Lecture 7
- Phasor 相量

A beginning of AC circuits

AC = alternating current;
Circuits driven by sinusoidal current or voltage sources are *AC circuits*

Outline

- Sinusoidal signals
- Phasor

Sinusoidal Signal (Current or Voltage)

$$v(t) = V_m \cos(\omega t + \theta)$$

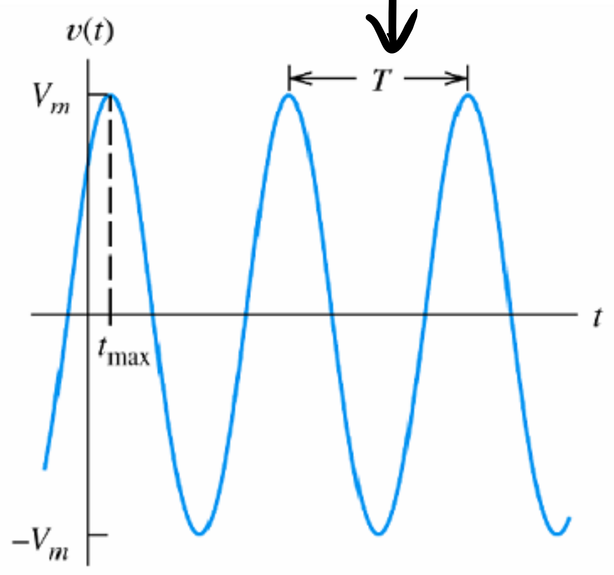
① ② ③

V_m is the **peak value** >0

ω is the **angular frequency** in radians per second >0 rad/s

$(\omega t + \theta)$ is the **phase angle**

θ initial phase angle



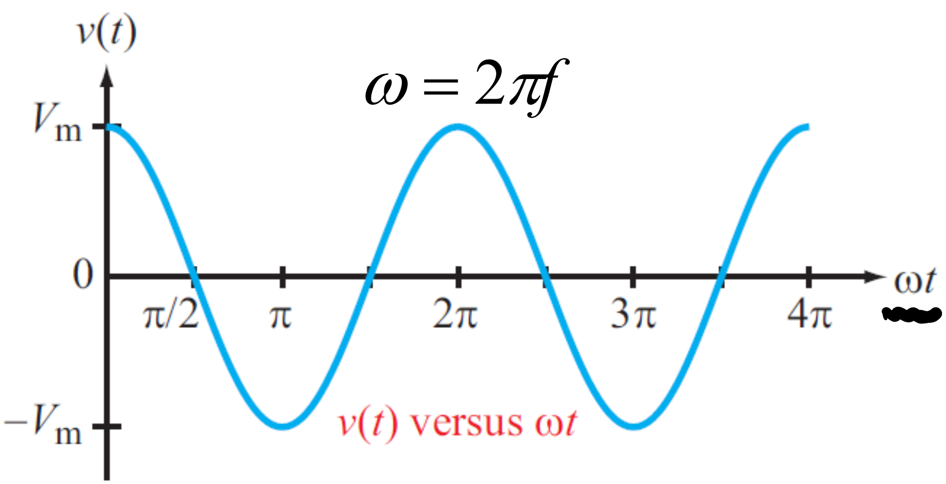
$$\omega = 2\pi f \qquad f = \frac{1}{T}$$

$|\theta| \leq 180°$

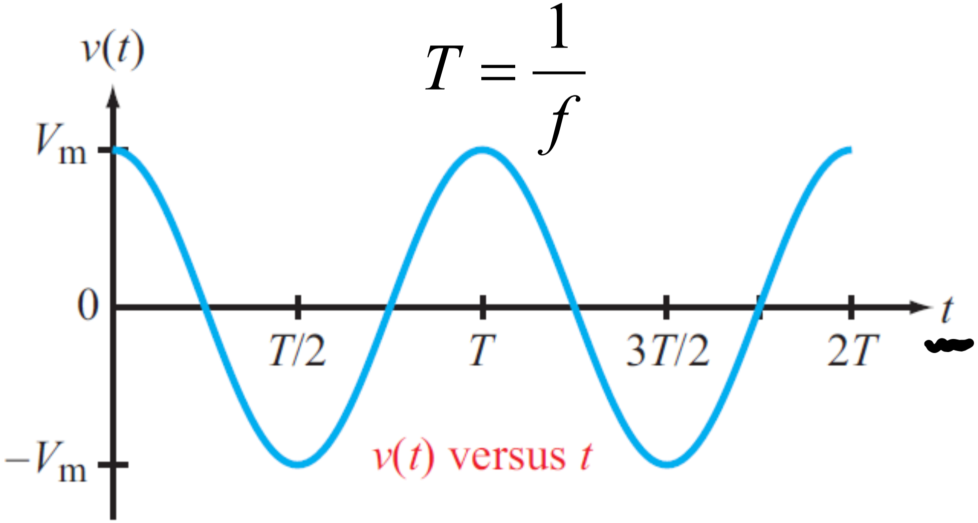
Sinusoidal Signals

$$v(t) = V_m \cos(\omega t + \theta)$$

$$\omega = 2\pi f$$



$$T = \frac{1}{f}$$



Useful relations

$$\sin x = \pm\cos(x \mp 90^\circ)$$
$$\cos x = \pm\sin(x \pm 90^\circ)$$
$$\sin x = -\sin(x \pm 180^\circ)$$
$$\cos x = -\cos(x \pm 180^\circ)$$
$$\sin(-x) = -\sin x$$
$$\cos(-x) = \cos x$$

