EE150 Signals and Systems – Part 1: Overview

February 15, 2022

Introduction to Signals

Signal:

a function of one or more independent variables (e.g., time and spatial variables); typically contains information about the behaviour or nature of some physical phenomena.

e.g. Voice (audio), TV (audio + video), voltage, current, stock price, etc.

e.g. Picture (brightness), air pressure, temperature, and wind speeds (altitude), etc.

This course focuses on signals involving a single independent variable, i.e., time.

Introduction to Systems

System:

responds to a particular signal input by producing another signal (output).

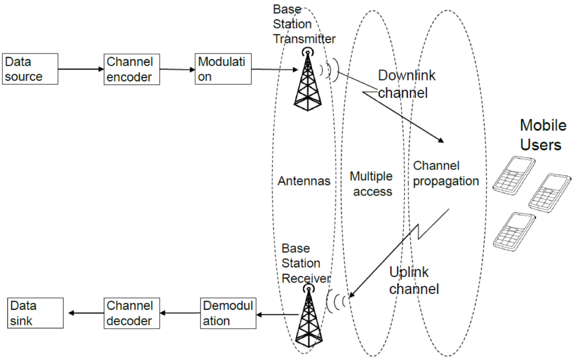
e.g. Biological sensory system, electronic circuits, automobile, etc.

$$\text{Input signal} \longrightarrow \boxed{\text{System}} \longrightarrow \text{Output signal}$$

Example: Cellular Communication Systems

Wired telephone network

Mobile telephone switching office



Objectives

System characterization

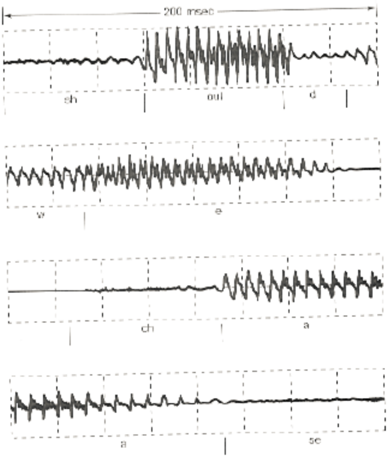
how it responds to input signal
(e.g. human auditory system)

System design

to process signal in a particular way
(e.g. signal restoration, signal identification, image processing)

Examples of Signals

Audio (intensity vs. time)
  characteristics: volume, rhythm, pitch

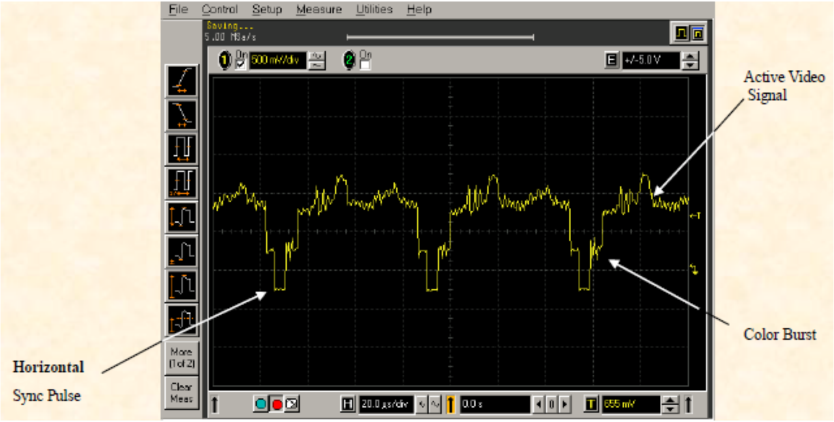


Examples of Signals cont.

TV signal (voltage vs. time)

modulated picture signal + audio signal; carrier signal;

system involves: antenna, tuner, CRT



Examples of Signals cont.

Biomedical signal (voltage vs. time)
e.g. Electrocardiogram

Traffic flow (quantity vs. time)
volume, composition, pattern, mobility;
system involved: traffic lights, roads, junctions

Network throughput
# of packets/sec., pattern

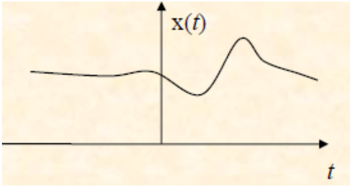
Examples of Signals cont.

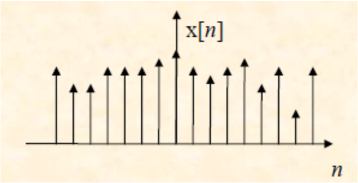
Stock price (index vs. time; $ vs. time)

Continuous vs. Discrete Time Signal

Continuous-time Signal
(independent variable: t)

Discrete-time Signal
(independent variable: n)

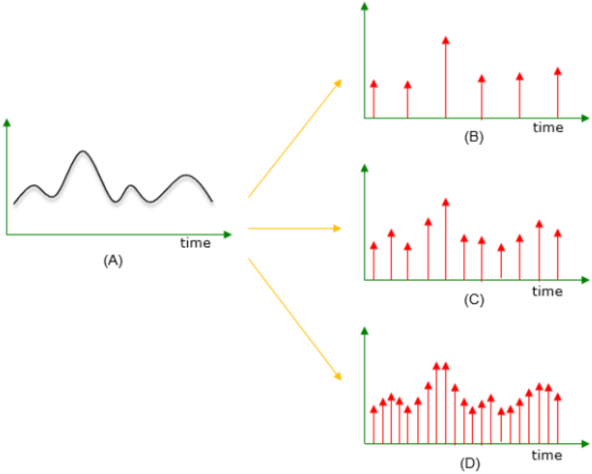




$x[n]$ is also referred as a <u>sequence</u>. Any particular one in $x[n]$ is called a <u>sample</u>.

