Discrete Mathematics: Lecture 24

Degree, Handshaking Theorem, Graph Transform, Graph Isomorphism,

Bipartite Graph, Matching

Xuming He
Associate Professor

School of Information Science and Technology
ShanghaiTech University

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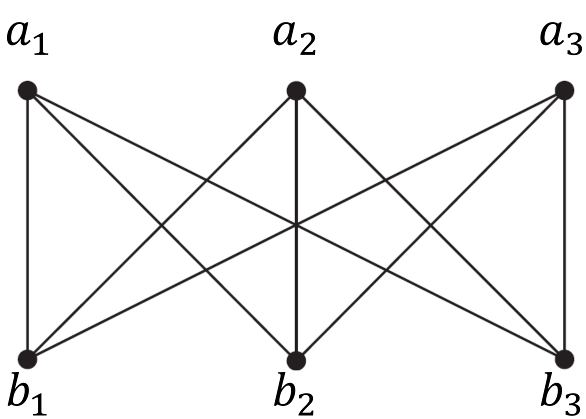
Notes by Prof. Liangfeng Zhang

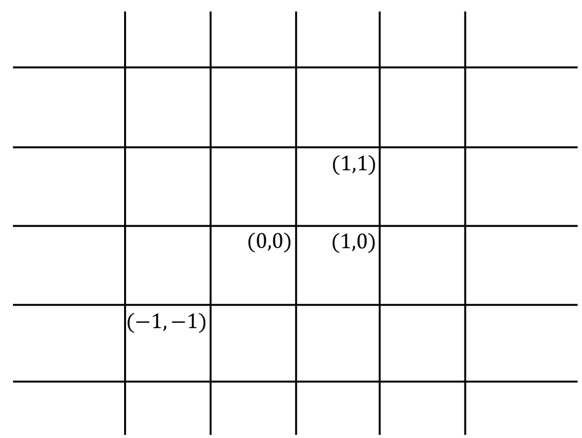
Graph

DEFINITION: A **graph** $G = (V, E)$ is defined by a nonempty set V of **vertices**顶点 and a set E of **edges**边, where each edge is associated with one or two vertices (called **endpoints**端点 of the edge).

- **Infinite Graph**无限图: $|V| = \infty$ or $|E| = \infty$
- **Finite Graph**有限图: $|V| < \infty$ and $|E| < \infty$; //$|V|$ is called the **order**阶数 of G



$$V = \{a_1, a_2, a_3, b_1, b_2, b_3\}$$
$$E = \left\{\{a_i, b_j\}: i, j = 1,2,3\right\}$$



$$V = \{(i, j): i, j \in \mathbb{Z}\}$$
$$E = \{\{(a, b), (c, d)\}: |a - c| = 1 \text{ or } |b - d| = 1\}$$

Types of Graphs

DEFINITION: Let $G = (V, E)$ be a graph with vertex set $V = \{v_1, \ldots, v_n\}$.

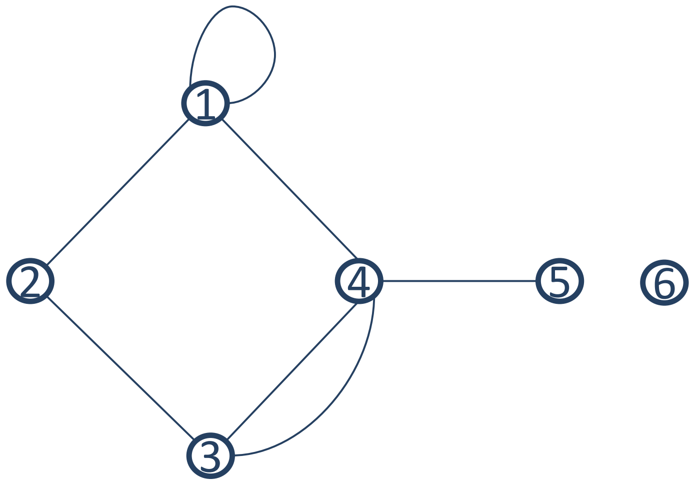
- **Question 1**: are the edges of G **directed**有向的?
  - No: G is an **undirected graph**无向图; the edge connecting v_i, v_j: $\{v_i, v_j\}$
  - Yes: G is a **directed graph**有向图; the edge starting at v_i and ending at v_j: (v_i, v_j)
- **Question 2**: are there **multiple edges**多重边connecting two different vertices v_i, v_j?
  - No: G is a **simple graph**简单图**;** Yes: G is a **multigraph**多重图
- **Question 3:** are there **loops**自环 connecting a vertex v_i to itself?
  - Yes: G is a **pseudograph**伪图

Type	Edges	Multiple Edges Allowed?	Loops Allowed?
Simple graph	undirected	No	No
Multigraph	undirected	Yes	No
Pseudograph	undirected	Yes	Yes
Simple directed graph	directed	No	No
Directed multigraph	directed	Yes	No
Mixed graph	undirected + directed	Yes	Yes

Degree

DEFINITION: Let $G = (V, E)$ be an <u>undirected</u> graph. We say that two vertices $u, v \in V$ are **adjacent**相邻的 (or **neighbors**邻居) if $\{u, v\} \in E$.

- **neighborhood**邻域 of v in G: $N(v) = \{u \in V: \{u, v\} \in E\}$
  - $N(A) = \cup_{v \in A} N(v)$ for $A \subseteq V$
- the **degree**度 $\deg(v)$ of $v \in V$ in G, is the number of edges incident with v
  - every loop from v to v contributes 2 to $\deg(v)$
- v is **isolated**孤立的 if $\deg(v) = 0$; v is **pendant**悬挂的 if $\deg(v) = 1$

