EE150 Signals and Systems – Part 3: Fourier Series Representation of Periodic Signals

Objective

- **Recall:**
  Previously, we use the weighted sum (integral) of shifted impulses to represent an input and then derive the convolution sum (integral).

★ In this chapter:
We use different basic signal, the complex exponential, to represent the input.

★ Why we use complex exponential?

Eigenfunction of LTI System

A signal for which the system output is just a constant (possibly complex) times the input is referred to as an eigenfunction of the system.

$$f(t) \longrightarrow \boxed{H} \longrightarrow C \cdot f(t)$$

C:constant $\rightarrow$ the eigenvalue

Objective

The output to an input $x(t)$ can be found easily if $x(t)$ can be expressed as weighted sum of the eigenfunctions.

Eigenfunction of CT LTI Systems

Consider an input $x(t) = \mathrm{e}^{st}$ and a CT LTI system with impulse response $h(t)$

$$
\begin{aligned}
y(t) &= \int_{-\infty}^{\infty} h(\tau)\mathrm{e}^{s(t-\tau)}\mathrm{d}\tau \\
&= \mathrm{e}^{st}\int_{-\infty}^{\infty} h(\tau)\mathrm{e}^{-s\tau}\mathrm{d}\tau
\end{aligned}
$$

Let $H(s) = \int_{-\infty}^{\infty} h(\tau)\mathrm{e}^{-s\tau}\mathrm{d}\tau$ (eigenvalue), then $y(t) = H(s)\mathrm{e}^{st}$. Hence, complex exponentials are eigenfunctions of LTI systems:

$$
\mathrm{e}^{st} \to H(s)\mathrm{e}^{st}
$$

Eigenfunction of DT LTI Systems

Consider an input $x[n] = z^n$ and a DT LTI system with impulse response $h[n]$

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]z^{n-k} = z^n \sum_{k=-\infty}^{\infty} h[k]z^{-k}$$

Let $H(z) = \sum_{k=-\infty}^{\infty} h[k]z^{-k}$ (eigenvalue), then $y[n] = H(z)z^n$.
Hence, complex exponentials are eigenfunctions of LTI systems:

$$z^n \rightarrow H(z)z^n$$

Usefulness of Complex Exponentials

Ex: If input $x(t) = a_1 e^{s_1 t} + a_2 e^{s_2 t} + a_3 e^{s_3 t}$, based on eigenfunction property and superposition property, the the response is
$y(t) = a_1 H(s_1) e^{s_1 t} + a_2 H(s_2) e^{s_2 t} + a_3 H(s_3) e^{s_3 t}$

Generally, for a CT LTI system, if the input is a linear combination of complex exponentials, then

$$x(t) = \sum_k a_k e^{s_k t} \to y(t) = \sum_k a_k H(s_k) e^{s_k t}.$$

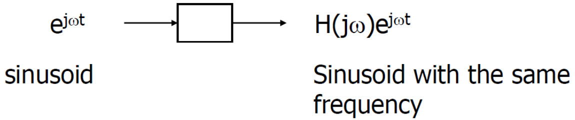
Similarly, for a DT LTI system,

$$x[n] = \sum_k a_k z_k^n \to y[n] = \sum_k a_k H(z_k) z_k^n.$$

Fourier Analysis

For Fourier analysis, we consider

1. CT: purely imaginary $s = j\omega$: $e^{j\omega t}$



2. DT: unit magnitude $z = e^{j\omega}$: $e^{j\omega n}$

$$e^{j\omega n} \to H(e^{j\omega})e^{j\omega n}$$

Periodic signals & Fourier Series Expansion

Jean Baptiste Joseph Fourier
March 21 1768 - May 16 1830
Born Auxerre, France. Died Paris, France.

- Using "trigonometric sum" to describe periodic signal can be tracked back to Babylonians who predicted astronomical events similarly.
- L. Euler (in 1748) and Bernoulli (in 1753) used the "normal mode" concept to describe the motion of a vibrating string; though JL Lagrange strongly criticized this concept.
- Fourier (in 1807) had found series of harmonically related sinusoids to be useful to describe the temperature distribution through body, and he claimed "any" periodic signal can be represented by such series.
- Dirichlet (in 1829) provide a precise condition under which a periodic signal can be represented by a Fourier series.

