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1. [10 marks] Solve the given initial value problem : [Assigned Homework #21 in Section 3.12]

$$\begin{cases} \frac{dx}{dt} + 5x + y = 0 \\ 4x - \frac{dy}{dt} - y = 0, \end{cases} \qquad x(1) = 0, \quad y(1) = 1.$$

$$\begin{cases} (D+5)x + y = 0 \quad \cdots ① \\ 4x - (D+1)y = 0 \quad \cdots ② \end{cases}$$

$① \times (D+1) + ②$: $(D+5)(D+1)x + 4x = 0$

$(D^2+6D+9)x = 0 \rightarrow D = -3$

$\boxed{x(t) = c_1 e^{-3t} + c_2 t e^{-3t}}$ 3

① gives $y = -(D+5)x$

$= -(-3c_1 e^{-3t} + c_2 e^{-3t} - 3c_2 t e^{-3t} + 5c_1 e^{-3t} + 5c_2 t e^{-3t})$

$= \boxed{-(2c_1 + c_2)e^{-3t} - 2c_2 t e^{-3t}}$ 3

$x(1) = 0 \rightarrow c_1 e^{-3} + c_2 e^{-3} = 0 \rightarrow c_1 + c_2 = 0 \rightarrow c_2 = -c_1$

$y(1) = 1 \rightarrow 1 = -(2c_1 + c_2)e^{-3} - 2c_2 e^{-3t}$

$1 = (-2c_1 - 3c_2)e^{-3} \rightarrow -(2c_1 + 3c_2) = e^3 \rightarrow$ $c_1 = e^3$, $\therefore c_2 = -e^3$ 2

Solution :
$$\begin{cases} x = e^3 \cdot e^{-3t} - e^3 \cdot t \cdot e^{-3t} = \boxed{e^{3-3t} - t e^{3-3t}} \\ y = -e^3 e^{-3t} + 2e^3 t e^{-3t} = \boxed{-e^{3-3t} + 2t e^{3-3t}} \end{cases}$$ 2

2. [10 marks] Find the Fourier series of the function f on the given interval.

[Shortened version of homework #9 in section 12.2]

$$f(x) = \begin{cases} 0 & -\pi < x \le 0 \\ \sin x & 0 \le x < \pi \end{cases}.$$

Hint : $a_0 = \dfrac{2}{\pi}, a_1 = 0, a_n = \dfrac{1+(-1)^n}{\pi(1-n^2)}, n = 2, 3, \ldots$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin\frac{n\pi}{\pi}x\,dx = \frac{1}{\pi}\int_0^{\pi} \sin x \cdot \sin nx\,dx$$

$\sin A \cdot \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$

$$= \frac{1}{\pi}\int_0^{\pi} \frac{1}{2}[\cos(1-n)x - \cos(1+n)x]\,dx \quad 2$$

$n \neq 1$

$$= \frac{1}{\pi}\cdot\frac{1}{2}\left[\frac{1}{1-n}\sin(1-n)x\Big]_0^{\pi} - \frac{1}{1+n}\sin(1+n)x\Big]_0^{\pi}\right]$$

$$= \frac{1}{2\pi}\left(\left[\cancel{\frac{1}{1-n}\sin(1-n)\pi}^{\,0} - \cancel{\frac{1}{1-n}\sin 0}^{\,0}\right] - \left[\frac{1}{1+n}\sin(1+n)\pi - 0\right]\right)$$

$$= 0, \quad n \ge 2 \quad 3$$

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$$b_1 = \frac{1}{\pi}\int_0^{\pi} \sin x \cdot \sin x\,dx = \frac{1}{\pi}\int_0^{\pi} \sin^2 x\,dx = \frac{1}{\pi}\int_0^{\pi} \frac{1-\cos 2x}{2}\,dx$$

$$= \frac{1}{\pi}\left[\frac{1}{2}x - \frac{1}{2}\cdot\frac{1}{2}\sin 2x\right]_0^{\pi} \quad 2$$

$$= \frac{1}{\pi}\left[\left(\frac{1}{2}\pi - \frac{1}{4}\sin 2\pi\right) - (0-0)\right] = \frac{1}{2}. \quad 1$$

