Convex Optimization Problems

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Outline

1. Optimization Problems

2. Convex Optimization

3. Quasi-Convex Optimization

4. Classes of Convex Problems: LP, QP, SOCP, SDP ✓

Optimization Problems in Standard Form I

$$\begin{aligned} \underset{\boldsymbol{x}}{\text{minimize}} \quad & f_0(\boldsymbol{x}) \\ \text{subject to} \quad & f_i(\boldsymbol{x}) \leq 0 \quad i = 1, \cdots, m \\ & \underbrace{h_i(\boldsymbol{x}) = 0}_{Ax = b} \quad i = 1, \cdots, p \end{aligned}$$

- $\boldsymbol{x} = (x_1, \cdots, x_n)$ is the optimization variable
- $f_0 : \mathbb{R}^n \to \mathbb{R}$ is the objective function
- $f_i : \mathbb{R}^n \to \mathbb{R} \quad i = 1, \cdots, m$ are the inequality constraint functions
- $h_i : \mathbb{R}^n \to \mathbb{R} \quad i = 1, \cdots, p$ are the equality constraint functions

Optimization Problems in Standard Form II

Feasibility:

- a point $\boldsymbol{x} \in \mathrm{dom}\, f_0$ is feasible if it satisfies all the constraints and infeasible otherwise
- a problem is feasible if it has at least one feasible point and infeasible otherwise.

Optimal value:

$$p^\star = \inf\{f_0(\boldsymbol{x}) \mid f_i(\boldsymbol{x}) \leq 0,\ i = 1, \cdots, m,\ h_i(\boldsymbol{x}) = 0,\ i = 1, \cdots, p\}$$

- $p^\star = \infty$ if problem is infeasible (no $\boldsymbol{x}$ satisfies the constraints)
- $p^\star = -\infty$ if problem is unbounded below

Optimal solution: $\boldsymbol{x}^\star$ such that $f(\boldsymbol{x}^\star) = p^\star$ (and $\boldsymbol{x}^\star$ feasible).

Global and Local Optimality

- A feasible $\boldsymbol{x}$ is **optimal** if $f_0(\boldsymbol{x}) = p^\star$; X_{opt} is the set of optimal points.
- A feasible $\boldsymbol{x}$ is **locally optimal** if it is optimal within a ball, i.e., there is an $R > 0$ such that $\boldsymbol{x}$ is optimal for

$$\begin{array}{ll} \underset{\boldsymbol{z}}{\text{minimize}} & f_0(\boldsymbol{z}) \\ \text{subject to} & f_i(\boldsymbol{z}) \leq 0 \quad i = 1, \cdots, m \\ & h_i(\boldsymbol{z}) = 0 \quad i = 1, \cdots, p \\ & \|\boldsymbol{z} - \boldsymbol{x}\|_2 \leq R \end{array}$$

Example:

- $f_0(x) = 1/x$, $\operatorname{dom} f_0 = \mathbb{R}_{++}$: $p^\star = 0$, no optimal point
- $f_0(x) = x^3 - 3x$: $p^\star = -\infty$, local optimum at $x = 1$.

Implicit Constraints

- The standard form optimization problem has an explicit constraint:

$$\boldsymbol{x} \in \mathcal{D} = \bigcap_{i=0}^{m} \operatorname{dom} f_i \ \cap \ \bigcap_{i=1}^{p} \operatorname{dom} h_i$$

- $\mathcal{D}$ is the domain of the problem
- The constraints $f_i(\boldsymbol{x}) \leq 0, h_i(\boldsymbol{x}) = 0$ are the explicit constraints
- A problem is unconstrained if it has no explicit constraints
- Example:

$$\underset{\boldsymbol{x}}{\text{minimize}} \quad \log(b - \boldsymbol{a}^T \boldsymbol{x})$$

is an unconstrained problem with implicit constraint $b > \boldsymbol{a}^T \boldsymbol{x}$

Feasibility Problem

- Sometimes, we don't really want to minimize any objective, just to find a feasible point:

$$\begin{array}{ll} \underset{\boldsymbol{x}}{\text{find}} & \boldsymbol{x} \\ \text{subject to} & f_i(\boldsymbol{x}) \le 0 \quad i = 1, \cdots, m \\ & h_i(\boldsymbol{x}) = 0 \quad i = 1, \cdots, p \end{array}$$

- This feasibility problem can be considered as a special case of a general problem:

