6. (a) $Y = X \cdot H = \frac{1}{2+j\omega} \cdot \frac{1}{(4+j\omega)^2}$

Assume $Y = \frac{A}{2+j\omega} + \frac{B}{4+j\omega} + \frac{C}{(4+j\omega)^2}$

法1: $= \frac{A(4+j\omega)^2 + B(2+j\omega)(4+j\omega) + C(2+j\omega)}{(2+j\omega)(4+j\omega)^2}$

$$\begin{cases} -A-B=0 \\ 8A+6B+C=0 \\ 16A+8B+2C=1 \end{cases} \longrightarrow \begin{cases} A=\frac{1}{4} \\ B=-\frac{1}{4} \\ C=-\frac{1}{2} \end{cases}$$

$$Y = \frac{\frac{1}{4}}{2+j\omega} + \frac{-\frac{1}{4}}{4+j\omega} + \frac{-\frac{1}{2}}{(4+j\omega)^2}$$

$$y = \frac{1}{4} e^{-2t} u(t) - \frac{1}{4} e^{-4t} u(t) - \frac{1}{2} t e^{-4t} u(t)$$

法2:

$$(2+j\omega)(4+j\omega)^2 Y = 1 = (4+j\omega)^2 A + (2+j\omega)(4+j\omega) B + (2+j\omega) C$$

令 $j\omega = -2$: $1 = 4A \Rightarrow A = \frac{1}{4}$

令 $j\omega = -4$: $1 = -2C \Rightarrow C = -\frac{1}{2}$

$$4^2 \cdot \left(\frac{1}{4}\right) + 8B + 2\left(-\frac{1}{2}\right) = 1$$

$$\Rightarrow B = -\frac{1}{4}$$

(b)

$$Y = X \cdot H = \frac{1}{(2+jw)^2} \cdot \frac{1}{(4+jw)^2} = \frac{A}{2+jw} + \frac{B}{(2+jw)^2} + \frac{C}{4+jw} + \frac{D}{(4+jw)^2}$$

$$= \frac{jw^3(-A-C) + w^2(-10A-B-8C-D) + jw(32A+20C+8B+4D) + (32A+16B+16C+4D)}{(2+jw)^2\,(4+jw)^2}$$

$$\begin{cases} -A-C=0 \\ -10A-B-8C-D=0 \\ 32A+8B+20C+4D=0 \\ 32A+16B+16C+4D=1 \end{cases} \Rightarrow \begin{cases} A=-\frac{1}{4} \\ B=\frac{1}{4} \\ C=\frac{1}{4} \\ D=\frac{1}{4} \end{cases}$$

$$y = -\frac{1}{4}e^{-2t}u(t) + \frac{1}{4}te^{-2t}u(t) + \frac{1}{4}e^{-4t}u(t) + \frac{1}{4}te^{-4t}u(t)$$

(c) $X = \frac{1}{1+jw}$ $\quad h(t) = x(-t) \rightarrow H = \frac{1}{1-jw}$

$$Y = \frac{1}{1+jw} \cdot \frac{1}{1-jw} = \frac{A}{1+jw} + \frac{B}{1-jw}$$

$$= \frac{(A+B) + jw(-A+B)}{(1+jw)(1-jw)} \Rightarrow \begin{cases} A=\frac{1}{2} \\ B=\frac{1}{2} \end{cases}$$

$$y = \frac{1}{2}e^{-t}u(t) + \frac{1}{2}e^{t}u(-t)$$

2. (a) $e^{-at}u(t) \longrightarrow \frac{1}{a+jw}$

So $\frac{1}{7+jw} \rightarrow e^{-7t}u(t)$

(b) $X = 2[\delta(w+7) + \delta(w-7)]$

$$x = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(w)e^{jwt}\,dw$$

$$= \frac{1}{2\pi} \cdot 2(e^{-j7t} + e^{j7t})$$

$$= \frac{1}{\pi}(e^{-j7t} + e^{j7t}) = \frac{2}{\pi}\cos 7t$$

(c) $e^{-a|t|} \longrightarrow \frac{2a}{a^2+w^2}$,

So $e^{-3|t|} \longrightarrow \frac{6}{w^2+9}$

$\frac{1}{6}e^{-3|t|} \longrightarrow \frac{1}{w^2+9}$ $\quad x = \frac{1}{6}e^{-3|t|}$

