First-Order Logic

AIMA Chapter 8, 9

Pros of propositional logic

- ☺ Propositional logic allows partial/disjunctive/negated information
  - (unlike most data structures and databases)

- ☺ Propositional logic is compositional:
  - meaning of $B_{1,1} \wedge P_{1,2}$ is derived from meaning of $B_{1,1}$ and of $P_{1,2}$

- ☺ Meaning in propositional logic is context-independent
  - (unlike natural language, where meaning depends on context)

Cons of propositional logic

- ☹ Hard to identify "individuals" (e.g., Mary, 3)
- ☹ Can't directly talk about properties of individuals or relations between individuals (e.g., "Bill is tall")
- ☹ Generalizations, patterns, regularities can't easily be represented (e.g., "all triangles have 3 sides")

First-order logic

- Whereas propositional logic assumes the world contains facts…
- First-order logic (like natural language) assumes the world contains
  - Objects: people, houses, numbers, colors, baseball games, wars, …
  - Relations: red, round, prime, bigger than, part of, comes between, …
  - Functions: father of, best friend of, one more than, …

- Also called first-order predicate logic

Syntax of FOL: Basic elements

- Logical symbols
  - Connectives $\neg, \Rightarrow, \wedge, \vee, \Leftrightarrow$
  - Quantifiers $\forall, \exists$
  - Variables x, y, a, b, ...
  - Equality =
- Non-logical symbols (ontology)
  - Constants KingArthur, 2, ShanghaiTech, ...
  - Predicates Brother, >, ...
  - Functions Sqrt, LeftLegOf, ...

Atomic sentences

Atomic sentence = *predicate* $(term_1,\ldots,term_n)$
or $term_1 = term_2$

Term = *constant* or *variable*
or *function* $(term_1,\ldots,term_n)$

Example:

Brother(KingJohn,RichardTheLionheart)

>(Length(LeftLegOf(Richard)), Length(LeftLegOf(KingJohn)))

Complex sentences

Complex sentences are made from atomic sentences using connectives

$$\neg S,\ \ S_1 \wedge S_2,\ \ S_1 \vee S_2,\ \ S_1 \Rightarrow S_2,\ \ S_1 \Leftrightarrow S_2,$$

Example:

$Sibling(KingJohn,Richard) \Rightarrow Sibling(Richard,KingJohn)$

$>(1,2) \vee \leq (1,2)$

$>(1,2) \wedge \neg >(1,2)$

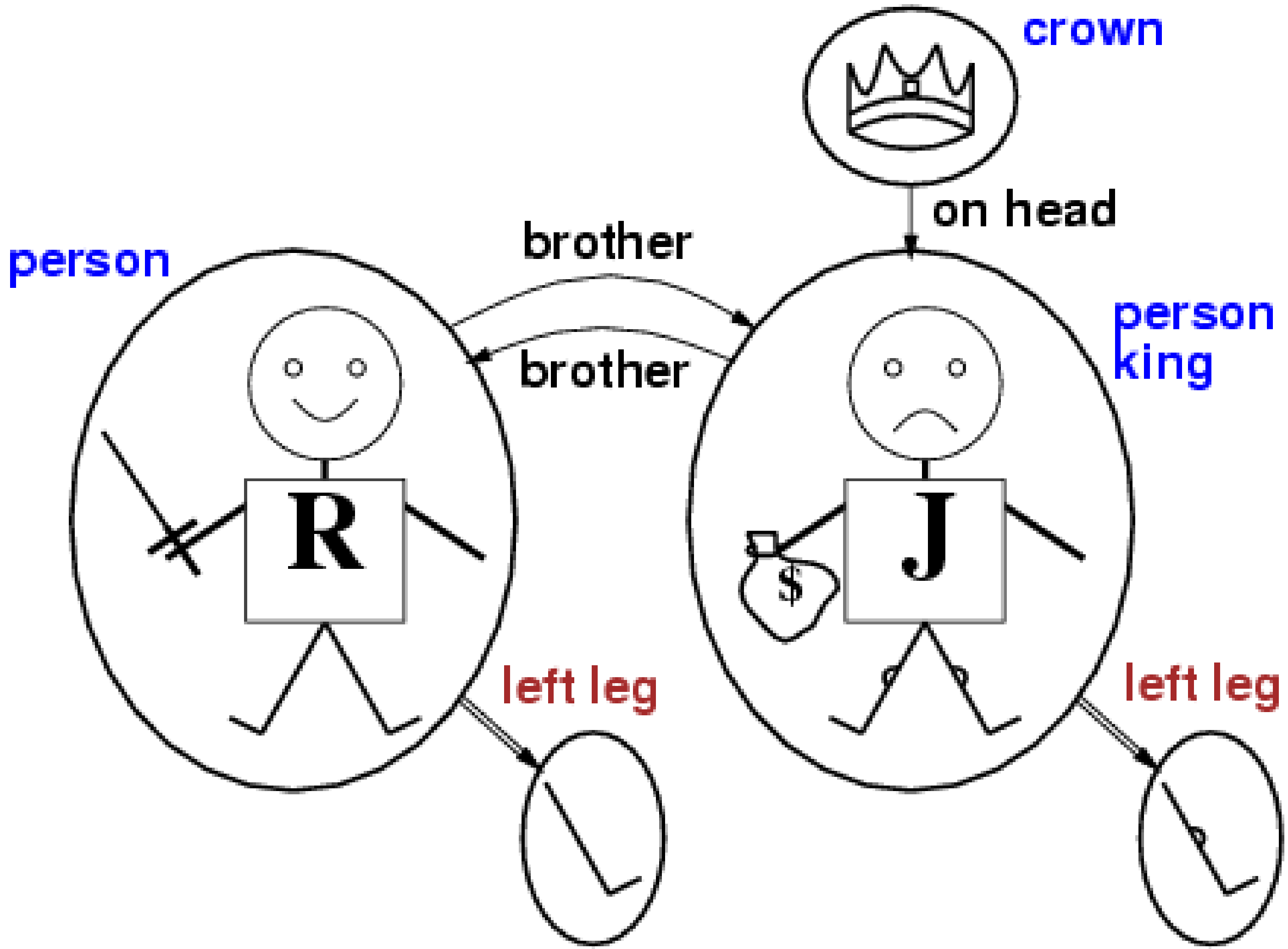
Semantics of FOL

- Sentences are true with respect to a model, which contains
  - Objects and relations among them
  - Interpretation specifying referents for

constant symbols	→	objects
predicate symbols	→	relations
function symbols	→	functional relations

- An atomic sentence $predicate(term_1,\ldots,term_n)$ is true iff the objects referred to by $term_1,\ldots,term_n$ are in the relation referred to by *predicate*

Models for FOL: Example



Models for FOL: Example

- Consider the interpretation:

  Richard → Person R

  John → Person J

  Brother → the brotherhood relation

  Under this interpretation, *Brother*(*Richard*, *John*) is true in the model.

Models for FOL

- How many models do we have? Infinite!

  Models vary in:
  - the number of objects (1 to ∞)
  - the relations among the objects
  - the mapping from constants to objects
  - the mapping from predicates to relations
  - … …

Quantifiers

- Allows us to express properties of collections of objects instead of enumerating objects by name
- Universal: “for all” $\forall$
- Existential: “there exists” $\exists$

Universal quantification

$\forall$<variables> <sentence>

Example: $\forall x\ \ At(x, STU) \Rightarrow Smart(x)$

(Everyone at ShanghaiTech is smart)

$\forall x\ P$ is true in a model m iff P is true with x being each possible object in the model

- Roughly speaking, equivalent to the conjunction of instantiations of P

$$
\begin{aligned}
& At(John, STU) \Rightarrow Smart(John) \\
\wedge\ & At(Richard, STU) \Rightarrow Smart(Richard) \\
\wedge\ & At(STU, STU) \Rightarrow Smart(STU) \\
\wedge\ & \ldots
\end{aligned}
$$

