SI231B - Matrix Computations, Spring 2022-23

Homework Set #4

Prof. Ziping Zhao

Acknowledgements:

1) Deadline: **2023-04-23 23:59:59**
2) Please submit your assignments via Gradescope.
3) You can write your homework using latex/word or you can write in handwriting and submit the scanned pdf.

Problem 1. (20 points)

1) Suppose $\mathbf{A}$ is a positive definite matrix. Prove that there exists a matrix $\mathbf{B}$ such that $\mathbf{A} = \mathbf{B}^2$. (10 points)
2) Prove that if we require $\mathbf{B}$ to be positive definite, then $\mathbf{B}$ is unique. (10 points)

Problem 2. (20 points)

For any graph G with vertex set $V = \{v_1, v_2, \ldots, v_n\}$, a walk from vertex u to vertex v (not necessarily distinct) is a sequence of vertices, not necessarily distinct, such that w_{i-1} and w_i are adjacent, and $w_0 = u$ and $w_k = v$. In this case, the walk is of length k. $\mathbf{A}$ is the adjacency matrix of G. Prove that the (i, j) entry of $\mathbf{A}^k$ is the number of walks from v_i to v_j of length k.

(Hint: You can use mathematical induction.)

Problem 3. (20 points)

Given

$$\mathbf{A} = \begin{bmatrix} 10 & 5 & 2 \\ 5 & 3 & 2 \\ 2 & 2 & 3 \end{bmatrix}.$$

1) Show that $\mathbf{A}$ is positive definite. (10 points)
2) Find the Cholesky factorization of $\mathbf{A}$. (10 points)

Problem 4. (20 points)

Prove the following propositions:

1) Suppose matrix $\mathbf{A} \in \mathbb{R}^n$ is positive definite, show that all diagonal entries $a_{ii} > 0$. (6 points)
2) Let $\mathbf{A} = \begin{bmatrix} 1 & a \\ a & b \end{bmatrix}$ and $a^2 < b$. Show that $\mathbf{A}$ is positive definite. (7 points)
3) Let matrix $\mathbf{A} \in \mathbb{R}^n, \mathbf{B} \in \mathbb{R}^m$ and $\mathbf{A}, \mathbf{B}$ is positive definite, show that $\begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{B} \end{bmatrix}$ is positive definite. (7 points)

Problem 5. (20 points)

1) Given a real symmetric matrix $\mathbf{M} \in \mathbb{S}^n$ and suppose $\mathbf{M}$ satisfies the following condition

$$m_{ii} \geq \sum_{j \neq i} |m_{ij}| \quad \text{for all } i, \tag{1}$$

prove that $\mathbf{M}$ is PSD. (7 points)

2) Given a real symmetric PSD matrix $\mathbf{M} \in \mathbb{S}^n$, does $\mathbf{M}$ always satisfy the condition in (1)? *(Note: Necessary explanations are needed.)* (5 points)

3) Given a real symmetric PSD matrix $\mathbf{M} \in \mathbb{S}^n$, Prove that $\mathbf{M}$ always satisfies the following condition

$$\sum_{i=1}^{n} m_{ii} \geq \frac{2}{n-1} \sum_{i=1}^{n} \sum_{j=1}^{i-1} m_{ij}.$$

(8 points)