CS244: Theory of Computation

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Outline

1. Linear temporal logic (LTL)
2. LTL model checking: Automata theoretical approach
3. Computation tree logic (CTL)
4. (Weak) alternating tree automata
5. CTL model checking: Automata theoretical approach

Temporal logics: The general background

A brief history

- Introduced by a philosopher Arthur Prior in 1950's (known as tense logic).
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Linear time versus branching time

- Linear time: Each moment has a unique future.
- Branching time: Each moment may have several possible futures.

Time point versus intervals

- Refer to the time by time points: Linear temporal logic, Computation tree logic, Modal μ-calculus,
- Refer to the time by time intervals: Interval temporal logics.

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Extensions

Timed, probabilistic, ...

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Syntax of LTL:

$$\varphi := p(p \in AP) \mid \varphi_1 \vee \varphi_2 \mid \neg\varphi_1 \mid X\varphi_1 \mid \varphi_1 U \varphi_2.$$

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Semantics of LTL:

Let $w \in (2^{AP})^\omega$ and φ be a LTL formula. Then

- $(w,i) \vDash p$ iff $p \in w_0$,
- $(w,i) \vDash \varphi_1 \vee \varphi_2$ iff $(w,i) \vDash \varphi_1$ or $(w,i) \vDash \varphi_2$,
- $(w,i) \vDash \neg\varphi_1$ iff not $(w,i) \vDash \varphi_1$,
- $(w,i) \vDash X\varphi_1$ iff $(w,i+1) \vDash \varphi_1$,
- $(w,i) \vDash \varphi_1 U \varphi_2$ iff $\exists j$ s.t. $j \geq i$, $(w,j) \vDash \varphi_2$ and $\forall k : i \leq k < j$, $(w,k) \vDash \varphi_1$.

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$w \vDash \varphi$ iff $(w,0) \vDash \varphi$.

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$L(\varphi)$: $\{w \in (2^{AP})^\omega \mid w \vDash \varphi\}$.

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Semantics of LTL:

Let $w \in (2^{AP})^\omega$ and φ be a LTL formula. Then

- $(w,i) \models p$ iff $p \in w_0$,
- $(w,i) \models \varphi_1 \vee \varphi_2$ iff $(w,i) \models \varphi_1$ or $(w,i) \models \varphi_2$,
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$w \models \varphi$ iff $(w,0) \models \varphi$.

$L(\varphi)$: $\{w \in (2^{AP})^\omega \mid w \models \varphi\}$.

Derived temporal operators:

$$\top := p \vee \neg p,\ F\varphi := \top\ U\varphi,\ G\varphi := \neg F \neg\varphi,\ \varphi_1 R \varphi_2 := \neg(\neg\varphi_1 U \neg\varphi_2),\ \ldots.$$

Remark: X: neXt, U: Until, F: Future, G: Global, R: Release.

Expressiveness of LTL

Examples: Xp, pUq, $G(p \to Fq)$, FGp, $GFp \to GFq$.

Proposition. The property "event p occurs at least at all even time points" is not expressible in LTL.

How about the formula $p \wedge G(p \to Xq) \wedge G(q \to Xp)$*?*

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How about the formula $p \wedge G(p \to Xq) \wedge G(q \to Xp)$*?*

Lemma. Let $AP = \{p\}$. Then for every LTL formula φ of size n over AP and every $m, m' \geq n$, $\{p\}^m(\varnothing\{p\})^\omega \vDash \varphi$ iff $\{p\}^{m'}(\varnothing\{p\})^\omega \vDash \varphi$.

Proof (Proposition).

For contradiction, suppose that "event p occurs at least at all even time points" can be defined by a LTL formula φ.
Let $n = |\varphi|$.
From the lemma, $\{p\}^n(\varnothing\{p\})^\omega \vDash \varphi$ iff $\{p\}^{n+1}(\varnothing\{p\})^\omega \vDash \varphi$.
On the other hand, either not $\{p\}^n(\varnothing\{p\})^\omega \vDash \varphi$ or not $\{p\}^{n+1}(\varnothing\{p\})^\omega \vDash \varphi$.
We get a contradiction. □

Theorem. $LTL \equiv FO[AP, +1, <]$ (monadic first-order logic of order).

Expressiveness of LTL

Proof of the lemma ($\{p\}^m(\varnothing\{p\})^\omega \vDash \varphi$ iff $\{p\}^{m'}(\varnothing\{p\})^\omega \vDash \varphi$).

Induction on the structure of φ.

- $\varphi = p$ and $m, m' \geq n = 1$: $\{p\}^m(\varnothing\{p\})^\omega \vDash p$ iff $\{p\}^{m'}(\varnothing\{p\})^\omega \vDash p$,
- $\varphi = \varphi_1 \vee \varphi_2$ or $\varphi = \neg\varphi_1$: easy,
- $\varphi = X\varphi_1$: $\{p\}^m(\varnothing\{p\})^\omega \vDash X\varphi_1$ iff $\{p\}^{m-1}(\varnothing\{p\})^\omega \vDash \varphi_1$ iff $\{p\}^{m'-1}(\varnothing\{p\})^\omega \vDash \varphi_1$ iff $\{p\}^{m'}(\varnothing\{p\})^\omega \vDash X\varphi_1$,
- $\varphi = \varphi_1 U \varphi_2$: By symmetry, it is sufficient to show $\{p\}^m(\varnothing\{p\})^\omega \vDash \varphi_1 U \varphi_2$ $\Rightarrow$ $\{p\}^{m'}(\varnothing\{p\})^\omega \vDash \varphi_1 U \varphi_2$. There are three situations.

□

$p^{m-i} \mid p^i(\emptyset\{p\})^\omega$

φ_2

$\forall j : 1 \leq j \leq m-i.$

$p^{i+j}(\emptyset\{p\})^\omega \models \varphi_1$

$p^m(\emptyset\{p\})^i \mid (\emptyset\{p\})^\omega$

φ_2

$\forall j : 0 \leq j < i.$

$\{p\}(\emptyset\{p\})^j(\emptyset\{p\})^\omega \models \varphi_1$

$(\emptyset\{p\})^{j+1}(\emptyset\{p\})^\omega \models \varphi_1$

$\forall j' : 1 \leq j' \leq m.$

$p^{j'}(\emptyset\{p\})^\omega \models \varphi_1$

$p^m(\emptyset\{p\})^i\emptyset \mid \{p\}(\emptyset\{p\})^\omega$

φ_2

$\forall j : 0 \leq j < i.$

$(\emptyset\{p\})^j\emptyset\{p\}(\emptyset\{p\})^\omega \models \varphi_1$

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$\forall j' : 0 \leq j' \leq m.$

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Expressiveness of LTL

Proof of the lemma ($\{p\}^m(\varnothing\{p\})^\omega \vDash \varphi$ iff $\{p\}^{m'}(\varnothing\{p\})^\omega \vDash \varphi$).

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$\square$

To exemplify the proof, consider the second situation:

$$(\varnothing\{p\})^\omega \vDash \varphi_1 \text{ and } \forall j' : 1 \leq j' \leq m. \{p\}^{j'}(\varnothing\{p\})^\omega \vDash \varphi_1.$$

Then

$$\{p\}^m(\varnothing\{p\})^\omega \vDash \varphi_1 \Rightarrow \forall n \leq j' \leq m'. \{p\}^{j'}(\varnothing\{p\})^\omega \vDash \varphi_1 \ \textit{(By IH)} \Rightarrow$$
$$\forall 1 \leq j' \leq m'. \{p\}^{j'}(\varnothing\{p\})^\omega \vDash \varphi_1 \Rightarrow \{p\}^{m'}(\varnothing\{p\})^\omega \vDash \varphi_1 U \varphi_2.$$

The arguments for the other two situations are similar.