5. [4 points] Consider a causal LTI system with frequency response

$$H(\omega) = \frac{1}{j\omega + 3}$$

Derive the input $x(t)$ so that the output of the system is

$$y(t) = e^{-3t}u(t) - e^{-4t}u(t)$$

$$y(t) \overset{FT}{\longleftrightarrow} Y(w) = \frac{1}{jw+3} - \frac{1}{jw+4} = \frac{1}{(jw+3)(jw+4)}$$

$$H(w) = \frac{Y(w)}{X(w)}$$

$$\Rightarrow X(w) = \frac{Y(w)}{H(w)} = \frac{\frac{1}{(jw+3)(jw+4)}}{\frac{1}{jw+3}} = \frac{1}{jw+4}$$

$$x(t) = F^{-1}\{X(w)\} = e^{-4t}u(t)$$

7. [5 points] Suppose $g(t) = x(t)cos(t)$ and the Fourier transform of $g(t)$ is

$$G(\omega) = \begin{cases} 1, & |\omega| \leq 2 \\ 0, & otherwise \end{cases}$$

Determine $x(t)$.

I $t \neq 0$

$$g(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(w) \cdot e^{jwt} \, dw$$

$$= \frac{1}{2\pi} \int_{-2}^{2} e^{jwt} \, dw$$

$$= \frac{\sin 2t}{\pi t}$$

$$x(t) = \frac{g(t)}{\cos(t)} = \frac{2 \sin t}{\pi t}$$

II $t = 0$

$$g(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(w) \cdot e^{jw0} \, dw$$

$$= \frac{1}{2\pi} \int_{-2}^{2} dw$$

$$= \frac{2}{\pi}$$

$$x(0) = \frac{g(0)}{\cos(0)} = \frac{2}{\pi}$$

$$x(t) = \begin{cases} \frac{2\sin(t)}{\pi t}, & t \neq 0 \\ \frac{2}{\pi}, & t = 0 \end{cases}$$

Definition: $$\text{Sinc}(t) = \begin{cases} \frac{\sin(\pi t)}{\pi t}, & t \neq 0 \\ 1, & t = 0 \end{cases}$$

3. [20 points]

(a) Let $x(t) = e^{-at}u(t)$, Using the linearity and scaling properties, derive the Fourier transform $e^{-a|t|} = x(t) + x(-t)$.

(b) Using part (a) and the duality property, determine the Fourier transform of $x(t) = \frac{1}{1+t^2}$

(c) If

$$r(t) = \frac{1}{1+(3t)^2}$$

find $R(\omega)$

(d) $x(t)$ is sketched in Figure 2, if $y(t) = x(\frac{t}{2})$, sketch $y(t), Y(\omega), X(\omega)$.

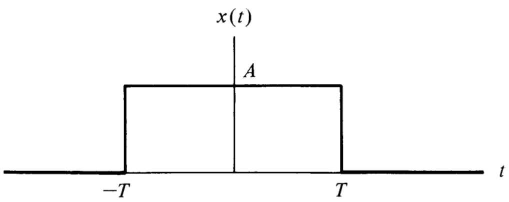


Figure 2: Problem 3.d

a)
$$e^{-a|t|} = e^{-at}u(t) + e^{at}u(-t)$$

$$= x(t) + x(-t)$$

$$\mathcal{F}\{x(t)\} = \int_{-\infty}^{\infty} x(t) e^{-j\omega t}\, dt$$

$$= \int_{0}^{\infty} e^{-at} e^{-j\omega t}\, dt$$

$$= \int_{0}^{\infty} e^{-(a+j\omega)t}\, dt$$

$$= \frac{1}{a+j\omega}$$

$$\mathcal{F}\{e^{-a|t|}\}$$

$$= \frac{1}{a+j\omega} + \frac{1}{a-j\omega}$$

$$= \frac{2a}{a^2+\omega^2}$$

b) Duality :

$$g(t) \xleftrightarrow{CTFT} G(j\omega) \Rightarrow G(t) \xleftrightarrow{CTFT} 2\pi g(-j\omega)$$

