SI231b - Matrix Computations, 2020-21 Spring

Homework Set #1

Prof. Ziping Zhao

Acknowledgements:

1) Deadline: **2021-03-11 23:59:59**
2) Submit your homework in pdf format to Email: zhangzp1@shanghaitech.edu.cn.
3) You can write your homework using latex/word or you can write in handwriting and submit the scanned pdf.

Problem 1. (10 points) Prove that if $\{\mathbf{a}_1, \ldots \mathbf{a}_n\} \subset \mathbb{R}^m$ is linearly independent, then $n \leq m$ must hold.

Problem 2. (20 points) Let $\mathcal{S}_1, \mathcal{S}_2 \subseteq \mathbb{R}^m$ be subspaces. Prove

$$\mathsf{dim}\left(\mathcal{S}_1 + \mathcal{S}_2\right) = \mathsf{dim}\mathcal{S}_1 + \mathsf{dim}\mathcal{S}_2 - \mathsf{dim}\left(\mathcal{S}_1 \cap \mathcal{S}_2\right).$$

Problem 3. (20 points) Prove $\mathsf{rank}(\mathbf{AB}) \leq \min\left\{\mathsf{rank}(\mathbf{A}), \mathsf{rank}(\mathbf{B})\right\}$ and discuss when the equality holds.

Problem 4. (20 points) Suppose a finite-dimensional subspace $\mathcal{U} \subset \mathcal{V}$. Given $\mathbf{v} \in \mathcal{V}$, $\mathbf{u} \in \mathcal{U}$, and a projection of $\mathbf{v}$ onto $\mathcal{U}$ denoted by $\Pi_{\mathcal{U}}(\mathbf{v})$, prove that

$$\|\mathbf{v} - \Pi_{\mathcal{U}}(\mathbf{v})\| \leq \|\mathbf{v} - \mathbf{u}\|,$$

where the equality holds if and only if $\mathbf{u} = \Pi_{\mathcal{U}}(\mathbf{v})$.

Problem 5. (10 points) Prove $\mathsf{det}\left(\mathbf{I}_m + \mathbf{AB}\right) = \mathsf{det}\left(\mathbf{I}_n + \mathbf{BA}\right)$, where $\mathbf{A} \in \mathbb{R}^{m \times n}$, $\mathbf{B} \in \mathbb{R}^{n \times m}$.

Problem 6. (20 points) Given a matrix $\mathbf{Q} = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$ and a vector $\mathbf{x} \in \mathbb{R}^2$,

1) explain what $\mathbf{Qx}$ does to $\mathbf{x}$ in terms of rotations.
2) explain what $\mathbf{Qx}$ does to $\mathbf{x}$ in terms of reflections (determine the reflection vector if you use the Householder reflection).