Practice guide: The Furuta pendulum

INTRODUCTION

The Furuta pendulum is a rotational version of the linear inverted pendulum, that is, it is a device consisting of an inverted pendulum whose lower end pivots on a rotating base as shown in figure 1.

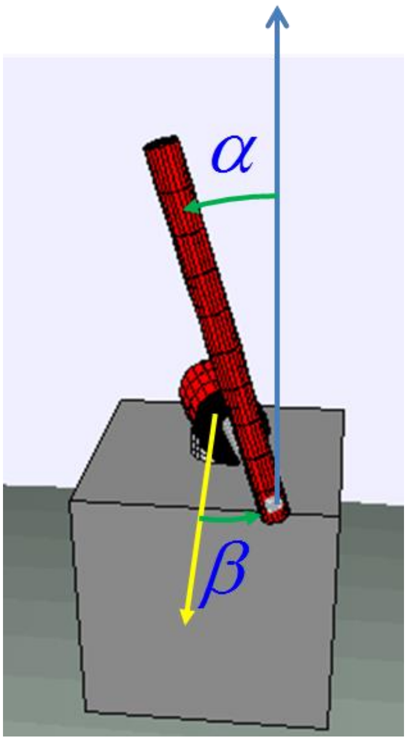

Fig. 1. Péndulo de Furuta.

The rotation of the base is actuated by means of an engine with an input voltage u that allows direct control of the angle of the base β . The Control of the angle of the base in turn establishes the position of the pivot of the pendulum, which allows to indirectly control the angle of the pendulum with respect to the vertical α. This device poses interesting control challenges since it exhibits a unstable behavior and "non-minimum phase" when it is controlled in a closed loop.

To carry out the practice, it is convenient to read carefully all the sections and carry out all proposed tasks. Some of these tasks, will be carried out theoretically, prior to the connection with the real system, most of them will be made in simulation mode, and many will be performed both in simulation and in remote mode. When finishing the experiences in simulation mode, you must send a report to the teacher with the experiences in which they must include the theoretical tasks. Finally, a final report will be prepared with all the results.

1. PRACTICE OBJETIVES

The objective of the practice is to analyze the behavior of the pendulum from the theoretical and simulation point of view and then develop non-linear control strategies for said pendulum. The development of the control will be carried out in an incremental way, following these steps:

A) Develop linear control law by state feedback capable of keeping the pendulum horizontally and upwards while the base of the same follows the changes made to the position reference.

B) Implement a control law capable of "lifting" the pendulum from the position from lower equilibrium to its position of superior equilibrium in order to subsequently apply control A.

The student must contrast the results obtained in simulation mode with the results obtained in remote mode.

2. SYSTEM MODEL

According to the diagram of figure 1 there are two degrees of freedom in the system α (rad) y β (rad) and a control action u (Volts). First, to model the physical system it would be necessary to express the kinetic and potential energies of the system in terms of the degrees of freedom α and β and their derivatives. Then, we calculate the Lagrangian system and the non-conservative generalized forces, which in our case, are the motor control torque τ and well rotational frictions that appear in the right terms on equations (1) and (2). The complete calculation and derivation of the Euler equations Lagrange are omitted for brevity. The result is shown below:

$$-\frac{m_p L_p L_r}{2}\cos(\alpha)\ddot{\alpha}+\left(m_p L_r^{2}-\frac{m_p L_p^{2}}{4}\left(\cos(\alpha)^{2}-1\right)+J_r\right)\ddot{\beta}+ \qquad (1)$$

$$+\frac{m_p L_p^{2}}{2}\cos(\alpha)\mathrm{sen}(\alpha)\dot{\alpha}\dot{\beta}+\frac{m_p L_p L_r}{2}\mathrm{sen}(\alpha)\dot{\alpha}^{2}+=\tau-B_r\dot{\beta}$$

$$\left(\frac{m_p L_p^{2}}{4}+J_p\right)\ddot{\alpha}-\frac{m_p L_p L_r}{2}\cos(\alpha)\ddot{\beta}-\frac{m_p L_p^{2}}{4}\cos(\alpha)\mathrm{sen}(\alpha)\dot{\beta}^{2}-\frac{m_p L_p}{2}g\,\mathrm{sen}(\alpha)=-B_p\dot{\alpha} \qquad (2)$$

Where m_p and L_p are the mass and length of the pendulum, L_r is the length of the movable arm from the base,, J_l and J_p are the moments of inertia of the base and the pendulum respectively and g is the gravitational acceleration. On the other hand, B_p y B_r are the coefficients of rotational friction of the pendulum and the base respectively.