Discrete-time signals are inherently discrete OR sampling of continuous-time signals

Sampling



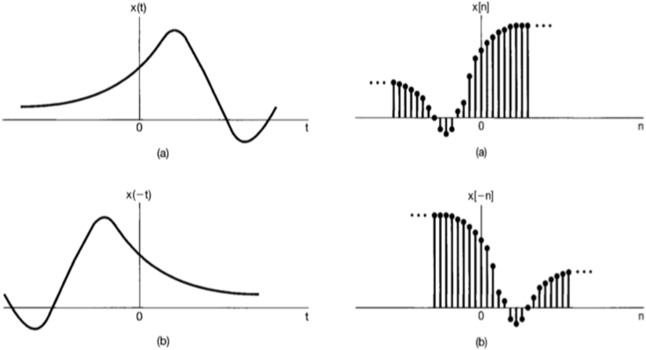
Transformation of Independent Variable

1. Time reflection: $x(t) \longleftrightarrow x(-t)$, $x[n] \longleftrightarrow x[-n]$
2. Time scaling: $x(t) \longleftrightarrow x(ct)$
3. Time shift: $x(t) \longleftrightarrow x(t - t_0)$, $x[n] \longleftrightarrow x[n - n_0]$

Transformation of Independent Variable cont.

Time reflection: reflect the signal at $t = 0$ or $n = 0$

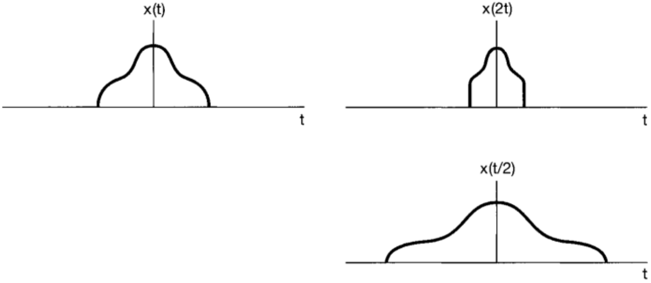
$$x(t) \longleftrightarrow x(-t)$$
$$x[n] \longleftrightarrow x[-n]$$



Transformation of Independent Variable cont.

Time scaling: stretched ($|c| < 1$) or compressed ($|c| > 1$) version of $x(t)$

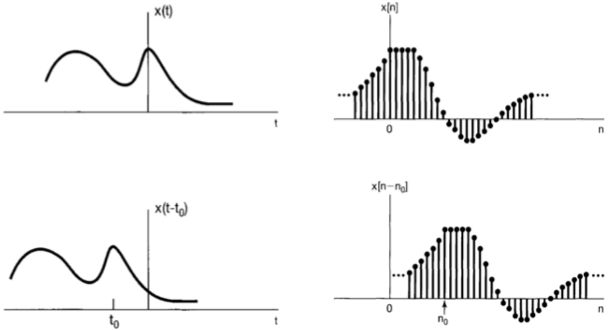
$$x(t) \longleftrightarrow x(ct)$$



Transformation of Independent Variable cont.

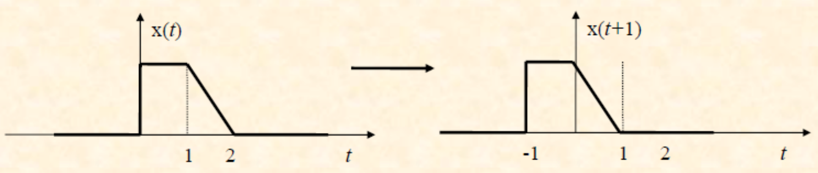
Time shift: delayed ($t_0 > 0$) or advanced ($t_0 < 0$) version of $x(t)$

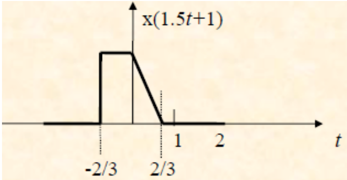
$$x(t) \longleftrightarrow x(t - t_0)$$
$$x[n] \longleftrightarrow x[n - n_0]$$



Transformation of Independent Variable cont.

To perform transformation $x(t) \rightarrow x(\alpha t + \beta)$,
You have to do time-shifting and then scaling.





Why this order?

$y(t) = x(1.5t + 1)$
Work out a few points:
$y(0) = x(1)$
$y(1) = x(2.5)$
$y(2) = x(4)$

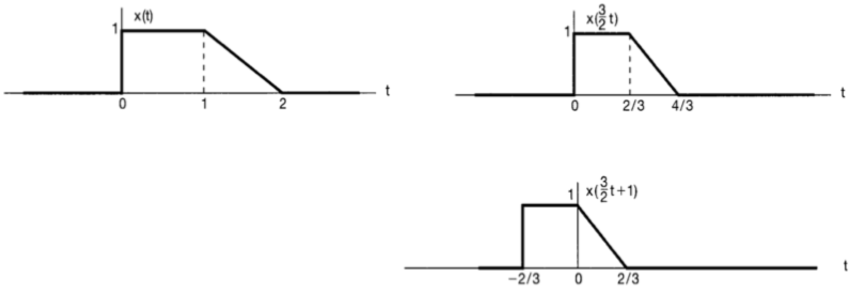
To get from y to x, we first scale t, then shift.

Therefore, to get from x to y, we first shift and then scale.

Q: What if we first scale and then shift?

Transformation of Independent Variable cont.

