CS244: Theory of Computation

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Fall 2022

Outline

Visibly pushdown automata (VPA)

Closure properties

Visibly pushdown grammar (VPG)

Logical characterization
- Equivalence of NFA and MSO
- Equivalence of VPA and MSO_μ

Decision problems

Motivation

Parenthesises in arithmetic expressions

$$(\ (\ (\ 5\ +\ x\ )\ *\ y\ +\ z\ )\ *\ (\ u\ -\ v\ )\ )\ /\ w$$

Motivation

Curly brackets in C Programs

```
for(...)
{
  if (...)
  {
    for (...)
    {
      ...
    }
  }
  else
  {
    ...
  }
}
```

Motivation

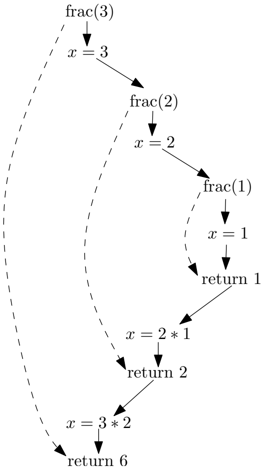
XML documents

⟨library⟩
⟨catalog⟩
⟨book⟩
⟨title⟩
Computational Complexity
⟨/title⟩
⟨author⟩
Fu Song
⟨/author⟩
⟨book⟩
⟨catalog⟩
⟨library⟩

Motivation

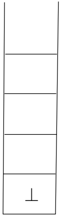
Recursive function calls and returns

```
frac(int y)
{
 int x = y;
 if (y >= 2)
 {
  x = y * frac(y - 1);
  return x;
 }
 else
 {
  return x;
 }
}
```



Motivation

$$(\; (\; (\; 5 \; + \; x \;) \; * \; y \; + \; z \;) \; * \; (\; u \; - \; v \;) \;) \; / \; w$$



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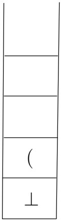

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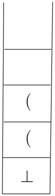

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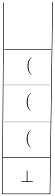

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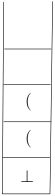

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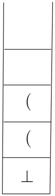

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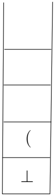

Motivation

Stack operations determined by the input symbol

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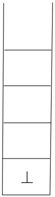
(: Push

): Pop



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  }
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    ...
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```

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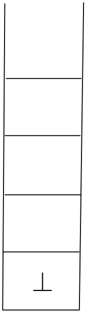

Motivation

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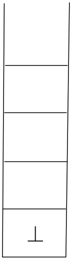
$\langle$library$\rangle$
$\langle$catalog$\rangle$
$\langle$book$\rangle$
$\langle$title$\rangle$
Computational Complexity
$\langle$/title$\rangle$
$\langle$author$\rangle$
Christos H. Papadimitriou
$\langle$/author$\rangle$
$\langle$book$\rangle$
$\langle$catalog$\rangle$
$\langle$library$\rangle$

$\langle tag \rangle$: Push

$\langle / tag \rangle$: Pop



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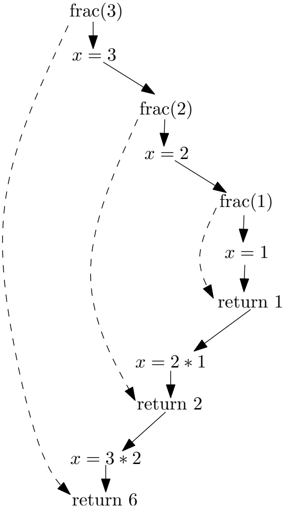
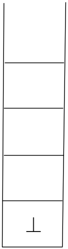


frac: Push
return: Pop



Visibly pushdown automata (VPA)

The alphabet Σ is partitioned into $\widetilde{\Sigma} = \langle \Sigma_c, \Sigma_r, \Sigma_l \rangle$

- Σ_c: finite set of calls,
- Σ_r: finite set of returns,
- Σ_l: finite set of local actions.

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A (nondeterministic) VPA $\mathcal{A}$ is a 7-tuple $(Q, \widetilde{\Sigma}, \Gamma, \delta, q_0, \perp, F)$, where

- Q is a finite set of states,
- $\widetilde{\Sigma}$ is the input alphabet,
- Γ is the stack alphabet,
- $\delta \subseteq Q \times \Sigma_c \times Q \times (\Gamma \setminus \{\perp\}) \;\cup\; Q \times \Sigma_r \times \Gamma \times Q \;\cup\; Q \times \Sigma_l \times Q$,
- $q_0 \in Q$ is the initial state,
- $\perp$ is the bottom symbol of the stack,
- $F \subseteq Q$ is the set of final states.

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Remark:

- No ε-transitions,
- Exactly one symbol is pushed in each call transition.

Visibly pushdown automata (VPA)

A deterministic VPA is a VPA $\mathcal{A} = (Q, \widetilde{\Sigma}, \Gamma, \delta, q_0, \perp, F)$ such that

Call: for every $(q, a) \in Q \times \Sigma_c$, there is at most one pair $(q', \gamma) \in Q \times (\Gamma \setminus \{\perp\})$ such that $(q, a, q', \gamma) \in \delta$,

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A deterministic VPA is complete if "at most" is replaced by "exactly".

Visibly pushdown automata (VPA): continued

A run of a VPA $\mathcal{A}$ over a word $w = a_1 \dots a_n$ is

a sequence $(q_0, \alpha_0)(q_1, \alpha_1) \dots (q_n, \alpha_n)$ *of configurations*

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Remark: Acceptance of VPAs are defined by final states, not by empty stack.

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Remark. As a result of the acceptance by final states,

VPAs over $\widetilde{\Sigma}$ *may accept non-well-matched words.*

Visibly pushdown languages (VPL)

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Theorem

VPL $\subsetneq$ CFL.

Embedding of CFL as VPLs

Proposition. For every CFL $L \subseteq \Sigma^*$, there are a VPL $L' \subseteq (\Sigma')^*$ with respect to some $\widetilde{\Sigma'}$ and a homomorphism $h : (\Sigma')^* \to \Sigma^*$ such that $L = h(L')$.

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Then $L = h(\mathcal{L}(\mathcal{A}'))$.

Outline

Visibly pushdown automata (VPA)

Closure properties

Visibly pushdown grammar (VPG)

Logical characterization

Equivalence of NFA and MSO

Equivalence of VPA and MSO_μ

Decision problems

Union and intersection

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Union.

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Proposition. VPLs with respect to $\widetilde{\Sigma}$ are closed under union and intersection. Let $\mathcal{A}_1 = (Q_1, \widetilde{\Sigma}, \Gamma_1, \delta_1, q_0^1, \bot_1, F_1)$ and $\mathcal{A}_2 = (Q_2, \widetilde{\Sigma}, \Gamma_2, \delta_2, q_0^2, \bot_2, F_2)$ be two VPAs.

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Intersection. Using the fact that $\mathcal{A}_1$ and $\mathcal{A}_2$, being VPAs, synchronize on the push and pop operations on the stack.

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Note that CFL are not closed under intersection

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- Do a subset construction but postpone handling the push-transitions that $\mathcal{A}$ does.
- Instead, we store the call actions and simulate the push-transitions corresponding to them later, namely at the time of the corresponding pop-transition.

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- Do a subset construction but postpone handling the push-transitions that $\mathcal{A}$ does.
- Instead, we store the call actions and simulate the push-transitions corresponding to them later, namely at the time of the corresponding pop-transition.
- The construction will have a component S that is a set of "summary" edges that keeps track of what state transitions are possible from a push-transition to the corresponding pop-transition.

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- Using the summary information, the set of reachable states is updated.

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  - R is the set of states reachable by $\mathcal{A}$ from any initial state on xa_1ya_2z

Illustration of the intuition of the proof of the Theorem

In an obviously way, we can define $(q, \alpha) \xrightarrow{w} (q', \alpha')$:

the reachability of the config. (q', α') *from* (q, α) *by reading* w.

Observation. Suppose $(q, \alpha) \xrightarrow{w} (q', \alpha')$ and w is well-matched, then $\alpha = \alpha'$.

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A well-matched word w *can be seen as a relation* $S_w \subseteq Q \times Q$,
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Point II.

Suppose w is well-matched.

- $S_\varepsilon = \mathrm{Id}_Q$.
- If $w = aw'$ with $a \in \Sigma_l$, then
  $$S_w = \{(q, q') \mid \exists q''.(q, a, q'') \in \delta, (q'', q') \in S_{w'}\}.$$
  Similarly for $w = w'a$.
- If $w = aw'b$ with $a \in \Sigma_c$ and $b \in \Sigma_r$, then
  $$S_w = \{(q, q') \mid \exists q_1, q_2, \gamma.\ (q, a, q_1, \gamma) \in \delta, (q_1, q_2) \in S_{w'}, (q_2, b, \gamma, q') \in \delta\}.$$

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Answer:

The set of states reachable from q_0 *after reading* w.

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What info. should be remembered
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Answer:

Let me think for a while . . .

