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Lecture 10 State Variable Model

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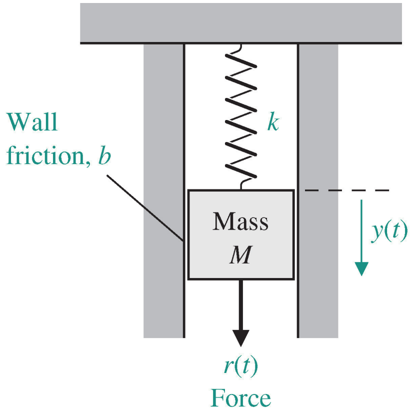
Introduction

Recall the spring-mass-damper system

$$M\frac{d^2y(t)}{dt^2} + b\frac{dy(t)}{dt} + ky(t) = r(t),$$



To obtain the time response $y(t)$

$$M\left(s^2Y(s) - sy(0^-) - \frac{dy}{dt}(0^-)\right) + b(sY(s) - y(0^-)) + kY(s) = R(s).$$

When

$$r(t) = 0, \quad \text{and} \quad y(0^-) = y_0, \quad \text{and} \quad \left.\frac{dy}{dt}\right|_{t=0^-} = 0,$$

We have

$$Ms^2Y(s) - Msy_0 + bsY(s) - by_0 + kY(s) = 0.$$

Limited to:
Linear
Time-invariant
SISO

Solving for $Y(s)$ yields

$$Y(s) = \frac{(Ms + b)y_0}{Ms^2 + bs + k} = \frac{p(s)}{q(s)}.$$

→ **Transfer Function**

Introduction

utilizing a set of ordinary differential equations in a convenient matrix-vector form.

In this chapter, we consider system modeling using time-domain methods.

- *Straight forward*
- *LTI SISO models, can be represented via state variable models. Powerful mathematical concepts from linear algebra and matrix-vector analysis, as well as effective computational tools, can be utilized.*
- *Readily extended to nonlinear, time-varying, and multiple input– output* systems.
- *Computer works in time-domain as well!*

For example, the mass of an airplane varies as a function of time as the fuel is expended during flight.

Outcomes:

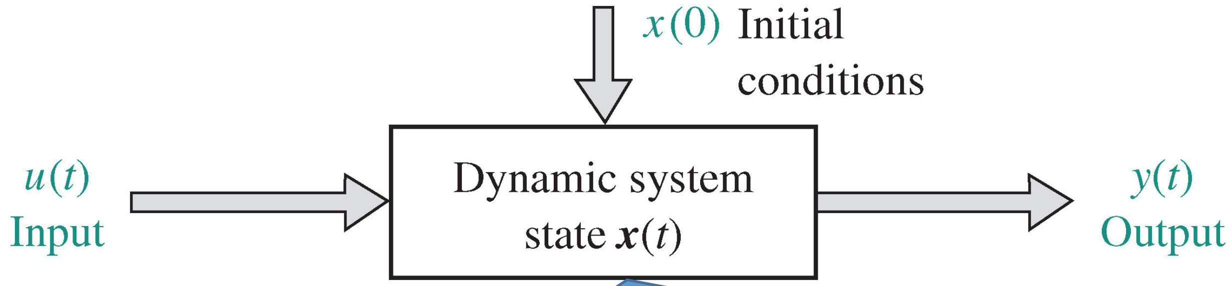
❑ *Understand state variables, state differential equations, and output equations.*

❑*Recognize that state variable models can describe the dynamic behavior of physical systems and can be represented by block diagrams*

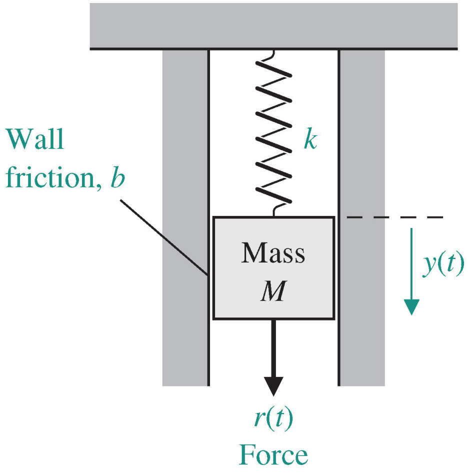
❑*Know how to obtain the transfer function model from a state variable model, and vice versa.*

❑*Be aware of solution methods for state variable models and the role of the state transition matrix in obtaining the time responses.*

State Space Model



the state of a system is described in terms of a set of state variables
$x(t) = (x_1(t), x_2(t), \ldots, x_n(t))$



A set of state variables sufficient to describe this system includes： the position and the velocity of the mass.