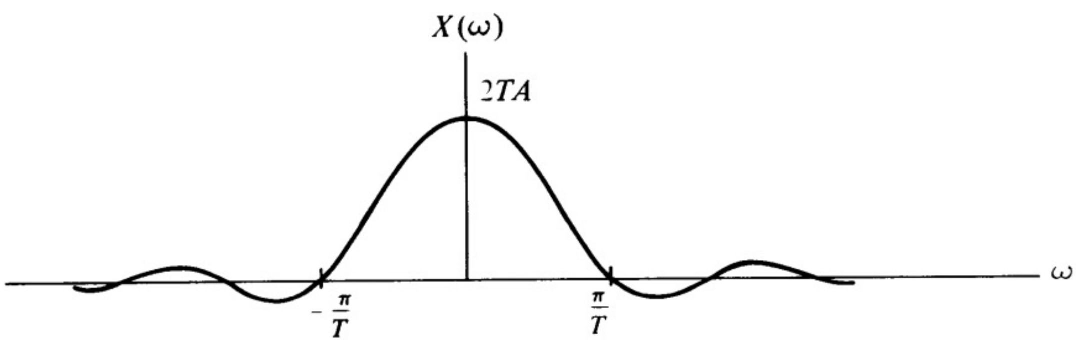
$$e^{-a|t|} \xleftrightarrow{\mathcal{F}} \frac{2a}{a^2+\omega^2}$$

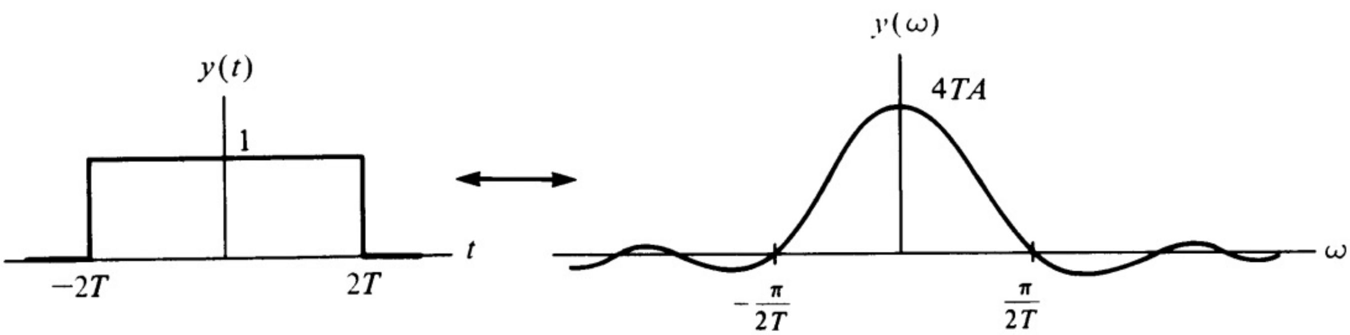
$$\Rightarrow \frac{1}{1+t^2} \xleftrightarrow{\mathcal{F}} \frac{1}{2}\cdot 2\pi e^{-1\cdot|j\omega|}$$

$$= \pi \cdot e^{-|\omega|}$$

c) $$\frac{1}{1+(3t)^2} \xleftrightarrow{\mathcal{F}} \frac{1}{3}\pi e^{-\left|\frac{\omega}{3}\right|}$$

d) $$X(j\omega) = A\int_{-T}^{T} e^{-j\omega t}\,dt = A\cdot\frac{2\sin(\omega T)}{\omega T}\,T$$





8.