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In an obviously way, we can define $(q,\alpha) \xrightarrow{w} (q',\alpha')$:

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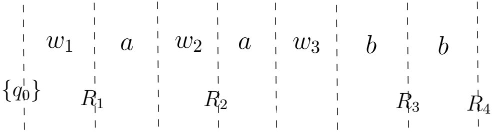
Consider $w_1 a w_2 a w_3 bb$ s.t. $a \in \Sigma_c, b \in \Sigma_r$ and w_1, w_2, w_3 are well-matched.



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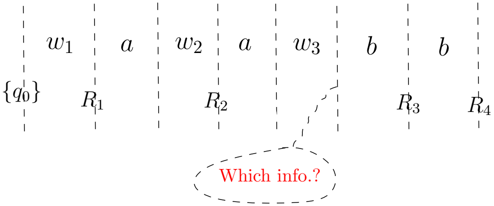

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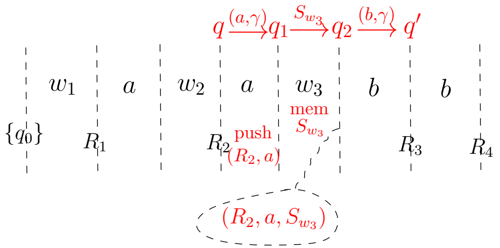

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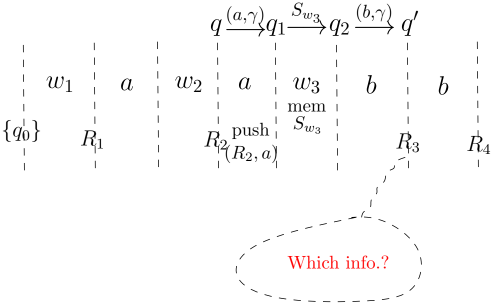

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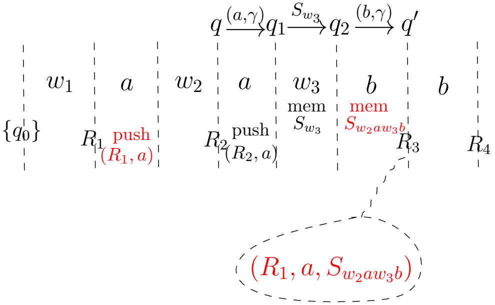

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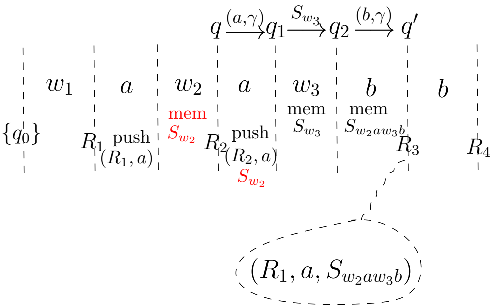

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- After reading $w = xa_1ya_2z$, the VPA $\mathcal{A}'$ we construct will reach the configuration
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- $S' = \mathrm{Id}_Q$ (note $\mathrm{Id}_Q = \{(q, q) \mid q \in Q\}$), the initialization of the summary;
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> If $a_3 \in \Sigma_c$, then $((S, R), a_3, (\mathrm{Id}_Q, R'), (S, R, a_3)) \in \delta'$,
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$$\boxed{\begin{array}{l}
((S,R), a_3, (S_2, R_2, a_2), (S', R')) \in \delta', \text{ where} \\
S' = \left\{ (q, q') \;\middle|\; \begin{array}{c} \exists q_1, q_2, q_3, \gamma \in \Gamma : (q, q_1) \in S_2, (q_2, q_3) \in S, \\ (q_1, a_2, q_2, \gamma) \in \delta, (q_3, a_3, \gamma, q') \in \delta \end{array} \right\}, \\
R' = \left\{ q' \;\middle|\; \begin{array}{c} \exists q_1, q_2, q_3, \gamma \in \Gamma : q_1 \in R_2, (q_2, q_3) \in S, \\ (q_1, a_2, q_2, \gamma) \in \delta, (q_3, a_3, \gamma, q') \in \delta \end{array} \right\}
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$(q, q_1) \in S_2$: $(q, \alpha) \stackrel{y}{\Longrightarrow}^* (q_1, \alpha)$

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or

$((S,R),a_3,\bot,(S',R')) \in \delta'$, where
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  - $R' = \{q' \mid \exists q \in R : (q,a_3,q') \in \delta\}$

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Determinization

We construct $\mathcal{A}' = (Q', \widetilde{\Sigma}, \Gamma', \delta', (\mathrm{Id}_Q, \{q_0\}), F')$:

- Q': (S, R) such that $S \subseteq Q \times Q, R \subseteq Q$,
- Γ': letters (S, R, a) such that $S \subseteq Q \times Q, R \subseteq Q, a \in \Sigma_c$,
- $F' = \{(S, R) \mid R \cap F \neq \emptyset\}$,
- δ':

Local if $a \in \Sigma_l$, then $((S, R), a, (S', R')) \in \delta'$, where
$R' = \{q' \mid \exists q \in R.(q, a, q') \in \delta\}$,
$S' = \{(q, q') \mid \exists q_1.(q, q_1) \in S, (q_1, a, q') \in \delta\}$.

Call if $a \in \Sigma_c$, then $((S, R), a, (\mathrm{Id}_Q, R'), (S, R, a)) \in \delta'$, where
$R' = \{q' \mid \exists q \in R, \gamma \in \Gamma.(q, a, q', \gamma) \in \delta\}$.

Return if $a \in \Sigma_r$, then $((S, R), a, (S'', R'', a'), (S', R')) \in \delta'$, where

$$S' = \left\{ (q, q') \,\middle|\, \begin{array}{c} \exists q_1, q_2, q_3, \gamma \in \Gamma. \\ (q, q_1) \in S'', (q_1, a', q_2, \gamma) \in \delta, (q_2, q_3) \in S, (q_3, a, \gamma, q') \in \delta \end{array} \right\},$$

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or $((S, R), a, \perp, (S', R')) \in \delta'$, where
$S' = \{(q, q') \mid \exists q''.(q, q'') \in S, (q'', a, \perp, q') \in \delta\}$,
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For a VPA $\mathcal{A}$, first construct a deterministic VPA $\mathcal{A}'$ such that $\mathcal{L}(\mathcal{A}) = \mathcal{L}(\mathcal{A}')$. Then, complement the set of final states.

Outline

Visibly pushdown automata (VPA)

Closure properties

Visibly pushdown grammar (VPG)

Logical characterization
- Equivalence of NFA and MSO
- Equivalence of VPA and MSO_{μ}

Decision problems

Visibly pushdown grammar (VPG)

A CFG $G = (\mathcal{N}, \Sigma, \mathcal{P}, S)$ is a VPG over $\widetilde{\Sigma}$ if $\mathcal{N}$ can be partitioned into $\mathcal{N}_0$ and $\mathcal{N}_1$, and each rule in $\mathcal{P}$ is of the following forms,

- $X \to \varepsilon$,
- $X \to aY$ such that if $X \in \mathcal{N}_0$, then $a \in \Sigma_l$, $Y \in \mathcal{N}_0$,
- $X \to aYbZ$ such that $a \in \Sigma_c$, $b \in \Sigma_r$, $Y \in \mathcal{N}_0$, and if $X \in \mathcal{N}_0$, then $Z \in \mathcal{N}_0$.

The non-terminals in $\mathcal{N}_0$ derive only well-matched words where there is a one-to-one correspondence between calls and returns.
The non-terminals in $\mathcal{N}_1$ derive words that can contain unmatched calls as well as unmatched returns.

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Example: Let $\widetilde{\Sigma} = (\{a\}, \{b\}, \emptyset)$. Then the VPG

$$S \to aSbC \mid aTbC,\ T \to \varepsilon,\ C \to \varepsilon,$$

such that $\mathcal{N}_0 = \{S, T, C\}$ defines $\{a^n b^n \mid n \geq 1\}$.

Equivalence of VPA and VPG

Theorem. $\mathrm{VPA} \equiv \mathrm{VPG}$.

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Let $\mathcal{A} = (Q, \widetilde{\Sigma}, \Gamma, \delta, q_0, \perp, F)$ be a VPA.

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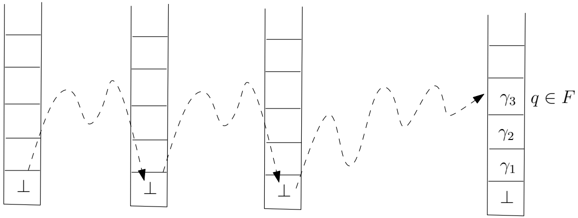
The intuition: Utilizing the nonterminals $[q, \gamma, p]$ with the meaning

the top symbol of the stack is γ,
and from state q,
by reading a well-matched word,
state p can be reached.



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  - $(q, \perp)$: the state is q and the stack is empty,
  - q: the state is q and the stack is nonempty.

