CS244: Theory of Computation

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Fall 2022

Outline

Intractability

Definition (Informal definition)

Computational problems that are solvable in principle, but the solutions require so much time or space that they can't be used in practice, are called intractable.

- Some problems thought to be intractable but none that have been proven to be intractable, e.g., SAT problem and all other NP-complete problems
- We give examples of problems that we can prove to be intractable
- Hierarchy theorems: relate the power of TM to the amount of time or space available for computation
- Relativization: Oracle Turing Machines
- Circuit complexity: complexity on a simplified model of the silicon chips

Outline

Space constructible function

Definition

A function $f : \mathbb{N} \to \mathbb{N}$ where $f(n) \geq O(\log n)$, is space constructible if there is a $O(f(n))$ space TM M that given an input (e.g., 1^n) of length n writes $f(n)$ in binary on the (work) tape.

- $f(n) = n^2$ is space constructible: given 1^n, compute the binary representation of n, then multiplying n with itself. The total space used is $O(n)$
- $f(n) = \log n$ (rounded down to the next lower integer for non-integers) is space constructible: given 1^n, compute the binary representation of n, then count the number of bits of n in binary. The total space used is $O(\log n)$
- $f(n) = \begin{cases} 2n & n \in A \\ 2n+1 & n \notin A \end{cases}$ for some A in $\text{SPACE}(n^2) \setminus \text{SPACE}(n)$ is not space constructible: as determining whether $n \in A$ or not needs $O(n^2)$ space.

Space hierarchy theorem

Theorem

For any space constructible function $f : \mathbb{N} \to \mathbb{N}$, there is a language A decidable in $O(f(n))$ space, but not in $o(f(n))$ space.

- $g(n) = O(f(n))$ if positive integers c and n_0 exist such that for every integer $n \geq n_0$, $g(n) \leq c \cdot f(n)$.
- $g(n) = o(f(n))$ means that for any real number $c > 0$, a number n_0 exists, where $g(n) < c \cdot f(n)$ for all $n \geq n_0$.

Proof (1)

Theorem

For any space constructible function $f : \mathbb{N} \to \mathbb{N}$, there is a language A decidable in $O(f(n))$ space, but not in $o(f(n))$ space.

- Let $f : \mathbb{N} \to \mathbb{N}$ be a space constructible function such that $f(n) \geq O(\log n)$.
- We want to define a language A that can be solved in space $O(f(n))$ but not in space $o(f(n))$.
- We define a $O(f(n))$ space TM D that decides A.
- We use the diagonalization method: if M is a $o(f(n))$ space TM, then D guarantees that A differs from M's language

Proof (2)

To design the $O(f(n))$ space TM D that decides A, we use the diagonalization method

D on input w:

1. $n = |w|$ [$O(\log n)$ space]
2. mark off $f(n)$ space (f is space constructible) [$O(f(n))$ space]
3. if $w \neq \langle M \rangle 10^*$ for some TM M, reject [$O(f(n))$ space]
4. simulate M on w: [$d\ g(n)$ space]
5. if the number of steps $> 2^{f(n)}$, reject [$O(f(n))$ space]
6. if M tries to use more than $f(n)$ space, reject [$O(f(n))$ space]
7. if M accepts, then reject
8. if M rejects, then accept

- Suppose M uses $g(n) = o(f(n))$ space, then exists n_0 such that $d\ g(n) < f(n)$ for all $n \geq n_0$
- If $|\langle M \rangle| \geq n_0$, then $D(\langle M \rangle) \neq M(\langle M \rangle)$
- If $|\langle M \rangle| < n_0$, then $D(\langle M \rangle 10^{n_0}) \neq M(\langle M \rangle 10^{n_0})$ (10^{n_0} gives enough space to D to simulate M)

Corollary

For any two functions $f_1, f_2 : \mathbb{N} \to \mathbb{N}$, *where* $f_1(n)$ *is* $o(f_2(n))$ *and* f_2 *is space constructible,*

$$\text{SPACE}(f_1(n)) \subset \text{SPACE}(f_2(n))$$

Separate various space complexity classes:

- $\text{SPACE}(n^{c_1}) \subset \text{SPACE}(n^{c_2})$ for all natural numbers $c_1 < c_2$
- $\text{SPACE}(n^{c_1}) \subset \text{SPACE}(n^{c_2})$ for all positive rational numbers $c_1 < c_2$
- $\text{SPACE}(n^{c_1}) \subset \text{SPACE}(n^{c_2})$ for all positive real numbers $c_1 < c_2$

Corollary

$co\mathbf{NL} = \mathbf{NL} \subset \text{PSPACE}$

$\mathbf{NL} = \text{NSPACE}(\log n) \subseteq \text{SPACE}(\log^2 n)$(Savitch's theorem) $\subset$ $\text{SPACE}(n) \subset \text{PSPACE}$

Space Intractable Problems

Definition

$$\textbf{EXPSPACE} = \bigcup_{k \in \mathbb{N}} \text{SPACE}(2^{n^k})$$

$$\text{SPACE}(n^k) \subset \text{SPACE}(n^{\log n}) \subset \text{SPACE}(2^n)$$

Theorem

PSPACE $\subset$ **EXPSPACE**

Time constructible function

Definition

A function $t : \mathbb{N} \to \mathbb{N}$ where $t(n) \geq O(n \log n)$, is time constructible if there is a $O(t(n))$ time TM M that given an input (e.g., 1^n) of length n writes $t(n)$ in binary on the (work) tape.

