CS244: Theory of Computation

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Context-free Grammar and Pushdown Automata

Outline

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Definition

A Context-free Grammar CFG is a formal grammar $(\mathcal{N}, \Sigma, \mathcal{P}, S)$ where

1. $\mathcal{N}$ is a finite set of non-terminals,
2. Σ is a finite set of terminals, disjoint from $\mathcal{N}$,
3. $\mathcal{P}$ is a finite set of production rules, of the form
$$A \to \alpha$$
with non-terminal $A \in \mathcal{N}$ and $\alpha \in (\mathcal{N} \cup \Sigma)^*$.
4. $S \in \mathcal{N}$ is the start symbolic.

Extended Backus-Naur form (EBNF) is a family of metasyntax notations, any of which can be used to express a context-free grammar

Example: CFG for arithmetic expressions,

$$\begin{array}{cc} E \to E + E & E \to E * E \\ E \to (E) & E \to id \end{array}$$

Derivations

Let α, β, γ be strings of non-terminals and terminals, and $A \rightarrow \gamma$ is a rule of the grammar. We say that $\alpha A \beta$ yields $\alpha\gamma\beta$, written

$$\alpha A \beta \Rightarrow \alpha\gamma\beta$$

Say that α derives β, written $\alpha \overset{*}{\Rightarrow} \beta$, if $\alpha = \beta$ or if a sequence $\gamma_1, \gamma_2, \ldots, \gamma_k$ exists for $k \geq 0$ and

$$\alpha \Rightarrow \gamma_1 \Rightarrow \gamma_2 \Rightarrow \ldots \Rightarrow \gamma_k \Rightarrow \beta.$$

The language $L(G)$ of the grammar G is

$$\{\alpha \in \Sigma^* \mid S \overset{*}{\Rightarrow} \alpha\},$$

which is a context-free language (CFL).

Example

Recalling the CFG for arithmetic expressions,

$$\begin{array}{cc} E \to E + E & E \to E * E \\ E \to (E) & E \to id \end{array}$$

$$E \Rightarrow E * E \Rightarrow (E) * E \Rightarrow (E + E) * E \Rightarrow (id + id) * id.$$

More Examples

1. The language $\{0^n1^n \mid n \geq 0\}$ has a grammar:

$$S_1 \to 0S_1 1|\varepsilon.$$

2. The language $\{1^n0^n \mid n \geq 0\}$ has a grammar:

$$S_2 \to 1S_2 0|\varepsilon.$$

3. The language

$$\{0^n1^n \mid n \geq 0\} \cup \{1^n0^n \mid n \geq 0\}$$

has a grammar:

$$\begin{aligned} S &\to S_1 \mid S_2 \\ S_1 &\to 0S_1 1|\varepsilon \\ S_2 &\to 1S_2 0|\varepsilon. \end{aligned}$$

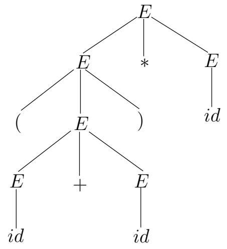
Derivation Trees

A derivation tree for a CFG $G = (\mathcal{N}, \Sigma, \mathcal{P}, S)$ is a labeled tree s.t.

- every vertex has a label from $\mathcal{N} \cup \Sigma \cup \{\varepsilon\}$,
- the root is labeled by S,
- every internal vertex is labeled by symbols from $\mathcal{N}$,
- for every vertex labeled by $A \in \mathcal{N}$,
  if its children are labeled by $X_1, \ldots, X_k$ (from left to right),
  then $A \to X_1 \ldots X_k$ is a rule of G.

Example: Derivation tree for $(id + id) * id$

$$\begin{array}{cc} E \to E + E & E \to E * E \\ E \to (E) & E \to id \end{array}$$



Leftmost Derivation and Rightmost Derivation

Leftmost derivation:

Each step in the derivation is applied to the leftmost nonterminal.

Roughly speaking, leftmost derivation corresponds to

a preorder DFS traversal of the derivation tree.

Rightmost derivation:

Each step in the derivation is applied to the rightmost nonterminal.

Roughly speaking, rightmost derivation corresponds to

a postorder DFS traversal of the derivation tree.

Leftmost Derivation Example

Example: Derivation tree for $(id + id) * id$

$$\begin{array}{cc} E \rightarrow E + E & E \rightarrow E * E \\ E \rightarrow (E) & E \rightarrow id \end{array}$$

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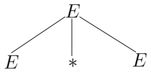

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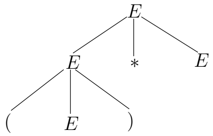

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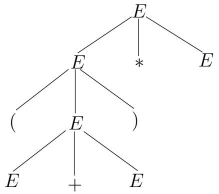

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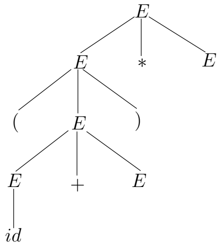

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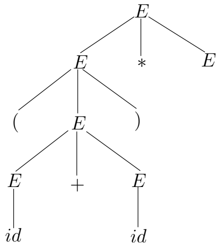

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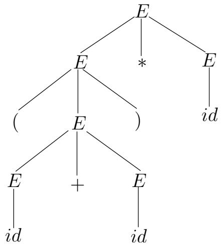


Outline

$$E \to E + E \quad E \to E * E$$
$$E \to (E) \qquad E \to id$$

The string $id + id * id$ have two different derivations:

1. $\langle E \rangle \Rightarrow \langle E \rangle * \langle E \rangle \Rightarrow \langle E \rangle + \langle E \rangle * \langle E \rangle \overset{*}{\Rightarrow} id + id * id$.
2. $\langle E \rangle \Rightarrow \langle E \rangle + \langle E \rangle \Rightarrow id + \langle E \rangle \overset{*}{\Rightarrow} id + id * id$.

Ambiguity

Definition
A string w is derived ambiguously in the context free grammar G if it has two or more different leftmost derivations.

Grammar G is ambiguous if it generates some string ambiguously.

$\{\mathtt{a}\}$ has two different grammars

1. $S \to S' \mid \mathtt{a}; S' \to \mathtt{a}$
2. $S \to \mathtt{a}$

The first is ambiguous, while the second is not.

$$\left\{\mathtt{a}^i\mathtt{b}^j\mathtt{c}^k \;\middle|\; i = j \text{ or } j = k\right\}$$

is inherently ambiguous, i.e., its every grammar is ambiguous.

Outline

Chomsky Normal Form

Definition

A context-free grammar is in Chomsky normal form if every rule is of the form

$$A \to BC$$
$$A \to a$$

where a is any terminal and A, B, and C are any non-terminals – except that B and C may be not the start variable.

In addition, we permit the rule $S \to \varepsilon$, where S is the start variable.

Theorem

For every CFG G, a Chomsky normal form CFG G' can be computed in linear time such that $L(G) = L(G')$.

Proof

1. Add a new start non-terminal S_0 with the rule $S_0 \to S$, where S is the original start variable.
2. Remove every $A \to \varepsilon$, where $A \neq S$. Then for each occurrence of A on the right-hand side of a rule, we add a new rule with that occurrence deleted. For $R \to A$, we add $R \to \varepsilon$ unless we had previously removed $R \to \varepsilon$.
3. Remove every $A \to B$. Then whenever a rule $B \to \alpha$ appears, where α is a string of non-terminal and terminals, we add the rule $A \to \alpha$ unless this was previously removed.
4. Replace each rule $A \to \sigma_1\sigma_2\cdots\sigma_k$ with $k \geq 3$ and each σ_i is a non-terminal or terminal with the rules

   $$A \to \sigma_1 A_1,\ A_1 \to \sigma_2 A_2,\ A_2 \to \sigma_3 A_3,\ \ldots,\ \text{and}\ A_{k-2} \to \sigma_{k-1}\sigma_k.$$

   The A_i's are new non-terminals. We replace any terminal σ_i with the new variable C_i and add $C_i \to \sigma_i$.

Outline

Pushdown automata

Definition

A pushdown automaton (PDA) M is a 6-tuple $(Q, \Sigma, \Gamma, \delta, q_0, F)$, where

1. Q is a finite set of states,
2. Σ is a finite set of input alphabet,
3. Γ is a finite set of stack alphabet,
4. $\delta : Q \times \Sigma_\varepsilon \times \Gamma_\varepsilon \to \mathcal{P}(Q \times \Gamma^*)$ is the transition function,
5. $q_0 \in Q$ is the start state, and
6. $F \subseteq Q$ is the set of accept states.