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- $\mathcal{N}_1 = \{(q, \perp) \mid q \in Q\} \cup \{q \mid q \in Q\}$,
  - $(q, \perp)$: the state is q and the stack is empty,
  - q: the state is q and the stack is nonempty.
- $\mathcal{P}$ is defined by the following rules,
  - if $(q, a, q') \in \delta$ s.t. $a \in \Sigma_I$, then
    $(q, \perp) \to a(q', \perp)$, $q \to aq'$, $[q, \gamma, p] \to a[q', \gamma, p]$.

Equivalence of VPA and VPG

From VPA to VPG.
Let $\mathcal{A} = (Q, \widetilde{\Sigma}, \Gamma, \delta, q_0, \perp, F)$ be a VPA.
Construct a VPG $(\mathcal{N}_0, \mathcal{N}_1, \widetilde{\Sigma}, \mathcal{P}, S)$ as follows.

- $S = (q_0, \perp)$,
- $\mathcal{N}_0 = \{[q, \gamma, p] \mid q, p \in Q, \gamma \in \Gamma \setminus \{\perp\}\}$,
- $\mathcal{N}_1 = \{(q, \perp) \mid q \in Q\} \cup \{q \mid q \in Q\}$,
  - $(q, \perp)$: the state is q and the stack is empty,
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- $\mathcal{P}$ is defined by the following rules,
  - if $(q, a, q') \in \delta$ s.t. $a \in \Sigma_l$, then
    $(q, \perp) \to a(q', \perp)$, $q \to aq'$, $[q, \gamma, p] \to a[q', \gamma, p]$.
  - if $(q, a, q', \gamma), (p', b, \gamma, p) \in \delta$ s.t. $a \in \Sigma_c, b \in \Sigma_r$, then
    $[q, \gamma_1, r] \to a[q', \gamma, p']b[p, \gamma_1, r]$, $(q, \perp) \to a[q', \gamma, p']b(p, \perp)$,
    $q \to a[q', \gamma, p']bp$.

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- $\mathcal{P}$ is defined by the following rules,
  - if $(q, a, q') \in \delta$ s.t. $a \in \Sigma_l$, then
    $(q, \perp) \rightarrow a(q', \perp)$, $q \rightarrow aq'$, $[q, \gamma, p] \rightarrow a[q', \gamma, p]$.
  - if $(q, a, q', \gamma), (p', b, \gamma, p) \in \delta$ s.t. $a \in \Sigma_c, b \in \Sigma_r$, then
    $[q, \gamma_1, r] \rightarrow a[q', \gamma, p']b[p, \gamma_1, r]$, $(q, \perp) \rightarrow a[q', \gamma, p']b(p, \perp)$,
    $q \rightarrow a[q', \gamma, p']bp$.
  - if $(q, a, q', \gamma) \in \delta$ s.t. $a \in \Sigma_c$, then
    $(q, \perp) \rightarrow aq'$, $q \rightarrow aq'$, $(q, \perp) \rightarrow a[q', \gamma, p]$, $q \rightarrow a[q', \gamma, p]$.

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Let $\mathcal{A} = (Q, \widetilde{\Sigma}, \Gamma, \delta, q_0, \perp, F)$ be a VPA.
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    $q \to a[q', \gamma, p']bp$.
  - if $(q, a, q', \gamma) \in \delta$ s.t. $a \in \Sigma_c$, then
    $(q, \perp) \to aq'$, $q \to aq'$, $(q, \perp) \to a[q', \gamma, p]$, $q \to a[q', \gamma, p]$.
  - if $(q, a, \perp, q') \in \delta$ s.t. $a \in \Sigma_r$, then $(q, \perp) \to a(q', \perp)$.

Equivalence of VPA and VPG

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Let $\mathcal{A} = (Q, \widetilde{\Sigma}, \Gamma, \delta, q_0, \perp, F)$ be a VPA.
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      $q \to a[q', \gamma, p']bp$.
    - if $(q, a, q', \gamma) \in \delta$ s.t. $a \in \Sigma_c$, then
      $(q, \perp) \to aq'$, $q \to aq'$, $(q, \perp) \to a[q', \gamma, p]$, $q \to a[q', \gamma, p]$.
    - if $(q, a, \perp, q') \in \delta$ s.t. $a \in \Sigma_r$, then $(q, \perp) \to a(q', \perp)$.
    - $\forall q \in Q.[q, \gamma, q] \to \varepsilon$,
    - $\forall q \in F.q \to \varepsilon, (q, \perp) \to \varepsilon$.

Equivalence of VPA and VPG: continued

From VPG to VPA.
Let $G = (\mathcal{N}_0, \mathcal{N}_1, \widetilde{\Sigma}, \mathcal{P}, S)$ be a VPG.
Construct VPA $\mathcal{A} = (\mathcal{N}, \widetilde{\Sigma}, \Sigma_r \times \mathcal{N} \cup \{\perp, \$\}, \delta, S, F)$ as follows.

- δ is defined by the following rules,
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  - if $X \to aY$ s.t. $a \in \Sigma_l$, then $(X, a, Y) \in \delta$,
  - if $X \to aY$ s,t. $a \in \Sigma_c$, then $(X, a, Y, \$) \in \delta$,

Equivalence of VPA and VPG: continued

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  - if $X \to aY$ s.t. $a \in \Sigma_r$, then $(X, a, \$, Y) \in \delta$ and $(X, a, \perp, Y) \in \delta$,

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  - if $X \to aYbZ$, then $(X, a, Y, (b, Z)) \in \delta$,

Equivalence of VPA and VPG: continued

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  - if $X \to aYbZ$, then $(X, a, Y, (b, Z)) \in \delta$,
  - if $X \to \varepsilon$ and $X \in \mathcal{N}_0$, then $(X, b, (b, Y), Y) \in \delta$.

Equivalence of VPA and VPG: continued

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  - if $X \to aYbZ$, then $(X, a, Y, (b, Z)) \in \delta$,
  - if $X \to \varepsilon$ and $X \in \mathcal{N}_0$, then $(X, b, (b, Y), Y) \in \delta$.
- $\mathcal{A}$ accepts if the state is in X s.t. $X \to \varepsilon$ and the top symbol is \$ or $\perp$.

Equivalence of VPA and VPG: continued

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Let $G = (\mathcal{N}_0, \mathcal{N}_1, \widetilde{\Sigma}, \mathcal{P}, S)$ be a VPG.
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- $\mathcal{A}$ accepts if the state is in X s.t. $X \to \varepsilon$ and the top symbol is \$ or $\perp$.

  Adapt $\mathcal{A}$ *into* $\mathcal{A}' = (\mathcal{N} \times \Gamma, \widetilde{\Sigma}, \Gamma, \delta', (S, \perp), \{(X, \gamma) \mid X \to \varepsilon, \gamma = \$, \perp\})$
  by adding the top symbol of the stack into the states.

Equivalence of VPA and VPG: continued

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Let $G = (\mathcal{N}_0, \mathcal{N}_1, \widetilde{\Sigma}, \mathcal{P}, S)$ be a VPG.
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  - if $X \to aY$ s.t. $a \in \Sigma_l$, then $\forall \gamma.((X, \gamma), a, (Y, \gamma)) \in \delta'$,

Equivalence of VPA and VPG: continued

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  - if $X \to aYbZ$, then $(X, a, Y, (b, Z)) \in \delta$,
  - if $X \to \varepsilon$ and $X \in \mathcal{N}_0$, then $(X, b, (b, Y), Y) \in \delta$.
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Equivalence of VPA and VPG: continued

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  - if $X \to aY$ s.t. $a \in \Sigma_l$, then $(X, a, Y) \in \delta$,
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- if $X \to aY$ s.t. $a \in \Sigma_r$, then
  $\forall \gamma.((X, \gamma), a, \bot, (Y, \bot)) \in \delta'$ and $\forall \gamma.((X, \$), a, (\$, \gamma), (Y, \gamma)) \in \delta'$,
- if $X \to aYbZ$, then $\forall \gamma.((X, \gamma), a, (Y, (b, Z)), ((b, Z), \gamma)) \in \delta'$,

Equivalence of VPA and VPG: continued

From VPG to VPA.
Let $G = (\mathcal{N}_0, \mathcal{N}_1, \widetilde{\Sigma}, \mathcal{P}, S)$ be a VPG.
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  $\forall\gamma.((X, \gamma), a, \perp, (Y, \perp)) \in \delta'$ and $\forall\gamma.((X, \$), a, (\$, \gamma), (Y, \gamma)) \in \delta'$,
- if $X \to aYbZ$, then $\forall\gamma.((X, \gamma), a, (Y, (b, Z)), ((b, Z), \gamma)) \in \delta'$,
- if $X \to \varepsilon$ and $X \in \mathcal{N}_0$, then $\forall\gamma.((X, (b, Z)), b, ((b, Z), \gamma), (Z, \gamma)) \in \delta'$.

Outline

Visibly pushdown automata (VPA)

Closure properties

Visibly pushdown grammar (VPG)

Logical characterization

- Equivalence of NFA and MSO
- Equivalence of VPA and MSO_μ

Decision problems

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Monadic Second-Order Logic (MSO)

Syntax.

