1. [4 marks] Evaluate $\mathcal{L}^{-1}\{\frac{1}{s^2(s^2+1)}\}$. **Hint :** Do not use the partial fractional decomposition.

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$$\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+1)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s}\cdot\frac{1}{s^2+1}\right\} = \int_0^t \sin\tau \, d\tau = -\cos\tau\Big]_0^t = -\cos t + 1$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+1)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s}\cdot\frac{1}{s(s^2+1)}\right\} = \int_0^t (1-\cos\tau)\, d\tau$$

$$= \tau - \sin\tau\Big]_0^t = \boxed{t - \sin t}$$

2. [4 marks] Find the Laplace transform of the periodic function : $f(t) = \begin{cases} 0 & 0 \le t < 1 \\ 1 & 1 \le t < 2 \end{cases}$.

page 238 The period $T=2$

$$\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-2s}}\int_0^2 e^{-st} f(t)\, dt$$

$$= \frac{1}{1-e^{-2s}}\left[\int_0^1 0\cdot e^{-st}\, dt + \int_1^2 1\cdot e^{-st}\, dt\right]$$

$$= \frac{1}{1-e^{-2s}}\left[\frac{e^{-st}}{-s}\right]_1^2$$

$$= \frac{1}{1-e^{-2s}}\left(\frac{e^{-2s}-e^{-s}}{-s}\right)$$

$$= \frac{e^{-s}-e^{-2s}}{s(1-e^{-2s})} = \frac{e^{-s}(1-e^{-s})}{s(1-e^{-s})(1+e^{-s})} = \boxed{\frac{e^{-s}}{s(1+e^{-s})}}$$

3. [8 marks] Use the Laplace transform to solve the given integral equation:

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$$t - 2f(t) = \int_0^t (e^\tau - e^{-\tau}) f(t-\tau)\,d\tau.$$

$$\mathcal{L}\{t\} - 2\mathcal{L}\{f\} = \mathcal{L}\{e^t - e^{-t}\}\mathcal{L}\{f\}$$

$$\frac{1}{s^2} - 2\mathcal{L}\{f\} = \left(\frac{1}{s-1} - \frac{1}{s+1}\right)\mathcal{L}\{f\}$$

$$\frac{1}{s^2} = \left(\frac{2}{s^2-1}\right)\mathcal{L}\{f\} + 2\mathcal{L}\{f\}$$

$$\therefore \mathcal{L}\{f\} = \frac{s^2-1}{2s^4} = \frac{1}{2}\frac{1}{s^2} - \frac{1}{12}\cdot\frac{3!}{s^4}$$

$$\therefore f(t) = \frac{1}{2}t - \frac{1}{12}t^3$$

4. [8 marks] Find the half-range cosine expansion of the function: $f(x) = \begin{cases} 1 & 0 < x < 1 \\ 2 - x & 1 \le x < 2 \end{cases}$.

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$$a_0 = \int_0^1 1\,dx + \int_1^2 (2-x)\,dx = x\Big]_0^1 + \left(2x - \tfrac{1}{2}x^2\right)\Big]_1^2$$

$$= 1 + (4-2) - \left(2 - \tfrac{1}{2}\right) = 3 - 2 + \tfrac{1}{2} = \boxed{\tfrac{3}{2}}$$

$$a_n = \int_0^1 1 \cos\frac{n\pi}{2}x\,dx + \int_1^2 \underbrace{(2-x)}_{u}\underbrace{\cos\frac{n\pi}{2}x\,dx}_{dv}$$

$$= \frac{2}{n\pi}\sin\frac{n\pi}{2}\Big]_0^1 + (2-x)\frac{2}{n\pi}\sin\frac{n\pi}{2}x\Big]_1^2 - \frac{2}{n\pi}\int_1^2 \sin\frac{n\pi}{2}x(-1)\,dx$$

$$= \frac{2}{n\pi} + 0 - \frac{2}{n\pi} + \frac{2}{n\pi}\cdot\left[\frac{-2}{n\pi}\cos\frac{n\pi}{2}x\right]_1^2$$