You can also first scale and then shift



Examples

Problem 1.5: Let $x(t)$ be a signal with $x(t) = 0$ for $t < 3$. For each signal given below, determine the values of t for which it is guaranteed to be zero

(a) $x(1-t)$
(b) $x(1-t) + x(2-t)$
(c) $x(3t)$

Problem 1.22: Describe how to obtain the following signals:

(a) $x[3-n]$
(b) $x[3n+1]$

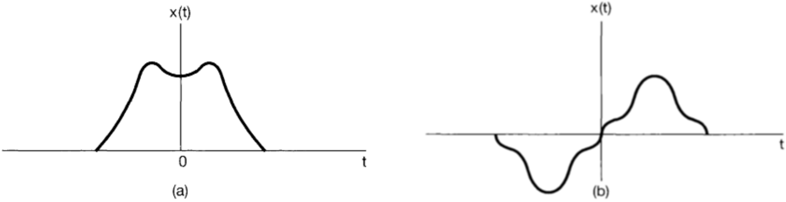
Even and Odd Functions

A signal is called an even signal (function) if

$$x(t) = x(-t), \quad x[n] = x[-n].$$

A signal is called an odd signal (function) if

$$x(t) = -x(-t), \quad x[n] = -x[-n].$$



Even and Odd Functions cont.

Any signal can be broken into sum of one even and one odd signal.

$$x(t) = Evenx(t) + Oddx(t)$$

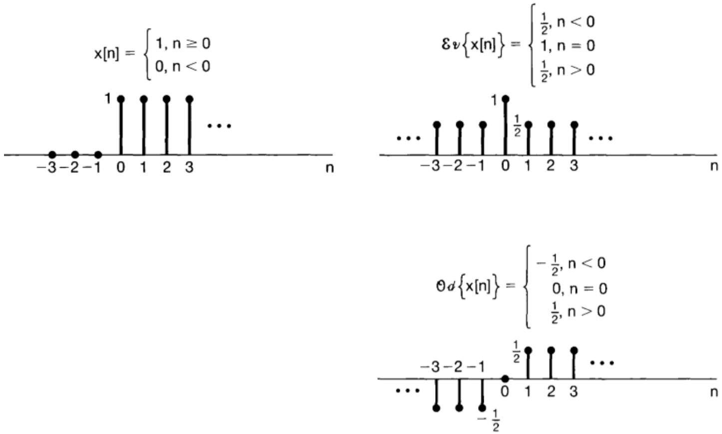
How?

$$\begin{aligned} x(t) &= Evenx(t) + Oddx(t), \\ x(-t) &= Evenx(-t) + Oddx(-t) = Evenx(t) - Oddx(t), \end{aligned}$$

$$\begin{aligned} \implies Evenx(t) &= \frac{1}{2}(x(t) + x(-t)), \\ Oddx(t) &= \frac{1}{2}(x(t) - x(-t)). \end{aligned}$$

Even and Odd Functions cont.

Example:



Periodic and Aperiodic Signal

IF $x(t)$ is periodic with period T, then
$x(t) = x(t + mT), \forall\ t$; m can be any integer

IF $x[n]$ is periodic with period N, then
$x[n] = x[n + mN], \forall\ n$; m can be any integer

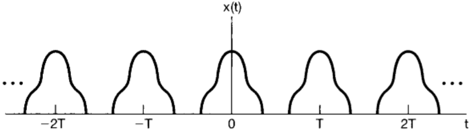
Fundamental period (T_0 or N_0):
the <u>smallest positive</u> value of (T or N) for which the above equation holds.

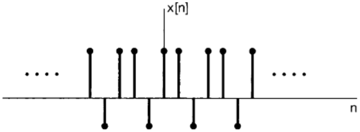
Aperiodic is also called Non-periodic

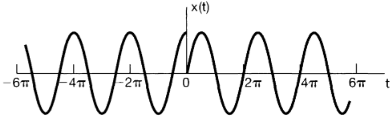
Periodic and Aperiodic Signal cont.

Q: What is the fundamental period of a constant function?

Examples:





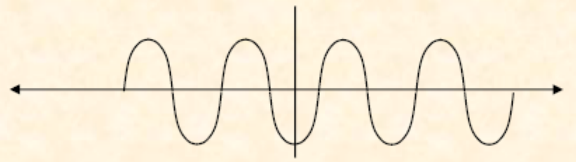


Exponential Signal

Real exponential: $x(t) = ce^{at}$

positive a: negative a:

Imaginary exponential: $x(t) = e^{j(\omega_0 t + \Phi)}$



Periodic and Sinusoidal Signal

Euler's Formula:

$$e^{j\omega_0 t} = \cos(\omega_0 t) + j \cdot \sin(\omega_0 t)$$

Is $x(t) = e^{j\omega_0 t}$ periodic? Suppose the period is T, then

$$x(t) = x(t+T), \implies e^{j\omega_0 t} = e^{j\omega_0 (t+T)}.$$

Hence $e^{j\omega_0 T} = 1$.

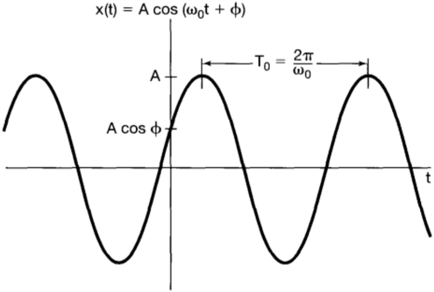
Fundamental period (T_0) is inversely proportional to fundamental frequency ($|\omega_0|$):

$$T_0 = 2\pi/|\omega_0|.$$

Periodic and Sinusoidal Signal cont.

