SI231B - Matrix Computations, Spring 2022-23

Homework Set #3

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Acknowledgements:

1) Deadline: **2023-04-08 23:59:59**
2) Please submit your assignments via Blackboard.
3) You can write your homework using latex/word or you can write in handwriting and submit the scanned pdf.

Problem 1. (20 points)

For the matrix below

$$\mathbf{A} = \begin{bmatrix} 5 & -1 & -1 \\ 3 & 1 & -1 \\ 4 & -2 & 1 \end{bmatrix}$$

1) Calculate the characteristic polynomial of $\mathbf{A}$. (5 points)
2) Find the eigenvalues of $\mathbf{A}$. (5 points)
3) Find a basis for each eigenspace of $\mathbf{A}$. (5 points)
4) Determine whether or not $\mathbf{A}$ is diagonalizable. If $\mathbf{A}$ is diagonalizable, then find an invertible matrix $\mathbf{V}$ and a diagonal matrix $\mathbf{\Lambda}$ such that $\mathbf{V}^{-1}\mathbf{A}\mathbf{V} = \mathbf{\Lambda}$. (5 points)

Solution:

1) The characteristic polynomial of $\mathbf{A}$ is

$$\det(\lambda\mathbf{I} - \mathbf{A}) = \begin{vmatrix} \lambda - 5 & 1 & 1 \\ -3 & \lambda - 1 & 1 \\ -4 & 2 & \lambda - 1 \end{vmatrix} = (\lambda - 3)(\lambda - 2)^2.$$

2) $\mathbf{A}$ has eigenvalues 3 and 2 , with algebraic multiplicities 1 and 2 respectively.
3) The eigenspace of $\mathbf{A}$ associated to the eigenvalue 3 is the null space of the matrix $3\mathbf{I} - \mathbf{A}$. To find a basis for the eigenspace, row reduce this matrix.

$$3\mathbf{I} - \mathbf{A} = \begin{pmatrix} -2 & 1 & 1 \\ -3 & 2 & 1 \\ -4 & 2 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 & 0 \\ -3 & 2 & 1 \\ -4 & 2 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Thus, the general solution to the equation $(3\mathbf{I} - \mathbf{A})\mathbf{x} = 0$ is $k_1(1, 1, 1)^T$.

The eigenspace of $\mathbf{A}$ associated to the eigenvalue 3 is the null space of the matrix $3\mathbf{I} - \mathbf{A}$. To find a basis for the eigenspace, row reduce this matrix.

$$2\mathbf{I}-\mathbf{A}=\begin{pmatrix} -3 & 1 & 1 \\ -3 & 1 & 1 \\ -4 & 2 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 0 \\ -3 & 1 & 1 \\ -4 & 2 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Thus, the general solution to the equation $(2\mathbf{I}-\mathbf{A})x=0$ is $k_2(1,1,2)^T$.

4) $\mathbf{A}$ is not diagonalizable since there is only one eigenvector of eigenvalue 2, whose algebraic multiplicity is 2.

Problem 2. (20 points)

1) Let $\mathbf{A}$ be the adjacency matrix of an undirected graph $G = (V, E)$ and λ_1 the largest eigenvalue of $\mathbf{A}$, and

$$a_{i,j} = \begin{cases} 1, & \text{if vertex } i \text{ and vertex } j \text{ are adjacent;} \\ 0, & \text{otherwise.} \end{cases}$$

Show that λ_1 is at least the average degree of the vertices in G. (10 points)

2) Let $\mathbf{A}$ be a symmetric matrix with the largest eigenvalue α_1. Let $\mathbf{B}$ be the matrix obtained by removing the last row and column from $\mathbf{A}$. And β_1 is the largest eigenvalue of $\mathbf{B}$. Show that $\alpha_1 \geq \beta_1$. (10 points)

(*Hint:* You can use the Rayleigh quotient to prove the two problems.)

