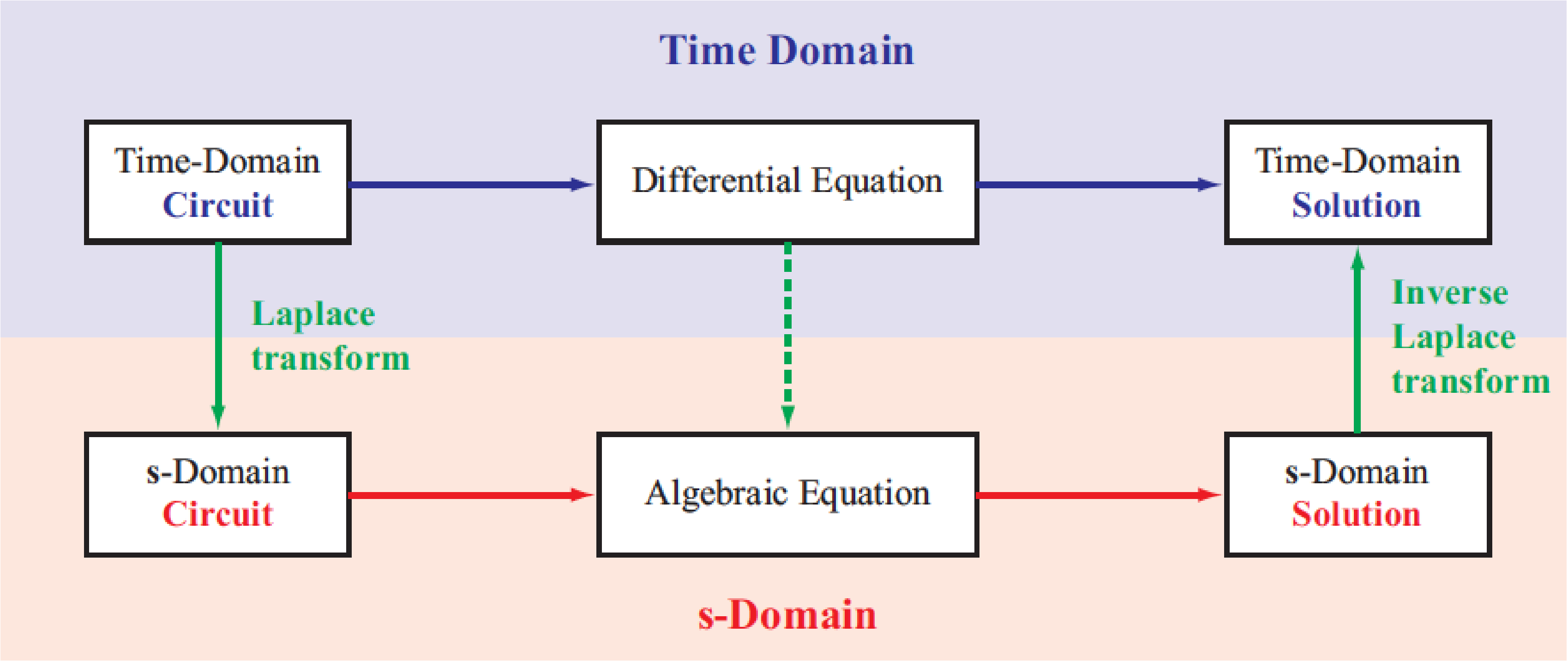
Lecture 13
- Laplace Transform

Analysis Techniques

Circuit Excitation	Method of Solution
dc (w/ switches)	DC/Transient analysis
ac	Phasor-domain analysis (Steady state only)
Periodic waveform	Fourier series + Phasor-domain (Steady state only)
Waveform	**Laplace transform (transient + steady state)**

[Source: Berkeley]

Laplace Transform Technique



[Source: Berkeley]

The French Newton Pierre-Simon Laplace (Late 1700)

- Developed mathematics in astronomy, physics, and statistics
- Began work in calculus which led to the Laplace Transform
- Focused later on celestial mechanics
  - One of the first scientists to suggest the existence of black holes

https://en.wikipedia.org/wiki/Pierre-Simon_Laplace

What are Laplace Transforms?

$$F(s) = \mathcal{L}[f(t)] = \int_{0_-}^{\infty} f(t)e^{-st}dt$$

- $f(t) \rightarrow F(s)$,
- t is real, being integrated
- s is variable complex; $s = \sigma + j\omega$.

- Note integral starts from 0_-

- Assume f(t)=0 for all $t < 0$

Inverse Laplace Transforms

$$f(t)=\mathcal{L}^{-1}[F(s)]=\frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty}F(s)e^{st}ds$$

- Conversely, F(*s*) $\rightarrow$ f(*t*), *t* is variable and s is integrated.

TABLE 12.1 An Abbreviated List of Laplace Transform Pairs

Type	$f(t)\ (t > 0-)$	$F(s)$
(step)	$u(t)$	$\frac{1}{s}$
(ramp)	t	$\frac{1}{s^2}$
(exponential)	e^{-at}	$\frac{1}{s+a}$
(sine)	$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
(cosine)	$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
(damped ramp)	te^{-at}	$\frac{1}{(s+a)^2}$
(damped sine)	$e^{-at} \sin \omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$
(damped cosine)	$e^{-at} \cos \omega t$	$\frac{s+a}{(s+a)^2+\omega^2}$