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Hence Fourier Series:

$$f(x) = \frac{1}{2}\left(\frac{2}{\pi}\right) + \left(a_1 \cos\frac{1\pi}{\pi}x + b_1 \sin\frac{1\pi}{\pi}x\right) + \sum_{n=2}^{\infty}\left(\frac{1+(-1)^n}{\pi(1-n^2)}\cdot\cos nx + 0\right)$$

$n=1$ 1

$$= \frac{1}{\pi} + \frac{1}{2}\sin x + \sum_{n=2}^{\infty}\frac{1}{\pi}\cdot\frac{1+(-1)^n}{1-n^2}\cdot\cos nx.$$

1 1

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3. [10 marks] Solve the integral equation :

$$f(t) = 3 + \int_0^t f(\tau)\ \cos(2(t-\tau))\, d\tau$$

[similar to Homework #42 in section 4.4]

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{3\} + \mathcal{L}\{f(t)\}\cdot\mathcal{L}\{\cos 2t\}$$

$$\mathcal{L}\{f\} = \frac{3}{s} + \mathcal{L}\{f\}\cdot\frac{s}{s^2+2^2}$$

$$\left(1-\frac{s}{s^2+4}\right)\mathcal{L}\{f\} = \frac{3}{s}$$

$$\mathcal{L}\{f\} = \frac{3(s^2+4)}{s(s^2-s+4)}$$

where $\frac{s^2+4}{s(s^2-s+4)} = \frac{A}{s} + \frac{Bs+C}{s^2-s+4}$

$\Rightarrow s^2+4 = A(s^2-s+4)+(Bs+C)s$

coeff of s^2: $1 = A+B \longrightarrow B=0$

coeff of s : $0 = -A+C \longrightarrow C=1$

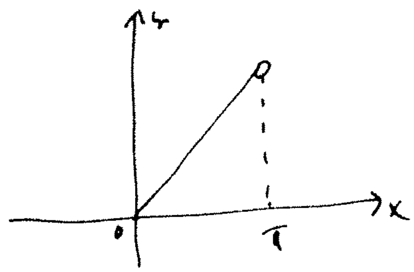
Const : $4 = 4A \longrightarrow A=1$

Hence $\frac{s^2+4}{s(s^2-s+4)} = \frac{1}{s} + \frac{1}{s^2-s+4}$

Therefore, $\mathcal{L}\{f\} = \frac{3}{s} + 3\cdot\frac{1}{(s-\frac{1}{2})^2+\frac{15}{4}} = \frac{3}{s} + 3\cdot\frac{2}{\sqrt{15}}\cdot\frac{\frac{\sqrt{15}}{2}}{(s-\frac{1}{2})^2+(\frac{\sqrt{15}}{2})^2}$

$$\therefore f(t) = \boxed{3 + \frac{6}{\sqrt{15}} e^{\frac{1}{2}t}\cdot\sin\left(\frac{\sqrt{15}}{2}\right)t}$$

5. [10 marks] Expand $f(x) = x,\ 0 < x < \pi$ in a Fourier series. [Assigned homework #36 in section 12.3]



$p = \frac{\pi}{2}$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x\,dx = \frac{2}{\pi}\cdot\frac{1}{2}\pi^2 = \pi \qquad 2$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x\cos 2nx\,dx = \frac{2}{\pi}\left[x\frac{1}{2n}\sin 2nx\Big]_0^{\pi} - \frac{1}{2n}\int_0^{\pi}\sin 2nx\,dx \right]$$

$$= \frac{2}{\pi}\left[\left(\cancel{\frac{\pi}{2n}\sin 2n\pi}^{\,0} - 0\right) + \frac{1}{2n}\cdot\frac{1}{2n}\cdot\cos 2nx\Big]_0^{\pi}\right] = 0 \qquad 3$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} x\sin 2nx\,dx = \frac{2}{\pi}\left[-x\cdot\frac{1}{2n}\cos 2nx\Big|_0^{\pi} + \frac{1}{2n}\int_0^{\pi}\cos 2nx\,dx \right]$$

