RYERSON UNIVERSITY

DEPARTMENT
OF
MATHEMATICS

MTH 314 | **Final Exam** | **April 25, 2007**

Total marks: 80 | **Time allowed: 3 Hours**

NAME (Print): ____________________ **STUDENT #:** ____________________

Instructions:

- You are allowed an $8\frac{1}{2} \times 11$ formula sheet written on both sides.
- No other aids allowed. Electronic devices such as calculators, cell-phones, pagers, MP3 players, etc, must be turned off and kept inaccessible during the test.
- Verify that your paper contains 9 questions on 10 pages.
- In every question show all your work. The correct answer alone may be worth nothing.
- If you need more space continue on the back of the page, directing marker where the answer continues with a bold sign.

1. Answer the following questions True or False. Circle your answer.
(No marks will be deducted for incorrect answers.)

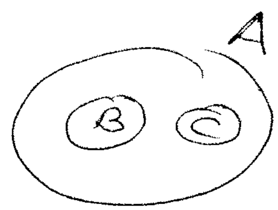
14 Mk

$\mathcal{P}(A)$ denotes the power set of A.

(a) The set of all irrational numbers $\mathbb{R} - \mathbb{Q}$, i.e. the set of all real numbers which cannot be expressed as quotients of integers, is countable. **T** **F**

(b) For every set A, $\emptyset \in A$ **T** **F**

(c) For every set A, $\emptyset \in \mathcal{P}(A)$ **T** **F**



(d) $(A \cup B = A \cup C) \to B = C$ **T** **F**

(e) For every regular expression R there is a finite state automaton which accepts the language denoted by R. **T** **F**

(f) A graph in which every vertex has even degree has an Euler circuit. **T** **F**

(Connected)

(g) A binary tree of height 5 can have 37 leaves (terminal vertices). **T** **F**

≤ 2

$\leq 2^2$

$\vdots$

$\leq 2^5 = 32$

2. (a) Find each of the following graphs if they exist. If they do not exist, explain briefly why not. Quote any theorems that you use in your explanation.

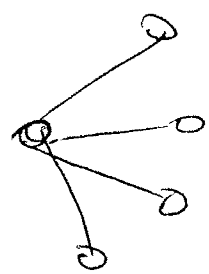
3 Mk

(a) A graph with 6 vertices with degrees 1, 2, 2, 3, 3, 4.

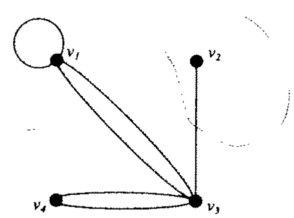
Odd number of odd vertices which is impossible

3 Mk

(b) A tree with five vertices with one of the vertices having degree 4.



(b) For the following graph determine the number of walks of length 3 starting and ending at v_1.



go to v_3 first : 4
use loop only : 1
use loop and then go to v_3 : 4

9

we can remove, since there is no way to visit them and come back in 3 steps.

4 Mk

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ 2 & 4 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 5 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 9 & \\ & \end{bmatrix}$$

4 Mk 3. (a) Given $L = (0 \vee 1)(10)^*$, find the six shortest strings in L.

(b) Consider $L = \{x \in \{0,1\}^* \mid x \text{ ends in } 01\}$

4 Mk i. Find a regular expression for L.

4 Mk ii. Give the state diagram for a finite state automaton M, which recognizes L.

8 Mk

4. Use the truth table below to determine whether the following argument form is valid. (Be sure to clearly indicate the premises, conclusions and critical rows and how you are using them to find your answer.)

$$\begin{array}{l} q \to r \\ p \vee q \\ \underline{r \vee \sim p} \\ \therefore\ r \end{array}$$

p	q	r	
T	T	T	
T	T	F	
T	F	T	
T	F	F	
F	T	T	
F	T	F	
F	F	T	
F	F	F	

5. Prove the following using the standard set identities. Be sure to clearly reference any identity you use.

8 Mk

For any sets A, B and C:

$$(A \cup B) - (C - A) = A \cup (B - C)$$

6. Consider the statement

$$(\forall x \in \mathbb{N})(\exists y \in \mathbb{N})(y < x)$$

8 Mk

(a) Translate the symbolic expression above into an English sentence. Your answer should not contain any mathematical symbols or variable names.

