Online Lecture Notes

Prof. Boris Houska

May 17, 2022

1 Fundamental Solutions of Linear Time-Varying Differential Equations

Our goal is to analyze the linear time-varying differential equation

$$\dot{x}(t) = A(t)x(t) + b(t) \quad \text{with} \quad x(t_0) = x_0$$

for time-varying coefficient functions $A : \mathbb{R} \to \mathbb{R}^{n_x \times n_x}$ and $b : \mathbb{R} \to \mathbb{R}^{n_x}$. The main difference to the time-invariant case is that, in general, we cannot find an explicit solution for $x(t)$.

1.1 Fundamental Solution

The main motivation for introduce a fundamental solution $G : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^{n_x}$ is to generalize the matrix exponential function. Here, the idea is to analyze the solution trajectories $x(t)$ in dependence on G, b, and x_0, where G only depends on A. The function G is defined by the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t} G(t,\tau) = A(t)G(t,\tau) \quad \text{and} \quad G(\tau,\tau) = I \; . \tag{1}$$

Due to the theorem of Picard-Lindelöf, the function G is well-defined by this differential equation. This means that existence and uniqueness of G is guaranteed, but we don't always have an explicit expression. Important properties of the function G are as follows:

1. The function G does not depend on b and x_0. It only depends on the time-varying function A.

2. The function G generalizes the matrix exponential in the sense that it shares many of its properties. We can compare the matrix exponential $X(t-\tau) = e^{A(t-\tau)}$ for a constant A with the function G for a time-varying A:

$X(t) = e^{At}$	$G(t,\tau)$ defined by (1)
$X(0) = I$	$G(\tau,\tau) = I$
$\dot{X}(t) = AX(t)$	$\frac{\mathrm{d}}{\mathrm{d}t} G(t,\tau) = A(t)G(t,\tau)$
$X(t_1+t_2) = X(t_1)X(t_2)$	$G(t_3,t_1) = G(t_3,t_2)G(t_2,t_1)$
$X(t)^{-1} = X(-t)$	$G(t,\tau)^{-1} = G(\tau,t)$

The properties in this table hold for all times $t, \tau \in \mathbb{R}$ and all $t_1, t_2, t_3 \in \mathbb{R}$. We will prove this below.

3. For the special $n_x = 1$ (scalar case) it is possible to find an explicit expression for G by using separation of variables:

$$\frac{\mathrm{d}}{\mathrm{d}t} G(t,\tau) = a(t) G(t,\tau) \qquad \Longrightarrow \qquad \frac{\frac{\mathrm{d}}{\mathrm{d}t} G(t,\tau)}{G(t,\tau)} = a(t)$$

$$\Longrightarrow \quad \log(G(t,\tau)) = \int_\tau^t a(\tau)\,\mathrm{d}t \qquad \Longrightarrow \qquad G(t,\tau) = \exp\left(\int_\tau^t a(\tau)\,\mathrm{d}t\right),$$

However, this formula cannot directly be generalized for the time-varying multivariate case. The separation of variables trick does not work in the matrix-valued case, since

$$\frac{\mathrm{d}}{\mathrm{d}t} G(t,\tau) G(t,\tau)^{-1} = A(t) \quad \text{does } \textbf{NOT} \text{ imply} \quad G(t,\tau)^{-1} \frac{\mathrm{d}}{\mathrm{d}t} G(t,\tau) = A(t) \ .$$

This means that we cannot simply integrate the expression on the left. Thus, the separation of variables in the matrix-valued case.

4. For the special that A is a constant matrix (time-invariant case), we have

$$G(t,\tau) = e^{A(t-\tau)} \ ,$$

this simply follows by checking that

$$\frac{\mathrm{d}}{\mathrm{d}t} G(t,\tau) = \frac{\mathrm{d}}{\mathrm{d}t} e^{A(t-\tau)} = A e^{A(t-\tau)} = A G(t,\tau) \quad \text{and} \quad G(\tau,\tau) = e^0 = I.$$

Again: recall that this expression cannot be generalized for the time-varying case, since we could have $A(t)A(t') \neq A(t')A(t)$.

5. The solution of the original differential equation for $x(t)$ is given by

$$x(t) = G(t,t_0)x_0 + \int_{t_0}^t G(t,\tau) b(\tau)\,\mathrm{d}\tau \ .$$

This can be proven by checking that

$$\begin{aligned}
\dot{x}(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\left[G(t,t_0)x_0 + \int_{t_0}^t G(t,\tau) b(\tau)\,\mathrm{d}\tau \right] \\
&= \frac{\mathrm{d}}{\mathrm{d}t} G(t,t_0)x_0 + G(t,t)b(t) + \int_{t_0}^t \frac{\mathrm{d}}{\mathrm{d}t} G(t,\tau) b(\tau)\,\mathrm{d}\tau \\
&= A(t)G(t,\tau)x_0 + b(t) + \int_{t_0}^t A(t) G(t,\tau) b(\tau)\,\mathrm{d}\tau \\
&= A(t)\left[G(t,t_0)x_0 + \int_{t_0}^t G(t,\tau) b(\tau)\,\mathrm{d}\tau \right] + b(t) \\
&= A(t)x(t) + b(t) \qquad (2)
\end{aligned}$$

and

$$x(t_0) = G(t_0,t_0)x_0 + \int_{t_0}^{t_0} G(t,\tau) b(\tau)\,\mathrm{d}\tau = I x_0 + 0 = x_0 \ . \qquad (3)$$