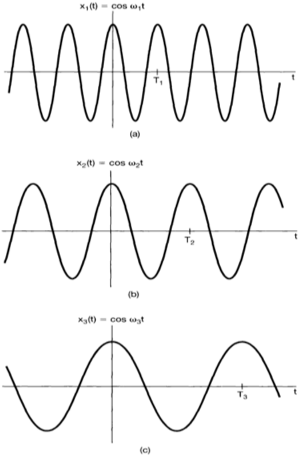
Sinusoidal signal: $x(t) = A\cos(\omega_0 t + \phi)$

- closely related to the periodic (T_0) complex exponential
- unit ω_0: radians/sec; ϕ: radians
- phase: $\omega_0 t + \phi$



Periodic and Sinusoidal Signal cont.

Fundamental frequency: $|\omega_1| > |\omega_2| > |\omega_3|$



Periodic and Sinusoidal Signal cont.

Complex exponential can be written in terms of sinusoidal signals with the same fundamental frequency

$$e^{j\omega_0 t} = \cos \omega_0 t + j \sin \omega_0 t$$

Sinusoidal signal can be written in terms of periodic complex exponentials with the same fundamental frequency

$$A \cos(\omega_0 t + \phi) = \frac{A}{2} e^{j\phi} e^{j\omega_0 t} + \frac{A}{2} e^{-j\phi} e^{-j\omega_0 t}$$

Energy and Power of Periodic Signals

Periodic signal: $x(t) = e^{j\omega_0 t}$
Total energy over a period

$$E_{\mathrm{period}} = \int_0^{T_0} |e^{j\omega_0 t}|^2 \mathrm{d}t = \int_0^{T_0} 1 \mathrm{d}t = T_0.$$

Average power over a period

$$P_{\mathrm{period}} = \frac{1}{T_0} E_{\mathrm{period}} = 1.$$

Complex periodic exponential signal has finite average power
Average power over a period

$$P_\infty = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |e^{j\omega_0 t}|^2 \mathrm{d}t = 1.$$

Harmonically related Complex Exponential

A set of periodic exponentials with fundamental frequencies that are all multiples of a single positive frequency $\omega_0 = \frac{2\pi}{T_0}$

$$\phi_k(t) = e^{jk\omega_0 t}, \quad k = 0, \pm 1, \pm 2, \ldots,$$

where $\phi_k(t)$ is periodic with fundamental frequency $|k|\omega_0$ and fundamental period

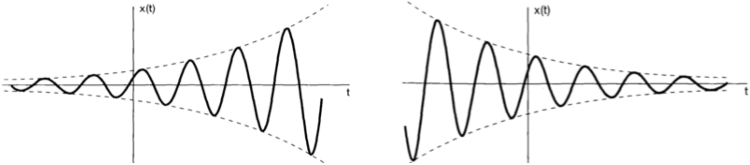
$$\frac{2\pi}{|k|\omega_0} = \frac{T_0}{|k|}.$$

General Complex Exponential

$$x(t) = C \cdot e^{(r+j\omega_0)t},$$

$$C = |C|e^{j\theta}, \quad r, \omega_0 \in \mathbb{R}$$

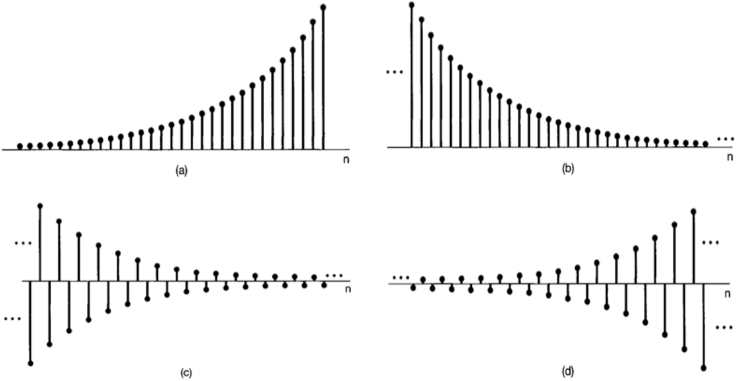
$$\implies x(t) = |C|e^{rt}e^{j(\omega_0 t+\theta)}$$



$Ce^{rt}\cos(\omega_0 t + \theta), r > 0$ $\qquad$ $Ce^{rt}\cos(\omega_0 t + \theta), r < 0$

Discrete-Time Complex Exponential

Real exponential signals: $x[n] = C\alpha^n$, C and α are real.

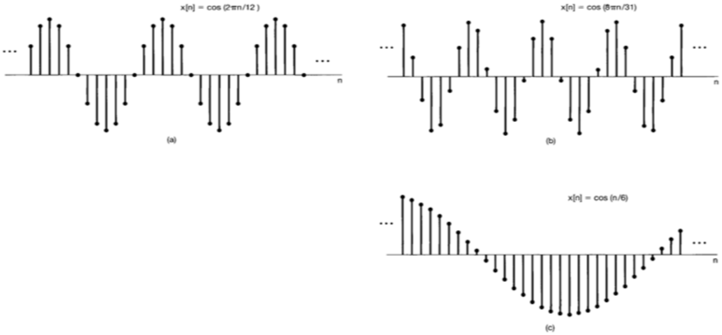


(a) $\alpha > 1$, (b) $0 < a < 1$, (c) $-1 < a < 0$, (d) $a < -1$

Discrete-Time Sinusoidal Signals

Sinusoidal signal: $x[n] = A\cos(\omega_0 n + \phi)$

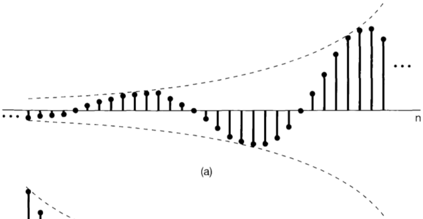
As $e^{j\omega_0 n} = \cos(\omega_0 n) + j\sin(\omega_0 n)$, we have
$A\cos(\omega_0 n + \phi) = \frac{A}{2}e^{j\phi}e^{j\omega_0 n} + \frac{A}{2}e^{-j\phi}e^{-j\omega_0 n}$