Configuration

Configuration

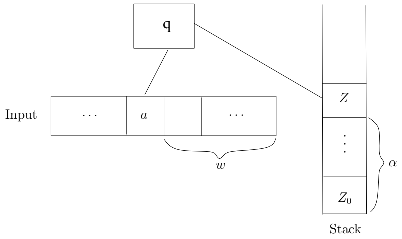
Given a PDA $M = \big(Q, \Sigma, \Gamma, \delta, q_0, F\big)$, a configuration is a tuple (q, w, γ) such that

- $q \in Q$,
- $w \in \Sigma^*$ (the suffix of the input not read yet),
- $\gamma \in \Gamma^*$ (the content of the stack, with the topmost symbol leftmost).

$\vdash_M$ Relation

The $\vdash_M$ relation between configurations is defined as follows:

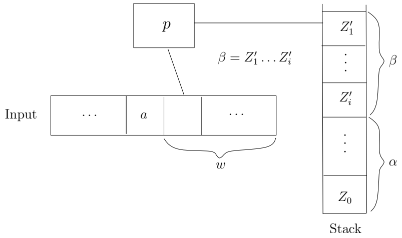
If $(p, \beta) \in \delta(q, a, Z)$ (where $a \in \Sigma_\varepsilon$), then $(q, aw, Z\alpha) \vdash_M (p, w, \beta\alpha)$.



$\vdash_M$ Relation

The $\vdash_M$ relation between configurations is defined as follows:

If $(p, \beta) \in \delta(q, a, Z)$ *(where* $a \in \Sigma_\varepsilon$*), then* $(q, aw, Z\alpha) \vdash_M (p, w, \beta\alpha)$.



Let $\vdash_M^*$ be reflexive and transitive relation of $\vdash_M^*$, namely,

- $(q, w, \gamma) \vdash_M^* (q, w, \gamma)$ for each configuration (q, w, γ);
- $(q, w, \gamma) \vdash_M^* (q'', w'', \gamma'')$ if $(q, w, \gamma) \vdash_M^* (q', w', \gamma')$ and $(q', w', \gamma') \vdash_M (q'', w'', \gamma'')$

Accepted Languages

Let $M = (Q, \Sigma, \Gamma, \delta, q_0, F)$ be a PDA.

- Define $L(M)$, the language accepted by accepting state, as follows,

$$\{w \in \Sigma^* \mid (q_0, w, \epsilon) \vdash_M^* (p, \varepsilon, \gamma) \text{ for } p \in F, \gamma \in \Gamma^*\}\,.$$

PDA for $\{0^n1^n \mid n \geq 0\}$

PDA $M = (Q, \Sigma, \Gamma, \delta, q_1, F)$ accepts $\{0^n1^n \mid n \geq 0\}$ by accepting states

$Q = \{q_1, q_2, q_3, q_4\}$,

$\Sigma = \{0, 1\}$,

$\Gamma = \{0, \$\}$,

$F = \{q_1, q_4\}$, and

δ is given by the following table, wherein blank entries signify $\emptyset$.

Input:	0			1			ε		
Stack:	0	\$	ε	0	\$	ε	0	\$	ε
q_1									$\{(q_2, \$)\}$
q_2			$\{(q_2, 0)\}$	$\{(q_3, \varepsilon)\}$					
q_3				$\{(q_3, \varepsilon)\}$				$\{(q_4, \varepsilon)\}$	
q_4									

Pushdown automata with start stack symbol

Definition

A pushdown automaton (PDA) M is a 7-tuple $\left(Q, \Sigma, \Gamma, \delta, q_0, Z_0, F\right)$, where

1. Q is a finite set of states,
2. Σ is a finite set of input alphabet,
3. Γ is a finite set of stack alphabet,
4. $\delta : Q \times \Sigma_\varepsilon \times \Gamma \to \mathcal{P}(Q \times \Gamma^*)$ is the transition function,
5. $q_0 \in Q$ is the start state,
6. $Z_0 \in \Gamma$ is a start stack symbol
7. $F \subseteq Q$ is the set of accept states.

- Define $L(M)$, the language accepted by accepting state, as follows,

$$\{w \in \Sigma^* \mid (q_0, w, Z_0) \vdash_M^* (p, \varepsilon, \gamma) \text{ for } p \in F, \gamma \in \Gamma^*\}.$$

- Define $N(M)$, the language accepted by empty stack, as follows,

$$\{w \in \Sigma^* \mid (q_0, w, Z_0) \vdash_M^* (p, \varepsilon, \varepsilon) \text{ for } p \in Q\}.$$

Quiz

Define a PDA $M = (\{q_0, q_1\}, \{0, 1\}, \{A, B, Z_0\}, \delta, q_0, Z_0, \emptyset)$ accepts $\{ww^R \mid w \in \{0, 1\}^*\}$ by empty stack.

Eliminating Start Stack Symbol

Let $M = (Q, \Sigma, \Gamma, \delta, q_0, Z_0, F)$ be a PDA with start stack symbol. The goal: Define a PDA without start stack symbol M' such that $L(M') = L(M)$. The intuition:

> *M' starts from a new state q_0', pushes Z_0 onto the stack, and goes to q_0.*

$M' = (Q \cup \{q_0'\}, \Sigma, \Gamma, \delta', q_0', F)$ is defined as follows. δ' includes all the elements of δ and the transition

- $\delta'(q_0', \varepsilon, \varepsilon) = \{(q_0, Z_0)\}$.

Adding Start Stack Symbol

Let $M = (Q, \Sigma, \Gamma, \delta, q_0, F)$ be a PDA without start stack symbol. The goal: Define a PDA with start stack symbol M' such that $L(M') = L(M)$. The intuition:

> *Add a new start symbol Z_0 to M which will not be popped from the stack, and enumerates all the possible topmost stack symbols for $\delta(q, \sigma, \varepsilon)$.*

$M' = (Q \cup \{q_0'\}, \Sigma, \Gamma \cup \{Z_0\}, \delta', q_0', Z_0, F)$ is defined as follows. δ' includes the transitions

- $\delta'(q_0', \varepsilon, Z_0) = \{(q_0, Z_0)\}$
- $\delta'(q, \sigma, Z) = \{(q, \gamma Z) \mid (q, \gamma) \in \delta(q, \sigma, \varepsilon), Z \in \Gamma \cup \{Z_0\}\}$
- $\delta'(q, \sigma, Z) = \delta(q, \sigma, Z)$

From accepting state to empty stack

Let $M = (Q, \Sigma, \Gamma, \delta, q_0, Z_0, F)$ be a PDA.
The goal: Define a PDA M' such that $N(M') = L(M)$.
The intuition:

if M accepts an input by a final state q, then from q, M' enters a new state q_e, empties the stack, and accepts.

In addition, M' includes a new start symbol Z_0' to deal with the following situation,

M reads all the input, enters a non-final state, and the stack is empty.

$M' = (Q \cup \{q_0', q_e\}, \Sigma, \Gamma \cup \{Z_0'\}, \delta', q_0', Z_0', \emptyset)$ is defined as follows.
δ' includes all the elements of δ and the transitions

- $\delta'(q_0', \varepsilon, Z_0') = \{(q_0, Z_0 Z_0')\}$,
- $\delta'(q, \varepsilon, X) = \delta(q, \varepsilon, X) \cup \{(q_e, X)\}$ for $q \in F$ and $X \in \Gamma$,
- $\delta'(q_e, \varepsilon, X) = \{(q_e, \varepsilon)\}$ for every $X \in \Gamma \cup \{Z_0'\}$.

From empty stack to accepting state

Let $M = (Q, \Sigma, \Gamma, \delta, q_0, Z_0, \emptyset)$ be a PDA.
The goal: Define a PDA M' such that $L(M') = N(M)$.
The intuition of M':

Add a new start symbol Z_0' to M, if Z_0' becomes the topmost symbol of the stack, then M' enters a final state.

$M' = (Q \cup \{q_0', q_f\}, \Sigma, \Gamma \cup \{Z_0'\}, \delta', q_0', Z_0', \{q_f\})$ is defined as follows.
δ' includes all the elements of δ and the transitions

- $\delta'(q_0', \varepsilon, Z_0') = \{(q_0, Z_0 Z_0')\}$,
- $(q_f, \varepsilon) \in \delta'(q, \varepsilon, Z_0')$ for every $q \in Q$.

Summary

Theorem

PDA without start stack symbol using accepting state

$$\equiv$$

PDA with start stack symbol using accepting state

$$\equiv$$

PDA with start stack symbol using empty stack

Note: PDA without start stack symbol using empty stack as accepting condition is weak than others, as its language always contains ε.

Outline

From CFG to PDA

The intuition

Use the stack to store the words reached in the leftmost derivation.

