8. [7 marks] Use the undetermined coefficient method to solve the given initial-value problem :

$$y'' - 4y' + 4y = t^3 e^{2t}, \quad y(0) = 0, \quad y'(0) = 0.$$

similar to #7 §3.4

$\lambda^2 - 4\lambda + 4 = 0 \rightarrow \lambda = 2$

$\therefore y_c = c_1 e^{2t} + c_2 t e^{2t}$ $\boxed{2}$

Assume $y_p = (\underbrace{At^5 + Bt^4 + Ct^3 + Dt^2}_{\text{let } M}) e^{2t}$. $\boxed{3}$

Then $y_p' = (\underbrace{5At^4 + 4Bt^3 + 3Ct^2 + 2Dt}_{\text{let } K}) e^{2t} + 2(\underbrace{At^5 + Bt^4 + Ct^3 + Dt^2}_{M}) e^{2t}$

$y_p'' = (20At^3 + 12Bt^2 + 6Ct + 2D)e^{2t} + 2(\underbrace{\quad}_{K})e^{2t}$
$+ 4(\underbrace{\quad}_{M})e^{2t} + 2(\underbrace{5At^4 + 4Bt^3 + 3Ct^2 + 2Dt}_{K})e^{2t}$

$\boxed{1}$ $\therefore y'' - 4y' + 4y = (20At^3 + 12Bt^2 + 6Ct + 2D)e^{2t} + 2Ke^{2t}$
$+ 4Me^{2t} + 2Ke^{2t} - 4Ke^{2t} - 8Me^{2t} + 4Me^{2t}$
$= (20At^3 + 12Bt^2 + 6Ct + 2D)e^{2t} \underset{\text{be}}{\overset{\text{must}}{=}} t^3 e^{2t}$

$\therefore 20A = 1, \quad B = C = D = 0$

$\therefore y_p = \frac{1}{20} t^5 \cdot e^{2t}$

$\therefore$ General solution $y = c_1 e^{2t} + c_2 t e^{2t} + \frac{1}{20} t^5 e^{2t}$

Since $y(0) = 0$, $y'(0) = 0$, $c_1 = c_2 = 0$

2 $\boxed{1}$ $\therefore$ Unique solution $y = \frac{t^5}{20} e^{2t}$.

2. [6 marks] Find the half-range sine expansion of the function: $f(x) = \begin{cases} 1 & 0 < x < 1 \\ 2 - x & 1 \le x < 2 \end{cases}$.

$b_n = \frac{2}{2}\int_0^2 f(x)\sin\frac{n\pi x}{2}\,dx$

$b_n = \int_0^1 \sin\frac{n\pi x}{2}\,dx + \int_1^2 (2-x)\sin\frac{n\pi x}{2}\,dx$

$= -\frac{2}{n\pi}\cos\frac{n\pi x}{2}\Big|_0^1 + \int_1^2 (2-x)\sin\frac{n\pi x}{2}\,dx$

$-\frac{2}{n\pi}\left(\cos\frac{n\pi}{2} - 1\right) + \frac{2}{n\pi}\cos\frac{n\pi}{2} + \frac{4}{n^2\pi^2}\sin\frac{n\pi}{2}$

$2 - x = u \Rightarrow -dx = du$

$\sin\frac{n\pi x}{2}dx = dv \Rightarrow v = -\frac{2}{n\pi}\cos\frac{n\pi x}{2}$

$\Rightarrow -\frac{(2-x)2}{n\pi}\cos\frac{n\pi x}{2} - \int_1^2 \frac{2}{n\pi}\cos\frac{n\pi x}{2}$

$\left(\frac{2(x-2)}{n\pi}\cos\frac{n\pi x}{2}\right|_1^2 - \frac{4}{n^2\pi^2}\sin\frac{n\pi x}{2}\Big]_1^2$