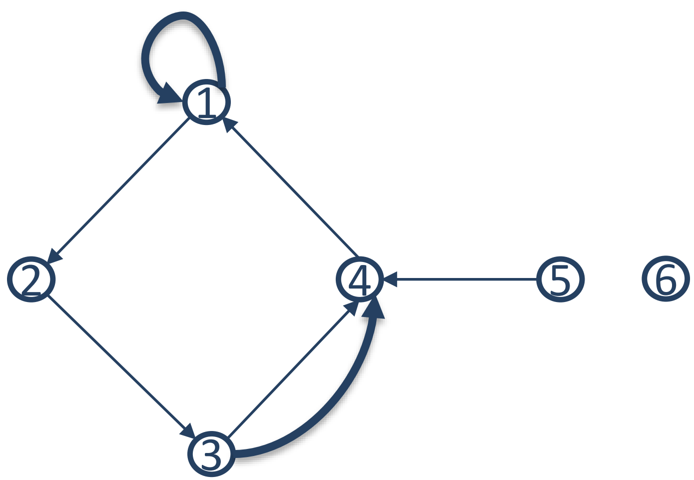


- 4 and 5 are adjacent
- $\{4,5\}$ is incident with 4 and 5
- $N(4) = \{1,3,5\}$; $N(\{1,4\}) = \{1,2,3,4,5\}$
- $\deg(1) = 4, \deg(2) = 2, \deg(3) = 3, \deg(4) = 4, \deg(5) = 1$
- 6 is isolated; 5 is pendant

Degree

DEFINITION: Let $G = (V, E)$ be a <u>directed</u> graph. If $(u, v) \in E$, we say that u is **adjacent to** v and v is **adjacent from** u.

- u is the **initial vertex**起始点 of (u, v); v is the **terminal vertex**终点 of (u, v)
  - $u = v$: u is the initial vertex and the terminal vertex
- **in-degree**入度 $\deg^-(v)$: the number of edges where v is the terminal vertex
- **out-degree**出度 $\deg^+(v)$: the number of edges where v is the initial vertex
  - $u = v$: the loop contributes 1 to $\deg^-(v)$ and 1 to $\deg^+(v)$



- 5 is adjacent to 4; 4 is adjacent from 5
- 5 is the initial vertex of (5,4)
- 4 is the terminal vertex of (5,4)
- 1 is the initial and terminal vertex of a loop
- $\deg^-(1) = 2\,; \deg^+(1) = 2$
- $\deg^-(4) = 3\,; \deg^+(4) = 1$

Handshaking Theorem

THEOREM: Let $G = (V, E)$ be an undirected graph. Then

$2|E| = \sum_{v \in V} \deg(v)$ and $|\{v \in V : \deg(v) \text{ is odd}\}|$ is even.

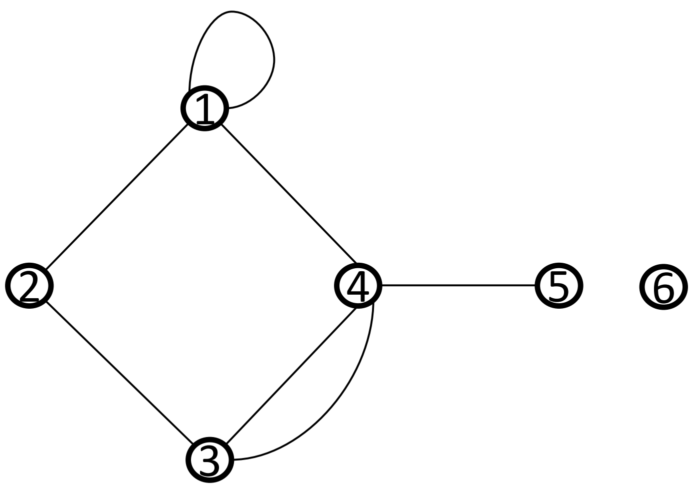
- Any edge $e \in E$ contribute 2 to the sum $\sum_{v \in V} \deg(v)$
  - $e = \{v_i, v_j\}$: e contributes 1 to $\deg(v_i)$ and 1 to $\deg(v_j)$
  - $e = \{v_i\}$: e contributes 2 to $\deg(v_i)$
- The m edges contribute $2|E|$ to $\sum_{v \in V} \deg(v)$.
  - Hence, $\sum_{v \in V} \deg(v) = 2|E|$
- $\sum_{v \in V} \deg(v) = \sum_{v \in V : 2 | \deg(v)} \deg(v) + \sum_{v \in V : 2 \nmid \deg(v)} \deg(v)$
  - $2 | \sum_{v \in V} \deg(v);\ 2 | \sum_{v \in V : 2 | \deg(v))} \deg(v)$
    - $2 | \sum_{v \in V : 2 \nmid \deg(v)} \deg(v)$
      - $|\{v \in V : \deg(v) \text{ is odd}\}|$ must be even

Handshaking Theorem

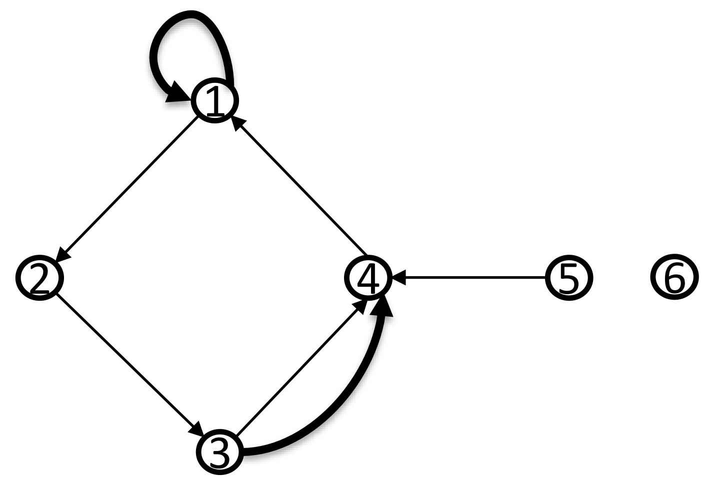
THEOREM: Let $G = (V, E)$ be a directed graph. Then

$$\sum_{v\in V} \deg^-(v) = \sum_{v\in V} \deg^+(v) = |E|.$$

- Every edge $e \in E$ contributes 1 to $\sum_{v\in V} \deg^-(v)$
  - $e = (v_i, v_j)$ contributes 1 to $\deg^-(v_i)$
- Hence, $\sum_{v\in V} \deg^-(v) = |E|$