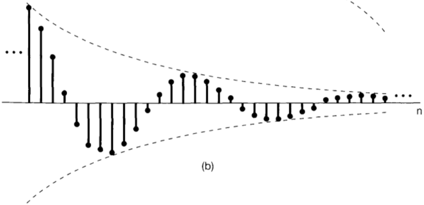
Discrete-Time General Complex Exponential

$$C = |C|e^{j\theta}, \alpha = |\alpha|e^{j\omega_0}$$

$$C\alpha^n = |C||\alpha|^n \cos(\omega_0 n + \theta) + j|C||\alpha|^n \sin(\omega_0 n + \theta)$$



(a)



(b)

Periodic Properties of Discrete-time Complex Exponential

For continuous-time complex exponential $x(t) = e^{j\omega_0 t}$

(1) the larger the ω_0, the higher the rate of oscillation

(2) $e^{j\omega_0 t}$ is periodic for any value of ω_0

Are the above two statements still valid for the discrete case $x[n] = e^{j\omega_0 n}$?

(1) oscillation: $e^{j(\omega_0+2\pi)n} = e^{j2\pi n}e^{j\omega_0 n} = e^{j\omega_0 n}$

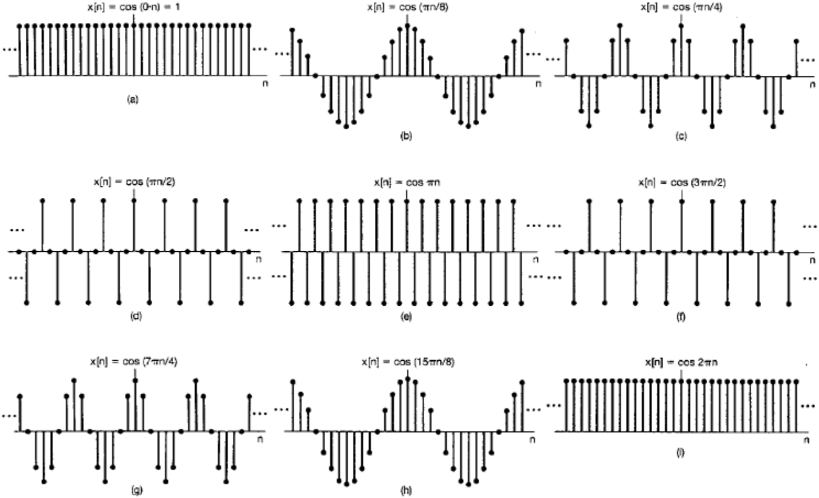


Figure 1.27 Discrete-time sinusoidal sequences for several different frequencies.

Periodic Properties of Discrete-time Complex Exponential

(1) For ω_0 within an interval $0 \leq \omega_0 \leq 2\pi$,
the frequency $\uparrow$ as $\omega_0 \uparrow$ for $0 \leq \omega_0 \leq \pi$,
the frequency $\downarrow$ as $\omega_0 \uparrow$ for $\pi \leq \omega_0 \leq 2\pi$.

(2) $e^{j\omega_0 n}$ might be non-periodic:

$$e^{j\omega_0(n+N)} = e^{j\omega_0 n}$$

implies $e^{j\omega_0 N} = 1$, hence $\omega_0 N$ needs to be a multiple of 2π:

$$\frac{\omega_0}{2\pi} = \frac{m}{N} \quad \text{is rational.}$$

Fundamental frequency of the periodic signal $e^{j\omega_0 n}$ is $\frac{2\pi}{N} = \frac{\omega_0}{m}$
Fundamental period is $N = m\left(\frac{2\pi}{\omega_0}\right)$

Example - Fundamental Period

What is the fundamental period of the discrete-time signal

$$x[n] = e^{j(2\pi/3)n} + e^{j(3\pi/4)n}$$

The fundamental period of the first term is 3;

The fundamental period of the second term is 8;

The fundamental period of the entire signal is 24.

Comparison of Signals $e^{j\omega_0 t}$ and $e^{j\omega_0 n}$

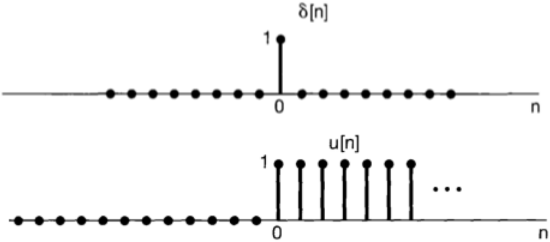
$e^{j\omega_0 t}$	$e^{j\omega_0 n}$
Distinct signals for distinct values of ω_0	Identical signals for values of ω_0 separated by multiples of 2π
Periodic for any choice of ω_0	Periodic only if $\omega_0 = 2\pi m/N$ for some integers $N > 0$ and m.
Fundamental frequency ω_0	Fundamental frequency* ω_0/m
Fundamental period $\omega_0 = 0$: undefined $\omega_0 \neq 0$: $\frac{2\pi}{\omega_0}$	Fundamental period* $\omega_0 = 0$: undefined $\omega_0 \neq 0$: $m\left(\frac{2\pi}{\omega_0}\right)$

*Assumes that m and N do not have any factors in common.

