EE150 Signals and Systems – Chapter 6: Time and Frequency Characterization of Signals and Systems

April 26, 2022

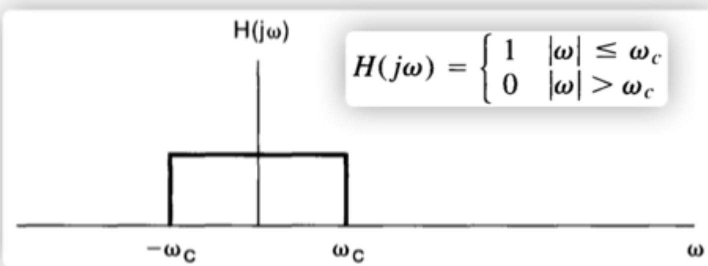
Why Time-Frequency Characterization

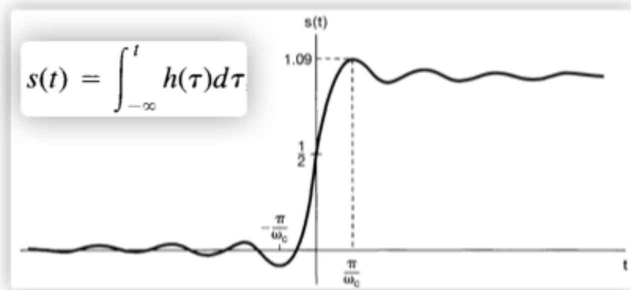
Example 1: Simplified operation

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t) = x(t) * h(t)$$

$$X(j\omega) \longrightarrow \boxed{H(j\omega)} \longrightarrow \boxed{Y(j\omega) = X(j\omega)H(j\omega)}$$

Example 2: Better Visualized





Magnitude-Phase Representation

- Continuous-time FT: $x(t) \xleftrightarrow{FT} \boxed{X(j\omega)} = |X(j\omega)|e^{j\angle X(j\omega)}$

  Magnitude: $|X(j\omega)|$
  Phase angle: $\angle X(j\omega)$

- Discrete-time FT: $x[n] \xleftrightarrow{FT} X(e^{j\omega}) = |X(e^{j\omega})|e^{j\angle X(e^{j\omega})}$

  Magnitude: $|X(e^{j\omega})|$
  Phase angle: $\angle X(e^{j\omega})$

Magnitude-Phase Representation

- Inverse Fourier Transform:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}\mathrm{d}\omega$$

- $|X(j\omega)|$ describes the basic frequency content of a signal
- $|X(j\omega)|^2$ is the energy-density spectrum of $x(t)$
- $|X(j\omega)|^2\mathrm{d}\omega/2\pi$ is the energy in signal $x(t)$ that lies in the infinitesimal frequency band between ω and $\omega + \mathrm{d}\omega$
- Parseval's equation

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\omega)|^2\mathrm{d}\omega = \int_{-\infty}^{\infty} |x(t)|^2\mathrm{d}t$$

Impact of Phase Angle

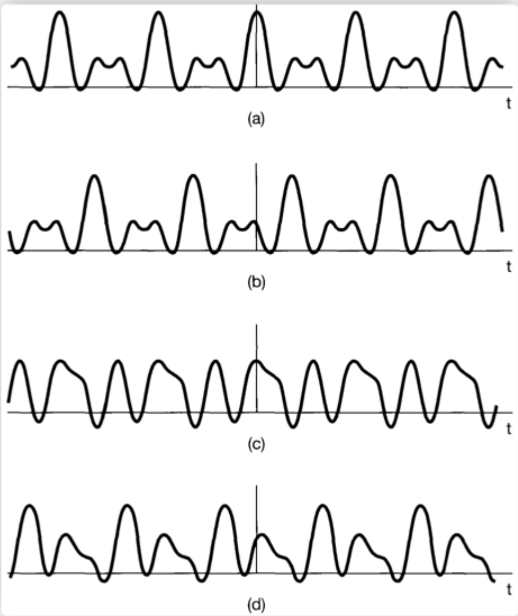
- $\angle X(j\omega)$ contains a substantial amount of information about the signal
- Changes of $\angle X(j\omega)$ lead to changes in the time-domain characteristics of signal $x(t)$, i.e., phase distortion
- Example 1: If $x(t)$ is real-valued tape recording, then $x(-t)$ represents the played backward

$$\mathcal{F}\{x(-t)\} = X(-j\omega) = |X(j\omega)|\boxed{e^{-j\angle X(j\omega)}}$$

$x(t)$ and $x(-t)$ have the same magnitude spectrum but different phase spectrum

Impact of Phase Angle

- Example 2:
  $x(t) = 1 + \frac{1}{2}\cos(2\pi t + \phi_1) + \cos(4\pi t + \phi_2) + \frac{2}{3}\cos(6\pi t + \phi_3)$



$\phi_1 = \phi_2 = \phi_3 = 0$

$\phi_1 = 4 \text{ rad}, \phi_2 = 8 \text{ rad}, \phi_3 = 12 \text{ rad}$

$\phi_1 = 6 \text{ rad}, \phi_2 = -2.7 \text{ rad}, \phi_3 = 0.93 \text{ rad}$

$\phi_1 = 1.2 \text{ rad}, \phi_2 = 4.1 \text{ rad}, \phi_3 = -7.02 \text{ rad}$

Magnitude and Phase Representation

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t) = x(t) * h(t)$$

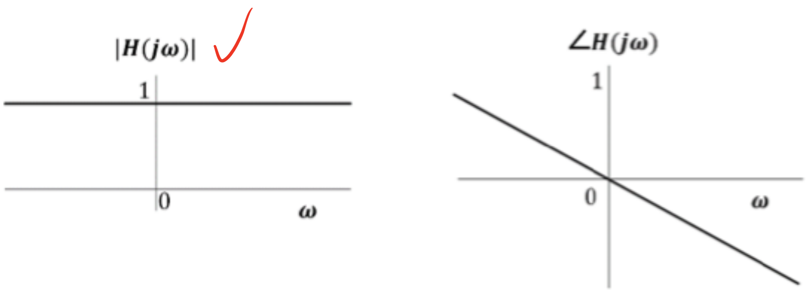
$$X(j\omega) \longrightarrow \boxed{H(j\omega)} \longrightarrow Y(j\omega) = X(j\omega)H(j\omega)$$

$$|Y(j\omega)| = |X(j\omega)||H(j\omega)|$$

$$\angle Y(j\omega) = \angle X(j\omega) + \angle H(j\omega)$$

- $|H(j\omega)|$ refers to the gain of the system
- $\angle H(j\omega)$ refers to the phase shift of the system

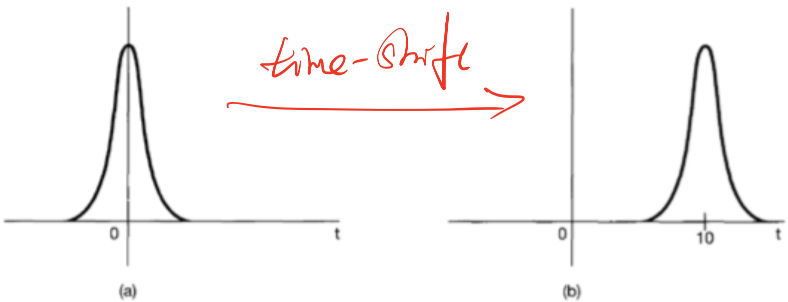
Linear Phase System



$$\boxed{H(j\omega) = e^{-j\omega t_0}}$$

$$|H(j\omega)| = 1$$

$$\angle H(j\omega) = -\omega t_0$$

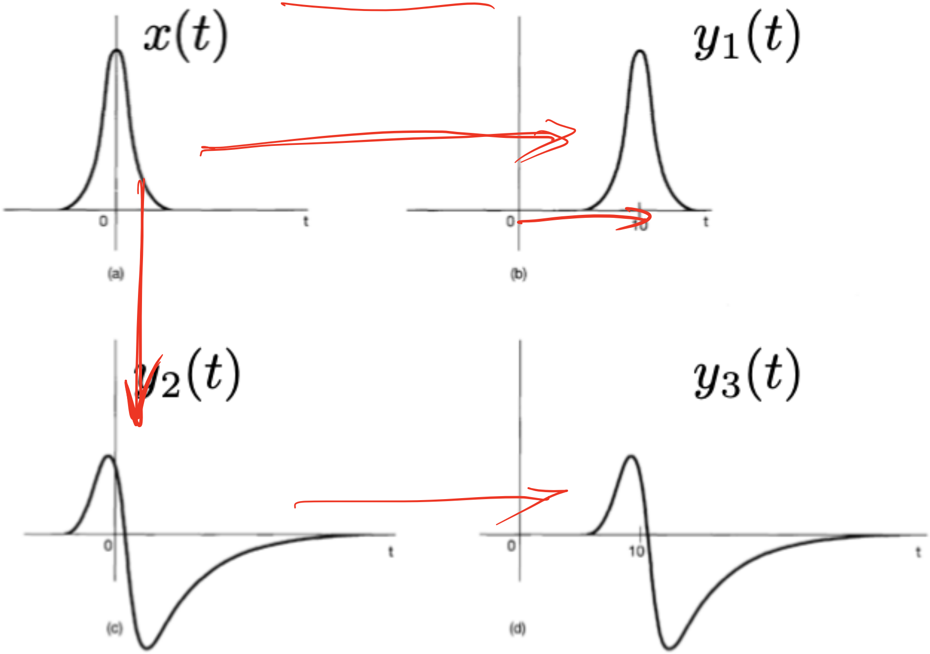


$$Y(j\omega) = X(j\omega)H(j\omega)$$
$$= X(j\omega)e^{-j\omega t_0}$$

$$y(t) = x(t - t_0)$$

Non-Linear Phase System

- $H_1(j\omega) = e^{-j\omega t_0}$ t_0
- $H_2(j\omega) = e^{j\angle H_2(j\omega)}$, where $\angle H_2(j\omega)$ is a non-linear function of ω
- $H_3(j\omega) = H_1(j\omega)H_2(j\omega)$, where $|H_3(j\omega)| = 1$ and $\angle H_3(j\omega) = -\omega t_0 + \angle H_2(j\omega)$



