Online Lecture Notes

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1 Repitition of Lecture 1

The goal of this lecture is to repeat a little bit the main results from Lecture 1 and the make the connection to the ongoing Lecture 5.

Lecture 1 was about scalar linear systems of the form

$$\dot{x}(t) = ax(t) + b \qquad \text{and} \qquad x(0) = x_0 \ .$$

In this case, the coefficients $a \in \mathbb{R}$ and $b \in \mathbb{R}$ are scalar, which means that the state is scalar, too. This means that $x : \mathbb{R} \to \mathbb{R}$. The main result from Lecture 1 that the solution of this ordinary differential equation (ODE) is unique and given by

$$x(t) = e^{at}(x_0 - x_{\mathrm{s}}) + x_{\mathrm{s}} \quad \text{with} \quad x_{\mathrm{s}} = -\frac{b}{a}$$

if $a \neq 0$ such that the steady x_{s} is well-defined. In the other case, if $a = 0$, then everything is trivial—we can just integrate on both sides to get,

$$x(t) = x_0 + \int_0^t b \,\mathrm{d}\tau = x_0 + bt \ .$$

If you have understood these formulas, you have basically understood everyting you need for Lecture 1 ! If you know, this already quite a few point in the mid-term exam—for instance in previous exams always some of the first questions are about simply linear scalar systems. Examples from the 2019 mid-term exam are

1. Solve the linear ODE

$$\dot{x}(t) = 2 \quad \text{and} \quad x(0) = -1 \ .$$

   Solution: in this case, we have $a = 0$, $b = 2$ and $x_0 = -1$. So, the above formula yields

$$x(t) = x_0 + bt = -1 + 2t$$

2. Solve the linear ODE

$$\dot{x}(t) = -x(t) + 1 \quad \text{and} \quad x(0) = 0 \ .$$

   Solution: in this case, we have $a = -1$, $b = 1$ and $x_0 = 0$. So, the above formula yields

$$x_{\mathrm{s}} = -\frac{b}{a} = 1, \qquad x(t) = e^{at}(x_0 - x_{\mathrm{s}}) + x_{\mathrm{s}} = 1 - e^{-t}$$

2 Multivariate Linear Systems

The whole point of Lecture 5 is to generalize the result from Lecture 1. The goal of Lecture 5 is to show that the solution of the multivariate ODE

$$\dot{x}(t) = Ax(t) + b \qquad \text{with} \qquad x(0) = x_0$$

with given coefficients $A \in \mathbb{R}^{n_x \times n_x}$ and $b \in \mathbb{R}^{n_x}$ is given by

$$x(t) = e^{At}(x_0 - x_\mathrm{s}) + x_\mathrm{s} \quad \text{with} \quad x_\mathrm{s} = -A^{-1}b$$

if A invertible. Notice that this formula is in complete analogy to the result from Lecture 1, with the only difference being that we replace a by A, the exponential function e^{at} by the matrix exponential e^{At} and $-\frac{b}{a}$ by $-A^{-1}b$. So, we could say that Lecture 5 contains Lecture 1 as a special case, which is obtained for $n_x = 1$.

2.1 Proof of the above formula if A is invertible

Let us assume that A is invertible. The first thing we can check is that the vector $x_\mathrm{s} = -A^{-1}b$ is really a steady-state. For this aim, we substitute this value into the right-hand side of the ODE, which gives

$$Ax_\mathrm{s} + b = A\left(-A^{-1}b\right) + b = -Ib + b = 0 \ , \tag{1}$$

because $AA^{-1} = I$. Now, the other things that we need to check is that $x(t)$ really satisfies the ODE. Let us start with computing the derivative of the function

$$x(t) = e^{At}(x_0 - x_\mathrm{s}) + x_\mathrm{s} \ .$$

It is given by

$$\dot{x}(t) = Ae^{At}(x_0 - x_\mathrm{s}) \ ,$$

since $\frac{\mathrm{d}}{\mathrm{d}t}e^{At} = Ae^{At}$. This means that

$$\begin{aligned} \dot{x}(t) &= Ae^{At}(x_0 - x_\mathrm{s}) = A\left[e^{At}(x_0 - x_\mathrm{s}) + x_\mathrm{s}\right] - Ax_\mathrm{s} = Ax(t) - Ax_\mathrm{s} \\ &\overset{(1)}{=} Ax(t) + b \ . \end{aligned} \tag{2}$$