- $t(n) = n^2$ is time constructible: given 1^n, compute the binary representation of n, then multiplying n with itself. The total time used is $O(n)$
- $t(n) = n \log n$ (rounded down to the next lower integer for non-integers) is is time constructible: given 1^n, compute the binary representation of n, then count the number of bits of n in binary. The total time used is $O(n)$
- $t(n) = \begin{cases} n^2 & n \in A \\ n^2 + 1 & n \notin A \end{cases}$ for some A in $\text{TIME}(n^3) \setminus \text{TIME}(n^2)$ is not time constructible: as determining whether $n \in A$ or not needs $O(n^3)$ time.

Time hierarchy theorem

Theorem

For any time constructible function $t : \mathbb{N} \to \mathbb{N}$, there is a language A decidable in $O(t(n))$ time, but not in $o(\frac{t(n)}{\log t(n)})$ time.

NOTE: We do not know how to prove that A is not in $o(t(n))$ time

- We construct a $O(t(n))$ time TM D that decides a language A
- show that A cannot be decided in $o(\frac{t(n)}{\log t(n)})$ time

Proof

To design the $O(t(n))$ time TM D that decides a language A, we use the diagonalization method

D on input w:

1. $n = |w|$ $[O(t(n))$ time$]$
2. Compute and store $\frac{t(n)}{\log t(n)}$ (t is time constructible) $[O(t(n))$ time$]$
3. if $w \neq \langle M \rangle 10^*$ for some TM M, reject $[O(t(n))$ time$]$
4. simulate M on w: $[d\ g(n)$ time$]$
5. if M uses more than $\frac{t(n)}{\log t(n)}$ steps, reject $[O(\log t(n))$ time$]$
6. if M accepts, then reject
7. if M rejects, then accept

- Suppose M uses $g(n) = o(\frac{t(n)}{\log t(n)})$ time, then exists n_0 such that $d\ g(n) < \frac{t(n)}{\log t(n)}$ for all $n \geq n_0$
- If $|\langle M \rangle| \geq n_0$, then $D(\langle M \rangle) \neq M(\langle M \rangle)$
- If $|\langle M \rangle| < n_0$, then $D(\langle M \rangle 10^{n_0}) \neq M(\langle M \rangle 10^{n_0})$ (10^{n_0} gives enough time to D to simulate M)

Corollary

For any two functions $t_1, t_2 : \mathbb{N} \to \mathbb{N}$, *where* $t_1(n)$ *is* $o(\frac{t_2(n)}{\log t_2(n)})$ *and* t_2 *is time constructible,*

$$\mathrm{TIME}(t_1(n)) \subset \mathrm{TIME}(t_2(n))$$

Separate various time complexity classes:

- $\mathrm{TIME}(n^{c_1}) \subset \mathrm{TIME}(n^{c_2})$ for all natural numbers $c_1 < c_2$
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- $\mathrm{TIME}(n^{c_1}) \subset \mathrm{TIME}(n^{c_2})$ for all positive real numbers $c_1 < c_2$

Definition

$\mathrm{EXPTIME} = \bigcup_{k \in \mathbb{N}} \mathrm{TIME}(2^{n^k})$

Corollary

$\mathbf{P} \subset \mathrm{EXPTIME}$.

EXPSPACE-complete

Definition

A language B is **EXPSPACE**-complete if

1. B is in **EXPSPACE**, and
2. every $A \in$ **EXPSPACE** is polynomial time reducible to B.

If B merely satisfies condition 2, then it is **EXPSPACE**-hard.

Regular expression with exponentiation

Syntax of regular expressions with exponentiation is defined by the following rules,

$$r := a \mid \varepsilon \mid \emptyset \mid r_1 \cup r_2 \mid r_1 \circ r_2 \mid r^* \mid r^k.$$

Semantics,

- $L(a) = \{a\}$, $L(\varepsilon) = \{\varepsilon\}$, $L(\emptyset) = \emptyset$,
- $L(r_1 \cup r_2) = L(r_1) \cup L(r_2)$,
- $L(r_1 \circ r_2) = L(r_1) \circ L(r_2)$,
- $L(r^*) = (L(r_1))^*$,
- $L(r^k) = (L(r))^k = \underbrace{L(r) \circ L(r) \cdots \circ L(r)}_{k}$

where

- $L_1 \circ L_2 = \{uv \mid u \in L_1, v \in L_2\}$ $(L \circ \emptyset = \emptyset \circ L = \emptyset)$,
- $L^* = \bigcup_{i=0}^{\infty} L^i$,
- $L^0 = \{\varepsilon\}$, $L^i = L^{i-1} \circ L$ for any $i > 0$.

Regular expression with exponentiation

Syntax of regular expressions with exponentiation is defined by the following rules,

$$r := a \mid \varepsilon \mid \emptyset \mid r_1 \cup r_2 \mid r_1 \circ r_2 \mid r^* \mid r^k.$$

$$EQ_{REX\uparrow} = \{\langle r_1, r_2\rangle \mid \quad r_1 \text{ and } r_2 \text{ are two equivalent regular expressions with exponentiation}\}$$

Theorem

$EQ_{REX\uparrow}$ is **EXPSPACE***-complete.*

$EQ_{REX^{\uparrow}}$ is in **EXPSPACE**

Naive idea:

1. transform REs with exponentiation r_1 and r_2 to standard REs r_1' and r_2' whose size is $n2^n$
2. construct DFA A from r_1' and DFA B from r_2', DFA has size at most 2^{n2^n}
3. construct a NFA C such that $L(C) = \left(L(A) \cap \overline{L(B)}\right) \cup \left(\overline{L(A)} \cap L(B)\right)$, cost 2^{n2^n} space
4. test emptiness of C

$EQ_{REX\uparrow}$ is in **EXPSPACE=NEXPSPACE**

An alternative idea:

1. transform REs with exponentiation r_1 and r_2 to standard REs r_1' and r_2' whose size is $n2^n$
2. construct NFA A from r_1' and NFA B from r_2', NFA has size at most $O(n2^n)$
3. test equivalent on A and B

N on input $\langle A, B \rangle$:

1. place a marker each of the start states of A and B
2. Repeat $2^{|A|+|B|}$ times: [only $2^{|A|+|B|}$ subsets of states]
3. Nondeterministically select an input symbol and change the positions of the markers on the states of A and B to simulate reading that symbol [nondeterministic linear space]
4. If at any point a marker was placed on an accepting state of one of the finite automaton and not on any accept state of the other finite automaton, accept. Otherwise, reject

$$L(A) = L(B) \iff \text{N rejects}$$

$EQ_{REX^{\uparrow}}$ is **EXPSPACE**-hard

- Consider a 2^{n^k} space TM M for some constant k which decides a language A
- For an input w
- r_1 is the language Δ^*, where $\Delta = Q \cup \Gamma \cup \{\sharp\}$
- r_2 is the language over Δ that are all non-rejecting computation histories of M on w
- r_1 and r_2 are equivalent iff M accepts w

$EQ_{REX^{\uparrow}}$ is **EXPSPACE**-hard

When a string over Δ fails to be a non-rejecting computation history?

1. not start with the start configuration: $r_{\text{bad-start}}$ [not a computation history]
2. not end with the rejecting configuration: $r_{\text{bad-reject}}$ [non rejecting computation history]
3. configuration does not follow from the preceding one: $r_{\text{bad-window}}$ [not a computation history]

$$r_2 = r_{\text{bad-start}} \cup r_{\text{bad-reject}} \cup r_{\text{bad-window}}$$

$EQ_{REX^{\uparrow}}$ is **EXPSPACE**-hard

When a string over Δ fails to be a non-rejecting computation history?

1. not start with the start configuration: [not a computation history]

$$r_{\text{bad-start}} = S_0 \cup S_1 \cup \cdots \cup S_n \cup S_b \cup S_\sharp$$

2. $S_0 = (\Delta \setminus \{q_0\})\Delta^*$ [not start with the starting state]
3. $S_i = \Delta^i(\Delta \setminus \{w_i\})\Delta^*$ for $1 \leq i \leq n$ [the $(i+1)$-th cell is not w_i]
4. $S_b = \Delta^{n+1}(\Delta \cup \{\varepsilon\})^{2^{n^k}-n-2}(\Delta \setminus \textvisiblespace)\Delta^*$ [some cell between $n+2$ and 2^{n^k} is not $\textvisiblespace$]
5. $S_\sharp = \Delta^{2^{n^k}}(\Delta \setminus \{\sharp\})\Delta^*$ [$\sharp$ is not in $(2^{n^k}+1)$-th cell]

$EQ_{REX^{\uparrow}}$ is **EXPSPACE**-hard

When a string over Δ fails to be a non-rejecting computation history?

1. not start with the start configuration:
   $r_{\text{bad-start}} = S_0 \cup S_1 \cup \cdots \cup S_n \cup S_b \cup S_\sharp$ [not a computation history]
2. not end with the rejecting configuration: $r_{\text{bad-reject}} = (\Delta \setminus \{q_{\text{accept}}\})^*$
   [non rejecting computation history]

$EQ_{REX^{\uparrow}}$ is **EXPSPACE**-hard

When a string over Δ fails to be a non-rejecting computation history?

1. not start with the start configuration:
   $r_{\text{bad-start}} = S_0 \cup S_1 \cup \cdots \cup S_n \cup S_b \cup S_\sharp$ [not a computation history]
2. not end with the rejecting configuration: $r_{\text{bad-reject}} = (\Delta \setminus \{q_{\text{accept}}\})^*$ [non rejecting computation history]
3. configuration does not follow from the preceding one: [not a computation history]

$$r_{\text{bad-window}} = \bigcup_{bad(avc,def)} \Delta^* abc \Delta^{2^{n^k}-2} def \Delta^*$$

where abc does not yield def according to the transition function (similar to Cook-Levin theorem)

$$r_2 = r_{\text{bad-start}} \cup r_{\text{bad-reject}} \cup r_{\text{bad-window}}$$

Size of r_2 is polynomial in n

EXPTIME-complete

Definition

A language B is EXPTIME-complete if

1. B is in EXPTIME, and
2. every $A \in$ EXPTIME is polynomial time reducible to B.

If B merely satisfies condition 2, then it is EXPTIME-hard.

Outline

Oracle Turing Machines

The meaning of the word Oracle according to Webster: a person (as a priestess of ancient Greece) through whom a deity is believed to speak; a shrine in which a deity reveals hidden knowledge or the divine purpose through such a person

Definition

Oracle machines are TMs O^A that are given access to a black box or oracle O_A that can solve the decision problem for some language A.

Definition

An oracle Turing machine (OTM) M^O is an ordinary TM with an extra tape called the oracle tape with the capability of querying whether the string on the oracle tape is in O or not.