1. Place the start symbol of the CFG as the start stack symbol of the PDA
2. Repeat the following steps forever.
   - 2.1 If the top of stack is a non-terminal A, nondeterministically select one of the rules for A and substitute A by the string on the right-hand side of the rule.
   - 2.2 If the top of stack is a terminal symbol a, read the next symbol from the input and compare it to a. If they match, repeat. If they do not match, reject on this branch of the nondeterminism.
   - 2.3 Doing so accepts the input if the stack becomes empty.

From CFG to PDA

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$$E$$

From CFG to PDA

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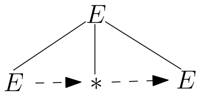

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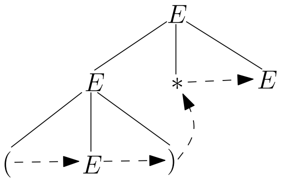

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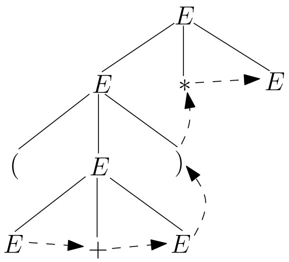

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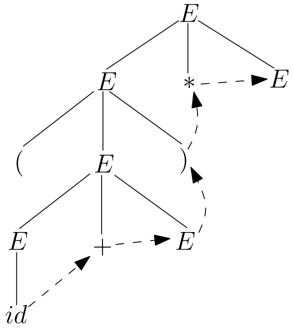

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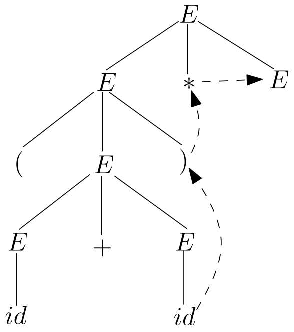

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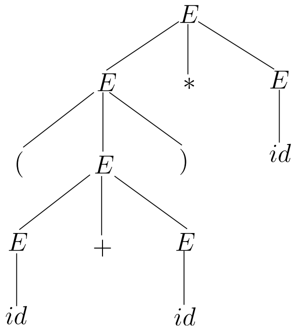

From CFG to PDA: continued

The construction

Let $G = (\mathcal{N}, \Sigma, \mathcal{P}, S)$ be a CFG.
Define a PDA $M = (\{q\}, \Sigma, \Sigma \cup \mathcal{N}, \delta, q, S, \emptyset)$ as follows.

- $\delta(q, \varepsilon, A) = \{(q, \alpha)\}$ for each production rule $A \to \alpha$,
- $\delta(q, a, a) = \{(q, \varepsilon)\}$ for every $a \in \Sigma$.

The correctness of the construction $N(M) = L(G)$

Claim.

$\forall x \in \Sigma^*, \alpha \in (\mathcal{N} \cup \Sigma)^*$,
$S \Rightarrow^* x\alpha$ by leftmost derivation in G iff $(q, x, S) \vdash_M^* (q, \varepsilon, \alpha)$.

Proof of the claim.

Induction on the derivation or computation steps.

From PDA to CFG

An illustration of the intuition:

Analyze the computation tree of PDA M for $\{ww^R \mid w \in \{0,1\}^*\}$ *over the input* 0110.

$\delta(q_0, 0, Z_0) = (q_0, AZ_0)$
$\delta(q_0, 1, A) = (q_0, BA)$
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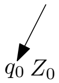

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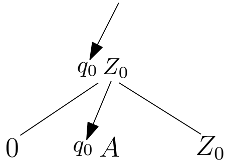

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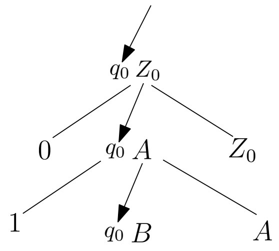

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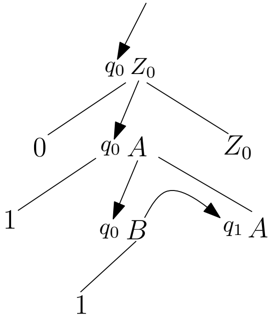

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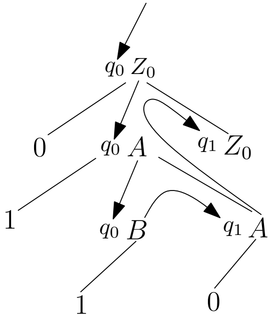

From PDA to CFG

An illustration of the intuition:

Analyze the computation tree of PDA M for $\{ww^R \mid w \in \{0,1\}^*\}$ *over the input* 0110.

$\delta(q_0, 0, Z_0) = (q_0, AZ_0)$
$\delta(q_0, 1, A) = (q_0, BA)$
$\delta(q_0, 1, B) = (q_1, \varepsilon)$
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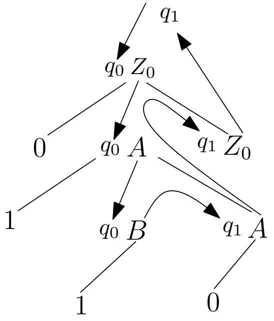

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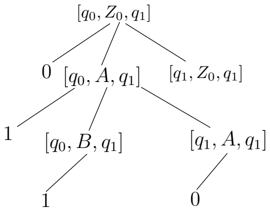


From PDA to CFG: continued

Let $M = (Q, \Sigma, \Gamma, \delta, q_0, Z_0, \emptyset)$ be a PDA.
The goal: Define a CFG $G = (\mathcal{N}, \Sigma, \mathcal{P}, S)$ such that $L(G) = N(M)$.

- $\mathcal{N} = Q \times \Gamma \times Q \cup \{S\}$.
- $\mathcal{P}$ is defined as follows.
  - $S \to [q_0, Z_0, q]$ for every $q \in Q$.
  - If $(p, B_1 \ldots B_k) \in \delta(q, a, X)$ (where $a \in \Sigma \cup \{\varepsilon\}$), then $[q, X, q_{k+1}] \to a[q_1, B_1, q_2] \ldots [q_k, B_k, q_{k+1}]$ such that $q_1 = p$.

The correctness of the construction

Claim. $[q, A, p] \Rightarrow^* x$ iff $(q, x, A) \vdash_M^* (p, \varepsilon, \varepsilon)$.
Proof of the claim.
Induction on the derivation or computation steps.

Outline

Outline

Example

$\left\{a^i b^i c^i \mid i \geq 1\right\}$ is not context free.

How to PROVE it is not context free?

Pumping lemma

Pumping lemma. For every CFL L, there is a natural number $p \geq 1$ (the pumping length) such that for every string $z \in L$ with $|z| \geq p$, z can be decomposed into $uvwxy$ satisfying that

- $|vx| \geq 1$,
- $|vwx| \leq p$,
- for every $i \in \mathbb{N}$, $uv^iwx^iy \in L$.

Let L be a CFL and $G = (\mathcal{N}, \Sigma, \mathcal{P}, S)$ be a CFG s.t. $L = L(G)$.

- Let k be the number of nonterminals in G
- Let b be the maximum number of symbols in the right side of a rule

In any parse tree using this grammar, every node can have no more than b children. So, if the height of the parse tree is at most h ($h = 0$ for root), the length of the string generated is at most b^h. Conversely, if a generated string is at least $b^h + 1$ long, each of its parse trees must be at least $h + 1$ high.

We choose the pumping length

$$p = b^{k+1}.$$

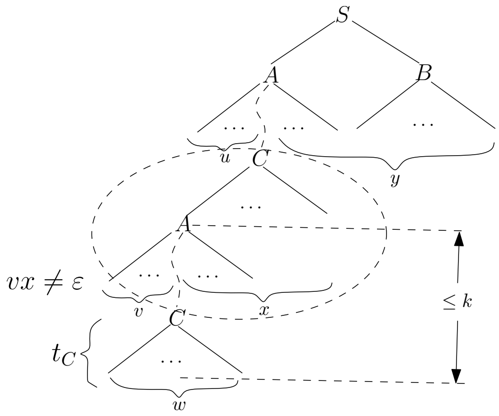
For any string $z \in L$ with $|z| \geq p$, any of its parse trees must be at least $k + 1$ high.

Pumping lemma: continued

The idea of the proof.

Analyzing the structure of the derivation trees of CFGs.

At least $k + 1$ high, then, a path from the root to a leaf of length $k + 1$ has $k + 2$ nodes. One non-terminal must appear at least twice in the last $k + 1$ non-terminal nodes on this path.

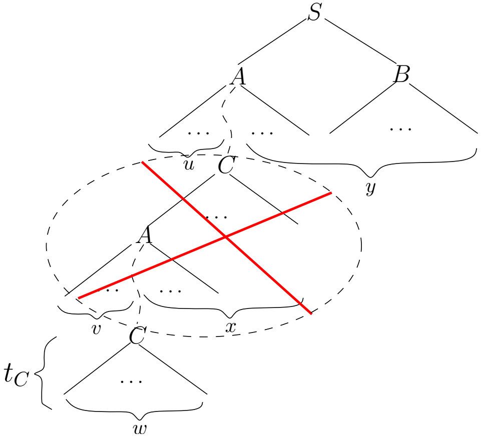


Pumping lemma: continued

The idea of the proof.

