The Dynamics of the Elastic Pendulum

How does the change in the magnitude of the mass affect the motion of a Damped Elastic Harmonic Oscillator?

***Subject**: Physics*

***Word Count**: 3980*

Contents

1 Introduction

I have chosen this research question because the fundamental relation between the ***mechanics of the damped elastic pendulum****, its* ***complex dynamics****, its* ***unstable chaotic behavior*** *and its* ***excruciating difficulty in possibly modeling*** *has led me to explore this field, as this phenomenon is exciting, unique and wonderful.*

2 Aim

To ***collect experimental raw data*** *relating to the variation in* ***radial displacement****, the variation in* ***angular displacement****, the variation in* ***angular frequency*** *and the variation in* ***absolute frequency*** *in a lab controlled setting of a* ***damped elastic harmonic oscillator*** *and to compare it with* ***computer simulations*** *of the same and hence study its* ***different states*** *and calculate the value the of* ***inconsistencies*** *in the* ***accuracy*** *of the* ***mathematically modeled computer simulation*** *to that of the* ***actual experiment*** *and vice-versa.*

3 Hypothesis

The ***system of the damped elastic pendula*** *would act very similar to the the* ***dynamics of the damped simple pendulum****, and that the damping that would take place would be* ***nearly equal*** *to the damped simple pendulum, which means that the* ***motion should be approximately similar to the simple pendulum in all degrees of freedom****.*

4 Background Research

Before beginning this investigation we must first know some important facts, formulae, laws and be familiar with the concepts that are to be incorporated in this investigation.

4.1 The Lagrangian function

Firstly, we must know that the fundamental concepts of ***Lagrangian mechanics*** *and* ***Lagrange's equations****.*

The ***Lagrangian*** *is a function that is mathematically defined as,*

$$L = T - V$$

Where, L is the Lagrangian, T is the ***kinetic energy*** *of the system and V is the* ***potential energy*** *of the system.*

The Lagrange's equation of second kind or the Lagrange-Euler Equation is mathematically defined as,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = Q_j$$

Where q_j *is the* ***generalized coordinates*** *and* Q_j *is the* ***generalized forces****.*

4.2 Generalized coordinates

According to our specific system that we are investigating, we have two generalized coordinates in ***polar coordinate representation****,* θ *and x.*

θ *represents the* ***angle formed by the spring component with the fictitious normal*** *of the system at any given time and is a function of time (time-dependent).*

x represents the ***extension in the spring component from the equilibrium length*** *at any given time and is a function of time (time - dependent).*

4.3 Generalized forces

According to our specific system that we are investigating, we only have one generalized forces (non-conservative force), the ***frictional force*** *acting on the path of the system in the form of drag or fluid resistance.*

Our frictional force can be embedded in Lagrange's equations in the form of the Q_j *term aided with the utilization of the* ***Rayleigh dissipation function****.*

As there isn't any other non-conservative force, the net total sum of the Lagrange's equations with the incorporation of the Rayleigh dissipation function would equal zero.

4.4 Air Drag/Fluid Resistance

Air Drag/Fluid Resistance is the force acting opposite to the relative motion of any object moving in any fluid medium. Drag force is proportional to the ***square***

of velocity, *as we are dealing with relatively high-speeds, which can be inferred from the small Reynolds's number.*

Drag forces decrease the fluid velocity relative to the solid mass in the fluid's path.

The type of Drag in play in this system is that of an underdamped ($\zeta < 1$) oscillator with viscous drag.

Note: *ζ here, in the context of Drag forces and resistive forces symbolizes the damping ratio.*

The general Drag equation is mathematically defined as,

$$F_D = \frac{1}{2}\rho v^2 C_D A$$

Where F_D is the ***Air/Fluid resistance*** *between the mass and the fluid, ρ is the* ***density of the fluid***, *v is the* ***speed of the object*** *relative to the fluid, C_D is* ***velocity decay constant*** *(damping constant) and A is the* ***cross sectional area***.

Note: *The Drag force can be embedded and modeled into the Euler-Lagrange equation by the aid of the Rayleigh dissipation function and can be generally modeled using Stokes Law, as we are using object masses that are spherical.*

4.5 Rayleigh dissipation function

Secondly, we must also know that,

If the frictional force on a particle with velocity $\vec{v}$ can be written as $\vec{F}_f = -\vec{k} \cdot \vec{v}$, the Rayleigh dissipation function can be defined for a system of ***n*** *particles as,*

$$R/D = \frac{1}{2}\sum_{i=0}^{n} C_D v_i^2 = \frac{1}{2}C_D \sum_{i=0}^{n} v_i^2$$

Where R or D is the Rayleigh dissipation function which is a function used to handle and model the effects of ***velocity-proportional frictional forces in Lagrangian mechanics***, *where C_D is the* ***velocity decay constant*** *(damping constant) and $\sum_{i=0}^{n} v_i^2$ is the sum of all velocities squared in all degrees of freedom pertaining to a mechanical system with n being the number of degrees of freedom in the system.*

Note: *An alternate representation could also be used to represent C_D in the form of a $n \times n$ matrix. Mathematically,*

$$C_D = \begin{bmatrix} k_i & 0 & 0 & \cdots & 0 \\ 0 & k_j & 0 & \cdots & 0 \\ 0 & 0 & k_k & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & 0 & k_n \end{bmatrix}$$

4.6 Stokes Law

We must know the Stroke's Law and the relation it defines to accurately model the change in Air Drag/Fluid Resistance with time.

Stroke's Law states a relation for the frictional force (drag force) exerted on ***spherical objects*** *with very* ***small Reynolds numbers*** *in a viscous fluid. Mathematically,*

$$F_D = 6\pi\mu R v_D$$

Note*: This relation is only applicable in this scenario because the mass that we are dealing with is spherical.*

Where F_D is the small Reynolds numbers between the mass and the fluid, μ is the ***dynamic viscosity*** *of the fluid, R is the* ***radius*** *of the mass and v_D is the* ***flow velocity*** *relative to the mass.*

4.7 Logarithmic Decrement

The system exhibits an interesting feature, that of constant logarithmic decrements, that is,

$$\ln\frac{x_1}{x_2} = \ln\frac{x_2}{x_3} = \ln\frac{x_3}{x_4} = \cdots\cdots\cdots$$

Where x_n and x_{n+1} are the amplitudes of any two successive peaks ($n \in \mathbb{R}$).

Also interestingly that for any two successive peaks of a graphical representation of any parameter versus time employed in this investigation, if we define,

$$\delta = \ln\left(\frac{x_n}{x_{n+1}}\right)$$

For a graphical representation of any parameter employed in this investigation.

Then,

$$\zeta = \frac{\delta}{\sqrt{\delta^2 + (2\pi)^2}}$$

With the above findings and definitions, we can create an expression for percentage overshoot, that is,

$$\textit{Percentage Overshoot} = 100 \cdot \exp\left(-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}\right)$$

*Consequently we can find an expression for ζ in terms of **percentage overshoot** (PO). That is,*

$$\zeta = \frac{-\ln\left(\frac{PO}{100}\right)}{\sqrt{\pi^2 + \ln^2\left(\frac{PO}{100}\right)}}$$

***Note**: ζ here, in the context of Drag forces and resistive forces symbolizes the damping ratio.*

4.8 Computer Simulation Software

*To **model** the **chaotic system of the Damped Harmonic Elastic Pendulum** we would need the aid of **computational technology**. Software's such MATLAB, Mathematica or some simple, eccentric computer programming language code in a emulatable script format, that would compute and yield solutions for necessary simulations that are required.*

*For the purpose of this investigation we shall be using a **Python script**, the source code of which can be found in **appendix A**, to model and simulate the dynamic and chaotic system.*

5 Materials Required

- *Pendulum Stand/Fixed Pivot*
- *Spherical Masses/Pendulum Masses of 1 kg, 2 kg, 3 kg, 4 kg, 5 kg, 6 kg, 7 kg, 8 kg, 9 kg and 10 kg*
- *Spring with spring stiffness constant of 100 N/m*

- *Weighing Scale (Digital/Analog)*

***Note**: In theory, the spring stiffness constant incorporated in research could be of any arbitrary value. For the purpose of this investigation, we shall specifically use springs of spring stiffness constant of 100 N/m.*

6 Variables

Physical Quantity	*Symbol*
Spring constant	k
Equilibrium spring length	l_0
Extension length	x
Angular displacement/Polar Angle	θ
Radius of spherical mass	r

Table 6.1: *General physical quantities employed in this investigation*

***Note**: In theory, the equilibrium spring length and the radius of the spherical mass incorporated in research could of any arbitrary value. For the purpose of this investigation, we shall specifically use springs of equilibrium spring length of 1 m and masses of radius* 5×10^{-2} *m.*

Independent Variable	*Dependent Variable*	*Controlled Variable*
Time	*Extension length*	*Temperature*
-	*Displacement*	*Fluid medium*
-	*Angular displacement*	*Reference point*
-	*Absolute frequency*	-

Table 6.2: *Segregation of employed variables as IV, DV or CV*

***Note**: The spring constant, equilibrium length and the radius of the mass is no* *longer variable as we have defined a set value to it.*

7 Procedure

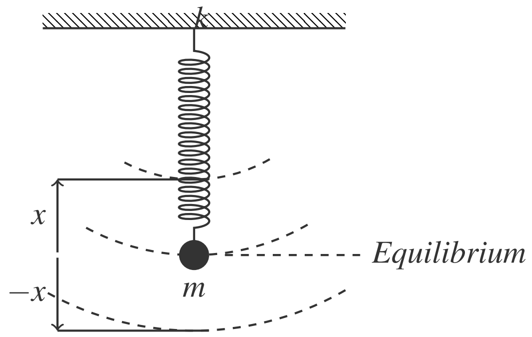

Figure 7.1: *Diagram of the experiment carried down in this investigation*

Using the materials specified in chapter 5, arrange the components as specified in figure 7.1

Following which, according to the ***specified initial conditions*** *specified in the next chapter carefully oscillate the pendulum.*

8 Initial Conditions

We define certain ***initial conditions*** *for this dynamic system and convert this system to an* ***initial value problem*** *(IVP) system.*

$$x(0) = 0$$

Where $x(0)$ *is the initial extension length of the spring at time t = 0*

$$\theta(0) = \pi/2$$

Where $\theta(0)$ is the initial angular displacement of the system at time t = 0

9 Equations

9.1 Equations of Motion

We know that,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} + \frac{\partial R}{\partial q_j} = 0$$

*Where L is the **Lagrangian**, q_i is the **generalized coordinates**, R is the **Rayleigh dissipation function** and "Q_j" is the **generalized forces**.*

According to our specific case that we are investigating, we have two generalized coordinates, "θ" and "x".

*Therefore the **coupled set of differential equations** that describe the motion of the damped elastic harmonic pendulum (Equations of Motion) with respect to the two generalized coordinates x and θ are,*

$$m \cdot \ddot{x} - m \cdot (l_0 + x)\dot{\theta}^2 + kx - g \cdot m \cdot \cos\theta + C_D \dot{x} = 0$$

$$m \cdot (l_0 + x)^2 \ddot{\theta} + 2m \cdot (l_0 + x)\dot{x}\dot{\theta} + g \cdot m \cdot (l_0 + x)\sin\theta + C_D \dot{\theta} = 0$$

With $\ddot{x}$ and $\ddot{\theta}$ isolated the equations are,

$$\ddot{x} = (l_0 + x)\dot{\theta}^2 - \frac{k}{m} \cdot x + g \cdot \cos\theta - \frac{C_D}{m}\dot{x}$$

$$\ddot{\theta} = -\frac{2}{(l_0 + x)}\dot{x}\dot{\theta} - \frac{g}{(l_0 + x)} \cdot \sin\theta - \frac{C_D}{m \cdot (l_0 + x)^2}\dot{\theta}$$

*Where, m is the **mass**, l_0 is the **rest length** of the spring, x is the **extension length** of the spring, θ is the **angle made between the spring and the normal**, g is the **gravitational acceleration**, k is the **spring stiffness constant** and C_D **damping coefficient**.*

***Note**: When damping is zero (Absolute Harmonic Motion), C_D is zero.*

These equations will be directly inputted into the python source code from ***appendix A*** *to solve these equations* ***numerically*** *with an accuracy of* $\pm 10^{-8}$ *for each time step of the* ***Runge-Kutta 4th Order*** *(RK4) algorithm.*

Note*: The numerical value of each time step involved in the simulation is* 10^{-4} *of a second.*

9.2 Drag Equation

We know that the general Drag equation is mathematically defined as,

$$F_D = \frac{1}{2}\rho v^2 C_D A$$

Where F_D *is the* ***Air/Fluid resistance*** *between the mass and the fluid,* ρ *is the* ***density of the fluid****, v is the* ***speed of the object*** *relative to the fluid,* C_D *is* ***velocity decay constant*** *(damping constant) and A is the* ***cross sectional area****.*

But as the fluid medium that we are dealing with is simply ***atmospheric air*** *at* 25° *and at 1 atm pressure,* $\rho = 1.1839 kg/m^3$. *Also that, the physical nature of the mass that we are dealing with is* ***spherical****, therefore* $C_D = 0.47$ *and* $A = \pi r^2$.

Therefore, we have,

$$F_D = \pi \cdot \frac{432165}{10^6} \cdot r^2 v^2$$

9.3 Stokes Law Equation

We know that,

$$F_D = 6\pi\mu R v_D$$

Where F_D *is the* ***Air/Fluid resistance*** *between the mass and the fluid,* μ *is the* ***dynamic viscosity*** *of the fluid, R is the* ***radius*** *of the mass and* v_D *is the* ***flow velocity*** *relative to the mass.*

But as the fluid medium that we are dealing with is simply ***atmospheric air*** *at* 25° *and at 1 atm pressure,* $\mu = 18.6\mu P \cdot s = 1.86 \times 10^{-5} P \cdot s$.

Also with close observation it can be inferred that the magnitude of ***flow velocity*** *relative to the mass is the same as the magnitude of the* ***absolute velocity*** *of the system. Therefore using* ***Newtonian vector*** *definition, we define,*

$$v = \sqrt{(\ddot{x})^2 + (\ddot{\theta})^2}$$

Therefore we can infer that,

$$v_D = \sqrt{(\ddot{x})^2 + (\ddot{\theta})^2}$$

Therefore, we finally have,

$$F_D = \frac{11.16}{10^5} \cdot \pi \sqrt{(\ddot{x})^2 + (\ddot{\theta})^2} \cdot R$$

9.4 Frequency Equations

The frequency of the system can be broken down into two types of frequencies if we consider it from the perspective of the two generalized coordinates. They are radial and angular frequencies

9.4.1 Radial frequency

Radial frequency is defined as the number of oscillations of the radial movement (Spring component of the system) per second. Physically and mathematically defined as,

$$\omega_r = \sqrt{\frac{k}{m}}$$

With observation, it can be inferred that, radial frequency remains constant whatsoever in the whole system for a particular case when ***m*** *is a constant.*

9.4.2 Angular frequency

Angular frequency is defined as the number of oscillations of the angular movement (Pendulum component of the system) per second. Physically and mathematically defined as,

$$\omega_\theta = \sqrt{\frac{g}{l_0 + x}}$$

With observation, it can be inferred that, angular frequency is a function of radial position and will be a variable all throughout in the whole system.

9.4.3 Absolute frequency

If we are to talk about absolute frequency, we can define absolute frequency using Newtonian vector definition as,

$$f = \sqrt{(\omega_r)^2 + (\omega_\theta)^2}$$

Therefore we have,

$$f = \sqrt{\frac{k}{m} + \frac{g}{l_0 + x}}$$

9.5 Fundamental Derived Equations

The fundamental derived equations we shall be using in this investigation are:

$$\ddot{x} = (l_0 + x)\dot{\theta}^2 - \frac{k}{m} \cdot x + g \cdot \cos\theta - \frac{C_D}{m}\dot{x} \tag{9.1}$$

$$\ddot{\theta} = -\frac{2}{(l_0 + x)}\dot{x}\dot{\theta} - \frac{g}{(l_0 + x)} \cdot \sin\theta - \frac{C_D}{m \cdot (l_0 + x)^2}\dot{\theta} \tag{9.2}$$

***Note**: The coefficient of damping (C_D) from now on will be equal to the numeric value of 0.47.*

$$F_D = \pi \cdot \frac{432165}{10^6} \cdot r^2 v^2 \tag{9.3}$$

$$F_D = \frac{11.16}{10^5} \cdot \pi \sqrt{(\ddot{x})^2 + (\ddot{\theta})^2} \cdot R \tag{9.4}$$

***Note**: R and r represent the same physical quantity, i.e., the radius of the mass used in a specific case.*

$$\omega_r = \sqrt{\frac{k}{m}} \tag{9.5}$$

$$\omega_\theta = \sqrt{\frac{g}{l_0 + x}} \tag{9.6}$$

$$f = \sqrt{\frac{k}{m} + \frac{g}{l_0 + x}} \tag{9.7}$$

***Note**: All the equations mentioned above shall be inputted algorithmically in an computer program/software and shall be numerically estimated to about an accuracy of approximately $\pm 10^{-8}$ of each time step interval.*

10 Experimental Data

10.1 Radial Displacement versus Time

Time / *Mass*	0	2.5	5.0	7.5	10.0	12.5	15.0	17.5	20.0
1	0	0.008	0.086	0.107	0.090	0.096	0.097	0.097	0.098
2	0	0.110	0.330	0.350	0.298	0.227	0.168	0.161	0.195
3	0	0.558	0.420	0.021	0.258	0.537	0.522	0.440	0.301
4	0	1.062	-0.003	0.831	0.262	0.2031	0.805	0.332	0.157
5	0	1.189	0.165	0.611	0.431	0.583	0.280	0.890	0.086
6	0	1.079	0.924	-0.012	1.313	0.228	0.759	0.596	0.396
7	0	0.854	1.686	0.223	0.182	1.556	0.211	0.652	1.334
8	0	0.579	1.832	1.230	-0.007	0.493	1.746	0.471	0.162
9	0	0.308	1.467	2.121	1.094	-0.114	0.455	1.825	1.403
10	0	0.062	0.923	2.062	2.204	1.304	-0.023	0.144	1.196

Table 10.1: *Experimental Data on **radial displacement** versus **time** from 0 to 20 seconds.*

***Note**: Mass is in **kilograms**, Time is in **seconds**, and the values of radial displacement versus time is in **meters per second**.*

10.2 Angular Displacement versus Time

Time / *Mass*	0	2.5	5.0	7.5	10.0	12.5	15.0	17.5	20.0
1	$\pi/2$	0.931	0.418	-0.045	-0.208	-0.097	0.029	0.052	0.018
2	$\pi/2$	0.942	0.462	0.282	0.287	0.365	0.418	0.365	0.237
3	$\pi/2$	0.364	-0.555	-1.005	-0.591	-0.191	0.082	0.219	0.335
4	$\pi/2$	0.052	-1.223	-0.222	0.704	0.683	0.022	-0.476	-0.604
5	$\pi/2$	-0.167	-0.948	0.491	0.588	-0.429	-0.608	0.163	0.684
6	$\pi/2$	-0.346	-0.395	1.155	-0.123	-0.720	0.355	0.415	-0.361
7	$\pi/2$	-0.493	-0.108	0.851	-0.714	0.006	0.625	-0.384	-0.157
8	$\pi/2$	-0.627	0.077	0.289	-0.949	0.5312	0.007	-0.404	0.526
9	$\pi/2$	-0.754	0.250	0.020	-0.395	0.808	-0.5195	0.0485	0.166
10	$\pi/2$	-0.879	0.464	-0.134	-0.062	0.290	-0.601	0.591	-0.115

Table 10.2: *Experimental Data on **angular displacement** versus **time** from 0 to 20 seconds.*

***Note**: Mass is in **kilograms**, Time is in **seconds**, and the values of angular displacement versus time is in **meters per second**.*

10.3 Angular Frequency versus Time

Time / *Mass*	0	2.5	5.0	7.5	10.0	12.5	15.0	17.5	20.0
1	3.130	3.118	3.006	2.976	2.994	2.991	2.989	2.989	2.988
2	3.128	2.971	2.716	2.696	2.749	2.828	2.897	2.906	2.870
3	3.129	2.509	2.630	3.099	2.794	2.526	2.539	2.607	2.771
4	3.130	2.181	3.136	2.314	2.790	2.859	2.332	2.715	2.925
5	3.128	2.117	2.906	2.471	2.627	2.492	2.768	2.274	2.996
6	3.129	2.175	2.261	3.153	2.060	2.830	2.363	2.481	2.667
7	3.133	2.303	1.912	2.834	2.883	1.960	2.848	2.538	2.086
8	3.130	2.493	1.861	2.103	3.148	2.566	1.890	2.581	2.980
9	3.131	2.741	1.995	1.773	2.168	3.331	2.597	1.865	2.046
10	3.131	3.040	2.255	1.790	1.750	2.643	3.170	2.936	2.127

Table 10.3: *Experimental Data on **angular frequency** versus **time** from 0 to 20 seconds.*

***Note**: Mass is in **kilograms**, Time is in **seconds**, and the values of angular frequency versus time is in **Hertz**.*

10.4 Absolute Frequency versus Time

Mass \ Time	0	2.5	5.0	7.5	10.0	12.5	15.0	17.5	20.0
1	10.478	10.474	10.442	10.434	10.440	10.438	10.437	10.437	10.437
2	7.7330	7.6700	7.5751	7.5677	7.5869	7.6157	7.6419	7.6452	7.6331
3	6.5675	6.2949	6.3448	6.5533	6.4140	6.3022	6.3072	6.3353	6.4018
4	5.8990	5.4550	5.9024	5.5098	5.7264	5.7581	5.5173	5.6895	5.7859
5	5.4588	4.9477	5.3319	5.1085	5.1840	5.1188	5.2601	5.0174	5.3830
6	5.1443	4.6250	4.6665	5.1564	4.5728	4.9661	4.7167	4.7776	4.8840
7	4.9075	4.4259	4.2359	4.7248	4.7560	4.25726	4.7335	4.4975	4.3170
8	4.7220	4.3255	3.9956	4.1124	4.7324	4.3660	4.0096	4.3801	4.6280
9	4.5728	4.3177	3.8847	3.7756	3.9763	4.7117	4.2269	3.8194	3.9060
10	4.4492	4.3860	3.8853	3.6340	3.6148	3.7780	4.476	4.3150	3.8050

Table 10.4: *Experimental Data on **absolute frequency** versus **time** from 0 to 20 seconds.*

Note: *Mass is in* ***kilograms***, *Time is in* ***seconds***, *and the values of absolute frequency versus time is in* ***Hertz***.

11 Simulation Data

11.1 Radial Displacement versus Time

Mass \ Time	0	2.5	5.0	7.5	10.0	12.5	15.0	17.5	20.0
1	0	0.0208	0.0579	0.0995	0.0884	0.0925	0.0967	0.0964	0.0973
2	0	0.1550	0.3814	0.3850	0.3373	0.2688	0.1916	0.1546	0.1778
3	0	0.7400	0.2550	0.0246	0.4000	0.5760	0.4740	0.3480	0.2620
4	0	1.0400	0.0120	0.9149	0.1368	0.356	0.7702	0.2000	0.1858
5	0	0.9870	0.3402	0.3793	0.6565	0.3680	0.4730	0.7240	0.1067
6	0	0.7830	1.1904	-0.0157	1.1406	0.4410	0.5061	0.8350	0.3361
7	0	0.5280	1.7330	0.4370	0.0078	1.4960	0.4510	0.3920	1.3653
8	0	0.2734	1.6078	1.5275	0.1501	0.2296	1.6615	0.7760	0.0770
9	0	0.0316	1.106	1.4240	0.0360	0.2080	1.6170	1.6120	1.4870
10	0	-0.1304	0.5778	1.7646	2.2672	1.6546	0.1937	-0.0095	1.0826

Table 11.1: *Simulation Data on* ***radial displacement*** *versus* ***time*** *from 0 to 20 seconds.*

Note: *Mass is in* ***kilograms***, *Time is in* ***seconds***, *and the values of radial displacement versus time is in* ***meters per second***.

11.2 Angular Displacement versus Time

Time / *Mass*	0	2.5	5.0	7.5	10.0	12.5	15.0	17.5	20.0
1	$\pi/2$	0.9511	0.5263	-0.1798	-0.1802	-0.1241	0.0058	0.0520	0.0224
2	$\pi/2$	0.7043	0.2804	0.1237	0.1480	0.2566	0.2573	0.3713	0.2580
3	$\pi/2$	0.1812	-0.7474	-0.9800	-0.4415	-0.0532	0.2189	0.3368	0.3938
4	$\pi/2$	-0.0913	-1.2379	-0.0741	0.8570	0.5420	-0.1042	-0.5949	-0.5705
5	$\pi/2$	-0.3086	-0.7538	0.6648	0.4344	-0.5780	-0.4749	0.2761	0.6362
6	$\pi/2$	0.4959	-0.2663	1.1854	-0.2326	-0.5803	0.4764	0.3063	-0.4548
7	$\pi/2$	-0.6630	-0.0158	0.6695	-0.8783	0.0931	0.5025	-0.5022	-0.1170
8	$\pi/2$	-0.8159	0.1684	0.1885	-0.8180	0.6686	-0.0651	-0.3160	0.5713
9	$\pi/2$	-0.9628	0.3679	-0.0456	-0.2958	0.6982	-0.6453	0.1078	0.1342
10	$\pi/2$	-1.0971	0.6205	-0.1991	0.0050	0.2226	-0.4875	0.6922	-0.1783

Table 11.2: *Simulation Data on **angular displacement** versus **time** from 0 to 20 seconds.*

***Note**: Mass is in **kilograms**, Time is in **seconds**, and the values of angular displacement versus time is in **meters per second**.*

11.3 Angular Frequency versus Time

Time / *Mass*	0	2.5	5.0	7.5	10.0	12.5	15.0	17.5	20.0
1	3.1321	3.1007	3.0449	2.9865	3.0024	2.9967	2.9914	2.9913	2.9000
2	3.1322	2.9163	2.6660	2.6620	2.7092	2.7827	2.8696	2.8698	2.8866
3	3.1322	2.3750	2.7964	3.0935	2.6483	2.4962	2.5802	2.6976	2.7872
4	3.1322	2.1929	3.1178	2.2637	2.9387	2.6881	2.3560	2.8647	2.8769
5	3.1322	2.2232	2.7084	2.6694	2.4352	2.6782	2.5813	2.3862	2.9762
6	3.1322	2.3482	2.1162	3.1574	2.1406	2.6132	2.5529	2.3122	2.7093
7	3.1320	2.5360	1.8960	2.6130	3.1200	1.9836	2.6070	2.6520	2.0378
8	3.1321	2.7760	1.9404	1.9713	2.9170	2.829	1.9190	2.3510	3.0182
9	3.1322	3.059	2.1570	1.7687	2.0156	3.0836	2.8540	1.9380	1.9835
10	3.1320	3.3590	2.4950	1.8890	1.7331	1.9212	2.8660	3.1475	2.1712

Table 11.3: *Simulation Data on **angular frequency** versus **time** from 0 to 20 seconds.*

Note: *Mass is in* ***kilograms****, Time is in* ***seconds****, and the values of angular frequency versus time is in* ***Hertz****.*

11.4 Absolute Frequency versus Time

Mass \ Time	0	2.5	5.0	7.5	10.0	12.5	15.0	17.5	20.0
1	10.479	10.470	10.453	10.436	10.441	10.439	10.437	10.437	10.437
2	7.7337	7.6492	7.5570	7.5556	7.5723	7.5987	7.6312	7.6484	7.6375
3	6.5684	6.2428	6.4151	6.5506	6.3528	6.2901	6.3238	6.3728	6.4113
4	5.9000	5.4600	5.8926	5.4884	5.8001	5.6766	5.5272	5.7622	5.7682
5	5.4599	4.9942	5.2282	5.2083	5.0917	5.2140	5.1638	5.0686	5.3725
6	5.1457	4.7102	4.5987	5.1613	4.6102	4.8473	4.8162	4.6935	4.9000
7	4.9087	4.5525	4.2284	4.5955	4.9023	4.2684	4.5896	4.6200	4.2935
8	4.7238	4.496	4.0329	4.0476	4.5860	4.5270	4.0235	4.246	4.6491
9	4.5740	4.5260	3.9705	3.7735	3.8951	4.543	4.3880	3.8562	3.8788
10	4.4509	4.6130	4.0281	3.6842	3.6063	3.7019	4.2690	4.4620	3.8556

Table 11.4: *Simulation Data on* ***absolute frequency*** *versus* ***time*** *from 0 to 20 seconds.*

Note: *Mass is in* ***kilograms****, Time is in* ***seconds****, and the values of absolute frequency versus time is in* ***Hertz****.*

12 Derived Data

We shall be modeling the ***viscous drag*** *with respect to time for all cases with the* ***aid of the simulation program*** *with the initial utility of equations 9.3 and 9.4, shall be called derived data, as it is derived from the values of* ***radial*** *and* ***angular displacement*** *versus* ***time****, and this is not the primary focus of this investigation.*

All data necessary relating to this chapter of the investigation can be found in ***appendix D***

13 Observations

Let ζ_e, τ_e, χ_e *and* ψ_e *be equal to the* ***sum of the experimental observations*** *from* $t = 0$ *to* $t = 20$ *seconds divided by the number of observations, for* ***radial***

displacement, ***angular displacement***, ***angular frequency*** *and* ***absolute frequency*** *respectively for each and every case that we are investigating.*

Let ζ_s, τ_s, χ_s *and* ψ_s *be equal to the* ***sum of the simulation observations*** *from* $t = 0$ *to* $t = 20$ *seconds divided by the number of observations, for* ***radial displacement***, ***angular displacement***, ***angular frequency*** *and* ***absolute frequency*** *respectively for each and every case that we are investigating.*

We further define, $\zeta 0_n = \frac{\zeta_e}{\zeta_s}$, $\tau 0_n = \frac{\tau_e}{\tau_s}$, $\chi 0_n = \frac{\chi_e}{\chi_s}$ *and* $\psi 0_n = \frac{\psi_e}{\psi_s}$ *for each and every case.*

Here, upon close observation we infer that, $100 \cdot \zeta 0_n$, $100 \cdot \tau 0_n$, $100 \cdot \chi 0_n$ *and* $100 \cdot \psi 0_n$ *are the* ***percentage accuracy*** *of the experimental values of radial displacement, angular displacement, angular frequency and absolute frequency respectively for each and every case that we are investigating.*

By using the above definitions we have,

13.1 Observation from Case 1 where m = 1 kg

$\zeta_{e_1} = 0.075444$ *and* $\zeta_{s_1} = 0.072167$

$\tau_{e_1} = 0.296533$ *and* $\tau_{s_1} = 0.293811$

$\chi_{e_1} = 3.02011$ *and* $\chi_{s_1} = 3.01622$

$\psi_{e_1} = 10.4463$ *and* $\psi_{s_1} = 10.4477$

Therefore,

$\zeta 0_1 = 1.04542$

$\tau 0_1 = 1.00927$

$\chi 0_1 = 1.00129$

$\psi 0_1 = 0.999872$

13.2 Observation from Case 2 where m = 2 kg

$\zeta_{e_2} = 0.204333$ *and* $\zeta_{s_2} = 0.227944$

$\tau_{e_2} = 0.547644$ *and* $\tau_{s_2} = 0.441155$

$\chi_{e_2} = 2.86233$ *and* $\chi_{s_2} = 2.83269$

$\psi_{e_2} = 7.62984$ *and* $\psi_{s_2} = 7.6204$

Therefore,

$\zeta_{0_2} = 0.896417$

$\tau_{0_2} = 1.24139$

$\chi_{0_2} = 1.01047$

$\psi_{0_2} = 1.00124$

13.3 Observation from Case 3 where m = 3 kg

$\zeta_{e_3} = 0.339667$ *and* $\zeta_{s_3} = 0.342178$

$\tau_{e_3} = 0.025422$ *and* $\tau_{s_3} = 0.053266$

$\chi_{e_3} = 2.64633$ *and* $\chi_{s_3} = 2.73407$

$\psi_{e_3} = 6.39122$ *and* $\psi_{s_3} = 6.39197$

Therefore,

$\zeta_{0_3} = 0.992661$

$\tau_{0_3} = 0.477259$

$\chi_{0_3} = 0.967911$

$\psi_{0_3} = 0.999884$

13.4 Observation from Case 4 where m = 4 kg

$\zeta_{e_4} = 0.405456$ *and* $\zeta_{s_4} = 0.401744$

$\tau_{e_4} = 0.056311$ *and* $\tau_{s_4} = 0.096322$

$\chi_{e_4} = 2.70911$ *and* $\chi_{s_4} = 2.71456$

$\psi_{e_4} = 5.69371$ *and* $\psi_{s_4} = 5.69726$

Therefore,

$\zeta_{0_4} = 1.00924$

$\tau_{0_4} = 0.584612$

$\chi_{0_4} = 0.997992$

$\psi_{0_4} = 0.999377$

13.5 Observation from Case 5 where m = 5 kg

$\zeta_{e_5} = 0.470556$ *and* $\zeta_{s_5} = 0.4483$

$\tau_{e_5} = 0.149422$ *and* $\tau_{s_5} = 0.163$

$\chi_{e_5} = 2.64211$ *and* $\chi_{s_5} = 2.64337$

$\psi_{e_5} = 5.20113$ *and* $\psi_{s_5} = 5.20013$

Therefore,

$\zeta_{0_5} = 1.04965$

$\tau_{0_5} = 0.916699$

$\chi_{0_5} = 0.999523$

$\psi_{0_5} = 1.00019$

13.6 Observation from Case 6 where m = 6 kg

$\zeta_{e_6} = 0.587$ *and* $\zeta_{s_6} = 0.4685$

$\tau_{e_6} = 0.2162$ *and* $\tau_{s_6} = 0.277866$

$\chi_{e_6} = 2.56878$ *and* $\chi_{s_6} = 2.56469$

$\psi_{e_6} = 4.83438$ *and* $\psi_{s_6} = 4.83146$

Therefore,

$\zeta_{0_6} = 1.25293$

$\tau_{0_6} = 0.778073$

$\chi_{0_6} = 1.00159$

$\psi_{0_6} = 1.0006$

13.7 Observation from Case 7 where m = 7 kg

$\zeta_{e_7} = 0.744222$ *and* $\zeta_{s_7} = 0.712233$

$\tau_{e_7} = 0.132977$ *and* $\tau_{s_7} = 0.073288$

$\chi_{e_7} = 2.49967$ *and* $\chi_{s_7} = 2.5086$

$\psi_{e_7} = 4.53948$ *and* $\psi_{s_7} = 4.55099$

Therefore,

$\zeta_{0_7} = 1.04491$

$\tau_{0_7} = 1.81444$

$\chi_{0_7} = 0.99644$

$\psi_{0_7} = 0.997471$

13.8 Observation from Case 8 where m = 8 kg

$\zeta_{e_8} = 0.722889$ *and* $\zeta_{s_8} = 0.700322$

$\tau_{e_8} = 0.113444$ *and* $\tau_{s_8} = 0.128066$

$\chi_{e_8} = 2.528$ *and* $\chi_{s_8} = 2.53933$

$\psi_{e_8} = 4.36351$ *and* $\psi_{s_8} = 4.37021$

Therefore,

$\zeta_{0_8} = 1.03222$

$\tau_{0_8} = 0.885824$

$\chi_{0_8} = 0.995538$

$\psi_{0_8} = 0.998467$

13.9 Observation from Case 9 where m = 9 kg

$\zeta_{e_9} = 0.951$ *and* $\zeta_{s_9} = 0.835733$

$\tau_{e_9} = 0.132755$ *and* $\tau_{s_9} = 0.103266$

$\chi_{e_9} = 2.40522$ *and* $\chi_{s_9} = 2.44351$

$\psi_{e_9} = 4.13234$ *and* $\psi_{s_9} = 4.15612$

Therefore,

$\zeta_{0_9} = 1.13792$

$\tau_{0_9} = 1.28556$

$\chi_{0_9} = 0.98433$

$\psi_{0_9} = 0.994278$

13.10 Observation from Case 10 where m = 10 kg

$\zeta_{e_{10}} = 0.874667$ *and* $\zeta_{s_{10}} = 0.822289$

$\tau_{e_{10}} = 0.124977$ *and* $\tau_{s_{10}} = 0.127677$

$\chi_{e_{10}} = 2.538$ *and* $\chi_{s_{10}} = 2.52378$

$\psi_{e_{10}} = 4.03814$ *and* $\psi_{s_{10}} = 4.07456$

Therefore,

$\zeta_{0_{10}} = 1.0637$

$\tau_{0_{10}} = 0.978853$

$\chi_{0_{10}} = 1.00563$

$\psi_{0_{10}} = 0.991062$

14 Observations from Derived Data

Upon close observation of the ***drag versus time*** *it is evident that, the drag periodically decreases with decrease in mass and increases in amplitude and frequency with steady increase in mass.*

As mentioned before, the viscous drag that is in play, exhibits underdamping, that is the damping ratio involved in this example is less than 1 ($\zeta < 1$).

The system exhibits an interesting feature, that of constant logarithmic decrements, that is,

$$\ln\frac{x_1}{x_2} = \ln\frac{x_2}{x_3} = \ln\frac{x_3}{x_4} = \cdots\cdots\cdots$$

Where x_n and x_{n+1} are the amplitudes of any two successive peaks ($n \in \mathbb{R}$).

