Test Paper of "SI231b: Matrix Computations"

Total points: 110

Your grade: min{your real grade, 100}

Problem 1 (15 points). Subspace problems: let $\mathcal{V} \subset \mathbb{R}^m$ be a subspace of dimension n, and let $v_1, v_2, \cdots, v_n$ be its basis. Define

$$\mathcal{T} = \{x | x = Ty, \ y \in \mathcal{V}\},$$

where $T \in \mathbb{R}^{m \times m}$ is nonsingular. Show that $\mathcal{T}$ is also a subspace and give its basis.

Problem 2 (20 points). LU factorization problems: for $A \in \mathbb{R}^{n \times n}$, suppose its LU factorization exists and its lower-triangular factor L has unit diagonal entries, i.e., $l_{ii} = 1$.

(1) (5 points). Show the uniqueness of this LU factorization;

(2) (5 points). Given $A = \begin{bmatrix} -3 & 1 & -2 \\ 1 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix}$, compute its LU factorization;

(3) (5 points). Solve the sequence of linear systems $Ax_i = b_i$ for $i = 1, 2, 3$, where $b_1 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$, $b_2 = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T$, $b_3 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$.

(4) (5 points). Compute the matrix condition norm $\kappa(A) = \|A\|\|A^{-1}\|$ using induced 1-norm and infinity norm.

Problem 3 (25 points). QR factorization problems: given a matrix

$$A = \begin{bmatrix} 1 & 3 \\ 1 & 3 \\ 1 & -1 \\ -1 & 1 \end{bmatrix},$$

(1) (5 points). Compute one orthonormal basis set for $\mathcal{R}(A)$;

(2) (5 points). Give the (reduced) QR factorization of A;

(3) (5 points). For $b = \begin{bmatrix} 6 & 6 & 8 & 8 \end{bmatrix}^T$, determine the optimal solution for $\min_x \|b - Ax\|_2$;

(4) (10 points). Give the orthogonal projector matrix that projects onto $\mathcal{N}(A^T)$.

Probme 4 (25 points). Eigenvalue problems:

(1) (5 points). For real symmetric matrix $A \in \mathbb{R}^{n\times n}$, prove that eigenvectors corresponding to distinct eigenvalues are orthogonal;

(2) (5 points). Given $A \in \mathbb{R}^{n\times n}$, and its eigen-pair $(\lambda,\ v)$. For any $d \in \mathbb{R}^n$, determine the eigen-pair of the matrix $A + vd^T$;

(3) (10 points). Given

$$A = \begin{bmatrix} \frac{1}{3} & \frac{1}{6} \\ -\frac{1}{3} & \frac{1}{3} \end{bmatrix},$$

compute $\lim_{k\to\infty} A^k$;

(4) (5 points). To compute the eigenvalues using the QR iteration, explain the benefit of Hessenberg reduction.

Problem 5 (15 points). Singular value decomposition problems:

(1) (5 points). For $A \in \mathbb{R}^{m\times n}$, if $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n$ are its nonzero singular values, show that

$$\min_{x\in\mathbb{R}^n,\ \|x\|_2=1} \|Ax\|_2 = \sigma_n;$$

(2) (10 points). For a real matrix $A \in \mathbb{R}^{m\times m}$, denote its singular value decomposition by $A = U\Sigma V^T$, determine the singular value decomposition of the matrix

$$\begin{bmatrix} 0 & A^T \\ A & 0 \end{bmatrix}.$$

Bonus (10 points). Show that for a real matrix $A \in \mathbb{R}^{n\times n}$, if $A = -A^T$, then $I - A$ is nonsingular and the matrix $(I - A)^{-1}(I + A)$ is orthogonal. This is known as the *Cayley transform* of A.

Reference Solution of SI231b: Matrix Computations of 2021 Fall

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Problem 1.

To show $\mathcal{T}$ is also a subspace, one needs to show

1. $0 \in \mathcal{T}$
2. $\forall x,\ y \in \mathcal{T},\ x + y \in \mathcal{T}$
3. $\forall \alpha \in \mathbb{R},\ \forall x \in \mathcal{T},\ \alpha x \in \mathcal{T}$

Item 1 accounts for 2 points, each of the second and third item accounts for 3 points.

One set of basis of $\mathcal{T}$ is $Tv_1,\ Tv_2,\ \cdots\cdots, Tv_n$. Next, one needs to show any vector from $\mathcal{T}$ can be represented by this set of basis, and the linear independence of these vectors need to be shown.

