SI251 Convex Optimization, Fall 2022
Quiz 1

Monday, Sep. 26

1. **Convex Set:** Describe the dual cone for each of the following cones:
   (a) $K = \mathbb{R}^2$. (10 points)
   (b) $K = \{(x_1, x_2) | x_1 + x_2 = 0\}$. (10 points)
2. **Convex Function:** Determine the convexity (i.e., convex, concave, or neither) of the following functions.
   (a) $f(x_1, x_2) = 1/(x_1 x_2)$. (10 points)
   (b) $f(x_1, x_2) = x_1^2/x_2$ on $\mathbb{R} \times \mathbb{R}_{++}$. (10 points)
3. **Convex Optimization:** Find all of the stationary points of the following functions. For each stationary point, determine if it is a local minimum, local maximum, or neither. Justify your answer.
   (a) $f_1(x, y) = \frac{x^2}{y^4 - 4y^2 + 5}$ on $\mathbb{R}^2$. (15 points)
   (b) $f_2(x, y) = 100(y - x^2)^2 - x^2$ on $\mathbb{R}^2$. (15 points)
4. **Duality:**
   (a) Derive the dual problems of the following primal problem:

$$\begin{array}{ll} \text{minimize} & \operatorname{Tr}(\mathbf{X}) \\ \text{subject to} & \mathbf{X} \succeq \mathbf{A} \\ & \mathbf{X} \succeq \mathbf{B} \end{array} \tag{1}$$

   where $\mathbf{A}, \mathbf{B} \in \mathbb{S}^n$. (15 points)

   (b) Consider the following compressive sensing problem via ℓ_1-minimization:

$$\begin{array}{ll} \underset{\boldsymbol{x}}{\text{minimize}} & \|\boldsymbol{x}\|_1 \\ \text{subject to} & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{z}, \end{array} \tag{2}$$

   where $\boldsymbol{A} \in \mathbb{R}^{m \times d}, \boldsymbol{x} \in \mathbb{R}^d$ and $\boldsymbol{z} \in \mathbb{R}^m$. Please write down the equivalent linear programming reformulation of problem (2), and then write down the dual problem of the reformulated linear program. (15 points)

Solution:

1. **Convex Set:** Describe the dual cone for each of the following cones:
   (a) $K^* = \{\mathbf{0}\}$. To see this, we need to identify the values of $\mathbf{y} \in \mathbb{R}^2$ for which $\mathbf{y}^T\mathbf{x} \geq 0$ for all $\mathbf{x} \in \mathbb{R}^2$. But given any $\mathbf{y} \neq 0$, consider the choice $\mathbf{x} = \mathbf{y}$, for which we have $\mathbf{y}^T\mathbf{x} = \|\mathbf{y}\|_2 < 0$. So the only possible choice is $\mathbf{y} = 0$ (which indeed satisfies $\mathbf{y}^T\mathbf{x} \geq 0$ for all $\mathbf{x} \in \mathbb{R}^2$).
   (b)

$$\begin{aligned} K^* &= \{(y_1, y_2) \mid x_1 y_1 + x_2 y_2 \geq 0 \text{ for all } \boldsymbol{x} \in K\} \\ &= \{(y_1, y_2) \mid x_1 (y_1 - y_2) \geq 0 \text{ for all } x_1\} \\ &= \{(y_1, y_2) \mid y_1 = y_2\} \end{aligned}$$

2. **Convex Function:** Determine the convexity (i.e., convex, concave, or neither) of the following functions.

(a)
$$\nabla^2 f(x_1, x_2) = \begin{bmatrix} \frac{2}{x_1^3 x_2} & \frac{1}{x_1^2 x_2^2} \\ \frac{1}{x_1^2 x_2^2} & \frac{2}{x_1 x_2^3} \end{bmatrix}.$$

When $x_1 x_2 > 0$, the Hessian of f is positive semidefinite, hence f is convex. When $x_1 x_2 < 0$, f is concave. Thus, f is neither convex nor concave when $x_1, x_2 \in \mathbb{R}$.

(b) Method 1:
$$\nabla^2 f(x_1, x_2) = \begin{bmatrix} \frac{2}{x_2} & -\frac{2x_1}{x_2^2} \\ -\frac{2x_1}{x_2^2} & \frac{2x_1^2}{x_2^3} \end{bmatrix} \succeq \mathbf{0}.$$

The Hessian of f is positive semidefinite, hence f is convex.
Method 2: The f is quadratic-over-linear function, and hence is convex.

3. **Convex Optimization:** Find all of the stationary points of the following functions. For each stationary point, determine if it is a local minimum, local maximum, or neither. Justify your answer.

(a)
$$\nabla f_1(x, y) = \left(\frac{2x}{y^4 - 4y^2 + 5}, \frac{-4x^2 y \left(y^2 - 2\right)}{\left(y^4 - 4y^2 + 5\right)^2} \right)$$

Noting that $y^4 - 4y^2 + 5 = \left(y^2 - 2\right)^2 + 1 > 0$ for all y, the denominator is never zero, and so the gradient vanishes iff $x = 0$. Finally, note that since the denominator is always positive and the numerator is nonnegative, $f_1(x, y) \geq 0$, with equality iff $x = 0$. It follows that every stationary point is a local minimum.