15 Analysis

15.1 Graphical Analysis

15.1.1 Graphs in relation to the Experimental values versus time

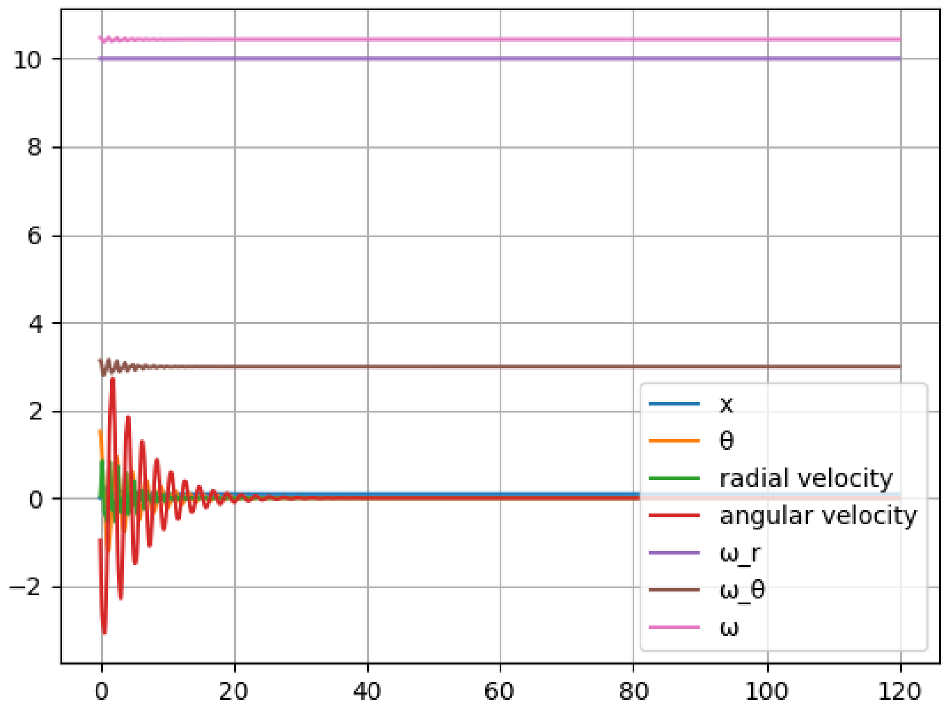

Figure 15.1: *Case 1 where m = 1 kg*

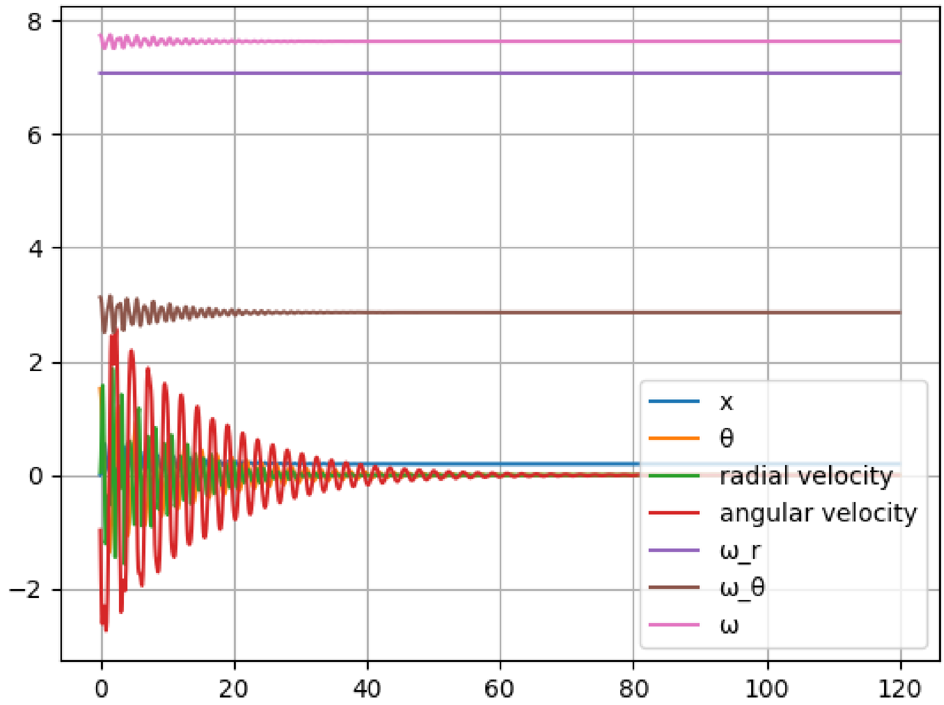

Figure 15.2: *Case 2 where m = 2 kg*

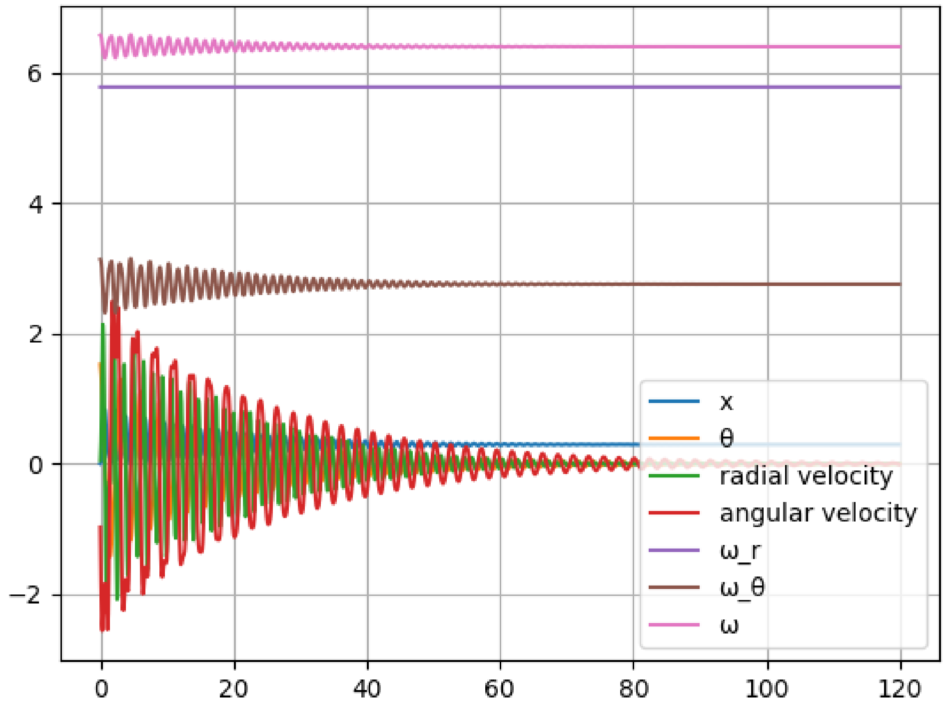

Figure 15.3: *Case 3 where m = 3 kg*

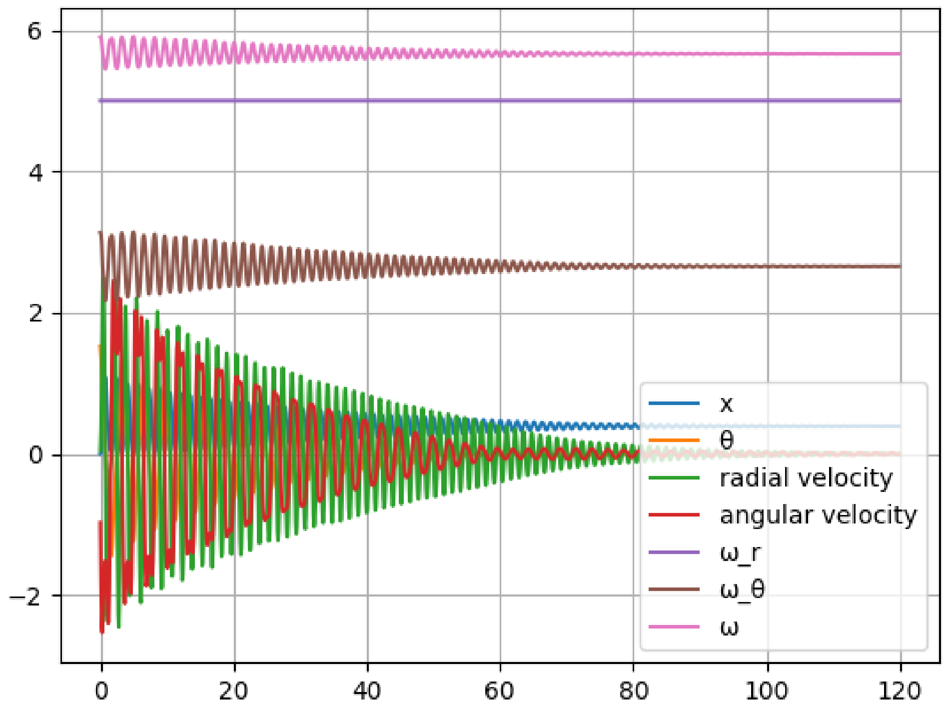

Figure 15.4: *Case 4 where m = 4 kg*

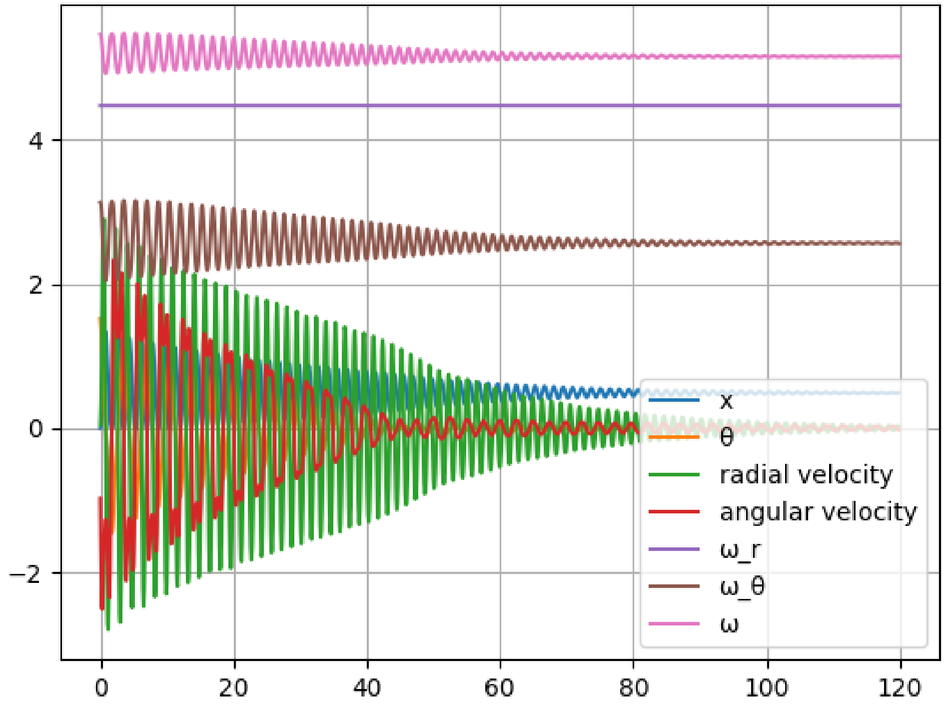

Figure 15.5: *Case 5 where m = 5 kg*

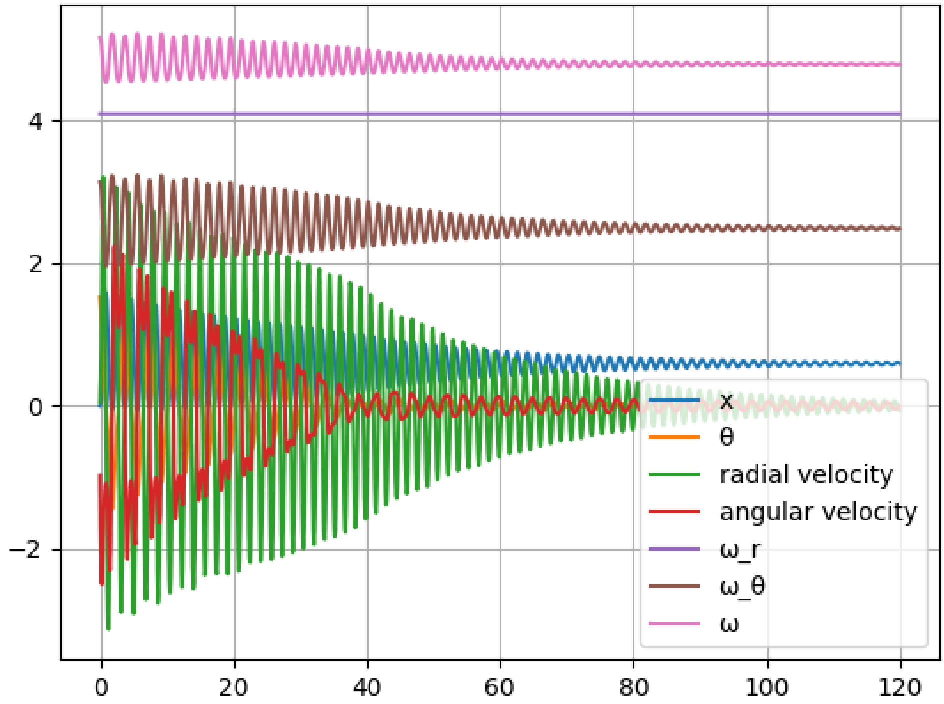

Figure 15.6: *Case 6 where m = 6 kg*

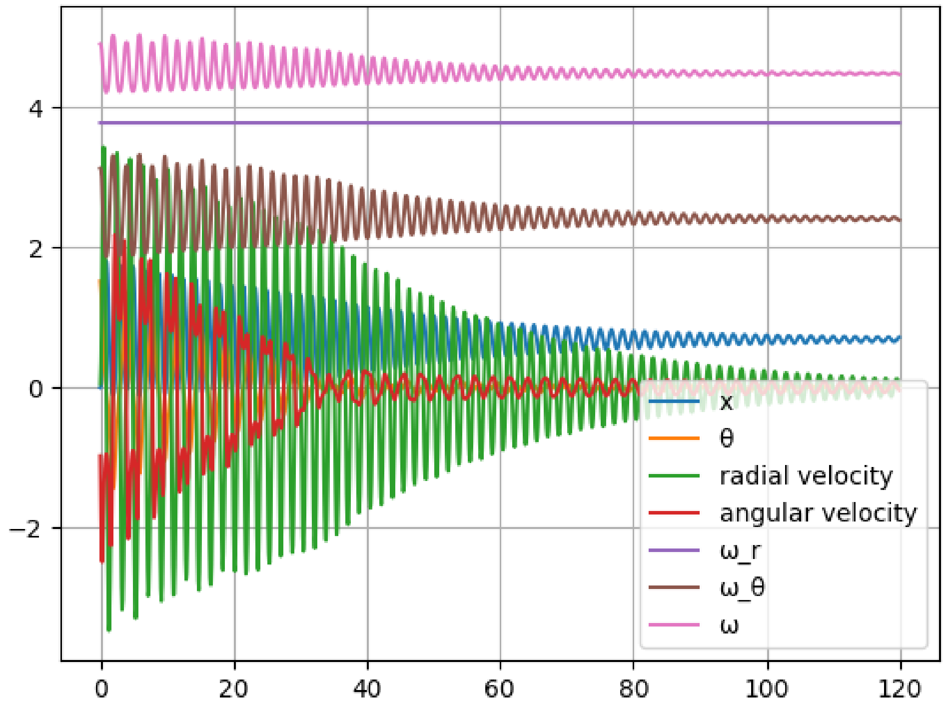

Figure 15.7: *Case 7 where m = 7 kg*

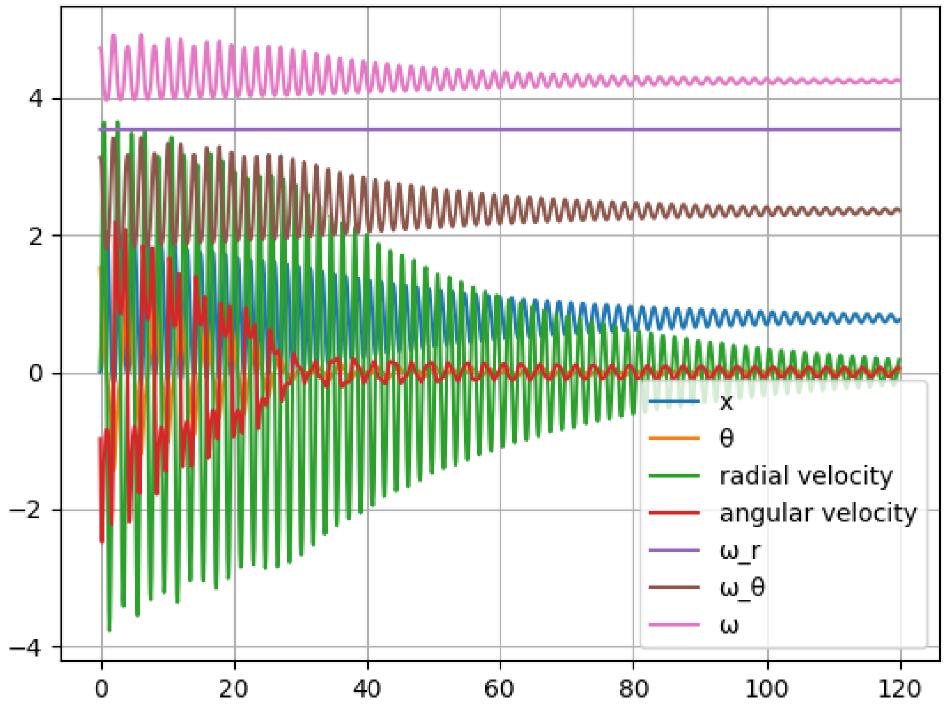

Figure 15.8: *Case 8 where m = 8 kg*

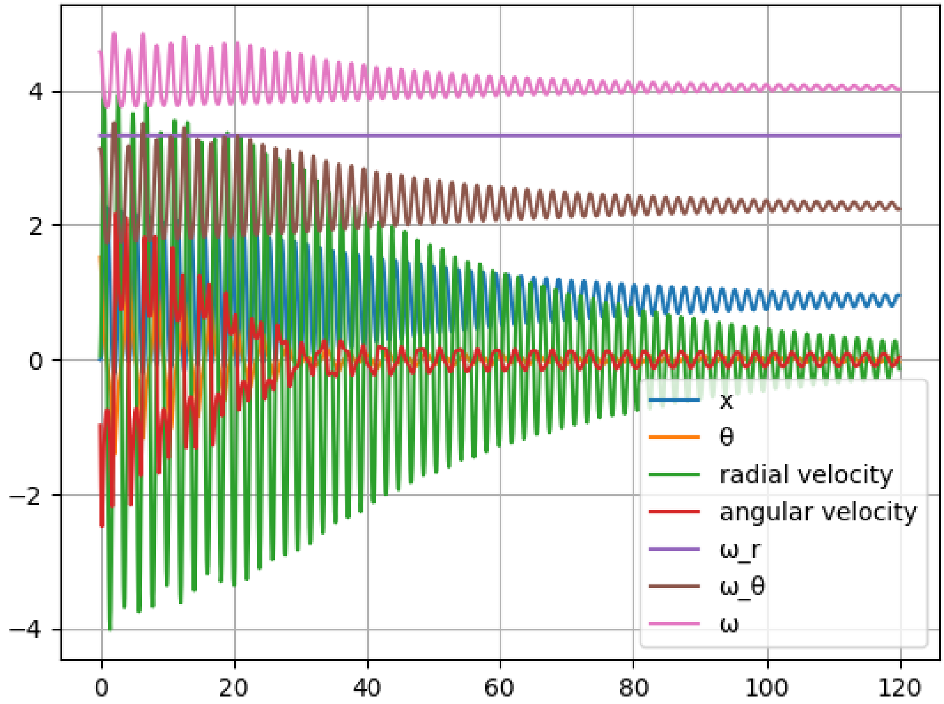

Figure 15.9: *Case 9 where m = 9 kg*

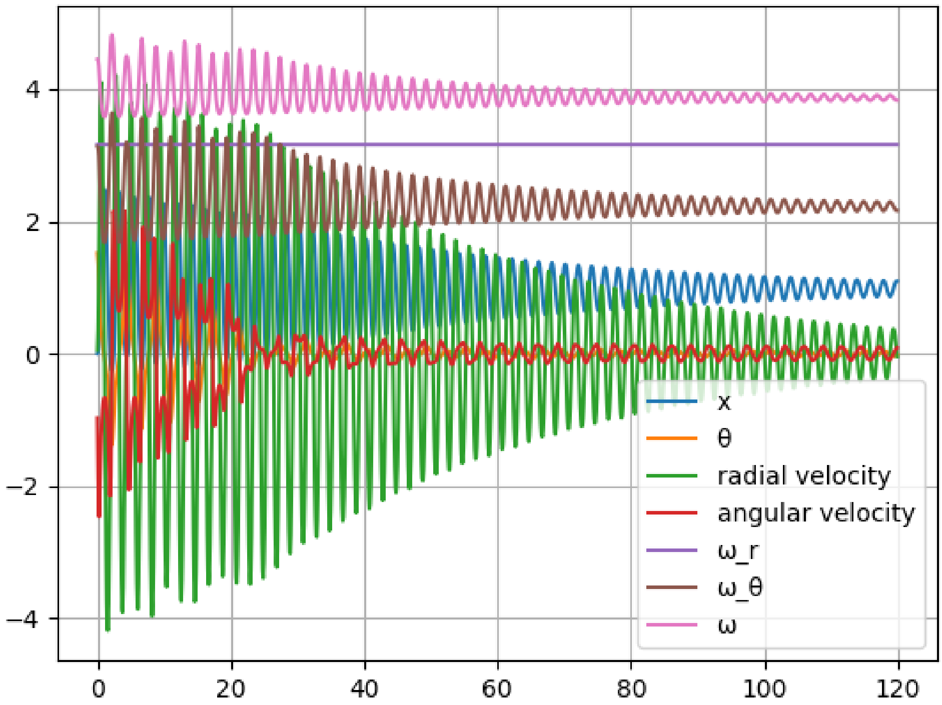

Figure 15.10: *Case 10 where m = 10 kg*

15.1.2 Graphs in relation to the Simulation values versus time

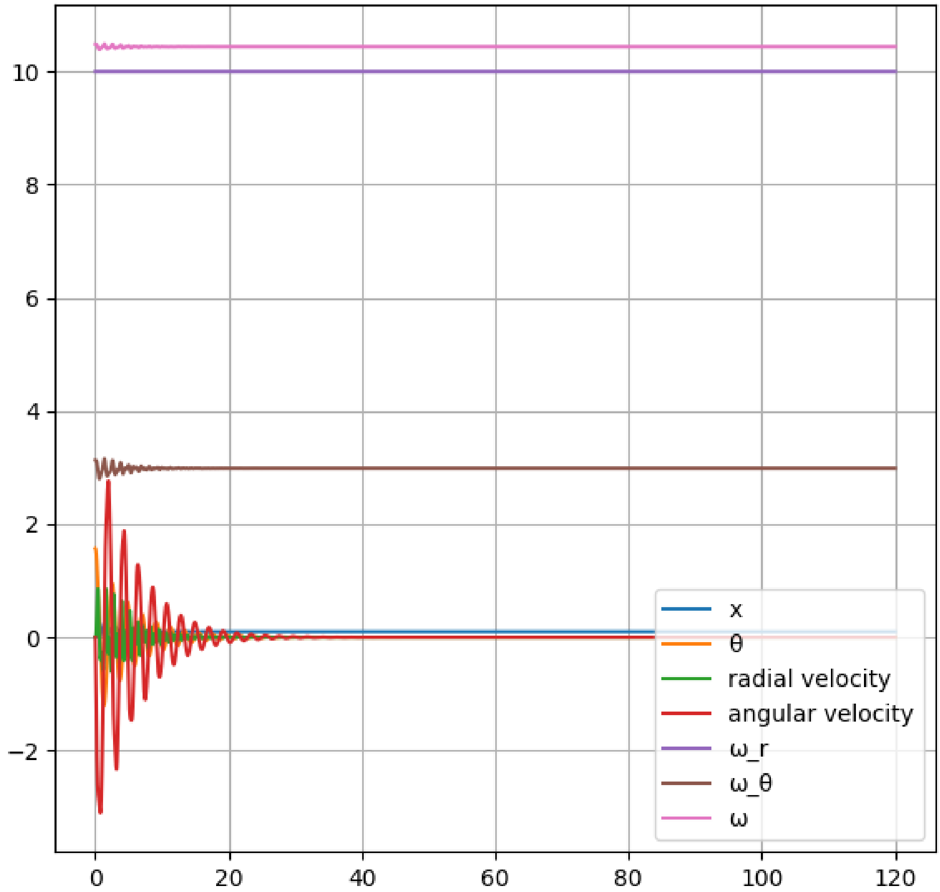

Figure 15.11: *Case 1 where m = 1 kg*

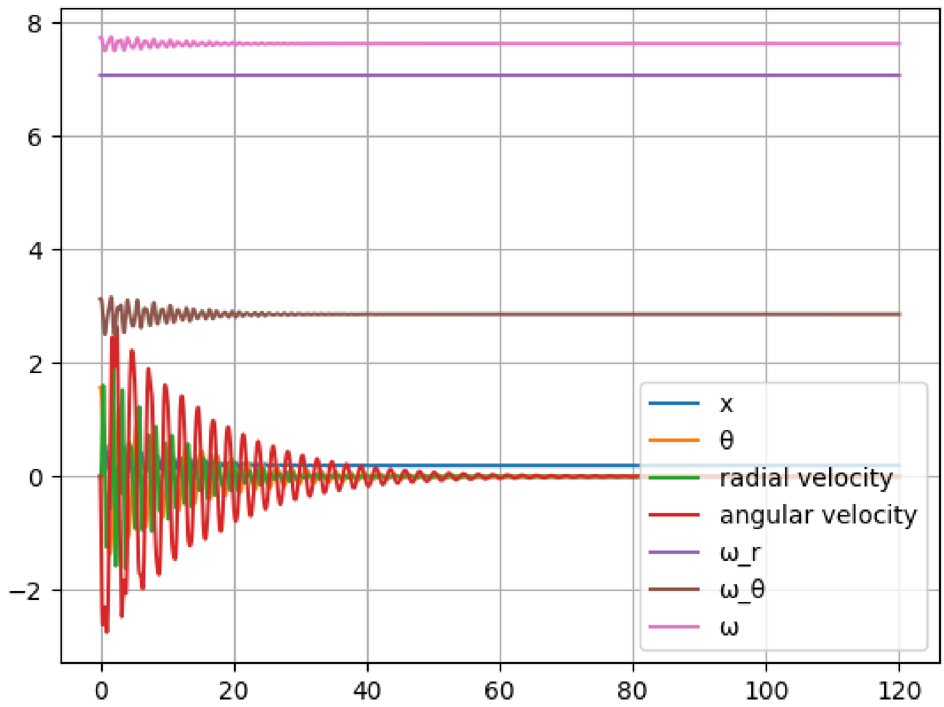

Figure 15.12: *Case 2 where m = 2 kg*

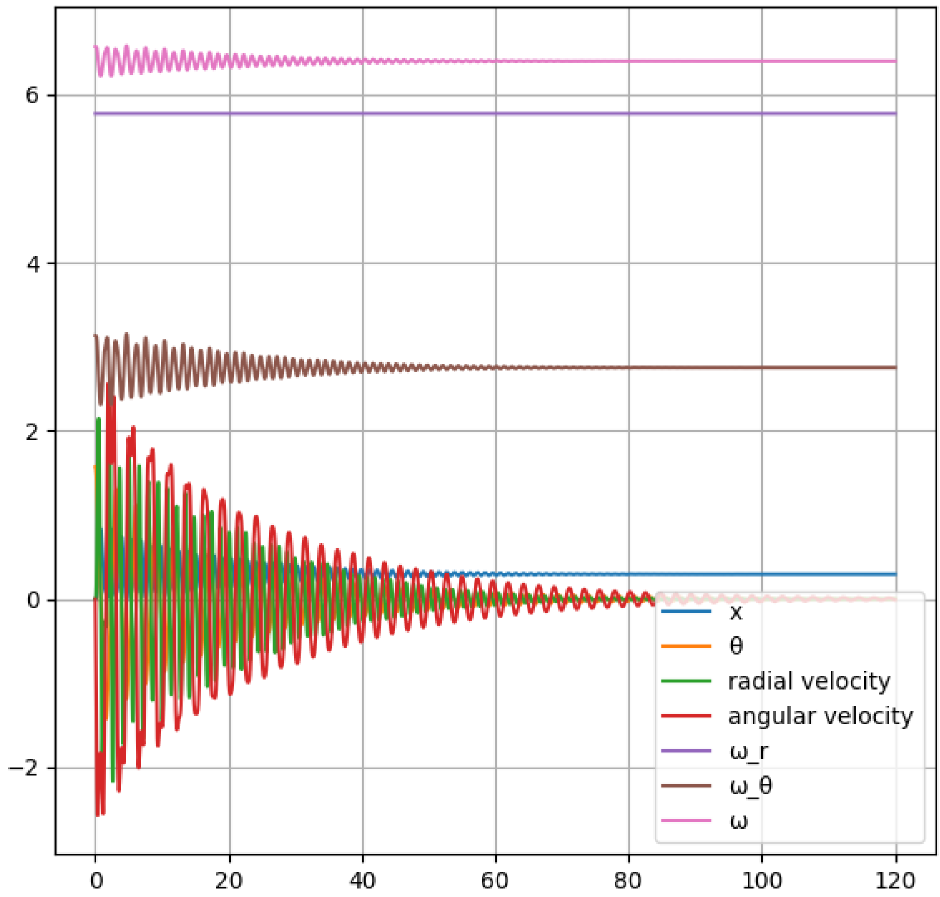

Figure 15.13: *Case 3 where m = 3 kg*

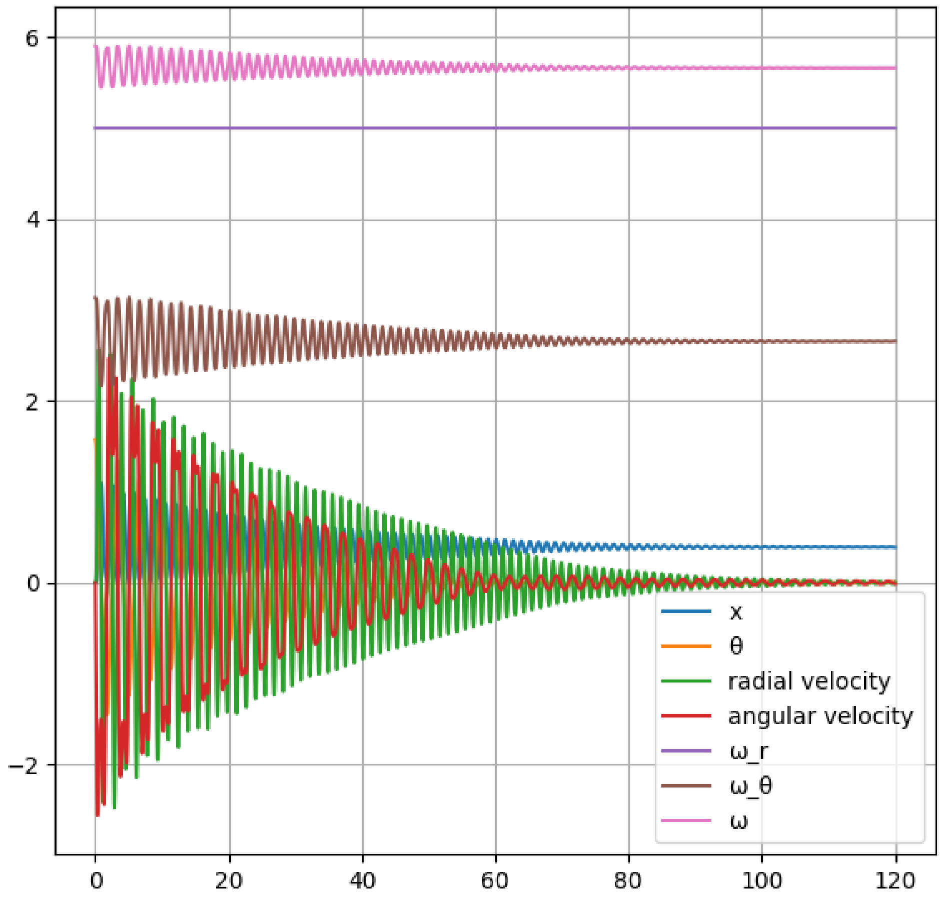

Figure 15.14: *Case 4 where m = 4 kg*

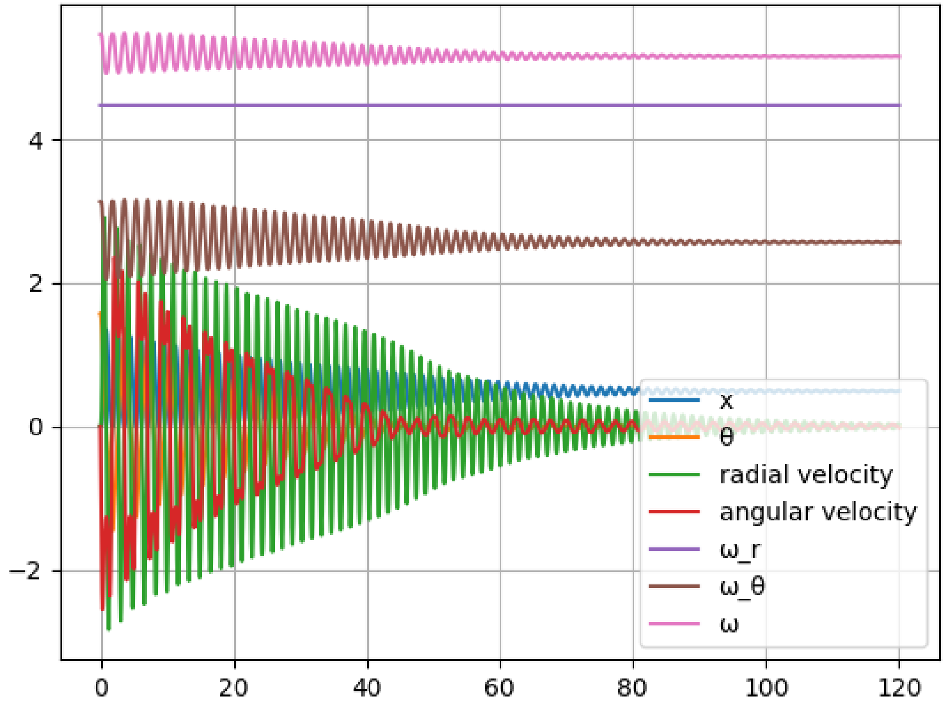

Figure 15.15: *Case 5 where m = 5 kg*

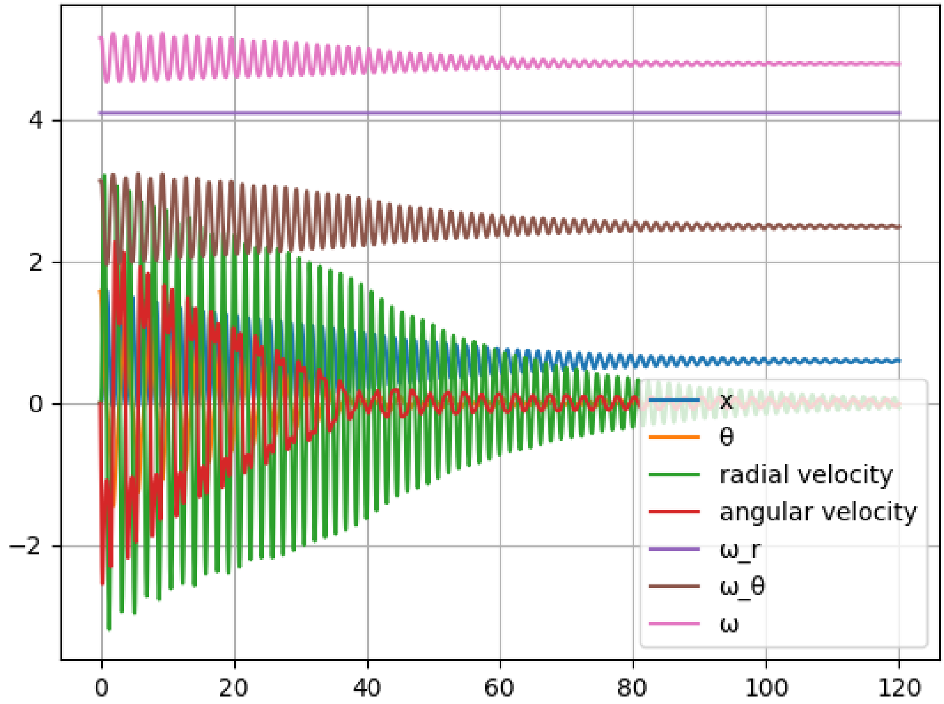

Figure 15.16: *Case 6 where m = 6 kg*

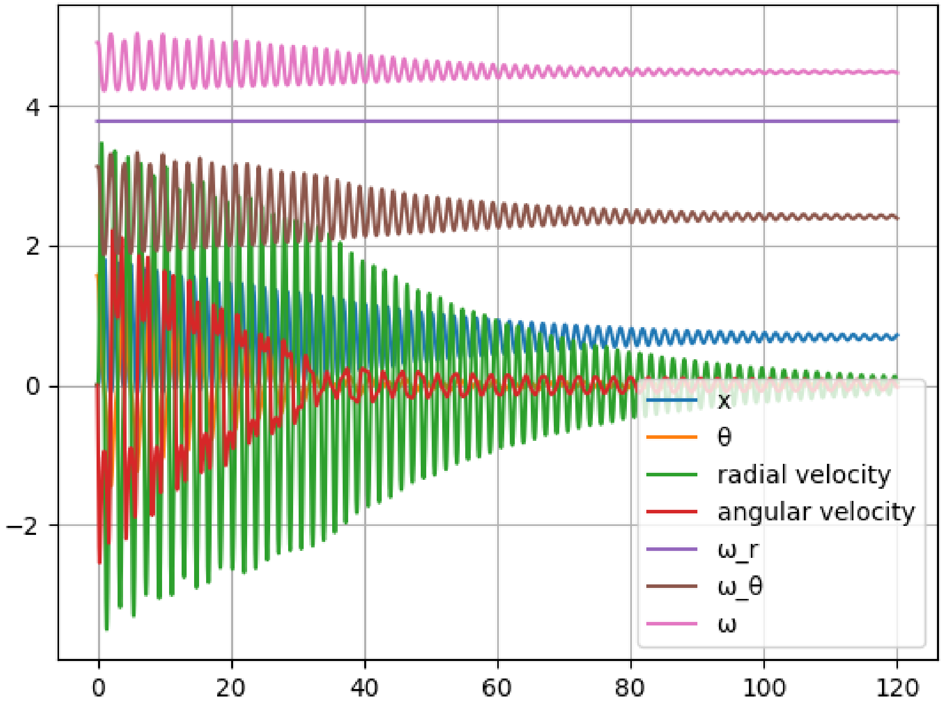

Figure 15.17: *Case 7 where m = 7 kg*

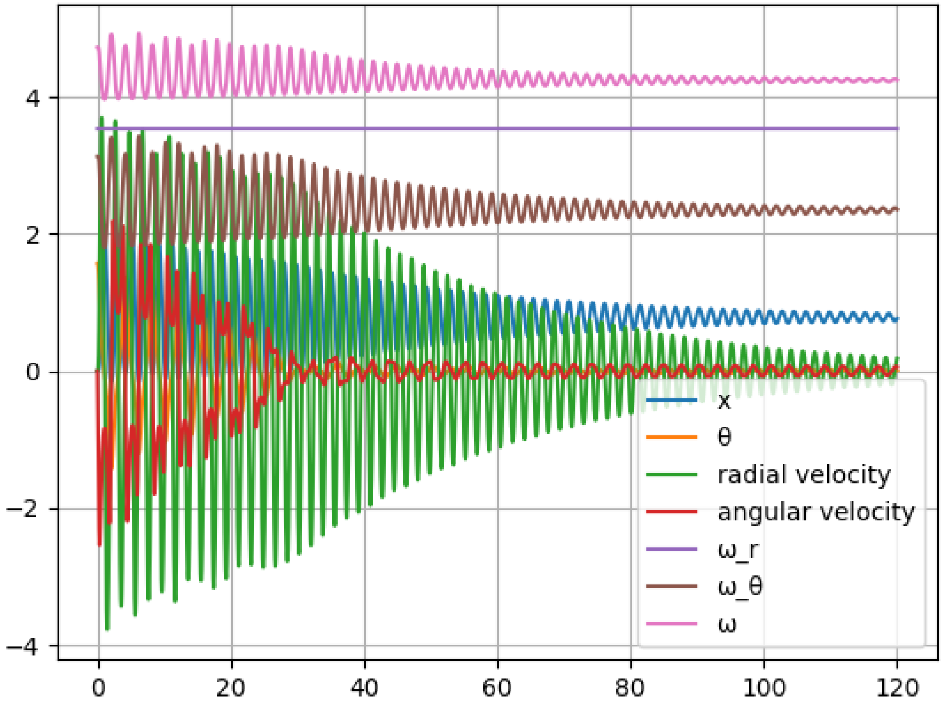

Figure 15.18: *Case 8 where m = 8 kg*

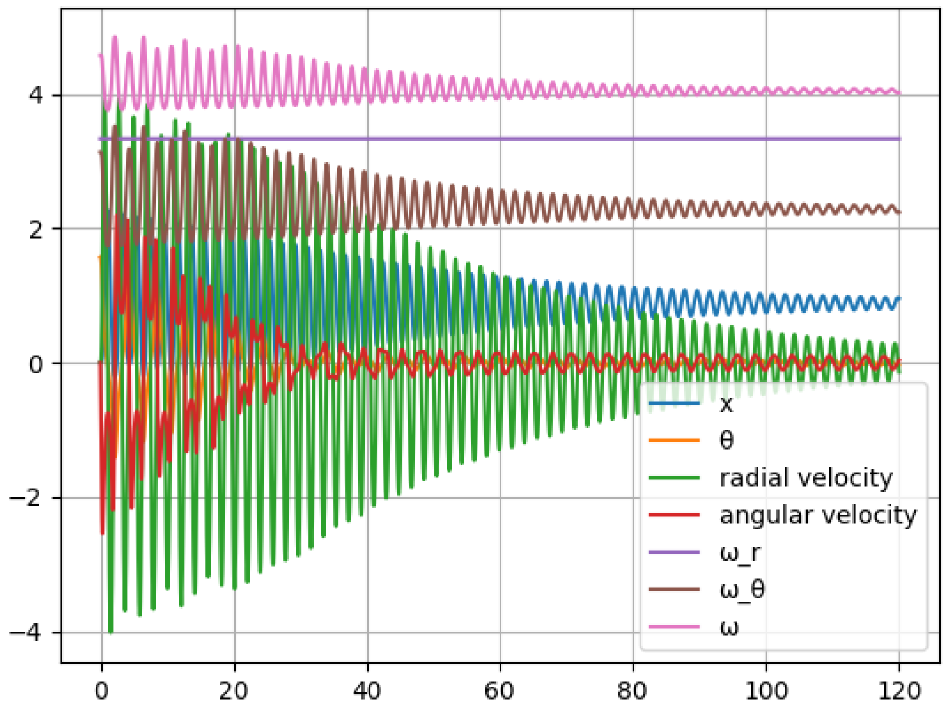

Figure 15.19: *Case 9 where m = 9 kg*

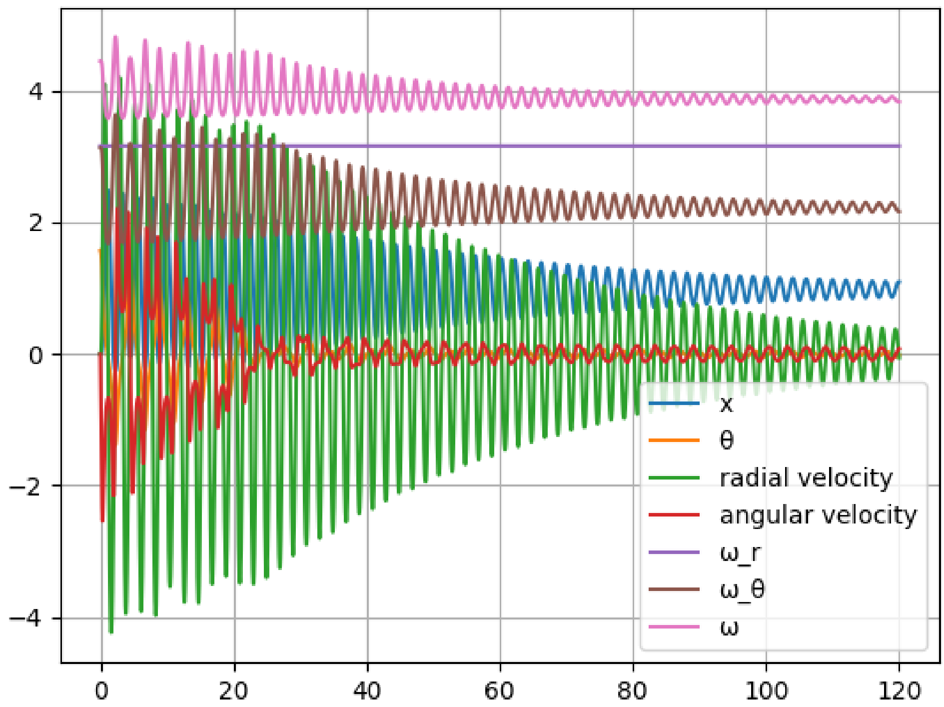

Figure 15.20: *Case 10 where m = 10 kg*

*Upon close visual observation, we see that the difference in the plotted values of that of the **simulation** and **experimental values** of the radial displacement, angular displacement, angular frequency and absolute frequency from the **x-t** graph are very **minute** to the extent that it would be right to say and consider that the experimental values are both **accurate** and **precise** in relation to that of the **literature/theoretical/simulation values**.*

*It is evident from studying the system **numerically** and **graphically** that the system exhibits an **exponential decay** with time for all parameters that are being investigated. This points out to **one conclusion**, that **all parameters of the system undergoes damping**, according to their various **inert energies**.*

*This **contradicts** the **initial hypothesis** laid out prior to beginning the investigation as if, the system had to model **similarly** to the simple pendulum, then the damping effect would take place on only the **angular motion** component of the system, but we see that this is not the case.*

We also observe that, with increase in mass, the ***amplitude*** *of the parameters employed in this investigation also increases, while the* ***frequency*** *of the parameters employed stays approximately constant.*

15.2 Numerical Analysis

15.2.1 Radial Displacement versus Time

We initially defined and conditionalized the initial value of the radial displacement versus time to be equal to 0.

Due to viscous underdamped motion, it gains energy and momentum and preserves it. With gradual but steady loss in its energy state and is in motion for more than 300 seconds before coming to a complete halt. ie. no motion.

15.2.2 Angular Displacement versus Time

We initially defined and conditionalized the initial value of the angular displacement versus time to be equal to $\pi/2$.