Outline

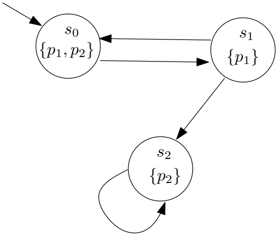
Kripke structure

A Kripke structure $\mathcal{S}$ is a tuple $(S, AP, \rightarrow, I, L)$, where

- S: the set of states,
- AP: the set of atomic propositions,
- $\rightarrow \subseteq S \times S$: the transition relation s.t. $\forall s \exists s'. s \rightarrow s'$,
- $I \subseteq S$: The set of initial states,
- $L : S \rightarrow 2^{AP}$: The labelling function.

A *path* π in $\mathcal{S}$: An infinite sequence of states $s_0 s_1 \ldots$ s.t. $\forall i. s_i \rightarrow s_{i+1}$.
A path $s_0 s_1 \ldots$ is *initial* if $s_0 \in I$.

$L(\mathcal{S}) = \{L(\pi) \mid \pi$ is an initial path in $\mathcal{S}\}$, where $L(\pi) = L(s_0)L(s_1)\ldots$ if $\pi = s_0 s_1 \ldots$.



LTL model checking

Let $\mathcal{S} = (S, AP, \to, I, L)$ be a Kripke structure and φ be an LTL formula. Then $\mathcal{S} \models \varphi$ iff for every initial path π in $\mathcal{S}$, $L(\pi) \models \varphi$.

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Model checking (MC) problem:

> *Given a Kripke structure $\mathcal{S}$ and an LTL formula φ, decide whether $\mathcal{S} \vDash \varphi$.*

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Automata-theoretical approach to MC problem

The idea:

> $\mathcal{S} = (S, AP, \rightarrow, I, L)$ *can be viewed as a Büchi automaton* $\mathcal{A}_{\mathcal{S}} = (S, 2^{AP}, \delta, I, S)$, *where* $(s, P, s') \in \delta$ *iff* $s \rightarrow s'$ *and* $P = L(s)$.

The algorithm:

1. Construct an equivalent Büchi automaton $\mathcal{A}_{\neg\varphi}$ from $\neg\varphi$.
2. Construct $\mathcal{A}'$ as a product of $\mathcal{A}_{\mathcal{S}}$ and $\mathcal{A}_{\neg\varphi}$ accepting $L(\mathcal{A}_{\mathcal{S}}) \cap L(\mathcal{A}_{\neg\varphi})$.
3. Decide whether $L(\mathcal{A}')$ is empty.

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3. Decide whether $L(\mathcal{A}')$ is empty.

Question: How to construct $\mathcal{A}_{\neg\varphi}$ from $\neg\varphi$?

Generalised Büchi automata (GBA)

A GBA $\mathcal{A}$ is a tuple $(Q, 2^{AP}, \delta, I, \mathcal{F})$, where

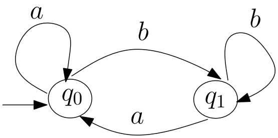
- Q: the set of states,
- δ: the transition relation,
- I: the set of initial states,
- $\mathcal{F} \subseteq 2^Q$: the acceptance component.

The runs of a GBA over ω-words are defined similarly to those of BA.

A run $r = q_0 q_1 \dots$ of a GBA $\mathcal{A}$ is accepting if $\forall F \in \mathcal{F}$, $\mathrm{Inf}(r) \cap F \neq \varnothing$.

Example:

$$\mathcal{F} = \{\{q_0\}, \{q_1\}\}$$



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Proof.

Let $\mathcal{A} = (Q, 2^{AP}, \delta, I, \mathcal{F})$ be a GBA.
Suppose $\mathcal{F} = \{F_1, \ldots, F_k\}$, we construct a BA $\mathcal{A}' = (Q', 2^{AP}, \delta', I', F')$ as follows.

- $Q' = Q \times \{0, \ldots, k\}$,
- $I' = I \times \{0\}$,
- $F' = Q \times \{k\}$,
- δ' is defined by the following rules,
  - for every $(q, P, q') \in \delta$ and every $i : 1 \leq i \leq k$ s.t. $q' \in F_i$, $((q, i-1), P, (q', i)) \in \delta'$,
  - for every $(q, P, q') \in \delta$, $((q, k), P, (q', 0)) \in \delta'$.

□

Closure of LTL formulas

For an LTL formula φ, let $\mathrm{sub}(\varphi)$ denote the set of subformulas of φ.

Given an LTL formula φ, the *closure* of φ, denoted by $\mathrm{cl}(\varphi)$, is

$\mathrm{sub}(\varphi) \cup \{\neg\psi \mid \psi \in \mathrm{sub}(\varphi)\}$ *(where $\neg\neg\psi$ and ψ are identified).*

Example:

Suppose $\varphi = G(p \to Fq) = \neg\,(true\; U \neg(\neg p \vee Fq))$. Then

$$\mathrm{cl}(\varphi) = \left\{ \begin{array}{c} p, \neg p, q, \neg q, true, \neg true, \\ Fq, \neg Fq, \\ \neg p \vee Fq, \neg(\neg p \vee Fq), \\ true\; U \neg(\neg p \vee Fq), \varphi \end{array} \right\},$$

where $true = p \vee \neg p, Fq = true\; U q$.

Elementary sets of formulas

Let φ be an LTL formula and $B \subseteq \mathrm{cl}(\varphi)$.

Then B is said to be *elementary* if B satisfies the following conditions,

- Consistency wrt. Boolean operators: For every $\psi_1 \vee \psi_2, \psi \in \mathrm{cl}(\varphi)$,
  - $\psi_1 \vee \psi_2 \in B$ iff $\psi_1 \in B$ or $\psi_2 \in B$,
  - if $\psi \in B$, then $\neg\psi \notin B$,
- Local consistency wrt. Until operators: For every $\psi_1 U \psi_2 \in \mathrm{cl}(\varphi)$,
  - if $\psi_2 \in B$, then $\psi_1 U \psi_2 \in B$,
  - if $\psi_1 U \psi_2 \in B$ and $\psi_2 \notin B$, then $\psi_1 \in B$,
- Maximality: For every $\psi \in \mathrm{cl}(\varphi)$, if $\psi \notin B$, then $\neg\psi \in B$.

Example:

Let $\varphi = G(p \to Fq) = \neg\,(true\ U \neg(\neg p \vee Fq))$.

Suppose $B = \{\neg p, q, true, Fq, \neg p \vee Fq, true\ U \neg(\neg p \vee Fq)\}$.

Then B is elementary.

- Boolean cosistency: $\neg p \in B \Rightarrow true, \neg p \vee Fq \in B, \ldots,$
- Local consistency wrt. Until: $q \in B \Rightarrow Fq \in B$,
  $true\ U \neg(\neg p \vee Fq) \in B$, $\neg(\neg p \vee Fq) \notin B \Rightarrow true \in B$,
- Maximality: $\varphi \notin B \Rightarrow true\ U \neg(\neg p \vee Fq) \in B, \ldots.$

From LTL to GBA

Theorem. Given an LTL formula φ, an equivalent GBA $\mathcal{A} = (Q, 2^{AP}, \delta, I, \mathcal{F})$ s.t. $|Q| = 2^{O(|\varphi|)}$ and $|\mathcal{F}| = O(|\varphi|)$ can be constructed.

Proof.

Let φ be an LTL formula.
Define a GBA $\mathcal{A} = (Q, 2^{AP}, \delta, I, \mathcal{F})$ as follows.