Aside: an orthonormal set

- Consider the set, S, of $x(t)$ satisfying $x(t) = x(t + T_0)$
- Dot-product (inner-product) defined as

$$< x_1(t), x_2(t) >= \frac{\omega_0}{2\pi} \int_{-\frac{\pi}{\omega_0}}^{\frac{\pi}{\omega_0}} x_1(t) x_2^*(t) dt$$

- Consider the set, B, of functions in S

$$\phi_k(t) = e^{jk\omega_0 t};\ \omega_0 = \frac{2\pi}{T_0}, k \in \mathbb{Z}$$

- Observe that they are orthonormal

$$\frac{\omega_0}{2\pi} \int_{-\frac{\pi}{\omega_0}}^{\frac{\pi}{\omega_0}} e^{jk_1\omega_0 t} e^{-jk_2\omega_0 t} dt = \begin{cases} 0 & k_1 \neq k_2 \\ 1 & k_1 = k_2 \end{cases}$$

Fourier's Idea

- The span of the orthonormal functions, B, covers most of S. i.e. $span(B) \approx S$
- More precisely, under mild assumptions: $x(t)$ is sum of sinusoids, i.e.

$$x(t) = \sum_k a_k e^{jk\omega_0 t}, \, where \; a_k = \frac{\omega_0}{2\pi} \int_{-\frac{\pi}{\omega_0}}^{\frac{\pi}{\omega_0}} x(\tau) e^{-jk\omega_0 \tau} d\tau$$

Example

Consider a periodic signal $x(t) = \sum_{k=-3}^{3} a_k \mathrm{e}^{jk2\pi t}$, where $a_0 = 1$, $a_1 = a_{-1} = 0.25$, $a_2 = a_{-2} = 0.5$, $a_3 = a_{-3} = 1/3$.

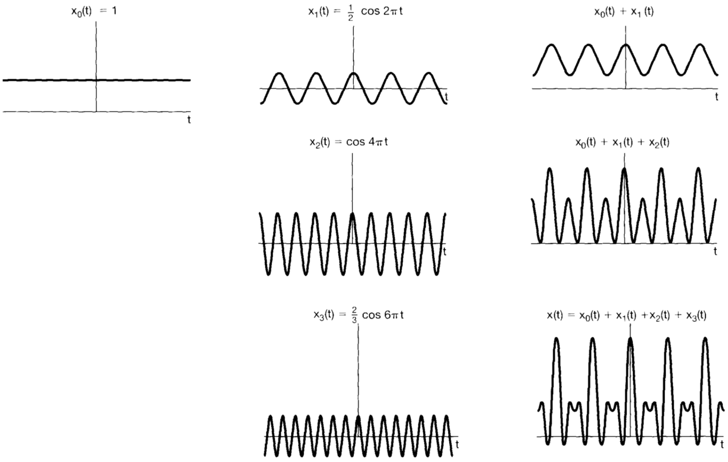
Collecting each of the harmonic components having the same fundamental frequency

$$x(t) = 1+\frac{1}{4}\left(\mathrm{e}^{j2\pi t} + \mathrm{e}^{-j2\pi t}\right)+\frac{1}{2}\left(\mathrm{e}^{j4\pi t} + \mathrm{e}^{-j4\pi t}\right)+\frac{1}{3}\left(\mathrm{e}^{j6\pi t} + \mathrm{e}^{-j6\pi t}\right)$$

Using Euler's relation

$$x(t) = 1 + \frac{1}{2}\cos(2\pi t) + \cos(4\pi t) + \frac{2}{3}\cos(6\pi t)$$

Example cont.



Periodic signals & Fourier Series Expansion cont.

