



Lecture 7 - Phasor

A beginning of AC circuits

AC = alternating current;

Circuits driven by sinusoidal current or voltage sources are *AC circuits*



Outline

- Sinusoidal signals
- Phasor



Sinusoidal Signal (Current or Voltage)

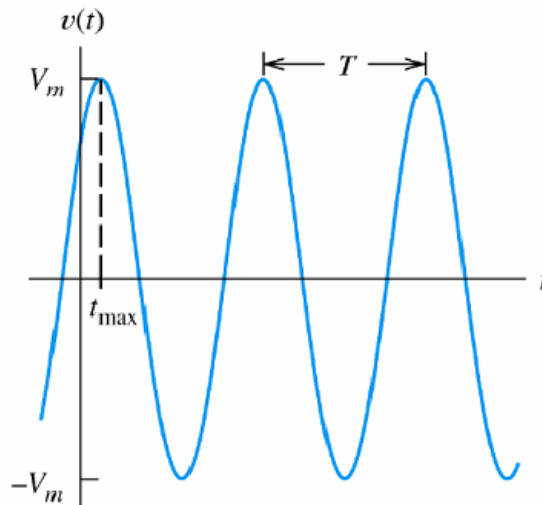
$$v(t) = V_m \cos(\omega t + \theta)$$

V_m is the **peak value**

ω is the **angular frequency** in radians per second

$(\omega t + \theta)$ is the **phase angle**

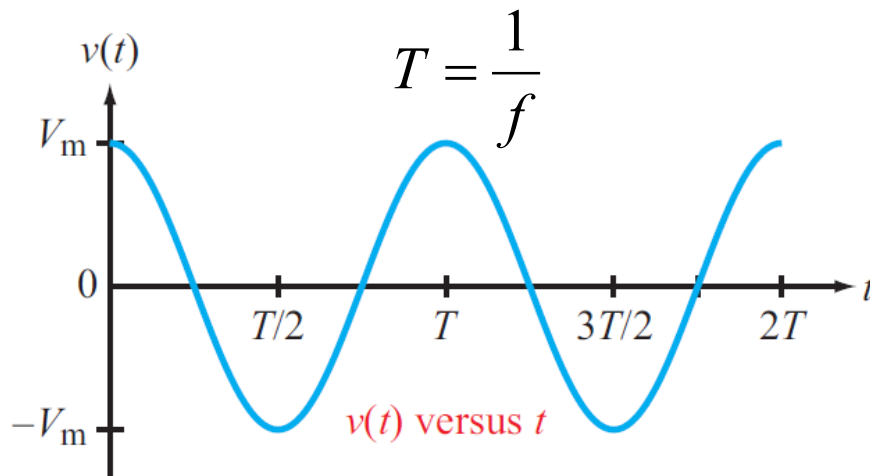
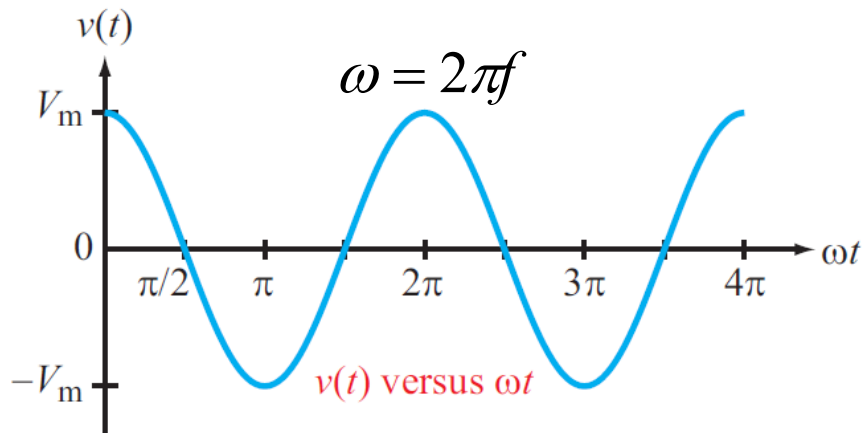
$$\omega = 2\pi f \quad f = \frac{1}{T}$$