$$= \frac{-4}{n^2\pi^2}\left(\cos n\pi - \cos\frac{n\pi}{2}\right) = \boxed{\frac{4}{n^2\pi^2}\left(\cos\frac{n\pi}{2} + (-1)^{n+1}\right)}$$

$$\therefore f(x) = \frac{3}{4} + \sum_{n=1}^{\infty} \frac{4}{n^2\pi^2}\left(\cos\frac{n\pi}{2} + (-1)^{n+1}\right)\cos\frac{n\pi}{2}x$$

5. [5 marks] Show that the set $\{\cos x, \cos 2x, \cos 3x \ldots\}$ is orthogonal on the interval $[-\pi, \pi]$. What is the norm?

For $m \neq n$

$$\int_{-\pi}^{\pi} \cos mx \cdot \cos nx \, dx = \frac{1}{2}\int_{-\pi}^{\pi} [\cos(m+n)x + \cos(m-n)x] \, dx$$

$$= \frac{1}{2}\left[\frac{\sin(m+n)x}{m+n} + \frac{\sin(m-n)x}{m-n}\right]_{-\pi}^{\pi} = 0$$

$\therefore$ the set is orthogonal.

$$\int_{-\pi}^{\pi} \cos^2 nx \, dx = \frac{1}{2}\int_{-\pi}^{\pi} [1 + \cos 2nx] \, dx = \frac{1}{2}\left[x + \frac{\sin 2nx}{2n}\right]_{-\pi}^{\pi}$$

$$= \pi$$

$\therefore$ the norm is $\sqrt{\pi}$

6. [5 marks] Find the vector that gives the direction in which $f(x, y) = xye^{x-y}$ increases most rapidely at $(2, 2)$. What is the maximum rate?

$$\nabla f(x,y) = [ye^{x-y} + xye^{x-y}]\,i + [xe^{x-y} - xye^{x-y}]\,j$$

$$\nabla f\Big|_{(2,2)} = (2+4)i + (2-4)j = \boxed{6i - 2j}$$

the maximum rate is $\|\nabla f_{(2,2)}\| = \sqrt{6^2+2^2} = \sqrt{40} = \boxed{2\sqrt{10}}$

7. [8 marks]

Use the Laplace transform to solve the given system of differential equations : $\begin{cases} \frac{dx}{dt} + x = \frac{dy}{dt} - y \\ \frac{dx}{dt} + \frac{dy}{dt} + 2y = 0 \end{cases}$ with initial conditions $x(0) = 0, y(0) = 1$.

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$$\begin{cases} s\mathcal{L}\{x\} + \mathcal{L}\{x\} - s\mathcal{L}\{y\} + 1 + \mathcal{L}\{y\} = 0 \\ s\mathcal{L}\{x\} + s\mathcal{L}\{y\} - 1 + 2\mathcal{L}\{y\} = 0 \end{cases}$$

$$\begin{cases} (s+1)\mathcal{L}\{x\} - (s-1)\mathcal{L}\{y\} = -1 \\ s\,\mathcal{L}\{x\} + (s+2)\mathcal{L}\{y\} = 1 \end{cases}$$

✓ $$-s(s-1)\mathcal{L}\{y\} - (s+1)(s+2)\mathcal{L}\{y\} = -s-(s+1)$$

$$\mathcal{L}\{y\}\cdot[-s^2+s-s^2-3s-2] = -2s-1$$

$$\mathcal{L}\{y\} = \frac{-2s-1}{-2s^2-2s-2} = \frac{s+\frac{1}{2}}{s^2+s+1} = \frac{(s+\frac{1}{2})}{(s+\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2}$$

$$\Rightarrow y(t) = \boxed{e^{-\frac{1}{2}t}\cos\frac{\sqrt{3}}{2}t}$$

✓ $$(s+2)(s+1)\mathcal{L}\{x\} + s(s-1)\mathcal{L}\{x\} = -(s+2)+(s-1)$$

$$\mathcal{L}\{x\}[s^2+3s+2+s^2-s] = -3$$

$$\mathcal{L}\{x\} = \frac{-3}{2s^2+2s+2} = \frac{-3/2}{(s+\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2} = -\sqrt{3}\cdot\frac{\sqrt{3}/2}{(s+\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2}$$

$$\Rightarrow x(t) = \boxed{-\sqrt{3}\,e^{-\frac{1}{2}t}\sin\frac{\sqrt{3}}{2}t}$$