Due to viscous underdamped motion, it gains energy and momentum and preserves it. With gradual but steady loss in its energy state and is in motion for more than 120 seconds before coming to a complete halt. ie. no motion.

15.2.3 Angular Frequency versus Time

We previously defined angular frequency (ω_θ*) to be as equation 9.6.*

The initial state ($\omega_\theta(0)$*) that follows is,*

$$\omega_\theta(0) = \sqrt{\frac{g}{l_0 + x(0)}} = \sqrt{\frac{g}{l_0}}$$

But we had previously mentioned that the magnitude of rest length of the spring would be 1 meter ($l_0 = 1m$*). Therefore we have,*

$$\omega_\theta(0) = \sqrt{g} \approx 3.13209195$$

This parameter keeps fluctuating through the time domain as this is a function of radial displacement which in turn is a function of time.

This initial state defined and derived above is uniform irrespective of the mass employed

15.2.4 Absolute Frequency versus Time

We previously defined absolute frequency (ω_r) to be as equation 9.5.

The initial state ($\omega_r(0)$) that follows is,

$$\omega_r(0) = \sqrt{\frac{k}{m}}$$

This parameter does not fluctuate through the time domain as this is not a function of any arbitrary variable that would be in turn is a function of time.

This initial state defined and derived above is constant for any particular case with a steady constant mass m, throught a case

16 Evaluation

We further define,

$$\zeta_0 = \overline{\zeta_{0_n}} = \frac{\sum_{n=1}^{n} \zeta_{0_n}}{n} = \frac{\zeta_{0_1} + \zeta_{0_2} + \zeta_{0_3} + \zeta_{0_4} + \zeta_{0_5} + \zeta_{0_6} + \zeta_{0_7} + \zeta_{0_8} + \zeta_{0_9} + \zeta_{0_{10}}}{10}$$

$$\tau_0 = \overline{\tau_{0_n}} = \frac{\sum_{n=1}^{n} \tau_{0_n}}{n} = \frac{\tau_{0_1} + \tau_{0_2} + \tau_{0_3} + \tau_{0_4} + \tau_{0_5} + \tau_{0_6} + \tau_{0_7} + \tau_{0_8} + \tau_{0_9} + \tau_{0_{10}}}{10}$$

$$\chi_0 = \overline{\chi_{0_n}} = \frac{\sum_{n=1}^{n} \chi_{0_n}}{n} = \frac{\chi_{0_1} + \chi_{0_2} + \chi_{0_3} + \chi_{0_4} + \chi_{0_5} + \chi_{0_6} + \chi_{0_7} + \chi_{0_8} + \chi_{0_9} + \chi_{0_{10}}}{10}$$

$$\psi_0 = \overline{\psi_{0_n}} = \frac{\sum_{n=1}^{n} \psi_{0_n}}{n} = \frac{\psi_{0_1} + \psi_{0_2} + \psi_{0_3} + \psi_{0_4} + \psi_{0_5} + \psi_{0_6} + \psi_{0_7} + \psi_{0_8} + \psi_{0_9} + \psi_{0_{10}}}{10}$$

Therefore we have,

$\zeta_0 = 1.0525068$

$\tau_0 = 0.997198$

$\chi_0 = 0.9960714$

$\psi_0 = 0.9982441$

Upon observation, we see that the value of ζ_0, τ_0, χ_0 and ψ we have found is not equal to 1, but is relatively very close, so we can say that we have some errors in calculating the experimental values of radial displacement, angular displacement, angular frequency and absolute frequency respectively versus time.

Percentage uncertainty** in measurement of radial displacement versus time is* $(1-\zeta_0)\cdot 100\% =$ ***5.25068%

Percentage uncertainty** in measurement of angular displacement versus time is* $(1-\tau_0)\cdot 100\% =$ ***0.2802%

Percentage uncertainty** in measurement of angular frequency versus time is* $(1-\chi_0)\cdot 100\% =$ ***0.39286%

Percentage uncertainty** in measurement of absolute frequency versus time is* $(1-\psi_0)\cdot 100\% =$ ***0.17559%

17 Limitations of Study

There are various limitations in this study/investigation as we have placed forth, certain strict initial conditions and other conditions that make this system so constrained and disables us to expand our researching capability of this chaotic phenomenon. Conditions such as,

- *Restricting the value of* $\theta(0)$ *to* $\pi/2$
- *Restricting the value of* $x(0)$ *to* 0
- *Restricting the value of* l_0 *to 1 m*
- *Restricting the value of the time domain from 0 to 20 seconds*
- *Restricting the value of the radius of the mass employed to* 5×10^{-2} *m*
- *Restricting the value of the spring stiffness constant to 100 N/m*

18 Safety Measures

*The **safety measures** that were taken during the experimentation as as follows:*

- *Proper Laboratory equipment was utilized to conduct and collect data*

- *All Laboratory equipment was used in the presence of Lab* ***instructors*** *and Lab* ***personnel***
- *The experiment was performed at a distance from the observer so as to, limit or eliminate the chances of any possible physical harm to the observer*
- *Any and all lab equipment was thoroughly examined for any defects that could potentially lead to safety hazards, before initiating experimentation*
- *It was made sure the experiment shall not be performed close to highly flammable materials that would result in combustion from the frictional force onto the fluid medium*

19 Sources of Error

There are various ways through which errors might have crept into our raw and ordered data, some of the possible sources of errors are:

1. *Raw experimental data presented here in the investigation report is collected through lab experimentation, and there are chances that the data collected may have slight discrepancy in it*
2. *Insignificant random human errors by the observer, ie. parallax errors*
3. *Uncertainties that cannot be minimized due to lack of highly sophisticated equipment and materials used in the experiments in this investigation*
4. *Assumptions and certain conditions put forth on the the system to model its chaotic behavior*

20 Conclusion

In this paper, I have shown the effects of changes in the ***mass*** *in an* ***elastic pendulum*** *oscillating in a fluid medium (Air(Damped motion)) on the* ***radial displacement***, ***angular displacement***, ***angular frequency*** *and* ***absolute frequency*** *by collecting raw data relating to the above parameters, under certain controlled conditions so as to completely study the* ***motion/dynamics of the elastic pendula motion***.

I have also investigated the ***validity*** *of our* ***initial hypothesis*** *and have come to a conclusion that our initial hypothesis was an* ***invalid statement***.

I have also shown the possible ***uncertainties in measurement*** *of the* ***radial displacement***, ***angular displacement***, ***angular frequency*** *and* ***absolute frequency***.

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Appendices

A Python Source Codes

A.1 Graphical Solution Codes

```
import numpy as np
from numpy.linalg import inv
from matplotlib import pyplot as plt
from math import cos, sin, tan, pi , sqrt

def G(y,t):

    x_d, θ_d, x, θ = y[0], y[1], y[2], y[3]

  x_dd = (l0+x) * θ_d**2 - k/m * x + g * cos(θ)
  θ_dd = -2.0/(l0+x) * x_d * θ_d - g/(l0+x) * sin(θ)

  return np.array([x_dd, θ_dd, x_d, θ_d])

def RK4_step(y, t, dt):
  k1 = G(y,t)
  k2 = G(y+0.5*k1*dt, t+0.5*dt)
  k3 = G(y+0.5*k2*dt, t+0.5*dt)
  k4 = G(y+k3*dt, t+dt)

  return dt * (k1 + 2*k2 + 2*k3 + k4) / 6

# variables

m = 2.0
r = 0.0
l0 = 1.0
g = 9.81
k = 100.0

delta_t = 1.0e-4
time = np.arange(0.0, 10.0, delta_t)

# initial state

y = np.array([0, 0.0, 0.0 , pi/2,])   # [velocity,
    displacement] [x, θ]

Y1 = []
Y2 = []
Y3 = []
```

```
Y4 = []
Y5 = []
Y6 = []
A1 = []

# time-stepping solution

for t in time:

  y = y + RK4_step(y, t, delta_t)

  Y1.append(y[2])
  Y2.append(y[3])
  Y3.append(y[0])
  Y4.append(y[1])
  Y5.append(sqrt(k/m))
  Y6.append(sqrt(g/(l0+y[2])))

# additional variables

# ploting the result

plt.plot(time,Y1)
plt.plot(time,Y2)
plt.plot(time,Y3)
plt.plot(time,Y4)
plt.plot(time,Y6)

plt.grid(True)
plt.legend(['x', 'θ', 'radial velocity', 'angular velocity',
    'ω_r'], loc='lower right')
plt.show()
```

Listing A.1: *Graphical Solution Python Script for system* ***without*** *Damping*

```
import numpy as np
from numpy.linalg import inv
from matplotlib import pyplot as plt
from math import cos, sin, tan, pi , sqrt

def G(y,t):

    x_d, θ_d, x, θ = y[0], y[1], y[2], y[3]

  x_dd = (l0+x) * θ_d**2 - k/m * x + g * cos(θ) - C/m * x_d
  θ_dd = -2.0/(l0+x) * x_d * θ_d - g/(l0+x) * sin(θ) - C/(m * (l0+x)**2) * θ_d

  return np.array([x_dd, θ_dd, x_d, θ_d])

def RK4_step(y, t, dt):
  k1 = G(y,t)
  k2 = G(y+0.5*k1*dt, t+0.5*dt)
  k3 = G(y+0.5*k2*dt, t+0.5*dt)
  k4 = G(y+k3*dt, t+dt)

  return dt * (k1 + 2*k2 + 2*k3 + k4) / 6

# variables

m = 2.0
r = 0.0
l0 = 1.0
g = 9.81
k = 100.0
C = 0.47

delta_t = 1.0e-4
time = np.arange(0.0, 10.0, delta_t)

# initial state

y = np.array([0, 0.0, 0.0 , pi/2,])   # [velocity, displacement] [x, θ]

Y1 = []
Y2 = []
Y3 = []
Y4 = []
Y5 = []
Y6 = []
A1 = []

# time-stepping solution
```

```
for t in time:

  y = y + RK4_step(y, t, delta_t)

  Y1.append(y[2])
  Y2.append(y[3])
  Y3.append(y[0])
  Y4.append(y[1])
  Y5.append(sqrt(k/m))
  Y6.append(sqrt(g/(l0+y[2])))

# additional variables

# ploting the result

plt.plot(time,Y1)
plt.plot(time,Y2)
plt.plot(time,Y3)
plt.plot(time,Y4)
plt.plot(time,Y6)

plt.grid(True)
plt.legend(['x', 'θ', 'radial velocity', 'angular velocity',
    'ω_r'], loc='lower right')
plt.show()
```

Listing A.2: *Graphical Solution Python Script for system* ***with*** *Damping*

A.2 Path Animation Codes

```
import pygame
import sys
from pygame.locals import *
from math import sin, cos, tan, pi
import numpy as np
from numpy.linalg import inv
from spring import spring

class Spring():
  def __init__(self, color, start, end, nodes, width, lead1, lead2):
    self.start = start
    self.end = end
    self.nodes = nodes
    self.width = width
    self.lead1 = lead1
    self.lead2 = lead2
    self.weight = 3
    self.color = color

  def update(self, start, end):
    self.start = start
    self.end = end

    self.x, self.y, self.p1, self.p2 = spring(self.start, self.end, self.nodes, self.width, self.lead1, self.lead2)
    self.p1 = (int(self.p1[0]), int(self.p1[1]))
    self.p2 = (int(self.p2[0]), int(self.p2[1]))

  def render(self):
    pygame.draw.line(screen, self.color, self.start, self.p1, self.weight)
    prev_point = self.p1
    for point in zip(self.x, self.y):
      pygame.draw.line(screen, self.color, prev_point, point, self.weight)
      prev_point = point
    pygame.draw.line(screen, self.color, self.p2, self.end, self.weight)

def G(y,t):

  x_d, θ_d, x, θ = y[0], y[1], y[2], y[3]

  x_dd = (10+x) * θ_d**2 - k/m * x + g * cos(θ)
  θ_dd = -2.0/(10+x) * x_d * θ_d - g/(10+x) * sin(θ)

```

```
  return np.array([x_dd, θ_dd, x_d, θ_d])

def RK4_step(y, t, dt):
  k1 = G(y,t)
  k2 = G(y+0.5*k1*dt, t+0.5*dt)
  k3 = G(y+0.5*k2*dt, t+0.5*dt)
  k4 = G(y+k3*dt, t+dt)

  return dt * (k1 + 2*k2 + 2*k3 + k4) / 6

def update(x, θ):
  x_coord = scale*(10+x) * sin(θ) + offset[0]
  y_coord = scale*(10+x) * cos(θ) + offset[1]

  return (int(x_coord), int(y_coord))

def render(point):
  x, y = point[0], point[1]

  if prev_point:
    pygame.draw.line(trace, LT_BLUE, prev_point, (x, y), 5)

  screen.fill(WHITE)
  if is_tracing:
    screen.blit(trace, (0,0))

  #pygame.draw.line(screen, BLACK, offset, (x,y), 5)
  s.update(offset, point)
  s.render()
  pygame.draw.circle(screen, BLACK, offset, 8)
  pygame.draw.circle(screen, BLUE, (x, y), int(m*10))

  return (x, y)

w, h = 1024, 768
WHITE = (255,255,255)
BLACK = (0,0,0)
RED = (255,0,0)
BLUE = (0,0,255)
LT_BLUE = (230,230,255)
offset = (w//2, h//4)
scale = 100
is_tracing = True

screen = pygame.display.set_mode((w,h))
screen.fill(WHITE)
trace = screen.copy()
pygame.display.update()
clock = pygame.time.Clock()
```

```python

# parameters
m = 6.0
l0 = 3.5
g = 9.81
k = 100.0

prev_point = None
t = 0.0
delta_t = 0.02
y = np.array([0.0, 0.0, 0.0, pi/2])

pygame.font.init()
myfont = pygame.font.SysFont('Comic Sans MS', 38)

s = Spring(BLACK, (0,0), (0,0), 25, 30, 90, 90)

while True:
  for event in pygame.event.get():
    if event.type == pygame.QUIT:
      sys.exit()

    if event.type == KEYDOWN:
      if event.key == K_t:
        is_tracing = not(is_tracing)
      if event.key == K_c:
        trace.fill(WHITE)

  point = update(y[2], y[3])
  prev_point = render(point)

  time_string = 'Time: {} seconds'.format(round(t,1))
  text = myfont.render(time_string, False, (0, 0, 0))
  screen.blit(text, (10,10))

  t += delta_t
  y = y + RK4_step(y, t, delta_t)

  clock.tick(60)
  pygame.display.update()
```
Listing A.3: *Path Animation Python Script for system* ***without*** *Damping*

```
import pygame
import sys
from pygame.locals import *
from math import sin, cos, tan, pi
import numpy as np
from numpy.linalg import inv
from spring import spring

class Spring():
  def __init__(self, color, start, end, nodes, width, lead1,
   lead2):
    self.start = start
    self.end = end
    self.nodes = nodes
    self.width = width
    self.lead1 = lead1
    self.lead2 = lead2
    self.weight = 3
    self.color = color

  def update(self, start, end):
    self.start = start
    self.end = end

    self.x, self.y, self.p1, self.p2 = spring(self.start,
   self.end, self.nodes, self.width, self.lead1, self.lead2)
    self.p1 = (int(self.p1[0]), int(self.p1[1]))
    self.p2 = (int(self.p2[0]), int(self.p2[1]))

  def render(self):
    pygame.draw.line(screen, self.color, self.start, self.p1,
    self.weight)
    prev_point = self.p1
    for point in zip(self.x, self.y):
      pygame.draw.line(screen, self.color, prev_point, point,
    self.weight)
      prev_point = point
    pygame.draw.line(screen, self.color, self.p2, self.end,
   self.weight)

def G(y,t):

  x_d, θ_d, x, θ = y[0], y[1], y[2], y[3]

  x_dd = (10+x) * θ_d**2 - k/m * x + g * cos(θ) - C/m * x_d

  θ_dd = -2.0/(10+x) * x_d * θ_d - g/(10+x) * sin(θ) - C/(m *
   (10+x)**2) * θ_d

```

```
  return np.array([x_dd, θ_dd, x_d, θ_d])

def RK4_step(y, t, dt):
  k1 = G(y,t)
  k2 = G(y+0.5*k1*dt, t+0.5*dt)
  k3 = G(y+0.5*k2*dt, t+0.5*dt)
  k4 = G(y+k3*dt, t+dt)

  return dt * (k1 + 2*k2 + 2*k3 + k4) / 6

def update(x, θ):
  x_coord = scale*(10+x) * sin(θ) + offset[0]
  y_coord = scale*(10+x) * cos(θ) + offset[1]

  return (int(x_coord), int(y_coord))

def render(point):
  x, y = point[0], point[1]

  if prev_point:
    pygame.draw.line(trace, LT_BLUE, prev_point, (x, y), 5)

  screen.fill(WHITE)
  if is_tracing:
    screen.blit(trace, (0,0))

  #pygame.draw.line(screen, BLACK, offset, (x,y), 5)
  s.update(offset, point)
  s.render()
  pygame.draw.circle(screen, BLACK, offset, 8)
  pygame.draw.circle(screen, BLUE, (x, y), int(m*10))

  return (x, y)

w, h = 1024, 768
WHITE = (255,255,255)
BLACK = (0,0,0)
RED = (255,0,0)
BLUE = (0,0,255)
LT_BLUE = (230,230,255)
offset = (w//2, h//4)
scale = 100
is_tracing = True

screen = pygame.display.set_mode((w,h))
screen.fill(WHITE)
trace = screen.copy()
pygame.display.update()
clock = pygame.time.Clock()
```

```python

# parameters
m = 6.0
l0 = 3.5
g = 9.81
k = 100.0
C = 0.47

prev_point = None
t = 0.0
delta_t = 0.02
y = np.array([0.0, 0.0, 0.0, pi/2])

pygame.font.init()
myfont = pygame.font.SysFont('Comic Sans MS', 38)

s = Spring(BLACK, (0,0), (0,0), 25, 30, 90, 90)

while True:
  for event in pygame.event.get():
    if event.type == pygame.QUIT:
      sys.exit()

    if event.type == KEYDOWN:
      if event.key == K_t:
        is_tracing = not(is_tracing)
      if event.key == K_c:
        trace.fill(WHITE)

  point = update(y[2], y[3])
  prev_point = render(point)

  time_string = 'Time: {} seconds'.format(round(t,1))
  text = myfont.render(time_string, False, (0, 0, 0))
  screen.blit(text, (10,10))

  t += delta_t
  y = y + RK4_step(y, t, delta_t)

  clock.tick(60)
  pygame.display.update()
```
Listing A.4: *Path Animation Python Script for system **with** Damping*

