EE101 Homework 2

Submit via Blackboard   Due: Oct.24th

Your name: __________   Student ID: _________________

Problem 1:

X-ray is a penetrating form of high-energy electromagnetic radiation, whose wavelength ranges from 10 picometers to 10 nanometers. Among the wide wavelength spectrum, X-rays with wavelength below 0.2-0.1nm are called hard X-rays, while those with longer wavelength are called soft X-rays.

(1) What is the relationship of X-ray's energy (E_X), wavelength (λ_X) and frequency (f_X)
(2) Given a hard X-ray with wavelength = 0.13 nm, what is its corresponding frequency and energy (in keV)? ($h = 6.626 * 10^{-34} m^2 kg\ s^{-1}, c = 3.000 * 10^8 ms^{-1}, e = 1.602 * 10^{-19} C$)
(3) Given a soft X-ray with frequency = $3.846 * 10^{17}\ Hz$, what is its corresponding wavelength and energy (in keV)?
(4) In medical imaging, X-ray with energy more than 60keV is often observed. Given an X-ray in medical imaging with energy = 69keV, what is its corresponding wavelength and frequency?
(5) (Bonus) Why do hard and soft x-ray have much longer wavelength and smaller energy compared to medical x-ray, but still can be applied in detecting crystal structure with resolution of a few nanometer, while medical images can only achieve resolution of a few micrometer?

Grading:
16 + 5 points, 4 points for (1)-(4), 5 bonus points for (5)

Solution:

(1) $\mathrm{E_X}, \lambda_X$ *and* f_X follows the equation as follows:

$$\mathrm{E_X} = hf_X = h\frac{c}{\lambda_X} \ (*)$$

(2) Take $\lambda_{\text{hard-X}} = 0.13\text{nm}$ into (*), we get:

$$\mathrm{f_{hard-X}} = \frac{c}{\lambda_{\text{hard-X}}} = 2.307 * 10^{18}\ Hz$$

$$\mathrm{E_{hard-X}} = h\frac{c}{\lambda_{hard-X}} = 9.5481\ \text{keV}$$

(3) Take $\mathrm{f_{soft-X}} = 3.846 * 10^{17}\ Hz$ into (*), we get:

$$\lambda_{\text{soft-X}} = \frac{c}{f_{soft-x}} = 0.78\ nm$$

$$\mathrm{E_{soft-X}} = h\frac{c}{\lambda_{soft-X}} = 1.5913\ keV$$

(4) Take $\mathrm{E_{med-X}} = 69keV$ into (*), we get:

$$\lambda_{\text{med-X}} = \frac{hc}{(E_{med-X})} = 0.018nm$$

$$\mathrm{f_{med-X}} = \frac{c}{\lambda_{\text{med-X}}} = 1.67 * 10^{19}\ Hz$$

(5)
In medical imaging, x-rays are regarded as particles and images are formed by the differentiated attenuation coefficient of different tissues. Therefore, the wavelength is only related to the energy of the particle. The higher the energy is, the higher the probability that particles can pass the tissue. But the resolution has little to do with wavelength.

However, in x-ray crystallography, x-ray takes on more of a wave. In this situation, the resolution is closely related to wavelength.

For example, when detecting the structure of crystals, since the size of each crystal cell resembles the length of the x-ray, crystal provides an appropriate place for x-ray diffraction. This process obeys the famous Bragg's Law:

$$2\text{d}\sin(\theta) = n\lambda$$

Where d is the spacing between diffracting planes, θ is the incident angle, n is any integer, and λ is the wavelength of the beam. Therefore, in the diffraction process, the resolution of the crystal structure (d) is comparable to λ, which is in the order of nanometer for both hard or soft X-ray.

Problem 2:

1. Starting from Equation (1), derive both Equation (2) and (3).

$$\Delta\lambda = \frac{\mathrm{h}}{\mathrm{m}_0 c}(1 - cos\theta) \ (1)$$

$$\Delta \mathrm{E} = \mathrm{E}_{\mathrm{X,inc}} - E_{X,scat} = \frac{hc}{\lambda_{inc}} - \frac{hc}{\lambda_{scat}} \ (2)$$

$$\mathrm{E}_{\mathrm{X,scat}} = \frac{E_{X,inc}}{1 + \left(\frac{E_{X,inc}}{mc^2}\right)(1 - cos\theta)} \ (3)$$

2. Given incident X-ray energy $\mathrm{E}_{\mathrm{X,inc}} = 67keV$, $(\mathrm{m_e c^2} = 511keV)$, calculate the scatter energy $\mathrm{E}_{\mathrm{X,scat}}$ when $\theta = 0, 30°, 60°, 90°$. Then plot $\mathrm{E}_{\mathrm{X,scat}}$ vs scatter angle.

