1. [4 marks] Use the method of undetermined coefficients to find the **form** of the particular solution of $y'' - 7y' + 10y = 6x^2 - \cos 2x + 2xe^{5x}$. **Hint** : Do not find the coefficients.

$$\lambda^2 - 7\lambda + 10 = 0 \to (\lambda - 2)(\lambda - 5) = 0 \to y_c = c_1 e^{2x} + c_2 e^{5x}$$

$$y_p = (Ax^2 + Bx + C) + (D\cos 2x + E\sin 2x) + (Fx^2 + Gx)e^{5x}$$

2. [4 marks] Evaluate $\mathcal{L}^{-1}\{\frac{6s}{s^2+4s+8}\}$. Similar to assigned question #15 §4.3

$$\text{Since } \frac{6s}{s^2+4s+8} = \frac{6s}{(s+2)^2+2^2} = \frac{6(s+2)-12}{(s+2)^2+2^2}$$

$$= 6\frac{s+2}{(s+2)^2+2^2} - 6\frac{2}{(s+2)^2+2^2},$$

$$\mathcal{L}^{-1}\left\{\frac{6s}{s^2+4s+8}\right\}$$

$$= 6\mathcal{L}^{-1}\left\{\frac{s+2}{(s+2)^2+2^2}\right\} - 6\mathcal{L}^{-1}\left\{\frac{2}{(s+2)^2+2^2}\right\}$$

$$= 6e^{-2t}\cos 2t - 6e^{-2t}\sin 2t$$

3. [8 marks] Use the variation of parameters to find the general solution of the differential equation:

$$y'' - y = \frac{2e^{2x}}{e^{2x}+1}.$$

Assigned H.W. #24, Ch 3 Review

Hint : Use $[\tan^{-1} u]' = \dfrac{u'}{1+u^2}$.

$$\lambda^2 - 1 = 0 \rightarrow y_c = c_1 e^{x} + c_2 e^{-x}$$

$$W = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -1-1 = -2$$

$$u_1' = -\frac{1}{2}\begin{vmatrix} 0 & e^{-x} \\ \frac{2e^{2x}}{e^{2x}+1} & -e^{-x} \end{vmatrix} = \frac{1}{2}\cdot\frac{2e^{x}}{e^{2x}+1} = \frac{e^{x}}{e^{2x}+1}$$

$$u_2' = -\frac{1}{2}\begin{vmatrix} e^x & 0 \\ e^x & \frac{2e^{2x}}{e^{2x}+1} \end{vmatrix} = -\frac{e^{3x}}{e^{2x}+1} = -\left[\frac{(e^{2x}+1)e^{x}-e^{x}}{e^{2x}+1}\right]$$

$$= \frac{e^{x}}{e^{2x}+1} - e^{x}$$

$$u_1 = \int \frac{e^{x}}{e^{2x}+1}\,dx \overset{u=e^x}{=\!=\!=} \tan^{-1}(e^{x}) \quad \text{by Hint}$$

and $u_2 = \int \left(\frac{e^{x}}{e^{2x}+1} - e^{x}\right)dx = \tan^{-1}(e^{x}) - e^{x}$ by Hint

$$\therefore y_p = e^{x}\cdot\tan^{-1}(e^{x}) + e^{-x}\tan^{-1}(e^{x}) - 1$$

the general solution

$$y = c_1 e^{x} + c_2 e^{-x} + e^{x}\tan^{-1}(e^{x}) + e^{-x}\tan^{-1}(e^{x}) - 1$$

4. [10 marks] Use the Laplace transform to solve the given initial-value problem :

$$y'' + 4y = f(t), \quad y(0) = 0, \quad y'(0) = -1,$$

where $f(t) = \begin{cases} 1 & 0 \le t < 1 \\ 0 & t \ge 1 \end{cases}$.

Assigned H.W. question: #66 §4.3

Since $\mathcal{L}\{f(t)\} = \mathcal{L}\{1 - \mathcal{U}(t-1)\} = \frac{1}{s} - \frac{e^{-s}}{s}$ (or by definition)

$$\mathcal{L}\{y''\} + \mathcal{L}\{4y\} = \mathcal{L}\{1 - \mathcal{U}(t-1)\}$$

$$s^2\mathcal{L}\{y\} - \cancel{sy(0)} - y'(0) + 4\mathcal{L}\{y\} = \frac{1}{s} - \frac{e^{-s}}{s}$$

$$\mathcal{L}\{y\}(s^2+4) = \frac{1 - e^{-s} - s}{s}$$

$$\therefore \mathcal{L}\{y\} = \frac{1-s}{s(s^2+4)} - \frac{e^{-s}}{s(s^2+4)}$$

where $\frac{1-s}{s(s^2+4)} = \frac{A}{s} + \frac{Bs+C}{s^2+4}$

basic equation: $1 - s = A(s^2+4) + (Bs+C)s$

coeff of s^2 : $0 = A + B \rightarrow B = -\frac{1}{4}$.

