EE150 Signals and Systems
– Part 4: Continuous-time Fourier Transform (CTFT)

Fourier Series and Fourier Transform

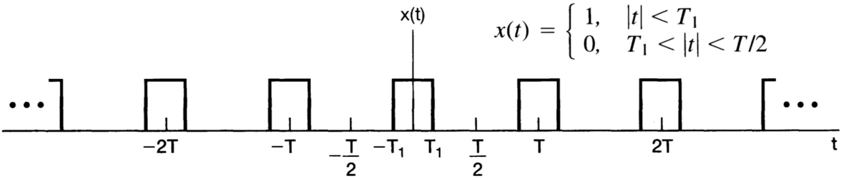
- Periodic signal: period T_0, fundamental frequency $\omega_0 = \frac{2\pi}{T_0}$

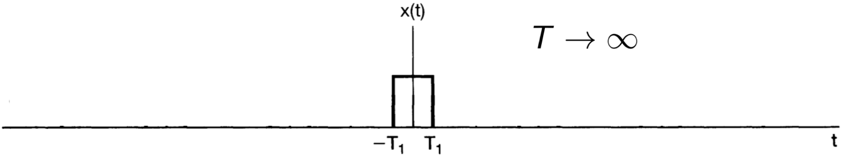
$$\text{Fourier series:} \qquad x(t) = \sum_{k=-\infty}^{\infty} a_k \mathrm{e}^{jk\omega_0 t}$$

- Aperiodic signal: periodic with $T_0 \to \infty$, fundamental frequency $\omega_0 \to 0$

$$\text{Inverse fourier transform:} \quad x(t) = \frac{1}{2\pi} \int_{\infty}^{\infty} X(j\omega) \mathrm{e}^{j\omega t} \mathrm{d}\omega$$

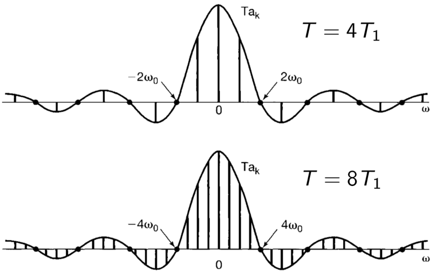
Graphical Illustration



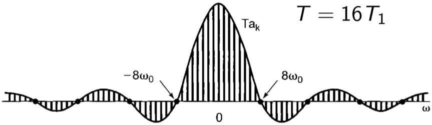


Graphical Illustration cont.

Consider ω as a continuous variable, $\frac{2\sin\omega T_1}{\omega}$ is the envelop of Ta_k

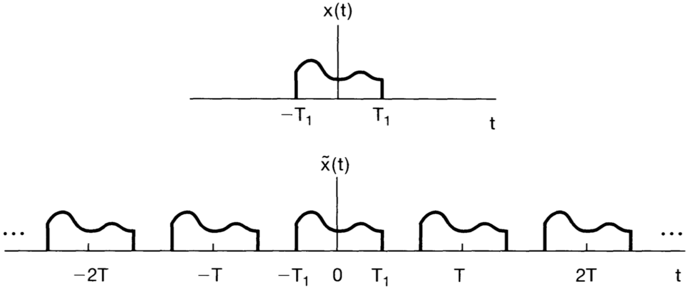


$$Ta_k = \left.\frac{2\sin\omega T_1}{\omega}\right|_{\omega = k\omega_0}$$



- $T \cdot a_k = 0$ if $k\omega_0 T_1 = m\pi, \forall m \in \mathbb{Z}$
- As T increases, the envelop is sampled with closer spacing
- As $T \to \infty$, $T \cdot a_k$ approaches the envelop

Fourier Series to Fourier Transform



$$\tilde{x}(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} \qquad a_k = \frac{1}{T}\int_{-T/2}^{T/2} \tilde{x}(t) e^{-jk\omega_0 t} dt$$

As $\tilde{x}(t) = x(t)$ for $|t| < T/2$, we have

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\mathrm{e}^{-jk\omega_0 t} = \frac{1}{T}\int_{-\infty}^{\infty} x(t)\mathrm{e}^{-jk\omega_0 t}$$

Fourier Series to Fourier Transform cont.