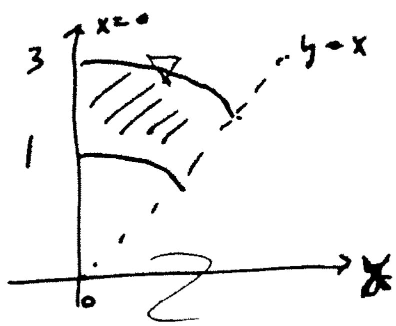
$+ \frac{2}{n\pi}\cos\frac{n\pi}{2} + \frac{4}{n^2\pi^2}\sin\frac{n\pi}{2}$

3. [7 marks]

Find the work done by the force $\boldsymbol{F} = -x^2 y\,\vec{\mathbf{i}} + y^2 x\,\vec{\mathbf{j}}$ along the closed cuve formed by

$$x = 0, x^2 + y^2 = 9, y = x, \text{ and } x^2 + y^2 = 1.$$

Assume counterclockwise orientation.




$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = y^2 + x^2$$

$$\oint_C P\,dx + Q\,dy$$

$$= \iint_R (x^2+y^2)\,dA$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_1^3 r^2 \cdot r\,dr\,d\theta$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1}{4} r^4 \Big]_1^3 d\theta$$

4

$$= 20\left(\frac{\pi}{2} - \frac{\pi}{4}\right)$$

$$= \boxed{5\pi}$$

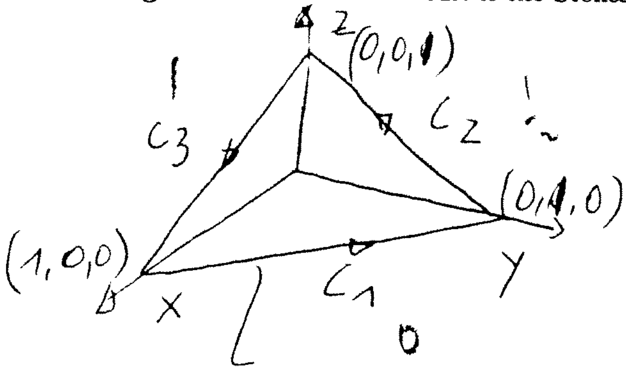
1

4. [8 marks] Using line integrals, evaluate $\oint_C \boldsymbol{F} \cdot d\boldsymbol{r}$ assuming $C$ is oriented counterclockwise, where

$$\boldsymbol{F} = (2z + x)\vec{\mathbf{i}} + (y - z)\vec{\mathbf{j}} + (x + y)\vec{\mathbf{k}}.$$

$C$ is the triangle with vertices $(1,0,0), (0,1,0), (0,0,1)$.

**Warning** : There will be no credit if the Stokes' Theorem is used.




$$\int_C \bar{F} \cdot d\bar{r} = \int_C P(x,y,z)dx + Q dy + R dz \quad 1$$

$P = (2z + x) \quad Q = (y - z)$

$R = (x + y)$

$C_1$ ; $y = 1 - x \Rightarrow dy = -dx$ , $z = 0$, $dz = 0$ 1

$$\int_{C_1} x\,dx - (1-x)\,dx + 0 = \int_1^0 (2x - 1)\,dx = \left(x^2 - x\right)_1^0 = 0 \quad 1$$

$C_2$ : $z = 1 - y \Rightarrow dz = -dy$ , $x = 0 \Rightarrow dx = 0$ 1

$$\int_{C_2} 0 + (y - 1 + y)\,dy - y\,dy = \int_1^0 (y - 1)\,dy = \left(\frac{y^2}{2} - y\right)_1^0 = \frac{1}{2} \quad 1$$