(b) The gradient reads
$$\nabla f_2(x, y) = \left(-400 \left(y - x^2\right) x - 2x, 200 \left(y - x^2\right)\right).$$
and vanishes only when $(x, y) = (0, 0)$, which is therefore the only stationary point. To characterize this point, we use the second derivative test and calculate the determinant of the Hessian

$$\begin{aligned} D(x, y) &= \frac{\partial^2 f_2}{\partial x^2} \frac{\partial^2 f_2}{\partial y^2} - \left(\frac{\partial^2 f_2}{\partial x \partial y} \right)^2 \\ &= 200 \left(-400y + 1200x^2 - 2\right) - (-400x)^2. \end{aligned}$$

Since $D(0, 0) = -400 < 0$, the Hessian is indefinite, the point $(0, 0)$ is a saddle point, i.e. neither a local minimum nor a local maximum.

4. **Duality:**

(a) Derive the dual problems of the following primal problem:

$$\begin{array}{ll} \text{minimize} & \operatorname{Tr}(\mathbf{X}) \\ \text{subject to} & \mathbf{X} \succeq \mathbf{A} \\ & \mathbf{X} \succeq \mathbf{B} \end{array} \tag{3}$$

where $\mathbf{A}, \mathbf{B} \in \mathbb{S}^n$.
The Lagrangian is
$$\begin{aligned} \mathcal{L}(\boldsymbol{X}, \boldsymbol{Z}, \boldsymbol{\Lambda}) &= \operatorname{Tr}(\boldsymbol{X}) + \operatorname{Tr}((\boldsymbol{A} - \boldsymbol{X})\boldsymbol{Z}) + \operatorname{Tr}((\boldsymbol{B} - \boldsymbol{X})\boldsymbol{\Lambda}) \\ &= \operatorname{Tr}(\boldsymbol{X}(\boldsymbol{I} - \boldsymbol{Z} - \boldsymbol{\Lambda})) + \operatorname{Tr}(\boldsymbol{A}\boldsymbol{Z}) + \operatorname{Tr}(\boldsymbol{B}\boldsymbol{\Lambda}) \end{aligned}$$
where $\boldsymbol{Z} \succeq \mathbf{0}$ and $\Lambda \succeq \mathbf{0}$ are the dual variables. The dual function is given by

$$g(\boldsymbol{Z}, \boldsymbol{\Lambda}) = \inf_{\boldsymbol{X}} \mathcal{L}(\boldsymbol{X}, \boldsymbol{Z}, \boldsymbol{\Lambda}) = \begin{cases} \operatorname{Tr}(\boldsymbol{A}\boldsymbol{Z}) + \operatorname{Tr}(\boldsymbol{B}\boldsymbol{\Lambda}) & \boldsymbol{Z} + \boldsymbol{\Lambda} = \boldsymbol{I} \\ -\infty & \text{otherwise} \end{cases}$$

The dual problem can be expressed as

$$\begin{array}{ll} \underset{\boldsymbol{Z}, \boldsymbol{\Lambda}}{\text{maximize}} & \operatorname{Tr}(\boldsymbol{A}\boldsymbol{Z}) + \operatorname{Tr}(\boldsymbol{B}\Lambda) \\ \text{subject to} & \boldsymbol{Z} + \boldsymbol{\Lambda} = \boldsymbol{I} \\ & \boldsymbol{Z} \succeq \mathbf{0}, \boldsymbol{\Lambda} \succeq \mathbf{0}. \end{array}$$

(b) the equivalent LP problem is

$$
\begin{array}{ll}
\underset{\boldsymbol{x}\in\mathbb{R}^d}{\text{minimize}} & \displaystyle\sum_{i=1}^{n} \boldsymbol{y}_i \\
\text{subject to} & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{z} \\
& \boldsymbol{y}_i \geq \boldsymbol{x}_i, i = 1, 2, \ldots, n \\
& \boldsymbol{y}_i \geq -\boldsymbol{x}_i, i = 1, 2, \ldots, n.
\end{array} \tag{4}
$$

The Lagrangian function of (4) is

$$
\begin{aligned}
L(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{\lambda}, \boldsymbol{u}, \boldsymbol{v}) &= \sum_{i=1}^{n} \boldsymbol{y}_i + \boldsymbol{\lambda}^T(\boldsymbol{A}\boldsymbol{x} - \boldsymbol{z}) + \boldsymbol{u}^T(\boldsymbol{x} - \boldsymbol{y}) - \boldsymbol{v}^T(\boldsymbol{x} + \boldsymbol{y}) \\
&= (\boldsymbol{1} - \boldsymbol{u} - \boldsymbol{v})^T \boldsymbol{y} + \left(\boldsymbol{\lambda}^T \boldsymbol{A} + \boldsymbol{u}^T - \boldsymbol{v}^T\right) \boldsymbol{x} - \boldsymbol{\lambda}^T \boldsymbol{z}
\end{aligned}
$$

The stationary condition of this function is

$$
\begin{aligned}
\frac{\partial L}{\partial \boldsymbol{y}} &= 1 - \boldsymbol{u} - \boldsymbol{v} = 0 \\
\frac{\partial L}{\partial \boldsymbol{x}} &= \boldsymbol{A}^T \boldsymbol{\lambda} + \boldsymbol{u} - \boldsymbol{v} = 0.
\end{aligned}
$$

So we have the dual problem of (4):

$$
\begin{array}{ll}
\underset{\boldsymbol{\lambda},\boldsymbol{u},\boldsymbol{v}}{\text{maximize}} & -\boldsymbol{\lambda}^T \boldsymbol{z} \\
\text{subject to} & \boldsymbol{u} \geq 0 \\
& \boldsymbol{v} \geq 0.
\end{array}
$$