- Define the envelop of $T \cdot a_k$ as $X(j\omega) = \int_{-\infty}^{\infty} x(t)\mathrm{e}^{-j\omega t}\mathrm{d}t$
- As $a_k = \frac{X(jk\omega_0)}{T}$, then

$$
\begin{aligned}
\tilde{x}(t) &= \sum_{k=-\infty}^{\infty} \frac{1}{T} X(jk\omega_0)\mathrm{e}^{jk\omega_0 t} \\
&= \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} X(jk\omega_0)\mathrm{e}^{jk\omega_0 t}\omega_0
\end{aligned}
$$

- As $T \to \infty$, $\tilde{x}(t) \to x(t)$

Fourier Transform pair

- Fourier transform defines a bijection (one-to-one, invertible) mapping (via):

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \quad \text{(Fourier Transform)}$$
$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega \quad \text{(Inverse FT)}$$

- This is valid as long as $x(t)$ is well-behaved, e.g. Schwartz function (wiki: Schwartz class)

Remarks

- Eigenfunctions (LTI system): $e^{j\omega t}$ all ω
- Dot-product (Inner-product)

$$< x_t(t), x_2(t) >= \lim_{T_0 \to \infty} \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x_1(t) x_2^*(t) dt$$

- (Show) $e^{j\omega t}$ are orthonormal
- Aperiodic signals can also be represented as a linear combination of complex exponentials, which occurs at a continuum of frequencies and have amplitude of $X(j\omega)(\mathrm{d}\omega/2\pi)$

Fourier Transform of $x(t)$

- Consider LTI system with impulse response $x(t)$
  -Know: $e^{j\omega t}$ is an eigen function
- Fourier transform: $X(j\omega)$ is eigenvalue corresponding to $e^{j\omega t}$
- Therefore

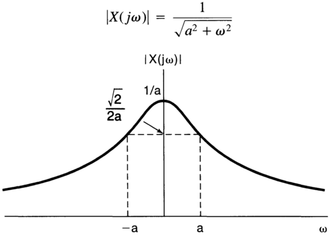
$$
\begin{aligned}
X(j\omega)e^{j\omega t} &= e^{j\omega t} * x(t) \\
&= \int_{-\infty}^{\infty} x(\tau)e^{j\omega(t-\tau)}d\tau \\
&= e^{j\omega t}\int_{-\infty}^{\infty} x(\tau)e^{-j\omega\tau}d\tau
\end{aligned}
$$

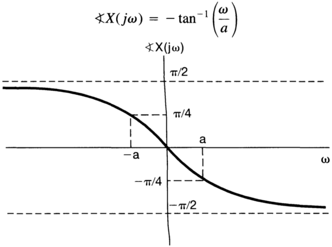
Example 4.1

Calculate the Fourier transform of signal $x(t) = \mathrm{e}^{-at}u(t), a > 0$

Solution:

$$X(j\omega) = \int_0^{\infty} \mathrm{e}^{-at}\mathrm{e}^{-j\omega t}\mathrm{d}t = \frac{1}{a + j\omega}, a > 0$$



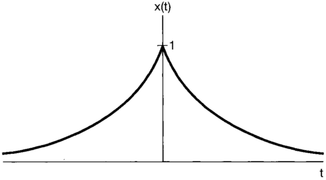


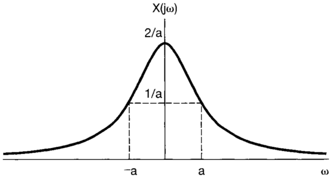
Example 4.2

Calculate the Fourier transform of signal $x(t) = \mathrm{e}^{-a|t|}$, $a > 0$

Solution:

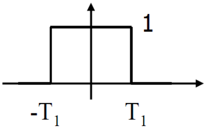
$$X(j\omega) = \int_{-\infty}^{\infty} \mathrm{e}^{-a|t|}\mathrm{e}^{-j\omega t}\mathrm{d}t = \int_{-\infty}^{0} \mathrm{e}^{at}\mathrm{e}^{-j\omega t}\mathrm{d}t + \int_{0}^{\infty} \mathrm{e}^{-at}\mathrm{e}^{-j\omega t}\mathrm{d}t$$
$$= \frac{1}{a - j\omega} + \frac{1}{a + j\omega} = \frac{2a}{a^2 + \omega^2}$$