Again， consider the spring-mass-damper system

Define a set of state variables

$$\boldsymbol{x}(t) = (x_1(t), x_2(t)).$$

$$x_1(t) = y(t) \quad \text{and} \quad x_2(t) = \frac{dy(t)}{dt}.$$

The differential equation describes the behavior of the system

$$M\frac{d^2y(t)}{dt^2} + b\frac{dy(t)}{dt} + ky(t) = u(t).$$

State Space Model

Substitute the state variables as *already defined and obtain*

$$M\frac{dx_2(t)}{dt} + bx_2(t) + kx_1(t) = u(t).$$

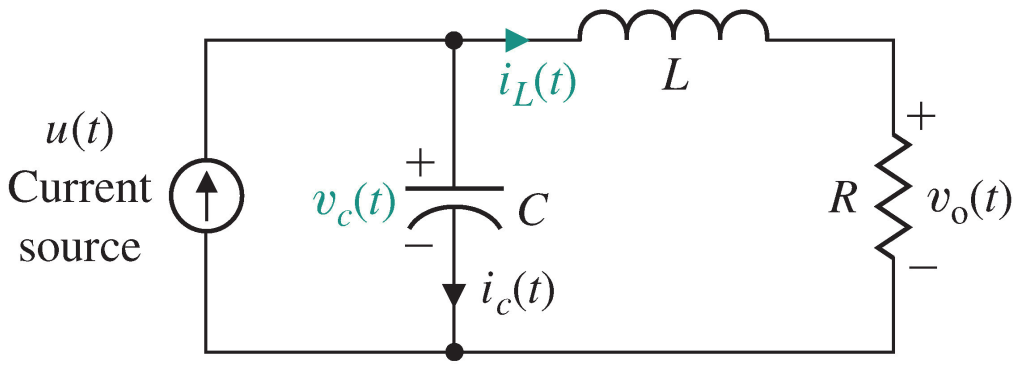
Further, write as the set of two first-order differential equations

$$\frac{dx_1(t)}{dt} = x_2(t)$$

$$\frac{dx_2(t)}{dt} = \frac{-b}{M}x_2(t) - \frac{k}{M}x_1(t) + \frac{1}{M}u(t).$$

State Space Model

Another example



State Space Model

Define a set of state variables

$$\mathbf{x}(t) = (x_1(t), x_2(t)).$$

where $x_1(t)$ is the capacitor voltage $v_c(t)$ and $x_2(t)$ is the inductor current $i_L(t)$

Utilizing Kirchhoff's current law at the junction, we obtain

$$i_c(t) = C\frac{dv_c(t)}{dt} = +u(t) - i_L(t).$$

$$L\frac{di_L(t)}{dt} = -Ri_L(t) + v_c(t).$$

The output of this system is represented by

$$v_o(t) = Ri_L(t).$$

State Space Model

$$\frac{dx_1(t)}{dt} = -\frac{1}{C}x_2(t) + \frac{1}{C}u(t),$$

$$\frac{dx_2(t)}{dt} = +\frac{1}{L}x_1(t) - \frac{R}{L}x_2(t).$$

$$y_1(t) = v_o(t) = Rx_2(t).$$

- *The engineer's interest is primarily in physical systems, where the variables typically are voltages, currents, velocities, positions, pressures, temperatures, and similar physical variables.*
- *The state variables that describe a system are not a unique set*

State Differential Equations

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general form

$$\frac{d}{dt}\begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix} = \begin{bmatrix} a_{11} & a_{12} \cdots & a_{1n} \\ a_{21} & a_{22} \cdots & a_{2n} \\ \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} \cdots & a_{nn} \end{bmatrix} \boxed{\begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}} + \begin{bmatrix} b_{11} \cdots & b_{1m} \\ \vdots & \vdots \\ b_{n1} \cdots & b_{nm} \end{bmatrix} \boxed{\begin{pmatrix} u_1(t) \\ \vdots \\ u_m(t) \end{pmatrix}}.$$

state vector *inputs*

compact notation of the state differential equation as

$$\boxed{\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t).}$$

output equation

$$\boxed{\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t),}$$

if we have n states, m inputs, r outputs, what's the dimension of A, B, C, D?

