CS244: Theory of Computation

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Outline

Why infinite words ?

Reactive systems: reacting continuously with the environment

- Operating systems,
- Communicating protocols,
- Control programs,
- Vending machines,
- ...

Salient feature of reactive systems:

Nonterminating

The behavior of reactive systems:

A set of infinite words.

Outline

Büchi automata (BA)

A Büchi automata $\mathcal{B}$ is a tuple $(Q, \Sigma, \delta, q_0, F)$ where

- Q: finite set of states, Σ: finite alphabet,
- q_0: initial state, $F \subseteq Q$: set of final states,
- $\delta \subseteq Q \times \Sigma \times Q$.

A run ρ of a Büchi automata $\mathcal{B}$ over an ω-word $w = a_1 a_2 \cdots \in \Sigma^\omega$ is

an infinite state sequence $q_0 q_1 \ldots$ such that $\forall i \geq 0.(q_i, a_{i+1}, q_{i+1}) \in \delta$.

$\mathrm{Inf}(\rho)$: the set of states occurring infinitely often in ρ.

A run is accepting iff $\mathrm{Inf}(\rho) \cap F \neq \emptyset$.

An ω-word w is accepted by $\mathcal{B}$ if there is an accepting run of $\mathcal{B}$ over w.

Let $\mathcal{L}(\mathcal{B})$ denote the set of ω-words accepted by $\mathcal{B}$.

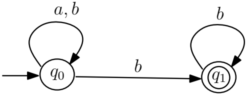
A deterministic Büchi automaton (DBA) $\mathcal{B}$ is a BA $(Q, \Sigma, \delta, q_0, F)$ s.t.

$\forall q \in Q, a \in \Sigma$, $\exists$ at most one $q' \in Q$ such that $(q, a, q') \in \delta$.

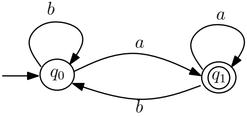
Then δ in a DBA can be seen as a partial function $\delta : Q \times \Sigma \to Q$.

Büchi automata: Example

"The letter *a* occurs only finitely often"



"The letter *a* occurs infinitely often"



Büchi automata: Several notations

Let $\mathcal{B} = (Q, \Sigma, \delta, q_0, F)$ be a BA, $q, q' \in Q$, and $w = a_1 \dots a_n \in \Sigma^*$.

A partial run of $\mathcal{B}$ over w from q to q' is a finite state sequence $q_1 q_2 \dots q_{n+1}$ such that

- $\forall i \leq n. (q_i, a_i, q_{i+1}) \in \delta$,
- $q_1 = q$, $q_{n+1} = q'$.

$q \xrightarrow{w} q'$: there is a partial run of $\mathcal{B}$ over w from q to q'.

$q \xrightarrow[F]{w} q'$: *there is a partial run of $\mathcal{B}$ over w from q to q' which contains an accepting state.*

ω-regular languages

Theorem. Let $L \subseteq \Sigma^\omega$. Then
L can be defined by a BA iff $L = \bigcup_{1 \leq i \leq n} U_i V_i^\omega$,
where $\forall i : 1 \leq i \leq n.\ U_i, V_i \subseteq \Sigma^*$ are regular and $\varepsilon \notin V_i$.

Proof.

Only if direction:
Suppose that L is defined by a BA $\mathcal{B} = (Q, \Sigma, \delta, q_0, F)$.
Let the NFA $L_{qq'} = \{w \in \Sigma^* \mid q \xrightarrow{w} q'\}$. Then $L = \bigcup_{q \in F} L_{q_0 q}(L_{qq} \setminus \{\varepsilon\})^\omega$.

If direction: Suppose $L = \bigcup_{1 \leq i \leq n} U_i V_i^\omega$.
Since Büchi automata are closed under union (which will be shown later), it is sufficient to prove that $U_i V_i^\omega$ can be defined by a BA.
Let $\mathcal{A}_1 = (Q_1, \Sigma, \delta_1, q_0^1, F_1)$ (resp. $\mathcal{A}_2 = (Q_2, \Sigma, \delta_2, q_0^2, F_2)$) define U_i (resp. V_i).
W.l.o.g. assume that there are no transitions (q, a, q_0^2) with $q \in Q_2$.
Then $\mathcal{B} = (Q_1 \cup Q_2, \Sigma, \delta, q_0^1, \{q_0^2\})$ defines L, where

$$\delta = \begin{array}{ll} \delta_1 \cup \delta_2 & \cup \left\{(q, a, q') \mid q \in F_1, (q_0^2, a, q') \in \delta_2\right\} \\ & \cup \{(q, a, q_0^2) \mid \exists q' \in F_2, (q, a, q') \in \delta_2\} \end{array}.$$

$\square$

Expressibility of DBA

Let $L \subseteq \Sigma^*$. Define $\overrightarrow{L} = \{w \in \Sigma^\omega \mid \exists^\omega n.\ w_1 \dots w_n \in L\}$.

Proposition. Let $L \subseteq \Sigma^\omega$. Then
L can be defined by a DBA iff $L = \overrightarrow{L'}$ for some regular language $L' \subseteq \Sigma^*$.

Proof.

Only if direction:
Suppose L is defined by the DBA $\mathcal{B} = (Q, \Sigma, \delta, q_0, F)$.
Let L' be defined by the DFA $\mathcal{A} = (Q, \Sigma, \delta, q_0, F)$, then $L = \overrightarrow{L'}$.
It is trivial that $L \subseteq \overrightarrow{L'}$.
$L \supseteq \overrightarrow{L'}$. Suppose $w \in \overrightarrow{L'}$. Then there exist infinitely many $n \in \mathbb{N}$ s.t. $w_1 \dots w_n \in L'$.
For each such n, let $q_0 \dots q_n$ be the accepting run of $\mathcal{A}$ over $w_1 \dots w_n$.
Then $q_0 \dots q_n \dots$ is an accepting run of $\mathcal{B}$ over w. Therefore, $w \in L$.

