

*EE Ver 1.01*

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*The Dynamics of the Elastic Pendulum*

*How does the change in the numeric value of the mass affect the motion of a Damped Elastic Harmonic Oscillator?*

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## Abstract

We will discuss the effects of change in the magnitude of **spring stiffness constant** on the magnitude of the **radial position**, **angular position** and the **integral frequency** of the system of the damped harmonic oscillator, by collecting raw data on the measurement of (**radial position** with respect to time and **angular position** with respect to time) under certain controlled spaces with **defined standard initial conditions**.

# **1** *Introduction*

*I have chosen this research question because the fundamental relation between the mechanics of the damped elastic pendulum, its complex dynamics, its unstable chaotic behaviour and its excruciating difficulty in possibly modeling this behaviour has led me to explore this field, as this phenomenon is exciting, unique and wonderful, with ample scope to carry out research and collect data both quantitatively and qualitatively.*

# **2** *Aim*

*To collect experimental raw data relating to the variation in radial position, the variation in angular position and the variation in absolute frequency in a lab controlled setting of a damped elastic harmonic oscillator and to compare it with computer simulations of the same and hence study its different states and further to study its sensitivity to initial conditions and calculate the value the inconsistencies in accuracy of the mathematically modeled computer simulation to that of the*

*actual experiment and vice-versa.*

## 3 *Background Research*

*Before beginning this investigation we must first know some important facts, formulae, laws and be familiar with the concepts that are to be incorporated in this investigation.*

### 3.1 *The Lagrangian function*

*Firstly, we must know that the fundamental concepts of Lagrangian mechanics and Lagrange's equations.*

*The Lagrangian is a function that is mathematically defined as,*

$$L = T - V$$

*Where, L is the Lagrangian, T is the kinetic energy of the system and V is the potential energy of the system.*

*The Lagrange's equation of second kind or the Lagrange-Euler Equation is mathematically defined as,*

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_j} \right) - \frac{\partial L}{\partial q_j} = Q_j$$

*Where  $q_i$  is the generalized coordinates and  $Q_j$  is the generalized forces.*

### 3.2 *Generalized coordinates*

*According to our specific system that we are investigating, we have two generalized coordinates in polar coordinate representation,  $\theta$  and  $x$ .*

$\theta$  represents the angle formed by the spring component with the fictitious normal of the system at any given time and is a function of time (time-dependent).

$x$  represents the extension in the spring component with the equilibrium length at any given time and is a function of time (time - dependent).

### 3.3 Generalized forces

According to our specific system that we are investigating, we only have one generalized forces (non-conservative force), the frictional force acting on the path of the system in the form of drag or fluid resistance.

Our frictional force can be embedded in Lagrange's equations in the form of the  $Q_j$  term aided with the utilisation of the Rayleigh dissipation function.

As there isn't any other non-conservative force, the net total sum of the Lagrange's equations with the incorporation of the Rayleigh dissipation function would equal zero.

### 3.4 Air Drag/Fluid Resistance

Air Drag/Fluid Resistance is the force acting opposite to the relative motion of any object moving in any fluid medium. Drag force is proportional to the square of velocity, as we are dealing with relatively high-speeds, which can be inferred from the huge Reynolds's number.

Drag forces decrease fluid velocity relative to the solid mass in the fluid's path.

The type of Drag in play in this system is that of an underdamped ( $\zeta < 1$ ) oscillator with viscous drag.

**Note:**  $\zeta$  here, in the context of Drag forces and resistive forces symbolizes the damping ratio.

The general Drag equation is mathematically defined as,

$$F_D = \frac{1}{2} \rho v^2 C_D A$$

Where  $F_D$  is the Air/Fluid resistance between the mass and the fluid,  $\rho$  is the density of the fluid,  $v$  is the speed of the object relative to the fluid,  $C_D$  is

velocity decay constant (damping constant) and  $A$  is the cross sectional area.

The Drag force can be embedded and modelled into the Euler-Lagrange equation by the aid of the Rayleigh dissipation function and can be generally modeled using Stroke's Law, as we are using object masses that are spherical.

### 3.5 Rayleigh dissipation function

Secondly, we must also know that,

If the frictional force on a particle with velocity  $\vec{v}$  can be written as  $\vec{F}_f = -\vec{k} \cdot \vec{v}$ , the Rayleigh dissipation function can be defined for a system of  $n$  particles as,

$$R/D = \frac{1}{2} \sum_{i=0}^n C_D v_i^2 = \frac{1}{2} C_D \sum_{i=0}^n v_i^2$$

Where  $R$  or  $D$  is the Rayleigh dissipation function which is a function used to handle and model the effects of velocity-proportional frictional forces in Lagrangian mechanics, where  $C_D$  is the velocity decay constant (damping constant) and  $\sum_{i=0}^n v_i^2$  is the sum of all velocities squared in all degrees of freedom pertaining to a mechanical system with  $i$  being the number of degrees of freedom in the system.

### 3.6 Stroke's Law

We must know the Stroke's Law and the relation it defines to accurately model the change in Air Drag/Fluid Resistance with time.

Stroke's Law states a relation for the frictional force (drag force) exerted on spherical objects with very small Reynolds numbers in a viscous fluid. Mathematically,

$$F_D = 6\pi\mu Rv_D$$

**Note:** This relation is only applicable in this scenario because the mass that we are dealing with is spherical.

Where  $F_D$  is the Air/Fluid resistance between the mass and the fluid,  $\mu$  is the dynamic viscosity of the fluid,  $R$  is the radius of the mass and  $v_D$  is the flow velocity relative to the mass.

### 3.7 Logarithmic Decrement

The system exhibits an interesting feature, that of constant logarithmic decrements, that is,

$$\ln \frac{x_1}{x_2} = \ln \frac{x_2}{x_3} = \ln \frac{x_3}{x_4} = \dots$$

Where  $x_n$  and  $x_{n+1}$  are the amplitudes of any two successive peaks ( $n \in \mathbb{R}$ ).

Also interestingly that for any two successive peaks, if we define,

$$\delta = \ln \left( \frac{x_n}{x_{n+1}} \right)$$

Then,

$$\zeta = \frac{\delta}{\sqrt{\delta^2 + (2\pi)^2}}$$

With the above findings and definitions, we can create an expression for percentage overshoot, that is,

$$\text{Percentage Overshoot} = 100 \cdot \exp \left( -\frac{\zeta \pi}{\sqrt{1 - \zeta^2}} \right)$$

Consequently we can find an expression for  $\zeta$  in terms of **percentage overshoot** (PO). That is,

$$\zeta = \frac{-\ln \left( \frac{PO}{100} \right)}{\sqrt{\pi^2 + \ln^2 \left( \frac{PO}{100} \right)}}$$

**Note:**  $\zeta$  here, in the context of Drag forces and resistive forces symbolizes the damping ratio.

### **3.8 Computer Simulation Software**

*To model the chaotic system of the Damped Harmonic Elastic Pendulum we would need the aid of **computational technology**. Software's such MATLAB, Mathematica or some simple, eccentric computer programming language code in a script format, that would compute and yield solutions and necessary simulations that are required.*

*For the purpose of this investigation we shall be using a **Python script**, the source code of which can be found in **appendix ??**, to model and simulate the dynamic and chaotic system.*

## **4 Materials Required**

- *Pendulum Stand/Fixed Pivot*
- *Cylindrical Masses/Pendulum Masses of 1 kg, 2 kg, 3 kg, 4 kg, 5 kg, 6 kg, 7 kg, 8 kg, 9 kg and 10 kg*
- *Spring with spring stiffness constant of 100 N/m*
- *Force Gauge*
- *Weighing Scale (Digital/Analog)*

**Note:** In theory, the spring stiffness constant incorporated in research could of any arbitrary value. For the purpose of this investigation, we shall specifically

use springs of spring stiffness constant of 100 N/m.

## 5 Variables

<i>Physical Quantity</i>	<i>Symbol</i>
<i>Spring constant</i>	$k$
<i>Equilibrium spring length</i>	$l_0$
<i>Extension length</i>	$x$
<i>Angular displacement/Polar Angle</i>	$\theta$
<i>Displacement from normal</i>	$r$

Table 5.1: General physical quantities employed in this investigation

**Note:** In theory, the equilibrium spring length incorporated in research could of any arbitrary value. For the purpose of this investigation, we shall specifically use springs of equilibrium spring length of 1 m.

<i>Independent Variable</i>	<i>Dependent Variable</i>	<i>Controlled Variable</i>
<i>Time</i>	<i>Extension length</i>	<i>Temperature</i>
-	<i>Displacement</i>	<i>Fluid resistance</i>
-	<i>Angular displacement</i>	<i>Reference point</i>
-	<i>Absolute frequency</i>	-

Table 5.2: Segregation of employed variables as IV, DV or CV

## 6 Procedure

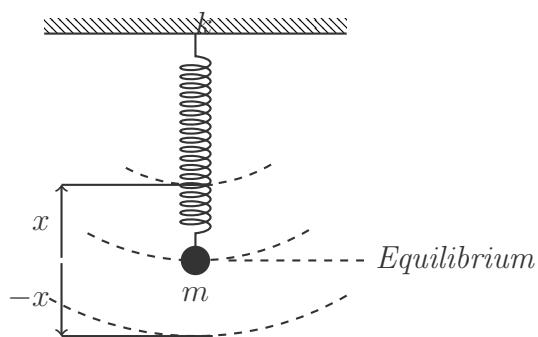


Figure 6.1: Diagram of the experiment carried down in this investigation

Using the materials specified in chapter ??, arrange the components as specified in figure ??

Following which, according to the **specified initial conditions** specified in the next chapter carefully oscillate the pendulum.

## 7 Initial Conditions

We define certain initial conditions for this dynamic system and convert this system to an initial value problem (IVP) system.

$$x(0) = 0$$

Where  $x(0)$  is the initial extension length of the spring at time  $t = 0$

$$\theta(0) = \pi/2$$

Where  $\theta(0)$  is the initial angular displacement of the system at time  $t = 0$

The radius of the mass employed in this investigation is approximately 5 cm  
( $r \approx 5 \cdot 10^{-2} m$ )

## 8 Equations

### 8.1 Equations of Motion

We know that,

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_j} \right) - \frac{\partial L}{\partial q_j} + \frac{\partial R}{\partial q_j} = 0$$

Where  $L$  is the Lagrangian,  $q_i$  is the generalized coordinates,  $R$  is the Rayleigh dissipation function and " $Q_j$ " is the generalized forces.

According to our specific case that we are investigating, we have two generalized coordinates, " $\theta$ " and " $x$ ".

Therefore the coupled set of differential equations that describe the motion of the damped elastic harmonic pendulum (Equations of Motion) with respect to the two generalized coordinates  $x$  and  $\theta$  are,

$$m \cdot \ddot{x} - m \cdot (l_0 + x)\dot{\theta}^2 + kx - g \cdot m \cdot \cos \theta + C_D \dot{x} = 0$$

$$m \cdot (l_0 + x)^2 \ddot{\theta} + 2m \cdot (l_0 + x)\dot{x}\dot{\theta} + g \cdot m \cdot (l_0 + x) \sin \theta + C_D \dot{\theta} = 0$$

With  $\ddot{x}$  and  $\ddot{\theta}$  isolated the equations are,

$$\ddot{x} = (l_0 + x)\dot{\theta}^2 - \frac{k}{m} \cdot x + g \cdot \cos \theta - \frac{C_D}{m} \dot{x}$$

$$\ddot{\theta} = -\frac{2}{(l_0 + x)} \dot{x} \dot{\theta} - \frac{g}{(l_0 + x)} \cdot \sin \theta - \frac{C_D}{m \cdot (l_0 + x)^2} \dot{\theta}$$

Where,  $m$  is the mass,  $l_0$  is the rest length of the spring,  $x$  is the extension length of the spring,  $\theta$  is the angle made between the spring and the normal,  $g$  is the gravitational acceleration,  $k$  is the spring stiffness constant and  $C_D$  damping coefficient.

**Note:** When damping is zero (Absolute Harmonic Motion),  $C_D$  is zero.

These equations will be directly inputted into the python source code from **appendix ??** to solve these equations numerically with an accuracy of  $\pm 10^{-8}$  for each time step of the **Runge-Kutta 4th Order** (RK4) algorithm.

## 8.2 Drag Equation

We know that the general Drag equation is mathematically defined as,

$$F_D = \frac{1}{2} \rho v^2 C_D A$$

Where  $F_D$  is the Air/Fluid resistance between the mass and the fluid,  $\rho$  is the density of the fluid,  $v$  is the speed of the object relative to the fluid,  $C_D$  is velocity decay constant (damping constant) and  $A$  is the cross sectional area.

But as the fluid medium that we are dealing with is simply atmospheric air at  $25^\circ$  and at 1 atm pressure,  $\rho = 1.1839 \text{ kg/m}^3$ . Also that, the physical nature of the mass that we are dealing with is spherical, therefore  $C_D = 0.47$  and  $A = \pi r^2$ .

Therefore, we have,

$$F_D = \pi \cdot \frac{432165}{10^6} \cdot r^2 v^2$$

## 8.3 Stokes's Law Equation

We know that,

$$F_D = 6\pi\mu R v_D$$

Where  $F_D$  is the Air/Fluid resistance between the mass and the fluid,  $\mu$  is the dynamic viscosity of the fluid,  $R$  is the radius of the mass and  $v_D$  is the flow velocity relative to the mass.

But as the fluid medium that we are dealing with is simply atmospheric air at  $25^\circ$  and at 1 atm pressure,  $\mu = 18.6 \mu P \cdot s = 1.86 \times 10^{-5} P \cdot s$ .

Also with close observation it can be inferred that the magnitude of flow velocity relative to the mass is the same as the magnitude of the absolute velocity of the system. Therefore using Newtonian vector definition, we define,

$$v = \sqrt{(\ddot{x})^2 + (\ddot{\theta})^2}$$

Therefore we can infer that,

$$v_D = \sqrt{(\ddot{x})^2 + (\ddot{\theta})^2}$$

Therefore, we finally have,

$$F_D = \frac{11.16}{10^5} \cdot \pi \sqrt{(\ddot{x})^2 + (\ddot{\theta})^2} \cdot R$$

## 8.4 Frequency Equations

The frequency of the system can be broken down into two types of frequencies if we consider it from the perspective of the two generalized coordinates. They are radial and angular frequencies

### 8.4.1 Radial frequency

Radial frequency is defined as the number of oscillations of the radial movement (Spring component of the system) per second. Physically and mathematically defined as,

$$\omega_r = \sqrt{\frac{k}{m}}$$

*With observation, it can be inferred that, radial frequency remains constant whatsoever in the whole system.*

#### 8.4.2 Angular frequency

*Angular frequency is defined as the number of oscillations of the angular movement (Pendulum component of the system) per second. Physically and mathematically defined as,*

$$\omega_\theta = \sqrt{\frac{g}{l_0 + x}}$$

*With observation, it can be inferred that, angular frequency is a function of radial position and will be a variable all throughout in the whole system.*

#### 8.4.3 Absolute frequency

*If we are to talk about absolute frequency, we can define absolute frequency using Newtonian vector definition as,*

$$f = \sqrt{(\omega_r)^2 + (\omega_\theta)^2}$$

*Therefore we have,*

$$f = \sqrt{\frac{k}{m} + \frac{g}{l_0 + x}}$$

### 8.5 Fundamental Derived Equations

*The fundamental derived equations we shall be using in this investigation are:*

$$\ddot{x} = (l_0 + x)\dot{\theta}^2 - \frac{k}{m} \cdot x + g \cdot \cos \theta - \frac{C_D}{m} \dot{x} \quad (8.1)$$

$$\ddot{\theta} = -\frac{2}{(l_0 + x)} \dot{x} \dot{\theta} - \frac{g}{(l_0 + x)} \cdot \sin \theta - \frac{C_D}{m \cdot (l_0 + x)^2} \dot{\theta} \quad (8.2)$$

**Note:** The coefficient of damping ( $C_D$ ) from now on will be equal to the numeric value of 0.47.

$$F_D = \pi \cdot \frac{432165}{10^6} \cdot r^2 v^2 \quad (8.3)$$

$$F_D = \frac{11.16}{10^5} \cdot \pi \sqrt{(\ddot{x})^2 + (\ddot{\theta})^2} \cdot R \quad (8.4)$$

**Note:**  $R$  and  $r$  represent the same physical quantity, i.e., the radius of the mass used in a specific case.

$$\omega_r = \sqrt{\frac{k}{m}} \quad (8.5)$$

$$\omega_\theta = \sqrt{\frac{g}{l_0 + x}} \quad (8.6)$$

$$f = \sqrt{\frac{k}{m} + \frac{g}{l_0 + x}} \quad (8.7)$$

**Note:** All the equations mentioned above shall be inputted algorithmically in an computer program/software and shall be numerically estimated to about

an accuracy of approximately  $\pm 10^{-8}$  of each time step interval.

## 9 *Experimental Data*

### 9.1 *Radial Displacement versus Time*

<i>Time</i> <i>Mass</i>	0	2.5	5.0	7.5	10.0	12.5	15.0	17.5	20.0
1	0	0.008	0.086	0.107	0.090	0.096	0.097	0.097	0.098
2	0	0.110	0.330	0.350	0.298	0.227	0.168	0.161	0.195
3	0	0.558	0.420	0.021	0.258	0.537	0.522	0.440	0.301
4	0	1.062	-0.003	0.831	0.262	0.2031	0.805	0.332	0.157
5	0	1.189	0.165	0.611	0.431	0.583	0.280	0.890	0.086
6	0	1.079	0.924	-0.012	1.313	0.228	0.759	0.596	0.396
7	0	0.854	1.686	0.223	0.182	1.556	0.211	0.652	1.334
8	0	0.579	1.832	1.230	-0.007	0.493	1.746	0.471	0.162
9	0	0.308	1.467	2.121	1.094	-0.114	0.455	1.825	1.403
10	0	0.062	0.923	2.062	2.204	1.304	-0.023	0.144	1.196

Table 9.1: *Experimental Data on radial displacement versus time from 0 to 20 seconds.*

## 9.2 *Angular Displacement versus Time*

<i>Time Mass</i>	0	2.5	5.0	7.5	10.0	12.5	15.0	17.5	20.0
1	$\pi/2$	0.931	0.418	-0.045	-0.208	-0.097	0.029	0.052	0.018
2	$\pi/2$	0.942	0.462	0.282	0.287	0.365	0.418	0.365	0.237
3	$\pi/2$	0.364	-0.555	-1.005	-0.591	-0.191	0.082	0.219	0.335
4	$\pi/2$	0.052	-1.223	-0.222	0.704	0.683	0.022	-0.476	-0.604
5	$\pi/2$	-0.167	-0.948	0.491	0.588	-0.429	-0.608	0.163	0.684
6	$\pi/2$	-0.346	-0.395	1.155	-0.123	-0.720	0.355	0.415	-0.361
7	$\pi/2$	-0.493	-0.108	0.851	-0.714	0.006	0.625	-0.384	-0.157
8	$\pi/2$	-0.627	0.077	0.289	-0.949	0.5312	0.007	-0.404	0.526
9	$\pi/2$	-0.754	0.250	0.020	-0.395	0.808	-0.5195	0.0485	0.166
10	$\pi/2$	-0.879	0.464	-0.134	-0.062	0.290	-0.601	0.591	-0.115

Table 9.2: *Experimental Data on angular displacement versus time from 0 to 20 seconds.*

## 9.3 *Angular Frequency versus Time*

<i>Time Mass</i>	0	2.5	5.0	7.5	10.0	12.5	15.0	17.5	20.0
1	3.130	3.118	3.006	2.976	2.994	2.991	2.989	2.989	2.988
2	3.128	2.971	2.716	2.696	2.749	2.828	2.897	2.906	2.870
3	3.129	2.509	2.630	3.099	2.794	2.526	2.539	2.607	2.771
4	3.130	2.181	3.136	2.314	2.790	2.859	2.332	2.715	2.925
5	3.128	2.117	2.906	2.471	2.627	2.492	2.768	2.274	2.996
6	3.129	2.175	2.261	3.153	2.060	2.830	2.363	2.481	2.667
7	3.133	2.303	1.912	2.834	2.883	1.960	2.848	2.538	2.086
8	3.130	2.493	1.861	2.103	3.148	2.566	1.890	2.581	2.980
9	3.131	2.741	1.995	1.773	2.168	3.331	2.597	1.865	2.046
10	3.131	3.040	2.255	1.790	1.750	2.643	3.170	2.936	2.127

Table 9.3: *Experimental Data on angular frequency versus time from 0 to 20 seconds.*

## 9.4 *Absolute Frequency versus Time*

<i>Time Mass</i>	0	2.5	5.0	7.5	10.0	12.5	15.0	17.5	20.0
1	10.478	10.474	10.442	10.434	10.440	10.438	10.437	10.437	10.437
2	7.7330	7.6700	7.5751	7.5677	7.5869	7.6157	7.6419	7.6452	7.6331
3	6.5675	6.2949	6.3448	6.5533	6.4140	6.3022	6.3072	6.3353	6.4018
4	5.8990	5.4550	5.9024	5.5098	5.7264	5.7581	5.5173	5.6895	5.7859
5	5.4588	4.9477	5.3319	5.1085	5.1840	5.1188	5.2601	5.0174	5.3830
6	5.1443	4.6250	4.6665	5.1564	4.5728	4.9661	4.7167	4.7776	4.8840
7	4.9075	4.4259	4.2359	4.7248	4.7560	4.25726	4.7335	4.4975	4.3170
8	4.7220	4.3255	3.9956	4.1124	4.7324	4.3660	4.0096	4.3801	4.6280
9	4.5728	4.3177	3.8847	3.7756	3.9763	4.7117	4.2269	3.8194	3.9060
10	4.4492	4.3860	3.8853	3.6340	3.6148	3.7780	4.476	4.3150	3.8050

Table 9.4: *Experimental Data on absolute frequency versus time from 0 to 20 seconds.*

# 10 *Simulation Data*

## 10.1 *Radial Displacement versus Time*

<i>Time</i> <i>Mass</i>	0	2.5	5.0	7.5	10.0	12.5	15.0	17.5	20.0
1	0	0.0208	0.0579	0.0995	0.0884	0.0925	0.0967	0.0964	0.0973
2	0	0.1550	0.3814	0.3850	0.3373	0.2688	0.1916	0.1546	0.1778
3	0	0.7400	0.2550	0.0246	0.4000	0.5760	0.4740	0.3480	0.2620
4	0	1.0400	0.0120	0.9149	0.1368	0.356	0.7702	0.2000	0.1858
5	0	0.9870	0.3402	0.3793	0.6565	0.3680	0.4730	0.7240	0.1067
6	0	0.7830	1.1904	-0.0157	1.1406	0.4410	0.5061	0.8350	0.3361
7	0	0.5280	1.7330	0.4370	0.0078	1.4960	0.4510	0.3920	1.3653
8	0	0.2734	1.6078	1.5275	0.1501	0.2296	1.6615	0.7760	0.0770
9	0	0.0316	1.106	1.4240	0.0360	0.2080	1.6170	1.6120	1.4870
10	0	-0.1304	0.5778	1.7646	2.2672	1.6546	0.1937	-0.0095	1.0826

Table 10.1: *Simulation Data on radial displacement versus time* from 0 to 20 seconds.