Group Delay

α

- $\angle H(j\omega) = -\phi - \alpha\omega$: non-linear function of ω
- $x(t)$: narrow band input
- $Y(j\omega) = X(j\omega)|H(j\omega)|e^{-j\phi}e^{-j\alpha\omega}$
- Time delay α is referred to as the group delay at $\omega = \omega_0$
- Group delay at different ω: $\tau(\omega) = -\frac{\mathrm{d}}{\mathrm{d}\omega}\{\angle H(j\omega)\}$ ✓

Group Delay - Example 1

Consider the impulse response of an all-pass system with a group delay that varies with frequency. The frequency response H (jw) for our example is the product of three factors; i.e.,

$$\boxed{H(j\omega)} = \prod_{i=1}^{3} H_i(j\omega)$$

$|H(j\omega)| = 1$

$\Uparrow$

$|H_i(j\omega)| = 1$

where

$$H_i(j\omega) = \frac{1 + (j\omega/\omega_i)^2 - 2j\epsilon_i(\omega/\omega_i)}{1 + (j\omega/\omega_i)^2 + 2j\epsilon_i(\omega/\omega_i)}$$

$\omega_1 = 315$ rad/sec and $\epsilon_1 = 0.066$
$\omega_2 = 943$ rad/sec and $\epsilon_1 = 0.033$
$\omega_3 = 1888$ rad/sec and $\epsilon_1 = 0.058$

Group Delay - Example 1

- As $|H_i(j\omega)| = 1, \forall\, i$, we have $|H(j\omega)| = 1$
- Phase of $H_i(j\omega)$ is

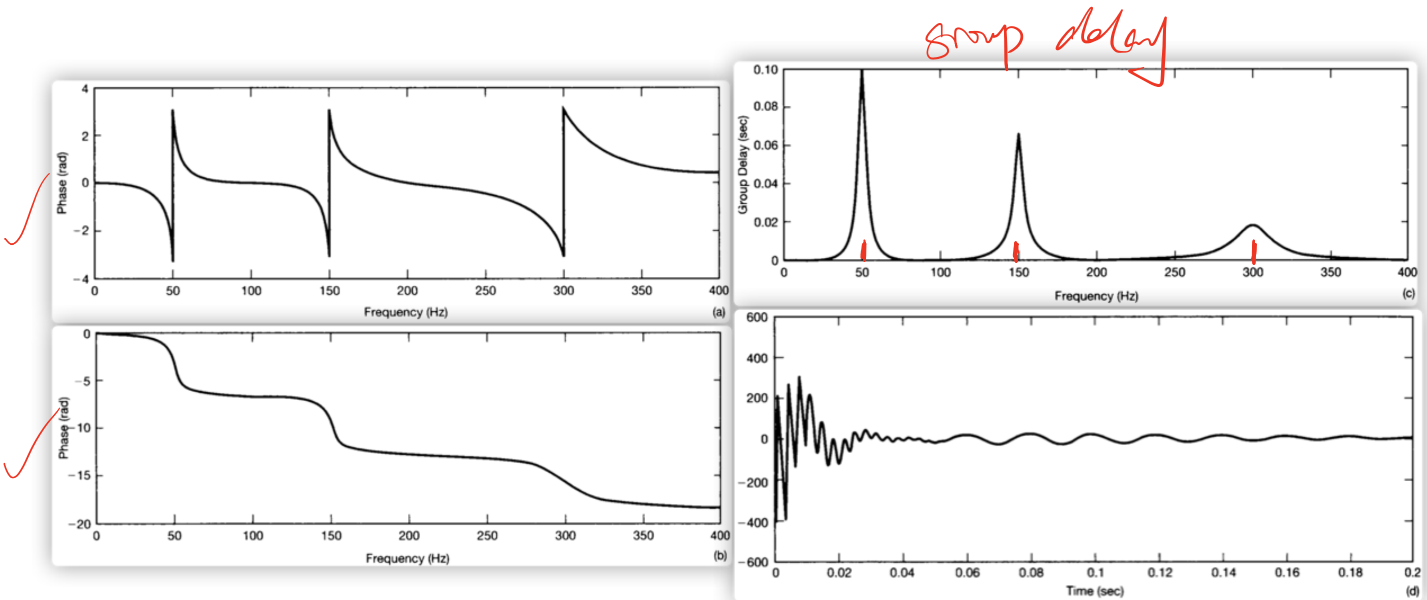
$$\angle H_i(j\omega) = -2 \arctan\left[\frac{2\epsilon_i(\omega/\omega_i)}{1-(\omega/\omega_i)^2}\right] \checkmark$$

- Phase of $H(j\omega)$ is then

$$\angle H(j\omega) = \sum_{i=1}^{3} \angle H_i(j\omega)$$

- Group delay: $\tau(\omega) = -\frac{\mathrm{d}}{\mathrm{d}\omega}\{\angle H(j\omega)\}$

Group Delay - Example 1



(a) Principle phase; (b) Unwrapped phase; (c) Group delay; (d) Impulse response.

Log-Magnitude and Bode Plots

$x(t) \rightarrow \boxed{h(t)} \rightarrow y(t)$

- Time domain: $y(t) = x(t) * h(t)$, Convolution ✓
- Frequency domain: $Y(j\omega) = X(j\omega)H(j\omega)$

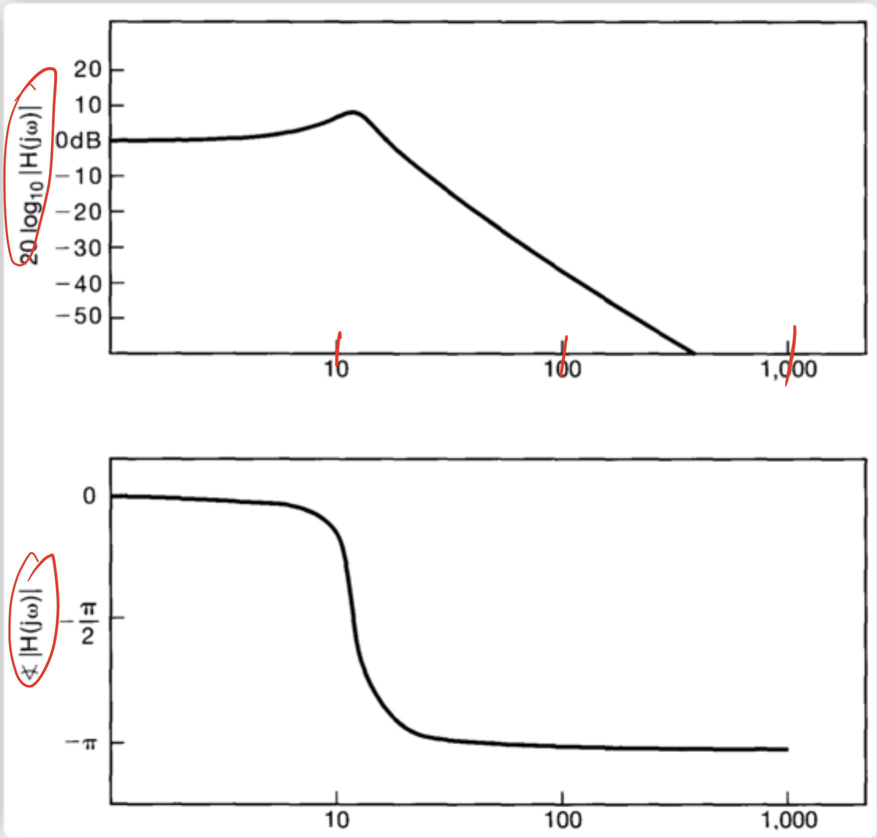
$$|Y(j\omega)| = |X(j\omega)||H(j\omega)|, \text{ Multiplication}$$
$$\angle Y(j\omega) = \angle X(j\omega) + \angle H(j\omega)$$

- Logarithmic amplitude: Addition

$$\underbrace{\log|Y(j\omega)|}_{dB} = \underbrace{\log|X(j\omega)|}_{dB} + \underbrace{\log|H(j\omega)|}_{dB}$$

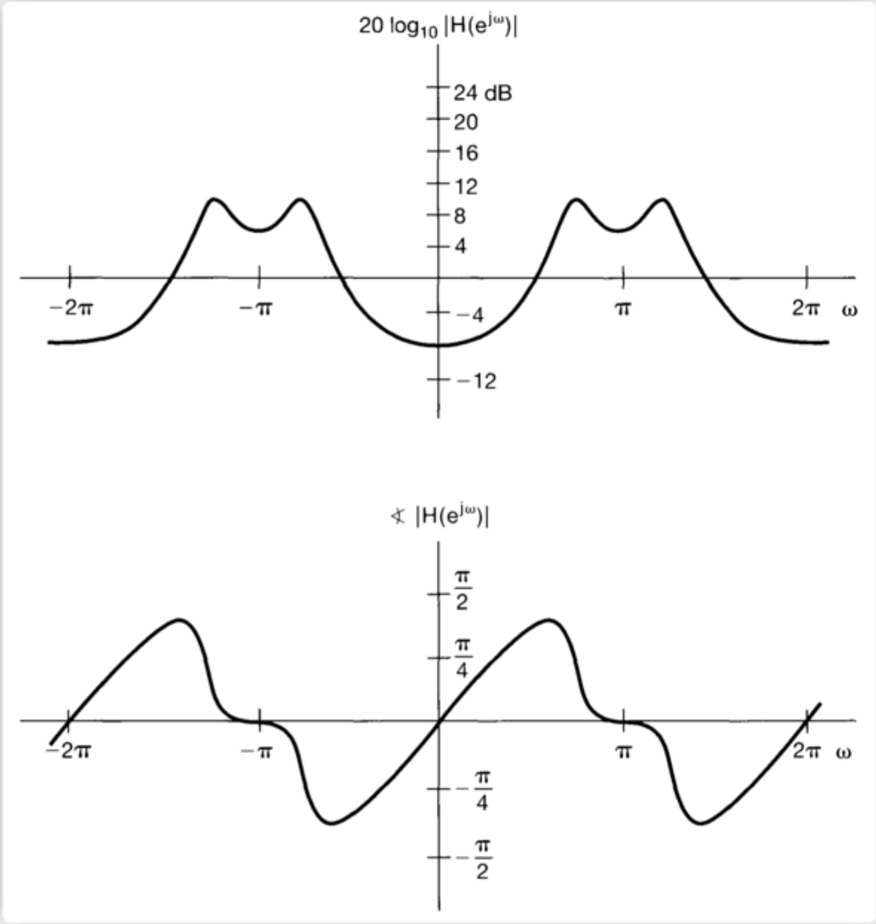
- Logarithmic amplitude scale: $20\log_{10}$, referred to as decibels (dB)
- Plots of $20\log_{10}|H(j\omega)|$ and $\angle|H(j\omega)|$ versus $\log_{10}(\omega)$ are referred to as Bode plots