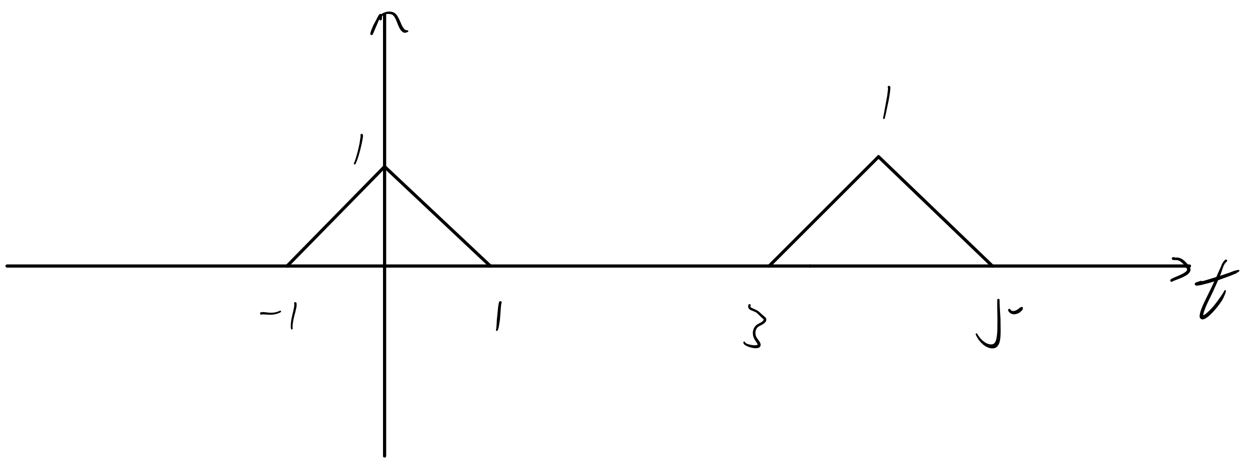
(a) $$X(jw) = \int_{-1}^{0} (t+1) e^{-jwt}\, dt + \int_{0}^{1} (-t+1) e^{-jwt}\, dt$$

$$= \frac{e^{jw}(jw-1)+1}{w^2} - \frac{1-e^{jw}}{jw} - \frac{jwe^{-jw}-1+e^{-jw}}{w^2} + \frac{1-e^{-jw}}{jw}$$

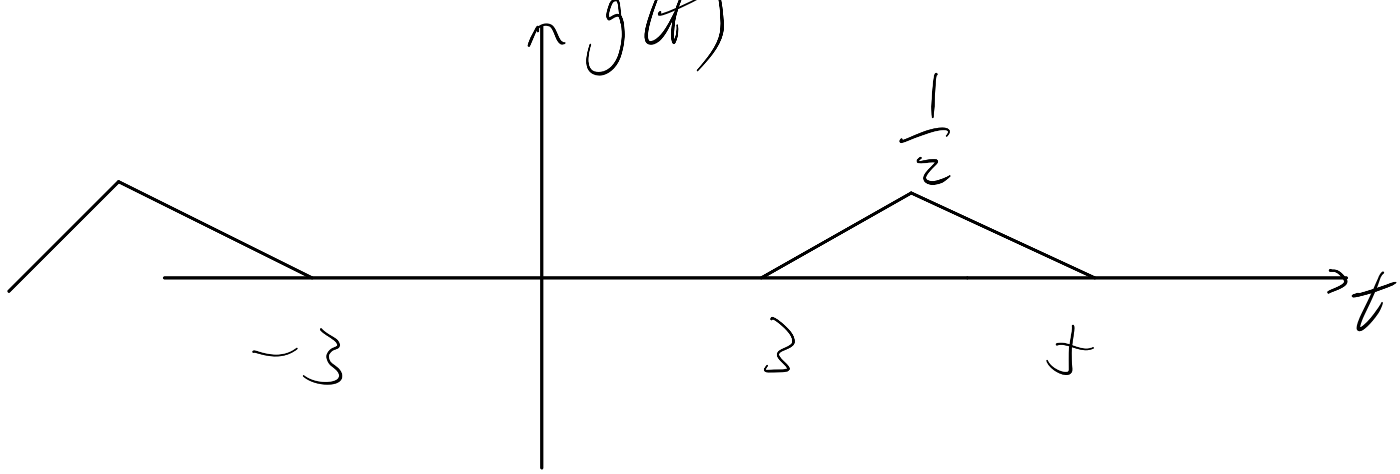
$$= \frac{4}{w^2} \sin^2\left(\frac{w}{2}\right)$$