$\varphi := P_\sigma(x) \mid x = y \mid \mathrm{suc}(x, y) \mid X(x) \mid \varphi_1 \vee \varphi_2 \mid \neg\varphi_1 \mid \exists x \varphi_1 \mid \exists X \varphi_1$,

where $\sigma \in \Sigma$, x, y are position (first-order) variables, X is a set (second-order) variables

An MSO sentence is a MSO formula without free variables.

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Semantics.

A structure $\mathcal{S}$ over Σ is

- a domain $S = \{1, \ldots, n\}$,
- an interpretation of all the unary predicates $P_\sigma \in \Sigma$ over S, denoted by $(P_\sigma)^{\mathcal{S}}$.

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- an interpretation of all the unary predicates $P_\sigma \in \Sigma$ over S, denoted by $(P_\sigma)^{\mathcal{S}}$.

Example. Let $\Sigma = \{a, b\}$. Then $\mathcal{S} = (\{1, 2, 3\}, (P_a)^{\mathcal{S}} = \{1\}, (P_b)^{\mathcal{S}} = \{2, 3\})$ is a structure over Σ.

Monadic Second-Order Logic (MSO)

Syntax.

$\varphi := P_\sigma(x) \mid x = y \mid \mathrm{suc}(x, y) \mid X(x) \mid \varphi_1 \vee \varphi_2 \mid \neg\varphi_1 \mid \exists x \varphi_1 \mid \exists X \varphi_1$,

where $\sigma \in \Sigma$, x, y are position (first-order) variables, X is a set (second-order) variables

An MSO sentence is a MSO formula without free variables.

Semantics.

A structure $\mathcal{S}$ over Σ is

- a domain $S = \{1, \ldots, n\}$,
- an interpretation of all the unary predicates $P_\sigma \in \Sigma$ over S, denoted by $(P_\sigma)^{\mathcal{S}}$.

Example. Let $\Sigma = \{a, b\}$. Then $\mathcal{S} = (\{1, 2, 3\}, (P_a)^{\mathcal{S}} = \{1\}, (P_b)^{\mathcal{S}} = \{2, 3\})$ is a structure over Σ.

A word $w = a_1 \ldots a_n$ can be seen as a structure $\mathcal{S}_w$ over Σ,

- the domain of $\mathcal{S}_w$, denoted by S_w, is $\{1, \ldots, n\}$,
- the interpretation of every $P_\sigma \in \Sigma$ is the set of positions with the letter σ in w.

Monadic Second-Order Logic (MSO)

Syntax.

$\varphi := P_\sigma(x) \mid x = y \mid \mathsf{suc}(x, y) \mid X(x) \mid \varphi_1 \vee \varphi_2 \mid \neg\varphi_1 \mid \exists x \varphi_1 \mid \exists X \varphi_1$,

where $\sigma \in \Sigma$, x, y are position (first-order) variables, X is a set (second-order) variables

An MSO sentence is a MSO formula without free variables.

Semantics. Given a MSO formula φ, a valuation of $\mathrm{free}(\varphi)$ over a structure $\mathcal{S}$ is a mapping $\mathcal{I}$ such that

- for every $x \in \mathrm{free}(\varphi)$, $\mathcal{I}(x) \in S$,
- for every $X \in \mathrm{free}(\varphi)$, $\mathcal{I}(X) \subseteq S$.

Monadic Second-Order Logic (MSO)

Syntax.

$\varphi := P_\sigma(x) \mid x = y \mid \mathrm{suc}(x,y) \mid X(x) \mid \varphi_1 \vee \varphi_2 \mid \neg\varphi_1 \mid \exists x \varphi_1 \mid \exists X \varphi_1$,

where $\sigma \in \Sigma$, x, y are position (first-order) variables, X is a set (second-order) variables

An MSO sentence is a MSO formula without free variables.

Semantics. A MSO formula φ is satisfied over a word $w = a_1 \ldots a_n$, with a valuation $\mathcal{I}$ of $\mathrm{free}(\varphi)$ over $\mathcal{S}_w$, denoted by $(w, \mathcal{I}) \models \varphi$, is defined as follows,

- $(w, \mathcal{I}) \models P_\sigma(x)$ iff $a_{\mathcal{I}(x)} = \sigma$,
- $(w, \mathcal{I}) \models x = y$ iff $\mathcal{I}(x) = \mathcal{I}(y)$,
- $(w, \mathcal{I}) \models \mathrm{suc}(x,y)$ iff $\mathcal{I}(x) + 1 = \mathcal{I}(y)$,
- $(w, \mathcal{I}) \models X(x)$ iff $\mathcal{I}(x) \in \mathcal{I}(X)$,
- $(w, \mathcal{I}) \models \varphi_1 \vee \varphi_2$ iff $(w, \mathcal{I}) \models \varphi_1$ or $(w, \mathcal{I}) \models \varphi_2$,
- $(w, \mathcal{I}) \models \neg\varphi_1$ iff not $(w, \mathcal{I}) \models \varphi_1$,
- $(w, \mathcal{I}) \models \exists x \varphi_1$ iff there is $j \in S_w$ such that $(w, \mathcal{I}[x \to j]) \models \varphi_1$,
- $(w, \mathcal{I}) \models \exists X \varphi_1$ iff there is $J \subseteq S_w$ such that $(w, \mathcal{I}[X \to J]) \models \varphi_1$.

Monadic Second-Order Logic (MSO)

Syntax.

$\varphi := P_\sigma(x) \mid x = y \mid \mathrm{suc}(x, y) \mid X(x) \mid \varphi_1 \vee \varphi_2 \mid \neg\varphi_1 \mid \exists x \varphi_1 \mid \exists X \varphi_1$,

where $\sigma \in \Sigma$, x, y are position (first-order) variables, X is a set (second-order) variables

An MSO sentence is a MSO formula without free variables.

Semantics.

Let φ be a MSO sentence.

The language defined by φ, denoted $\mathcal{L}(\varphi)$: The set of words satisfying φ.

A language $L \subseteq \Sigma^*$ is MSO-definable

if

there is a MSO sentence φ such that $\mathcal{L}(\varphi) = L$.

Monadic Second-Order Logic (continued)

Abbreviations.

- $\varphi_1 \wedge \varphi_2 = \neg(\neg\varphi_1 \vee \neg\varphi_2)$,
- $\varphi_1 \rightarrow \varphi_2 = \neg\varphi_1 \vee \varphi_2$,
- $\forall x \varphi_1 = \neg\exists x(\neg\varphi_1)$,
- $x < y = \begin{array}{c} \neg x = y \ \wedge \\ \forall X\ (\ (X(x) \wedge \forall z_1 \forall z_2 (X(z_1) \wedge \mathrm{suc}(z_1, z_2) \rightarrow X(z_2))) \rightarrow X(y)\ ) \end{array}$,
- $\mathrm{first}(x) = \forall y(x = y \vee x < y)$,
- $\mathrm{last}(x) = \forall y(x = y \vee y < x)$.

Monadic Second-Order Logic (continued)

Abbreviations.

- $\varphi_1 \wedge \varphi_2 = \neg(\neg\varphi_1 \vee \neg\varphi_2)$,
- $\varphi_1 \rightarrow \varphi_2 = \neg\varphi_1 \vee \varphi_2$,
- $\forall x \varphi_1 = \neg\exists x(\neg\varphi_1)$,
- $x < y = \begin{array}{c} \neg x = y \wedge \\ \forall X \left(\left(X(x) \wedge \forall z_1 \forall z_2 (X(z_1) \wedge \mathrm{suc}(z_1, z_2) \rightarrow X(z_2)) \right) \rightarrow X(y) \right) \end{array}$,
- $\mathrm{first}(x) = \forall y(x = y \vee x < y)$,
- $\mathrm{last}(x) = \forall y(x = y \vee y < x)$.

Example.

$$\neg\exists x \ \mathrm{first}(x), \ \text{i.e.,} \ \varepsilon$$

Monadic Second-Order Logic (continued)

Abbreviations.

- $\varphi_1 \wedge \varphi_2 = \neg(\neg\varphi_1 \vee \neg\varphi_2)$,
- $\varphi_1 \rightarrow \varphi_2 = \neg\varphi_1 \vee \varphi_2$,
- $\forall x \varphi_1 = \neg\exists x(\neg\varphi_1)$,
- $x < y = \begin{array}{c} \neg x = y \wedge \\ \forall X \; (\; (X(x) \wedge \forall z_1 \forall z_2 (X(z_1) \wedge \mathrm{suc}(z_1, z_2) \rightarrow X(z_2))) \rightarrow X(y) \;) \end{array}$,
- $\mathrm{first}(x) = \forall y(x = y \vee x < y)$,
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Example.

$$\neg\exists x \; \mathrm{first}(x), \text{ i.e., } \varepsilon$$

$$\exists x \exists y (P_a(x) \wedge P_b(y) \wedge x < y),$$

i.e., all the words that has two positions where b occurs later than a

Monadic Second-Order Logic (continued)

Abbreviations.

