Discrete Mathematics: Lecture 18

logically equivalent, rule of replacement, tautological implications

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Logically Equivalent

DEFINITION: Let A and B be WFFs in propositional variables $p_1, \ldots, p_n$.

- A and B are **logically equivalent** (等值) if they always have the same truth value for every truth assignment (of $p_1, \ldots, p_n$)
  - Notation: $A \equiv B$

THEOREM: $A \equiv B$ if and only if $A \leftrightarrow B$ is a tautology.

- $A \equiv B$
- iff for any truth assignment, A, B take the same truth values
- iff for any truth assignment, $A \leftrightarrow B$ is true
- iff $A \leftrightarrow B$ is a tautology

THEOREM: $A \equiv A$; If $A \equiv B$, then $B \equiv A$; If $A \equiv B, B \equiv C$, then $A \equiv C$

QUESTION: How to prove $A \equiv B$?

Proving $A \equiv B$

EXAMPLE: $P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$ **//distributive law**

- Idea: Show that A, B have the same truth table.

P	Q	R	$Q \vee R$	$P \wedge (Q \vee R)$	$P \wedge Q$	$P \wedge R$	$(P \wedge Q) \vee (P \wedge R)$
T	**T**	**T**	**T**	**T**	**T**	**T**	**T**
T	**T**	**F**	**T**	**T**	**T**	**F**	**T**
T	**F**	**T**	**T**	**T**	**F**	**T**	**T**
T	**F**	**F**	**F**	**F**	**F**	**F**	**F**
F	**T**	**T**	**T**	**F**	**F**	**F**	**F**
F	**T**	**F**	**T**	**F**	**F**	**F**	**F**
F	**F**	**T**	**T**	**F**	**F**	**F**	**F**
F	**F**	**F**	**F**	**F**	**F**	**F**	**F**

REMARK: $P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$ can be shown similarly.

Logical Equivalences

Name	Logical Equivalences	NO.
Double Negation Law 双重否定律	$\neg(\neg P) \equiv P$	1
Identity Laws 同一律	$P \wedge \mathbf{T} \equiv P$	2
	$P \vee \mathbf{F} \equiv P$	3
Idempotent Laws 等幂律	$P \vee P \equiv P$	4
	$P \wedge P \equiv P$	5
Domination Laws 零律	$P \vee \mathbf{T} \equiv \mathbf{T}$	6
	$P \wedge \mathbf{F} \equiv \mathbf{F}$	7
Negation Laws 补余律	$P \vee \neg P \equiv \mathbf{T}$	8
	$P \wedge \neg P \equiv \mathbf{F}$	9

Logical Equivalences

Name	Logical Equivalences	NO.
Commutative Laws 交换律	$P \vee Q \equiv Q \vee P$	10
	$P \wedge Q \equiv Q \wedge P$	11
Associative Laws 结合律	$P \vee (Q \vee R) \equiv (P \vee Q) \vee R$	12
	$P \wedge (Q \wedge R) \equiv (P \wedge Q) \wedge R$	13
Distributive Laws 分配律	$P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$	14
	$P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$	15
De Morgan’s Laws 摩根律	$\neg(P \wedge Q) \equiv (\neg P) \vee (\neg Q)$	16
	$\neg(P \vee Q) \equiv (\neg P) \wedge (\neg Q)$	17
Absorption Laws 吸收律	$P \vee (P \wedge Q) \equiv P$	18
	$P \wedge (P \vee Q) \equiv P$	19

Logical Equivalences

Name	Logical Equivalences	NO.
Laws Involving Implication $\rightarrow$	$P \rightarrow Q \equiv \neg P \vee Q$	20
	$P \rightarrow Q \equiv \neg Q \rightarrow \neg P$	21
	$(P \rightarrow R) \wedge (Q \rightarrow R) \equiv (P \vee Q) \rightarrow R$	22
	$P \rightarrow (Q \rightarrow R) \equiv (P \wedge Q) \rightarrow R$	23
	$P \rightarrow (Q \rightarrow R) \equiv Q \rightarrow (P \rightarrow R)$	24
Laws Involving Bi-Implication $\leftrightarrow$	$P \leftrightarrow Q \equiv (P \rightarrow Q) \wedge (Q \rightarrow P)$	25
	$P \leftrightarrow Q \equiv (\neg P \vee Q) \wedge (P \vee \neg Q)$	26
	$P \leftrightarrow Q \equiv (P \wedge Q) \vee (\neg P \wedge \neg Q)$	27
	$P \leftrightarrow Q \equiv \neg P \leftrightarrow \neg Q$	28

Proving $A \equiv B$

Rule of Replacement: (替换规则) Replacing a sub-formula in a formula F with a logically equivalent sub-formula gives a formula logically equivalent to the formula F.