(b) $$\tilde{x}(t) = \int_{-\infty}^{\infty} x(t-\tau) \cdot \sum_{k=-\infty}^{\infty} \delta(\tau - 4k)\, d\tau$$

$$= \sum_{k=-\infty}^{\infty} x(t-4k)$$



(c) $\tilde{x}(t) = \int_{-\infty}^{\infty} g(t-\tau) \cdot \sum_{k=-\infty}^{\infty} \delta(\tau - 4k)\, d\tau = \sum_{k=-\infty}^{\infty} g(t-4k)$



(d) $\sum_{n=-\infty}^{\infty} \delta(t - nT) \xleftrightarrow{FT} \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta\left(w - \frac{2\pi k}{T}\right)$

$\therefore \tilde{X}(jw) = X(jw) \cdot \frac{\pi}{2} \sum_{k=-\infty}^{\infty} \delta\left[w - \frac{\pi}{2}k\right]$

$= G(jw) \cdot \frac{\pi}{2} \sum_{k=-\infty}^{\infty} \delta\left[w - \frac{\pi}{2}k\right]$

1. [15 points] Find the Fourier transform of each of the following signals, derive and sketch their magnitude and phase as a function of frequency, both positive and negative frequency required.

(a)$\delta(t-5)$
(b)$e^{-at}u(t)$, a real and positive
(c)$e^{(-1+j2)t}u(t)$

Solution:

(a) $X(\omega)=\int_{-\infty}^{\infty}x(t)e^{-j\omega t}dt=\int_{-\infty}^{\infty}\delta(t-5)e^{-j\omega t}\,dt=e^{-j5\omega}=\cos 5\omega-j\sin 5\omega$,(当然也可用平移原理把 1 进行平移来得到。)

i.e.$\delta(t)\leftrightarrow 1$.

By the shifting property of the unit impulse.

$X(\omega)=1\cdot e^{-j5\omega}$.

$|X(\omega)|=|e^{-5\omega}|=1\quad$ for all ω,

$\angle X(\omega)=\tan^{-1}\left[\frac{\mathrm{Im}\{X(\omega)\}}{\mathrm{Re}\{X(\omega)\}}\right]=\tan^{-1}\left(\frac{-\sin 5\omega}{\cos 5\omega}\right)=-5$

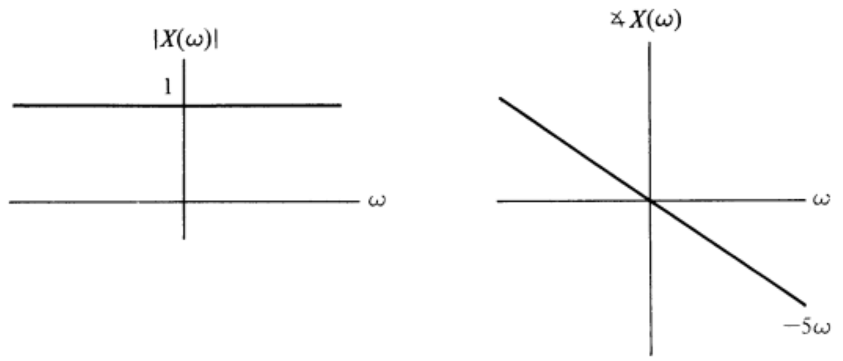


Figure S8.2-1

(b) $X(\omega)=\int_{-\infty}^{\infty}x(t)e^{-j\omega t}dt=\int_{0}^{\infty}e^{-at}e^{-j\omega t}dt=\int_{0}^{\infty}e^{-(a+j\omega)t}dt=\frac{-1}{a+j\omega}e^{-(a+j\omega)t}\Big|_{0}^{\infty}=\frac{1}{a+j\omega}$

(或者直接采用变换表进行)

$$|X(\omega)|=[X(\omega)X^*(\omega)]^{1/2}=\left[\frac{1}{a+j\omega}\left(\frac{1}{a-j\omega}\right)\right]^{1/2}\frac{1}{\sqrt{a^2+\omega^2}},$$

$$Re\{X(\omega)\}=\frac{X(\omega)+X^*(\omega)}{2}=\frac{a}{a^2+\omega^2},$$

$$Im\{X(\omega)\}=\frac{X(\omega)-X^*(\omega)}{2}=\frac{-\omega}{a^2+\omega^2},$$

$$\measuredangle X(\omega)=\tan^{-1}\left[\frac{Im\{X(\omega)\}}{Re\{X(\omega)\}}\right]=-\tan^{-1}\frac{\omega}{a}$$

The magnitude and angle of $X(\omega)$ are shown in Figure S8.2-2.

