EE160 Introduction to Control: Homework 7

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Deadline: June 3, 2022

1. *(3 points) Adjoint time-varying differential equation.* Let $A : \mathbb{R} \to \mathbb{R}^{n_x \times n_x}$ be a given function and x the solution of the linear time-varying differential equation

$$\forall t \in [0,T], \quad \dot{x}(t) = A(t)x(t) \quad x(0) = x_0 \ .$$

Moreover, let λ denote the solutions of the associated adjoint differential equation, given by

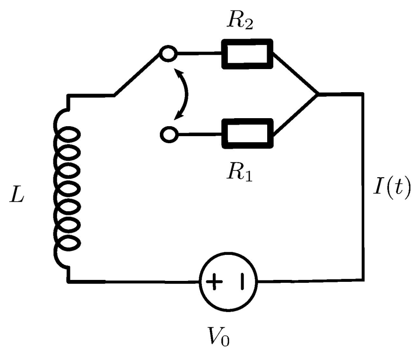
$$\forall t \in [0,T], \quad \dot{\lambda}(t) = A(T-t)^\top \lambda(t) \quad \lambda(0) = \lambda_0 \ .$$

Prove that the equation

$$\lambda_0^\top x(T) = \lambda(T)^\top x_0$$

holds for all $T \in \mathbb{R}$.

2. *(6 points) Electric circuit with periodic switch.* The electric circuit in the figure below consists of a battery with constant voltage $V_0 > 0$, an inductor with inductance $L > 0$, two resistors with resistance $R_1, R_2 > 0$, respectively, as well as a switch.



We assume that the switch changes its position every second. Thus, the period time is $T = 2s$. The current in the circuit at time t is denoted by $I(t)$. The following relations are known.

- The voltage V_0 at the battery is constant.
- The induced voltage at the inductor is given by $V_L(t) = L\dot{I}(t)$.
- The voltage at the resistors is $V_R(t) = R_1(t)I(t)$ if the switch is at time t at Position 1. Otherwise, if the switch is at Position 2, the voltage at the resistor is $V_R(t) = R_2(t)I(t)$.
- Due to Kirchhoff's voltage law, we have $V_L(t) + V_R(t) + V_0 = 0$.

(a) Derive a linear time-varying differential equation for the current $I(t)$.

(b) Find an explicit expression for the monodromy matrix $G(T, 0)$ that is associated with the differential equation for the current $I(t)$.

(c) Work out an explicit expression for the periodic limit orbit $I_p(t)$ and prove that we have

$$\lim_{t \to \infty} (I(t) - I_p(t)) = 0$$

independent of the initial value $I(0) = I_0$.

3. *(6 points) Stability analysis of dynamical systems.* Determine the equilibrium points and their stability properties of the following dynamical system for $t > 0$,

   (a) $\dot{x} = x(x-1)$

   (b) $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

   (c) $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & e^{2t} \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$