Bode Plots: CT Systems



$\log_{10}(\omega)$

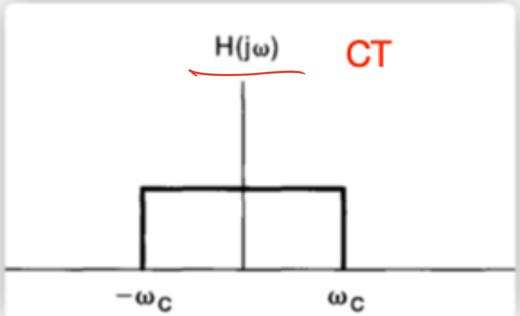
Bode Plots: DT Systems



Ideal Frequency-Selective Filters

- Frequency-selective filters
  - Low-pass filter
  - High-pass filter
  - Band-pass filter
- We focus on low-pass filter, similar concepts and results hold for high-pass and band-pass filters

Ideal Low-Pass Filter: Zero Phase

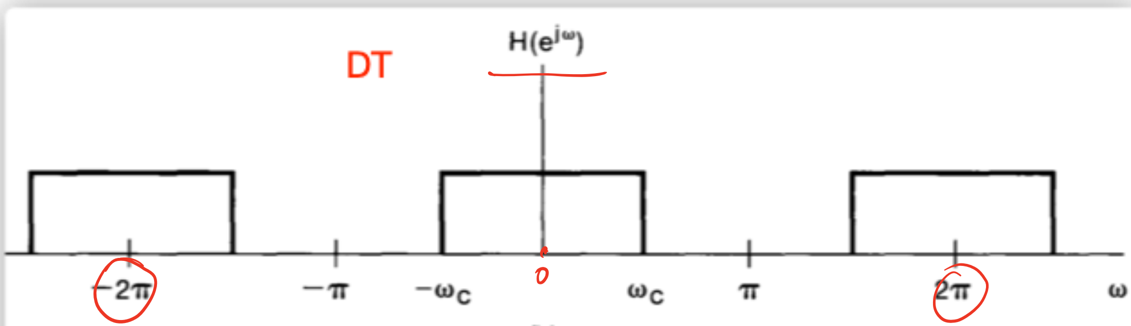


CT

$$H(j\omega) = \begin{cases} 1 & |\omega| \le \omega_c \\ 0 & |\omega| > \omega_c \end{cases}.$$

DT

$$H(e^{j\omega}) = \begin{cases} 1 & |\omega| \le \omega_c \\ 0 & \omega_c < |\omega| \le \pi \end{cases}$$



(b)

Ideal Low-Pass Filter: Zero Phase

$$h(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(j\omega)e^{j\omega t}d\omega$$

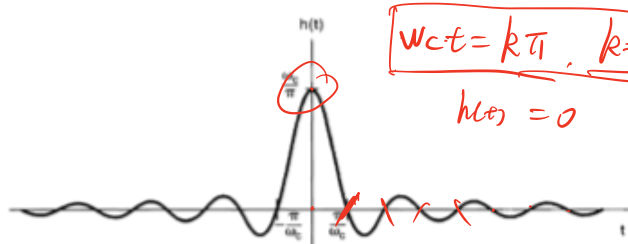
$$= \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} 1 \cdot e^{j\omega t}d\omega$$

$$= \frac{1}{2\pi}\cdot\frac{1}{jt}e^{j\omega t}\Big|_{-\omega_c}^{\omega_c}$$

$$= \frac{1}{2\pi}\cdot\frac{1}{jt}\cdot 2j\sin(\omega_c t)$$

$$= \frac{\sin\omega_c t}{\pi t} = \frac{\omega_c}{\pi}\mathrm{sinc}(\omega_c t)$$

$$h(n) = \frac{\sin\omega_c n}{\pi n} = \frac{\omega_c}{\pi}\mathrm{sinc}(\omega_c n)$$



$h(t) = 0,\ \omega_c t = k\pi, k \neq 0.\ \ \omega_c\uparrow$, width of $h(t)\downarrow$

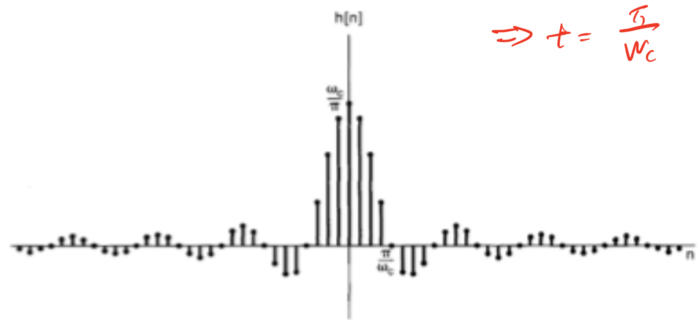
(a)

$\boxed{w_c t = k\pi,\ k \neq 0}$

$h(t) = 0$

$w_c t = \pi$

$\Rightarrow t = \frac{\pi}{w_c}$



(b)

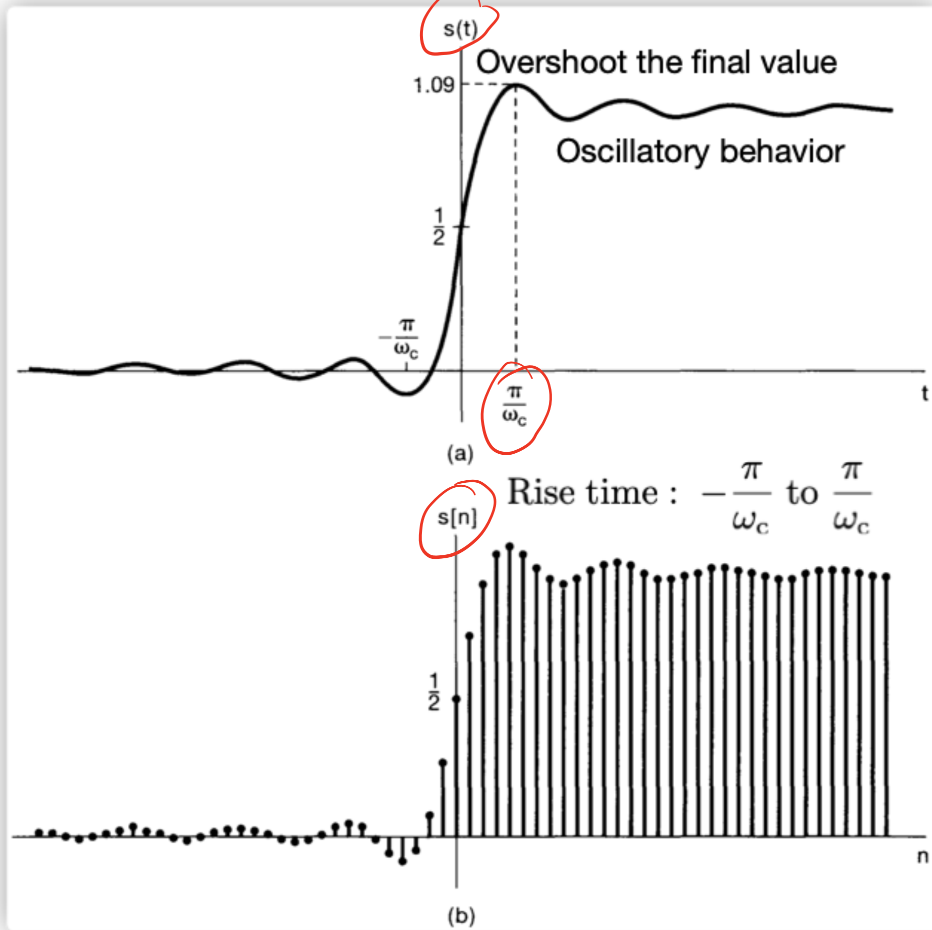
Ideal Low-Pass Filter: Zero Phase

$\delta(t) \rightarrow \boxed{h(t)} \rightarrow h(t)$

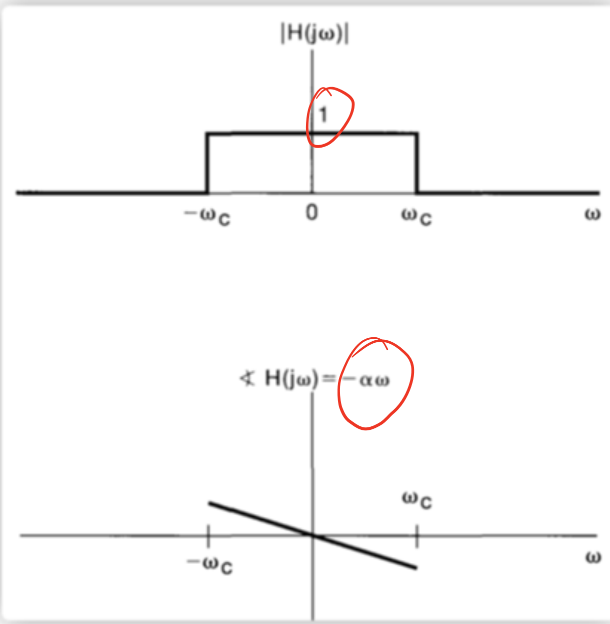
$u(t) \rightarrow \boxed{h(t)} \rightarrow s(t)$