If direction:
Let $L = \overrightarrow{L'}$ and $\mathcal{A} = (Q, \Sigma, \delta, q_0, F)$ be a DFA defining L'.
Then the DBA $\mathcal{B} = (Q, \Sigma, \delta, q_0, F)$ defines L. $\square$

Expressibility of DBA

Let $L \subseteq \Sigma^*$. Define $\overrightarrow{L} = \{w \in \Sigma^\omega \mid \exists^\omega n.\ w_1 \ldots w_n \in L\}$.

Proposition. Let $L \subseteq \Sigma^\omega$. Then
L can be defined by a DBA iff $L = \overrightarrow{L'}$ for some regular language $L' \subseteq \Sigma^*$.

Proposition. BA is strictly more expressive than DBA.

Proof.

The language L "The letter a occurs only finitely often"
is not expressible in DBA.

For contradiction, assume that L is defined by a DBA $\mathcal{B}$.
Consider ab^ω. The run of $\mathcal{B}$ over ab^ω is accepting. Let $n_1 \in \mathbb{N}$ s.t. $q_0 \xrightarrow[F]{ab^{n_1}} q_1$.

Consider $ab^{n_1}ab^\omega$. Let $n_2 \in \mathbb{N}$ s.t. $q_0 \xrightarrow[F]{ab^{n_1}} q_1 \xrightarrow[F]{ab^{n_2}} q_2$.
Continue like this, we can get an ω-word $ab^{n_1}ab^{n_2}\ldots$ which is accepted by $\mathcal{B}$,

$$q_0 \xrightarrow[F]{ab^{n_1}} q_1 \xrightarrow[F]{ab^{n_2}} q_2 \xrightarrow[F]{ab^{n_3}} q_3 \xrightarrow[F]{ab^{n_4}} q_4 \cdots$$

while on the other hand contains infinitely many a's, a contradiction. $\square$

Recap

Closure Properties

	Union	Intersection	Complement	Concatenation	Kleene-$*$
Regular	YES	YES	YES	YES	YES
CFL	YES	NO	NO	YES	YES
DCFL	NO	NO	YES	NO	NO
VPL	YES	YES	YES	YES	YES

Decision problems

	Emptiness	Universality/Equivalence	Inclusion
NFA	**NL**	PSPACE	PSPACE
PDA	**P**	Undecidable	Undecidable
DPDA	**P**	Decidable	Undecidable
VPA	**P**	EXPTIME	EXPTIME

- NFA=MSO
- VPA=MSO_μ
- ω-regular language L: L can be defined by a BA iff $L = \bigcup_{1 \leq i \leq n} U_i V_i^\omega$
- DBA$\subseteq$NBA

Outline

Union and intersection

Proposition. The class of ω-regular languages is closed under union and intersection.

Proof.

Let $\mathcal{A}_1 = (Q_1, \Sigma, \delta_1, q_0^1, F_1), \mathcal{A}_2 = (Q_2, \Sigma, \delta_2, q_0^2, F_2)$ define resp. L_1, L_2.

Union:

The BA $\mathcal{A} = (Q_1 \cup Q_2 \cup \{q_0\}, \Sigma, \delta, q_0, F_1 \cup F_2)$ defines $L_1 \cup L_2$, where

$$\delta = \delta_1 \cup \delta_2 \cup \{(q_0, a, q) \mid (q_0^1, a, q) \in \delta_1\} \cup \{(q_0, a, q) \mid (q_0^2, a, q) \in \delta_2\}.$$

Intersection:

The BA $\mathcal{A} = (Q_1 \times Q_2 \times \{0, 1, 2\}, \Sigma, \delta, (q_0^1, q_0^2, 0), Q_1 \times Q_2 \times \{2\})$ defines $L_1 \cap L_2$, where δ is defined as follows,

Suppose $(q_1, a, q_1') \in \delta_1$ and $(q_2, a, q_2') \in \delta_2$.

- If $q_1' \notin F_1$, then $((q_1, q_2, 0), a, (q_1', q_2', 0)) \in \delta$,
  otherwise, $((q_1, q_2, 0), a, (q_1', q_2', 1)) \in \delta$.
- If $q_2' \notin F_2$, then $((q_1, q_2, 1), a, (q_1', q_2', 1)) \in \delta$,
  otherwise, $((q_1, q_2, 1), a, (q_1', q_2', 2)) \in \delta$.
- $((q_1, q_2, 2), a, (q_1', q_2', 0)) \in \delta$.

$\square$

Complementation

Theorem. The class of ω-regular languages is closed under complementation.

Let $L \subseteq \Sigma^\omega$ defined by a BA $\mathcal{B} = (Q, \Sigma, \delta, q_0, F)$.
Define a congruence $\sim_\mathcal{B}$ over Σ^* as follows:

$$u \sim_\mathcal{B} v \text{ iff } \forall q, q' \in Q.(q \xrightarrow{u} q' \Leftrightarrow q \xrightarrow{v} q') \text{ and } (q \xrightarrow[F]{u} q' \Leftrightarrow q \xrightarrow[F]{v} q').$$

Let $[u]$ denote the equivalence class of u under $\sim_\mathcal{B}$.

Lemma. $\sim_\mathcal{B}$ is of finite index.

- Repeatedly partition sets of words for each pair of states q, q'.
- The number of pairs of states is finite.

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Let $[u]$ denote the equivalence class of u under $\sim_\mathcal{B}$.

Lemma. $\sim_\mathcal{B}$ is of finite index.

Lemma. $\sim_\mathcal{B}$ saturates L, namely,

for every $u, v \in \Sigma^*$, $[u][v]^\omega \cap L \neq \emptyset$ implies that $[u][v]^\omega \subseteq L$.

Proof

Lemma. $\sim_{\mathcal{B}}$ saturates L, namely,

for every $u, v \in \Sigma^*$, $[u][v]^\omega \cap L \neq \emptyset$ implies that $[u][v]^\omega \subseteq L$.

Proof.

Suppose $u_1 v_1 v_2 \dots \in L$ s.t. $u_1 \in [u]$ and $v_1, v_2, \dots \in [v]$.