## 10.2 *Angular Displacement versus Time*

<i>Time Mass</i>	0	2.5	5.0	7.5	10.0	12.5	15.0	17.5	20.0
1	$\pi/2$	0.9511	0.5263	-0.1798	-0.1802	-0.1241	0.0058	0.0520	0.0224
2	$\pi/2$	0.7043	0.2804	0.1237	0.1480	0.2566	0.2573	0.3713	0.2580
3	$\pi/2$	0.1812	-0.7474	-0.9800	-0.4415	-0.0532	0.2189	0.3368	0.3938
4	$\pi/2$	-0.0913	-1.2379	-0.0741	0.8570	0.5420	-0.1042	-0.5949	-0.5705
5	$\pi/2$	-0.3086	-0.7538	0.6648	0.4344	-0.5780	-0.4749	0.2761	0.6362
6	$\pi/2$	0.4959	-0.2663	1.1854	-0.2326	-0.5803	0.4764	0.3063	-0.4548
7	$\pi/2$	-0.6630	-0.0158	0.6695	-0.8783	0.0931	0.5025	-0.5022	-0.1170
8	$\pi/2$	-0.8159	0.1684	0.1885	-0.8180	0.6686	-0.0651	-0.3160	0.5713
9	$\pi/2$	-0.9628	0.3679	-0.0456	-0.2958	0.6982	-0.6453	0.1078	0.1342
10	$\pi/2$	-1.0971	0.6205	-0.1991	0.0050	0.2226	-0.4875	0.6922	-0.1783

Table 10.2: *Simulation Data on angular displacement versus time from 0 to 20 seconds.*

## 10.3 *Angular Frequency versus Time*

<i>Time Mass</i>	0	2.5	5.0	7.5	10.0	12.5	15.0	17.5	20.0
1	3.1321	3.1007	3.0449	2.9865	3.0024	2.9967	2.9914	2.9913	2.9000
2	3.1322	2.9163	2.6660	2.6620	2.7092	2.7827	2.8696	2.8698	2.8866
3	3.1322	2.3750	2.7964	3.0935	2.6483	2.4962	2.5802	2.6976	2.7872
4	3.1322	2.1929	3.1178	2.2637	2.9387	2.6881	2.3560	2.8647	2.8769
5	3.1322	2.2232	2.7084	2.6694	2.4352	2.6782	2.5813	2.3862	2.9762
6	3.1322	2.3482	2.1162	3.1574	2.1406	2.6132	2.5529	2.3122	2.7093
7	3.1320	2.5360	1.8960	2.6130	3.1200	1.9836	2.6070	2.6520	2.0378
8	3.1321	2.7760	1.9404	1.9713	2.9170	2.829	1.9190	2.3510	3.0182
9	3.1322	3.059	2.1570	1.7687	2.0156	3.0836	2.8540	1.9380	1.9835
10	3.1320	3.3590	2.4950	1.8890	1.7331	1.9212	2.8660	3.1475	2.1712

Table 10.3: *Simulation Data on angular frequency versus time from 0 to 20 seconds.*

## 10.4 *Absolute Frequency versus Time*

<i>Time Mass</i> \ <i>Time</i>	0	2.5	5.0	7.5	10.0	12.5	15.0	17.5	20.0
1	10.479	10.470	10.453	10.436	10.441	10.439	10.437	10.437	10.437
2	7.7337	7.6492	7.5570	7.5556	7.5723	7.5987	7.6312	7.6484	7.6375
3	6.5684	6.2428	6.4151	6.5506	6.3528	6.2901	6.3238	6.3728	6.4113
4	5.9000	5.4600	5.8926	5.4884	5.8001	5.6766	5.5272	5.7622	5.7682
5	5.4599	4.9942	5.2282	5.2083	5.0917	5.2140	5.1638	5.0686	5.3725
6	5.1457	4.7102	4.5987	5.1613	4.6102	4.8473	4.8162	4.6935	4.9000
7	4.9087	4.5525	4.2284	4.5955	4.9023	4.2684	4.5896	4.6200	4.2935
8	4.7238	4.496	4.0329	4.0476	4.5860	4.5270	4.0235	4.246	4.6491
9	4.5740	4.5260	3.9705	3.7735	3.8951	4.543	4.3880	3.8562	3.8788
10	4.4509	4.6130	4.0281	3.6842	3.6063	3.7019	4.2690	4.4620	3.8556

Table 10.4: *Simulation Data on absolute frequency versus time from 0 to 20 seconds.*

## 11 *Derived Data*

We shall be modelling the viscous drag with respect to time for all cases with the aid of the simulation program with the initial utility of equations ?? and ??, and shall call it derived data, as it is derived from the values of radial and angular displacement versus time, and this is not the primary focus of this investigation.

All data necessary relating to this chapter of the investigation can be found

*in appendix ??*

## 12 Observations

Let  $\zeta_e$ ,  $\tau_e$ ,  $\chi_e$  and  $\psi_e$  be equal to the sum of the experimental observations from  $t = 0$  to  $t = 20$  seconds divided by the number of observations, for radial displacement, angular displacement, angular frequency and absolute frequency respectively for each and every case that we are investigating.

Let  $\zeta_s$ ,  $\tau_s$ ,  $\chi_s$  and  $\psi_s$  be equal to the sum of the simulation observations from  $t = 0$  to  $t = 20$  seconds divided by the number of observations, for radial displacement, angular displacement, angular frequency and absolute frequency respectively for each and every case that we are investigating.

We further define,  $\zeta_{0_n} = \frac{\zeta_e}{\zeta_s}$ ,  $\tau_{0_n} = \frac{\tau_e}{\tau_s}$ ,  $\chi_{0_n} = \frac{\chi_e}{\chi_s}$  and  $\psi_{0_n} = \frac{\psi_e}{\psi_s}$  for each and every case.

Here, upon close observation we infer that,  $100 \cdot \zeta_{0_n}$ ,  $100 \cdot \tau_{0_n}$ ,  $100 \cdot \chi_{0_n}$  and  $100 \cdot \psi_{0_n}$  are the percentage accuracy of the experimental values of radial displacement, angular displacement, angular frequency and absolute frequency respectively for each and every case that we are investigating.

By using the above definitions we have,

### 12.1 Observation from Case 1 where $m = 1$ kg

$$\zeta_{e_1} = 0.075444 \text{ and } \zeta_{s_1} = 0.072167$$

$$\tau_{e_1} = 0.296533 \text{ and } \tau_{s_1} = 0.293811$$

$$\chi_{e_1} = 3.02011 \text{ and } \chi_{s_1} = 3.01622$$

$$\psi_{e_1} = 10.4463 \text{ and } \psi_{s_1} = 10.4477$$

Therefore,

$$\zeta_{0_1} = 1.04542$$

$$\tau_{0_1} = 1.00927$$

$$\chi_{0_1} = 1.00129$$

$$\psi_{0_1} = 0.999872$$

## 12.2 *Observation from Case 2 where m = 2 kg*

$$\zeta_{e_2} = 0.204333 \text{ and } \zeta_{s_2} = 0.227944$$

$$\tau_{e_2} = 0.547644 \text{ and } \tau_{s_2} = 0.441155$$

$$\chi_{e_2} = 2.86233 \text{ and } \chi_{s_2} = 2.83269$$

$$\psi_{e_2} = 7.62984 \text{ and } \psi_{s_2} = 7.6204$$

Therefore,

$$\zeta_{0_2} = 0.896417$$

$$\tau_{0_2} = 1.24139$$

$$\chi_{0_2} = 1.01047$$

$$\psi_{0_2} = 1.00124$$

## 12.3 *Observation from Case 3 where m = 3 kg*

$$\zeta_{e_3} = 0.339667 \text{ and } \zeta_{s_3} = 0.342178$$

$$\tau_{e_3} = 0.025422 \text{ and } \tau_{s_3} = 0.053266$$

$$\chi_{e_3} = 2.64633 \text{ and } \chi_{s_3} = 2.73407$$

$$\psi_{e_3} = 6.39122 \text{ and } \psi_{s_3} = 6.39197$$

Therefore,

$$\zeta_{0_3} = 0.992661$$

$$\tau_{0_3} = 0.477259$$

$$\chi_{0_3} = 0.967911$$

$$\psi_{0_3} = 0.999884$$

## 12.4 *Observation from Case 4 where m = 4 kg*

$$\zeta_{e_4} = 0.405456 \text{ and } \zeta_{s_4} = 0.401744$$

$$\tau_{e_4} = 0.056311 \text{ and } \tau_{s_4} = 0.096322$$

$$\chi_{e_4} = 2.70911 \text{ and } \chi_{s_4} = 2.71456$$

$$\psi_{e_4} = 5.69371 \text{ and } \psi_{s_4} = 5.69726$$

*Therefore,*

$$\zeta_{0_4} = 1.00924$$

$$\tau_{0_4} = 0.584612$$

$$\chi_{0_4} = 0.997992$$

$$\psi_{0_4} = 0.999377$$

## 12.5 *Observation from Case 5 where m = 5 kg*

$$\zeta_{e_5} = 0.470556 \text{ and } \zeta_{s_5} = 0.4483$$

$$\tau_{e_5} = 0.149422 \text{ and } \tau_{s_5} = 0.163$$

$$\chi_{e_5} = 2.64211 \text{ and } \chi_{s_5} = 2.64337$$

$$\psi_{e_5} = 5.20113 \text{ and } \psi_{s_5} = 5.20013$$

*Therefore,*

$$\zeta_{0_5} = 1.04965$$

$$\tau_{0_5} = 0.916699$$

$$\chi_{0_5} = 0.999523$$

$$\psi_{0_5} = 1.00019$$

## 12.6 *Observation from Case 6 where m = 6 kg*

$\zeta_{e_6} = 0.587$  and  $\zeta_{s_6} = 0.4685$

$\tau_{e_6} = 0.2162$  and  $\tau_{s_6} = 0.277866$

$\chi_{e_6} = 2.56878$  and  $\chi_{s_6} = 2.56469$

$\psi_{e_6} = 4.83438$  and  $\psi_{s_6} = 4.83146$

Therefore,

$\zeta_{0_6} = 1.25293$

$\tau_{0_6} = 0.778073$

$\chi_{0_6} = 1.00159$

$\psi_{0_6} = 1.0006$

## 12.7 *Observation from Case 7 where m = 7 kg*

$\zeta_{e_7} = 0.744222$  and  $\zeta_{s_7} = 0.712233$

$\tau_{e_7} = 0.132977$  and  $\tau_{s_7} = 0.073288$

$\chi_{e_7} = 2.49967$  and  $\chi_{s_7} = 2.5086$

$\psi_{e_7} = 4.53948$  and  $\psi_{s_7} = 4.55099$

Therefore,

$\zeta_{0_7} = 1.04491$

$\tau_{0_7} = 1.81444$

$\chi_{0_7} = 0.99644$

$\psi_{0_7} = 0.997471$

## 12.8 *Observation from Case 8 where m = 8 kg*

$\zeta_{e_8} = 0.722889$  and  $\zeta_{s_8} = 0.700322$

$\tau_{e_8} = 0.113444$  and  $\tau_{s_8} = 0.128066$

$\chi_{e_8} = 2.528$  and  $\chi_{s_8} = 2.53933$

$\psi_{e_8} = 4.36351$  and  $\psi_{s_8} = 4.37021$

Therefore,

$$\zeta_{0_8} = 1.03222$$

$$\tau_{0_8} = 0.885824$$

$$\chi_{0_8} = 0.995538$$

$$\psi_{0_8} = 0.998467$$

## 12.9 *Observation from Case 9 where m = 9 kg*

$\zeta_{e_9} = 0.951$  and  $\zeta_{s_9} = 0.835733$

$\tau_{e_9} = 0.132755$  and  $\tau_{s_9} = 0.103266$

$\chi_{e_9} = 2.40522$  and  $\chi_{s_9} = 2.44351$

$\psi_{e_9} = 4.13234$  and  $\psi_{s_9} = 4.15612$

Therefore,

$$\zeta_{0_9} = 1.13792$$

$$\tau_{0_9} = 1.28556$$

$$\chi_{0_9} = 0.98433$$

$$\psi_{0_9} = 0.994278$$

## 12.10 *Observation from Case 10 where m = 10 kg*

$\zeta_{e_{10}} = 0.874667$  and  $\zeta_{s_{10}} = 0.822289$

$\tau_{e_{10}} = 0.124977$  and  $\tau_{s_{10}} = 0.127677$

$\chi_{e_{10}} = 2.538$  and  $\chi_{s_{10}} = 2.52378$

$\psi_{e_{10}} = 4.03814$  and  $\psi_{s_{10}} = 4.07456$

Therefore,

$$\zeta_{0_{10}} = 1.0637$$

$$\tau_{0_{10}} = 0.978853$$

$$\chi_{0_{10}} = 1.00563$$

$$\psi_{0_{10}} = 0.991062$$

## 13 *Observations from Derived Data*

Upon close observation of the **drag versus time** it is evident that, the drag periodically decreases with decrease in mass and increases in amplitude and frequency with steady increase in mass

As mentioned before, the viscous drag that is in play, exhibits underdamping that is the damping ratio involved in this example is less than 1 ( $\zeta < 1$ ).

The system exhibits an interesting feature, that of constant logarithmic decrements, that is,

$$\ln \frac{x_1}{x_2} = \ln \frac{x_2}{x_3} = \ln \frac{x_3}{x_4} = \dots \dots \dots$$

Where  $x_n$  and  $x_{n+1}$  are the amplitudes of any two successive peaks ( $n \in \mathbb{R}$ ).

Also interestingly that for any two successive peaks, if we define,

$$\delta = \ln \left( \frac{x_n}{x_{n+1}} \right)$$

Then,

$$\zeta = \frac{\delta}{\sqrt{\delta^2 + (2\pi)^2}}$$

With the above findings and definitions, we can create an expression for percentage overshoot, that is,

$$\text{Percentage Overshoot} = 100 \cdot \exp \left( -\frac{\zeta \pi}{\sqrt{1 - \zeta^2}} \right)$$

Consequently we can find an expression for  $\zeta$  in terms of **percentage overshoot** (PO). That is,

$$\zeta = \frac{-\ln \left( \frac{PO}{100} \right)}{\sqrt{\pi^2 + \ln^2 \left( \frac{PO}{100} \right)}}$$

**Note:**  $\zeta$  here, in the context of Drag forces and resistive forces symbolizes

the damping ratio.