For every natural number, there exists a smaller natural number

(b) Give a counter-example to show that the statement above is not valid.

(For us (this year) $\mathbb{N}$ starts from 1.)

Counter example $x = 1$.

8 Mk

7. Show that if the difference of two squares is divisible by 2 then their difference is divisible by 4.

$$\text{i.e. } \forall n, m \in \mathbb{Z},\ 2 \mid (n^2 - m^2) \longrightarrow 4 \mid (n^2 - m^2)\,.$$

8. The following "proof" is incorrect. Clearly indicate the first line where an incorrect statement or inference is made and below give a brief explanation of what is wrong.

4 Mk

Every postage greater than or equal to 19¢ can be made up from only 5¢ and 7¢ stamps.

Let $P(n)$ denote the predicate that $n = 5\ell + 7m$ for some $\ell, m \in \mathbb{N}$.

Base Cases:

$$\begin{array}{rcrcrl} 19 & = & 5 & + & 2 \times 7, & \ell = 1,\ m = 2 \\ 20 & = & 4 \times 5 & & , & \ell = 4,\ m = 0 \\ 21 & = & & & 3 \times 7, & \ell = 0,\ m = 3 \\ 22 & = & 3 \times 5 & + & 1 \times 7, & \ell = 3,\ m = 1 \end{array}$$

Inductive Step:

Let $n = k \geq 22$,
Assume that $P(19), P(20), \ldots, P(k-1), P(k)$ are all true.
We will prove $P(k+1)$.
$\underline{P(k-4)}$ is true by the inductive hypothesis, since $k - 4 \leq k$.
Thus $k - 4 = 5\ell + 7m$ for some $\ell, m \in \mathbb{N}$.
$\Rightarrow k + 1 - 5 = 5\ell + 7m$ (Addition)
$\Rightarrow k + 1 = 5\ell + 7m + 5$ (Cancellation)
$\Rightarrow k + 1 = 5(\ell + 1) + 7m$ (Distributivity)
Thus $P(k+1)$ is true.

[**Note:** We assume that the set of natural numbers $\mathbb{N}$ includes 0.]

If $k = 22$, $k - 4 = 18 < 19$.
Hence, we cannot use ind. hyp. for $P(k-4)$ in this case!

8 Mk

9. Let f_n be the recursive sequence defined as follows:

$$f_0 = 1,\ f_1 = 2,\ f_2 = 4, \quad f_{n+1} = 2f_n - f_{n-1} + 2f_{n-2} \text{ for } n \geq 2.$$

Prove that $f_n = 2^n$, $\forall n \in \mathbb{N}$.

RYERSON UNIVERSITY

DEPARTMENT
OF
MATHEMATICS

MTH 314 **Final Exam** **April 17, 2009**

Total marks: 80 **Time allowed: 3 Hours**

NAME (Print): ______________________ **STUDENT #:** ______________________

Instructions:

- You are allowed an $8\frac{1}{2} \times 11$ formula sheet written on both sides.
- No other aids allowed. Electronic devices such as calculators, cell-phones, pagers, MP3 players, etc. must be turned off and kept inaccessible during the test.
- Verify that your paper contains 8 questions on 9 pages.
- In every question show all your work. The correct answer alone may be worth nothing.
- If you need more space continue on the back of the page, directing marker where the answer continues with a bold sign.

1. Use the truth table below to determine whether the equivalence below is true. Be sure to show how you are determining equivalence.

10 Mk

$$(q \wedge r) \to \sim p \equiv \sim p \vee \sim q \vee \sim r$$

p	q	r	$q \wedge r$	$\sim p$		$\sim q$	$\sim r$	
T	T	T	T	F	F	F	F	F
T	T	F	F	F	T	F	T	T
T	F	T	F	F	T	T	F	T
T	F	F	F	F	T	T	T	T
F	T	T	T	T	T	F	F	T
F	T	F	F	T	T	F	T	T
F	F	T	F	T	T	T	F	T
F	F	F	F	T	T	T	T	T

8 Mk

2. (a) Consider the statement

$$\forall x[x \in \mathbb{N} \to (\exists y(y \in \mathbb{N} \wedge x < y))]$$

Translate the statement above into an English sentence. Your answer should not contain any mathematical symbols or variable names.

~~The~~ For every natural number there exists a larger natural number.