We prove that $u'_1 v'_1 v'_2 \dots \in L$ for every $u'_1 \in [u]$ and $v'_1, v'_2, \dots \in [v]$.

There exists an accepting run ρ of $\mathcal{B}$ over $u_1 v_1 v_2 \dots$.

Let $q_1, q_2, \dots$ be the states in ρ such that $q_0 \xrightarrow{u_1} q_1$, $\forall i \geq 1. q_i \xrightarrow{v_i} q_{i+1}$.

Then there are $i_1 < i_2 < \dots$ s.t.

$$q_1 \xrightarrow[F]{v_1 \dots v_{i_1}} q_{i_1+1},\ \forall j \geq 1. q_{i_j+1} \xrightarrow[F]{v_{i_j+1} \dots v_{i_{j+1}}} q_{i_{j+1}+1}.$$

By def. of $\sim_{\mathcal{B}}$, $q_0 \xrightarrow{u'_1} q_1$, $q_1 \xrightarrow[F]{v'_1 \dots v'_{i_1}} q_{i_1+1}$, and $\forall j \geq 1. q_{i_j+1} \xrightarrow[F]{v'_{i_j+1} \dots v'_{i_{j+1}}} q_{i_{j+1}+1}$.

Therefore, $u'_1 v'_1 v'_2 \dots$ is accepted by $\mathcal{B}$, thus in L. □

Complementation

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for every $u, v \in \Sigma^*$, $[u][v]^\omega \cap L \neq \emptyset$ implies that $[u][v]^\omega \subseteq L$.

Lemma. $\forall w \in \Sigma^\omega$, $\exists u, v \in \Sigma^*$ s.t. $w \in [u][v]^\omega$.

Proof.

For a pair (i, j) such that $i < j$, assign a color $[w_i \dots w_{j-1}]$, i.e., the equivalence class of $w_i \dots w_{j-1}$.

From Ramsey theorem,

$\exists$ a color $[v]$ and an infinite sequence $1 \leq i_1 < i_2 < \dots$ s.t.

$\forall j < k$, the pair (i_j, i_k) is assigned the color $[v]$.

Let $u = w_1 \dots w_{i_1 - 1}$. Then

$w = (w_1 \dots w_{i_1-1})(w_{i_1} \dots w_{i_2-1})(w_{i_2} \dots w_{i_3-1}) \cdots \in [u][v]^\omega$. $\square$

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Lemma. $\forall w \in \Sigma^\omega$, $\exists u, v \in \Sigma^*$ s.t. $w \in [u][v]^\omega$.

Lemma. $\forall u \in \Sigma^*$ s.t. $[u]$ is regular.

Proof.

It is sufficient to prove that $L_{qq'} = \left\{ w \mid q \xrightarrow{w} q' \right\}$ and $L^F_{qq'} = \left\{ w \mid q \xrightarrow[F]{w} q' \right\}$ are regular for all q, q'.

$L_{qq'}$ is regular: Obvious, the NFA $(Q, \Sigma, \delta, q, q')$

$L^F_{qq'}$ is regular: Defined by the NFA $(Q \times \{0,1\}, \Sigma, \delta', (q,0), (q',1))$, where

$\forall p, p' \in Q$, if $(p, a, p') \in \delta$, then $((p,1), a, (p',1)) \in \delta'$, and if $p' \notin F$, then $((p,0), a, (p',0)) \in \delta'$, otherwise, $((p,0), a, (p',1)) \in \delta'$.

□

Complementation

Theorem. The class of ω-regular languages is closed under complementation.

Lemma. $\sim_{\mathcal{B}}$ is of finite index.

Lemma. $\sim_{\mathcal{B}}$ saturates L, namely,

for every $u, v \in \Sigma^*$, $[u][v]^\omega \cap L \neq \emptyset$ implies that $[u][v]^\omega \subseteq L$.

Lemma. $\forall w \in \Sigma^\omega$, $\exists u, v \in \Sigma^*$ s.t. $w \in [u][v]^\omega$.

Lemma. $\forall u \in \Sigma^*$ s.t. $[u]$ is regular.

Proof of the theorem.

Let $S = \{([u],[v]) \mid [u][v]^\omega \cap L \neq \emptyset\}$. Then $\overline{L} = \bigcup_{([u],[v]) \notin S} [u][v]^\omega$.

- $\bigcup_{([u],[v]) \notin S} [u][v]^\omega \subseteq \overline{L}$: If $([u],[v]) \notin S$, then $[u][v]^\omega \cap L = \emptyset$, so $[u][v]^\omega \subseteq \overline{L}$.
- $\overline{L} \subseteq \bigcup_{([u],[v]) \notin S} [u][v]^\omega$: For every $w \in \overline{L}$, there are $[u],[v]$ such that $w \in [u][v]^\omega$. If $([u],[v]) \in S$, then $w \in [u][v]^\omega \subseteq L$, it follows $([u],[v]) \notin S$.

$\square$

Complementation

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Lemma. $\sim_{\mathcal{B}}$ is of finite index.

Lemma. $\sim_{\mathcal{B}}$ saturates L, namely,

for every $u, v \in \Sigma^*$, $[u][v]^\omega \cap L \neq \emptyset$ implies that $[u][v]^\omega \subseteq L$.

Lemma. $\forall w \in \Sigma^\omega$, $\exists u, v \in \Sigma^*$ s.t. $w \in [u][v]^\omega$.

Lemma. $\forall u \in \Sigma^*$ s.t. $[u]$ is regular.

Complexity analysis

The automaton $\mathcal{B}'$ defining $\overline{L}$:

The union of the BAs for the languages $[u][v]^\omega$ with $([u], [v]) \notin S$.

The BA for $[u][v]^\omega$ can be easily obtained from the NFAs for resp. $[u]$ and $[v]$.

[u] is determined by $(\{(q, q') \mid q \xrightarrow{u} q'\}, \{(q, q') \mid q \xrightarrow[F]{u} q'\}) \Rightarrow$

$2^{2|Q|^2}$ *equivalence classes* $\Rightarrow 2^{2|Q|^2}$ *states in the NFA for $[u]$ and $[v]$.*

Conclusion: There are $2^{O(|Q|^2)}$ states in $\mathcal{B}'$.