## 14 Analysis

### 14.1 Graphical Analysis

#### 14.1.1 Graphs in relation to the Experimental values versus time

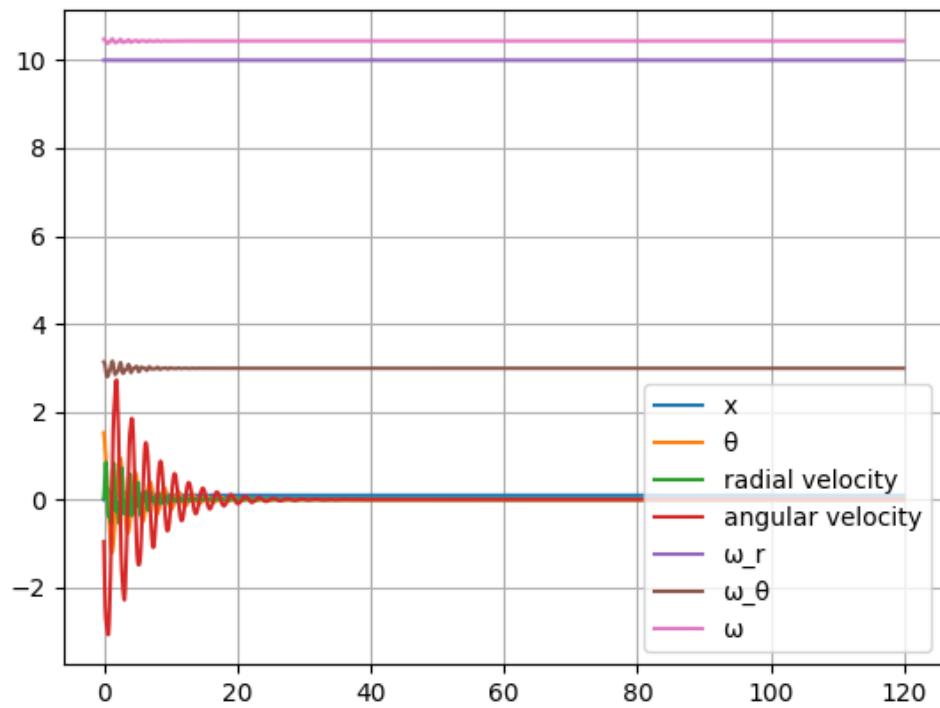


Figure 14.1: Case 1 where  $m = 1 \text{ kg}$

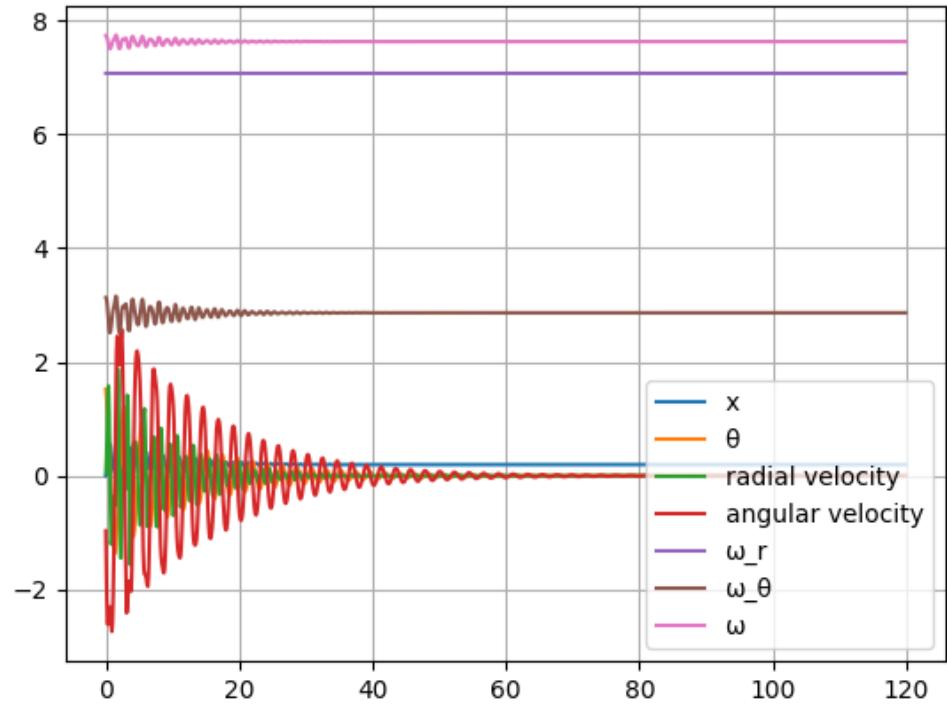


Figure 14.2: Case 2 where  $m = 2 \text{ kg}$

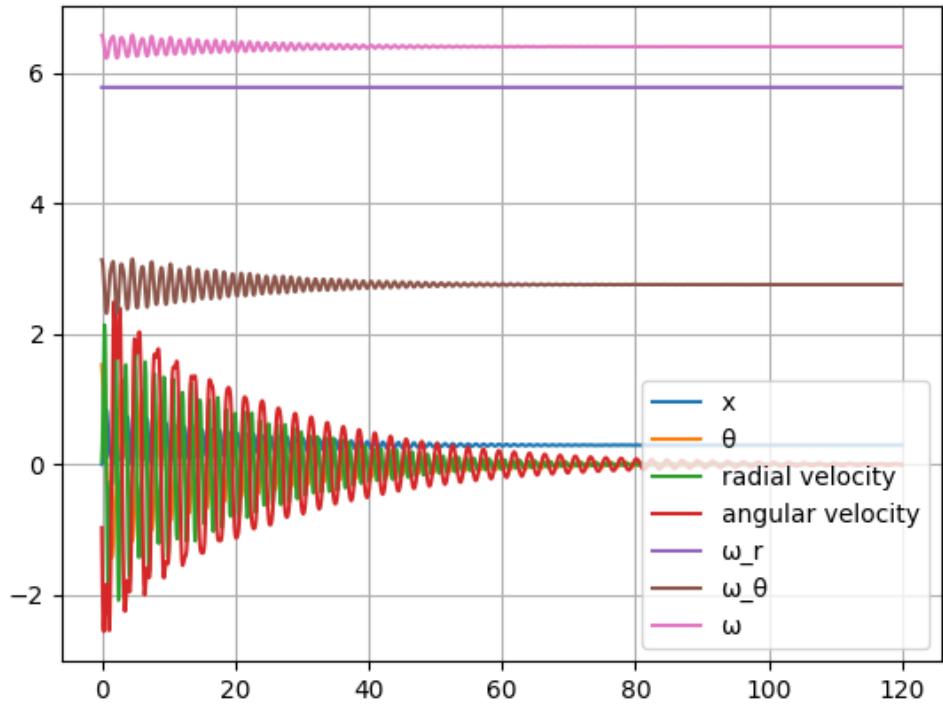


Figure 14.3: Case 3 where  $m = 3 \text{ kg}$

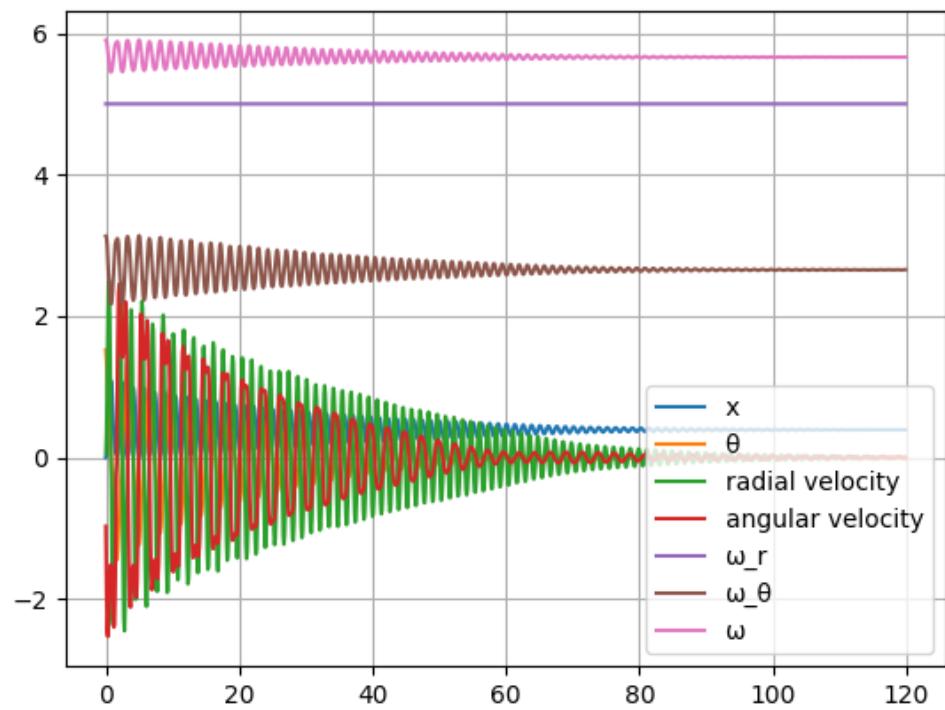


Figure 14.4: Case 4 where  $m = 4 \text{ kg}$

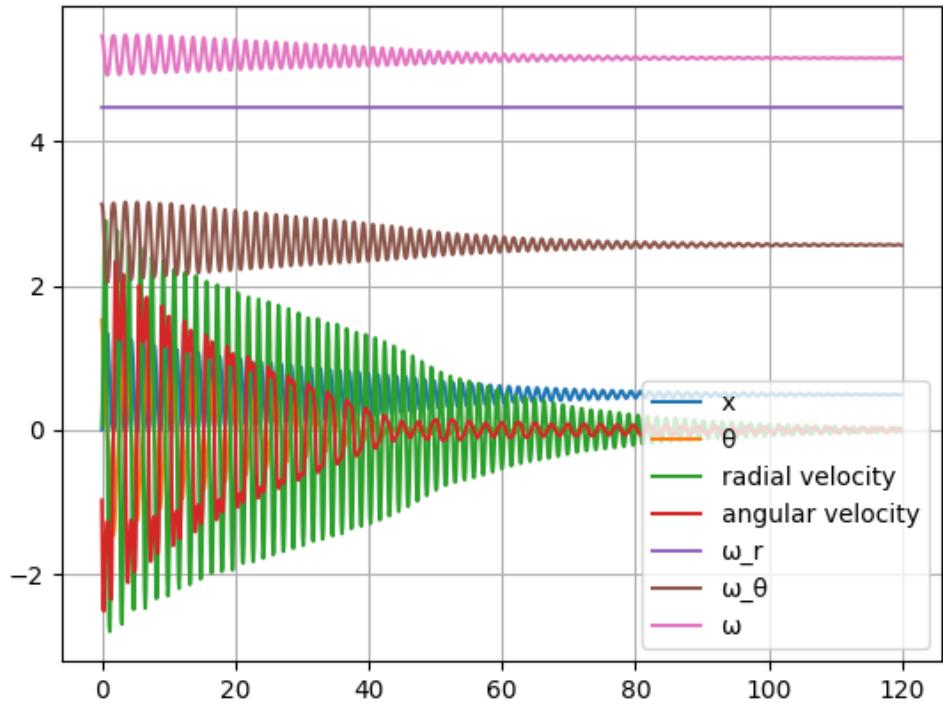


Figure 14.5: Case 5 where  $m = 5 \text{ kg}$

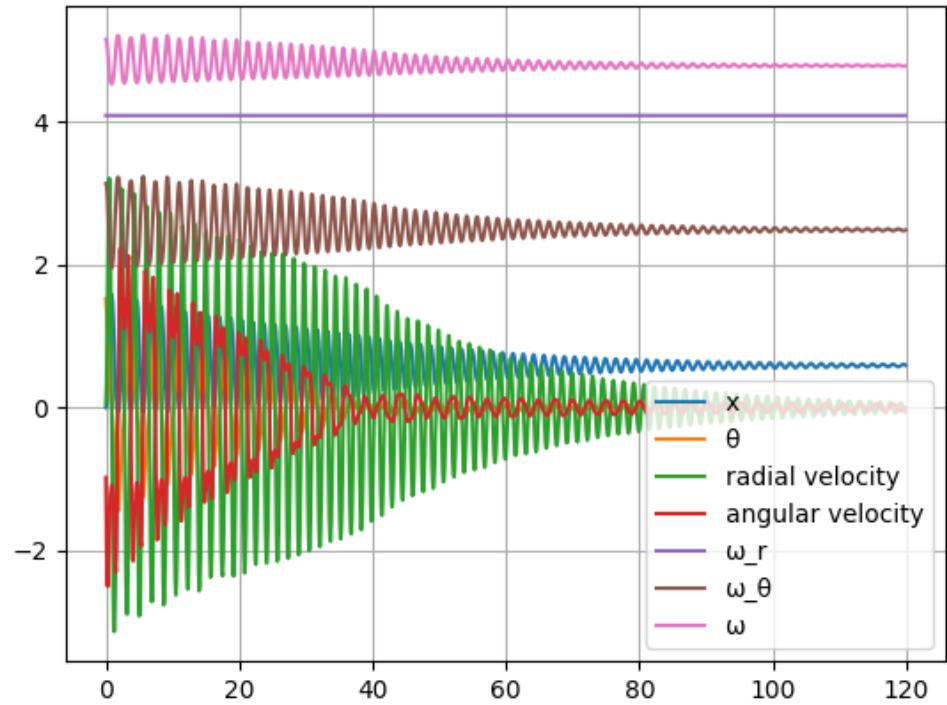


Figure 14.6: Case 6 where  $m = 6 \text{ kg}$

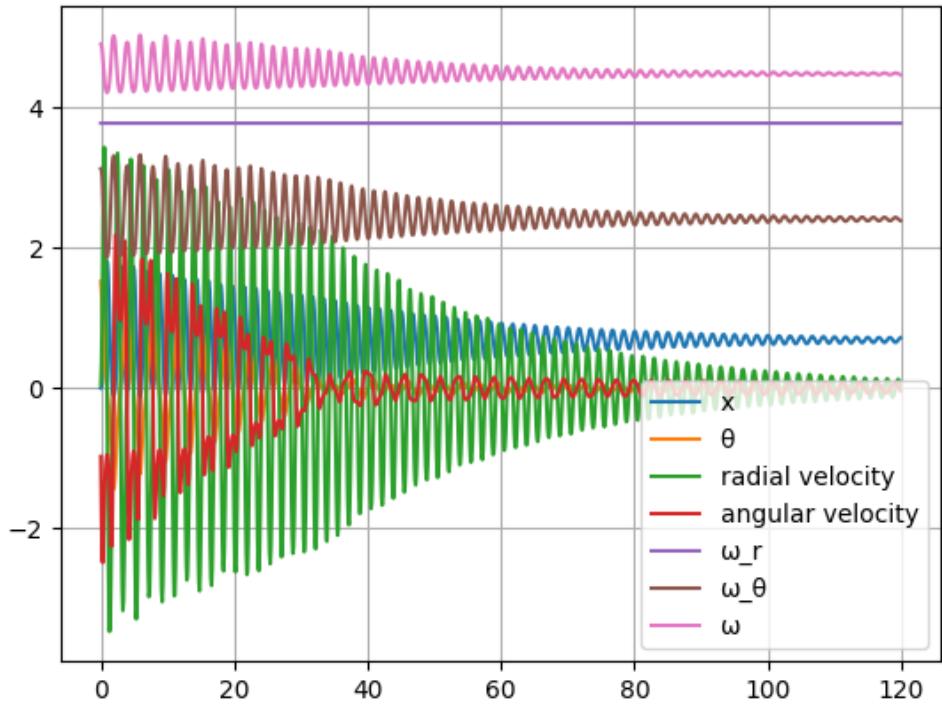


Figure 14.7: Case 7 where  $m = 7 \text{ kg}$

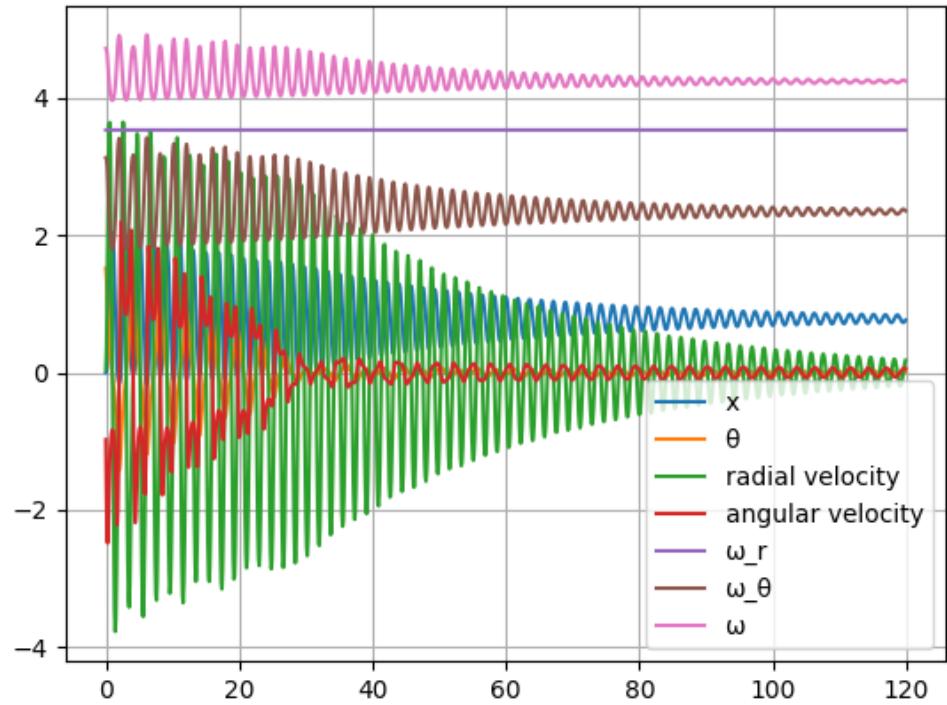


Figure 14.8: Case 8 where  $m = 8 \text{ kg}$

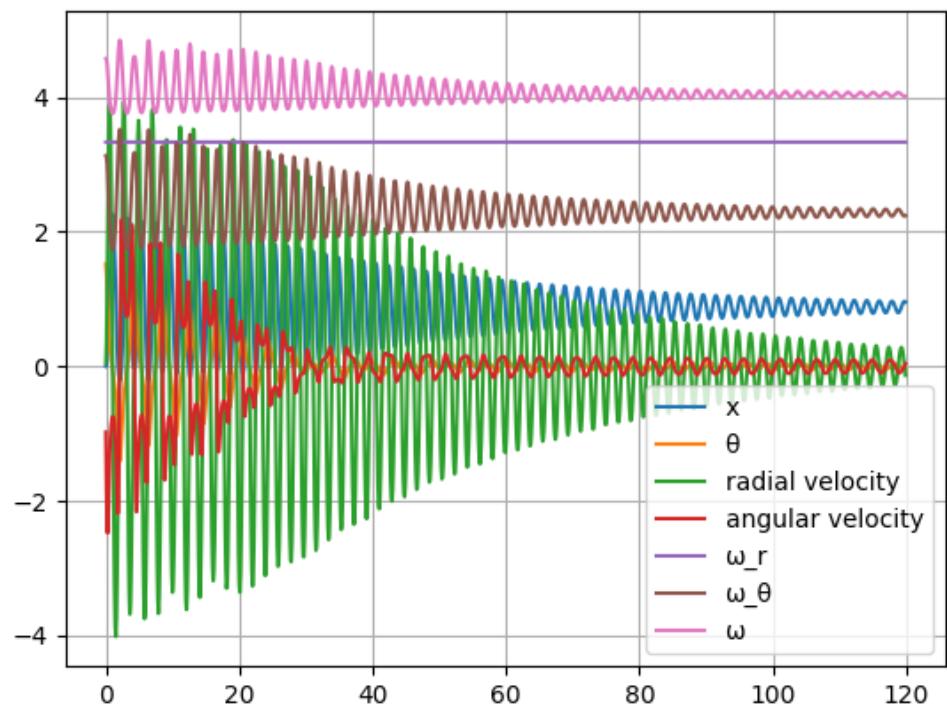


Figure 14.9: Case 9 where  $m = 9 \text{ kg}$

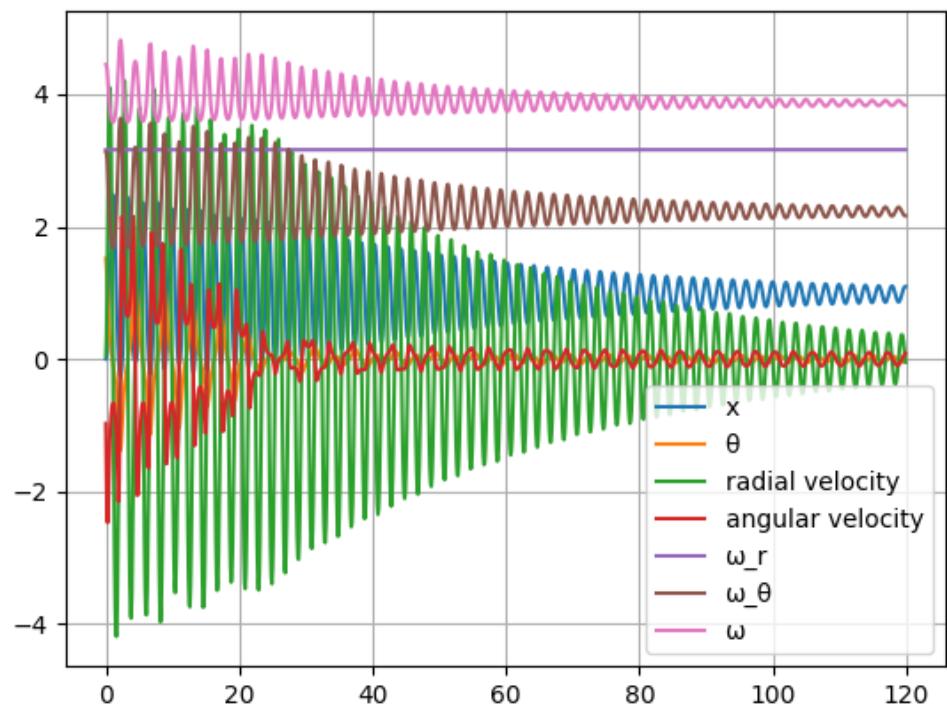


Figure 14.10: Case 10 where  $m = 10 \text{ kg}$

### 14.1.2 Graphs in relation to the Simulation values versus time

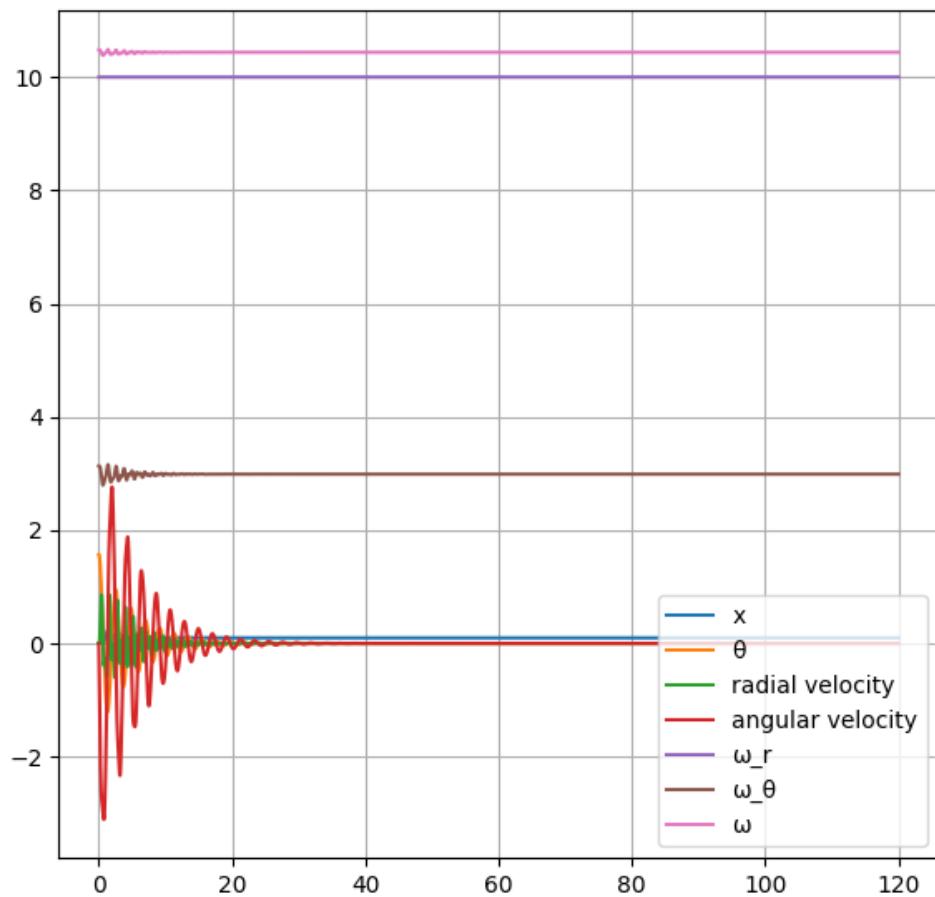


Figure 14.11: Case 1 where  $m = 1 \text{ kg}$

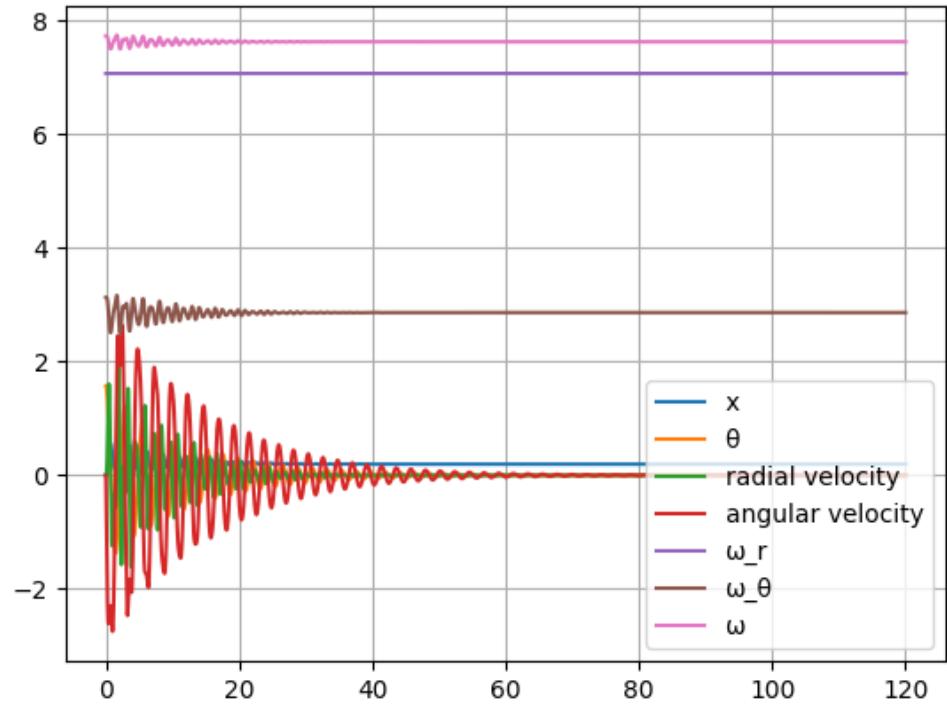


Figure 14.12: Case 2 where  $m = 2 \text{ kg}$

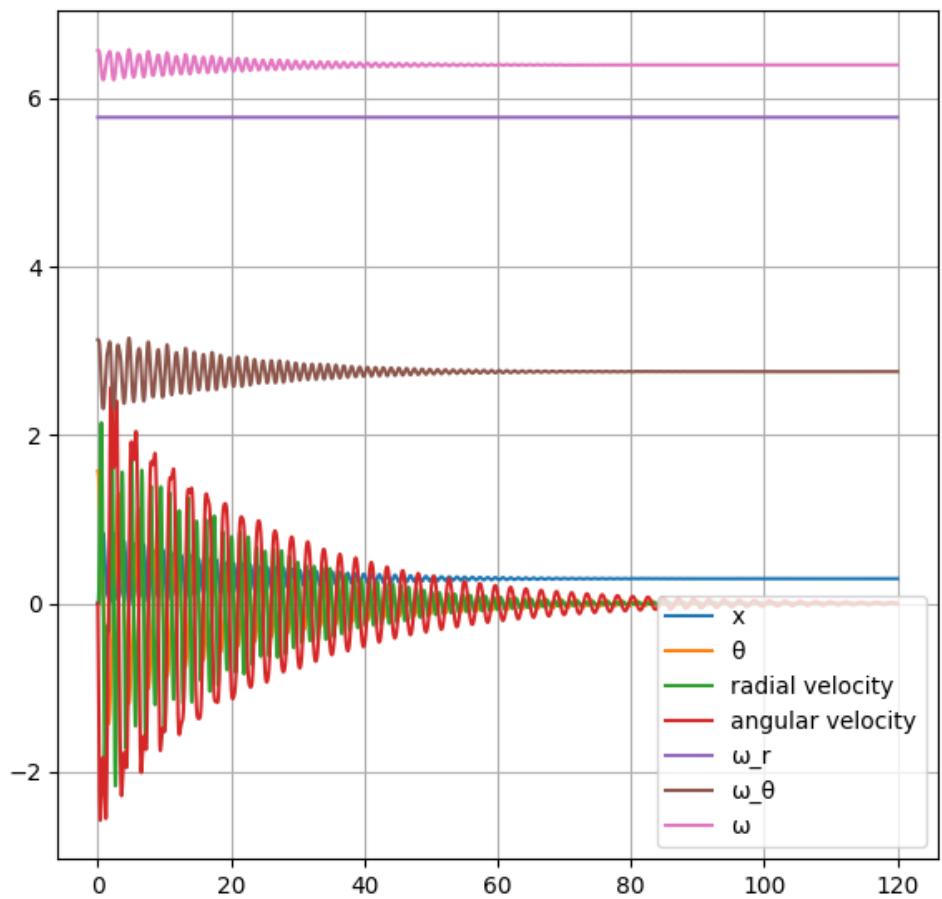


Figure 14.13: Case 3 where  $m = 3 \text{ kg}$

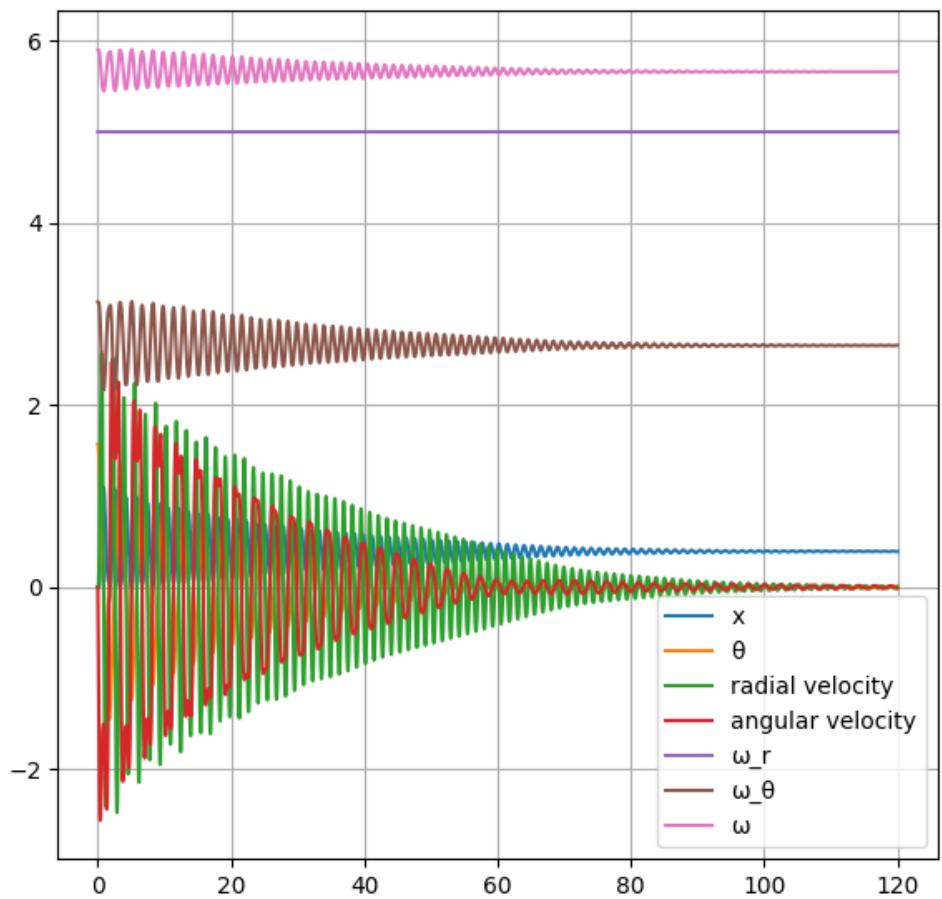


Figure 14.14: *Case 4 where  $m = 4 \text{ kg}$*

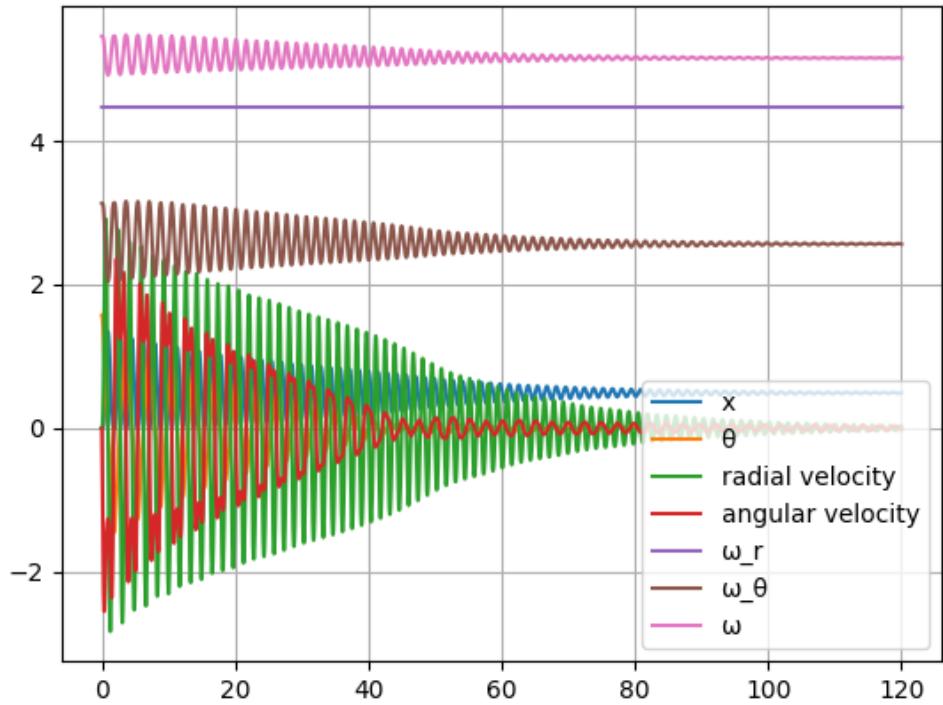


Figure 14.15: Case 5 where  $m = 5 \text{ kg}$

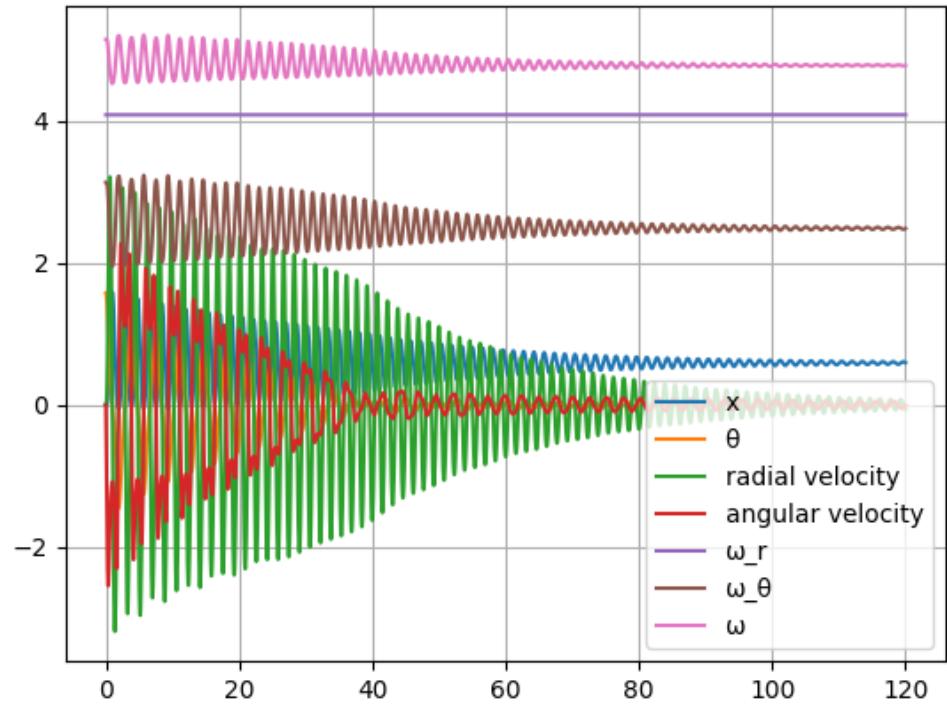


Figure 14.16: Case 6 where  $m = 6 \text{ kg}$

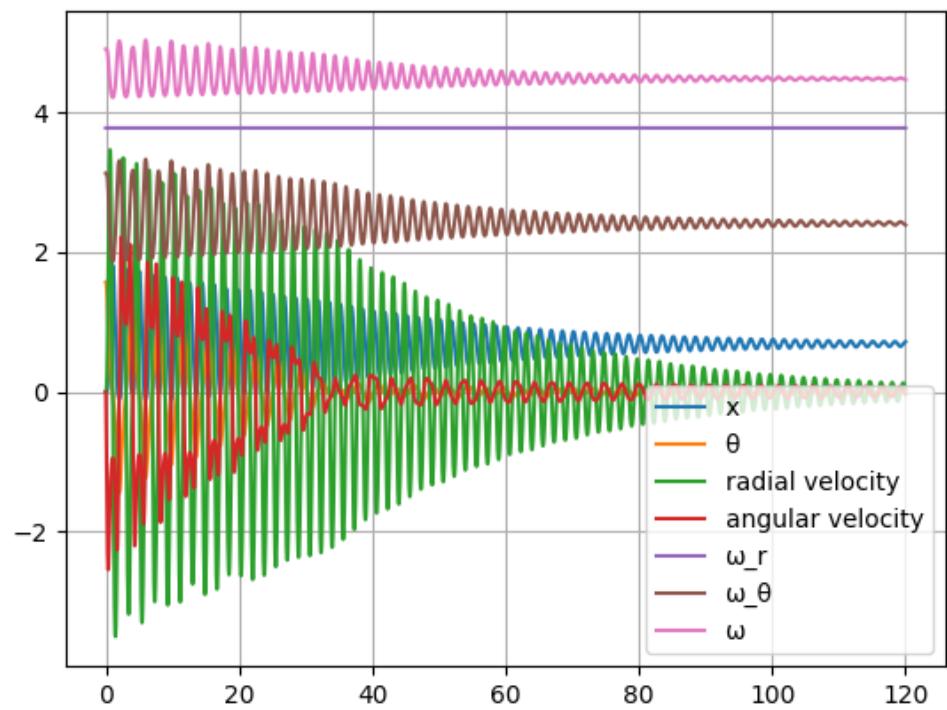


Figure 14.17: Case 7 where  $m = 7 \text{ kg}$

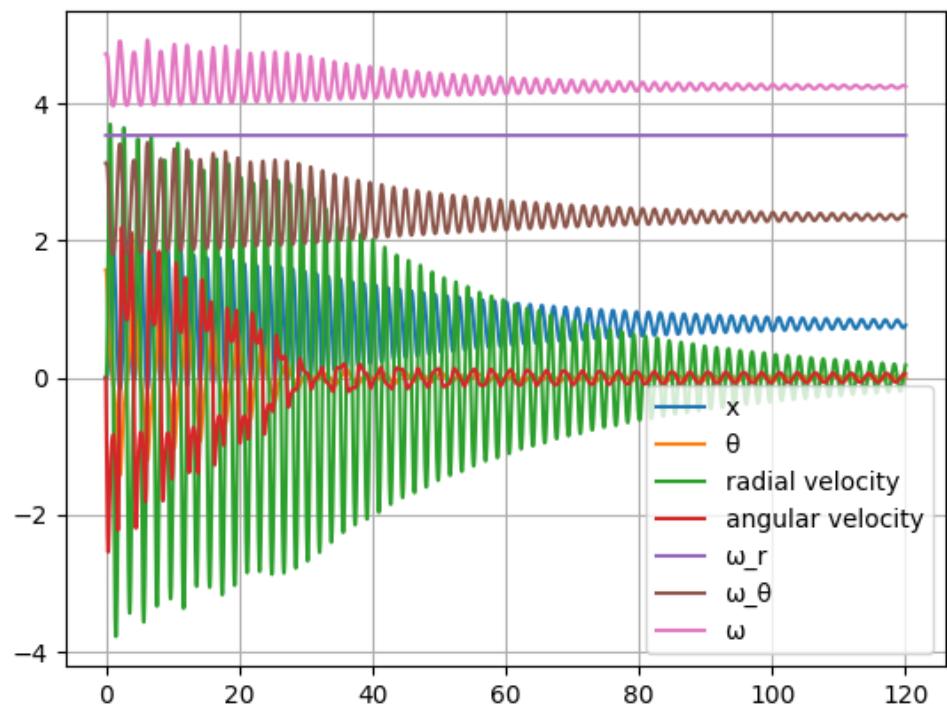


Figure 14.18: Case 8 where  $m = 8 \text{ kg}$

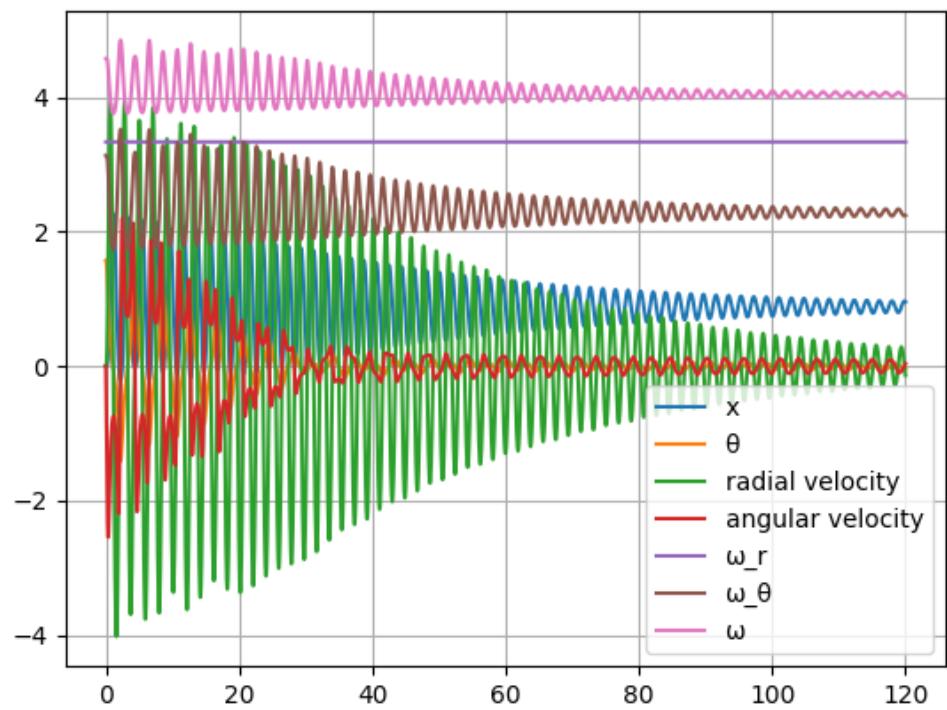


Figure 14.19: Case 9 where  $m = 9 \text{ kg}$

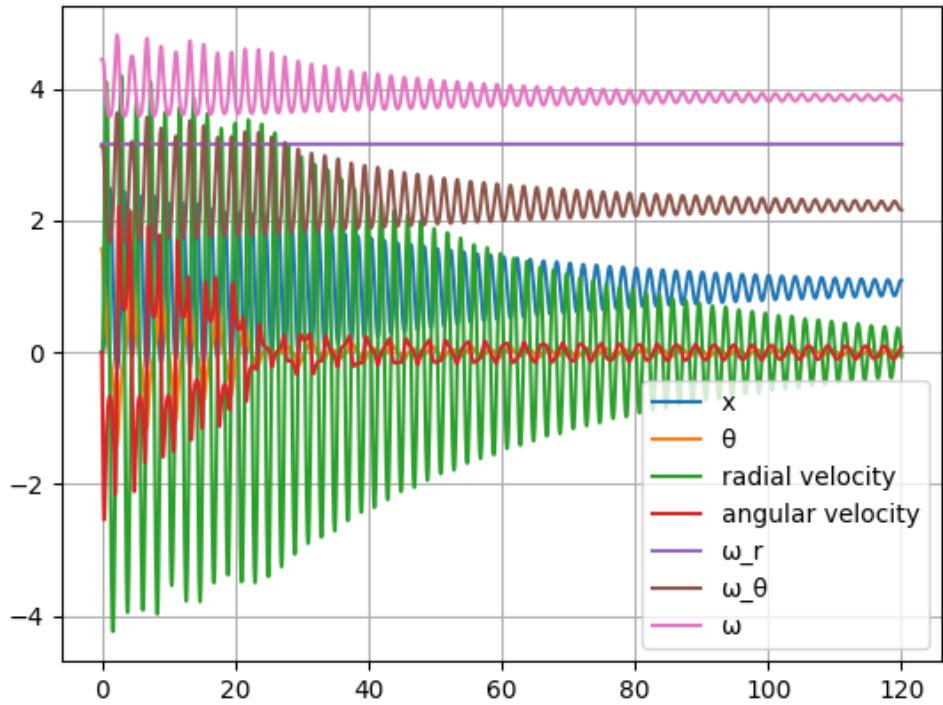


Figure 14.20: *Case 10 where  $m = 10 \text{ kg}$*

