2023Fall Probability & Statistics for EECS 2023/10/18

Homework 3

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1. Please reinterpret the following story from the Bayesian perspective.

   狼来了：从前有个放羊娃，每天都把羊群带到山上去吃草, 山里有狼出没. 第一天，放羊娃觉得无聊，想要作弄山下耕作的村民. 他朝着山下大喊“狼来了!狼来了”，村民们信以为真，冲上山来准备帮助他，发现被欺骗了，大家很生气. 第二天，放羊娃故技重施，村民们虽然有点迟疑，但还是冲上山来准备打狼，结果又一次 发现被欺骗了，大家非常生气. 第三天，狼真的来了，此时放羊娃慌了，哭着向山下大喊“狼来了!狼来了!”，请 求村民的帮助. 但这一次村民们认为他又在撒谎，无人相信他. 最后他所有的羊都被狼吃掉了.

   

2. A fair die is rolled repeatedly, and a running total is kept (which is, at each time, the total of all the rolls up until that time). Let p_n be the probability that the running total is ever exactly n (assume the die will always be rolled enough times so that the running total will eventually exceed n, but it may or may not ever equal n).

   (a) Write down a recursive equation for p_n (relating p_n to earlier terms p_k in a simple way). Your equation should be true for all positive integers n, so give a definition of p_0 and p_k for $k < 0$ so that the recursive equation is true for small values of n.

   (b) Find p_7.

   (c) Give an intuitive explanation for the fact that $p_n \to 1/3.5 = 2/7$ as $n \to \infty$.

3. A sequence of $n \geq 1$ independent trials is performed, where each trial ends in "success" or "failure" (but not both). Let p_i be the probability of success in the i^{th} trial, $q_i = 1 - p_i$, and $b_i = q_i - 1/2$, for $i = 1, 2, \ldots, n$. Let A_n be the event that the number of successful trials is even.

   (a) Show that for $n = 2$, $P(A_2) = 1/2 + 2b_1 b_2$.

(b) Show by induction that

$$P(A_n) = 1/2 + 2^{n-1} b_1 b_2 \dots b_n$$

(This result is very useful in cryptography. Also, note that it implies that if n coins are flipped, then the probability of an even number of Heads is 1/2 if and only if at least one of the coins is fair.) *Hint*: Group some trials into a super-trial.

(c) Check directly that the result of (b) is true in the following simple cases: $p_i = 1/2$ for some i; $p_i = 0$ for all i; $p_i = 1$ for all i.

4. A message is sent over a noisy channel. The message is a sequence $x_1, x_2, \dots, x_n$ of n bits ($x_i \in \{0, 1\}$). Since the channel is noisy, there is a chance that any bit might be corrupted, resulting in an error (a 0 becomes a 1 or vice versa). Assume that the error events are independent. Let p be the probability that an individual bit has an error ($0 < p < 1/2$). Let $y_1, y_2, \dots, y_n$ be the received message (so $y_i = x_i$ if there is no error in that bit, but $y_i = 1 - x_i$ if there is an error there).

   To help detect errors, the nth bit is reserved for a parity check: x_n is defined to be 0 if $x_1 + x_2 + \dots + x_{n-1}$ is even, and 1 if $x_1 + x_2 + \dots + x_{n-1}$ is odd. When the message is received, the recipient checks whether y_n has the same parity as $y_1 + y_2 + \dots + y_{n-1}$. If the parity is wrong, the recipient knows that at least one error occurred; otherwise, the recipient assumes that there were no errors.

   (a) For $n = 5, p = 0.1$, what is the probability that the received message has errors which go undetected?

   (b) For general n and p, write down an expression (as a sum) for the probability that the received message has errors which go undetected.

   (c) Give a simplified expression, not involving a sum of a large number of terms, for the probability that the received message has errors which go undetected.

5. For x and y binary digits (0 or 1), let $x \bigoplus y$ be 0 if $x = y$ and 1 if $x \neq y$ (this operation is called exclusive or (often abbreviated to XOR), or addition mod 2).

   (a) Let $X \sim \text{Bern}(p)$ and $Y \sim \text{Bern}(1/2)$, independently. What is the distribution of $X \bigoplus Y$?

   (b) With notation as in sub-problem (a), is $X \bigoplus Y$ independent of X? Is $X \bigoplus Y$ independent of Y? Be sure to consider both the case $p = 1/2$ and the case $p \neq 1/2$.

   (c) Let $X_1, \dots, X_n$ be i.i.d. (*i.e.*, independent and identically distributed) Bern(1/2) R.V.s. For each nonempty subset J of $\{1, 2, \dots, n\}$, let

   $$Y_J = \bigoplus_{j \in J} X_j.$$

   Show that $Y_J \sim \text{Bern}(1/2)$ and that these $2^n - 1$ R.V.s are pairwise independent, but not independent.

6. (**Optional Challenging Problem**) By LOTP for problems with recursive structure, we generate many difference equations. To solve the difference equation in the form of

$$f_{i+1} = b \cdot f_i + a \cdot f_{i-1}, i \geq 1.$$

where a and b are constants, we turn to the so-called characteristic equation:

$$x^2 = bx + a.$$

If such equation has two distinct roots r_1 and r_2, then the general form of f_i is

$$f_i = c \cdot r_1^i + d \cdot r_2^i,$$

If there is only one distinct root r, then the general form of f_i is

$$f_i = c \cdot r^i + d \cdot i \cdot r^i.$$

Show the mathematical principle behind the method of characteristic equation.