The dynamics of a DC motor can be modeled through:

$$\tau=\frac{k_t\left(u-\dot{\beta}k_m\right)}{R} \qquad (3)$$

Where: R is the resistance of the motor coil, kt the constant that regulates the relationship between the torque and the primary current and km the constant that determines the voltage induced in the primary by its rotation speed with respect to the secondary.

It is easy to verify that this model corresponds to the general structure of an under-manipulated robot manipulator since it can be easily expressed as:

$$D(q)\ddot{q}+C(q,\dot{q})\dot{q}+G(q)=u \qquad (4)$$

Where $q=[\alpha,\beta]^T$ are the generalized coordinates, D is the matrix that represents the inertia of the system, C represents the coriolis and friction forces and G the effect of gravity on the structure.

Since the differential equations are of second order in α and β to put the model in the form of equations of the state space in the standard form $\dot{x} = f(x,u)$ the state is defined as follows:

$$x=\begin{bmatrix}\alpha\\ \beta\\ \dot{\alpha}\\ \dot{\beta}\end{bmatrix} \qquad (5)$$

If equation (3) is replaced in (2), the resulting equations are expressed in terms of the states and the derivative of the state is cleared in terms of the state We finally get:

$$\begin{aligned}
\frac{d\alpha}{dt}&=\dot{\alpha}\\
\frac{d\beta}{dt}&=\dot{\beta}\\
\frac{d\dot{\alpha}}{dt}&=\frac{K\mathrm{c}u+\left(a_1-a_2\mathrm{c}^2\right)\left(G\mathrm{s}-d_2\dot{\alpha}\right)-\left(d_1+2a_2\mathrm{cs}\dot{\alpha}\right)\mathrm{c}\dot{\beta}+a_2\left(a_1-a_2\mathrm{c}^2\right)\mathrm{cs}\dot{\beta}^2-\mathrm{cs}\dot{\alpha}^2}{a_1a_3-\left(a_2a_3+1\right)\mathrm{c}^2} \qquad (6)\\
\frac{d\dot{\beta}}{dt}&=\frac{Ka_3u+\mathrm{cs}G-d_2\mathrm{c}\dot{\alpha}-a_3\left(d_1+2a_2\mathrm{cs}\dot{\alpha}\right)\dot{\beta}+a_2\mathrm{c}^2\mathrm{s}\dot{\beta}^2-a_3\mathrm{s}\dot{\alpha}^2}{a_1a_3-\left(a_2a_3+1\right)\mathrm{c}^2}
\end{aligned}$$

Where, to abbreviate, c = cos (α), s = sin (α). The values of the numerical constants of the model are:

$$m_p=0.024\mathrm{Kg},\quad m_r=0.095\mathrm{Kg},\quad L_p=0.129\mathrm{m},\quad L_r=0.085\mathrm{m},$$
$$J_r=\mathrm{m_r L_r^2/12},\quad J_p=\mathrm{m_p L_p^2/12},\quad B_r=0.00105\ \mathrm{Nms/rad},\quad B_p=0.0000263\ \mathrm{Nms/rad},$$
$$k_t=0.042\mathrm{Nm/A},\quad k_m=0.042\mathrm{k_t/R_m},\quad R=8.4\ \Omega \quad y \quad g=9.81m/s^2.$$

So the new constants turn out to be the following:

a_1=$2L_r/L_p+2J_r/(m_pL_pL_r)+L_p/(2L_r)$=2.51

a_2=$L_p/(2L_r)$=0.76

a_3=$L_p/(2L_r)+2J_p/(m_pL_pL_r)$=1.01

$d_1=(2k_mk_t)/(R_mm_pL_pL_r)+(2B_r)/(m_pL_pL_r)=0.08\ s^{-1}rad^{-1}$ (7)

$d_2=2B_p/(m_pL_pL_r)=0.2\ s^{-1}rad^{-1}$

$K=2k_t/(R_mm_pL_pL_r)=38\ V^{-1}s^{-2}$

$G=g/L_r=115\ s^{-2}$

3. TASKS

Tarea 1: Analysis of the model.