(b) Is the statement true or false? (Circle the correct answer.)

True **False**

(c) Translate the following statement into a predicate formula using **only** the symbols from the list

$$<, \mathbb{R}, \in, \exists, \forall, \to, \sim, \wedge, \vee, (,), x, y, z, \text{ 'such that'}$$

For any two real numbers with one strictly greater than the other, some real number lies strictly between them.

$$\forall_{x\in\mathbb{R}} \forall_{y\in\mathbb{R}} \left(x<y \longrightarrow \exists_{z\in\mathbb{R}} (x<z) \wedge (z<y)\right)$$

Proof (Extra!)

Let $x, y \in \mathbb{R}$ s.t. $x<y$.

Then, take $z = \dfrac{x+y}{2}$ . . . .

10 Mk

3. Prove the following using the standard set identities such as DeMorgans laws, distributivity, etc.:

$$\forall A, B \subseteq U, \quad (A \cap B^c)^c \cap B^c = B^c - A.$$

$$LHS = (A \cap B^c)^c \cap B^c = \cancel{\left((A \cap B^c) \cup B\right)^c}$$

$$= (A^c \cup B) \cap B^c$$

$$= (A^c \cap B^c) \cup (B \cap B^c)$$

$$= (B^c \cap A^c) \cup \emptyset$$

$$= B^c \cap A^c$$

$$= B^c \setminus A = RHS$$

10 Mk

4. Prove that

$$\forall n \in \mathbb{N}, \quad \text{if 3 does not divide } n \text{ then 3 divides } n^2 - 1.$$

[<u>Hint:</u> look at possible remainders when n is divided by 3.]

Case I: $n = 3k+1, \; k \in \mathbb{Z}$

$$n^2 - 1 = (3k+1)^2 - 1 = 9k^2 + 6k + 1 - 1$$
$$= 3(\underbrace{3k^2+2k}_{\in \mathbb{Z}})$$

Case II: $n = 3k+2, \; k \in \mathbb{Z}$

$$n^2 - 1 = (3k+2)^2 - 1 = 9k^2 + 12k + 4 - 1$$
$$= 3(\underbrace{3k^2+4k+1}_{\in \mathbb{Z}})$$

10 Mk

5. Given the sequence a_n defined by the recurrence relation

$$a_0 = 0,$$
$$a_n = 2(a_{n-1} + 1),\ n \geq 1$$

(a) Calculate a_1, a_2, a_3, and a_4.

$$a_1 = 2(a_0+1) = 2(0+1) = 2$$
$$a_2 = 2(2+1) = 6 = 2^2+2^1$$
$$a_3 = 2(6+1) = 14 = 2^3+2^2+2$$
$$a_4 = 2(14+1) = 30 = 2^4+2^3+2^2+2$$

(b) Using iteration, solve the recurrence relation (i.e. find an explicit formula for a_n). Simplify your answer as much as possible. Your final solution should not contain sums (this means that if your final solution still contains sums, you will not get full marks for this question, but you may get part marks depending on the correctness of your answer).

$$a_n = 2^n + 2^{n-1} + \ldots + 2$$
$$= (2^n + 2^{n-1} + \ldots + 2 + 1) - 1$$
$$= (2^{n+1} - 1) - 1 = \underline{2^{n+1} - 2}$$

10 Mk

6. Define a sequence $a_1, a_2, a_3, \ldots$ as follows:

$$a_1 = 1, a_2 = 3, \text{ and } a_k = a_{k-1} + a_{k-2},$$

for all integers $k \geq 3$. Use strong mathematical induction to prove that

$$a_n \leq (\frac{7}{4})^n, \text{ for all integers } n \geq 1.$$

Base cas~~e~~(s): $a_1 = 1 \leqslant \left(\frac{7}{4}\right)^1$ ✓

$a_2 = 3 \leqslant \left(\frac{7}{4}\right)^2$ ✓

Assume: $a_k \leqslant \left(7/4\right)^k$ for all $k \leqslant n$

To show: $a_{k+1} \leqslant \left(7/4\right)^{k+1}$

$$LHS = a_{k+1} = a_k + a_{k-1} \underset{\text{ind. hyp}}{\leqslant} \left(7/4\right)^k + \left(7/4\right)^{k-1}$$

$$= \left(7/4\right)^{k-1}\left(7/4 + 1\right) = \left(7/4\right)^{k-1} \frac{11}{4}$$

$$\leqslant \left(7/4\right)^{k-1}\left(7/4\right)^2 = \left(7/4\right)^{k+1},$$

Since $\frac{11}{4} = \frac{11 \cdot 4}{4^2} = \frac{44}{4^2} < \frac{49}{4^2} = \frac{7^2}{4^2} = \left(7/4\right)^2$