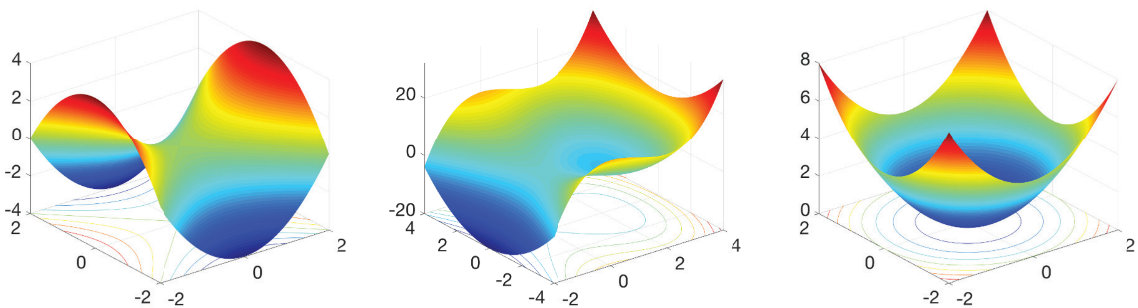
$$\begin{array}{ll} \underset{\boldsymbol{x}}{\text{minimize}} & 0 \\ \text{subject to} & f_i(\boldsymbol{x}) \le 0 \quad i = 1, \cdots, m \\ & h_i(\boldsymbol{x}) = 0 \quad i = 1, \cdots, p \end{array}$$

where $p^\star = 0$ if constraints are feasible and $p^\star = \infty$ otherwise.

Stationary Points

Given a smooth function $f : \mathbb{R}^n \to \mathbb{R}$, a point $\boldsymbol{x} \in \mathbb{R}^n$ is called

- A **stationary point**, if $\nabla f(\boldsymbol{x}) = \mathbf{0}$;
- A **local minimum**, if $\boldsymbol{x}$ is a stationary point and there exists a neighborhood $\mathcal{B} \subseteq \mathbb{R}^n$ of $\boldsymbol{x}$ such that $f(\boldsymbol{x}) \leq f(\boldsymbol{y})$ for any $\boldsymbol{y} \in \mathcal{B}$;
- A **global minimum**, if $\boldsymbol{x}$ is a stationary point and $f(\boldsymbol{x}) \leq f(\boldsymbol{y})$ for any $\boldsymbol{y} \in \mathbb{R}^n$;
- **Saddle point**, if $\boldsymbol{x}$ is a stationary point and for any neighborhood $\mathcal{B} \subseteq \mathbb{R}^n$ of $\boldsymbol{x}$, there exist $\boldsymbol{y}, \boldsymbol{z} \in \mathcal{B}$ such that $f(\boldsymbol{z}) \leq f(\boldsymbol{x}) \leq f(\boldsymbol{y})$ and $\lambda_{\min}(\nabla^2 f(\boldsymbol{x})) \leq 0$.



(a) strict saddle　　(b) local minimum　　(c) global minimum

$\nabla f(x) = 0$

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Convex Optimization Problem

- Convex optimization problem in standard form:

$$\begin{aligned} \underset{\boldsymbol{x}}{\text{minimize}} \quad & f_0(\boldsymbol{x}) \\ \text{subject to} \quad & f_i(\boldsymbol{x}) \leq 0 \quad i = 1, \cdots, m \\ & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \end{aligned}$$

  where $f_0, f_1, \cdots, f_m$ are convex and equality constraints are affine.

- **Local and global optima:** any locally optimal point of a convex problem is globally optimal

- Most problems are not convex when formulated

- Reformulating a problem in convex form is an art, there is no systematic way

Example

- The following problem is nonconvex (why not?):

$$\begin{array}{ll} \underset{\boldsymbol{x}}{\text{minimize}} & x_1^2 + x_2^2 \\ \text{subject to} & x_1/(1+x_2^2) \leq 0 \\ & (x_1+x_2)^2 = 0 \end{array}$$

- The objective is convex.
- The equality constraint function is not affine; however, we can rewrite it as $x_1 = -x_2$ which is then a linear equality constraint.
- The inequality constraint function is not convex; however, we can rewrite it as $x_1 \leq 0$ which again is linear.
- We can rewrite it as

$$\begin{array}{ll} \underset{\boldsymbol{x}}{\text{minimize}} & x_1^2 + x_2^2 \\ \text{subject to} & x_1 \leq 0 \\ & x_1 = -x_2 \end{array}$$

x^* is a local minimum. $\forall y$. there exist a small $0<\theta<1$

$$f(x^*) < f(x^* + \theta(y - x^*)) \quad \checkmark \qquad ①$$

$$f(x^* + \theta(y - x^*)) = f(\theta y + (1-\theta) x^*)$$

$$\leq \theta f(y) + (1-\theta) f(x^*) \quad ②$$

$$f(x^*) < \theta f(y) + (1-\theta) f(x^*)$$

$$\Rightarrow \quad \theta f(x^*) < \theta f(y). \quad \forall y.$$

$$f(x^*) < f(y). \quad \forall y$$

$f(x^*)$. is a globally minimizer

Global and Local Optimality

Any locally optimal point of a convex problem is globally optimal.
Proof: Suppose $\boldsymbol{x}$ is locally optimal (around a ball of radius R) and $\boldsymbol{y}$ is optimal with $f_0(\boldsymbol{y}) < f_0(\boldsymbol{x})$. We will show this cannot be.