1A) Check that the model in the state space is not singular since the The denominator is never canceled regardless of the physical parameters of the system. **Pista:** Calculate the extreme values (when $c^2=0$ y $c^2=1$) and replace the values of the constants so that the result is always positive.

1B) Once this is done, determine the equilibrium points by matching the derivatives from (6) to zero. Show that for any value of β you have two points of equilibrium (α = 0 and α = **π**).

1C) Linearize equation (8) around the upper equilibrium point α = β = 0 obtaining a linear model of the form:

$$\dot{x} = Ax + Bu \qquad (8)$$

1D) Demonstrate using the first method of Lyapunov (eigenvalues of the linear approximation) that equilibrium points with α = 0 are unstable.

Question 1: Why does not this method work for us at the lower equilibrium point?

1E) Check that if β is kept **constant** the equilibrium points with π=α are stable. To do this substitute in (6) the condition $\frac{d\beta}{dt}==\beta$. From the fourth equation clear uy substitute it in the third equation, in this way check that the resulting system of eliminating β is:

$$\frac{d\alpha}{dt} = \dot{\alpha}$$
$$\frac{d\dot{\alpha}}{dt} = \frac{G\sin(\alpha) - d_2\dot{\alpha}}{a_3}$$

Which is the equation of a simple pendulum with damping. Then, using LaSalle's theorem and Lyapunov's function: $V=(1-\cos(\alpha')) + c\alpha'^2$, where $\alpha' = \alpha - \pi$ and c is a y c constant that will have to be chosen properly for eliminate the terms of undefined sign, show that the equilibrium point π=α (α'=0) is stable.

Task 2: Design of the linear control:

In this task, a linear control must be designed by state feedback able to keep the pendulum in horizontal position and upwards while the base of it follows position reference changes β_r. To do this, define the error vector as follows:

$$e = x - x_r = \begin{bmatrix} \alpha - 0 \\ \beta - \beta_r \\ \dot{\alpha} - 0 \\ \dot{\beta} - 0 \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta - \beta_r \\ \dot{\alpha} \\ \dot{\beta} \end{bmatrix} \qquad (9)$$

And design a vector of gains Kc=[k_{c1},k_{c2},k_{c3},k_{c4}]^T such that the linear control by feedback of states:

$$u = -K_c e = -k_{c1}\alpha - k_{c2}\left(\beta - \beta_r\right) - k_{c3}\dot{\alpha} - k_{c4}\dot{\beta} \qquad (10)$$

is able to stabilize the linear system (8). For this, any desired linear design technique: assignment of zeros and poles, LQR, place of roots, etc ... (Remember that place , lqr and rltool commands from Matlab can help you do the design).

As a reference, the closed-loop poles of the reference controller are approximately `-53,-5+3i,-5-3i and -9.`

Task 3: Analysis of the linear control:

3A) First, to check that the closed loop system is stable tet's analyze the error dynamics according to the linear model (8):

$$\begin{aligned} \dot{e} = \dot{x} = Ax + Bu = \\ = A(e + x_c) - BK_c e \\ = (A - BK_c)e + Ax_c \\ = (A - BK_c)e = A_{eq}e \end{aligned} \qquad (11)$$

Question 2: Why does the term Ax_cdissapear?

To see that the system is really stable, calculate the eigenvalues of A_{eq} and show that they are all in the left half plane of the complex plane. Then, solve the Lyapunov equation to get the symmetric and positive definite matrix, P, such that:

$$PA_{eq} + A_{eq}^T P = -\mathbf{1}_{4x4} \qquad (12)$$

Tip: The Matlab lyap command can be useful to solve this equation.

Defining the function of Lyapunov

$$V = e^T P e \quad (13)$$

If applied to the linearized system (8) it allows to verify that it is asymptotically stable since:

$$\dot{V} = e^T (PA_{eq} + A_{eq}^T P)e = -e^T e = -\|e\|^2 \quad (14)$$

That is defined negative.

So far the function of lyapunov (13) has not been very useful since it only confirms what we already knew through the eigenvalues. The interesting thing is to apply this function of lyapunov to the nonlinear system to see how far the stability is maintained.

