SI211 Homework 6

Prof. Boris Houska

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1. *Bound on Norm of Matrix Inverse.* Consider matrix $A \in \mathbb{R}^{n\times n}$ with $\|A\| < 1$. Prove that matrix $I - A$ is invertible and

$$\|(I-A)^{-1}\| \leq \frac{1}{1-\|A\|}$$

   where I is the identity matrix.

2. *Condition Number.* Consider Hilbert's matrix of size $n \times n$, given by

$$H_n = \begin{pmatrix} 1 & \frac{1}{2} & \cdots & \frac{1}{n} \\ \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{n} & \frac{1}{n+1} & \cdots & \frac{1}{2n-1} \end{pmatrix}$$

   (a) Solve the linear equation system with $b = (1,1,1)^{\mathsf{T}}$

$$H_3 x_a = b \qquad (H_3 + \delta H) x_b = b \qquad H_3 x_c = b + \delta b$$

   numerically, with $\delta H = I/1000$, $\delta b =$ `2e-3 * [1;-1;1]`. Compute $\mathrm{cond}_\infty(H_3)$, $\frac{\|x_b - x_a\|_\infty}{\|x_a\|_\infty}$, $\frac{\|x_c - x_a\|_\infty}{\|x_a\|_\infty}$, , $\frac{\|\delta H\|_\infty}{\|H_3\|_\infty}$, $\frac{\|\delta b\|_\infty}{\|b\|_\infty}$ and interpret your result.

   (b) Write a computer program to plot $\mathrm{cond}_\infty(H_n)$ for $n =$ `2:10`.

3. *LR Decomposition.* Consider the matrix

$$A = \begin{pmatrix} 9 & 1 & 9 \\ 3 & 4 & 10 \\ 8 & 1 & 2 \end{pmatrix}$$

   (a) Write out the Gauss elimination process for A. Write out the Forbenius matrices G_1 and G_2 explicitly.

   (b) Implement an efficient computer progra6m to solve $Ax = b_i$ for

$$b_1 = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} \qquad b_2 = \begin{pmatrix} 3 \\ 2 \\ 3 \end{pmatrix} \qquad b_3 = \begin{pmatrix} 10 \\ 4 \\ 9 \end{pmatrix}$$

   using the LR decomposition result calculated in previous sub-question. You need to implement two functions: `s = forwardSubstitution(M, v)` and `s = backwardSubstitution(M, v)` by yourself.

4. *Cholesky Decomposition.* Write a computer program to calculate a Cholesky decomposition $A = LDL^{\mathsf{T}}$ for

$$A = \begin{pmatrix} 353 & -51 & -11 & -96 & 10 \\ -51 & 485 & -52 & -18 & -1 \\ -11 & -52 & 331 & 69 & 21 \\ -96 & -18 & 69 & 572 & 63 \\ 10 & -1 & 21 & 63 & 582 \end{pmatrix}$$