EE150 Signals and Systems
– Part 2: Linear Time-Invariant (LTI) System

Outline of LTI System

- Linearity & Time Invariance
- Convolution Sum & Convolution Integral
- Representation of Signals in terms of Impulses
- Convolution operator

Linear Systems

A system is linear if the following condition holds for any two inputs $x_1(t)$ and $x_2(t)$:

If

$$x_1(t) \longrightarrow \boxed{\text{System}} \longrightarrow y_1(t), \qquad x_2(t) \longrightarrow \boxed{\text{System}} \longrightarrow y_2(t),$$

then

$$ax_1(t) + bx_2(t) \longrightarrow \boxed{\text{System}} \longrightarrow ay_1(t) + by_2(t)$$

Time (shift) Invariance

A system is time-invariant if the following holds for $x(t)$:

If

$$x(t) \longrightarrow \boxed{\text{System}} \longrightarrow y(t),$$

then

$$x(t-t_0) \longrightarrow \boxed{\text{System}} \longrightarrow y(t-t_0)$$

Time (shift) Invariance

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$$x(t-t_0) \longrightarrow \boxed{\text{System}} \longrightarrow y(t-t_0)$$

1. A shift in the input produces the same shift in the output
2. The system has no internal way to keep time

Why LTI is Important?

First, many physical processes possess the properties of linearity and time invariance;

Second, if we can represent any input by a summation of basic signal ($\delta(t)$ or $\delta[n]$) and its time-shifted version, then by using LTI property, the output can be found from the summation or integration of the individual output of the system in terms of its reponses to basic signal and its time-shifted version.

Representation of Discrete-Time Signals

Disctete-time unit impulse can be used to construct any discrete-time signal, because

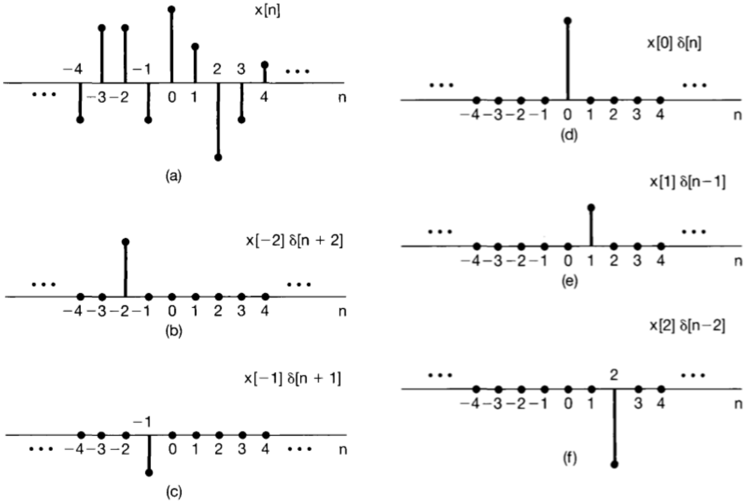
$$\delta[n-k]x[k] = \begin{cases} x[k], & n = k \\ 0, & n \neq k \end{cases}$$

$x[n]$ is represented as a linear combination of shifted unit impulses $\delta[n-k]$, where the weights are $x[k]$:

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k].$$

Note: for any value of n, only one of the terms on the right-hand side (RHS) is nonzero.

Representation of Signal in terms of Impulses cont.



Goal: compute output of LTI system

Use linearity and time invariance as before

- Let output of $\delta[n]$ be $h[n]$,
  therefore output of $\delta[n-k]$ is $h[n-k]$ (time-invariance)
- We know that
  $x[n] = \sum_{-\infty}^{\infty} x[k]\delta[n-k]$ (linear comb. of delta sequences)
  therefore $y[n] = \sum_{-\infty}^{\infty} x[k]h[n-k]$ (linearity)