3B) Since both the actual plant model (6) and the control law (10) are continuous and derivable, if equation (6) is rewritten by substituting u and x in terms of the error variables we get to the form:

$$\dot{e} = f(e) \quad (15)$$

Exapanding in Taylor series we have:

$$\dot{e} = f(0) + \frac{\partial f}{\partial e} e + o\left(\|e\|^2\right) = A_{eq} e + o\left(\|e\|^2\right) \quad (16)$$

Since the first term is null and the second is the linear approximation that as we know it is given by (11). It is finally obtained:

$$\begin{aligned} \dot{V} &= \left(A_{eq} e + o\left(\|e\|^2\right)\right)^T P e + e^T P \left(A_{eq} e + o\left(\|e\|^2\right)\right) \\ &= -\|e\|^2 + o(\|e\|^3) \le -\|e\|^2 + M\|e\|^3 \end{aligned} \quad (17)$$

3C) It is clear then that the quadratic term in e will be greater than the cubic term in e in the proximity to the origin (where M || e || < 1). So, the non-linear system is stable (which we also knew from the linearization theorem of Lyapunov).

Write a program that explicitly calculates the derivative of V using the equations (6) and (10) and estimate by the Monte Carlo method the value of the constant M. To do this use the following algorithm:

1) $M_{est}=0$
2) Take a random value of e within the range $||e||<e_{max}$.
3) Obtain dV/dt and $\quad M_1 = \left| \frac{\dot{V} + \|e\|^2}{\|e\|^3} \right| \quad$ (be careful with division by zero)
4) Update $M=\max(M,M_1)$
5) Go back to 2) until you have a sufficient number of points (for example, 1000000 points).
6) Finally, apply a security margin, M=1.1*M

Once the value of M is calculated, estimate the region of attraction. For this, calculate the minimum value of V (V_0) when ||e||=1/M.

This guarantees that if $V<V_0$ then ||e||<1/M and therefore V decreases. In other wors, the region $V<V_0$ is a region of attraction of the equilibrium point (applying LaSalle).

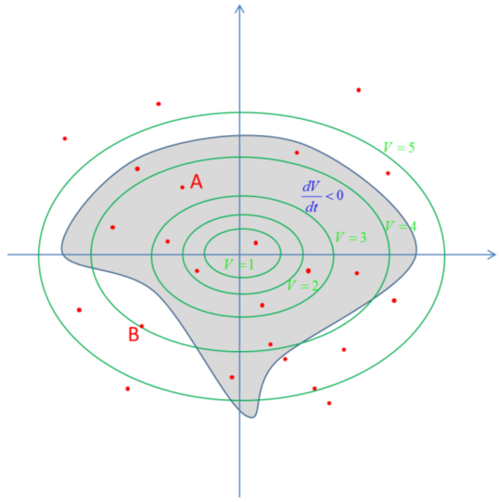

Fig. 2. Monte Carlo.

Question 3: Up to what maximum angle, α_{max}, you can tilt the pendulum assuming that the other errors are zero before leaving the region of attraction $V<V_0$?

3D) We previously said that the system was non-minimal phase but we have not mentioned it until now.

From the linear model (8) and the control action (10), calculate the function of Beta transfer with respect to its closed-loop reference. Check that this function contains zeros in the right half-plane of the complex plane (it is non-minimum phase).

Pregunta 4: How can **I** know that, if the gain, K_c, has been choosen by **you** arbitrarily?

Task 4: Checking the linear control in simulation.

This task consists in checking the theoretical results of the previous points through the simulation of the system. For this, the control previously designed will be implemented and used, and the next things must be verified:

4A) Whether the control is able to stabilize the pendulum when it abandons the proximities to the equilibrium point, by carrying it next to such point again.

4B) Whether the control is able to maintain the pendulum in the right position for a small change in the reference.

4C) When a disturbance that makes the pendulum get out of the region of attraction is applied, the controller does not work properly anymore and the pendulum ends up falling. Compare the theoretical region of attraction obtained previously, α_{max},with the result in the simulation.

4D) The pendulum presents an inverse response in β (due to the non-minimum phase). To check it, plot the response of β to a step change in the reference β_c.