State Differential Equations

RLC

$$\frac{dx_1(t)}{dt} = -\frac{1}{C}x_2(t) + \frac{1}{C}u(t),$$

$$\frac{dx_2(t)}{dt} = +\frac{1}{L}x_1(t) - \frac{R}{L}x_2(t).$$

$$y_1(t) = v_o(t) = Rx_2(t).$$

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & \frac{-1}{C} \\ \frac{1}{L} & \frac{-R}{L} \end{bmatrix}\mathbf{x}(t) + \begin{bmatrix} \frac{1}{C} \\ 0 \end{bmatrix}u(t)$$

$$y(t) = [0 \quad R]\mathbf{x}(t).$$

When $R = 3$, $L = 1$, and $C = 1/2$, we have

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & -2 \\ 1 & -3 \end{bmatrix}\mathbf{x}(t) + \begin{bmatrix} 2 \\ 0 \end{bmatrix}u(t)$$

$$y(t) = [0 \quad 3]\mathbf{x}(t).$$

$y(\mathrm{t}) = ?$

Solution of Differential Equations

Consider the first-order differential equation

$$\dot{x}(t) = ax(t) + bu(t),$$

Take the Laplace transform

$$sX(s) - x(0) = aX(s) + bU(s);$$

therefore,

$$X(s) = \frac{x(0)}{s-a} + \frac{b}{s-a}U(s).$$

The inverse Laplace transform

$$x(t) = e^{at}x(0) + \int_0^t e^{+a(t-\tau)}bu(\tau)d\tau.$$

We expect the solution of the general state differential equation to be similar and to be of exponential form.

The ***matrix exponential*** *function is defined by in a similar Taylor series form*

$$e^{\mathbf{A}t} = \exp(\mathbf{A}t) = \mathbf{I} + \mathbf{A}t + \frac{\mathbf{A}^2t^2}{2!} + \cdots + \frac{\mathbf{A}^kt^k}{k!} + \cdots,$$

Solution of Differential Equations

matrix exponentials is a DEFINITION

$$X(t) = e^{tA} = \sum_{i=0}^{\infty} \frac{1}{i!} [tA]^i \ .$$

- We have $X(0) = I$.
- $\boxed{\text{The time derivative of } X \text{ satisfies } \dot{X}(t) = A \cdot X(t).}$
- $\boxed{X(t) \text{ commutes with } A, \text{ i.e., } AX(t) = X(t)A.}$
- If $A \cdot B = B \cdot A$, then $e^{A+B} = e^A \cdot e^B$.
- $\boxed{\textbf{But in general } e^{A+B} \neq e^A \cdot e^B \ \textbf{!!!}}$
- $X(t_1) \cdot X(t_2) = X(t_1 + t_2)$ for all $t_1, t_2 \in \mathbb{R}$.
- The function $X(t) = e^{tA}$ is invertible, $X(t)^{-1} = e^{-tA}$.

Conclusion: The solution of the state differential equation

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$

is found to be

$$\mathbf{x}(t) = \exp(\mathbf{A}t)\mathbf{x}(0) + \int_0^t \exp[\mathbf{A}(t-\tau)]\mathbf{B}\mathbf{u}(\tau)d\tau.$$

Proof:

- *Uniqueness of Solutions*

If we have two solutions $x_1, x_2 : \mathbb{R} \to \mathbb{R}^{n_x}$, then $y = x_1 - x_2$ satisfies

$$\dot{y}(t) = Ay(t) \quad \text{with} \quad y(0) = 0\,.$$

The auxiliary function $v(t) = e^{-At}y(t)$ satisfies

$$\begin{aligned} \dot{v}(t) &= -Ae^{-At}y(t) + e^{-At}Ay(t) = -Ae^{-At}y(t) + Ae^{-At}y(t) = 0 \\ v(0) &= 0\,, \end{aligned}$$

$$\Longrightarrow \quad v(t) = y(t) = 0 \quad \Longrightarrow \quad x_1 = x_2\,.$$

Solution of Differential Equations

Conclusion: The solution of the state differential equation

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$

is found to be

$$\mathbf{x}(t) = \exp(\mathbf{A}t)\mathbf{x}(0) + \int_0^t \exp[\mathbf{A}(t-\tau)]\mathbf{B}\mathbf{u}(\tau)d\tau.$$

Proof:

- *Verify the ODE*

Generalized Leibniz integral rule.