- Q is the set of elementary set of formulas $B \subseteq \mathrm{cl}(\varphi)$,
- $I = \{B \mid \varphi \in B\}$,
- δ is the set of tuples (B, P, B') s.t.
  - $P = \{p \in AP \mid p \in B\}$,
  - for every $\psi, X\psi \in \mathrm{cl}(\varphi)$, $X\psi \in B$ iff $\psi \in B'$,
  - for every $\psi_1 U \psi_2 \in \mathrm{cl}(\varphi)$,

$$\psi_1 U \psi_2 \in B \Leftrightarrow (\psi_2 \in B \text{ or } (\psi_1 \in B, \psi_1 U \psi_2 \in B')).$$

- $\mathcal{F} = \{F_{\psi_1 U \psi_2} \mid \psi_1 U \psi_2 \in \mathrm{cl}(\varphi)\}$, where

$$F_{\psi_1 U \psi_2} = \{B \in Q \mid \psi_1 U \psi_2 \in B \Rightarrow \psi_2 \in B\}.$$

Claim. For every $w \in (2^{AP})^{\omega}$, $w \models \varphi$ iff $w \in L(\mathcal{A})$. $\square$

From LTL to GBA

Claim. For every $w \in (2^{AP})^\omega$, $w \models \varphi$ iff $w \in L(\mathcal{A})$.

Proof.

"Only if" direction: Suppose $w \models \varphi$.
For every $i \in \mathbb{N}$, let $B_i = \{\psi \in \mathrm{cl}(\varphi) \mid (w, i) \models \psi\}$.
Then $B_0 B_1 \dots$ is a run of $\mathcal{A}$ over w.
$B_0 B_1 \dots$ is also an accepting run:
For every $\psi_1 U \psi_2 \in \mathrm{cl}(\varphi)$,

- if $\exists i. \forall j : j \geq i.\ \psi_1 U \psi_2 \notin B_j$, then
  $\forall j : j \geq i.\ B_j \in F_{\psi_1 U \psi_2} \Rightarrow \mathrm{Inf}(B_0 B_1 \dots) \cap F_{\psi_1 U \psi_2} \neq \emptyset$,
- if $\exists$ infinitely many i s.t. $\psi_1 U \psi_2 \in B_i$, in other words, $(w, i) \models \psi_1 U \psi_2$, then
  $\exists$ infinitely many i' s.t. $(w, i') \models \psi_2$,
  thus, $\psi_2, \psi_1 U \psi_2 \in B_{i'}$, *so,* $B_{i'} \in F_{\psi_1 U \psi_2}$
  $\Rightarrow$
  $\mathrm{Inf}(B_0 B_1 \dots) \cap F_{\psi_1 U \psi_2} \neq \emptyset$.

□

From LTL to GBA

Claim. For every $w \in (2^{AP})^\omega$, $w \models \varphi$ iff $w \in L(\mathcal{A})$.

Proof.

"If" direction: Suppose $w \in L(\mathcal{A})$.
Then there is an accepting run $B_0 B_1 \ldots$ of $\mathcal{A}$ over w.
It is sufficient to show that for every $\psi \in \mathrm{cl}(\varphi)$, the following holds,

$$\text{for every } i \in \mathbb{N} \text{ s.t. } \psi \in B_i,\ (w,i) \models \psi.$$

Induction on the structure of formulas.

- $\psi = p$: Then $p \in B_i$, so $p \in w_i$ (from the construction of $\mathcal{A}$), $(w,i) \models \psi$,
- $\psi = \psi_1 \vee \psi_2$ or $\psi = \neg\psi_1$: Easy.
- $\psi = X\psi_1$: Then $\psi_1 \in B_{i+1}$, so $(w, i+1) \models \psi_1$ (by induction hypothesis), $(w,i) \models X\psi_1$.
- $\psi = \psi_1 U \psi_2$: Then either $\psi_2 \in B_i$ or ($\psi_1 \in B_i$ and $\psi_1 U \psi_2 \in B_{i+1}$).
  From $\mathrm{Inf}(B_0 B_1 \ldots) \cap F_{\psi_1 U \psi_2} \neq \emptyset$, we know
  $$\exists j : j \geq i.\ \psi_2 \in B_j \text{ and } \forall k : i \leq k < j.\ \psi_1 \in B_k.$$
  By induction hypothesis, $(w,j) \models \psi_2$ and $\forall k : i \leq k < j.\ (w,k) \models \psi_1$.
  We deduce that $(w,i) \models \psi_1 U \psi_2$.

From LTL to GBA: Quiz

Let $\varphi = G(p \to Fq) = \neg\,(true\; U \neg(\neg p \vee Fq))$.

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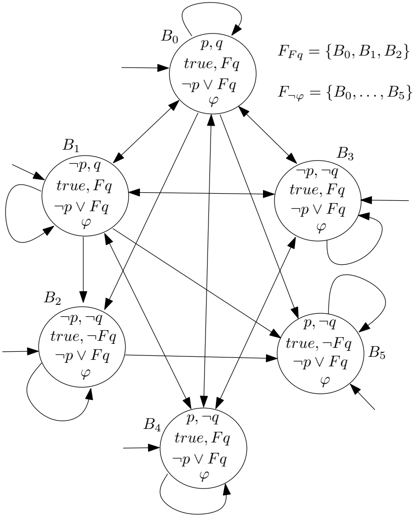

Outline

CTL

Syntax:

$$\varphi := p(p \in AP) \mid \varphi_1 \vee \varphi_2 \mid \neg\varphi_1 \mid EX\varphi_1 \mid AX\varphi_1 \mid E\varphi_1 U\varphi_2 \mid A\varphi_1 U\varphi_2$$

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Syntax:

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Semantics:

Given a Kripke structure $\mathcal{S} = (S, AP, \to, I, L)$ and a CTL formula φ,

- $(\mathcal{S}, s) \models p$ iff $p \in L(s)$,
- $(\mathcal{S}, s) \models \varphi_1 \vee \varphi_2$ iff $(\mathcal{S}, s) \models \varphi_1$ or $(\mathcal{S}, s) \models \varphi_2$,
- $(\mathcal{S}, s) \models \neg\varphi_1$ iff not $(\mathcal{S}, s) \models \varphi_1$,
- $(\mathcal{S}, s) \models EX\varphi_1$ iff there exists s' s.t. $s \to s'$ and $(\mathcal{S}, s') \models \varphi_1$,
- $(\mathcal{S}, s) \models AX\varphi_1$ iff for all s' s.t. $s \to s'$, it holds $(\mathcal{S}, s') \models \varphi_1$,
- $(\mathcal{S}, s) \models E\varphi_1 U \varphi_2$ iff there exists a path π of $\mathcal{S}$ starting from s s.t. $\pi \models \varphi_1 U \varphi_2$,
- $(\mathcal{S}, s) \models A\varphi_1 U \varphi_2$ iff for every path π of $\mathcal{S}$ starting from s, $\pi \models \varphi_1 U \varphi_2$,

where $\pi \models \varphi_1 U \varphi_2$ iff $\exists i \geq 0$, $(\mathcal{S}, \pi(i)) \models \varphi_2$ and $\forall j : 0 \leq j < i$, $(\mathcal{S}, \pi(j)) \models \varphi_1$.