B Raw Experimental Data

B.1 Radial Displacement versus Time

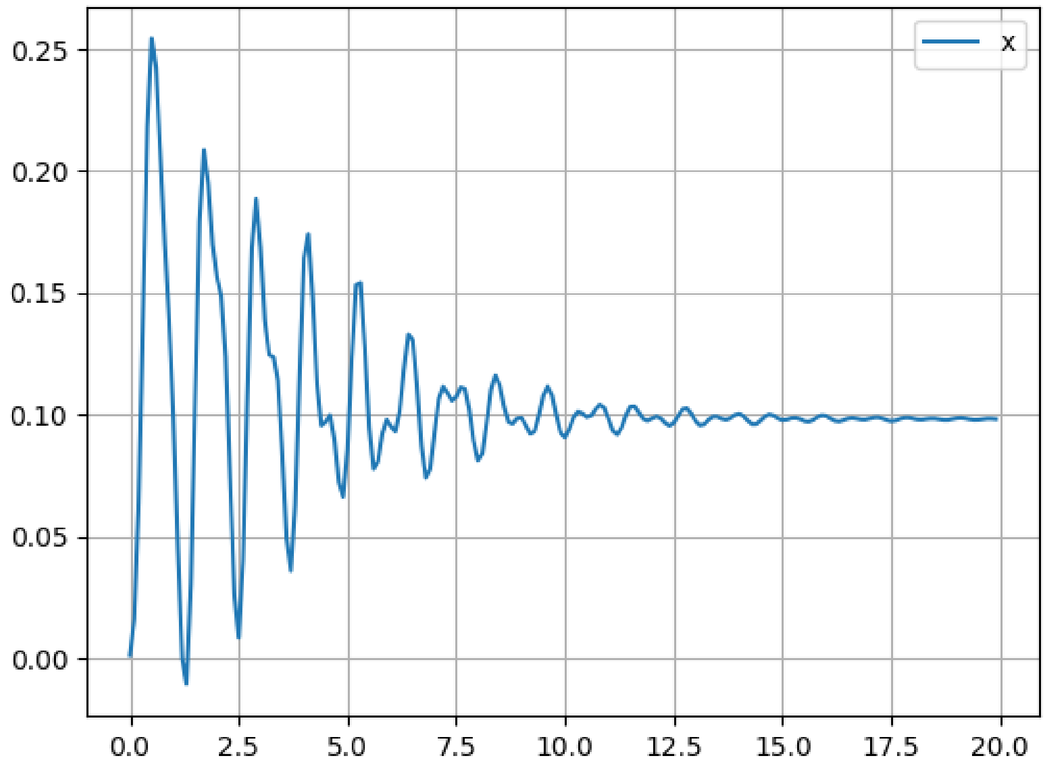

Figure B.1: *Case 1 where m = 1 kg*

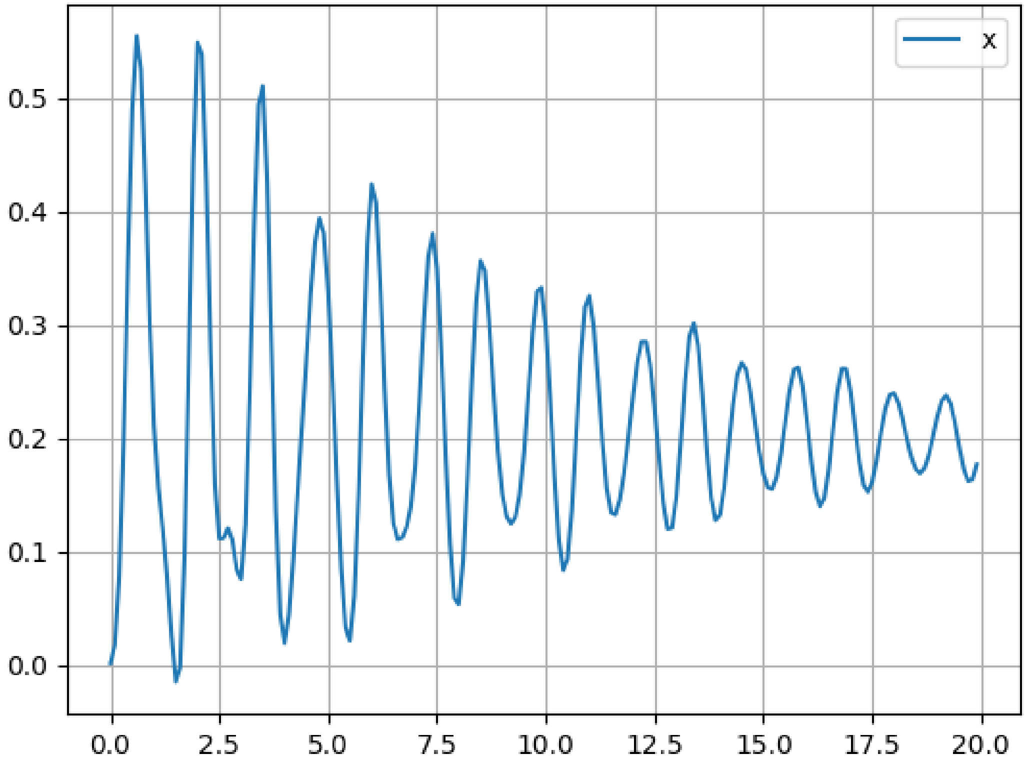

Figure B.2: *Case 2 where m = 2 kg*

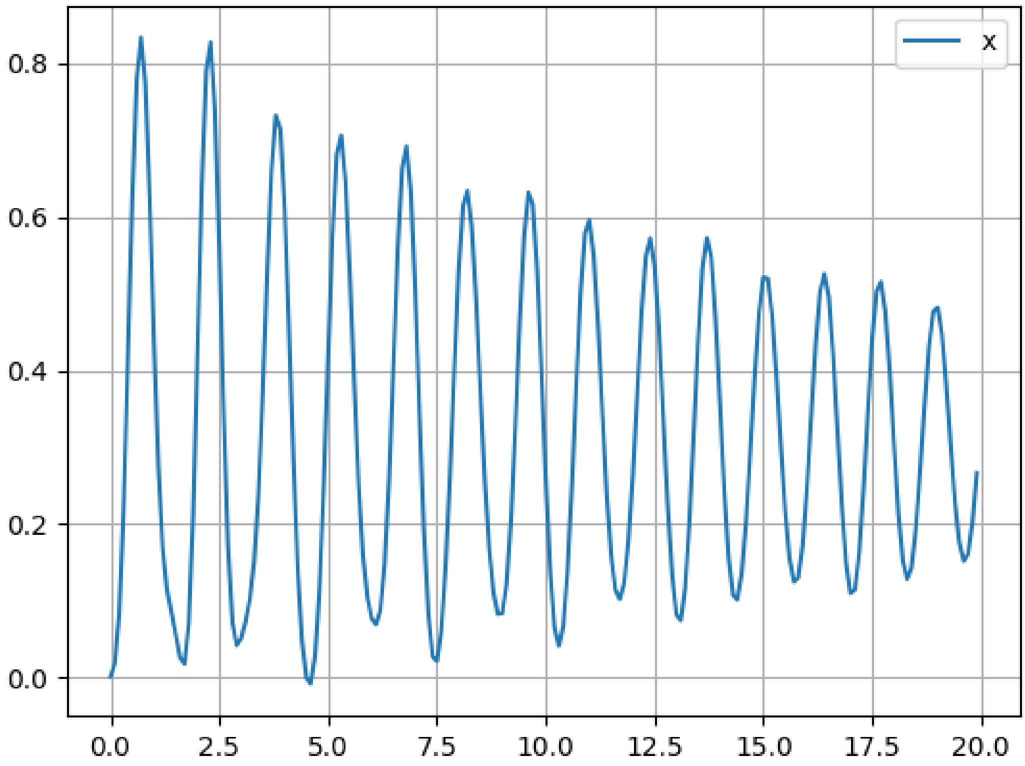

Figure B.3: *Case 3 where m = 3 kg*

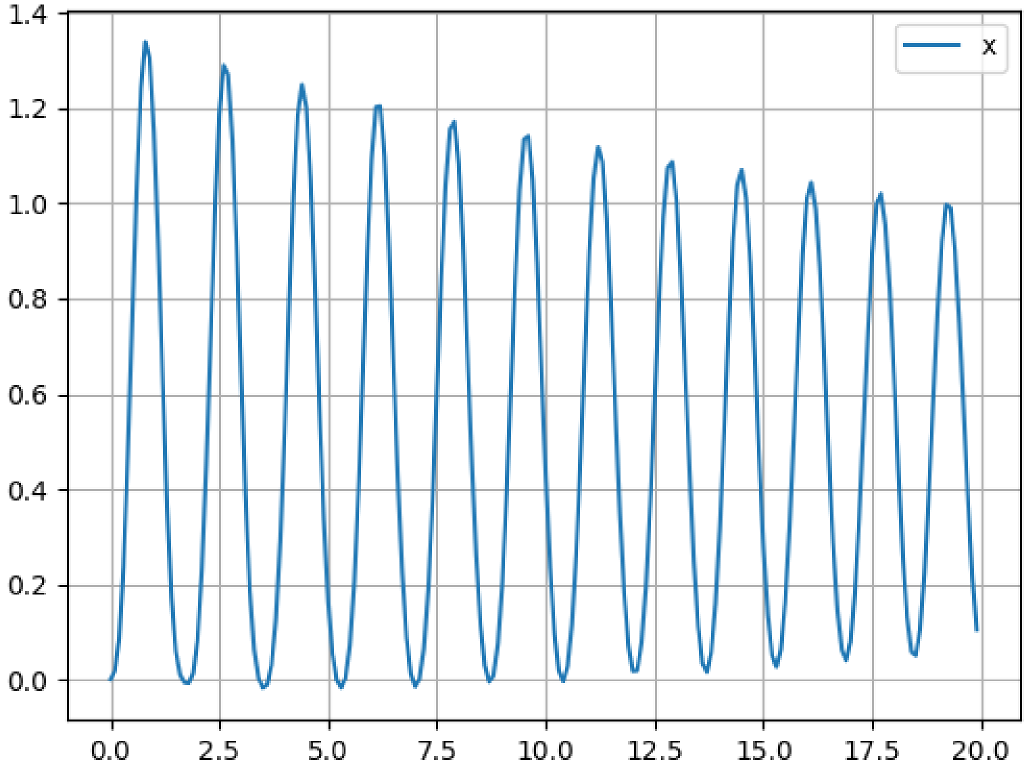

Figure B.4: *Case 4 where m = 4 kg*

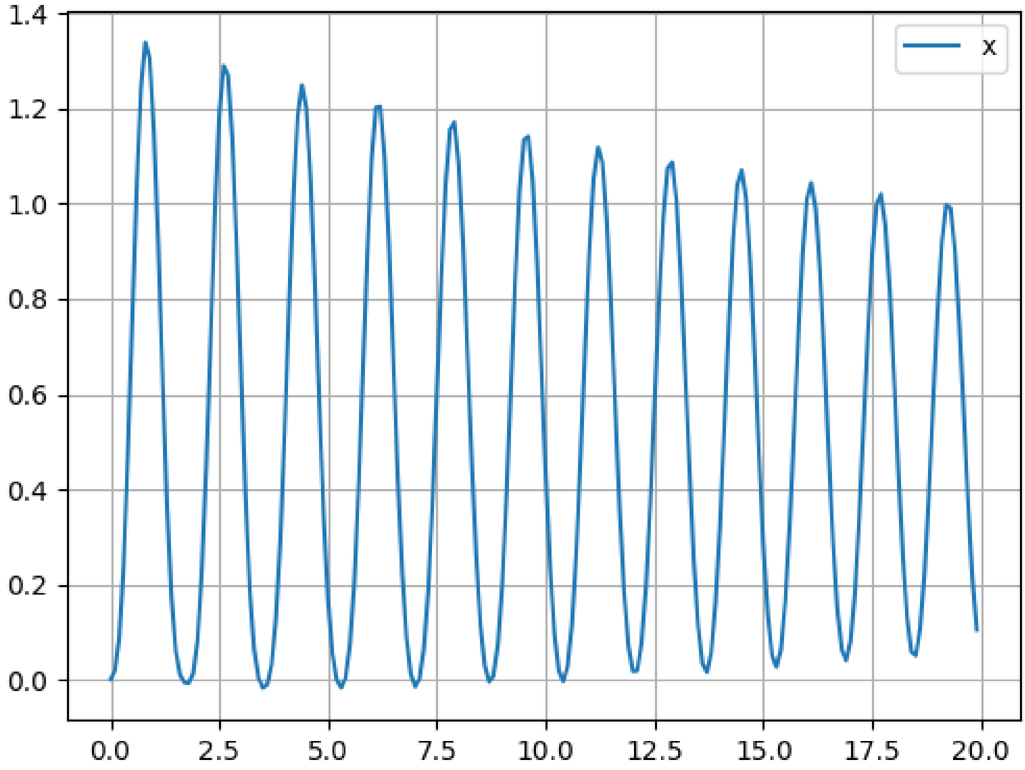

Figure B.5: *Case 5 where m = 5 kg*

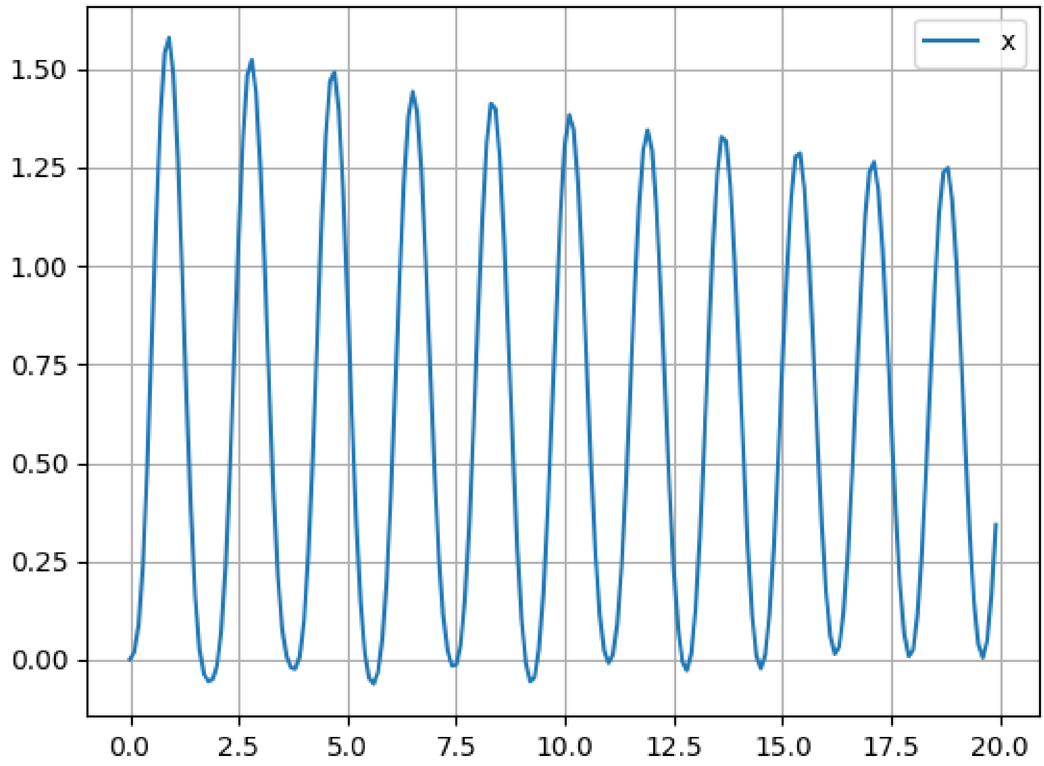

Figure B.6: *Case 6 where m = 6 kg*

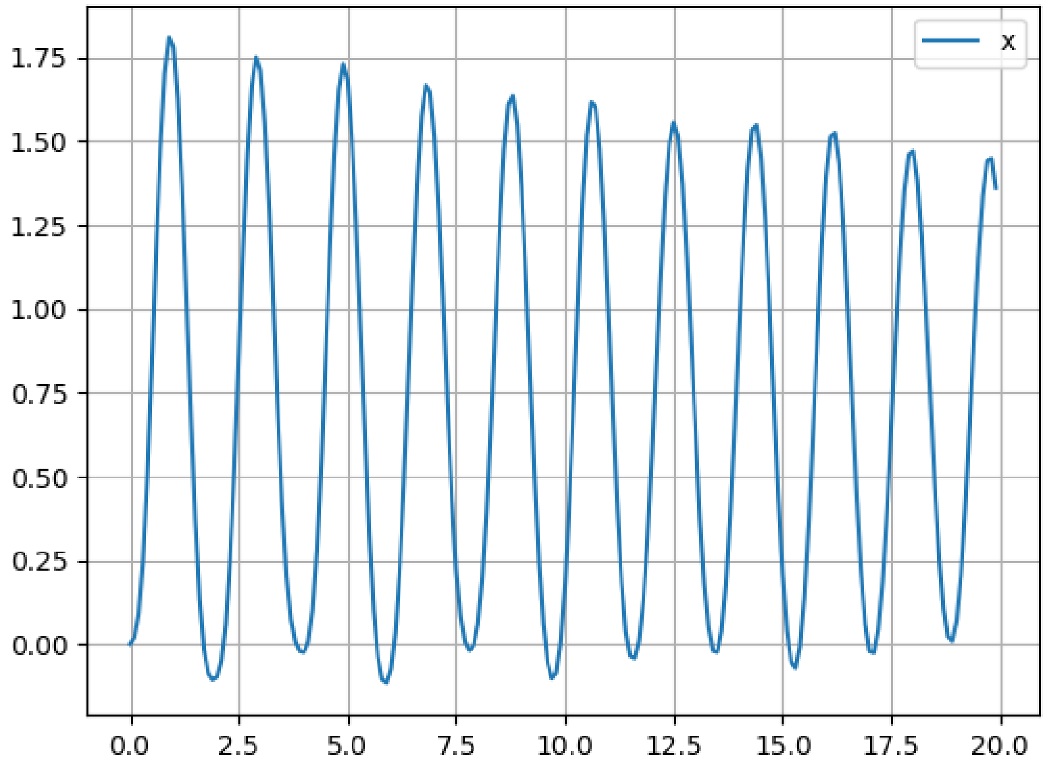

Figure B.7: *Case 7 where m = 7 kg*

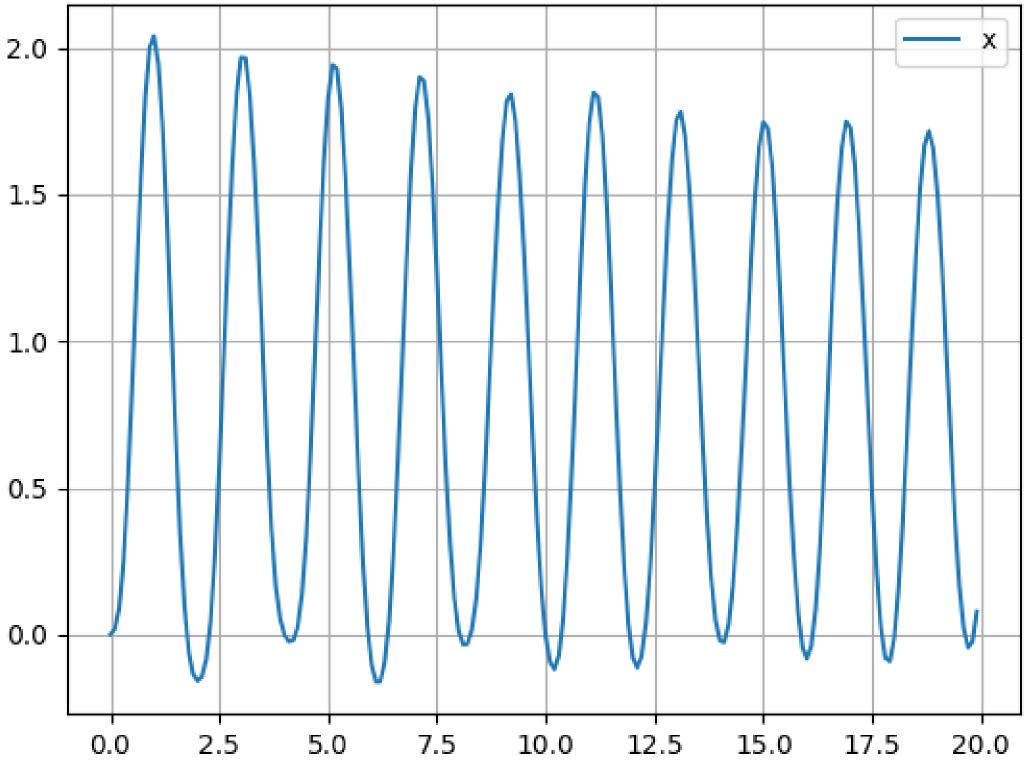

Figure B.8: *Case 8 where m = 8 kg*

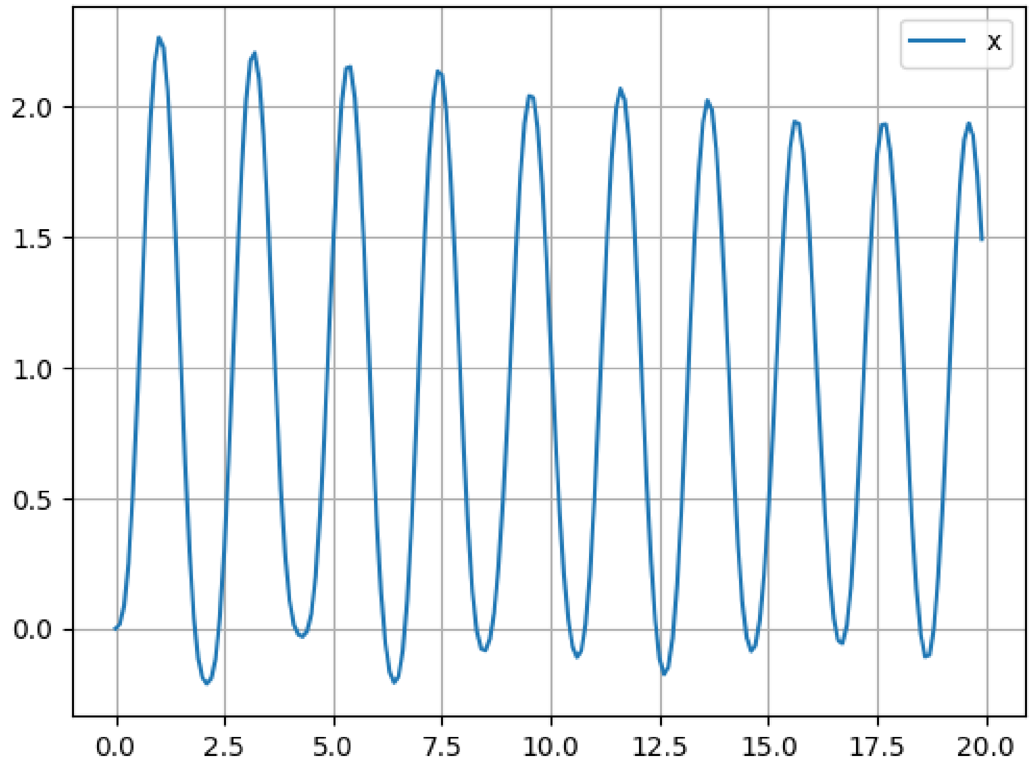

Figure B.9: *Case 9 where m = 9 kg*

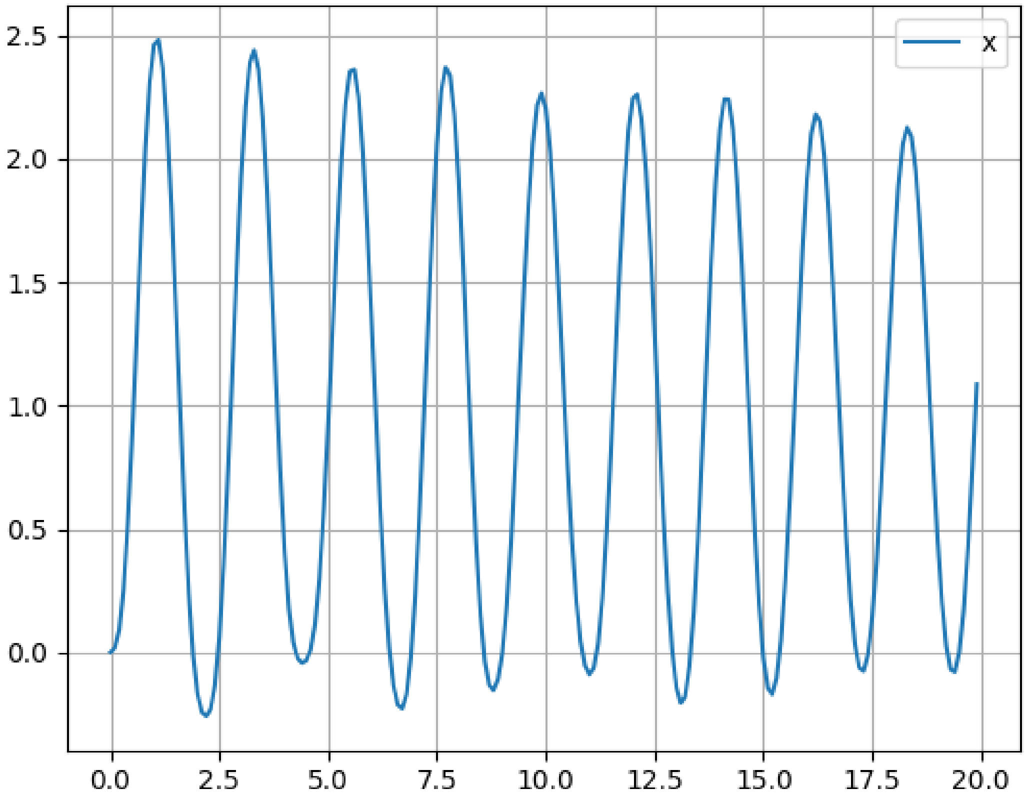

Figure B.10: *Case 10 where m = 10 kg*

B.2 Angular Displacement versus Time

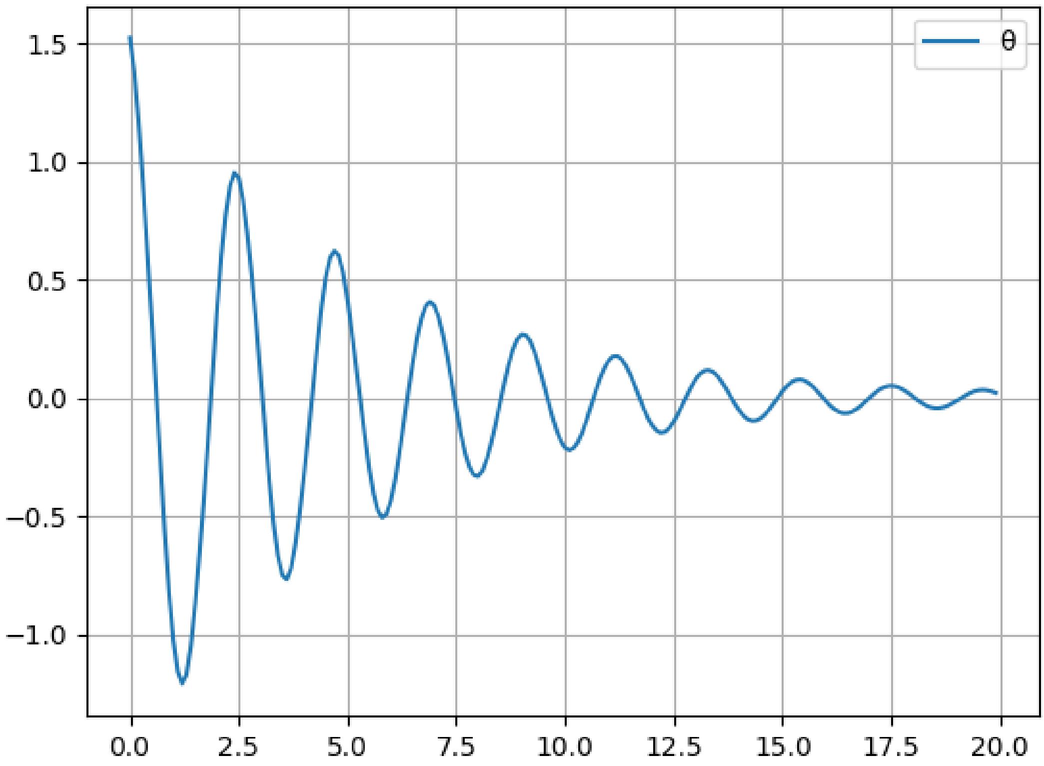

Figure B.11: *Case 1 where m = 1 kg*

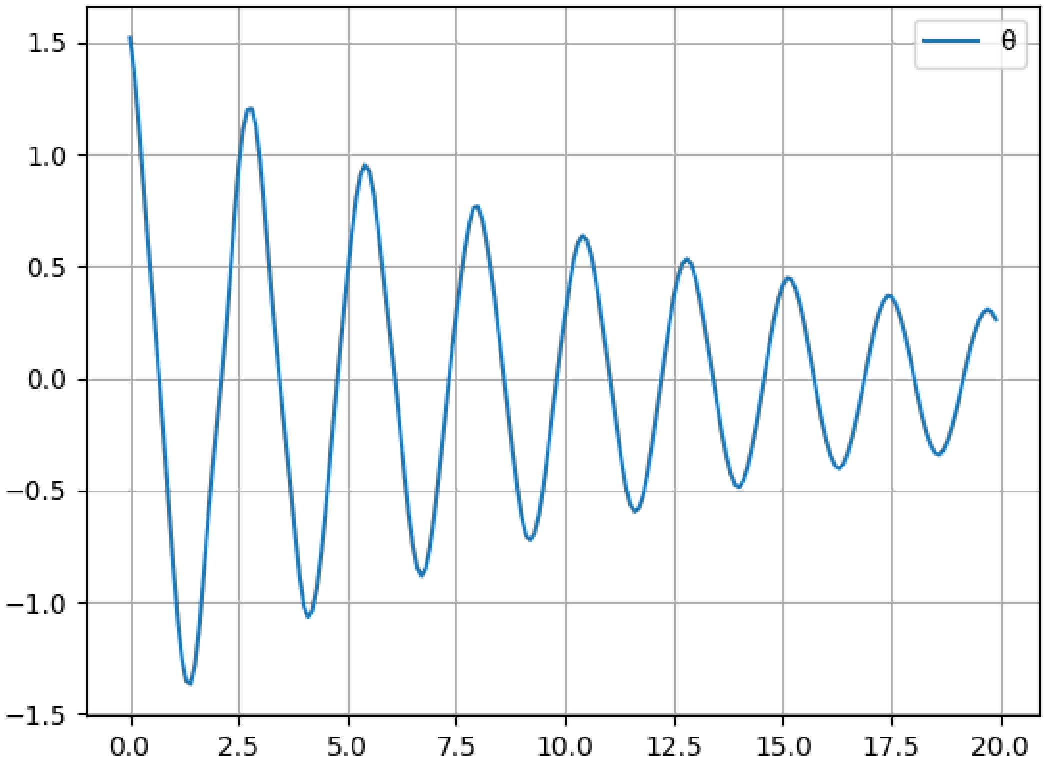

Figure B.12: *Case 2 where m = 2 kg*

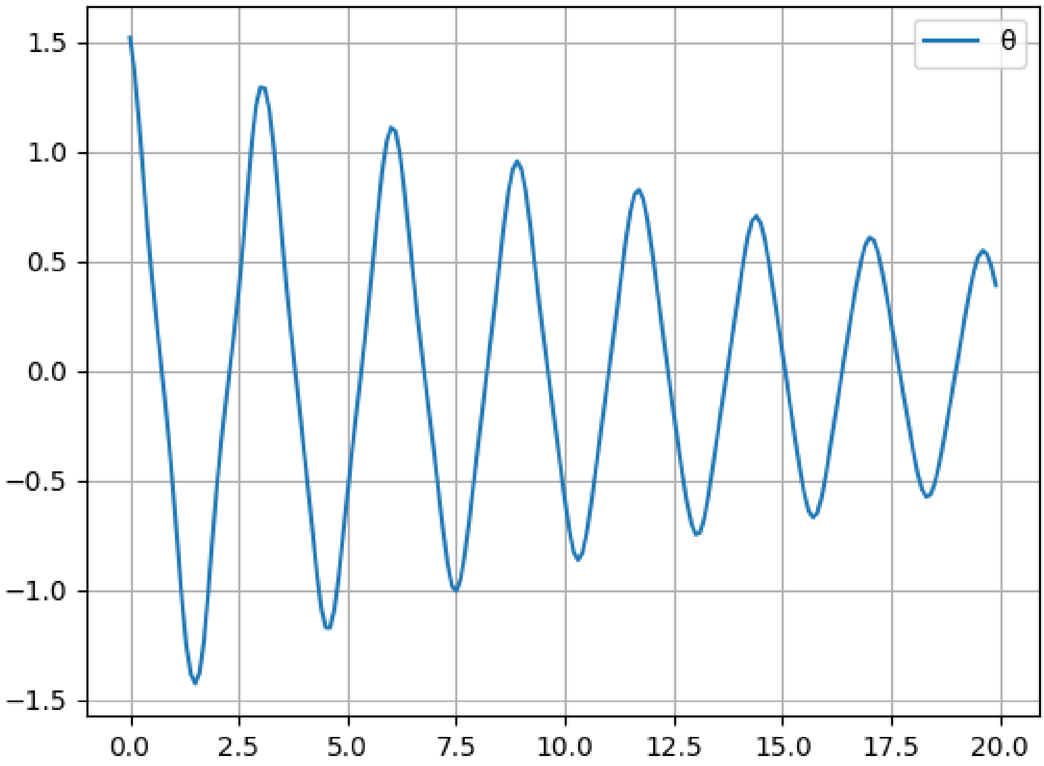

Figure B.13: *Case 3 where m = 3 kg*

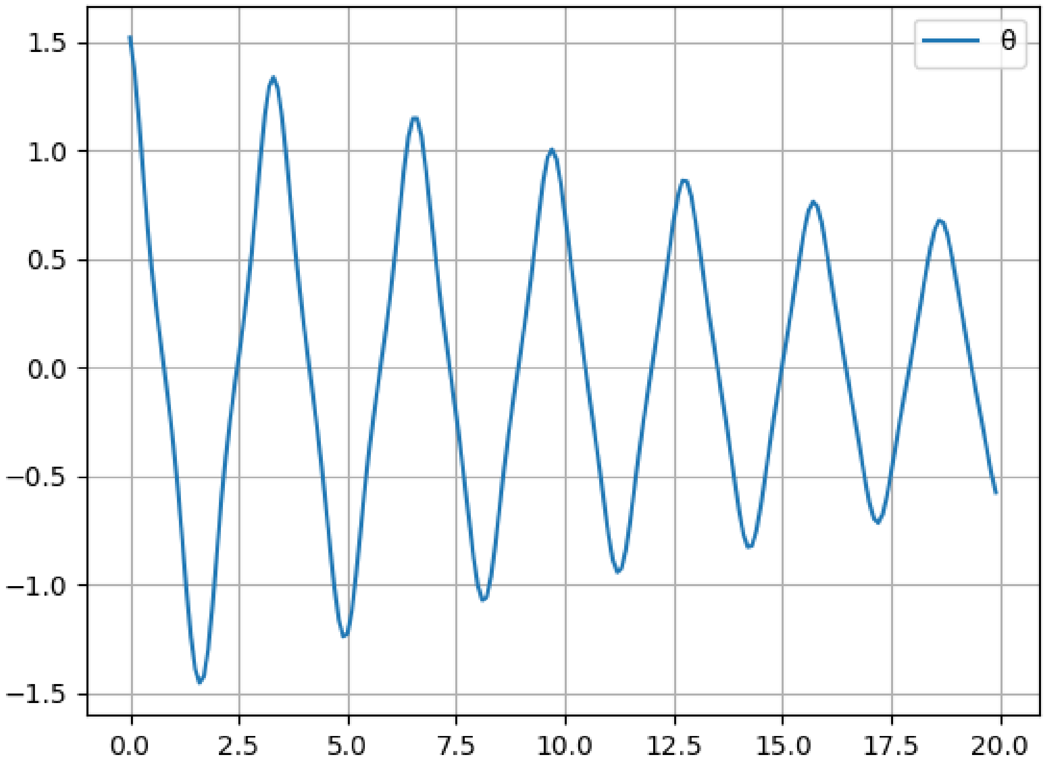

Figure B.14: *Case 4 where m = 4 kg*

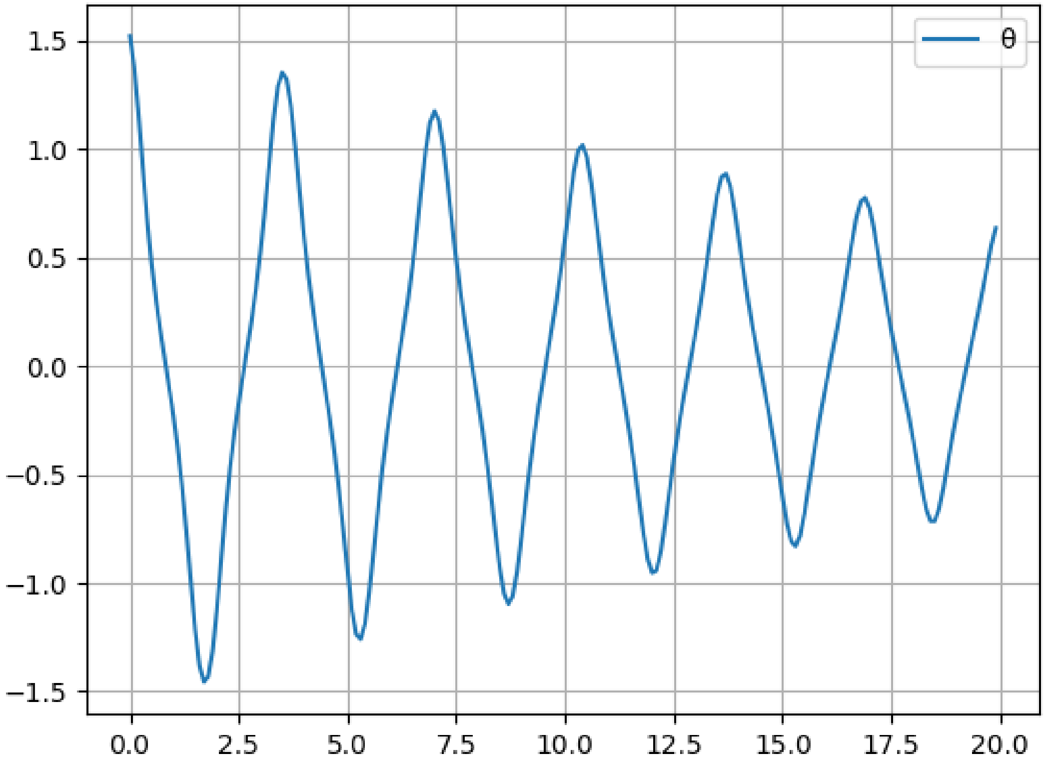

Figure B.15: *Case 5 where m = 5 kg*

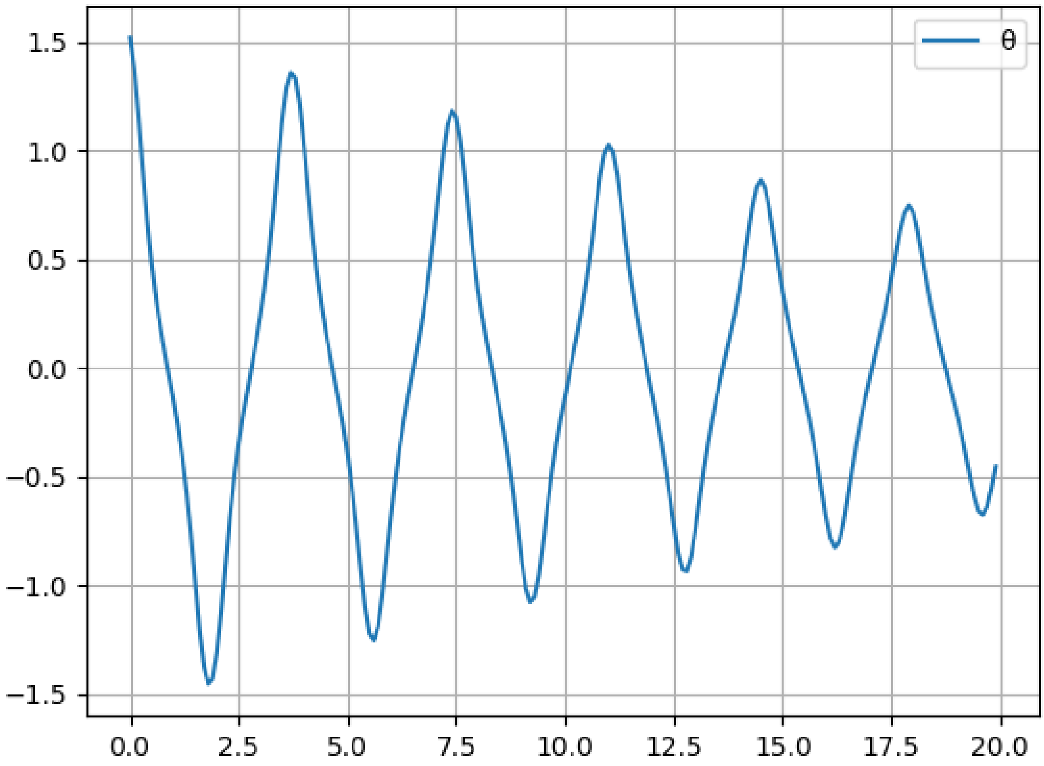

Figure B.16: *Case 6 where m = 6 kg*

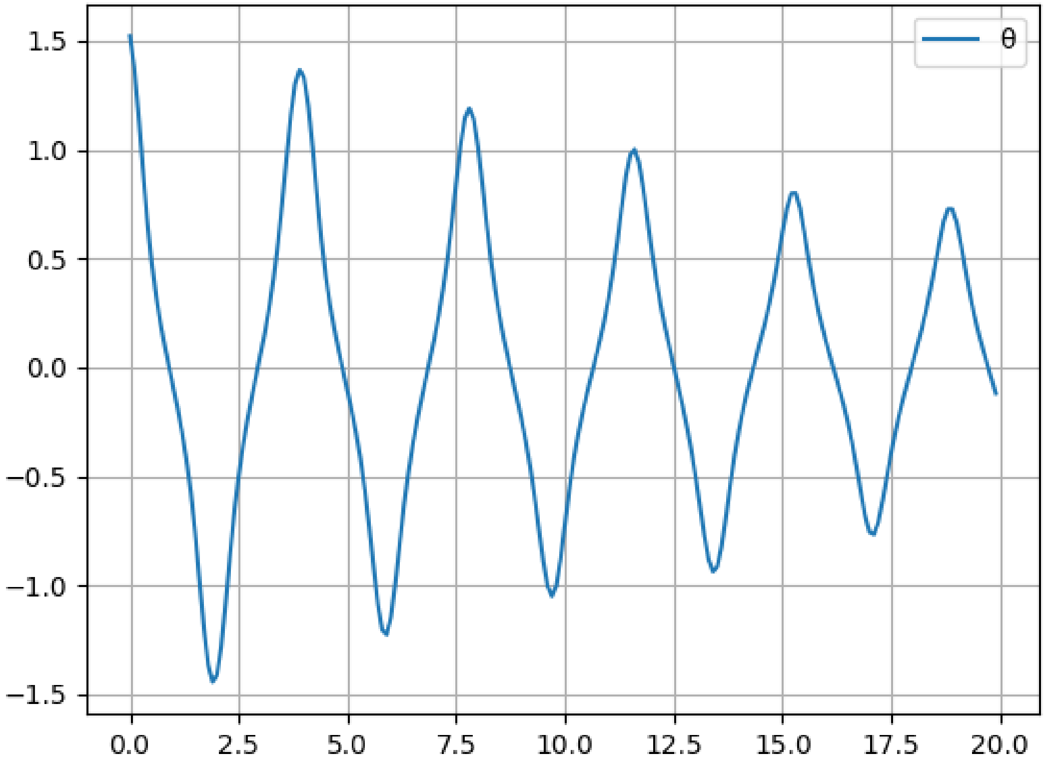

Figure B.17: *Case 7 where m = 7 kg*

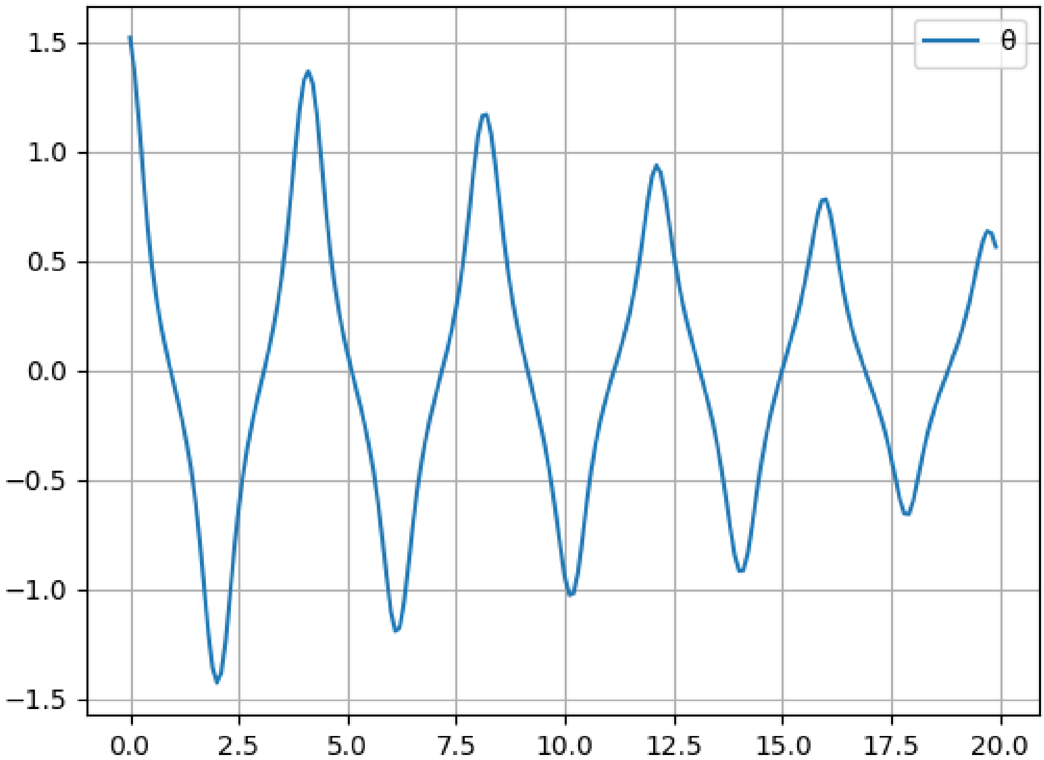