10 Mk

7. Let L be the language given by

$$L = \{x \in \{0,1\}^* \mid x \text{ contains exactly one } 0\}.$$

(a) Construct a finite automaton A which accepts L:

(b) Write a regular expression r such that $L(r) = L$ (i.e. a regular expression generating L):

12 Mk

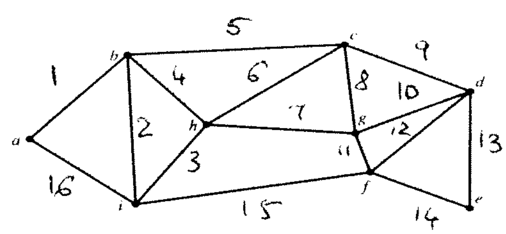
8\. (a) Does there exist a **simple** graph with vertices of degrees

$$1, 2, 3, 4, 4?$$

If it does, give an example of such a graph; if it does not, prove its non-existence.

We have two vertices of degree 4.
Each vertex is adjacent to everyone else.
Hence, the minimum degree is at least 2.
But there is one vertex of degree 1.
So such a graph <u>cannot</u> exist.

(b) Does the following graph have an Euler circuit? If it does, construct such a circuit; if it does not, prove that such a circuit cannot exist.



$$a_{k+1} = e^{\log a_{k+1}} \leq e^{\log(7/4)^{k+1}} = e^{(k+1)\log(7/4)}$$

$\cancel{\log(}$

$$k+1 \leq k + k - 1$$

$$k \leq 2k-2$$

$$k \leq 2(k-1)$$

$$\left(\frac{7}{4}\right)^{k+1} \leq \left(\frac{7}{4}\right)^{k} \cancel{\leq} \cancel{+}\left(\frac{7}{4}\right)^{k-1}$$

$\geq$

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MTH 314 **Final Exam** **April 23, 2012**

Total marks: 80 **Time allowed: 3 Hours**

NAME (Print): ______________ **STUDENT #:** ______________

Instructions:

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10 Mk

1. Use the truth table below to determine whether the following argument form is valid. (Be sure to clearly indicate the premises, conclusions and critical rows and how you are using them to find your answer.)

$$\begin{array}{l} q \to r \\ p \vee q \\ \underline{r \vee \sim p} \\ \therefore\ r \end{array}$$

p	q	r	q → r	p ∨ q	~p	r ∨ ~p	
T	T	T	T	T	F	T	* T ✓
T	T	F	F	T	F	F	
T	F	T	T	T	F	T	* T ✓
T	F	F	T	T	F	F	
F	T	T	T	T	T	T	* T ✓
F	T	F	F	T	T	T	
F	F	T	T	F	T	T	
F	F	F	T	F	T	T	

2. Prove the following using the standard set identities. Be sure to clearly reference any identity you use.

10 Mk

For any sets A, B and C:

$$(A \cup B) - (C - A) = A \cup (B - C)$$

$$LHS = (A \cup B) \setminus (C \setminus A) = (A \cup B) \setminus (C \cap A^c)$$

$$= (A \cup B) \cap (C \cap A^c)^c = (A \cup B) \cap (C^c \cup A)$$

$$= (A \cup B) \cap (A \cup C^c) = A \cup (B \cap C^c)$$

$$= A \cup (B \setminus C) = RHS$$

10 Mk

3. Show that if the difference of two squares is divisible by 2 then their difference is divisible by 4.

$$\text{i.e. } \forall n, m \in \mathbb{Z},\ 2 \mid (n^2 - m^2) \longrightarrow 4 \mid (n^2 - m^2)\,.$$

$$2 \mid (n-m)(n+m) \longrightarrow 4 \mid (n-m)(n+m)$$

I n even, m even

$n = 2k, k \in \mathbb{Z}$, $m = 2l, l \in \mathbb{Z}$

$(n-m)(n+m) = (2k)\cdot(2l) = 4(kl)$, $kl \in \mathbb{Z}$

conclusion is true ! ✓

II n even, m odd n − m is odd

n + m is odd

(n − m)(m + n) is odd

hence, the assumption is false ✓

III n odd, m even ⋮ ✓

IV n, m odd n − m is even

n + m is even

(n − m)(m + n) is divisible by 4

✓

10 Mk 4. Using mathematical induction, show that

$$2^n < (n+1)!$$

for all $n \geq 2$.