New Classes: $\mathbf{P}^O$ and $\mathbf{NP}^O$

Definition

For every oracle O, $\mathbf{P}^O$ is the set of all languages that can be decided by a poly-time deterministic TM with oracle access to O.

$\mathbf{NP}^O$ is the set of all languages that can be decided by a poly-time non-deterministic TM with oracle access to O.

Theorem

$\overline{\text{SAT}}$ *is in* $\mathbf{P}^{\text{SAT}}$, *where* $\overline{\text{SAT}}$ *denotes the language of unsatisfiable formulae.*

1. Given oracle access to SAT,
2. A deterministic poly-time M^O can query its oracle whether $\phi \in \text{SAT}$
3. Then give the opposite answer.

Theorem

$\mathbf{P} = \mathbf{P}^O$ *for all* $O \in \mathbf{P}$

1. It trivially follows $\mathbf{P} \subseteq \mathbf{P}^O$.
2. If $O \in \mathbf{P}$, then replace each oracle call with a deterministic poly-time computation of O.
3. Number of oracle calls can be polynomial and product of two polynomials is another polynomial. Hence, $\mathbf{P}^O \subseteq \mathbf{P}$.

Theorem

EXPTIME $= \mathbf{P}^O = \mathbf{NP}^O$ *for all* $O \in$ EXPTIME

1. It trivially follows EXPTIME $\subseteq \mathbf{P}^O \subseteq \mathbf{NP}^O$.
2. If $O \in$ EXPTIME, then replace each oracle call with a deterministic exp-time computation of O.
3. Number of oracle calls can be exponential and product of two exponentials is another exponential. Hence, $\mathbf{NP}^O \subseteq$ EXPTIME.

Theorem

$\mathbf{NP} \subseteq \mathbf{P}^{\mathrm{SAT}}$ *and* $co\mathbf{NP} \subseteq \mathbf{P}^{\mathrm{SAT}}$.

1. It trivially follows $\mathbf{NP} \subseteq \mathbf{P}^{\mathrm{SAT}}$.
2. $co\mathbf{NP} \subseteq co(\mathbf{P}^{\mathrm{SAT}}) = \mathbf{P}^{\mathrm{SAT}}$

Quiz

Two Boolean formulas ϕ and ψ are equivalent if the formulas have the same value on any assignment to the boolean variables. A formula is minimal if no smaller formula is equivalent to it.

$$\text{NONMIN} = \{\langle\phi\rangle \mid \phi \text{ is not a minimal Boolean formula}\}$$

Theorem
NONMIN $\in \mathbf{NP}^{\text{SAT}}$.

Relativization

We go back to diagonalization and its limits. For that, we use the concept of relativization.

Definition (Relativization)

- The OTM M^O introduced earlier allows us to magically solve the SAT problem in one step if our O was SAT.
- Irrespective of the relation between **P** and **NP**, this OTM can solve any **NP** problem in deterministic polynomial time because every problem in **NP** is poly-time reducible to SAT.
- Such a OTM is said to be computing relative to SAT. This is the significance of the term relativization.

Limits of the Diagonalization Method

Our goal is to show the limits of diagonalization in proving the **P** vs **NP** question. To that end, we will prove the following theorem:

Theorem

There exist oracles A and B such that $\mathbf{P}^A \neq \mathbf{NP}^A$ and $\mathbf{P}^B = \mathbf{NP}^B$.

- But, how does the above theorem help?
- The diagonalization method is a simulation of one TM M by another TM U. U can determine the behaviour of M and then behave differently.
- Suppose, M and U were given identical oracles O
- Now, whenever, M queries the oracle, so can U and the simulation can proceed as before
- So, any theorem proved about TMs using only diagonalization would still hold if both machines were given the same oracle.

Limits of the Diagonalization Method

Theorem

There exist oracles A and B such that $\mathbf{P}^A \neq \mathbf{NP}^A$ and $\mathbf{P}^B = \mathbf{NP}^B$.

- If we could show $\mathbf{P} \neq \mathbf{NP}$ by the diagonalization method, we conclude that they are different relative to any oracle as well, namely $\mathbf{P}^B \neq \mathbf{NP}^B$
- But $\mathbf{P}^B = \mathbf{NP}^B$, so the conclusion is false
- On the other side, no proof that relies on simulation (as in diagonalization) can show that $\mathbf{P} = \mathbf{NP}$ because $\mathbf{P}^A \neq \mathbf{NP}^A$.
- The relativization method tells us that to solve the $\mathbf{P}$ versus $\mathbf{NP}$ question, we must analyze computations, not just simulate them.

Proof

Theorem

There exist oracles A and B such that $\mathbf{P}^A \neq \mathbf{NP}^A$ and $\mathbf{P}^B = \mathbf{NP}^B$.

1. How to fix B
2. We have shown that EXPTIME $= \mathbf{P}^O = \mathbf{NP}^O$ for all $O \in$ EXPTIME
3. O can be any EXPTIME/PSPACE-complete language
4. $\mathbf{NP}^{\mathbf{TGBF}} \subseteq$ NPSPACE $\subseteq$ PSPACE $\subseteq \mathbf{P}^{\mathbf{TGBF}}$, where **TGBF** is PSPACE-complete

Proof

Theorem

There exist oracles A and B such that $\mathbf{P}^A \neq \mathbf{NP}^A$ and $\mathbf{P}^B = \mathbf{NP}^B$.

