1. [5 marks] Find points on the surface $x^2 + 4x + y^2 + z^2 - 2z = 11$ at which the tangent plane is [parallel to yz-plane] ~~vertical~~.   #27, sec. 9.6

$\nabla F = (2x+4)\,\mathbf{i} + 2y\,\mathbf{j} + (2z-2)\,\mathbf{k}$, so a normal to the surface at (x_0, y_0, z_0) is $(2x_0+4)\,\mathbf{i} + 2y_0\,\mathbf{j} + (2z_0-2)\,\mathbf{k}$.

A vertical plane has normal $c\,\mathbf{i}$ for $c \neq 0$

Thus $\begin{cases} 2x_0+4 = c \rightarrow x_0 = \frac{1}{2}(c-4) \\ 2y_0 = 0 \longrightarrow y_0 = 0 \\ 2z_0 - 2 = 0 \rightarrow z_0 = 1 \end{cases}$

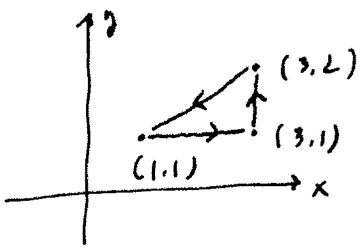
Since (x_0, y_0, z_0) is on the surface, $\left[\frac{1}{2}(c-4)\right]^2 + 4\cdot\left[\frac{1}{2}(c-4)\right] + 0 + 1 - 2 = 11$

$(c-4)^2 + 8(c-4) - 48 = 0$

$[(c-4)+12][(c-4)-4] = 0$

$\therefore \begin{cases} c-4 = -12 \rightarrow c = -8 \\ c-4 = 4 \rightarrow c = 8 \end{cases}$ The points are $(-6, 0, 1), (2, 0, 1)$

2. [5 marks] Find the work done by the force $\boldsymbol{F}(x, y) = (312x + 2y)\,\vec{\mathbf{i}} + (2016y - 2x)\,\vec{\mathbf{j}}$ acting counterclockwise once around the triangle with vertices $(3, 2)$, $(1, 1)$ and $(3, 1)$.   # 33, sec 9.8



$P(x,y) = 312x + 2y$ and $Q(x,y) = 2016y - 2x$

$\frac{\partial P}{\partial y} = 2$ and $\frac{\partial Q}{\partial x} = -2$

$$\oint_C (312x+2y)\,dx + (2016y - 2x)\,dy$$

$$= \iint_R (-2-2)\,dA$$

$$= -4 \underbrace{\iint_R dA}_{\text{Area of Triangle}}$$

$$= -4 \cdot \frac{1}{2} \cdot 2 \cdot 1 = \boxed{-4}$$

* It can be solved by line integrals.

3. [6 marks] Evaluate $\displaystyle\int_{(1,0,0)}^{(2,-\frac{\pi}{2},1)} (2x\sin y + e^{3z})\,dx + (x^2\cos y)\,dy + (3xe^{3z}+5)\,dz.$

#21, sec 9.9

$\frac{\partial\phi}{\partial x} = 2x\sin y + e^{3z}$ implies $\phi(x,y,z) = x^2\sin y + e^{3z}\cdot x + f(y,z)$.

$\frac{\partial\phi}{\partial y} \overset{\text{want}}{\underset{\text{be}}{=}} x^2\cos y$ implies $x^2\cos y + f_y(y,z) \overset{\text{want}}{\underset{\text{be}}{=}} x^2\cos y$

so $f_y(y,z)=0$

$f(y,z) = g(z)$

Therefore $\phi(x,y,z) = x^2\sin y + xe^{3z} + g(z)$

$\frac{\partial\phi}{\partial z} \overset{\text{want}}{\underset{\text{be}}{=}} 3xe^{3z}+5$ means $3xe^{3z}+g'(z) \overset{\text{want}}{\underset{\text{be}}{=}} 3xe^{3z}+5$

$g'(z)=5$

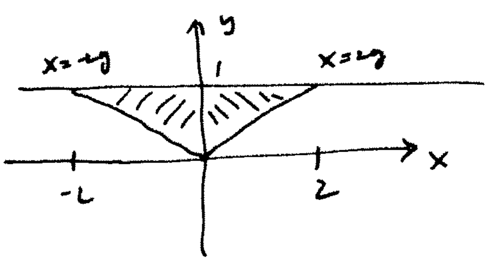
$g(z)=5z$

Thus $\phi(x,y,z) = x^2\sin y + xe^{3z} + 5z$

$$\int_{(1,0,0)}^{(2,-\frac{\pi}{2},1)} \cdots = x^2\sin y + xe^{3z} + 5z\Big]_{(1,0,0)}^{(2,-\frac{\pi}{2},1)} = \left(4\cdot\sin(-\tfrac{\pi}{2}) + 2e^3 + 5\right) - \left(1\cdot\sin 0 + 1e^0 + 0\right) = \boxed{2e^3}$$