*Upon close visual observation, we see that the difference in the plotted values of that of the simulation and experimental values of the radial displacement, angular displacement, angular frequency and absolute frequency from the  $x-t$  graph are very minute to the extent that it would be right to say and consider that the experimental values are both accurate and precise in relation to that of the literature/theoretical/simulation values.*

*We also observe that, with increase in mass, the **amplitude** of the parameters employed in this investigation also increases while the **frequency** of the parameters employed stays approximately constant.*

## 14.2 Numerical Analysis

### 14.2.1 Radial Displacement versus Time

We initially defined and conditionalized the initial value of the radial displacement versus time to be equal to 0.

Due to viscous underdamped motion, it gains energy and momentum and preserves it, with gradual but steady loss int its energy state and is in motion for more than 300 seconds before coming to a complete halt. ie. no motion.

### 14.2.2 Angular Displacement versus Time

We initially defined and conditionalized the initial value of the angular displacement versus time to be equal to  $\pi/2$ .

Due to viscous underdamped motion, it gains energy and momentum and preserves it, with gradual but steady loss int its energy state and is in motion for more than 120 seconds before coming to a complete halt. ie. no motion.

### 14.2.3 Angular Frequency versus Time

We previously defined angular frequency ( $\omega_\theta$ ) to be as equation ??.

The initial state ( $\omega_\theta(0)$ ) that follows is,

$$\omega_\theta(0) = \sqrt{\frac{g}{l_0 + x(0)}} = \sqrt{\frac{g}{l_0}}$$

But we had previously mentioned that the magnitude of rest length of the spring would be 1 meter ( $l_0 = 1m$ ). Therefore we have,

$$\omega_\theta(0) = \sqrt{g} \approx 3.13209195$$

This parameter keeps fluctuating through the time domain as this is a function of radial displacement which in turn is a function of time.