v	1	2	3	4	5	6
$\deg(v)$	4	2	3	4	1	0



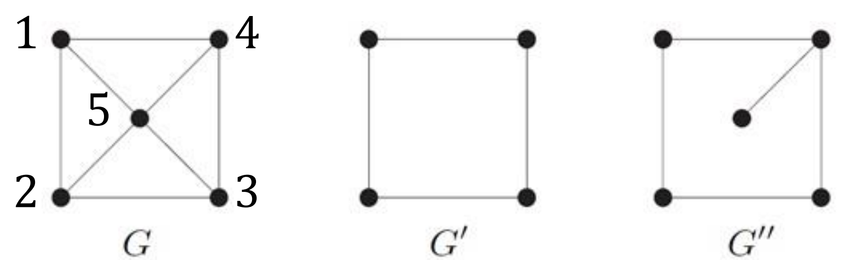
v	1	2	3	4	5	6
$\deg^-(v)$	2	1	1	3	0	0
$\deg^+(v)$	2	1	2	1	1	0

Subgraph

DEFINITION: Let $G = (V, E)$ be a simple graph. $H = (W, F)$ is a **subgraph**子图 of G if $W \subseteq V$ and $F \subseteq E$.

- **proper subgraph**真子图: H is a subgraph of G and $H \neq G$.
- The **subgraph induced**导出子图 by $W \subseteq V$ is (W, F), where $F = \{e: e \in E, e \subseteq W\}$.
  //Notation: $G[W]$
- The **subgraph induced**导出子图 by $F \subseteq E$ is (W, F), where $W = \{v: v \in V, v \in e \text{ for some } e \in F\}$. //Notation: $G[F]$

EXAMPLE: Let G, G', G'' be three graphs as below.

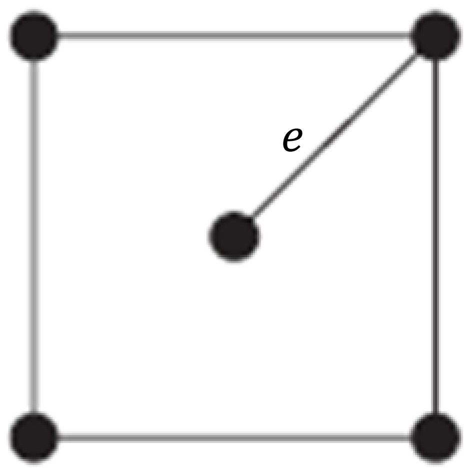


- G', G'' are subgraphs of G; G', G'' are proper subgraphs of G
- G' is a subgraph induced by $W = \{1,2,3,4\}$, i.e., $G' = G[W]$
- G'' is a subgraph induced by $F = \{\{1,2\}, \{2,3\}, \{3,4\}, \{4,1\}, \{4,5\}\}$, i.e., $G'' = G[F]$