Sinusoidal Signals

$$v(t) = V_m \cos(\omega t + \theta)$$



Useful relations

$$\sin x = \pm \cos(x \mp 90^\circ)$$

$$\cos x = \pm \sin(x \pm 90^\circ)$$

$$\sin x = -\sin(x \pm 180^\circ)$$

$$\cos x = -\cos(x \pm 180^\circ)$$

$$\sin(-x) = -\sin x$$

$$\cos(-x) = \cos x$$

$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$$

$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$$

$$2 \sin x \sin y = \cos(x - y) - \cos(x + y)$$

$$2 \sin x \cos y = \sin(x + y) + \sin(x - y)$$

$$2 \cos x \cos y = \cos(x + y) + \cos(x - y)$$

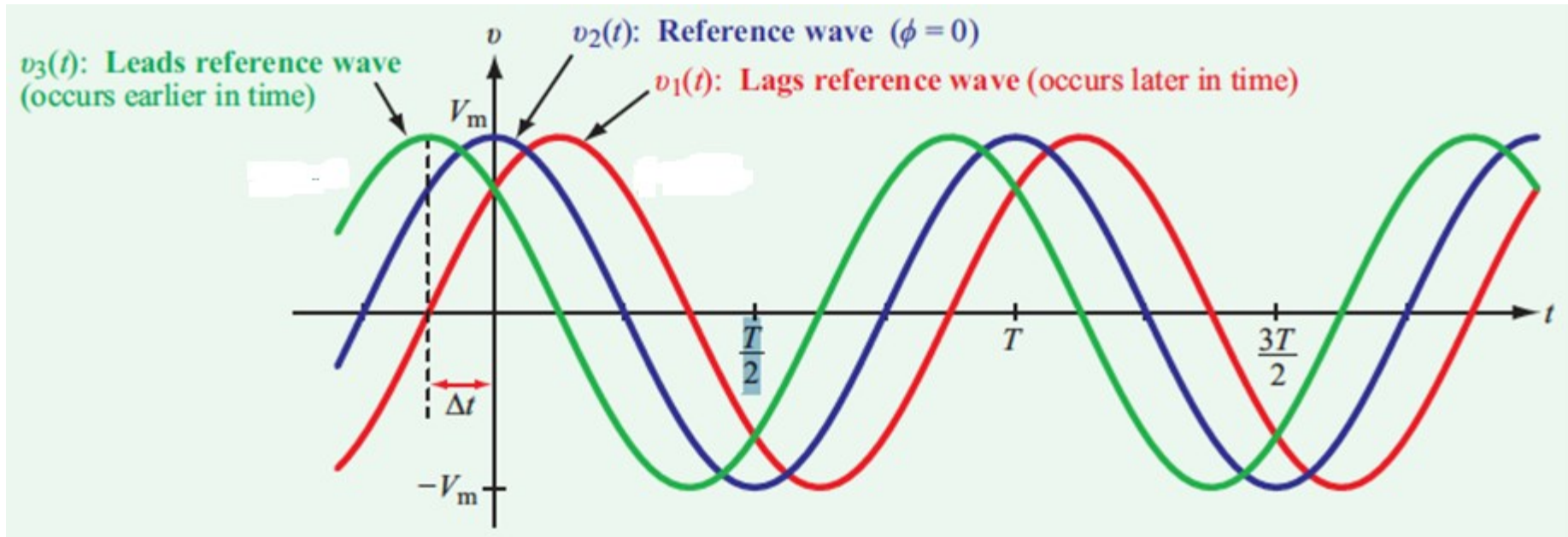


Phase Lead/Lag

$$V_m \cos \frac{2\pi t}{T}$$

$$V_m \cos \left(\frac{2\pi t}{T} - \frac{\pi}{4} \right)$$

$$V_m \cos \left(\frac{2\pi t}{T} + \frac{\pi}{4} \right)$$





Why Sinusoidal signals?

- Numbers of natural phenomenon are sinusoidal in nature.
 - Motion of a pendulum, vibration of a string, ripples on ocean surface
- A very easy signal to generate and transmit
 - Dominant form of signal in communication/electric power industries
 - In the late 1800's there was a battle between proponents of DC and AC. AC won out due to its efficiency for long distance transmission.
- Lastly, they are very easy to handle mathematically.
 - Derivative and integral are also sinusoids.
- Through *Fourier analysis*, any practical periodic function can be represented as sum of sinusoids.