4. [4 marks] Rewrite the iterated integral $\displaystyle\int_0^1\int_{-2y}^{2y} \cancel{\frac{312y}{2016x}}\,dx\,dy$ by reversing the order of integration. Do **not eveluate the integral.**

#36, sec 9.10

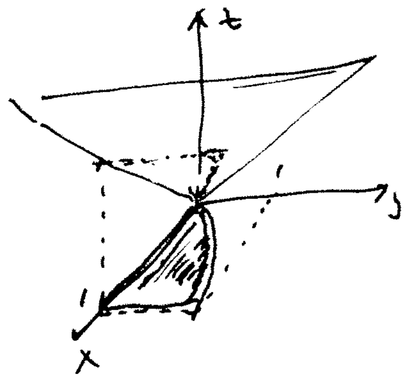


$$\int_0^1\int_{-2y}^{2y} f(x,y)\,dx\,dy$$

$$= \int_{-2}^{0}\int_{-\frac{1}{2}x}^{1} f(x,y)\,dy\,dx + \int_0^2\int_{\frac{1}{2}x}^{1} f(x,y)\,dy\,dx$$

5. [7 marks] Find the volume of the solid in the first octant bounded by the graphs of the equations : $z = x + y, y = x^2, x = 1, y = 0,$ and $z = 0$.

#45, sec 9.10



$$V = \iint_R f(x,y)\,dA$$

$$= \int_0^1 \int_0^{x^2} (x+y)\,dy\,dx$$

$$= \int_0^1 xy + \tfrac{1}{2}y^2 \Big]_0^{x^2} dx$$

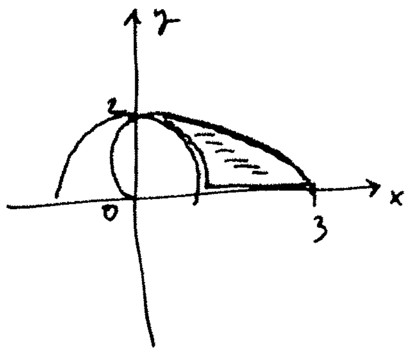
$$= \int_0^1 \left(x^3 + \tfrac{1}{2}x^4\right) dx$$

$$= \tfrac{1}{4}x^4 + \tfrac{1}{10}x^5 \Big]_0^1$$

$$= \tfrac{1}{4} + \tfrac{1}{10}$$

$$= \frac{14}{40} = \boxed{\frac{7}{20}}$$

6. [8 marks] Find the area of the region in the first quadrant bounded by the graphs of the equations : $r = 2, r = 2 + 2\cos\theta$, and $y = 0$. #15, sec 9.11



$$A = \iint_R dA$$

$$= \int_0^{\frac{\pi}{2}} \int_2^{2+2\cos\theta} r\,dr\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{2}r^2\Big]_2^{2+2\cos\theta} d\theta$$

$$= \int_0^{\frac{\pi}{2}} \left[\frac{1}{2}\left(2^2(1+\cos\theta)^2\right) - \frac{1}{2}2^2\right] d\theta$$

$$= \int_0^{\frac{\pi}{2}} \left[2\left(1+2\cos\theta + \frac{1+\cos 2\theta}{2}\right) - 2\right] d\theta$$

$$= \int_0^{\frac{\pi}{2}} (4\cos\theta + 1 + \cos 2\theta)\,d\theta$$

$$= 4\sin\theta + \theta + \frac{1}{2}\sin 2\theta\Big]_0^{\frac{\pi}{2}}$$

$$= \boxed{4 + \frac{\pi}{2}}$$

7. [6 marks] Show that $\int_C -ydx + xdy = ad - bc$, where C is the line segment from the point (a, b) to (c, d).