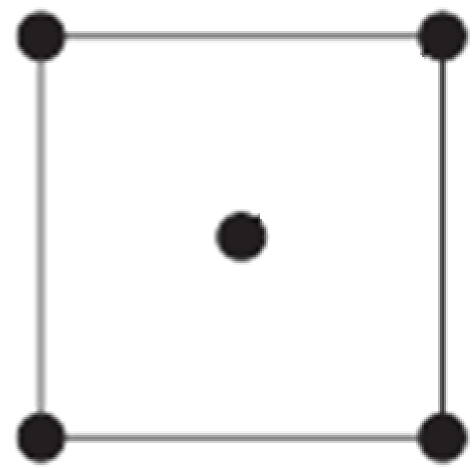
Removing An Edge

DEFINITION: Let $G = (V, E)$ be a simple graph and $e \in E$. Define

$$G - e = (V, E - \{e\})$$



$G = (V, E)$

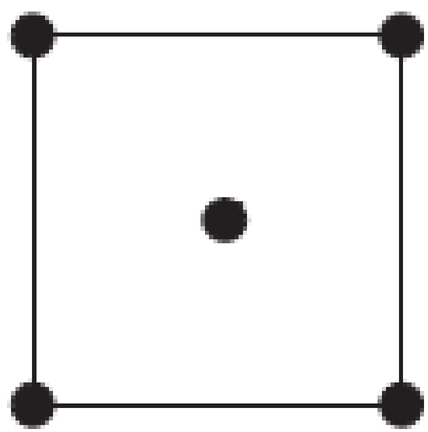


$G - e = (V, E - \{e\})$

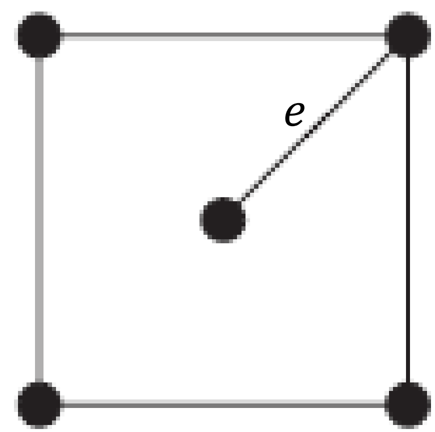
Adding An Edge

DEFINITION: Let $G = (V, E)$ be a simple graph and $e \notin E$. Define

$$G + e = (V, E \cup \{e\})$$



$G = (V, E)$

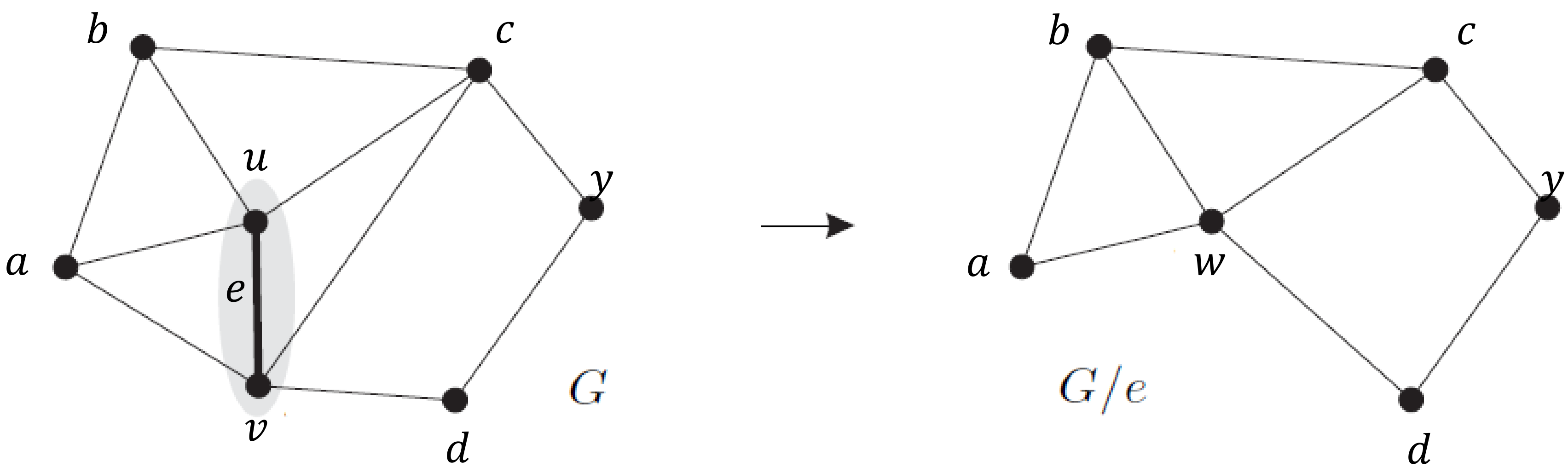


$G + e = (V, E \cup \{e\})$

Edge Contraction

DEFINITION: Let $G = (V, E)$ be a simple graph and $e = \{u, v\} \in E$. Define $G/e = (V', E')$, where $V' = (V - \{u, v\}) \cup \{w\}$ and

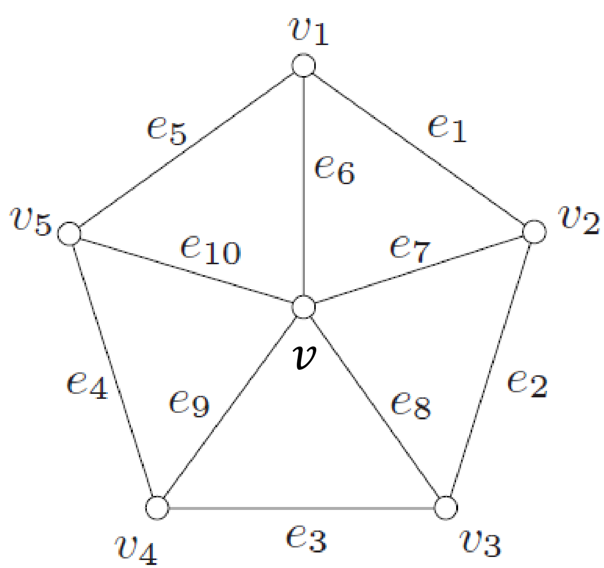
$$E' = \{e' \in E : e' \cap e = \emptyset\} \cup \left\{\{w, x\} : \{u, x\} \in E \text{ or } \{v, x\} \in E\right\}$$



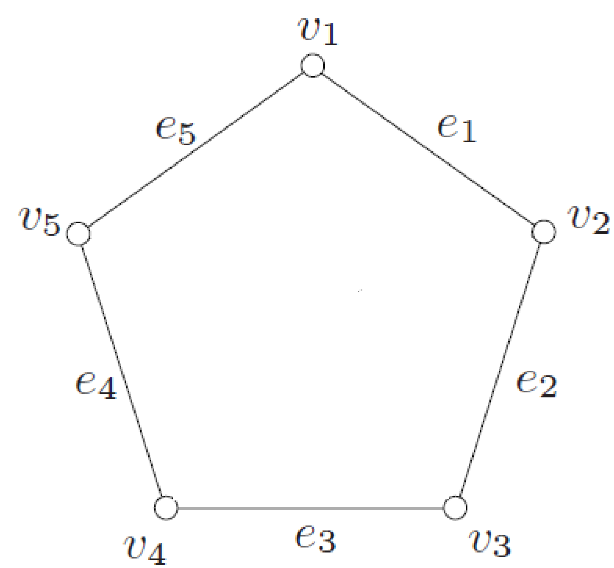
Removing A Vertex

DEFINITION: Let $G = (V, E)$ be a simple graph and let $v \in V$. Define

$$G - v = (V - \{v\}, E'), \text{ where } E' = \{e \in E : v \notin e\}$$



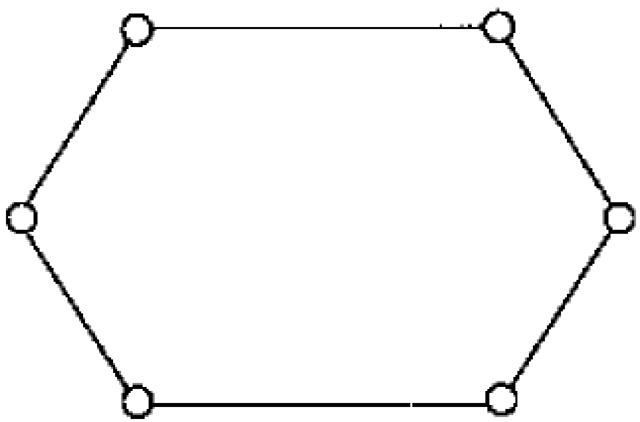
$G = (V, E)$



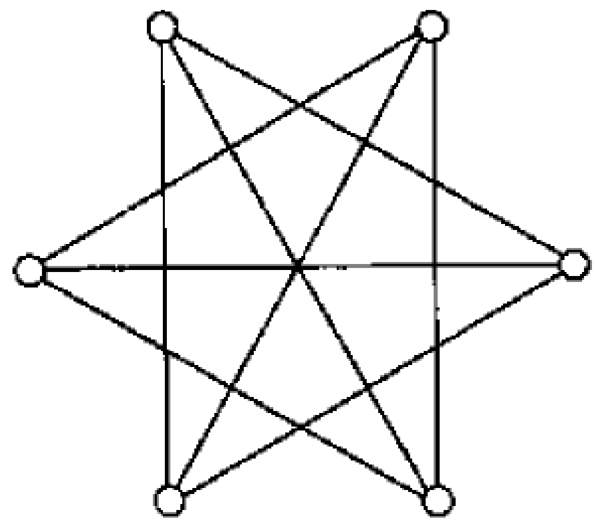
$G - v$

Complement

DEFINITION: Let $G = (V, E)$ be a simple graph of order n. Define the **complement graph**补图 of G as $\bar{G} = (V, E')$, where $E' = \left\{\{u, v\}: u, v \in V,\ u \neq v, \{u, v\} \notin E\right\}$