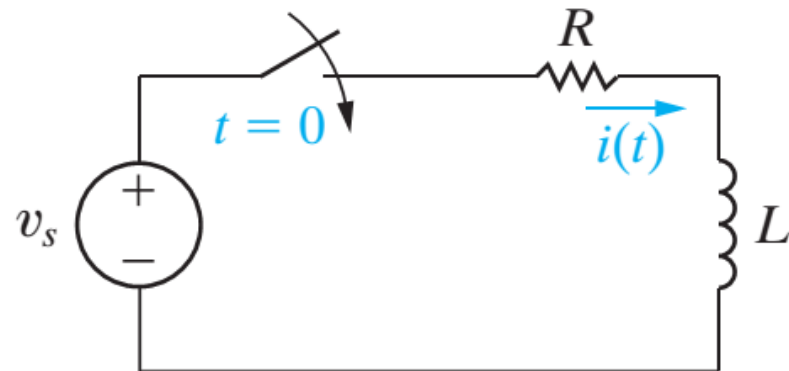
The Sinusoidal Response

$$v_s = V_m \cos(\omega t + \phi), i(0^-) = 0.$$

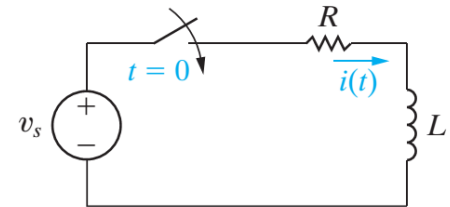
Find $i(t)$, $t \geq 0$.

$$L \frac{di}{dt} + Ri = V_m \cos(\omega t + \phi)$$

Ordinary differential equation



Sinusoidal Steady-State Response



$$v_s = V_m \cos(\omega t + \phi)$$

$$i = \left[\frac{-V_m}{\sqrt{R^2 + \omega^2 L^2}} \cos(\phi - \theta) e^{-(R/L)t} \right] + \left[\frac{V_m}{\sqrt{R^2 + \omega^2 L^2}} \cos(\omega t + \phi - \theta) \right]$$



Transient response



Steady-state response

- Steady-state solution is sinusoidal
- Response frequency = source frequency
- **Magnitude & phase** (initial phase angle) of **S.S.** response differs from that of source



Outline

- Sinusoidal signals
- Phasor



Phasor

$$v(t) = V \cos(\omega t + \phi) = \operatorname{Re}\{V e^{j\phi} e^{j\omega t}\} = \operatorname{Re}(\mathbf{V} e^{j\omega t})$$

$$\mathbf{V} = V e^{j\phi}$$

- As Euler's formula:

$$e^{\pm j\phi} = \cos \phi \pm j \sin \phi$$

$$\cos \phi = \operatorname{Re}(e^{j\phi})$$

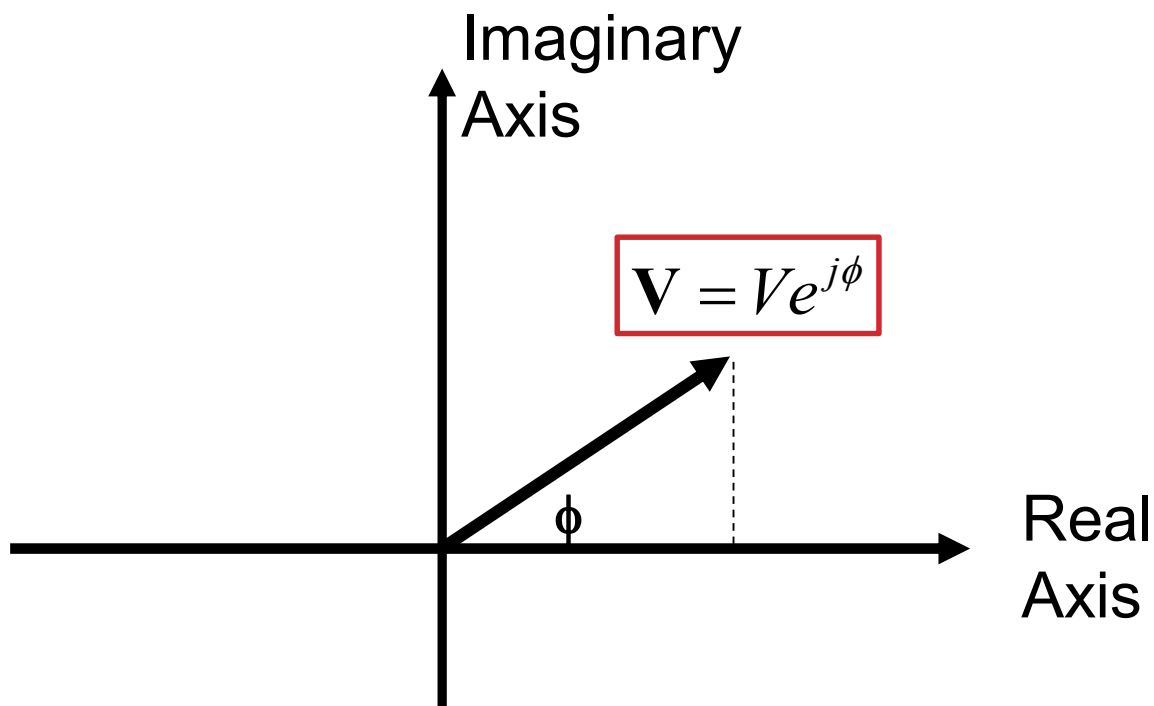
$$\sin \phi = \operatorname{Im}(e^{j\phi})$$