Just take the segment from $\boldsymbol{x}$ to $\boldsymbol{y}$: $\boldsymbol{z} = \theta \boldsymbol{y} + (1-\theta)\boldsymbol{x}$.
Obviously the objective function is strictly decreasing along the segment since $f_0(\boldsymbol{y}) < f_0(\boldsymbol{x})$:

$$\theta f_0(\boldsymbol{y}) + (1-\theta) f_0(\boldsymbol{x}) < f_0(\boldsymbol{x}) \qquad \theta \in (0,1]$$

Using now the convexity of the function, we can write

$$f_0(\theta \boldsymbol{y} + (1-\theta)\boldsymbol{x}) < f_0(\boldsymbol{x}) \qquad \theta \in (0,1]$$

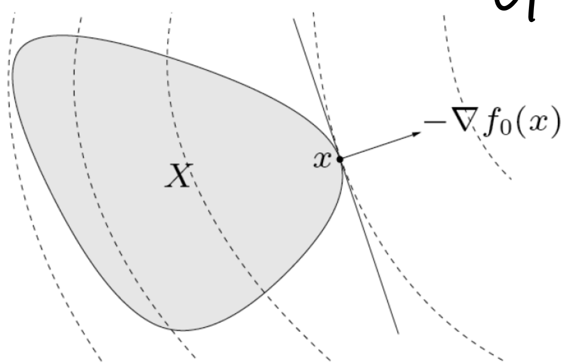
Finally, just choose θ sufficiently small such that the point $\boldsymbol{z}$ is in the ball of local optimality of $\boldsymbol{x}$, arriving at a contradiction.

Optimality Criterion for Differentiable f_0 I

Minimum Principle: A feasible point x is optimal if and only if

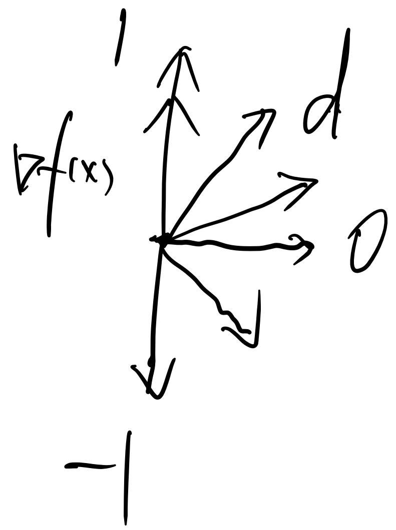
$$\nabla f_0(\boldsymbol{x})^T(\boldsymbol{y}-\boldsymbol{x}) \geq 0 \qquad \text{for all feasible } \boldsymbol{y}$$

$d = y - x$



$\nabla f_0^T(x) \cdot d \geq 0$

$f_0(y) \geq f_0(x) + \nabla f_0^T(x)(y-x), \quad \forall y \in \operatorname{dom} f_0$



1
d
$\nabla f(x)$
0
-1

f_0 is convex and differentiable

① $f_0(x) \le f_0(y), \ \forall y \in \operatorname{dom} f_0$

② $\nabla f_0^T(x)(y-x) \geq 0. \ \forall y \in \operatorname{dom} f_0$

③ $\nabla f_0(x) = 0$ unconstrained

Proof:

②⇒① convex definition ....

①⇒②. suppose $f_0(x) \le f_0(y), \ \forall y \in \operatorname{dom} f_0$

$\exists y .. \ \nabla f_0^T(x)(y-x) < 0$

$$\nabla f_0^T(x) = \lim_{\theta \to 0} \frac{f(x+\theta(y-x)) - f(x)}{\theta(y-x)} (y-x)$$

$z = x + \theta(y-x), \qquad \theta \in [0,1]$

$$\frac{df(z)}{d\theta} = \frac{d}{d\theta}\left(f(x+\theta(y-x))\right)\Big|_{\theta=0}$$

$$= \nabla f_0^T(x)(y-x) < 0 \quad \downarrow$$

$f(z) < f(x)$

③ ⇒ ② Clearly.