This operation is called *convolution sum*

$$y[n] = x[n] * h[n]$$

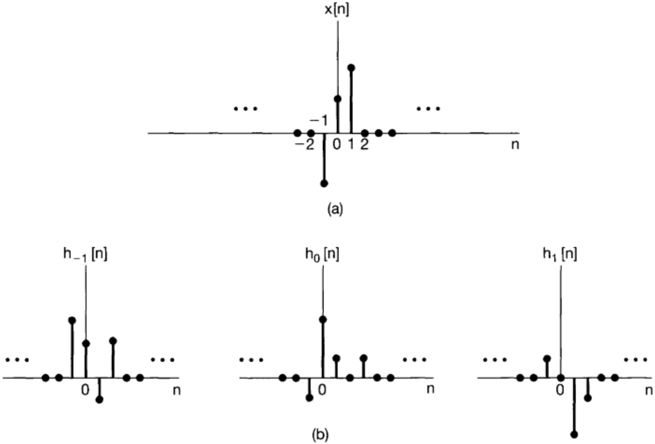
Impulse response

- $\delta[n]$ or $\delta(t)$ is called the impulse function
- When input is $\delta[n]$ (or $\delta(t)$), the output of a system, $h[n]$ (or $h(t)$), is called impulse response
- Unit impulse response completely characterizes an LTI system.

  As we have seen

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

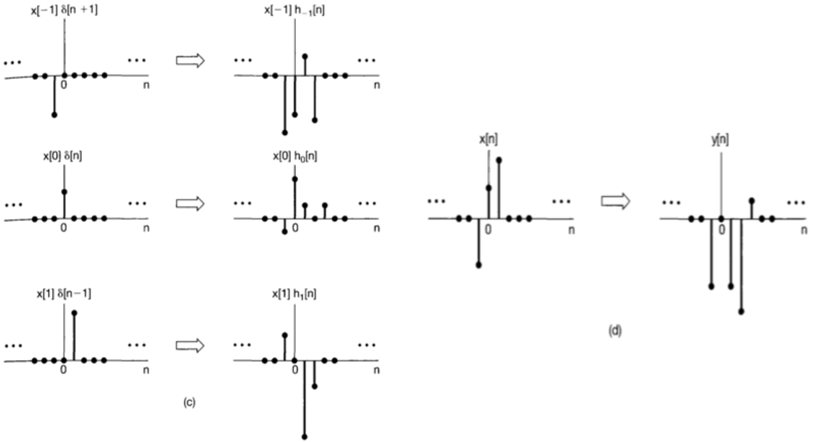
Graphical Interpretation



(a)

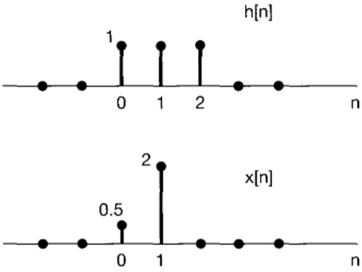
(b)

Graphical Interpretation cont.



Discrete-Time LTI System

Example 1: Consider an LTI system with impulse response $h[n]$ and input $x[n]$

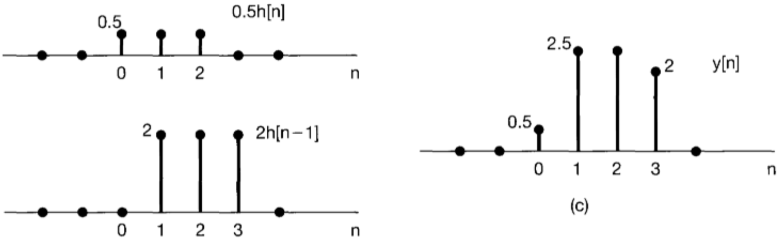


(a)

Discrete-Time LTI System cont.

View as functions of n

$$y[n] = x[0]h[n] + x[1]h[n-1] = 0.5h[n] + 2h[n-1]$$

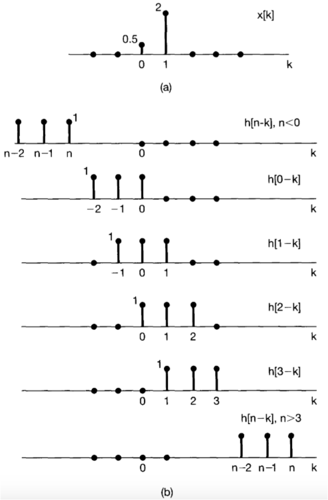


(b)

(c)