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{a(t)}^{b(t)} g(t,\tau)\,\mathrm{d}\tau = g(t,b(t))\,\dot{b}(t) - g(t,a(t))\,\dot{a}(t) + \int_{a(t)}^{b(t)} g_t(t,\tau)\,\mathrm{d}\tau\,.$$

$$\dot{x}(t) = A\,e^{At}x(0) + e^{A(t-t)}Bu(t) + \int_0^t A\,e^{A(t-\tau)}Bu(\tau)d\tau$$

$$= A\left[e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau\right] + Bu(t)$$

$$= Ax(t) + Bu(t)$$

Solution of Differential Equations

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Specially, the solution of an unforced system

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t)$$

is found to be

$$\mathbf{x}(t) = \exp(\mathbf{A}t)\mathbf{x}(0)$$

The matrix exponential function describes the unforced response of the system and is called the fundamental or state transition matrix $\Phi(t, 0)$.

Thus, the general solution can be written as

$$\mathbf{x}(t) = \boldsymbol{\Phi}(t)\mathbf{x}(0) + \int_0^t \boldsymbol{\Phi}(t-\tau)\mathbf{B}\mathbf{u}(\tau)d\tau.$$

NOTE, up to now, we are talking about LTI system, for nonlinear or time-varying, there is NO nice general solution form.

Solution of Differential Equations

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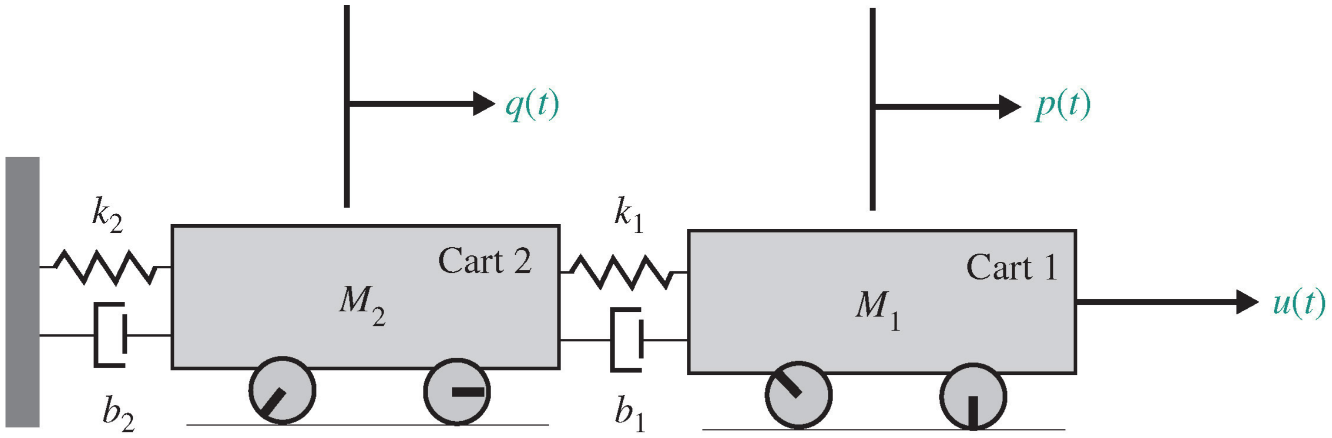
Example: *Two rolling carts*



M_1, M_2 = mass of carts

$p(t), q(t)$ = position of carts

$u(t)$ = external force acting on system

k_1, k_2 = spring constants

b_1, b_2 = damping coefficients

We assume that the carts have negligible rolling friction

we use Newton's second law

$$M_1\ddot{p}(t) + b_1\dot{p}(t) + k_1 p(t) = u(t) + k_1 q(t) + b_1\dot{q}(t),$$

$$M_2\ddot{q}(t) + (k_1 + k_2)q(t) + (b_1 + b_2)\dot{q}(t) = k_1 p(t) + b_1\dot{p}(t).$$

by defining

$$x_1(t) = p(t),$$
$$x_2(t) = q(t).$$
$$x_3(t) = \dot{x}_1(t) = \dot{p}(t),$$
$$x_4(t) = \dot{x}_2(t) = \dot{q}(t).$$

Choose the position difference between Cart1 and Cart2 as the output. Write the state space model of the system in a compact form, i.e. identify the mtrices A,B,C,D

Block Diagram and Canonical Form

There are several block diagrams that could represent the same transfer function.