② ⇒ ③. $y = x - \theta \nabla f_0(x)$. $\theta > 0$ small

$$\nabla f_0^T(x)(y-x) = -\theta \|\nabla f_0(x)\|^2 \leq 0$$

≥ 0

$\Rightarrow \quad \nabla f_0(x) = 0$. $\square$

Optimality Criterion for Differentiable f_0 II

- **Unconstrained problem:** $\boldsymbol{x}$ is optimal iff

$$\boldsymbol{x} \in \operatorname{dom} f_0, \qquad \nabla f_0(\boldsymbol{x}) = 0$$

- **Equality constrained problem:** $\min_{\boldsymbol{x}} f_0(\boldsymbol{x}) \quad \text{s.t. } \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$
  $\boldsymbol{x}$ is optimal iff

$$\boldsymbol{x} \in \operatorname{dom} f_0, \qquad \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b},\ \nabla f_0(\boldsymbol{x}) + \boldsymbol{A}^T \boldsymbol{\nu} = \boldsymbol{0}$$

- **Minimization over nonnegative orthant:** $\min_{\boldsymbol{x}} f_0(\boldsymbol{x}) \quad \text{s.t. } \boldsymbol{x} \succeq \boldsymbol{0}$ $\boldsymbol{x}$ is optimal iff

$\nabla f(x)^T \cdot x = 0$

$$\boldsymbol{x} \in \operatorname{dom} f_0, \qquad \boldsymbol{x} \succeq 0, \begin{cases} \nabla_i f_0(\boldsymbol{x}) \geq 0 & x_i = 0 \\ \nabla_i f_0(\boldsymbol{x}) = 0 & x_i > 0 \end{cases}$$

$\nabla f_0^T(x)(y - x) \geq 0, \quad \forall y \geq 0,$

$\Updownarrow$

$x_i \cdot \nabla_i f_0(x_i) = 0, \quad i = 1, \cdots, n$

$$L(x,\nu) = f_0(x) + \nu^T(Ax-b)$$

$$\frac{\partial L}{\partial x} = \nabla f_0(x) + A^T \nu = 0$$

$$\frac{\partial L}{\partial \nu} = Ax - b = 0$$

Equivalent Reformulations I

- **Eliminating/introducing equality constraints:**

$$
\begin{array}{ll}
\underset{\boldsymbol{x}}{\text{minimize}} & f_0(\boldsymbol{x}) \\
\text{subject to} & f_i(\boldsymbol{x}) \leq 0 \quad i = 1, \cdots, m \\
 & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}
\end{array}
$$

is equivalent to

$$
\begin{array}{ll}
\underset{\boldsymbol{z}}{\text{minimize}} & f_0(\boldsymbol{F}\boldsymbol{z} + \boldsymbol{x}_0) \\
\text{subject to} & f_i(\boldsymbol{F}\boldsymbol{z} + \boldsymbol{x}_0) \leq 0 \quad i = 1, \cdots, m
\end{array}
$$

where $\boldsymbol{F}$ and $\boldsymbol{x}_0$ are such that $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \Longleftrightarrow \boldsymbol{x} = \boldsymbol{F}\boldsymbol{z} + \boldsymbol{x}_0$ for some $\boldsymbol{z}$.

$$A(Fz + X_0) = b$$

$$\Rightarrow AFz + AX_0 = b$$

$\underbrace{Fz}$.

$N(A) = \{ x / R^n \mid Ax = 0 \}$

$\underbrace{Fz \in N(A)}$

Equivalent Reformulations II

❧ **Introducing slack variables for linear inequalities:**

$$\begin{array}{ll} \underset{\boldsymbol{x}}{\text{minimize}} & f_0(\boldsymbol{x}) \\ \text{subject to} & \boldsymbol{a}_i^T \boldsymbol{x} \leq b_i \quad i = 1, \cdots, m \end{array}$$

is equivalent to

$$\begin{array}{ll} \underset{\boldsymbol{x}, \boldsymbol{s}}{\text{minimize}} & f_0(\boldsymbol{x}) \\ \text{subject to} & \boldsymbol{a}_i^T \boldsymbol{x} + s_i = b_i \quad i = 1, \cdots, m \\ & s_i \geq 0 \end{array}$$

Equivalent Reformulations III

❧ **Epigraph form:** a standard form convex problem is equivalent to

$$
\begin{array}{lll}
\underset{\boldsymbol{x},t}{\text{minimize}} & t & \\
\text{subject to} & f_0(\boldsymbol{x}) - t \leq 0 & \\
& f_i(\boldsymbol{x}) \leq 0 & i = 1, \cdots, m \\
& \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} &
\end{array}
$$

P_2

$\Updownarrow$

P_1

$$
\begin{array}{lll}
\min\limits_{x} & f_0(x) & \\
\text{s.t.} & f_i(x) \leq 0, & i = 1, \cdots, m \\
& Ax = b &
\end{array}
$$

Proof: $P_1 \Rightarrow P_2$

suppose x^* is the minimizer of P_1.

then $f_0(x^*) \leq f_0(x)$, $\forall x \in \text{dom} f_0$.

$$\underbrace{f_0(x^*)}_{\Delta} \leq f_0(x) \leq \underset{\sim}{t}$$

$f_0^*(x)$ is the minimizer for t.