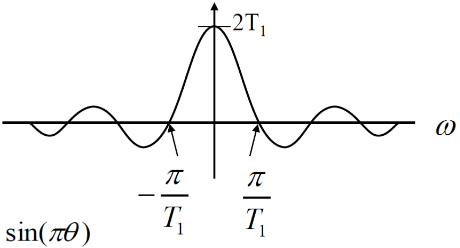
Square pulse and "sinc" function

Example (1). Square Pulse



$\xleftrightarrow{\mathrm{F}}$

$$X(j\omega) = \int_{-T_1}^{T_1} e^{-j\omega t}\,dt = \frac{2\sin\omega T_1}{\omega}$$

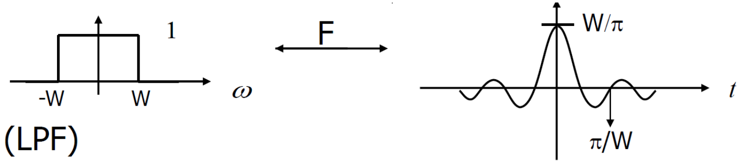


Define $\quad \mathrm{sinc}(\theta) \equiv \dfrac{\sin(\pi\theta)}{\pi\theta}$

Then $\quad X(j\omega) = 2T_1\,\mathrm{sinc}\left(\dfrac{\omega T_1}{\pi}\right)$ for square pulse.

Square pulse and "sinc" function

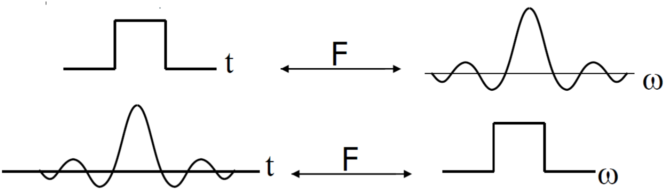
Example (2). Frequency-domain



$$x(t) = \frac{1}{2\pi}\int_{-W}^{W} e^{j\omega t}\,d\omega = \frac{\sin(Wt)}{\pi t} = \frac{W}{\pi} sinc\left(\frac{Wt}{\pi}\right)$$

Duality property of Fourier Transform

Note:

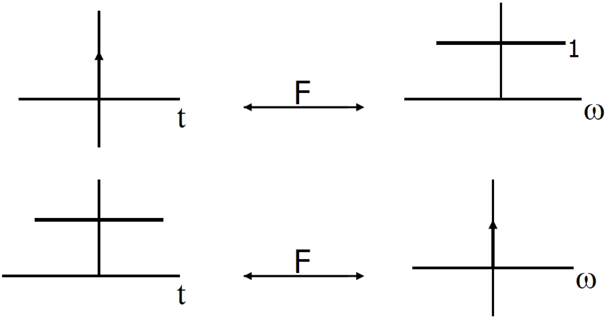


Duality property of Fourier Transform

Example (3).

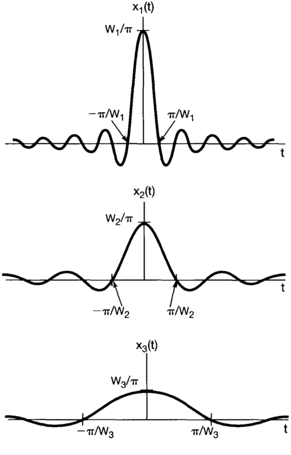
$$x(t) = \delta(t) \xleftrightarrow{FT} X(j\omega) = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt = 1$$

$$(Note : \int_{-\infty}^{\infty} \delta(t - t_0) \cdot f(t) dt = f(t_0))$$



$$\boxed{\delta(t) \xleftrightarrow{F} 1}$$

Remarks



x1(t)
W1/π
−π/W1
π/W1
t
X1(jω)
1
−W1
W1
ω
x2(t)
W2/π
−π/W2
π/W2
t
X2(jω)
1
−W2
W2
ω
x3(t)
W3/π
−π/W3
π/W3
t
X3(jω)
1
−W3
W3
ω

Fourier Transform for Periodic Signal

Fourier transform can be applied to periodic signal

Consider $x(t)$ and its FT, $X(j\omega)$.