Phasor:

$$v(t) = V \cos(\omega t + \phi) = \operatorname{Re}\{V e^{j\phi} e^{j\omega t}\} = \operatorname{Re}(\mathbf{V} e^{j\omega t}) \quad \boxed{\mathbf{V} = V e^{j\phi}}$$

Complex representation of the magnitude and phase of a **sinusoid**





Phasor : Complex Numbers

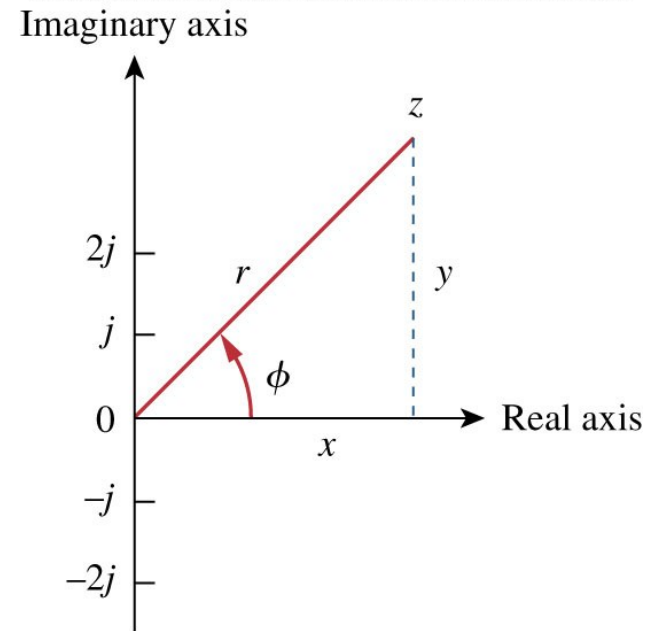
- A powerful method for representing sinusoids is the phasor, a complex expression as well.
- A complex number z can be represented in *rectangular form* as:

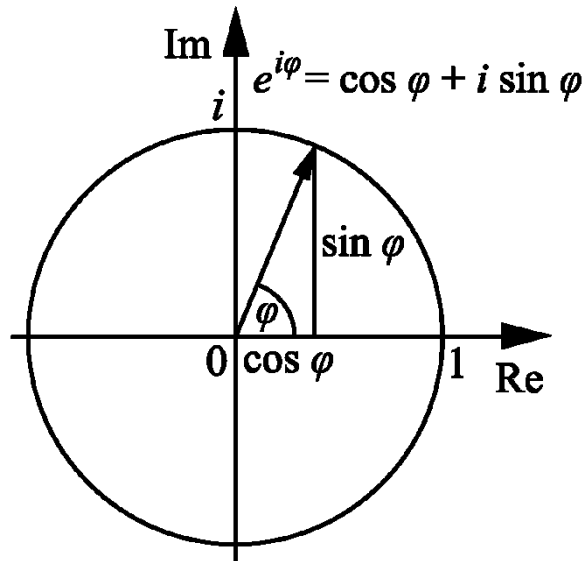
$$z = x + jy \quad \begin{aligned} \operatorname{Re}(z) &= x \\ \operatorname{Im}(z) &= y \end{aligned}$$

- It can also be written in *polar* or *exponential* form as:

$$z = r \angle \phi = re^{j\phi}$$

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$$1 = 1e^{j0} = 1\angle 0^\circ$$

