CS244: Theory of Computation

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Outline

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Can a machine reproduce itself?

Self-Reference

Lemma

There is a computable function $f : \Sigma^* \to \Sigma^*$, where if w is any string, $f(w)$ is the description of a Turing machine P_w that prints out w and then halts.

P_f on input w: (computable function $f : \Sigma^* \to \Sigma^*$)

1. Construct a TM P_w:

   P_w on input x:

   1. Erase input.
   2. Write w on the tape.
   3. Halt.

2. Output $\langle P_w \rangle$.

Self-Reproduce TM

A TM SELF such that $\langle \text{SELF} \rangle = \langle AB \rangle$

- A produces $\langle B \rangle$ and passes control to B
- B produces $\langle A \rangle$
- AB produces $\langle AB \rangle$

How to implement A and B?

$$A = P_{\langle B \rangle} \text{ and } B = P_{\langle A \rangle}?$$

Self-Reproduce TM

- $A = P_{\langle B \rangle}$, A writes $\langle B \rangle$ into the tape
- B constructs A based on the output $\langle B \rangle$ of A

B on input $\langle M \rangle$:

1. Compute $f(\langle M \rangle)$ that is $P_{\langle M \rangle}$. ($P_{\langle B \rangle} = A$ when $M = B$)
2. Combine $P_{\langle M \rangle}$ with $\langle M \rangle$ to make a complete TM SELF.
3. Output $\langle \text{SELF} \rangle$

Behavior of SELF

- $A = P_{\langle B \rangle}$, A writes $\langle B \rangle$ into the tape
- B constructs A based on the output $\langle B \rangle$ of A

B on input $\langle M \rangle$:

1. Compute $q(\langle M \rangle)$ that is $P_{\langle M \rangle}$. ($P_{\langle B \rangle} = A$ when $M = B$)
2. Combine $P_{\langle M \rangle}$ with $\langle M \rangle$ to make a complete TM SELF.
3. Output $\langle \text{SELF} \rangle$

Behavior of SELF

1. First A runs. It prints $\langle B \rangle$ on the tape.
2. B starts. It looks at the tape and finds its input, $\langle B \rangle$.
3. B calculates $f(\langle B \rangle)$ that is $\langle P_{\langle B \rangle} \rangle = \langle A \rangle$.
4. B combines that with $\langle A \rangle$ and $\langle B \rangle$ into the TM SELF.
5. B prints $\langle \text{SELF} \rangle$ and halts.

Recursion Theorem

Similar to SELF,
a TM M has ability to refer to its own description $\langle M \rangle$

This idea can be generalized into recursion theorem which allows a TM M to obtain its own description $\langle M \rangle$ and perform computation with $\langle M \rangle$ instead of just printing $\langle M \rangle$

Theorem (Recursion theorem)

Let T be a Turing machine that computes a function $t : \Sigma^* \times \Sigma^* \to \Sigma^*$. There is a Turing machine R that computes a function $r : \Sigma^* \to \Sigma^*$, where for every w,

$$r(w) = t(\langle R \rangle, w)$$

Recursion Theorem

Theorem (Recursion theorem)

Let T be a Turing machine that computes a function $t : \Sigma^* \times \Sigma^* \to \Sigma^*$. There is a Turing machine R that computes a function $r : \Sigma^* \to \Sigma^*$, where for every w,

$$r(w) = t(\langle R \rangle, w)$$

R on input w:

- $A = P_{\langle BT \rangle}$, A writes $\langle BT \rangle$ into the tape following w
- B constructs A based on the output $\langle BT \rangle$ of A

B on input $\langle M \rangle$:

1. Compute $f(\langle M \rangle)$ that is $P_{\langle M \rangle}$. ($P_{\langle BT \rangle} = A$ when $M = BT$)
2. Combine $P_{\langle M \rangle}$ with $w\langle BT \rangle$ and write $\langle R, w \rangle$ into the tap
3. Pass control to T

Applications of Recursion Theorem

Theorem

A_{TM} is not decidable.