Grading:
16 points in total:
4 points for question 1, with each formula 2 points.
12 points for question 2, with 8 points for calculating θ, 4 points for the plots.

Solution:

1. The step from equation (1) to (2) simply involves de Broglie's relationship $\mathrm{E} = \mathrm{hf} = \frac{\mathrm{hc}}{\lambda}$. Therefore, the incident and scattered X-ray energy can be represented as:

$$\mathrm{E}_{\mathrm{X,inc}} = hf_{inc} = \frac{hc}{\lambda_{inc}} \ (4)$$

$$\mathrm{E}_{\mathrm{X,scat}} = hf_{scat} = \frac{hc}{\lambda_{scat}} \ (5)$$

Hence:

$$\Delta \mathrm{E} = \mathrm{E}_{\mathrm{X,inc}} - E_{X,scat} = \frac{hc}{\lambda_{inc}} - \frac{hc}{\lambda_{scat}} \ (6)$$

From (1), we know that:

$$\lambda_{\mathrm{scat}} = \lambda_{inc} + \Delta\lambda = \lambda_{inc} + \frac{h}{m_0 c}(1 - \cos\theta) \ (7)$$

Then taking (7) into (5), we can write:

$$\mathrm{E}_{\mathrm{X,scat}} = \frac{hc}{\lambda_{inc} + \frac{h}{m_0 c}(1 - \cos\theta)} = \frac{\frac{hc}{\lambda_{inc}}}{1 + \frac{\frac{hc}{\lambda_{inc}}}{m_0 c^2}(1 - \cos\theta)}$$

$$= \frac{E_{X,inc}}{1 + \left(\frac{E_{X,inc}}{mc^2}\right)(1 - cos\theta)} \ (8)$$

Equation (6) and (8) is the desired result.

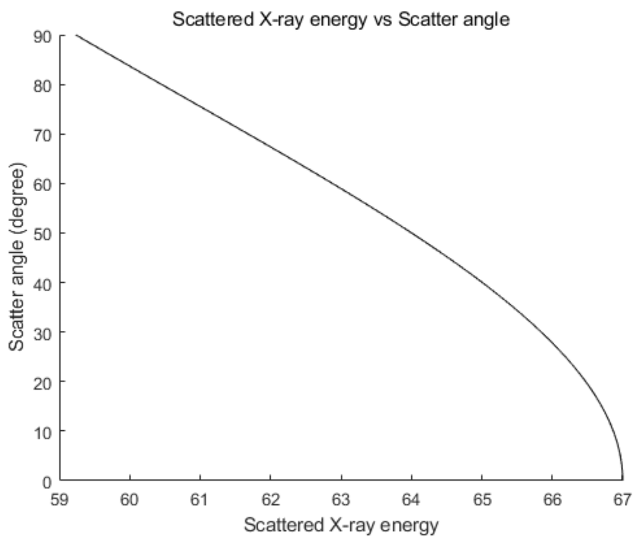
2. Take $E_{X,inc} = 67keV$ and $m_e c^2 = 511keV$ into (8), the following relationship is obtained:

$$E_{X,scat} = \frac{67keV}{1 + \left(\frac{67}{511}\right)(1 - cos\theta)}$$

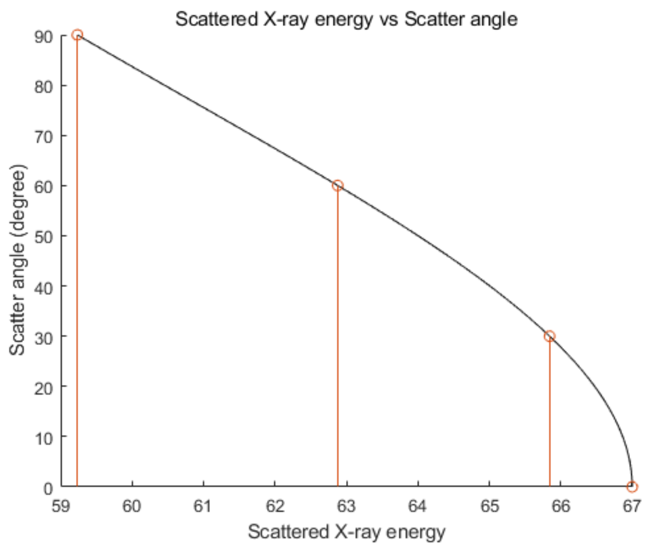
Take $\theta = 0, 30°, 60°, 90°$, we can get:

$$E_{X,scat} = \begin{cases} 67.0000keV, & \theta = 0° \\ 65.8434keV, & \theta = 30° \\ 62.8779keV, & \theta = 60° \\ 59.2336keV, & \theta = 90° \end{cases}$$

The $E_{X,scat}$ vs scatter angle plot is as follows:



The $E_{X,scat}$ calculated can be marked on the plot:



Problem 3:

$6.5 * 10^{16}$ X-ray photons with energy of $50keV$ were incident into a 4cm thickness phantom, and finally reached the detector. The length of lead strip is 4cm and the separation of lead strip is 1cm. Assuming that Compton scattering will at most happen once during the travelling, and it can happen anywhere. Figure 1 shows several possibilities of X-ray propagation. Provided the linear attenuation of Phantom 4cm^{-1}, calculate the energy range of X-ray which was detected by detector. (If you need any physical constant for the calculation, please refer to the course PPT)

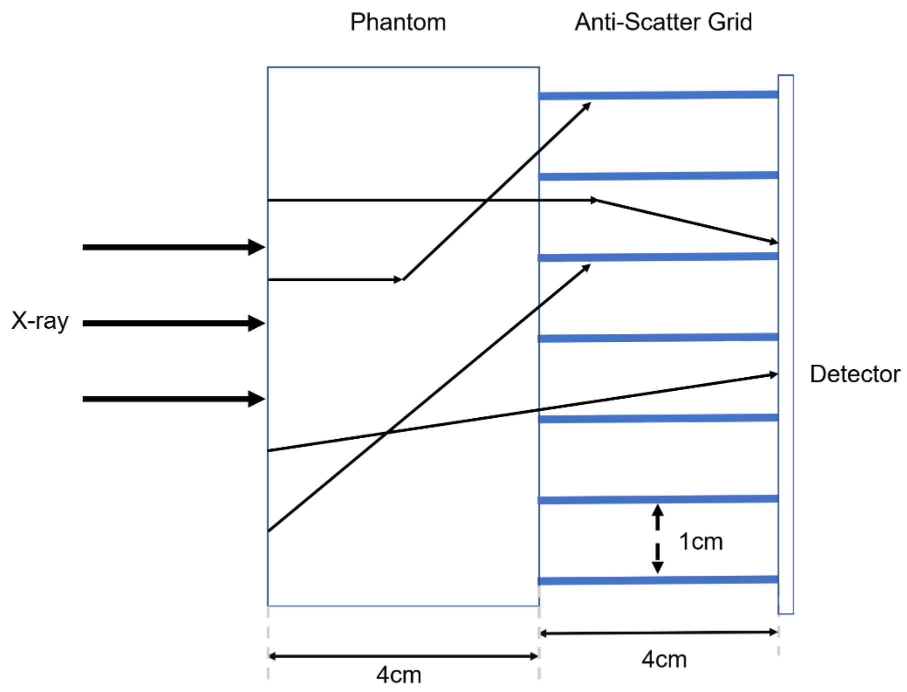
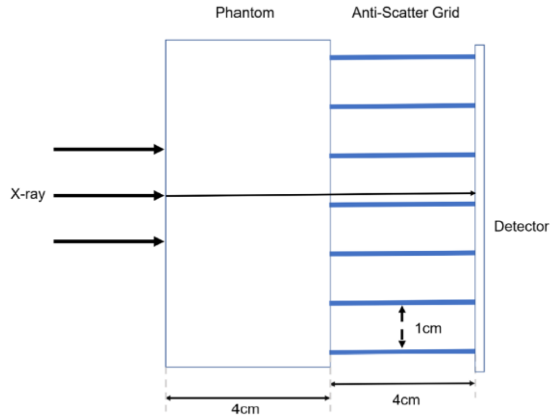