- Theorem (for reasonable functions):
  $x(t)$ may be expressed as a Fourier series:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k \cdot e^{jk\omega_0 t}$$

(sum of sinusoids whose frequencies are multiple of ω_0, the "fundamental frequency".)
Where a_k can be obtained by

$$a_k = \frac{1}{T_0} \int_{T_0} x(\tau) e^{-jk\omega_0 \tau} d\tau$$ - *Fourier series coefficient.*

Note: $e^{jk\omega_0 t}$, for $k = -\infty$ to ∞, are orthonormal function. (Normal basic signal)

Example 3.4

Let $x(t) = 1 + \sin(\omega_0 t) + 2\cos(\omega_0 t) + \cos(2\omega_0 t + \frac{\pi}{4})$ has fundamental frequency ω_0.

First, expanding $x(t)$ in terms of complex exponentials

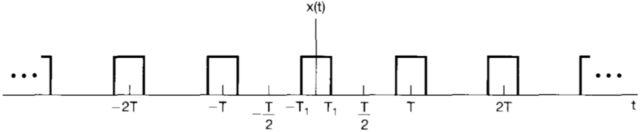
$$x(t) = 1 + \left(1 + \frac{1}{2j}\right) e^{j\omega_0 t} + \left(1 - \frac{1}{2j}\right) e^{-j\omega_0 t} + \frac{1}{2} e^{j\frac{\pi}{4}} e^{j2\omega_0 t} + \frac{1}{2} e^{-j\frac{\pi}{4}} e^{-j2\omega_0 t}$$

Then, the Fourier series coefficients can be directly obtained.

Example 3.5 Square Wave

For a periodic square wave, the definition over one period is

$$x(t) = \begin{cases} 1, & |t| < T_1, \\ 0, & T_1 < |t| < \frac{T}{2}. \end{cases}$$



Periodic with fundamental period T and fundamental frequency $\omega_0 = \frac{2\pi}{T}$

Example 3.5 Square Wave cont.

For $k = 0$,

$$a_0 = \frac{1}{T}\int_{-T_1}^{T_1} \mathrm{d}t = \frac{2T_1}{T}.$$

For $k \neq 0$,

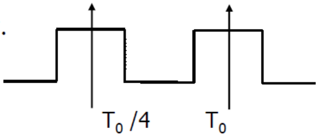
$$a_k = \frac{1}{T}\int_{-T_1}^{T_1} \mathrm{e}^{-jk\omega_0 t}\mathrm{d}t = \frac{2\sin(k\omega_0 T_1)}{k\omega_0 T} = \frac{\sin(k\omega_0 T_1)}{k\pi}.$$

When $T_1 = \frac{1}{4}T \rightarrow$ 50% duty-cycle square wave

$$a_k = \frac{sin(\frac{k\pi}{2})}{k\pi}.$$

Example 3.5 Square Wave cont.

$\therefore$



$$\equiv \frac{1}{2} + \frac{2}{\pi}\cos\omega_0 t - \frac{2}{3\pi}\cos 3\omega_0 t + \cdots$$

k	...	-5	-4	-3	-2	-1	0	1	2	3	4	5
a_k	...	$1/5\pi$	0	$-1/3\pi$	0	$1/\pi$	$1/2$	$1/\pi$	0	$-1/3\pi$	0	$1/5\pi$

Fourier Series

$$
\begin{aligned}
x(t) &= \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \\
&= \sum_{k=-\infty}^{\infty} (a_k cos(k\omega_0 t) + j a_k sin(k\omega_0 t)) \\
&= a_0 + \sum_{k>0} ((a_k + a_{-k}) cos(k\omega_0 t) + j(a_k - a_{-k}) sin(k\omega_0 t))
\end{aligned}
$$

In general, a_k is complex. Therefore, this is not the real and imaginary decomposition.

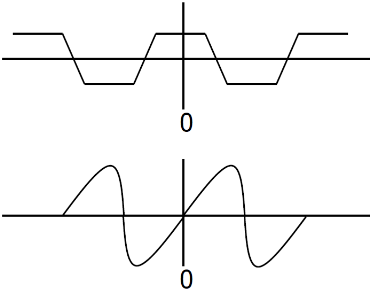
However, this is the even and odd decomposition

Fourier Series cont.