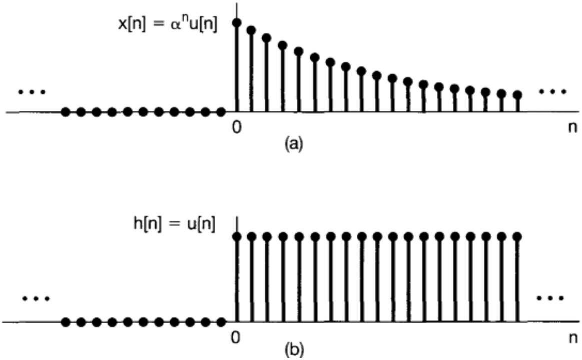
Discrete-Time LTI System cont.

View as functions of k

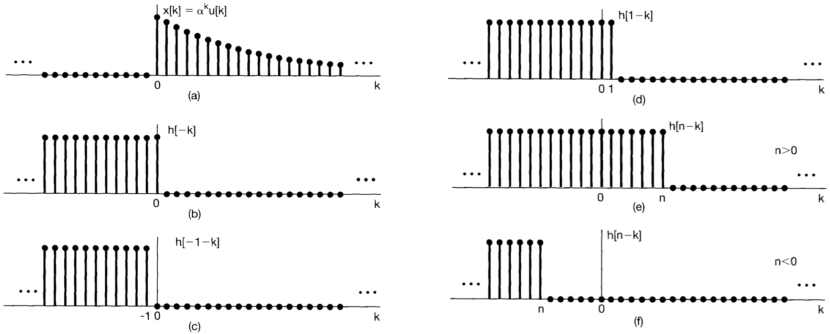


Discrete-Time LTI System cont.

Example 2: Consider an input $x[n]$ and a unit impulse response $h[n]$ given by $x[n] = \alpha^n u[n]$ and $h[n] = u[n]$ with $0 < \alpha < 1$.



Discrete-Time LTI System cont.



Discrete-Time LTI System cont.

For $n < 0$, we have $x[k]h[n-k] = 0, \forall\, k$;
For $n \geq 0$, we have

$$x[k]h[n-k] = \begin{cases} \alpha^k, & 0 \leq k \leq n \\ 0, & \text{otherwise} \end{cases}$$

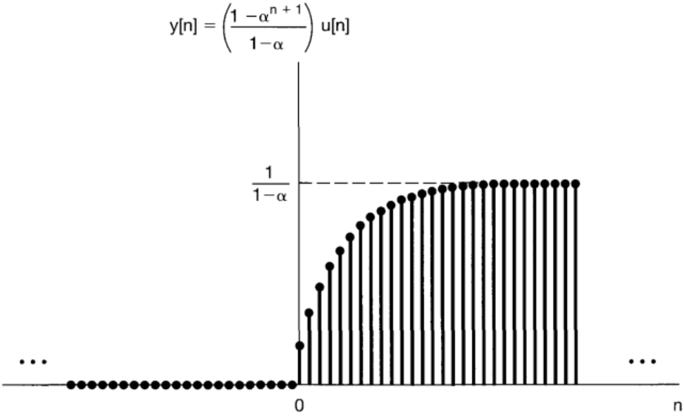
Hence,

$$y[n] = \sum_{k=0}^{n} \alpha^k = \frac{1-\alpha^{n+1}}{1-\alpha}, \qquad n \geq 0$$

For all n,

$$y[n] = \left(\frac{1-\alpha^{n+1}}{1-\alpha}\right) u[n].$$

Discrete-Time LTI System cont.

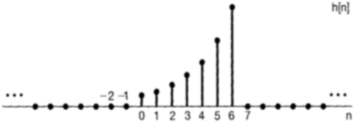


Discrete-Time LTI System cont.