Outline

MSO over infinite words

Syntax.

$\varphi := P_\sigma(x) \mid x = y \mid \mathsf{suc}(x, y) \mid X(x) \mid \varphi_1 \vee \varphi_2 \mid \neg\varphi_1 \mid \exists x \varphi_1 \mid \exists X \varphi_1$,

where $\sigma \in \Sigma$.

A MSO formula φ is satisfied over an ω-word $w = a_1 \dots a_n \dots$, with a valuation $\mathcal{I}$ of $\mathrm{Free}(\varphi)$ over $\mathcal{S}_w$, denoted by $(w, \mathcal{I}) \models \varphi$, is defined as follows,

- $(w, \mathcal{I}) \models P_\sigma(x)$ iff $a_{\mathcal{I}(x)} = \sigma$,
- $(w, \mathcal{I}) \models x = y$ iff $\mathcal{I}(x) = \mathcal{I}(y)$,
- $(w, \mathcal{I}) \models \mathsf{suc}(x, y)$ iff $\mathcal{I}(x) + 1 = \mathcal{I}(y)$,
- $(w, \mathcal{I}) \models X(x)$ iff $\mathcal{I}(x) \in \mathcal{I}(X)$,
- $(w, \mathcal{I}) \models \varphi_1 \vee \varphi_2$ iff $(w, \mathcal{I}) \models \varphi_1$ or $(w, \mathcal{I}) \models \varphi_2$,
- $(w, \mathcal{I}) \models \neg\varphi_1$ iff not $(w, \mathcal{I}) \models \varphi_1$,
- $(w, \mathcal{I}) \models \exists x \varphi_1$ iff there is $j \in S_w$ such that $(w, \mathcal{I}[x \to j]) \models \varphi_1$,
- $(w, \mathcal{I}) \models \exists X \varphi_1$ iff there is $J \subseteq S_w$ such that $(w, \mathcal{I}[X \to J]) \models \varphi_1$.

BA $\equiv$ MSO

From BA to MSO
Let $\mathcal{A} = (Q, \Sigma, \delta, q_0, F)$ be a BA. Let $Q = \{q_0, q_1, \ldots, q_n\}$.
Construct the MSO formula φ as follows,

$$\exists X_{q_0} \ldots X_{q_n} (\varphi_{unique} \wedge \varphi_{init} \wedge \varphi_{trans} \wedge \varphi_{final}),$$

where

- X_q stands for the positions where the run is in state q,
- $\varphi_{unique} = \bigwedge_{q \neq q'} \forall x \neg (X_q(x) \wedge X_{q'}(x))$
- $\varphi_{init} = \exists x (\mathrm{First}(x) \wedge \bigvee_{(q_0, a, q) \in \delta} (P_a(x) \wedge X_q(x))),$
- $\varphi_{trans} = \forall x \forall y (\mathrm{suc}(x, y) \to \bigvee_{(q, a, q') \in \delta} X_q(x) \wedge P_a(y) \wedge X_{q'}(y)),$
- $\varphi_{final} = \forall x \exists y \left(x < y \wedge \bigvee_{q \in F} X_q(y) \right).$

Then $\mathcal{L}(\varphi) = \mathcal{L}(\mathcal{A})$.

From MSO to BA
Similar to the construction of an NFA from a MSO formula.

Outline

Nonemptiness

Input: Büchi automaton $\mathcal{B} = (Q, \Sigma, \delta, q_0, F)$.

Question: Is $\mathcal{L}(\mathcal{B}) \neq \emptyset$?

Find a SCC (strongly-connected-component) C satisfying the following conditions,

- C contains an accepting state,
- C is reachable from q_0.

Proposition. Nonemptiness of Büchi automata can be decided in linear time.

SCCs of a directed graph can be found in linear time by a DFS search.

Language inclusion

Input: Büchi automata $\mathcal{B}_1$ and $\mathcal{B}_2$.

Question: Is $\mathcal{L}(\mathcal{B}_1) \subseteq \mathcal{L}(\mathcal{B}_2)$?

Theorem. Language inclusion of Büchi automata is PSPACE-complete.

Upper bound.

Construct $\mathcal{B}_2'$ defining $\overline{\mathcal{L}(\mathcal{B}_2)}$ and test the emptiness of $\mathcal{L}(\mathcal{B}_1 \cap \mathcal{B}_2')$.

There are $|Q_1|2^{O(|Q_2|^2)}$ *states in* $\mathcal{B}_1 \cap \mathcal{B}_2' \Rightarrow$
The nonemptiness of $\mathcal{B}_1 \cap \mathcal{B}_2'$ *can be decided in PSPACE*

- *Nondeterministically guess on the fly a path from the initial state to a cycle containing an accepting state.*
- *NPSPACE* $\equiv$ *PSPACE.*

Language inclusion

Input: Büchi automata $\mathcal{B}_1$ and $\mathcal{B}_2$.

Question: Is $\mathcal{L}(\mathcal{B}_1) \subseteq \mathcal{L}(\mathcal{B}_2)$?

Theorem. Language inclusion of Büchi automata is PSPACE-complete.

Lower bound.

Universality of Büchi automata ($\mathcal{L}(\mathcal{B}) = \Sigma^\omega$) is PSPACE-hard.

> *Reduction from the membership problem of PSPACE TMs.*
> *Use BA to describe the unsuccessful computations of PSPACE TMs.*

Let $M = (Q, \Sigma, \Gamma, \delta, q_0, B, F)$ be a linear space (say cn) TM.
In addition, let $\widehat{\Gamma} = \Gamma \cup Q \cup \{\$\}$.

A successful computation of M over w: $\$C_1\$C_2\$\dots\$C_m\$\left(\widehat{\Gamma} \setminus \{\$\}\right)^\omega$ s.t.