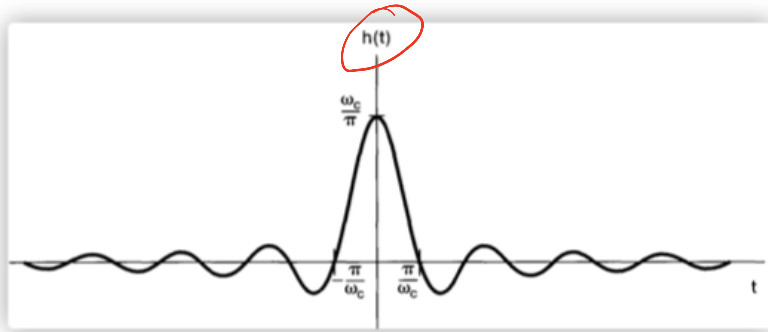
$$s(t) = \int_{-\infty}^{t} h(\tau) d\tau$$

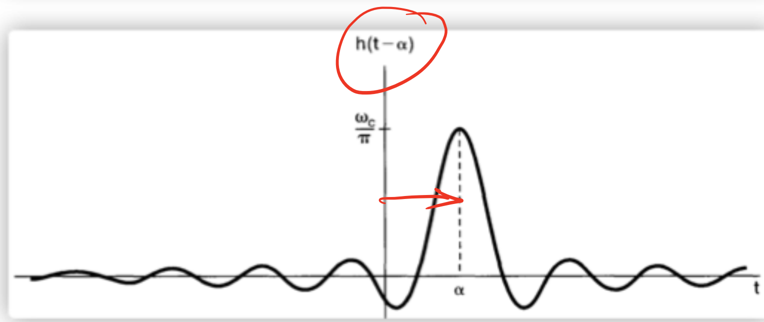
$$s(n) = \sum_{m=-\infty}^{n} h(m)$$



Ideal Low-Pass Filter: Linear Phase

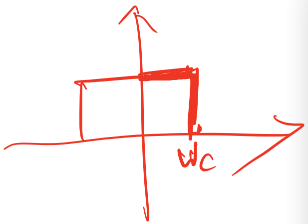


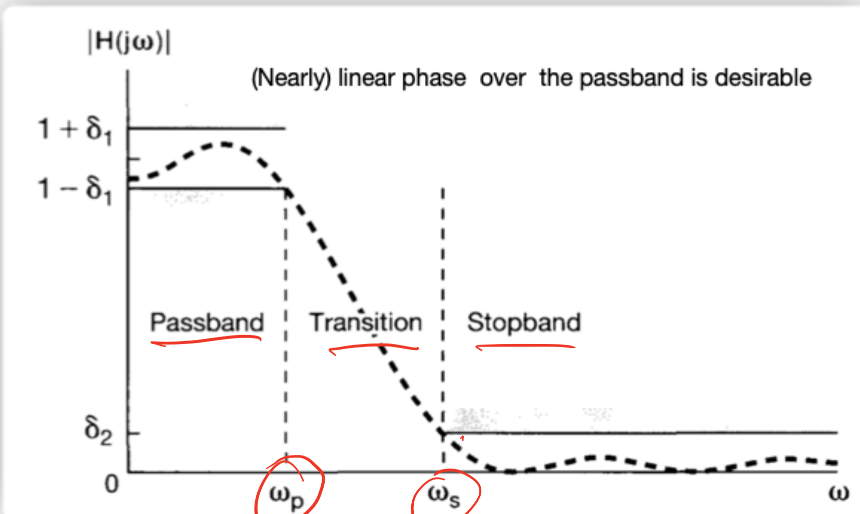




Non-Ideal Filters: Frequency Domain

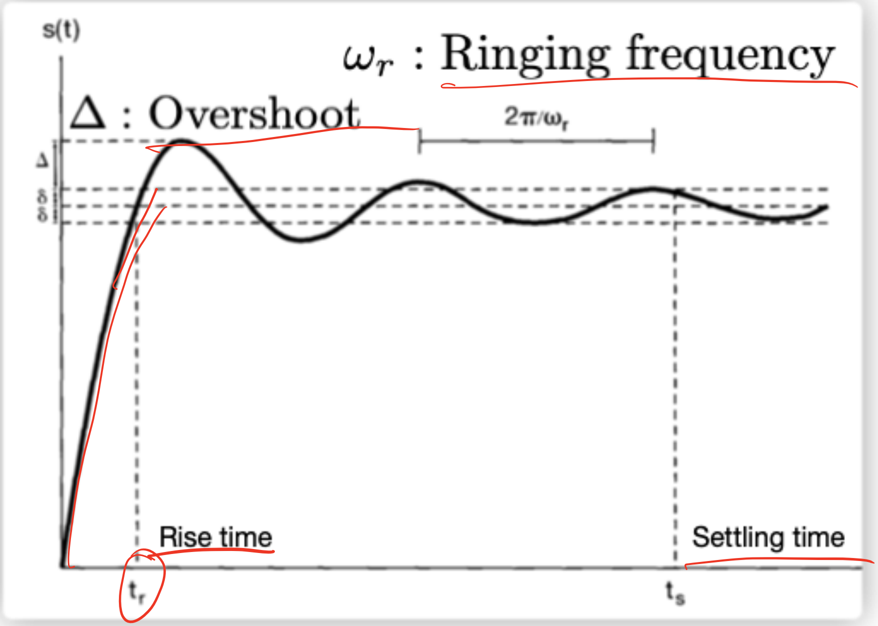
- Ideal low-pass filter is not implementable
- Gradual transition band is sometimes preferable





δ_1 : passband ripple $\quad \delta_2$: stopband ripple

Non-Ideal Filters: Time Domain

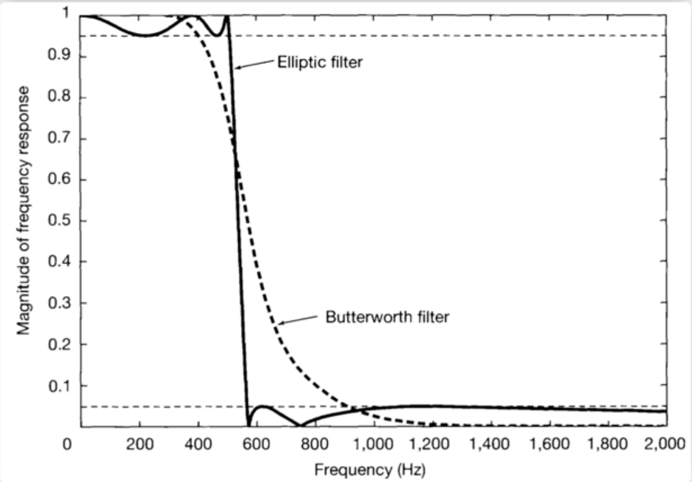


Step response of a CT low-pass filter

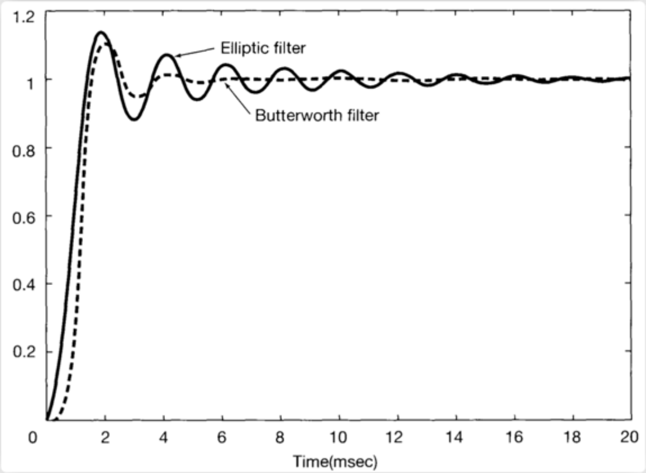
Non-Ideal Filters: Example

- Fifth-order Butterworth filter and a fifth-order elliptic filter
- Same cutoff frequency
- Same passband and stopband ripple
- Tradeoff between time-domain characteristics (t_s) and frequency-domain characteristics ($\omega_s - \omega_p$)

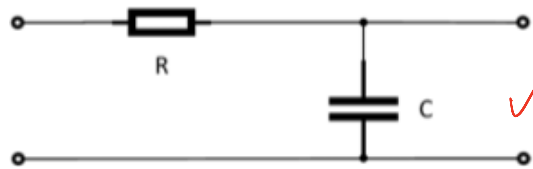
Magnitude



Step response



First-Order CT System



- Differential equation:

$$C\frac{\mathrm{d}y(t)}{\mathrm{d}t} = \frac{x(t) - y(t)}{R}$$

$$\tau\frac{\mathrm{d}y(t)}{\mathrm{d}t} + y(t) = x(t), \tau = RC$$

- Frequency response:

$$Y(j\omega) = H(j\omega) \cdot X(j\omega)$$

$$\tau j\omega Y(j\omega) + Y(j\omega) = X(j\omega)$$

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{1}{j\omega\tau + 1}$$

First-Order CT System

$e^{-at}u(t) \xleftrightarrow{FT} \dfrac{1}{j\omega+a}$

- Impulse response:
  $h(t) = \mathcal{F}^{-1}\{H(j\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{j\omega\tau+1}e^{j\omega t}\mathrm{d}\omega$ ✓