$C_3$ : $z = 1 - x \Rightarrow dz = -dx$ , $y = 0$ $dy = 0$ 1

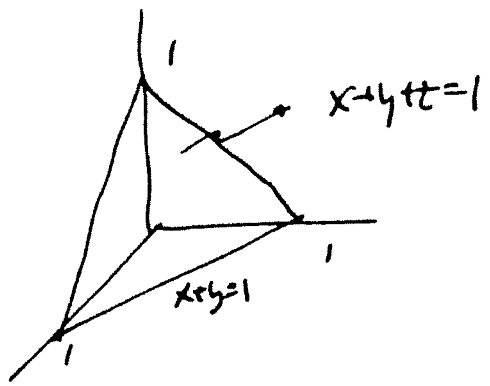
$$\int_{C_3} (2(1-x) + x)\,dx + (x)(-dx) = \int_0^1 (2 - 2x)\,dx = \left(2x - x^2\right)_0^1 = 1 \quad 1$$

$$\int_{C_1} + \int_{C_2} + \int_{C_3} = \frac{1}{2} + 1 = 3/2 \quad 1$$

5. [8 marks] Evaluate $\oint_C \boldsymbol{F} \cdot d\boldsymbol{r}$ using Stokes Theorem assuming $C$ is oriented counterclockwise, where

$$\boldsymbol{F} = (2z + x)\,\vec{\mathbf{i}} + (y - z)\,\vec{\mathbf{j}} + (x + y)\,\vec{\mathbf{k}}.$$

$C$ is the triangle with vertices $(1, 0, 0), (0, 1, 0), (0, 0, 1)$.




part 1/

$$\operatorname{curl} \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2z+x & y-z & x+y \end{vmatrix}$$

$$= i(1+1) + j(2-1) + k(0)$$

$$= 2i + j \qquad 2$$

part 2/ $g(x.y.z) = x+y+z-1=0$

$$\nabla g = i + j + k, \quad \|\nabla g\| = \sqrt{3} \qquad 2$$

$$\vec{n} = \frac{1}{\sqrt{3}}(i + j + k) \qquad \operatorname{curl} F \cdot \vec{n} = \frac{1}{\sqrt{3}}(2+1)$$

part 3/ $f(x,y) = 1-x-y \longrightarrow f_x = -1, \; f_y = -1$

$$\therefore \; ds = \sqrt{3}\, dA \qquad 2$$

part 4/ $\iint_S \operatorname{curl} F \cdot \vec{n})\, ds = \iint_R \frac{3}{\sqrt{3}} \cdot \sqrt{3}\, dA = 3 \iint_R dA = 3 \cdot \frac{1}{2} = \frac{3}{2}$

$$\text{or} = 3 \int_0^1 \int_0^{1-x} dy\, dx = \frac{3}{2} \qquad 2$$

$$\text{or} = 3 \int_0^1 \int_0^{1-y} dx\, dy = \frac{3}{2}$$

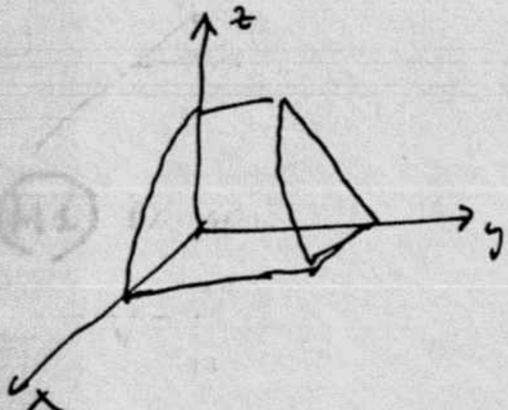
6. [8 marks]

Using the divergence theorem, find the outward flux $\iint_S (\boldsymbol{F}\cdot\vec{\mathbf{n}})dS$ for the vector field

$$\boldsymbol{F} = (5x^2 - e^y \tan^{-1} z)\,\vec{\mathbf{i}} + (x+y)^2\,\vec{\mathbf{j}} - (2yz + x^{2014})\,\vec{\mathbf{k}}$$

and a solid rgeion in the first octant bounded by

$$z = 1 - x^2, z = 0, z = 2 - y, \text{ and } y = 0.$$



part1/ $\operatorname{div} \vec{F} = 10x + 2(x+y) - 2y = 12x$ 2

2 part2/ $\iint_S \vec{F}\cdot\vec{n}\,dS = \iiint_D \operatorname{div}\vec{F}\cdot dV = 12\iiint x \, dy\, dz\, dx$