- $\forall i$, $C_i \in \Gamma^j Q \Gamma^{cn-j}$ for some j,
- $\forall i < m$, $C_i \vdash_M C_{i+1}$,
- $C_1 = q_0 w B^{cn-n}$, $C_m \in \Gamma^* F \Gamma^*$.

Indeed, Universality problem of NFA is PSPACE-complete.

Outline

Various acceptance conditions

Acceptance conditions of ω-automata

- Muller condition: $(Q, \Sigma, \delta, q_0, \mathcal{F})$, where $\mathcal{F} \subseteq 2^Q$,

  A run ρ is accepting iff $\mathrm{Inf}(\rho) \in \mathcal{F}$.

- Rabin condition: $(Q, \Sigma, \delta, q_0, (U_i, V_i)_{1 \le i \le k})$, where $\forall i.\ U_i, V_i \subseteq Q$,

  A run ρ is accepting iff $\exists i.\ \mathrm{Inf}(\rho) \cap U_i = \emptyset \wedge \mathrm{Inf}(\rho) \cap V_i \neq \emptyset$.

- Streett condition: $(Q, \Sigma, \delta, q_0, (U_i, V_i)_{1 \le i \le k})$, where $\forall i.\ U_i, V_i \subseteq Q$,

  A run ρ is accepting iff $\forall i.\ \mathrm{Inf}(\rho) \cap V_i \neq \emptyset \rightarrow \mathrm{Inf}(\rho) \cap U_i \neq \emptyset$.

- Parity condition: $(Q, \Sigma, \delta, q_0, c)$, where $c : Q \rightarrow \{1, \ldots, k\}$,

  A run ρ is accepting iff $\min(\{c(q) \mid q \in \mathrm{Inf}(\rho)\})$ *is even.*

- Rabin chain condition: A Rabin condition $(U_i, V_i)_{1 \le i \le k}$ s.t.
  $U_1 \subseteq V_1 \subseteq U_2 \subseteq V_2 \subseteq \cdots \subseteq U_k \subseteq V_k$.

Observation. Parity $\equiv$ Rabin chain.
Parity $\Rightarrow$ Rabin chain: $c : Q \rightarrow \{1, \ldots, 2k+1\}$
$\forall i : 1 \le i \le k.\ U_i = \{q \mid c(q) \le 2i - 1\},\ V_i = \{q \mid c(q) \le 2i\}$.
Rabin chain $\Rightarrow$ Parity: $\forall i : 1 \le i \le k.\ c(U_i \setminus V_{i-1}) = 2i - 1,\ c(V_i \setminus U_i) = 2i$.

Equivalence of all the acceptance conditions

- Muller condition: $(Q, \Sigma, \delta, q_0, \mathcal{F})$, where $\mathcal{F} \subseteq 2^Q$,
  A run ρ is accepting iff $\mathrm{Inf}(\rho) \in \mathcal{F}$.
- Rabin condition: $(Q, \Sigma, \delta, q_0, (U_i, V_i)_{1 \le i \le k})$, where $\forall i.\ U_i, V_i \subseteq Q$,
  A run ρ is accepting iff $\exists i.\ \mathrm{Inf}(\rho) \cap U_i = \emptyset \wedge \mathrm{Inf}(\rho) \cap V_i \neq \emptyset$.
- Streett condition: $(Q, \Sigma, \delta, q_0, (U_i, V_i)_{1 \le i \le k})$, where $\forall i.\ U_i, V_i \subseteq Q$,
  A run ρ is accepting iff $\forall i.\ \mathrm{Inf}(\rho) \cap V_i \neq \emptyset \rightarrow \mathrm{Inf}(\rho) \cap U_i \neq \emptyset$.
- Parity condition: $(Q, \Sigma, \delta, q_0, c)$, where $c : Q \rightarrow \{1, \ldots, k\}$,
  A run ρ is accepting iff $\min(\{c(q) \mid q \in \mathrm{Inf}(\rho)\})$ is even.

From Büchi to the other conditions:
Let $\mathcal{B} = (Q, \Sigma, \delta, q_0, F)$ be a BA.

- Muller: $(Q, \Sigma, \delta, q_0, \mathcal{F})$ with $\mathcal{F} = \{P \mid P \cap F \neq \emptyset\}$,
- Rabin: $(Q, \Sigma, \delta, q_0, (\emptyset, F))$,
- Streett: $(Q, \Sigma, \delta, q_0, (F, Q))$,
- Parity: $(Q, \Sigma, \delta, q_0, c)$ with $c(F) = 2$ and $c(Q \setminus F) = 3$.

Equivalence of all the acceptance conditions

- Streett condition: $(Q, \Sigma, \delta, q_0, (U_i, V_i)_{1 \le i \le k})$, where $\forall i.\ U_i, V_i \subseteq Q$,

  A run ρ is accepting iff $\forall i.\ \mathrm{Inf}(\rho) \cap V_i \neq \emptyset \to \mathrm{Inf}(\rho) \cap U_i \neq \emptyset$.

- Parity condition: $(Q, \Sigma, \delta, q_0, c)$, where $c : Q \to \{1, \ldots, k\}$,

  A run ρ is accepting iff $\min(\{c(q) \mid q \in \mathrm{Inf}(\rho)\})$ is even.

From Parity to Streett:
Let $\mathcal{A} = (Q, \Sigma, \delta, q_0, c)$ be a Parity automaton and $c : Q \to \{1, \ldots, 2k+1\}$.
Then $\mathcal{A}$ is equivalent to the Streett automaton $(Q, \Sigma, \delta, q_0, (U_i, V_i)_{0 \le i \le k})$,
where $U_i = \{q \mid c(q) \le 2i\}$, $V_i = \{q \mid c(q) \le 2i + 1\}$.

Equivalence of all the acceptance conditions

- Muller condition: $(Q, \Sigma, \delta, q_0, \mathcal{F})$, where $\mathcal{F} \subseteq 2^Q$,

  A run ρ is accepting iff $\mathrm{Inf}(\rho) \in \mathcal{F}$.