  $a = \frac{1}{\tau}$

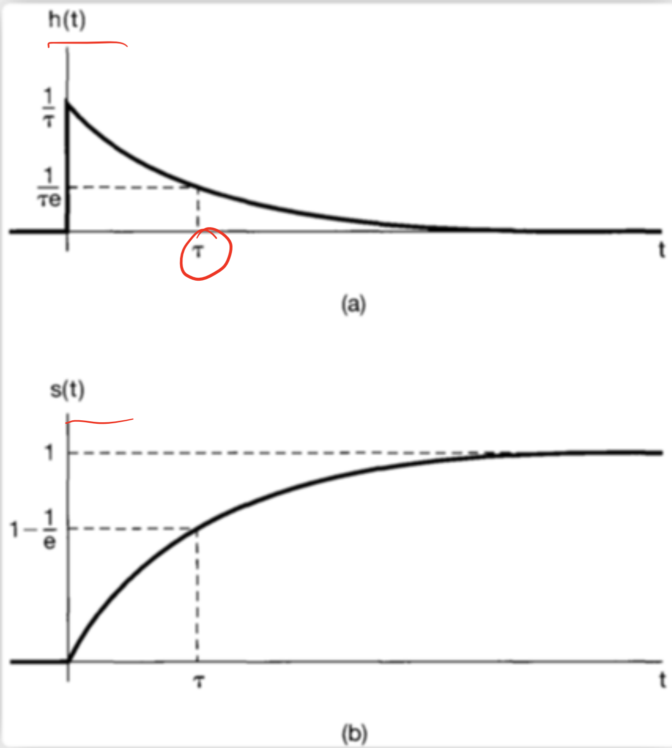
  - Consider: $x(t) = e^{-at}u(t), a > 0$, we have

$$X(j\omega) = \int_0^{\infty} e^{-at}e^{-j\omega t}\mathrm{d}t = \int_0^{\infty} e^{-(a+j\omega)t} = \mathrm{d}t = \frac{1}{j\omega + a}$$

  - As $H(j\omega) = \frac{1}{j\omega\tau+1} = \frac{1/\tau}{j\omega+1/\tau}$, we have $h(t) = \frac{1}{\tau}e^{-t/\tau}u(t)$
- Step response:

$$s(t) = \int_{-\infty}^{t} h(t)\mathrm{d}t = \frac{1}{\tau}\int_0^{t} e^{-t/\tau}\mathrm{d}t = (1 - e^{-t/\tau})u(t)$$

First-Order CT System



τ : time constant

$$h(\tau) = \frac{1}{\tau e}$$

$$s(\tau) = 1 - 1/e$$

$\tau \downarrow$, $h(t)$ decays more sharply

$s(t)$ rises more sharply

First-Order CT System

- Frequency response:
  $H(j\omega) = \frac{1}{j\omega\tau+1} = \frac{1-j\omega\tau}{(1+j\omega\tau)(1-j\omega\tau)} = \frac{1-j\omega\tau}{1+(\omega\tau)^2}$
- Logarithmic amplitude:

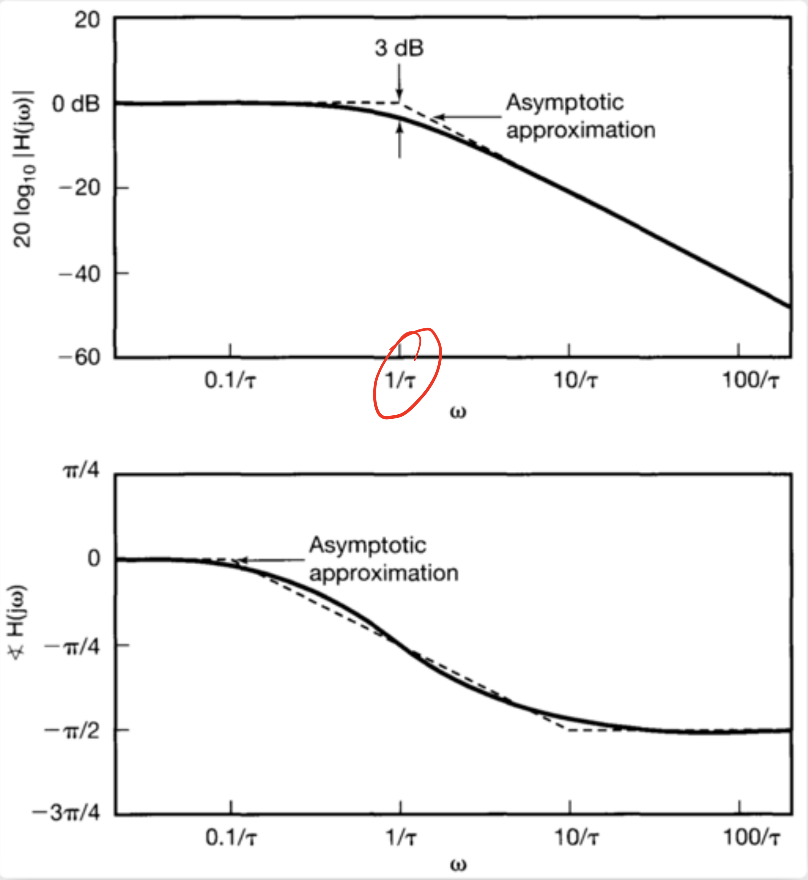
$$
\begin{aligned}
20\log_{10}|H(j\omega)| &= -10\log_{10}[(\omega\tau)^2+1] \\
&= \begin{cases} 0, & \omega \ll 1/\tau \\ -20\log_{10}(\omega) - 20\log_{10}(\tau), & \omega \gg 1/\tau \end{cases}
\end{aligned}
$$

  Break frequency: $\omega = 1/\tau$, $20\log_{10}|H(j\omega)| \approx 3$ dB
- Phase shift:

$$
\begin{aligned}
\angle H(j\omega) &= -\tan^{-1}(\omega\tau) \\
&\approx \begin{cases} 0 & \omega \le 0.1/\tau \\ -\frac{\pi}{4}[\log_{10}(\omega\tau)+1] & 0.1/\tau \le \omega \le 10/\tau \\ -\frac{\pi}{2} & \omega \ge 10/\tau \end{cases}
\end{aligned}
$$

  Break frequency: $\omega = 1/\tau$, $\angle H(j\omega) = -\pi/4$

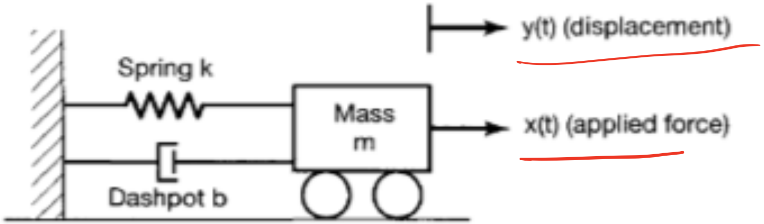
First-Order CT System



Second-Order CT System

Differential equation

$$\frac{d^2y(t)}{dt} + 2\zeta\omega_n \frac{dy(t)}{dt} + {\omega_n}^2 y(t) = {\omega_n}^2 x(t)$$



$${\omega_n}^2 = \frac{k}{m}$$

$$\omega_n = \sqrt{\frac{k}{m}}$$

$$2\zeta\omega_n = \frac{b}{m}$$

$$\zeta = \frac{b}{2\sqrt{km}}$$

$$m\frac{d^2y(t)}{dt} = x(t) - ky(t) - b\frac{dy(t)}{dt}$$

$$\frac{d^2y(t)}{dt} + \left(\frac{b}{m}\right)\frac{dy(t)}{dt} + \left(\frac{k}{m}\right)y(t) = \frac{1}{m}x(t)$$

Second-Order CT System

$\frac{dy(t)}{dt} \longleftrightarrow j\omega Y(j\omega)$

$\frac{d^2y(t)}{dt^2} \longleftrightarrow (j\omega)^2 Y(j\omega)$

Frequency response $\frac{d^2y(t)}{dt} + 2\zeta\omega_n \frac{dy(t)}{dt} + \omega_n{}^2 y(t) = \omega_n{}^2 x(t)$

$$(j\omega)^2 Y(j\omega) + 2\zeta\omega_n (j\omega) Y(j\omega) + \omega_n{}^2 Y(j\omega) = \omega_n{}^2 X(j\omega)$$

$$H(j\omega) = \frac{\omega_n{}^2}{(j\omega)^2 + 2\zeta\omega_n(j\omega) + \omega_n{}^2}$$

⬇ partial-fraction expansion

Impulse response $H(j\omega) = \frac{\omega_n{}^2}{(j\omega - c_1)(j\omega - c_2)} = \frac{M_1}{(j\omega - c_1)} - \frac{M_2}{(j\omega - c_2)}$

c_1, c_2: roots of $(j\omega)^2 + 2\zeta\omega_n(j\omega) + \omega_n{}^2 = 0$

$\zeta \neq 1$ $\quad c_1 = -\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1}, \;\; c_2 = -\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}$

$$M_1(j\omega - c_2) - M_2(j\omega - c_1) = \omega_n{}^2 \longrightarrow M(c_1 - c_2) = \omega_n{}^2$$

$$M_1 = M_2 = M = \frac{\omega_n}{2\sqrt{\zeta^2 - 1}} \longrightarrow h(t) = M[e^{c_1 t} - e^{c_2 t}]u(t)$$

Second-Order CT System

Frequency response $\frac{d^2y(t)}{dt} + 2\zeta\omega_n\frac{dy(t)}{dt} + {\omega_n}^2 y(t) = {\omega_n}^2 x(t)$

$$(j\omega)^2 Y(j\omega) + 2\zeta\omega_n(j\omega)Y(j\omega) + {\omega_n}^2 Y(j\omega) = {\omega_n}^2 X(j\omega)$$

$$H(j\omega) = \frac{{\omega_n}^2}{(j\omega)^2 + 2\zeta\omega_n(j\omega) + {\omega_n}^2}$$