Analyzing the structure of the derivation trees of CFGs.

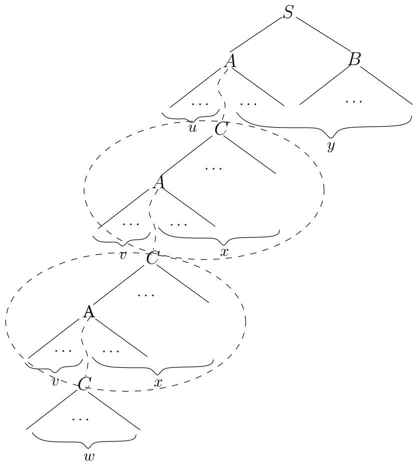
If $vx = \varepsilon$, then consider the reduced derivation tree instead



Pumping lemma: continued

The idea of the proof.

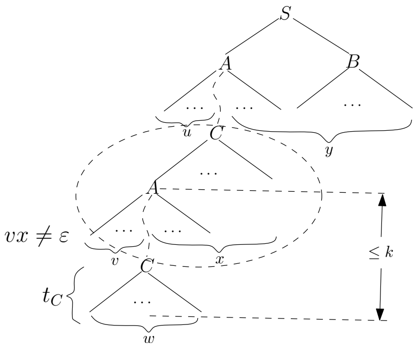
Analyzing the structure of the derivation trees of CFGs.



Pumping lemma: continued

The idea of the proof.

Analyzing the structure of the derivation trees of CFGs.



$|vwx|$: # leaves of a tree of height $\leq k+1$: $\leq b^{k+1}$.

Note: $p := b^{k+1}$.

Applications of pumping lemma

Proposition. $L = \{a^i b^i c^i \mid i \geq 1\}$ is not a CFL.

To the contrary, suppose that L is a CFL.
Then there is a CFG G such that $L = L(G)$.
Let $p \geq 1$ be the pumping length.
Consider $z = a^p b^p c^p$.
Then $z = uvwxy$ s.t. $|vx| \geq 1$ and $|vwx| \leq p$. $\exists$ the following situations

- $vwx = a^i$ or $vwx = b^i$ or $vwx = c^i$ for some $i : 1 \leq i \leq p$.
  Consequences: $uv^2wx^2y \notin L$, a contradiction.
- $vwx = a^i b^j$ or $vwx = b^i c^j$ for some $i, j : 1 \leq i, j, i + j \leq p$.
  Consequences: $uv^2wx^2y \notin L$, a contradiction.

Outline

Closure properties

Theorem. The class of CFLs is

- closed under union,
- not closed under intersection,
- closed under intersection with regular languages,
- not closed under complementation.

Union

Suppose $G_1 = (\mathcal{N}_1, \Sigma, \mathcal{P}_1, S_1)$ and $G_2 = (\mathcal{N}_2, \Sigma, \mathcal{P}_2, S_2)$.

Then G: Add the rules $S \to S_1, S \to S_2$.

It is easy to see that $L(G) = L(G_1) \cup L(G_2)$.

Intersection

$L_1 = \{a^i b^i c^j \mid i, j \geq 1\}$ and $L_2 = \{a^j b^i c^i \mid i, j \geq 1\}$ are both CFL.
On the other hand, $L_1 \cap L_2 = \{a^i b^i c^i \mid i \geq 1\}$ is not a CFL.

Intersection with regular languages

Let

- $M = (Q, \Sigma, \Gamma, \delta, q_0, Z_0, F)$ be a PDA for L_1,
- $\mathcal{A} = (Q', \Sigma, \delta', q_0', F')$ be a NFA for L_2.

Then $L_1 \cap L_2 = L(M'')$ such that

$$M'' = (Q \times Q', \Sigma, \Gamma, \delta'', (q_0, q_0'), Z_0, F \times F'),$$

where

- $\delta''((q, q'), a, X) = \{((p, p'), \gamma) \mid (p, \gamma) \in \delta(q, a, X), \delta'(q', a) = p'\}$ for $a \in \Sigma$,
- $\delta''((q, q'), \varepsilon, X) = \{((p, q'), \gamma) \mid (p, \gamma) \in \delta(q, \varepsilon, X)\}$.

Complementation

If the class of CFLs is closed under complementation, then
from the fact that $L_1 \cap L_2 = \overline{(\overline{L_1} \cup \overline{L_2})}$,
the class of CFLs is closed under intersection, a contradiction.

Outline

Outline

Deterministic pushdown automata

Definition

A deterministic pushdown automaton (DPDA) is a 6-tuple $\big(Q, \Sigma, \Gamma, \delta, q_0, F\big)$, where

1. Q is a finite set of states,
2. Σ is a finite set of input alphabet,
3. Γ is a finite set of stack alphabet,
4. $\delta : Q \times \Sigma_\varepsilon \times \Gamma_\varepsilon \to \big(Q \times \Gamma_\varepsilon\big) \cup \{\emptyset\}$ is the transition function,
5. $q_0 \in Q$ is the start state, and
6. $F \subseteq Q$ is the set of accept states.

For every $q \in Q$, $a \in \Sigma$, and $x \in \Gamma$, exactly one of the values

$$\delta(q, a, x), \delta(q, a, \varepsilon), \delta(q, \varepsilon, x), \text{ and } \delta(q, \varepsilon, \varepsilon)$$

is not $\emptyset$.

Intuition: at most one action at any configuration

Deterministic Context-free Language

1. The language of a DPDA is a deterministic context-free language (DCFL),
2. A language can be represented by a DPDA is not inherently ambiguous

   $$\text{Regular} \subset \text{DCFL} \subset \text{Inherently Unambiguous} \subset \text{CFL}$$

3. $L = \{a^n b^n c^m d \mid m, n \geq 0\} \cup \{a^n b^m c^m e \mid m, n \geq 0\}$ is Inherently Unambiguous but NOT DCFL
4. $\{0^i 1^i \mid i \geq 0\}$ is DCFL
5. $\{a^i b^j c^k \mid i, j, k \geq 0, i = j \vee i = k\}$ is CFL but not DCFL(have to guess whether $i = j$ or $i = k$)
6. $\{ww^R \mid w \in \Gamma^*\}$ is not DCFL (have to guess when w^R starts)

DPDA may reject inputs by failing to read the entire input string, but such DPDA introduces messy cases.

Lemma

Every DPDA has an equivalent DPDA that always reads the entire input string.

Proof (1)

Let $P = \left(Q, \Sigma, \Gamma, \delta, q_0, F\right)$ be a DPDA. P may fail to read the entire input for two reasons:

1. It tries to pop an empty stack – hanging.
2. It makes an endless sequence of ε-input moves – looping.

To solve 1, we initialize the stack with \$. If \$ is popped from the stack before the end of the input, P reads to the end of the input and rejects.

For 2, we identify the looping situations, i.e., those from which no further input symbol is ever read, and reprogramming P so that it reads and rejects the input instead of looping.

Proof (2)

Add new states:

$$q_{\text{start}},\ q_{\text{accept}},\ q_{\text{reject}}.$$

(i) Once P enters an accept state, it remains in accepting states until it reads the next input symbol:

- Add a new accept state q_a for every $q \in Q$.
- For every $q \in Q$, if $\delta(q, \varepsilon, x) = (r, y)$, then set $\delta(q_a, \varepsilon, x) = (r_a, y)$. If $q \in F$, also change $\delta(q, \varepsilon, x) = (r_a, y)$.
- For each $q \in Q$ and $a \in \Sigma$, if $\delta(q, a, x) = (r, y)$, then $\delta(q_a, a, x) = (r, y)$.
- Let F' be the set of new and old accept states.

Proof (3)

(ii) P rejects if it tries to pop an empty stack.

- P initializes the stack with the symbol \$ by $\delta(q_{\mathrm{start}}, \varepsilon, \varepsilon) = (q_0, \$)$.
- If P detects \$ while in a non-accepting state, it enters q_{rejects} and scans the input to the end.
- If P detects \$ while in an accept state, it enters q_{accept}. Then if any input remains unread, it enters q_{reject} and scans the input to the end.
- More precisely, for $x \in \Gamma$ and $\delta(q, a, x) \neq \emptyset$,
  - if $a \neq \varepsilon$ or $q \notin F'$, then set $\delta(q, a, \$) = (q_{\mathrm{reject}}, \varepsilon)$.
  - if $q \in F'$ and $a = \varepsilon$, then set $\delta(q, \varepsilon, \$) = (q_{\mathrm{accept}}, \varepsilon)$.
  - for $a \in \Sigma$, set $\delta(q_{\mathrm{reject}}, a, \varepsilon) = (q_{\mathrm{reject}}, \varepsilon)$ and $\delta(q_{\mathrm{accept}}, a, \varepsilon) = (q_{\mathrm{reject}}, \varepsilon)$.