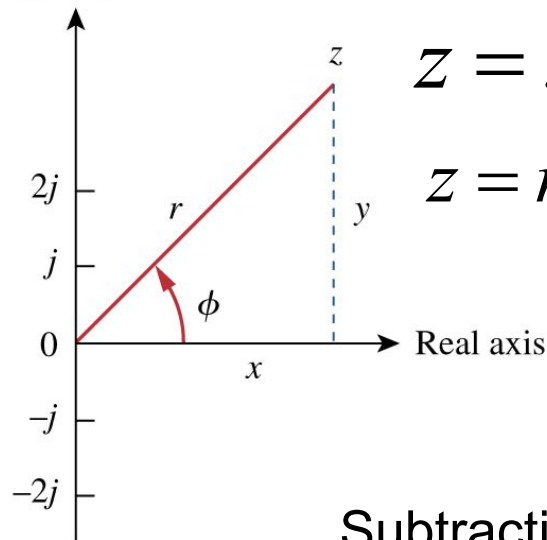
$$j = 1e^{j\frac{\pi}{2}} = 1\angle 90^\circ$$



Arithmetic With Complex Numbers

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Imaginary axis



$$z = x + jy$$

$$z = r \angle \phi = re^{j\phi}$$

Addition

$$z_1 + z_2 = (x_1 + x_2) + j(y_1 + y_2)$$

Subtraction

$$z_1 - z_2 = (x_1 - x_2) + j(y_1 - y_2)$$

Multiplication

$$z_1 z_2 = r_1 r_2 \angle (\phi_1 + \phi_2)$$

Division

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} \angle (\phi_1 - \phi_2)$$

Reciprocal

$$\frac{1}{z} = \frac{1}{r} \angle (-\phi)$$

Square Root

$$z^{1/2} = \pm |z|^{1/2} e^{j\theta/2}$$

Complex Conjugate

$$z^* = x - jy = r \angle -\phi = re^{-j\phi}$$



Relations for Complex Numbers

| | |
|--|--|
| Euler's Identity: $e^{j\theta} = \cos \theta + j \sin \theta$ | |
| $\sin \theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$ | $\cos \theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$ |
| $\mathbf{z} = x + jy = \mathbf{z} e^{j\theta}$ | $\mathbf{z}^* = x - jy = \mathbf{z} e^{-j\theta}$ |
| $x = \Re(\mathbf{z}) = \mathbf{z} \cos \theta$ | $ \mathbf{z} = \sqrt[4]{\mathbf{z}\mathbf{z}^*} = \sqrt{x^2 + y^2}$ |
| $y = \Im(\mathbf{z}) = \mathbf{z} \sin \theta$ | $\theta = \tan^{-1}(y/x)$ |
| $\mathbf{z}^n = \mathbf{z} ^n e^{jn\theta}$ | $\mathbf{z}^{1/2} = \pm \mathbf{z} ^{1/2} e^{j\theta/2}$ |
| $\mathbf{z}_1 = x_1 + jy_1$ | $\mathbf{z}_2 = x_2 + jy_2$ |
| $\mathbf{z}_1 = \mathbf{z}_2$ iff $x_1 = x_2$ and $y_1 = y_2$ | $\mathbf{z}_1 + \mathbf{z}_2 = (x_1 + x_2) + j(y_1 + y_2)$ |
| $\mathbf{z}_1 \mathbf{z}_2 = \mathbf{z}_1 \mathbf{z}_2 e^{j(\theta_1 + \theta_2)}$ | $\frac{\mathbf{z}_1}{\mathbf{z}_2} = \frac{ \mathbf{z}_1 }{ \mathbf{z}_2 } e^{j(\theta_1 - \theta_2)}$ |
| $-1 = e^{j\pi} = e^{-j\pi} = 1 \angle \pm 180^\circ$ | |
| $j = e^{j\pi/2} = 1 \angle 90^\circ$ | $-j = e^{-j\pi/2} = 1 \angle -90^\circ$ |
| $\sqrt{j} = \pm e^{j\pi/4} = \pm \frac{(1+j)}{\sqrt{2}}$ | $\sqrt{-j} = \pm e^{-j\pi/4} = \pm \frac{(1-j)}{\sqrt{2}}$ |