⬇ partial-fraction expansion

Impulse response

$\zeta = 1$ $\quad c_1 = c_1 = -\omega_n \quad H(j\omega) = \frac{{\omega_n}^2}{(j\omega + \omega_n)^2}$

$$e^{-at}u(t) \xrightarrow{\mathcal{F}} H(j\omega) = \frac{1}{j\omega + a}$$

$$te^{-at}u(t) \xrightarrow{\mathcal{F}} H(j\omega) = \frac{1}{(j\omega + a)^2}$$

$$h(t) = {\omega_n}^2 te^{-\omega_n t}u(t)$$

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Second-Order CT System

Impulse response

$\zeta \neq 1$ $$h(t) = M[e^{c_1 t} - e^{c_2 t}]u(t)$$

$0 < \zeta < 1$ $$h(t) = \frac{\omega_n}{2\sqrt{\zeta^2 - 1}}[e^{\left(-\zeta\omega_n + \omega_n\sqrt{\zeta^2-1}\right)t} - e^{\left(-\zeta\omega_n - \omega_n\sqrt{\zeta^2-1)}\right)t}]\, u(t)$$

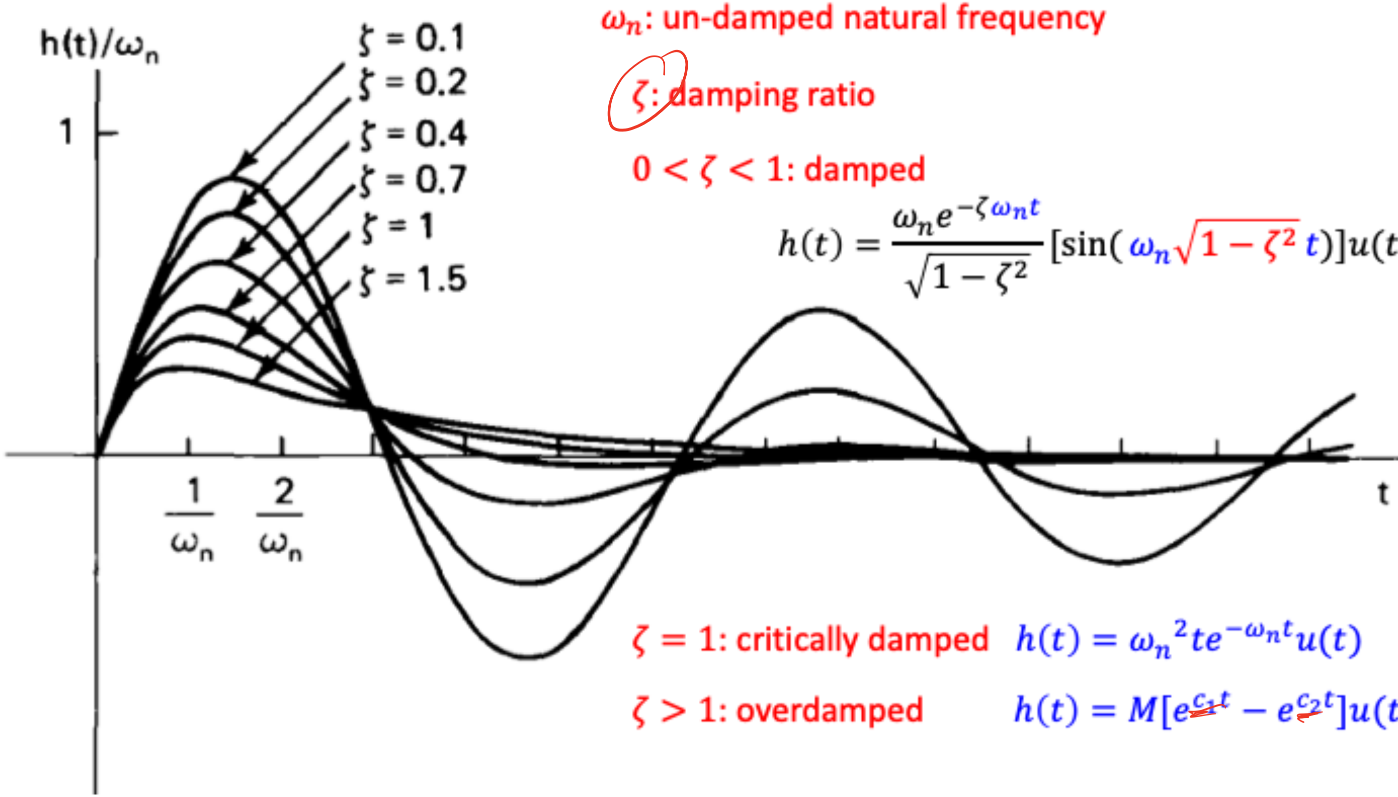
$$= \frac{\omega_n e^{-\zeta\omega_n t}}{2\sqrt{\zeta^2 - 1}}\left[e^{j\omega_n\sqrt{1-\zeta^2}t} - e^{-j\omega_n\sqrt{1-\zeta^2}t}\right]u(t)$$

$$= \frac{\omega_n e^{-\zeta\omega_n t}}{2\sqrt{\zeta^2 - 1}}[2j\sin(\omega_n\sqrt{1-\zeta^2}\,t)]u(t)$$

$$h(t) = \frac{\omega_n e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}[\sin(\omega_n\sqrt{1-\zeta^2}\,t)]u(t)$$

$\zeta > 1$?

Second-Order CT System



ω_n: un-damped natural frequency

ζ: damping ratio

$0 < \zeta < 1$: damped

$$h(t) = \frac{\omega_n e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}[\sin(\omega_n\sqrt{1-\zeta^2}\,t)]u(t)$$

$\zeta = 1$: critically damped $\quad h(t) = \omega_n{}^2 t e^{-\omega_n t}u(t)$

$\zeta > 1$: overdamped $\quad h(t) = M[e^{c_1 t} - e^{c_2 t}]u(t)$

Second-Order CT System

$\zeta \neq 1$ $\quad h(t) = M[e^{c_1 t} - e^{c_2 t}]u(t)$

$$s(t) = \underbrace{\int_{-\infty}^{t} h(t)\,dt} = M\int_0^t (e^{c_1 t} - e^{c_2 t})\,dt$$

$$= \begin{Bmatrix} 0, t < 0 \\ M\left(\frac{e^{c_1 t}}{c_1} - \frac{e^{c_2 t}}{c_2}\right)\Big|_0^t = 1 + M\left[\frac{e^{c_1 t}}{c_1} - \frac{e^{c_2 t}}{c_2}\right], t \geq 0 \end{Bmatrix} = \left\{1 + M\left[\frac{e^{c_1 t}}{c_1} - \frac{e^{c_2 t}}{c_2}\right]\right\}u(t)$$

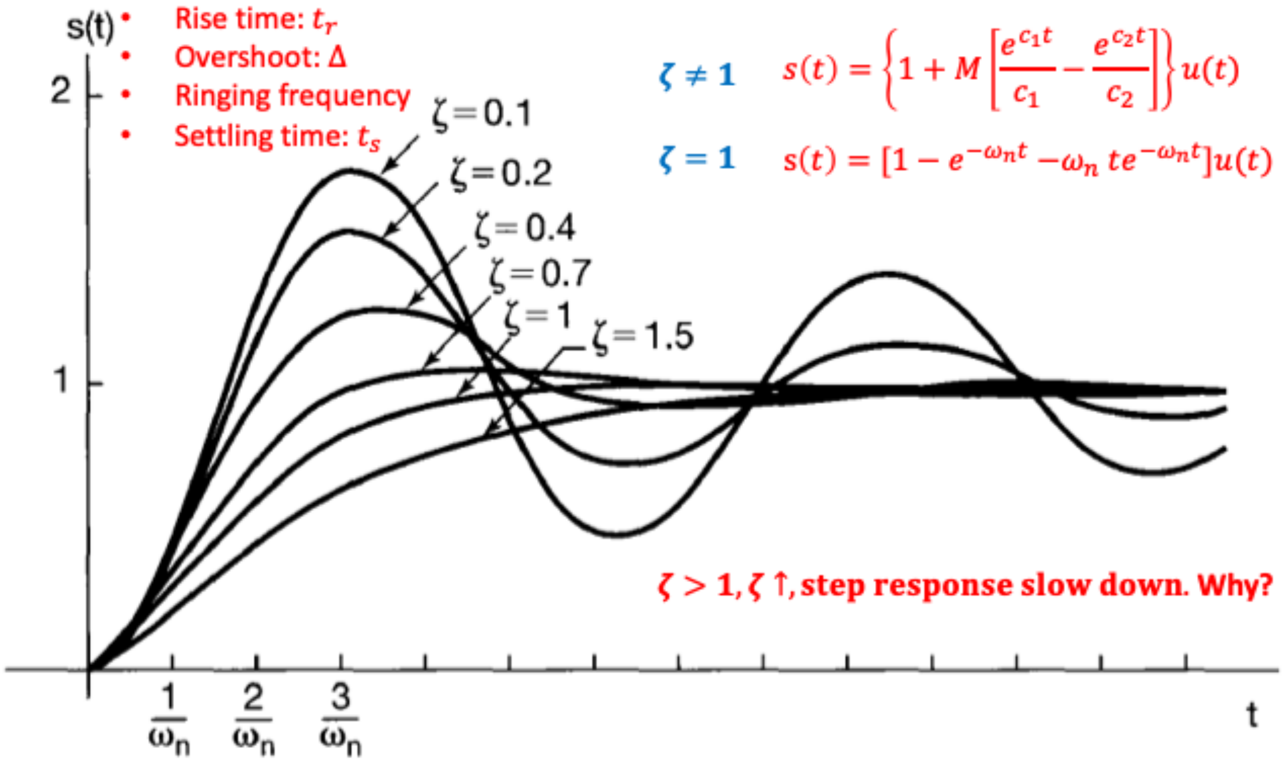
$\zeta = 1$ $\quad h(t) = \omega_n{}^2 t e^{-\omega_n t}u(t)$

$$s(t) = \int_0^t \omega_n{}^2 t e^{-\omega_n t}\,dt = -\omega_n \int_0^t t\, e^{-\omega_n t} d(-\omega_n t) = -\omega_n \int_0^t t\, de^{-\omega_n t}$$

$$= \begin{Bmatrix} 0, t < 0 \\ -\omega_n t e^{-\omega_n t}\Big|_0^t - \int_0^t e^{-\omega_n t}\, d(-\omega_n t) = 1 - e^{-\omega_n t} - \omega_n\, t e^{-\omega_n t}, t \geq 0 \end{Bmatrix}$$