Assume $X(j\omega) = 2\pi\delta(\omega - \omega_0)$. Find $x(t)$.

$$\begin{aligned} x(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi \cdot \delta(\omega - \omega_0) e^{j\omega t} d\omega \\ &= e^{j\omega_0 t} \end{aligned}$$

Fourier Transform for Periodic Signal

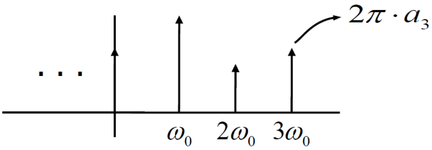
Now for more general case,

$$X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0) \rightarrow x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

exactly Fourier Series representation of a periodic signal.

$\Longrightarrow$ We can find the FT for a periodic signal by

$$x(t) \xleftrightarrow{FS} a_k \rightarrow X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi \cdot a_k \cdot \delta(\omega - k\omega_0)$$

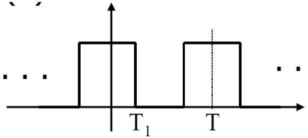


Fourier Transform for Periodic Signal

Note: If $x(t)$ is periodic with period T

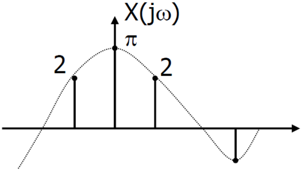
$\rightarrow X(j\omega)$ is discrete, with frequency spacing$= \omega_0 = \frac{2\pi}{T}$

e.g. (1)



$$\xleftrightarrow{\text{F.S.}} a_k = \frac{\sin(k\omega_0 T_1)}{\pi k}$$

$$X(j\omega) = \sum_{k=-\infty}^{\infty} 2 \cdot \frac{\sin(k\omega_0 T_1)}{k} \delta(\omega - k\omega_0)$$



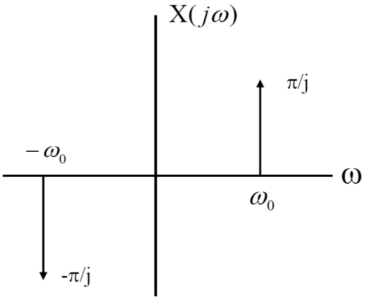
(for T_1=T/4 case)

Fourier Transform of $\sin(\omega_0 t)$

E.g. (2)

$$x(t) = \sin(\omega_0 t) \xleftrightarrow{FS} a_1 = \frac{1}{2j}, a_{-1} = \frac{1}{-2j}$$

$$\& \ a_k = 0 \ \textit{for all other } k$$

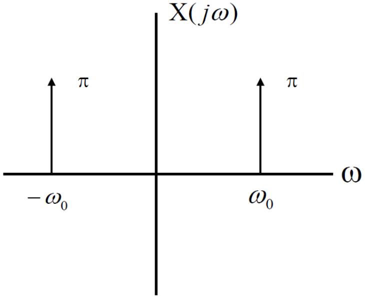


Fourier Transform of $\cos(\omega_0 t)$

E.g. (3)

$$x(t) = \cos(\omega_0 t) \xleftrightarrow{FS} a_1 = a_{-1} = \frac{1}{2}$$

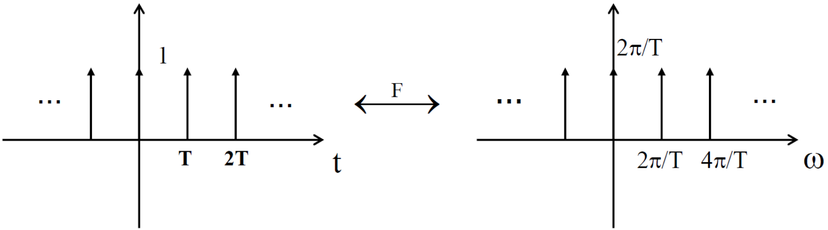
$$\& \ a_k = 0 \ \textit{for all other } k$$



E.g. (4)

$$x(t) = \sum_{-\infty}^{\infty} \delta(t - kT) \xleftrightarrow{FS} a_k = \frac{1}{T} \int_{-T/2}^{T/2} \delta(t) \cdot e^{-jk\omega_0 t} dt = \frac{1}{T}$$

$$\therefore X(j\omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\omega - \frac{2\pi k}{T})$$



Properties of Fourier Transform

Notation: $X(j\omega) = \mathcal{F}\{x(t)\}$ *or* $x(t) \xleftrightarrow{FT} X(j\omega)$