Figure B.18: *Case 8 where m = 8 kg*

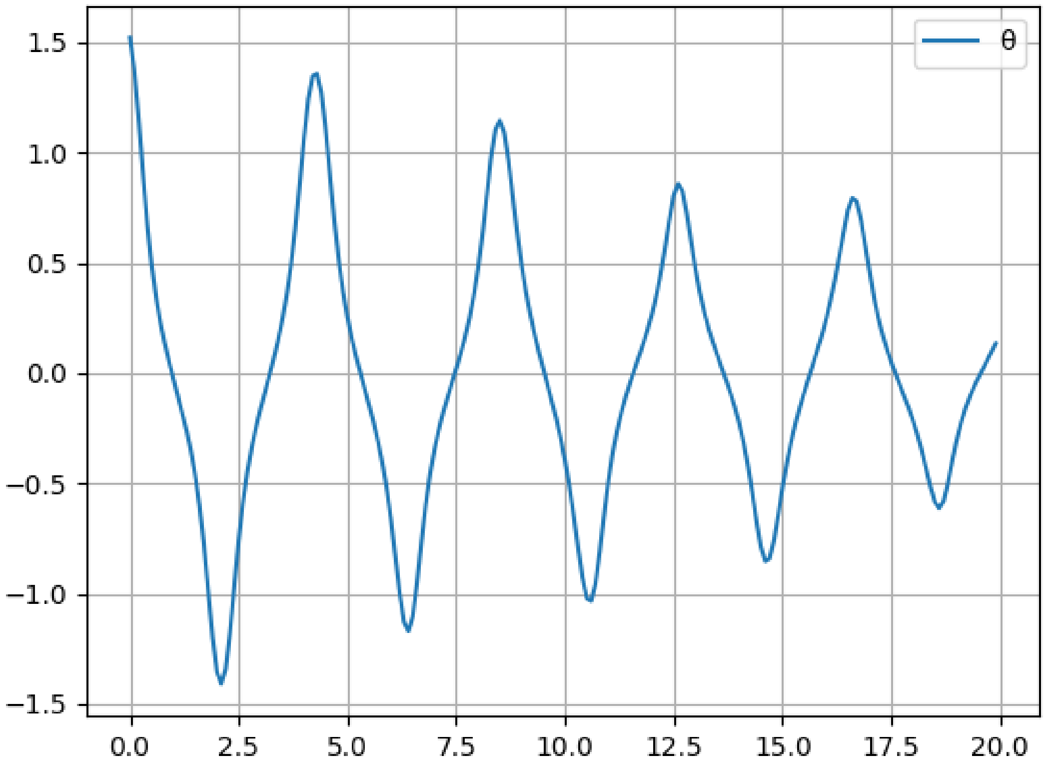

Figure B.19: *Case 9 where m = 9 kg*

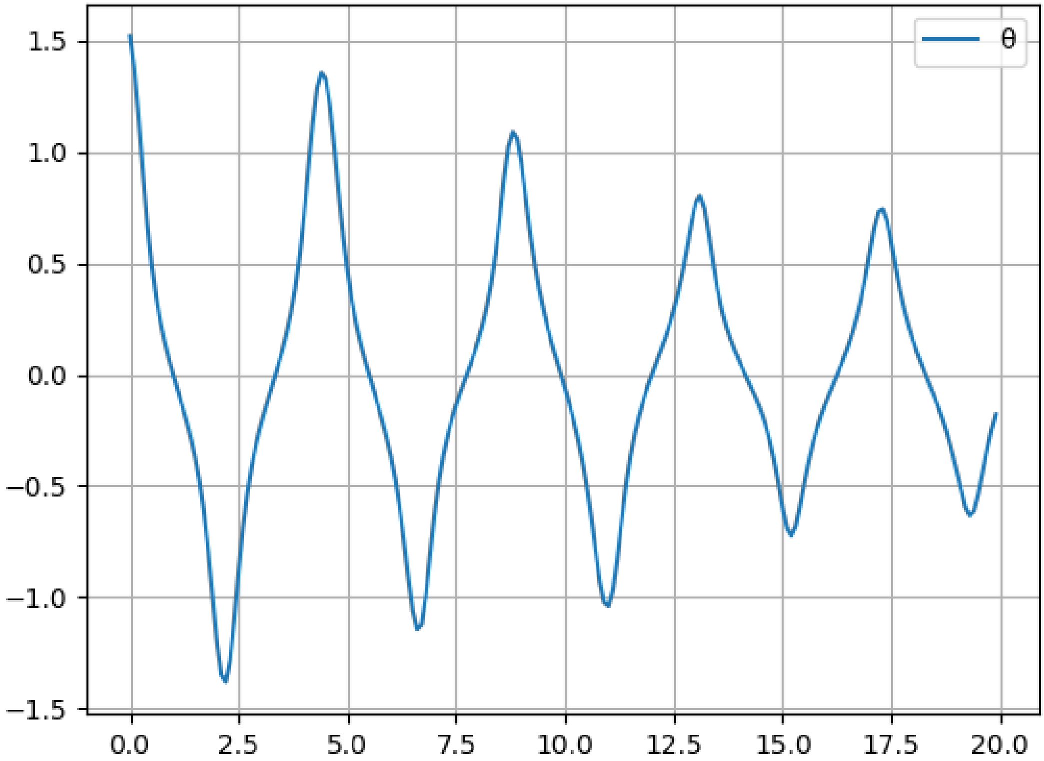

Figure B.20: *Case 10 where m = 10 kg*

B.3 Angular Frequency versus Time

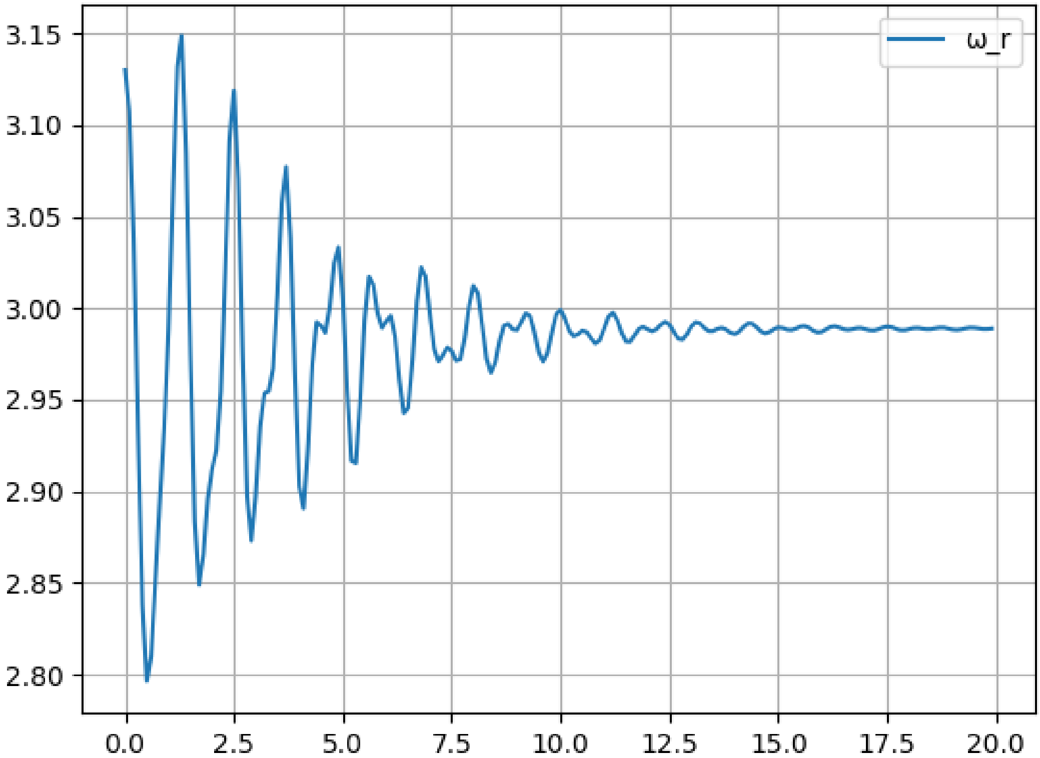

Figure B.21: *Case 1 where m = 1 kg*

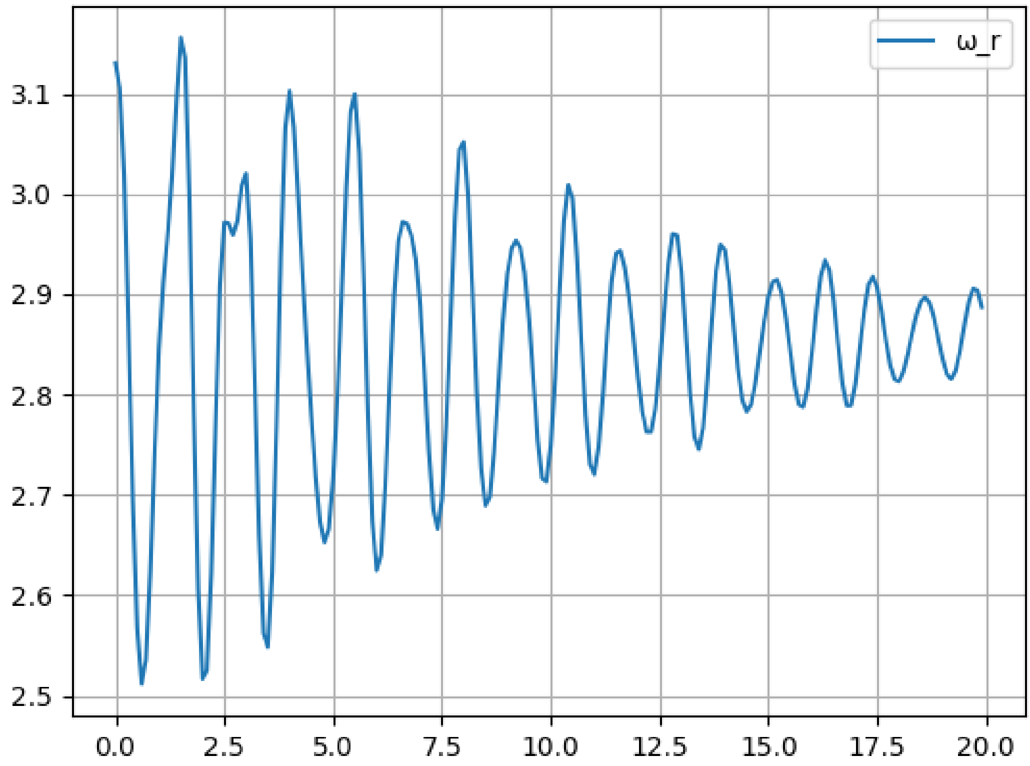

Figure B.22: *Case 2 where m = 2 kg*

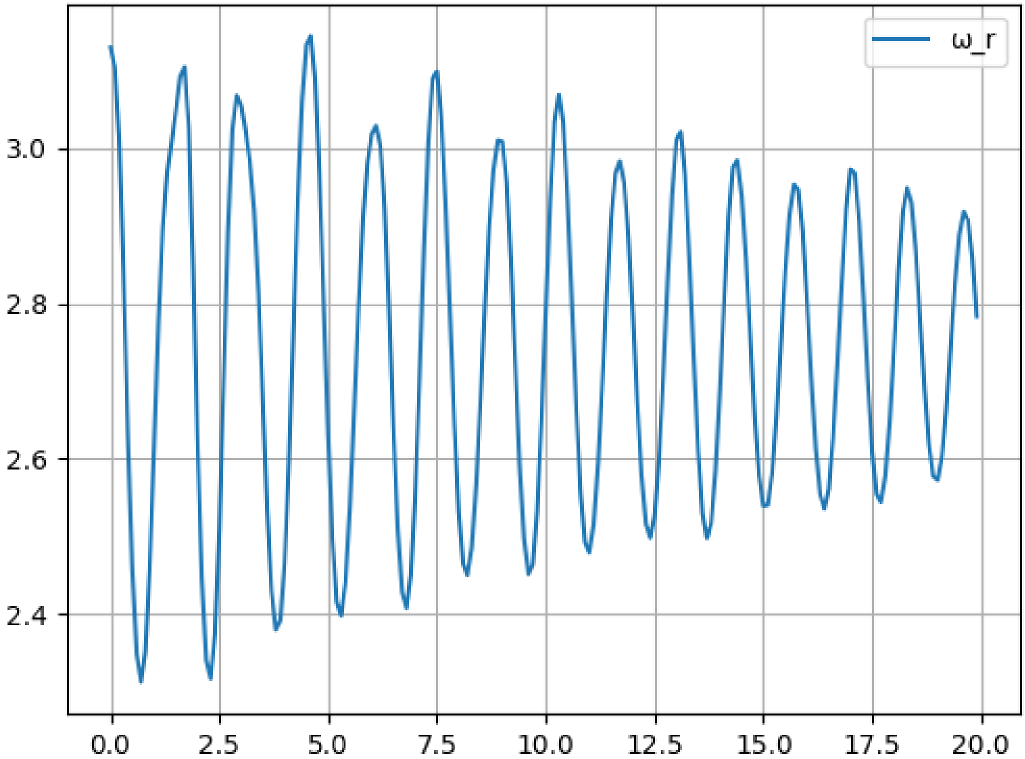

Figure B.23: *Case 3 where m = 3 kg*

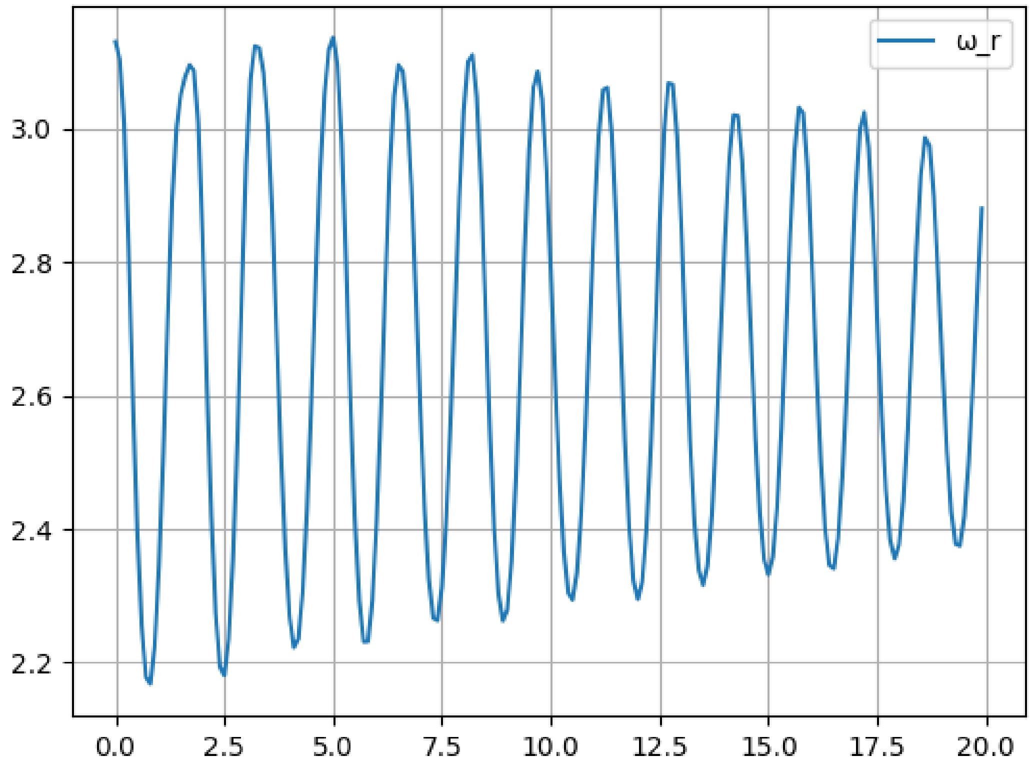

Figure B.24: *Case 4 where m = 4 kg*

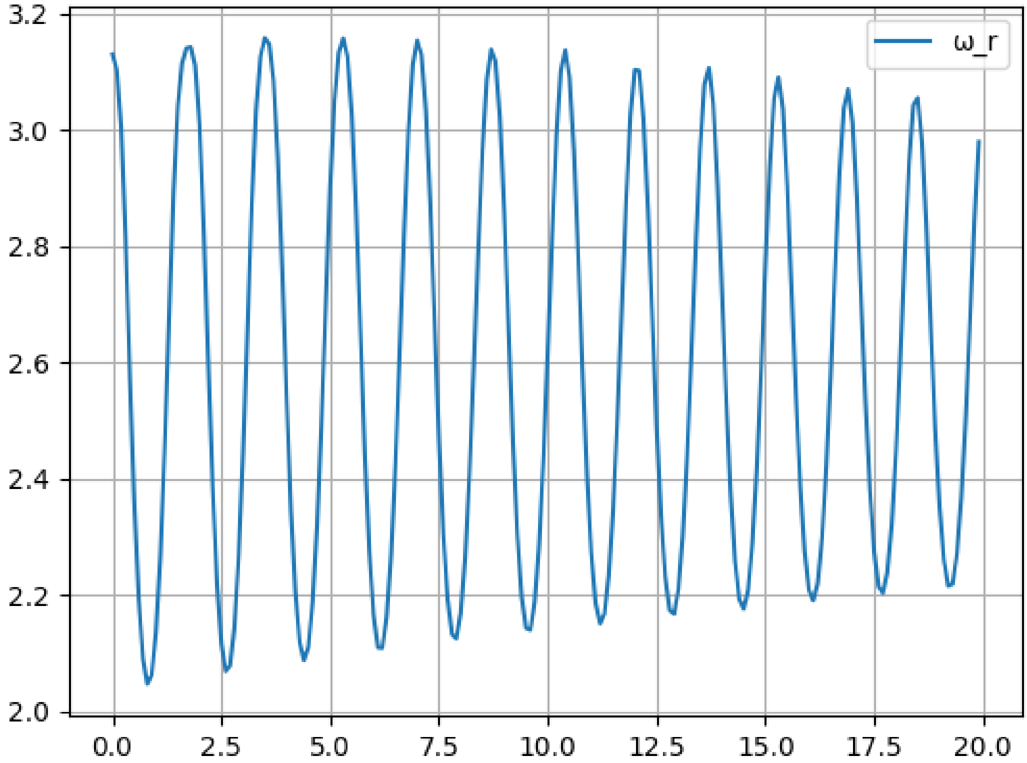

Figure B.25: *Case 5 where m = 5 kg*

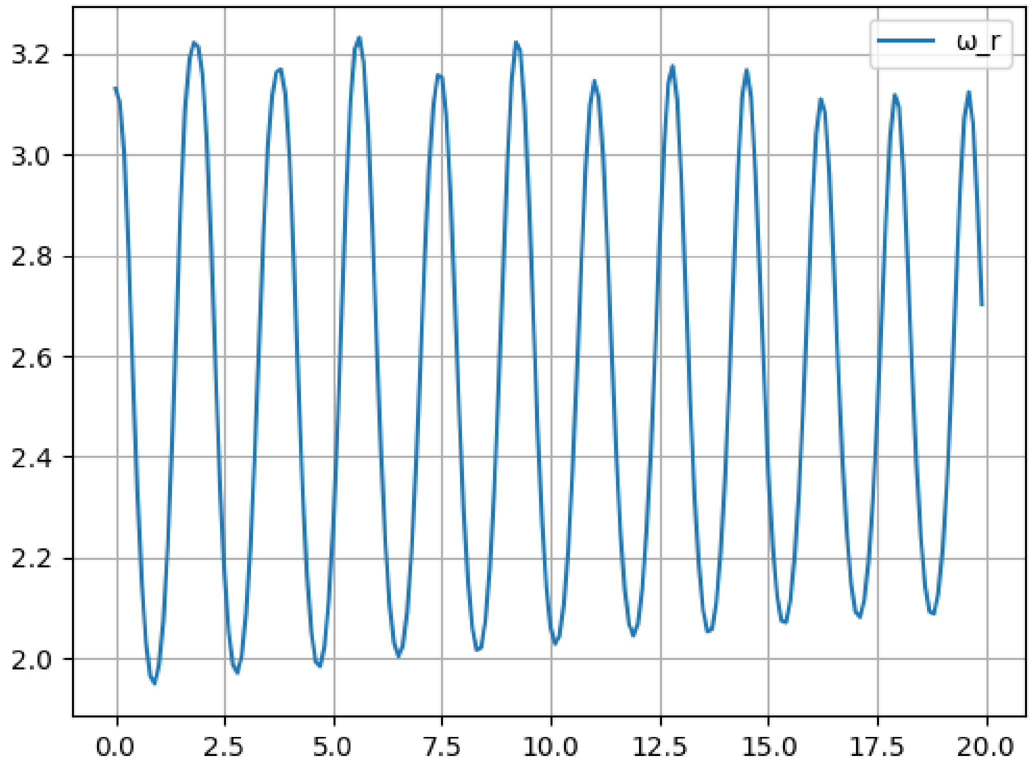

Figure B.26: *Case 6 where m = 6 kg*

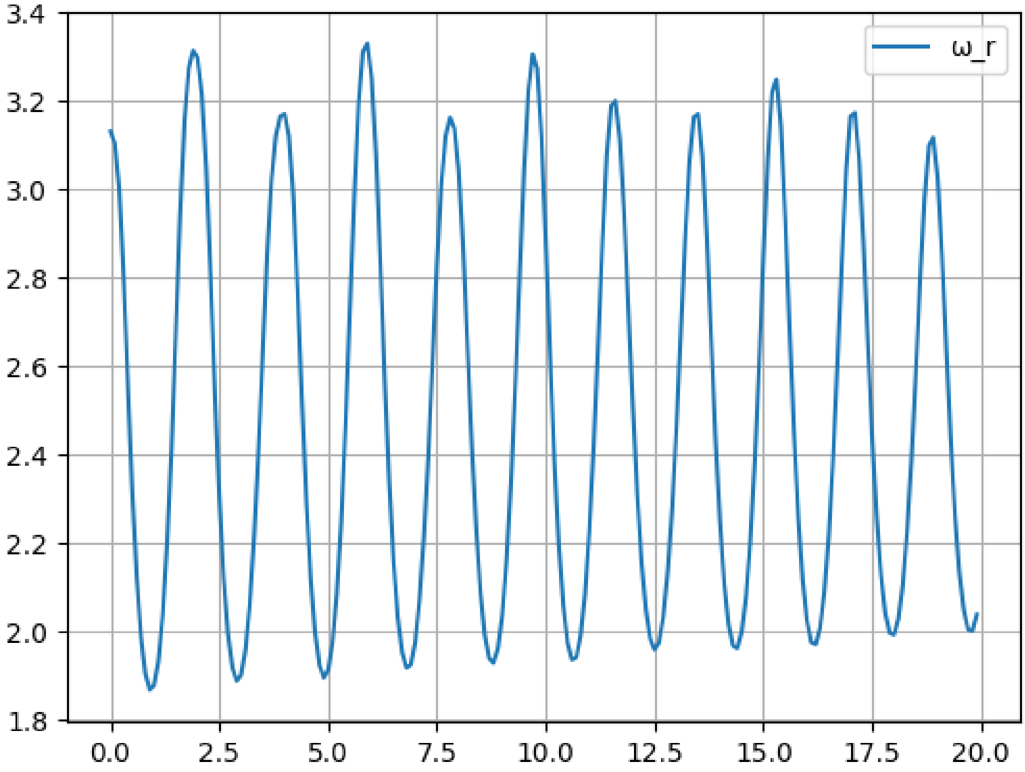

Figure B.27: *Case 7 where m = 7 kg*

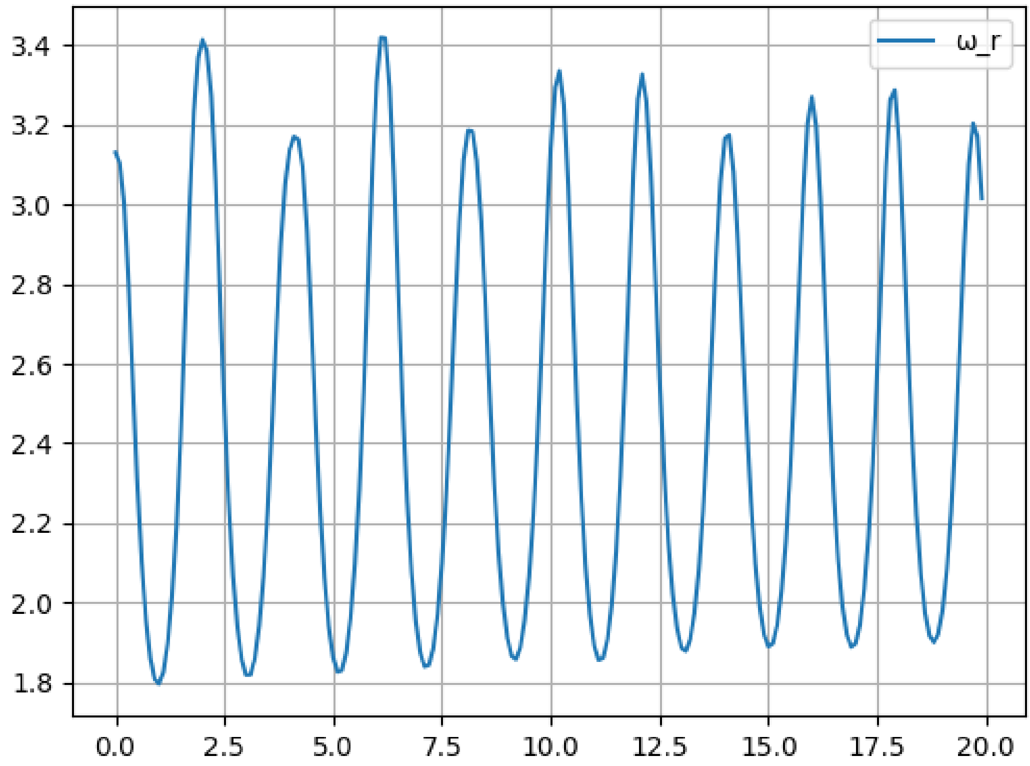

Figure B.28: *Case 8 where m = 8 kg*

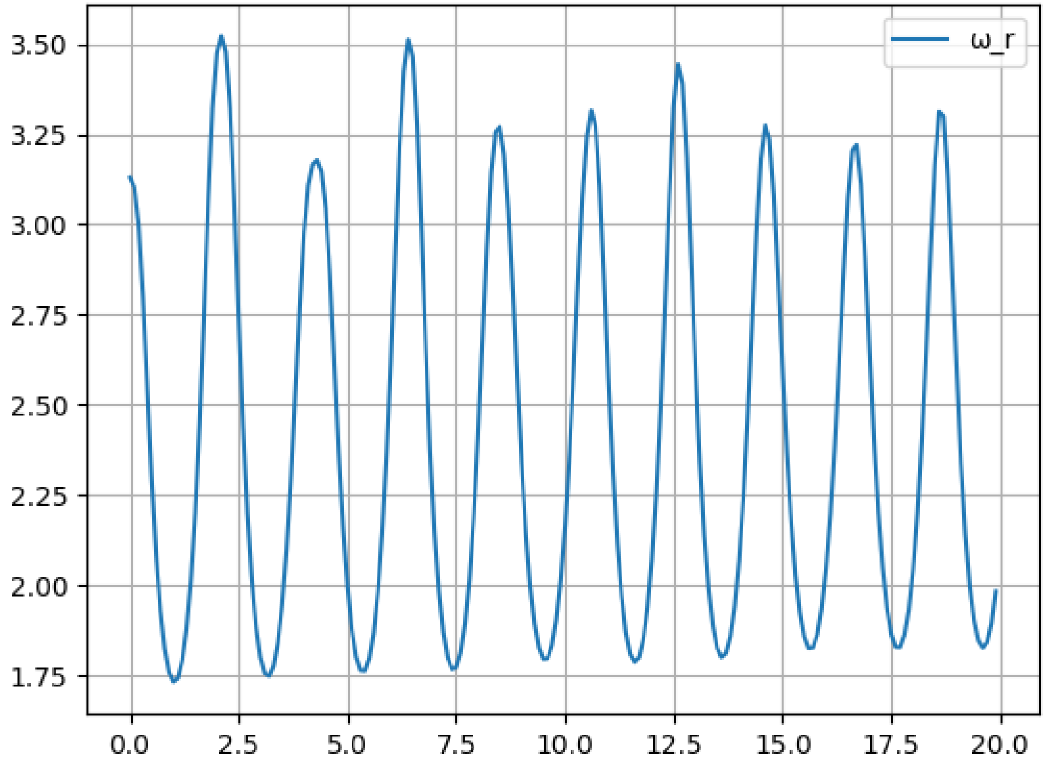

Figure B.29: *Case 9 where m = 9 kg*

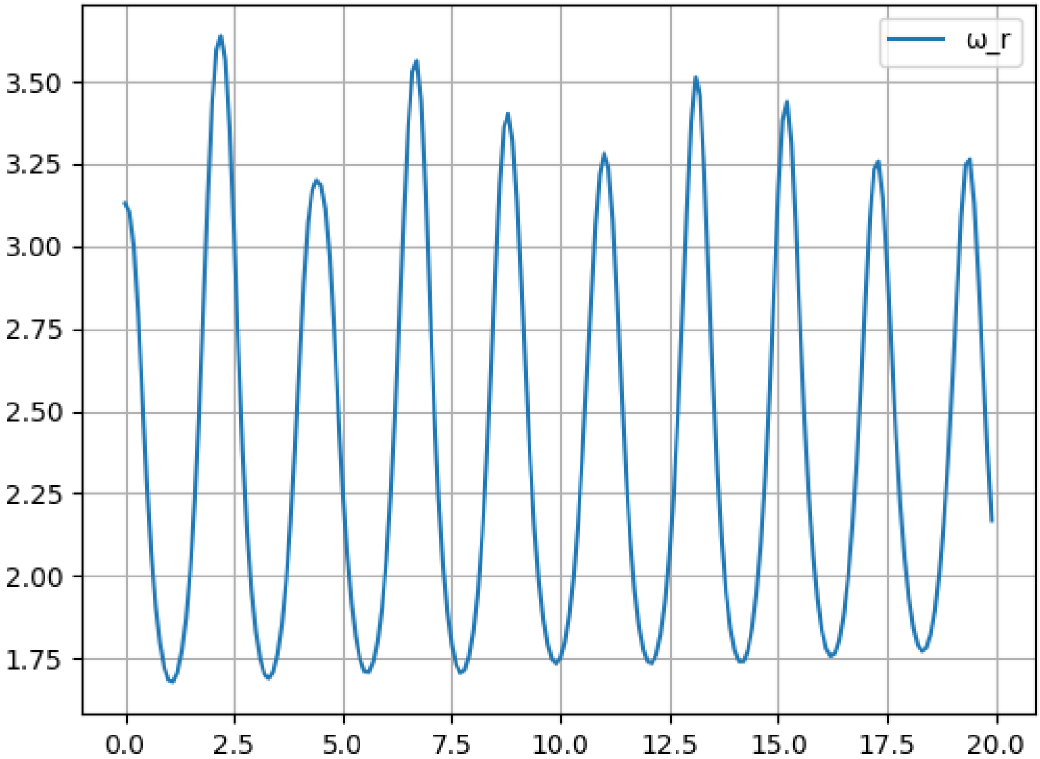

Figure B.30: *Case 10 where m = 10 kg*

B.4 Absolute Frequency versus Time

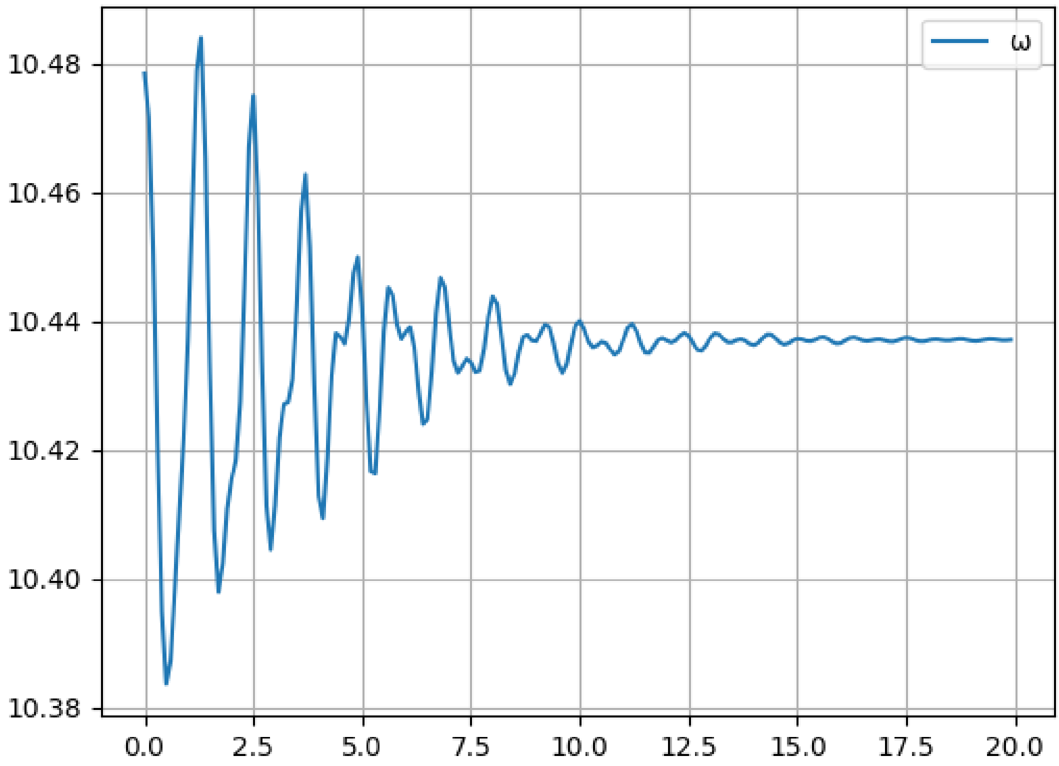

Figure B.31: *Case 1 where m = 1 kg*

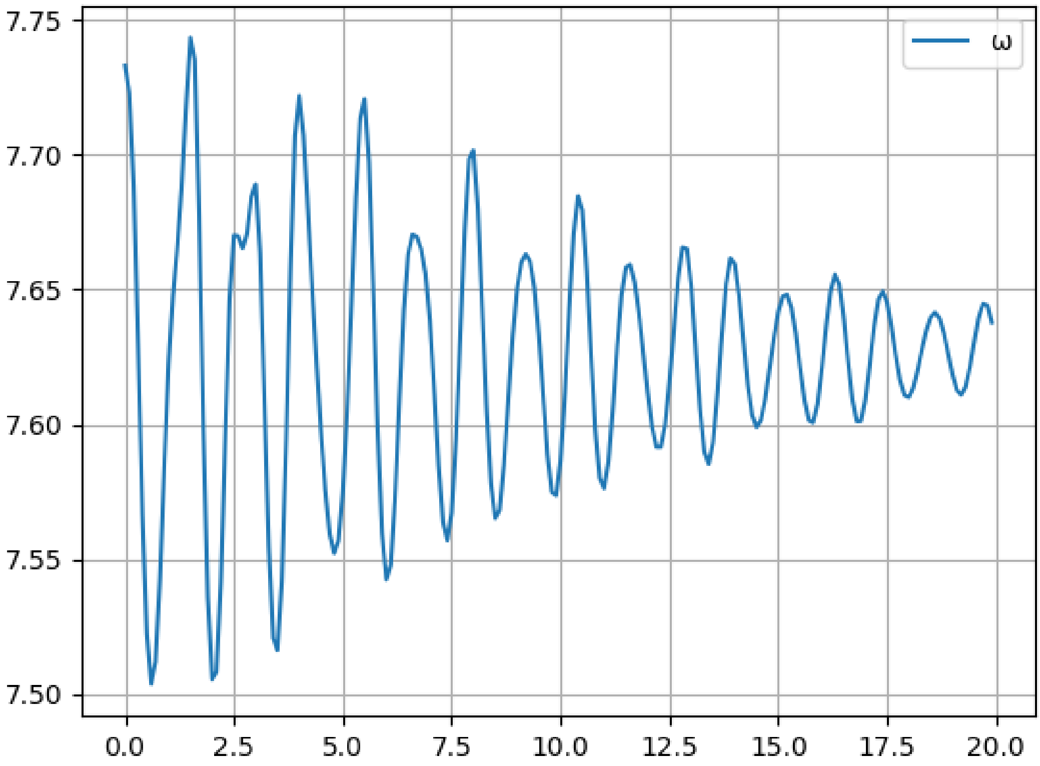

Figure B.32: *Case 2 where m = 2 kg*

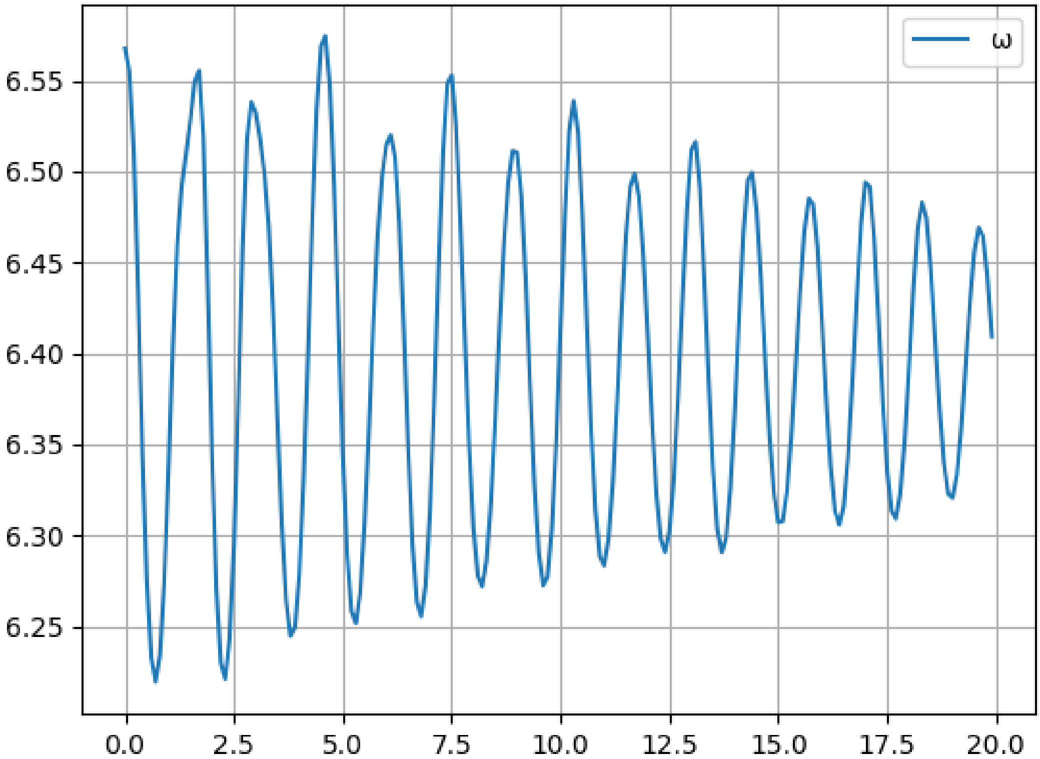

Figure B.33: *Case 3 where m = 3 kg*

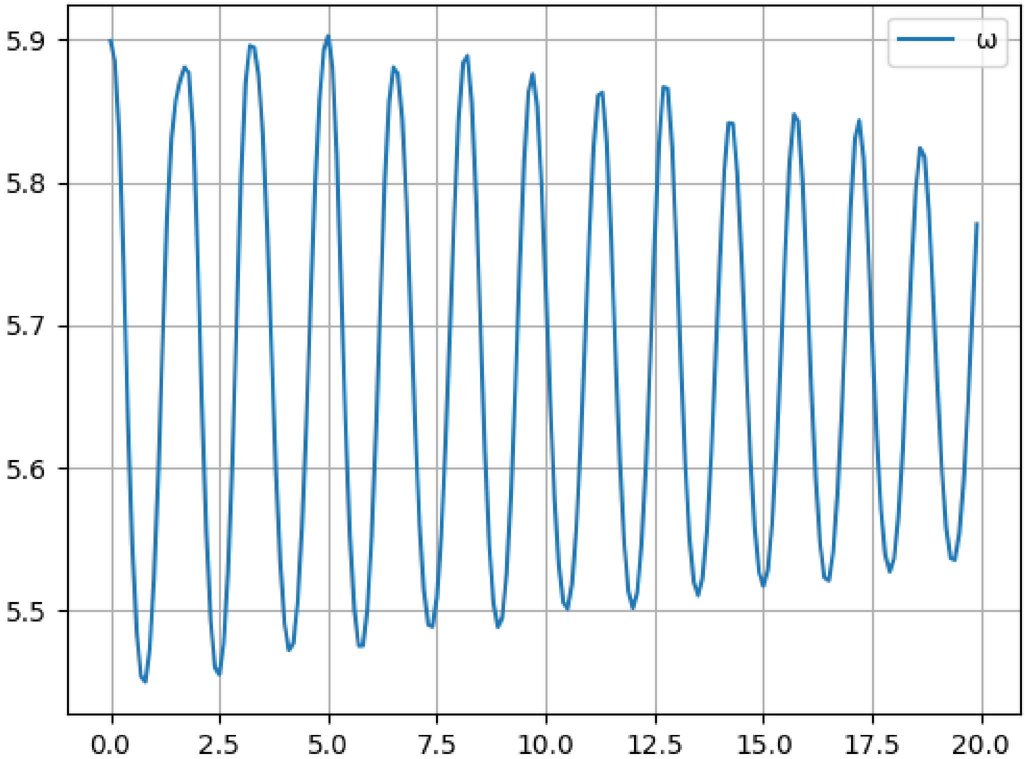

Figure B.34: *Case 4 where m = 4 kg*

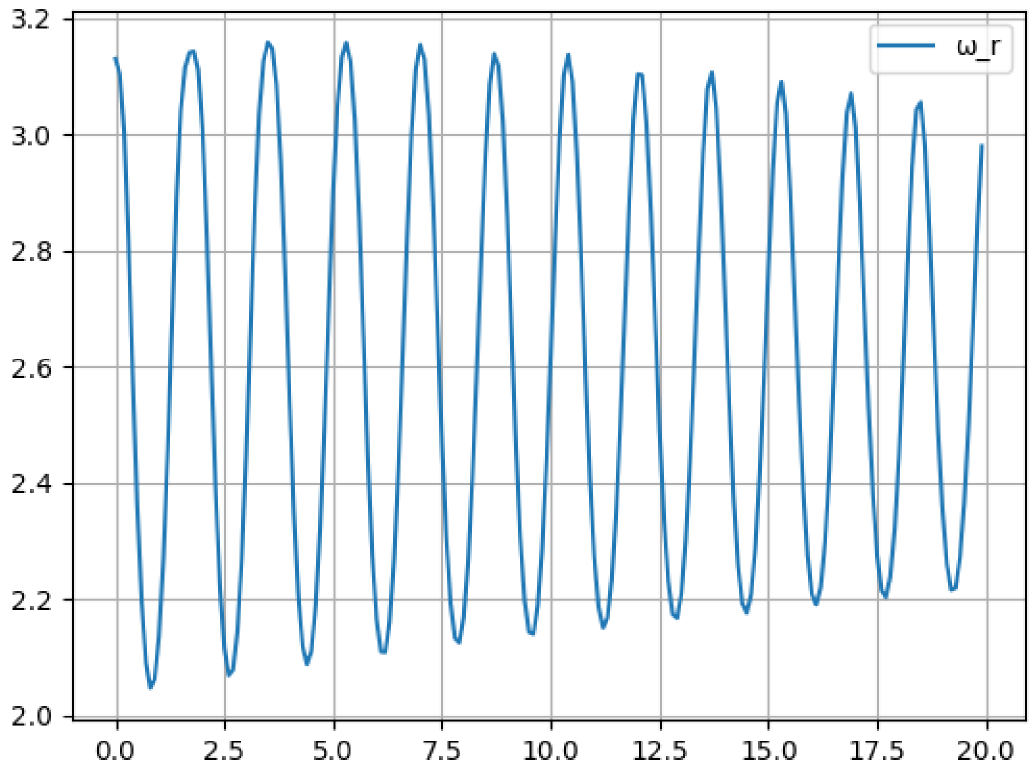

Figure B.35: *Case 5 where m = 5 kg*

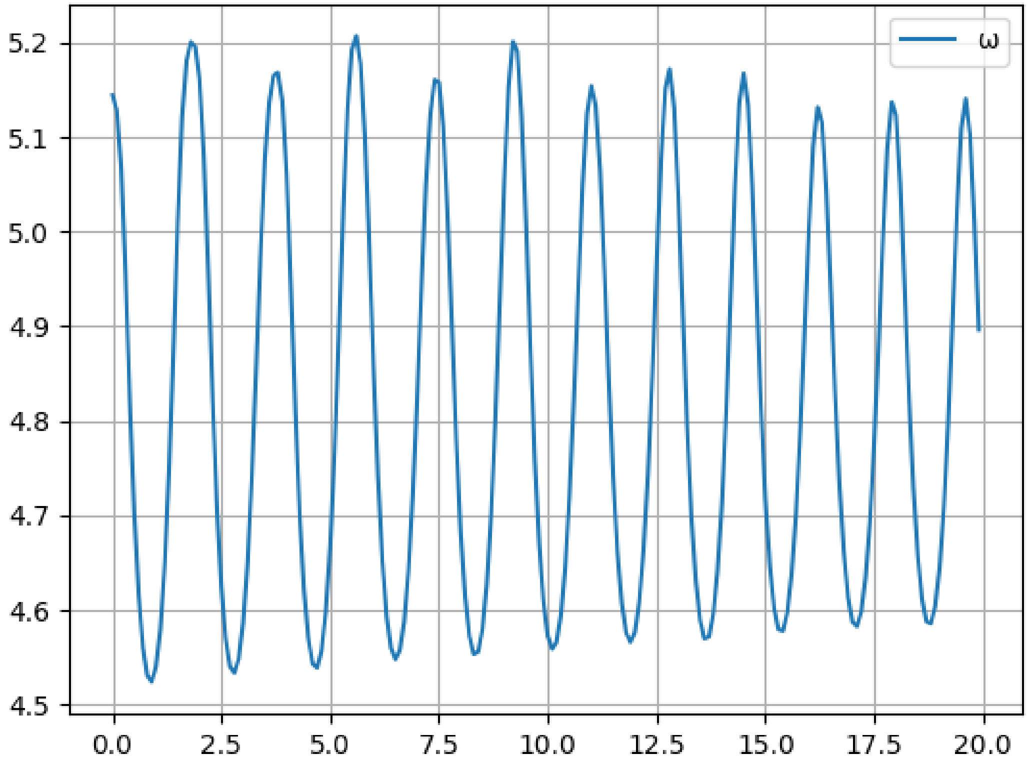