Example 3: Consider two sequences

$$x[n] = \begin{cases} 1, & 0 \le n \le 4 \\ 0, & \text{otherwise} \end{cases} \qquad h[n] = \begin{cases} \alpha^n, & 0 \le n \le 6 \\ 0, & \text{otherwise} \end{cases}$$

(a)



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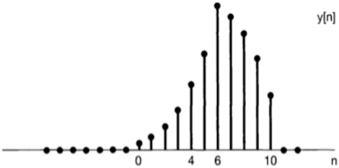
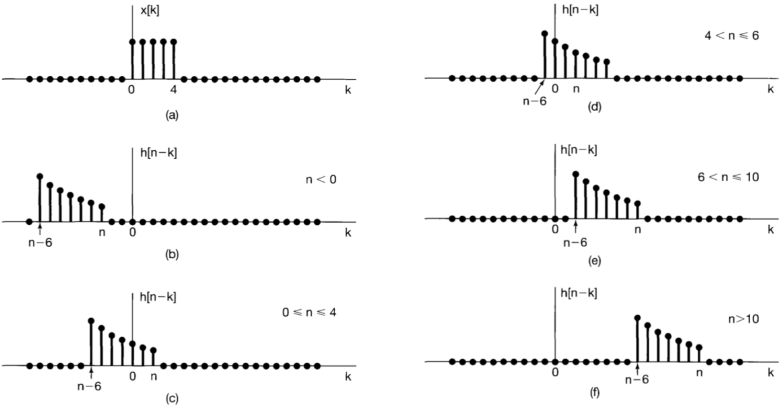


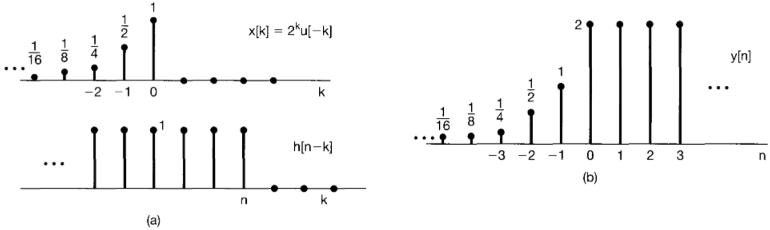
Discrete-Time LTI System cont.



Discrete-Time LTI System cont.

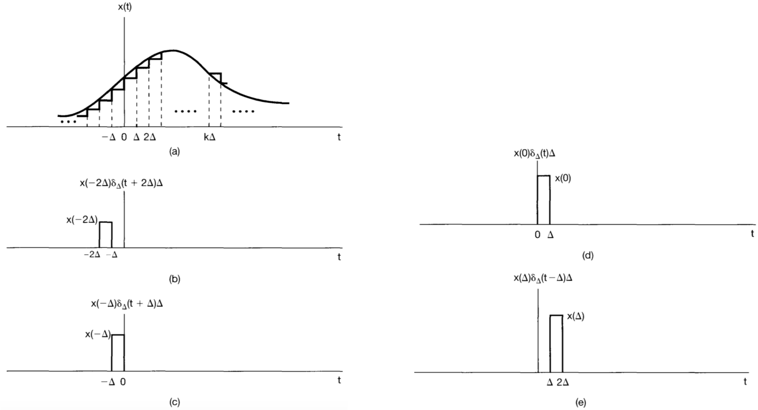
Example 4: Consider an LTI system with input $x[n]$ and unit impulse response $h[n]$ specified as follows

$$x[n] = 2^n u[-n] \qquad h[n] = u[n]$$



Representation of Continuous-Time Signals

$$\delta_\Delta(t) = \begin{cases} \frac{1}{\Delta}, 0 \le t < \Delta \\ 0, \ \text{otherwise} \end{cases} \qquad \hat{x}(t) = \sum_{k=-\infty}^{\infty} x(k\Delta)\delta_\Delta(t - k\Delta)\Delta$$



Representation of Continuous-Time Signals

When Δ approaches 0, the approximation $\hat{x}(t)$ becomes better and better, and in the limit equals $x(t)$:

$$x(t) = \lim_{\Delta \to 0} \sum_{k=-\infty}^{\infty} x(k\Delta)\delta_{\Delta}(t - k\Delta)\Delta$$