$P_2 \Rightarrow P_1$: suppose (x^*, t^*) is minimizer of P_2

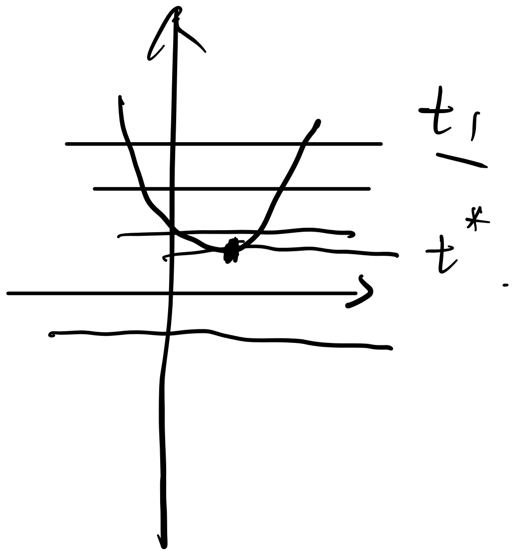
$$f(x^*) \leq t^* \leq t \quad \forall t$$

If $f(x^*)$ is not the minimizer of P_1.

then $\exists x'$. $f(x') < f(x^*)$

Let $t = f(x')$. $\quad t < f(x^*) \leq t^* \leq t$

conflict!



t_1
t^*

Equivalent Reformulations IV

- **Minimizing over some variables:**

$$\begin{array}{ll} \underset{\boldsymbol{x},\boldsymbol{y}}{\text{minimize}} & f_0(\boldsymbol{x},\boldsymbol{y}) \\ \text{subject to} & f_i(\boldsymbol{x}) \leq 0 \quad i = 1, \cdots, m \end{array}$$

is equivalent to

$$\begin{array}{ll} \underset{\boldsymbol{x}}{\text{minimize}} & \tilde{f}_0(\boldsymbol{x}) \\ \text{subject to} & f_i(\boldsymbol{x}) \leq 0 \quad i = 1, \cdots, m \end{array}$$

where $\tilde{f}_0(\boldsymbol{x}) = \inf_{\boldsymbol{y}} f_0(\boldsymbol{x},\boldsymbol{y})$

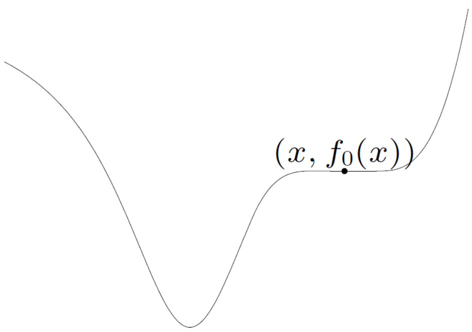
Outline

Quasiconvex Optimization

$$\begin{array}{ll} \underset{\boldsymbol{x}}{\text{minimize}} & f_0(\boldsymbol{x}) \\ \text{subject to} & f_i(\boldsymbol{x}) \leq 0 \quad i = 1, \cdots, m \\ & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \end{array}$$

where $f_0 : \mathbb{R}^n \longrightarrow \mathbb{R}$ is quasiconvex and $f_1, \cdots, f_m$ are convex

- Observe that it can have locally optimal points that are not (globally) optimal:



Quasiconvex Optimization

- **Convex representation** of sublevel sets of a quasiconvex function f_0: there exists a family of convex functions $\phi_t(\boldsymbol{x})$ for fixed t such that

$$f_0(\boldsymbol{x}) \le t \quad \Longleftrightarrow \quad \phi_t(\boldsymbol{x}) \le 0$$

- **Example:**

$$f_0(\boldsymbol{x}) = \frac{p(\boldsymbol{x})}{q(\boldsymbol{x})} \le t$$

with p convex, q concave, and $p(\boldsymbol{x}) \ge 0$, $q(\boldsymbol{x}) > 0$ on $\operatorname{dom} f_0$. We can choose:

$$\phi_t(\boldsymbol{x}) = p(\boldsymbol{x}) - t q(\boldsymbol{x})$$

  - for $t \ge 0$, $\phi_t(\boldsymbol{x})$ is convex in $\boldsymbol{x}$
  - $p(\boldsymbol{x})/q(\boldsymbol{x}) \le t$ if and only if $\phi_t(\boldsymbol{x}) \le 0$

Quasiconvex Optimization

Solving a quasiconvex problem via convex feasibility problems: the idea is to solve the epigraph form of the problem with a sandwich technique in t:

- for fixed t the epigraph form of the original problem reduces to a feasibility convex problem

$$\phi_t(\boldsymbol{x}) \leq 0, \quad f_i(\boldsymbol{x}) \leq 0 \forall i, \quad \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}$$

- if t is too small, the feasibility problem will be infeasible
- if t is too large, the feasibility problem will be feasible
- start with upper and lower bounds on t (termed u and l, resp.) and use a sandwich technique (bisection method): at each iteration use $t = (l + u)/2$ and update the bounds according to the feasibility or infeasibility of the problem.

