SI231B - Matrix Computations, Spring 2022-23

Homework Set #1

Prof. Ziping Zhao

Acknowledgements:

1) Deadline: **2023-03-13 23:59:59**
2) Please submit your assignments via Blackboard.
3) You can write your homework using latex/word or you can write in handwriting and submit the scanned pdf.

Problem 1. (20 points)

1) Prove that $\mathsf{rank}(\mathbf{A}+\mathbf{B}) \leq \mathsf{rank}(\mathbf{A}) + \mathsf{rank}(\mathbf{B})$. (10 points)
2) Prove that $\mathsf{rank}(\mathbf{AB}) \leq \min\{\mathsf{rank}(\mathbf{A}), \mathsf{rank}(\mathbf{B})\}$ and discuss when the equality holds. (10 points)
   (*Hint*: It suffices to show dim $\mathcal{R}(\mathbf{AB}) \leq$ dim $\mathcal{R}(\mathbf{A})$ and dim $\mathcal{R}(\mathbf{B}^T\mathbf{A}^T) \leq$ dim $\mathcal{R}(\mathbf{B}^T)$.)

Problem 2. (20 points)

From the slides, we know that the direct sum of two subspaces $\mathcal{S}_1$ and $\mathcal{S}_2$ is defined as $\mathcal{S}_3 = \mathcal{S}_1 \oplus \mathcal{S}_2$ if $\mathcal{S}_1 + \mathcal{S}_2 = \mathcal{S}_3$ and $\mathcal{S}_1 \cap \mathcal{S}_2 = \{\mathbf{0}\}$.

1) Suppose $\mathbf{a}_1 + \mathbf{a}_2 = \mathbf{0}$, $\mathbf{a}_1 \in \mathcal{S}_1$, $\mathbf{a}_2 \in \mathcal{S}_2$. Prove that $\mathcal{S}_3 = \mathcal{S}_1 \oplus \mathcal{S}_2$ iff $\mathbf{a}_1 = \mathbf{a}_2 = \mathbf{0}$. (10 points)
2) Suppose $\mathbf{a} = \mathbf{a}_1 + \mathbf{a}_2$, $\mathbf{a} \in \mathcal{S}_1 + \mathcal{S}_2$, $\mathbf{a}_1 \in \mathcal{S}_1$, $\mathbf{a}_2 \in \mathcal{S}_2$. Prove that $\mathcal{S}_3 = \mathcal{S}_1 \oplus \mathcal{S}_2$ iff $\mathbf{a}_1$ and $\mathbf{a}_2$ are unique. (10 points)

   (*Hint*: You can use the conclusion in 1).)

Problem 3. (20 points)

Given a matrix as follows:

$$\mathbf{A} = \begin{bmatrix} -1 & 3 & 3 & 2 \\ -1 & 4 & 7 & 10 \\ 2 & -4 & -1 & 5 \end{bmatrix}.$$

derive its LU decomposition.

Problem 4. (20 points)

Given a matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$, suppose that the LDM (LDU) decomposition of $\mathbf{A}$ exists, prove that

1) The LDM (LDU) decomposition of $\mathbf{A}$ is *uniquely* determined; (10 points)
2) If $\mathbf{A}$ is a symmetric matrix, then its LDM (LDU) decomposition must be $\mathbf{A} = \mathbf{LDL}^T$, which is called LDL (LDL^T) decomposition in this case. (10 points)

(*Hints*: The existence of the LDM (LDU) decomposition implies the non-singularity of the matrix.)

Problem 5. (20 points)

Consider matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$ in the following form,

$$\mathbf{A} = \begin{bmatrix} b_1 & c_1 & 0 & 0 & 0 & 0 \\ a_2 & b_2 & c_2 & 0 & 0 & 0 \\ 0 & a_3 & b_3 & c_3 & 0 & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & a_{n-1} & b_{n-1} & c_{n-1} \\ 0 & 0 & \cdots & 0 & a_n & b_n \end{bmatrix},$$

where a_j, b_j, and c_j are non-zero entries. The matrix in such form is known as a **Tridiagonal Matrix** in the sense that it contains three diagonals.

1) LU decomposition is particularly efficient in the case of tridiagonal matrices. Find the LU decomposition of $\mathbf{A}$ (derivation is expected) and try to complete the Algorithm 1. (15 points)

Algorithm 1: LU decomposition for tridiagonal matrices

Input : Tridiagonal matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$.

Output: LU decomposition of $\mathbf{A}$.

1 *Complete the algorithm here...*

2) Consider symmetric tridiagonal matrices

$$\mathbf{A} = \begin{bmatrix} a & a & 0 \\ a & a+b & b \\ 0 & b & b+c \end{bmatrix},$$

and give the LU decompositions of matrix $\mathbf{A}$. (5 points)