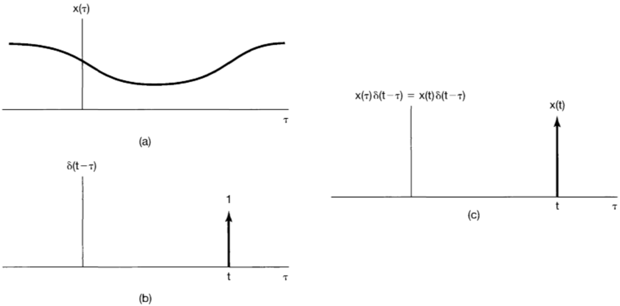
As $\Delta \to 0$, we have

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\mathrm{d}\tau, \qquad \textbf{Sifting Property}$$

Representation of Continuous-Time Signals

A continuous-time signal is the superposition of scaled and shifted pulses

$$\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\mathrm{d}\tau = \int_{-\infty}^{\infty} x(t)\delta(t-\tau)\mathrm{d}\tau = x(t)\int_{-\infty}^{\infty} \delta(t-\tau)\mathrm{d}\tau$$
$$= x(t)$$



Convolution Integral Representation of LTI Systems

Response $\hat{y}(t)$ of an LTI system is the superposition of the responses to the scaled and shifted versions of $\delta_\Delta(t)$.

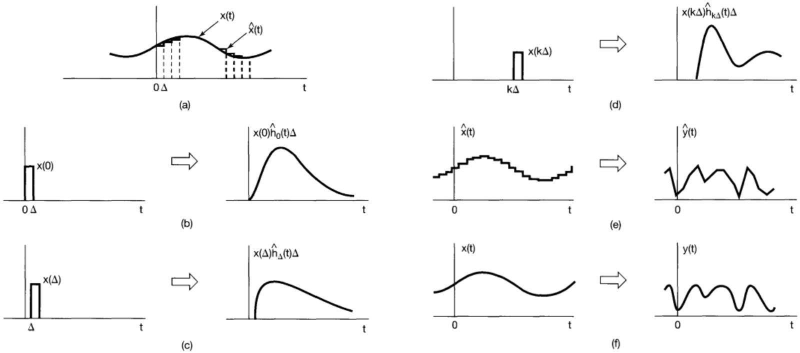
Denote $\hat{h}_{k\Delta}(t)$ as the response to input $\delta_\Delta(t - k\Delta)$

$$\hat{y}(t) = \sum_{k=-\infty}^{\infty} x(k\Delta)\hat{h}_{k\Delta}(t)\Delta$$

As $\Delta \to 0$, convolution integral is

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} x(\tau)h_\tau(t)\mathrm{d}\tau \\ &= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\mathrm{d}\tau \end{aligned}$$

Graphical Interpretation



Continuous time LTI systems

- As seen before

$$x(t) = \int_{-\infty}^{+\infty} x(\tau)\delta(t-\tau)d\tau$$

- Let $h(t)$ be the output when input is $\delta(t)$
- Then

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau = x(t) * h(t)$$

Continuous-Time LTI Systems cont.

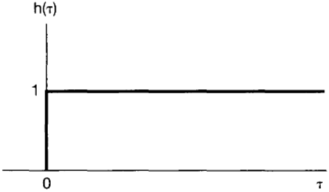
Example 5: Let $x(t)$ be the input to an LTI system with unit impulse response $h(t)$, where

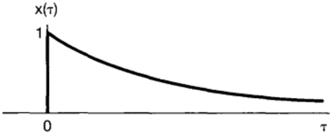
$$x(t) = e^{-at}u(t), \quad a > 0$$

and

$$h(t) = u(t)$$

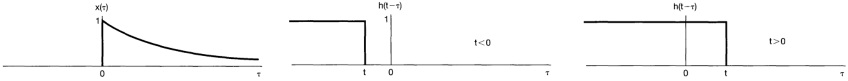
.





Continuous-Time LTI Systems cont.

Solution:



For $t > 0$,

$$x(\tau)h(t-\tau) = \begin{cases} e^{-a\tau}, 0 < \tau < t \\ 0, \qquad \text{otherwise} \end{cases}$$

Then

$$y(t) = \int_0^t e^{-a\tau} \mathrm{d}\tau = \frac{1}{a}\left(1 - e^{-at}\right) u(t)$$

Continuous-Time LTI Systems cont.