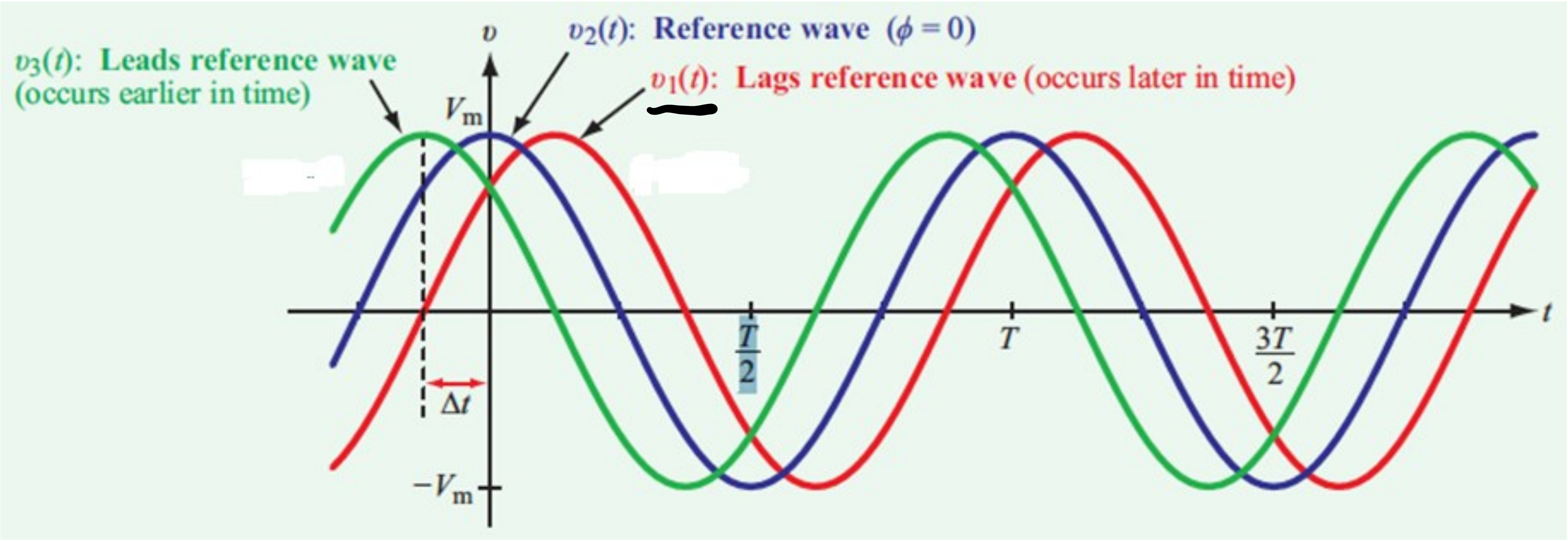
$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$$
$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$$

$$2\sin x \sin y = \cos(x - y) - \cos(x + y)$$
$$2\sin x \cos y = \sin(x + y) + \sin(x - y)$$
$$2\cos x \cos y = \cos(x + y) + \cos(x - y)$$

Phase Lead/Lag

$$V_m \cos\frac{2\pi t}{T} \qquad V_m \cos\left(\frac{2\pi t}{T} - \frac{\pi}{4}\right) \qquad V_m \cos\left(\frac{2\pi t}{T} + \frac{\pi}{4}\right)$$

2 1 3 Leads



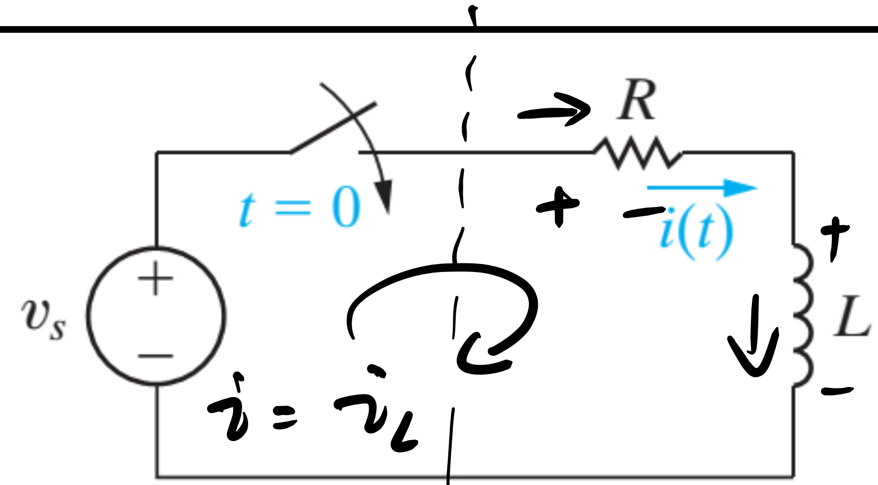
[Source: Berkeley]

Why Sinusoidal signals?

- Numbers of natural phenomenon are sinusoidal in nature.
  - Motion of a pendulum, vibration of a string, ripples on ocean surface
- A very easy signal to generate and transmit
  - Dominant form of signal in communication/electric power industries
  - In the late 1800's there was a battle between proponents of DC and AC. AC won out due to its efficiency for long distance transmission.
- Lastly, they are very easy to handle mathematically.
  - Derivative and integral are also sinusoids.
- Through *Fourier analysis*, any practical periodic function can be represented as sum of sinusoids.

[Source: Berkeley]

The Sinusoidal Response



$v_s = V_m \cos(\omega t + \phi)$, $i(0^-) = 0$.

Find $i(t)$, $t \geq 0$.