CTL

Syntax:

$$\varphi := p(p \in AP) \mid \varphi_1 \vee \varphi_2 \mid \neg\varphi_1 \mid EX\varphi_1 \mid AX\varphi_1 \mid E\varphi_1 U\varphi_2 \mid A\varphi_1 U\varphi_2$$

Semantics:

Given a Kripke structure $\mathcal{S} = (S, AP, \rightarrow, I, L)$ and a CTL formula φ,

- $(\mathcal{S}, s) \models p$ iff $p \in L(s)$,
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where $\pi \models \varphi_1 U\varphi_2$ iff $\exists i \geq 0$, $(\mathcal{S}, \pi(i)) \models \varphi_2$ and $\forall j : 0 \leq j < i$, $(\mathcal{S}, \pi(j)) \models \varphi_1$.

$\mathcal{S} \models \varphi$ iff for every $s_0 \in I$, $(\mathcal{S}, s_0) \models \varphi$.

CTL

Syntax:

$$\varphi := p(p \in AP) \mid \varphi_1 \vee \varphi_2 \mid \neg\varphi_1 \mid EX\varphi_1 \mid AX\varphi_1 \mid E\varphi_1 U \varphi_2 \mid A\varphi_1 U \varphi_2$$

Semantics:

Given a Kripke structure $\mathcal{S} = (S, AP, \rightarrow, I, L)$ and a CTL formula φ,

- $(\mathcal{S}, s) \vDash p$ iff $p \in L(s)$,
- $(\mathcal{S}, s) \vDash \varphi_1 \vee \varphi_2$ iff $(\mathcal{S}, s) \vDash \varphi_1$ or $(\mathcal{S}, s) \vDash \varphi_2$,
- $(\mathcal{S}, s) \vDash \neg\varphi_1$ iff not $(\mathcal{S}, s) \vDash \varphi_1$,
- $(\mathcal{S}, s) \vDash EX\varphi_1$ iff there exists s' s.t. $s \rightarrow s'$ and $(\mathcal{S}, s') \vDash \varphi_1$,
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where $\pi \vDash \varphi_1 U \varphi_2$ iff $\exists i \geq 0$, $(\mathcal{S}, \pi(i)) \vDash \varphi_2$ and $\forall j : 0 \leq j < i$, $(\mathcal{S}, \pi(j)) \vDash \varphi_1$.

$\mathcal{S} \vDash \varphi$ iff for every $s_0 \in I$, $(\mathcal{S}, s_0) \vDash \varphi$.

Example: AFq, $AG(p \rightarrow AFq)$.

Positive normal form (PNF) of CTL

Recall: R (Release) operator, $\varphi_1 R \varphi_2 = \neg(\neg\varphi_1 U \neg\varphi_2)$.
Let $w \in (2^{AP})^\omega$ and $\varphi_1 R \varphi_2$ be a LTL formula, then $(w, i) \models \varphi_1 R \varphi_2$ iff

- either for every $j : i \leq j$, $(w, j) \models \varphi_2$,
- or there exists $j : i \leq j$ s.t. $(w, j) \models \varphi_1$ and for every $k : i \leq k \leq j$, $(w, k) \models \varphi_2$.

Fact. $\neg(\varphi_1 U \varphi_2) \equiv (\neg\varphi_1) R (\neg\varphi_2)$ and $\neg(\varphi_1 R \varphi_2) \equiv (\neg\varphi_1) U (\neg\varphi_2)$.

Positive normal form for CTL:

$$\varphi := \begin{array}{c} true \mid false \mid p \mid \neg p \mid \varphi_1 \vee \varphi_2 \mid \varphi_1 \wedge \varphi_2 \mid EX\varphi_1 \mid AX\varphi_1 \mid \\ E\varphi_1 U \varphi_2 \mid A\varphi_1 U \varphi_2 \mid E\varphi_1 R \varphi_2 \mid A\varphi_1 R \varphi_2 \end{array}$$

Proposition. Every CTL formula can be transformed into an equivalent formula in positive normal form.

Proof.

The idea: Push $\neg$ to the front of atomic positions.
For instance, $\neg(E\varphi_1 U \varphi_2) \equiv A(\neg\varphi_1) R (\neg\varphi_2)$, $\neg(E\varphi_1 R \varphi_2) \equiv A(\neg\varphi_1) U (\neg\varphi_2)$. $\square$

Outline

1. Linear temporal logic (LTL)
2. LTL model checking: Automata theoretical approach
3. Computation tree logic (CTL)
4. (Weak) alternating tree automata
5. CTL model checking: Automata theoretical approach

Alternating automata over binary trees

A notation:

Let X be a finite set. Then $\mathcal{B}^{+}(X)$ is the positive Boolean combinations of elements of X, formally,

$\varphi := true \mid false \mid x (x \in X) \mid \varphi_1 \vee \varphi_2 \mid \varphi_1 \wedge \varphi_2$

An *alternating Büchi automaton* over infinite binary trees (ABTA) $\mathcal{A}$ is a tuple $(Q, 2^{AP}, \delta, q_0, F)$, where

- Q, q_0, F are similar to those of nondeterministic Büchi automata,
- $\delta \subseteq Q \times 2^{AP} \to \mathcal{B}^{+}(\{0,1\} \times Q)$.

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- $\delta \subseteq Q \times 2^{AP} \to \mathcal{B}^+(\{0,1\} \times Q)$.

A *run* of a ABTA $\mathcal{A} = (Q, 2^{AP}, \delta, q_0, F)$ over a binary tree $t = (D, L)$ is an infinite tree $r_{\mathcal{A},t} = (D_r, L_r)$, where $D_r \subseteq \mathbb{N}^*$ is a tree domain and $L_r : D_r \to D \times Q$ satisfying the following conditions.

$\forall y \in D_r$ s.t. $L_r(y) = (x, q)$ and $\delta(q, L(x)) = \theta$.
Then there is $S = \{(b_0, q_0), \ldots, (b_n, q_n)\} \subseteq \{0,1\} \times Q$ s.t.
$S \models \theta$, *and* $\forall i : 0 \leq i \leq n$, $yi \in D_r$ *and* $L_r(yi) = (xb_i, q_i)$.

In particular, if $\delta(q, L(x)) = true$, then S can be empty.

A run $r_{\mathcal{A},t}$ is *accepting* if for every infinite path π in $r_{\mathcal{A},t}$, $\mathrm{Inf}(L_r(\pi)) \cap F \neq \emptyset$.

ABTA over binary trees: Example

$AG(p_1 \rightarrow AFp_2)$

$\mathcal{A} = (Q, 2^{\{p_1,p_2\}}, \delta, q_0, F)$

$Q = \{q_0, q_1\} \qquad F = \{q_0\}$

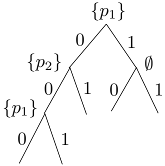
$$\left.\begin{array}{r}\delta(q_0,\emptyset)\\ \delta(q_0,\{p_2\})\\ \delta(q_0,\{p_1,p_2\})\end{array}\right. = (0,q_0) \wedge (1,q_0)$$

$$\delta(q_0,\{p_1\}) = (0,q_0) \wedge (0,q_1) \wedge (1,q_0) \wedge (1,q_1)$$

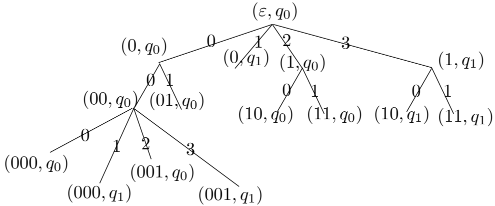
$$\left.\begin{array}{r}\delta(q_1,\emptyset)\\ \delta(q_1,\{p_1\})\end{array}\right. = (0,q_1) \wedge (1,q_1)$$

$$\delta(q_1,\{p_2\}) = \delta(q_1,\{p_1,p_2\}) = true$$

t



$r_{\mathcal{A},t}$



Finitely-branching trees

Recall: A tree domain $D \subseteq \mathbb{N}^*$ s.t.