Discrete Time Unit Step and Unit Impulse Sequence

Unit Impulse function $\delta[n]$, and Unit Step function $u[n]$:

$$\delta[n] = \begin{cases} 0, & n \neq 0 \\ 1, & n = 0 \end{cases}, \qquad u[n] = \begin{cases} 0, & n < 0 \\ 1, & n \geq 0 \end{cases}$$



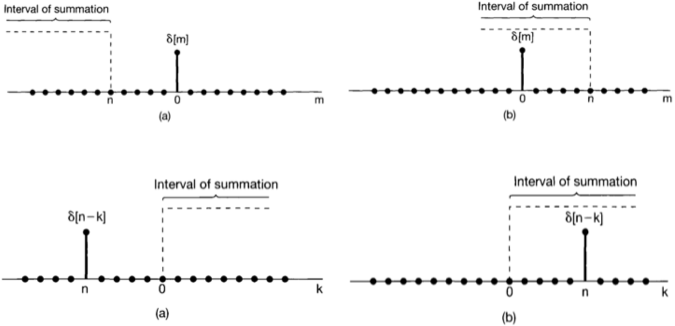
Note: $u[n]$ at $n = 0$ is defined.

Discrete Time Unit Step and Unit Impulse Sequence cont.

Unit impulse: first difference of unit step $\delta[n] = u[n] - u[n-1]$

Unit step: running sum of unit impulse

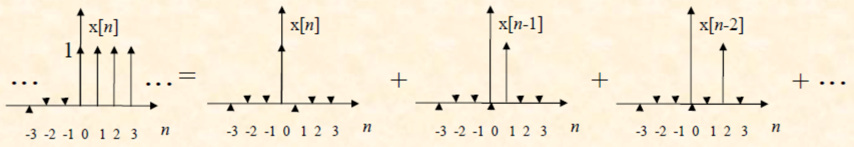
$$u[n] = \sum_{m=-\infty}^{n} \delta[m] = \sum_{k=0}^{+\infty} \delta[n-k]$$



Discrete Time Unit Step and Unit Impulse Sequence cont.

Running sum of unit sample: superposition of delayed impulses

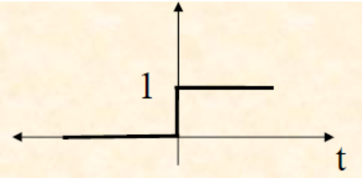
$$u[n] = \sum_{k=0}^{+\infty} \delta[n-k]$$

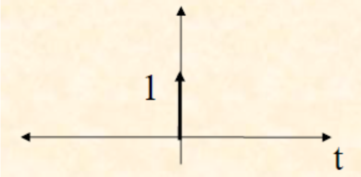


Unit Step and Unit Impulse Function

Unit Step function $u(t)$, and Unit Impulse function $\delta(t)$:

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases}, \qquad \delta(t) = \frac{d}{dt}u(t).$$



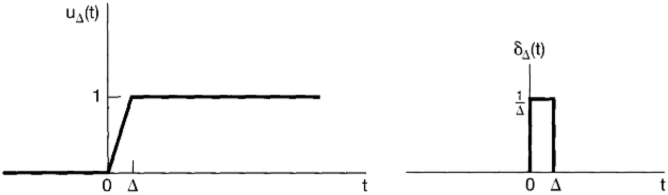


$$\text{Running integral} : u(t) = \int_{-\infty}^{t} \delta(\tau)d\tau.$$

A system can be characterized by its unit step response or unit impulse response.

Unit Step and Unit Impulse Function cont.

Unit step is discontinuous at $t = 0$.
Unit step can be approximated as $u_\Delta(t)$

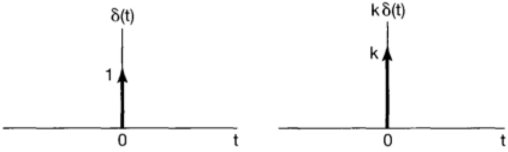


$$\delta_\Delta(t) = \frac{\mathrm{d}u_\Delta(t)}{\mathrm{d}t}, \qquad \delta(t) = \lim_{\Delta \to 0} \delta_\Delta(t)$$

Note that $\delta_\Delta(t)$ is a short pulse, with duration Δ and with unit area for any value of Δ.

Unit Step and Unit Impulse Function cont.

$\delta(t)$ has no duration but unit area.
The arrow at $t = 0$ indicates that the area of the pulse is concentrated at $t = 0$.
The height of the arrow and the "1" next to the arrow are used to represent the area of the impulse.