$V_L + V_R = V_s$

$$\Rightarrow \boxed{L\frac{di}{dt} + Ri = V_m \cos(\omega t + \phi)}$$

Ordinary differential equation

$$\frac{di}{dt} + \frac{R}{L} i = \frac{V_m}{L} \cos(\omega t + \phi)$$

G.S. : $i' = A \cdot e^{-\frac{R}{L} \cdot t}$

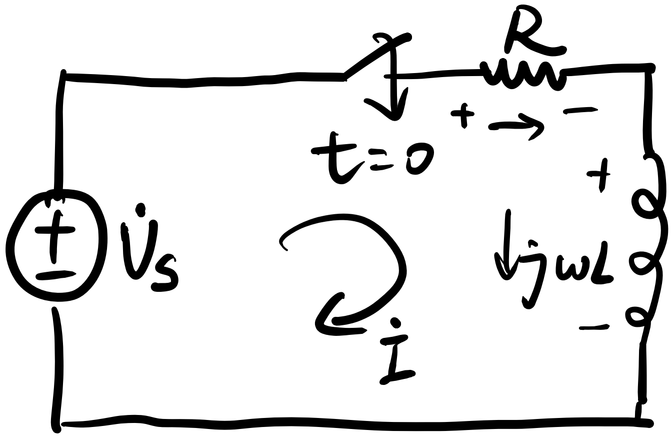
P.S. : $i'' = B \cdot \cos(\omega t + C)$

$$-B\omega\sin(\omega t+C)+\frac{R}{L}\cdot B\cdot\cos(\omega t+C)$$
$$=\frac{V_m}{L}\cos(\omega t+\phi)$$

$$\Rightarrow\begin{cases} B=\dfrac{V_m}{\sqrt{R^2+\omega^2L^2}} \\ C=\phi-\arctan\dfrac{\omega L}{R}\end{cases}$$

$$i(t)=i'+i''$$
$$=A\cdot e^{-R/L\cdot t}+B\cdot\cos(\omega t+C)$$

$i(t=0)=0$ to determine A.



$$v_s(t) = V_m \cos(\omega t + \phi)$$

$$\dot{V}_s = V_m \angle \phi$$

$$Z_R = R$$

$$Z_L = j\omega L$$

$$\dot{V}_s = \dot{I} \cdot R + \dot{I} \cdot j\omega L$$

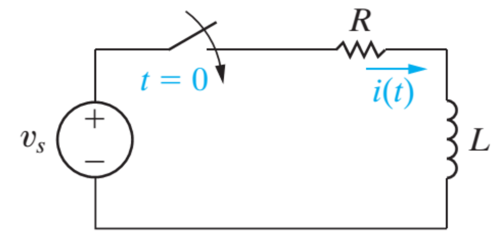
$$V_m \angle \phi = \dot{I}(R + j\omega L)$$

$$\Rightarrow \quad \dot{I} = \frac{V_m \angle \phi}{R + j\omega L}$$

$$= \frac{V_m}{\sqrt{R^2 + \omega^2 L^2}} \angle\left(\phi - \arctan\frac{\omega L}{R}\right)$$

$$\Rightarrow i(t) = \frac{V_m}{\sqrt{\cdots}} \cos\left(\omega t + \phi - \arctan\frac{\omega L}{R}\right)$$

Sinusoidal Steady-State Response



$$v_s = V_m \cos(\omega t + \phi)$$

$$i = \boxed{\frac{-V_m}{\sqrt{R^2 + \omega^2 L^2}} \cos(\phi - \theta) e^{-(R/L)t}} + \boxed{\frac{V_m}{\sqrt{R^2 + \omega^2 L^2}} \cos(\omega t + \phi - \theta)}$$

Transient response

Steady-state response

- Steady-state solution is sinusoidal
- Response frequency = source frequency
- **Magnitude & phase** (initial phase angle) of S.S. response differs from that of source

Outline

- Sinusoidal signals
- Phasor

Phasor

$$\mathrm{Re}\{V \cdot e^{j(\omega t+\phi)}\}$$

$$v(t) = V\cos(\omega t + \phi) = \underline{\mathrm{Re}\{Ve^{j\phi}e^{j\omega t}\}} = \mathrm{Re}(\dot{\mathbf{V}}e^{j\omega t})$$

$$\boxed{\dot{\mathbf{V}} = Ve^{j\phi}}$$

$$\dot{V} = V \cdot e^{j\phi}$$

- As Euler's formula:

$$e^{\pm j\phi} = \cos\phi \pm j\sin\phi$$

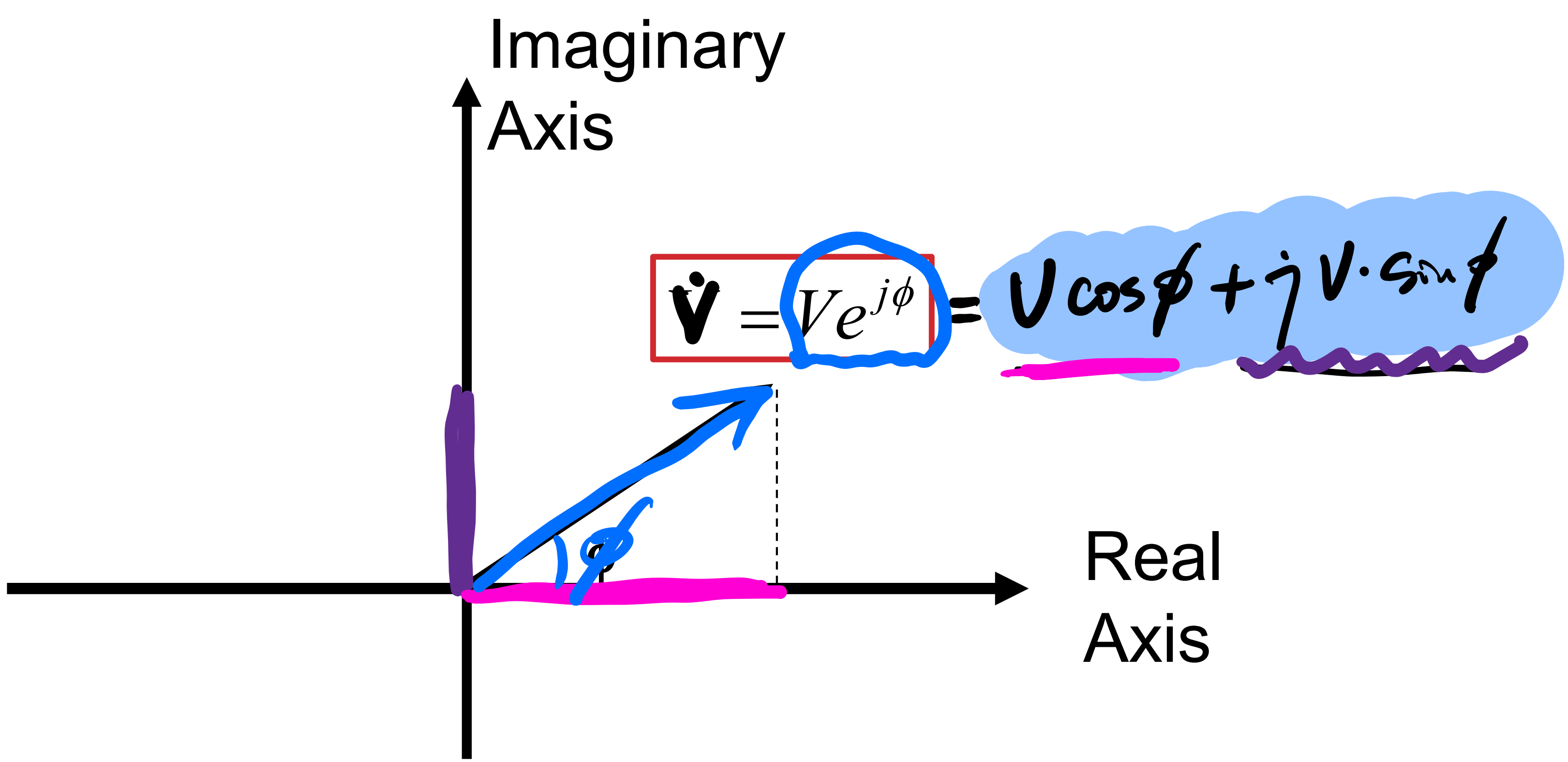
$$\cos\phi = \mathrm{Re}(e^{j\phi})$$

$$\sin\phi = \mathrm{Im}(e^{j\phi})$$

Phasor:

$$v(t) = V\cos(\omega t + \phi) = \mathrm{Re}\left\{Ve^{j\phi}e^{j\omega t}\right\} = \mathrm{Re}\left(\mathbf{V}e^{j\omega t}\right) \qquad \boxed{\mathbf{V} = Ve^{j\phi}}$$

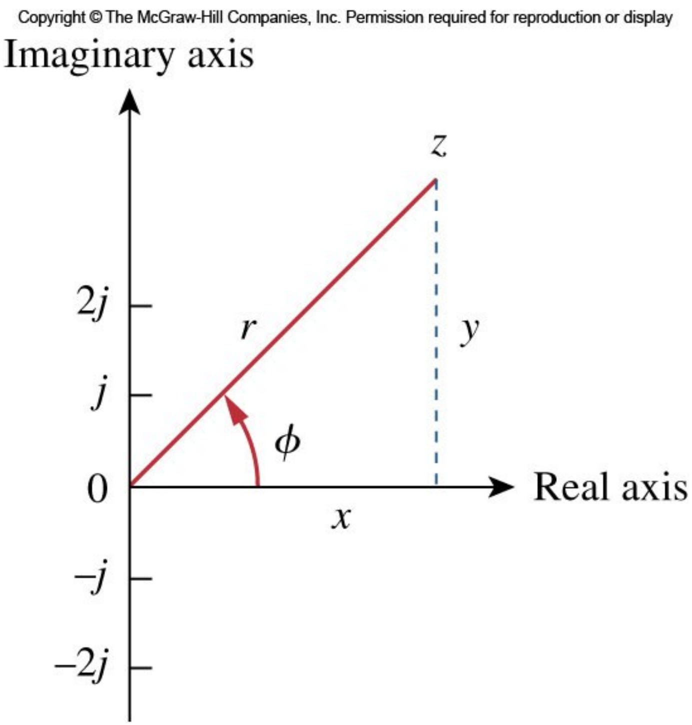
Complex representation of the magnitude and phase of a **sinusoid**



Phasor : Complex Numbers

- A powerful method for representing sinusoids is the phasor, a complex expression as well.
- A complex number z can be represented in *rectangular form* as:

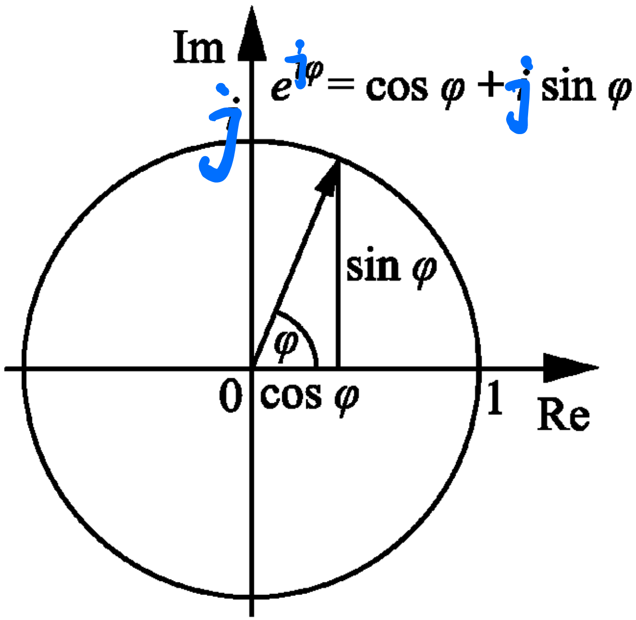
$$z = x + jy \qquad \mathrm{Re}(z) = x \qquad \mathrm{Im}(z) = y$$



