Stability Analysis

- Introduction
- Stability analysis for LTI systems
- Floquet theory
- Lyapunov theory

Contents

Stability

The linear differential equation

$$\dot{x}(t) = A(t)x(t) \quad \text{with} \quad x(0) = x_0$$

is called stable, if there exists for every $\epsilon > 0$ a $\delta > 0$ such that for every $x_0 \in \mathbb{R}^{n_x}$ with $\|x_0\| \leq \delta$ the function $x(t)$ satisfies $\|x(t)\| \leq \epsilon$ for all $t \geq 0$.

Asymptotic Stability

The linear differential equation

$$\dot{x}(t) = A(t)x(t) \quad \text{with} \quad x(0) = x_0$$

is called asymptotically stable, if it is stable and additionally satisfies

$$\lim_{t \to \infty} x(t) \; = \; 0 \; .$$

Contents

Stability of LTI systems

LTI system $\dot{x}(t) = Ax(t)$ is stable if and only if

- all eigenvalues of the matrix A have non-positive real part and
- all purely imaginary eigenvalues have algebraic multiplicity 1.

Stability of LTI systems

Proof:

- **Step 1:** Write A in Jordan normal form:

$$A = T\mathrm{diag}(J_1, \ldots, J_{n_j})T^{-1}$$

such that

$$x(t) = e^{At}x_0 = T\,\mathrm{diag}\left(e^{J_1 t}, \ldots, e^{J_{n_j} t}\right) T^{-1}x_0\ .$$

Stability of LTI systems

Proof:

- **Step 2:** The block matrices

$$e^{J_i t} = e^{\lambda_i t} \begin{pmatrix} 1 & t & \dots & \frac{t^{m_i-1}}{(m_i-1)!} \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & 1 & t \\ 0 & \dots & 0 & 1 \end{pmatrix}$$

are uniformly bounded for all $t \geq 0$ if either λ_i has a strictly negative real part or if λ_i is purely imaginary and $m_i = 1$.

Stability of LTI systems

Proof:

- **Step 3:** Stability follows then form the estimate

$$\|x(t)\| \leq \left\|e^{At}\right\| \|x_0\| \leq \epsilon$$

for all x_0 with $\|x_0\| \leq \delta$ and $\delta = \frac{\epsilon}{\max_{t \geq 0} \|e^{At}\|}$, since the maximum exists.

Stability of LTI systems

Proof:

- **Step 4 (other direction):** if there exists an imaginary eigenvalue with algebraic multiplicity larger then 1 or an eigenvalue with strictly positive real part, we can find $0 \neq c \in \mathbb{R}^{n_x}$ with

$$\lim_{t \to \infty} \|e^{At} c\| \to \infty \, . \qquad \text{(why?)}$$

  This implies that the system is unstable.

Asymptotic Stability of LTI systems

A linear time invariant system is asymptotically stable if and only if all eigenvalues of A have strictly negative real part.

Proof: Exercise.

Contents

Stability analysis for periodic systems

Let A be periodic, $A(t+T) = A(t)$. The linear time varying system

$$\dot{x}(t) = A(t)x(t)$$

is stable if and only if

- the eigenvalues of the associated monodromy matrix $G(T, 0)$ are contained in the closed unit disk and
- all eigenvalues that are on the unit circle have algebraic multiplicity 1.
- The eigenvalues of $G(T, 0)$ are also called "Floquet multipliers".

Stability analysis for periodic systems

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Stability analysis for periodic systems

Proof (sketch): Introduce the Jordan normal form

$$G(T,0) = T \operatorname{diag}\left(J_1, \ldots, J_{n_j}\right) T^{-1}$$

Since we have

$$x(kT) = G(T,0)^k x_0$$

for all integer $k \in \mathbb{N}$, we are interested in analyzing the k-th power of the monodromy matrix, which can be written as

$$G(T,0)^k = T \operatorname{diag}\left(J_1^k, \ldots, J_{n_j}^k\right) T^{-1} \ .$$

Stability analysis for periodic systems

Proof (sketch):

The k-th power of the i-th Jordan block can be worked out explicitly

$$J_i^k = \begin{pmatrix} \lambda_i^k & k\lambda_i^{k-1} & \dots & \frac{k!\lambda_i^{k-m_i+1}}{(k-m_i+1)!} \\ 0 & \lambda_i^k & \ddots & \vdots \\ \vdots & \ddots & \ddots & k\lambda_i^{k-1} \\ 0 & \dots & 0 & \lambda_i^k \end{pmatrix}.$$

(from here the proof is straightforward)

Contents

Lyapunov function

Idea: Instead of analyzing A directly, introduce a Lyapunov function $V : \mathbb{R}^{n_x} \to \mathbb{R}$ that decreasing along the trajectories of the system.

- V is assumed to be differentiable; Notation:

$$\dot{V}(x(t)) = \nabla_x V(x(t))^\intercal \dot{x}(t)$$

- $x(t)$ denotes the solution of a differential equation
- $\nabla_x V$ denotes the gradient of V

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Lyapunov theory

Main idea of Lyapunov theory: formulate conditions on the functions V and $\dot{V}(x(t))$ which imply desired stability or boundedness properties of the state trajectory x.