- $\forall xi \in \mathbb{N}^*$, if $xi \in D$, then $x \in D$ as well,
- $\forall xi \in \mathbb{N}^*$, if $xi \in D$, then $xj \in D$ for every $j : 0 \leq j < i$.

A tree domain D is *finitely branching* if

$$\forall x \in D,\ \exists n \in \mathbb{N}\ s.t.\ \forall m \geq n,\ xm \notin D.$$

A *finitely-branching tree* t over 2^{AP} is a pair (D, L) s.t.

$$D \text{ is a finitely branching tree domain and } L : D \to 2^{AP}.$$

Alternating automata over finitely-branching trees

Transition conditions over Q (TC^Q):

- $true, false \in TC^Q$,
- $\forall p \in AP$, $p, \neg p \in TC^Q$,
- for every $q_1, q_2 \in Q$, $q_1 \vee q_2, q_1 \wedge q_2 \in TC^Q$,
- for every $q \in Q$, $q, \Diamond q, \Box q \in TC^Q$.

Alternating automata over finitely-branching trees

An *alternating Büchi automaton* over finitely-branching trees (ABTA) $\mathcal{A}$ is a tuple $(Q, 2^{AP}, \delta, q_0, F)$ where $\delta : Q \to TC^Q$.

A run of an ABTA $\mathcal{A}$ over a (finitely-branching) tree $t = (D, L)$ is a winning strategy for Player 0 in the Büchi game $\mathcal{G} = (V_0, V_1, E, F \cup \{q_\top\})$, where

- $V_0 \subseteq D \times Q \cup \{q_\top\}$ s.t. $q_\top \in V_0$, and $(x, q) \in V_0$ iff
  - $\delta(q) = false$, or
  - $\delta(q) = p$ and $p \notin L(x)$, or
  - $\delta(q) = \neg p$ and $p \in L(x)$, or
  - $\delta(q) = q'$, or
  - $\delta(q) = q_1 \vee q_2$, or
  - $\delta(q) = \Diamond q'$.
- $V_1 \subseteq D \times Q \cup \{q_\perp\}$ s.t. $q_\perp \in V_1$, and $(x, q) \in V_1$ iff
  - $\delta(q) = true$, or
  - $\delta(q) = p$ and $p \in L(x)$, or
  - $\delta(q) = \neg p$ and $p \notin L(x)$, or
  - $\delta(q) = q_1 \wedge q_2$, or
  - $\delta(q) = \Box q'$.

Alternating automata over finitely-branching trees

An *alternating Büchi automaton* over finitely-branching trees (ABTA) $\mathcal{A}$ is a tuple $(Q, 2^{AP}, \delta, q_0, F)$ where $\delta : Q \to TC^Q$.

A run of an ABTA $\mathcal{A}$ over a (finitely-branching) tree $t = (D, L)$ is a winning strategy for Player 0 in the Büchi game $\mathcal{G} = (V_0, V_1, E, F \cup \{q_\top\})$, where

- E is defined as follows: $(q_\perp, q_\perp), (q_\top, q_\top) \in E$, and for every $(x, q) \in V_0 \cup V_1$,
  - if $\delta(q) = false$, or $\delta(q) = p$ and $p \notin L(x)$, or $\delta(q) = \neg p$ and $p \in L(x)$, then $((x, q), q_\perp) \in E$,
  - if $\delta(q) = true$, or $\delta(q) = p$ and $p \in L(x)$, or $\delta(q) = \neg p$ and $p \notin L(x)$, then $((x, q), q_\top) \in E$,
  - if $\delta(q) = q'$, then $((x, q), (x, q')) \in E$,
  - if $\delta(q) = q_1 \vee q_2$ (or $q_1 \wedge q_2$), then $((x, q), (x, q_1)), ((x, q), (x, q_2)) \in E$,
  - if $\delta(q) = \Diamond q'$ (or $\Box q'$), then for every children xi of x, $((x, q), (xi, q')) \in E$.

Remark: (V_0, V_1, E) defined above may not be a bipartite graph.

Acceptance:

> $\mathcal{A}$ *accepts* t *iff Player 0 has a winning strategy in* $\mathcal{G}$ *starting from* (ε, q_0).

ABTA over finitely-branching trees: Example

$AG(p_1 \to AFp_2)$

$\mathcal{A} = (Q, 2^{\{p_1,p_2\}}, \delta, q_0, F)$

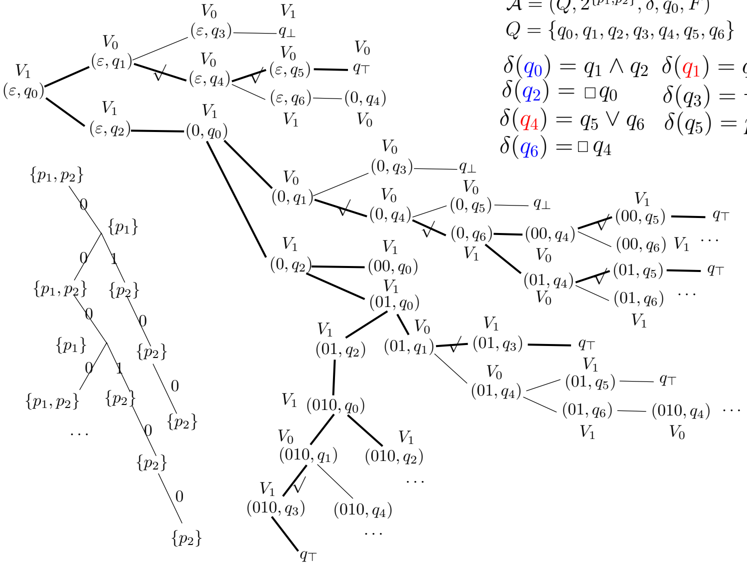
$Q = \{q_0, q_1, q_2, q_3, q_4, q_5, q_6\}$ $F = \{q_0\}$

$\delta(q_0) = q_1 \wedge q_2$ $\delta(q_1) = q_3 \vee q_4$

$\delta(q_2) = \Box q_0$ $\delta(q_3) = \neg p_1$

$\delta(q_4) = q_5 \vee q_6$ $\delta(q_5) = p_2$

$\delta(q_6) = \Box q_4$



Unwinding of Kripke structures

Let $\mathcal{S} = (S, AP, \to, \{s_0\}, L)$ be a Kripke structure.
$\forall s \in S$, let $suc(s)$ denote the set of successors of s.
Moreover, we assume that the states in $suc(s)$ are ordered.

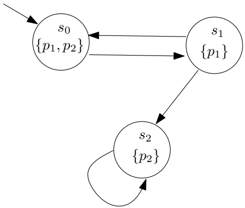
$\mathcal{S}$ can be seen as an infinite tree $T_{\mathcal{S}} = (D_{\mathcal{S}}, L_{\mathcal{S}})$ as follows.

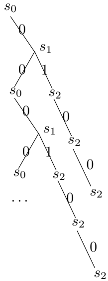
- $L_{\mathcal{S}}(\varepsilon) = s_0$,
- for every $y \in D_{\mathcal{S}}$, if $L_{\mathcal{S}}(y) = s$ and $suc(s) = \{s'_0, \dots, s'_k\}$, then for every $i : 0 \le i \le k$, $yi \in D_{\mathcal{S}}$ and $L_{\mathcal{S}}(yi) = s'_i$.

We can also view $T_{\mathcal{S}}$ as a tree over the alphabet 2^{AP}:

Replace $L_{\mathcal{S}}(y) = s$ with $L_{\mathcal{S}}(y) = L(s)$.