- It can also be written in *polar* or *exponential* form as:

$$z = r\angle\phi = re^{j\phi}$$

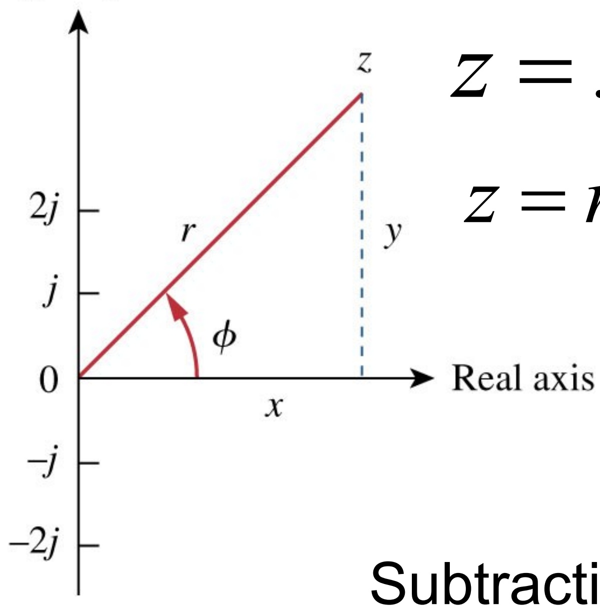
$r = \sqrt{x^2 + y^2}$



$$1 = 1e^{j0} = 1\angle 0°$$

$$j = 1e^{j\frac{\pi}{2}} = 1\angle 90°$$

Arithmetic With Complex Numbers



$$z = x + jy$$

$$z = r\angle\phi = re^{j\phi}$$

Addition

$$z_1 + z_2 = (x_1 + x_2) + j(y_1 + y_2)$$

Subtraction

$$z_1 - z_2 = (x_1 - x_2) + j(y_1 - y_2)$$

Multiplication

$$z_1 z_2 = r_1 r_2 \angle(\phi_1 + \phi_2)$$

Division

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\angle(\phi_1 - \phi_2)$$

Reciprocal

$$\frac{1}{z} = \frac{1}{r}\angle(-\phi)$$

Square Root

$$\boxed{\mathbf{z}^{1/2} = \pm|\mathbf{z}|^{1/2} e^{j\theta/2}}$$

Complex Conjugate

$$z^* = x - jy = r\angle -\phi = re^{-j\phi}$$

Relations for Complex Numbers

Euler's Identity: $e^{j\theta} = \cos\theta + j\sin\theta$

$\sin\theta = \dfrac{e^{j\theta} - e^{-j\theta}}{2j}$	$\cos\theta = \dfrac{e^{j\theta} + e^{-j\theta}}{2}$
$\mathbf{z} = x + jy = \lvert\mathbf{z}\rvert e^{j\theta}$	$\mathbf{z}^* = x - jy = \lvert\mathbf{z}\rvert e^{-j\theta}$
$x = \mathfrak{Re}(\mathbf{z}) = \lvert\mathbf{z}\rvert\cos\theta$	$\lvert\mathbf{z}\rvert = \sqrt[+]{\mathbf{z}\mathbf{z}^*} = \sqrt[+]{x^2 + y^2}$
$y = \mathfrak{Im}(\mathbf{z}) = \lvert\mathbf{z}\rvert\sin\theta$	$\theta = \tan^{-1}(y/x)$
$\mathbf{z}^n = \lvert\mathbf{z}\rvert^n e^{jn\theta}$	$\mathbf{z}^{1/2} = \pm\lvert\mathbf{z}\rvert^{1/2} e^{j\theta/2}$
$\mathbf{z}_1 = x_1 + jy_1$	$\mathbf{z}_2 = x_2 + jy_2$
$\mathbf{z}_1 = \mathbf{z}_2$ iff $x_1 = x_2$ and $y_1 = y_2$	$\mathbf{z}_1 + \mathbf{z}_2 = (x_1 + x_2) + j(y_1 + y_2)$
$\mathbf{z}_1\mathbf{z}_2 = \lvert\mathbf{z}_1\rvert\lvert\mathbf{z}_2\rvert e^{j(\theta_1+\theta_2)}$	$\dfrac{\mathbf{z}_1}{\mathbf{z}_2} = \dfrac{\lvert\mathbf{z}_1\rvert}{\lvert\mathbf{z}_2\rvert} e^{j(\theta_1-\theta_2)}$
$-1 = e^{j\pi} = e^{-j\pi} = 1\angle\pm180^\circ$	
$j = e^{j\pi/2} = 1\angle 90^\circ$	$-j = e^{-j\pi/2} = 1\angle -90^\circ$
$\sqrt{j} = \pm e^{j\pi/4} = \pm\dfrac{(1+j)}{\sqrt{2}}$	$\sqrt{-j} = \pm e^{-j\pi/4} = \pm\dfrac{(1-j)}{\sqrt{2}}$

[Source: Berkeley]