$s(t) = [1 - e^{-\omega_n t} - \omega_n\, t e^{-\omega_n t}]u(t)$

Second-Order CT System



- Rise time: t_r
- Overshoot: Δ
- Ringing frequency
- Settling time: t_s

$\zeta \neq 1$ $\quad s(t) = \left\{1 + M\left[\frac{e^{c_1 t}}{c_1} - \frac{e^{c_2 t}}{c_2}\right]\right\}u(t)$

$\zeta = 1$ $\quad s(t) = [1 - e^{-\omega_n t} - \omega_n t e^{-\omega_n t}]u(t)$

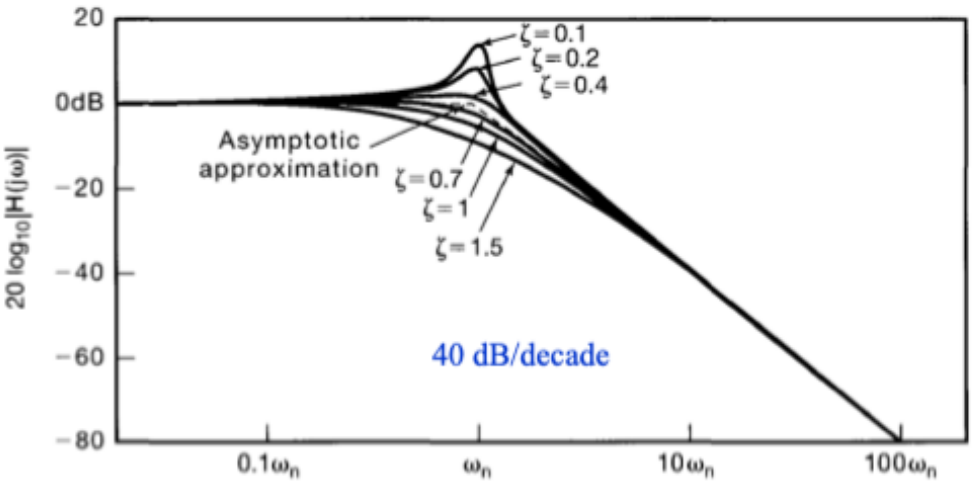
$\zeta > 1, \zeta \uparrow$, step response slow down. Why?

Second-Order CT System

$$H(j\omega) = \frac{{\omega_n}^2}{(j\omega)^2 + 2\zeta\omega_n(j\omega) + {\omega_n}^2} = \frac{1}{(j\omega/\omega_n)^2 + 2\zeta(j\omega/\omega_n) + 1}$$

$$20log_{10}|H(j\omega)| = -20\log_{10}|(j\omega/\omega_n)^2 + 2\zeta(j\omega/\omega_n) + 1|$$

$$= -10\log_{10}\left\{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + 4\zeta^2\left(\frac{\omega}{\omega_n}\right)^2\right\} \simeq \begin{cases} 0, & \omega \ll \omega_n \\ -40\log_{10}\omega + 40\log_{10}\omega_n, & \omega \gg \omega_n \end{cases}$$



$$\omega_{max} = \omega_n\sqrt{1-2\zeta^2}$$

$\zeta<0.707$

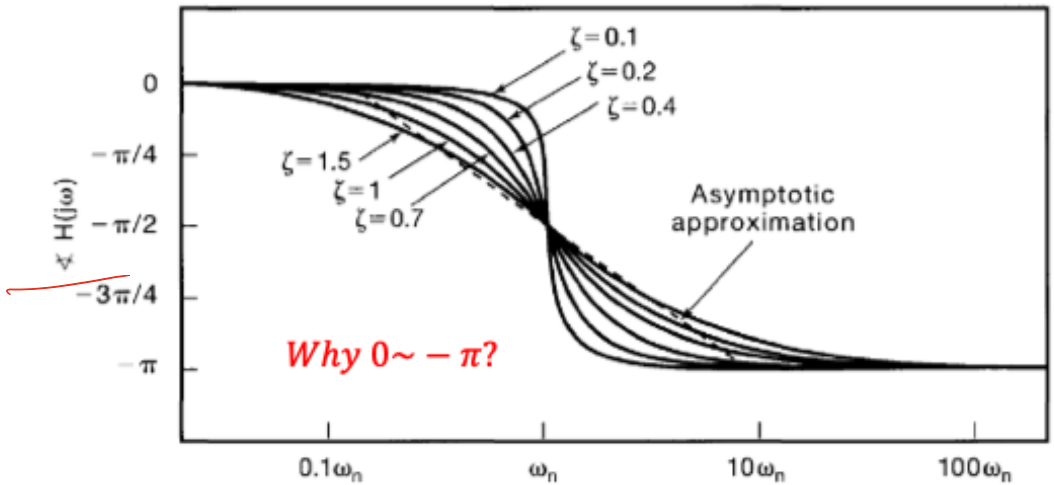
$$|H(j\omega_{max})| = \frac{1}{2\zeta\sqrt{1-\zeta^2}}$$

$$\text{Quality } Q = \frac{1}{2\zeta}$$

Second-Order CT System

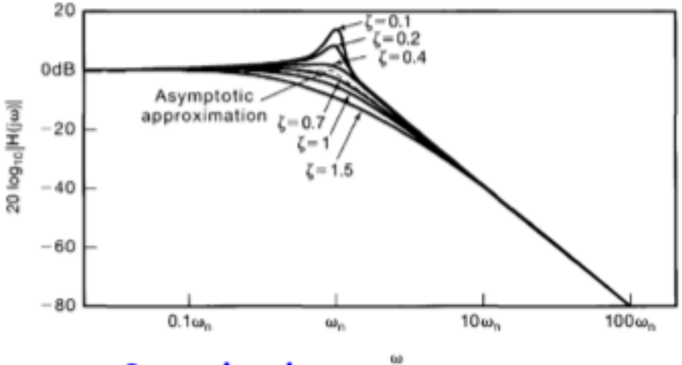
$$H(j\omega) = \frac{{\omega_n}^2}{(j\omega)^2 + 2\zeta\omega_n(j\omega) + {\omega_n}^2} = \frac{1}{(j\omega/\omega_n)^2 + 2\zeta(j\omega/\omega_n) + 1}$$

$$\angle H(j\omega) = -\tan^{-1}\left[\frac{2\zeta(\omega/\omega_n)}{1-(\omega/\omega_n)^2}\right] \simeq \begin{cases} 0, & \omega \le 0.1\omega_n \\ -\frac{\pi}{2}\left[\log_{10}\left(\frac{\omega}{\omega_n}\right)+1\right], & 0.1\omega_n \le \omega \le 10\omega_n \\ -\pi, & \omega \ge 10\omega_n \end{cases}$$



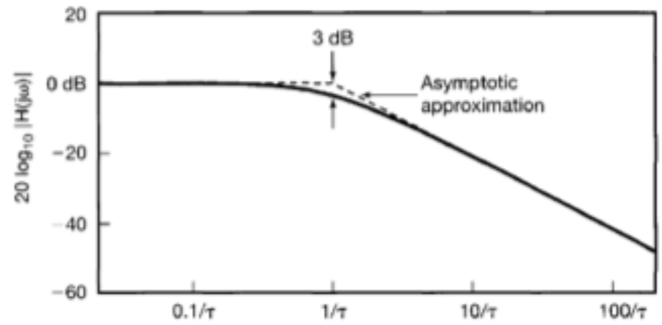
$$\angle H(j\omega_n) = -\frac{\pi}{2}$$