Figure B.36: *Case 6 where m = 6 kg*

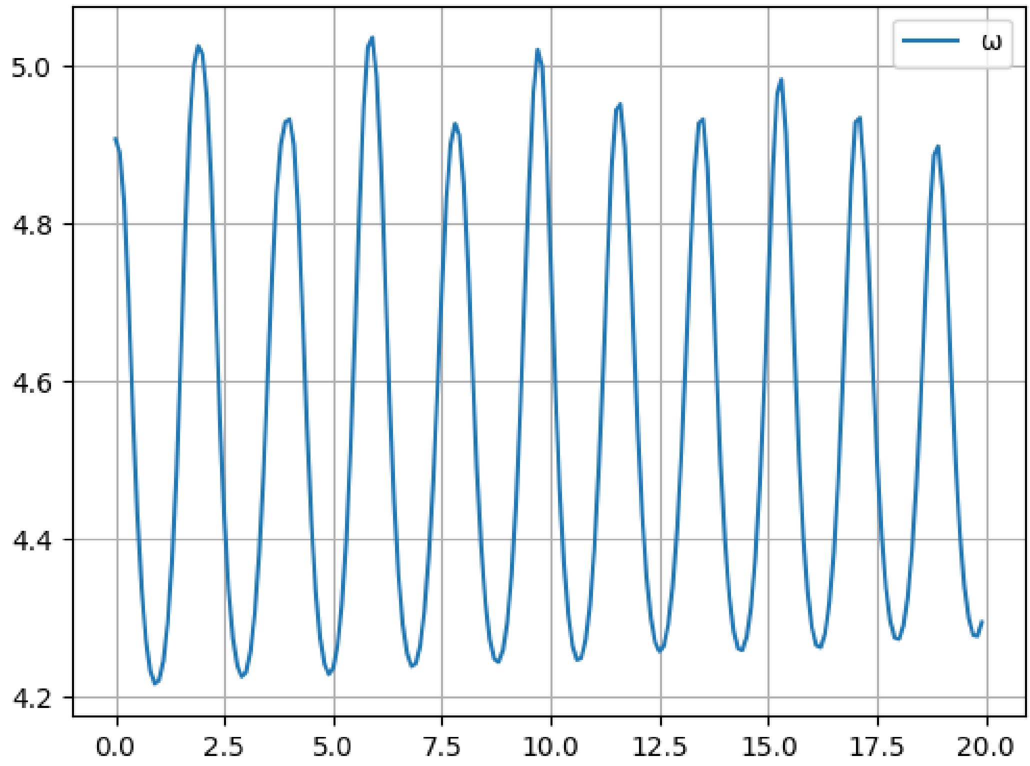

Figure B.37: *Case 7 where m = 7 kg*

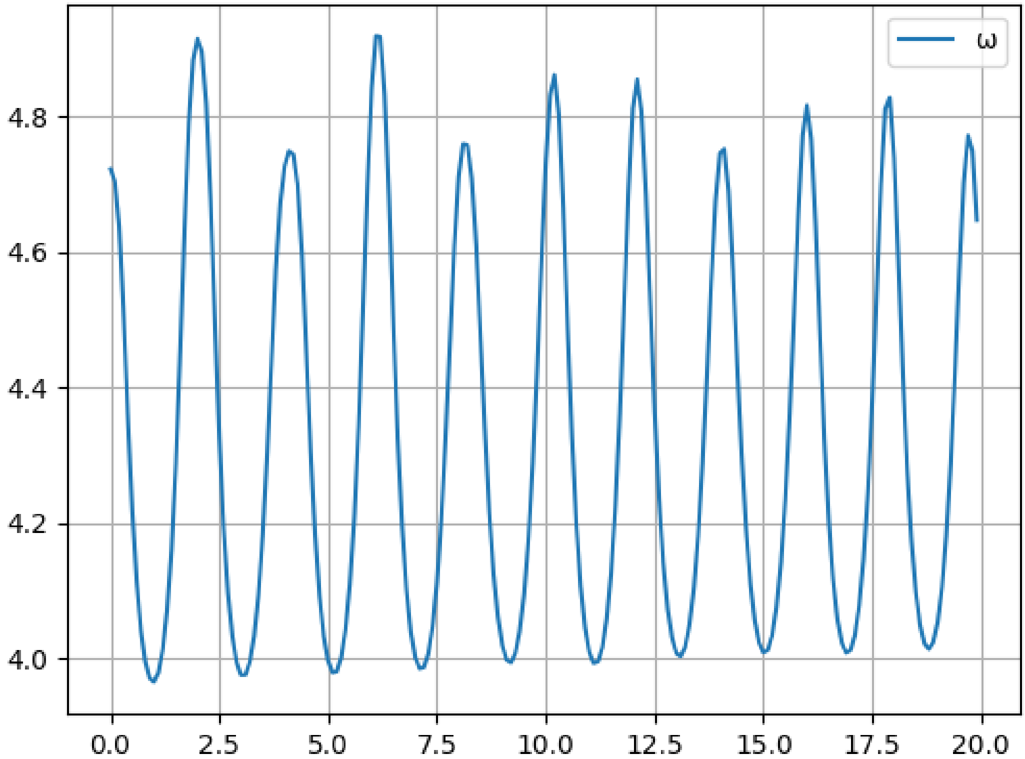

Figure B.38: *Case 8 where m = 8 kg*

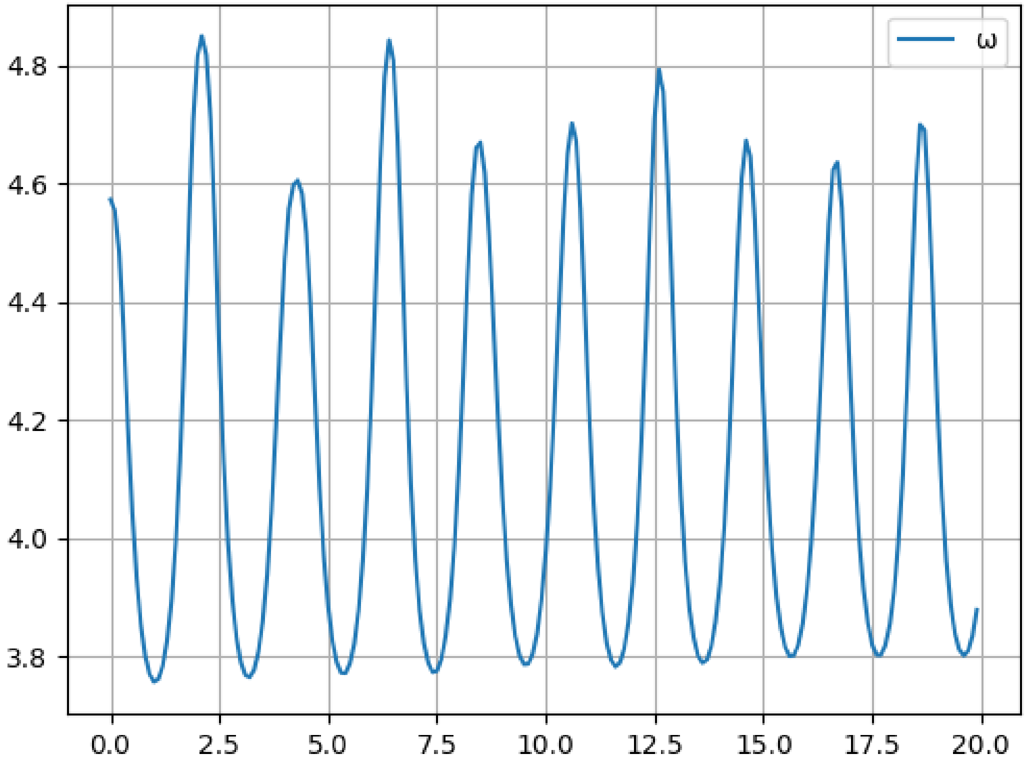

Figure B.39: *Case 9 where m = 9 kg*

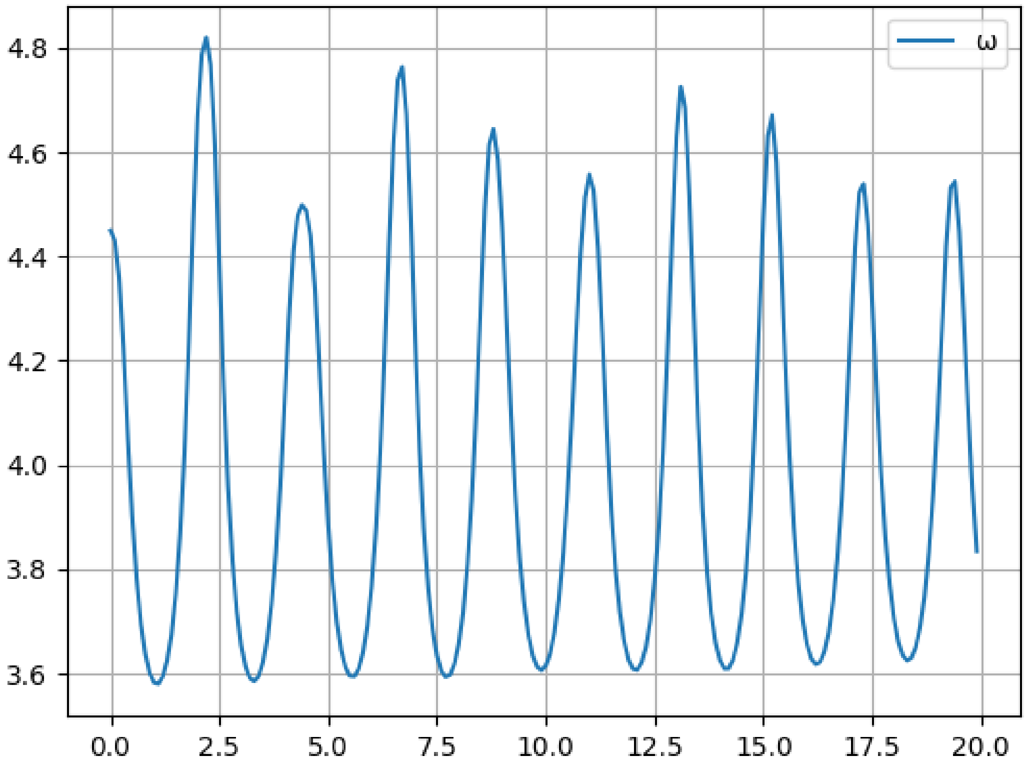

Figure B.40: *Case 10 where m = 10 kg*

C Raw Simulation Data

C.1 Radial Displacement versus Time

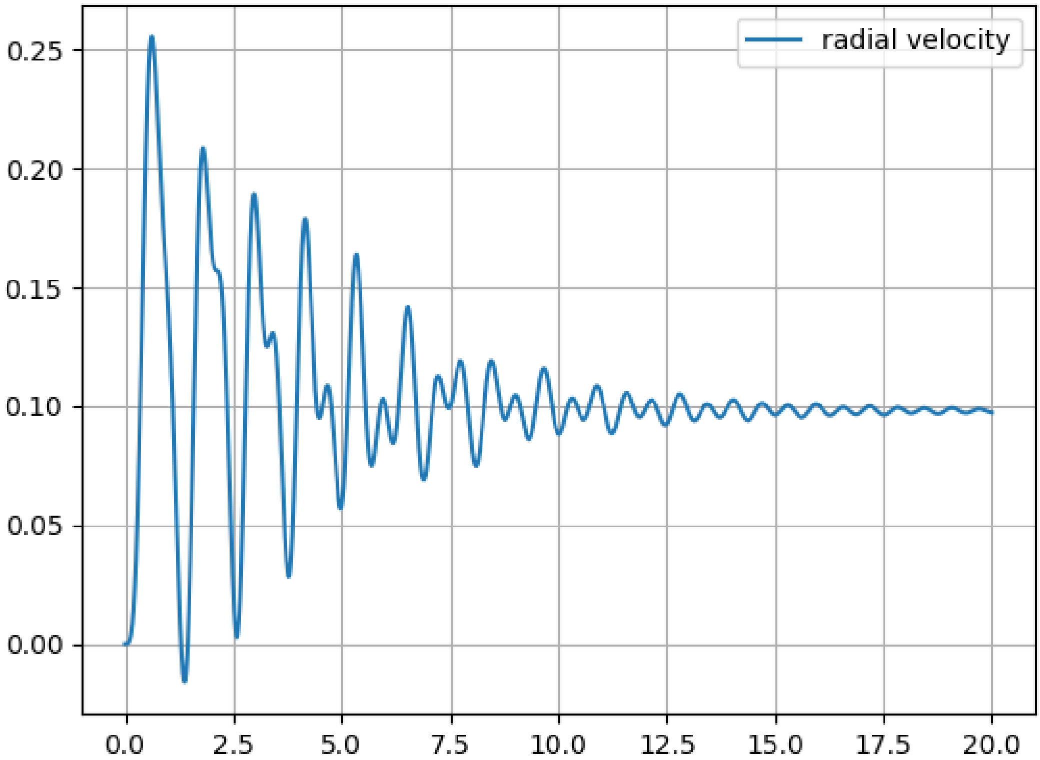

Figure C.1: *Case 1 where m = 1 kg*

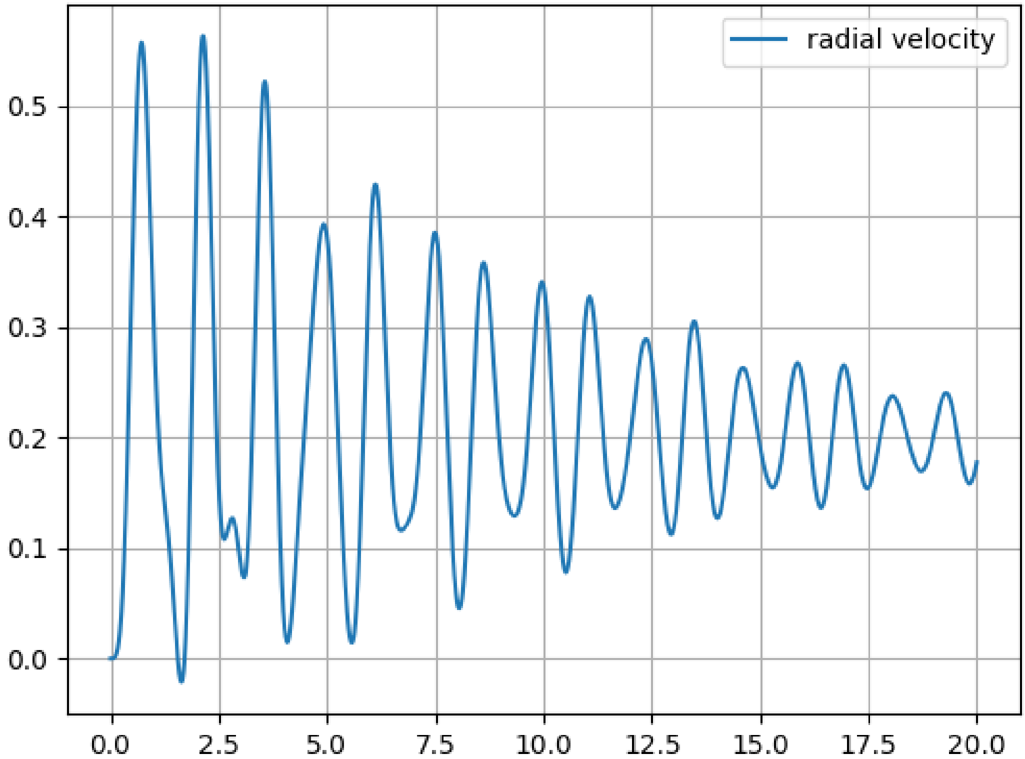

Figure C.2: *Case 2 where m = 2 kg*

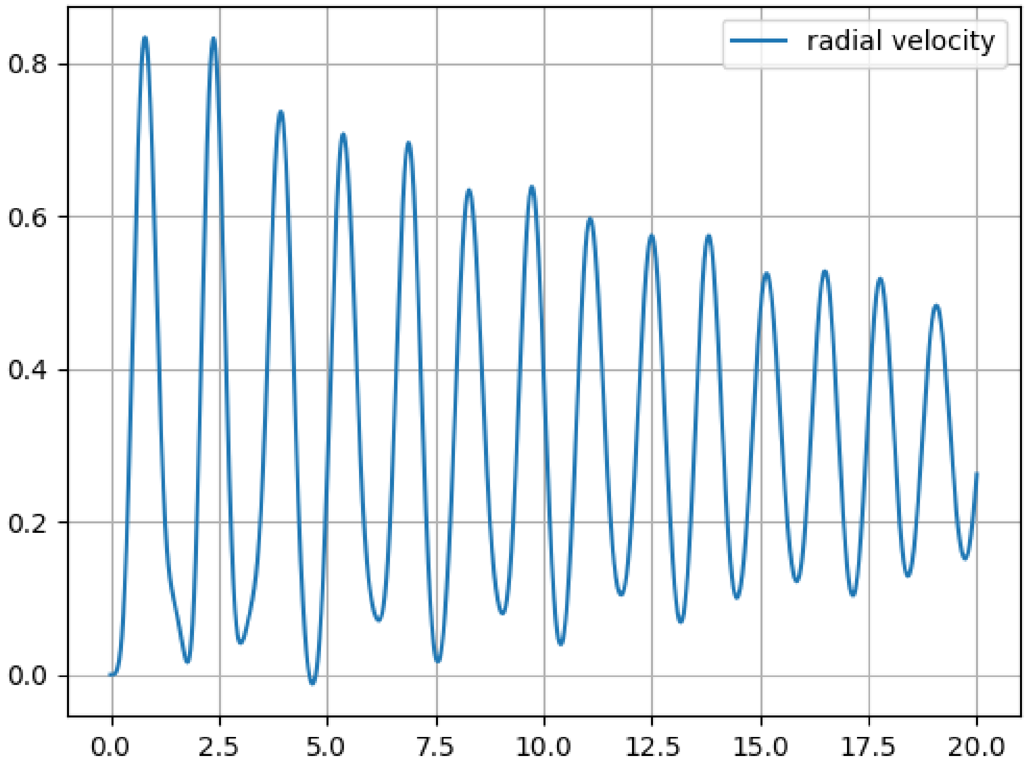

Figure C.3: *Case 3 where m = 3 kg*

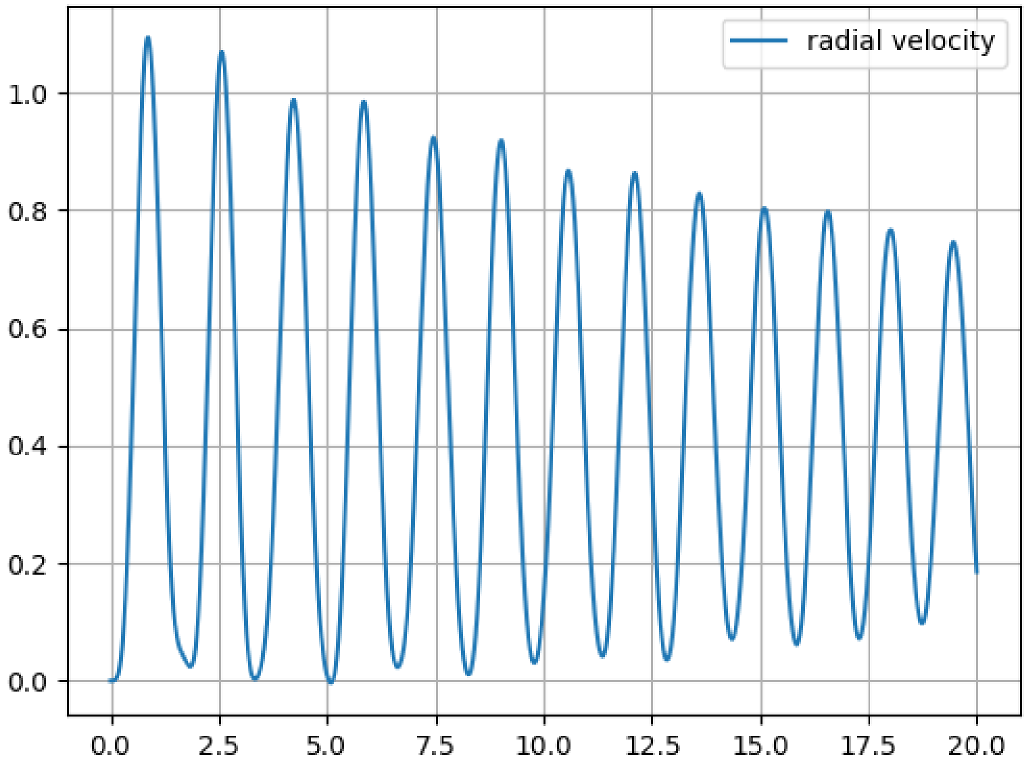

Figure C.4: *Case 4 where m = 4 kg*

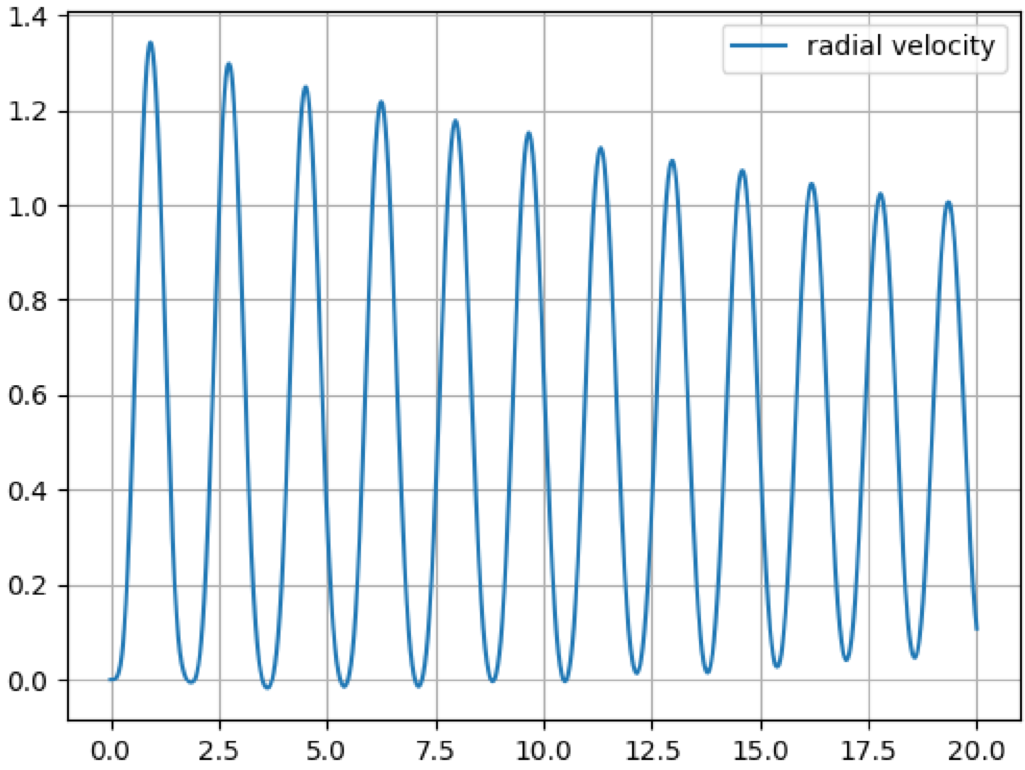

Figure C.5: *Case 5 where m = 5 kg*

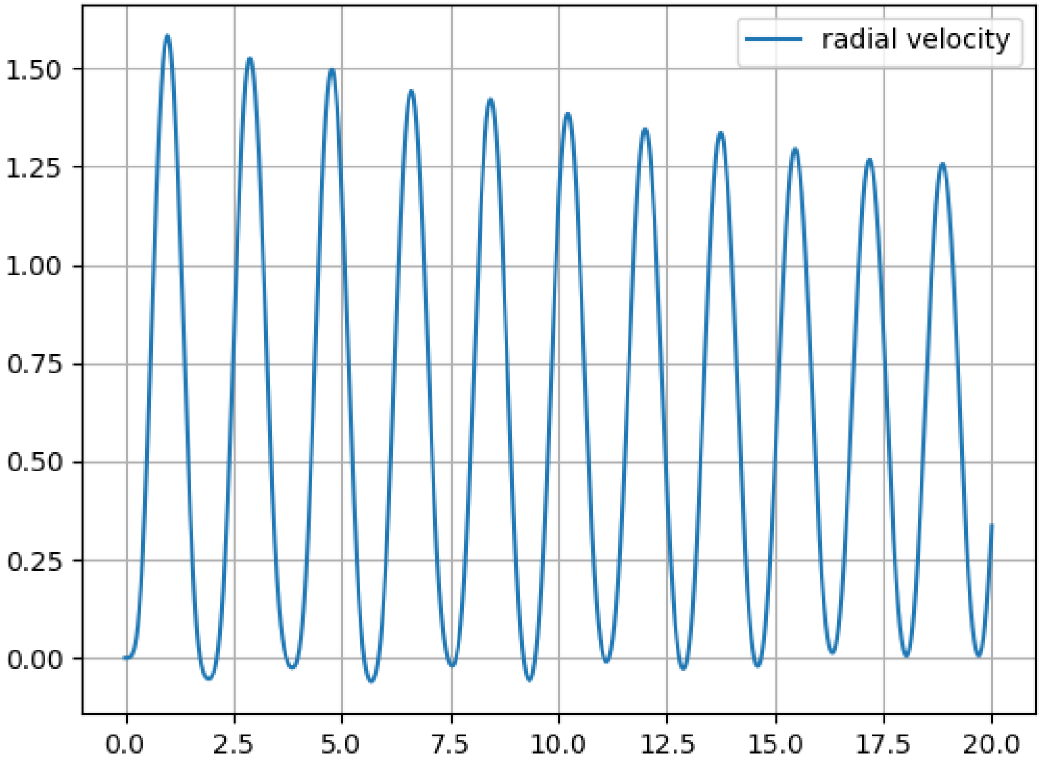

Figure C.6: *Case 6 where m = 6 kg*

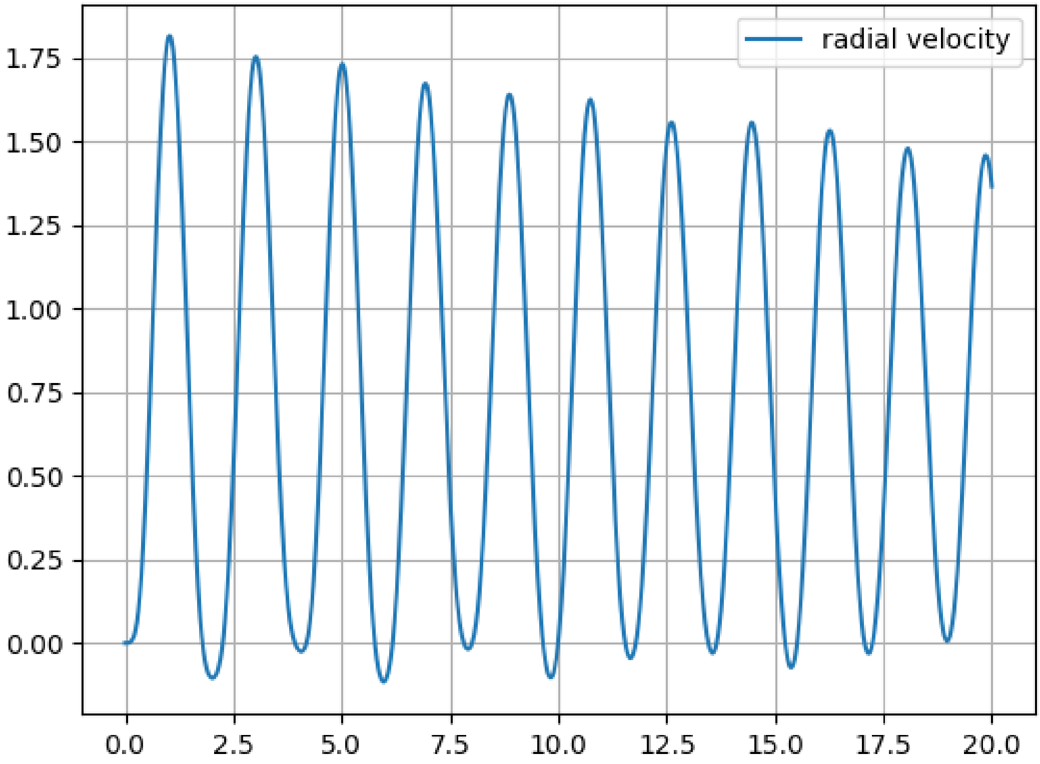

Figure C.7: *Case 7 where m = 7 kg*

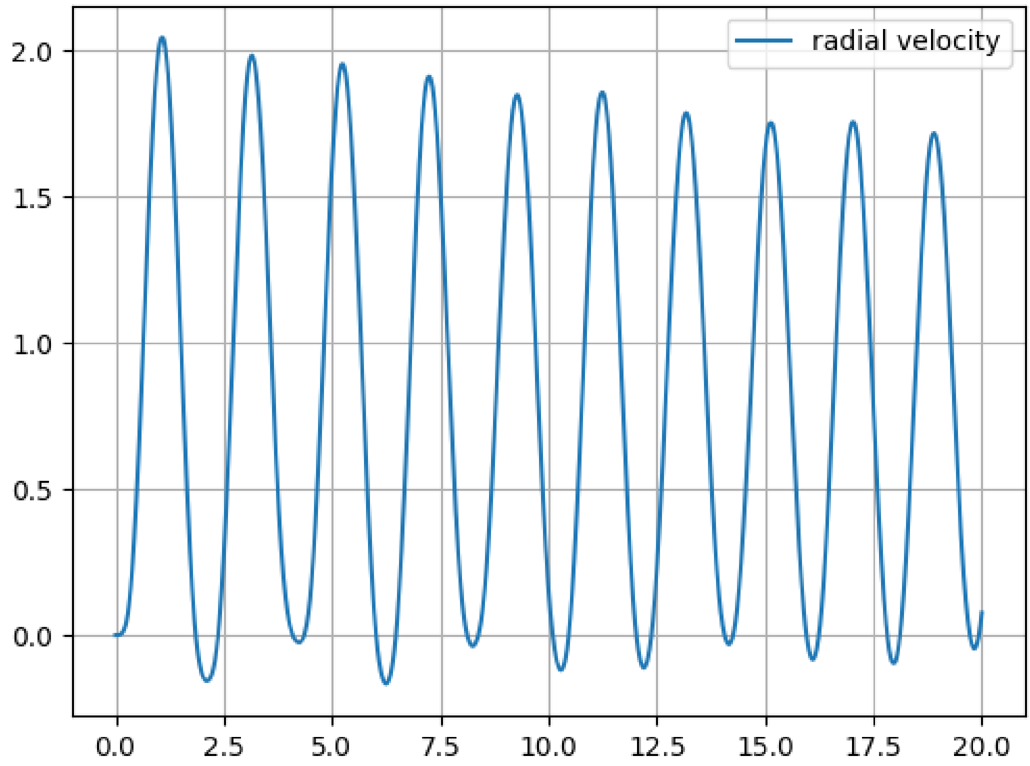

Figure C.8: *Case 8 where m = 8 kg*

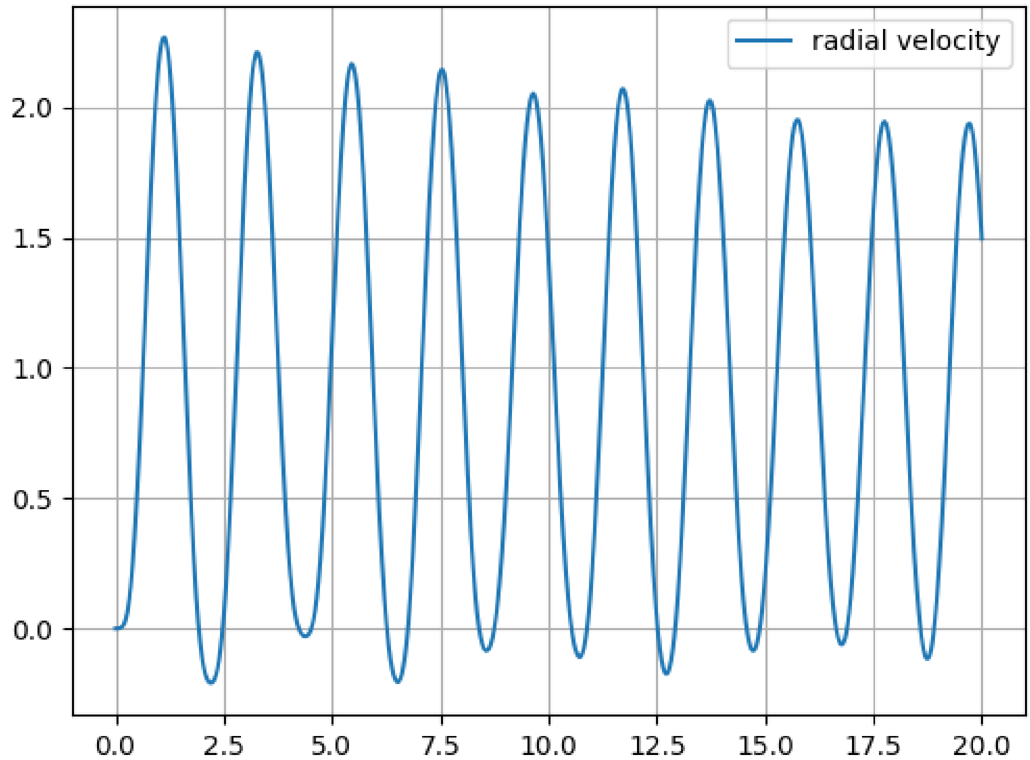

Figure C.9: *Case 9 where m = 9 kg*

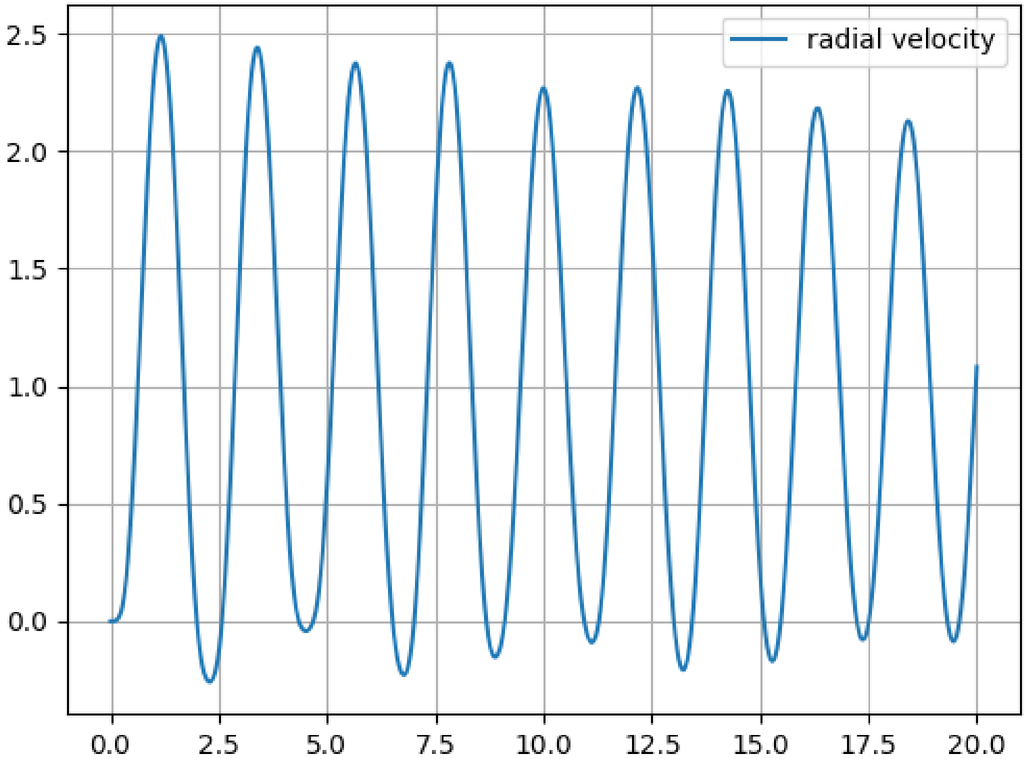

Figure C.10: *Case 10 where m = 10 kg*

C.2 Angular Displacement versus Time

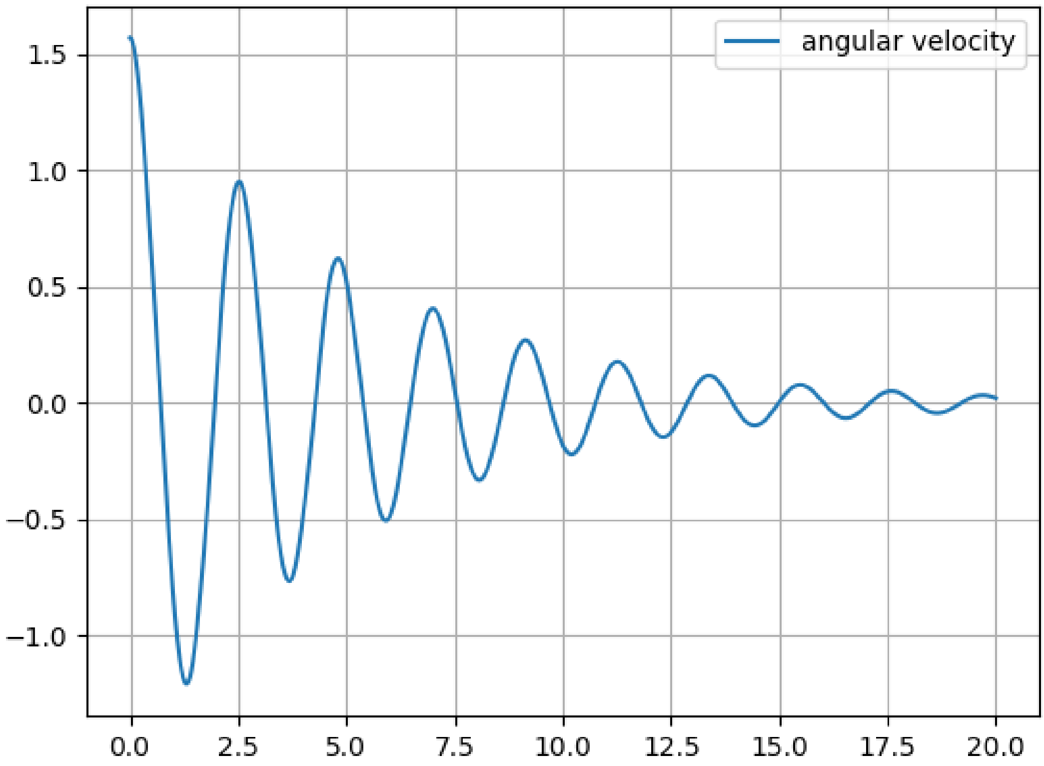

Figure C.11: *Case 1 where m = 1 kg*

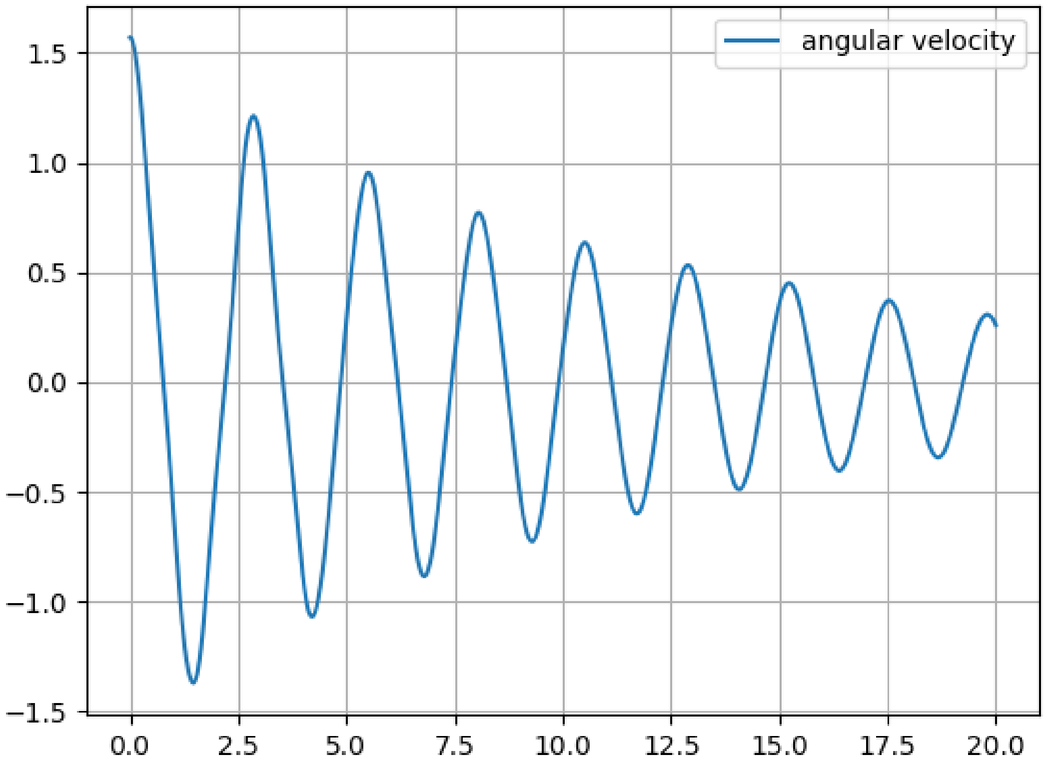

Figure C.12: *Case 2 where m = 2 kg*

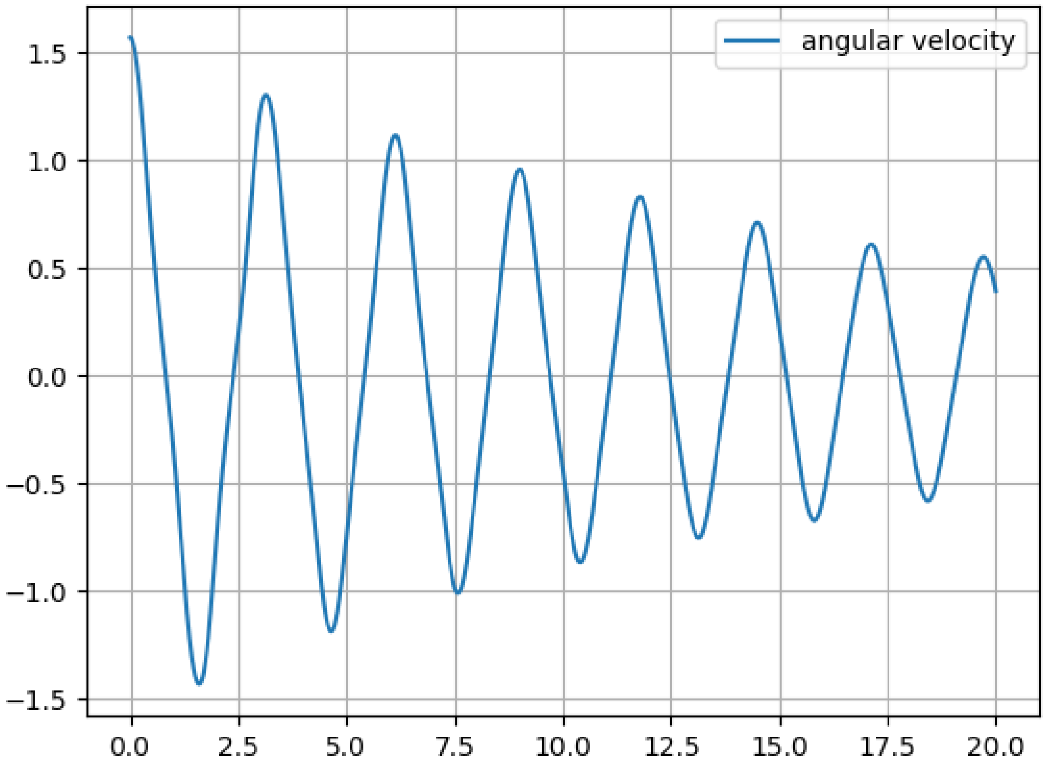

Figure C.13: *Case 3 where m = 3 kg*

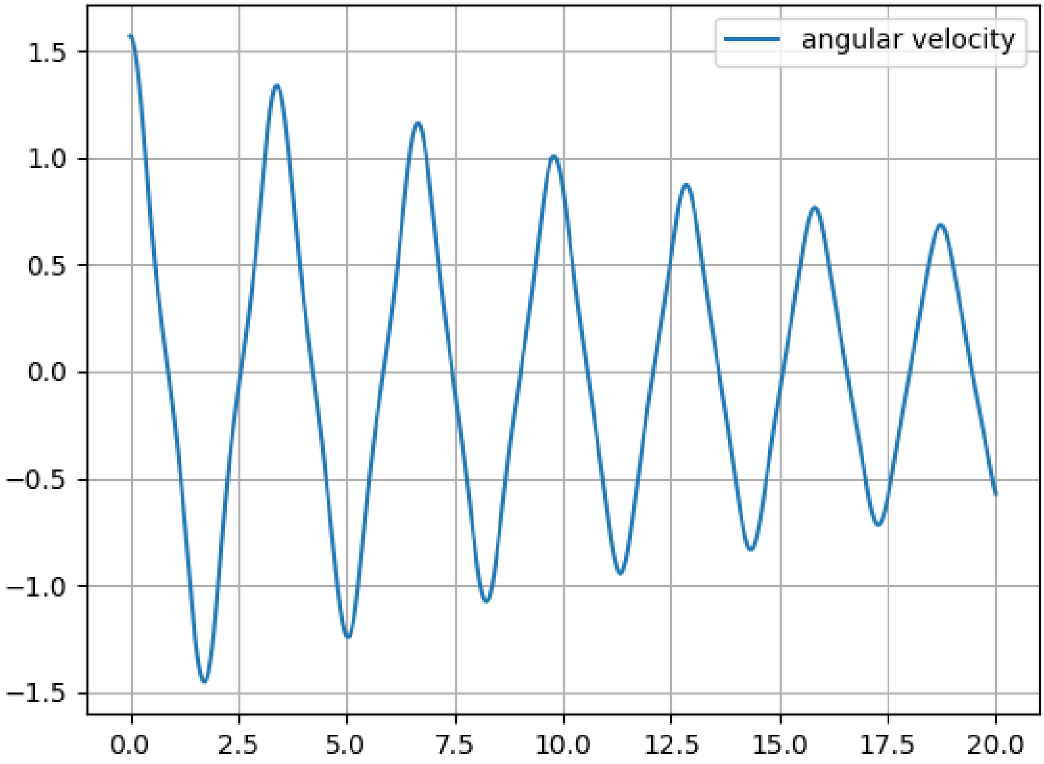

