Online Lecture Notes

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1 Example 2: Chained Spring-Mass Damper System

Let us consider a system with N chained trolleys that are connected by springs.

1. The first trolley is attached to a wall via a spring with spring constant D,
2. The second trolley is attached to the first trolley via a spring with the same spring constant,
3. and so on, ...
4. and the N-th trolley is only attached to $(N-1)$th trolley but its other end is free.

Two tasks: 1) Develop a physical model for this system. 2) Rewrite the equations into standard form.

1. Step 1: Introduce variable names for the states of the system: $p_i(t)$ will denote the position of the i-th trolley. We use the notation $v_i(t)$ to denote its velocity.
2. Step 2: Work out the differential equation. Questions: What is x? What is A? What is b? (what is x_0?)

   (a) We need to stack all positions and velocities into one big vector:

   $$x = \begin{pmatrix} p_1 \\ v_1 \\ p_2 \\ v_2 \\ \vdots \\ p_m \\ v_m \end{pmatrix} : \mathbb{R} \to \mathbb{R}^{2N}$$

   (b) In order to figure out what are A and b are, we need to write out our differential equations first. Newton's equations of motion:

$$\begin{aligned}
\dot{p}_1(t) &= v_1(t) && (1)\\
\dot{v}_1(t) &= -\frac{D}{m}p_1(t) + \frac{D}{m}(p_2(t) - p_1(t)) && (2)\\
\dot{p}_2(t) &= v_2(t) && (3)\\
\dot{v}_2(t) &= -\frac{D}{m}(p_2(t) - p_1(t)) + \frac{D}{m}(p_3(t) - p_2(t)) && (4)\\
&\vdots && (5)\\
\dot{p}_m(t) &= v_m(t) && (6)\\
\dot{v}_m(t) &= -\frac{D}{m}(p_m(t) - p_{m-1}(t)) && (7)
\end{aligned}$$

One things that we can see directly now is that $b = 0$. In order to figure out what A is, we need to put all the coefficient into a matrix format:

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ -\frac{2D}{m} & 0 & \frac{D}{m} & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ \frac{D}{m} & 0 & -\frac{2D}{m} & 0 & \frac{D}{m} & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & -\frac{D}{m} & 0 \end{pmatrix} \in \mathbb{R}^{2N \times 2N}$$

and finally, the initial state x_0 stacks all intial positions and velocities,

$$x_0 = \begin{pmatrix} p_1^0 \\ v_1^0 \\ p_2^0 \\ v_2^0 \\ \vdots \\ p_m^0 \\ v_m^0 \end{pmatrix} \in \mathbb{R}^{2N}.$$

2 Matrix Exponential

Recall from Lecture 1: the solution of the scalar linear ODE, $\dot{x}(t) = ax(t)$ is given by

$$x(t) = e^{at} x_0$$

In the multivariate case, we are trying to generalize this result. Basically, what we would like to do is to use the same formula also for matrices and vectors. This means that we should define the matrix exponential such that the solution of the multivariate differential eqaution $\dot{x}(t) = Ax(t)$ is given by

$$x(t) = e^{At} x_0 \ .$$

BUT: now, the question is what is "e^{At}" ??? In order to answer this question, we use the Taylor series expansion of the e-function. In the scalar case it is given by

$$e^{at} = 1 + at + \frac{(at)^2}{2!} + \frac{(at)^3}{3!} + \frac{(at)^4}{4!} + \ldots$$

This is an infinite sum, which converges uniformly for all $t \in \mathbb{R}$. Now, the main idea to use the same formula to define what we mean by e^{At}. Here, the idea is to literally replace a by the capital A. This gives us the definition

$$e^{At} \stackrel{\text{def}}{=} I + At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^3 + \frac{1}{4!}A^4t^4 + \ldots = \sum_{i=0}^{\infty} \frac{1}{i!} A^i t^i \ .$$

Here, the matrix powers "A^i" are computed by standard matrix multiplication:

$$A^i \stackrel{\text{def}}{=} \underbrace{A \cdot A \cdot A \cdot \ldots \cdot A}_{i \text{ times}} \quad \text{and} \quad A^0 \stackrel{\text{def}}{=} I \ .$$

It can be proven that this series expansion converges uniformly for all $t \in \mathbb{R}$.

2.1 Example: Diagonal Matrices

Let us consider the case that A is diagonal,

$$A = \begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix} \in \mathbb{R}^{2 \times 2} \ .$$

In this case it is particularly easy to work out the powers of A, since we have

$$\forall i \in \mathbb{N}, \qquad A^i = \begin{pmatrix} A_{11}^i & 0 \\ 0 & A_{22}^i \end{pmatrix} \ .$$

This means that the matrix exponential of a diagonal matrix is given by

$$
\begin{aligned}
e^{At} &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix} t + \frac{t^2}{2!} \begin{pmatrix} A_{11}^2 & 0 \\ 0 & A_{22}^2 \end{pmatrix} + \dots \\
&= \begin{pmatrix} 1 + A_{11}t + \frac{1}{2!}A_{11}^2 t^2 + \frac{1}{3!}A_{11}^3 t^3 & 0 \\ 0 & 1 + A_{22}t + \frac{1}{2!}A_{22}^2 t^2 + \frac{1}{3!}A_{22}^3 t^3 \end{pmatrix} \\
&= \begin{pmatrix} e^{A_{11}t} & 0 \\ 0 & e^{A_{22}t} \end{pmatrix} \qquad (8)
\end{aligned}
$$

This means that we can simply get the exponential of a diagonal matrix by taking the exponential function of all of its diagonal elements. Of course, this works for any dimension:

$$
A = \begin{pmatrix} A_{11} & & \\ & \ddots & \\ & & A_{nn} \end{pmatrix} \implies e^{At} = \begin{pmatrix} e^{A_{11}t} & & \\ & \ddots & \\ & & e^{A_{nn}t} \end{pmatrix}.
$$