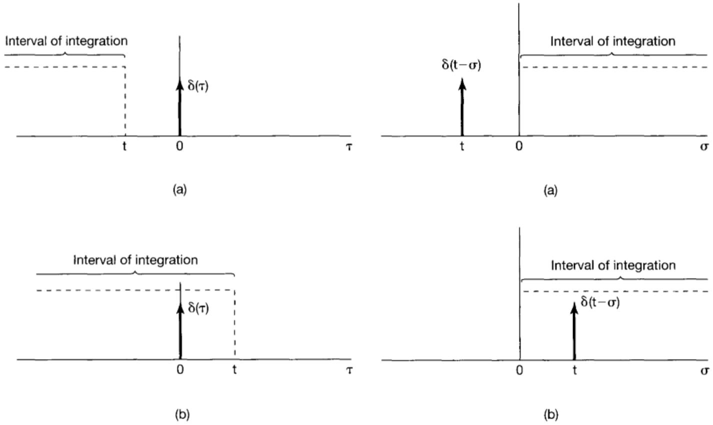
$k\delta(t)$ has an area of k

$$\int_{-\infty}^{t} k\delta(\tau)\mathrm{d}\tau = ku(t)$$

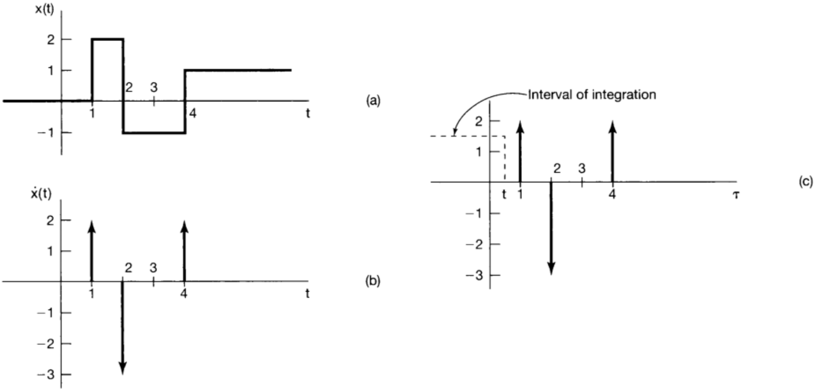
.

Unit Step and Unit Impulse Function cont.

Running integral $u(t) = \int_{-\infty}^{t} \delta(\tau)\mathrm{d}\tau = \int_{0}^{\infty} \delta(t-\sigma)\mathrm{d}\sigma$

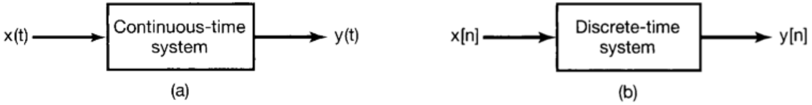


Unit Step and Unit Impulse Function cont.



Overview of System

System: any process that results in the transformation of signal

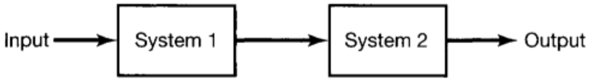


A system is continuous-time if both input and output are continuous-time.

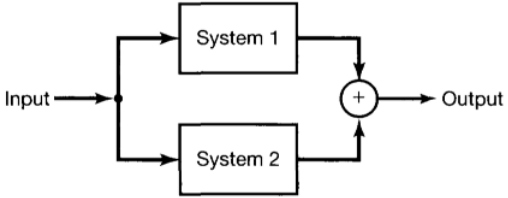
A system is discrete-time if both input and output are discrete-time.

Interconnection of Systems

(1) Cascade (Series): the output of System 1 is the input of System 2.

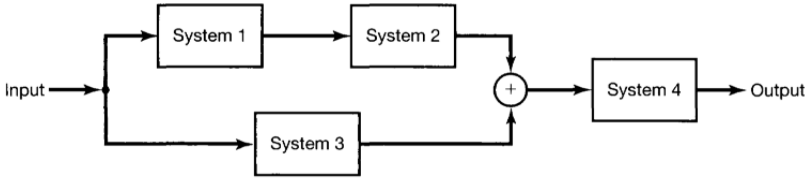


(2) Parallel: the same input is applied to Systems 1 and 2; the final output is the sum of the outputs of Systems 1 and 2.

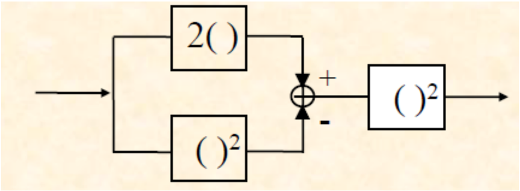


Interconnection of Systems cont.

(3) Series/Parallel

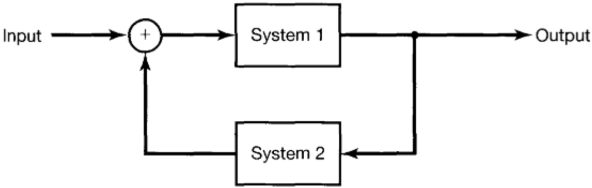


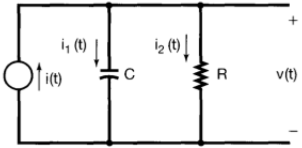
Ex. $y[n] = (2x[n] - x[n]^2)^2$

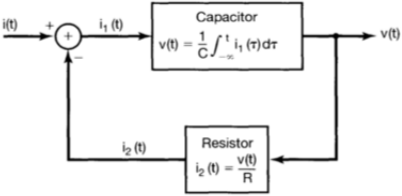


Interconnection of Systems cont.

(4) Feedback: Output of System 1 is the input to System 2; Output of System 2 is fed back and added to the external input to produce the actual input to System 1.







Properties of System

Properties of System

1. Memory/Memoryless
2. Invertibility and Inverse System
3. Causality
4. Stability
5. Time-invariance
6. Linearity

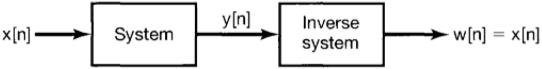
(1). Memory and Memoryless

A system is memoryless if the output only depends on input at the same time.

Ex. (1) $y[n] = \big(2x[n] - x^2[n]\big)^2$.
(2) Resister is a memoryless component: $v(t) = Ri(t)$.
(3) With memory: e.g. $y[n] = \sum_{k=-\infty}^{n} x[k]$.
(4) Delay system: $y[n] = x[n-1]$.
(5) Capacitor: $y(t) = \frac{1}{C}\int_{-\infty}^{t} x(\tau)\mathrm{d}\tau$.