Verify that the values of β start moving in one direction and then go in the opposite direction.

Question 5: Could it be the other way around? That is, could we make β to approach to β_c from the beginning without making β go closer to β_c first?

Tip: If, for example, we want to move β to the right and we start moving the base to the right, would not the pendulum fall to the left? Make simulations until you can explain the behavior. (In this section we are asking for a physical interpretation of what happens to the pendulum and not a rigorous mathematical demonstration).

Advice: If the sampling period is lowered (even more), the simulation gets slower and allows to see better the behavior of the pendulum.

4E) As you can see in simulation it is "dangerous" to get away from the equilibrium point, since the control does not work properly and the behavior of the pendulum is violent. Modify the control law so that when the angle exceeds 30 degrees, the controller turns off (u = 0).

Task 5: Checking the linear control in the real system:

In this section it will be verified that the control works not only in the simulated system, but also in the real pendulum. For this purpose, perform the following tests:

5A) Check if the control is able to stabilize the pendulum when it abandons the proximities to the equilibrium point, by carrying it next to such point again.

5B) Check if the control is able to maintain the pendulum in the right position for a small change in the reference.

5C) Compare the theoretical region of attraction obtained previously, α_{max}, with the result in the real system.

5D) The pendulum presents an inverse response in β (due to the non-minimum phase).

In all the analysis we have carried out untill now, we have assumed that the action over the plant is continuous and instantaneous. However, any real implementation carried out with a computer suffers from certain delays.

In particular, in the case that concerns us, the plant is being sampled in a periodic manner. This means that, within a sampling period, the control action remains constant. In a way, it is as if the pendulum is in an "open loop". Since the system is inherently unstable, the bigger the period of sampling is, the worse the control law will work. If the sampling period keeps growing, the system will eventually leave the region of attraction and the pendulum will fall.

5E) Determine empirically what is the **máximum** sampling period that allows to hold the pendulum with the desgined control.

NOTE: During the real test with the remote lab, do not forget to take all the data that you think it is necessary to write the lab report (such as screenshots of the interface or the data of the graphs). Do not forget that you can save the graphs by clicking on with the right mouse button. Do not forget to save the controllers that you are testing.

Task 6: Design a control law for the swing-up:

So far we have designed a control that is able to hold the pendulum When we are close to the equilibrium point, the objective of this section is to design a control that is capable of bringing us close to that equilibrium position from any initial position.

The objective of this section is for the student to be able to read and understand a classical nonlinear control present in the literature. A superficial reading of a article may give the feeling of having understood it, but for the brevity inherent in the publications there are many details that are omitted and only reaches Understand in depth the content of an article when trying repeat the results and implement it in a real plant.

In this section we propose a possible design scheme based on the control of Åström pendulum energy (which is the technique used in the controller of example) although the student is totally free to implement any control technique that you deem convenient as long as it works properly.

The steps to follow for the student who chooses to implement control of Åström are the following:

6A) Read the first article "Pendulum Equations", in particular section 2 where model a Furuta pendulum and compare our equations (1) and (2) with the equation (8) of said article. From this information, deduce the relationship between our variables α, β and theirs θ, φ, also note that in their model they use T instead of τ.

6B) Put our equations in the form (9), using our known constants (7).

6C) Now that you know how to manipulate the pendulum equations and put them in shape Simple read the second article, "Swing-up Pendulum Energy Control". How can see the pendulum equations are even simpler! But the IDEA exposed is the same for our pendulum.

NOTA: Åström considers the angle null when the pendulum is up and us when it is down, pay attention also to its criterion of signs…

6E) Repeat the steps taken for the control of literature and try to apply the law of control (7) when α is big (for example, $|\alpha|>10^{o}$) and switch to the linear control designed in the previous case when α is small (for example, $|\alpha|\leq 10º$). Note that to implement the control you will have to replace your "u" control signal (pivot acceleration) by our "u" control signal (motor voltage) by changing the variables we did in step 2).

Task 7: Checking the non-linear control in Simulation:

In this task, the control designed in the real pendulum will be implemented checking which is able to lift and stabilize the pendulum in its upper position from any initial condition.

Task 8: Checking the non-linear control in the real system.

In this task, the tests with the real pendulum will be repeated to check its functioning.