Let us initially consider the fourth-order transfer function

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_0}{s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0}$$

Rearranging the terms in above Equation and taking the inverse Laplace transform yields

$$\frac{d^4(y(t)/b_0)}{dt^4} + a_3\frac{d^3(y(t)/b_0)}{dt^3} + a_2\frac{d^2(y(t)/b_0)}{dt^2} + a_1\frac{d(y(t)/b_0)}{dt} + a_0(y(t)/b_0) = u(t).$$

Define the four state variables as follows:

$$\begin{aligned} x_1(t) &= y(t)/b_0 \\ x_2(t) &= \dot{x}_1(t) = \dot{y}(t)/b_0 \\ x_3(t) &= \dot{x}_2(t) = \ddot{y}(t)/b_0 \\ x_4(t) &= \dot{x}_3(t) = \dddot{y}(t)/b_0. \end{aligned}$$

$\Rightarrow$

$$\begin{aligned} \dot{x}_1(t) &= x_2(t), \\ \dot{x}_2(t) &= x_3(t), \\ \dot{x}_3(t) &= x_4(t), \\ \dot{x}_4(t) &= -a_0 x_1(t) - a_1 x_2(t) - a_2 x_3(t) - a_3 x_4(t) + u(t); \\ y(t) &= b_0 x_1(t). \end{aligned}$$

Block Diagram and Canonical Form

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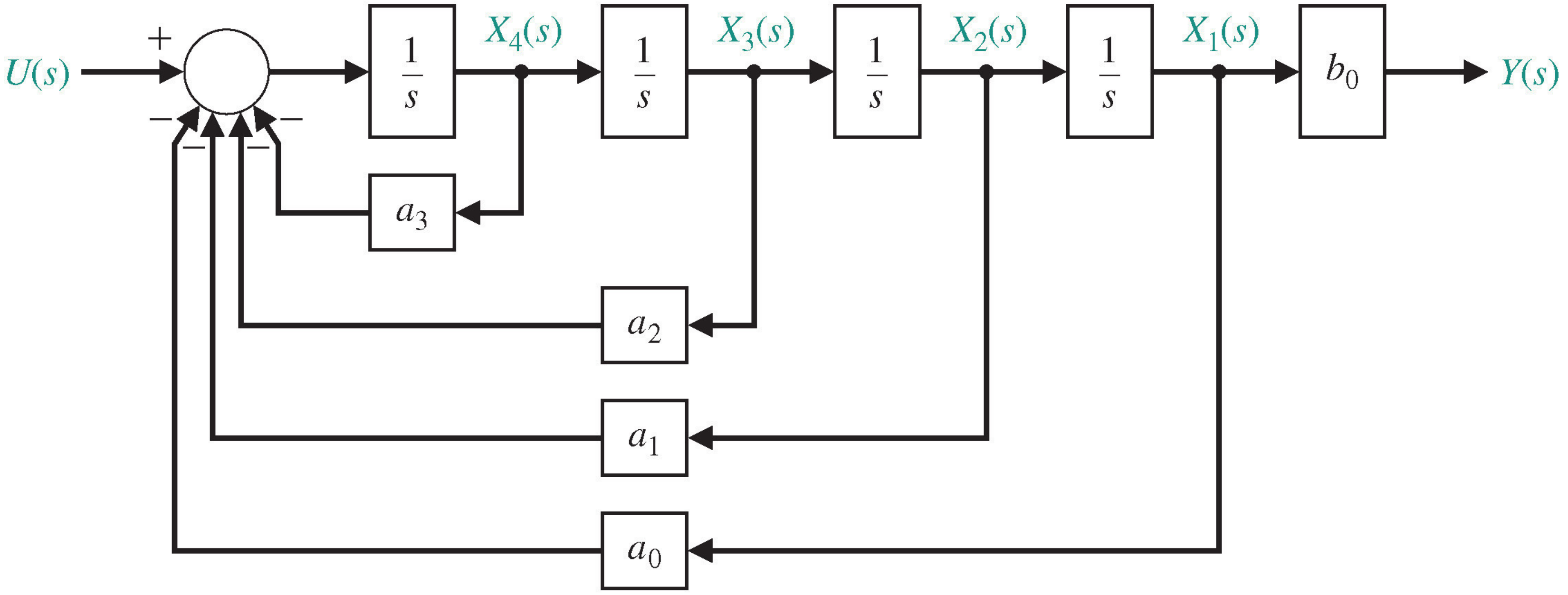
$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_0}{s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0}$$

$$\begin{aligned}
\dot{x}_1(t) &= x_2(t), \\
\dot{x}_2(t) &= x_3(t), \\
\dot{x}_3(t) &= x_4(t), \\
\dot{x}_4(t) &= -a_0 x_1(t) - a_1 x_2(t) - a_2 x_3(t) - a_3 x_4(t) + u(t); \\
y(t) &= b_0 x_1(t).
\end{aligned}$$