This initial state defined and derived above is uniform irrespective of the mass employed

#### 14.2.4 Absolute Frequency versus Time

We previously defined absolute frequency ( $\omega_r$ ) to be as equation ??.

The initial state ( $\omega_r(0)$ ) that follows is,

$$\omega_r(0) = \sqrt{\frac{k}{m}}$$

This parameter does not fluctuate through the time domain as this is not a function of any arbitrary variable that would be in turn is a function of time.

This initial state defined and derived above is constant for any particular case with a steady constant mass  $m$ , throught a case

## 15 Evaluation

We further define,

$$\zeta_0 = \overline{\zeta_{0_n}} = \frac{\sum_{n=1}^n \zeta_{0_n}}{n} = \frac{\zeta_{0_1} + \zeta_{0_2} + \zeta_{0_3} + \zeta_{0_4} + \zeta_{0_5} + \zeta_{0_6} + \zeta_{0_7} + \zeta_{0_8} + \zeta_{0_9} + \zeta_{0_{10}}}{10}$$

$$\tau_0 = \overline{\tau_{0_n}} = \frac{\sum_{n=1}^n \tau_{0_n}}{n} = \frac{\tau_{0_1} + \tau_{0_2} + \tau_{0_3} + \tau_{0_4} + \tau_{0_5} + \tau_{0_6} + \tau_{0_7} + \tau_{0_8} + \tau_{0_9} + \tau_{0_{10}}}{10}$$

$$\chi_0 = \overline{\chi_{0_n}} = \frac{\sum_{n=1}^n \chi_{0_n}}{n} = \frac{\chi_{0_1} + \chi_{0_2} + \chi_{0_3} + \chi_{0_4} + \chi_{0_5} + \chi_{0_6} + \chi_{0_7} + \chi_{0_8} + \chi_{0_9} + \chi_{0_{10}}}{10}$$

$$\psi_0 = \overline{\psi_{0_n}} = \frac{\sum_{n=1}^n \psi_{0_n}}{n} = \frac{\psi_{0_1} + \psi_{0_2} + \psi_{0_3} + \psi_{0_4} + \psi_{0_5} + \psi_{0_6} + \psi_{0_7} + \psi_{0_8} + \psi_{0_9} + \psi_{0_{10}}}{10}$$

Therefore we have,

$$\zeta_0 = 1.0525068$$

$$\tau_0 = 0.997198$$

$$\chi_0 = 0.9960714$$

$$\psi_0 = 0.9982441$$

Upon observation, we see that the value of  $\zeta_0$ ,  $\tau_0$ ,  $\chi_0$  and  $\psi$  we have found is not equal to 1, but is relatively very close, so we can say that we have some errors in calculating the experimental values of radial displacement, angular displacement, angular frequency and absolute frequency respectively versus time.

**Percentage uncertainty** in measurement of radial displacement versus time is  $(1 - \zeta_0) \cdot 100\% = 5.25068\%$

**Percentage uncertainty** in measurement of angular displacement versus time is  $(1 - \tau_0) \cdot 100\% = 0.2802\%$

**Percentage uncertainty** in measurement of angular frequency versus time is  $(1 - \chi_0) \cdot 100\% = 0.39286\%$

**Percentage uncertainty** in measurement of absolute frequency versus time is  $(1 - \psi_0) \cdot 100\% = 0.17559\%$

## 16 Limitations of Study

There are various limitations in this study/investigation as we have placed forth, certain strict initial conditions and other conditions that make this system so constrained and disables us to expand our researching capability of this chaotic phenomenon. Conditions such as,

- Restricting the value of  $\theta(0)$  to  $\pi/2$
- Restricting the value of  $x(0)$  to 0
- Restricting the value of  $l_0$  to 1 m

- Restricting the value of the time domain from 0 to 20 seconds

## 17 Sources of Error

*There are various ways through which errors might have crept into our raw and ordered data, some of the possible sources of errors are:*

1. Raw experimental data presented here in the investigation report is collected through lab experimentation, and there are chances that the data collected may have slight discrepancy in it
2. Insignificant random human errors by the observer, ie. parallax errors
3. Uncertainties that cannot be minimized due to lack of highly sophisticated equipment and materials used in the experiments in this investigation
4. Assumptions and certain conditions put forth on the system to model its chaotic behavior

## 18 Conclusion

*In this paper, I have shown the effects of changes in the **mass** in an **elastic pendulum** oscillating in a fluid medium (Air(Damped motion)) on the **radial displacement**, **angular displacement**, **angular frequency** and **absolute frequency** by collecting raw data relating to the above parameters, under certain controlled conditions. Also I have shown the possible **uncertainties in measurement** of the **radial displacement**, **angular displacement**, **angular frequency** and **absolute frequency**.*

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# A *Python Source Codes*

## A.1 *Graphical Solution Codes*

```
1 import numpy as np
2 from numpy.linalg import inv
3 from matplotlib import pyplot as plt
4 from math import cos, sin, tan, pi , sqrt
5
6 def G(y,t):
7
8     x_d, theta_d, x, theta = y[0], y[1], y[2], y[3]
9
10    x_dd = (10+x) * theta_d**2 - k/m * x + g * cos(theta)
11    theta_dd = -2.0/(10+x) * x_d * theta_d - g/(10+x) * sin(theta)
12
13    return np.array([x_dd, theta_dd, x_d, theta_d])
14
15 def RK4_step(y, t, dt):
16     k1 = G(y,t)
17     k2 = G(y+0.5*k1*dt, t+0.5*dt)
18     k3 = G(y+0.5*k2*dt, t+0.5*dt)
19     k4 = G(y+k3*dt, t+dt)
20
21     return dt * (k1 + 2*k2 + 2*k3 + k4) / 6
22
23 # variables
24
25 m = 2.0
26 r = 0.0
27 l0 = 1.0
28 g = 9.81
29 k = 100.0
30
31 delta_t = 1.0e-4
32 time = np.arange(0.0, 10.0, delta_t)
33
```

```

34 # initial state
35
36 y = np.array([0, 0.0, 0.0 , pi/2,])    # [velocity,
37     displacement] [x, θ]
38 Y1 = []
39 Y2 = []
40 Y3 = []
41 Y4 = []
42 Y5 = []
43 Y6 = []
44 A1 = []
45
46 # time-stepping solution
47
48 for t in time:
49
50     y = y + RK4_step(y, t, delta_t)
51
52     Y1.append(y[2])
53     Y2.append(y[3])
54     Y3.append(y[0])
55     Y4.append(y[1])
56     Y5.append(sqrt(k/m))
57     Y6.append(sqrt(g/(l0+y[2])))
58
59 # additional variables
60
61 # plotting the result
62
63 plt.plot(time,Y1)
64 plt.plot(time,Y2)
65 plt.plot(time,Y3)
66 plt.plot(time,Y4)
67 plt.plot(time,Y6)
68
69 plt.grid(True)
70 plt.legend(['x', 'θ', 'radial velocity', 'angular velocity',
71             'ω_r'], loc='lower right')
72 plt.show()

```

Listing A.1: *Graphical Solution Python Script for system **without** Damping*

```

1 import numpy as np
2 from numpy.linalg import inv
3 from matplotlib import pyplot as plt
4 from math import cos, sin, tan, pi , sqrt
5
6 def G(y,t):
7
8     x_d , theta_d , x , theta = y[0] , y[1] , y[2] , y[3]
9
10    x_dd = (10+x) * theta_d**2 - k/m * x + g * cos(theta) - C/m * x_d
11    theta_dd = -2.0/(10+x) * x_d * theta_d - g/(10+x) * sin(theta) - C/(m *
12        (10+x)**2) * theta_d
13
14    return np.array([x_dd , theta_dd , x_d , theta_d])
15
16 def RK4_step(y, t, dt):
17     k1 = G(y,t)
18     k2 = G(y+0.5*k1*dt, t+0.5*dt)
19     k3 = G(y+0.5*k2*dt, t+0.5*dt)
20     k4 = G(y+k3*dt, t+dt)
21
22     return dt * (k1 + 2*k2 + 2*k3 + k4) / 6
23
24 # variables
25 m = 2.0
26 r = 0.0
27 l0 = 1.0
28 g = 9.81
29 k = 100.0
30 C = 0.47
31
32 delta_t = 1.0e-4
33 time = np.arange(0.0, 10.0, delta_t)
34
35 # initial state
36
37 y = np.array([0, 0.0, 0.0 , pi/2,]) # [velocity,
38      displacement] [x, theta]
39 Y1 = []
40 Y2 = []
41 Y3 = []
42 Y4 = []
43 Y5 = []
44 Y6 = []
45 A1 = []
46
47 # time-stepping solution

```

```

48
49 for t in time:
50
51     y = y + RK4_step(y, t, delta_t)
52
53     Y1.append(y[2])
54     Y2.append(y[3])
55     Y3.append(y[0])
56     Y4.append(y[1])
57     Y5.append(sqrt(k/m))
58     Y6.append(sqrt(g/(l0+y[2])))

59
60 # additional variables
61
62 # plotting the result
63
64 plt.plot(time,Y1)
65 plt.plot(time,Y2)
66 plt.plot(time,Y3)
67 plt.plot(time,Y4)
68 plt.plot(time,Y6)
69
70 plt.grid(True)
71 plt.legend(['x', 'θ', 'radial velocity', 'angular velocity',
72             'ω_r'], loc='lower right')
72 plt.show()

```

Listing A.2: *Graphical Solution Python Script for system **with** Damping*

## A.2 Path Animation Codes

```
1 import pygame
2 import sys
3 from pygame.locals import *
4 from math import sin, cos, tan, pi
5 import numpy as np
6 from numpy.linalg import inv
7 from spring import spring
8
9 class Spring():
10     def __init__(self, color, start, end, nodes, width, lead1,
11                  lead2):
12         self.start = start
13         self.end = end
14         self.nodes = nodes
15         self.width = width
16         self.lead1 = lead1
17         self.lead2 = lead2
18         self.weight = 3
19         self.color = color
20
21     def update(self, start, end):
22         self.start = start
23         self.end = end
24
25         self.x, self.y, self.p1, self.p2 = spring(self.start,
26                                                self.end, self.nodes, self.width, self.lead1, self.lead2)
27         self.p1 = (int(self.p1[0]), int(self.p1[1]))
28         self.p2 = (int(self.p2[0]), int(self.p2[1]))
29
30     def render(self):
31         pygame.draw.line(screen, self.color, self.start, self.p1,
32                           self.weight)
33         prev_point = self.p1
34         for point in zip(self.x, self.y):
35             pygame.draw.line(screen, self.color, prev_point, point,
36                               self.weight)
37             prev_point = point
38         pygame.draw.line(screen, self.color, self.p2, self.end,
39                           self.weight)
40
41     def G(y,t):
42
43         x_d, theta_d, x, theta = y[0], y[1], y[2], y[3]
44
45         x_dd = (10+x) * theta_d**2 - k/m * x + g * cos(theta)
46         theta_dd = -2.0/(10+x) * x_d * theta_d - g/(10+x) * sin(theta)
```

```

42
43     return np.array([x_dd, theta_dd, x_d, theta_d])
44
45 def RK4_step(y, t, dt):
46     k1 = G(y,t)
47     k2 = G(y+0.5*k1*dt, t+0.5*dt)
48     k3 = G(y+0.5*k2*dt, t+0.5*dt)
49     k4 = G(y+k3*dt, t+dt)
50
51     return dt * (k1 + 2*k2 + 2*k3 + k4) / 6
52
53 def update(x, theta):
54     x_coord = scale*(10+x) * sin(theta) + offset[0]
55     y_coord = scale*(10+x) * cos(theta) + offset[1]
56
57     return (int(x_coord), int(y_coord))
58
59 def render(point):
60     x, y = point[0], point[1]
61
62     if prev_point:
63         pygame.draw.line(trace, LT_BLUE, prev_point, (x, y), 5)
64
65     screen.fill(WHITE)
66     if is_tracing:
67         screen.blit(trace, (0,0))
68
69     #pygame.draw.line(screen, BLACK, offset, (x,y), 5)
70     s.update(offset, point)
71     s.render()
72     pygame.draw.circle(screen, BLACK, offset, 8)
73     pygame.draw.circle(screen, BLUE, (x, y), int(m*10))
74
75     return (x, y)
76
77 w, h = 1024, 768
78 WHITE = (255,255,255)
79 BLACK = (0,0,0)
80 RED = (255,0,0)
81 BLUE = (0,0,255)
82 LT_BLUE = (230,230,255)
83 offset = (w//2, h//4)
84 scale = 100
85 is_tracing = True
86
87 screen = pygame.display.set_mode((w,h))
88 screen.fill(WHITE)
89 trace = screen.copy()
90 pygame.display.update()

```

```

91  clock = pygame.time.Clock()
92
93 # parameters
94 m = 6.0
95 l0 = 3.5
96 g = 9.81
97 k = 100.0
98
99 prev_point = None
100 t = 0.0
101 delta_t = 0.02
102 y = np.array([0.0, 0.0, 0.0, pi/2])
103
104 pygame.font.init()
105 myfont = pygame.font.SysFont('Comic Sans MS', 38)
106
107 s = Spring(BLACK, (0,0), (0,0), 25, 30, 90, 90)
108
109 while True:
110     for event in pygame.event.get():
111         if event.type == pygame.QUIT:
112             sys.exit()
113
114         if event.type == KEYDOWN:
115             if event.key == K_t:
116                 is_tracing = not(is_tracing)
117             if event.key == K_c:
118                 trace.fill(WHITE)
119
120         point = update(y[2], y[3])
121         prev_point = render(point)
122
123         time_string = 'Time: {} seconds'.format(round(t,1))
124         text = myfont.render(time_string, False, (0, 0, 0))
125         screen.blit(text, (10,10))
126
127         t += delta_t
128         y = y + RK4_step(y, t, delta_t)
129
130         clock.tick(60)
131         pygame.display.update()

```

Listing A.3: *Path Animation Python Script for system **without** Damping*



```

43     return np.array([x_dd, theta_dd, x_d, theta_d])
44
45 def RK4_step(y, t, dt):
46     k1 = G(y, t)
47     k2 = G(y+0.5*k1*dt, t+0.5*dt)
48     k3 = G(y+0.5*k2*dt, t+0.5*dt)
49     k4 = G(y+k3*dt, t+dt)
50
51     return dt * (k1 + 2*k2 + 2*k3 + k4) / 6
52
53 def update(x, theta):
54     x_coord = scale*(10+x) * sin(theta) + offset[0]
55     y_coord = scale*(10+x) * cos(theta) + offset[1]
56
57     return (int(x_coord), int(y_coord))
58
59 def render(point):
60     x, y = point[0], point[1]
61
62     if prev_point:
63         pygame.draw.line(trace, LT_BLUE, prev_point, (x, y), 5)
64
65     screen.fill(WHITE)
66     if is_tracing:
67         screen.blit(trace, (0,0))
68
69     #pygame.draw.line(screen, BLACK, offset, (x,y), 5)
70     s.update(offset, point)
71     s.render()
72     pygame.draw.circle(screen, BLACK, offset, 8)
73     pygame.draw.circle(screen, BLUE, (x, y), int(m*10))
74
75     return (x, y)
76
77 w, h = 1024, 768
78 WHITE = (255,255,255)
79 BLACK = (0,0,0)
80 RED = (255,0,0)
81 BLUE = (0,0,255)
82 LT_BLUE = (230,230,255)
83 offset = (w//2, h//4)
84 scale = 100
85 is_tracing = True
86
87 screen = pygame.display.set_mode((w,h))
88 screen.fill(WHITE)
89 trace = screen.copy()
90 pygame.display.update()
91 clock = pygame.time.Clock()

```

```

92
93 # parameters
94 m = 6.0
95 l0 = 3.5
96 g = 9.81
97 k = 100.0
98 C = 0.47
99
100 prev_point = None
101 t = 0.0
102 delta_t = 0.02
103 y = np.array([0.0, 0.0, 0.0, pi/2])
104
105 pygame.font.init()
106 myfont = pygame.font.SysFont('Comic Sans MS', 38)
107
108 s = Spring(BLACK, (0,0), (0,0), 25, 30, 90, 90)
109
110 while True:
111     for event in pygame.event.get():
112         if event.type == pygame.QUIT:
113             sys.exit()
114
115         if event.type == KEYDOWN:
116             if event.key == K_t:
117                 is_tracing = not(is_tracing)
118             if event.key == K_c:
119                 trace.fill(WHITE)
120
121     point = update(y[2], y[3])
122     prev_point = render(point)
123
124     time_string = 'Time: {} seconds'.format(round(t,1))
125     text = myfont.render(time_string, False, (0, 0, 0))
126     screen.blit(text, (10,10))
127
128     t += delta_t
129     y = y + RK4_step(y, t, delta_t)
130
131     clock.tick(60)
132     pygame.display.update()