After k iteration, the interval is $\frac{u-l}{2^k} \leq \varepsilon$

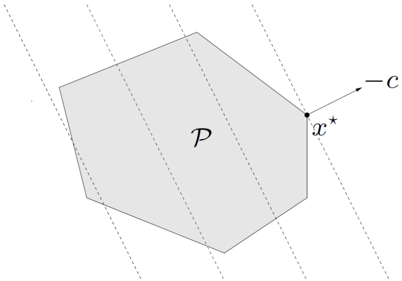
$\Rightarrow \quad k \geq \log_2 \frac{u-l}{\varepsilon}$

Outline

Linear Programming (LP)

$$\begin{array}{ll} \underset{\boldsymbol{x}}{\text{minimize}} & \boldsymbol{c}^T \boldsymbol{x} + d \\ \text{subject to} & \boldsymbol{G}\boldsymbol{x} \leq \boldsymbol{h} \\ & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \end{array}$$

- Convex problem: affine objective and constraint functions.
- Feasible set is a polyhedron:



ℓ_1- and ℓ_∞- Norm Problems as LPs I

- **ℓ_∞-norm minimization:**

$$\begin{array}{ll} \underset{\boldsymbol{x}}{\text{minimize}} & \|\boldsymbol{x}\|_\infty \\ \text{subject to} & \boldsymbol{G}\boldsymbol{x} \leq \boldsymbol{h} \\ & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \end{array}$$

is equivalent to the LP

$$\begin{array}{ll} \underset{t,\boldsymbol{x}}{\text{minimize}} & t \\ \text{subject to} & -t\mathbf{1} \preceq \boldsymbol{x} \preceq t\mathbf{1} \\ & \boldsymbol{G}\boldsymbol{x} \leq \boldsymbol{h} \\ & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \end{array}$$

ℓ_1- and ℓ_∞- Norm Problems as LPs II

- **ℓ_1-norm minimization:**

$$\begin{array}{ll} \underset{\boldsymbol{x}}{\text{minimize}} & \|\boldsymbol{x}\|_1 \\ \text{subject to} & \boldsymbol{G}\boldsymbol{x} \leq \boldsymbol{h} \\ & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \end{array}$$

is equivalent to the LP

$$\begin{array}{ll} \underset{\boldsymbol{t},\boldsymbol{x}}{\text{minimize}} & \sum_i t_i \\ \text{subject to} & -\boldsymbol{t} \preceq \boldsymbol{x} \preceq \boldsymbol{t} \\ & \boldsymbol{G}\boldsymbol{x} \leq \boldsymbol{h} \\ & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \end{array}$$

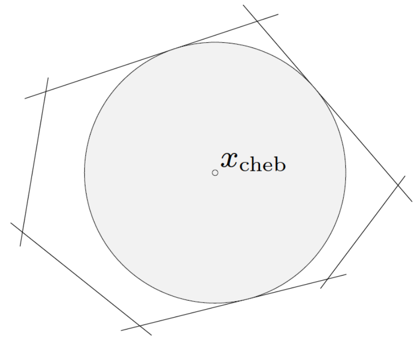
Examples: Chebyshev Center of a Polyhedron I

- Chebyshev center of a polyhedron

$$\mathcal{P} = \{\boldsymbol{x} \mid \boldsymbol{a}_i^T \boldsymbol{x} \le b_i,\ i = 1, \cdots, m\}$$

is center of largest inscribed ball

$$\mathcal{B} = \{\boldsymbol{x}_c + \boldsymbol{u} \mid \|\boldsymbol{u}\| \le r\}$$



- Let's solve the problem

$$\begin{array}{ll} \underset{r, \boldsymbol{x}_c}{\text{maximize}} & r \\ \text{subject to} & \boldsymbol{x} \in \mathcal{P} \quad \text{for all} \quad \boldsymbol{x} = \boldsymbol{x}_c + \boldsymbol{u} \text{ with } \|\boldsymbol{u}\| \le r \end{array}$$

- Observe that $\boldsymbol{a}_i^T \boldsymbol{x} \le b_i$ for all $\boldsymbol{x} \in \mathcal{B}$ if and only if

$$\sup_{\boldsymbol{u}} \{\boldsymbol{a}_i^T (\boldsymbol{x}_c + \boldsymbol{u}) \mid \|\boldsymbol{u}\| \le r\} \le b_i$$