1. Odd/Even periodic functions

Even

Odd



$\Rightarrow$ $a_k = a_{-k}$

$\Rightarrow$ all sine terms vanish

$\Rightarrow$ $a_k = -a_{-k}$

Fourier Series cont.

2. Approximation by Truncating Higher Harmonics

$$\textit{If} \quad a_k(\textit{for } |k| > N) \textit{ are small}, x(t) \approx \hat{x}(t) \equiv \sum_{-N}^{N} a_k e^{jk\omega_0 t}$$

The approximation error is

$$e(t) = x(t) - \hat{x}(t)$$

How good is the approximation?

- Metric: relative energy in the error over a period

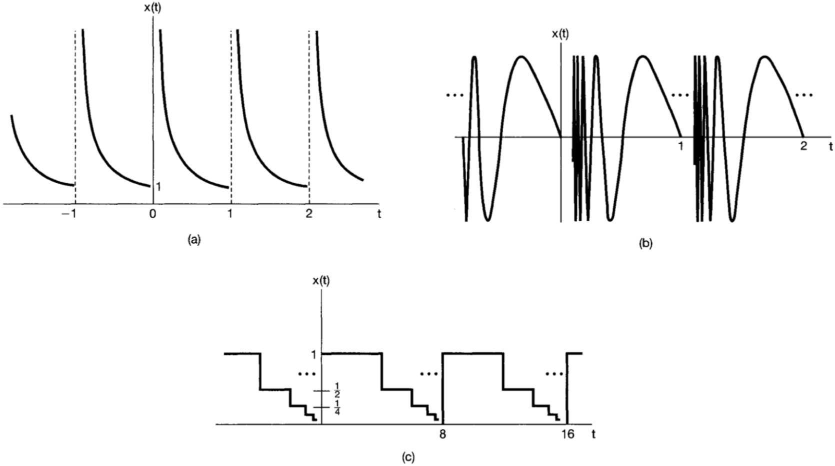
$$err = \frac{< e(t), e(t) >}{< x(t), x(t) >} = \frac{\int_{\frac{-\pi}{\omega_0}}^{\frac{\pi}{\omega_0}} e(t)e^*(t)dt}{\int_{\frac{-\pi}{\omega_0}}^{\frac{\pi}{\omega_0}} x(t)x^*(t)dt} = \frac{\sum_{|k|>N} |a_k|^2}{\sum_{k=-\infty}^{\infty} |a_k|^2}$$

Convergence of Fourier Series

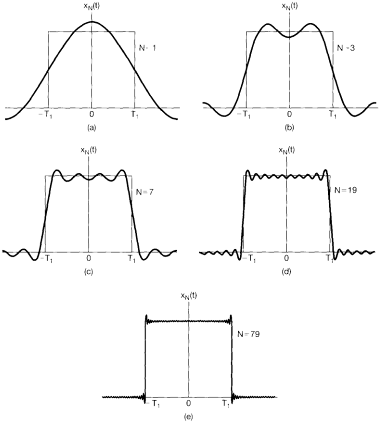
- Conditions for convergence of Fourier series is a deep subject
  - Finite energy over a single period
  - Dirichlet conditions:
    - Over any period, $x(t)$ must be absolutely integrable
    - Finite number of extrema during any single period
    - Finite number of discontinuities in any finite interval of time; each of these discontinuities is finite.

Check: `https://en.wikipedia.org/wiki/Dini_test`

Signals Violating Dirichlet Conditions



Gibbs Phenomenon



Overshoot $\approx 9\%$ as N goes to ∞

Properties of Continuous-Time Fourier Series

Assume $x(t)$ is periodic with period T and fundamental frequency $\omega_0 = 2\pi/T$.
$x(t)$ and its Fourier-series coefficients a_k are denoted by

$$x(t) \overset{FS}{\longleftrightarrow} a_k$$

Assume $x(t) \overset{FS}{\longleftrightarrow} a_k, \quad y(t) \overset{FS}{\longleftrightarrow} b_k$ (using same T)

1. Linearity: $z(t) = \alpha x(t) + \beta y(t) \overset{FS}{\longleftrightarrow} \alpha a_k + \beta b_k$
2. Time-shift: $x(t - t_0) \overset{FS}{\longleftrightarrow} e^{-jk\omega_0 t_0} a_k$

Properties of C-T Fourier Series cont.