(2). Invertibility and Inverse System

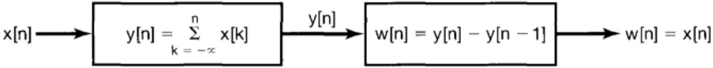
A system is invertible if distinct inputs lead to distinct outputs.



If $w[n] = x[n]$, then system 2 is the inverse system of system 1.

E.g. $y(t) = 2x(t)$, then $w(t) = 0.5y(t)$

$y[n] = \sum_{k=-\infty}^{n} x[k]$, then $w[n] = y[n] - y[n-1]$



(3). Causality

A system is causal if the output at any time only depends on the input at the present time and before.

E.g. $y[n] = x[n] - x[n-1]$: causal
$y(t) = x(t+1)$: non-causal

Note: All memoryless are causal

Causal property is more important for real-time processing.

But for some applications, such as image-processing, no need to process the data causally.

$$y[n] = \frac{1}{2M+1} \sum_{k=-M}^{M} x[n-k].$$

(3). Causality cont.

Examples

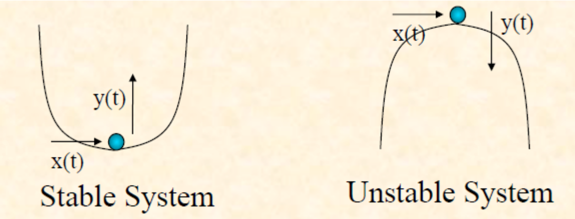
1. $y[n] = x[-n]$
2. $y(t) = x(t)\cos(t+1)$

(4). Stability

A system is stable if bounded input gives bounded output.

BIBO stable

E.g. $x(t)$: the horizontal force; $y(t)$: vertical displacement



(4). Stability cont.

Examples

1. $y(t) = tx(t)$
2. $y(t) = e^{x(t)}$

(4). Stability cont.

$y(t) = \frac{d}{dt}x(t)$ is not stable

$$\text{Let } x(t) = \begin{cases} \sqrt{t}, & t \in [0,1] \\ -\sqrt{-t}, & t \in [-1,0] \\ \text{bounded and with proper derivatives}, & \text{otherwise} \end{cases}$$

Then $y(t) \to \infty$ as $t \to 0$.

(5). Time-Invariance

A system is time-invariant if a time shift in the input only causes a time shift in the output.

i.e. If $x[n] \to y[n]$, then $x[n-n_0] \to y[n-n_0]$

Ex. $y(t) = \sin(x(t))$

Let $y_1(t) = sin(x_1(t))$, $x_2(t) = x_1(t-t_0)$
Then $y_2(t) = sin(x_2(t)) = sin(x_1(t-t_0)) = y_1(t-t_0)$

Hence time-invariant (T.I.)

(5). Time-invariance cont.

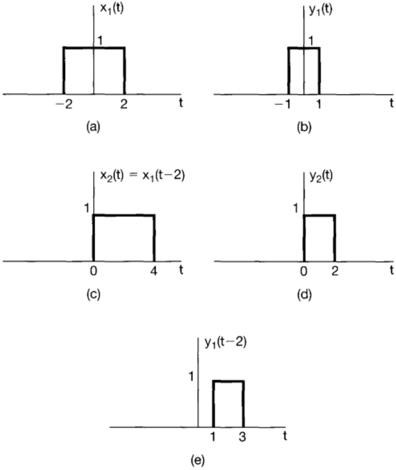
Ex. $y[n] = nx[n]$

Let $y_1[n] = nx_1[n]$, $x_2[n] = x_1[n - n_0]$
Then $y_2[n] = nx_2[n] = nx_1[n - n_0]$
However $y_1[n - n_0] = (n - n_0)x_1[n - n_0] \neq y_2[n]$

Hence, $y[n]$ is not time-invariant (T.I.)

(5). Time-invariance cont.

Consider the system $y(t) = x(2t)$



(6). Linearity

A system is linear if

1. the response to $x_1(t) + x_2(t)$ is $y_1(t) + y_2(t)$

— additivity

2. the response to $a \cdot x_1(t)$ is $a \cdot y_1(t)$, where a is any complex constant.

— scaling

Combine the above two properties, we can conclude

$$ax_1(t) + bx_2(t) \implies ay_1(t) + by_2(t)$$

— superposition property

For discrete-time: $ax_1[n] + bx_2[n] \implies ay_1[n] + by_2[n]$

(6). Linearity cont.

If linear, zero input gives zero output.

Q: Is $y[n] = 2x[n] + 3$ linear?
A: No, because it violates zero-in zero-out property.

However, this system is an "incremental linear system": difference of output is a linear function of difference of input.

$$y_1[n] - y_2[n] = 2x_1[n] + 3 - (2x_2[n] + 3) = 2(x_1[n] - x_2[n])$$

Exercise

(1). $y[n] = x^2[n]$
non-linear; time-invariant

(2). $y[n] = nx[n]$
memoryless; causal; linear; not time-invariant; not stable

(3). $y(t) = x(\sin(t))$

(4). $y(t) = \frac{d}{dt}x(t)$

(5). $y(t) = \int_{-\infty}^{2t} x(\tau)d\tau$

(6). $y(t) = x(t-2) + x(2-t)$

Exercise cont.

(7). $y(t) = x(t)\cos(8t)$

(8). $y[n] = x[3n]$

(9). $y(t) = x(\frac{t}{3})$

(10) $y[n] = x[n-2] - 2x[n-8]$

(11) $y[n] = x[4n+1]$

Summary

1. Continuous-time and discrete-time signals
2. Transformation of independent variables
3. Complex exponential and sinusoidal signals
4. Unit impulse and unit step functions
5. Interconnection of systems