Figure 1

Grading:
16 points in total, 8 points for the upper and lower limit respectively

Solution: $[49.85\text{keV}, 50\text{keV}]$
Extreme case 1: (No Compton Scattering)

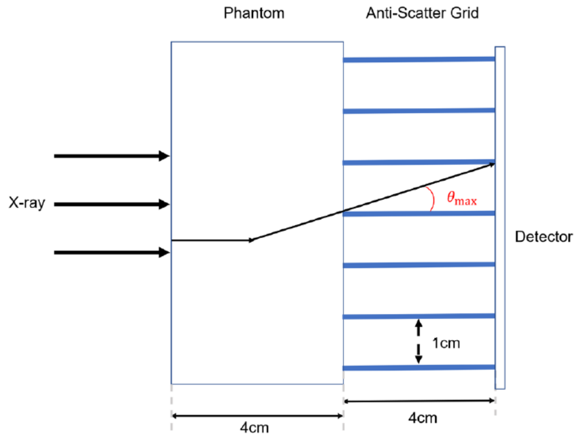


$$\mathrm{E}_{max} = \mathrm{E}_{\mathrm{inc}} = 50\text{keV}$$

Extreme case 2:

Compton scattering function: $\mathrm{E}_{\mathrm{X,scat}} = \frac{E_{X,inc}}{1+\left(\frac{E_{X,inc}}{m_e c^2}\right)(1-cos\theta)}$

In order to minimize $\mathrm{E}_{\mathrm{scat}}$, we should find the maximum value of θ, which is shown below:



$$\theta_{\max} = \arctan\left(\frac{1}{4}\right) \approx 14.04°$$

$$\cos(\theta_{max}) = \frac{4}{\sqrt{4^2+1^2}} = \frac{4}{\sqrt{17}}$$

$$E_{scat\_min} = \frac{E_{X,inc}}{1+\left(\frac{E_{X,inc}}{m_e c^2}\right)(1-\cos\theta_{max})} \approx 49.86\text{keV}$$

Problem 4:

Digital X-ray mammography is used to detect small tumors or microcalcifications in the breast and proves to be effective detection of breast cancer. In mammographic examinations, the upper plate keeps going down to compress the breast, as shown in the figure below. Answer the following with a brief explanation

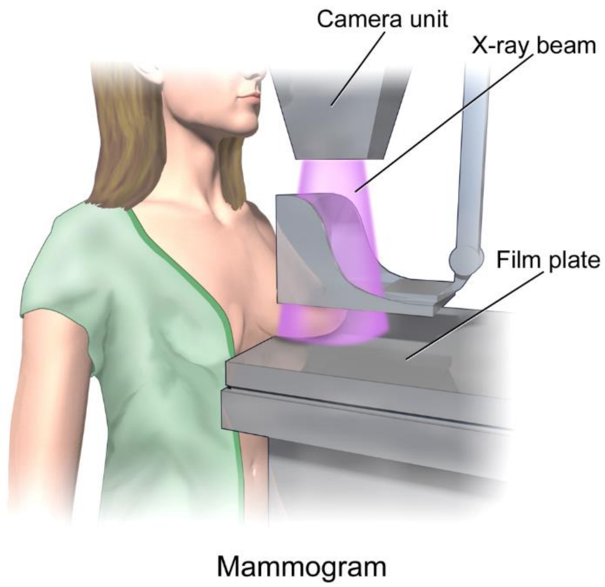
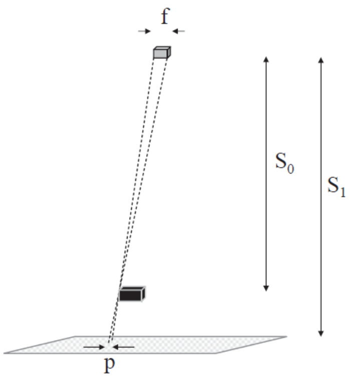


Figure 2: Illustration of the compression during X-ray mammography (from Wikipedia)

(a) Will the spatial resolution become better or worse by compression? Why?
(b) Will the image contrast become better or worse? Why
(c) Is the required X-ray dose for a given image SNR higher or lower with compression?

Grading:
15 points in total, 5 points for each question.