Recall that we prove this via the diagonalization method. We can prove this via the recursion theorem

B on input w:

1. Obtain via the recursion theorem, $\langle B \rangle$.
2. Run the decider R of A_{TM} on $\langle B, w \rangle$.
3. If R accepts, then reject. If R rejects, then accept.

Applications of Recursion Theorem

$$\text{MIN}_{\text{TM}} = \{\langle M \rangle \mid M \text{ is a minimal TM that is equivalent to } M\}.$$

Theorem

MIN_{TM} is not Turing-recognizable.

C on input w:

1. Obtain via the recursion theorem, $\langle C \rangle$.
2. Run the enumerator E of MIN_{TM} until a TM D appears with $|\langle D \rangle| > |\langle C \rangle|$.
3. Simulate D on w.

- D and C are equivalent
- $|\langle D \rangle| > |\langle C \rangle|$
- Then, $\langle D \rangle \notin \text{MIN}_{\text{TM}}$, contradiction with Item 2

Applications of Recursion Theorem

Theorem

Let $f : \Sigma^* \to \Sigma^*$ be a computable function. There exists a TM M such that $f(\langle M \rangle)$ describes a TM equivalent to M.

M on input w:

1. Obtain via the recursion theorem, $\langle M \rangle$.
2. Compute $f(\langle M \rangle)$ to obtain a description of a TM M'.
3. Simulate M' on w.

Then, M and M' are equivalent.

Outline

Logical Theories

Definition (Model)

A model M is a tuple $(U, P_1, \cdots, P_k)$, where

- U is the universe
- $P_i : X^r \to \{\text{Ture,False}\}$ for $1 \leq i \leq k$ is a r-arity relation for some $r \in \mathbb{N}$

Definition (Logical formulae)

Formulae over M are defined by the following syntax:

$$\phi ::= P_i(x_1, \cdots, x_r) \mid \phi_1 \wedge \phi_2 \mid \phi_1 \vee \phi_2 \mid (\phi) \mid \neg\phi \mid \exists x.[\phi] \mid \forall x.[\phi]$$

where $x, x_1, \cdots, x_r$ are variables over U and P_i is a r-arity relation in M.

Logical Theories

Definition (Sentence)

A variable x in a formula ϕ is called free variable if it is not bound within the scope of a quantifier.

A formula ϕ is called sentence if it does not have any free variables.

Definition (PNF)

A formula ϕ is in prenex normal form if

$$\phi = Q_1 x_1 . Q_2 x_2 . \cdots . Q_k x_k . [\psi]$$

where $Q_1, \cdots Q_k \in \{\forall, \exists\}$ and ψ is a formula without quantifiers.

Language of Logical Theories

$$L(M) = \{\phi \mid \phi \text{ is true in the model } M\}$$

- $\forall x.\forall y.[x \leq y \lor y \leq x]$ is true in $M = (\mathbb{N}, \leq)$
- $\forall x.\forall y.[x < y \lor y < x]$ is false in $M = (\mathbb{N}, <)$

Language of Logical Theories

Let P be a 3-arity relation such that $P(x_1, x_2, x_3) \equiv x_1 + x_2 = x_3$

$$L(N_+) = \{\phi \mid \phi \text{ is true in the model } N_+ = (\mathbb{N}, P)\}$$

E.g., $\forall x.\exists y.[x + x = y]$ is true in N_+,
but, $\forall x.\exists y.[y + y = x]$ is false in N_+.

Theorem

$L(N_+)$ is decidable.

We construct an NFA N such that:

$$\phi \text{ is true} \iff \varepsilon \in L(N)$$

Proof (1)

Theorem

$L(N_+)$ is decidable.

