Online Lecture Notes

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1 General comments

The first half of this course is all about Lectures 1,2,3,4,5, and 6. The idea is that the mid-term will cover these six lecture. Some comments are:

1. We are aware that the epidemic situation is not ideal. We are open the postponing the mid-exam a little bit such that we have time to prepare all the material and do enough exercises before the exam. The initial plan was to run the mid-term exam on April 7, but we will most likely postpone by one or two week.

2. Some students mentioned that it's difficult to follow some of the lectures. Generally, it would help us if you let us know, which are parts are difficult.

   (a) So one thing we can do is that we repeat some aspect of lecture in the coming weeks.

   (b) We can also do exercises together if you like. So just let us know which parts you would like to repeat or train more.

   (c) The current homework are not too excessive in order to give you to decide how much training you need or many exercises you want to solve by yourself. There is plenty of exercises in the lecture notes.

   (d) Additionally, there are exercise session on Friday evening; please join the exercises—they are provided by our TA Yuxuan Guo—you can ask all question in Chinese.

   (e) If you have difficulties following my English, there is several things you can do: first, I'll try to write up all I say in the lecture notes. Secondly, ask more questions either to me or to our TA (if you like Chinese, Yuxuan can help to translate).

3. We will have one set of homework problems per lecture. We just sent Homework 4.

4. In order to train for the exams, it will be important to go over previous exam problems. Especially, look at the ones from 2019 and 2020. Of course, during this course, our exam problem will look very similar to ones from previous years. I'll say much more about this in the coming weeks.

5. If you want to check current learning status: make sure that you understand how to work with scalar systems. The most important part from Lecture 1-4 are:

   (a) Standard form of a scalar linear system and its explicit solution. There is definitely always at least one mid-term exam question on this. (Lecture 1). We are revisiting Lecture 1 as part of Lecture 5, which discusses the multivariate case.

   (b) Make sure that you understand the explicit solution for an open-loop controlled scalar system. Basically, if we provide an input function u, you need to know how to find the corresponding state x and interpret solution. (Lecture 2)

   (c) Make sure that you understand how to analyze a linear system with linear feedback (proportional controller; see Lecture 3). We will come back to this in Lecture 6, which will be about the multivariate case.

   (d) Also recall the techniques from Lecture 4 about the (mostly scalar) nonlinear systems. We discussed many aspects, this includes *Separation of Variables, see Homework 4, Exercise 1*, *Proofs of Existence and Uniqueness; a bit less important for the exam—you only need to be able to know the results, it's not required to the proof by yourself.* For the Picard iteration, see Homework 4, Exercise 2. And last but not least, make sure that you understand basic numerical integration schemes such as Taylor model based integration and Runge-Kutta integrators. Towards the end of Lecture 4 we have also discussed how to use this to linearize nonlinear ODEs close to their steady-state or periodic. This is a lot of material: let us know if you want to repeat parts of this, or if you need more exercises.

   The experience from previous years (I gave this course many times before) is that most students have no problems following Lecture 1-4, but Lecture 5 and 6 are bit more difficult as we need methods from Linear Algebra. I have been trying to slow down a bit during these lectures, while Lecture 1-4 was may be a bit faster—let us know if it was too fast. Please keep in mind we have enough time during this semester to get through with all the material. There is plenty of time to recall things that were to fast.

6. Homework 4b:

   (a) Send an email to Yuxuan (our TA), summarizing all part of the lecture that we difficult for you. Deadline: the exercise session tomorrow evening!!! Please let us know: for me it is difficult to estimate which part of the lecture are to fast—I can even see you during the online teaching. If you don't send feedback we are running blind...

2 Matrix Exponential (continue)

The goal of this section to work out a general computationally tractable expression for the matrix exponential. We proceed in two steps. First we work out the matrix exponential for the simpler case that A is diagonalizable. And in a second step, we discuss the general case.