Example

- Evaluate these complex numbers

(a) $(40\angle 50^\circ + 20\angle -30^\circ)^{1/2}$

(b) $\frac{10\angle -30^\circ + (3 - j4)}{(2 + j4)(3 - j5)^*}$



Exercise

- Evaluate the following complex numbers

(a) $[(5 + j2)(-1 + j4) - 5\angle 60^\circ]^*$

(b) $\frac{10 + j5 + 3\angle 40^\circ}{-3 + j4} + 10\angle 30^\circ + j5$



Phasors

$$v(t) = V_m \cos(\omega t + \phi)$$

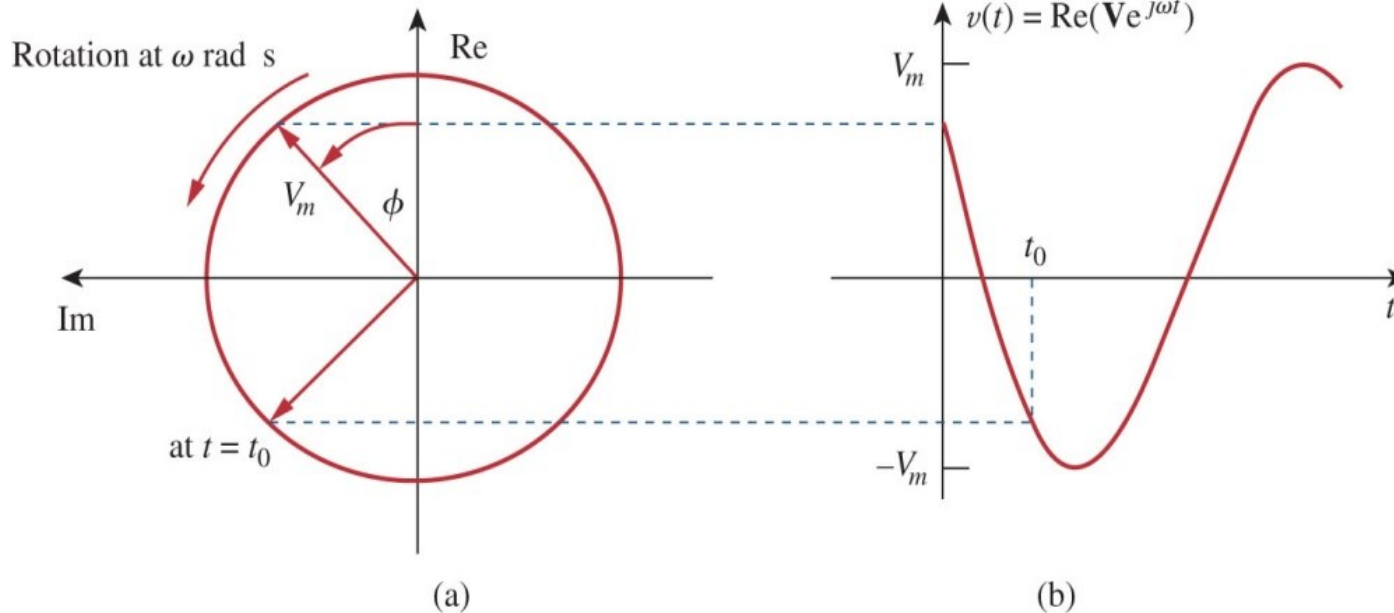
(Time-domain
representation)



$$\mathbf{V} = V_m \angle \phi$$

(Phasor-domain
representation)

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Example

- Transform these sinusoids to phasors

(a) $i = 6 \cos(50t - 40^\circ) \text{ A}$

(b) $v = -4 \sin(30t + 50^\circ) \text{ V}$



Example

- Find the sinusoids represented by these phasors

$$\mathbf{I} = -3 + j4 \text{ A}$$

$$\mathbf{V} = j8e^{-j20^\circ} \text{ V}$$

$$\mathbf{V} = -25 \angle 40^\circ \text{ V}$$

$$\mathbf{I} = j(12 - j5) \text{ A}$$



Phasor Relationships for Resistors

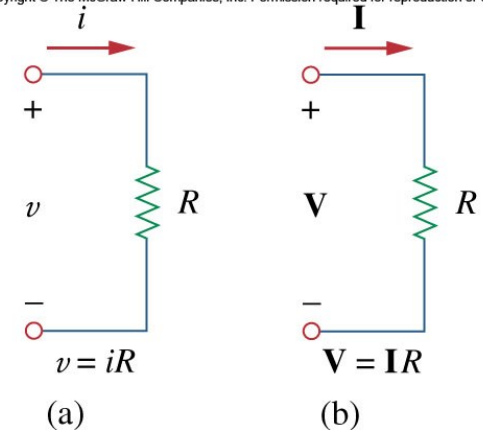
- For the resistor, the voltage and current are related via Ohm's law. As such, the voltage and current are in phase with each other.