Figure C.14: *Case 4 where m = 4 kg*

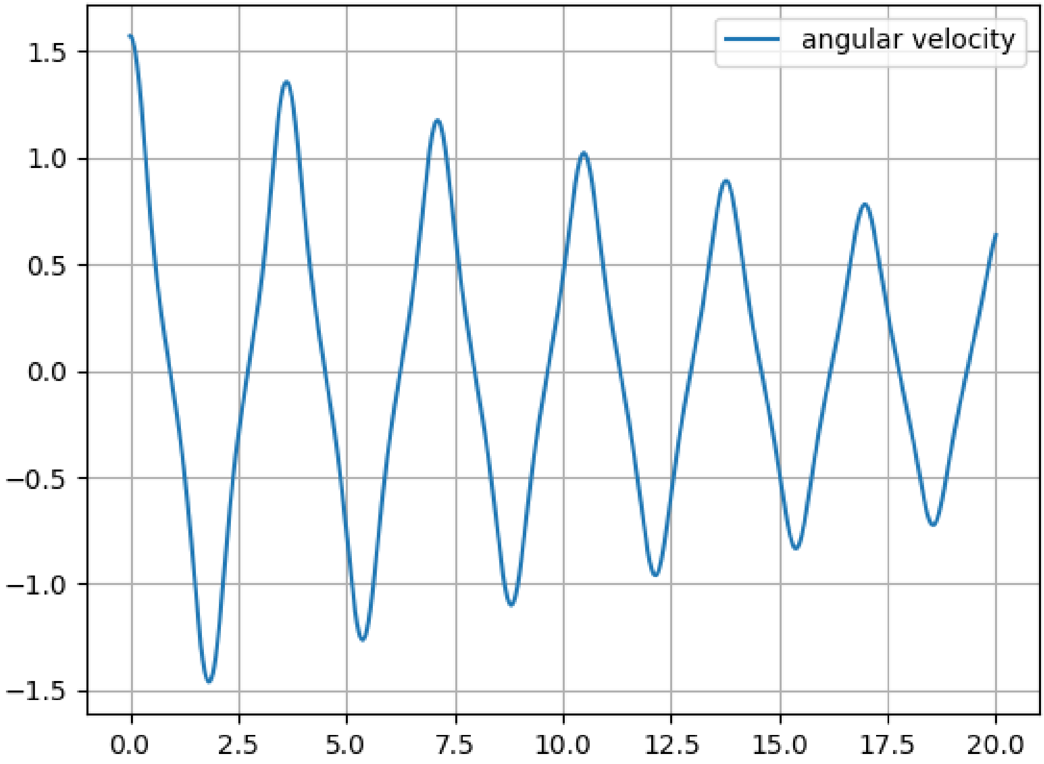

Figure C.15: *Case 5 where m = 5 kg*

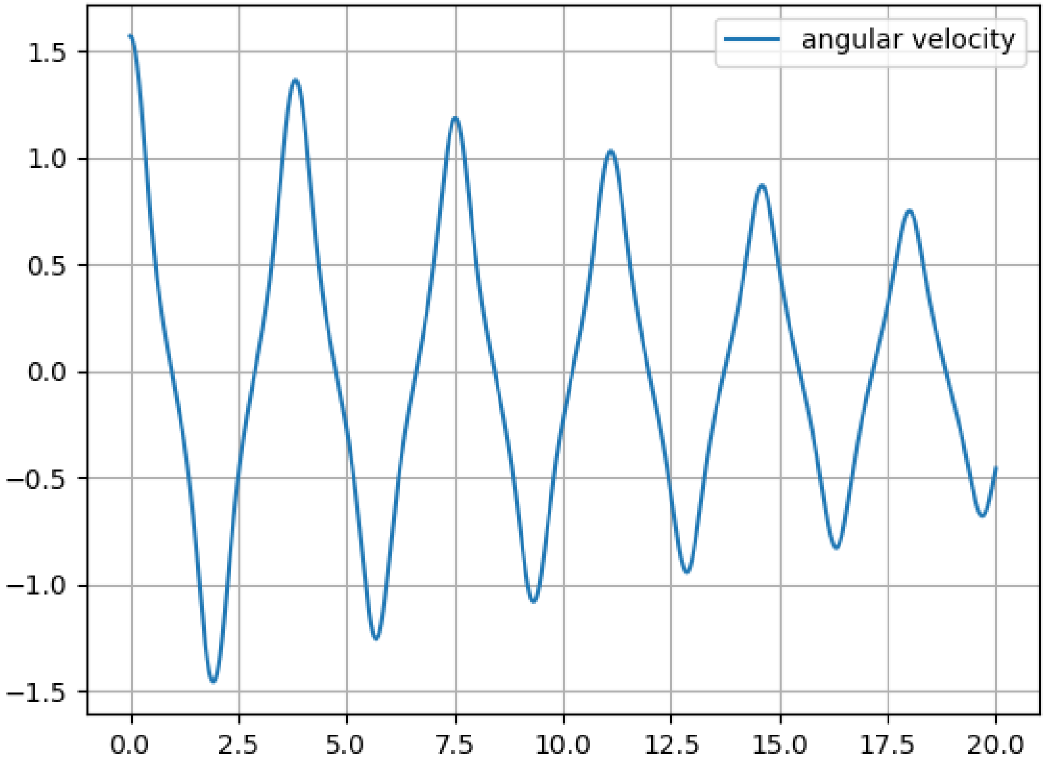

Figure C.16: *Case 6 where m = 6 kg*

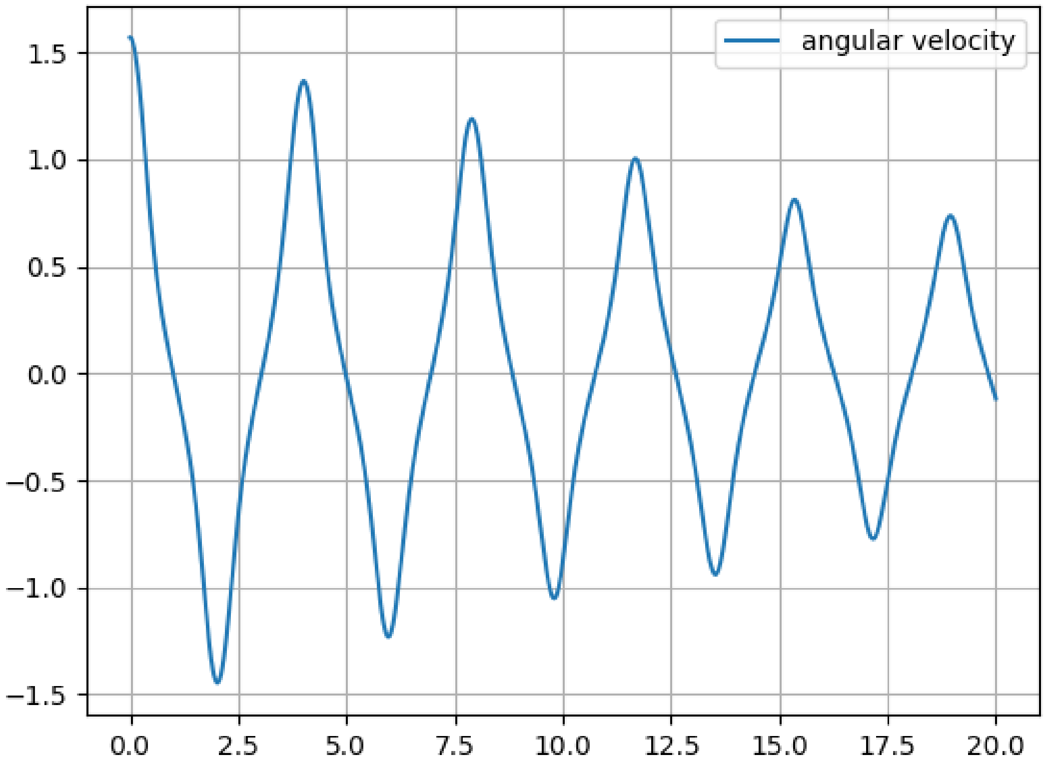

Figure C.17: *Case 7 where m = 7 kg*

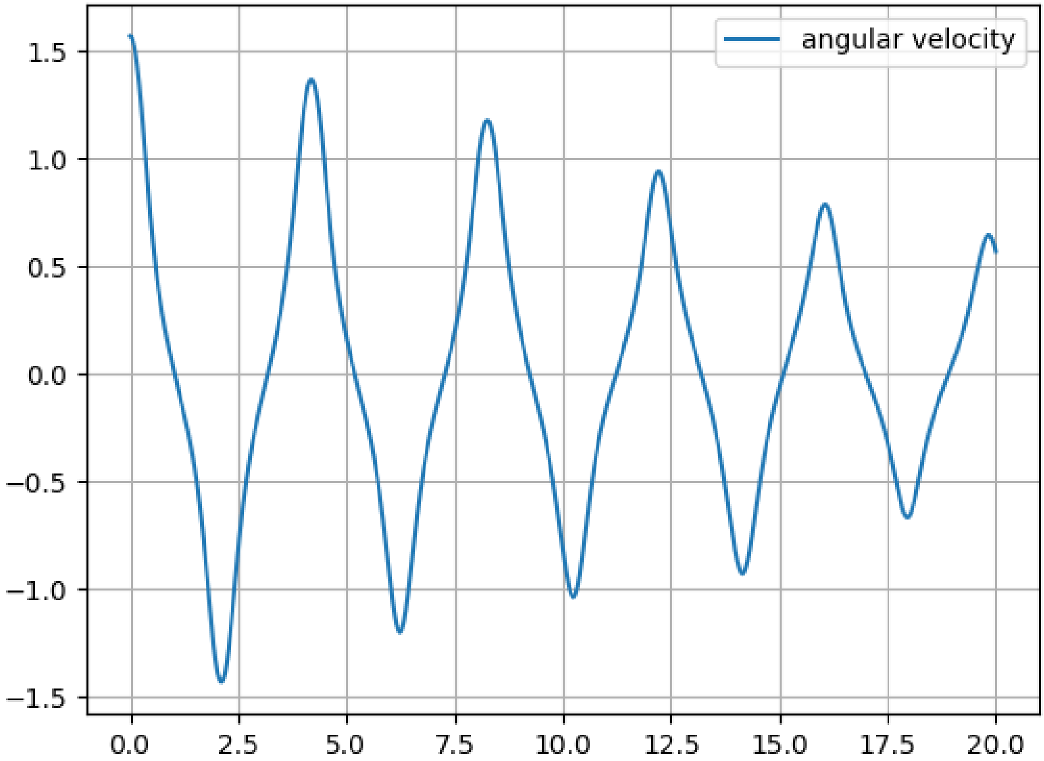

Figure C.18: *Case 8 where m = 8 kg*

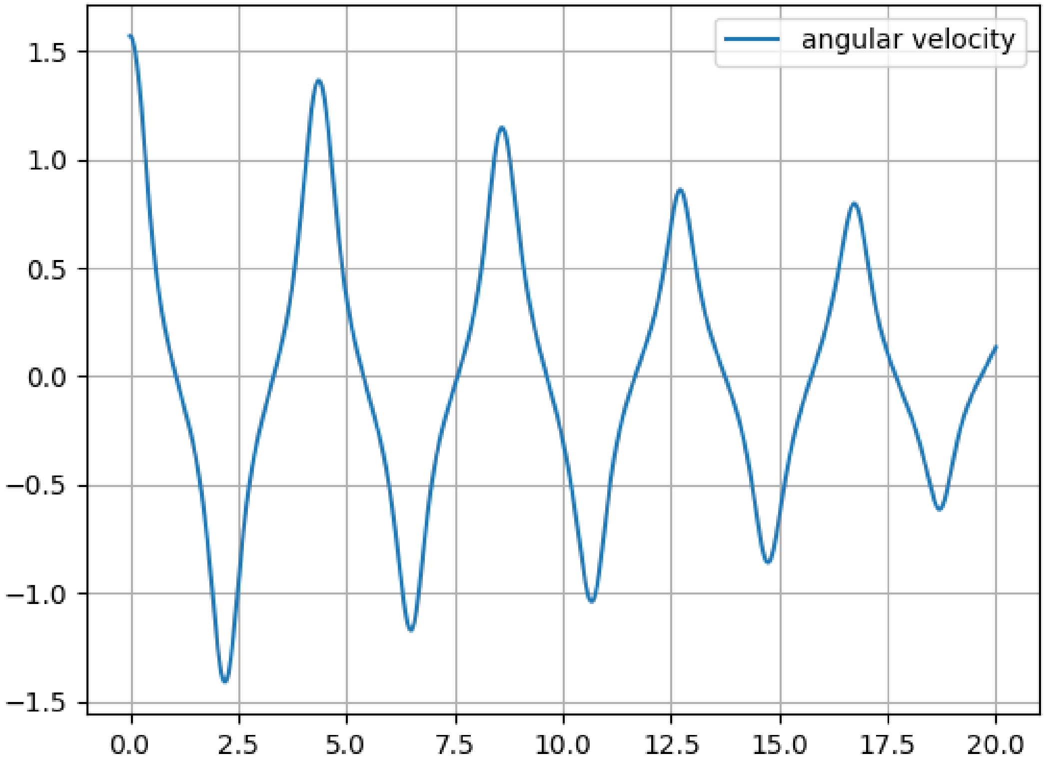

Figure C.19: *Case 9 where m = 9 kg*

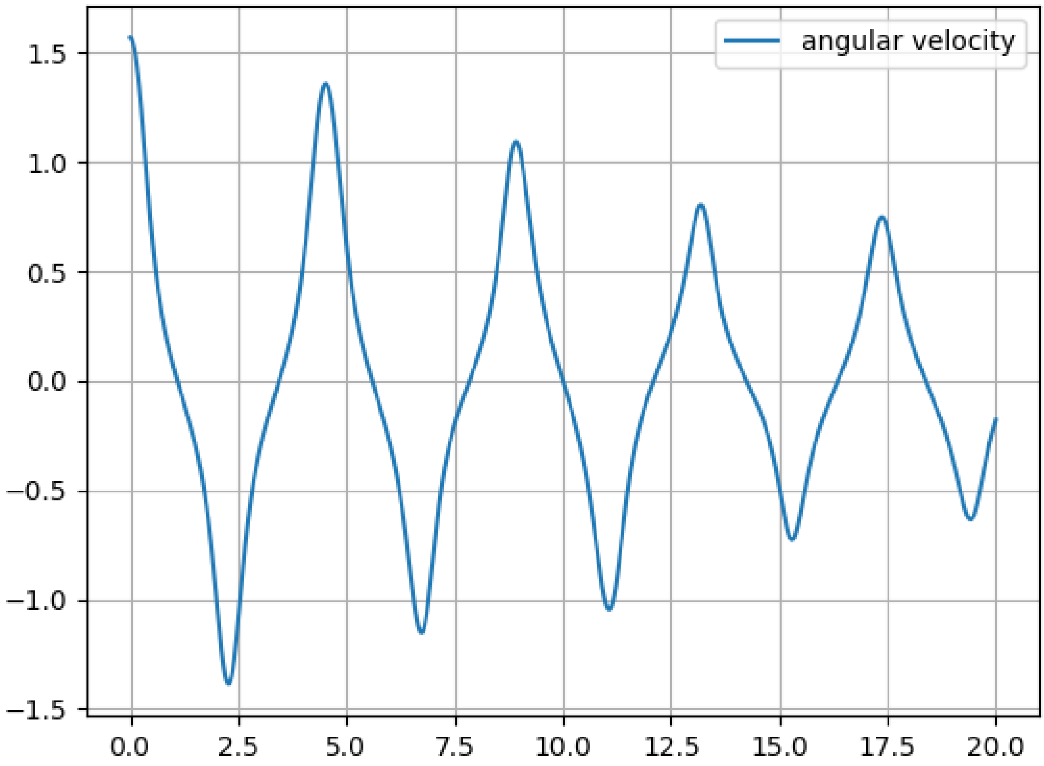

Figure C.20: *Case 10 where m = 10 kg*

C.3 Angular Frequency versus Time

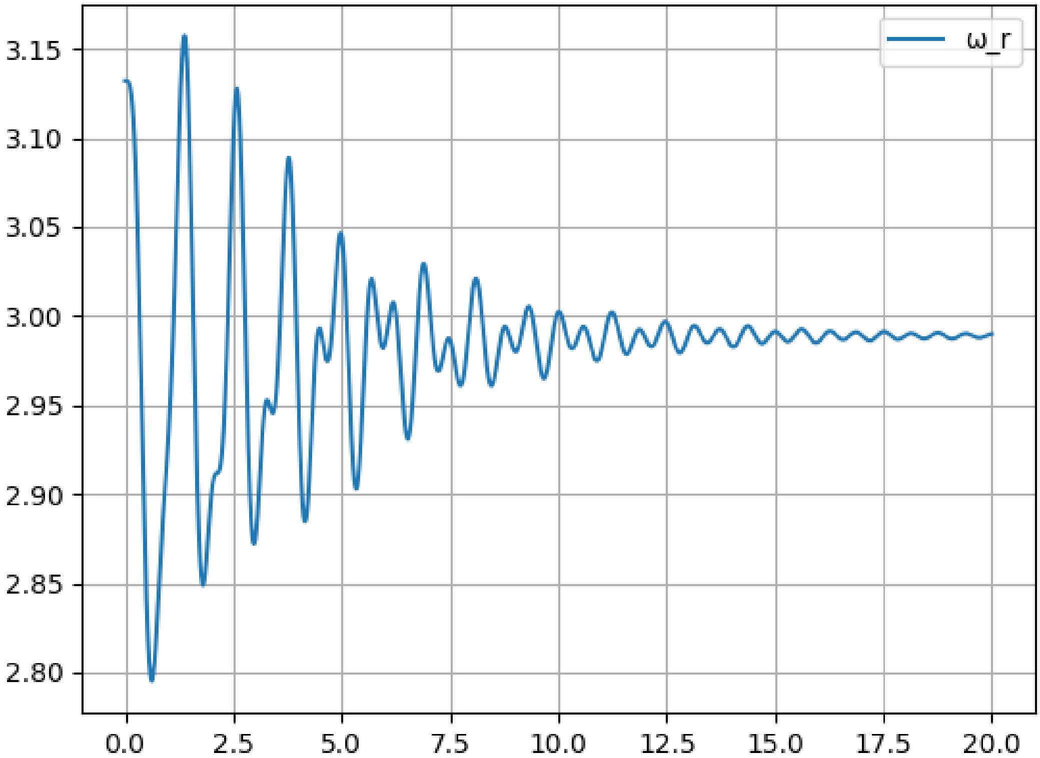

Figure C.21: *Case 1 where m = 1 kg*

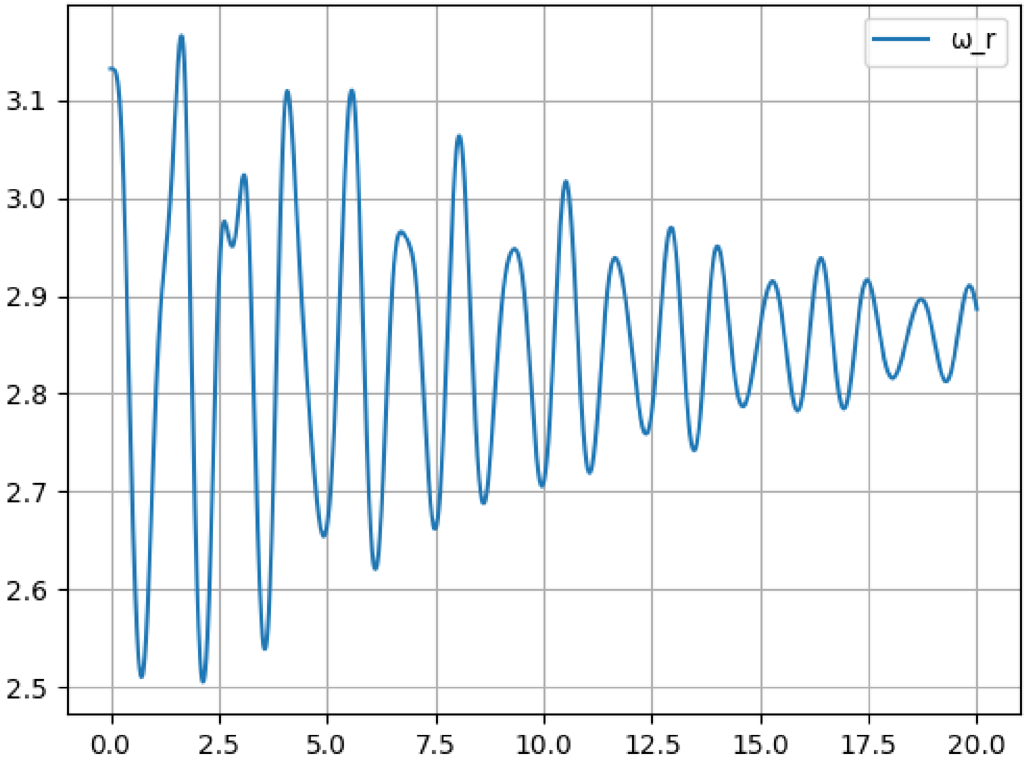

Figure C.22: *Case 2 where m = 2 kg*

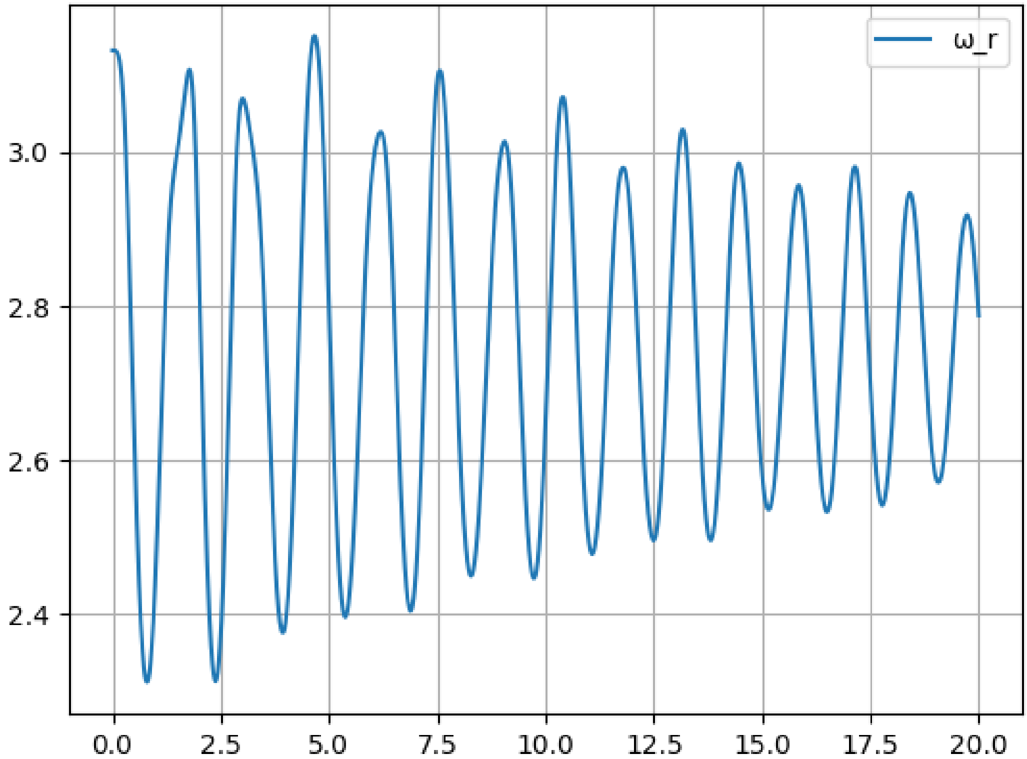

Figure C.23: *Case 3 where m = 3 kg*

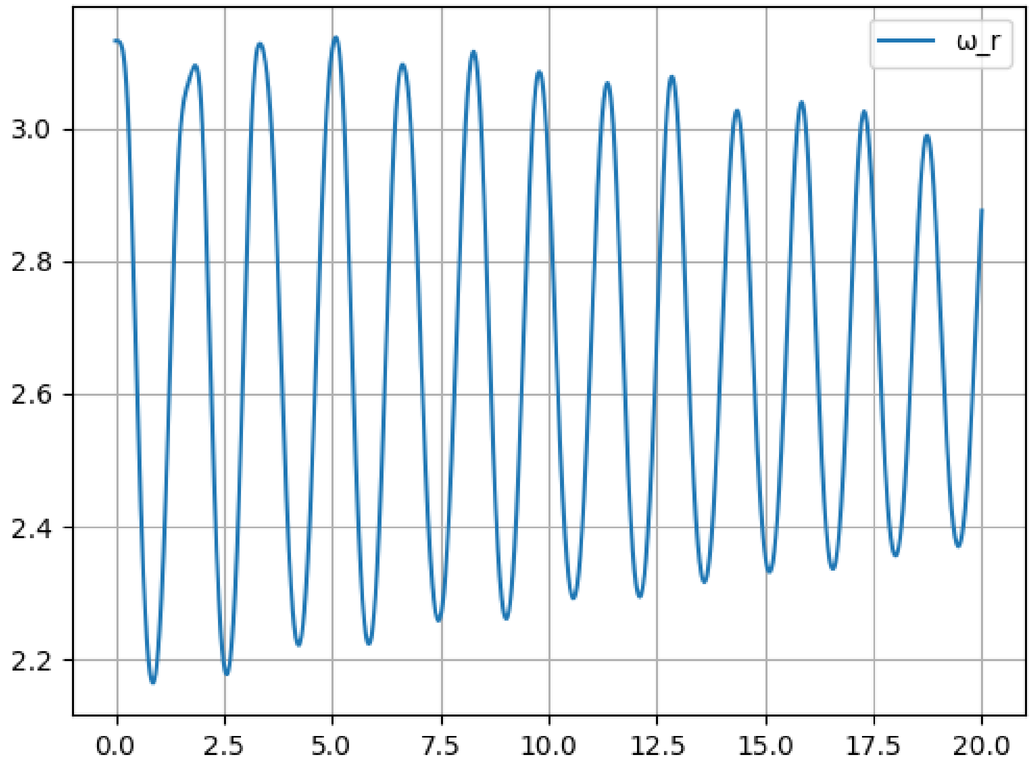

Figure C.24: *Case 4 where m = 4 kg*

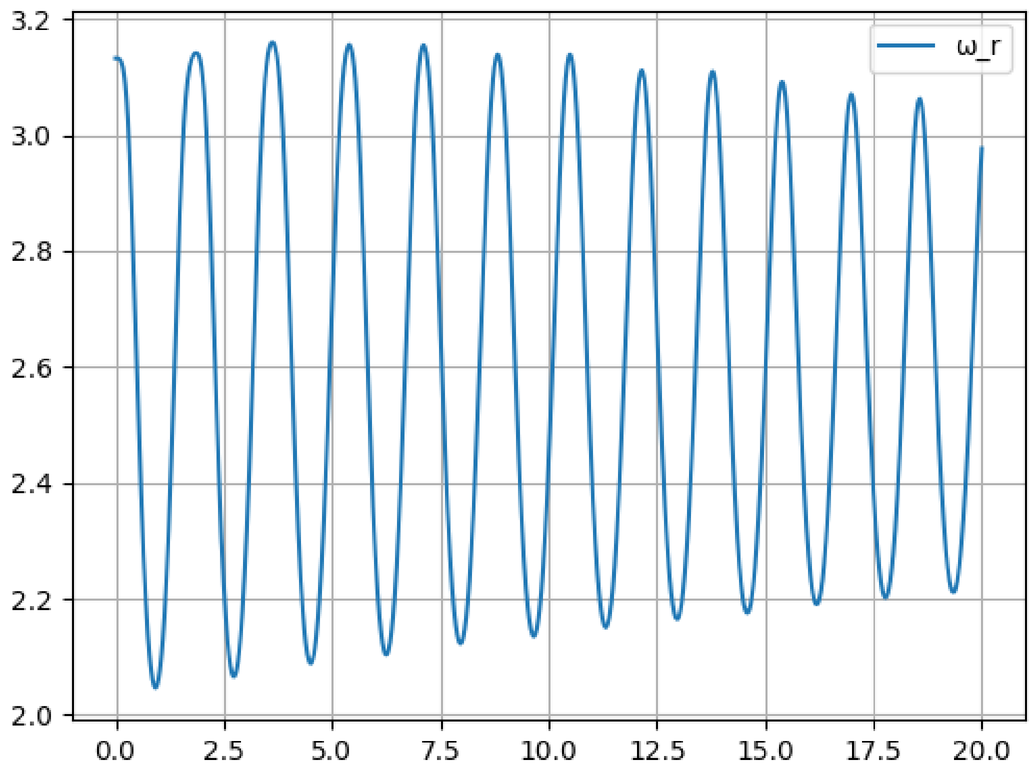

Figure C.25: *Case 5 where m = 5 kg*

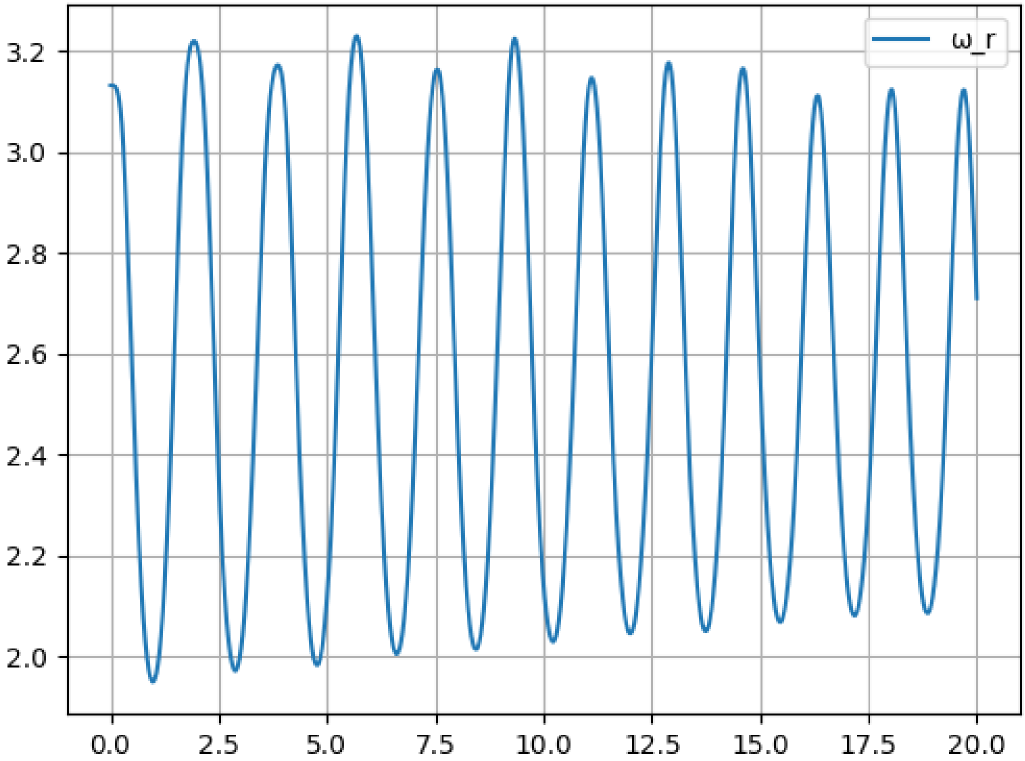

Figure C.26: *Case 6 where m = 6 kg*

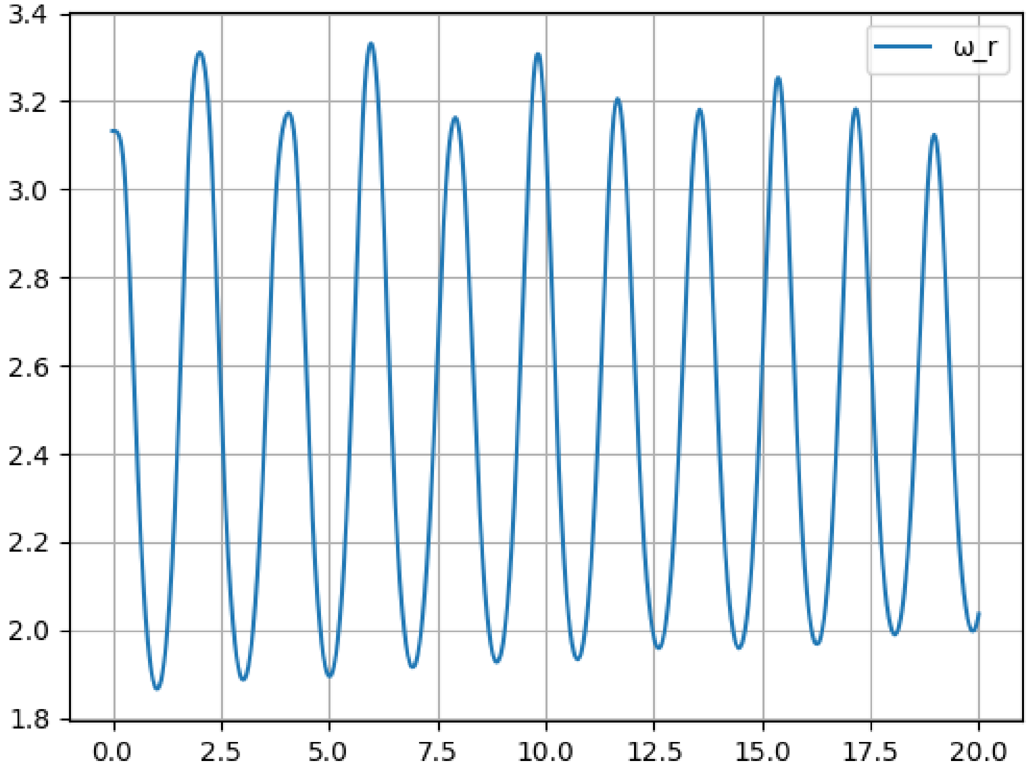

Figure C.27: *Case 7 where m = 7 kg*

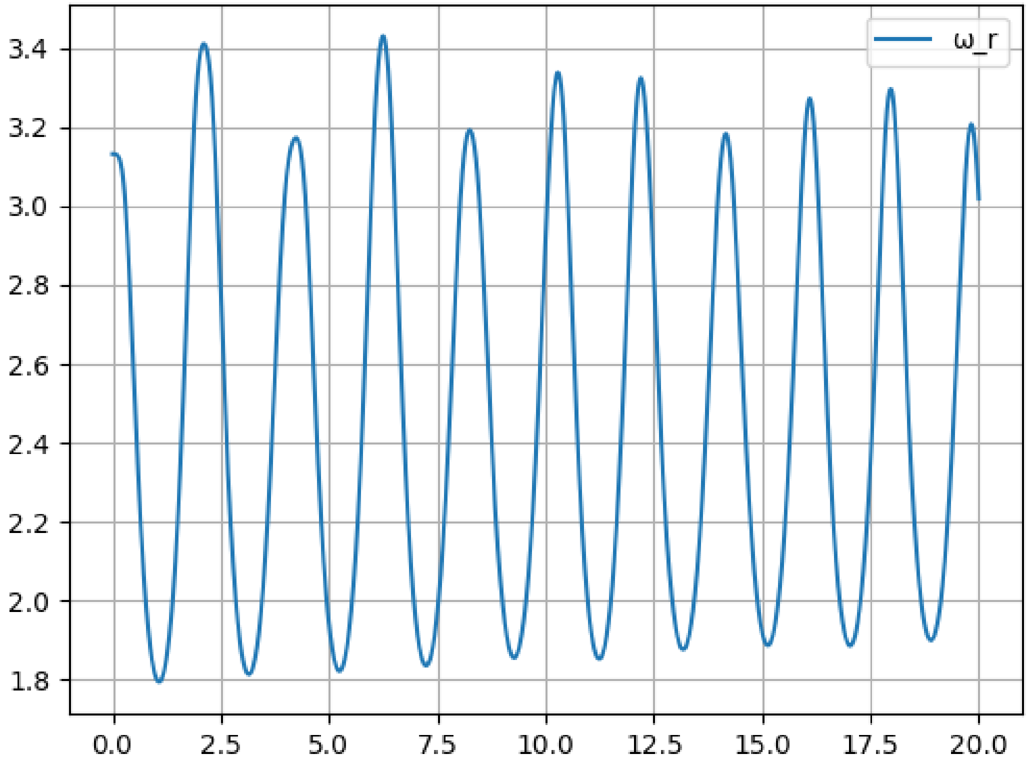

Figure C.28: *Case 8 where m = 8 kg*

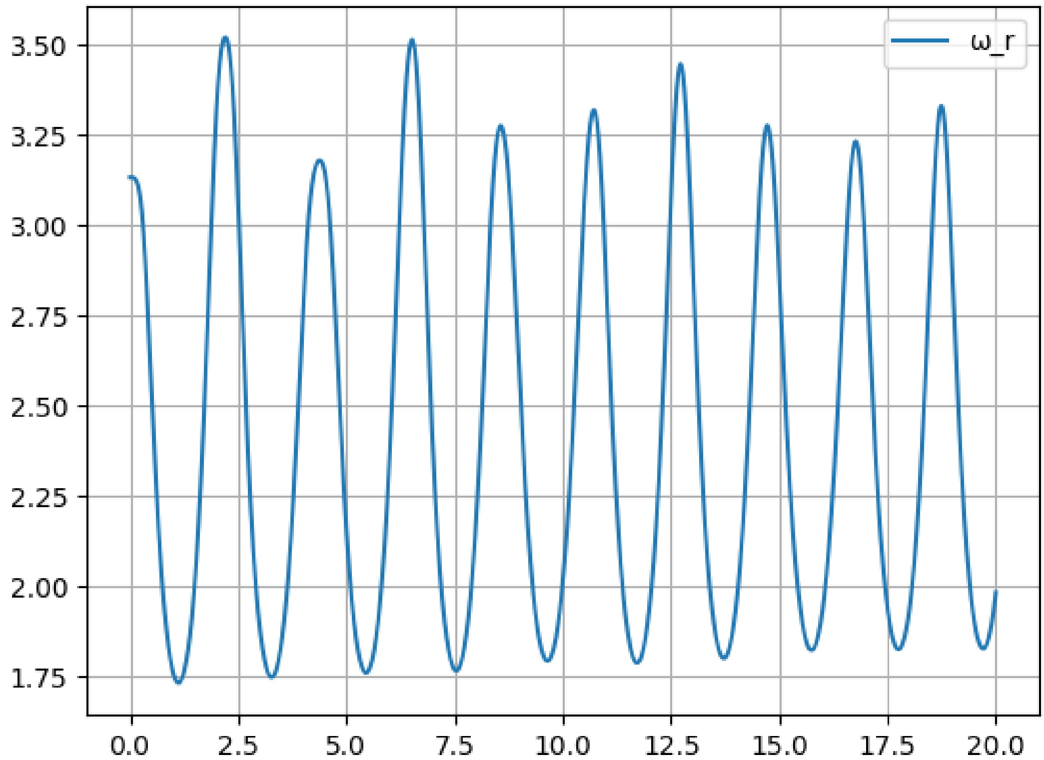

Figure C.29: *Case 9 where m = 9 kg*

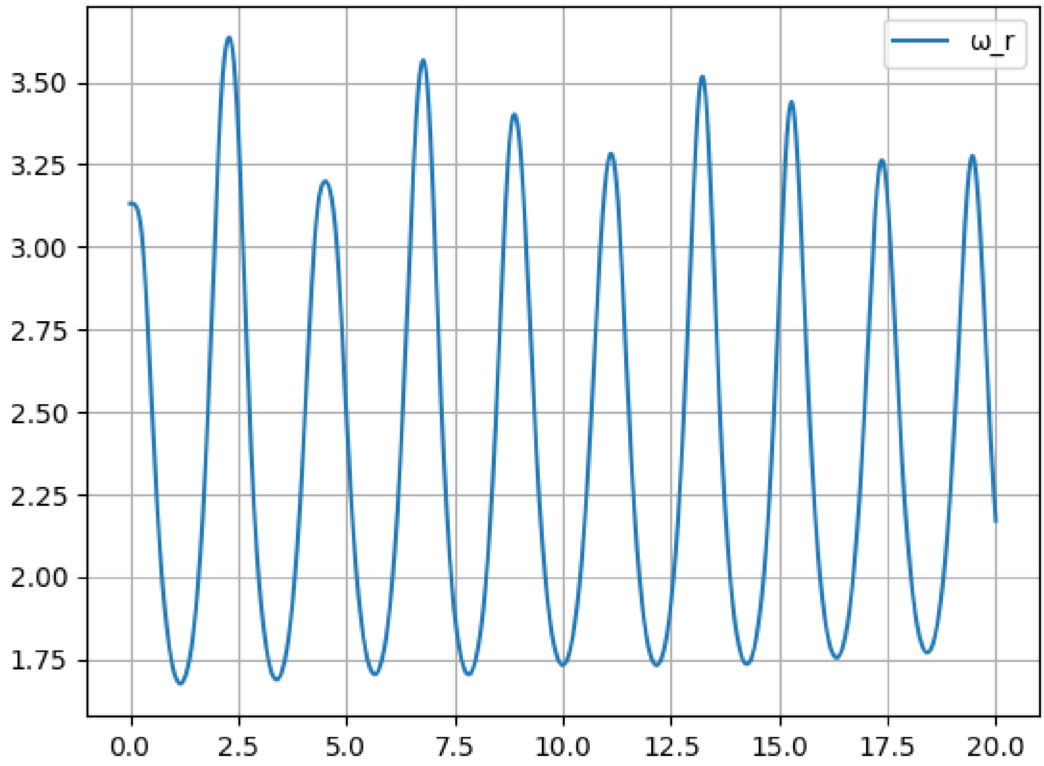

Figure C.30: *Case 10 where m = 10 kg*

C.4 Absolute Frequency versus Time

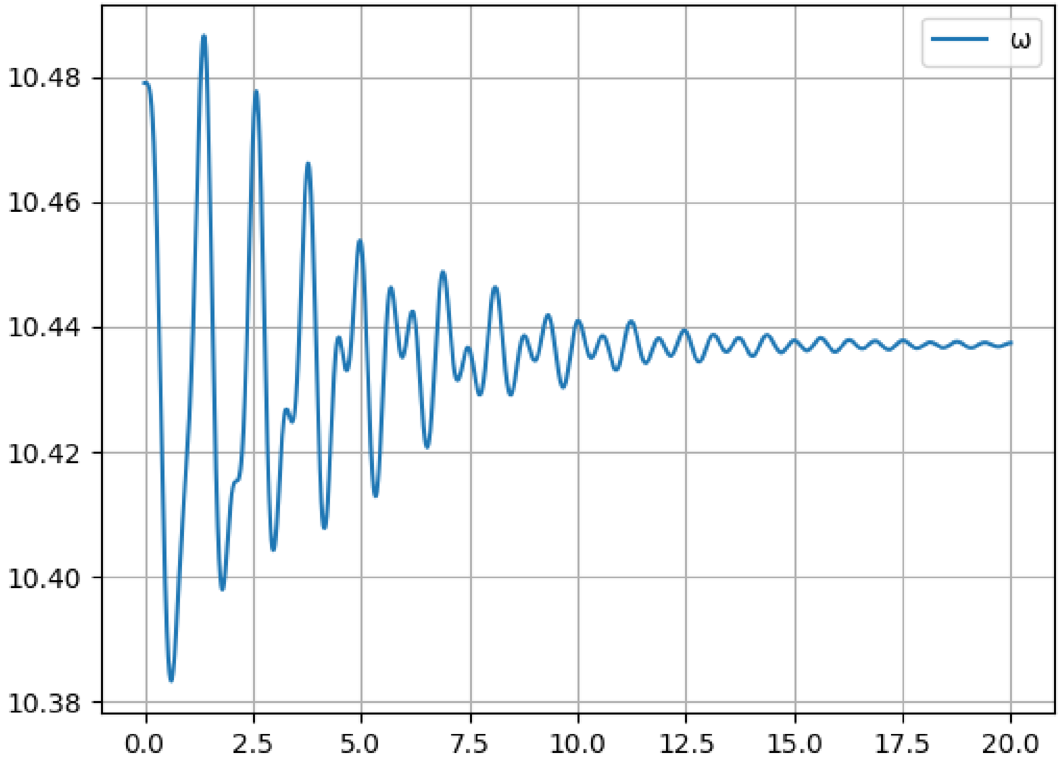

Figure C.31: *Case 1 where m = 1 kg*