G

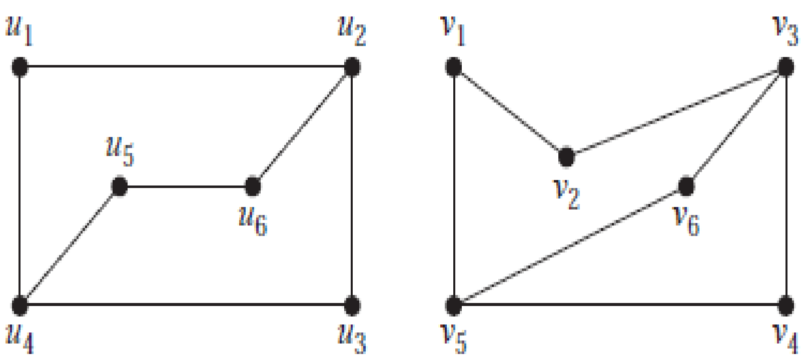


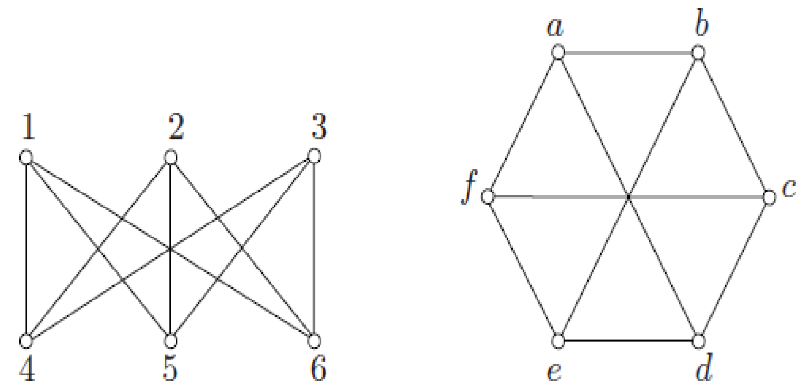
$\bar{G}$

Graph Isomorphism

DEFINITION: The simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are **isomorphic**同构 if there is a bijection $\sigma: V_1 \to V_2$ such that $\{u, v\} \in E_1 \Leftrightarrow \{\sigma(u), \sigma(v)\} \in E_2$.

- σ is called an **isomorphism**同构映射
- **nonisomorphic:** not isomorphic





u_1	u_2	u_3	u_4	u_5	u_6
v_6	v_3	v_4	v_5	v_1	v_2

Isomorphism σ

1	2	3	4	5	6
a	c	e	b	d	f

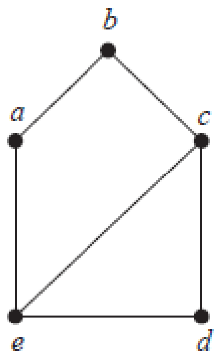
Isomorphism σ

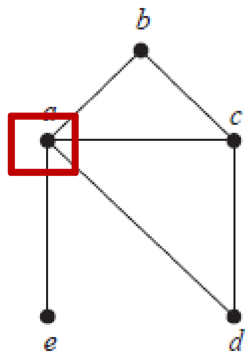
Graph Invariants

DEFINITION: **Graph invariants** are properties preserved by graph isomorphism. For example,

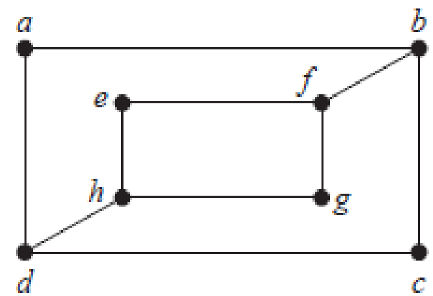
- The number of vertices
- The number of edges
- The number of vertices of each degree

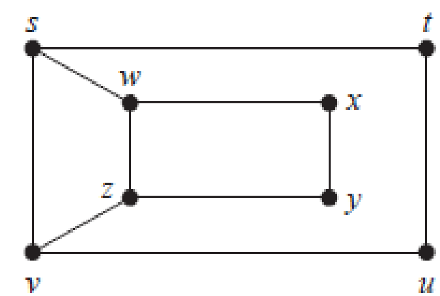
REAMRKS: The graph invariants can be used to determine if two graphs are isomorphic or not.

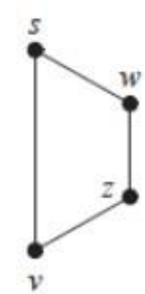




There is no vertex of degree 4 in the 1st graph







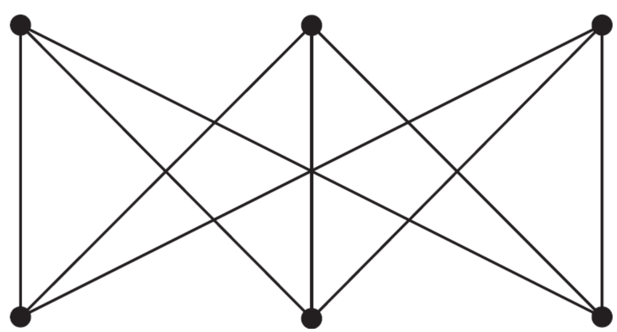
Induced subgraphs are not isomorphic. Original graphs are not isomorphic.

The subgraphs induced by the vertices of degree 3 must be isomorphic to each other.