```

Listing A.4: *Path Animation Python Script for system **with** Damping*

## B *Raw Experimental Data*

### B.1 *Radial Displacement versus Time*

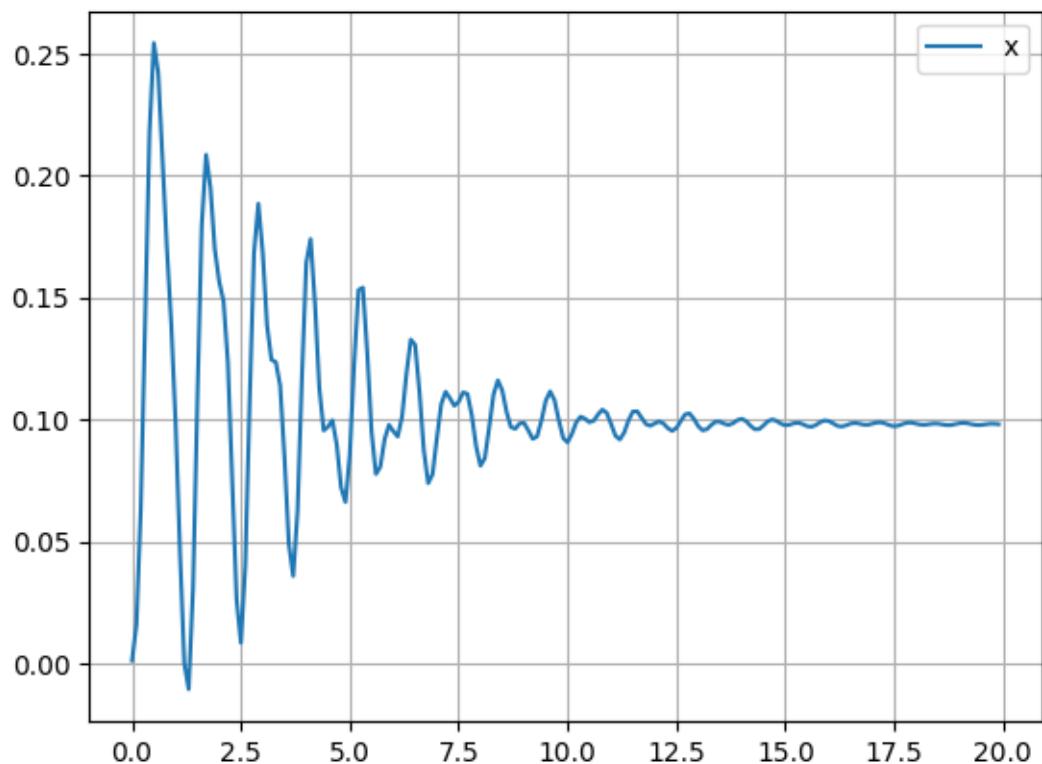


Figure B.1: *Case 1 where  $m = 1 \text{ kg}$*

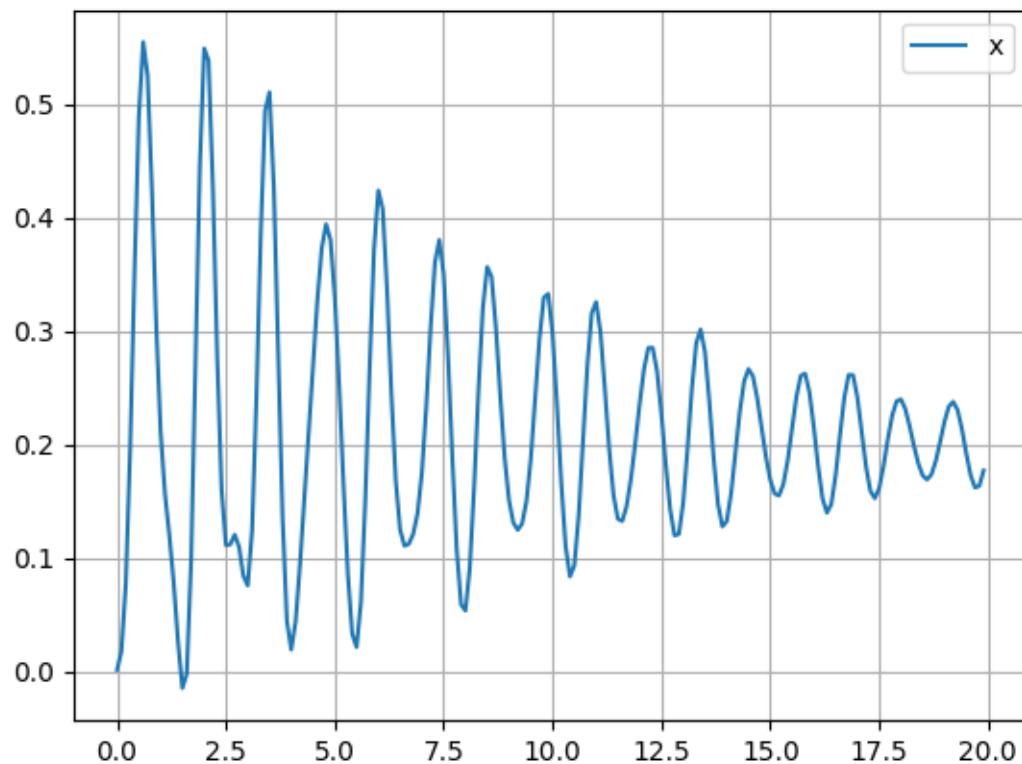


Figure B.2: Case 2 where  $m = 2 \text{ kg}$

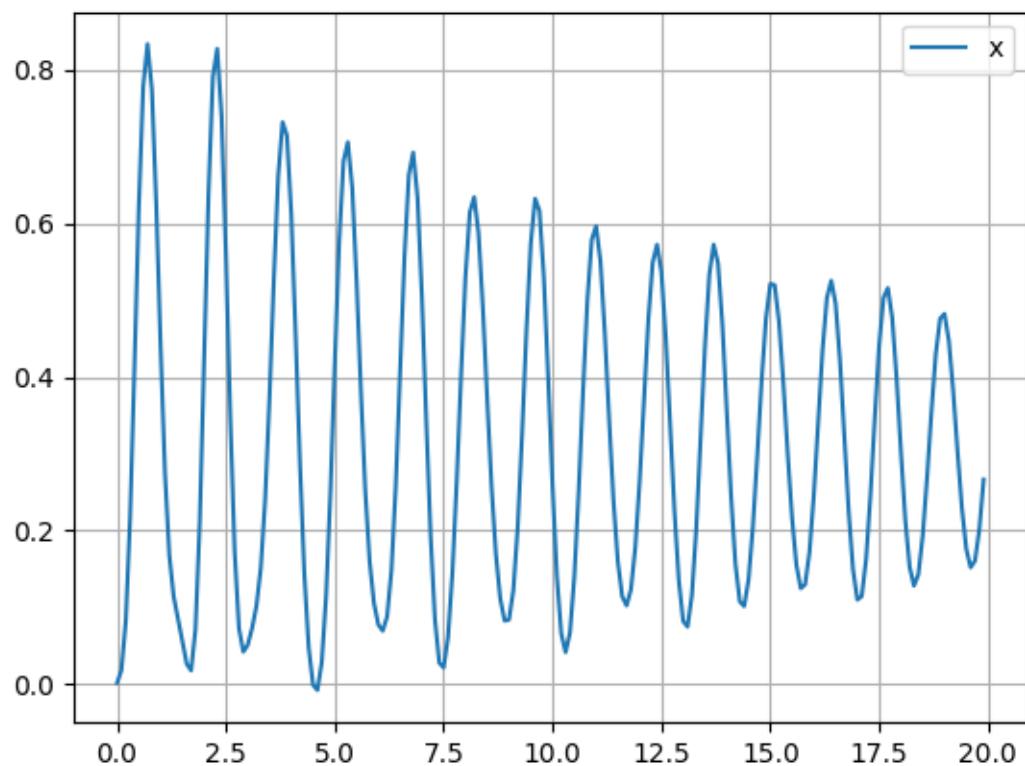


Figure B.3: Case 3 where  $m = 3 \text{ kg}$

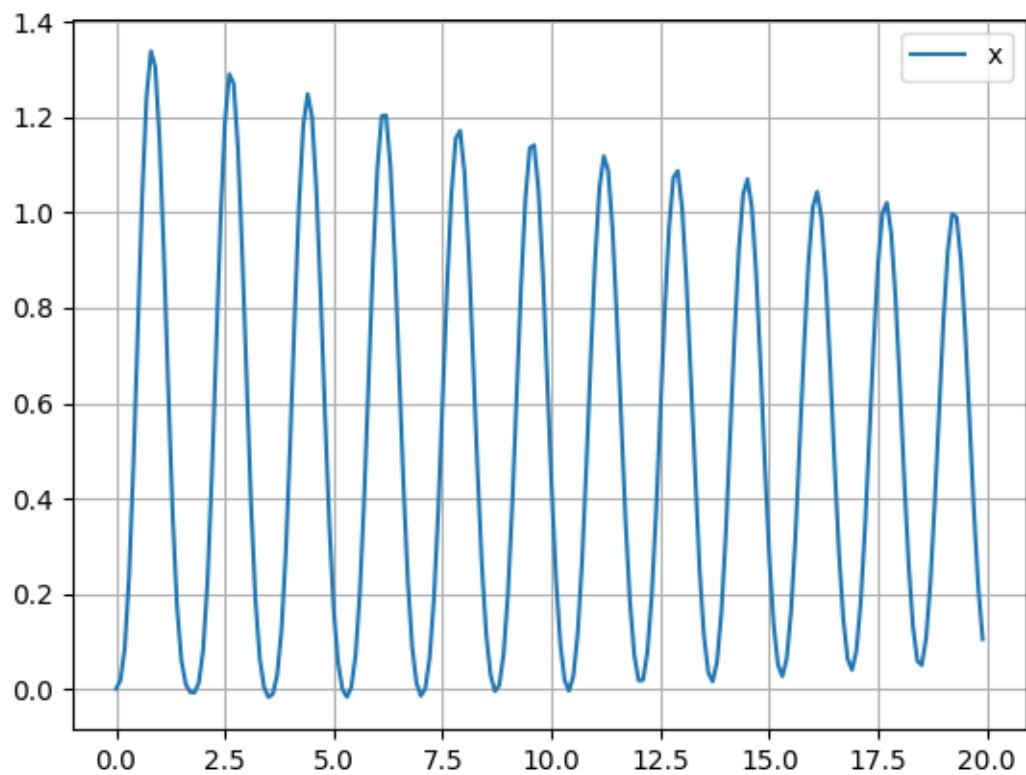


Figure B.4: Case 4 where  $m = 4 \text{ kg}$

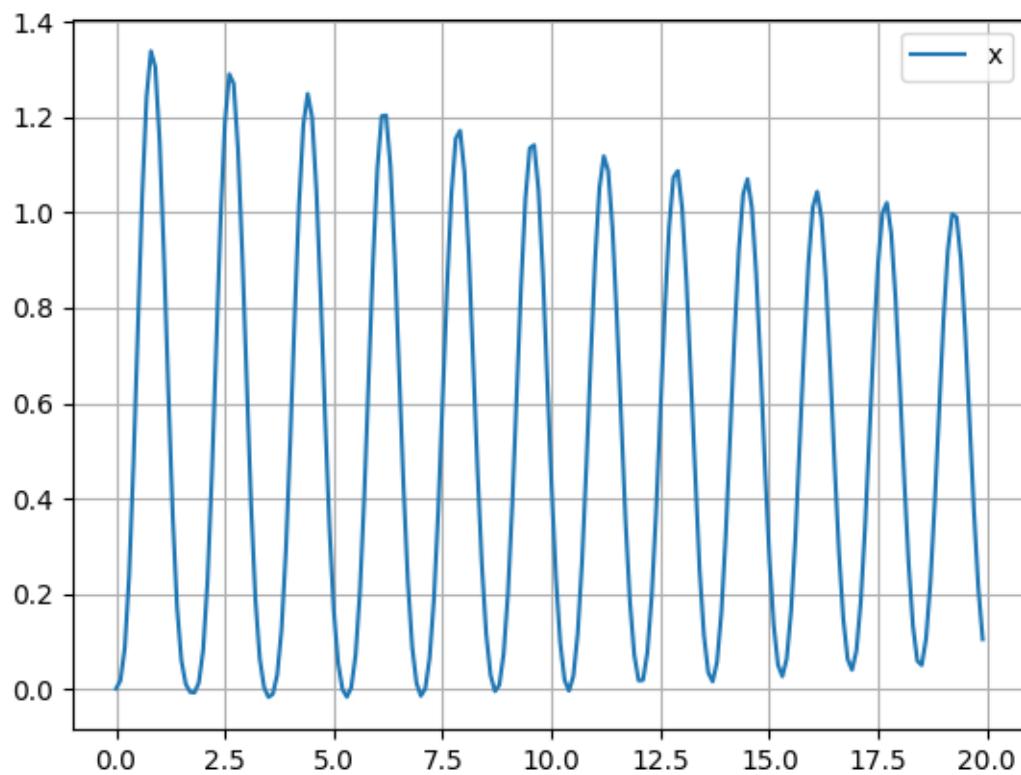


Figure B.5: Case 5 where  $m = 5 \text{ kg}$

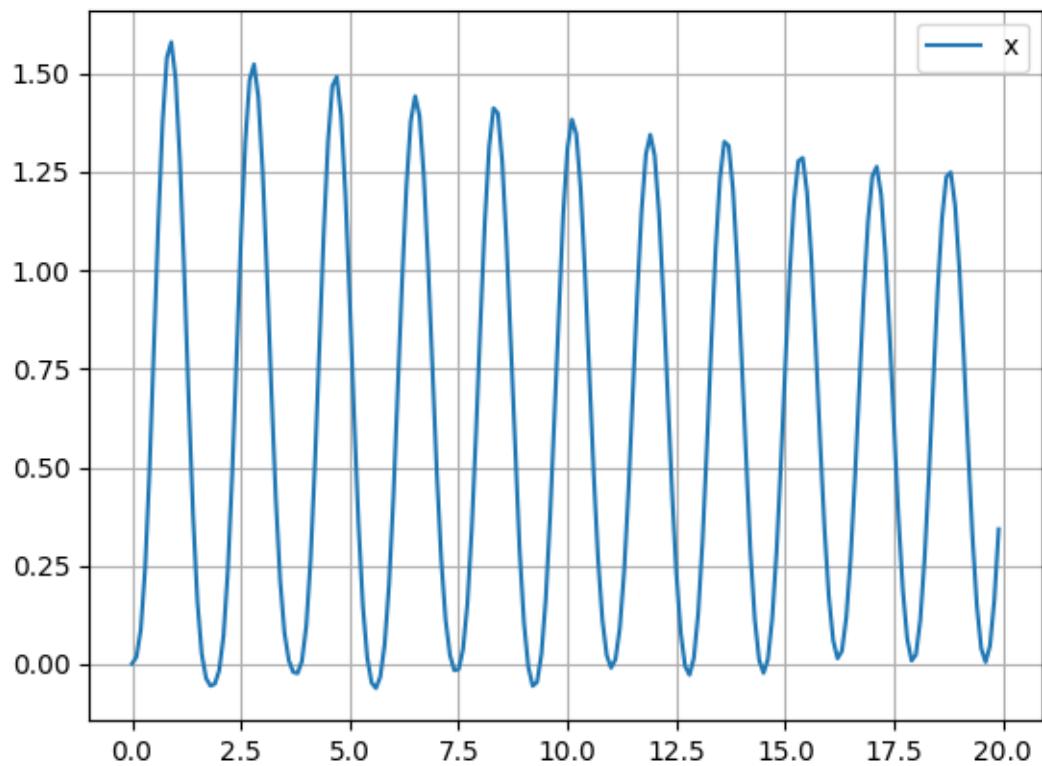


Figure B.6: Case 6 where  $m = 6 \text{ kg}$

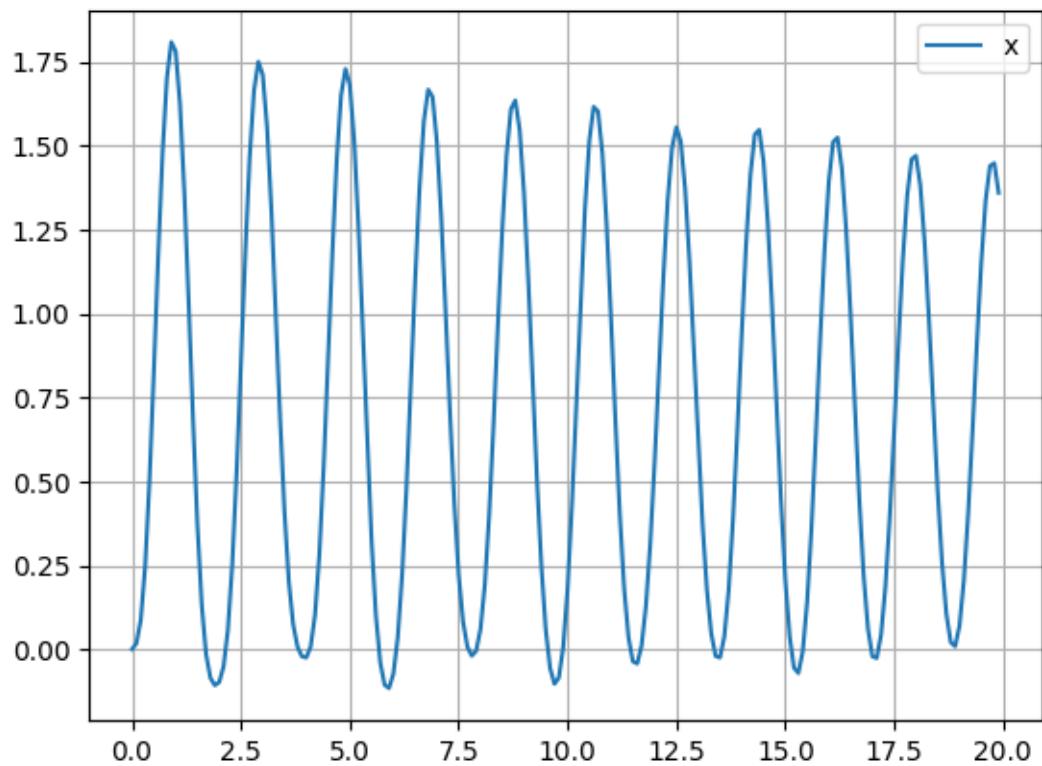


Figure B.7: Case 7 where  $m = 7 \text{ kg}$

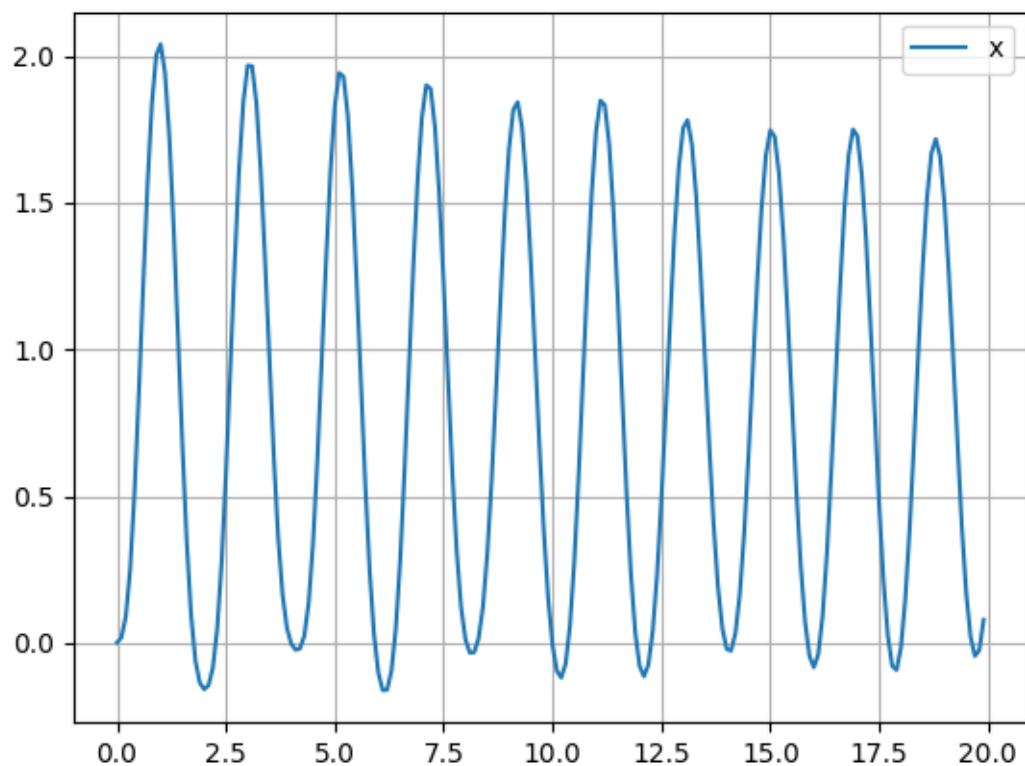


Figure B.8: Case 8 where  $m = 8 \text{ kg}$