- $\varphi_1 \wedge \varphi_2 = \neg(\neg\varphi_1 \vee \neg\varphi_2)$,
- $\varphi_1 \to \varphi_2 = \neg\varphi_1 \vee \varphi_2$,
- $\forall x \varphi_1 = \neg\exists x(\neg\varphi_1)$,
- $x < y = \begin{array}{c} \neg x = y \wedge \\ \forall X \left((X(x) \wedge \forall z_1 \forall z_2 (X(z_1) \wedge \mathrm{suc}(z_1, z_2) \to X(z_2))) \to X(y) \right) \end{array}$,
- $\mathrm{first}(x) = \forall y(x = y \vee x < y)$,
- $\mathrm{last}(x) = \forall y(x = y \vee y < x)$.

Example.

$$\neg\exists x\ \mathrm{first}(x),\ \text{i.e., } \varepsilon$$

$$\exists x \exists y (P_a(x) \wedge P_b(y) \wedge x < y),$$

i.e., all the words that has two positions where b occurs later than a

$$\exists X \left(\begin{array}{c} \exists x(\mathrm{first}(x) \wedge X(x)) \wedge \\ \forall x \forall y \forall z (\mathrm{suc}(x, y) \wedge \mathrm{suc}(y, z) \wedge X(x) \to X(z)) \\ \wedge \forall x (X(x) \to P_a(x)) \end{array} \right),$$

i.e., non-empty words and all the even positions have a

From NFA to MSO
Let $\mathcal{A} = (Q, \Sigma, \delta, q_0, F)$ be a NFA.

From NFA to MSO

Let $\mathcal{A} = (Q, \Sigma, \delta, q_0, F)$ be a NFA. Let $Q = \{q_0, q_1, \ldots, q_n\}$.
Construct the MSO formula φ as follows,

$$\exists X_{q_0} \ldots X_{q_n} (\varphi_{unique} \wedge \varphi_{init} \wedge \varphi_{trans} \wedge \varphi_{final}),$$

where

- X_q stands for the positions where the run is in state q,
- $\varphi_{unique} = \bigwedge_{q \neq q'} \forall x \neg (X_q(x) \wedge X_{q'}(x))$
- $\varphi_{init} = \exists x (\mathrm{first}(x) \wedge \bigvee\limits_{(q_0, a, q) \in \delta} (P_a(x) \wedge X_q(x))),$
- $\varphi_{trans} = \forall x \forall y (\mathrm{suc}(x, y) \rightarrow \bigvee\limits_{(q, a, q') \in \delta} X_q(x) \wedge P_a(y) \wedge X_{q'}(y)),$
- $\varphi_{final} = \exists x (\mathrm{last}(x) \wedge \bigvee\limits_{q \in F} X_q(x)).$

Then $\mathcal{L}(\varphi) = \mathcal{L}(\mathcal{A})$ if $q_0 \notin F$ and $\mathcal{L}(\varphi \vee \forall x (\neg \mathrm{first}(x))) = \mathcal{L}(\mathcal{A})$ if $q_0 \in F$.

From MSO to NFA.

A normal form for MSO formulas

New modalities,

$X \subseteq Y$, $\mathrm{Singleton}(X)$, $suc(X, Y)$.

Then a MSO formula φ can be transformed into a normal form φ' by the following rules,

- if $\varphi = P_\sigma(x)$, then $\varphi' = \mathrm{Singleton}(X) \wedge X \subseteq P_\sigma$,
- if $\varphi = x = y$, then $\varphi' = \mathrm{Singleton}(X) \wedge \mathrm{Singleton}(Y) \wedge X \subseteq Y \wedge Y \subseteq X$,
- if $\varphi = \mathrm{suc}(x, y)$, then $\varphi' = \mathrm{suc}(X, Y)$,
- if $\varphi = Z(x)$, then $\varphi' = \mathrm{Singleton}(X) \wedge X \subseteq Z$,
- if $\varphi = \varphi_1 \vee \varphi_2$, then $\varphi' = \varphi_1' \vee \varphi_2'$,
- if $\varphi = \neg\varphi_1$, then $\varphi' = \neg\varphi_1'$,
- if $\varphi = \exists x \varphi_1$, then $\varphi' = \exists X(\mathrm{Singleton}(X) \wedge \varphi_1')$,
- if $\varphi = \exists X \varphi_1$, then $\varphi' = \exists X \varphi_1'$.

From MSO to NFA.

$$\varphi := X \subseteq P_\sigma \mid P_\sigma \subseteq X \mid X \subseteq Y \mid \mathrm{Singleton}(X) \mid \mathsf{suc}(X,Y) \mid \varphi_1 \vee \varphi_2 \mid \neg\varphi_1 \mid \exists X \varphi_1.$$

Let $\varphi(X_1, \ldots, X_k)$ be a MSO formula in the prefix normal form.
We construct a NFA $\mathcal{A} = (Q, \Sigma \times \{0,1\}^k, \delta, q_0, F)$ as follows.
Consider $\varphi(X_1, \cdots, X_k)$ with $X_1, \cdots, X_k$ free variables, $(a, b_1 \cdots b_k)$ at position p such that $a \in \Sigma$ and $b_i \in \{0,1\}$ denotes that

- a is the symbol at position p,
- $b_i = 1$ denotes that $p \in X_i$
- $b_i = 0$ denotes that $p \notin X_i$

From MSO to NFA.

$$\varphi := X \subseteq P_\sigma \mid P_\sigma \subseteq X \mid X \subseteq Y \mid \mathrm{Singleton}(X) \mid \mathsf{suc}(X,Y) \mid \varphi_1 \vee \varphi_2 \mid \neg\varphi_1 \mid \exists X \varphi_1.$$

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- a is the symbol at position p,
- $b_i = 1$ denotes that $p \in X_i$
- $b_i = 0$ denotes that $p \notin X_i$

input	a_1	a_2	a_3	a_4	a_5	$a_6 \cdots$
X_1	1	0	1	0	1	$0 \cdots$
X_2	0	0	1	1	0	$1 \cdots$

X_1: even positions
X_2: prime positions

NFA≡MSO

From MSO to NFA.

$\varphi := X \subseteq P_\sigma \mid P_\sigma \subseteq X \mid X \subseteq Y \mid \mathrm{Singleton}(X) \mid \mathsf{suc}(X, Y) \mid \varphi_1 \vee \varphi_2 \mid \neg\varphi_1 \mid \exists X \varphi_1.$

Let $\varphi(X_1, \ldots, X_k)$ be a MSO formula in the prefix normal form.
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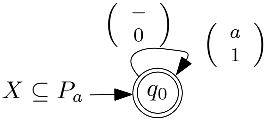

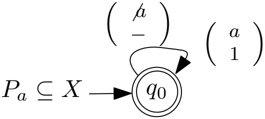

NFA≡MSO

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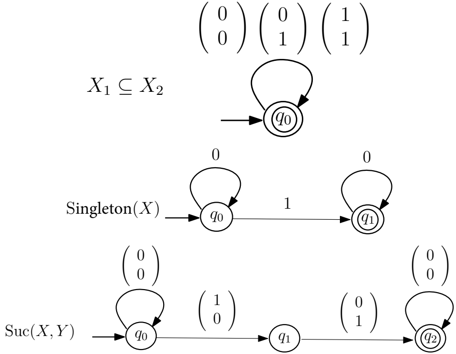

NFA≡MSO

From MSO to NFA.

$\varphi := X \subseteq P_\sigma \mid P_\sigma \subseteq X \mid X \subseteq Y \mid \mathrm{Singleton}(X) \mid \mathsf{suc}(X, Y) \mid \varphi_1 \vee \varphi_2 \mid \neg\varphi_1 \mid \exists X \varphi_1.$

Let $\varphi(X_1, \dots, X_k)$ be a MSO formula in the normal form.
We construct a NFA $\mathcal{A} = (Q, \Sigma \times \{0,1\}^k, \delta, q_0, F)$ as follows.

- $\varphi = \varphi_1 \vee \varphi_2$
  NFAs are closed under union,
- $\varphi = \neg\varphi_1$
  NFAs are closed under complementation,
- $\varphi = \exists X_1 \varphi_1$
  NFAs are closed under projection (a special case of homomorphisms), e.g. $(b_1, \dots, b_k) \to (b_2, \dots, b_k)$.

Then $\mathcal{L}(\varphi) = \mathcal{L}(\mathcal{A})$

Outline

Visibly pushdown automata (VPA)

Closure properties

Visibly pushdown grammar (VPG)

Logical characterization

- Equivalence of NFA and MSO
- Equivalence of VPA and MSO_μ

Decision problems

MSO_μ

Fix $\widetilde{\Sigma}$.