- Suppose $\phi = Q_1 x_1 . Q_2 x_2 . \cdots . Q_k x_k . [\psi]$.
- Let $\phi_i = Q_{i+1} x_{i+1} . Q_{i+2} x_{i+2} . \cdots . Q_k x_k . [\psi]$ for $0 \leq i \leq k$, where $\phi_k = \psi$.
- Then ϕ_i contains free variables $x_1, x_2 \cdots, x_i$.
- We show how to construct a NFA N_k from ϕ_k, i.e., ψ.
- Then, we show how to construct an NFA N_{k-1} from N_k.
- Finally, N_0 accepts a word iff ϕ is true.

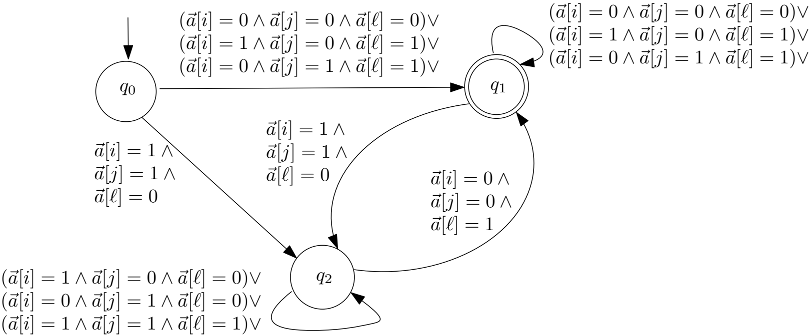
Proof (2)

$$\text{Let } \Sigma_i = \left\{ \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 1 \\ 1 \end{bmatrix}, \cdots, \begin{bmatrix} 1 \\ \cdot \\ \cdot \\ \cdot \\ 1 \\ 1 \end{bmatrix} \right\}$$

- Each column is size i and $\Sigma_0 = \{[]\}$
- A sequence $\vec{a}_1 \vec{a}_2 \cdots \vec{a}_m$ of symbols from Σ_i denotes a valuation of variables $x_1, x_2, \cdots, x_i$
- The value of x_j is the reverse of the binary sequence $\vec{a}_1[j]\vec{a}_2[j] \cdots \vec{a}_m[j]$ of the j-th row in $\vec{a}_1 \vec{a}_2 \cdots \vec{a}_m$

Proof (3)

Consider $P(x_i, x_j, x_\ell)$: Construct $N_{P(x_i, x_j, x_\ell)}$



Proof (4)

- For $\neg P(x_i, x_j, x_\ell)$: construct $\overline{N}_{P(x_i, x_j, x_\ell)}$
- For $P(x_i, x_j, x_\ell) \wedge P'(x_{i'}, x_{j'}, x_{\ell'})$: construct $N_{P(x_i, x_j, x_\ell)} \cap N_{P'(x_{i'}, x_{j'}, x_{\ell'})}$
- For $P(x_i, x_j, x_\ell) \vee P'(x_{i'}, x_{j'}, x_{\ell'})$: construct $N_{P(x_i, x_j, x_\ell)} \cup N_{P'(x_{i'}, x_{j'}, x_{\ell'})}$
- Finally, we get the NFA N_k

Lemma

N_k accepts a word w (that is an assignment of variables $x_1, x_2, \cdots, x_k$) iff ψ is true under w.

Proof (5)

- Suppose the NFA N_i for ϕ_i
- Construct a NFA N_{i-1} for

$$\phi_{i-1} = \exists x_i . Q_{i+1} x_{i+1} . Q_{i+2} x_{i+2} . \cdots Q_k x_k . [\psi]$$

- N_{i-1} is same as N_i, except that

$$\text{If } q \xrightarrow{\begin{bmatrix} b_1 \\ \cdot \\ \cdot \\ \cdot \\ b_{i-1} \\ b_i \end{bmatrix}} q' \text{ in } N_i \text{, then, } N_{i-1} \text{ has } q \xrightarrow{\begin{bmatrix} b_1 \\ \cdot \\ \cdot \\ \cdot \\ b_{i-1} \end{bmatrix}} q'$$

Lemma

N_{i-1} accepts a word w (that is an assignment of variables $x_1, x_2, \cdots, x_{i-1}$) iff ψ_{i-1} is true under w.