Proof (4)

(iii) Modify P to reject instead of making an endless sequence of ε-input moves.

- For every $q \in Q$ and $x \in \Gamma$, call (q, x) a looping situation if, when P is started in state q with $x \in \Gamma$ on the top of the stack, if it never pops anything below x and it never reads an input symbol.
- A loop situation is accepting, if P enters an accept state during its subsequent moves, and otherwise rejecting.
- If (q, x) is an accepting looping situation, set $\delta(q, \varepsilon, x) = (q_{\mathrm{accept}}, \varepsilon)$.
- If (q, x) is a rejecting looping situation, set $\delta(q, \varepsilon, x) = (q_{\mathrm{reject}}, \varepsilon)$.

Note that: for $a \in \Sigma$, set $\delta(q_{\mathrm{reject}}, a, \varepsilon) = (q_{\mathrm{reject}}, \varepsilon)$ and $\delta(q_{\mathrm{accept}}, a, \varepsilon) = (q_{\mathrm{reject}}, \varepsilon)$.

Outline

Theorem

The class of DCFLs is not closed under union and intersection.

- $L_1 = \{a^i b^j c^k \mid i, j, k \geq 0, i = j\}$ is DCFL
- $L_2 = \{a^i b^j c^k \mid i, j, k \geq 0, i = k\}$ is DCFL
- $L_1 \cup L_2 = \{a^i b^j c^k \mid i, j, k \geq 0, i = j \vee i = k\}$ is CFL but not DCFL (have to guess whether $i = j$ or $i = k$)
- $L_1 \cap L_2 = \{a^i b^j c^k \mid i, j, k \geq 0, i = j = k\}$ is not CFL hence not DCFL

Theorem

The class of DCFLs is closed under complementation. That is, if A is a DCFL, then

$$\Sigma^* - A = \left\{ s \in \Sigma^* \;\middle|\; s \notin A \right\}$$

is also a DCFL.

Proof (1)

We cannot simply swap the accept and non-accept states of a DPDA. The DPDA may accept its input by entering both accept and non-accept states in a sequence of moves at the end of the input string.

For example, $(q_0, w, \varepsilon) \rightarrow^* (q, \varepsilon, \gamma) \rightarrow^+ (q', \varepsilon, \gamma')$, $q \notin F$, but $q' \in F$

Assume $P = \big(Q, \Sigma, \Gamma, \delta, q_0, F\big)$ is a DPDA for A which always reads the whole input string. Moreover, once P enters an accept state, it remains in accept states until it reads the next input symbol.

Proof (2)

We designate some states as reading states which never pop symbols from the stack

- If P in a state q reads an $a \in \Sigma$ without popping the stack, i.e., $\delta(q, a, \varepsilon) \neq \emptyset$, then q is a reading state.
- If $\delta(q, a, x) = (r, y)$, then add a new state q_x and modify δ as

  $$\delta(q, \varepsilon, x) = (q_x, \varepsilon) \quad \text{and} \quad \delta(q_x, a, \varepsilon) = (r, y).$$

  Let q_x be a reading state, and it is an accept state if $q \in F$.

Remove the accepting state designation from any state which isn't a reading state, namely, only reading states can be accepting states

Proof (3)

Now, invert which reading states are classified as accepting. The resulting DPDA recognizes the complementary language.

Corollary

Any CFL whose complement is not a CFL is not a DCFL.

Intuition: Consider the CFL L such that $\overline{L}$ is not a CFL. Assume L is DCFL, then, $\overline{L}$ is a DCFL, hence CFL, contradiction.

Example

$\left\{a^i b^j c^k \;\middle|\; i \neq j \text{ or } j \neq k\right\}$ is a CFL but not a DCFL.

Endmarked languages

Definition

For any language A the endmarked language $A\dashv$ is defined by

$$\{w\dashv \mid w \in A\}.$$

Here $\dashv$ is the special endmarker symbol denoting the end of input.

Theorem

A is a DCFL if and only if $A\dashv$ is a DCFL.

Proof (1)

Let P be an DPDA recognizing A. Then DPDA P' recognizes $A\dashv$ by:

1. Simulating P until P' reads $\dashv$.
2. P' accepts if P had entered an accept state during the previous symbol.
3. P' does not read any symbols after $\dashv$.

Assume that only reading states in P can be accepting states and once enters an accepting state, it remains in accepting states until it reads the next input symbol.
For every $q \in F$, $\delta'(q, \dashv, x) = (q_f, \varepsilon)$ for all $x \in \Gamma_\varepsilon$ and q_f is the unique accepting state.
Why do we need the above assumption? To be deterministic, reading states q gives $\delta(q, a, x) \neq \emptyset$, then $\delta(q, \varepsilon, x) = \emptyset$

Proof (2)

Let DPDA $P = (Q, \Sigma \cup \{\dashv\}, \Gamma, \delta, q_0, F)$ recognize $A\dashv$. Construct a DPDA P' recognizing A.

Intuition:

- P' reads input and simulates P
- Before reading the next input symbol, P' determines whether P would accept if that input symbol is $\dashv$
- If yes, then P' enters an accepting state
- Note: P may eventually enters an accepting state after reading $\dashv$ by manipulating the stack
- How to predicate future states are accepting or not?
- We will store additional information (R) on the stack of P' which allows P' to determine immediately whether P would accept
- R contains states of P from which P eventually enters an accepting state but without reading further input

Proof (3)

Let DPDA $P = (Q, \Sigma \cup \{\dashv\}, \Gamma, \delta, q_0, F)$ recognize $A\dashv$.

Modify P so that each of its moves does exactly one of the following operations:

- read an input symbol, i.e., $\delta(q, a, \varepsilon) = (q', \varepsilon)$;
- push a symbol onto the stack, i.e., $\delta(q, \varepsilon, \varepsilon) = (q', \gamma)$;
- or pop a symbol from the stack, i.e., $\delta(q, \varepsilon, x) = (q', \varepsilon)$.

We construct a P' to simulates P. Every time P' pushes a stack symbol of P', then it pushes a symbol representing a subset of P's states, i.e., $\Gamma' = \Gamma \cup \mathcal{P}(Q)$. Then, the stack content of P' would be $R_{k+1}\gamma_k \cdots R_2\gamma_1 R_1\gamma_0 R_0$, by starting P in any state of R_{k+1} with stack content $\gamma_k \cdots \gamma_1\gamma_0$ will eventually accept without reading any more input.

Initially, P' pushes the set R_0 defined by

$$R_0 = \left\{ q \in Q \;\middle|\; (q, \varepsilon, \varepsilon) \vdash_M^* (q', \varepsilon, u) \text{ for some } q' \in F, u \in \Gamma^* \right\}$$

when P is started in q with an empty stack, it eventually accepts without reading any input symbols

Proof (4)

Then P' begins simulating P.

- To simulate a pop move, P' first pops and discards the set of states on the top of the stack, then it pops again to obtain the symbol in order to simulate P, i.e., if $\delta(q, \varepsilon, x) = (q', \varepsilon)$, then $\delta'(q, \varepsilon, R) = (q_x, \varepsilon)$ and $\delta'(q_x, \varepsilon, x) = (q', \varepsilon)$
- To simulate a push move $\delta(q, \varepsilon, \varepsilon) = (q', x)$, P' examines the set of states R on the top of its stack, and then it pushes x and the set of states

$$\left\{q'' \;\middle|\; q'' \in F \text{ or } (\delta(q'', \varepsilon, x) = (r, \varepsilon) \text{ and } r \in R)\right\}.$$

- P' simulates a read move $\delta(q, a, \varepsilon) = (q', \varepsilon)$, by examining the set R on the top of the stack and entering an accept state if $q' \in R$.
  - If P' is at the end of the input string, i.e., $a = \dashv$, then it accepts.
  - Otherwise i.e., $a \neq \dashv$,, it will continue simulating P, so this accept state must also record P's state.

Outline

Our goal is to define deterministic context-free grammars (DCFG), the counterpart to deterministic pushdown automata.

We will show that these two models are equivalent on endmarked languages.

From derivations to reductions

Derivations in CFGs begin with the start non-terminal and proceed top down with a series of substitutions according to the grammar's rules, until the derivation obtains a string of terminals.