Block Diagram and Canonical Form

Now consider when the numerator is a polynomial in s, so that we have

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_3 s^3 + b_2 s^2 + b_1 s + b_0}{s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0} \, \frac{Z(s)}{Z(s)}.$$

the intermediate variable

Equating the numerator and denominator polynomials yields

$$Y(s) = [b_3 s^3 + b_2 s^2 + b_1 s + b_0]Z(s)$$

$$U(s) = [s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0]Z(s).$$

$$y(t) = b_3 \frac{d^3 z(t)}{dt^3} + b_2 \frac{d^2 z(t)}{dt^2} + b_1 \frac{dz(t)}{dt} + b_0 z(t)$$

$$u(t) = \frac{d^4 z(t)}{dt^4} + a_3 \frac{d^3 z(t)}{dt^3} + a_2 \frac{d^2 z(t)}{dt^2} + a_1 \frac{dz(t)}{dt} + a_0 z(t).$$

Define the four state variables as follows:

$$\begin{aligned} x_1(t) &= z(t) \\ x_2(t) &= \dot{x}_1(t) = \dot{z}(t) \\ x_3(t) &= \dot{x}_2(t) = \ddot{z}(t) \\ x_4(t) &= \dot{x}_3(t) = \dddot{z}(t). \end{aligned}$$

$$\begin{aligned} \dot{x}_1(t) &= x_2(t), \\ \dot{x}_2(t) &= x_3(t), \\ \dot{x}_3(t) &= x_4(t), \\ \dot{x}_4(t) &= -a_0 x_1(t) - a_1 x_2(t) - a_2 x_3(t) - a_3 x_4(t) + u(t), \end{aligned}$$

$$y(t) = b_0 x_1(t) + b_1 x_2(t) + b_2 x_3(t) + b_3 x_4(t).$$

Block Diagram and Canonical Form

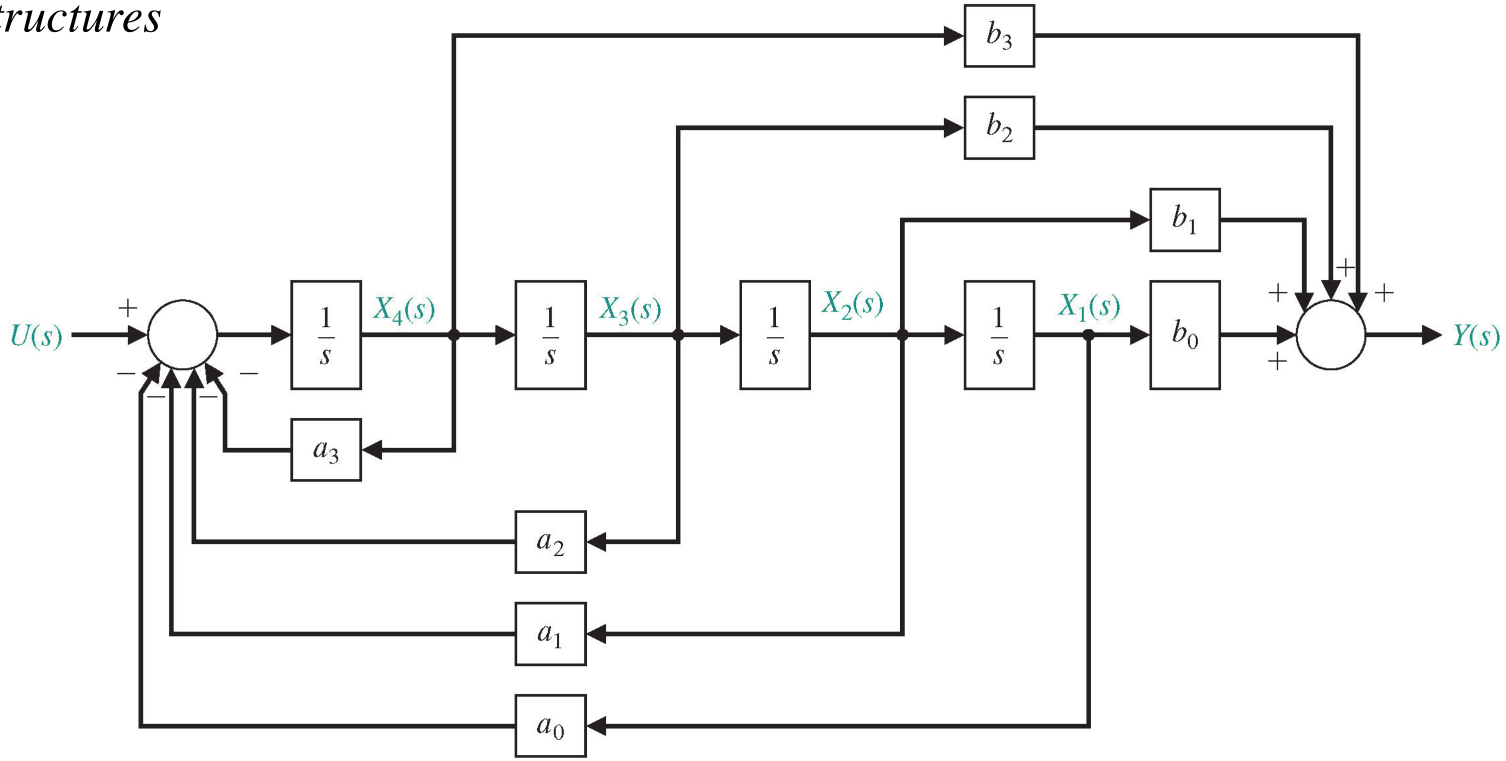
In matrix form, we can represent the system