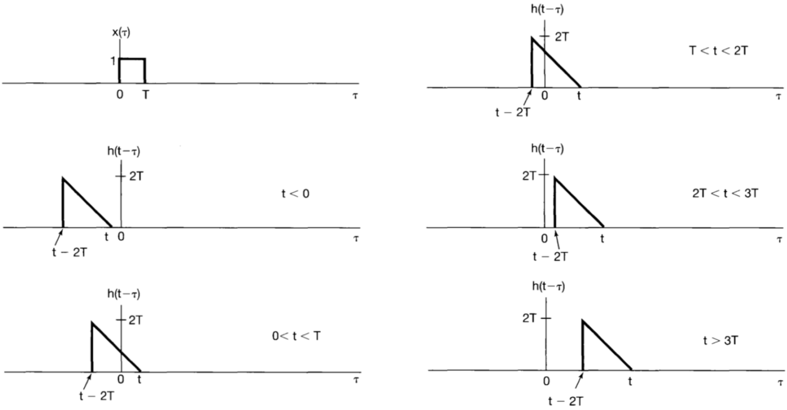
Example 6: Consider the convolution of the following two signals

$$x(t) = \begin{cases} 1, & 0 < t < T \\ 0, & \text{otherwise} \end{cases}$$

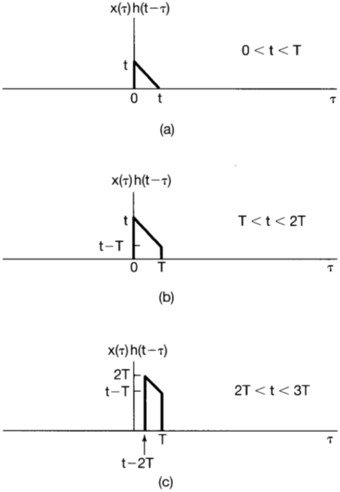
and

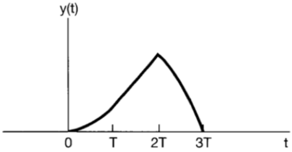
$$h(t) = \begin{cases} t, & 0 < t < 2T \\ 0, & \text{otherwise} \end{cases}$$

Continuous-Time LTI Systems cont.



Continuous-Time LTI Systems cont.





Continuous-Time LTI Systems cont.

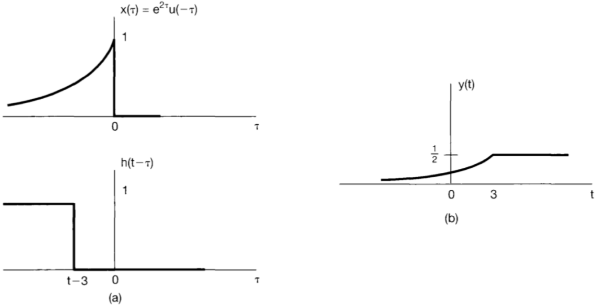
Example 7: Let $y(t)$ denote the convolution of the following two signals

$$x(t) = e^{2t}u(-t)$$

and

$$h(t) = u(t-3).$$

Continuous-Time LTI Systems cont.



When $t-3<0$, $y(t)=\int_{-\infty}^{t-3} e^{2\tau}\mathrm{d}\tau=\frac{1}{2}e^{2(t-3)}$.

When $t-3\geq 0$, $y(t)=\int_{-\infty}^{0} e^{2\tau}\mathrm{d}\tau=\frac{1}{2}$.

The convolution operation

- Convolution of two signals $x(t)$ and $h(t)$, denoted by $x(t) * h(t)$, is defined by

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau = x(t) * h(t)$$

- For discrete-time

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k] = x[n] * h[n]$$

Properties of convolution

1. Commutative: $x(t) * h(t) = h(t) * x(t)$
2. Bi-linear: $(ax_1(t) + bx_2(t)) * h(t) = a(x_1 * h) + b(x_2 * h)$,
   $x * (ah_1 + bh_2) = a(x * h_1) + b(x * h_2)$
3. Shift: $x(t - \tau) * h(t) = x(t) * h(t - \tau)$
4. Identity: $\delta(t)$ is the identity signal,

   $$x * \delta = x = \delta * x$$

   Identity is unique: $i(t) = i(t) * \delta(t) = \delta(t)$
5. Associative: $x_1 * (x_2 * x_3) = (x_1 * x_2) * x_3$