2.1 Matrix exponential of diagonalizable matrices

Last lecture we disucssed the case that A is diagonalizable. This means that we write A in the form

$$A = TDT^{-1} \ .$$

Here, D is a diagonal matrix, whose diagonal components are the eigenvalues of A. They can be found by solving the characteristic equation

$$\det(A - \lambda I) = 0 \ ,$$

which has n_x complex-valued solutions for λ, the eigenvalues of A, Here, n_x is the number of states, $A \in \mathbb{R}^{n_x \times n_x}$. The corresponding eigenvectors (assuming they exist) satisfy

$$AT_i = \lambda_i T_i$$

with $T_i \in \mathbb{C}^{n_x}$. The matrix T is composed by sorting all of the eigenvectors of A into the matrix T. Finally, the matrix exponential is given by

$$e^{At} = \sum_{i=0}^{\infty} \frac{1}{i!} t^i [TDT^{-1}]^i = T \left[\sum_{i=0}^{\infty} \frac{1}{i!} t^i D^i \right] T^{-1} = e^{Dt}$$

The diagonal component of the matrix e^{Dt} are given by

$$e^{D_{ii}t} = e^{\lambda_i t} = e^{(\mathrm{Re}(\lambda_i) + \sqrt{-1} \cdot \mathrm{Im}(\lambda_i))t} = e^{\mathrm{Re}(\lambda_i)t} \left[\cos(\mathrm{Im}(\lambda_i)t) + \sqrt{-1} \cdot \sin(\mathrm{Im}(\lambda_i))t) \right]$$

Thus, if we introduce the shorthands $\sigma_i = \mathrm{Re}(\lambda_i)$ and $\omega_i = \mathrm{Im}(\lambda_i)$, we can write this as

$$e^{\lambda_i t} = e^{\sigma_i t} \left[\cos(\omega_i t) + \sqrt{-1} \cdot \sin(\omega_i t) \right]$$

This means that σ_i can be interpreted as a growth or decay factor. There are three cases:

1. If $\sigma_i > 0$, this means that the amplitude of the oscillation will increase exponentially over time.

2. If $\sigma_i = 0$, the system keep on oscillating forever. In this case we have purely imaginary eigenvalues, which means that there is only a oscillation that is periodic. Example: an ideal spring without damping would oscillate forever under the (in practice unrealistic) assumption that no energy is dissipated.

3. If $\sigma_i < 0$ the amplidute of the oscillation will decrease exponentially.

2.2 General Matrix Exponentials

Notice that all matrices $A \in \mathbb{R}^{n_x \times n_x}$ are diagonalizable, although the set of matrices that are not diagonalizable has Lebesgue measure zero—in other words almost all matrices are diagonalizable. One important special case though is the double integrator example, where we have

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} .$$

In this example, the eigenvalues of A are both equal to zero, because

$$0 = \det(A - \lambda I) = \det\left(\begin{pmatrix} -\lambda & 1 \\ 0 & -\lambda \end{pmatrix} \right) = \lambda^2 \qquad \Longleftrightarrow \qquad \lambda_{1,2} = 0 .$$

If we try to search for an eigenvector, we need to solve

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} T_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \Longleftrightarrow \qquad T_{12} = 0$$

This means that all possible eigenvectors are given by

$$T_1 = \begin{pmatrix} T_{11} \\ 0 \end{pmatrix} = T_{11} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{for any} \quad T_{11} \in \mathbb{C}$$

But all of these eigenvectors are linearly dependent!!! Basically, this means that we cannot find a basis of eigenvector that spans $\mathbb{R}^2$. In summary, there exist matrices A that are not diagonalizable (even if this special case does not occur all too frequently in practice).

In order to deal with this case, we need to recall a basic result from linear algebra, which is saying that every matrix $A \in \mathbb{R}^{n_x \times n_x}$ can be written in Jordan normal form,

$$A = T(D + N)T^{-1},$$

where T is invertible, D is diagonal, N is nilpotent, $N^m = 0$ for at least one $m \leq n_x$, and D and N commute; that is

$$DN = ND .$$

This is a bit of an advanced result from linear algebra—we'll do a few examples and exercises about this later. For now you just need to remember that such a decomposition is possible (in principle this can be found in standard linear algebra books, but we recall more later).