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t),$$

Controllable Canonical Form

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u(t). \qquad y(t) = \mathbf{C}\mathbf{x}(t) = [b_0 \quad b_1 \quad b_2 \quad b_3]\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}.$$

The graphical structures



Block Diagram and Canonical Form

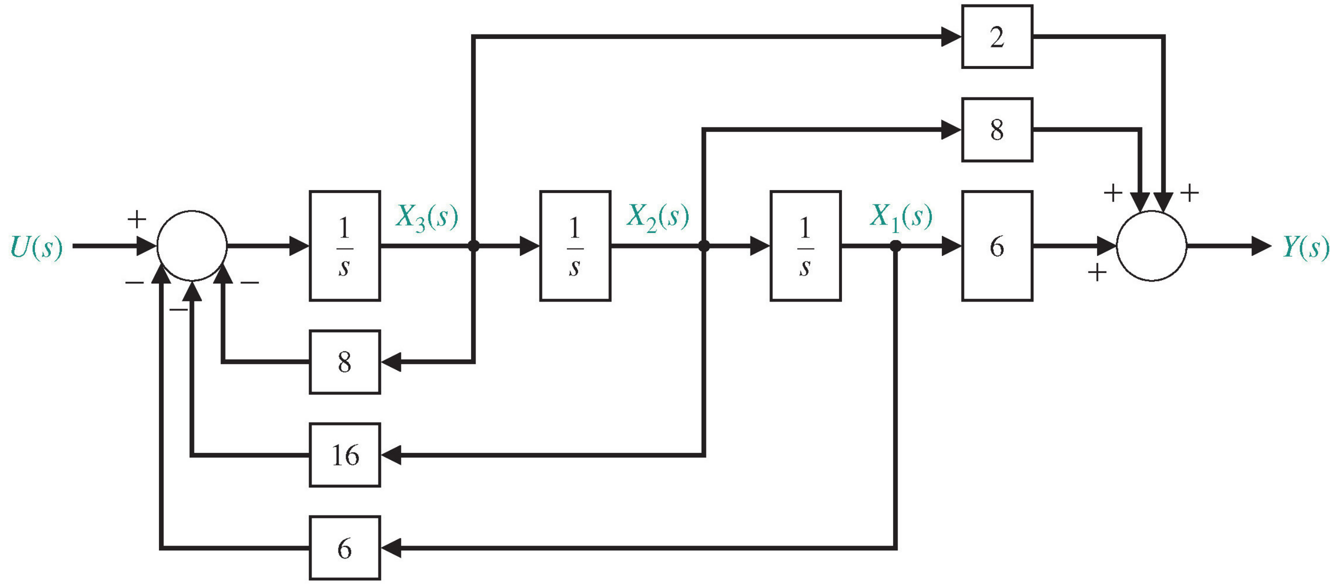
Example:

$$T(s) = \frac{Y(s)}{U(s)} = \frac{2s^2 + 8s + 6}{s^3 + 8s^2 + 16s + 6}.$$

Write down the state space model and corresponding block diagram

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -16 & -8 \end{bmatrix}\mathbf{x}(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}u(t),$$

$$y(t) = [6 \quad 8 \quad 2]\mathbf{x}(t).$$



Block Diagram and Canonical Form

A second form of the model we need to consider is the decoupled response modes.