- Rabin condition: $(Q, \Sigma, \delta, q_0, (U_i, V_i)_{1 \le i \le k})$, where $\forall i.\ U_i, V_i \subseteq Q$,

  A run ρ is accepting iff $\exists i.\ \mathrm{Inf}(\rho) \cap U_i = \emptyset \wedge \mathrm{Inf}(\rho) \cap V_i \neq \emptyset$.

- Streett condition: $(Q, \Sigma, \delta, q_0, (U_i, V_i)_{1 \le i \le k})$, where $\forall i.\ U_i, V_i \subseteq Q$,

  A run ρ is accepting iff $\forall i.\ \mathrm{Inf}(\rho) \cap V_i \neq \emptyset \rightarrow \mathrm{Inf}(\rho) \cap U_i \neq \emptyset$.

From Rabin and Streett to Muller:
Let $\mathcal{A} = (Q, \Sigma, \delta, q_0, (U_i, V_i)_{1 \le i \le k})$ be a Rabin (resp. Streett) automaton.
Then $\mathcal{A}$ is equivalent to the Muller automaton $(Q, \Sigma, \delta, q_0, \mathcal{F})$, where
$\mathcal{F} = \{F \mid \exists i. F \cap U_i = \emptyset \wedge F \cap V_i \neq \emptyset\}$
(resp. $\mathcal{F} = \{F \mid \forall i. F \cap V_i \neq \emptyset \rightarrow F \cap U_i \neq \emptyset\}$).

Equivalence of all the acceptance conditions

Muller condition: $(Q, \Sigma, \delta, q_0, \mathcal{F})$, where $\mathcal{F} \subseteq 2^Q$,

A run ρ is accepting iff $\mathrm{Inf}(\rho) \in \mathcal{F}$.

From Muller to Büchi

Let $\mathcal{A} = (Q, \Sigma, \delta, q_0, \mathcal{F})$ be a Muller automaton s.t.

$\mathcal{F} = \{F_1, \dots, F_k\}$ and $\forall i : 1 \leq i \leq k.\ F_i = \{q_i^1, \dots, q_i^{l_i}\}$.

Construct a Büchi automaton $\mathcal{B} = (Q', \Sigma, \delta', q_0', F')$ as follows.

- $Q' = Q \cup \{(q, i, j) \mid q \in Q, 1 \leq i \leq k, 0 \leq j \leq |F_i|\}$,
- $q_0' = q_0$,
- $F' = \{(q, i, |F_i|) \mid q \in Q, 1 \leq i \leq k\}$,
- δ' is defined as follows,
  - δ' contains all the transitions in δ,
  - for every transition $(q, a, q') \in \delta$ and every $i : 1 \leq i \leq k$ such that $q' \in F_i$, $(q, a, (q', i, 0)) \in \delta'$, guess F_i
  - for every transition $(q, a, q') \in \delta$,
    - if $q, q' \in F_i$ and $q' = q_i^{j+1}$, then $((q, i, j), a, (q', i, j+1)) \in \delta'$, increase the counter,
    - if $q, q' \in F_i$ and $q' \neq q_i^{j+1}$, then $((q, i, j), (q', i, j)) \in \delta'$,
  - for every transition $(q, a, q') \in \delta$, if $q, q' \in F_i$, then $((q, i, l_i), a, (q', i, 0)) \in \delta'$, reset the counter,

Transformation between deterministic automata

Theorem. Deterministic Muller, Rabin, Streett and Parity automata are expressively equivalent.

From Parity to Rabin and Streett, from Rabin and Streett to Muller:

Same as the nondeterministic automata.

Transformation between deterministic automata

- Muller condition: $(Q, \Sigma, \delta, q_0, \mathcal{F})$, where $\mathcal{F} \subseteq 2^Q$,

  A run ρ is accepting iff $\mathrm{Inf}(\rho) \in \mathcal{F}$.

- Rabin condition: $(Q, \Sigma, \delta, q_0, (U_i, V_i)_{1 \leq i \leq k})$, where $\forall i.\ U_i, V_i \subseteq Q$,

  A run ρ is accepting iff $\exists i.\ \mathrm{Inf}(\rho) \cap U_i = \emptyset \wedge \mathrm{Inf}(\rho) \cap V_i \neq \emptyset$.

- Rabin chain condition: A Rabin condition $(U_i, V_i)_{1 \leq i \leq k}$ s.t.
  $U_1 \subseteq V_1 \subseteq U_2 \subseteq V_2 \subseteq \cdots \subseteq U_k \subseteq V_k$.

From deterministic Muller to deterministic Parity (Rabin chain):
Let $\mathcal{A} = (Q, \Sigma, \delta, q_0, \mathcal{F})$ be a deterministic Muller automaton.
Suppose $Q = \{q_0, \ldots, q_n\}$.

The main idea.

Latest appearance record (LAR)

$$q_{i_0} q_{i_1} \ldots q_{i_r} \sharp q_{i_{r+1}} \ldots q_{i_n}$$

$$\Big\downarrow\ \delta(q_{i_n}, a) = q_{i_s}$$

$$q_{i_0} q_{i_1} \ldots q_{i_{s-1}} \sharp q_{i_{s+1}} \ldots q_{i_n} q_{i_s}$$

Transformation between deterministic automata

- Muller condition: $(Q, \Sigma, \delta, q_0, \mathcal{F})$, where $\mathcal{F} \subseteq 2^Q$,

  A run ρ is accepting iff $\mathrm{Inf}(\rho) \in \mathcal{F}$.

- Rabin condition: $(Q, \Sigma, \delta, q_0, (U_i, V_i)_{1 \leq i \leq k})$, where $\forall i.\ U_i, V_i \subseteq Q$,

  A run ρ is accepting iff $\exists i.\ \mathrm{Inf}(\rho) \cap U_i = \emptyset \wedge \mathrm{Inf}(\rho) \cap V_i \neq \emptyset$.

- Rabin chain condition: A Rabin condition $(U_i, V_i)_{1 \leq i \leq k}$ s.t.
  $U_1 \subseteq V_1 \subseteq U_2 \subseteq V_2 \subseteq \cdots \subseteq U_k \subseteq V_k$.

