Numerical Optimization, 2022 Fall
Homework 4 Solution

1 Lagrange

Please use Lagrange to give the dual problems of the following

1. [15pts]

$$\begin{aligned} \min \quad & 2x_1 - x_2 \\ \text{s.t.} \quad & 2x_1 - x_2 - x_3 \geq 3 \\ & x_1 - x_2 + x_3 \geq 2 \\ & x_i \geq 0, \quad i = 1, 2, 3. \end{aligned} \tag{1}$$

The Lagrangian is

$$\begin{aligned} \mathcal{L}(\boldsymbol{x}, \boldsymbol{\lambda}, \boldsymbol{\mu}) &= 2x_1 - x_2 + \lambda_1(3 - 2x_1 + x_2 + x_3) + \lambda_2(2 - x_1 + x_2 - x_3) - \mu_1 x_1 - \mu_2 x_2 - \mu_3 x_3 \\ &= (-2\lambda_1 - \lambda_2 - \mu_1 + 2)x_1 + (\lambda_1 + \lambda_2 - \mu_2 - 1)x_2 + (\lambda_1 - \lambda_2 - \mu_3)x_3 + (3\lambda_1 + 2\lambda_2) \end{aligned} \tag{2}$$

where $\boldsymbol{\lambda} = (\lambda_1, \lambda_2)^T \geq \mathbf{0}$ and $\boldsymbol{\mu} = (\mu_1, \mu_2, \mu_3)^T \geq \mathbf{0}$. The dual objective is

$$g(\boldsymbol{\lambda}, \boldsymbol{\mu}) = \min_{\boldsymbol{x}} \mathcal{L}(\boldsymbol{x}, \boldsymbol{\lambda}, \boldsymbol{\mu}) \tag{3}$$

Since we only have interests in the case that $g(\boldsymbol{\lambda}, \boldsymbol{\mu}) > -\infty$, each coefficient in front of primal variable x_i should be set as 0. Hence, the dual problem is

$$\begin{aligned} \max \quad & 3\lambda_1 + 2\lambda_2 \\ \text{s.t.} \quad & -2\lambda_1 - \lambda_2 - \mu_1 + 2 = 0 \\ & \lambda_1 + \lambda_2 - \mu_2 - 1 = 0 \\ & \lambda_1 - \lambda_2 - \mu_3 = 0 \\ & \lambda_i \geq 0, \ i = 1, 2 \\ & \mu_j \geq 0, \ j = 1, 2, 3 \end{aligned} \tag{4}$$

We can remove the redundant μ_j's. Therefore, the final form is

$$\begin{aligned} \max \quad & 3\lambda_1 + 2\lambda_2 \\ \text{s.t.} \quad & 2\lambda_1 + \lambda_2 \leq 2 \\ & \lambda_1 + \lambda_2 \geq 1 \\ & \lambda_1 - \lambda_2 \geq 0 \\ & \lambda_i \geq 0, \ i = 1, 2. \end{aligned} \tag{5}$$

to show the double dual, formulate the lagrange again in the same way.

2. [15pts]

$$
\begin{aligned}
\min \quad & 0 \cdot x_1 + 0 \cdot x_2 \\
\text{s.t.} \quad & -2x_1 + 2x_2 \leq -1 \\
& 2x_1 - x_2 \leq 2 \\
& -4x_2 \leq 3 \\
& -15x_1 - 12x_2 \leq -2 \\
& 12x_1 + 20x_2 \leq -1.
\end{aligned} \tag{6}
$$

The Lagrangian is

$$
\begin{aligned}
\mathcal{L}(\boldsymbol{x}, \boldsymbol{\lambda}) &= \lambda_1(1 - 2x_1 + 2x_2) + \lambda_2(2x_1 - x_2 - 2) + \lambda_3(-4x_2 - 3) + \lambda_4(2 - 15x_1 - 12x_2) \\
&\quad + \lambda_5(1 + 12x_1 + 20x_2) \\
&= (-2\lambda_1 + 2\lambda_2 - 15\lambda_4 + 12\lambda_5)x_1 + (2\lambda_1 - \lambda_2 - 4\lambda_3 - 12\lambda_4 + 20\lambda_5)x_2 \\
&\quad + (\lambda_1 - 2\lambda_2 - 3\lambda_3 + 2\lambda_4 + \lambda_5)
\end{aligned} \tag{7}
$$

where $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5)^T \geq \mathbf{0}$. The dual objective is

$$
g(\boldsymbol{\lambda}) = \min_{\boldsymbol{x}} \mathcal{L}(\boldsymbol{x}, \boldsymbol{\lambda}) \tag{8}
$$

Since we only have interests in the case that $g(\boldsymbol{\lambda}) > -\infty$, each coefficient in front of primal variable x_i should be set as 0. Hence, the dual problem is

$$
\begin{aligned}
\max \quad & \lambda_1 - 2\lambda_2 - 3\lambda_3 + 2\lambda_4 + \lambda_5 \\
\text{s.t.} \quad & -2\lambda_1 + 2\lambda_2 - 15\lambda_4 + 12\lambda_5 = 0 \\
& 2\lambda_1 - \lambda_2 - 4\lambda_3 - 12\lambda_4 + 20\lambda_5 = 0 \\
& \lambda_i \geq 0, \ i = 1, 2, \ldots, 5.
\end{aligned} \tag{9}
$$

to show the double do, formulate the lagrange again in the same way.

2 Primal-Dual Feasibility

From the lecture we know that the primal and dual of an LP problem may be both infeasible, please write a specific example of this situation and then briefly explain why are both problems infeasible. [20pts]

The following LP problem has no feasible solution

$$
\begin{aligned}
\min \quad & x_1 - 2x_2 \\
\text{s.t.} \quad & x_1 - x_2 \geq 2 \\
& -x_1 + x_2 \geq -1 \\
& x_1, x_2 \geq 0
\end{aligned} \tag{10}
$$

and neither does its dual

$$
\begin{aligned}
\min \quad & 2\lambda_1 - \lambda_2 \\
\text{s.t.} \quad & \lambda_1 - \lambda_2 \leq 1 \\
& -\lambda_1 + \lambda_2 \leq -2 \\
& \lambda_1, \lambda_2 \geq 0
\end{aligned} \tag{11}
$$

Reason: The two constraints of each problem can't be true at the same time.