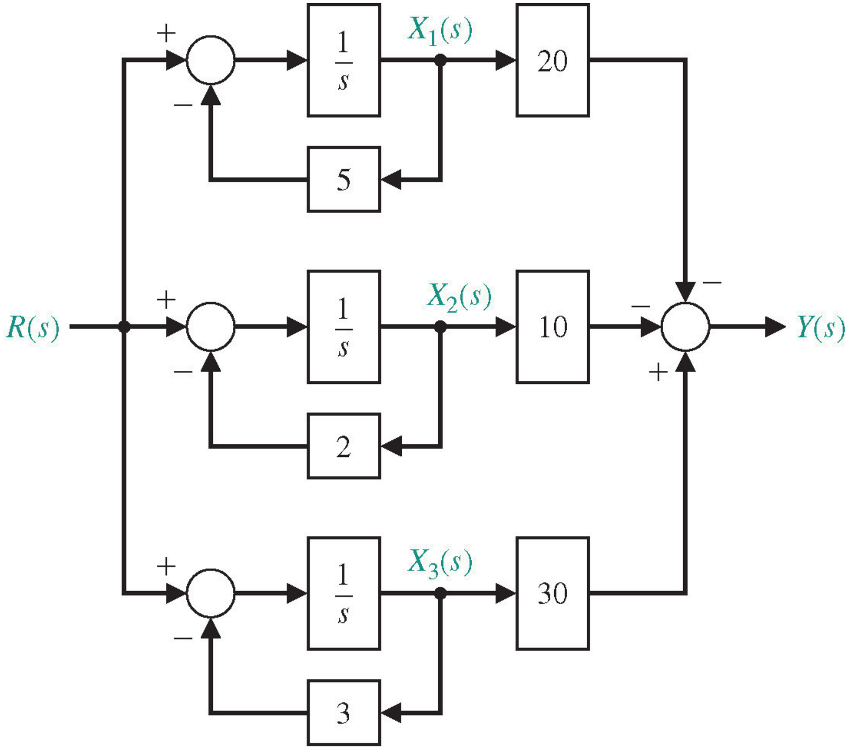
$$\frac{Y(s)}{R(s)} = T(s) = \frac{k_1}{s+5} + \frac{k_2}{s+2} + \frac{k_3}{s+3},$$

where we find that $k_1 = -20$, $k_2 = -10$, and $k_3 = 30$.

The state variable matrix differential equation and block diagram are

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} -5 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} r(t)$$

$$y(t) = [-20 \quad -10 \quad 30]\mathbf{x}(t).$$



this format is often called the diagonal canonical form.

Block Diagram and Canonical Form

The state space model is NOT unique in the sense that

Any invertible linear matrix transformation is represented by z = Mx can transforms the x-vector into the z-vector by means of the M matrix.

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

$$\xrightarrow{z = Mx}$$

$$\dot{z} = M\dot{x}$$
$$= MAM^{-1}z + MBu$$
$$y = CM^{-1}z + Du$$

The Laplace transforms

$$s\mathbf{X}(s) = \mathbf{A}\mathbf{X}(s) + \mathbf{B}U(s)$$

$$Y(s) = \mathbf{C}\mathbf{X}(s) + \mathbf{D}U(s)$$

$$\boldsymbol{X}(s) = (s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{B}U(s)$$
$$Y(s) = [\boldsymbol{C}(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{B} + \boldsymbol{D}]U(s)$$

$$G(s) = \frac{Y(s)}{U(s)} = \mathrm{CM^{-1}(sI - MAM^{-1})^{-1}MB + D}$$

$$(\boldsymbol{AB})^{-1} = \boldsymbol{B}^{-1}\boldsymbol{A}^{-1}$$

$$G(s) = \frac{Y(s)}{U(s)} = \mathrm{C(sI - A)^{-1}B + D}$$

Block Diagram and Canonical Form

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

$$\Leftrightarrow$$

$$G(s) = \frac{Y(s)}{U(s)} = \mathrm{C}(s\mathrm{I} - \mathrm{A})^{-1}\mathrm{B} + \mathrm{D} = \frac{k_p N(s)}{M(s)}$$

This indicates that

The location of poles of the system depends on the dynamic matrix A, to be more specific, the eigenvalues of A.

Naturally, a new stability criterion arise

The system described by *the state space mode (A,B,C,D)* is said to be *Hurwitz or stable if all eigenvalues of A have negative real parts.*

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Thanks!

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