SI231b: Matrix Computations

Lecture 14: Eigenvalue Revealing Decomposition

Yue Qiu

qiuyue@shanghaitech.edu.cn

School of Information Science and Technology

ShanghaiTech University

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Algebraic Multiplicity

- Characteristic polynomial $p(\lambda) = \prod_{i=1}^{n} (\lambda - \lambda_i)$
- denote μ_i as the number of repeated eigenvalues of λ_i $(i = 1, \ldots, k)$

$$p(\lambda) = (\lambda - \lambda_1)^{\mu_1} (\lambda - \lambda_2)^{\mu_2} \cdots (\lambda - \lambda_k)^{\mu_k},$$

  with $\mu_1 + \mu_2 + \cdots + \mu_k = n$ and λ_i is distinct with λ_j.
- μ_i is called the algebraic multiplicity of the eigenvalue λ_i

Geometric Multiplicity

- every λ_i can have more than one eigenvector (scaling not counted)
- eigenspace $\mathcal{E}_{\lambda_i}$ associated with λ_i, $\mathcal{E}_{\lambda_i} = \mathcal{N}(\mathrm{A} - \lambda_i \mathrm{I})$
- $\gamma_i = \dim(\mathcal{E}_{\lambda_i})$ is called the geometric multiplicity of the eigenvalue λ_i

Fact: $\mu_i \geq \gamma_i$ for each λ_i.

Outline

- Diagonalization
- Similarity Transformation
- Schur Decomposition
- Eigenvalues of Hermitian Matrices

Theorem 1: An $n \times n$ matrix A is nondefective if and only if it has an eigenvalue decomposition

$$\mathsf{A} = \mathsf{V}\Lambda\mathsf{V}^{-1},$$

with $\Lambda = \mathrm{diag}(\lambda_1,\ \lambda_2,\ \cdots,\ \lambda_n)$ and the k-th column of V being the eigenvector v_k associated with λ_k.

Hint: you need the following lemma to prove the theorem

Lemma 1: Let $\mathsf{A} \in \mathbb{C}^{n\times n}$, and suppose that the eigenvalues $\lambda_1, \ldots, \lambda_n$ are ordered such that $\{\lambda_1, \ldots \lambda_k\}$, $k \leq n$, is the set of all distinct eigenvalues of A. Also, let v_i be *any* eigenvector associated with λ_i. Then $\mathsf{v}_1, \ldots, \mathsf{v}_k$ must be linearly independent.

From **Theorem 1**, another term for nondefective is diagonalizable.

If A admits an eigenvalue decomposition, the following properties can be shown (easily):

- $\det(\mathrm{A}) = \prod_{i=1}^{n} \lambda_i$
- $\mathrm{tr}(\mathrm{A}) = \sum_{i=1}^{n} \lambda_i$
- the eigenvalues of A^k are $\lambda_1^k, \ldots, \lambda_n^k$
- A is nonsingular if and only if A does not have zero eigenvalues
- suppose that A is also nonsingular. Then, $\mathrm{A}^{-1} = \mathrm{V}\boldsymbol{\Lambda}^{-1}\mathrm{V}^{-1}$

Note: the first three properties does not require the eigenvalue decomposition to prove.

Similarity Transformation

For $A \in \mathbb{R}^{n \times n}$ (or $\mathbb{C}^{n \times n}$), if $T \in \mathbb{R}^{n \times n}$ (or $\mathbb{C}^{n \times n}$) is nonsingular, the map $A \mapsto T^{-1}AT$ is called a similarity transformation of A.

Theorem 2 If T is nonsingular, then A and $T^{-1}AT$ have the same

- characteristic polynomial
- eigenvalues
- algebraic multiplicity
- geometric multiplicity

Hint: using characteristic polynomial to show.

Schur Decomposition

Let $A \in \mathbb{C}^{n \times n}$, the Schur decomposition of A is given by

$$A = QTQ^H,$$

where Q is unitary ($Q^H Q = I$), and T is upper-triangular.

Property: Since A and T are similar, the eigenvalues of A appear on the diagonal of T.

Theorem 3: Every square matrix A has a Schur decomposition.

Hint: applying induction to prove.

Real Eigenvalues

Let $A \in \mathbb{C}^{n\times n}$ be Hermitian ($A = A^H$), then

1. the eigenvalues $\lambda_1, \ldots, \lambda_n$ of A are real
2. suppose that λ_i's are ordered such that $\{\lambda_1, \ldots, \lambda_k\}$ is the set of all distinct eigenvalues of A. Also, let v_i be *any* eigenvector associated with λ_i. Then $v_1, \ldots, v_k$ must be orthonormal.

Remark:

- the above results apply to real symmetric matrices, recall that

$$A = A^T \Rightarrow A = A^H.$$

Corollary:

- for a real symmetric matrix, all eigenvectors $v_1, \ldots, v_n$ can be chosen as real

Diagonalization of Hermitian Matrices

Theorem 4: Every Hermitian matrix $A \in \mathbb{C}^{n\times n}$ has an eigenvalue decomposition given by

$$A = V\mathbf{\Lambda}V^H,$$

where $V \in \mathbb{C}^{n\times n}$ is unitary ($V^H V = I$), $\mathbf{\Lambda} = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ with $\lambda_i \in \mathbb{R}$ for all i. Also, if $A \in \mathbb{R}^{n\times n}$ is symmetric, V is an orthogonal matrix.

Hint: can you use Schur decomposition to prove this?

Remark:

- does not require the assumption of $\mu_i = \gamma_i$ for all λ_i

Corollary:

- If A is Hermitian or real symmetric, $\mu_i = \gamma_i$ for all λ_i (no. of repeated eigenvalues = no. of linearly independent eigenvectors)

Factorize a matrix to a form in which eigenvalues are explicitly displayed

- Diagonalization, $A = V\Lambda V^{-1}$, exists if and only if A is nondefective.
- Schur decompositon, $A = QTQ^H$ always exists.
- Jordan canonical form, $A = SJS^{-1}$ always exists (will not be introduced in our lecture), where

$$J = \begin{bmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_k \end{bmatrix}$$

with

$$J_i = \begin{bmatrix} \lambda_i & & & \\ & \lambda_i & & \\ & & \ddots & \\ & & & \lambda_i \end{bmatrix}, \quad \text{or} \quad J_i = \begin{bmatrix} \lambda_i & 1 & & \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{bmatrix}$$

Readings

You are supposed to read

- Gene H. Golub and Charles F. Van Loan. *Matrix Computations*, Johns Hopkins University Press, 2013.

  Chapter 7.1, 8.1