EXAMPLE: $P \rightarrow Q \equiv \neg Q \rightarrow \neg P$

$$P \rightarrow Q \equiv \neg P \vee Q \equiv Q \vee \neg P \equiv \neg(\neg Q) \vee \neg P \equiv \neg Q \rightarrow \neg P$$

EXAMPLE: $P \leftrightarrow Q \equiv (\neg P \vee Q) \wedge (P \vee \neg Q)$

$$P \leftrightarrow Q \equiv (P \rightarrow Q) \wedge (Q \rightarrow P) \equiv (\neg P \vee Q) \wedge (\neg Q \vee P)$$
$$\equiv (\neg P \vee Q) \wedge (P \vee \neg Q)$$

EXAMPLE: $P \rightarrow (Q \rightarrow R) \equiv (P \wedge Q) \rightarrow R$

$$P \rightarrow (Q \rightarrow R) \equiv \neg P \vee (\neg Q \vee R) \equiv (\neg P \vee \neg Q) \vee R \equiv \neg(P \wedge Q) \vee R$$
$$\equiv (P \wedge Q) \rightarrow R$$

Logically Equivalent

THEOREM: Let $A^{-1}(\mathbf{T})$ be the set of truth assignments such that A is true. Then $A \equiv B$ if and only if $A^{-1}(\mathbf{T}) = B^{-1}(\mathbf{T})$.

- $A \equiv B$ if and only if $A^{-1}(\mathbf{F}) = B^{-1}(\mathbf{F})$

Proving $A \equiv B$

EXAMPLE: $P \wedge Q \equiv Q \wedge P$ **//commutative law**

- Idea: Show that $A^{-1}(\mathbf{T}) = B^{-1}(\mathbf{T})$.
- $A = P \wedge Q; B = Q \wedge P$
  - $A = \mathbf{T}$ if and only if $(P, Q) = (\mathbf{T}, \mathbf{T})$
    - $A^{-1}(\mathbf{T}) = \{(\mathbf{T}, \mathbf{T})\}$
  - $B = \mathbf{T}$ if and only if $(Q, P) = (\mathbf{T}, \mathbf{T})$
    - $B^{-1}(\mathbf{T}) = \{(\mathbf{T}, \mathbf{T})\}$
- $A^{-1}(\mathbf{T}) = B^{-1}(\mathbf{T})$
- $A \equiv B$

REMARK: $P \wedge (Q \wedge R) \equiv (P \wedge Q) \wedge R$ can be shown similarly.

- **Associative law**

Proving $A \equiv B$

EXAMPLE: $P \vee Q \equiv Q \vee P$ **//commutative law**

- Idea: Show that $A^{-1}(\mathbf{F}) = B^{-1}(\mathbf{F})$.
- $A = P \vee Q; B = Q \vee P$
  - $A = \mathbf{F}$ if and only if $(P, Q) = (\mathbf{F}, \mathbf{F})$
    - $A^{-1}(\mathbf{F}) = \{(\mathbf{F}, \mathbf{F})\}$
  - $B = \mathbf{F}$ if and only if $(Q, P) = (\mathbf{F}, \mathbf{F})$
    - $B^{-1}(\mathbf{F}) = \{(\mathbf{F}, \mathbf{F})\}$
- $A^{-1}(\mathbf{F}) = B^{-1}(\mathbf{F})$
- $A \equiv B$

REMARK: $P \vee (Q \vee R) \equiv (P \vee Q) \vee R$ can be shown similarly.

- **Associative law**

Tautological Implications

DEFINITION: Let A and B be WFFs in propositional variables $p_1, \ldots, p_n$.

- A **tautologically implies** (重言蕴涵) B if every truth assignment that causes A to be true causes B to be true.
  - Notation: $A \Rightarrow B$, called a **tautological implication**
  - $A^{-1}(\mathbf{T}) \subseteq B^{-1}(\mathbf{T})$; $B^{-1}(\mathbf{F}) \subseteq A^{-1}(\mathbf{F})$

THEOREM: $A \Rightarrow B$ iff $A \to B$ is a tautology.

- $A \Rightarrow B$ iff $A^{-1}(\mathbf{T}) \subseteq B^{-1}(\mathbf{T})$ iff $A \to B$ is a tautology

THEOREM: $A \Rightarrow B$ iff $A \land \neg B$ is a contradiction.

- $A \to B \equiv \neg A \lor B \equiv \neg(A \land \neg B)$

Proving $\boldsymbol{A \Rightarrow B}$: (1) $A^{-1}(\mathbf{T}) \subseteq B^{-1}(\mathbf{T})$; (2) $B^{-1}(\mathbf{F}) \subseteq A^{-1}(\mathbf{F})$;

(3) $A \to B$ is a tautology; (4) $A \land \neg B$ is a contradiction

Proving $A \Rightarrow B$

EXAMPLE: Show the tautological implication "$p \wedge (p \rightarrow q) \Rightarrow q$".