Solution:

1) Let $\mathbf{d}$ denote the vector of degrees of vertices in G , so $\mathbf{d}(i)$ is the degree of vertex i . Now, take the Rayleigh quotient of the all-1s vector. We find

$$\lambda_1 = \max_x \frac{\mathbf{x}^T\mathbf{A}\mathbf{x}}{\mathbf{x}^T\mathbf{x}} \geq \frac{\mathbf{1}^T\mathbf{A}\mathbf{1}}{\mathbf{1}^T\mathbf{1}} = \frac{\mathbf{1}^T\mathbf{d}}{n} = \frac{1}{n}\sum_i \mathbf{d}(i).$$

2) For any vector $\mathbf{y} \in \mathbb{R}^{n-1}$, we have

$$\mathbf{y}^T\mathbf{B}\mathbf{y} = \begin{pmatrix} \mathbf{y} \\ 0 \end{pmatrix}^T \mathbf{A} \begin{pmatrix} \mathbf{y} \\ 0 \end{pmatrix}$$

So, for $\mathbf{y}$ an eigenvector of $\mathbf{B}$ of eigenvalue β_1 ,

$$\beta_1 = \frac{\mathbf{y}^T\mathbf{B}\mathbf{y}}{\mathbf{y}^T\mathbf{y}} = \frac{\begin{pmatrix} \mathbf{y} \\ 0 \end{pmatrix}^T \mathbf{A} \begin{pmatrix} \mathbf{y} \\ 0 \end{pmatrix}}{\begin{pmatrix} \mathbf{y} \\ 0 \end{pmatrix}^T \begin{pmatrix} \mathbf{y} \\ 0 \end{pmatrix}} \leq \max_{\mathbf{x}\in\mathbb{R}^n} \frac{\mathbf{x}^T\mathbf{A}\mathbf{x}}{\mathbf{x}^T\mathbf{x}} = \alpha_1$$

Problem 3. (20 points)

Let $\mathbf{A}$ be an $n \times n$ square matrix.

1) Suppose that $\mathbf{A}^{-1}$ exists. Prove: if λ is an eigenvalue of $\mathbf{A}$, then $1/\lambda$ is an eigenvalue of $\mathbf{A}^{-1}$. (5 points)
2) Prove that if $\mathbf{A}^2 = \mathbf{I}$, then the eigenvalue of $\mathbf{A}$ must be 1 or -1. (5 points)
3) Suppose that λ_1 and λ_2 are two distinct eigenvalues of $\mathbf{A}$. And suppose that $\mathbf{x}_1$ is an eigenvector of $\mathbf{A}$ under λ_1, and $\mathbf{x}_2$ is an eigenvector of $\mathbf{A}$ under λ_2. Prove that there dose not exists any real number t such that $t\mathbf{x}_1 = \mathbf{x}_2$. (5 points)
4) Suppose that λ_1 and λ_2 are two distinct eigenvalues of $\mathbf{A}$. And suppose that $\mathbf{x}_1$ is an eigenvector of $\mathbf{A}$ under λ_1, and $\mathbf{x}_2$ is an eigenvector of $\mathbf{A}$ under λ_2. Prove that $\mathbf{x}_1 + \mathbf{x}_2$ is not an eigenvector of $\mathbf{A}$. (5 points)

Solution:

1) Suppose λ is an eigenvalue of $\mathbf{A}$, there exists an vector $\mathbf{x}$ satisfies

$$\begin{aligned} \mathbf{A}\mathbf{x} &= \lambda\mathbf{x} \quad \Longrightarrow \\ \mathbf{A}^{-1}\mathbf{A}\mathbf{x} &= \lambda\mathbf{A}^{-1}\mathbf{x} \quad \Longrightarrow \\ \frac{1}{\lambda}\mathbf{x} &= \mathbf{A}^{-1}\mathbf{x} \end{aligned}$$

2) Consider any eigenvalue λ of $\mathbf{A}$, and let $\mathbf{x}$ be an arbitrary eigenvector of $\mathbf{A}$ corresponding to λ. It gets

$$\begin{aligned} \mathbf{A}\mathbf{x} &= \lambda\mathbf{x} \quad \Longrightarrow \\ \mathbf{A}^2\mathbf{x} &= \lambda\mathbf{A}\mathbf{x} \quad \Longrightarrow \\ \mathbf{I}\mathbf{x} &= \lambda\mathbf{A}\mathbf{x} \quad \Longrightarrow \\ \mathbf{x} &= \lambda^2\mathbf{x} \quad \Longrightarrow \\ \mathbf{0} &= (\lambda^2 - 1)\mathbf{x} \end{aligned}$$

As we know that $\mathbf{x} \neq 0$, we have $\lambda^2 = 1$, i.e., $\lambda = -1$ or 1.

