldsymbol{a}_i \in \boldsymbol{\xi}_i, i = 1, \ldots, m \end{aligned} \tag{6}$$

   where $\boldsymbol{c}, \boldsymbol{x}, \boldsymbol{a}_i \in \mathbb{R}^d$, $b_i \in \mathbb{R}$ and $\boldsymbol{\xi}_i = \{\bar{\boldsymbol{a}}_i + \boldsymbol{P}_i \boldsymbol{u} | \|\boldsymbol{u}\| \leq 1\}$ with $\boldsymbol{a}_i \in \mathbb{R}^d$, $\boldsymbol{u} \in \mathbb{R}^m$ and $\boldsymbol{P}_i \in \mathbb{R}^{d \times m}$. (10 points)

3. Consider the following optimization problem

$$\begin{aligned} \underset{\boldsymbol{x}}{\text{minimize}} \quad & \boldsymbol{x}^\top \boldsymbol{C} \boldsymbol{x} + 2\boldsymbol{c}^\top \boldsymbol{x} \\ \text{subject to} \quad & \boldsymbol{x}^\top \boldsymbol{A}_i \boldsymbol{x} + 2\boldsymbol{a}_i^\top \boldsymbol{x} \geq b_i, \ \ i = 1, \ldots, m, \end{aligned} \tag{7}$$

   where $\boldsymbol{x}, \boldsymbol{a}_i, \boldsymbol{c} \in \mathbb{R}^n$, $\boldsymbol{C}, \boldsymbol{A}_i \in \mathbb{R}^{n \times n}$ and $b_i \in \mathbb{R}$. (10 points)

III. **Largrange Duality**

The Boolean linear program is an optimization problem of the form

$$\begin{aligned} \underset{\boldsymbol{x}}{\text{minimize}} \quad & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{subject to} \quad & \boldsymbol{A}\boldsymbol{x} \preceq \boldsymbol{b} \\ & x_i \in \{0, 1\}, \ \ i = 1, \ldots, n, \end{aligned} \tag{8}$$

where $\boldsymbol{c}, \boldsymbol{x} \in \mathbb{R}^n$, $\boldsymbol{b} \in \mathbb{R}^m$ and $\boldsymbol{A} \in \mathbb{R}^{m \times n}$. In general, it is very difficult to solve. Hence, we consider the Linear Programming (LP) relaxation of this problem,

$$\begin{aligned} \underset{\boldsymbol{x}}{\text{minimize}} \quad & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{subject to} \quad & \boldsymbol{A}\boldsymbol{x} \preceq \boldsymbol{b} \\ & 0 \leq x_i \leq 1, \ \ i = 1, \ldots, n, \end{aligned} \tag{9}$$

which is far easier to solve, and gives a lower bound on the optimal value of the Boolean LP. In this problem we derive another lower bound for the Boolean LP, and work out the relation between the two lower bounds.

1. The Boolean LP can be reformulated as the problem (10 points)

$$\begin{aligned} \underset{\boldsymbol{x}}{\text{minimize}} \quad & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{subject to} \quad & \boldsymbol{A}\boldsymbol{x} \preceq \boldsymbol{b} \\ & x_i(1 - x_i) = 0, \ \ i = 1, \ldots, n, \end{aligned} \tag{10}$$

   which has quadratic equality constraints. Find the Lagrange dual problem of this problem.

2. Show that the lower bound obtained via Lagrangian relaxation, and via the LP relaxation are the same. Hint: Derive the dual problem of the LP relaxation (9). (10 points)

IV. **Convex Optimization Algorithms**

Consider the LASSO (Least absolute shrinkage and selection operator) problem

$$\min_{\boldsymbol{\beta}} \quad \frac{1}{2}\|\boldsymbol{y} - \boldsymbol{X}\boldsymbol{\beta}\|_2^2 + \lambda\|\boldsymbol{\beta}\|_1 \tag{11}$$

where $\boldsymbol{y} \in \mathbb{R}^n$, $\boldsymbol{X} \in \mathbb{R}^{n\times p}$, $\boldsymbol{\beta} \in \mathbb{R}^p$ and $\lambda \in \mathbb{R}^+$.

1. For this subproblem, consider a special case that $\boldsymbol{X}$ is an identity matrix. You are required to describe how to implement a subgradient method to solve the LASSO problem. (Hint: A closed-form solution can be can be derived in this case.) (10 points)
2. In this subproblem, you are required to solve the LASSO problem with the proximal gradient descent method, and write down the proximal gradient updates. (10 points)
3. In this subproblem, you are required to write down the exact Alternating Direction Method of Multipliers (ADMM) update steps for the LASSO problem. (10 points)