Base case: $n=2$ LHS $= 2^2 = 4$

RHS $= (2+1)! = 3! = 6$

LHS $<$ RHS

Assume: $2^n < (n+1)!$ for some $n \geq 2$.

To show: $2^{n+1} < (n+2)!$

(by ind hyp)

RHS $= (n+2)! = (n+1)!\,(n+2) > 2^n(n+2)$

$> 2^n \cdot 2 = 2^{n+1} =$ LHS

since $n+2 > 2$

10 Mk

5. Let f_n be the recursive sequence defined as follows:

$$f_0 = 1,\ f_1 = 2,\ f_2 = 4,\quad f_{n+1} = 2f_n - f_{n-1} + 2f_{n-2} \text{ for } n \geq 2.$$

Prove that $f_n = 2^n$, $\forall n \in \mathbb{N}$.

Strong induction!

Base case(s): $f_0 = 1 = 2^0$ ✓

$f_1 = 2 = 2^1$ ✓

$f_2 = 4 = 2^2$ ✓

Assumption: $f_k = 2^k$ for all $k \leq n$ for some $n \in \mathbb{N}$, $n \geq 2$

To show: $f_{n+1} = 2^{n+1}$

$$LHS = f_{n+1} = 2f_n - f_{n-1} + 2f_{n-2}$$

ind. hyp

$$= 2 \cdot 2^n - \cancel{2^{n-1}} + \cancel{2 \cdot 2^{n-2}}$$

$$= 2^{n+1} = RHS$$

6. Define a binary relation R on $\mathbb{Z}$ as follows:

$$xRy \Leftrightarrow 2|(x+y).$$

10 Mk

(a) Show that R is an equivalence relation on $\mathbb{Z}$.

reflexive: Let $x \in \mathbb{Z}$. Since $x+x=2x$, $x+x$ is div. by 2
Hence, xRx

Symmetric: Suppose that xRy; that is, $2|(x+y)$.
Since $x+y=y+x$, $2|(y+x)$. So yRx.

transitivity: Suppose that xRy and yRz.

$2|(x+y)$ $\Rightarrow$ $x+y=2k, k\in\mathbb{Z}$

$2|(y+z)$ $\Rightarrow$ $y+z=2l, l\in\mathbb{Z}$

Now, $x+z$
$= (x+y)+(y+z)-2y$
$= 2k+2l-2y = 2(k+l-y)$ $\in \mathbb{Z}$

Hence xRz

(b) List three elements in the equivalence class of 0, [0].

0, 2, 4

(c) Describe the distinct equivalence classes of R.

$[2] = [0] = \{\ldots, -4, -2, 0, 2, 4, \ldots\}$ = even

$[1] = \{\ldots, -3, -1, 1, 3, 5, \ldots\}$ = odd

10 Mk

7. Let L be the language given by

$$L = \{x \in \{0,1\}^* \mid x \text{ contains an even number of } 1\text{'s}\}.$$

(a) Construct a finite automaton A which accepts L:

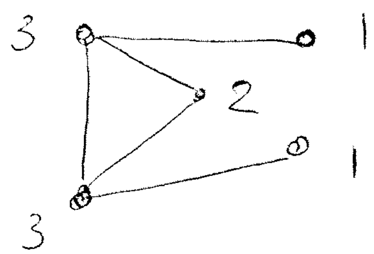
(b) Write a regular expression r such that $L(r) = L$ (i.e. a regular expression generating L):

10 Mk

8. (a) Does there exist a **simple** graph with vertices of degrees

$$1, 1, 2, 3, 3?$$

If it does, give an example of such a graph; if it does not, prove its non-existence.



(b) Does the following graph have a Hamiltonian circuit? If it does, construct such a circuit; if it does not, prove that such a circuit cannot exist.