From deterministic Muller to deterministic Parity (Rabin chain):
Let $\mathcal{A} = (Q, \Sigma, \delta, q_0, \mathcal{F})$ be a deterministic Muller automaton.
Suppose $Q = \{q_0, \ldots, q_n\}$.

Construct a Parity automaton $\mathcal{A}' = (Q', \Sigma, \delta', q_0', (U_i, V_i)_{0 \leq i \leq n})$ as follows.

- Q' is the set of sequences $u \sharp v$ s.t. uv is a permutation of $q_0 \ldots q_n$.
- $q_0' = \sharp q_n q_{n-1} \ldots q_0$.
- if $\delta(q_{i_n}, a) = q_{i_s}$, then

  $$\delta'(q_{i_0} \ldots q_{i_r} \sharp q_{i_{r+1}} \ldots q_{i_n}, a) = q_{i_0} \ldots q_{i_{s-1}} \sharp q_{i_{s+1}} \ldots q_{i_n} q_{i_s}.$$

  In particular, if $\delta(q_{i_n}, a) = q_{i_n}$, then

  $$\delta'(q_{i_0} \ldots q_{i_r} \sharp q_{i_{r+1}} \ldots q_{i_n}, a) = q_{i_0} \ldots \sharp q_{i_n}.$$

- $U_i = \{u \sharp v \mid |u| < i\}$, $V_i = U_i \cup \{u \sharp v \mid |u| = i, \exists F \in \mathcal{F}.\ F = v\}$.
  $U_1 \subseteq V_1 \subseteq \cdots \subseteq U_n \subseteq V_n$.

Transformation between deterministic automata

Construct a Parity automaton $\mathcal{A}' = (Q', \Sigma, \delta', q'_0, (U_i, V_i)_{0 \leq i \leq n})$ as follows.

- Q' is the set of sequences $u \sharp v$ s.t. uv is a permutation of $q_0 \dots q_n$.
- $q'_0 = \sharp q_n q_{n-1} \dots q_0$.
- if $\delta(q_{i_n}, a) = q_{i_s}$, then

  $$\delta'(q_{i_0} \dots q_{i_r} \sharp q_{i_{r+1}} \dots q_{i_n}, a) = q_{i_0} \dots q_{i_{s-1}} \sharp q_{i_{s+1}} \dots q_{i_n} q_{i_s}.$$

  In particular, if $\delta(q_{i_n}, a) = q_{i_n}$, then

  $$\delta'(q_{i_0} \dots q_{i_r} \sharp q_{i_{r+1}} \dots q_{i_n}, a) = q_{i_0} \dots \sharp q_{i_n}.$$

- $U_i = \{u \sharp v \mid |u| < i\}$, $V_i = U_i \cup \{u \sharp v \mid |u| = i, \exists F \in \mathcal{F}.\ F = v\}$.

  $U_1 \subseteq V_1 \subseteq \dots \subseteq U_n \subseteq V_n$.

Correctness of the construction.

($\Rightarrow$) Let $w \in \Sigma^\omega$ and ρ be the accepting run of $\mathcal{A}$ over w. Then $\mathrm{Inf}(\rho) = F \in \mathcal{F}$.

Consider the run ρ' of $\mathcal{A}'$ corresponding to ρ.

$\exists j$ s.t. after the position j in ρ, only the states in $\mathrm{Inf}(\rho)$ appear $\Longrightarrow$
$\exists j' \geq j$ s.t. after the position j' in ρ',
all the states in $\mathrm{Inf}(\rho)$ are on the right side of $\sharp$ in LARs $\Longrightarrow$
$\exists i$ s.t. after the position j' in ρ', all the LARs $u \sharp v$ satisfy $|u| \geq i$,
and $\exists^\omega u \sharp v$ s.t. $|u| = i$ and $v = \mathrm{Inf}(\rho) = F \Longrightarrow$
$\mathrm{Inf}(\rho') \cap U_i = \emptyset$ and $\mathrm{Inf}(\rho') \cap V_i \neq \emptyset$, ρ' is an accepting run of $\mathcal{A}'$

Transformation between deterministic automata

Construct a Parity automaton $\mathcal{A}' = (Q', \Sigma, \delta', q_0', (U_i, V_i)_{0 \le i \le n})$ as follows.

- Q' is the set of sequences $u\sharp v$ s.t. uv is a permutation of $q_0 \dots q_n$.
- $q_0' = \sharp q_n q_{n-1} \dots q_0$.
- if $\delta(q_{i_n}, a) = q_{i_s}$, then

  $$\delta'(q_{i_0} \dots q_{i_r} \sharp q_{i_{r+1}} \dots q_{i_n}, a) = q_{i_0} \dots q_{i_{s-1}} \sharp q_{i_{s+1}} \dots q_{i_n} q_{i_s}.$$

  In particular, if $\delta(q_{i_n}, a) = q_{i_n}$, then

  $$\delta'(q_{i_0} \dots q_{i_r} \sharp q_{i_{r+1}} \dots q_{i_n}, a) = q_{i_0} \dots \sharp q_{i_n}.$$

- $U_i = \{u\sharp v \mid |u| < i\}$, $V_i = U_i \cup \{u\sharp v \mid |u| = i, \exists F \in \mathcal{F}.\ F = v\}$.

  $$U_0 \subseteq V_0 \subseteq U_1 \subseteq V_1 \subseteq \dots \subseteq U_n \subseteq V_n$$

Correctness of the construction.

($\Leftarrow$) Let $w \in \Sigma^\omega$ and ρ' be the accepting run of $\mathcal{A}'$ over w.