Example:





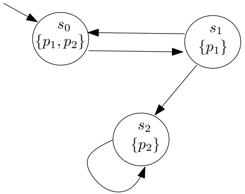
ABTA interpreted over Kripke structures

Suppose $\mathcal{A} = (Q, 2^{AP}, \delta, q_0, F)$ be an ABTA over finitely-branching trees and $\mathcal{S} = (S, AP, \rightarrow, s_0, L)$ be a Kripke structure.

A run of $\mathcal{A}$ over $\mathcal{S}$ is a run of $\mathcal{A}$ over $T_{\mathcal{S}}$.

As a matter of fact, a run of $\mathcal{A}$ over $\mathcal{S}$ can be defined by the winning strategies of Player 0 in the Büchi game $\mathcal{G}' = (V_0', V_1', E', (S \times F) \cup \{q_\top\})$, where

- $V_0' \subseteq S \times Q \cup \{q_\top\}$ and $V_1' \subseteq S \times Q \cup \{q_\bot\}$ are defined similar to V_0 and V_1 in $\mathcal{G}$,
- E is defined as follows: $(q_\bot, q_\bot), (q_\top, q_\top) \in E$, and for every $(s, q) \in V_0' \cup V_1'$,
  - if $\delta(q) = false$, or $\delta(q) = p$ and $p \notin L(s)$, or $\delta(q) = \neg p$ and $p \in L(s)$, then $((s, q), q_\bot) \in E$,
  - if $\delta(q) = true$, or $\delta(q) = p$ and $p \in L(s)$, or $\delta(q) = \neg p$ and $p \notin L(s)$, then $((s, q), q_\top) \in E$,
  - if $\delta(q) = q'$, then $((s, q), (s, q')) \in E$,
  - if $\delta(q) = q_1 \vee q_2$ (or $q_1 \wedge q_2$), then $((s, q), (s, q_1)), ((s, q), (s, q_2)) \in E$,
  - if $\delta(q) = \Diamond q'$ (or $\Box q'$), then for every successor s' of s, $((s, q), (s', q')) \in E$.



$AG(p_1 \to AFp_2)$

$\mathcal{A} = (Q, 2^{\{p_1,p_2\}}, \delta, q_0, F)$

$Q = \{q_0, q_1, q_2, q_3, q_4, q_5, q_6\}$ $F = \{q_0, q_2\}$

$\delta(q_0) = q_1 \wedge q_2$ $\delta(q_1) = q_3 \vee q_4$

$\delta(q_2) = \Box q_0$ $\delta(q_3) = \neg p_1$

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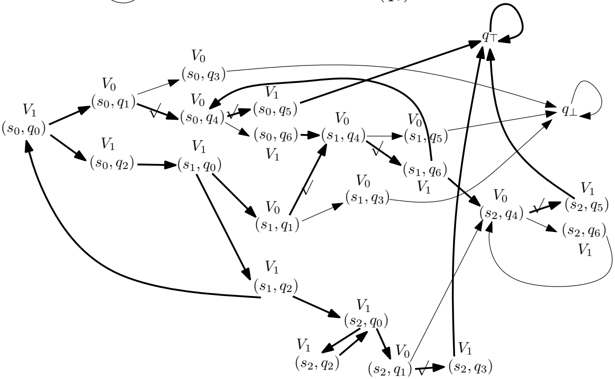

Weak alternating Büchi tree automata (WABTA)

A WABTA $\mathcal{A}$ (over Kripke structures) is a ABTA $(Q, 2^{AP}, \delta, q_0, F)$ s.t.

- Q is partitioned into n pairwise-disjoint subsets $Q_1, \ldots, Q_n$,
- there is partial order $\leq$ among $Q_1, \ldots, Q_n$ s.t.
  $\forall q \in Q_i, q' \in Q_j$, if q' occurs in $\delta(q)$, then $Q_j \leq Q_i$,
- for every Q_i, either $Q_i \subseteq F$ or $Q_i \cap F = \emptyset$.

Observation.

Every infinite path in a run finally get trapped in some Q_i.
The infinite path satisfies the acceptance condition iff $Q_i \subseteq F$.

Example:

The ABTA $\mathcal{A}$ for $AG(p_1 \to AFp_2)$ is in fact a WABTA:

- $\delta(q_0) = q_1 \wedge q_2$, $\delta(q_1) = q_3 \vee q_4$, $\delta(q_2) = \Box(q_0)$, $\delta(q_3) = \neg p_1$,
- $\delta(q_4) = q_5 \vee q_6$, $\delta(q_5) = p_2$, $\delta(q_6) = \Box q_4$,
- $F = \{q_0, q_2\}$.

The partition and the partial order:

$$Q_1 = \{q_0, q_2\} \geq Q_2 = \{q_1\} \geq \begin{array}{c} Q_3 = \{q_3\} \\ Q_4 = \{q_4, q_6\} \geq Q_5 = \{q_5\} \end{array}.$$

Outline

1. Linear temporal logic (LTL)
2. LTL model checking: Automata theoretical approach
3. Computation tree logic (CTL)
4. (Weak) alternating tree automata
5. CTL model checking: Automata theoretical approach

CTL model checking: Automata-theoretic approach

W.l.o.g. in CTL model checking problem for $\mathcal{S} = (S, AP, \rightarrow, I, L)$ and φ, we assume that I is a singleton.

Automata-theoretical approach to CTL model checking:

Let $\mathcal{S} = (S, AP, \rightarrow, s_0, L)$ be a Kripke structure and φ be a CTL formula.

1. construct a WABTA $\mathcal{A}_\varphi = (Q, 2^{AP}, \delta, q_0, F)$ from φ in linear time,
2. construct the Büchi game $\mathcal{G}' = (V_0', V_1', E', (S \times F) \cup \{q_\top\})$ in time $O(||\mathcal{A}_\varphi|| \times ||\mathcal{S}||)$,
3. decide whether Player 0 has a winning strategy in $\mathcal{G}'$ starting from (s_0, q_0) in time $O(||\mathcal{G}'||)$.

Remark: In the third step above, the fact that $\mathcal{A}_\varphi$ is a WABTA is used.

Therefore, by using automata-theoretic approach, we get the following result.

Theorem. Given a Kripke structure $\mathcal{S}$ and a CTL formula φ, the problem whether $\mathcal{S} \models \varphi$ can be decided in time $O(||\mathcal{S}|| \times |\varphi|)$.

From CTL to WABTA

Proposition. Given a CTL formula φ, a WABTA $\mathcal{A}_\varphi$ can be constructed in linear time s.t. $L(\mathcal{A}_\varphi)$ is the set of Kripke structures satisfying φ.

Proof.

$\mathcal{A}_\varphi = (sub(\varphi), 2^{AP}, \delta, q_0, F)$, where

- $q_0 = \varphi$, $F = \{\psi_1 R \psi_2 \mid \psi_1 R \psi_2 \in \mathrm{cl}(\varphi)\}$,
- $\{\varphi_1\} \leq \{\varphi_2\}$ iff $\varphi_1 \in sub(\varphi_2)$,
- and δ is defined as follows:
  - $\delta(true) = true$, $\delta(false) = false$,
  - $\delta(p) = p$, $\delta(\neg p) = \neg p$,
  - $\delta(\varphi_1 \vee \varphi_2) = \varphi_1 \vee \varphi_2$, $\delta(\varphi_1 \wedge \varphi_2) = \varphi_1 \wedge \varphi_2$,
  - $\delta(EX\varphi_1) = \Diamond\varphi_1$, $\delta(AX\varphi_1) = \Box\varphi_1$,
  - $\delta(E\varphi_1 U \varphi_2) = \varphi_2 \vee (\varphi_1 \wedge \Diamond E\varphi_1 U \varphi_2)$, $\delta(A\varphi_1 U \varphi_2) = \varphi_2 \vee (\varphi_1 \wedge \Box A\varphi_1 U \varphi_2)$,
  - $\delta(E\varphi_1 R \varphi_2) = \varphi_2 \wedge (\varphi_1 \vee \Diamond E\varphi_1 R \varphi_2)$, $\delta(A\varphi_1 R \varphi_2) = \varphi_2 \wedge (\varphi_1 \vee \Box A\varphi_1 R \varphi_2)$.