TABLE 12.2 An Abbreviated List of Operational Transforms

Operation	$f(t)$	$F(s)$
Multiplication by a constant	$Kf(t)$	$KF(s)$
Addition/subtraction	$f_1(t) + f_2(t) - f_3(t) + \cdots$	$F_1(s) + F_2(s) - F_3(s) + \cdots$
First derivative (time)	$\dfrac{df(t)}{dt}$	$sF(s) - f(0^-)$
Second derivative (time)	$\dfrac{d^2f(t)}{dt^2}$	$s^2F(s) - sf(0^-) - \dfrac{df(0^-)}{dt}$
nth derivative (time)	$\dfrac{d^nf(t)}{dt^n}$	$s^nF(s) - s^{n-1}f(0^-) - s^{n-2}\dfrac{df(0^-)}{dt} - s^{n-3}\dfrac{df^2(0^-)}{dt^2} - \cdots - \dfrac{d^{n-1}f(0^-)}{dt^{n-1}}$
Time integral	$\displaystyle\int_0^t f(x)\,dx$	$\dfrac{F(s)}{s}$
Translation in time	$f(t-a)u(t-a), a > 0$	$e^{-as}F(s)$
Translation in frequency	$e^{-at}f(t)$	$F(s+a)$
Scale changing	$f(at), a > 0$	$\dfrac{1}{a}F\left(\dfrac{s}{a}\right)$
First derivative (s)	$tf(t)$	$-\dfrac{dF(s)}{ds}$
nth derivative (s)	$t^nf(t)$	$(-1)^n\dfrac{d^nF(s)}{ds^n}$
s integral	$\dfrac{f(t)}{t}$	$\displaystyle\int_s^\infty F(u)\,du$

Homogeneity and Additivity

$$\mathcal{L}[a_1 f_1(t)] = a_1 \mathcal{L}[f_1(t)] = a_1 F_1(s)$$

$$\mathcal{L}[a_1 f_1(t) + a_2 f_2(t)] = a_1 \mathcal{L}[f_1(t)] + a_2 \mathcal{L}[f_2(t)] = a_1 F_1(s) + a_2 F_2(s)$$

here a_1 and a_2 are constants

Important implication:

$$\sum_{k=1}^{k} i_k(t) = 0 \iff \sum_{k=1}^{k} I_k(s) = 0$$

$$\sum_{k=1}^{k} u_k(t) = 0 \iff \sum_{k=1}^{k} U_k(s) = 0$$

Time Differentiation

$$\mathcal{L}\left[\frac{d}{dt}f(t)\right] = sF(s) - f(0\_)$$

Initial and final value

$$\mathcal{L}\left[\frac{d}{dt}f(t)\right] = sF(s) - f(0\_)$$

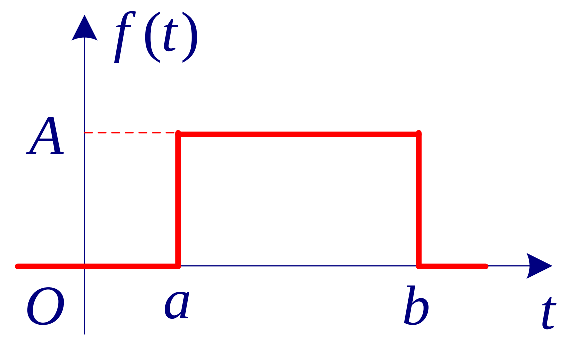
Time integral

$$\mathscr{L}\left[\int_{0_-}^{t} f(\tau)d\tau\right] = \frac{1}{s}F(s)$$

Translation in the Time Domain

$$\mathcal{L}[f(t-\tau)\,u(t-\tau)] = e^{-s\tau}F(s)$$

- Example



$$f(t) = A[u(t-a) - u(t-b)]$$

$$F(s) = A\mathcal{L}[u(t-a) - u(t-b)] = \frac{A}{s}(e^{-as} - e^{-bs})$$

Translation in Frequency domain

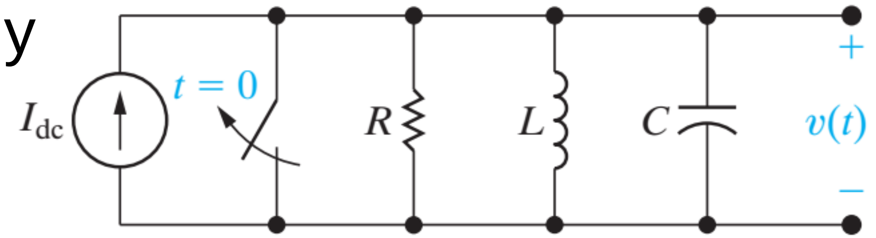
$$\mathcal{L}[e^{\alpha t} f(t)] = F(s\text{-}\alpha)$$

- Example

$$\mathcal{L}[\sin \omega t] = \frac{\omega}{s^2 + \omega^2} \qquad \mathcal{L}[e^{-\alpha t} \sin \omega t] = \frac{\omega}{(s+\alpha)^2 + \omega^2}$$

Applying the Laplace Transform

- We assume no initial energy stored at t=0



$$\frac{v(t)}{R} + \frac{1}{L}\int_0^t v(x)\,dx + C\frac{dv(t)}{dt} = I_{dc}u(t).$$

$$\frac{V(s)}{R} + \frac{1}{L}\frac{V(s)}{s} + C[sV(s) - v(0^-)] = I_{dc}\left(\frac{1}{s}\right)$$

$$V(s)\left(\frac{1}{R} + \frac{1}{sL} + sC\right) = \frac{I_{dc}}{s}$$

$$V(s) = \frac{I_{dc}/C}{s^2 + (1/RC)s + (1/LC)}. \longrightarrow \boxed{v(t) = \mathscr{L}^{-1}\{V(s)\}.}$$