1. Linearity: $a \cdot x(t) + b \cdot y(t) \xleftrightarrow{FT} a \cdot X(j\omega) + b \cdot Y(j\omega)$
2. Time-shift: $x(t - t_0) \xleftrightarrow{FT} e^{-j\omega t_0} \cdot X(j\omega)$
3. Conjugation: $x^*(t) \xleftrightarrow{FT} X^*(-\omega)$

   Conjugate symmetry: if $x(t)$ is real $\rightarrow X(-j\omega) = X^*(j\omega)$

Example 4.9

- Express $x(t)$ as a linear combination of $x_1(t)$ and $x_2(t)$

$$x(t) = \frac{1}{2}x_1(t-2.5) + x_2(t-2.5)$$

- As derived in Ex. 4.4, $X_1(j\omega) = \frac{2\sin(\omega/2)}{\omega}$, $X_2(j\omega) = \frac{2\sin(3\omega/2)}{\omega}$
- Using linearity and time-shift properties

$$X(j\omega) = \mathrm{e}^{-j5\omega/2}\left[\frac{\sin(\omega/2) + 2\sin(3\omega/2)}{\omega}\right]$$

Properties of Fourier Transform

4. Differentiation & integration

$$\frac{dx(t)}{dt} = \frac{1}{2\pi}\int_{-\infty}^{\infty}(j\omega)X(j\omega)e^{j\omega t}d\omega$$

$$\therefore \frac{dx(t)}{dt} \xleftrightarrow{FT} j\omega \cdot X(j\omega)$$

$$\int_{-\infty}^{t} x(\tau)d\tau \xleftrightarrow{FT} \frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega)$$

Example 4.11

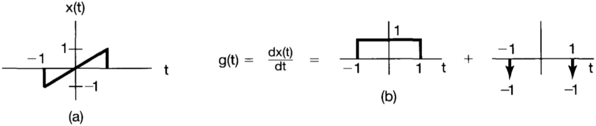
Derive the Fourier Transform of the unit step $x(t) = u(t)$

$$g(t) = \delta(t) \xleftrightarrow{FT} G(j\omega) = 1$$

Based on integration property and $x(t) = \int_{-\infty}^{t} g(\tau)\mathrm{d}\tau$

$$\begin{aligned} X(j\omega) &= \frac{G(j\omega)}{j\omega} + \pi G(0)\delta(\omega) \\ &= \frac{1}{j\omega} + \pi\delta(\omega) \end{aligned}$$

Example 4.12



$$G(j\omega) = \frac{2\sin\omega}{\omega} - e^{j\omega} - e^{-j\omega}$$

With $G(0) = 0$ and using Integration property

$$X(j\omega) = \frac{2\sin\omega}{j\omega^2} - \frac{2\cos\omega}{j\omega}$$

Properties of Fourier Transform

5. Time and Frequency Scaling

$$x(at) \xleftrightarrow{FT} \frac{1}{|a|} X(\frac{j\omega}{a})$$

$$x(-t) \xleftrightarrow{FT} X(-j\omega)$$

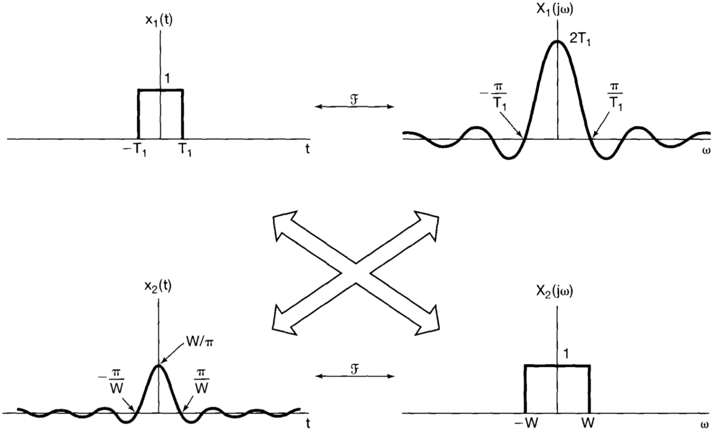
6. Duality

$$g(t) \xleftrightarrow{FT} G(j\omega) \implies G(t) \xleftrightarrow{FT} 2\pi \cdot g(-j\omega)$$

7. Parseval's Relation

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega$$

Properties of Fourier Transform



Properties of Fourier Transform

Proof of Parseval's Relation:

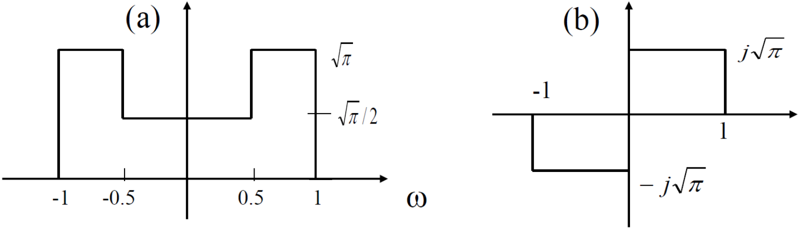
Proof.

$$\begin{aligned}
\int_{-\infty}^{\infty} |x(t)|^2 dt &= \int_{-\infty}^{\infty} x(t) \cdot x^*(t) dt \\
&= \int_{-\infty}^{\infty} x(t) \cdot \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(j\omega) e^{-j\omega t} d\omega \right] dt \\
\text{Change order:} \quad &= \frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(j\omega) \cdot \left[\int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \right] d\omega \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega
\end{aligned}$$

□

Ex. Find $E = \int_{-\infty}^{\infty} |x(t)|^2 dt$ and $D = \frac{dx(t)}{dt}|_{t=0}$

for the following two $X(j\omega)$



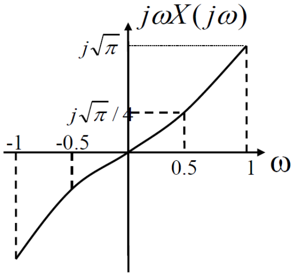
$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega = \begin{cases} \frac{5}{8} & for\ (a) \\ 1 & for\ (b) \end{cases}$$

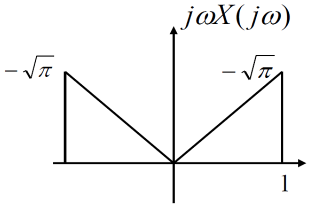
Properties of Fourier Transform

For D, remember $g(t) = \frac{d}{dt}x(t) \xleftrightarrow{FT} j\omega \cdot X(j\omega) = G(j\omega)$
Also note that

$$g(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(j\omega)d\omega = D$$

$$\Longrightarrow D = \frac{1}{2\pi}\int_{-\infty}^{\infty} j\omega X(j\omega)d\omega = \begin{cases} 0 & \text{for } (a) \\ -\frac{\sqrt{\pi}}{2\pi} & \text{for } (b) \end{cases}$$





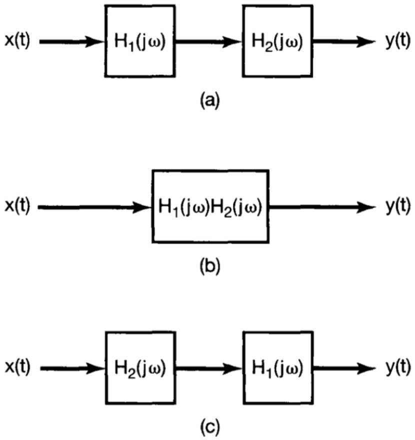
Convolution Property

$$y(t) = h(t) * x(t) \xleftrightarrow{FT} Y(j\omega) = H(j\omega) \cdot X(j\omega)$$

where $h(t)$ is system impulse response, $H(j\omega)$ is the frequency response

$$\begin{aligned}
Y(j\omega) &= \int_{-\infty}^{\infty} y(t)e^{-j\omega t}dt \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(\tau)h(t-\tau)e^{-j\omega t}d\tau dt \\
&= \int_{-\infty}^{\infty} x(\tau)e^{-j\omega\tau}\left(\int_{-\infty}^{\infty} h(t-\tau)e^{-j\omega(t-\tau)}dt\right)d\tau \\
&= \int_{-\infty}^{\infty} x(\tau)e^{-j\omega\tau}H(j\omega)d\tau \\
&= X(j\omega)H(j\omega)
\end{aligned}$$

Convolution Property



Utilization of Convolution Property

Ex. Assume $x(t) = \frac{\sin(\omega_i t)}{\pi t}$ is the input and is filtered by an ideal LPF with cut-off frequency ω_c. Find the output, $y(t)$

Ideal LPF: $h(t) = \frac{\sin(\omega_c t)}{\pi t}$

$$y(t) = h(t) * x(t) = \frac{\sin(\omega_c t)}{\pi t} * \frac{\sin(\omega_i)t}{\pi t} \implies \text{difficult to find}$$