Thus, our differential equation is satisfied. Moreover, the initial value condition

$$x(0) = e^{0t}(x_0 - x_\mathrm{s}) + x_\mathrm{s} = I(x_0 - x_\mathrm{s}) + x_\mathrm{s} = x_0 - x_\mathrm{s} + x_\mathrm{s} = x_0$$

is satisfied, too. In summary, we have shown that the function

$$\dot{x}(t) = e^{At}(x_0 - x_\mathrm{s}) + x_\mathrm{s}$$

indeed satisfies the multivariate linear differential equation under the assumption that A is invertible. We still need to discuss whether this the only solution and how this can be generalized for the case that A is not invertible. However, please keep in mind that the above is extremely simple and in complete analogy to Lecture 1 after replacing a with A.

2.2 Examples

The best way to understand the result from Lecture 5 is to do as many examples as you can! Just come up with a matrix A and a vector b and try yourself! Here in this lecture, we discuss the harmonic oscillator example

$$\dot{x}_1(t) = x_2(t), \quad \dot{x}_2(t) = -x_1(t) \quad \text{and} \quad x_1(0) = 1 \quad \text{and} \quad x_2(0) = 0 \, .$$

In order to solve this ODE, we need to pass through the following steps:

1. What are A, b, and x_0? Here, we sort the coefficients of the ODE into a matrix,

   $$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{and} \quad x_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \, .$$

   Notice that the matrix A is invertible, since $\det(A) = 1 \neq 0$. Alternatively, we could check that the eigenvalues of A, given by i and $-i$, are non-zero.

2. In the second step, we need to work out the steady-state. Here, we find

   $$x_{\mathrm{s}} = -A^{-1}b = -\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \, .$$

3. In the third step, we need to work the matrix exponential e^{At}. Fortunately, we already discussed this example in the previous lectures, where we found that

   $$e^{At} = \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix} \, .$$

   Recall that there are several ways to derive this result. For example, one could plug A into the series expansion of the e-function and collect the terms. Or, alternatively, we could find this result by using the diagonalization of A. We discussed both ways in the previous lectures!

4. The fourth and final step is to substitute the previous into the formula for $x(t)$, which is given by

   $$\begin{aligned} x(t) &= e^{At}(x_0 - x_{\mathrm{s}}) + x_{\mathrm{s}} \\ &= e^{At} x_0 \\ &= \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} \cos(t) \\ -\sin(t) \end{pmatrix} \, . \end{aligned} \tag{3}$$

   This is an explicit expression to the ODE that we wanted to solve!

What would happen if we consider the slightly more complicated harmonic oscillator with offset:

$$\dot{x}_1(t) = x_2(t) + 1, \quad \dot{x}_2(t) = -x_1(t) + 1 \quad \text{and} \quad x_1(0) = 1 \quad \text{and} \quad x_2(0) = 0\ ?$$

In this example, the matrix A is still the same as above, but now the vector b is equal to

$$b = \begin{pmatrix} 1 \\ 1 \end{pmatrix} .$$

This means that the steady-state would now be given by

$$x_{\mathrm{s}} = -A^{-1}b = -\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = -\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} .$$

Finally, our solution to the differential equation is given by

$$\begin{aligned} x(t) &= e^{At}(x_0 - x_{\mathrm{s}}) + x_{\mathrm{s}} \\ &= \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix} \begin{pmatrix} 1-1 \\ 0-(-1) \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\ &= \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\ &= \begin{pmatrix} \sin(t)+1 \\ \cos(t)-1 \end{pmatrix} . \end{aligned} \tag{4}$$

Thus, also in this case, we have found an epxlicit expression for the solution trajectory $x(t)$.