- Let $A = p \wedge (p \rightarrow q)$; $B = q$. Need to show that "$A \Rightarrow B$"
- $A^{-1}(\mathbf{T}) = \{(\mathbf{T}, \mathbf{T})\}$; $B^{-1}(\mathbf{T}) = \{(\mathbf{T}, \mathbf{T}), (\mathbf{F}, \mathbf{T})\}$: $A^{-1}(\mathbf{T}) \subseteq B^{-1}(\mathbf{T})$.

p	q	$p \rightarrow q$	A	B
T	**T**	**T**	<u>**T**</u>	<u>**T**</u>
T	**F**	**F**	**F**	**F**
F	**T**	**T**	**F**	**T**
F	**F**	**T**	**F**	**F**

- $A \rightarrow B \equiv \neg\big(p \wedge (p \rightarrow q)\big) \vee q$
  $\equiv \big(\neg p \vee \neg(p \rightarrow q)\big) \vee q$
  $\equiv (\neg p \vee q) \vee \neg(p \rightarrow q)$
  $\equiv (p \rightarrow q) \vee \neg(p \rightarrow q)$
  $\equiv \mathbf{T}$
- $A \wedge \neg B \equiv \big(p \wedge (p \rightarrow q)\big) \wedge \neg q$
  $\equiv (\neg q \wedge p) \wedge (p \rightarrow q)$
  $\equiv \neg(p \rightarrow q) \wedge (p \rightarrow q)$
  $\equiv \mathbf{F}$

Tautological Implications

Name	Tautological Implication	NO.
Conjunction(合取)	$(P) \wedge (Q) \Rightarrow P \wedge Q$	1
Simplification(化简)	$P \wedge Q \Rightarrow P$	2
Addition(附加)	$P \Rightarrow P \vee Q$	3
Modus ponens(假言推理)	$P \wedge (P \rightarrow Q) \Rightarrow Q$	4
Modus tollens(拒取)	$\neg Q \wedge (P \rightarrow Q) \Rightarrow \neg P$	5
Disjunctive syllogism(析取三段论)	$\neg P \wedge (P \vee Q) \Rightarrow Q$	6
Hypothetical syllogism(假言三段论)	$(P \rightarrow Q) \wedge (Q \rightarrow R) \Rightarrow (P \rightarrow R)$	7
Resolution (归结)	$(P \vee Q) \wedge (\neg P \vee R) \Rightarrow Q \vee R$	8

Proofs for 5 and 6

EXAMPLE: $\neg Q \wedge (P \rightarrow Q) \Rightarrow \neg P$

- $A = \neg Q \wedge (P \rightarrow Q), B = \neg P.$
- $A \rightarrow B \equiv \neg\big(\neg Q \wedge (P \rightarrow Q)\big) \vee \neg P$

$$\equiv \big(Q \vee \neg(P \rightarrow Q)\big) \vee \neg P$$
$$\equiv (\neg P \vee Q) \vee \neg(P \rightarrow Q)$$
$$\equiv \mathbf{T}$$

EXAMPLE: $\neg P \wedge (P \vee Q) \Rightarrow Q$

- $A = \neg P \wedge (P \vee Q), B = Q.$
- $A \rightarrow B \equiv \neg\big(\neg P \wedge (P \vee Q)\big) \vee Q$

$$\equiv \big(P \vee \neg(P \vee Q)\big) \vee Q$$
$$\equiv (\neg(P \vee Q) \vee P) \vee Q$$
$$\equiv \neg(P \vee Q) \vee (P \vee Q)$$
$$\equiv \mathbf{T}$$

Proofs for 7 and 8

EXAMPLE: $(P \to Q) \wedge (Q \to R) \Rightarrow P \to R$

- $A = (P \to Q) \wedge (Q \to R);\ B = (P \to R).$
- $A \wedge \neg B \equiv (\neg P \vee Q) \wedge (\neg Q \vee R) \wedge (P \wedge \neg R)$

$$\equiv \big((\neg P \vee Q) \wedge P\big) \wedge \big((\neg Q \vee R) \wedge \neg R\big)$$

$$\equiv \big((\neg P \wedge P) \vee (Q \wedge P)\big) \wedge \big((\neg Q \wedge \neg\, R) \vee (R \wedge \neg R)\big)$$

$$\equiv (Q \wedge P) \wedge (\neg Q \wedge \neg R)$$

$$\equiv \mathbf{F}$$

EXAMPLE: $(P \vee Q) \wedge (\neg P \vee R) \Rightarrow Q \vee R$

- $A = (P \vee Q) \wedge (\neg P \vee R); B = (Q \vee R).$
- $A \wedge \neg B \equiv (P \vee Q) \wedge (\neg P \vee R) \wedge (\neg Q \wedge \neg R)$

$$\equiv \big((P \vee Q) \wedge \neg Q\big) \wedge ((\neg P \vee R) \wedge \neg R)$$

$$\equiv (P \wedge \neg Q) \wedge (\neg P \wedge \neg R)$$

$$\equiv \mathbf{F}$$

More Examples

EXAMPLE: $(P \leftrightarrow Q) \wedge (Q \leftrightarrow R) \Rightarrow (P \leftrightarrow R)$