On the other hand, $Y(j\omega) = H(j\omega) \cdot X(j\omega)$

Utilization of Convolution property

$$X(j\omega) = \begin{cases} 1 & |\omega| \leq \omega_i \\ 0 & \text{otherwise} \end{cases} \quad H(j\omega) = \begin{cases} 1 & |\omega| \leq \omega_c \\ 0 & \text{otherwise} \end{cases}$$

$$\rightarrow Y(j\omega) = \begin{cases} 1 & |\omega| \leq \omega_0 \\ 0 & elsewhere \end{cases}$$

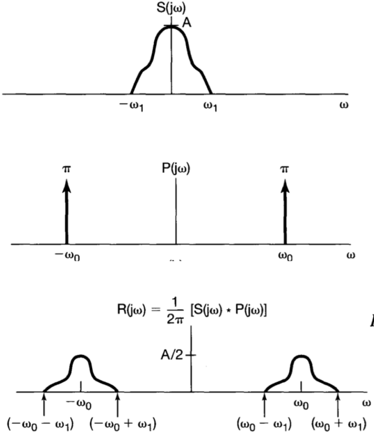
where ω_0 is the smaller one of ω_i and ω_c

$$\implies y(t) = \begin{cases} \dfrac{\sin(\omega_c t)}{\pi t} & if\ \omega_c \leq \omega_i \\ \dfrac{\sin(\omega_i t)}{\pi t} & if\ \omega_i \leq \omega_c \end{cases}$$

Multiplication

multiplication in time $\longleftrightarrow$ convolution in frequency

$$r(t) = s(t) \cdot p(t) \xleftrightarrow{FT} R(j\omega) = \frac{1}{2\pi}[S(j\omega) * P(j\omega)]$$



$$p(t) = \cos\omega_0 t.$$

$$P(j\omega) = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0),$$

$$r(t) = s(t)p(t)$$

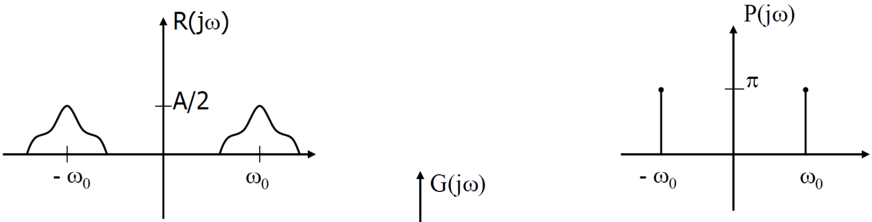
$$R(j\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} S(j\theta)P(j(\omega - \theta))d\theta$$
$$= \frac{1}{2}S(j(\omega - \omega_0)) + \frac{1}{2}S(j(\omega + \omega_0))$$

(This problem illustrates the "modulation process" that is discussed in Principle Comm.)

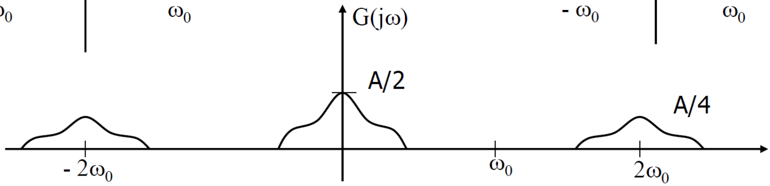
Multiplication

Ex: Assume $g(t) = r(t) \cdot p(t)$ where:

FT of $r(t)$ is: FT of $p(t) = \cos(\omega_0 t)$ is:



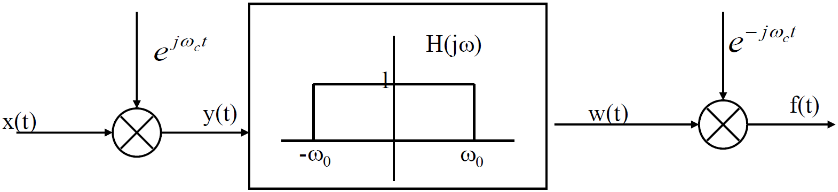
$\Rightarrow$



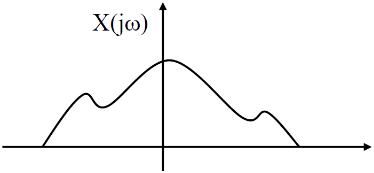
(This problem illustrates the "demodulation process" that is discussed in Principle Comm.)