Given a word $w = a_1 \ldots a_n \in \Sigma^*$, a binary relation $\mu(x, y)$ can be defined such that

$$\mu(i,j) \textit{ iff } a_i \textit{ is a call and } a_j \textit{ is a matching return.}$$

MSO$_\mu$

Fix $\widetilde{\Sigma}$.
Given a word $w = a_1 \ldots a_n \in \Sigma^*$, a binary relation $\mu(x, y)$ can be defined such that

$\mu(i, j)$ *iff* a_i *is a call and* a_j *is a matching return.*

Example. In the word "(( )) ( ) ((", $\mu(1, 4), \mu(2, 3), \mu(5, 6)$ hold.

MSO_μ

Fix $\widetilde{\Sigma}$.
Given a word $w = a_1 \ldots a_n \in \Sigma^*$, a binary relation $\mu(x, y)$ can be defined such that

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Example. In the word "(( )) ( ) ((", $\mu(1, 4), \mu(2, 3), \mu(5, 6)$ hold.

Syntax of MSO_μ over $\widetilde{\Sigma}$.

$$\varphi := P_\sigma(x) \mid x = y \mid \mathrm{suc}(x, y) \mid X(x) \mid \mu(x, y) \mid \varphi_1 \vee \varphi_2 \mid \neg\varphi_1 \mid \exists x \varphi_1 \mid \exists X \varphi_1,$$

where $\sigma \in \Sigma$.

MSO_μ

Fix $\widetilde{\Sigma}$.
Given a word $w = a_1 \ldots a_n \in \Sigma^*$, a binary relation $\mu(x, y)$ can be defined such that

$$\mu(i,j) \textit{ iff } a_i \textit{ is a call and } a_j \textit{ is a matching return.}$$

Example. In the word "(( )) ( ) ((", $\mu(1,4), \mu(2,3), \mu(5,6)$ hold.

Syntax of MSO_μ **over** $\widetilde{\Sigma}$.

$$\varphi := P_\sigma(x) \mid x = y \mid \mathrm{suc}(x,y) \mid X(x) \mid \mu(x,y) \mid \varphi_1 \vee \varphi_2 \mid \neg\varphi_1 \mid \exists x \varphi_1 \mid \exists X \varphi_1,$$

where $\sigma \in \Sigma$.

Semantics of $\mathbf{MSO}_\mu$ **over** $\widetilde{\Sigma}$.

- $(w, \mathcal{I}) \models \mu(x,y)$ iff $\mu(\mathcal{I}(x), \mathcal{I}(y))$ holds on w.

MSO$_\mu$

Fix $\widetilde{\Sigma}$.
Given a word $w = a_1 \ldots a_n \in \Sigma^*$, a binary relation $\mu(x, y)$ can be defined such that

$\mu(i, j)$ iff a_i is a call and a_j is a matching return.

Example. In the word "(( )) ( ) ((", $\mu(1, 4), \mu(2, 3), \mu(5, 6)$ hold.

Syntax of *MSO$_\mu$* over $\widetilde{\Sigma}$.

$$\varphi := P_\sigma(x) \mid x = y \mid \mathrm{suc}(x, y) \mid X(x) \mid \mu(x, y) \mid \varphi_1 \vee \varphi_2 \mid \neg\varphi_1 \mid \exists x \varphi_1 \mid \exists X \varphi_1,$$

where $\sigma \in \Sigma$.

Semantics of MSO$_\mu$ over $\widetilde{\Sigma}$.

- $(w, \mathcal{I}) \models \mu(x, y)$ iff $\mu(\mathcal{I}(x), \mathcal{I}(y))$ holds on w.

Example. Let $\widetilde{\Sigma} = (\{a\}, \{b\}, \{c\})$

$$\forall x(P_a(x) \rightarrow \exists y \exists z(P_b(y) \wedge P_c(z) \wedge x < z \wedge z < y \wedge \mu(x, y)))$$

From VPA to MSO$_\mu$

Let $\mathcal{A} = (Q, \widetilde{\Sigma}, \Gamma, \delta, q_0, \perp, F)$ be a VPA, $Q = \{q_0, \ldots, q_n\}$, $\Gamma = \{\gamma_1, \ldots, \gamma_k\}$.

From VPA to MSO$_\mu$

Let $\mathcal{A} = (Q, \widetilde{\Sigma}, \Gamma, \delta, q_0, \perp, F)$ be a VPA, $Q = \{q_0, \ldots, q_n\}$, $\Gamma = \{\gamma_1, \ldots, \gamma_k\}$.
Define $\varphi := \exists X_{q_0} \ldots X_{q_n} P_{\gamma_1} \ldots P_{\gamma_k} (\varphi_{unique} \wedge \varphi_{init} \wedge \varphi_{trans} \wedge \varphi_{final})$ as follows,

From VPA to MSO_μ

Let $\mathcal{A} = (Q, \widetilde{\Sigma}, \Gamma, \delta, q_0, \perp, F)$ be a VPA, $Q = \{q_0, \ldots, q_n\}$, $\Gamma = \{\gamma_1, \ldots, \gamma_k\}$.
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- X_q stands for the positions where the run is in state q

From VPA to MSO$_\mu$

Let $\mathcal{A} = (Q, \widetilde{\Sigma}, \Gamma, \delta, q_0, \perp, F)$ be a VPA, $Q = \{q_0, \ldots, q_n\}$, $\Gamma = \{\gamma_1, \ldots, \gamma_k\}$.
Define $\varphi := \exists X_{q_0} \ldots X_{q_n} P_{\gamma_1} \ldots P_{\gamma_k} (\varphi_{unique} \wedge \varphi_{init} \wedge \varphi_{trans} \wedge \varphi_{final})$ as follows,

- X_q stands for the positions where the run is in state q
- P_γ stands for the positions where r is pushed/popped

From VPA to MSO$_\mu$

Let $\mathcal{A} = (Q, \widetilde{\Sigma}, \Gamma, \delta, q_0, \perp, F)$ be a VPA, $Q = \{q_0, \ldots, q_n\}$, $\Gamma = \{\gamma_1, \ldots, \gamma_k\}$.
Define $\varphi := \exists X_{q_0} \ldots X_{q_n} P_{\gamma_1} \ldots P_{\gamma_k} (\varphi_{unique} \wedge \varphi_{init} \wedge \varphi_{trans} \wedge \varphi_{final})$ as follows,

- X_q stands for the positions where the run is in state q
- P_γ stands for the positions where r is pushed/popped
- $\varphi_{unique} = \bigwedge_{q \neq q'} \forall x \neg (X_q(x) \wedge X_{q'}(x)) \wedge \bigwedge_{\gamma \neq \gamma'} \forall x \neg (P_\gamma(x) \wedge P_{\gamma'}(x))$

From VPA to MSO_μ

Let $\mathcal{A} = (Q, \widetilde{\Sigma}, \Gamma, \delta, q_0, \perp, F)$ be a VPA, $Q = \{q_0, \ldots, q_n\}$, $\Gamma = \{\gamma_1, \ldots, \gamma_k\}$.
Define $\varphi := \exists X_{q_0} \ldots X_{q_n} P_{\gamma_1} \ldots P_{\gamma_k} (\varphi_{unique} \wedge \varphi_{init} \wedge \varphi_{trans} \wedge \varphi_{final})$ as follows,

- X_q stands for the positions where the run is in state q
- P_γ stands for the positions where r is pushed/popped
- $\varphi_{unique} = \bigwedge_{q \neq q'} \forall x \neg (X_q(x) \wedge X_{q'}(x)) \wedge \bigwedge_{\gamma \neq \gamma'} \forall x \neg (P_\gamma(x) \wedge P_{\gamma'}(x))$
- $$\varphi_{init} = \exists x \left(\mathrm{first}(x) \wedge \left(\begin{array}{c} \bigvee_{(q_0,a,q) \in \delta} (P_a(x) \wedge X_q(x)) \bigvee \\ \bigvee_{(q_0,a,q,\gamma) \in \delta} (P_a(x) \wedge X_q(x) \wedge P_\gamma(x)) \bigvee \\ \bigvee_{(q_0,a,\perp,q) \in \delta} (P_a(x) \wedge X_q(x) \wedge P_\perp(x)) \end{array} \right) \right),$$

From VPA to MSO$_\mu$

Let $\mathcal{A} = (Q, \widetilde{\Sigma}, \Gamma, \delta, q_0, \perp, F)$ be a VPA, $Q = \{q_0, \ldots, q_n\}$, $\Gamma = \{\gamma_1, \ldots, \gamma_k\}$.
Define $\varphi := \exists X_{q_0} \ldots X_{q_n} P_{\gamma_1} \ldots P_{\gamma_k} (\varphi_{unique} \wedge \varphi_{init} \wedge \varphi_{trans} \wedge \varphi_{final})$ as follows,

From VPA to MSO$_\mu$

Let $\mathcal{A} = (Q, \widetilde{\Sigma}, \Gamma, \delta, q_0, \bot, F)$ be a VPA, $Q = \{q_0, \ldots, q_n\}$, $\Gamma = \{\gamma_1, \ldots, \gamma_k\}$.
Define $\varphi := \exists X_{q_0} \ldots X_{q_n} P_{\gamma_1} \ldots P_{\gamma_k} (\varphi_{unique} \wedge \varphi_{init} \wedge \varphi_{trans} \wedge \varphi_{final})$ as follows,

- $\varphi_{trans} = \forall x \forall y \, (\mathrm{suc}(x, y) \to \psi_{call} \vee \psi_{return} \vee \psi_{local})$,

From VPA to MSO$_\mu$

Let $\mathcal{A} = (Q, \widetilde{\Sigma}, \Gamma, \delta, q_0, \perp, F)$ be a VPA, $Q = \{q_0, \ldots, q_n\}$, $\Gamma = \{\gamma_1, \ldots, \gamma_k\}$.