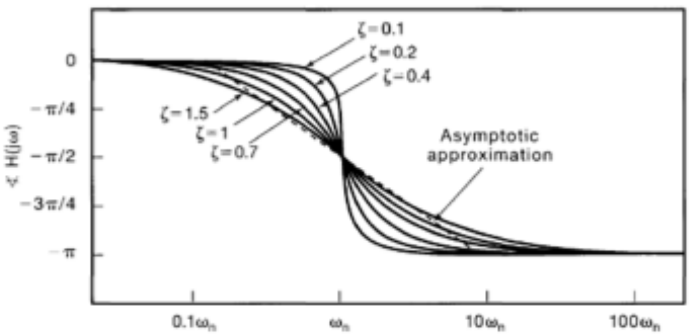
Comparison

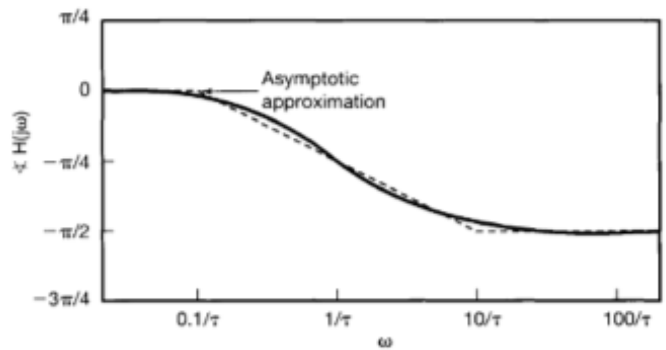


Second-order

First-order

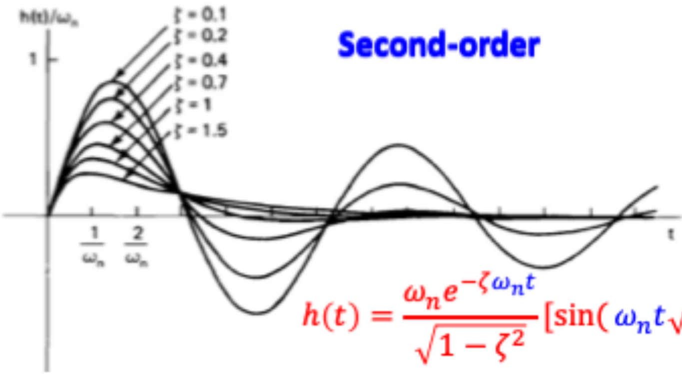






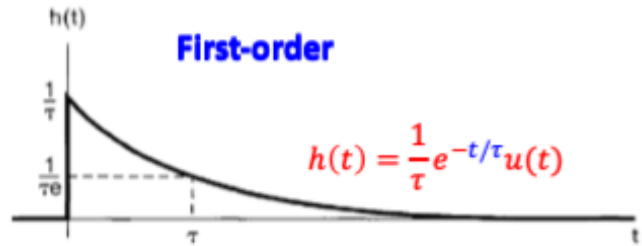
Comparison

Second-order



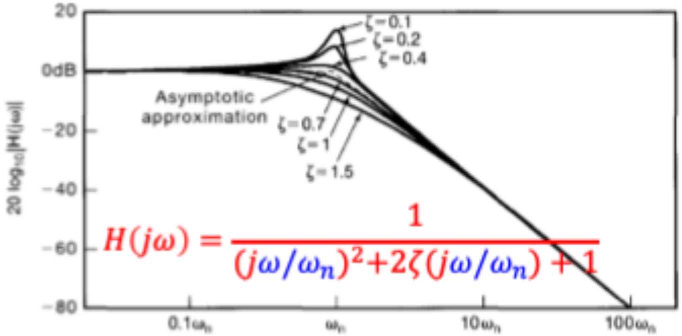
$$h(t) = \frac{\omega_n e^{-\zeta \omega_n t}}{\sqrt{1-\zeta^2}}[\sin(\omega_n t\sqrt{1-\zeta^2})]u(t)$$

First-order



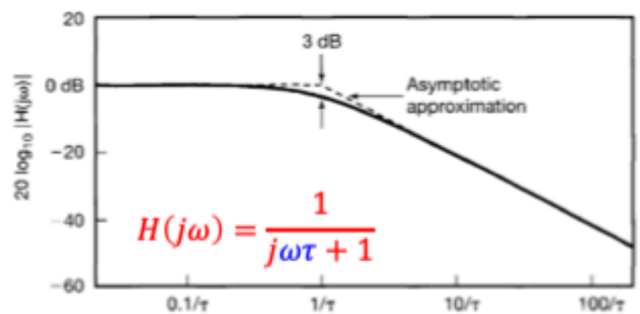
$$h(t) = \frac{1}{\tau}e^{-t/\tau}u(t)$$

ω_n: time frequency scaling



$$H(j\omega) = \frac{1}{(j\omega/\omega_n)^2 + 2\zeta(j\omega/\omega_n) + 1}$$

τ: time frequency scaling



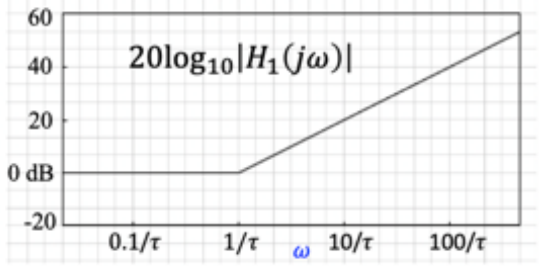
$$H(j\omega) = \frac{1}{j\omega\tau + 1}$$

Bode Plots for Rational Frequency Responses

$H_1(j\omega) = j\omega\tau + 1$

$20\log_{10}|H_1(j\omega)| = -20\log_{10}\left|\frac{1}{H_1(j\omega)}\right|$

$\angle H_1(j\omega) = -\angle\left[\frac{1}{H_1(j\omega)}\right]$



$H_2(j\omega) = (j\omega/\omega_n)^2 + 2\zeta(j\omega/\omega_n) + 1$

$20\log_{10}|H_2(j\omega)| = -20\log_{10}\left|\frac{1}{H_2(j\omega)}\right|$

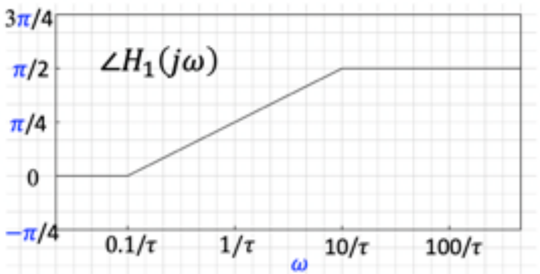
$\angle H_2(j\omega) = -\angle\left[\frac{1}{H_2(j\omega)}\right]$

$H_3(j\omega) = K$

If $K > 0$, $K = |K|e^{j0}$, if $K < 0$, $K = |K|e^{j\pi}$

$20\log_{10}|H_3(j\omega)| = 20\log_{10}|K|$

$\angle H_3(j\omega) = \begin{cases} 0, K > 0 \\ \pi, K < 0 \end{cases}$



Bode Plots for Rational Frequency Responses

$$H(j\omega) = \frac{2\times 10^4}{(j\omega)^2 + 100j\omega + 10^4}$$

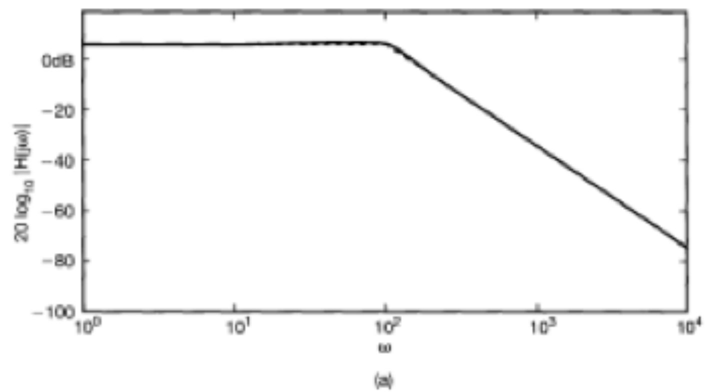
$$\hat{H}(j\omega) = \frac{1}{(j\omega/\omega_n)^2 + 2\zeta(j\omega/\omega_n) + 1}$$

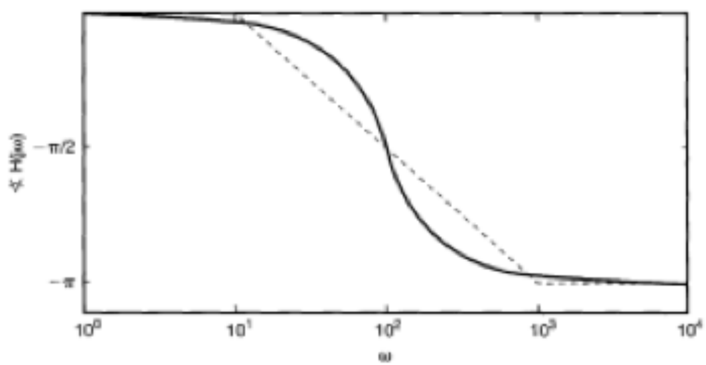
$$H(j\omega) = 2\times\frac{1}{(j\omega/100)^2 + j(\omega/100) + 1}$$

$$\omega_n = 100, \qquad \zeta = 0.5$$

$$20\log_{10}|H(j\omega)| = 20\log_{10}2 + 20\log_{10}|\hat{H}(j\omega)|$$

$$\angle H(j\omega) = \angle\hat{H}(j\omega)$$



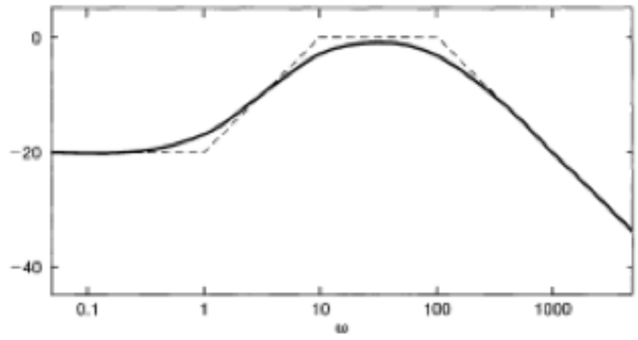


Bode Plots for Rational Frequency Responses

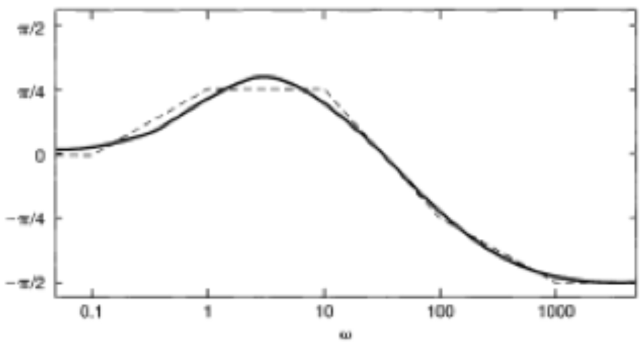
$$H(j\omega) = \frac{100(1 + j\omega)}{(10 + j\omega)(100 + j\omega)}$$

$$= \left(\frac{1}{10}\right)\left(\frac{1}{1 + j\omega/10}\right)\left(\frac{1}{1 + j\omega/100}\right)(1 + j\omega)$$

$\omega_r = 1/\tau$: 10 100 1



(a)



(b)

$$\rightarrow y[n] - a\,y[n-1] = x[n]$$

$$Y(e^{j\omega}) - a \cdot Y(e^{j\omega}) \cdot e^{-j\omega} = X(e^{j\omega})$$

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{1}{1 - a e^{-j\omega}}$$

$$\downarrow$$

$$h[n]$$

$$y[n] - 2r\cos\theta\, y[n-1] + r^2 y[n-2] = x[n]$$

$$Y(e^{j\omega}) - 2r\cos\theta\, Y(e^{j\omega}) \cdot e^{-j\omega} + r^2 \cdot Y(e^{j\omega}) \cdot e^{-j2\omega} = X(e^{j\omega})$$

$$\boxed{H(e^{j\omega})} = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{1}{1 - 2r\cos\theta\, e^{-j\omega} + r^2 e^{-j2\omega}}$$

$$\downarrow$$

$$h[n]$$