Give the set of basis vectors accounts for 1 point; Show any vector from $\mathcal{T}$ can be represented by this set of vectors accounts for 3 points; Show the linear independence of these vectors account for 3 points.

Problem 2.

(1) To show the uniqueness of this LU factorization, one can assume that there exists two distinct factorization which gives

$$A = L_1U_1 = L_2U_2.$$

This gives $L_2^{-1}L_1 = U_2U_1^{-1}$. Since the product of lower-triangular matrices are lower-triangular, and the product of upper-triangular matrices are upper triangular. This means that $L_2^{-1}L_1 = U_2U_1^{-1}$ equals a diagonal matrix.

Since L_1 and L_2 are lower-triangular matrices with unit diagonals, it's easy to verify that L_2^{-1} also has unit diagonals and also $L_2^{-1}L_1$, this in turn states that

$$L_2^{-1}L_1 = U_2U_1^{-1} = I,$$

which gives $L_1 = L_2$ and $U_1 = U_2$

Only giving $L_2^{-1}L_1 = U_2U_1^{-1} = I$ without explanation gets 3 points.

(2) $L = \begin{bmatrix} 1 & & \\ -\frac{1}{3} & 1 & \\ -\frac{1}{3} & -\frac{1}{2} & 1 \end{bmatrix}$, $U = \begin{bmatrix} -3 & 1 & -2 \\ & \frac{4}{3} & \frac{1}{3} \\ & & -\frac{1}{2} \end{bmatrix}$

Both correct L and U gets 5 points; 1 correct gets 3 points; none correct, but the results are lower-triangular and upper triangular gets 1 point.

(3) $x_1 = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -1 \end{bmatrix}$, $x_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$, $x_3 = \begin{bmatrix} \frac{3}{2} \\ \frac{1}{2} \\ -2 \end{bmatrix}$.

All correct gets 5 points; 2 correct answers get 4 points; 1 correct answer gets 2 points.

(4) According to (3), $A^{-1} = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 1 & \frac{3}{2} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ -1 & -1 & -2 \end{bmatrix}$.

$\|A\|_1 = 5$, $\|A\|_\infty = 6$, $\|A^{-1}\|_1 = 4$, $\|A^{-1}\|_\infty = 4$.

Therefore,

$$\kappa_1(A) = \|A\|_1 \|A^{-1}\|_1 = 20,$$
$$\kappa_\infty(A) = \|A\|_\infty \|A^{-1}\|_\infty = 24.$$

Correct A^{-1} gets 2 points, each correct norm for both A and A^{-1} gets 1 point, and the 2 correct condition numbers get 1 point.

Problem 3.

(1) The orthonormal basis for $\mathcal{R}(A)$ is

$$q_1 = \frac{a_1}{\|a_1\|_2} = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}.$$

$$q_2 = a_2 - < a_2, q_1 > q_1 = \begin{bmatrix} 2 \\ 2 \\ -2 \\ 2 \end{bmatrix}, \quad q_2 = \frac{q_2}{\|q_2\|_2} = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}.$$

Correct q_1 gets 2 points, correct q_2 gets 3 points. Wrong q_1 or q_2 but correct formula gets 1 point each.

(2) $Q = [q_1,\ q_2] = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & -1 \\ -1 & 1 \end{bmatrix}$, $R = \begin{bmatrix} 2 & 2 \\ & 4 \end{bmatrix}$.

Correct Q gets 2 points and R gets 3 points.

(3) The least square solution is given by solving

$$QRx = QQ^T b,$$

i.e., $x = R^{-1}Q^T b = \begin{bmatrix} \frac{3}{2} \\ \frac{3}{2} \end{bmatrix}$.

If numerical result of x is wrong but correct formula of $x = R^{-1}Q^T b$ or $x = (A^T A)^{-1} A^T b$ is given, this case gets 2 points.

(4) The orthogonal projector onto $\mathcal{R}(A)$ is given by $P = QQ^T$. Since

$$\mathcal{R}(A) + \mathcal{N}(A^T) = \mathbb{R}^4, \quad \mathcal{R}(A) \perp \mathcal{N}(A^T).$$

The complementary projector $I - P$ is the orthogonal projector that projects onto $\mathcal{N}(A^T)$, which is

$$I - P = \frac{1}{2} \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Without stating $\mathcal{N}(A^T)$ being the orthogonal complement of $\mathcal{R}(A)$ but just give $I-P$ gets 6 points; Explain the above logic in detail without the final numerical results gets 8 points; All correct gets 10 points.

Problem 4.