One things I want to discuss today is that we can use the Jordan normal form to work out a general expression for the matrix exponential, namely,

$$e^{At} = \sum_{i=0}^{\infty} \frac{1}{i!} t^i \left[T(D+N)T^{-1} \right]^i \tag{1}$$

$$= T \left[\sum_{i=0}^{\infty} \frac{1}{i!} t^i \left[(D+N) \right]^i \right] T^{-1} = T e^{(D+N)t} T^{-1} \tag{2}$$

This steps is exactly the same as for the case that A is diagonalizable! But next, we can use that D and N commute, which means that we can apply the addition theorem for the matrix exponential,

$$e^{(D+N)t} = e^{Dt} e^{Nt} \ .$$

This is fortunate, because we have already anlyzed terms of the form e^{Dt} (see above). The term e^{Nt} is easy to analyze, too, since N is nilpotent,

$$e^{Nt} = \sum_{i=0}^{\infty} \frac{1}{i!} t^i N^i = \sum_{i=0}^{m-1} \frac{1}{i!} t^i N^i \ .$$

This is a polynomial of order $m-1$ in t. This means that our matrix exponential has in general severar terms,

$$e^{At} = T e^{Dt} \left[\sum_{i=0}^{m-1} \frac{1}{i!} t^i N^i \right] T^{-1} \ .$$

This means that the coefficients of e^{At} are either

1. exponential functions (if the real part of some eigenvalues of A is non-zero), or
2. sine and cosine function (if the imaginary part of some eigenvalues of A is non-zero), or
3. polynomials (if A has a non-trivial Jordan block), or
4. products of the above functions (if all of these things happen simultaneously), or
5. linear combinations of all of these functions (since the matrix T constant).

This is a very interesting result, since this means that there are no other functions (apart from 1-5) are possible. There can't be suddenly a log-function or squareroot or other complicated stuff. With this we have completely undertsood the matrix exponentials for linear systems! The above linear algebra based strategy is completely general; it works for all matrices A.

3 Applications of Matrix Exponentials to Linear Systems

The goal of the next lecture will be to show that matrix exponentials can be used to find the general solution of a linear time-invariant differential equation system. In order to understand the main idea, we consider the homogeneous system

$$\dot{x}(t) \;=\; Ax(t) \qquad \text{with} \qquad x(0) \;=\; x_0,$$

with $x_0 \in \mathbb{R}^{n_x}$ and $A \in \mathbb{R}^{n_x \times n_x}$ being given. It is now easy to see that we can directly write down the solution of this equation by using matrix exponentials, namely, we have

$$x(t) = e^{At} x_0 \; .$$

This is in complete analogy to our result from Lecture 1. The only things that we have done here is to replace the scalar "a" with the matrix "A". This can be proven by recalling the following two main properties of the matrix exponential:

$$e^0 \;=\; I \tag{3}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} e^{At} \;=\; Ae^{At} \; . \tag{4}$$

This implies that

$$\frac{\mathrm{d}}{\mathrm{d}t} x(t) \;=\; \frac{\mathrm{d}}{\mathrm{d}t}\left[e^{At} x_0\right] \overset{(4)}{=} Ae^{At} x_0 = Ax(t) \tag{5}$$

$$x(0) \;=\; e^0 x_0 \overset{(3)}{=} I x_0 = x_0 \; . \tag{6}$$

With this, we have shown that $x(t) = e^{At} x_0$ is indeed a solution to the above linear differential equation.

4 Summary and Outlook

In this lecture we have completed our analysis of matrix exponentials. We now know how to compute them by using tools from linear algebra. Moreover, we have motivated why these matrix exponentials are extremely powerful for solving linear differential equations. In the next lecture, we will

1. Discuss how to solve general linear differential equations by using matrix exponentials.
2. We will show that the solution is unique, and
3. We will discuss many examples!

In addition, please remember to send us feedback about this lecture. Let us know which parts are difficult, where you need more exercises or summaries, or more material...