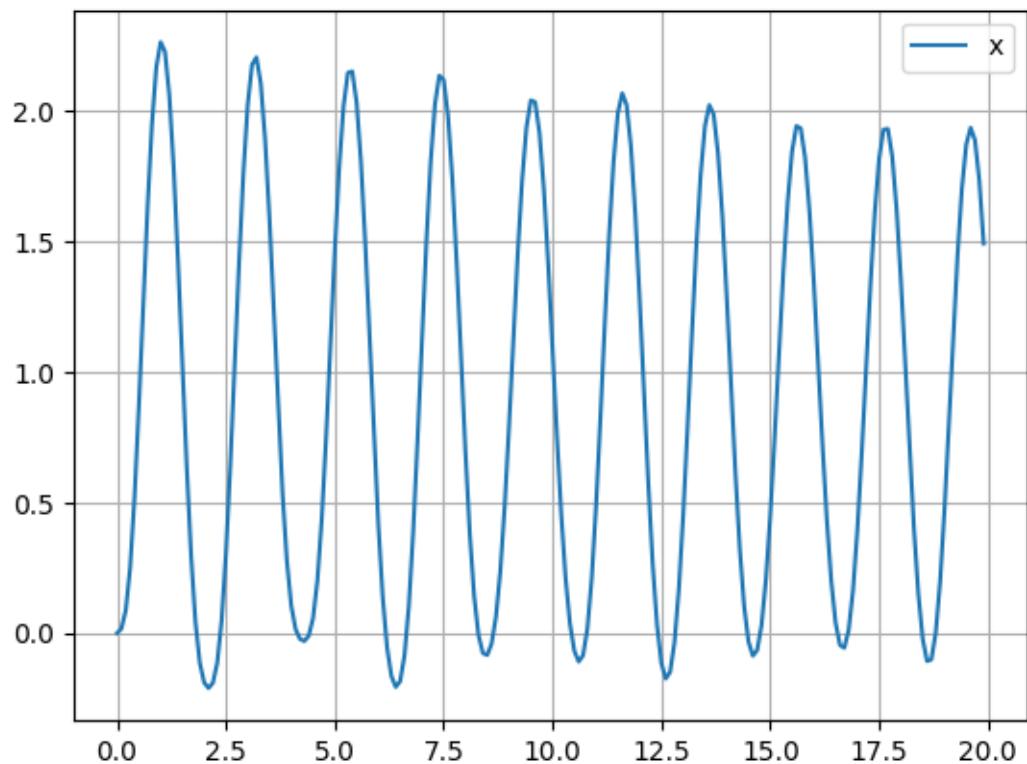


Figure B.9: *Case 9 where  $m = 9 \text{ kg}$*

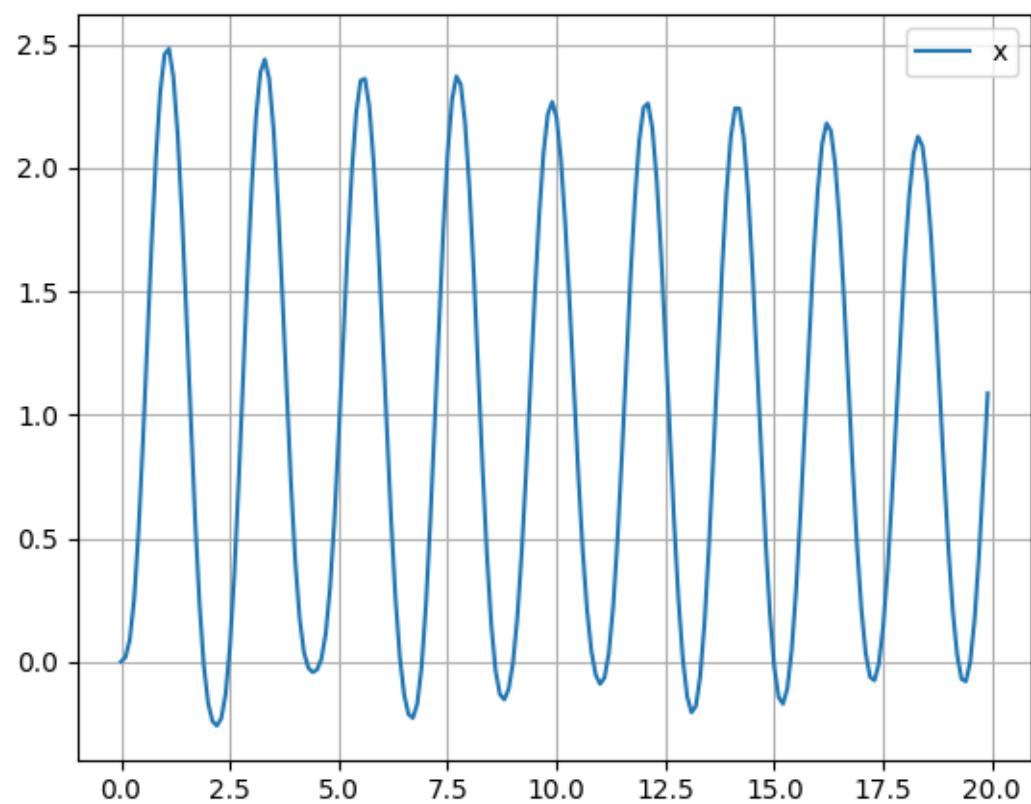


Figure B.10: *Case 10 where  $m = 10 \text{ kg}$*

## B.2 *Angular Displacement versus Time*

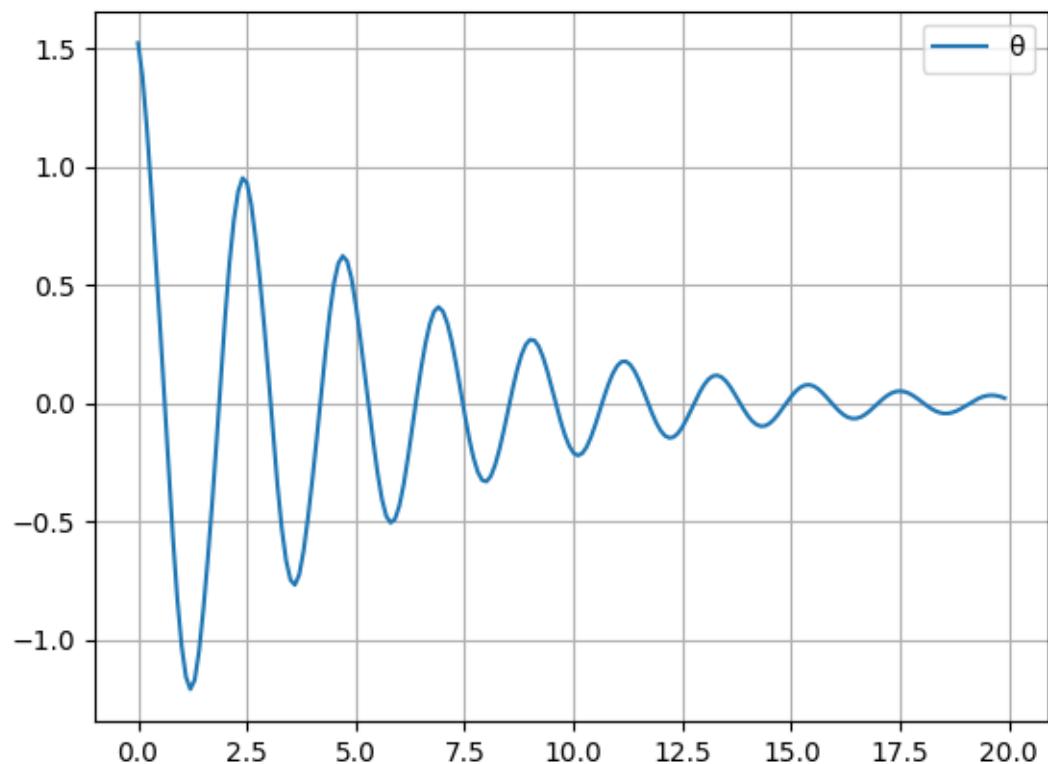


Figure B.11: *Case 1 where  $m = 1 \text{ kg}$*

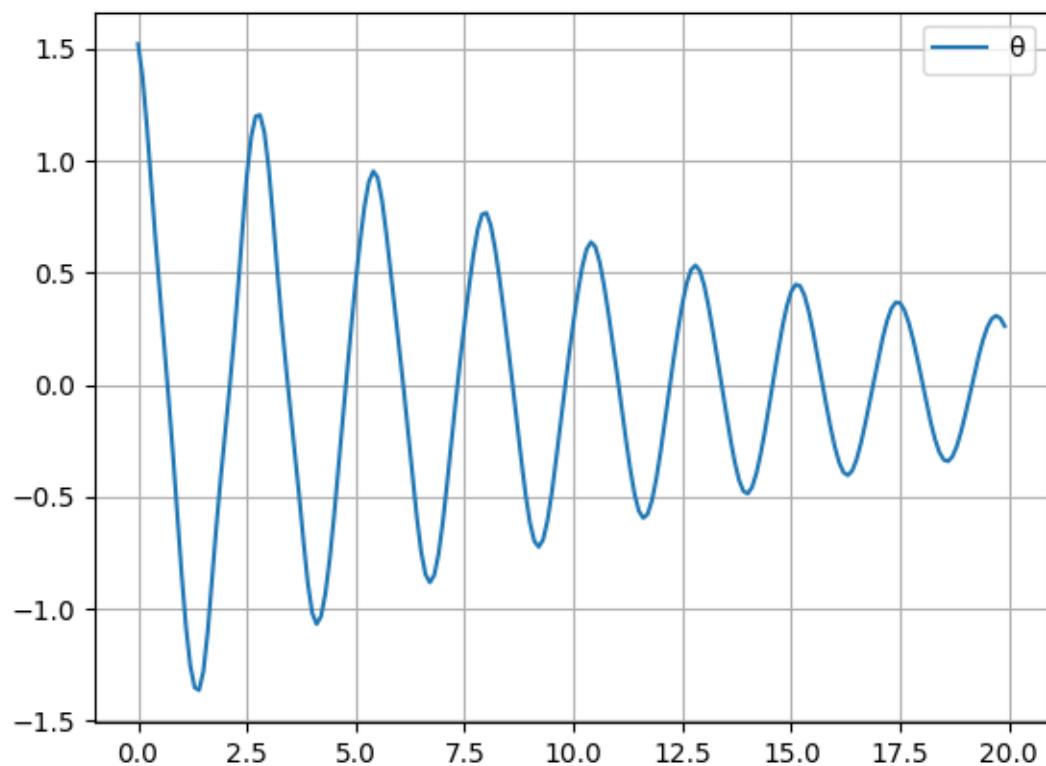


Figure B.12: *Case 2 where  $m = 2 \text{ kg}$*

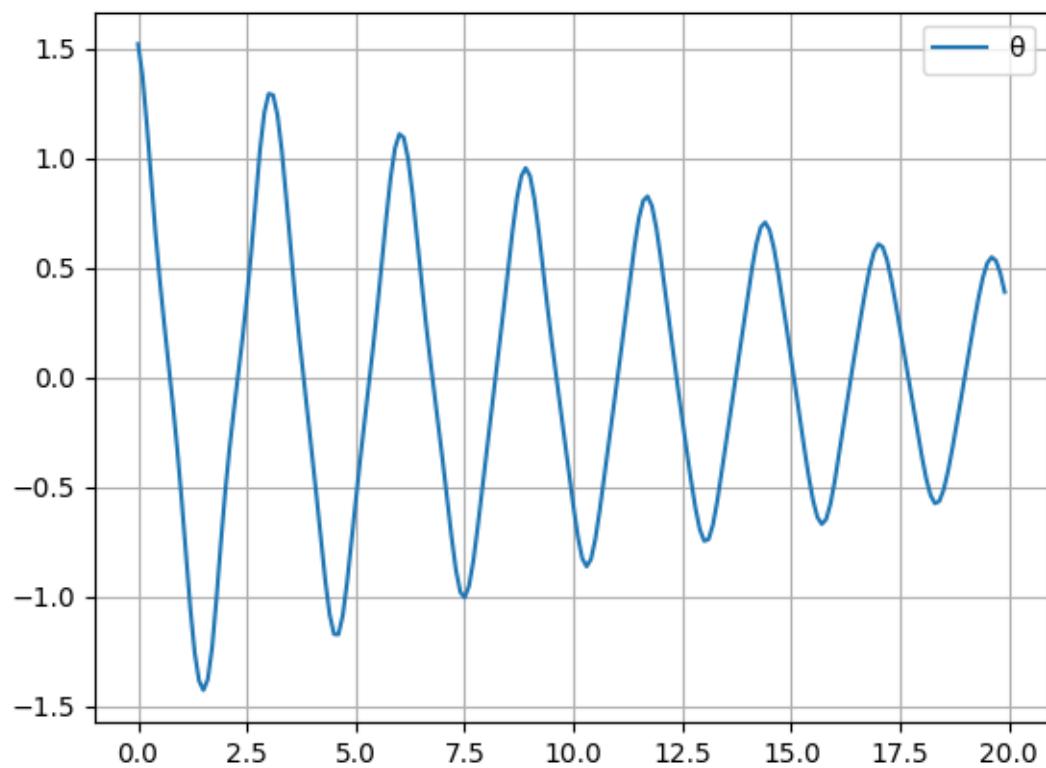


Figure B.13: *Case 3 where  $m = 3 \text{ kg}$*

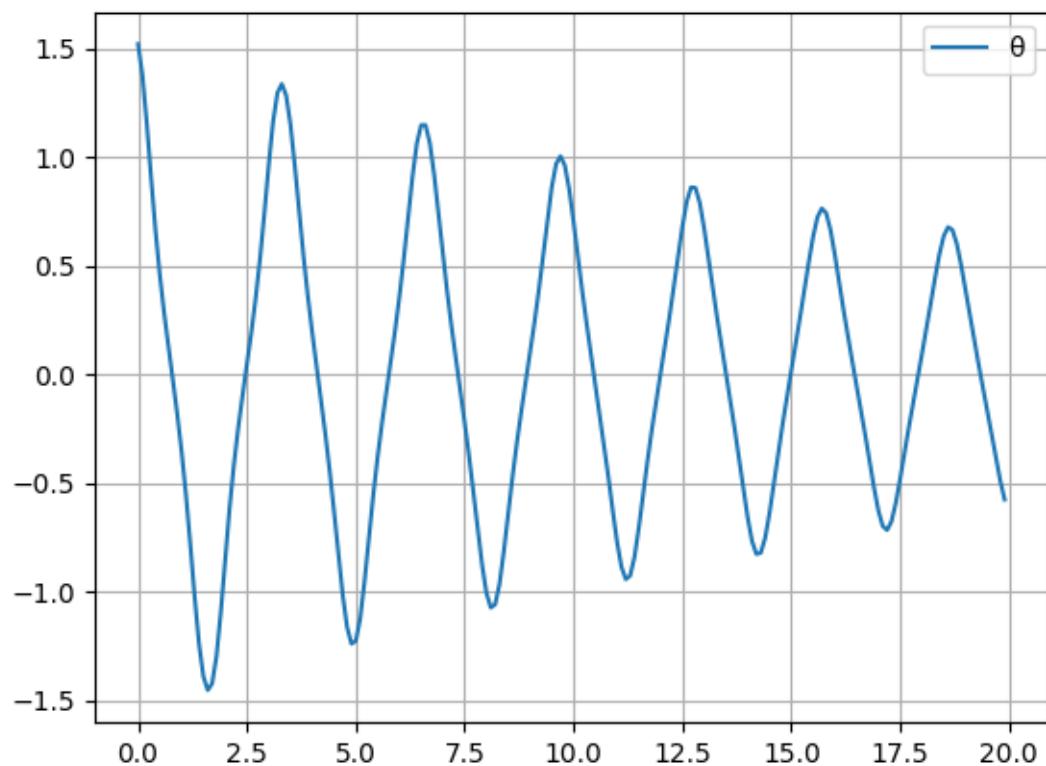


Figure B.14: *Case 4 where  $m = 4 \text{ kg}$*

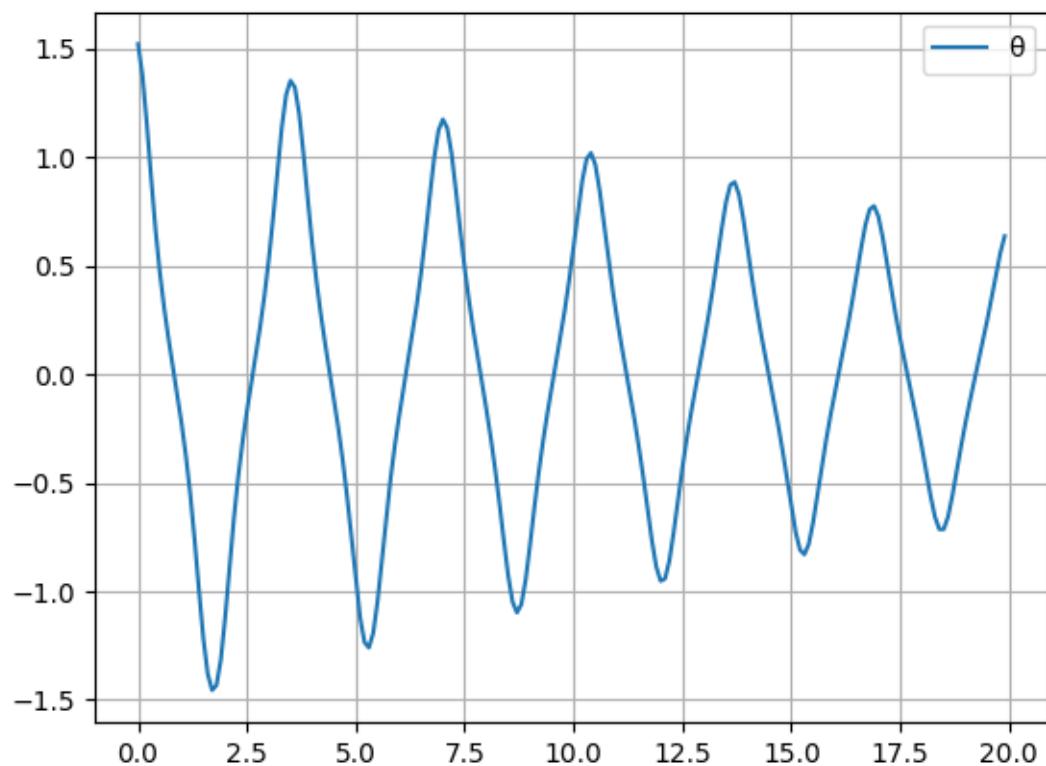


Figure B.15: *Case 5 where  $m = 5 \text{ kg}$*

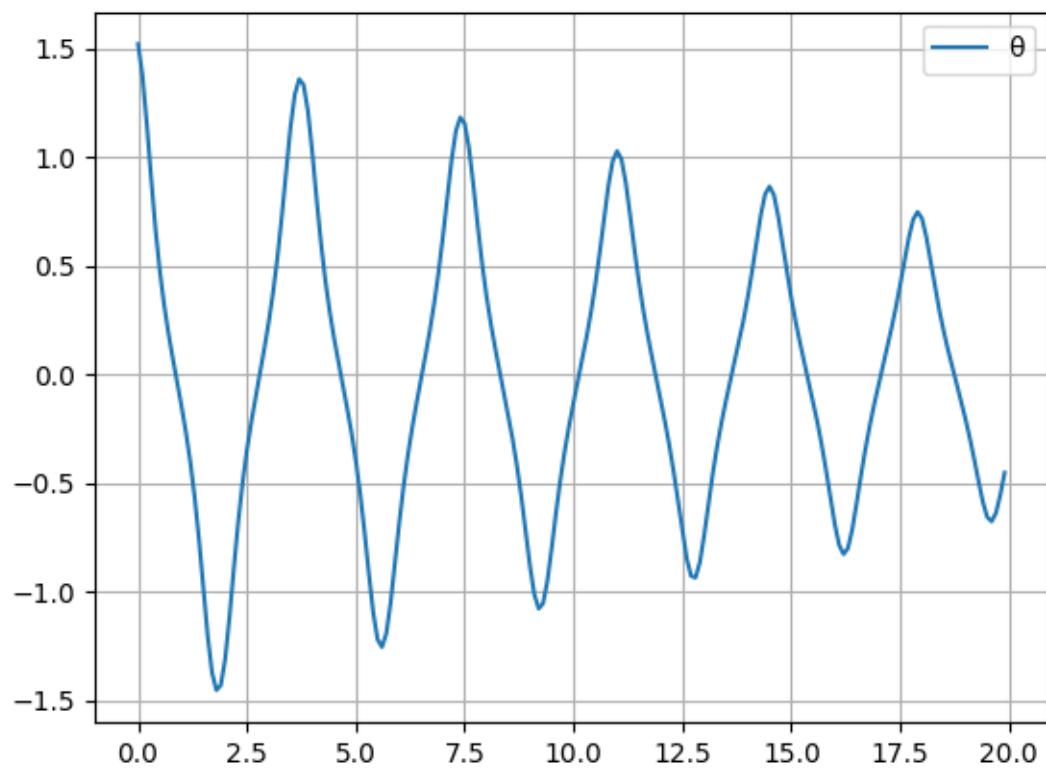


Figure B.16: *Case 6 where  $m = 6 \text{ kg}$*

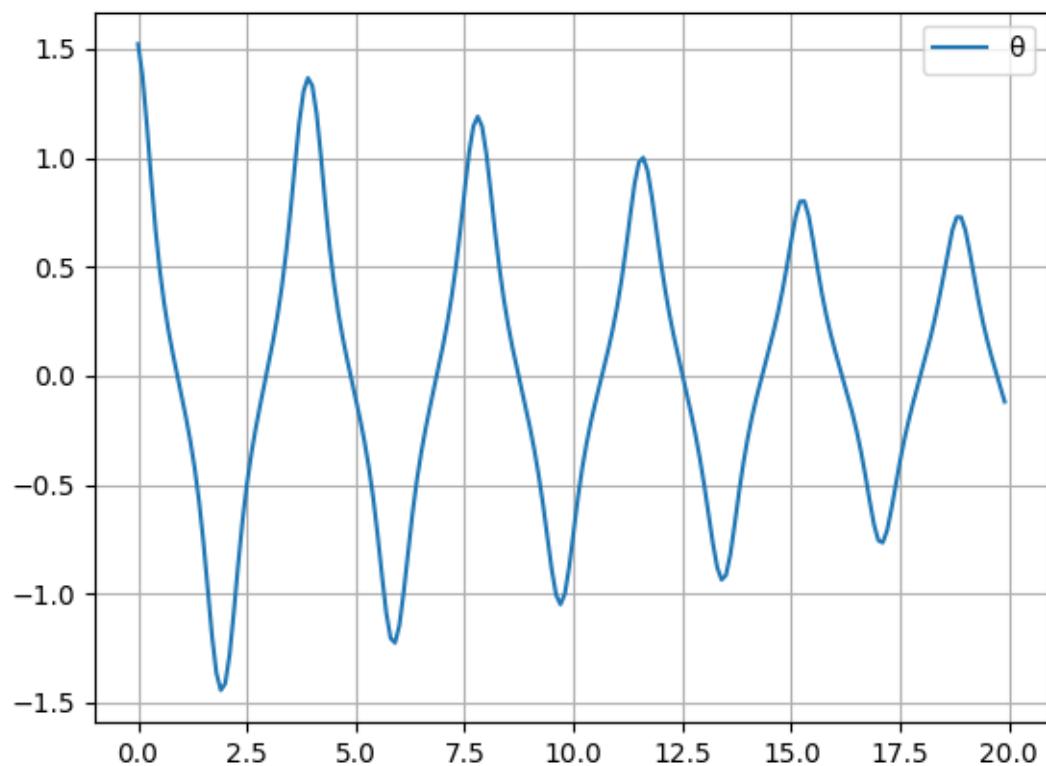


Figure B.17: Case 7 where  $m = 7 \text{ kg}$

