1. [5 marks] Use the method of undetermined coefficients to find only the **form** of a particular solution of $y^{(4)} + y''' = 312x^2e^{-x} + 2015$. **Hint :** Do not find the coefficients.

[Similar to Problem #1 in the posted Sample test]

c.e.: $\lambda^4+\lambda^3=0 \rightarrow \lambda^3(\lambda+1)=0 \rightarrow \lambda=-1, 0$ (multiplicity 3)

$y_c = c_1 + c_2x + c_3x^2 + c_4e^{-x}$

Trial particular solution

Duplications!

$y_p = (Ax^2+Bx+C)e^{-x} + D$

Hence $y_p = \boxed{(Ax^3+Bx^2+Cx)e^{-x} + Dx^3}$

2. [5 marks] Given that $y_1(x) = x^2$ is a solution of the differential equation $x^2y'' + 2xy' - 6y = 0$, where $0 < x < \infty$, find the general solution.

[problem #10 in section 3.2]

Standard form : $y'' + \frac{2}{x}y' - \frac{6}{x^2} = 0$ gives $p(x) = \frac{2}{x}$

$$y_2 = x^2 \int \frac{e^{-\int \frac{2}{x}dx}}{x^4}\, dx$$

$$= x^2 \int \frac{e^{-2\ln x}}{x^4}\, dx$$

$$= x^2 \int \frac{x^{-2}}{x^4}\, dx$$

$$= x^2 \int x^{-6} dx$$

$$= x^2 \cdot \left(-\tfrac{1}{5}x^{-5}\right) = -\tfrac{1}{5}x^{-3}$$

G.S.: $\boxed{y = c_1x^2 + c_2x^{-3}}$

3. [6 marks] Given that the functions $\{x, x^{-2}, x^{-2}\ln x\}$ are solutions of the differential equation $x^3y''' + 6x^2y'' + 4xy' - 4y = 0$, where $0 < x < \infty$, determine if they form a fundamental set of solutions.

[problem #29 in section 3.1]

$$W(x, x^{-2}, x^{-2}\ln x) = \begin{vmatrix} x & x^{-2} & x^{-2}\ln x \\ 1 & -2x^{-3} & -2x^{-3}\ln x + x^{-2}\cdot\frac{1}{x} \\ 0 & 6x^{-4} & 6x^{-4}\ln x - 2x^{-3}\cdot\frac{1}{x} - 3x^{-4} \end{vmatrix}$$

$$= x\begin{vmatrix} -2x^{-3} & -2x^{-3}\ln x + x^{-3} \\ 6x^{-4} & 6x^{-4}\ln x - 5x^{-4} \end{vmatrix} - \begin{vmatrix} x^{-2} & x^{-2}\ln x \\ 6x^{-4} & 6x^{-4}\ln x - 5x^{-4} \end{vmatrix}$$

$$= x(-12x^{-7}\ln x + 10x^{-7} + 12x^{-7}\ln x - 6x^{-7})$$
$$- (6x^{-6}\ln x - 5x^{-6} - 6x^{-6}\ln x)$$

$$= x(4x^{-7}) - (-5x^{-6})$$

$$= 9x^{-6} \neq 0$$

Since $W(x, x^{-2}, x^{-2}\ln x)$ is not zero,

$\{x, x^{-2}, x^{-2}\ln x\}$ is a linearly independent set.

Hence $\{x, x^{-2}, x^{-2}\ln x\}$ forms a fundamental set.

4. [12 marks] Use the variation of parameters to find the unique solution of given Initial Value Problem:

$$y'' + 2y' + y = e^{-x}\ln x \qquad y(1) = 0, y'(1) = 0$$

[Based on the assigned homework problem #15 in section 3.5]

step1/ $\lambda^2 + 2\lambda + 1 = 0 \rightarrow \lambda = -1$ (multiplicity 2)

$\therefore\ y_c = c_1 e^{-x} + c_2 x e^{-x}$

step2/ $W(e^{-x}, xe^{-x}) = \begin{vmatrix} e^{-x} & xe^{-x} \\ -e^{-x} & e^{-x} - xe^{-x} \end{vmatrix} = e^{-2x} - xe^{-2x} + xe^{-2x} = e^{-2x}$

$u_1' = e^{2x}\begin{vmatrix} 0 & xe^{-x} \\ e^{-x}\ln x & e^{-x} - xe^{-x} \end{vmatrix} = e^{2x}(-xe^{-x} \cdot e^{-x}\ln x) = -x\ln x$

$u_2' = e^{2x} \cdot \begin{vmatrix} e^{-x} & 0 \\ -e^{-x} & e^{-x}\ln x \end{vmatrix} = e^{2x} \cdot e^{-2x} \cdot \ln x = \ln x$

$u_1 = -\int x\ln x\, dx \xrightarrow[dv = x dx]{u = \ln x} -\frac{1}{2}x^2 \cdot \ln x + \frac{1}{2}\int x^2 \cdot \frac{1}{x}\, dx = -\frac{1}{2}x^2 \ln x + \frac{1}{4}x^2$

$u_2 = \int \ln x\, dx \xrightarrow[dv = x dx]{u = \ln x} x\ln x - \int x \frac{1}{x}\, dx = x\ln x - x$

Therefore, $y_p = (-\frac{1}{2}x^2\ln x + \frac{1}{4}x^2)e^{-x} + x(x\ln x - x)e^{-x}$