Proof (6)

- Suppose the NFA N_i for ϕ_i
- Construct an NFA N_{i-1} for

$$\phi_{i-1} = \forall x_i . Q_{i+1} x_{i+1} . Q_{i+2} x_{i+2} . \cdots Q_k x_k . [\psi]$$

- Construct an NFA $\overline{N}_{i-1}$ for

$$\neg\phi_{i-1} = \exists x_i . \overline{Q}_{i+1} x_{i+1} . \overline{Q}_{i+2} x_{i+2} . \cdots \overline{Q}_k x_k . [\neg\psi]$$

- N_{i-1} is the complement of $\overline{N}_{i-1}$

Lemma

N_{i-1} accepts a word w (that is an assignment of variables $x_1, x_2, \cdots, x_{i-1}$) iff ψ_{i-1} is true under w.

Presburger Arithmetic

A more general decidable theory: Presburger Arithmetic

- universe: integer $\mathbb{Z}$
- atomic formula: $\sum_{i=1}^{n} a_i x_i \bowtie c$,
  - a_i's and c are integer constants
  - x_i's are integer variables
  - $\bowtie \in \{=, \neq, <, >, \leq, \geq, \equiv_m\}$
- Boolean connectors: $\wedge, \vee, \neg$
- Quantifiers: $\forall, \exists$

Its complexity lies between 2-NEXPTime and 3-EXPTime/2-EXPSpace .[1]

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[1] Antoine Durand-Gasselin, Peter Habermehl: On the Use of Non-deterministic Automata for Presburger Arithmetic. CONCUR 2010: 373-387.

An Undecidable Theory: Peano arithmetic

Let P_1 be a 3-arity relation such that $P(x_1, x_2, x_3) \equiv x_1 + x_2 = x_3$
Let P_2 be a 3-arity relation such that $P(x_1, x_2, x_3) \equiv x_1 \times x_2 = x_3$

$$L(N_{+,\times}) = \{\phi \mid \phi \text{ is true in the model } N_{+,\times} = (\mathbb{N}, P_1, P_2)\}$$

Theorem

$L(N_{+,\times})$ is undecidable.

Proof idea: A_{TM} can be reduced to $L(N_{+,\times})$

1. For a TM M and an input w, construct a formula $\psi_{M,w}$ encoding an accepting computation history of M on w
2. $\psi_{M,w}$ contains a free variable x
3. $\exists x.\psi_{M,w}$ is true iff M accepts w

Gödel's Incompleteness Theorem

In any reasonable system of formalizing the notion of provability in number theory, some true statements are unprovable

Definition (Formal Proof)

A formal proof of a statement ϕ is a sequence of statements, $S_1, S_2, \cdots, S_n$ such that

- For every $1 \leq i \leq n$, S_i follows from the preceding statements and certain basic axioms about numbers, using simple and precise rules of implication
- $S_n = \phi$

Gödel's Incompleteness Theorem

The following two reasonable properties of proofs hold:

- The correctness of a proof of a statement can be checked by machine. Formally, $L_{provable} = \{\langle \phi, \pi \rangle | \pi \text{ is a proof of } \phi\}$ is decidable.
- The system of proofs is sound. That is, if a statement is provable (i.e., has a proof), it is true.

Theorem

$L(N_{+,\times})$ is Turing-recognizable.

Proof

Theorem
$L(N_{+,\times})$ is Turing-recognizable.

$P^{TR}_{+,\times}$ on input ϕ:

1. For each π possible proof of length $1, 2, \cdots$.
2. Run the proof checker R of $L_{provable}$ on $\langle \phi, \pi \rangle$.
3. If R accepts, then accept. If R rejects, then continue Item 2.

Theorem

Some true statements in $L(N_{+,\times})$ is not provable.

Assume that all true statements in $L(N_{+,\times})$ are provable. We reduce from $L(N_{+,\times})$.