For defining DCFGs we take a bottom up approach, by starting with a string of terminals and processing the derivation in reverse, employing a series of reduce steps until reaching the start variable.

- Each reduce step is a reversed substitution, whereby the string of terminals and non-terminals on the right-hand side of a rule is replaced by the non-terminal on the corresponding left-hand side.
- The string replaced is the reducing string.
- The entire reversed derivation a reduction.

Formal definition of reductions

If u and v are strings of non-terminals and terminals, write $u \rightarrowtail v$ to mean that v can be obtained from u by a reduce step, i.e., $v \Rightarrow u$.

A reduction from u to v is a sequence

$$u = u_1 \rightarrowtail u_2 \rightarrowtail \ldots \rightarrowtail u_k = v$$

and we say that u is reducible to v, written $u \overset{*}{\rightarrowtail} v$, equivalent $v \overset{*}{\Rightarrow} u$.

A reduction from u is a reduction from u to the start non-terminal.

In a leftmost reduction, each reducing string is reduced only after all other reducing strings that lie entirely to its left.

This is the reverse of rightmost derivation

Let $w \in L(G)$, and let u_i appear in a leftmost reduction of w. In the reduce step $u_i \rightarrowtail u_{i+1}$, say that the rule $T \to h$ was applied in reverse.

Therefore,

$$u_i = xhy \quad \text{and} \quad u_{i+1} = xTy,$$

where h is the reducing string.

$(h, T \to h)$ is a handle of u_i. A string that appears in a leftmost reduction of some string in $L(G)$ is called a valid string. We define handles only for valid strings.

Valid string may have several handles, but only if the grammar is ambiguous.

Unambiguous grammars generate strings by one parse tree only, and therefore the leftmost reductions, and hence the handles, are also unique. In that case, we may refer to the handle of a valid string.

In

$$u_i = xhy \quad \text{and} \quad u_{i+1} = xTy,$$

y is always a string of terminals.

Example (1)

Consider the grammar G:

$$\begin{aligned} R &\to S \mid T \\ S &\to aSb \mid ab \\ T &\to aTbb \mid abb \end{aligned}$$

Then

$$\begin{aligned} L(G) &= B \cup C \\ \text{where } B &= \{a^m b^m \mid m \geq 1\} \\ \text{and } C &= \{a^m b^{2m} \mid m \geq 1\}. \end{aligned}$$

Some leftmost reductions:

$$aa\underline{ab}bb \rightarrowtail a\underline{aSb}b \rightarrowtail \underline{aSb} \rightarrowtail \underline{S} \rightarrowtail R,$$

$$aa\underline{abb}bbbb \rightarrowtail a\underline{aTbb}bb \rightarrowtail \underline{aTbb} \rightarrowtail \underline{T} \rightarrowtail R.$$

Example (2)

Consider the grammar G:

$$\begin{aligned} S &\to T\dashv \\ T &\to T(T) \mid \varepsilon \end{aligned}$$

A leftmost reduction:

$$\_()()\dashv \rightarrowtail T(\_)()\dashv \rightarrowtail \underline{T(T)}()\dashv \rightarrowtail T(\_)\dashv \rightarrowtail \underline{T(T)}\dashv \rightarrowtail \underline{T\dashv} \rightarrowtail S.$$

Definition

A handle h of a valid string $v = xhy$ is a forced handle if h is the unique handle in every valid string $xh\hat{y}$ where $\hat{y} \in \Sigma^*$.

Definition

A deterministic context-free grammar (DCFG) is a context-free grammar such that every valid string has a forced handle.

The above definition does not tell us how to decide whether a given CFG is a DCFG.

DK-test (Donald Knuth)

A CFG G is DCFG iff G passes DK-test.

For any CFG we construct an associated DFA DK that can identify handles. DK accepts its input z if

1. z is the prefix of some valid string $v = zy$, and
2. z ends with a handle of v.

Moreover, each accept state of DK indicates the associated reducing rule(s). In a general CFG, multiple reducing rules may apply, depending on which valid v extends z. But in a DCFG, each accept state corresponds to exactly one reducing rule.

Two provisos

1. The start non-terminal of a CFG does not appear on the right-hand side of any rule.
2. Every non-terminal appears in a reduction of some string in the grammar's language, i.e., no useless production rules.

The plan

To construct DFA DK, we will construct an equivalent NFA K and convert K to DK via the subset construction.

To understand K, we introduce an NFA J which

> *accepts every input string that ends with the right-hand side of any rule.*

The NFA J

1. guesses a rule $B \to u$ to use,
2. guesses at which point to start matching the input with u,
3. keeps track of its progress through u.

We represent this progress by placing a dot in the corresponding point in the rule, yielding the following dotted rules:

$$\begin{aligned}
&B \to .u_1 u_2 \cdots u_{k-1} u_k \\
&B \to u_1 . u_2 \cdots u_{k-1} u_k \\
&\quad \vdots \\
&B \to u_1 u_2 \cdots u_{k-1} . u_k \\
&B \to u_1 u_2 \cdots u_{k-1} u_k .
\end{aligned}$$

Dotted rules

1. Each dotted rule corresponds to one state of J, i.e., for $B \to u.v$ we have a state $\boxed{B \to u.v}$.

2. The accept states $\boxed{\boxed{B \to u.}}$ correspond to the complete rules.

3. We add a separate start state with a self-loop on all symbols and an ε-move to $\boxed{B \to .u}$ for each rule $B \to u$.

Thus, J accepts if the match completes successfully at the end of the input. If a mismatch occurs or if the end of the match does not coincide with the end of the input, this branch of J's computation rejects.

The NFA K

- Like J, the states of K correspond to all dotted rules.
- It has a special start state that has an ε-move to $\boxed{S_1 \to .u}$ for every rule with S_1 being the start variable.
- Shift-moves: for a rule $B \to uav$ where a can be a terminal or non-terminal

$$\boxed{B \to u.av} \xrightarrow{a} \boxed{B \to ua.v}$$

- ε-moves: for a rule $B \to uCv$ and $C \to r$

$$\boxed{B \to u.Cv} \xrightarrow{\varepsilon} \boxed{C \to .r}$$

- The accept states are all $\boxed{\boxed{B \to u.}}$.

Lemma

K may enter state $\boxed{T \to u.v}$ on reading input z if and only if $z = xu$ and $xuvy$ is a valid string with handle uv and reducing rule $T \to uv$, for some $y \in \Sigma^*$.

Proof (1)

First step is to compute xu

Assume K enters state $\boxed{T \to u.v}$ on reading input z, whose path from the start state is viewed as runs of shift-moves separated by ε-moves.

- The shift-moves are transitions between states sharing the same rule, shifting the dot rightward over symbols read from the input.
- In the ith run (ε-move), the rule is $S_i \to u_i S_{i+1} v_i$, where S_{i+1} is the non-terminal expanded in the next run.
- The penultimate run is for rule $S_\ell \to u_\ell T v_\ell$, and the final run has rule $T \to uv$.

$$S_1 \xrightarrow{\varepsilon} .u_1 S_2 v_1 \xrightarrow{u_1}{}^{*} u_1.S_2 v_1 \xrightarrow{\varepsilon} .u_2 S_3 v_2 \xrightarrow{u_2}{}^{*} u_2.S_3 v_2 \to^{*} \cdots$$
$$\xrightarrow{u_{\ell-1}}{}^{*} u_{\ell-1}.S_\ell v_{\ell-1} \xrightarrow{\varepsilon} .u_\ell T v_\ell \xrightarrow{u_\ell}{}^{*} u_\ell.T v_\ell \xrightarrow{\varepsilon} .uv \xrightarrow{u}{}^{*} u.v$$

- So the input $z = u_1 u_2 \ldots u_\ell u = xu$, because the strings u_i and u were the shift-move symbols read from the input.

Proof (2)

Second step is to compute a valid string $xuvy$.

$$S_1 \xrightarrow{\varepsilon} .u_1 S_2 v_1 \overset{u_1}{\rightarrow^*} u_1.S_2 v_1 \xrightarrow{\varepsilon} .u_2 S_3 v_2 \overset{u_2}{\rightarrow^*} u_2.S_3 v_2 \rightarrow^* \cdots$$
$$\overset{u_{\ell-1}}{\rightarrow^*} u_{\ell-1}.S_\ell v_{\ell-1} \xrightarrow{\varepsilon} .u_\ell T v_\ell \overset{u_\ell}{\rightarrow^*} u_\ell.T v_\ell \xrightarrow{\varepsilon} .uv \overset{u}{\rightarrow^*} u.v$$

Let $y' = v_\ell \ldots v_2 v_1$, then $xuvy'$ $(= u_1 u_2 \ldots u_\ell uvv_\ell \ldots v_2 v_1)$ is derivable in G.