8. [8 marks] Solve the given system of differential equations : $\begin{cases} \frac{dx}{dt} + x = \frac{dy}{dt} - y \\ \frac{dx}{dt} + \frac{dy}{dt} + 2y = 0 \end{cases}$

with initial conditions $x(0) = 0, y(0) = 1$. Justify your answer.

Hint 1: Rewrite the system in terms of the operator D.

Hint 2: One of solutions, $\boxed{y(t) = e^{-\frac{1}{2}t}\cos\frac{\sqrt{3}}{2}t}$ is given. Find $x(t)$ only.

Warning : There will be no credit if it is solved using the Laplace transform.

Method 1/

$$(*)\begin{cases} (D+1)x - (D-1)y = 0 \\ Dx + (D+2)y = 0 \end{cases}$$

$\therefore (D+2)(D+1)x + D(D-1)x = 0$

$(D^2+3D+2+D^2-D)x = 0 \longrightarrow (2D^2+2D+2)x = 0$

Hence, $x'' + x' + x = 0 \longrightarrow m^2+m+1 = 0 \rightarrow m = \frac{-1 \pm \sqrt{3}\,i}{2}$

$x(t) = e^{-\frac{1}{2}t}[c_1 \cos\frac{\sqrt{3}}{2}t + c_2 \sin\frac{\sqrt{3}}{2}t]$

Since $x(0)=0$, $0 = 1[c_1+0] \longrightarrow c_1 = 0$

Therefore, $x(t) = \boxed{c_2 \sin\frac{\sqrt{3}}{2}t \cdot e^{-\frac{1}{2}t}}$

$\frac{dx}{dt} = -\frac{1}{2}e^{-\frac{1}{2}t}c_2 \sin\frac{\sqrt{3}}{2}t + e^{-\frac{1}{2}t}\cdot c_2 \cdot \frac{\sqrt{3}}{2}\cdot\cos\frac{\sqrt{3}}{2}t$ and

$\frac{dy}{dt} + 2y = -\frac{1}{2}e^{-\frac{1}{2}t}\cos\frac{\sqrt{3}}{2}t - e^{-\frac{1}{2}t}\frac{\sqrt{3}}{2}\sin\frac{\sqrt{3}}{2}t + 2e^{-\frac{1}{2}t}\cos\frac{\sqrt{3}}{2}t$

Since $\frac{dx}{dt} = -(\frac{dy}{dt} + 2y)$, by comparing the coefficients of $e^{-\frac{1}{2}t}\sin\frac{\sqrt{3}}{2}t$,

$-\frac{1}{2}c_2 = \frac{\sqrt{3}}{2} \longrightarrow c_2 = -\sqrt{3}$

$\therefore x(t) = \boxed{-\sqrt{3}e^{-\frac{1}{2}t}\sin\frac{\sqrt{3}}{2}t}$

Method 2/ $(*)$ gives $x - 2Dy - y = 0 \quad \therefore x = 2Dy + y$

$x = 2[-\frac{1}{2}e^{-\frac{1}{2}t}\cos\frac{\sqrt{3}}{2}t - \frac{\sqrt{3}}{2}e^{-\frac{1}{2}t}\sin\frac{\sqrt{3}}{2}t] + e^{-\frac{1}{2}t}\cos\frac{\sqrt{3}}{2}t$

$= \boxed{-\sqrt{3}e^{-\frac{1}{2}t}\sin\frac{\sqrt{3}}{2}t}$