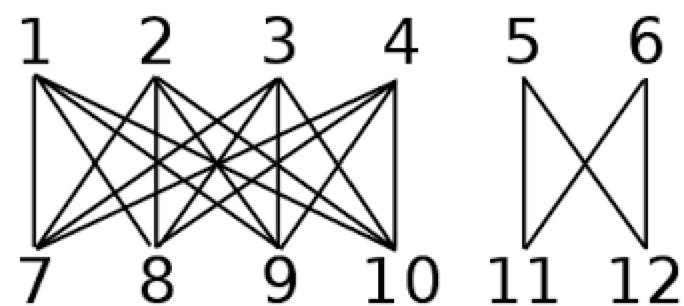
Bipartite Graph

DEFINITION: $G = (V, E)$ is a **bipartitie graph**二分图、二部图 if V has a partition $\{V_1, V_2\}$ such that $E \subseteq \left\{\{u_1, u_2\}: u_1 \in V_1, u_2 \in V_2\right\}$.

- (V_1, V_2) is a **bipartition**二划分 of the vertex set V.



A bipartite graph of order 6



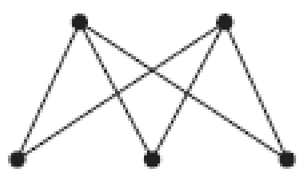
A bipartite graph of order 12

- $V_1 = \{1,2,3,4,5,6\}$
- $V_2 = \{7,8,9,10,11,12\}$

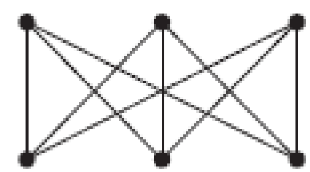
Complete Bipartite Graph

DEFINITION: A **complete bipartite graph**完全二部图 is a graph $K_{m,n} = (V, E)$ with $V = \{x_1, \dots, x_m\} \cup \{y_1, \dots, y_n\}$ and $E = \left\{\{x_i, y_j\}: i \in [m], j \in [n]\right\}$

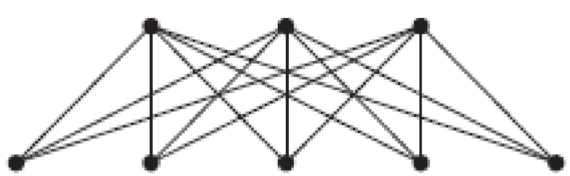
- Every vertex in V_1 is adjacent to every vertex in V_2



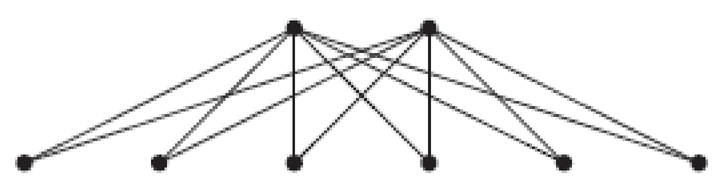
$K_{2,3}$



$K_{3,3}$



$K_{3,5}$



$K_{2,6}$

Bipartite Graph

Theorem

A simple graph is bipartite if and only if it is possible to assign one of two different colors to each vertex such that no two adjacent vertices have the same color.

Proof:

- If $G = (V, E)$ is bipartite, $V = V_1 \cup V_2$. Assign color c_1 to vertices of V_1 and color c_2 to vertices of V_2.
- Reversely, suppose we can assign colors c_1 and c_2 to the vertices such that no two adjacent have the same. Let V_i be the set of vertices of color c_i, for $i = 1, 2$. Then $V = V_1 \cup V_2$. By assumption there are no edges connecting two vertices of V_1 or two vertices of V_2, so each edge connects one vertex of V_1 with one vertex of V_2. $\square$

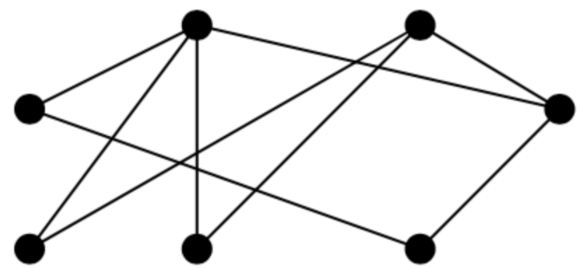
Bipartite Graph*

THEOREM: A simple graph $G = (V, E)$ is a bipartite graph iff there is a map $f: V \to \{1,2\}$ such that "$\{x, y\} \in E \Rightarrow f(x) \neq f(y)$"

- Only if: $G = (V_1 \cup V_2,\ E)$, where $V_1 \cap V_2 = \emptyset$.
  - Define $f: V \to \{1,2\}$ such that $f(x) = \begin{cases} 1 & \text{if } x \in V_1 \\ 2 & \text{if } x \in V_2 \end{cases}$
  - $\{x, y\} \in E \Rightarrow x \in V_1, y \in V_2$ or $x \in V_2, y \in V_1$
    - $f(x) \neq f(y)$
- If: $f\colon V \to \{1,2\}$ is a map such that "$\{x, y\} \in E \Rightarrow f(x) \neq f(y)$"
  - Let $V_1 = f^{-1}(1), V_2 = f^{-1}(2)$
    - $V = V_1 \cup V_{2,} V_1 \cap V_2 = \emptyset$
      - $\{V_1, V_2\}$ is a bipartition of V
  - $\{x, y\} \in E \Rightarrow f(x) \neq f(y) \Rightarrow x \in V_1, y \in V_2$ or $x \in V_2, y \in V_1$
    - G is a bipartite graph.

Bipartite Graph

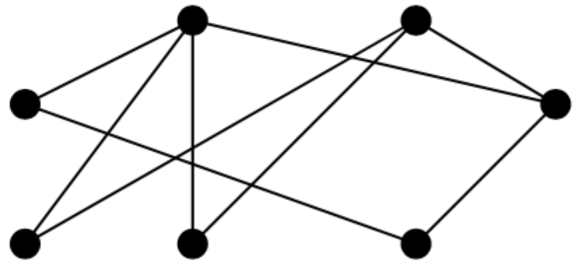
Example: Is the graph G bipartite?