Example

- Evaluate these complex numbers

(a) $(40\angle 50^\circ + 20\angle -30^\circ)^{1/2}$

$\pm|47.72|^{\frac{1}{2}} \angle \frac{25.62}{2}$

(b) $\dfrac{10\angle -30^\circ + (3 - j4)}{(2 + j4)(3 - j5)^*}$

$0.565 \angle -160.13^\circ$

Exercise

- Evaluate the following complex numbers

(a) $[(5 + j2)(-1 + j4) - 5\angle 60^\circ]^*$

(b) $\dfrac{10 + j5 + 3\angle 40^\circ}{-3 + j4} + 10\angle 30^\circ + j5$

Phasors

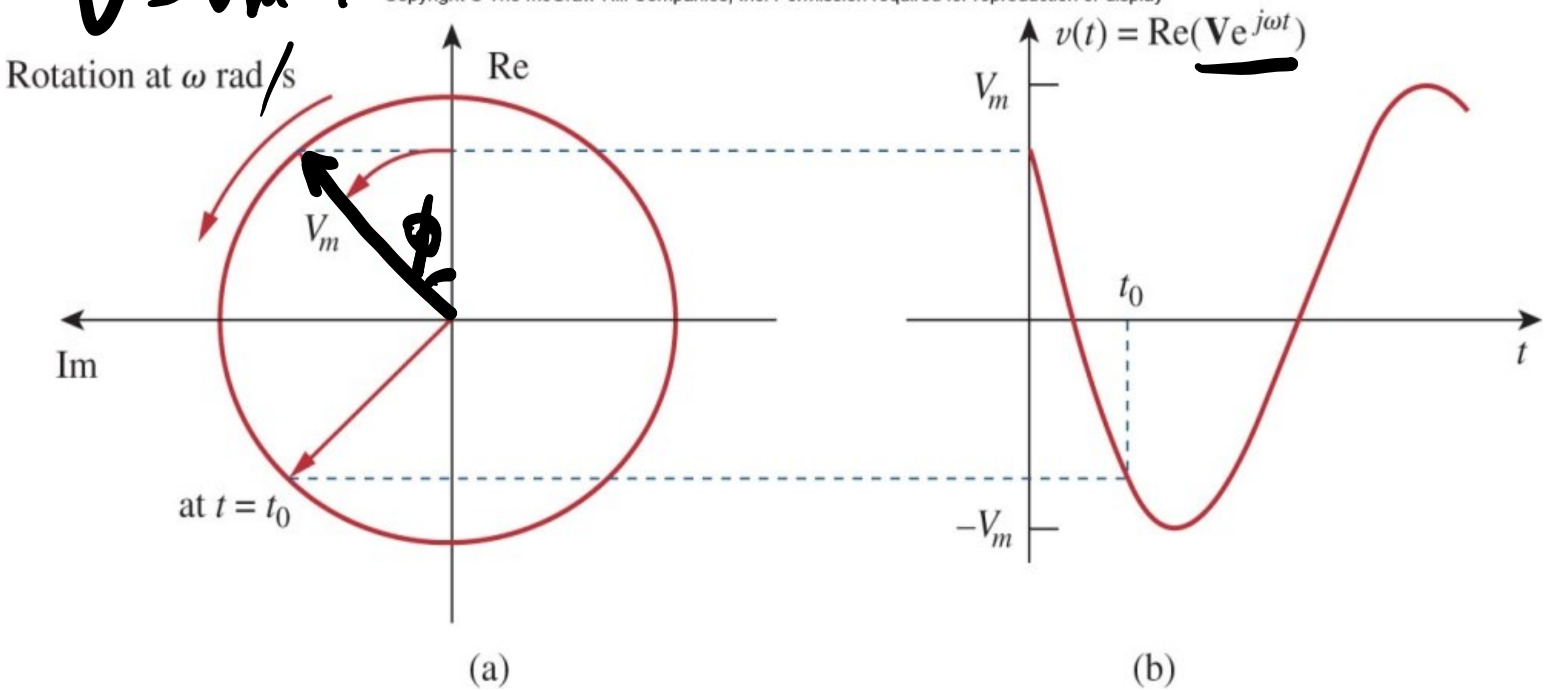
相量 (ω)

$$v(t) = V_m \cos(\omega t + \phi) \quad \Leftrightarrow \quad \dot{\mathbf{V}} = V_m \angle \phi$$

(Time-domain representation) (Phasor-domain representation)

$\dot{V} = V_m \angle \phi$

$= V_m \cos(\omega t + \phi)$



Example

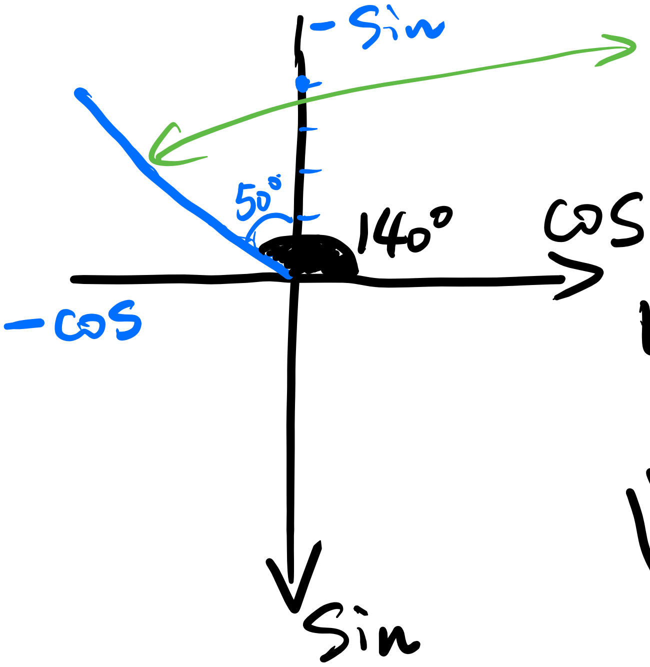
- Transform these sinusoids to phasors

(a) $i = \underline{6}\cos(\underline{50}t \underline{- 40^\circ})$ A
(b) $v = -4\sin(30t + 50^\circ)$ V