$P^{NP}_{+,\times}$ on input ϕ:

1. Run the prover of $L(N_{+,\times})$ on ϕ and $\neg\phi$ in parallel.
2. If ϕ is true, then accept.
3. If $\neg\phi$ is true, then reject.

Then $L(N_{+,\times})$ will be decidable, contradicting the fact that $L(N_{+,\times})$ is undecidable.

Outline

Turing reducibility

Definition

Language A is mapping reducible to language B, written $A \leq_{\mathrm{m}} B$, if there is a computable function $f : \Sigma^* \to \Sigma^*$, where for every $w \in \Sigma^*$

$$w \in A \iff f(w) \in B.$$

A_{TM} and $\overline{A}_{\mathsf{TM}}$ are reducible to one another because a solution to either could be used to solve the other by simply reversing the answer

Definition

An oracle for a language B is an external device that is capable of reporting whether any string $w \in B$. An oracle Turing machine is a modified Turing machine that has the additional capability of querying an oracle. We write M^B to describe an oracle Turing machine that has an oracle for language B.

Turing reducibility

An oracle Turing machine can decide more languages than an ordinary Turing machine can

$T^{A_{\mathsf{TM}}}$ on input $\langle M \rangle$:

1. Construct the following TM N:
   N on any input:
   1.1 Run M in parallel on all strings in Σ^*
   1.2 If M accepts any of these strings, accept
2. Query the oracle to determine whether $\langle N, 0 \rangle \in L(A_{\mathsf{TM}})$
3. If the oracle answers NO, accept ; if YES, reject

- If $L(M) \neq \emptyset$, then $L(N) = \Sigma^*$, then $T^{A_{\mathsf{TM}}}$ rejects
- If $L(M) = \emptyset$, then $L(N) = \emptyset$, then $T^{A_{\mathsf{TM}}}$ accepts

$T^{A_{\mathsf{TM}}}$ is a decider of E_{TM}.

Turing reducibility

Definition

Language A is Turing reducible to language B, written $A \leq_T B$, if A is decidable relative to B, i.e., A uses an oracle of the language B

Theorem

If $A \leq_T B$ and B is decidable, then A is decidable.

Replace the oracle of the language B in orale TM T^B by the decider of B yields a standard decidable TM for A

Turing reducibility

Oracle TM solve many problems that are not solvable by ordinary Turing machines. But even such a powerful machine cannot decide all languages

Theorem

Let $A'_{\mathsf{TM}} = \{\langle M^{A_{\mathsf{TM}}}, w\rangle \mid M^{A_{\mathsf{TM}}}$ *is an oracle TM and accepts* $w\}$*.* A'_{TM} *is undecidable relative to* A_{TM}*.*

Turing reducibility

Theorem

Let $A'_{\mathsf{TM}} = \{\langle M^{A_{\mathsf{TM}}}, w\rangle \mid M^{A_{\mathsf{TM}}}$ is an oracle TM and accepts $w\}$. A'_{TM} is undecidable relative to A_{TM}.

Assume A'_{TM} is decidable relative to A_{TM}, let $T^{A_{\mathsf{TM}}}$ be the decider of A'_{TM}.

$D^{A_{\mathsf{TM}}}$ on input $\langle M\rangle$:

1. Simulate the decider $T^{A_{\mathsf{TM}}}$ of A'_{TM} on $\langle M, \langle M\rangle\rangle$
2. If $T^{A_{\mathsf{TM}}}$ accepts, then reject
3. If $T^{A_{\mathsf{TM}}}$ rejects, then accept

- $D^{A_{\mathsf{TM}}}$ accepts $\langle M\rangle$ iff $T^{A_{\mathsf{TM}}}$ rejects $\langle M, \langle M\rangle\rangle$ iff M rejects $\langle M\rangle$
- Let M be $D^{A_{\mathsf{TM}}}$
- $D^{A_{\mathsf{TM}}}$ accepts $\langle D^{A_{\mathsf{TM}}}\rangle$ iff $D^{A_{\mathsf{TM}}}$ rejects $\langle D^{A_{\mathsf{TM}}}\rangle$