3) Assume, on the contrary, there exists a such t. Since $\mathbf{A}\mathbf{x}_1 = \lambda_1\mathbf{x}_1$, we have $\mathbf{A}(c_1\mathbf{x}_1) = \lambda(c\mathbf{x}_1)$, which leads to $\mathbf{A}\mathbf{x}_2 = \lambda_1\mathbf{x}_2$.

   On the other hand, we have $\mathbf{A}\mathbf{x}_2 = \lambda_2\mathbf{x}_2$. Thus, $\lambda_1\mathbf{x}_2 = \mathbf{A}\mathbf{x}_2 = \lambda_2\mathbf{x}_2$, that implies $\lambda_1 = \lambda_2$, which is a contradiction.

4) Assume, on the contrary, that $\mathbf{x}_1 + \mathbf{x}_2$ is an eigenvector of $\mathbf{A}$ corresponding to some eigenvalue λ_3. Then, we have

$$\begin{aligned} \mathbf{A}(\mathbf{x}_1 + \mathbf{x}_2) &= \lambda_3(\mathbf{x}_1 + \mathbf{x}_2) \quad \Longrightarrow \\ \mathbf{A}\mathbf{x}_1 + \mathbf{A}\mathbf{x}_2 &= \lambda_3\mathbf{x}_1 + \lambda_3\mathbf{x}_2 \quad \Longrightarrow \\ \lambda_1\mathbf{x}_1 + \lambda_2\mathbf{x}_2 &= \lambda_3\mathbf{x}_1 + \lambda_3\mathbf{x}_2 \quad \Longrightarrow \\ (\lambda_1 - \lambda_3)\mathbf{x}_1 &= (\lambda_3 - \lambda_2)\mathbf{x}_2 \end{aligned}$$

As $\lambda_1 \neq \lambda_2$, there at least one of $\lambda_1 - \lambda_3$ and $\lambda_3 - \lambda_2$ is non-zero. Without loss of generality, suppose $\lambda_1 - \lambda_3 \neq 0$, which gives

$$\mathbf{x}_1 = \frac{\lambda_3 - \lambda_2}{\lambda_1 - \lambda_3}\mathbf{x}_2$$

According to 3), the above is impossible, which is a contradiction. Proof done.

Problem 4. (20 points) For $\mathbf{A} \in \mathbb{R}^{m\times n}$, prove that $\mathbf{A}^T\mathbf{A}$ and $\mathbf{A}^T$ have the same range space.

Solution:

For $\mathbf{x} \in \mathcal{N}(\mathbf{A})$, we have $\mathbf{A}\mathbf{x} = \mathbf{0}$ for some $\mathbf{x} \in \mathbb{R}^n$, which means $\mathbf{A}\mathbf{A}^T\mathbf{x} = 0$ also holds and $\mathbf{x} \in \mathcal{N}(\mathbf{A}^T\mathbf{A})$, hence $\mathcal{N}(\mathbf{A}) \subseteq \mathcal{N}(\mathbf{A}^T\mathbf{A})$. For $\mathbf{x} \in \mathcal{N}(\mathbf{A}^T\mathbf{A})$, we have $\mathbf{A}^T\mathbf{A}\mathbf{x} = 0$, which means $(\mathbf{A}\mathbf{x})^T\mathbf{A}\mathbf{x} = 0$ and $\mathbf{A}\mathbf{x} = \mathbf{0}$, hence $\mathbf{x} \in \mathcal{N}(\mathbf{A})$ and $\mathcal{N}(\mathbf{A}^T\mathbf{A}) \subseteq \mathcal{N}(\mathbf{A})$. Therefore, we can conclude that $\mathcal{N}(\mathbf{A}) = \mathcal{N}(\mathbf{A}^T\mathbf{A})$ and $\mathsf{rank}(\mathbf{A}^T) = \mathsf{rank}(\mathbf{A}) = \mathsf{rank}(\mathbf{A}^T\mathbf{A})$, indicating that $\mathsf{dim}(\mathcal{R}(\mathbf{A}^T\mathbf{A})) = \mathsf{dim}(\mathcal{R}(\mathbf{A}^T))$.