1. How to fix A?
2. Consider $L_A = \{1^n \mid \exists w \in A.|w| = n\}$, L_A is in $\mathbf{NP}^A$ for any A
3. It remains to show that $L_A \notin \mathbf{P}^A$
4. Let $M_1, M_2, \cdots$ (countable) be all the possible polynomial time oracle (A) TMs with time complexity $f_1, f_2, \cdots$
5. Let $A = \bigcup_{i \geq 1} A_i$, where $A_0 = \emptyset$ and $n_0 = 1$, for every $i \geq 1$,
   - run M_i on 1^{n_i} for the minimum n_i such that $2^{n_i} > f_i(n_i)$ and $n_i > n_{i-1}$ (M_i cannot query its oracle on all strings of length n_i)
   - let S be the set of strings with length n_i that are queried by M_i to the oracle, let $x \in \{0,1\}^{n_i} \setminus S$ (such x must exist), i.e., x is not queried to the oracle
   - If M_i accepts 1^{n_i}, $A_i = A_{i-1}$ $\quad [\nexists w \in A_i \text{ with } |w| = n_i]$
   - If M_i rejects 1^{n_i}, $A_i = A_{i-1} \cup \{x\}$ $\quad [x \in A_i \wedge |x| = n_i]$
6. Thus, no i exists such that M_i recognize L_A.

Outline

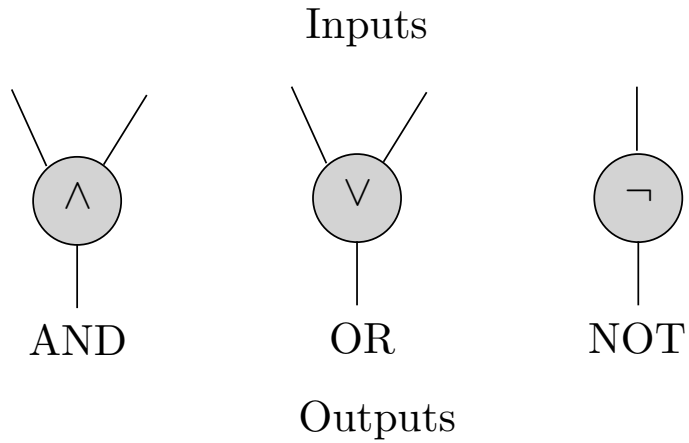
Boolean circuits

- Computers are built from electronic devices wired together in a design called a digital circuit.
- We can also simulate theoretical models, such as Turing machines, with the theoretical counterpart to digital circuits, called Boolean circuits.
- Two purposes are served by establishing the connection between TMs and Boolean circuits.
  - First, researchers believe that circuits provide a convenient computational model for attacking the **P** versus **NP** and related questions.
  - Second, circuits provide an alternative proof of the Cook-Levin theorem that SAT is **NP**-complete.

Boolean circuits

Definition

A Boolean circuit is a collection of gates and inputs connected by wires. Cycles aren't permitted. Gates take three forms: AND gates, OR gates, and NOT gates



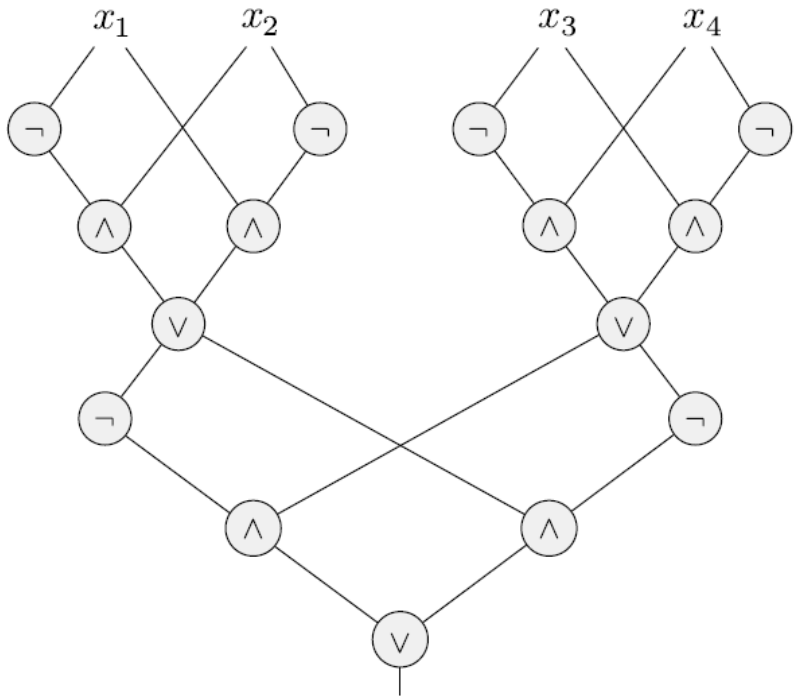
- A Boolean circuit C with n inputs and 1 outputs can be modeled as a function $f_C : \{0,1\}^n \to \{0,1\}$
- A Boolean circuit C with n inputs and k outputs can be modeled as a function $f_C : \{0,1\}^n \to \{0,1\}^k$

Example

The n-input parity function $\texttt{parity}_n : \{0,1\}^n \to \{0,1\}$ outputs 1 if an odd number of 1's appear in the input variables.

$$\texttt{parity}_n(x_1, \cdots, x_n) = \bigoplus_{i=1}^{n} x_i$$

$$x \oplus y = (\neg x \wedge y) \vee (x \wedge \neg y)$$



Boolean family

Definition

A circuit family C is an infinite list of circuits, $(C_0, C_1, C_2, \cdots)$, where C_n has n input variables. We say that C decides a language A over $\{0, 1\}$ if for every string w,

$$w \in A \iff C_n(w) = 1$$

where n is the length of w.

