# Numerical Optimization, 2022 Fall
# Homework 2

参考答案

Due 14:59 (CST), Sep. 22, 2022
(NOTE: Homework will not be accepted after this due for any reason.)
(Only latex version is accepted)

## 1 LP Convertion

Convert the following problem to a linear program in standard form [30pts]

$$\begin{aligned} \min \quad & |x| + |y| + |z| \\ \text{s.t.} \quad & x + y \leq 1 \\ & 2x + z = 3. \end{aligned} \tag{1}$$

**Solution:**

The standard form is:

$$\begin{aligned} \text{minimize} \quad & x^+ + x^- + y^+ + y^- + z^+ + z^- \\ \text{subject to} \quad & x^+ - x^- + y^+ - y^- + s = 1 \\ & 2x^+ - 2x^- + z^+ - z^- = 3 \\ & x^+, x^-, y^+, y^-, z^+, z^-, s \geq 0 \end{aligned}$$

**Deduction:**

Let
$$\begin{aligned} x = x^+ - x^-, \quad & x^+ = max(x, 0), x^- = max(-x, 0) \\ y = y^+ - y^-, \quad & y^+ = max(y, 0), y^- = max(-y, 0) \\ z = z^+ - z^-, \quad & z^+ = max(z, 0), z^- = max(-z, 0) \end{aligned}$$

Then we can get the standard form above.

## 2 Polyhedra

A polyhedron $P$ is a set that can be described in the form:

$$\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}, \quad \boldsymbol{C}\boldsymbol{x} = \boldsymbol{d},$$

where $\boldsymbol{A} \in \mathbb{R}^{m\times n}$, $\boldsymbol{C} \in \mathbb{R}^{p\times n}$. Prove the polyhedron $P$ is a convex set.[20pts]

Note: Do not simply claim it is intersection of hyper-planes and half-spaces. Prove it by definition of convex set.

**Prove:**

In order to prove that P is a convex set, we need to prove that

If for any $\boldsymbol{x_1}, \boldsymbol{x_2} \in \boldsymbol{x}$, which satisfy

$$\begin{aligned} \boldsymbol{A}\boldsymbol{x_1} \leq \boldsymbol{b}, \quad \boldsymbol{C}\boldsymbol{x_1} = \boldsymbol{d} \\ \boldsymbol{A}\boldsymbol{x_2} \leq \boldsymbol{b}, \quad \boldsymbol{C}\boldsymbol{x_2} = \boldsymbol{d} \end{aligned}$$

then for any $\lambda \in [0, 1]$ satisfies that [condition1]

$$\begin{aligned} \boldsymbol{A}(\lambda\boldsymbol{x_1} + (1-\lambda)\boldsymbol{x_2}) \leq \boldsymbol{b} \\ \boldsymbol{C}(\lambda\boldsymbol{x_1} + (1-\lambda)\boldsymbol{x_2}) = \boldsymbol{d} \end{aligned}$$

So when

$$\begin{aligned} \boldsymbol{A}\boldsymbol{x_1} \leq \boldsymbol{b}, \quad \boldsymbol{C}\boldsymbol{x_1} = \boldsymbol{d} \\ \boldsymbol{A}\boldsymbol{x_2} \leq \boldsymbol{b}, \quad \boldsymbol{C}\boldsymbol{x_2} = \boldsymbol{d} \end{aligned}$$

Then for any $\lambda \in [0, 1]$

$$\begin{aligned} \boldsymbol{A}\lambda\boldsymbol{x_1} \leq \lambda\boldsymbol{b}, \quad \boldsymbol{C}\lambda\boldsymbol{x_1} = \lambda\boldsymbol{d} \\ \boldsymbol{A}(1-\lambda)\boldsymbol{x_2} \leq (1-\lambda)\boldsymbol{b}, \quad \boldsymbol{C}(1-\lambda)\boldsymbol{x_2} = (1-\lambda)\boldsymbol{d} \end{aligned}$$

Then we add the first line and the second line and get

$$\begin{aligned} \boldsymbol{A}\lambda\boldsymbol{x_1} + \boldsymbol{A}(1-\lambda)\boldsymbol{x_2} \leq \lambda\boldsymbol{b} + (1-\lambda)\boldsymbol{b} = \boldsymbol{b} \\ \boldsymbol{C}\lambda\boldsymbol{x_1} + \boldsymbol{C}(1-\lambda)\boldsymbol{x_2} = \lambda\boldsymbol{d} + (1-\lambda)\boldsymbol{d} = \boldsymbol{d} \end{aligned}$$

So [condition1] has been proved

So P is a convex set.