For $\mathbf{y} \in \mathcal{R}(\mathbf{A}^T\mathbf{A})$, there exists $\mathbf{x} \in \mathbb{R}^n$ such that $\mathbf{A}^T\mathbf{A}\mathbf{x} = \mathbf{y}$, we can always find $\tilde{\mathbf{x}} = \mathbf{A}\mathbf{x}$ that satisfies $\mathbf{A}^T\tilde{\mathbf{x}} = \mathbf{y}$ which means $\mathbf{y} \in \mathcal{R}(\mathbf{A}^T)$ and $\mathcal{R}(\mathbf{A}^T\mathbf{A}) \subseteq \mathcal{R}(\mathbf{A}^T)$. Therefore $\mathcal{R}(\mathbf{A}^T) = \mathcal{R}(\mathbf{A}^T\mathbf{A})$.

Problem 5. (20 points) Time-of-arrival (TOA) based source localization is a scenario in which the location of a target sensor is determined based on the TOA measurements of the target sensor collected by many anchors.

Define $\mathbf{x}^* \in \mathbb{R}^n$ be the unknown true position of the target sensor. Define $\{\mathbf{a}_i\}_{i=1}^m \subseteq \mathbb{R}^n$ be the known position of the ith anchor and suppose that the vectors $\{\mathbf{a}_i - \mathbf{a}_1\}_{i=1}^m$ span $\mathbb{R}^n$ ($m \geq n+1$). Then the TOA based *range measurement* between the target and the ith anchor is modeled as

$$r_i = \|\mathbf{x}^* - \mathbf{a}_i\|_2 + \omega_i, \quad i = 1, ..., m,$$

where $\{\omega_i\}_{i=1}^m \subseteq \mathbb{R}$ is the noise.

1) Suppose there is no noise, i.e, $\omega_i = 0$ for $i = 1, ..., m$. Prove that $\mathbf{x}^*$ can be recovered based on the following *linear least squares* problem:

$$\mathbf{x}_{\text{LS}} = \arg\min_{\mathbf{x}\in\mathbb{R}^n} \|\mathbf{A}\mathbf{x} - \mathbf{b}\|_2^2,$$

i.e., $\mathbf{x}_{\text{LS}} = \mathbf{x}^*$ in the noiseless case, where

$$\mathbf{A} = \begin{bmatrix} (\mathbf{a}_2 - \mathbf{a}_1)^T \\ \vdots \\ (\mathbf{a}_m - \mathbf{a}_{m-1})^T \end{bmatrix}, \quad \mathbf{b} = \frac{1}{2}\begin{bmatrix} \|\mathbf{a}_2\|_2^2 - \|\mathbf{a}_1\|_2^2 + r_1^2 - r_2^2 \\ \vdots \\ \|\mathbf{a}_m\|_2^2 - \|\mathbf{a}_{m-1}\|_2^2 + r_{m-1}^2 - r_m^2 \end{bmatrix}.$$

Besides, derive the solution of the estimator $\mathbf{x}_{\text{LS}}$. (10 points)

2) Suppose the noise ω_i satisfies $\omega_i \ll \|\mathbf{x}^* - \mathbf{a}_i\|_2$ for $i = 1, ..., m$. We can estimate $\mathbf{x}^*$ based on the following *nonlinear least squares* problem:

$$\hat{\mathbf{x}} = \arg\min_{\mathbf{x}\in\mathbb{R}^n} \sum_{i=1}^m \left(\|\mathbf{x} - \mathbf{a}_i\|_2 - r_i\right)^2.$$

Define $\boldsymbol{\omega} = [\omega_1, ..., \omega_m]^T$ and suppose $\|\boldsymbol{\omega}\|_2 \leq c\sqrt{m}\sigma$ where $c, \sigma > 0$ are some constants. Prove that the following result holds:

$$\|\hat{\mathbf{x}} - \mathbf{x}^*\|_2 \leq K_1\sqrt{m}\sigma + K_2 m\sigma^2,$$

for some constants $K_1, K_2 > 0$ which are determined by $\{\mathbf{a}_i\}_{i=1}^m$, c, and $\mathbf{x}^*$. (10 points)