(a) $i = \cdots$

$\dot{I} = 6\angle -40^\circ$ A

(b)



$$v = \underline{-4\sin}(30t + 50^\circ)$$

$$v = 4\cos(30t + 140^\circ)$$

$$\dot{V} = 4\angle 140^\circ$$

Example

- Find the sinusoids represented by these phasors

$$\mathbf{I} = -3 + j4 \text{ A}$$

$$\mathbf{V} = j8e^{-j20^\circ} \text{ V}$$

$$v(t) = 8\cos(\omega t + 70^\circ)$$

$$\mathbf{V} = -25\angle 40^\circ \text{ V}$$

$$\mathbf{I} = j(12 - j5) \text{ A}$$

$$\dot{V} = 1\angle 90^\circ \times 8\angle -20^\circ = 8\angle 70^\circ$$

Phasor Relationships for Resistors

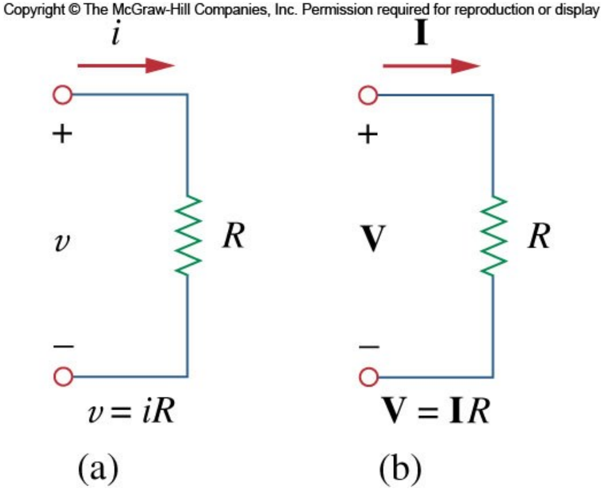
- For the resistor, the voltage and current are related via Ohm's law. As such, the voltage and current are in phase with each other.

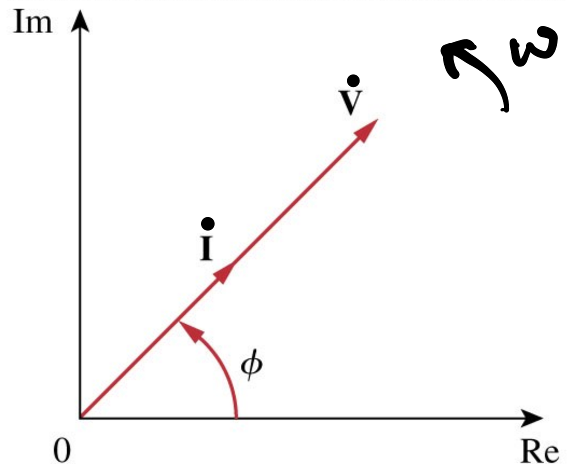
$$i = I_m \cos(\omega t + \phi) \rightarrow \dot{\mathbf{I}} = I_m \angle \phi$$

$$v = RI_m \cos(\omega t + \phi) \rightarrow \dot{\mathbf{V}} = RI_m \angle \phi$$

$v(t) = i(t) \cdot R$

$$\boxed{\dot{\mathbf{V}} = R\dot{\mathbf{I}}}$$



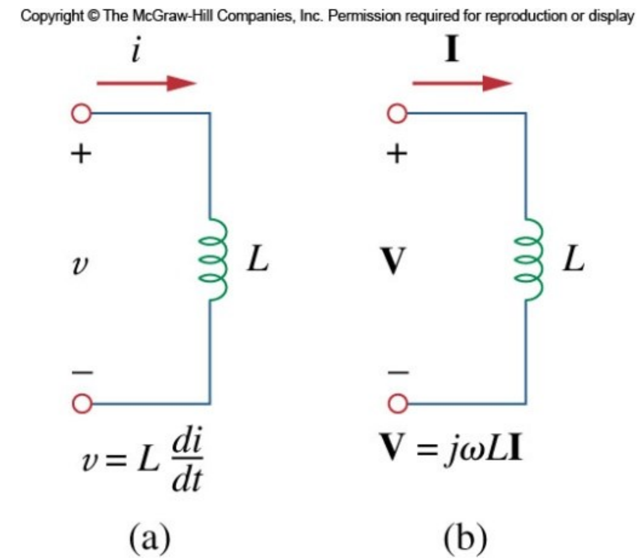


Phasor Relationships for Inductors

$$i = I_m \cos(\omega t + \phi) \rightarrow \dot{\mathbf{I}} = I_m \angle \phi$$

$$v = L\frac{di}{dt} = -\omega L I_m \sin(\omega t + \phi)$$

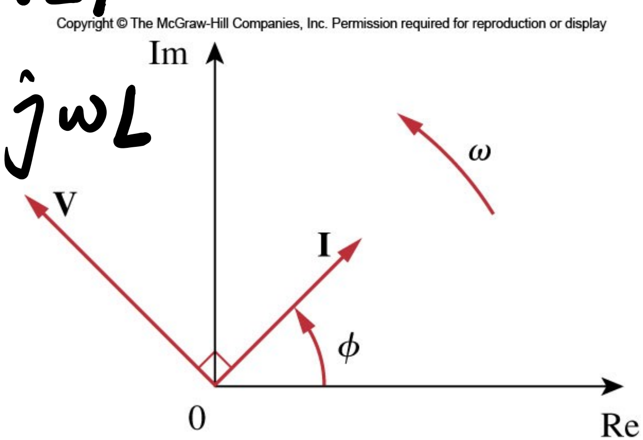
$$= \omega L I_m \cos(\omega t + \phi + 90°)$$



$$\dot{\mathbf{V}} = \omega L I_m \angle \phi + 90° = \omega L I_m e^{j(\phi + 90°)} = \omega L \underbrace{e^{j90°}}_{1\angle 90°} \cdot I_m e^{j\phi} = j\omega L \cdot \dot{\mathbf{I}}$$

$$\Rightarrow \boxed{\dot{\mathbf{V}} = j\omega L \dot{\mathbf{I}}}$$

$$Z_L = \dot{V}/\dot{I} = j\omega L$$



- The voltage leads the current by 90° (phase shift = 90°)

