Numerical Optimization, 2022 Fall
Homework 3 Solution

1 Pivot

Show that if the vectors $\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_m$ are a basis in E^m, the vectors $\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_{p-1}, \boldsymbol{a}_q, \boldsymbol{a}_{p+1}, \ldots, \boldsymbol{a}_m$ also are a basis if and only if $\bar{a}_{pq} \neq 0$, where $\bar{a}_{pq}$ is defined by the tableau shown in Table 1. [10pts]

x_1	x_2	x_3	$\cdots$	x_m	x_{m+1}	x_{m+2}	$\cdots$	x_n	
1	0	0	$\cdots$	0	$\bar{a}_{1(m+1)}$	$\bar{a}_{1(m+2)}$	$\cdots$	$\bar{a}_{1n}$	$\bar{a}_{10}$
0	1	0	$\cdots$	0	$\bar{a}_{2(m+1)}$	$\bar{a}_{2(m+2)}$	$\cdots$	$\bar{a}_{2n}$	$\bar{a}_{20}$
0	0	1	$\cdots$	0	$\bar{a}_{3(m+1)}$	$\bar{a}_{3(m+2)}$	$\cdots$	$\bar{a}_{3n}$	$\bar{a}_{30}$
.	.	.		.	.	.		.	.
.	.	.		.	.	.		.	.
.	.	.		.	.	.		.	.
0	0	0	$\cdots$	1	$\bar{a}_{m(m+1)}$	$\bar{a}_{m(m+2)}$	$\cdots$	$\bar{a}_{mn}$	$\bar{a}_{m0}$

表 1: Tableau

This proof is equivalent to proving the fact that the linear independence condition holds if and only if $\bar{a}_{pq} \neq 0$.

1. If $\bar{a}_{pq} \neq 0$, the vectors $\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_{p-1}, \boldsymbol{a}_q, \boldsymbol{a}_{p+1}, \ldots, \boldsymbol{a}_m$ are linearly independent:
   - The proof is provided on the lecture slides (Lecture 3, Page 19)
2. If the vectors $\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_{p-1}, \boldsymbol{a}_q, \boldsymbol{a}_{p+1}, \ldots, \boldsymbol{a}_m$ are linearly independent, $\bar{a}_{pq} \neq 0$:
   - This can be proved by proving its contrapositive, i.e. if $\bar{a}_{pq} = 0$, the vectors $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_q, \ldots, \boldsymbol{a}_m$ are linearly dependent. The proof can be derived from the following equation

$$
\begin{aligned}
\boldsymbol{a}_q = \boldsymbol{B}\boldsymbol{y}_q &= \sum_{i=1}^{m} \bar{a}_{iq}\boldsymbol{a}_i \\
&= \bar{a}_{1q}\boldsymbol{a}_1 + \cdots + \bar{a}_{(p-1)q}\boldsymbol{a}_{p-1} + \boxed{0 \cdot \boldsymbol{a}_q} + \bar{a}_{(p+1)q}\boldsymbol{a}_{p+1} + \cdots + \bar{a}_{mq}\boldsymbol{a}_m
\end{aligned} \tag{1}
$$

2 Reduced Cost

If $r_j > 0$ for every j corresponding to a variable x_j that is not basic, show that the corresponding basic feasible solution is the unique optimal solution. [10pts]
We first show that the BFS is optimal and then show its uniqueness.

1. Because $r_j \geq 0$ for every j corresponding to a nonbasic variable, the corresponding basic feasible solution is an optimal solution by the optimal criterion listed on lecture slides (Lecture 3, Page 11).
2. Since $r_j > 0$, change the basis will strictly increase the objective function value. Therefore, the optimal solution is unique.

3 Two-Phase Simplex

Use the two-phase simplex procedure to solve the following problem [10pts]

$$
\begin{aligned}
\min \quad & -3x_1 + x_2 + 3x_3 - x_4 \\
\text{s.t.} \quad & x_1 + 2x_2 - x_3 + x_4 = 0 \\
& 2x_1 - 2x_2 + 3x_3 + 3x_4 = 9 \\
& x_1 - x_2 + 2x_3 - x_4 = 6 \\
& x_i \geq 0, \quad i = 1,2,3,4.
\end{aligned} \tag{2}
$$

For phase I, we add artificial variables x_5, x_6, x_7 and change the objective function to $x_5 + x_6 + x_7$. The resulting tableau is

1	2	-1	1	1	0	0	0
2	-2	3	3	0	1	0	9
1	-1	2	-1	0	0	1	6
0	0	0	0	1	1	1	0

This does not represent a basic feasible solution yet because we need zeros under the basic columns (we need to three 1's in the bottom row to be 0). Using elementary row operations, we can eliminate the unwanted 1's. Notice that this is identical to pivoting on the 1's in the artificial columns. The resulting basic feasible tableau is:

$\boxed{1}$	2	-1	1	1	0	0	0
2	-2	3	3	0	1	0	9
1	-1	2	-1	0	0	1	6

$$
\begin{array}{ccccccc|c}
\hline
-4 & 1 & -4 & -3 & 0 & 0 & 0 & -15
\end{array}
$$

Using the simplex method, we have

$$
\begin{array}{ccccccc|c}
1 & 2 & -1 & 1 & 1 & 0 & 0 & 0 \\
0 & -6 & \boxed{5} & 1 & -2 & 1 & 0 & 9 \\
0 & -3 & 3 & -2 & -1 & 0 & 1 & 6 \\
\hline
0 & 9 & -8 & 1 & 4 & 0 & 0 & -15
\end{array}
$$

$$\downarrow$$

$$
\begin{array}{ccccccc|c}
1 & 4/5 & 0 & 6/5 & 3/5 & 1/5 & 0 & 9/5 \\
0 & -6/5 & 1 & 1/5 & -2/5 & 1/5 & 0 & 9/5 \\
0 & \boxed{3/5} & 0 & -13/5 & 1/5 & -3/5 & 1 & 3/5 \\
\hline
0 & -3/5 & 0 & 13/5 & 4/5 & 8/5 & 0 & -3/5
\end{array}
$$

$$\downarrow$$

$$
\begin{array}{ccccccc|c}
1 & 0 & 0 & 14/3 & 1/3 & 1 & -4/3 & 1 \\
0 & 0 & 1 & -5 & 0 & -1 & 2 & 3 \\
0 & 1 & 0 & -13/3 & 1/3 & -1 & 5/3 & 1 \\
\hline
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0
\end{array}
$$

which shows the final phase I tableau.

For phase II, we replace the last row with the true objective function. Zero out the elements under the basic variables (same as pivoting on elements $(1,1)$; $(3,2)$ and $(2,3)$). The resulting tableau is (without artificial variables)

$$
\begin{array}{cccc|c}
1 & 0 & 0 & 14/3 & 1 \\
0 & 0 & 1 & -5 & 3 \\
0 & 1 & 0 & -13/3 & 1 \\
\hline
-3 & 1 & 3 & -1 & 0
\end{array}
$$

$$\downarrow$$

$$
\begin{array}{cccc|c}
1 & 0 & 0 & 14/3 & 1 \\
0 & 0 & 1 & -5 & 3 \\
0 & 1 & 0 & -13/3 & 1 \\
\hline
0 & 0 & 0 & 97/3 & -7
\end{array}
$$

Since the last row is nonnegative, we are done. The solution is $(x_1, x_2, x_3, x_4) = (1, 1, 3, 0)$. Value of objective function is $-(-7) = 7$.