Hint for 2):

- *You can find the upper bound of* $\|\mathbf{x}_{LS} - \mathbf{x}^*\|_2$ *and* $\|\mathbf{x}_{LS} - \hat{\mathbf{x}}\|_2$ *and then combine them to get the result.*
- *Based on 1), you may need to define* $r_i^* = \|\mathbf{x}^* - \mathbf{a}_i\|_2$ *(resp.* $\hat{r}_i = \|\hat{\mathbf{x}} - \mathbf{a}_i\|_2$*) and let* $\mathbf{b}^*$ *(resp.* $\hat{\mathbf{b}}$*) be the vector obtained by replacing* r_i *with* r_i^* *(resp.* $\hat{r}_i$*) in* $\mathbf{b}$*, and then* $\mathbf{x}^*$ *(resp.* $\hat{\mathbf{x}}$*) satisfies* $\mathbf{x}^* = \arg\min_{\mathbf{x}\in\mathbb{R}^n} \|\mathbf{A}\mathbf{x} - \mathbf{b}^*\|_2^2$ *(resp.* $\hat{\mathbf{x}} = \arg\min_{\mathbf{x}\in\mathbb{R}^n} \left\|\mathbf{A}\mathbf{x} - \hat{\mathbf{b}}\right\|_2^2$*).*
- *You may need the relation* $f(\hat{\mathbf{x}}) = \sum_{i=1}^m \left(\|\hat{\mathbf{x}} - \mathbf{a}_i\|_2 - r_i\right)^2 \leq f(\mathbf{x}^*)$.

Solution:

1) Suppose there is no noise, i.e, $\omega_i = 0$ for $i = 1, ..., m$, we have

$$r_i = \|\mathbf{x}^* - \mathbf{a}_i\|_2 \text{ for } i = 1, ..., m,$$

or, equivalently,

$$r_i^2 = \|\mathbf{x}^* - \mathbf{a}_i\|_2^2 = \|\mathbf{x}^*\|_2^2 - 2\mathbf{a}_i^T\mathbf{x}^* + \|\mathbf{a}_i\|_2^2 \text{ for } i = 1, ..., m.$$

Subtracting the ith equation from the $(i+1)$th equation, we have

$$(\mathbf{a}_{i+1} - \mathbf{a}_i)^T\mathbf{x}^* = \frac{1}{2}\left(\|\mathbf{a}_{i+1}\|_2^2 - \|\mathbf{a}_i\|_2^2 + r_i^2 - r_{i+1}^2\right) \text{ for } i = 2, ..., m.$$

Combing the $m-1$ equations, we have

$$\mathbf{A}\mathbf{x}^* = \mathbf{b}, \tag{1}$$

where

$$\mathbf{A} = \begin{bmatrix} (\mathbf{a}_2 - \mathbf{a}_1)^T \\ \vdots \\ (\mathbf{a}_m - \mathbf{a}_{m-1})^T \end{bmatrix}, \quad \mathbf{b} = \frac{1}{2}\begin{bmatrix} \|\mathbf{a}_2\|_2^2 - \|\mathbf{a}_1\|_2^2 + r_1^2 - r_2^2 \\ \vdots \\ \|\mathbf{a}_m\|_2^2 - \|\mathbf{a}_{m-1}\|_2^2 + r_{m-1}^2 - r_m^2 \end{bmatrix}.$$

Since the vectors $\{\mathbf{a}_i - \mathbf{a}_1\}_{i=1}^m$ span $\mathbb{R}^n$, we have $\{\mathbf{a}_{i+1} - \mathbf{a}_i\}_{i=1}^m$ span $\mathbb{R}^n$ and hence $\mathbf{A}$ has full column rank. $\mathbf{x}^*$ is the solution to the linear system or equivalently the following linear least squares (LS) problem

$$\mathbf{x}_{\text{LS}} = \arg\min_{\mathbf{x}\in\mathbb{R}^n} \|\mathbf{A}\mathbf{x} - \mathbf{b}\|_2^2,$$

and the LS solution is given by

$$\mathbf{x}_{\text{LS}} = \left(\mathbf{A}^T\mathbf{A}\right)^{-1}\mathbf{A}^T\mathbf{b}.$$