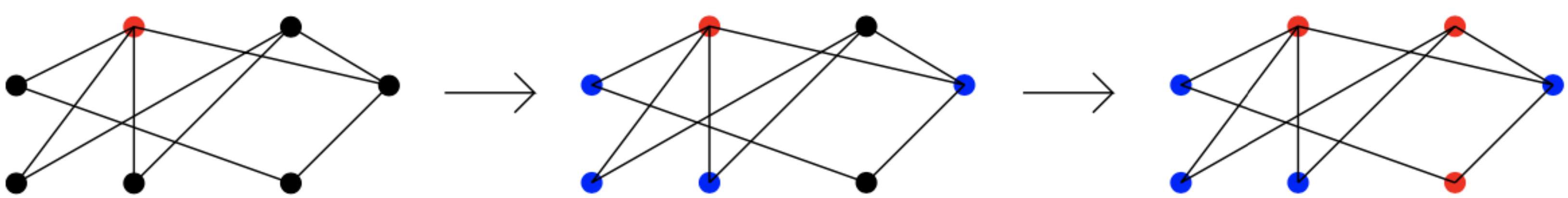
G

Bipartite Graph

Example: Is the graph G bipartite?



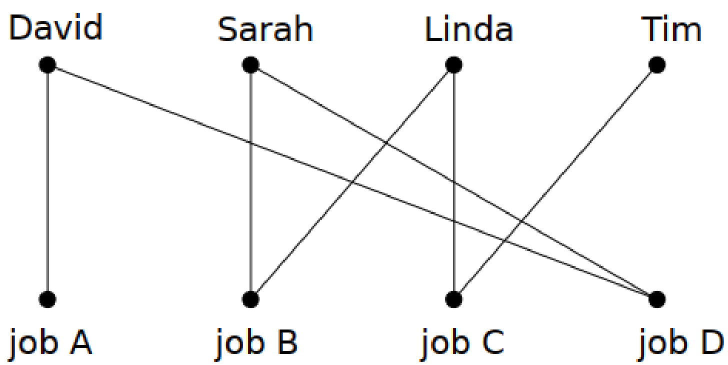
G



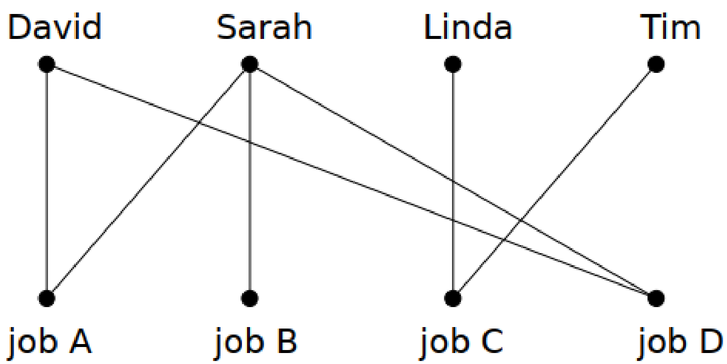
Motivation: Job Assignment

Suppose there are m employees and n different jobs to be done, with $m \geq n$.

1



2



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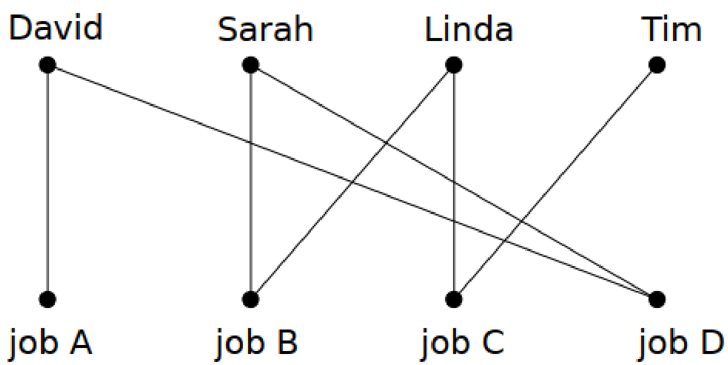
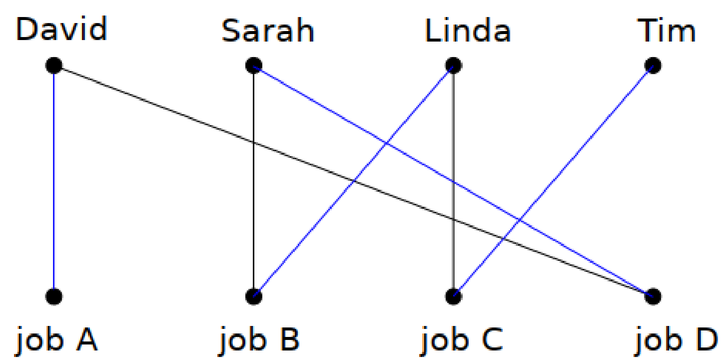
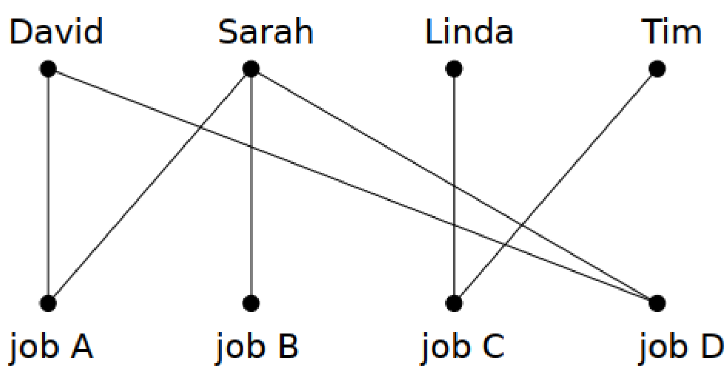




Possible solution for situation 1

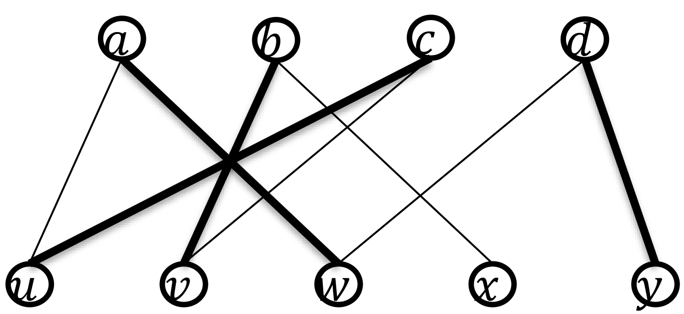
2



Matching

DEFINITION: Let $G = (V, E)$ be a simple graph. $M \subseteq E$ is a **matching**匹配 if $e \cap e' = \emptyset$ for every $e, e' \in M$. A vertex $v \in V$ is **matched** in M if $\exists e \in M$ such that $v \in e$, otherwise, v is **not matched.**

- **maximum matching**最大匹配**:** a matching with largest number of edges.
- In a bipartite graph $G = (A \cup B, E)$, $M \subseteq E$ is a **complete matching**完全匹配 from A to B if every $u \in A$ is matched.