Complexity of Boolean family

- The size of a circuit C is the number of gates that it contains
- A circuit is size minimal if no smaller circuit is equivalent to it
  - The problem of minimizing circuits has obvious engineering applications but is very difficult to solve in general
  - Even the problem of testing whether a particular circuit is minimal does not appear to be solvable in **P** or in **NP**.
- A circuit family is minimal if every C_i on the list is a minimal
- The size complexity of a circuit family $(C_0, C_1, C_2, \cdots)$ is the function $f : \mathbb{N} \to \mathbb{N}$, where $f(n)$ is the size of C_n
- The depth of a circuit is the length (number of wires) of the longest path from an input variable to the output gate. Depth minimal circuits and circuit families, and the depth complexity of circuit families are similar

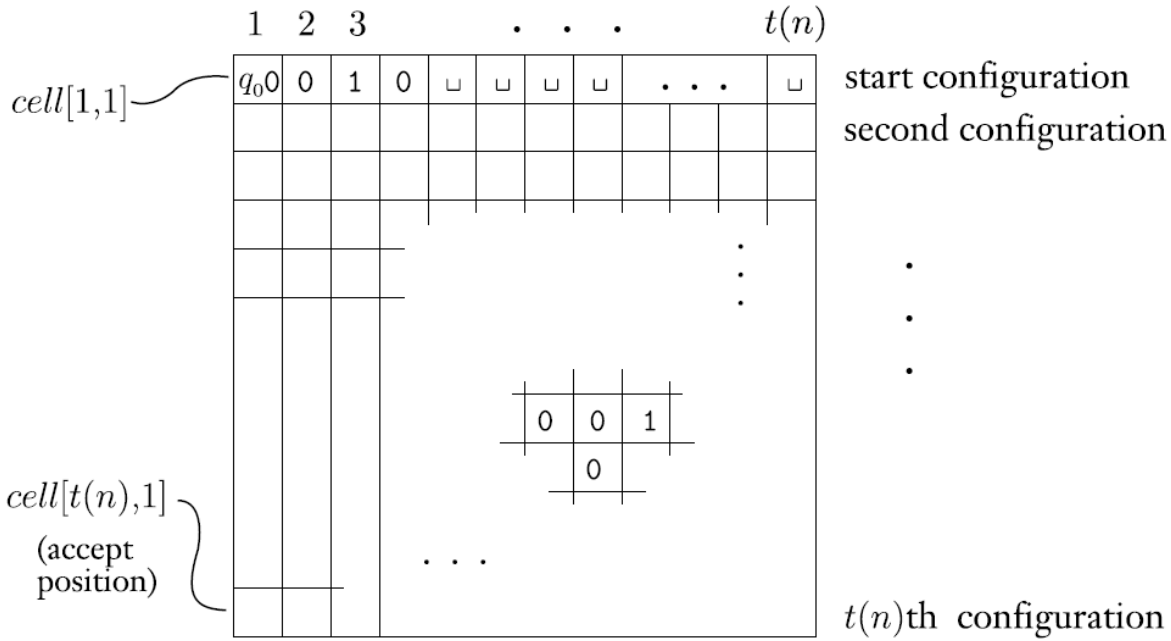
Definition

The circuit complexity of a language is the size complexity of a minimal circuit family for that language.
The circuit depth complexity of a language is defined similarly, using depth instead of size.

Theorem

Let $t : \mathbb{N} \to \mathbb{N}$ *be a function, where* $t(n) \geq n$. *If* $A \in \mathsf{TIME}(t(n))$, *then* A *has circuit complexity* $O(t^2(n))$.

Proof (1)



1. M be a TM that decides A in time $t(n)$ and assume that the tape alphabet of M is $\{0, 1\}$
2. M moves head onto the leftmost and writes $\sqcup$ on that cell prior to entering the accept state, $q_{accept}\sqcup$, and stays here forever, so that we can designate a gate in the final row of the circuit to be the output gate
3. The state and the tape symbol under the tape head by a single composite character, e.g., $q_0 0$

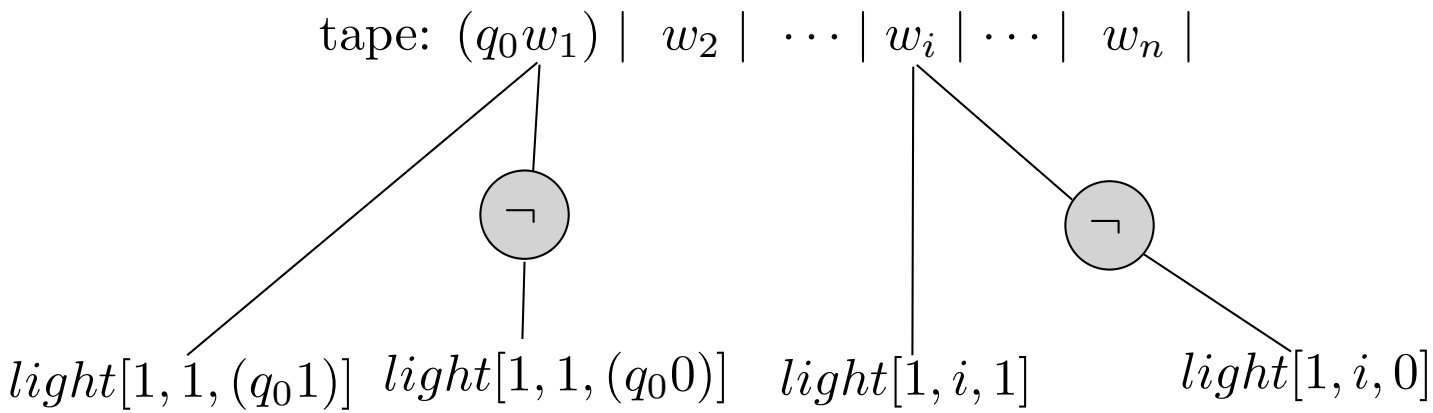
Proof (2)