#21, sec 9.12

curve C can be expressed by $y - b = \frac{d-b}{c-a}(x-a)$ → (Method 1)

or $\begin{cases} x = a + (c-a)t \\ y = b + (d-b)t \end{cases}$ $0 \le t \le 1$. → (Method 2)

Method 1/

$$\int_C -ydx + xdy = \int_a^c \left[-\frac{d-b}{c-a}(x-a) - b\right] \cdot dx + x\underbrace{\left(\frac{d-b}{c-a}\right)dx}_{dy}$$

$$= \int_a^c \left(\frac{d-b}{c-a} \cdot a - b\right) dx$$

$$= \left(\frac{d-b}{c-a}a - b\right)x\Big]_a^c = \left(\frac{d-b}{c-a}a - b\right)(c-a) = (d-b)a - b(c-a)$$

$$= ad - ba - bc + ba$$

$$= \boxed{ad - bc}$$

Method 2/

$$\int_C -ydx + xdy = \int_0^1 \left[\left(-b-(d-b)t\right)\cdot(c-a) + \left(a + (c-a)t\right)\cdot(d-b)\right]dt$$

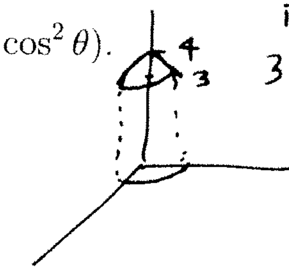
$$= \int_0^1 \left[-b(c-a) - (d-b)(c-a)t + a(d-b) + (d-b)(c-a)t\right]dt$$

$$= \int_0^1 \left(-b(c-a) + a(d-b)\right)dt = -bc + ab + ad - ab$$

$$= \boxed{ad - bc}$$

8. [9 marks] Find the upward flux $\iint_S (\boldsymbol{F}\cdot\vec{\mathbf{n}})\,d\boldsymbol{S}$ for the vector field $\boldsymbol{F}(x,y,z)=\frac{1}{2}x^2\,\vec{\mathbf{i}}+\frac{1}{2}y^2\,\vec{\mathbf{j}}+z\,\vec{\mathbf{k}}$ through the surface S in the first octant given by that portion of the paraboloid $z=4-x^2-y^2$ for $3\le z\le 4$. #33 sec 9.13

Hint : $\cos^3\theta=\cos\theta(1-\sin^2\theta)$ and $\sin^3\theta=\sin\theta(1-\cos^2\theta)$.



intersection
$3=4-x^2-y^2$
$x^2+y^2=1$

part 1/ $g(x,y,z)=x^2+y^2+z-4$ gives

$$\nabla g = 2x\,i+2y\,j+k$$

$$\vec{n}=\frac{2x\,i+2y\,j+k}{\sqrt{4x^2+4y^2+1}}$$

part 2/ $$\vec{F}\cdot\vec{n}=\frac{x^3+y^3+z}{\sqrt{4x^2+4y^2+1}}$$

part 3/ $f(x,y)=4-x^2-y^2 \longrightarrow \begin{cases} f_x=-2x \\ f_y=-2y\end{cases} \longrightarrow ds=\sqrt{1+4x^2+4y^2}\,dA$

part 4/ $$\text{Flux}=\iint_S \vec{F}\cdot\vec{n}\,ds=\iint_R (x^3+y^3+z)\,dA=\iint_R (x^3+y^3+(4-x^2-y^2))\,dA$$

$$=\int_0^{\frac{\pi}{2}}\int_0^1 (r^3\cos^3\theta+r^3\sin^3\theta+4-r^2)\,r\,dr\,d\theta$$

$$=\int_0^{\frac{\pi}{2}} \frac{1}{5}r^5\cos^3\theta+\frac{1}{5}r^5\sin^3\theta+2r^2-\frac{1}{4}r^4\Big]_0^1\,d\theta$$

$$=\int_0^{\frac{\pi}{2}}\left(\frac{1}{5}\cos^3\theta+\frac{1}{5}\sin^3\theta+\frac{7}{4}\right)d\theta$$

$$=\int_0^{\frac{\pi}{2}}\left(\frac{1}{5}\left[\cos\theta(1-\sin^2\theta)+\sin\theta(1-\cos^2\theta)\right]+\frac{7}{4}\right)d\theta$$

$$=\int_0^{\frac{\pi}{2}}\left[\frac{1}{5}\left(\cos\theta-\sin^2\theta(\cos\theta)+\sin\theta+\cos^2\theta(-\sin\theta)\right)+\frac{7}{4}\right]d\theta$$

$$=\frac{1}{5}\left[\sin\theta-\frac{1}{3}\sin^3\theta-\cos\theta+\frac{1}{3}\cos^3\theta\right]+\frac{7}{4}\theta\Big]_0^{\frac{\pi}{2}}$$

$$=\left(\frac{1}{5}\left(1-\frac{1}{3}-0+0\right)+\frac{7}{4}\cdot\frac{\pi}{2}\right)-\left(\frac{1}{5}\left(-1+\frac{1}{3}\right)\right)$$

$$=\frac{1}{5}\cdot\frac{2}{3}+\frac{7}{8}\pi+\frac{1}{5}\cdot\frac{2}{3}=\boxed{\frac{4}{15}+\frac{7}{8}\pi}$$