Solution:



(a) The schematic of the X-ray imaging process is presented on the left. The size of the penumbra region P can be determined by S_1, S_0 and f by:

$$P = \frac{f(S_1 - S_0)}{S_0}$$

When compressing the breast in mammographic examination, S_1 remains unchanged and S_0 is reduced. Thus, P will decrease, indicating a better spatial resolution of the image.

(b) The contrast of X-ray images is predominantly determined by Compton scatter. Compression makes the object thinner in the direction of X-ray transmission, and therefore the number of Compton scattered X-rays decrease, contributing to the enhancement of image contrast.

(c) Firm compression reduces overlapping anatomy, decreases tissue thickness, it results in fewer scattered X-rays, less geometric blurring of anatomic structures, and lower radiation dose to the breast tissues. So, for a given image SNR, when with compression, the required dose decreases.

Problem 5:

Compute the CT number of the following materials at X-ray energy $= 50\text{keV}(\text{cm}^{-1})$

TABLE3-1 MATERIAL DENSITY, ELECTRONS PER MASS, ELECTRON DENSITY, AND THE LINEAR ATTENUATION COEFFIENT (AT 50 keV) FOR SEVERAL MATERIALS

MATERIAL	DENSITY(g/cm^3)	ELECTRONS PER MASS(e/g) $\times 10^{23}$	ELECTRON DENSITY (e/cm^3) $\times 10^{23}$	μ @ 50keV(cm^{-1})
Hydrogen gas	0.000084	5.97	0.0005	0.000028
Water vapor	0.000598	3.34	0.002	0.000128
Air	0.00129	3.006	0.0038	0.00029
Fat	0.91	3.34	3.04	0.193
Ice	0.917	3.34	3.06	0.196
Water	1	3.34	3.34	0.214
Muscle	1	3.36	3.36	0.214
Compact bone	1.85	3.192	5.91	0.573

Figure 3: Table from PPT-04, slide 26

Grading:
12 points, 3 points for the CT number formula, 9 points for correct CT number

Solution:

CT number is closely related to the attenuation coefficient of the material and is given by:

$$\text{CT number} = \frac{\mu - \mu_{H_2O}}{\mu_{H_2O}} * 1000 \ (*)$$

Take all the attenuation coefficients into (*), we get:

Material	CT number (HU)
Hydrogen gas	-999.9
Water vapor	-999.4
Air	-998.6
Fat	-98.1
Ice	-84.1
Water	0
Muscle	0
Compact bone	1509.3

Problem 6:

For the object shown in Figure 4(a), assume that a darker area corresponds to an area of higher signal and the detailed geometry relationship is shown as in figure 4(b). Please answer the following questions.

(1) Draw projections that would be acquired at angles $\phi = 0°, 45°, 90°, 135°$ and $180°$ (ignore beam hardening)

(2) Sketch the sinogram for values of ϕ from 0 to 360°.

(3) Do back projection to the sinogram to reconstruct the image.