$= -\frac{1}{2}x^2 e^{-x}\ln x + \frac{1}{4}x^2 e^{-x} + x^2 e^{-x}\ln x - x^2 e^{-x}$

step3/ $y = c_1 e^{-x} + c_2 x e^{-x} + \frac{1}{2}x^2 e^{-x}\ln x - \frac{3}{4}x^2 e^{-x}$

step4/ $y' = -c_1 e^{-x} + c_2 e^{-x} - c_2 x e^{-x} + xe^{-x}\ln x - \frac{1}{2}x^2 e^{-x}\ln x + \frac{1}{2}xe^{-x} - \frac{3}{2}xe^{-x} + \frac{3}{4}x^2 e^{-x}$

I.C. $y(1) = 0$: $\quad 0 = c_1 e^{-1} + c_2 e^{-1} - \frac{3}{4}e^{-1} \longrightarrow c_1 + c_2 = \frac{3}{4}$

$y'(1) = 0$: $\quad 0 = -c_1 e^{-1} + c_2 e^{-1} - c_2 e^{-1} + 0 - 0 + \frac{1}{2}e^{-1} - \frac{3}{2}e^{-1} + \frac{3}{4}e^{-1}$

$\rightarrow -c_1 e^{-1} - \frac{1}{4}e^{-1} = 0 \rightarrow c_1 = \frac{-1}{4}$

$c_2 = 1$

Unique solution: $\boxed{y = -\frac{1}{4}e^{-x} + xe^{-x} + \frac{1}{2}x^2 e^{-x}\ln x - \frac{3}{4}x^2 e^{-x}}$

5. [10 marks] Solve the given differential equation;

$$y'' + \omega^2 y = k \sin \omega x$$

where ω, k are nonzero constants.

[shortened version of homowork #33 in section 3.4]

Step 1/ $\lambda^2 + w^2 = 0 \longrightarrow \lambda = \pm wi \longrightarrow y_c = c_1 \cos wx + c_2 \sin wx$

Step 2/ $y_p = Ax \cos wx + Bx \sin wx$ ($\because$ there is a duplication)

$$y_p' = A \cos wx - Awx \sin wx + B \sin wx + Bwx \cos wx$$

$$= A \cos wx + B \sin wx + x(-Aw \sin wx + Bw \cos wx)$$

$$y_p'' = -Aw \sin wx + Bw \cos wx + (-Aw \sin wx + Bw \cos wx) + \underbrace{x(-Aw^2 \cos wx - Bw^2 \sin wx)}_{-w^2 y_p}$$

D.E.: $-2Aw \sin wx + 2Bw \cos wx - w^2 y_p + w^2 y_p \overset{\text{must}}{\underset{\text{be}}{=\!=}} k \sin wx$

$$\begin{cases} -2Aw = k \longrightarrow A = \dfrac{-k}{2w} \\ 2Bw = 0 \longrightarrow B = 0 \end{cases}$$

Step 3

G.S.: $\boxed{y = c_1 \cos wx + c_2 \sin wx - \dfrac{k}{2w} x \cos wx}$

6. [12 marks] Solve the given Initial Value Problem:

$$y'' + 9y = g(x), \quad y(0) = 0, \quad y'(0) = 3,$$

$$\text{where } g(x) = \begin{cases} 18 & 0 \le x \le \pi \\ 0 & x > \pi \end{cases}.$$

[simplified version of homework #42 in section 3.4]

step1/ $\lambda^2 + 9 = 0 \longrightarrow \lambda = \pm 3i \longrightarrow y_c = C_1 \cos 3X + C_2 \sin 3X$

step 2/ $\boxed{\text{on } [0,\pi]}$ $y_p \overset{\text{guess}}{=\!=} A$, $y_p' = y_p'' = 0$

D.E.: $0 + 9A \overset{\text{must be}}{=\!=} 18 \longrightarrow A = 2$ Therefore $y_p = 2$

step3/ $y = C_1 \cos 3X + C_2 \sin 3X + 2$

step4/ $y' = -3C_1 \sin 3X + 3C_2 \cos 3X$

I.C. $y(0) = 0 \longrightarrow 0 = C_1 + 2 \longrightarrow C_1 = -2$

I.C. $y'(0) = 3 \longrightarrow 3 = 0 + 3C_2 \longrightarrow C_2 = 1$

on $[0,\pi]$, its solution $y = \boxed{-2\cos 3X + \sin 3X + 2}$, $y' = 6\sin 3X + 3\cos 3X$

$\boxed{\text{on } (\pi, \infty)}$ its general solution $y = \boxed{C_3 \cos 3X + C_4 \sin 3X}$ $y' = -3C_3 \sin 3X + 3C_4 \cos 3X$

at $x = \pi$, y must be continuous

$$-2\cos 3\pi + \sin 3\pi + 2 \overset{\text{must be}}{=\!=} C_3 \cos 3\pi + C_4 \sin 3\pi$$

$$2 + 2 = -C_3 \longrightarrow C_3 = -4$$

at $x = \pi$, y' must be continuous

$$6\sin 3\pi + 3\cos 3\pi \overset{\text{must be}}{=\!=} -3C_3 \sin 3\pi + 3C_4 \cos 3\pi$$

$$-3 = -3C_4 \longrightarrow C_4 = 1$$

on (π, ∞) its solution

$$y = -4\cos 3X + \sin 3X$$

Hence

the Unique solution $y(x) = \begin{cases} -2\cos 3X + \sin 3X + 2 & 0 \le X \le \pi \\ -4\cos 3X + \sin 3X & X > \pi \end{cases}$