(1) Assume λ_1 and λ_2 are two distinct eigenvalues of A with corresponding eigenvectors v_1 and v_2, then

$$Av_1 = \lambda_1 v_1, \quad Av_2 = \lambda_1 v_2.$$

This gives

$$\lambda_1 v_2^T v_1 = v_2^T A v_1, \quad \lambda_2 v_1^T v_2 = v_1^T A v_2,$$

i.e., $(\lambda_1 - \lambda_2) v_2^T v_1 = 0$ due to the symmetry of the real matrix A. Since λ_1 and λ_2 are distinct, therefore, v_1 and v_2 are orthogonal.

Using the Schur decomposition to prove this result is also possible.

(2) The eigenpair is given by $(\lambda + d^T v, v)$.

(3) $|\lambda(A)| \leq \|A\|_1 = \frac{2}{3}$, therefore, $\lim_{k\to\infty} A^k = 0$.

Using $|\lambda(A)| \leq \|A\|_\infty = \frac{2}{3}$ to give the result is also correct; Or by computing the eigenvalues of A to give the result is also right.

(4) QR iteration with Hessenberg reduction can preserve the upper Hessenberg structure for each iteration, therefore it reduces the computational complexity of each step from $\mathcal{O}(n^3)$ to $\mathcal{O}(n^2)$.

Without stating that the upper Hessenberg structure preserving at each iteration loses 2 points; Without quantifying the complexity loses 1 points.

Problem 5

(1) The SVD of A given by $A = U\Sigma V^T$ with $U \in \mathbb{R}^{m\times m}$, $\Sigma \in \mathbb{R}^{m\times n}$, $V \in \mathbb{R}^{n\times n}$, then

$$\begin{aligned}
\|Ax\|_2^2 &= x^T A^T A x \\
&= x^T V \Sigma^T \Sigma \underbrace{V^T x}_{y} \\
&= y^T \Sigma^T \Sigma y \\
&= y^T \begin{bmatrix} \sigma_1^2 & & & \\ & \sigma_2^2 & & \\ & & \dots & \\ & & & \sigma_n^2 \end{bmatrix} y
\end{aligned}$$

Associating with $\|y\|_2 = \|x\|_2 = 1$, one can finish this proof.

(2) It's easy to show that

$$\begin{bmatrix} & A^T \\ A & \end{bmatrix} = \begin{bmatrix} & V \\ U & \end{bmatrix} \begin{bmatrix} \Sigma & \\ & \Sigma \end{bmatrix} \begin{bmatrix} V^T & \\ & U^T \end{bmatrix}$$

To show this form is an SVD, one needs to show the orthogonality of the matrix $\begin{bmatrix} & V \\ U & \end{bmatrix}$ and $\begin{bmatrix} V^T & \\ & U^T \end{bmatrix}$, one also needs to point out that $\begin{bmatrix} \Sigma & \\ & \Sigma \end{bmatrix}$ is a diagonal matrix with positive diagonal entries.

Only giving this form gets 7 points; each statement of the orthogonality and positiveness of the diagonal matrix gets 1 point

Bonus

To show the non-singularity of the matrix $I - A$, one can show that $\mathcal{N}(I - A) = \{0\}$.

$\forall x \in \mathcal{N}(I - A)$, $(I - A)x = 0$, i.e., $x = Ax$.

Next, one can/need to show that $x \in \mathcal{R}(I - A^T)$. It's easy to see that $x = \frac{1}{2}(I - A^T)x$ due to the fact $x = Ax$ and $A = -A^T$. This gives $x \in \mathcal{R}(I - A^T)$.

Together one gets $x \in \mathcal{R}(I - A^T) \cap \mathcal{N}(I - A) = \{0\}$, which means $\mathcal{N}(I - A) = \{0\}$.

To show the orthogonality of the matrix $(I - A)^{-1}(I + A)$, one need to show

$(I-A)^{-1}(I+A)((I-A)^{-1}(I+A))^T = I$, which is easy because

$$\begin{aligned}
(I-A)^{-1}(I+A)((I-A)^{-1}(I+A))^T &= (I-A)^{-1}(I+A)(I+A^T)(I-A^T)^{-1} \\
&= (I-A)^{-1}(I+A)(I-A)(I+A)^{-1} \\
&= (I-A)^{-1}(I-A^2)(I+A)^{-1} \\
&= (I-A)^{-1}(I-A)(I+A)(I+A)^{-1} \\
&= I
\end{aligned}$$

Showing the non-singularity gets 5 points and the orthogonality gets 5 points.

There are different approaches to solve this problem, here we only give the easy one.