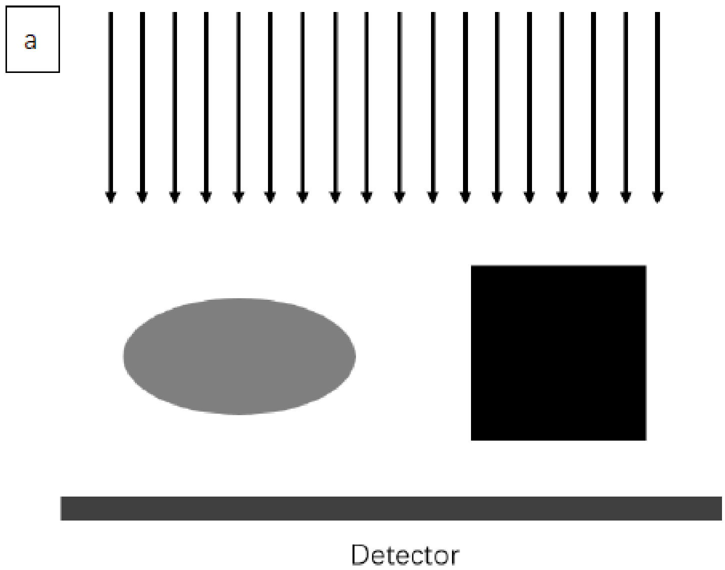


Figure 4(a): Diagram of the object and the direction of X-ray

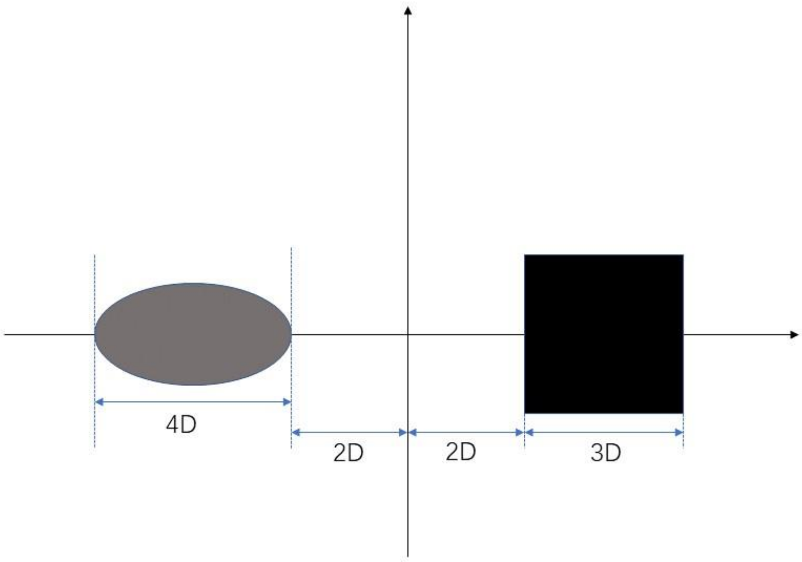
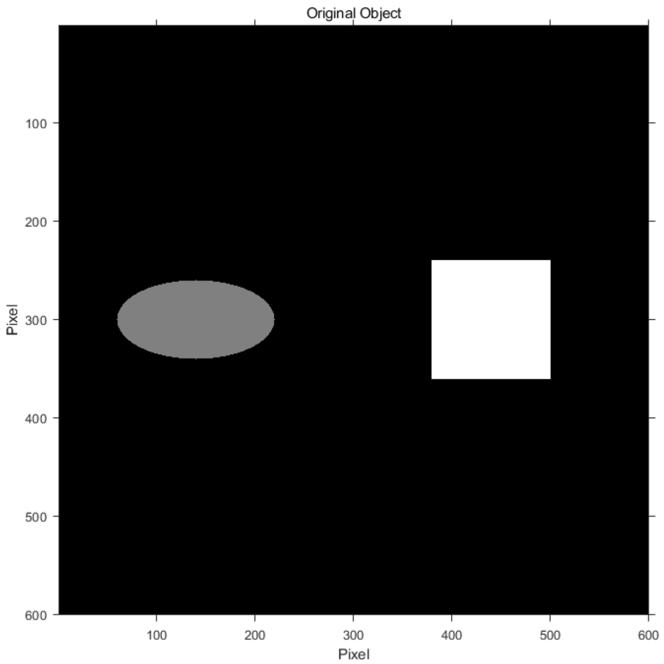