$a_i^T u \le \|a_i\| \cdot \|u\|$ $a_i^T(x_c+u) \le b_i$, $\|u\| \le r$

$= r \cdot \|a_i\|$

Examples: Chebyshev Center of a Polyhedron II

- Using Schwartz inequality, the supremum condition can be rewritten as
$$\boldsymbol{a}_i^T \boldsymbol{x}_c + r\|\boldsymbol{a}_i\|_2 \le b_i$$
- Hence, the Chebyshev center can be obtained by solving:

$$\begin{array}{ll} \underset{r, \boldsymbol{x}_c}{\text{maximize}} & r \\ \text{subject to} & \boldsymbol{a}_i^T \boldsymbol{x}_c + r\|\boldsymbol{a}_i\|_2 \le b_i, \quad i = 1, \cdots, m \end{array}$$

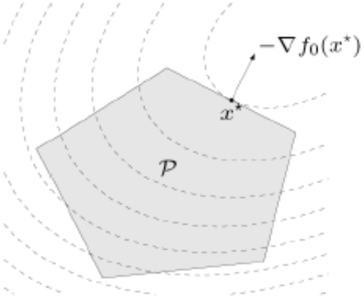
which is an LP.

Quadratic Programming (QP)

$$\begin{array}{ll} \underset{\boldsymbol{x}}{\text{minimize}} & (1/2)\,\boldsymbol{x}^T\boldsymbol{P}\boldsymbol{x} + \boldsymbol{q}^T\boldsymbol{x} + r \\ \text{subject to} & \boldsymbol{G}\boldsymbol{x} \le \boldsymbol{h} \\ & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \end{array}$$

- Convex problem (assuming $\boldsymbol{P} \in \mathbb{S}^n_+ \succeq \mathbf{0}$): convex quadratic objective and affine constraint functions.

- Minimization of a convex quadratic function over a polyhedron:



Quadratically Constrained QP (QCQP)

$$\begin{array}{lll} \underset{\boldsymbol{x}}{\text{minimize}} & (1/2)\, \boldsymbol{x}^T \boldsymbol{P}_0 \boldsymbol{x} + \boldsymbol{q}_0^T \boldsymbol{x} + r_0 & \\ \text{subject to} & (1/2)\, \boldsymbol{x}^T \boldsymbol{P}_i \boldsymbol{x} + \boldsymbol{q}_i^T \boldsymbol{x} + r_i \leq 0 & i = 1, \cdots, m \;\checkmark \\ & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} & \end{array}$$

- Convex problem (assuming $\boldsymbol{P}_i \in \mathbb{S}_+^n \succeq \mathbf{0}$): convex quadratic objective and constraint functions.

Second-Order Cone Programming (SOCP)

$$
\begin{array}{ll}
\underset{\boldsymbol{x}}{\text{minimize}} & \boldsymbol{f}^T \boldsymbol{x} \\
\text{subject to} & \underbrace{\|\boldsymbol{A}_i \boldsymbol{x} + \boldsymbol{b}_i\| \le \boldsymbol{c}_i^T \boldsymbol{x} + d_i}_{} \quad i = 1, \cdots, m \\
& \boldsymbol{F}\boldsymbol{x} = \boldsymbol{g}
\end{array}
$$

$$-(c_i^T x + d_i) \le A_i x + b_i \le c_i^T x + d_i$$

- Convex problem: linear objective and second-order cone constraints
- For $\boldsymbol{A}_i$ row vector, it reduces to an LP
- For $\boldsymbol{c}_i = \boldsymbol{0}$, it reduces to a QCQP
- More general than QCQP and LP

$$\underbrace{(A_i x + b_i)^T (A_i x + b_i)}_{}$$

$$= (x^T A_i^T + b_i^T)(A_i x + b_i)$$

Robust LP as an SOCP

- Sometimes, the parameters of an optimization problem are imperfect

- Consider the robust LP:

$$\begin{array}{ll} \underset{\boldsymbol{x}}{\text{minimize}} & \boldsymbol{c}^T\boldsymbol{x} \\ \text{subject to} & \boldsymbol{a}_i^T\boldsymbol{x} \leq b_i \quad \forall \boldsymbol{a}_i \in \mathcal{E}_i,\ i = 1, \cdots, m \end{array}$$

where $\mathcal{E}_i = \{\bar{\boldsymbol{a}}_i + \boldsymbol{P}_i\boldsymbol{u} \mid \|\boldsymbol{u}\| \leq 1\}$

- It can be rewritten as the SOCP:

$$\begin{array}{ll} \underset{\boldsymbol{x}}{\text{minimize}} & \boldsymbol{c}^T\boldsymbol{x} \\ \text{subject to} & \bar{\boldsymbol{a}}_i^T\boldsymbol{x} + \|\boldsymbol{P}_i^T\boldsymbol{x}\|_2 \leq b_i \quad i = 1, \cdots, m \end{array}$$