2) First, let $r_i^* = \|\mathbf{x}^* - \mathbf{a}_i\|_2$ for $i = 1, \ldots, m$ and let $\mathbf{b}^*$ be the vector obtained by replacing r_i in $\mathbf{b}$ with r_i^*. Based on the LS result, we have $\mathbf{x}^* = \left(\mathbf{A}^T\mathbf{A}\right)^{-1}\mathbf{A}^T\mathbf{b}^*$ and

$$\|\mathbf{x}_{\text{LS}} - \mathbf{x}^*\|_2 = \left\|\left(\mathbf{A}^T\mathbf{A}\right)^{-1}\mathbf{A}^T(\mathbf{b} - \mathbf{b}^*)\right\|_2 \le \left\|\left(\mathbf{A}^T\mathbf{A}\right)^{-1}\mathbf{A}^T\right\|_2 \|\mathbf{b} - \mathbf{b}^*\|_2,$$

where $\left\|\left(\mathbf{A}^T\mathbf{A}\right)^{-1}\mathbf{A}^T\right\|_2$ is a constant. Based on the range measurement equation, we have $r_i = \|\mathbf{x}^* - \mathbf{a}_i\|_2 + \omega_i = r_i^* + \omega_i$ and hence $r_i^2 - (r_i^*)^2 = 2r_i^*\omega_i + \omega_i^2$ for $i = 1, \ldots, m$. Then,

$$\|\mathbf{b} - \mathbf{b}^*\|_2 = \frac{1}{2}\left\|\begin{bmatrix} r_1^2 - (r_1^*)^2 - (r_2^2 - (r_2^*)^2) \\ \vdots \\ r_{m-1}^2 - \left(r_{m-1}^*\right)^2 - (r_m^2 - (r_m^*)^2) \end{bmatrix}\right\|_2 = \left\|\begin{bmatrix} r_1^*\omega_1 + \frac{1}{2}\omega_1^2 - (r_2^*\omega_2 + \frac{1}{2}\omega_2^2) \\ \vdots \\ r_{m-1}^*\omega_{m-1} + \frac{1}{2}\omega_{m-1}^2 - (r_m^*\omega_m + \frac{1}{2}\omega_m^2) \end{bmatrix}\right\|_2$$

$$= \left\|\begin{bmatrix} r_1^*\omega_1 - r_2^*\omega_2 + \frac{1}{2}(\omega_1^2 - \omega_2^2) \\ \vdots \\ r_{m-1}^*\omega_{m-1} - r_m^*\omega_m + \frac{1}{2}(\omega_{m-1}^2 - \omega_m^2) \end{bmatrix}\right\|_2 \le C_0\|\boldsymbol{\omega}\|_2 + \frac{1}{2}\|\tilde{\boldsymbol{\omega}}\|_2$$

for some constant $C_0 > 0$, where

$$\tilde{\boldsymbol{\omega}} := \begin{bmatrix} \omega_1^2 - \omega_2^2 \\ \vdots \\ \omega_{m-1}^2 - \omega_m^2 \end{bmatrix}.$$

Since $\|\tilde{\boldsymbol{\omega}}\|_2 = \sqrt{\sum_{i=2}^m |\omega_{i-1}^2 - \omega_i^2|^2} \le \sum_{i=2}^m |\omega_{i-1}^2 - \omega_i^2| = \|\tilde{\boldsymbol{\omega}}\|_1 \le 2\|\boldsymbol{\omega}\|_2^2$, our assumption on $\|\boldsymbol{\omega}\|_2$ yields

$$\|\mathbf{x}_{\text{LS}} - \mathbf{x}^*\|_2 \le C_1\sqrt{m}\sigma + C_2 m\sigma^2,$$

for some constants $C_1, C_2 > 0$.