- Fully expand all non-terminals that appear in y' until each non-terminal derives some string of terminals, and let y be the resulting string.
- The string $xuvy$ is valid because it occurs in a leftmost reduction of $w \in L(G)$ of terminals obtained by expanding all non-terminals in $xuvy'$.
- uv is the handle in the reduction and its reducing rule is $T \rightarrow uv$.

This proves the direction from left to right.

Proof (3)

Assume that string $xuvy$ is a valid string with handle uv and reducing rule $T \to uv$. We need to show that K may enter $\boxed{T \to u.v}$ on reading input xu.

1. The parse tree is rooted at the start variable S_1 and it must contain the variable T, as $T \to uv$ is the first reduce step in the reduction of $xuvy$.
2. Let $S_2, \ldots, S_\ell$ be the non-terminals on the path from S_1 to T. All non-terminals in the parse tree that appear leftward of this path must be unexpanded, or else uv would not be the handle.
3. Each S_i leads to S_{i+1} by some rule $S_i \to u_i S_{i+1} v_i$:

$$
\begin{gathered}
S_1 \to u_1 S_2 u_k \\
S_2 \to u_2 S_3 u_2 \\
\vdots \\
S_\ell \to u_\ell T v_\ell \\
T \to uv.
\end{gathered}
$$

Proof (4)

On input $z = xu$, the path from K's start state to $\boxed{T \to u.v}$ is:

1. K makes an ε-move to $\boxed{S_1 \to .u_1 S_2 v_1}$.
2. Reading the symbols of u_1, it performs the corresponding shift-moves until it enters $\boxed{S_1 \to u_1 . S_2 v_1}$ at the end of u_1.
3. It makes an ε-move to $\boxed{S_2 \to .u_2 S_3 v_2}$ and continues with shift-moves on reading u_2 until it reaches $\boxed{S_2 \to u_2 . S_3 v_2}$. ......
4. After reading u_ℓ it enters $\boxed{S_\ell \to u_\ell . T v_\ell}$ which leads by an ε-move to $\boxed{T \to .uv}$.
5. Finally after reading u it is in $\boxed{T \to u.v}$.

Lemma

K may enter state $\boxed{T \to u.v}$ on reading input z if and only if $z = xu$ and $xuvy$ is a valid string with handle uv and reducing rule $T \to uv$, for some $y \in \Sigma^*$.

Corollary

K may enter state $\boxed{T \to h.}$ on reading input z if and only if $z = xh$ and h is a handle of some valid string xhy with reducing rule $T \to h$.

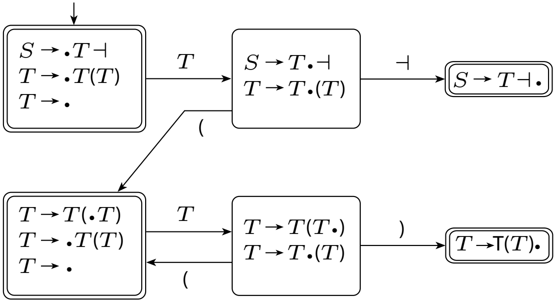
From K to DK

We convert NFA K to DFA DK by the subset construction. All states unreachable from the start state are removed.

Thus, each of DK's states thus contains one of more dotted rules. Each accept state contains at least one completed rule.

An example (LR(0) automaton in CS131)

$$S \to T\dashv$$
$$T \to T(T) \mid \varepsilon$$



The DK-test

Starting with a CFG G, construct the associated DFA DK. Determine whether G is deterministic by examining DK's accept states. G passes the DK-test if every accept state contains

1. exactly one completed rule, and
2. no dotted rule in which a terminal symbol immediately follows the dot, i.e., no dotted rule of the form $B \to u.av$ for $a \in \Sigma$.

Theorem

G passes the DK-test if and only if G is a DCFG.

Proof ($\Rightarrow$)

Assume that G is not deterministic and show that it fails the DK-test.

- Take a valid string xhy that has an unforced handle h.
- Some valid string xhy' has a different handle $\hat{h} \neq h$, where y' is a string of terminals, i.e., $y' \in \Sigma^*$. Thus $xhy' = \hat{x}\hat{h}\hat{y}$.
- If $xh = \hat{x}\hat{h}$, then input xh sends DK to a state with two completed rules ($T \rightarrow h.$ and $\hat{T} \rightarrow \hat{h}.$) , failing the DK-test.
- If $xh \neq \hat{x}\hat{h}$, by symmetry, we assume that xh is a proper prefix of $\hat{x}\hat{h}$. Then $y' \in \Sigma^+$.
  - Let q be the state that DK enters on input xh, which must be accepting as h is a handle of xhy.
  - A transition arrow must exit q as $\hat{x}\hat{h}$ sends DK to an accept state via q. That transition arrow is labeled with a terminal symbol since $y' \in \Sigma^+$.

  Hence q contains a dotted rule with a terminal symbol immediately following the dot, violating the DK-test.

Proof ($\Leftarrow$)

Assume G fails the DK-test at some accept state q. We show that G has an unforced handle.

q has a complete rule $T \to h.$, for it is accepting. Let z be a string that leads DK to q. Then $z = xh$ where some valid string xhy has handle h with reducing rule $T \to h$, for $y \in \Sigma^*$.

- If q has another completed rule $B \to \hat{h}.$. Then some valid string xhy' must have a different handle $\hat{h}$ with reducing rule $B \to \hat{h}$. Hence, h is not a forced handle.
- q contains a rule $B \to u.av$ with $a \in \Sigma$.

Proof ($\Leftarrow$)

q has a complete rule $T \to h.$, for it is accepting. Let z be a string that leads DK to q. Then $z = xh$ where some valid string xhy has handle h with reducing rule $T \to h$, for $y \in \Sigma^*$.

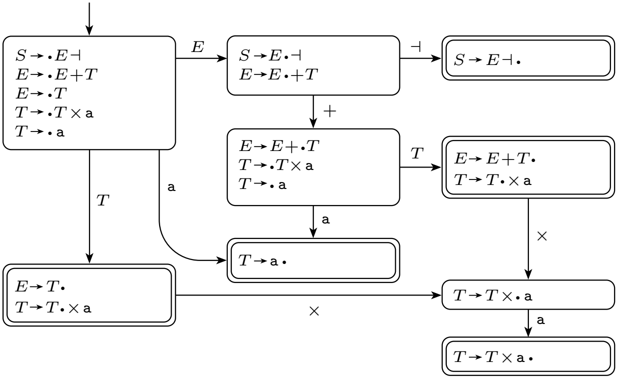
- q contains a rule $B \to u.av$ with $a \in \Sigma$.
  - xh takes DK to q, so $xh = \hat{x}u$, where $\hat{x}uav\hat{y}$ is a valid string and has a handle uav with reducing rule $B \to uav$, for some $\hat{y} \in \Sigma^*$.
  - Expand all non-terminals in v to get $v' \in \Sigma^*$, and let $y' = av'\hat{y}$ with $y' \in \Sigma^+$.
  - The following is a leftmost reduction

$$xhy' = xhav'\hat{y} = \hat{x}uav'\hat{y} \overset{*}{\rightarrowtail} \hat{x}uav\hat{y} \rightarrowtail \hat{x}B\hat{y} \overset{*}{\rightarrowtail} S.$$

  - $\hat{x}uav\hat{y}$ is valid, and we can obtain $\hat{x}uav'\hat{y}$ from it using a rightmost derivation, so $\hat{x}uav'\hat{y}$ is also valid.
  - The handle of $\hat{x}uav'\hat{y}$ either lies inside v' (if $v \neq v'$) or is uav (if $v = v'$). In either case, the handle includes or follows a and thus cannot be h because h fully precedes a. Hence h is a unforced handle.

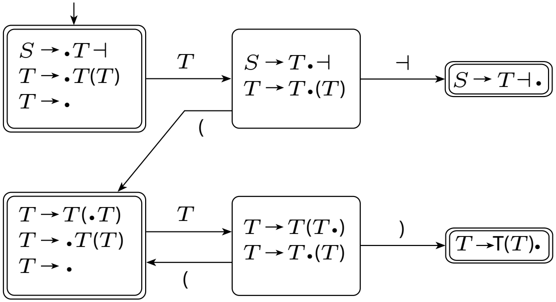
An example failing the *DK*-test

$$
\begin{aligned}
S &\to E\dashv \\
E &\to E + T \mid T \\
T &\to T \times \mathtt{a} \mid \mathtt{a}
\end{aligned}
$$



An example passing the *DK*-test

$$S \to T\dashv$$
$$T \to T(T) \mid \varepsilon$$



Outline

Relationship of DPDAs and DCFGs

- For every DCFG G of any context-free language, we can construct a DPDA P such that $L(P) = L(G)$
- For every DPDA P of any endmarked language, we can construct a DCFG G such that $L(P) = L(G)$
- Endmarkers do not affect the class of languages that DPDAs recognize, but they do affect the class of languages of DCFG
- Without endmarkers, DCFG only generated a strict subclass (prefix-free) of DCFL

Theorem

An endmarked language is generated by a deterministic context-free grammar if and only if it is deterministic context free.