# 3 LP Comprehensive Problem

Consider minimizing a linear function over a polyhedron $P$ as follows:

$$\begin{aligned} \min \quad & -x_1 - x_2 - x_3 \\ \text{s.t.} \quad & x_1 + x_2 + 2x_3 \leq 8 \\ & x_2 + 6x_3 \leq 12 \\ & x_1 \leq 4 \\ & x_2 \leq 6 \\ & x_1, x_2, x_3 \geq 0. \end{aligned} \tag{2}$$

(a) convert the above LP problem to a standard form[30pts]

(b) give all extreme points (basic feasible solutions) and the optimal solution[20pts]

**Solution:**

(a) The standard form is

$$\begin{aligned} \text{minimize} \quad & -x_1 - x_2 - x_3 \\ \text{subject to} \quad & x_1 + x_2 + 2x_3 + s_1 = 8 \\ & x_2 + 6x_3 + s_2 = 12 \\ & x_1 + s_3 = 4 \\ & x_2 + s_4 = 6 \\ & x_1, x_2, x_3, s_1, s_2, s_3, s_4 \geq 0 \end{aligned}$$

(b) Solve the standard form:

$$\begin{aligned} & x_1 + x_2 + 2x_3 + s_1 = 8 \\ & x_2 + 6x_3 + s_2 = 12 \\ & x_1 + s_3 = 4 \\ & x_2 + s_4 = 6 \\ & \text{with all variables} \geq 0 \end{aligned}$$

we get the basic solution that

(a) $x_1 = 0, x_2 = 0, x_3 = 0, s_1 = 8, s_2 = 12, s_3 = 4, s_4 = 6$ feasible

(b) $x_1 = 0, x_2 = 0, x_3 = 4, s_1 = 0, s_2 = -12, s_3 = 4, s_4 = 6$ not feasible

(c) $x_1 = 0, x_2 = 0, x_3 = 2, s_1 = 4, s_2 = 0, s_3 = 4, s_4 = 6$ feasible

(d) $x_1 = 0, x_2 = 8, x_3 = 0, s_1 = 0, s_2 = 4, s_3 = 4, s_4 = -2$ not feasible

(e) $x_1 = 0, x_2 = 12, x_3 = 0, s_1 = -4, s_2 = 0, s_3 = 4, s_4 = 6$ not feasible

(f) $x_1 = 0, x_2 = 6, x_3 = 0, s_1 = 2, s_2 = 6, s_3 = 4, s_4 = 0$ feasible

(g) $x_1 = 0, x_2 = 6, x_3 = 1, s_1 = 0, s_2 = 0, s_3 = 4, s_4 = 0$ feasible

(h) $x_1 = 8, x_2 = 0, x_3 = 0, s_1 = 0, s_2 = 12, s_3 = -4, s_4 = 6$ not feasible

(i) $x_1 = 4, x_2 = 0, x_3 = 0, s_1 = 4, s_2 = 12, s_3 = 0, s_4 = 6$ feasible

(j) $x_1 = 4, x_2 = 0, x_3 = 2, s_1 = 0, s_2 = 0, s_3 = 0, s_4 = 6$ feasible

(k) $x_1 = -4, x_2 = 12, x_3 = 0, s_1 = 0, s_2 = 0, s_3 = 8, s_4 = -6$ not feasible

(l) $x_1 = 4, x_2 = 4, x_3 = 0, s_1 = 0, s_2 = 8, s_3 = 0, s_4 = 2$ feasible

(m) $x_1 = 2, x_2 = 6, x_3 = 0, s_1 = 0, s_2 = 6, s_3 = 2, s_4 = 0$ feasible

(n) $x_1 = 4, x_2 = 12, x_3 = 0, s_1 = -8, s_2 = 0, s_3 = 0, s_4 = -6$ not feasible

(o) $x_1 = 4, x_2 = 0, x_3 = 2, s_1 = 0, s_2 = 0, s_3 = 0, s_4 = 6$ feasible

(p) $x_1 = 4, x_2 = 6, x_3 = 1, s_1 = -4, s_2 = 0, s_3 = 0, s_4 = 0$ not feasible

So the bfs are (0,0,0),(0,0,2),(0,6,0),(0,6,1),(4,0,0),(4,0,2),(4,4,0),(2,6,0),(4,0,2)
And the optimal solution is -8 when $x = 4, y = 4, z = 0$ or $x = 2, y = 6, z = 0$