Phasor Relationships for Capacitors

$$v = V_m \cos(\omega t + \phi) \rightarrow \dot{\mathbf{V}} = V_m \angle \phi$$

$$\rightarrow i = C\frac{dv}{dt} = -\omega C V_m \sin(\omega t + \phi)$$

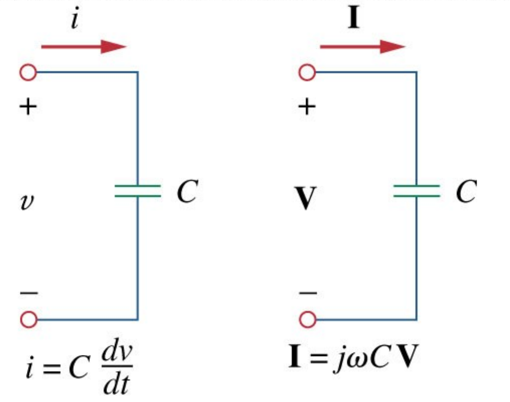
$$= \omega C V_m \cos(\omega t + \phi + 90°)$$

$$\dot{\mathbf{I}} = \omega C V_m \angle \phi + 90° = \cdots = j\omega C \cdot \dot{\mathbf{V}}$$

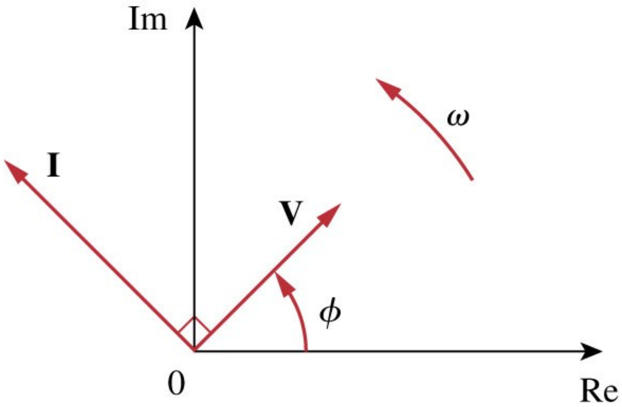
$$\Rightarrow \boxed{\dot{\mathbf{V}} = \frac{\dot{\mathbf{I}}}{j\omega C}}$$

$$Z_c = \dot{V}/\dot{I} = \frac{1}{j\omega C}$$

- The voltage lags the current by 90°.



(a) (b)



Impedance

- The voltage-current relations for R, L and C elements are

$$\mathbf{V} = R\mathbf{I} \qquad \mathbf{V} = j\omega L\mathbf{I} \qquad \mathbf{V} = \frac{\mathbf{I}}{j\omega C}$$

$$\frac{\mathbf{V}}{\mathbf{I}} = R \qquad \frac{\mathbf{V}}{\mathbf{I}} = j\omega L \qquad \frac{\mathbf{V}}{\mathbf{I}} = \frac{1}{j\omega C}$$

- Phasors allow us to express the relationship between current and voltage using a formula like Ohm's law:

$$\mathbf{V} = \mathbf{I}\,\boldsymbol{Z} \quad \text{or} \quad \mathbf{Z} = \frac{\mathbf{V}}{\mathbf{I}}$$

- **Z** is called impedance, measured in ohms.
  - <u>Impedance is not a phasor</u>! But it is (often) a complex number.
  - Impedance depends on the frequency ω.

Admittance

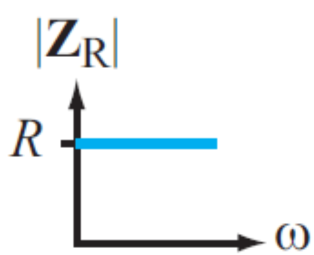
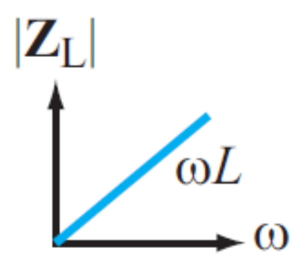
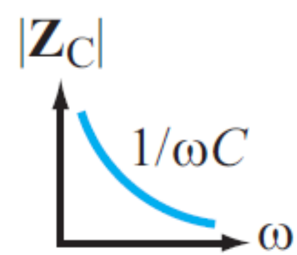
S

Admittance is simply the inverse of impedance, unit: Simens.

$$\mathbf{Y} = \frac{\mathbf{I}}{\mathbf{V}}$$

Element	Impedance	Admittance
R	$\mathbf{Z} = R$	$\mathbf{Y} = \dfrac{1}{R}$
L	$\mathbf{Z} = j\omega L$	$\mathbf{Y} = \dfrac{1}{j\omega L}$
C	$\mathbf{Z} = \dfrac{1}{j\omega C}$	$\mathbf{Y} = j\omega C$

Summary of *R, L, C*

Property	R	L	C
v–i	$v = Ri$	$v = L\dfrac{di}{dt}$	$i = C\dfrac{dv}{dt}$
V–**I**	$\mathbf{V} = R\mathbf{I}$	$\mathbf{V} = j\omega L\mathbf{I}$	$\mathbf{V} = \dfrac{\mathbf{I}}{j\omega C}$
Z	R	$j\omega L$	$\dfrac{1}{j\omega C}$
dc equivalent	R	Short circuit	Open circuit
High-frequency equivalent	R	Open circuit	Short circuit
Frequency response			

[Source: Berkeley]