$$= \frac{2}{\pi}\left[-\frac{\pi}{2n}(\cos 2n\pi - 0\cos 0) + \cancel{\frac{1}{2n}\cdot\frac{1}{2n}\sin 2n\pi}^{\,0}\Big|_0^{\pi} \right] \qquad 3$$

$$= \frac{2}{\pi}\left[-\frac{\pi}{2n}\cdot 1\right] = \boxed{-\frac{1}{n}}$$

$$f(x) = \boxed{\frac{\pi}{2} - \sum_{n=1}^{\infty}\frac{1}{n}\sin 2nx} \qquad 2$$

5. [10 marks] Use the Laplace Transform to solve the given Initial Value Problem; [similar to Homework #66 in section 4.3]

$$y'' + 9y = g(t), \quad y(0) = 0, \quad y'(0) = 3,$$

where $g(t) = \begin{cases} 18 & 0 \le t \le \pi \\ 0 & t > \pi \end{cases}$ $\Rightarrow 18(1 - \mathcal{U}(t-\pi))$

$$[s^2 \mathcal{L}\{y\} - s \cdot 0 - 3] + 9\mathcal{L}\{y\} = \mathcal{L}\{18(1-\mathcal{U}(t-\pi))\} \quad 2/2$$

$$(s^2+9)\mathcal{L}\{y\} = 3 + \frac{18}{s} - \frac{18e^{-\pi s}}{s}$$

$$\therefore \mathcal{L}\{y\} = \frac{3}{s^2+9} + 18\left[\frac{1}{s(s^2+9)} - \frac{e^{-\pi s}}{s(s^2+9)}\right] \quad 1/1$$

$\boxed{M1}$

where $\frac{1}{s(s^2+9)} = \frac{A}{s} + \frac{Bs+C}{s^2+9} \Rightarrow 1 = A(s^2+9) + (Bs+C)s$ 1/1

$$\begin{cases} B + A = 0 \rightarrow B = -\frac{1}{9} \quad 1/1 \\ C = 0 \\ 9A = 1 \rightarrow A = \frac{1}{9} \quad 1/1 \end{cases}$$

$$\therefore \frac{1}{s(s^2+9)} = \frac{1}{9}\frac{1}{s} - \frac{1}{9}\frac{s}{s^2+9}$$

$$\mathcal{L}\{y\} = \frac{3}{s^2+9} + \left[\frac{2}{s} - \frac{2s}{s^2+9}\right] - \left[\frac{2}{s} - \frac{2s}{s^2+9}\right]e^{-\pi s} \quad 2/2$$

$$y = \boxed{\sin 3t + [2 - 2\cos 3t] - [2 - 2\cos 3(t-\pi)]\,\mathcal{U}(t-\pi)}$$

that is, $y = \begin{cases} \boxed{\sin 3t + 2 - 2\cos 3t} & 0 < t \le \pi \\ \sin 3t + 2 - 2\cos 3t - 2 + 2\cos(3t - 3\pi) & \end{cases}$

$$= \sin 3t - 2\cos 3t + 2(\cos 3t \cdot \cos(3\pi))$$

$$= \boxed{\sin 3t - 4\cos 3t} \quad t > \pi \quad 3$$

$\boxed{M2}$ since $\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+9)}\right\}$

$$= \frac{1}{3}\int_0^t \sin 3\tau \, d\tau = -\frac{1}{9}\cos 3\tau \Big|_0^t = \frac{-\cos 3t + 1}{9}, \quad 1/1 \quad 2/2$$

$$\therefore y = \underbrace{\sin 3t - 2\cos 3t + 2}_{1/1} \underbrace{- 2[1 - \cos 3(t-\pi)]\,\mathcal{U}(t-\pi)}_{2/2}$$