$\exists i$ s.t. $\mathrm{Inf}(\rho') \cap U_i = \emptyset$ and $\mathrm{Inf}(\rho') \cap V_i \neq \emptyset \Longrightarrow$

$\exists F \in \mathcal{F}$ and j' s.t. $u\sharp v$ in the position j' of ρ' satisfies $|u| = i, v = F$,

and after the position j' in ρ',

all $u'\sharp v'$ satisfy $|u'| \ge i$, and $\exists^\omega u'\sharp v'$, $|u'| = i, v' = F \Longrightarrow$

Consider the run ρ of $\mathcal{A}$ over w: After the position j' in ρ,

only states in F occur (o.w. $u'\sharp v'$ s.t. $|u'| < i$ occurs after j' in ρ'),

and every state in F occur infinitely often (o.w. $\exists j'' > j'$, all $u'\sharp v'$ after j'' satisfy $|u'| > i$, thus $\mathrm{Inf}(\rho') \cap V_i = \emptyset$).

Therefore, ρ is accepting.

Outline

Deterministic Muller automata (DMA)

Muller condition: $(Q, \Sigma, \delta, q_0, \mathcal{F})$, where $\mathcal{F} \subseteq 2^Q$,

A run ρ is accepting iff $\mathrm{Inf}(\rho) \in \mathcal{F}$.

Proposition. The class of languages recognized by DMA is closed under all Boolean operations.

- Union: $\mathcal{A}_1 = (Q_1, \Sigma, \delta_1, q_0^1, \mathcal{F}_1)$ and $\mathcal{A}_2 = (Q_2, \Sigma, \delta_2, q_0^2, \mathcal{F}_2)$.

  $\mathcal{A} = (Q_1 \times Q_2, \Sigma, \delta, (q_0^1, q_0^2), \mathcal{F})$, where
  - $\delta((q_1, q_2), a) = (\delta_1(q_1, a), \delta_2(q_2, a))$,
  - $\mathcal{F} = \{S \subseteq Q_1 \times Q_2 \mid \mathrm{proj}_2(S) \in \mathcal{F}_2\} \cup \{S \subseteq Q_1 \times Q_2 \mid \mathrm{proj}_1(S) \in \mathcal{F}_1\}$.
- Intersection: $\mathcal{A}_1 = (Q_1, \Sigma, \delta_1, q_0^1, \mathcal{F}_1)$ and $\mathcal{A}_2 = (Q_2, \Sigma, \delta_2, q_0^2, \mathcal{F}_2)$.

  $\mathcal{A} = (Q_1 \times Q_2, \Sigma, \delta, (q_0^1, q_0^2), \mathcal{F})$, where
  - $\delta((q_1, q_2), a) = (\delta_1(q_1, a), \delta_2(q_2, a))$,
  - $\mathcal{F} = \{S \subseteq Q_1 \times Q_2 \mid \mathrm{proj}_1(S) \in \mathcal{F}_1, \mathrm{proj}_2(S) \in \mathcal{F}_2\}$.
- Complementation: $\mathcal{A} = (Q, \Sigma, \delta, q_0, \mathcal{F}) \Rightarrow \mathcal{B} = (Q, \Sigma, \delta, q_0, 2^Q \setminus \mathcal{F})$.

Expressibility of DMA

Recall $\overrightarrow{W} = \{w \in \Sigma^\omega \mid \exists^\omega n.\ w_1 \dots w_n \in W\}$.

Proposition. L can be defined by a DBA iff $L = \overrightarrow{L'}$ for some regular language $L' \subseteq \Sigma^*$.

Theorem. An ω-language L is definable by a DMA iff L is a Boolean combination of sets $\overrightarrow{W}$ for regular $W \subseteq \Sigma^*$.

Proof.

"If" direction:

- $\overrightarrow{W}$ is recognized by a deterministic Büchi automaton,
- The class of languages recognized by DMAs is closed under all Boolean combinations.

"Only if" direction:

Suppose L is defined by a DMA $\mathcal{A} = (Q, \Sigma, \delta, q_0, \mathcal{F})$.

For every $q \in Q$, let W_q denote the language defined by DFA $(Q, \Sigma, \delta, q_0, \{q\})$.

Then

$$L = \bigcup_{F \in \mathcal{F}} \left(\bigcap_{q \in F} \overrightarrow{W_q} \cap \bigcap_{q \notin F} \overline{\overrightarrow{W_q}} \right).$$

□

Mcnaughton's theorem: NBA $\equiv$ DMA

Theorem. From every nondeterministic Büchi automaton, an equivalent DMA can be constructed.

NBA $\Rightarrow$ Semi-deterministic Büchi automata (SDBA) $\Rightarrow$ DMA

Using the slides and lecture notes by Bernd Finkbeiner.

NBA $\Rightarrow$ SDBA:

- Slides: `http://www.react.uni-saarland.de/teaching/automata-games-verification-12/downloads/intro6.pdf`
- Lecture notes: `http://www.react.uni-saarland.de/teaching/automata-games-verification-12/downloads/notes5.pdf`

SDBA $\Rightarrow$ DMA:

- Slides: `http://www.react.uni-saarland.de/teaching/automata-games-verification-12/downloads/intro7.pdf`
- Lecture notes: `http://www.react.uni-saarland.de/teaching/automata-games-verification-12/downloads/notes6.pdf`

Outline

ω-regular $\equiv$ WMSO

WMSO:

The same syntax as MSO, with the interpretations of set variables restricted to finite sets.

WMSO to MSO=NBA=DMA: WMSO $\varphi \Rightarrow$ MSO $\overline{\varphi}$

$$\overline{\exists X \eta} = \exists X(\exists y \forall x(X(x) \to x \leq y) \land \overline{\eta}).$$

From DMA to WMSO:

It is sufficient to show that $\overrightarrow{W}$ with W regular can be defined by a WMSO sentence φ.

W is regular $\Rightarrow \exists$ a MSO sentence ψ on finite words equivalent to W.

Then $\overrightarrow{W}$ is defined by $\forall x \exists y(x < y \land \psi_{\leq y})$, where $\psi_{\leq y}$ is obtained from ψ as follows:

- Replace every subformula $\exists X \eta$ with $\exists X(\forall x'(X(x') \to x' \leq y) \land \eta_{\leq y})$.
- Replace every subformula $\exists x' \eta$ with $\exists x'(x' \leq y \land \eta_{\leq y})$.