2.3 Proof of the uniqueness of solutions

As in the scalar case, the linear ODE

$$\dot{x}(t) = Ax(t) + b \qquad \text{with} \qquad x(0) = x_0$$

has as at most one solution. In order prove this, assume that we have two solutions $x(t)$ and $y(t)$, which both satisfy the ODE,

$$\dot{x}(t) = Ax(t) + b,\ \dot{y}(t) = Ay(t) + b, \qquad \text{with} \qquad x(0) = x_0,\ y(0) = x_0 \ .$$

Then, we can construct their difference function

$$z(t) = x(t) - y(t)$$

satisfying

$$\dot{z}(t) = \dot{x}(t) - \dot{z}(t) = A(x(t) - y(t)) = Az(t) \qquad \text{and} \quad z(0) = x(0) - y(0) = 0 \ ,$$

Now, similar to the corresponding proof of uniqueness in Lecture 1, we introduce the auxiliary function

$$v(t) = e^{-At}z(t) \qquad \Longleftrightarrow \qquad z(t) = e^{At}v(t) \ .$$

This function is well-defined, since we know that the matrix exponential is invertible. Next, we verify that

$$\begin{aligned} \dot{v}(t) &= -Ae^{-At}z(t) + e^{-At}\dot{z}(t) \\ &= -Ae^{-At}z(t) + e^{-At}Az(t) \ . \end{aligned} \tag{5}$$

Now, we need to recall that the matrices A and e^{-At} commute (since A and $-A$ commute). This means that

$$\begin{aligned} \dot{v}(t) &= -Ae^{-At}z(t) + e^{-At}Az(t) \\ &= -Ae^{-At}z(t) + Ae^{-At}z(t) \\ &= 0 \ . \end{aligned} \tag{6}$$

Moreover, we have that

$$v(0) = e^0 z(0) = 0 \ .$$

Thus, all coefficients of the vector-valued function v are constant and equal to 0,

$$\forall t \in \mathbb{R}, \qquad v(t) = 0 \ .$$

This implies that

$$z(t) = e^{At}v(t) = 0 \ ,$$

which, in turn, implies that $x(t) = y(t)$. Thus, whenever we have two solutions to the same ODE, they must be equal. This is the same as saying that any solution to the ODE is unique.

2.4 Construction of General Solutions to Linear ODEs

In the most general case, we can write down an ODE of the form

$$\dot{x}(t) = Ax(t) + b \qquad \text{with} \qquad x(0) = x_0$$

with A being a general matrix that may be not invertible. In this case, there exists in general no steady-state. This is in analalogy to the case "$a = 0$" in the scalar case. However, in the multivariate things are a bit more complicated, since some eigenvalues of A may be equal to 0, while others are not. The main idea for writing down a general solution is to introduce two matrix valued functions, namely,

$$X(t) = e^{At} \qquad \text{and} \qquad Y(t) = X(t) \int_0^t X(\tau)^{-1}\, \mathrm{d}\tau \ .$$

The function $Y(t)$ is well-defined, since the matrix exponential function $X(t)$ is invertible. It satisfies

$$\begin{aligned} \dot{Y}(t) &= \dot{X}(t) \int_0^t X(\tau)^{-1}\, \mathrm{d}\tau + X(t)X(t)^{-1} \\ &= AX(t) \int_0^t X(\tau)^{-1}\, \mathrm{d}\tau + I \\ &= AY(t) + I \end{aligned} \tag{7}$$

and

$$Y(0) = X(0) \int_0^0 X(\tau)^{-1}\, \mathrm{d}\tau = 0 \ .$$

Next, we claim that the general solution of the above ODE is given by

$$x(t) = X(t)x_0 + Y(t)b \ .$$

This is now easy to check, since we have

$$\begin{aligned} \dot{x}(t) &= \dot{X}(t)x_0 + \dot{Y}(t)b = AX(t)x_0 + (AY(t) + I)b \\ &= A(X(t)x_0 + Y(t)b) + b \\ &= Ax(t) + b \end{aligned} \tag{8}$$

as well as

$$x(0) = X(0)x_0 + Y(0)b = I \cdot x_0 + 0 \cdot b = x_0 \ .$$

In summary, we have found a completely general expression for $x(t)$ even if A is not invertible. The only problem is that in practice it can be cumbersome to work out the integral explicitly, but there are theory problems.

3 Summary

This completes all the material of Lectures 1 and 5. We have now a general expression for solving linear ODEs. In the next lecture on Thursday we will do more examples! Moreover, I plan to repeat a bit of the material from Lectures 2 and 3 (which will be closely related to Lecture 6).