- $A = (P \leftrightarrow Q) \wedge (Q \leftrightarrow R); B = (P \leftrightarrow R).$
- $A = \mathbf{T}$ iff $(P \leftrightarrow Q) = \mathbf{T}$ and $(Q \leftrightarrow R) = \mathbf{T}$ iff $P = Q$ and $Q = R$
  - $A^{-1}(\mathbf{T}) = \{(\mathbf{T}, \mathbf{T}, \mathbf{T}), (\mathbf{F}, \mathbf{F}, \mathbf{F})\}$
- $B = \mathbf{T}$ iff $P = R$
  - $B^{-1}(\mathbf{T}) = \{(\mathbf{T}, \mathbf{T}, \mathbf{T}), (\mathbf{T}, \mathbf{F}, \mathbf{T}), (\mathbf{F}, \mathbf{T}, \mathbf{F}), (\mathbf{F}, \mathbf{F}, \mathbf{F})\}$
- $A^{-1}(\mathbf{T}) \subseteq B^{-1}(\mathbf{T})$

EXAMPLE: $(Q \rightarrow R) \Rightarrow ((P \vee Q) \rightarrow (P \vee R))$

- $A = Q \rightarrow R; B = ((P \vee Q) \rightarrow (P \vee R)).$
- $A = \mathbf{F}$ iff $(Q, R) = (\mathbf{T}, \mathbf{F})$
  - $A^{-1}(\mathbf{F}) = \{(\mathbf{T}, \mathbf{T}, \mathbf{F}), (\mathbf{F}, \mathbf{T}, \mathbf{F})\}$
- $B = \mathbf{F}$ iff $(P \vee Q, P \vee R) = (\mathbf{T}, \mathbf{F})$ iff $(P, Q) \neq (\mathbf{F}, \mathbf{F})$ and $(P, R) = (\mathbf{F}, \mathbf{F})$
  - $B^{-1}(\mathbf{F}) = \{(\mathbf{F}, \mathbf{T}, \mathbf{F})\}$
- $A^{-1}(\mathbf{F}) \supseteq B^{-1}(\mathbf{F})$

More Examples

EXAMPLE: $(P \rightarrow R) \wedge (Q \rightarrow S) \wedge (P \vee Q) \Rightarrow R \vee S$

- $A = (P \rightarrow R) \wedge (Q \rightarrow S) \wedge (P \vee Q); B = R \vee S$
- $A \wedge \neg B \equiv (P \rightarrow R) \wedge (Q \rightarrow S) \wedge (P \vee Q) \wedge \neg(R \vee S)$

$$
\begin{aligned}
&\equiv (\neg P \vee R) \wedge (\neg Q \vee S) \wedge (P \vee Q) \wedge (\neg R \wedge \neg S) \\
&\equiv ((\neg P \vee R) \wedge \neg R)) \wedge ((\neg Q \vee S) \wedge \neg S) \wedge (P \vee Q) \\
&\equiv ((\neg P \wedge \neg R) \vee (R \wedge \neg R)) \wedge ((\neg Q \wedge \neg S) \vee (S \wedge \neg S)) \wedge (P \vee Q) \\
&\equiv ((\neg P \wedge \neg R) \vee \mathbf{F}) \wedge ((\neg Q \wedge \neg S) \vee \mathbf{F}) \wedge (P \vee Q) \\
&\equiv (\neg P \wedge \neg R) \wedge (\neg Q \wedge \neg S) \wedge (P \vee Q) \\
&\equiv \neg R \wedge (\neg Q \wedge \neg S) \wedge (\neg P \wedge (P \vee Q)) \\
&\equiv \neg R \wedge (\neg Q \wedge \neg S) \wedge ((\neg P \wedge P) \vee (\neg P \wedge Q)) \\
&\equiv \neg R \wedge (\neg Q \wedge \neg S) \wedge (\mathbf{F} \vee (\neg P \wedge Q)) \\
&\equiv \neg R \wedge (\neg Q \wedge \neg S) \wedge (\neg P \wedge Q) \\
&\equiv \neg R \wedge \neg S \wedge \neg P \wedge (\neg Q \wedge Q) \\
&\equiv \neg R \wedge \neg S \wedge \neg P \wedge \mathbf{F} \\
&\equiv \mathbf{F}
\end{aligned}
$$