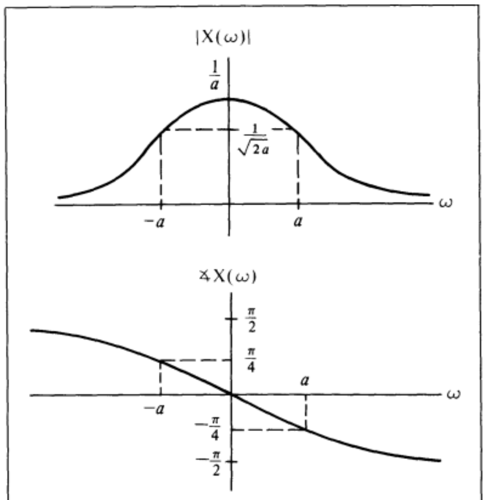


Figure S8.2-2

(c)

(c) $X(\omega) = \int_{-\infty}^{\infty} e^{(-1+j2)t}u(t)e^{-j\omega t}\,dt = \int_0^{\infty} e^{(-1+j2)t}e^{-j\omega t}\,dt$

$$= \frac{1}{-1+j(2-\omega)} e^{[-1+j(2-\omega)]t}\Big|_0^{\infty}$$

Since $Re\{-1+j(2-\omega)\} < 0$, $\lim_{t\to\infty} e^{[-1+j(2-\omega)]t} = 0$. Therefore,

$$X(\omega) = \frac{1}{1+j(\omega-2)}$$

$$|X(\omega)| = [X(\omega)X^*(\omega)]^{1/2} = \frac{1}{\sqrt{1+(\omega-2)^2}}$$

$$Re\{X(\omega)\} = \frac{X(\omega)+X^*(\omega)}{2} = \frac{1}{1+(\omega-2)^2}$$

$$Im\{X(\omega)\} = \frac{X(\omega)-X^*(\omega)}{2} \frac{-(\omega-2)}{1+(\omega-2)^2}$$

$$\sphericalangle X(\omega) = \tan^{-1}\left[\frac{Im\{X(\omega)\}}{Re\{X(\omega)\}}\right] = -\tan^{-1}(\omega-2)$$

The magnitude and angle of $X(\omega)$ are shown in Figure S8.2-3.

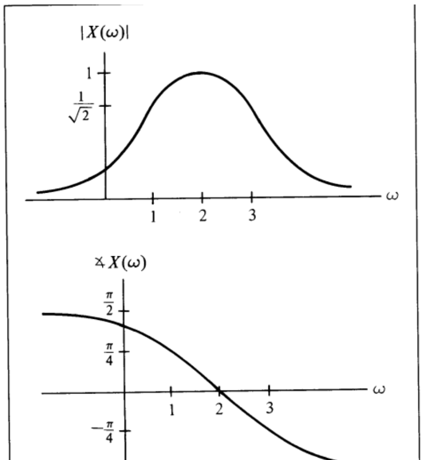


Figure S8.2-3

Note that there is no symmetry about $\omega = 0$ since $x(t)$ is not real.

4. [6 points] If $X(\omega)$ is the Fourier transform of $x(t)$, prove the following equations.

(a) $x(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)d\omega$

(b) $X(0) = \int_{-\infty}^{\infty} x(t)dt$

Solution:

(a) $x(t) = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\, d\omega$

Substituting $t = 0$ in the preceding equation, we get

$$x(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, d\omega$$

(b) $X(\omega) = \displaystyle\int_{-\infty}^{\infty} x(t)e^{-j\omega t}\, dt$

Substituting $\omega = 0$ in the preceding equation, we get

$$X(0) = \int_{-\infty}^{\infty} x(t)\, dt$$