1. Construct a circuit C_n that simulates M on inputs of length n
   - Start configuration $cell[1,1], cell[1,2], \cdots, cell[1,n]$
   - Accepting configuration look at $cell[t[n],1]$
   - Configuration moving using windows of tableau, $cell[i,j-1], cell[i,j], cell[i,j+1]$ decides $cell[i+1,j]$ from transition of M
2. The gates of C_n are organized in rows, one row for one configuration
3. Each row is wired into the previous row so that it can calculate its configuration from the previous row's configuration
4. $|Q \times \Gamma| + |\Gamma|$ lights for each cell
5. $light[i,j,s]$ is on iff $cell[i,j]$ contains the symbol s
6. then $O(t^2(n))$ lights

Proof (3)

Start configuration $cell[1,1], cell[1,2], \cdots, cell[1,n]$

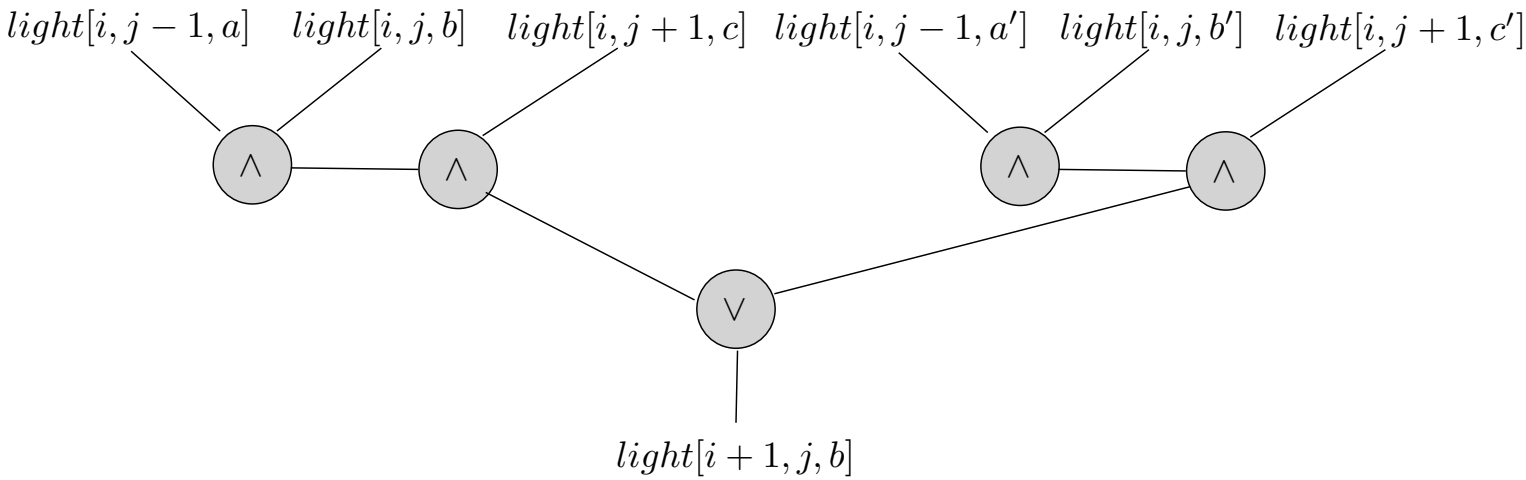
$$w = w_1 w_2 \cdots w_n \text{ over } \{0,1\}$$



Proof (4)