Remark: $\delta(E\varphi_1 U \varphi_2) = \varphi_2 \vee (\varphi_1 \wedge \Diamond E\varphi_1 U \varphi_2)$ are abbrev. of transitions $\delta(E\varphi_1 U \varphi_2) = \varphi_2 \vee q$, $\delta(q) = \varphi_1 \wedge q'$, $\delta(q') = \Diamond E\varphi_1 U \varphi_2$, where q, q' are new introduced states in the same partition as $E\varphi_1 U \varphi_2$. □

The special structure of Büchi game $\mathcal{G}'$

Let $\mathcal{S} = (S, AP, \rightarrow, s_0, L)$ be a Kripke structure and $\mathcal{A}_\varphi = (sub(\varphi), 2^{AP}, \delta, q_0, F)$ be a WABTA.

The special structure of $\mathcal{A}_\varphi$ induces a special structure of the game $\mathcal{G}' = (V_0', V_1', E', (S \times F) \cup \{q_\top\})$:

- $V_0' \cup V_1'$ can be partitioned into $(S \times \{\psi\})_{\psi \in sub(\varphi)}, \{q_\bot\}, \{q_\top\}$,
- $S \times \{\psi_1\} \leq S \times \{\psi_2\}$ iff $\{\psi_1\} \leq \{\psi_2\}$, $\forall \psi \in sub(\varphi), q_\top, q_\bot \leq S \times \{\psi\}$,
- E' is non-increasing wrt. $\leq$.

Weak Büchi game:

A Büchi game (V_0, V_1, E, F) is weak if $V_0 \cup V_1$ can be partitioned into subsets $V_1', \ldots, V_n'$ s.t.

- $\forall q \in V_i', q' \in V_j'.\ (q, q') \in E$ implies $V_j' \leq V_i'$.
- $\forall i.$ either $V_i' \subseteq F$ or $V_i' \cap F = \varnothing$.

Theorem. Weak Büchi game can be solved in linear time.

Solving weak Büchi game in linear time

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Proof.

Let $\mathcal{G} = (V_0, V_1, E, F)$ be a weak Büchi game with partitions $V_1', \ldots, V_n'$.
W.l.o.g. we assume that

- for every $v \in V_0 \cup V_1$, $vE \neq \varnothing$,
- for every i, j, if $V_i' \geq V_j'$, then $i \leq j$.

□

Solving weak Büchi game in linear time

Theorem. Weak Büchi game can be solved in linear time.

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Let $\mathcal{G} = (V_0, V_1, E, F)$ be a weak Büchi game with partitions $V_1', \ldots, V_n'$.

The algorithm.

Compute $I : V_0 \cup V_1 \rightarrow \{true, false\}$ as follows.

Initially, set $I(v) = \bot$ (undefined) for every $v \in V_0 \cup V_1$.

For i from n to 1, do the following computation.

1. *For every $v \in V_i'$ s.t. $I(v) = \bot$, set $I(v) = true$ iff $V_i' \subseteq F$.*
2. *Repeat the following procedure until $I(v)$'s no more updated:*
   For every $v \in V_0 \cup V_1$,
   - *$v \in V_0$:*
     - *if $\exists$ a successor of v, say v', s.t. $I(v') = true$, then set $I(v) = true$,*
     - *if every successor v' of v satisfy $I(v') = false$, then set $I(v) = false$.*
   - *$v \in V_1$:*
     - *if $\exists$ a successor v' of v satisfy $I(v') = false$, then set $I(v) = false$,*
     - *if every successor v' of v satisfy $I(v') = true$, then set $I(v) = true$.*

Claim. Player 0 has a winning strategy in $\mathcal{G}$ starting from q_0 iff $I(q_0) = true$.

□

Solving weak Büchi game: Example

$F = \{q_0, q_2\}$

$Q_1 = \{q_0, q_2\} \geq Q_2 = \{q_1\} \geq$ $Q_3 = \{q_3\}$

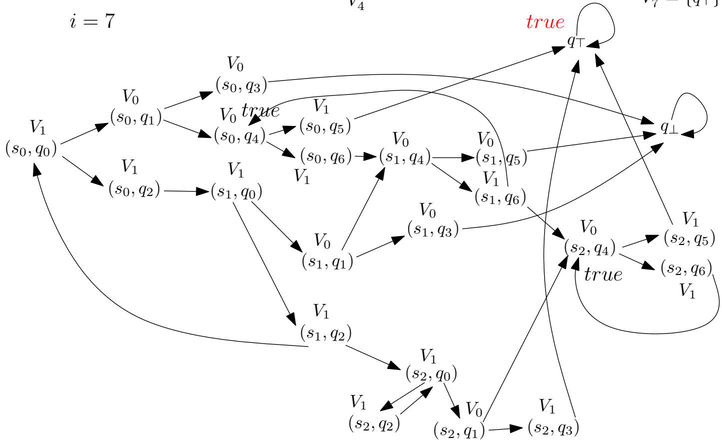
$Q_4 = \{q_4, q_6\} \geq Q_5 = \{q_5\}$

V_1' V_2' V_3' V_4' V_5'

$V_6' = \{q_\perp\}$

$V_7' = \{q_\top\}$

$i = 7$



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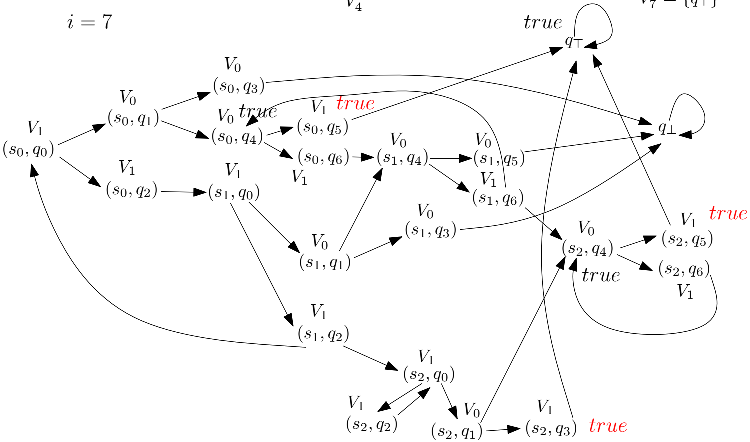

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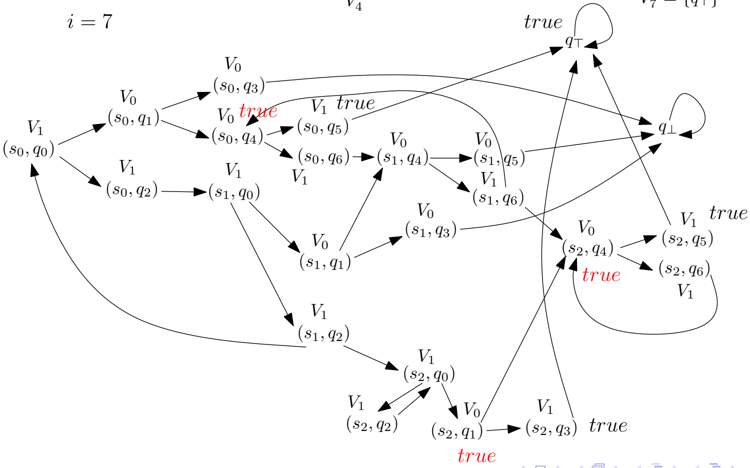