Next, define $\hat{r}_i = \|\hat{\mathbf{x}} - \boldsymbol{a}_i\|_2$ for $i = 1, \ldots, m$ and let $\hat{\mathbf{b}}$ be the vector obtained by replacing r_i in $\mathbf{b}$ with $\hat{r}_i$. Based on the LS result, we have $\hat{\mathbf{x}} = \left(\mathbf{A}^T\mathbf{A}\right)^{-1}\mathbf{A}^T\hat{\mathbf{b}}$ and

$$\|\mathbf{x}_{\text{LS}} - \hat{\mathbf{x}}\|_2 = \left\|\left(\mathbf{A}^T\mathbf{A}\right)^{-1}\mathbf{A}^T\left(\mathbf{b} - \hat{\mathbf{b}}\right)\right\|_2 \leq \left\|\left(\mathbf{A}^T\mathbf{A}\right)^{-1}\mathbf{A}^T\right\|_2 \left\|\mathbf{b} - \hat{\mathbf{b}}\right\|_2.$$

We also have

$$r_i^2 - \hat{r}_i^2 = 2r_i\left(r_i - \hat{r}_i\right) - \left(r_i - \hat{r}_i\right)^2 = 2\left(r_i^* + \omega_i\right)\left(r_i - \hat{r}_i\right) - \left(r_i - \hat{r}_i\right)^2$$

and then

$$\left\|\mathbf{b} - \hat{\mathbf{b}}\right\|_2 = \frac{1}{2}\left\|\begin{bmatrix} r_1^2 - \hat{r}_1^2 - (r_2^2 - \hat{r}_2^2) \\ \vdots \\ r_{m-1}^2 - \hat{r}_{m-1}^2 - (r_m^2 - \hat{r}_m^2) \end{bmatrix}\right\|_2$$

$$= \left\|\begin{bmatrix} \left(r_1^* + \omega_1\right)\left(r_1 - \hat{r}_1\right) - \frac{1}{2}\left(r_1 - \hat{r}_1\right)^2 - \left(\left(r_2^* + \omega_2\right)\left(r_2 - \hat{r}_2\right) - \frac{1}{2}\left(r_2 - \hat{r}_2\right)^2\right) \\ \vdots \\ \left(r_{m-1}^* + \omega_{m-1}\right)\left(r_{m-1} - \hat{r}_{m-1}\right) - \frac{1}{2}\left(r_{m-1} - \hat{r}_{m-1}\right)^2 - \left(\left(r_m^* + \omega_m\right)\left(r_m - \hat{r}_m\right) - \frac{1}{2}\left(r_m - \hat{r}_m\right)^2\right) \end{bmatrix}\right\|_2$$

$$= \left\|\begin{bmatrix} \left(r_1^* + \omega_1\right)\left(r_1 - \hat{r}_1\right) - \left(r_2^* + \omega_2\right)\left(r_2 - \hat{r}_2\right) + \frac{1}{2}\left(\left(r_2 - \hat{r}_2\right)^2 - \left(r_1 - \hat{r}_1\right)^2\right) \\ \vdots \\ \left(r_{m-1}^* + \omega_{m-1}\right)\left(r_{m-1} - \hat{r}_{m-1}\right) - \left(r_m^* + \omega_m\right)\left(r_m - \hat{r}_m\right) + \frac{1}{2}\left(\left(r_m - \hat{r}_m\right)^2 - \left(r_{m-1} - \hat{r}_{m-1}\right)^2\right) \end{bmatrix}\right\|_2$$

$$\leq \left\|\begin{bmatrix} \left(r_1^* + \omega_1\right)\left(r_1 - \hat{r}_1\right) \\ \vdots \\ \left(r_{m-1}^* + \omega_{m-1}\right)\left(r_{m-1} - \hat{r}_{m-1}\right) \end{bmatrix}\right\|_2 + \left\|\begin{bmatrix} \left(r_2^* + \omega_2\right)\left(r_2 - \hat{r}_2\right) \\ \vdots \\ \left(r_m^* + \omega_m\right)\left(r_m - \hat{r}_m\right) \end{bmatrix}\right\|_2 + \frac{1}{2}\|\tilde{\mathbf{r}}\|_2$$

where

$$\tilde{\mathbf{r}} := \begin{bmatrix} \left(r_2 - \hat{r}_2\right)^2 - \left(r_1 - \hat{r}_1\right)^2 \\ \left(r_3 - \hat{r}_3\right)^2 - \left(r_2 - \hat{r}_2\right)^2 \\ \vdots \\ \left(r_m - \hat{r}_m\right)^2 - \left(r_{m-1} - \hat{r}_{m-1}\right)^2 \end{bmatrix}.$$