Let us additionally prove that we have

$$G(t_3,t_1) = G(t_3,t_2)G(t_2,t_1)$$

for all t_1, t_2, t_3 as claimed above. Here, the main idea is to consider the time-varying differential equations

$$\begin{aligned}
\forall t \in [t_1,t_2], \quad \dot{x}(t) &= A(t)x(t) \quad \text{with} \quad x(t_1) = x_0 && (4)\\
\forall t \in [t_2,t_3], \quad \dot{y}(t) &= A(t)y(t) \quad \text{with} \quad y(t_2) = x(t_2) && (5)\\
\forall t \in [t_1,t_3], \quad \dot{z}(t) &= A(t)z(t) \quad \text{with} \quad z(t_1) = x_0 && (6)
\end{aligned}$$

Notice that this construction is such that $y(t_3) = z(t_3)$, since the solutions of all of these linear differential equations are unique. Moreover, the solutions at the time point t_1, t_2, t_3 of the trajectories x, y, z are given by

$$\begin{aligned}
x(t_2) &= G(t_2,t_1)x_0 && (7)\\
y(t_3) &= G(t_3,t_2)x(t_2) && (8)\\
z(t_3) &= G(t_3,t_1)x_0 && (9)
\end{aligned}$$

By substituting the solution for $x(t_2)$ from the first equation into the second equation and using $y(t_3) = z(t_3)$, we obtain the equation

$$G(t_3,t_1)x_0 = z(t_3) = y(t_3) = G(t_3,t_2)x(t_2) = G(t_3,t_2)G(t_2,t_1)x_0 \ .$$

Since this equation holds for all initial values x_0 and since G does not depend on x_0, this yields that

$$G(t_3,t_1) = G(t_3,t_2)G(t_2,t_1) \ .$$

In particular, if we substitute $t_1 = t_3 = t$ and $t_2 = \tau$, we find that

$$I = G(t,t) = G(t,\tau)G(\tau,t) \ .$$

This implies that

$$G(t,\tau)^{-1} = G(\tau,t)$$

as claimed above, too. Additionally, notice that the adjoint differential equation for the differential for the fundamental solution is given by

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\tau}G(t,\tau) &= \frac{\mathrm{d}}{\mathrm{d}\tau}G(\tau,t)^{-1}\\
&= -G(\tau,t)^{-1}\left[\frac{\mathrm{d}}{\mathrm{d}\tau}G(\tau,t)\right]G(\tau,t)^{-1}\\
&= -G(\tau,t)^{-1}A(\tau)G(\tau,t)G(\tau,t)^{-1}\\
&= -G(t,\tau)A(\tau) \ . && (10)
\end{aligned}$$

This is called the adjoint fundamential differential equation. Notice that A is now multiplied from the right and also the minus is changing compared to the nominal differential equation for $G(t,\tau)$.

2 Periodic Orbits

Let us consider the linear time varying differential equation

$$\dot{x}(t) = A(t)x(t) + b(t)$$

for time-varying periodic coefficient functions $A : \mathbb{R} \to \mathbb{R}^{n_x \times n_x}$ and $b : \mathbb{R} \to \mathbb{R}^{n_x}$ with

$$\forall t \in \mathbb{R}, \qquad A(t+T) = A(t) \quad \text{and} \quad b(t+T) = b(t)$$

for a given period time $T > 0$. In this case, it is often possible to find periodic solution trajectory $x_\mathrm{p}(t)$, which satisfies the above differential equation and $x_\mathrm{p}(t+T) = x_\mathrm{p}(t)$. This solution trajectory can be found by solving the implicit equation

$$x_\mathrm{p}(t+T) = G(t+T,t)x_p(t) + \int_t^{t+T} G(T+t,\tau)b(\tau)\,\mathrm{d}\tau = x_\mathrm{p}(t)\ .$$

By resorting terms, this equation is equivalent to

$$\left[I - G(t+T,t)\right] x_p(t) = \int_t^{t+T} G(T+t,\tau)b(\tau)\,\mathrm{d}\tau\ .$$

Thus, if the matrix $I - G(t+T,t)$ is invertible, we can find the periodic solution trajectory

$$x_p(t) = \left[I - G(t+T,t)\right]^{-1} \int_t^{t+T} G(T+t,\tau)b(\tau)\,\mathrm{d}\tau\ .$$

Let us have a closer look at the matrix

$$I - G(t+T,t) \quad = \quad I - G(t+T,T)G(T,0)G(0,t) \tag{11}$$

Due to periodicity of A we have that $G(t+T,T) = G(t,0)$. Thus,

$$\begin{aligned}
I - G(t+T,t) \quad &= \quad I - G(t,0)G(T,0)G(0,t) &(12)\\
&= \quad G(t,0)G(0,t) - G(t,0)G(T,0)G(0,t) &(13)\\
&= \quad G(t,0)\left[I - G(T,0)\right]G(0,t) &(14)\\
&= \quad G(t,0)\left[I - G(T,0)\right]G(t,0)^{-1} &(15)
\end{aligned}$$

Thus, we see that $I - G(t+T,t)$ is invertible if and only if the matrix $I - G(T,0)$ is invertible. This makes the analysis a bit easier as now $G(T,0)$ is only depending on T but not on t.

In the next, we will see how the monodromy matrix can be used to analyze the limit behavior of periodic linear systems.