Advantages:

- the concepts can be applied to nonlinear systems
- Lyapunov functions are closely related to invariant sets

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Examples for conditions on V

- *Positive definiteness.* V is called positive definite, if $V(x) \geq 0$ for all $x \in \mathbb{R}$ but $V(x) = 0$ if and only if $x = 0$.
- *Monotonicity.* V is called monotonically decreasing, if

$$\dot{V}(x(t)) \leq 0$$

  for all initial values x_0.
- *Strict monotonicity.* V is called strictly monotonically decreasing, if

$$\dot{V}(x(t)) < 0$$

  for $x(t) \neq 0$ and all initial values x_0.

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Examples for conditions on V

- *Unboundedness.* V is called unbounded, if $\lim_{\|x\|\to\infty} V(x) = \infty$.
  Equivalent: all sublevel sets of V are bounded.
- *Positive quadratic.* V is called positive quadratic, if there exists $P \in \mathbb{S}^{n_x}_{++}$ such that $V(x) = x^\mathsf{T} P x$.
- *Exponential contractivity.* V is called exponentially contractive, if there exists a $\alpha > 0$ such that $\dot{V}(x(t)) \leq -\alpha V(x(t))$.

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Examples for Lyapunov statements

- If V is monotonically decreasing, then all sublevel sets of the form

$$S_\alpha(t) = \{s \in \mathbb{R}^{n_x} \mid V(s) \leq \alpha\}$$

are for all $\alpha \in \mathbb{R}$ invariant sets of the differential equation for x.

Proof: Use the inequality

$$V(x(t)) \;=\; V(x(0)) + \underbrace{\int_0^t \dot{V}(x(\tau))\,\mathrm{d}\tau}_{\leq 0} \;\leq\; V(x(0))\,.$$

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Examples for Lyapunov statements

- If the function V is positive definite, unbounded, and monotonically decreasing, then $x(t)$ is bounded.

Proof: Use that the invariant sublevel set

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is bounded; i.e., $x(t)$ is uniformly bounded for all $t \in \mathbb{R}$.

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Examples for Lyapunov statements

- If V is positive definite, unbounded, and strictly monotonically decreasing, then $x(t)$ is bounded and converges to zero for $t \to \infty$, $\lim_{t\to\infty} x(t) = 0$.

Proof: Boundedness is already established. Since $V(x(t))$ is strictly monotonically decreasing and bounded, $V_\infty = \lim_{t\to\infty} V(x(t))$ must exist. If we would have $V_\infty > 0$, then

$$\lim_{t\to\infty} V(x(t)) \;=\; V(x(0)) + \lim_{t\to\infty} \int_0^t \dot{V}(x(\tau))\,\mathrm{d}\tau \;=\; -\infty \;,$$

which is a contradiction to the positive definiteness of V. Thus, $\lim_{t\to\infty} V(x(t)) = 0$, which implies $\lim_{t\to\infty} x(t) = 0$.

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Examples for Lyapunov statements

- If V is positive quadratic, $V(x) = x^\intercal P x$ with $\lambda_{\min}(P) > 0$, and exponentially contractive with dissipation rate $\alpha > 0$, $\dot{V} \leq -\alpha V$, then x is exponentially stable. I.e., there exists a constant

$$\|x(t)\|_2 \;\leq\; \sqrt{\frac{V(x(0))}{\lambda_{\min}(P)}}\, e^{-\alpha t/2} \;\leq\; \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}}\, \|x(0)\|\, e^{-\alpha t/2}$$

for all $t \geq 0$.

Proof: Use the estimate

$$\lambda_{\min}(P)\|x(t)\|_2^2 \;\leq\; V(x(t)) \;\leq\; V(x(0))e^{-\alpha t}\,.$$

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Lyapunov theory for LTI systems

The LTI system $\dot{x}(t) = Ax(t)$ is stable if and only if there exists a positive definite quadratic Lyapunov function $V(x) = x^T P x$, which proves it, i.e.,

$$\exists P \succ 0 : \quad P = P^\mathsf{T} \quad \text{and} \quad A^\mathsf{T} P + PA \preceq 0 \; .$$

Proof: If we can find a positive definite P with $A^\mathsf{T} P + PA \preceq 0$, we have

$$\dot{V} = \dot{x}^T P x + x^T P \dot{x} = x^\mathsf{T}(A^\mathsf{T} P + PA)x \leq 0 \; ,$$

i.e., V is a Lyapunov function proving stability.

The other direction is slightly more difficult to prove...

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Lyapunov theory for LTI systems

We can always write A in Jordan normal form

$A = T\mathrm{diag}(J_1, \ldots, J_{n_j})T^{-1}$.

$$A^\intercal P + PA \preceq 0 \qquad \Leftrightarrow \qquad J_i^\intercal Q_i + Q_i J_i \preceq 0$$

for all $i \in \{1, \ldots, n_j\}$ with $P = (T^\intercal)^{-1}\,\mathrm{diag}(Q_1, \ldots, Q_{n_j})T^{-1}$.

- **Case 1:** If A stable and $m_i = 1$, we have $J_i = \lambda_i \leq 0$, i.e., $J_i^\intercal Q_i + Q_i J_i \preceq 0$ is satisfied for any $Q_i > 0$.

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Lyapunov theory for LTI systems

- **Case 2:** If A stable but $m_i > 1$, we must have $\lambda_i < 0$. In this case, the integral

$$Q_i = \int_0^\infty \left(e^{J_i t}\right)^\mathsf{T} e^{J_i t}\,\mathrm{d}t$$

exist and satisfies $J_i^\mathsf{T} Q_i + Q_i J_i = -I$.

Thus, if A is stable, we can construct a positive definite solution of $A^\mathsf{T} P + PA \preceq 0$. This completes the proof.

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