Define $\varphi := \exists X_{q_0} \ldots X_{q_n} P_{\gamma_1} \ldots P_{\gamma_k} (\varphi_{unique} \wedge \varphi_{init} \wedge \varphi_{trans} \wedge \varphi_{final})$ as follows,

- $\varphi_{trans} = \forall x \forall y \, (\mathrm{suc}(x, y) \rightarrow \psi_{call} \vee \psi_{return} \vee \psi_{local})$, where
  - $\psi_{call} = \bigvee_{(q,a,q',\gamma)\in\delta} (X_q(x) \wedge P_a(y) \wedge X_{q'}(y) \wedge P_\gamma(y))$,

From VPA to MSO$_\mu$

Let $\mathcal{A} = (Q, \widetilde{\Sigma}, \Gamma, \delta, q_0, \bot, F)$ be a VPA, $Q = \{q_0, \ldots, q_n\}$, $\Gamma = \{\gamma_1, \ldots, \gamma_k\}$.
Define $\varphi := \exists X_{q_0} \ldots X_{q_n} P_{\gamma_1} \ldots P_{\gamma_k} (\varphi_{unique} \wedge \varphi_{init} \wedge \varphi_{trans} \wedge \varphi_{final})$ as follows,

- $\varphi_{trans} = \forall x \forall y \, (\mathrm{suc}(x, y) \to \psi_{call} \vee \psi_{return} \vee \psi_{local})$, where
  - $\psi_{call} = \bigvee_{(q,a,q',\gamma) \in \delta} (X_q(x) \wedge P_a(y) \wedge X_{q'}(y) \wedge P_\gamma(y))$,
  - $\psi_{local} = \bigvee_{(q,a,q') \in \delta} (X_q(x) \wedge P_a(y) \wedge X_{q'}(y))$,

From VPA to MSO$_\mu$

Let $\mathcal{A} = (Q, \widetilde{\Sigma}, \Gamma, \delta, q_0, \perp, F)$ be a VPA, $Q = \{q_0, \dots, q_n\}$, $\Gamma = \{\gamma_1, \dots, \gamma_k\}$.
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  - $\psi_{return} =$
  $$\bigvee_{(q,a,\gamma,q')\in\delta} (X_q(x) \wedge P_a(y) \wedge X_{q'}(y) \wedge P_\gamma(y) \wedge \exists z(\mu(z, y) \wedge P_\gamma(z))) \bigvee$$
  $$\bigvee_{(q,a,\perp,q')\in\delta} (q(x) \wedge P_a(y) \wedge X_{q'}(y) \wedge P_\perp(y) \wedge \neg\exists z(\mu(z, y))) .$$

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- $\varphi_{final} = \exists x \left(\mathrm{last}(x) \wedge \bigvee_{q\in F} X_q(x) \right)$.

From VPA to MSO$_\mu$

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- $\varphi_{final} = \exists x \left(\mathrm{last}(x) \wedge \bigvee_{q\in F} X_q(x) \right)$.

Then $\mathcal{L}(\varphi) = \mathcal{L}(\mathcal{A})$ if $q_0 \notin F$ and $\mathcal{L}(\varphi \vee \forall x(\neg \mathrm{first}(x))) = \mathcal{L}(\mathcal{A})$ if $q_0 \in F$.

VPA $\equiv$ MSO$_\mu$: continued

From MSO$_\mu$ to VPA.

$$\varphi := \begin{array}{l} X \subseteq P_\sigma \mid P_\sigma \subseteq X \mid X \subseteq Y \mid \mathrm{Singleton}(X) \mid \\ \mathsf{suc}(X, Y) \mid \mu(X, Y) \mid \varphi_1 \vee \varphi_2 \mid \neg\varphi_1 \mid \exists X \varphi_1 \end{array} ,$$

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We construct the VPA by structural induction.

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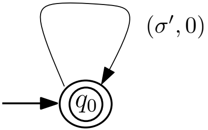
A word w' over Σ' encodes a word w over Σ with valuations of all X_i's.

$$\sigma', \sigma'' \neq \sigma : \begin{array}{ll} (\sigma', 0), \downarrow (\sigma', 0) & (\sigma', 0), \uparrow (\sigma'', 0) \\ (\sigma', 0), \uparrow (\sigma, 1) & (\sigma', 0), \uparrow \perp (\sigma', 0), \uparrow (\sigma, 0) \\ (\sigma, 1), \downarrow (\sigma, 1) & (\sigma, 0), \downarrow (\sigma, 0) \end{array}$$

$X \subseteq P_\sigma : \sigma \in \Sigma_c$



where $\downarrow$=push, $\uparrow$=pop. If push, $\sigma' \in \Sigma_c$; If pop, $\sigma' \in \Sigma_r$; otherwise $\sigma' \in \Sigma_l$; If σ'' in stack, $\sigma'' \in \Sigma_c$.

From MSO$_\mu$ to VPA.

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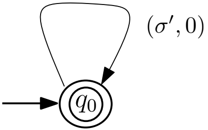

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VPA $\equiv$ MSO$_\mu$: continued

From MSO$_\mu$ to VPA.

$$\varphi := \begin{array}{l} X \subseteq P_\sigma \mid P_\sigma \subseteq X \mid X \subseteq Y \mid \mathrm{Singleton}(X) \mid \\ \mathsf{suc}(X, Y) \mid \mu(X, Y) \mid \varphi_1 \vee \varphi_2 \mid \neg\varphi_1 \mid \exists X \varphi_1 \end{array},$$

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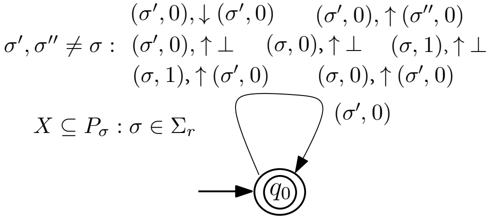
We construct the VPA by structural induction.

Consider a MSO$_\mu$ $\phi(X_1, \cdots, X_k)$ with free variables $X_1, \cdots, X_k$.

Define, $\widetilde{\Sigma'} = \langle \Sigma'_c, \Sigma'_r, \Sigma'_l \rangle$, where $\Sigma'_s = \Sigma_s \times \{0,1\}^k$ for $s \in \{c, r, l\}$.

A word w' over Σ' encodes a word w over Σ with valuations of all X_i's.



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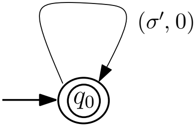
$X \subseteq P_\sigma : \sigma \in \Sigma_r$



$P_\sigma \subseteq X$ can be handled similarly!

VPA $\equiv$ MSO$_\mu$: continued

From MSO$_\mu$ to VPA.
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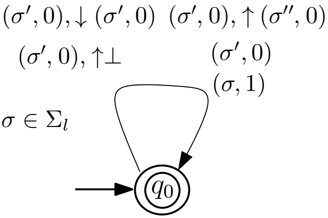

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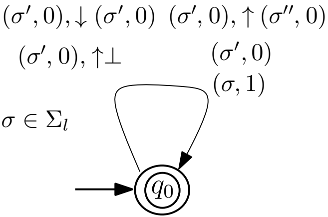
$\sigma', \sigma'' \neq \sigma :$

$X \subseteq P_\sigma : \sigma \in \Sigma_l$



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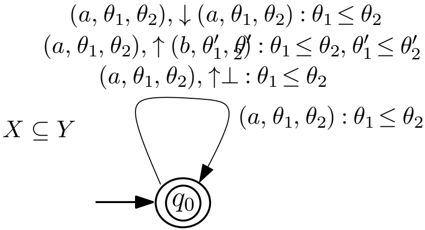

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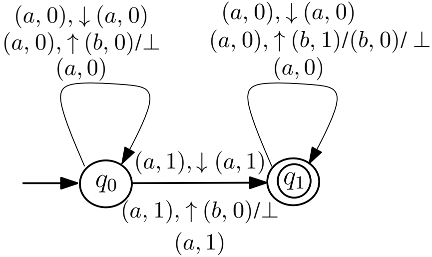
$\mathrm{Singleton}(X)$



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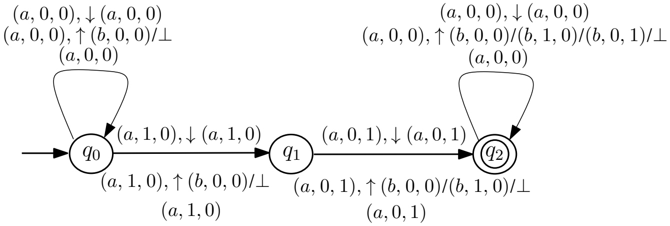
$\mathrm{suc}(X, Y)$



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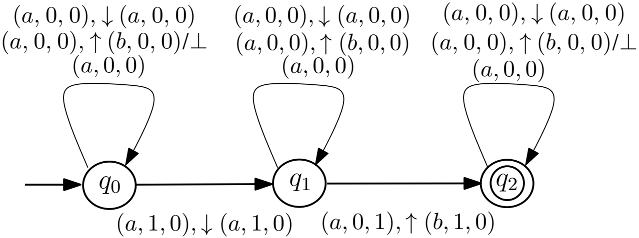
$\mu(X, Y)$



Satisfiability of MSO and MSO_μ

- A MSO formula ϕ over finite words is satisfiable iff the NFA $\mathcal{A}_\phi$ is nonempty.