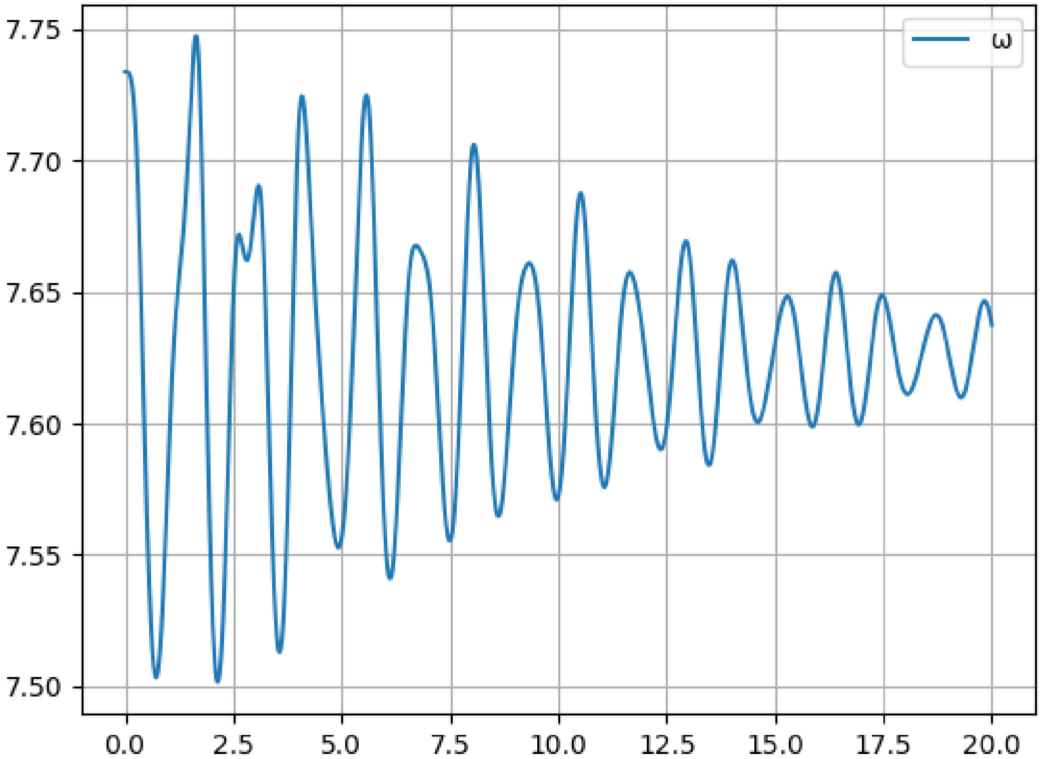

Figure C.32: *Case 2 where m = 2 kg*

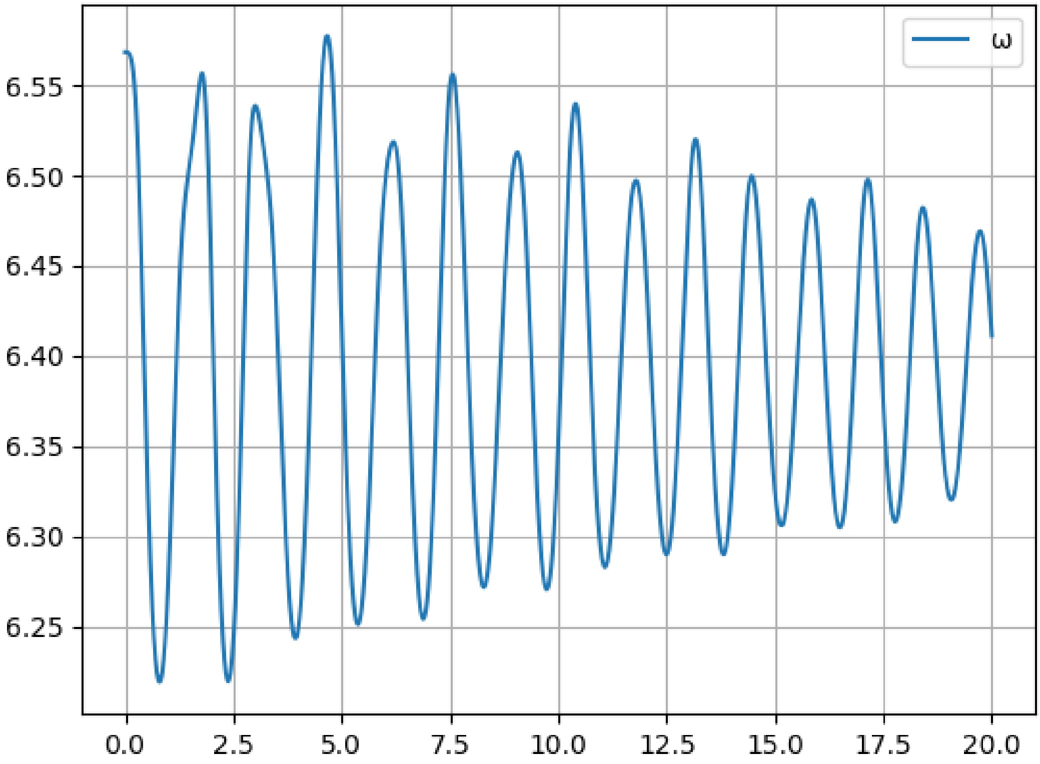

Figure C.33: *Case 3 where m = 3 kg*

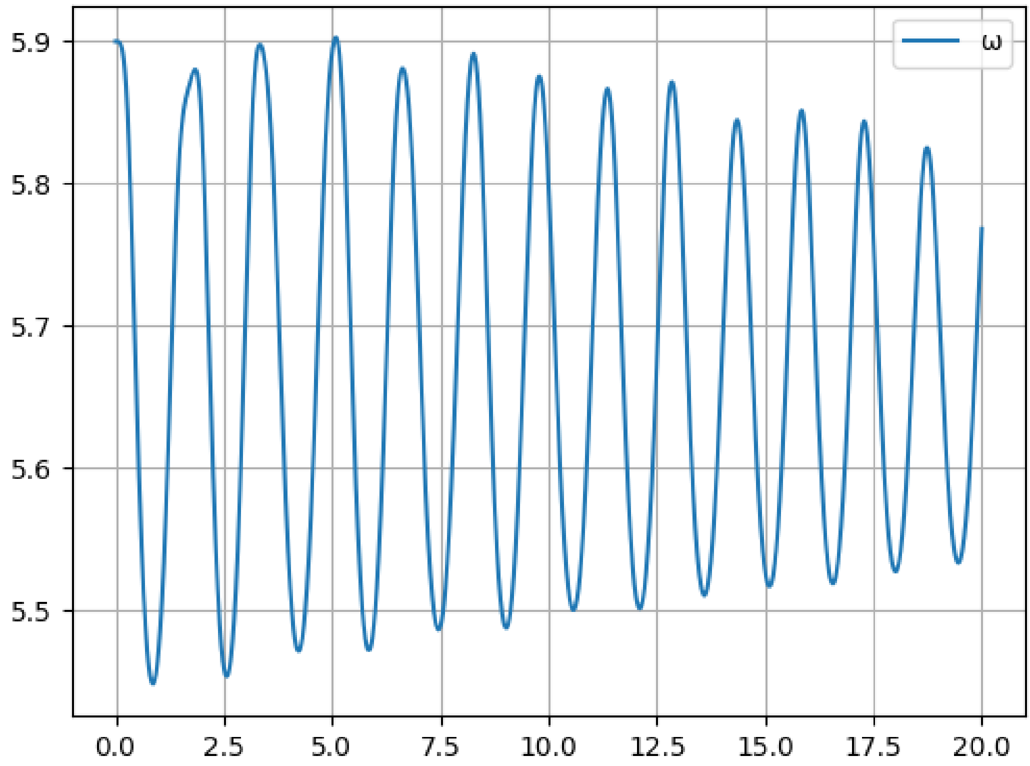

Figure C.34: *Case 4 where m = 4 kg*

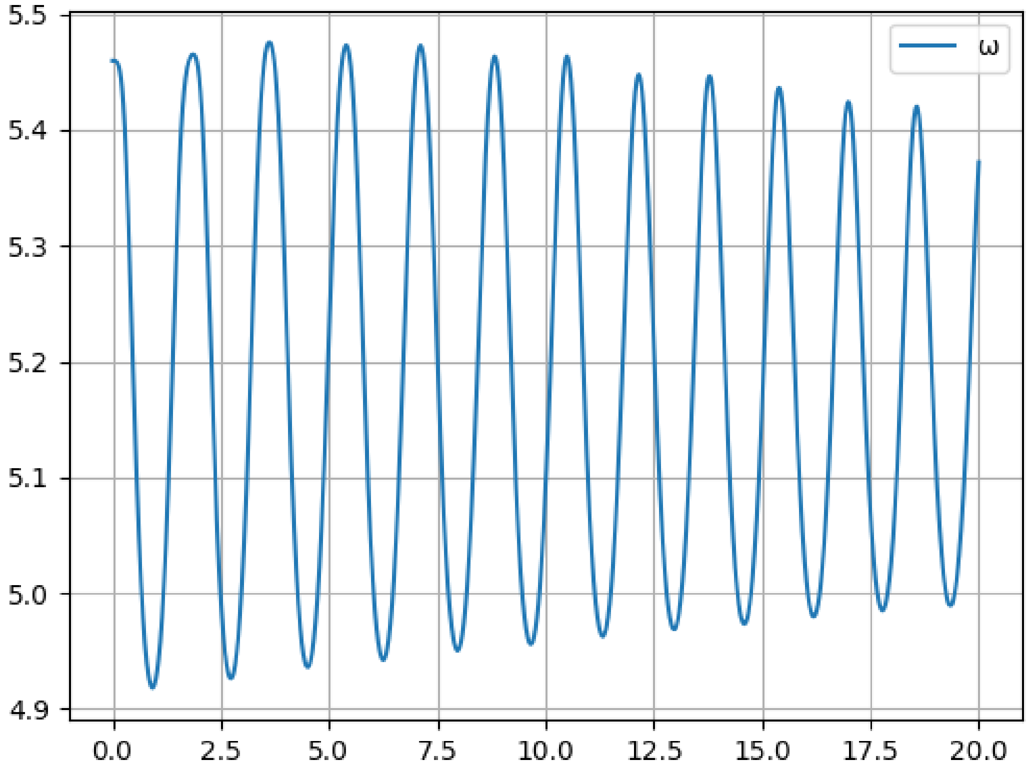

Figure C.35: *Case 5 where m = 5 kg*

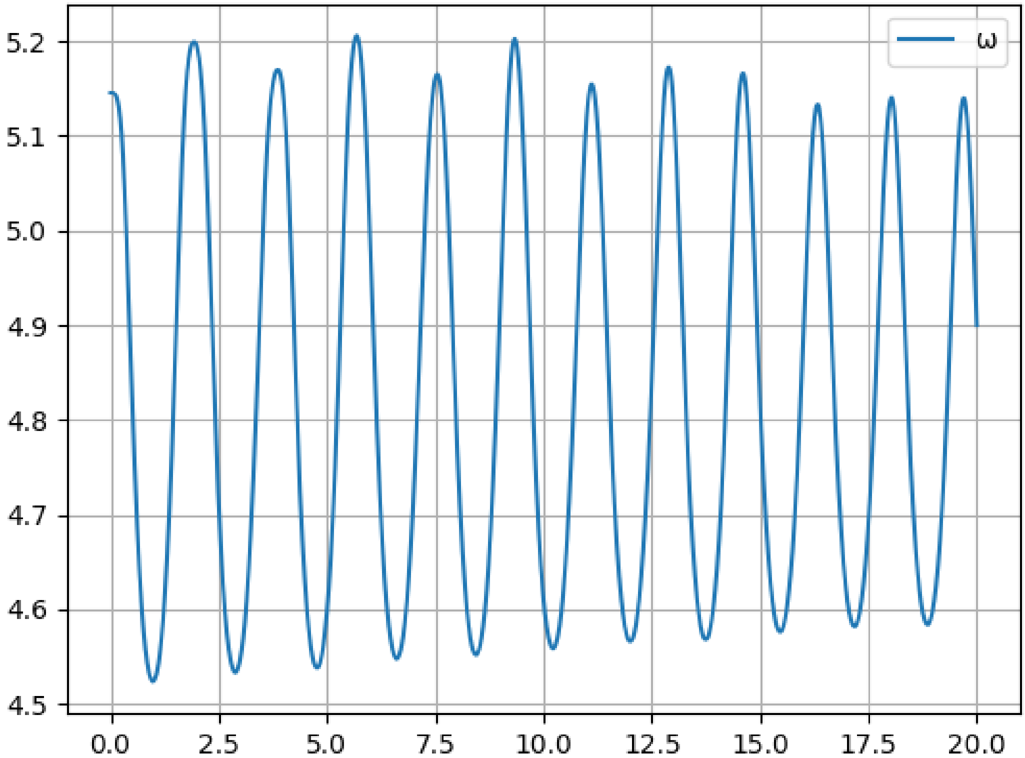

Figure C.36: *Case 6 where m = 6 kg*

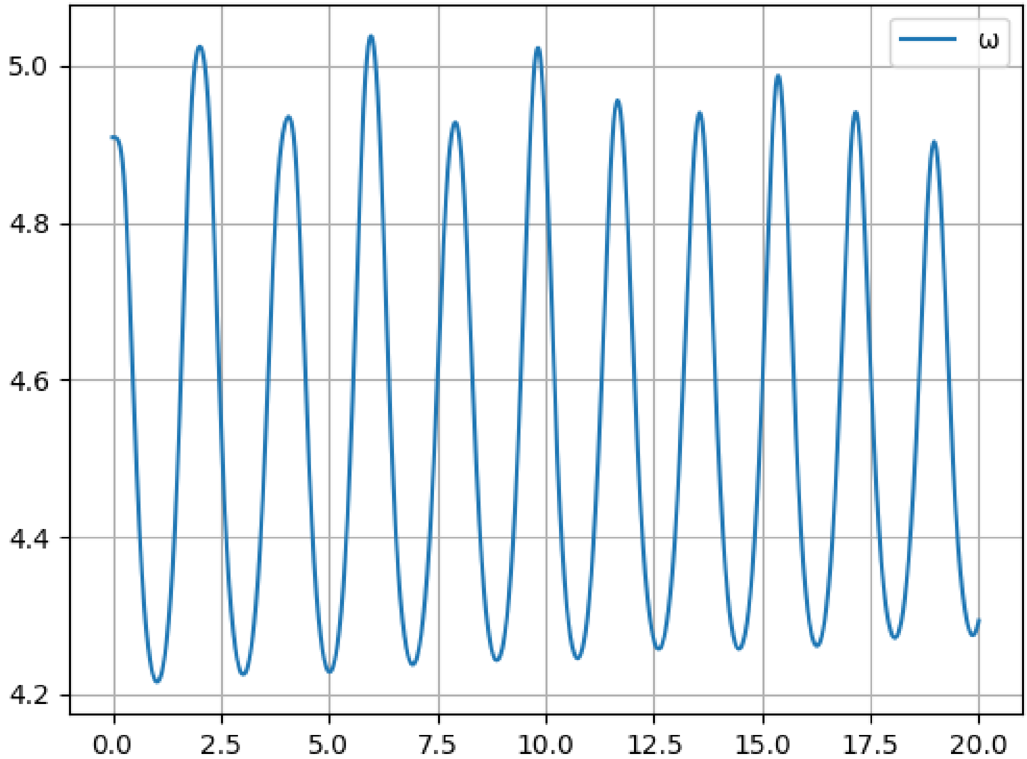

Figure C.37: *Case 7 where m = 7 kg*

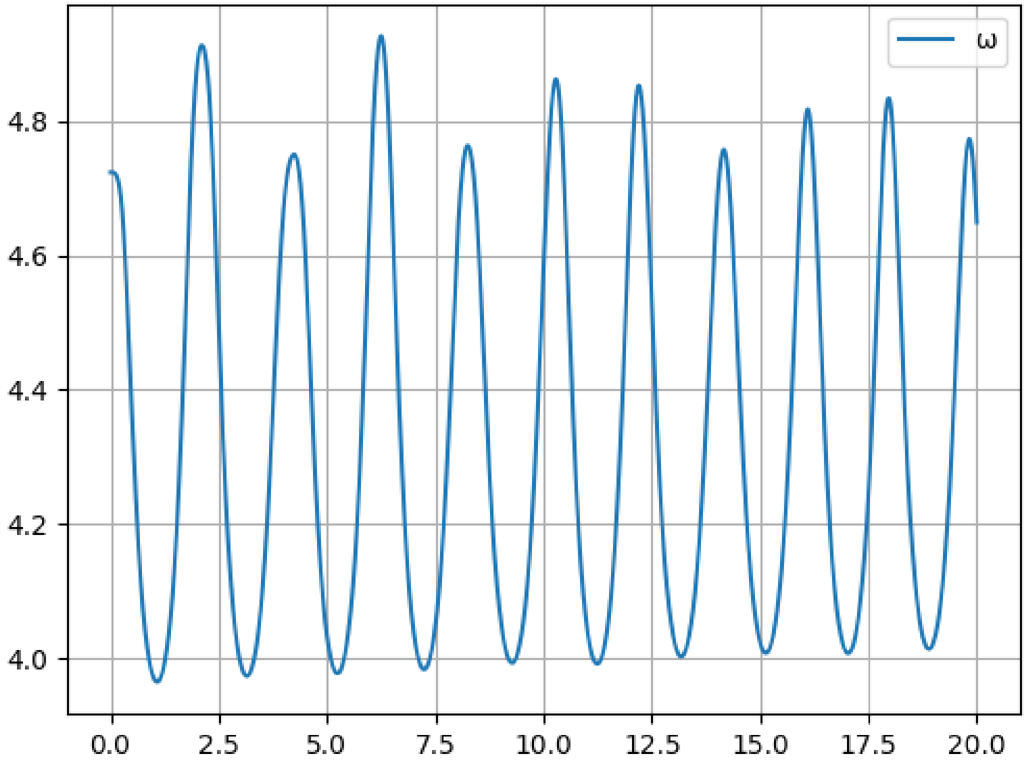

Figure C.38: *Case 8 where m = 8 kg*

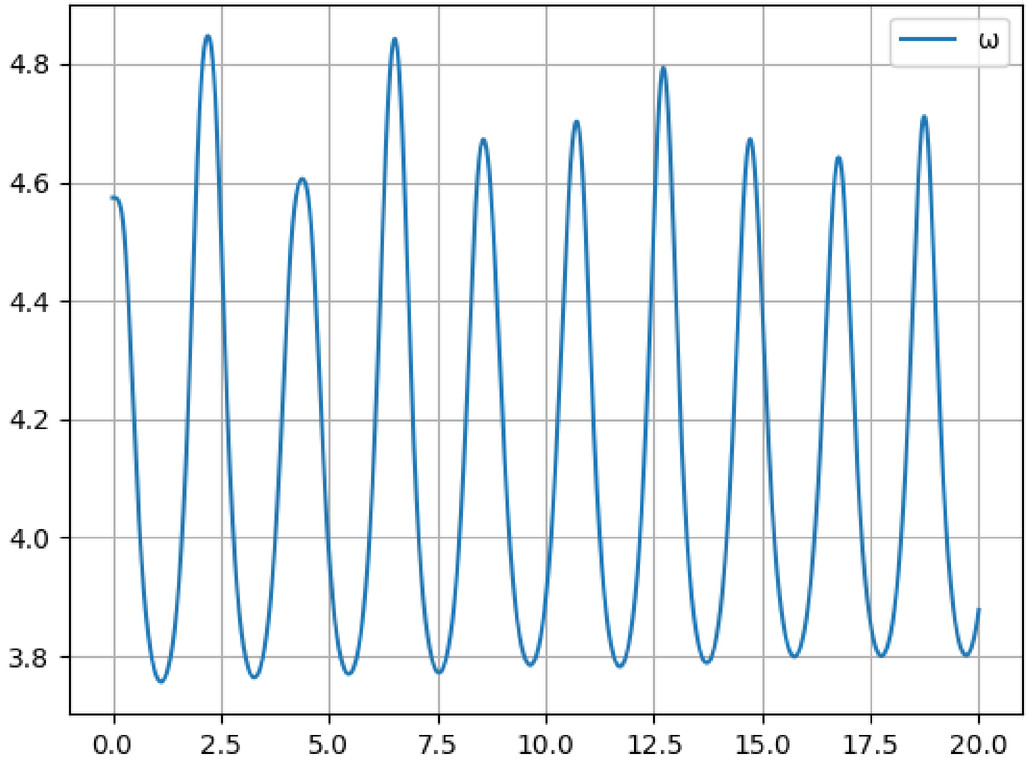

Figure C.39: *Case 9 where m = 9 kg*

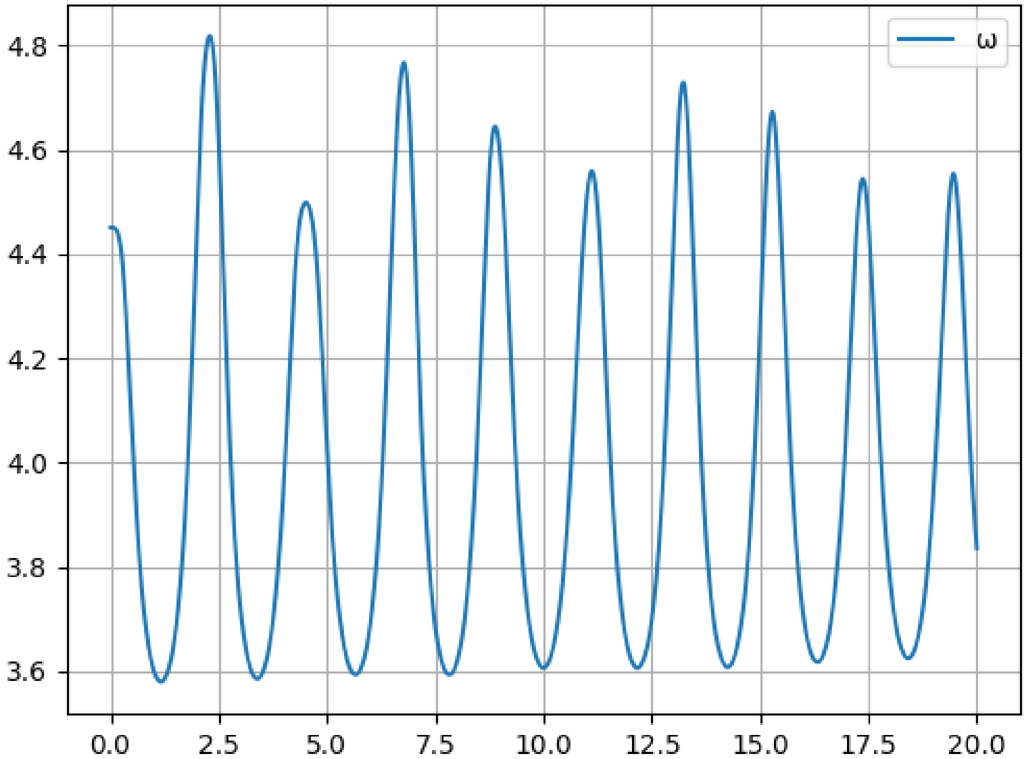

Figure C.40: *Case 10 where m = 10 kg*

D Derived Data

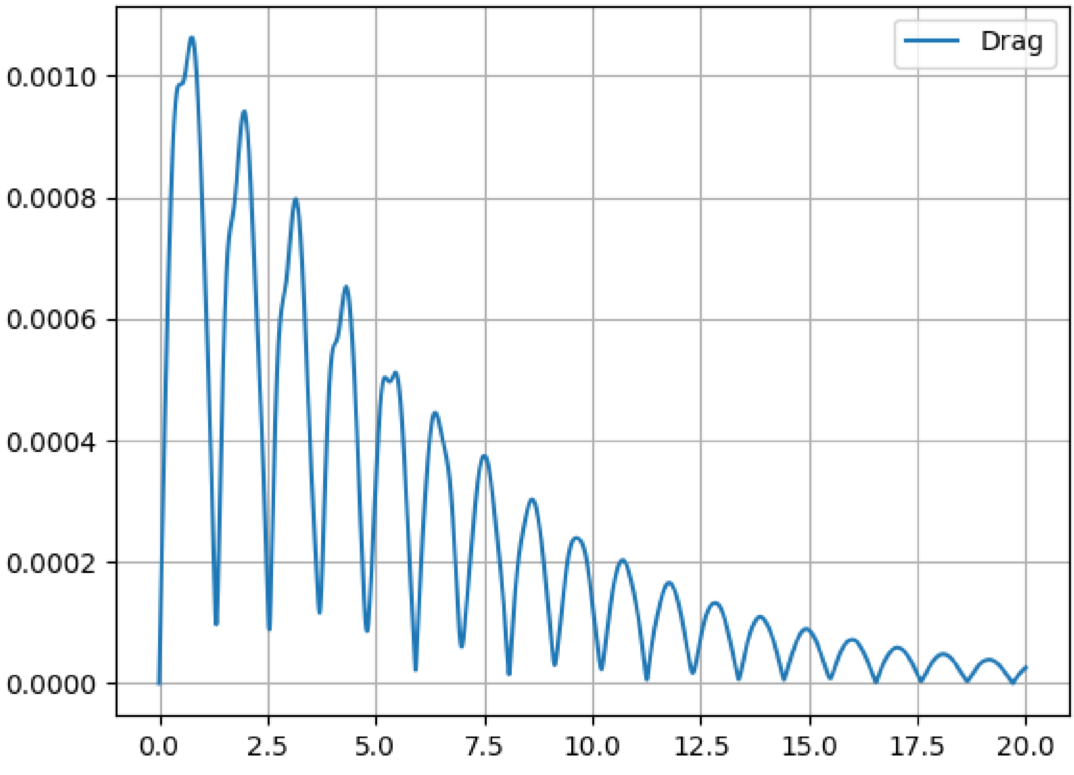

Figure D.1: *Case 1 where m = 1 kg*

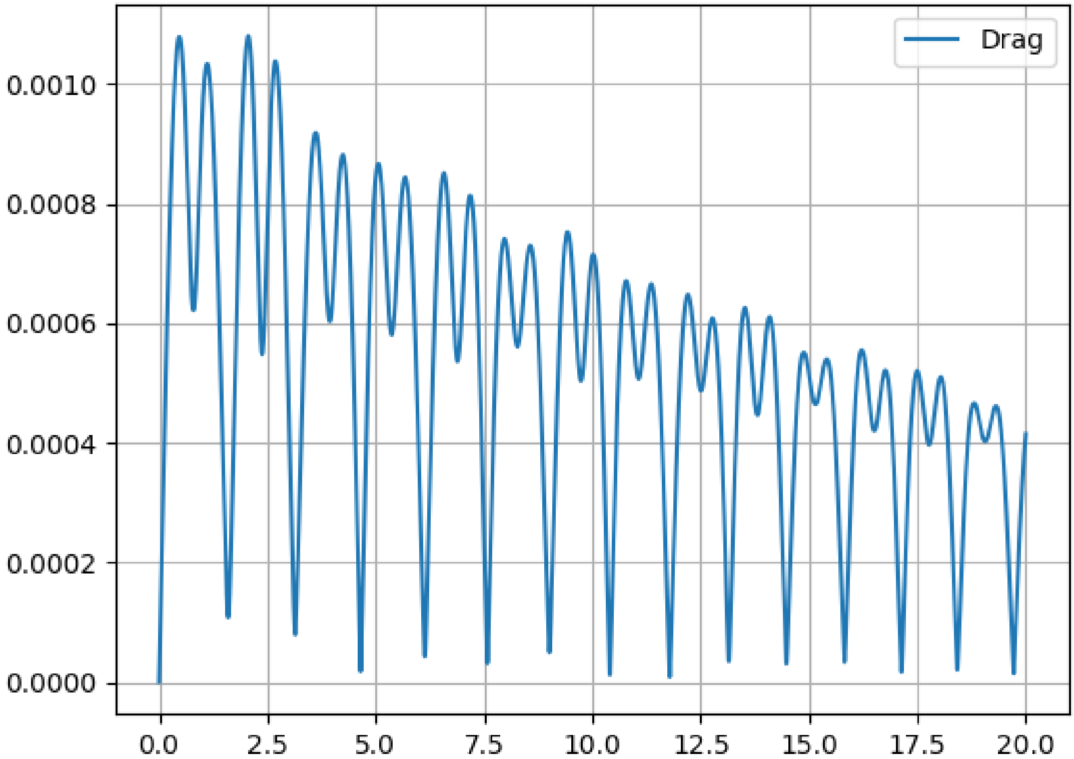

Figure D.2: *Case 2 where m = 2 kg*

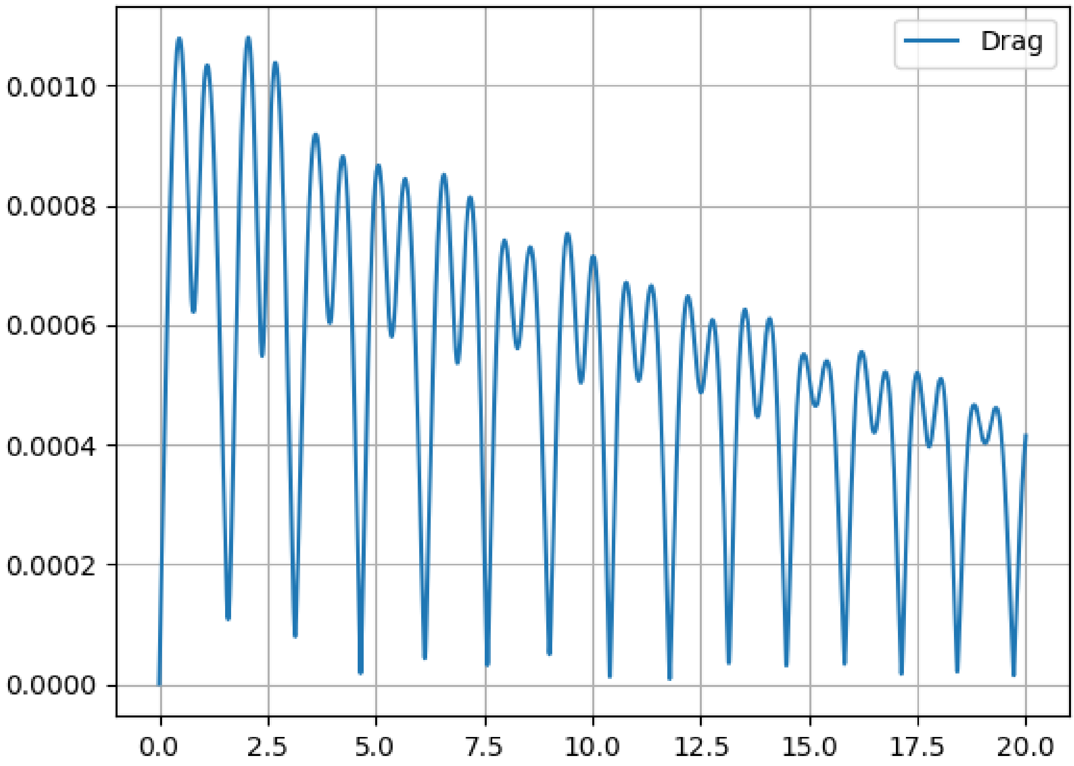

Figure D.3: *Case 3 where m = 3 kg*

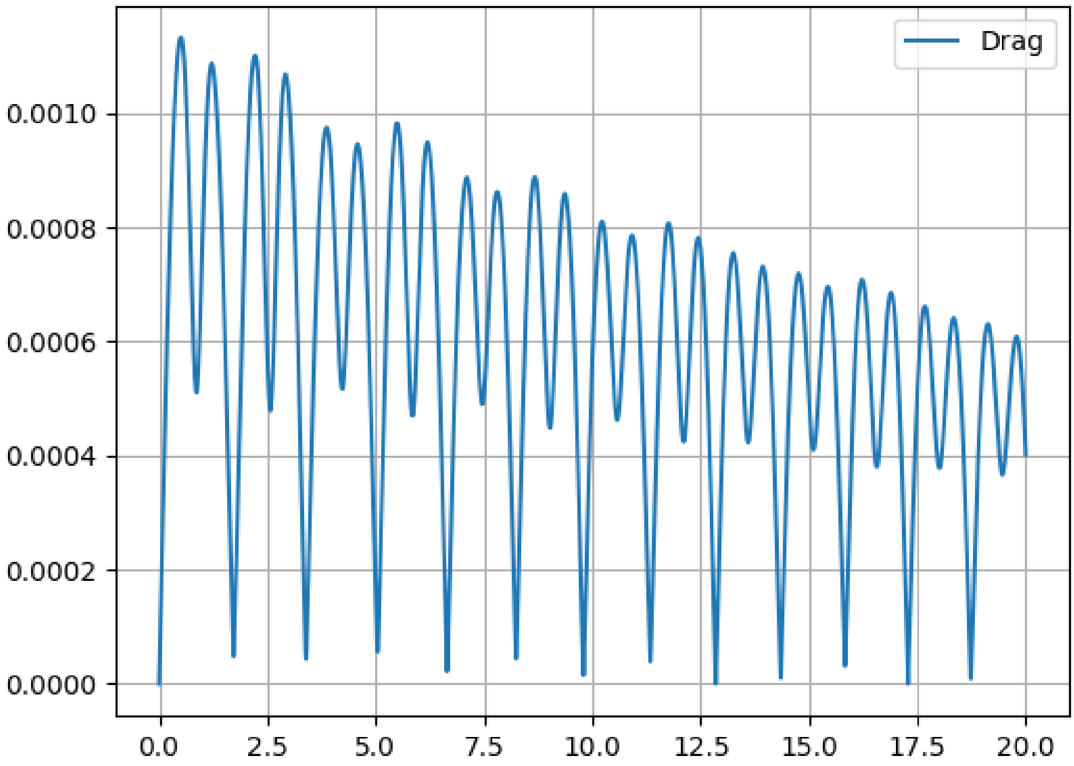

Figure D.4: *Case 4 where m = 4 kg*

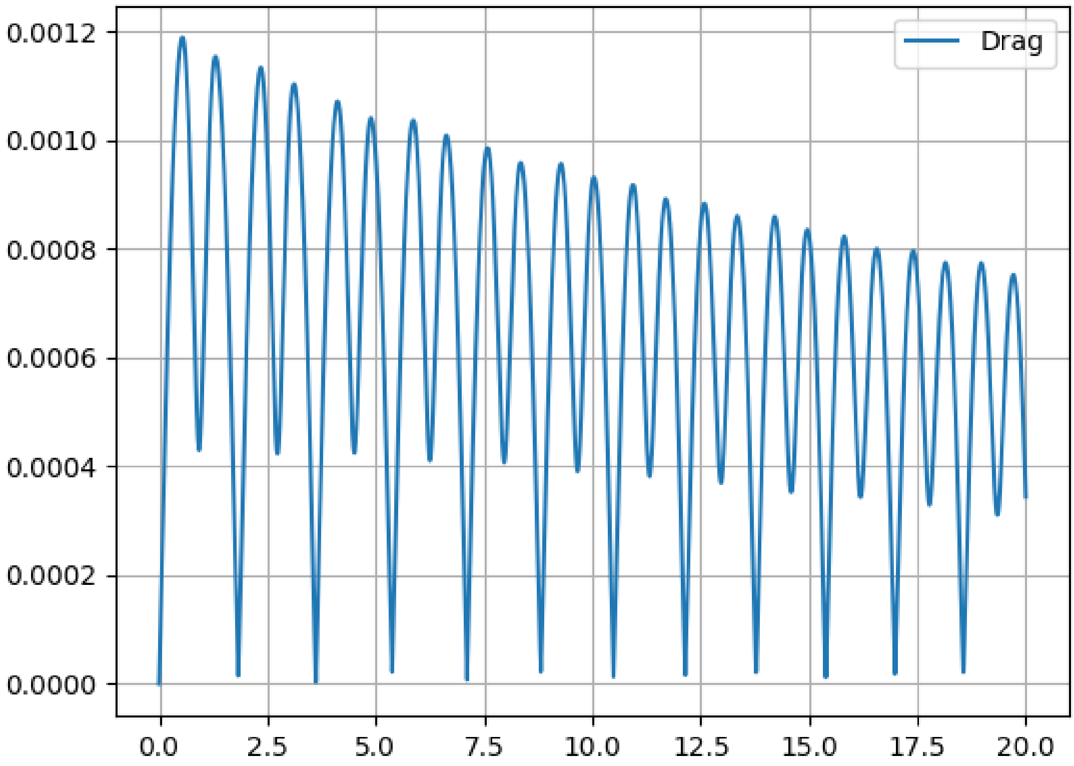

Figure D.5: *Case 5 where m = 5 kg*

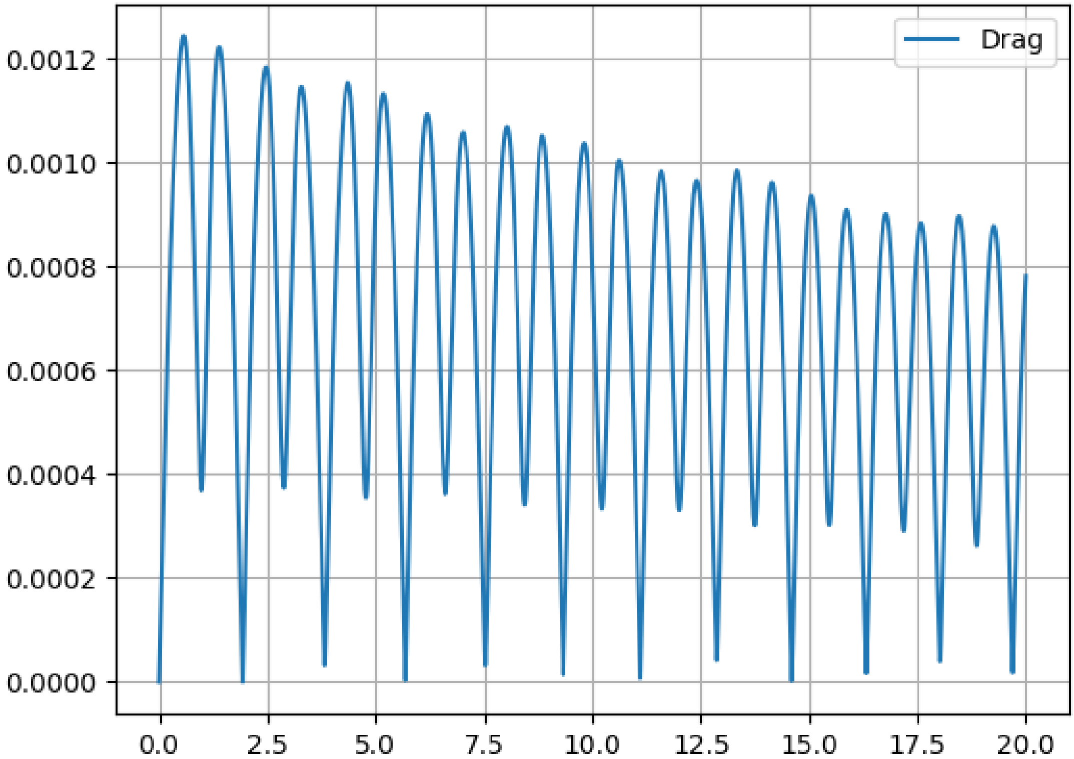

Figure D.6: *Case 6 where m = 6 kg*

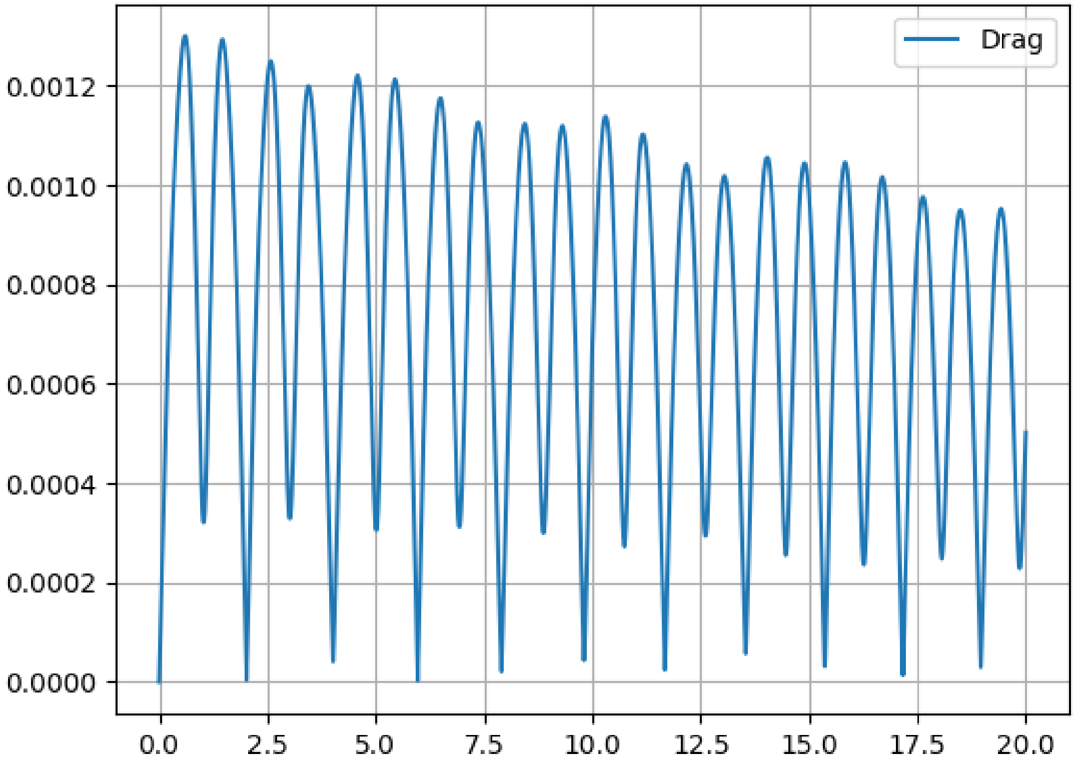

Figure D.7: *Case 7 where m = 7 kg*

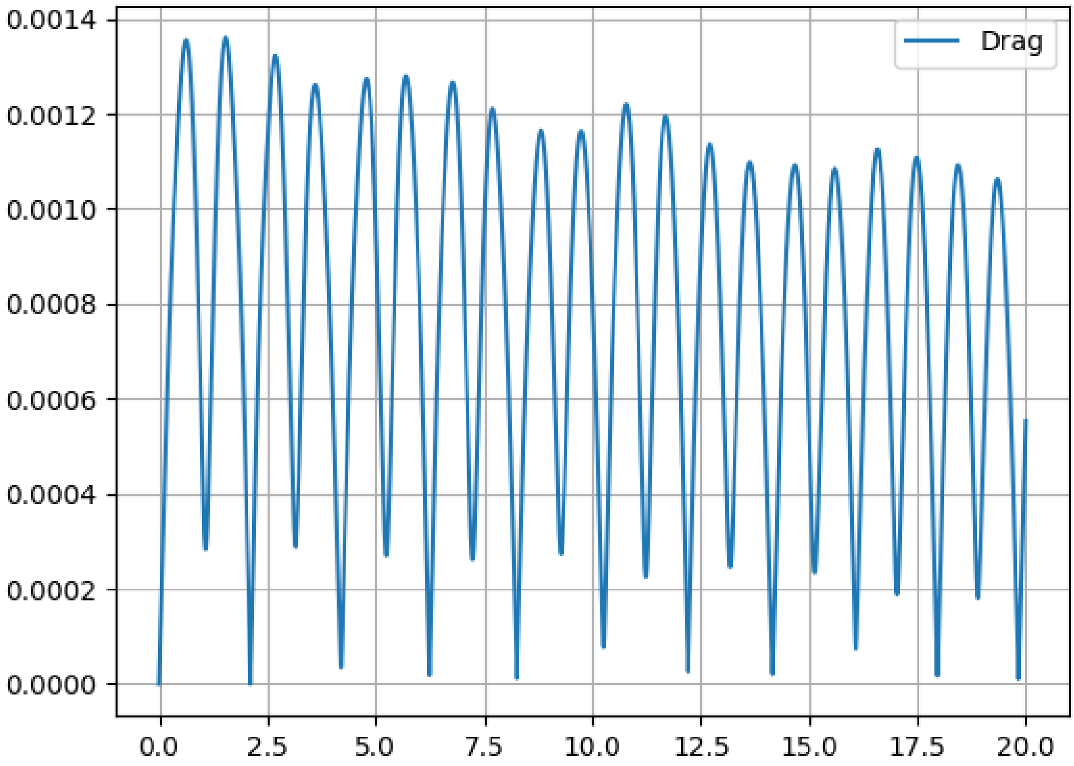

Figure D.8: *Case 8 where m = 8 kg*

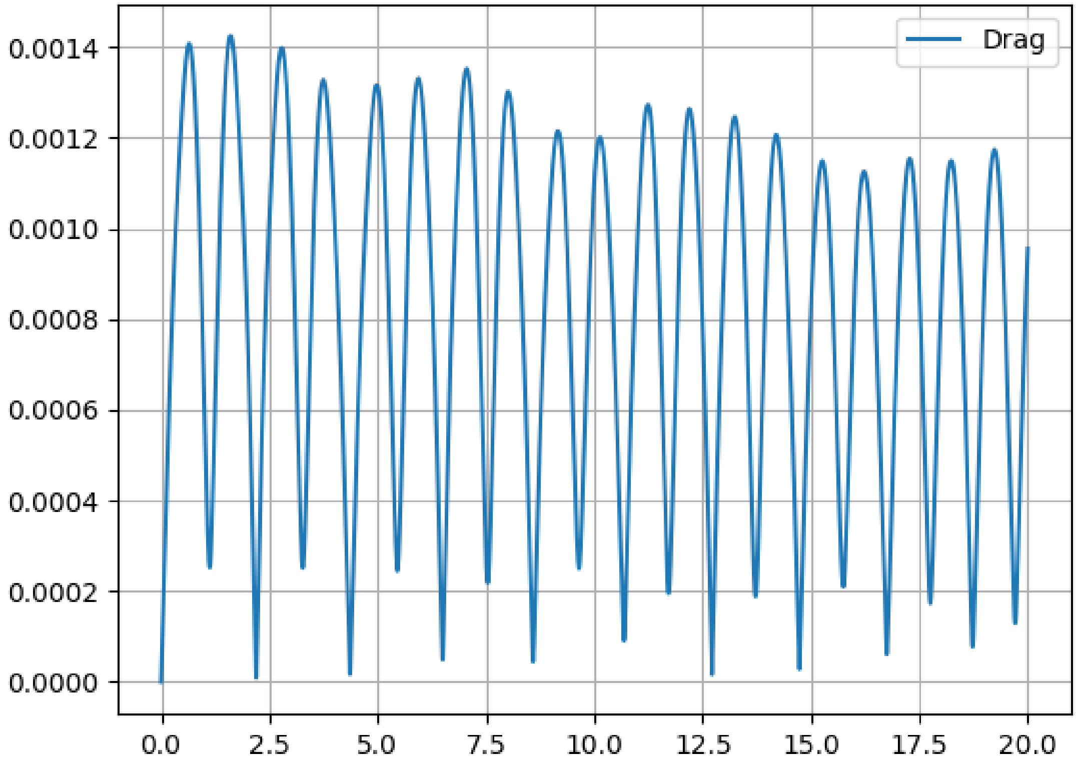

Figure D.9: *Case 9 where m = 9 kg*

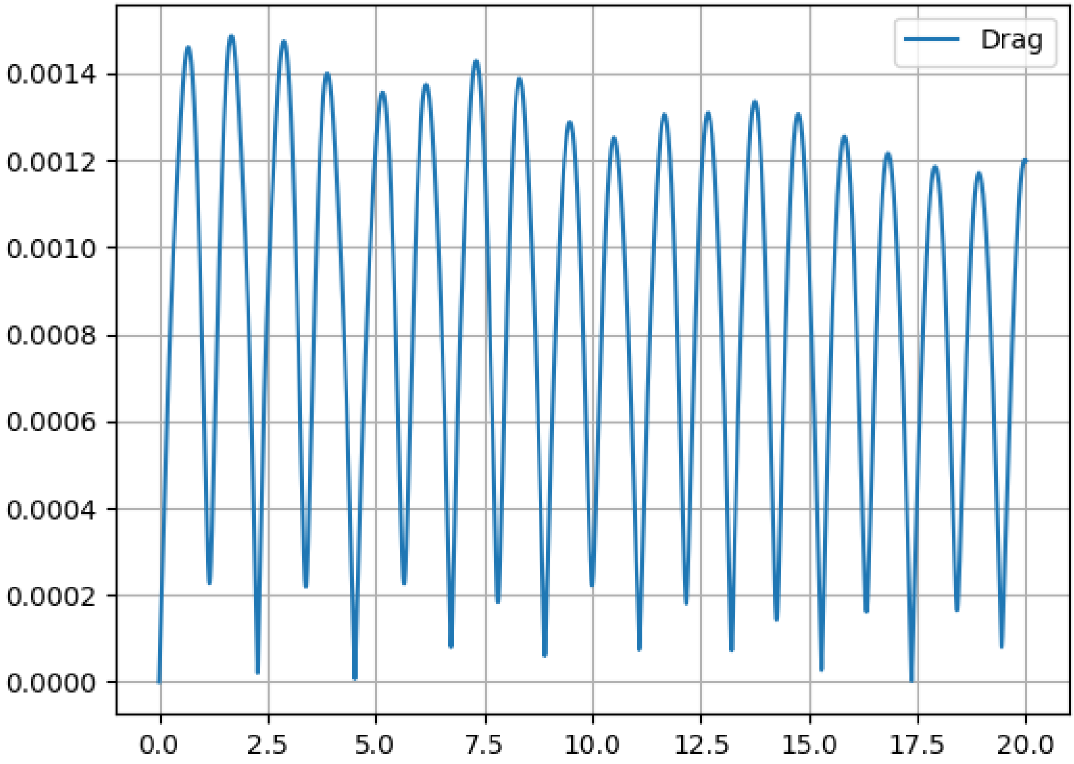

Figure D.10: *Case 10 where m = 10 kg*