Commutative Property

In discrete time

$$x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{r=-\infty}^{\infty} h[r]x[n-r] = h[n] * x[n]$$

In continuous time

$$x(t) * h(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)\mathrm{d}\tau$$

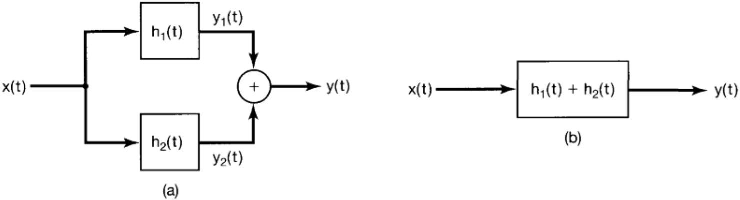
Distributive Property

In discrete time

$$x[n] * (h_1[n] + h_2[n]) = x[n] * h_1[n] + x[n] * h_2[n]$$

In continuous time

$$x(t) * [h_1(t) + h_2(t)] = x(t) * h_1(t) + x(t) * h_2(t)$$



Distributive Property cont.

In discrete time

$$(x_1[n] + x_2[n]) * h[n] = x_1[n] * h[n] + x_2[n] * h[n]$$

In continuous time

$$[x_1(t) + x_2(t)] * h(t) = x_1(t) * h(t) + x_2(t) * h(t)$$

Ex. Let $y[n]$ denote the convolution of the following two sequences

$$x[n] = 0.5^n u[n] + 2^n u[-n],$$
$$h[n] = u[n].$$

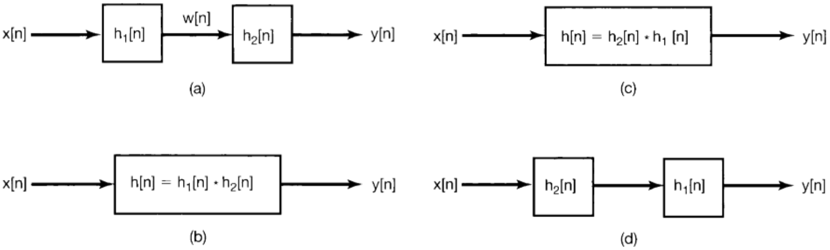
Associative Property

In discrete time

$$x[n] * (h_1[n] * h_2[n]) = (x[n] * h_1[n]) * h_2[n]$$

In continuous time

$$x(t) * [h_1(t) * h_2(t)] = [x(t) * h_1(t)] * h_2(t)$$



Memoryless LTI Systems

A system is memoryless if its output at any time depends only on the value of the input at that same time

In discrete time

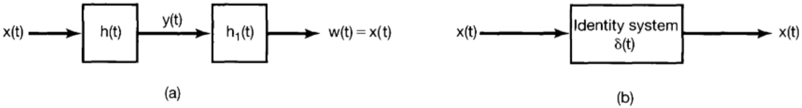
$$h[n] = 0, n \neq 0.$$

In continuous time

$$h(t) = 0, t \neq 0.$$

Invertibility of LTI Systems

A system is invertible only if an inverse system exists.



In continuous time

$$h(t) * h_1(t) = \delta(t).$$

In discrete time

$$h[n] * h_1[n] = \delta[n].$$

Invertibility of LTI Systems cont.

Ex. Derive the impulse response of the inverse system

1. $h(t) = \delta(t - t_0)$
2. $h[n] = u[n]$

Causality of LTI Systems

For a discrete-time LTI system to be casual, $y[n]$ must not depend on $x[k]$ for $k > n$. Hence, the impulse response

$$h[n] = 0, \text{for } n < 0$$

Thus, convolution sum

$$y[n] = \sum_{k=-\infty}^{n} x[k]h[n-k]$$

or equivalently

$$y[n] = \sum_{k=0}^{\infty} h[k]x[n-k]$$

Causality of LTI Systems cont.