" " s : $-1 = C$

const : $1 = 4A \rightarrow A = \frac{1}{4}$

and $\frac{1}{s(s^2+4)} = \frac{A}{s} + \frac{Bs+C}{s^2+4}$

basic equation : $1 = A(s^2+4) + (Bs+C)s$

$A + B = 0$, $C = 0$, $4A = 1$

Therefore

$$\mathcal{L}\{y\} = \frac{1}{4}\frac{1}{s} - \frac{1}{4}\frac{s}{s^2+4} - \frac{1}{s^2+4} - e^{-s}\left[\frac{1}{4}\frac{1}{s} - \frac{1}{4}\frac{s}{s^2+4}\right]$$

$$y(t) = \frac{1}{4}t - \frac{1}{4}\cos 2t - \frac{1}{2}\sin 2t - \left[\frac{1}{4} - \frac{1}{4}\cos 2(t-1)\right]\mathcal{U}(t-1)$$

5. [5 marks] Given that $y_1 = x^{\frac{1}{2}} \ln x$ is a solution of $4x^2y'' + y = 0$ on the invterval $(0, \infty)$, use reduction of order to find a second solution y_2.

Similar to Assigled H.W #14 §3.2

$$y_2 = x^{\frac{1}{2}} \ln x \int \frac{e^{\int 0 dx}}{x \ln^2 x} dx = x^{\frac{1}{2}} \ln x \int \frac{1}{x(\ln x)^2} dx$$

$$= x^{\frac{1}{2}} \ln x \left(-(\ln x)^{-1}\right)$$

$$= -x^{\frac{1}{2}}$$

$$\therefore y = x^{\frac{1}{2}}$$

6. [5 marks] Evaluate $\mathcal{L}^{-1}\{\frac{se^{\frac{-\pi s}{2}}}{s^2+4}\}$.

Similar to Assigned H.W #45 §4.3

by the formular

$$\mathcal{L}^{-1}\left\{\frac{e^{-\frac{\pi s}{2}} \cdot s}{s^2+4}\right\} = \cos 2\left(t - \frac{\pi}{2}\right) \mathcal{U}\left(t - \frac{\pi}{2}\right)$$

$$\overset{or}{=} -\cos 2t\, \mathcal{U}\left(t - \frac{\pi}{2}\right)$$

7. [7 marks] Use the Laplace transform to solve the given initial-value problem :

Assigned H.W
#24 §4.3

$$y'' - 4y' + 4y = t^3e^{2t}, \quad y(0) = 0, \quad y'(0) = 0.$$

$$\mathcal{L}\{y''\} - 4\mathcal{L}\{y'\} + 4\mathcal{L}\{y\} = \mathcal{L}\{t^3e^{2t}\}$$

$$s^2\mathcal{L}\{y\} - 4s\mathcal{L}\{y\} + 4\mathcal{L}\{y\} = \left.\frac{3!}{s^4}\right|_{s-2}$$

$$\therefore \mathcal{L}\{y\}(s^2-4s+4) = \frac{6}{(s-2)^4}$$

$$\therefore \mathcal{L}\{y\} = \frac{6}{(s-2)^6}$$

$$\therefore y = \mathcal{L}^{-1}\left\{\frac{6}{(s-2)^6}\right\} = \frac{1}{20}\mathcal{L}^{-1}\left\{\frac{5!}{(s-2)^6}\right\} = \frac{1}{20}e^{2t}\cdot t^5$$

8. [7 marks] Use the undetermined coefficient method to solve the given initial-value problem :

$$y'' - 4y' + 4y = t^3e^{2t}, \quad y(0) = 0, \quad y'(0) = 0.$$

similar to #7 §3.4

$\lambda^2 - 4\lambda + 4 = 0 \rightarrow \lambda = 2$

$\therefore y_c = c_1 e^{2t} + c_2 t e^{2t}$

Assume $y_p = (\underbrace{At^5 + Bt^4 + ct^3 + Dt^2}_{M})e^{2t}$.

Then $y_p' = (\underbrace{5At^4 + 4Bt^3 + 3ct^2 + 2Dt}_{K})e^{2t} + 2(\underbrace{At^5 + Bt^4 + ct^3 + Dt^2}_{M})e^{2t}$

$y_p'' = (20At^3 + 12Bt^2 + 6ct + 2D)e^{2t} + 2(\underbrace{\quad}_{K})e^{2t}$
$+ 4(\underbrace{\quad}_{M})e^{2t} + 2(\underbrace{5At^4 + 4Bt^3 + 3ct^2 + 2Dt}_{K})e^{2t}$

$\therefore y'' - 4y' + 4y = (20At^3 + 12Bt^2 + 6ct + 2D)e^{2t} + 2Ke^{2t}$
$+ 4Me^{2t} + 2Ke^{2t} - 4Ke^{2t} - 8Me^{2t} + 4Me^{2t}$
$= (20At^3 + 12Bt^2 + 6ct + 2D)e^{2t} \underset{\text{be}}{\overset{\text{must}}{=\!=}} t^3e^{2t}$

$\therefore 20A = 1, \quad B = C = D = 0$

$\therefore y_p = \frac{1}{20}t^5 \cdot e^{2t}$

$\therefore$ General solution $y = c_1 e^{2t} + c_2 t e^{2t} + \frac{1}{20}t^5 e^{2t}$

Since $y(0) = 0, \; y'(0) = 0, \quad c_1 = c_2 = 0$

$\therefore$ Unique solution $y = \frac{t^5}{20}e^{2t}$.