$$i = I_m \cos(\omega t + \phi) \quad \mathbf{I} = I_m \angle \phi$$

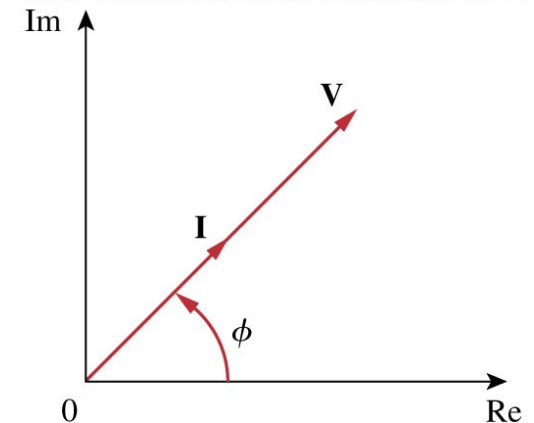
$$v = RI_m \cos(\omega t + \phi) \quad \mathbf{V} = RI_m \angle \phi$$

$$\mathbf{V} = R\mathbf{I}$$

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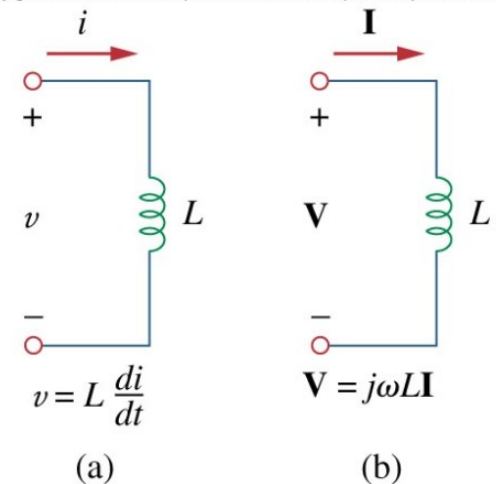


Phasor Relationships for Inductors

$$i = I_m \cos(\omega t + \phi) \quad \mathbf{I} = I_m \angle \phi$$

$$v = L \frac{di}{dt} = -\omega L I_m \sin(\omega t + \phi) \\ = \omega L I_m \cos(\omega t + \phi + 90^\circ)$$

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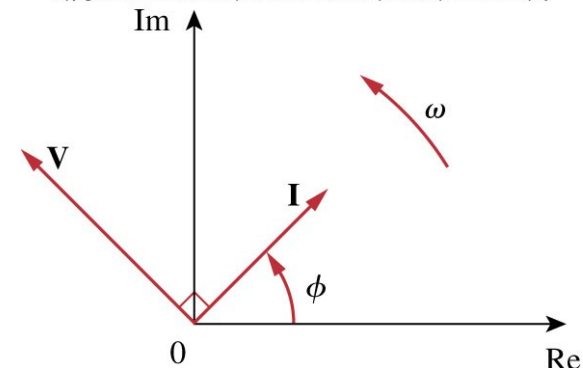


$$\mathbf{V} = \omega L I_m \angle \phi + 90^\circ = \omega L I_m e^{j(\phi + 90^\circ)} = \omega L e^{j90^\circ} \cdot I_m e^{j\phi} = j\omega L \cdot \mathbf{I}$$

$$\mathbf{V} = j\omega L \mathbf{I}$$

- The voltage leads the current by 90° (phase shift = 90°)

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Phasor Relationships for Capacitors

$$v = V_m \cos(\omega t + \phi) \quad \mathbf{V} = V_m \angle \phi$$

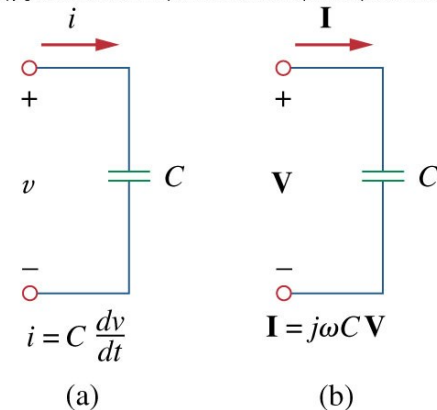
$$i = C \frac{dv}{dt} = -\omega C V_m \sin(\omega t + \phi) \\ = \omega C V_m \cos(\omega t + \phi + 90^\circ)$$

$$\mathbf{I} = \omega C V_m \angle \phi + 90^\circ = \dots = j\omega C \cdot \mathbf{V}$$