3. Time-reverse: $x(-t) \overset{FS}{\longleftrightarrow} a_{-k}$

4. Time-scaling: $x(\alpha t) = \sum_{k=-\infty}^{\infty} a_k e^{jk(\alpha\omega_0)t}$

5. Multiplication:

$$x(t)y(t) \overset{FS}{\longleftrightarrow} h_k = \sum_{l=-\infty}^{\infty} a_l b_{k-l} \quad - \textit{Convolution}!$$

Properties of C-T Fourier Series cont.

6. conjugation & conjugate symmetry:

$$x^* \xleftrightarrow{FS} a^*_{-k}$$

If $x(t)$ real $\rightarrow x(t) = x^*(t)$

$\rightarrow a_k^* = a_{-k}$

If $x(t)$ is real and even,

$\rightarrow a_k = a_{-k} = a_k^*$

$\rightarrow$ Fourier Series Coefficients are real & even

If $x(t)$ is real and odd,

$\rightarrow a_k = -a_{-k} = -a_k^*$

$\rightarrow$ Fourier Series Coefficients are purely imaginary & odd

Properties of C-T Fourier Series cont.

7. $\frac{dx(t)}{dt} \xleftrightarrow{FS} jk\omega_0 a_k \qquad \int_{-\infty}^{t} x(\tau)d\tau \xleftrightarrow{FS} \frac{a_k}{jk\omega_0}$

8. Parseval's Identity: $\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |a_k|^2$

Proof.

$$\begin{aligned}
\frac{1}{T}\int_T |x(t)|^2 dt &= \frac{1}{T}\int_T \sum_{k_1,k_2} a_{k_1} a_{k_2}^* e^{j(k_1-k_2)\omega_0 t} dt \\
&= \sum_{k_1,k_2} a_{k_1} a_{k_2}^* \delta[k_1-k_2] \\
&= \sum_k |a_k|^2
\end{aligned}$$

$\square$

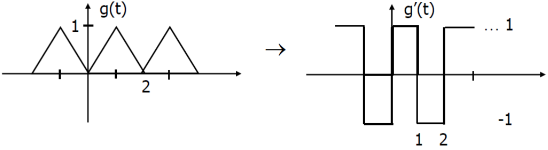
Example:

a. $x(t) = \cos \omega_0 t$

$$\rightarrow x(t) = \frac{1}{2}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right) = \frac{1}{2}e^{j\omega_0 t} + \frac{1}{2}e^{-j\omega_0 t}$$

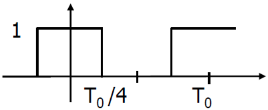
$$\therefore a_0 = 0, a_1 = \frac{1}{2}, a_{-1} = \frac{1}{2}, a_k = 0 \quad \textit{otherwise}$$

b.



Example: cont.

Recall from the previous example, we have

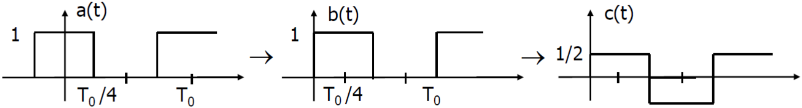


$\Rightarrow$

$$a_0 = \frac{1}{2}$$

$$a_k = \frac{\sin(k\omega_0 \cdot \frac{T_0}{4})}{k\pi} = \frac{\sin(\frac{k\pi}{2})}{k\pi}$$

By changing of variable, we can obtain



$$c(t) = a(t - \frac{T_0}{4}) - \frac{1}{2}$$

Example: cont.