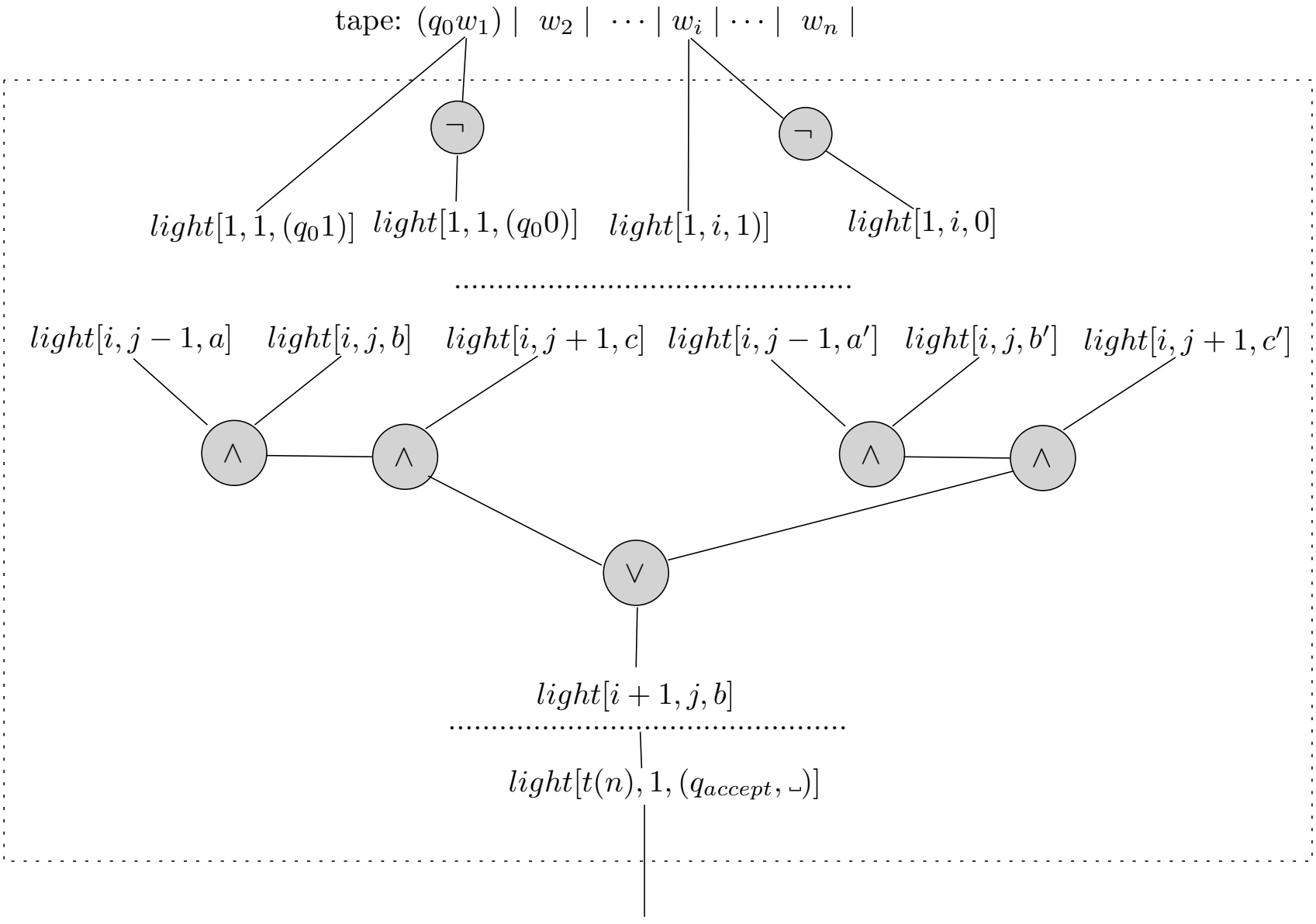
Configuration moving: $cell[i, j-1], cell[i, j], cell[i, j+1]$ decides $cell[i+1, j]$

If $cell[i, j-1], cell[i, j], cell[i, j+1]$ contain a, b, c or a', b', c', then, $cell[i+1, j]$ contains b according to δ



Proof (5)

Circuit C_n



Circuit-satisfiability

A Boolean circuit is satisfiable if some setting of the inputs causes the circuit to output 1.

Definition

The circuit-satisfiability problem tests whether a circuit is satisfiable. Let

$$\text{CIRCUIT-SAT} = \{\langle C \rangle \mid \text{C is a satisfiable Boolean circuit}\}$$

Theorem

CIRCUIT-SAT is **NP***-complete.*

Circuit-satisfiability

Theorem

CIRCUIT-SAT is **NP***-complete.*

1. It is easy to show that CIRCUIT-SAT is in **NP**, whether C is true or not under an assignment of inputs can be done in PTIME
2. **NP**-hardness. Consider a language A is **NP**-complete. Construct a polynomial-time reduction $f : A \to$ CIRCUIT-SAT such that

$$w \in A \iff f(w) \in \text{CIRCUIT-SAT}$$

   2.1 A is in **NP**, then it has a polynomial time verifier V whose input has the form $\langle w, c \rangle$, where c is certificate
   2.2 Construct a circuit C from V as previously in polynomial time
   2.3 Fill in the inputs to the circuit C that correspond to w
   2.4 Only remaining inputs to the circuit correspond to the certificate c

   C is satisfiable (the assignment is a certificate c) iff $w \in A$

An alternative proof of the Cook-Levin theorem

Theorem

*3*SAT *is* **NP***-complete.*

- 3SAT is obviously in **NP**
- We show that all languages in **NP** are polynomial-time reducible to 3SAT by reducing CIRCUIT-SAT to 3SAT

Proof

Theorem

*3*SAT *is* **NP**-*complete.*

- Given a circuit C. Let $x_1, \cdots, x_n$ are inputs of C and $g_1, \cdots, g_m$ are gates of C
- Construct a 3SATformula over variables $y_1, \cdots, y_n, y_{n+1}, \cdots, y_{n+m}$
- For each NOT-agate $y_j = \neg y_i$: we have

$$(y_i \to \neg y_j) \wedge (\neg y_i \to y_j) \equiv (y_i \vee y_j) \wedge (\neg y_i \vee \neg y_j)$$

- For each AND-agate $y_k = y_i \wedge y_j$: we have

$$(y_i \wedge y_j \leftrightarrow y_k) \equiv (\neg y_i \vee \neg y_j \vee y_k) \wedge (\neg y_k \vee y_i) \wedge (\neg y_k \vee y_j)$$

- For each OR-agate $y_k = y_i \vee y_j$: we have

$$(y_i \vee y_j \leftrightarrow y_k) \equiv (y_i \vee y_j \vee \neg y_k) \wedge (\neg y_i \vee y_k) \wedge (\neg y_j \vee y_k)$$