For the first term, we have

$$\left\|\begin{bmatrix} \left(r_1^* + \omega_1\right)\left(r_1 - \hat{r}_1\right) \\ \vdots \\ \left(r_{m-1}^* + \omega_{m-1}\right)\left(r_{m-1} - \hat{r}_{m-1}\right) \end{bmatrix}\right\|_2 = \sqrt{\sum_{i=2}^{m}\left|r_{i-1}^* + \omega_{i-1}\right|^2\left|r_{i-1} - \hat{r}_{i-1}\right|^2}$$

$$\leq \sqrt{\sum_{i=2}^{m}\left(r_{i-1}^* + |\omega_{i-1}|\right)^2\left|r_{i-1} - \hat{r}_{i-1}\right|^2} \leq \sqrt{\sum_{i=2}^{m}\left(2r_{i-1}^*\right)^2\left|r_{i-1} - \hat{r}_{i-1}\right|^2} \leq C_3\sqrt{\sum_{i=1}^{m}\left|r_i - \hat{r}_i\right|^2},$$

where in the second inequality we have used $|\omega_i| \ll \|\mathbf{x}^* - \mathbf{a}_i\|_2 = r_i^*$ for $i = 1, ..., m$ and a similar result holds for the second term. For the third term, we have $\frac{1}{2}\|\tilde{\mathbf{r}}\|_2 \leq \frac{1}{2}\|\tilde{\mathbf{r}}\|_1 \leq \sum_{i=1}^{m}|r_i - \hat{r}_i|^2$.

Defining $f(\mathbf{x}) = \sum_{i=1}^{m} \left(\|\mathbf{x} - \mathbf{a}_i\|_2 - r_i\right)^2$, we have

$$f(\hat{\mathbf{x}}) = \sum_{i=1}^{m} \left(\|\hat{\mathbf{x}} - \mathbf{a}_i\|_2 - r_i\right)^2 = \sum_{i=1}^{m} \left(\hat{r}_i - r_i\right)^2 \leq f\left(\mathbf{x}^*\right) = \sum_{i=1}^{m} \left(\|\mathbf{x}^* - \mathbf{a}_i\|_2 - r_i\right)^2 = \sum_{i=1}^{m} \omega_i^2 = \|\boldsymbol{\omega}\|_2^2 \leq c^2 m\sigma^2,$$

where the first inequality is due to the fact that $\hat{\mathbf{x}}$ is the optimal solution of $f(\mathbf{x})$. Then we have

$$\|\mathbf{b} - \hat{\mathbf{b}}\|_2 \leq C_4 \sqrt{\sum_{i=1}^{m} |r_i - \hat{r}_i|^2} + \sum_{i=1}^{m} |r_i - \hat{r}_i|^2 \leq C_5\sqrt{m}\sigma + c^2 m\sigma^2$$

for some constants $C_4, C_5 > 0$. This gives

$$\|\mathbf{x}_{\mathrm{LS}} - \hat{\mathbf{x}}\| \leq C_6\sqrt{m}\sigma + C_7 m\sigma^2$$

for some constants $C_7, C_8 > 0$.

Finally, the desired result then follows

$$\begin{aligned}
\|\hat{\mathbf{x}} - \mathbf{x}^*\|_2 &= \|(\mathbf{x}_{\mathrm{LS}} - \mathbf{x}^*) - (\mathbf{x}_{\mathrm{LS}} - \hat{\mathbf{x}})\|_2 \\
&\leq \|\mathbf{x}_{\mathrm{LS}} - \mathbf{x}^*\|_2 + \|\mathbf{x}_{\mathrm{LS}} - \hat{\mathbf{x}}\|_2 \\
&\leq C_1\sqrt{m}\sigma + C_2 m\sigma^2 + C_6\sqrt{m}\sigma + C_7 m\sigma^2 \\
&= K_1\sqrt{m}\sigma + K_2 m\sigma^2,
\end{aligned}$$

with $K_1 = C_1 + C_6 > 0$ and $K_2 = C_2 + C_7 > 0$. Along this proof, it is easy to see that K_1 and K_2 depend on $\{\mathbf{a}_i\}_{i=1}^{m}$, c, and $\mathbf{x}^*$. In other words, the estimation accuracy depends on the position of the anchors, the position of the target sensor, and the noise level.