Assume: $a(t) \xleftrightarrow{FS} a_k, \quad b(t) \xleftrightarrow{FS} b_k, \quad c(t) \xleftrightarrow{FS} c_k$

$$\Rightarrow c_k = \begin{cases} a_0 - \frac{1}{2}, & k = 0 \\ a_k e^{-jk\omega_0 \frac{T_0}{4}}, & k \neq 0 \end{cases} \quad \textit{where } \omega_0 = 2\pi \frac{1}{T_0}$$

$$\therefore c_k = \begin{cases} 0 \qquad , & k = 0 \\ \frac{\sin(k\frac{\pi}{2})}{k\pi} \times e^{-j\frac{k\pi}{2}}, & k \neq 0 \end{cases}$$

Example: cont.

Assume: $g^{'}(t) \overset{FS}{\longleftrightarrow} d_k$

$$\because g^{'}(t) = 2 \cdot c(t) \quad with \;\; T_0 = 2$$
$$\therefore d_k = 2 \cdot c_k$$

Assume: $g(t) \overset{FS}{\longleftrightarrow} e_k$

$$\therefore e_k = \frac{d_k}{jk\omega_0} = \frac{2c_k}{jk\omega_0} = \frac{2c_k}{jk\pi\frac{2}{T_0}}$$

For $k \neq 0 \Rightarrow e_k = \frac{2\sin(\frac{k\pi}{2})}{j(k\pi)^2} e^{-j\frac{k\pi}{2}} \quad (T_0 = 2)$

For $k = 0 \quad e_0 = \frac{The \;\; area \;\; under \;\; g(t) \;\; in \;\; one \;\; period}{period} = \frac{1}{2}$

Example 3.9

Suppose we have the following facts about a signal $x(t)$

1. $x(t)$ is a real signal
2. $x(t)$ is periodic with period $T = 4$, and it has Fourier series coefficient a_k
3. $a_k = 0$ for $k > 1$
4. The signal with Fourier coefficient $c_k = \mathrm{e}^{-j\pi k/2} a_{-k}$ is odd
5. $\frac{1}{4} \int_4 |x(t)|^2 \mathrm{d}t = \frac{1}{2}$

Fourier Series for Discrete-Time Periodic Signal

- Difference:
  1. continuous-time: infinite series
     discrete-time: finite series
  2. no convergence issue in discrete-time

★ Recall: $x[n]$ is periodic with period N if $x[n] = x[n+N]$

The fundamental period is the smallest positive N which satisfies the above eq.; and $\omega_0 = 2\pi/N$ is the fundamental frequency.

An Orthonormal Set

- Consider the set, T, of $x[n]$ satisfying $x[n] = x[n+N]$
- Dot-product (inner-product) defined as

$$< x_1[n], x_2[n] >= \frac{1}{N}\sum_{n=0}^{N-1} x_1[n]x_2^*[n]$$

- Consider the set, C, of N functions in T

$$\mu_k[n] = e^{jk\omega_0 n}; \omega_0 = \frac{2\pi}{N}, 0 \leq k \leq N-1$$

- Observe that they are orthonormal

$$\frac{1}{N}\sum_{m=0}^{N-1} e^{jk_1\omega_0 m} e^{-jk_2\omega_0 m} = \begin{cases} 0 & k_1 \neq k_2 \\ 1 & k_1 = k_2 \end{cases}$$

Fourier Series for Discrete-Time Periodic Signal

Theorem

$$x[n] = \sum_{k=0}^{N-1} a_k e^{jk\omega_0 n}, \ \ where \ a_k = \frac{1}{N}\sum_{m=0}^{N-1} x[m]e^{-jk\omega_0 m}$$

Proof:

$$x[n] = \sum_{k=0}^{N-1}\left(\frac{1}{N}\sum_{m=0}^{N-1} x[m]e^{-jk\omega_0 m}\right)e^{jk\omega_0 n},$$

$$\Leftrightarrow \quad x[n] = \frac{1}{N}\sum_{m=0}^{N-1} x[m]\sum_{k=0}^{N-1} e^{-jk\omega_0 m}e^{jk\omega_0 n},$$