For a continuous-time LTI system to be casual, the impulse response

$$h(t) = 0, \text{for } t < 0$$

Thus, convolution integral

$$y(t) = \int_{-\infty}^{t} x(\tau)h(t-\tau)\mathrm{d}\tau$$

or equivalently

$$y(t) = \int_{0}^{\infty} h(\tau)x(t-\tau)\mathrm{d}\tau$$

Stability of LTI Systems

A system is stable if every bounded input produces a bounded output.

In discrete-time, consider an input $x[n]$ that is bounded in magnitude $|x[n]| < B, \forall n$
The magnitude of the output

$$|y[n]| = \left| \sum_{k=-\infty}^{\infty} h[k]x[n-k] \right| \leq \sum_{k=-\infty}^{\infty} |h[k]| \, |x[n-k]| \leq B \sum_{k=-\infty}^{\infty} |h[k]|$$

Thus, if $\sum_{k=-\infty}^{\infty} |h[k]| < \infty$, then $y[n]$ is bounded, and the system is stable.

In continuous time, the system is stable if the impulse response is absolutely integrable $\int_{-\infty}^{\infty} |h(\tau)| \mathrm{d}\tau < \infty$.

Convolving $\delta(t)$ with itself

$\delta(t) * \delta(t)$

- Answer: It is $\delta(t)$.
  Let $y(t) = \delta(t) * \delta(t)$, then check that
  $y(0) = 0,\ t \neq 0,$
  $\int_{-\infty}^{\infty} y(t)dt = 1.$
- Another reason, if $x(t)$ is smooth

$$x(t) * (\delta(t) * \delta(t)) = (x(t) * \delta(t)) * \delta(t) = x(t) * \delta(t) = x(t)$$

  Therefore $\delta(t) * \delta(t)$ is also an identity function.
  Since identity is unique $\delta(t) * \delta(t) = \delta(t)$.

Derivatives

- If $y(t) = x(t) * h(t)$ then note that (use linearity and time invariance)

$$\frac{y(t+\epsilon) - y(t)}{\epsilon} = \frac{x(t+\epsilon) - x(t)}{\epsilon} * h(t)$$

Hence

$$y'(t) = x'(t) * h(t) = x(t) * h'(t)$$

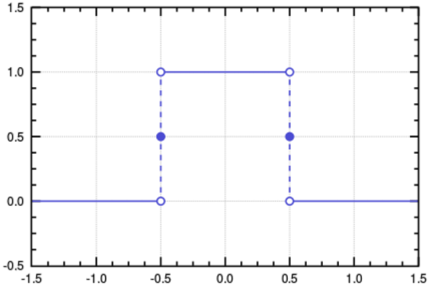
Convolution is smoothing

- If $x(t)$ has k derivatives and $h(t)$ has r derivatives, then $y(t)$ has at least $k + r$ derivatives
- $y^{(1)}(t) = x^{(1)}(t) * h(t) = x(t) * h^{(1)}(t)$
- $y^{(2)}(t) = x^{(2)}(t) * h(t) = x^{(1)}(t) * h^{(1)}(t) = x(t) * h^{(2)}(t)$
- ......
- $y^{(k+r)}(t) = x^{(k)}(t) * h^{(r)}(t)$

Examples on convolution

Rectangular function: $rect(t)$

$$rect(t) = u(t+\frac{1}{2}) - u(t-\frac{1}{2}) = \begin{cases} 1, & |t| < \frac{1}{2} \\ 0, & |t| > \frac{1}{2} \end{cases}.$$

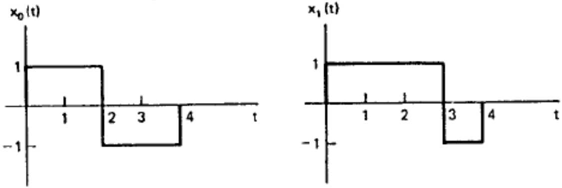


$$rect * rect = ?$$

Examples on convolution cont.

- $rect * rect = ?$: sliding window
- $rect * rect = \int \frac{d}{dt}(rect * rect)dt = ?$
- $rect * \int rect(t)dt = ?$

Examples on convolution cont.



$x_0 * x_1 = ?$