( If dzdxdy, there should be two $\iiint$s )

2 $$= 12\int_0^1\int_0^{1-x^2}\int_0^{2-z} x\, dy\, dz\, dx$$

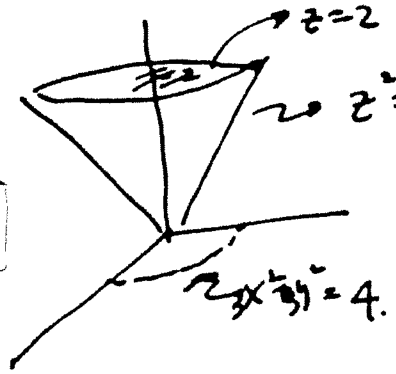
$$= 12\int_0^1\int_0^1 x(2-z)\,dz\,dx = 12\int_0^1 \left. 2xz - \tfrac{1}{2}xz^2 \right]_0^{1-x^2} dx$$

2 $$= 12\int_0^1 \left[2x(1-x^2) - \tfrac{1}{2}x(1-x^2)^2\right]dx = -6(1-x^2)^2 + \tfrac{1}{3}(1-x^2)^3\Big]_0^1 = 5$$

$$\cancel{= 12\left[x^2 - \tfrac{2}{4}x^4 - \tfrac{1}{3}(1-x^2)^3\right]_0^1} = \boxed{5}$$

7. [7 marks]

Find the volume of the solid in the first octant that is bounded by the graphs of the given equations : $z^2 = 3x^2 + 3y^2$, $z = 2$, $x = 0$, $y = 0$.




$z=2 \rightarrow \boxed{\rho = \sec\phi}$ 1

$z^2 = 3x^2+3y^2$ : $\rho^2\cos^2\phi = 3\rho^2 \cdot \sin^2\phi$

1

$\frac{\sin\phi}{\cos\phi} = \frac{1}{\sqrt{3}}$

$\tan\phi = \frac{1}{\sqrt{3}}$

1 $\boxed{\phi = \frac{\pi}{6}}$

(M1) By sphrecal c.s. 3

$$V = \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{6}} \int_0^{2\sec\phi} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{6}} \frac{1}{3}\rho^3 \sin\phi \Big|_0^{2\sec\phi} d\phi \, d\theta$$

2

$$= \frac{8}{3} \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{6}} \sec^3\phi \sin\phi \, d\phi \, d\theta$$

$$= \frac{8}{3} \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{6}} \sec^2\phi \cdot \tan\phi \, d\phi \, d\theta$$

$$= \frac{8}{3} \int_0^{\frac{\pi}{2}} \frac{1}{2}\tan^2\phi \Big]_0^{\frac{\pi}{6}} d\theta$$

1

$$= \frac{4}{3} \int_0^{\frac{\pi}{2}} \frac{1}{3} \, d\theta = \boxed{\frac{2}{9}\pi}$$

(M2)

Volume of cone:

$\cancel{\frac{1}{3} \cdot \pi \cdot \left(\frac{4}{3}\right) \cdot 2 = \frac{8}{9}\pi}$

~~In $1^{st}$ octant, $\frac{2}{9}\pi$~~

(M3) By Cylindrical M. 4

$$V = \int_0^{\frac{\pi}{2}} \int_0^{\frac{2}{\sqrt{3}}} \int_{\sqrt{3}r}^{2} dz \, r \, dr \, d\theta$$