$$\Leftrightarrow \quad x[n] = \sum_{m=0}^{N-1} x[m]\delta[n-m].$$

Example 3.12

Discrete-time periodic square wave

$$a_k = \frac{1}{N}\sum_{n=-N_1}^{N_1} \mathrm{e}^{-jk(2\pi/N)n}$$

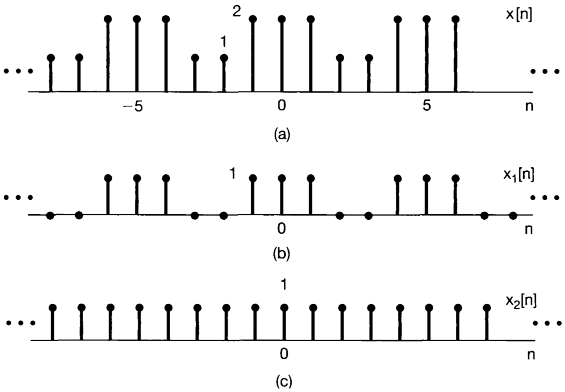
$$= \begin{cases} \frac{1}{N}\frac{\sin[2\pi k(N_1+1/2)/N]}{\sin(\pi k/N)}, & k \neq 0, \pm N, \pm 2N, \ldots \\ \frac{2N_1+1}{N}, & k = 0, \pm N, \pm 2N, \ldots \end{cases}$$

Properties of Discrete-Time Fourier Series

$x[n]$ and $y[n]$ are periodic with period N, a_k and b_k are periodic with period N

1. Linearity: $Ax[n] + By[n] \overset{FS}{\longleftrightarrow} Aa_k + Bb_k$
2. Time shifting: $x[n - n_0] \overset{FS}{\longleftrightarrow} a_k \mathrm{e}^{-jk(2\pi/N)n_0}$
3. Frequency shifting: $\mathrm{e}^{jM(2\pi/N)n} x[n] \overset{FS}{\longleftrightarrow} a_{k-M}$
4. Conjugation: $x^*[n] \overset{FS}{\longleftrightarrow} a^*_{-k}$
5. Time reversal: $x[-n] \overset{FS}{\longleftrightarrow} a_{-k}$
6. Periodic convolution: $\sum_{r=<N>} x[r]y[n-r] \overset{FS}{\longleftrightarrow} Na_k b_k$
7. Multiplication: $x[n]y[n] \overset{FS}{\longleftrightarrow} \sum_{l=<N>} a_l b_{k-l}$
8. First difference: $x[n] - x[n-1] \overset{FS}{\longleftrightarrow} (1 - \mathrm{e}^{-jk(2\pi/N)})a_k$
9. Parseval's relation: $\frac{1}{N} \sum_{n=<N>} |x[n]|^2 = \sum_{k=<N>} |a_k|^2$

Example 3.13



As $x[n] = x_1[n] + x_2[n]$, we have $a_k = b_k + c_k$

Fourier Series and LTI-System

- Recall: eigenfunction

$$x(t)=e^{st} \longrightarrow \boxed{H} \longrightarrow y(t)=H(s)\, e^{st}$$

where $H(s)=\int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau$

$h(\tau)$: impulse response of the system

For discrete-time, similarly we have

$$x[n]=z^{n} \longrightarrow \boxed{H} \longrightarrow y[n]=H(z)\, z^{n}$$

where $H(z)=\sum_{k=-\infty}^{\infty} h[k]z^{-k}$

Fourier Series and LTI-System cont.

$H(s)$ and $H(z)$ are referred to as "system function" (transfer functions).

$(C-T)$ continuous-time: $\quad Re\{s\}=0 \rightarrow s=j\omega$

$(D-T)$ discrete-time: $\quad |z|=1 \rightarrow z=e^{j\omega}$

$$\Rightarrow \begin{cases} H(j\omega)=\int_{-\infty}^{\infty} h(t)e^{-j\omega t}dt, & C-T, \\ H(e^{j\omega})=\sum\limits_{n=-\infty}^{\infty} h[n]e^{-j\omega n}, & D-T. \end{cases}$$

$H(j\omega)$ and $H(e^{j\omega})$ are the "frequency response" of the continuous time and discrete-time system, respectively.