Multiplication

Ex: (Frequency Selective Filtering with variable Central Frequency)

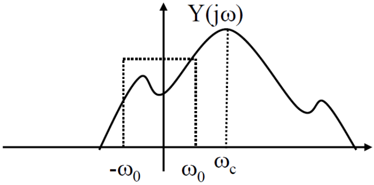


Assume

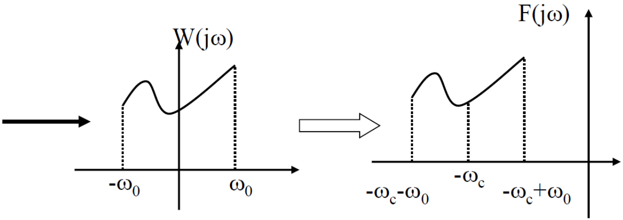


Multiplication

Then



$$Y(j\omega) = \delta(\omega - \omega_c) * X(j\omega)$$
$$= X(j(\omega - \omega_c))$$



$$F(j\omega) = W(j(\omega + \omega_c))$$

Properties of Fourier Transform

TABLE 4.1 PROPERTIES OF THE FOURIER TRANSFORM

Section	Property	Aperiodic signal	Fourier transform
		$x(t)$ $y(t)$	$X(j\omega)$ $Y(j\omega)$
4.3.1	Linearity	$ax(t) + by(t)$	$aX(j\omega) + bY(j\omega)$
4.3.2	Time Shifting	$x(t - t_0)$	$e^{-j\omega t_0} X(j\omega)$
4.3.6	Frequency Shifting	$e^{j\omega_0 t} x(t)$	$X(j(\omega - \omega_0))$
4.3.3	Conjugation	$x^*(t)$	$X^*(-j\omega)$
4.3.5	Time Reversal	$x(-t)$	$X(-j\omega)$
4.3.5	Time and Frequency Scaling	$x(at)$	$\frac{1}{\|a\|} X\left(\frac{j\omega}{a}\right)$
4.4	Convolution	$x(t) * y(t)$	$X(j\omega)Y(j\omega)$
4.5	Multiplication	$x(t)y(t)$	$\frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\theta)Y(j(\omega - \theta))d\theta$
4.3.4	Differentiation in Time	$\frac{d}{dt}x(t)$	$j\omega X(j\omega)$
4.3.4	Integration	$\int_{-\infty}^{t} x(t)dt$	$\frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega)$
4.3.6	Differentiation in Frequency	$tx(t)$	$j\frac{d}{d\omega}X(j\omega)$
4.3.3	Conjugate Symmetry for Real Signals	$x(t)$ real	$\begin{cases} X(j\omega) = X^*(-j\omega) \\ \mathcal{R}e\{X(j\omega)\} = \mathcal{R}e\{X(-j\omega)\} \\ \mathcal{I}m\{X(j\omega)\} = -\mathcal{I}m\{X(-j\omega)\} \\ \|X(j\omega)\| = \|X(-j\omega)\| \\ \sphericalangle X(j\omega) = -\sphericalangle X(-j\omega) \end{cases}$
4.3.3	Symmetry for Real and Even Signals	$x(t)$ real and even	$X(j\omega)$ real and even
4.3.3	Symmetry for Real and Odd Signals	$x(t)$ real and odd	$X(j\omega)$ purely imaginary and odd
4.3.3	Even-Odd Decomposition for Real Signals	$x_e(t) = \mathcal{E}v\{x(t)\}$ [$x(t)$ real] $x_o(t) = \mathcal{O}d\{x(t)\}$ [$x(t)$ real]	$\mathcal{R}e\{X(j\omega)\}$ $j\mathcal{I}m\{X(j\omega)\}$
4.3.7	Parseval's Relation for Aperiodic Signals	$\int_{-\infty}^{+\infty} \|x(t)\|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \|X(j\omega)\|^2 d\omega$	

Summary

- Developed Fourier transformation representation of continuous-time signals.
- Aperiodic signal as the limit of periodic signal with period $\rightarrow \infty$
- Derive FT from FS for periodic signal.
- Properties of C-T Fourier Transform.
- Basic FT pairs