Generalized Inequality Constraints

- Convex problem with generalized inequality constraints:

$$\begin{array}{ll} \underset{\boldsymbol{x}}{\text{minimize}} & f_0(\boldsymbol{x}) \\ \text{subject to} & \boldsymbol{f}_i(\boldsymbol{x}) \preceq_{K_i} \mathbf{0} \quad i = 1, \cdots, m \\ & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \end{array}$$

  where f_0 is convex and f_i are K_i-convex w.r.t. proper cone K_i
- It has the same properties as a standard convex problem
- **Conic form problem:** special case with affine objective and constraints:

$$\begin{array}{ll} \underset{\boldsymbol{x}}{\text{minimize}} & \boldsymbol{c}^T \boldsymbol{x} \\ \text{subject to} & \boldsymbol{F}\boldsymbol{x} + \boldsymbol{g} \preceq_K \mathbf{0} \\ & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \end{array}$$

Semidefinite Programming (SDP)

$$\begin{array}{ll} \underset{\boldsymbol{x}}{\text{minimize}} & \boldsymbol{c}^T \boldsymbol{x} \\ \text{subject to} & x_1 \boldsymbol{F}_1 + x_2 \boldsymbol{F}_2 + \cdots + x_n \boldsymbol{F}_n \preceq \boldsymbol{G} \\ & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \end{array}$$

- Inequality constraint is called linear matrix inequality (LMI)
- Convex problem: linear objective and linear matrix inequality (LMI) constraints
- Observe that multiple LMI constraints can always be written as a single one

SDP I

❧ LP and equivalent SDP:

$$\begin{array}{ll} \underset{\boldsymbol{x}}{\text{minimize}} & \boldsymbol{c}^T\boldsymbol{x} \\ \text{subject to} & \boldsymbol{A}\boldsymbol{x} \preceq \boldsymbol{b} \end{array} \qquad \begin{array}{ll} \underset{\boldsymbol{x}}{\text{minimize}} & \boldsymbol{c}^T\boldsymbol{x} \\ \text{subject to} & \text{diag}(\boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}) \preceq \boldsymbol{0} \end{array}$$

❧ SOCP and equivalent SDP:

$$\begin{array}{ll} \underset{\boldsymbol{x}}{\text{minimize}} & \boldsymbol{f}^T\boldsymbol{x} \\ \text{subject to} & \|\boldsymbol{A}_i\boldsymbol{x} + \boldsymbol{b}_i\|^2 \le (\boldsymbol{c}_i^T\boldsymbol{x} + d_i,)^2 \quad i = 1, \cdots, m \end{array}$$

$$\Updownarrow$$

$$\begin{array}{ll} \underset{\boldsymbol{x}}{\text{minimize}} & \boldsymbol{f}^T\boldsymbol{x} \\ \text{subject to} & \begin{bmatrix} (\boldsymbol{c}_i^T\boldsymbol{x} + d_i)\boldsymbol{I} & \boldsymbol{A}_i\boldsymbol{x} + \boldsymbol{b}_i \\ \boldsymbol{A}_i\boldsymbol{x} + \boldsymbol{b}_i & \boldsymbol{c}_i^T\boldsymbol{x} + d_i \end{bmatrix} \succeq \boldsymbol{0}, \quad i = 1, \cdots, m \end{array}$$

$$X = \begin{bmatrix} A & B \\ D & C \end{bmatrix} \succeq 0 \iff A - BC^{-1}D \succeq 0,\ C \succ 0 \iff C - DA^{-1}B \succeq 0,\ A \succ 0$$

SDP II

- **Eigenvalue minimization:**

$$\underset{\boldsymbol{x}}{\text{minimize}} \quad \lambda_{\max}(\boldsymbol{A}(\boldsymbol{x}))$$

where $\boldsymbol{A}(\boldsymbol{x}) = \boldsymbol{A}_0 + x_1\boldsymbol{A}_1 + \cdots + x_n\boldsymbol{A}_n$, is equivalent to SDP

$$\begin{array}{ll} \underset{\boldsymbol{x},t}{\text{minimize}} & t \\ \text{subject to} & \boldsymbol{A}(\boldsymbol{x}) \preceq t\boldsymbol{I} \end{array}$$

- It follows from

$$\lambda_{\max}(\boldsymbol{A}(\boldsymbol{x})) \leq t \iff \boldsymbol{A}(\boldsymbol{x}) \preceq t\boldsymbol{I}$$

Reference

Chapter 4 of:

- Stephen Boyd and Lieven Vandenberghe, *Convex Optimization*. Cambridge, U.K.: Cambridge University Press, 2004.