Satisfiability of MSO and MSO$_\mu$

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- A MSO$_\mu$ formula ϕ over visibly pushdown words is satisfiable iff the VPA $\mathcal{A}_\phi$ is nonempty.

Satisfiability of MSO and MSO$_\mu$

- A MSO formula ϕ over finite words is satisfiable iff the NFA $\mathcal{A}_\phi$ is nonempty.
- A MSO$_\mu$ formula ϕ over visibly pushdown words is satisfiable iff the VPA $\mathcal{A}_\phi$ is nonempty.
- Upper Bound [Büchi, Elgot, Trakhtenbrot, 1957-8 (independently)]: Nonelementary Growth $2^{\cdot^{\cdot^{\cdot^{2^n}}}}$, (tower of height $O(n)$), due to complementation
- Lower Bound [Stockmeyer, 1974]: Satisfiability of FO over finite words is nonelementary (no bounded height tower).

Outline

Visibly pushdown automata (VPA)

Closure properties

Visibly pushdown grammar (VPG)

Logical characterization

- Equivalence of NFA and MSO
- Equivalence of VPA and MSO_{μ}

Decision problems

Nonemptiness

Theorem. The nonemptiness of VPA can be solved in $O(n^3)$ time.

A VPA can be transformed into an equivalent VPG in $O(n^3)$ time.

The emptiness of a CFG can be solved in linear time.

Language inclusion and universality problems

Theorem. The language inclusion problem and universality problem of VPA is EXPTIME-complete.

Upper bound.

Given two VPAs $\mathcal{A}_1$ and $\mathcal{A}_2$,

- determinize $\mathcal{A}_2$ into $\mathcal{A}_2'$,
- complement $\mathcal{A}_2'$ into $\mathcal{B}$,
- test whether $\mathcal{L}(\mathcal{A}_1) \cap \mathcal{L}(\mathcal{B}) = \emptyset$.

Language inclusion and universality problems

Theorem. The language inclusion problem and universality problem of VPA is EXPTIME-complete.

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$$\mathcal{L}(\mathcal{A}_1) \subseteq \mathcal{L}(\mathcal{A}_2) \iff \mathcal{L}(\mathcal{A}_1) \cap \mathcal{L}(\mathcal{B}) = \emptyset$$

$$\mathcal{L}(\mathcal{A}_2) = \widetilde{\Sigma}^* \iff \mathcal{L}(\mathcal{B}) = \emptyset$$

The determinization procedure can be fulfilled in EXPTIME.

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The determinization procedure can be fulfilled in EXPTIME. To show EXPTIME-hardness, we show that universality of VPA is EXPTIME-hard.

$$\mathcal{L}(\mathcal{A}) = \widetilde{\Sigma}^* \iff \mathcal{L}(\mathcal{A}) \subseteq \mathcal{L}(\mathcal{A}')$$

where $\mathcal{L}(\mathcal{A}') = \widetilde{\Sigma}^*$

Language inclusion and universality problems

Theorem. The language inclusion of VPA is EXPTIME-complete.
Lower bound.

The universality of VPA is EXPTIME*-hard.*

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Result from complexity theory: **APSPACE**= EXPTIME.

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Result from complexity theory: **APSPACE**= EXPTIME.

An alternating TM (ATM) is a TM $M = (Q_\exists, Q_\forall, \Sigma, \Gamma, \delta, q_0, B, F)$ such that

- the state set is divided into two disjoint subsets, $Q_\exists$ ("existential" state), $Q_\forall$ ("universal" state),
- for every $q \in Q$ and $a \in \Gamma$, $|\delta(q,a)| = 2$.

A run of an ATM M over an input $w \in \Sigma^*$ is a configuration tree s.t.

- the root of the tree is the initial configuration,
- we assume that for every node (configuration) $\alpha q \beta$ in the tree, if $q \in Q_\exists$, then

  $\alpha q \beta$ has one of its successor config. as its unique child in the tree,
- for every node (configuration) $\alpha q \beta$ in the tree, if $q \in Q_\forall$, then

  the two successor config. of $\alpha q \beta$ are both its children in the tree.

APSPACE: The class of languages accepted by ATMs using polynomial space.

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Reduction from

the membership problem of alternating TMs using polynomial space.

Let $M = (Q_\exists, Q_\forall, \Sigma, \Gamma, \delta, q_0, B, F)$ be an ATM using linear space, say cn.

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- We will construct a VPA $\mathcal{B}$ that accepts all the non-accepting computation histories of M over an input w, then

$$\mathcal{L}(\mathcal{B}) = \widetilde{\Sigma}^* \iff M \text{ does not accept } w$$

- To construct $\mathcal{B}$, we first construct a deterministic VPA $\mathcal{A}$ that accepts all the accepting computation histories of M over an input w, then complement $\mathcal{A}$

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Use C_x*'s (where* $x \in \{0,1\}^*$*) to denote the nodes of* t*,*
e.g. the root is C_ε, while the left child of the root is C_0, and so on.

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Use C_x's *(where* $x \in \{0,1\}^*$*) to denote the nodes of* t*,*
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Encode t by a word θ which is generated by a DFS traversal of t.

Language inclusion and universality problems

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Lower bound.

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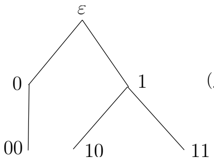
Initially set $\theta = \varepsilon$.

1. The traversal starts from the root C_ε.
2. When a node C_x is visited for the first time, then $\theta = \theta(fC_x)$,
3. When a node C_x is visited again by backtracking from its right-child, then $\theta = \theta(b\overline{C_x})^r$.
4. Each leaf is an accepting configuration.



$$(fC_\varepsilon)(fC_0)(fC_{00})(b\overline{C_{00}})^r(b\overline{C_0})^r(fC_1)(fC_{10})(b\overline{C_{10}})^r(fC_{11})(b\overline{C_{11}})^r(b\overline{C_1})^r(b\overline{C_\varepsilon})^r$$

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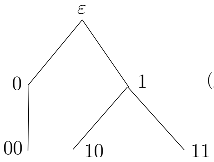

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Let $\Gamma' = \Gamma \cup Q \cup \overline{\Gamma} \cup \overline{Q} \cup \{f, b\}$, $\widetilde{\Gamma'} = \langle \Gamma \cup Q \cup \{f\}, \overline{\Gamma} \cup \overline{Q} \cup \{b\} \rangle$.

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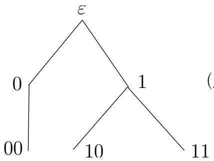

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The format of a successful computation θ, *e.g. well-matched call-returns.*

Language inclusion and universality problems

Theorem. The language inclusion of VPA is EXPTIME-complete.

Lower bound.

The universality of VPA is EXPTIME-hard.

The set of unsuccessful computations of M

can be accepted by a nondeterministic VPA $\mathcal{B}$ of polynomial size.

$$M \textit{ does not accept } w \textit{ iff } \mathcal{L}(\mathcal{B}) = (\Gamma')^*.$$

Equivalence problem

Theorem. The equivalence of VPA is EXPTIME-complete.

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Upper bound.

Invoke two times of inclusion testing.

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Invoke two times of inclusion testing.

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$$\mathcal{L}(\mathcal{A}) = \widetilde{\Sigma}^* \iff \mathcal{L}(\mathcal{A}) = \mathcal{L}(\mathcal{A}')$$

where $\mathcal{L}(\mathcal{A}') = \widetilde{\Sigma}^*$

Summary

Closure Properties

	Union	Intersection	Complement	Concatenation	Kleene-$*$
Regular	YES	YES	YES	YES	YES
CFL	YES	NO	NO	YES	YES
DCFL	NO	NO	YES	NO	NO
VPL	YES	YES	YES	YES	YES

Summary

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VPL	YES	YES	YES	YES	YES

Decision problems

	Emptiness	Universality/Equivalence	Inclusion
NFA	**NL**	PSPACE	PSPACE
PDA	**P**	Undecidable	Undecidable
DPDA	**P**	Decidable	Undecidable
VPA	**P**	EXPTIME	EXPTIME