- $M = \{au, bv\}$ is a matching
  - a, b, u, v are matched in M
  - c, d, x, y are not matched in M
  - M is not a maximum matching
- $M' = \{aw, bv, cu, dy\}$ is a maximum matching
- M' is a complete matching from V_1 to V_2

- $V = \{a, b, c, d, u, v, w, x, y\}$
- $V_1 = \{a, b, c, d\}$;
- $V_2 = \{u, v, w, x, y\}$
- $E = \{au, aw, bv, bx, cu, cv, dw, dy\}$

Matching

DEFINITION: Let $G = (V, E)$ be a simple graph. $M \subseteq E$ is a **matching**匹配 if $e \cap e' = \emptyset$ for every $e, e' \in M$. A vertex $v \in V$ is **matched** in M if $\exists e \in M$ such that $v \in e$, otherwise, v is **not matched.**

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Example: Marriages. Suppose there are m men and n women on an island. Each person has a list of people of the opposite gender acceptable as a spouse $\Rightarrow$ bipartite graph.

- matching $\Leftrightarrow$ marriages
- maximum matching $\Leftrightarrow$ largest possible number of marriages
- complete matching from women to men $\Leftrightarrow$ marriages such that every women is married but possibly not all men.

Hall's Theorem

EXAMPLE: Marriage on an Island

- There are m boys $X = \{x_1, \ldots, x_m\}$ and n girls $Y = \{y_1, \ldots, y_n\}$
- $G = \left(X \cup Y, E = \left\{\{x_i, y_j\}: x_i \text{ and } y_j \text{ are willing to get married}\right\}\right)$
- What is the largest number of couples that can be formed?

THEOREM (Hall 1935): A bipartitie graph $G = (X \cup Y, E)$ has a complete matching from X to Y iff $|N(A)| \geq |A|$ for any $A \subseteq X$.

- $\Rightarrow$: Let $\{\{x_1, y_1\}, \ldots, \{x_m, y_m\}\}$ be a complete matching from X to Y
  - For any $A = \{x_{i_1}, \ldots, x_{i_s}\} \subseteq X$, $N(A) \supseteq \{y_{i_1}, \ldots, y_{i_s}\}$
    - $|N(A)| \geq s = |A|$
- $\Leftarrow$: suppose that $|N(A)| \geq |A|$ for any $A \subseteq X$. Find a complete matching M.
  - By induction on $|X|$
  - $|X| = 1$: Let $X = \{x\}$.
    - $|N(X)| \geq 1$
      - $\exists y \in Y$ such that $e = \{x, y\} \in E$.
        - $M = \{e\}$ is a complete matching from X to Y

Hall's Theorem

- **Induction hypothesis**: "$\forall A \subseteq X, |N(A)| \geq |A| \Rightarrow \exists$ complete matching" is true when $|X| \leq k$
- Prove that "$\forall A \subseteq X, |N(A)| \geq |A| \Rightarrow \exists$ complete matching" when $|X| = k + 1$
  - Let $X = \{x_1, \dots, x_k, x_{k+1}\}$.
  - **Case 1:** $\forall\ A \subseteq X$ with $1 \leq |A| \leq k$, $|N_G(A)| \geq |A| + 1$
    - $N_G(A)$: A's neighborhood in G
    - Say $y_{k+1} \in N_G(\{x_{k+1}\})$.
    - Let $V' = (X \setminus \{x_{k+1}\}) \cup (Y \setminus \{y_{k+1}\})$; $E' = \{e \in E: e \subseteq V' \times V'\}$
    - Let $G' = (V', E') = G - \{x_{k+1}\} - \{y_{k+1}\}$.
      - $\forall\ A \subseteq \{x_1, \dots, x_k\}, |N_{G'}(A)| \geq |N_G(A)| - |\{y_{k+1}\}| \geq |A| + 1 - 1 = |A|$
        - $\exists$ a complete matching M' from $X - \{x_{k+1}\}$ to $Y - \{y_{k+1}\}$ in G' (IH)
    - $M = M' \cup \left\{\{x_{k+1}, y_{k+1}\}\right\}$ is a complete matching from X to Y in G

Hall's Theorem

- **Case 2:** $\exists A \subseteq X, 1 \leq |A| \leq k$ such that $|N_G(A)| = |A|$
  - Say $A = \{x_1, \dots, x_j\}$ and $N_G(A) = \{y_1, \dots, y_j\}$, where $1 \leq j \leq k$
  - Let $V' = A \cup N_G(A), E' = \{e \in E: e \subseteq V' \times V'\}$ and $G' = (V', E')$
    - $\forall A' \subseteq A,\ |N_{G'}(A')| = |N_G(A')| \geq |A'|$
    - There is a complete matching M' from A to $N_G(A)$ in G' (IH)
  - Let $V'' = (X \setminus A) \cup \big(Y \setminus N_G(A)\big), E'' = \{e \in E: e \subseteq V'' \times V''\}$,
  - Let $G'' = (V'', E'') = G - A - N_G(A)$
    - Then $\forall\ A'' \subseteq X \backslash A, |N_{G''}(A'')| \geq |A''|$.
      - Otherwise, $|N_G(A'' \cup A)| = |N_{G''}(A'')| + |N_G(A)| < |A''| + |A|$
        - $\exists$ a complete matching M'' from $X \setminus A$ to $Y \setminus N_G(A)$ (IH)
  - $M = M' \cup M''$ is a complete matching from X to Y