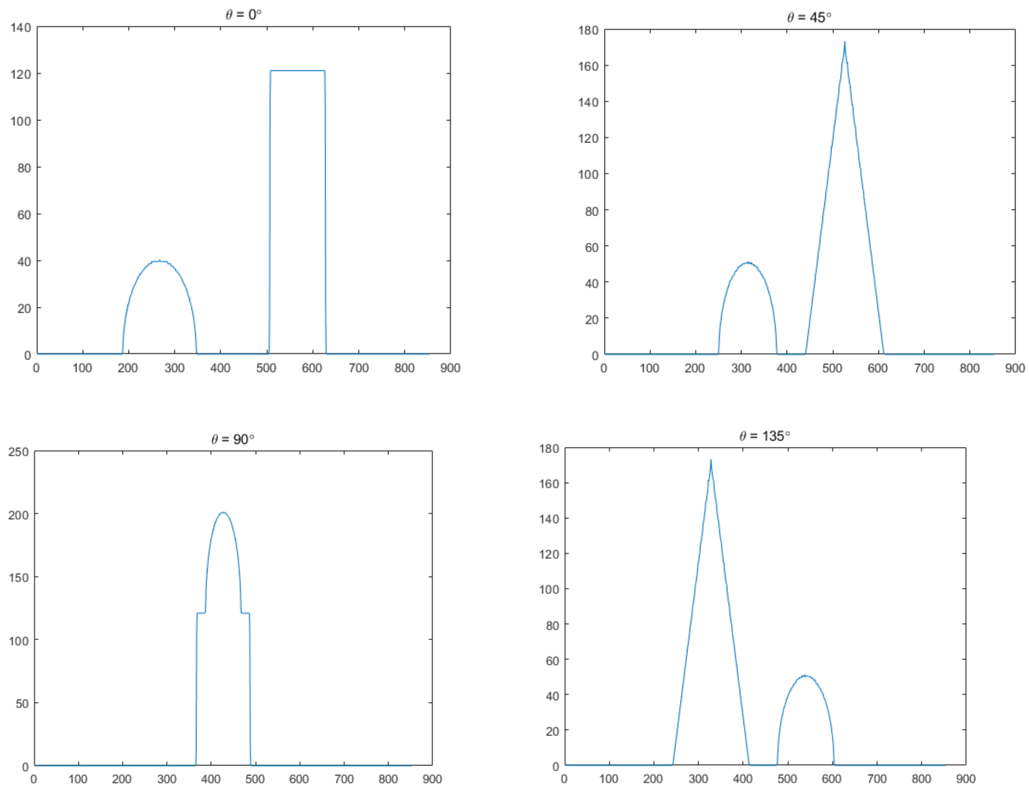
Figure 4(b): detailed geometric relation

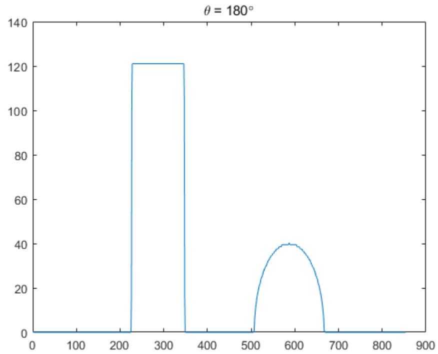
Grading:
25 points in total, 9 points for question 1, 8 points for question 2, 8 points for question 3

Solution:
The original figure is as follows:

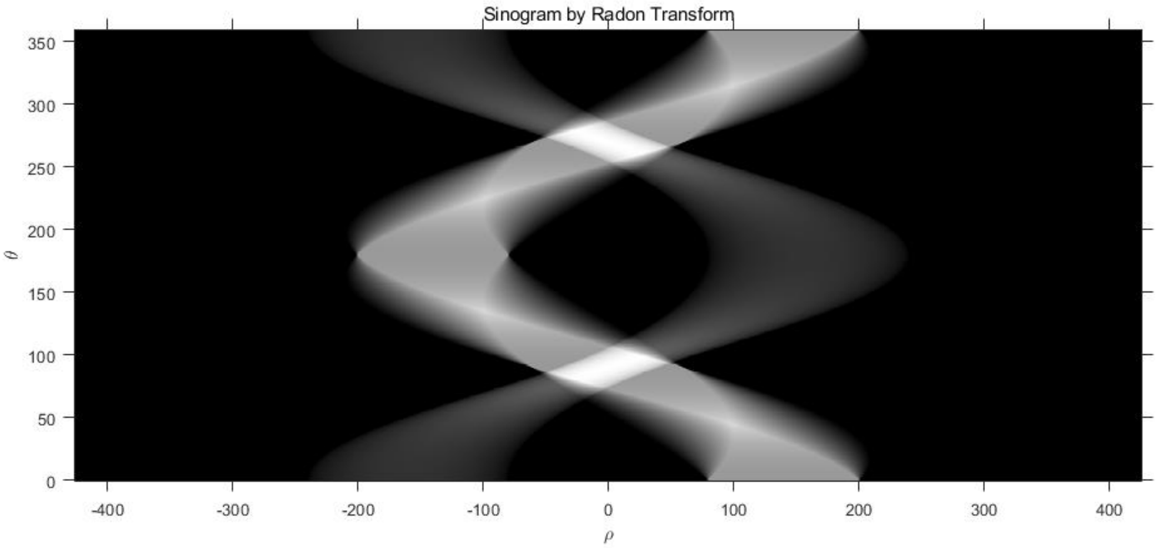


Sampling at different angles:





The sinogram by radon transform:



The reconstructed image by back projection without filtering is shown as follows: