Numerical Optimization, 2021 Fall

Homework 5 Solution

1 Activity Location

There is in general a strong connection between the theories of optimization and free competition, which is illustrated by an idealized model of activity location. Suppose there are n economic activities (various factories, homes, stores, etc.) that are to be individually located on n distinct parcels of land. If activity i is located on parcel j that activity can yield s_{ij} units (dollars) of value.

If the assignment of activities to land parcels is made by a central authority, it might be made in such a way as to maximize the total value generated. In other words, the assignment would be made so as to maximize $\sum_i \sum_j s_{ij}x_{ij}$ where

$$x_{ij} = \begin{cases} 1 & \text{if activity } i \text{ is assigned to parcel } j \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

More explicitly this approach leads to the optimization problem

$$\begin{aligned} \max \quad & \sum_i \sum_j s_{ij}x_{ij} \\ \text{s.t.} \quad & \sum_j x_{ij} = 1, \quad i = 1, 2, \ldots, n \\ & \sum_i x_{ij} = 1, \quad j = 1, 2, \ldots, n \\ & x_{ij} \geq 0, \quad x_{ij} = 0 \text{ or } 1. \end{aligned} \tag{2}$$

Actually, it can be shown that the final requirement ($x_{ij} = 0$ or 1) is automatically satisfied at any extreme point of the set defined by the other constraints, so that in fact the optimal assignment can be found by using the simplex method of linear programming.

If one considers the problem from the viewpoint of free competition, it is assumed that, rather than a central authority determining the assignment, the individual activities bid for the land and thereby establish prices.

(1) Show that there exists a set of activity prices p_i, $i = 1, 2, \ldots, n$ and land prices q_j, $j = 1, 2, \ldots, n$ such that

$$p_i + q_j \geq s_{ij}, \quad i = 1, 2, \ldots, n, \quad j = 1, 2, \ldots, n \tag{3}$$

with equality holding if in an optimal assignment activity i is assigned to parcel j. [20pts]

We write the dual of the problem as

$$\begin{aligned} \min \quad & \sum_i p_i + \sum_j q_j \\ \text{s.t.} \quad & p_i + q_j \geq s_{ij}, \quad i, j \in \{1, 2, \ldots, n\} \\ & p_i, q_j \text{ free.} \end{aligned} \tag{4}$$

To show that there exists i and j for which $p_i + q_j \geq s_{ij}$, it is enough to show that the primal (the original maximization problem) has a solution. To do this, it is enough to show that the primal objective function is bounded. Note that since the sum of x_{ij} over i and(or) j is 1 according to primal constraints, $\sum_i \sum_j s_{ij}$ is an upper bound for the objective function. Therefore, primal has a solution and this means that the dual problem is feasible. That is, there exist i and j for which $p_i + q_j \geq s_{ij}$.
If in an optimal assignment activity i is assigned to parcel j, we have $x_{ij} = 1$. By complementary slackness, we have $p_i + q_j = s_{ij}$.

(2) Show that Part (1) implies that if activity i is optimally assigned to parcel j and if j' is any other parcel

$$s_{ij} - q_j \geq s_{ij'} - q_{j'} \tag{5}$$

Give an economic interpretation of this result and explain the relation between free competition and optimality in this context. [20pts]
By Part (1), we have $p_i + q_j = s_{ij}$ and $p_i + q_{j'} \geq s_{ij'}$. Hence, $s_{ij} - q_j = p_i \geq s_{ij'} - q_{j'}$.
s_{ij} is the value created by locating activity i at parcel j, and q_j is the price of land j. Their difference is the net profit generated by locating activity i at parcel j.
Therefore, choosing j such that

$$s_{ij} - q_j \geq s_{ij'} - q_{j'} \tag{6}$$

is to choose the location for activity i with the maximum net profit.
The equilibrium in free competition achieves both primal and dual optimality. Primal objective value (where the central authority maximizes its total revenue) is equal to the dual objective value (where the individual activities minimize their total price/cost).

(3) Assuming that each s_{ij} is positive, show that the prices can all be assumed to be nonnegative. [20pts]
Consider change the constraints $\sum_i x_{ij} = 1, \sum_j x_{ij} = 1$ to $\sum_i x_{ij} \leq 1, \sum_j x_{ij} \leq 1$ in the primal. Then equality and inequality are equivalent if $s_{ij} > 0$. In the latter case, the dual variables are non-negative.

2 Production Plan

A textile firm is capable of producing three products—x_1, x_2, x_3. Its production plan for next month must satisfy the constraints

$$\begin{aligned} x_1 + 2x_2 + 2x_3 &\leq 12 \\ 2x_1 + 4x_2 + x_3 &\leq f \\ x_1, x_2, x_3 &\geq 0. \end{aligned} \tag{7}$$

The first constraint is determined by equipment availability and is fixed. The second constraint is determined by the availability of cotton. The net profits of the products are 2, 3, and 3, respectively, exclusive of the cost of cotton and fixed costs.

(1) Find the shadow price λ_2 of the cotton input as a function of f, then plot $\lambda_2(f)$ and the net profit $z(f)$ exclusive of the cost for cotton. [30pts]
The LP can be formulated as

$$\begin{aligned} \max \quad & 2x_1 + 3x_2 + 3x_3 \\ \text{s.t.} \quad & x_1 + 2x_2 + 2x_3 \leq 12 \\ & 2x_1 + 4x_2 + x_3 \leq f \\ & x_1, x_2, x_3 \geq 0. \end{aligned} \tag{8}$$

We can solve this LP with simplex method. Transform the above LP to its standard form

$$\begin{aligned}
\min \quad & -2x_1 - 3x_2 - 3x_3 \\
\text{s.t.} \quad & x_1 + 2x_2 + 2x_3 + y_1 = 12 \\
& 2x_1 + 4x_2 + x_3 + y_2 = f \\
& x_1, x_2, x_3 \geq 0 \\
& y_1, y_2 \geq 0.
\end{aligned} \tag{9}$$

Then we have our first tableau

$$\begin{array}{ccccc|c}
1 & 2 & 2 & 1 & 0 & 12 \\
2 & 4 & 1 & 0 & 1 & f \\
\hline
-2 & -3 & -3 & 0 & 0 & 0
\end{array}$$

Now we have two situations to consider

(a) Suppose $\frac{f}{4} < \frac{12}{2}$, i.e. $\boldsymbol{f < 24}$, then we pivot around the blue 4 element

$$\begin{array}{ccccc|c}
0 & 0 & 3/2 & 1 & -1/2 & 12 - f/2 \\
1/2 & 1 & 1/4 & 0 & 1/4 & f/4 \\
\hline
-1/2 & 0 & -9/4 & 0 & 3/4 & 3f/4
\end{array}$$

Now we got two more situations to think about

i. Suppose $\frac{f/4}{1/4} < \frac{12-f/2}{3/2}$, i.e. $\boldsymbol{f < 6}$, then we pivot around the blue $1/4$ element and we are done

$$\begin{array}{ccccc|c}
-3 & -6 & 0 & 1 & -2 & 12 - 2f \\
2 & 4 & 1 & 0 & 1 & f \\
\hline
4 & 9 & 0 & 0 & 3 & \boldsymbol{3f}
\end{array}$$

ii. Suppose $\frac{f/4}{1/4} \geq \frac{12-f/2}{3/2}$, i.e. $\boldsymbol{f \geq 6}$, then we pivot around the green $3/2$ element

$$\begin{array}{ccccc|c}
0 & 0 & 1 & 2/3 & -1/3 & 8 - f/3 \\
\boxed{1/2} & 1 & 0 & -1/6 & 1/3 & f/3 - 2 \\
\hline
-1/2 & 0 & 0 & 3/2 & 0 & 18
\end{array}$$

Pivot around the boxed $1/2$ element and we are done

$$\begin{array}{ccccc|c}
0 & 0 & 1 & 2/3 & -1/3 & 8 - f/3 \\
1 & 2 & 0 & -1/3 & 2/3 & 2f/3 - 4 \\
\hline
0 & 1 & 0 & 4/3 & 1/3 & \boldsymbol{16 + f/3}
\end{array}$$

(b) Suppose $\frac{f}{4} \geq \frac{12}{2}$, i.e. $\boldsymbol{f \geq 24}$, then we pivot around the green 2 element

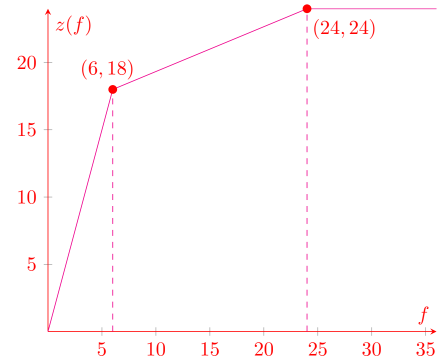
$$\begin{array}{ccccc|c}
\boxed{1/2} & 1 & 1 & 1/2 & 0 & 6 \\
0 & 0 & -3 & -2 & 1 & f - 24 \\
\hline
-1/2 & 0 & 0 & 3/2 & 0 & 18
\end{array}$$

Pivot around the boxed $1/2$ element and we are done

$$\begin{array}{ccccc|c} 1 & 2 & 2 & 1 & 0 & 12 \\ 0 & 0 & -3 & -2 & 1 & f-24 \\ \hline 0 & 1 & 1 & 1/2 & 0 & \mathbf{24} \end{array}$$

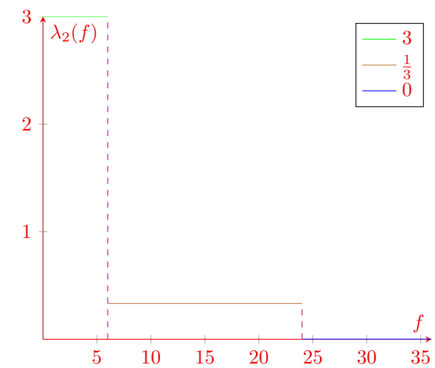
Therefore, we have the net profit exclusive of the cost for cotton as

$$z(f) = \begin{cases} 24 & \text{if } f \geq 24 \\ \dfrac{f}{3} + 16 & \text{if } 6 \leq f < 24 \\ 3f & \text{if } f < 6. \end{cases} \tag{10}$$



and the shadow price as

$$\lambda_2(f) = \begin{cases} 0 & \text{if } f > 24 \\ \left[0, \dfrac{1}{3}\right] & \text{if } f = 24 \\ \dfrac{1}{3} & \text{if } 6 < f < 24 \\ \left[\dfrac{1}{3}, 3\right] & \text{if } f = 6 \\ 3 & \text{if } f < 6. \end{cases} \tag{11}$$



(2) The firm may purchase cotton on the open market at a price of 1/6. However, it may acquire a limited amount s at a price of 1/12 from a major supplier that it purchases from frequently. Determine the net profit of the firm $\pi(f)$ as a function of f, according to the limited amount s. [10pts]
Consider cases according to the relation between s and f

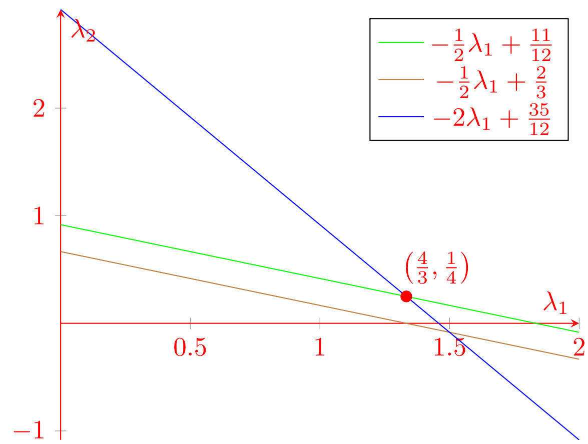
(a) If $f < s$, then the textile firm can purchase all cotton from the major supplier, with a cheaper price. Hence, the new LP can be formulated as

$$\begin{aligned} \max \quad & 2x_1 + 3x_2 + 3x_3 - \frac{1}{12}(2x_1 + 4x_2 + x_3) \\ \text{s.t.} \quad & x_1 + 2x_2 + 2x_3 \leq 12 \\ & 2x_1 + 4x_2 + x_3 \leq f \\ & x_1, x_2, x_3 \geq 0. \end{aligned} \tag{12}$$

Write its dual as

$$
\begin{aligned}
\min \quad & 12\lambda_1 + f\lambda_2 \\
\text{s.t.} \quad & \lambda_1 + 2\lambda_2 \geq \frac{11}{6} \\
& 2\lambda_1 + 4\lambda_2 \geq \frac{8}{3} \\
& 2\lambda_1 + \lambda_2 \geq \frac{35}{12} \\
& \lambda_1, \lambda_2 \geq 0.
\end{aligned} \tag{13}
$$

We can use graph to help us quickly solve the problem



Note the gradient of the objective is $(12, f)$ and we can move in the opposite direction of the gradient to minimize the objective of the dual. Finally, we have the net profit for this case as

$$
\pi_a(f) = \begin{cases} 22 & \text{if } f \geq 24 \\ \dfrac{f}{4} + 16 & \text{if } 6 \leq f < 24 \\ \dfrac{35f}{12} & \text{if } f < 6. \end{cases} \tag{14}
$$

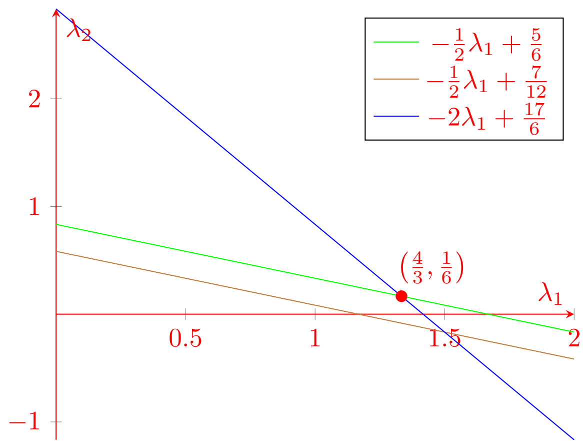
(b) If $f \geq s$, then the textile firm can purchase the first s cotton from the major supplier with a cheaper price and purchase the rest of cotton with a higher price. Intuitively, the new LP can be formulated as

$$
\begin{aligned}
\max \quad & 2x_1 + 3x_2 + 3x_3 - \frac{1}{12}s - \frac{1}{6}(2x_1 + 4x_2 + x_3 - s) \\
\text{s.t.} \quad & x_1 + 2x_2 + 2x_3 \leq 12 \\
& 2x_1 + 4x_2 + x_3 \leq f \\
& x_1, x_2, x_3 \geq 0.
\end{aligned} \tag{15}
$$

Ignore the standalone s term for now, write the dual as

$$
\begin{aligned}
\min \quad & 12\lambda_1 + f\lambda_2 \\
\text{s.t.} \quad & \lambda_1 + 2\lambda_2 \geq \frac{5}{3} \\
& 2\lambda_1 + 4\lambda_2 \geq \frac{7}{3} \\
& 2\lambda_1 + \lambda_2 \geq \frac{17}{6} \\
& \lambda_1, \lambda_2 \geq 0.
\end{aligned} \tag{16}
$$

We can use graph to help us quickly solve the problem



Note the gradient of the objective is $(12, f)$ and we can move in the opposite direction of the gradient to minimize the objective of the dual. Now, add back the $\frac{s}{12}$ term, and we have a "net profit" for this case as

$$\pi_{b_0}(f) = \begin{cases} \frac{s}{12} + 20 & \text{if } f \geq 24 \\ \frac{2f+s}{12} + 16 & \text{if } 6 \leq f < 24 \\ \frac{34f+s}{12} & \text{if } f < 6. \end{cases} \tag{17}$$

However, this is not the end of this case. If you take a deeper look at LP (15), you will notice that it assumes all provided cotton would be used. What if only part of the cotton is used? Is that tricky case likely to happen?

Unfortunately, yes, it happens and results in the partial usage of cotton with $x_1 = 12, x_2 = 0, x_3 = 0$ for all $f \geq 24$ (*Verify this by using Simplex Method to solve the LP!*), which means only 24 units of cotton will be used if $f \geq 24$. Thus, when $f \geq s \geq 24$, the LP should be formulated as LP (12), and results in an optimum of 22, in which case all cotton is purchased with a cheaper price.

Therefore, the net profit for this case should be

$$\pi_b(f) = \begin{cases} 22 & \text{if } f \geq s \geq 24 \\ \frac{s}{12} + 20 & \text{if } f \geq 24 \text{ and } s < 24 \\ \frac{2f+s}{12} + 16 & \text{if } 6 \leq f < 24 \\ \frac{34f+s}{12} & \text{if } f < 6. \end{cases} \tag{18}$$