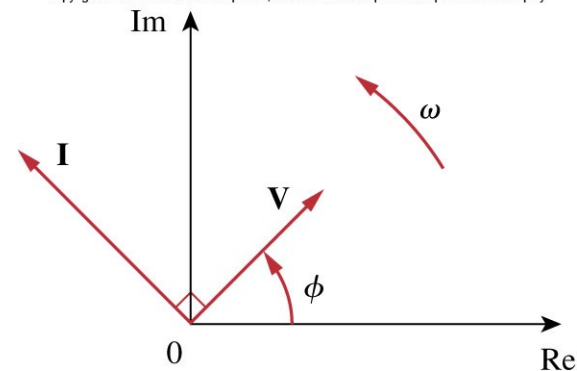
$$\mathbf{V} = \frac{\mathbf{I}}{j\omega C}$$

- The voltage lags the current by 90° .

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Impedance

- The voltage-current relations for R, L and C elements are

$$\begin{array}{ccc} \mathbf{V} = R\mathbf{I} & \mathbf{V} = j\omega L\mathbf{I} & \mathbf{V} = \frac{\mathbf{I}}{j\omega C} \\ & \downarrow & \\ \frac{\mathbf{V}}{\mathbf{I}} = R & \frac{\mathbf{V}}{\mathbf{I}} = j\omega L & \frac{\mathbf{V}}{\mathbf{I}} = \frac{1}{j\omega C} \end{array}$$

- Phasors allow us to express the relationship between current and voltage using a formula like Ohm's law:

$$\mathbf{V} = \mathbf{I}\mathbf{Z} \quad \text{or} \quad \mathbf{Z} = \frac{\mathbf{V}}{\mathbf{I}}$$

- \mathbf{Z} is called **impedance**, measured in ohms.
 - Impedance is not a phasor! But it is (often) a complex number.
 - Impedance depends on the frequency ω .



Admittance





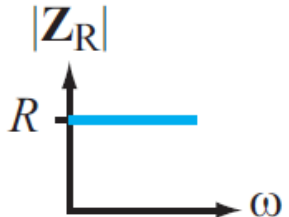
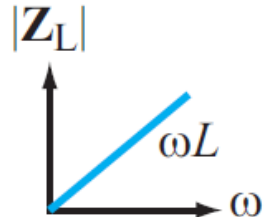
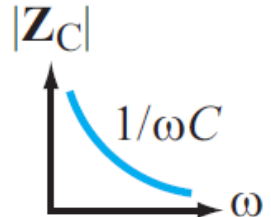
Admittance is simply the inverse of impedance, unit: Simens.

$$\mathbf{Y} = \frac{\mathbf{I}}{\mathbf{V}}$$

| Element | Impedance | Admittance |
|---------|------------------------------------|------------------------------------|
| R | $\mathbf{Z} = R$ | $\mathbf{Y} = \frac{1}{R}$ |
| L | $\mathbf{Z} = j\omega L$ | $\mathbf{Y} = \frac{1}{j\omega L}$ |
| C | $\mathbf{Z} = \frac{1}{j\omega C}$ | $\mathbf{Y} = j\omega C$ |



Summary of R , L , C

| Property | R | L | C |
|---------------------------|--|--|--|
| $v-i$ | $v = Ri$ | $v = L \frac{di}{dt}$ | $i = C \frac{dv}{dt}$ |
| $\mathbf{V-I}$ | $\mathbf{V} = R\mathbf{I}$ | $\mathbf{V} = j\omega L\mathbf{I}$ | $\mathbf{V} = \frac{\mathbf{I}}{j\omega C}$ |
| \mathbf{Z} | R | $j\omega L$ | $\frac{1}{j\omega C}$ |
| dc equivalent | R |  Short circuit |  Open circuit |
| High-frequency equivalent | R |  Open circuit |  Short circuit |
| Frequency response |  |  |  |