Recall Endmarked languages

Definition
For any language A the endmarked language $A\dashv$ is defined by

$$\left\{ w\dashv \;\middle|\; w \in A \right\}.$$

Here $\dashv$ is the special endmarker symbol.

Theorem
A is a DCFL if and only if $A\dashv$ is a DCFL.

Prefix-free languages

Definition

$A \subseteq \Sigma^*$ is prefix-free if for every $w \in A$ and every proper prefix w' of w we have $w' \notin A$.

Lemma

Every $A\dashv$ is prefix-free.

Lemma

Every DCFG generates a prefix-free language.

Prefix-free languages (cont'd)

The grammar

$$\begin{aligned} S &\to T\dashv \\ T &\to T(T) \mid \varepsilon \end{aligned}$$

is DCFG, which recognizes some $A\dashv$

A cannot be generated by DCFG, since it is not prefix-free.

Lemma

Every DCFG has an equivalent DPDA.

Intuition:

- Construct the DFA DK for a DCFG G
- The DPDA P simulates the run of DK on an input string
  - DK reads an input symbol a and moves from state q_1 to q_2 iff P reads a when the top of the stack is q_1 and pushes a and q_2 into the stack
  - When DK reaches an accepting state q, it necessarily contains a unique dotted rule $T \to u.$ and the projection of top-$2|u|$ symbols in the stack to the terminals and non-terminals is u,
  - pops the top-$2|u|$ symbols from the stack,
  - suppose now q_1 is the state in the top of the stack, DK moves from q_1 to q_2 after reading T, then pushes T and q_2 into the stack
  - continues the above steps until the start non-terminal S is pushed onto the stack, then P enters the accepting state

Lemma

Every DPDA that recognizes an endmarked language has an equivalent DCFG.

Proof (1)

Let $P = \big(Q, \Sigma, \Gamma, \delta, q_0, \{q_{\mathrm{accept}}\}\big)$ be a DPDA. For each pair of states p and q, let A_{pq} be a non-terminal which generates

all strings taking P from p with an empty stack to q with an empty stack.

We modify P such that:

1. It has a single accept state q_{accept}.
2. It empties its stack before accepting, which cannot be deterministic without reading the endmark $\dashv$.
3. Each transition either pushes a symbol onto the stack or pops one off the stack, but it does not do both at the same time.

Proof (2)

1. For every $p, q, r, s, \in Q$, $u \in \Gamma$, and $a, b \in \Sigma_\varepsilon$, if $\delta(p, a, \varepsilon) = (r, u)$ and $\delta(s, b, u) = (q, \varepsilon)$, then add $A_{pq} \to aA_{rs}b$.
2. For every $p, q, r \in Q$, add $A_{pq} \to A_{pr}A_{rq}$.
3. For every $p \in Q$, add $A_{pp} \to \varepsilon$.

To avoid ambiguity, we combine rules of types 1 and 2 into:

1-2. For every $p, q, r, s, t \in Q$, $u \in \Gamma$, and $a, b \in \Sigma_\varepsilon$, if $\delta(r, a, \varepsilon) = (s, u)$ and $\delta(t, b, u) = (q, \varepsilon)$, add the rule $A_{pq} \to A_{pr}aA_{st}b$.

Let G be the resulting grammar.

Proof (3)

In a derivation of the original grammar, for each substitution due to a type 2 rule $A_{pq} \to A_{pr} A_{rq}$, we can assume that

r is P's state when it is at the rightmost point where the stack becomes empty midway.

Then the subsequent substitution of A_{rq} must expand it using a type 1 rule $A_{rq} \to a A_{st} b$.

We can combine these two substitutions into a single type 1-2 rule

$$A_{pq} \to A_{pr} a A_{st} b.$$

In a derivation using the modified grammar G, if we replace each type 1-2 rule $A_{pq} \to A_{pr} a A_{st} b$ by the type 2 rule $A_{pq} \to A_{pr} A_{rq}$ followed by the type 1 rule $A_{rq} \to a A_{st} b$, we get the same result.

Proof (4)

We use DK-test to show that G is deterministic.

- We need to analyze how P operates on valid strings by extending its input alphabet and transition function to process non-terminals in addition to terminal symbols.
- Add all symbols A_{pq} to P's input alphabet and extend its δ by defining
$$\delta(p, A_{pq}, \varepsilon) = (q, \varepsilon).$$
Set all other transitions involving A_{pq} to $\emptyset$.
- To preserve P' deterministic behavior, if P reads A_{pq} from the input then disallow an ε-move.

Proof (5)

Consider the derivation:

$$A_{q_0,q_{\mathrm{accept}}} = v_0 \Rightarrow v_1 \Rightarrow \cdots \Rightarrow v_i \Rightarrow \cdots \Rightarrow v_k = w.$$

Claim

If P reads v_i containing a non-terminal A_{pq}, it enters state p just prior to reading A_{pq}.

If $i = 0$, then $v_i = A_{q_0,q_{\mathrm{accept}}}$ and P starts in q_0.

Assume the claim is true for some $i \geq 1$.

- $v_i = xA_{pq}y$ and A_{pq} is the non-terminal substituted in the step $v_i \Rightarrow v_{i+1}$. By IH, P enters state p after reading x, prior to reading A_{pq}. By the construction of G, the substitution rules may of two types:
  1. $A_{pq} \to A_{pr} a A_{st} b$ or
  2. $A_{pp} \to \varepsilon$.

Proof (6)

- Thus either $v_{i+1} = xA_{pr}aA_{st}by$ or $v_{i+1} = xy$.
- In the first case, when P reads $A_{pr}aA_{st}b$ in v_{i+1}, we know it starts in state p, because it has just finished reading x.

  As P reads $A_{pr}aA_{st}b$ in v_{i+1}, it enters the sequence of states r, s, t, and q, due to the substitution rule's construction.

  - Therefore, it enters state p just prior to reading A_{pr} and it enters state s just prior to reading A_{st}.
  - The claim holds on non-terminals in the y part because, after reading b, P enters state q and then it reads y ($A_{pq} \to A_{pr}aA_{st}b$). On input v_i, it also enters q just before reading y, so the computations agree on the y parts of v_i and v_{i+1}.

  Obviously, the computations agree on the x parts.
- In the second case, no new non-terminals are introduced, so we only need to observe that the computations agree on the x and y parts of v_i and v_{i+1}.

Proof (7)

Claim

G passes the DK-test.

Select one of the accept states of DK for G. It contains a completed rule R which can be

1. $A_{pq} \rightarrow A_{pr} a A_{st} b.$ or
2. $A_{pp} \rightarrow .$

We need to show in both cases that the state cannot contain

a. another complete rule, and

b. a dotted rule that has a terminal symbol immediately after the dot.

In each case, we start by considering a string z on which DK goes to the above accept state.

Case 1a.

Suppose R is a completed type 1-2 rule: $A_{pq} \to A_{pr} a A_{st} b$. For any rule in this accept state, z must end with $A_{pr} a A_{st} b$ because DK goes to that state on z. Hence the symbols preceding the dot must be consistent in all such rules.

- These symbols are $A_{pr} a A_{st} b$ in R so any other type 1-2 completed rule must have exactly the same symbols on the right-hand side
- The non-terminal on the left-hand side must also agree, so the rules must be the same.

Case 1a. (cont'd)

- Assume that the accept state contains R and some type 3 completed ε-rule: $T = A_{ss} \rightarrow ..$
- From R we know that z ends with $A_{pr}aA_{st}b$.
- P pops its stack at the end of z because a pop occurs at that point in R, due to G's construction. $\delta(t, b, u) = (q, \epsilon)$
- An exception occurs at DK's start state, where this dot may occur at the beginning of the rule, but this accept state cannot be the start state because it contains a completed type 1-2 rule.
- According to the way we build DK, a completed ε-rule in a state must derive from a dotted rule that resides in the same state, where the dot isn't at the very beginning and the dot immediately precedes some non-terminals. $A \rightarrow A_{q_1,q_2}x.A_{ss}y.$
- In G, that means T derives from a type 1-2 dotted rule where the dot precedes the second non-terminal. From G's construction a push occurs just before the dot. $\delta(q_2, x, \epsilon) = (s, u')$
- This implies that (deterministic) PDA P does a push move at the very end of z, contradicting our previous statement.