Remarks

The transfer function, $H(j\omega)$, is then

$$H(j\omega) \equiv |H(j\omega)|\, e^{j\phi} \qquad \textit{with} \quad \phi \equiv \angle H(j\omega)$$

The output is $\rightarrow \quad H(j\omega)e^{j\omega t} = |H(j\omega)|\, e^{j(\omega t+\phi)}$

Fourier Series and LTI-System

Note: For C.T., periodic signal, then

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \rightarrow y(t) = \sum_{k=-\infty}^{\infty} a_k H(jk\omega_0) e^{jk\omega_0 t}$$

Similarly,

$$x[n] = \sum_{k=<N>} a_k e^{jk(2\pi/N)n} \rightarrow y[n] = \sum_{k=<N>} a_k H(e^{j2\pi k/N}) e^{jk(2\pi/N)n}$$

Remarks cont.

1. Sinusoids are eigenfunctions for any LTI system.
2. $H(j\omega)$ characterizes the "frequency response" of a linear system.

$$H(j\omega) \equiv \int_{-\infty}^{\infty} h(\tau) e^{-j\omega\tau} d\tau$$

$H(j\omega)$ is said to be the Fourier Transform of the time domain function $h(t)$.

Both $h(t)$ and $H(j\omega)$ can be used to find the output for a particular input.

Filter

- Filtering: A process that changes the relative amplitude (or phase) of some frequency components.

★ e.g.

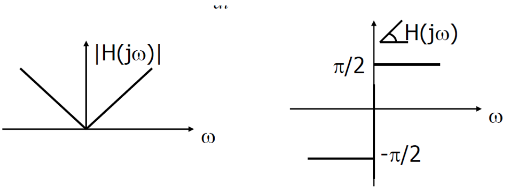
frequency-shaping filter
(like equalizer in a Hi-Fi system)

frequency-selective filter
(like low-pass, band-pass, high-pass filters)

Frequency Shaping Filter

E.g. Differentiator (a HPF or LPF?)

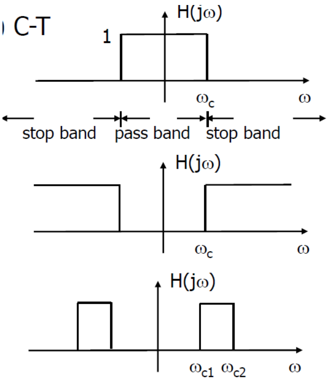
$$y(t) = \frac{dx(t)}{dt} \rightarrow H(j\omega) = j\omega$$



$\rightarrow$high frequency component is amplified & low frequency component is suppressed.$\rightarrow$ HPF (high pass filter)
(a.k.a. as edge-enhancement filter in image processing)

Frequency-Selective Filter

Select some bands of frequencies and reject others.

1. 

Ideal low-pass filter (LPF)

$$H(j\omega) = \begin{cases} 1 \text{ in pass band} \\ 0 \text{ in stop band} \end{cases}$$

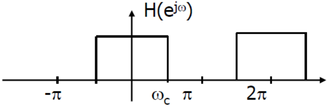
Ideal high-pass filter (HPF)

Ideal band-pass filter (BPF)

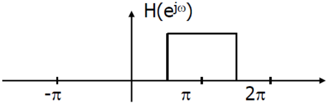
Frequency-Selective Filter cont.

2. D-T

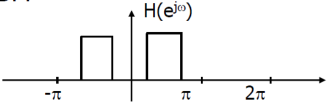
LPF



HPF



BPF



Frequency-Selective Filter cont.

For D-T: $H(e^{j\omega})$ is periodic with period 2π

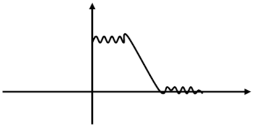
low frequencies: at around $\omega = 0, \pm 2\pi, \pm 4\pi, \ldots$

high frequencies: at around $\omega \pm \pi, \pm 3\pi, \ldots$

Note: ideal filters are not realizable;

Practical filters have transition band, and may have ripple in stopband and passband

e.g. LPF



Summary

- Developed Fourier series representation for both C-T and D-T systems.
- Properties of Fourier Series
- Eigenfunction and Eigenvalue of LTI systems
- System function (transfer function) and frequency response
- Filtering