A common mistake to avoid

- Typically, $\Rightarrow$ is the main connective with $\forall$
- Common mistake: using $\wedge$ as the main connective with $\forall$:

  $\forall x \;\; At(x,STU) \wedge Smart(x)$

  means "Everyone is at STU and everyone is smart"

Existential quantification

$\exists$<variables> <sentence>

Example: $\exists x\ \ At(x, STU) \land Smart(x)$

(Someone at ShanghaiTech is smart)

$\exists x\ P$ is true in a model m iff P is true with x being some possible object in the model

- Roughly speaking, equivalent to the disjunction of instantiations of P

$$
\begin{aligned}
& (At(John, STU) \land Smart(John)) \\
\lor\ & (At(Richard, STU) \land Smart(Richard)) \\
\lor\ & (At(STU, STU) \land Smart(STU)) \\
\lor\ & \ldots
\end{aligned}
$$

Another common mistake to avoid

- Typically, $\wedge$ is the main connective with $\exists$
- Common mistake: using $\Rightarrow$ as the main connective with $\exists$:

  $\exists x\ At(x, STU) \Rightarrow Smart(x)$

  is true if there is anyone who is not at STU!

Properties of quantifiers

- $\forall x\ \forall y$ is the same as $\forall y\ \forall x$
- $\exists x\ \exists y$ is the same as $\exists y\ \exists x$
- $\exists x\ \forall y$ is not the same as $\forall y\ \exists x$
  - $\exists x\ \forall y$ Loves(x,y)

    "There is a person who loves everyone in the world"
  - $\forall y\ \exists x$ Loves(x,y)

    "Everyone in the world is loved by at least one person"
- Quantifier duality: each can be expressed using the other

$$\forall x\ \text{Likes}(x,\text{IceCream}) \quad \equiv \quad \neg\exists x\ \neg\text{Likes}(x,\text{IceCream})$$

$$\exists x\ \text{Likes}(x,\text{Broccoli}) \quad \equiv \quad \neg\forall x\ \neg\text{Likes}(x,\text{Broccoli})$$

Sentences with variables

- A variable is free in a formula if it is not quantified
  - e.g., $\forall x \; P(x,y)$
- A variable is bound in a formula if it is quantified
  - e.g., $\forall x \; \exists y \; P(x,y)$
- In a FOL sentence, every variable must be bound.

FOL example: kinship

- Brothers are siblings

  $\forall x,y\ Brother(x,y) \Rightarrow Sibling(x,y).$
- "Sibling" is symmetric

  $\forall x,y\ Sibling(x,y) \Leftrightarrow Sibling(y,x).$
- One's mother is one's female parent

  $\forall x,y\ Mother(x,y) \Leftrightarrow (Female(x) \wedge Parent(x,y)).$
- A first cousin is a child of a parent's sibling

  $\forall x,y\ FirstCousin(x,y) \Leftrightarrow \exists p,ps\ Parent(p,x) \wedge Sibling(ps,p) \wedge Parent(ps,y)$

FOL example: kinship

- Siblings are people with the same parents

  $\forall x,y\ Sibling(x,y) \Leftrightarrow \exists m,f\ \ Parent(m,x) \land Parent(f,x) \land Parent(m,y) \land Parent(f,y)$

Is this correct?

Equality

$term_1 = term_2$ is true under a given interpretation if and only if $term_1$ and $term_2$ refer to the same object

Example: Siblings are people with the same parents:

$\forall x,y\ Sibling(x,y) \Leftrightarrow [\neg(x = y) \wedge \exists m,f\ \neg(m = f) \wedge Parent(m,x) \wedge Parent(f,x) \wedge Parent(m,y) \wedge Parent(f,y)]$

TODO: no truth-value without model

- Give a FOL sentence that looks wrong on the surface
- Hilbert: “One must be able to say at all times—instead of points, straight lines, and planes—tables, chairs, and beer mugs.”