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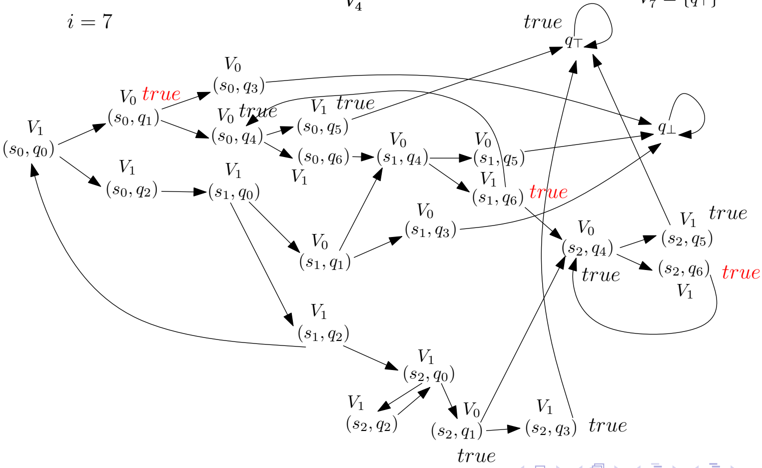

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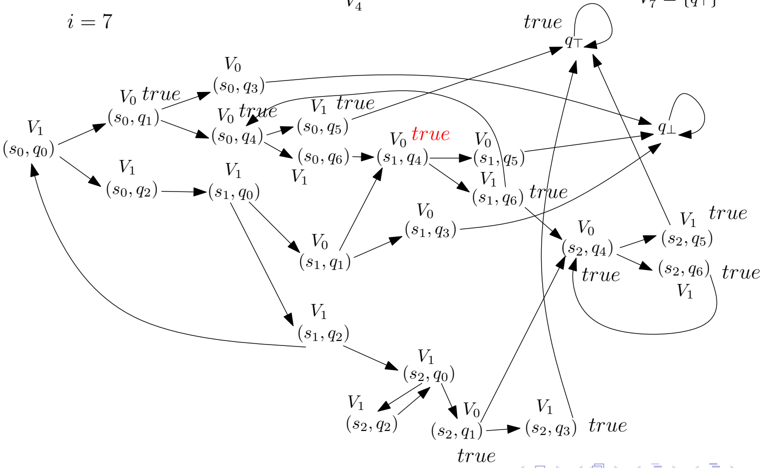

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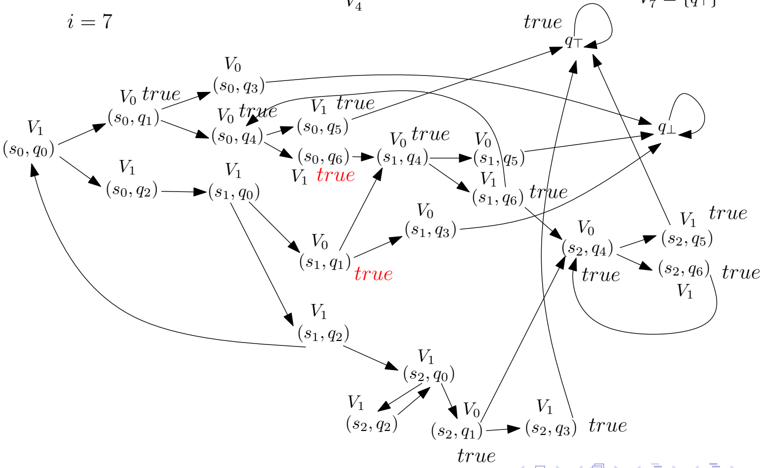

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V'_3 V'_5

V'_4

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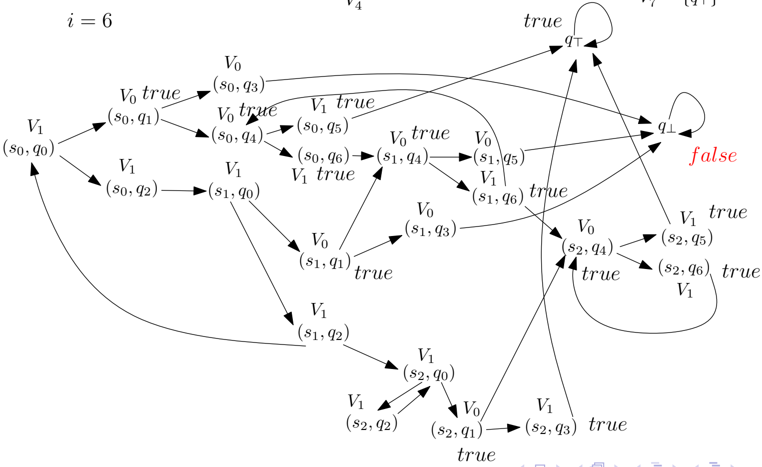
$i = 6$



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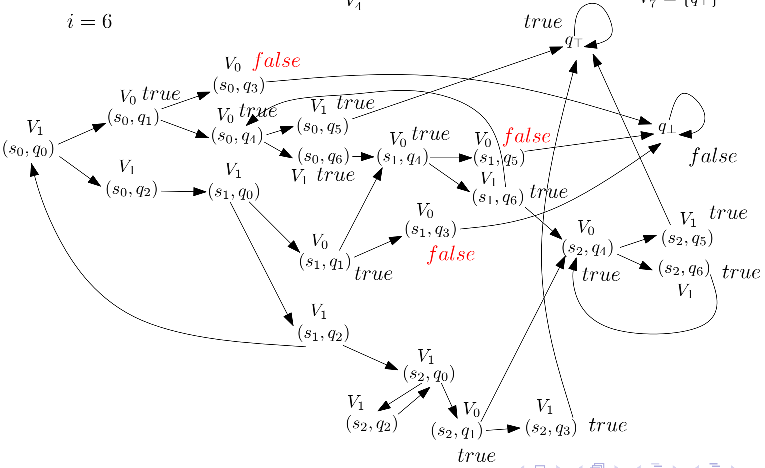

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V_1' V_2' V_3' V_4' V_5'

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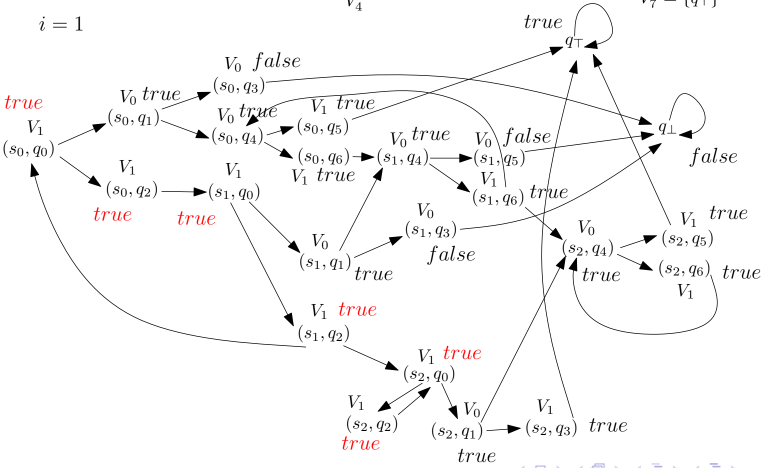
$i = 1$



References

The main references for these two lectures.

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