4

$$= \int_0^{\frac{\pi}{2}} \int_0^{\frac{2}{\sqrt{3}}} (2-\sqrt{3}r) r \, dr \, d\theta$$

$$= \int_0^{\frac{\pi}{2}} r^2 - \frac{\sqrt{3}}{3} r^3 \Big]_0^{\frac{2}{\sqrt{3}}} d\theta$$

$$= \frac{\pi}{2} \left( \frac{4}{3} - \frac{\cancel{\sqrt{3}}}{3} \cdot \frac{8}{3\cancel{\sqrt{3}}} \right)$$

$$= \frac{\pi}{2} \left( \frac{12-8}{9} \right) = \boxed{\frac{2}{9}\pi}$$ 1

8. [5 marks]

Find points on the surface $x^2 + 4x + y^2 + z^2 - 2z = 11$ at which the tangent plane is horizontal.

The gradient of $F(x,y,z) = x^2 + 4x + y^2 + z^2 - 2x$ is

1 $\nabla F = (2x+4)\,i + 2y\,j + (2z-2)\,k$

so a normal to the surface at $(x_0, y_0, z_0)$ is

1 $(2x_0+4)\,i + 2y_0\,j + (2z_0-2)\,k$.

A horizontal plane has normal $ck$ for $c \neq 0$.

Thus, we want $2x_0 + 4 = 0$, $2y_0 = 0$, $2z_0 - 2 = c$ or

1 $x_0 = -2$, $y_0 = 0$, $z_0 = c+1$.

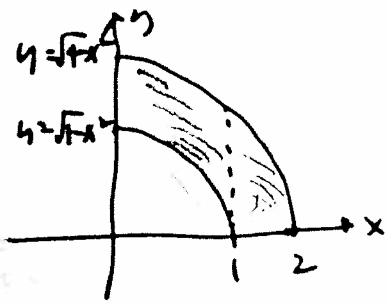
Since $(x_0, y_0, z_0)$ is on the surface, 1

$(-2)^2 + 4(-2) + (c+1)^2 - 2(c+1) = c^2 - 5 = 11$ $\therefore$ $c = \pm 4$

the points are $(-2, 0, 5)$, $(-2, 0, -3)$ 1

9. [5 marks]

Evaluate the given iterated integral $\displaystyle\int_0^1 \int_{\sqrt{1-x^2}}^{\sqrt{4-x^2}} dy\,dx + \int_1^2 \int_0^{\sqrt{4-x^2}} dy\,dx$ by changing to polar coordinates.




$$A = \int_0^{\frac{\pi}{2}} \int_1^2 r\,dr\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} \tfrac{1}{2} r^2 \Big]_1^2 \, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \tfrac{3}{2} \, d\theta = \boxed{\frac{3\pi}{4}}$$

$$\boxed{\frac{4\pi - \pi}{4}}$$

10. [8 marks]

Find the work done by the force $\boldsymbol{F}(x,y,z) = 8xy^3z\,\vec{\mathbf{i}} + 12x^2y^2z\,\vec{\mathbf{j}} + 4x^2y^3\,\vec{\mathbf{k}}$ acting along the helix $\boldsymbol{r}(t) = 2\cos t\,\vec{\mathbf{i}} + 2\sin t\,\vec{\mathbf{j}} + t\,\vec{\mathbf{k}}$ from $(2,0,0)$ to $(1,3,\pi)$.

$$P_y = 24xy^2z = Q_x$$

$$Q_z = 12x^2y^2 = R_y \quad \text{and}$$

$$R_x = 8xy^3 = P_z$$

$\therefore\ \vec{F}$ is conservative. $\boxed{3}$

Thus the work done between two points is independent of the path.

We obtain

$\phi = 4x^2y^3z$ which is a potential ft for $\vec{F}$. $\boxed{3}$

then

$$\int_{(2,0,0)}^{(1,3,\pi)} F\cdot dr = 4x^2y^3z\Big|_{(2,0,0)}^{(1,3,\pi)} = 4\cdot 1\cdot 27\cdot\pi = \boxed{108\pi}$$

$\boxed{2}$