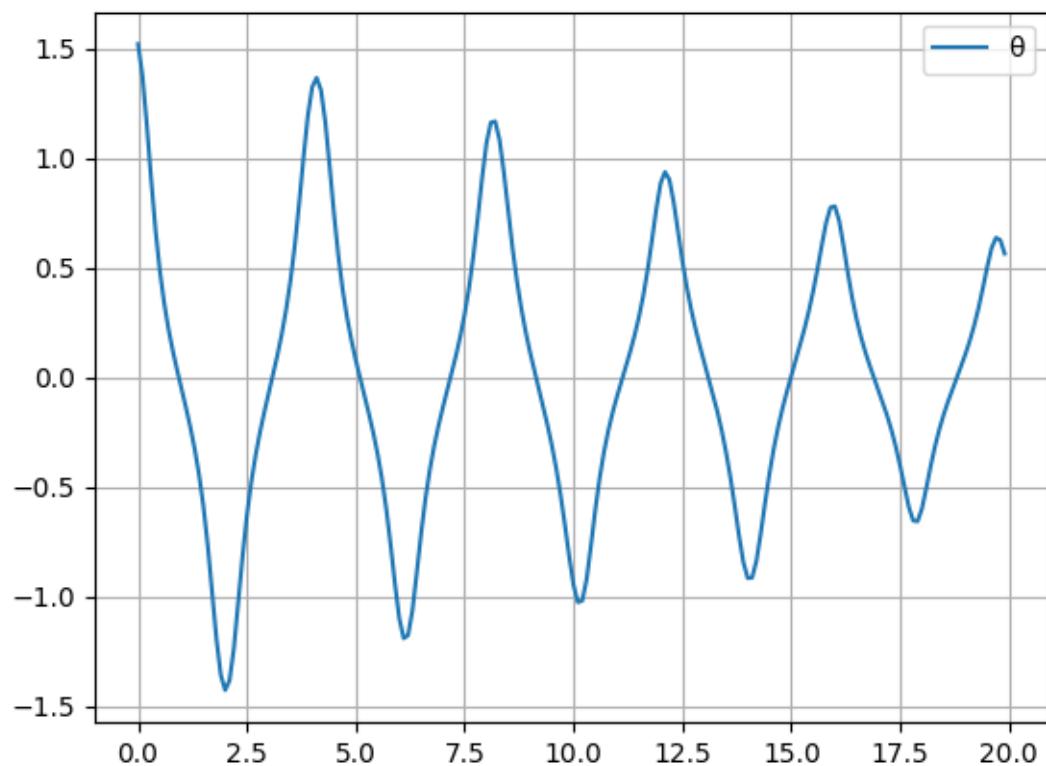


Figure B.18: *Case 8 where  $m = 8 \text{ kg}$*

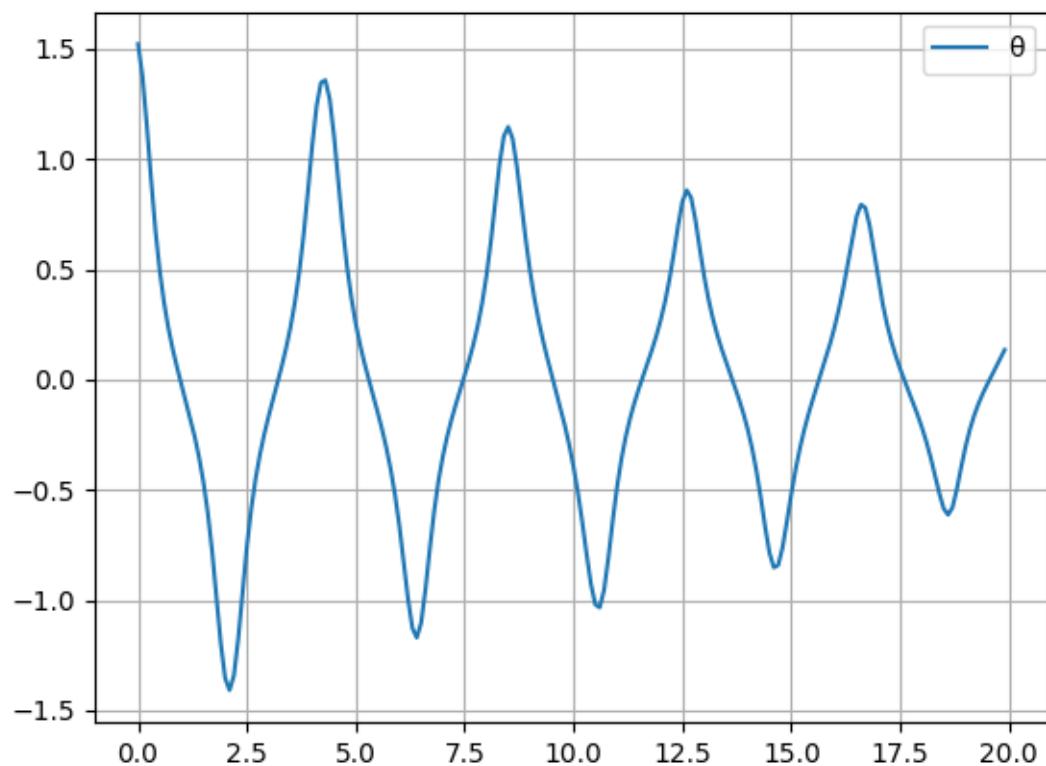


Figure B.19: *Case 9 where  $m = 9 \text{ kg}$*

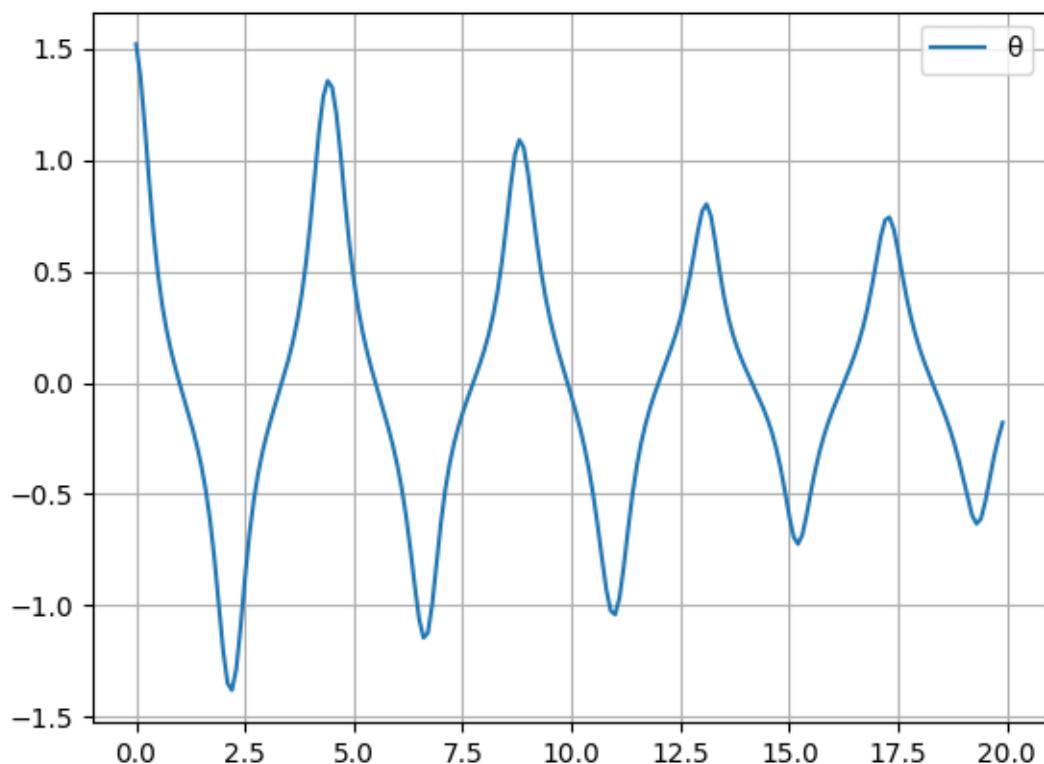


Figure B.20: *Case 10 where  $m = 10 \text{ kg}$*

### B.3 Angular Frequency versus Time

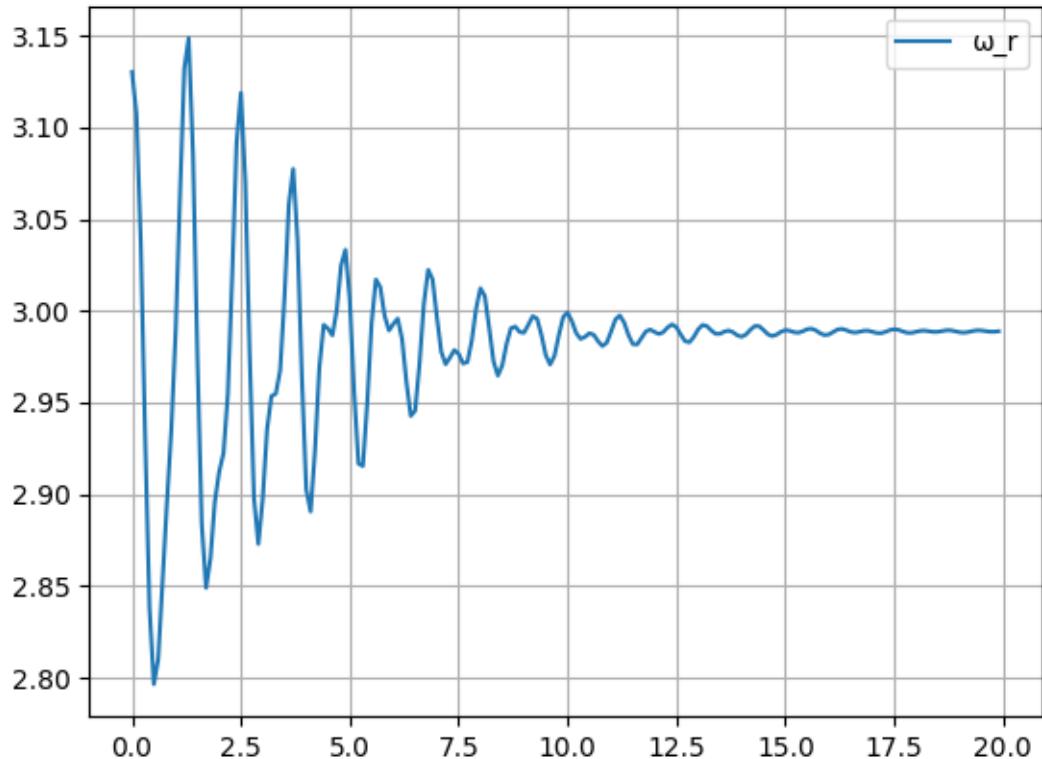


Figure B.21: *Case 1 where  $m = 1 \text{ kg}$*

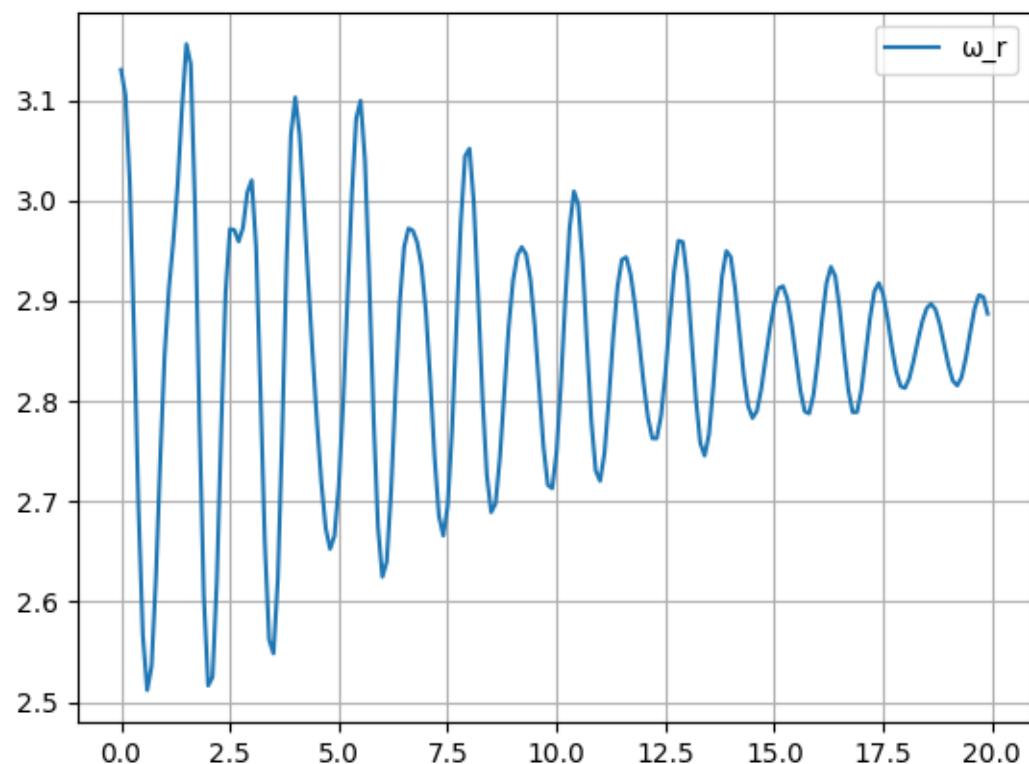


Figure B.22: *Case 2 where  $m = 2 \text{ kg}$*

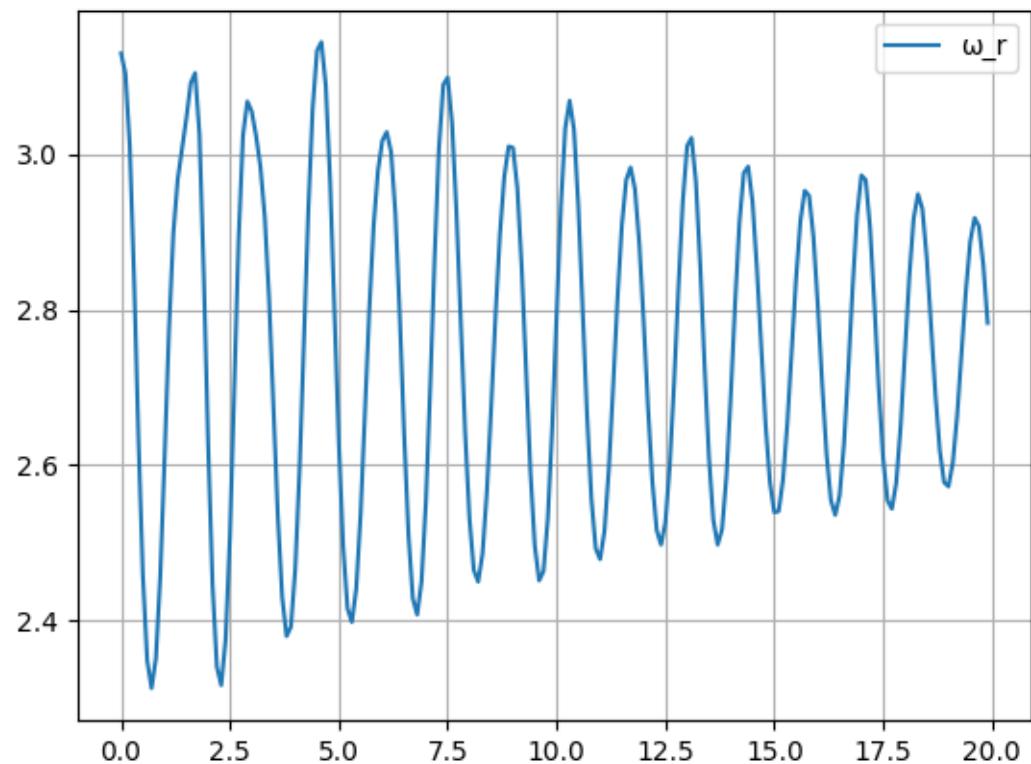


Figure B.23: *Case 3 where  $m = 3 \text{ kg}$*

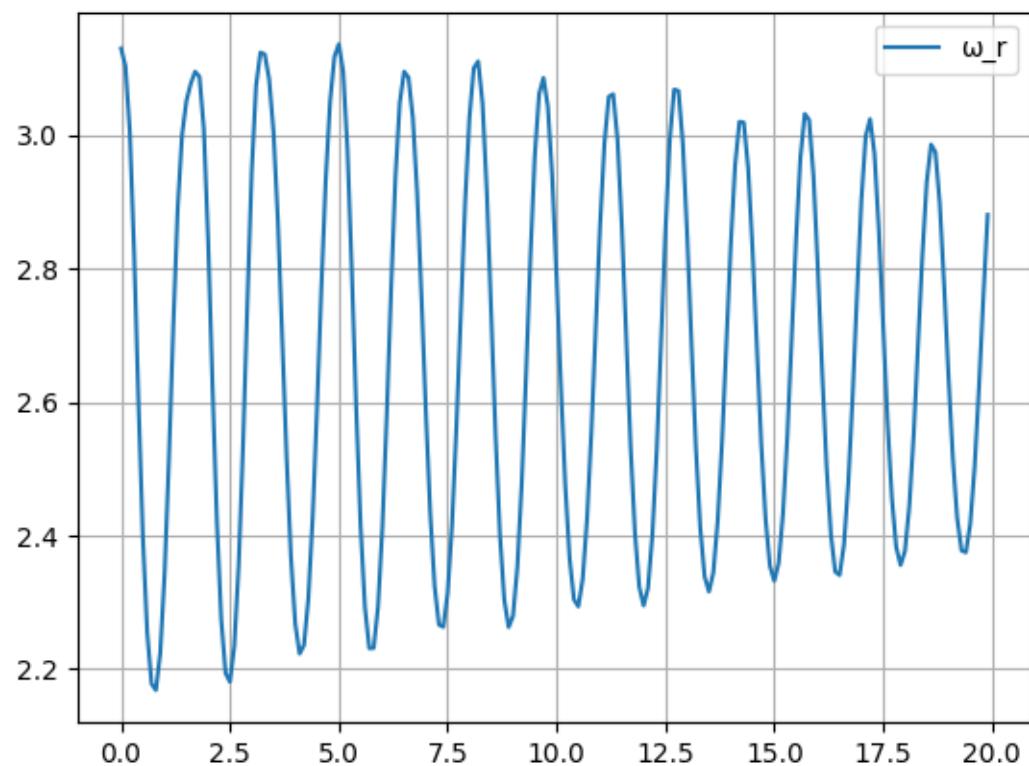


Figure B.24: *Case 4 where  $m = 4 \text{ kg}$*

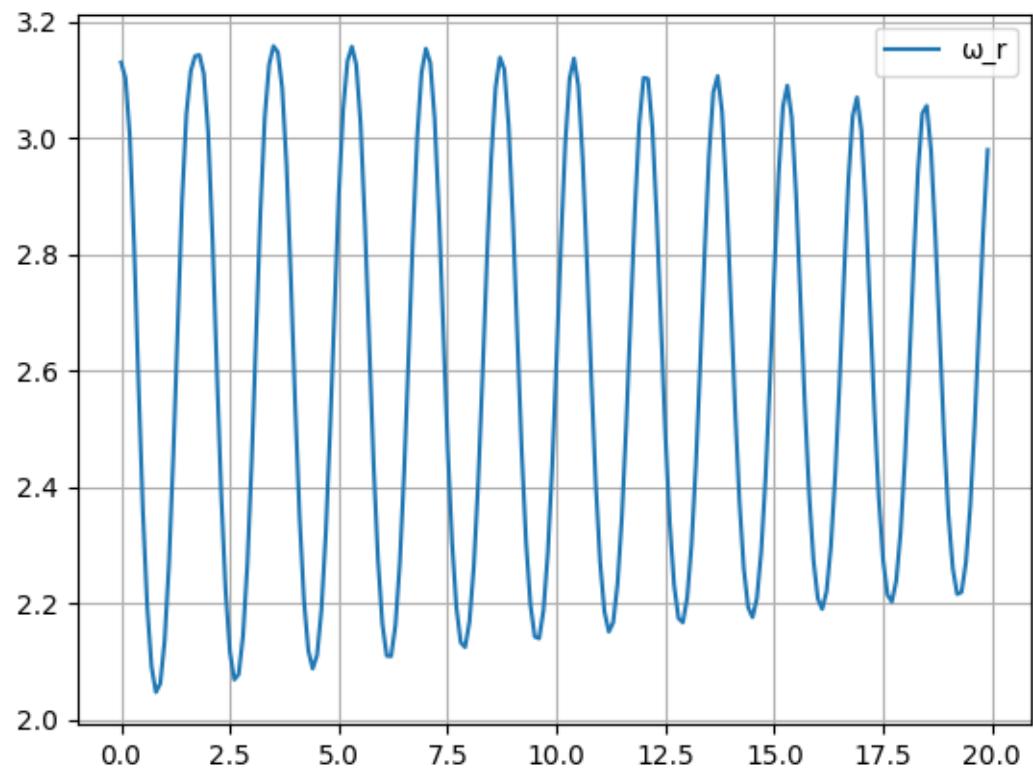


Figure B.25: Case 5 where  $m = 5 \text{ kg}$

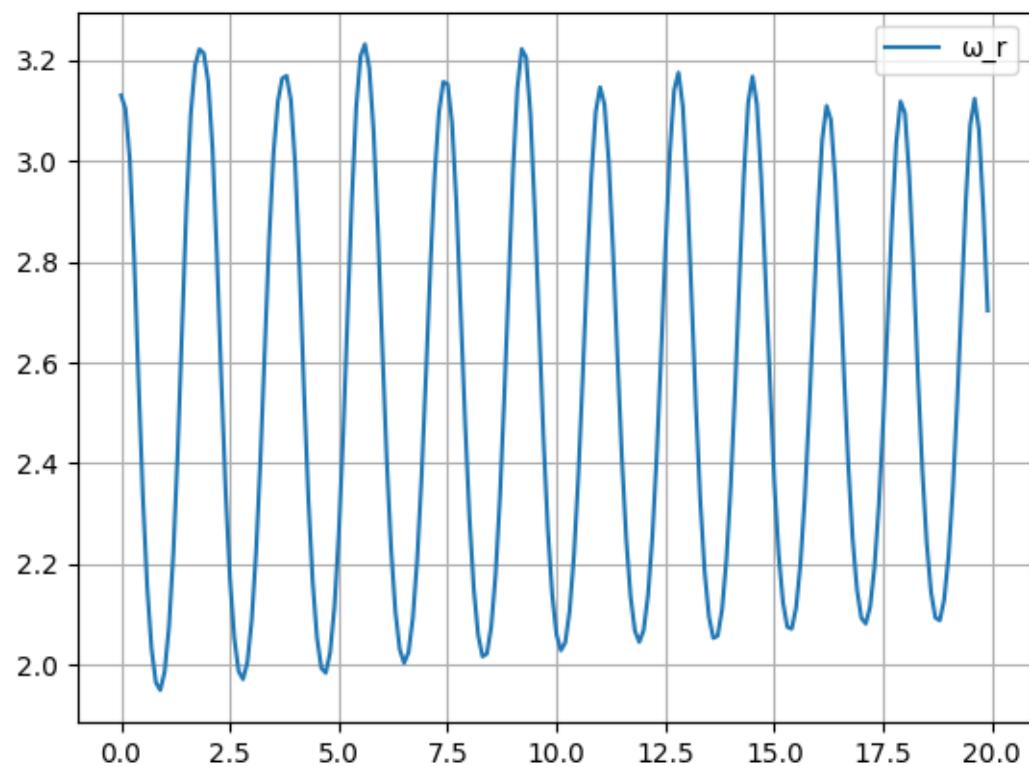


Figure B.26: Case 6 where  $m = 6 \text{ kg}$

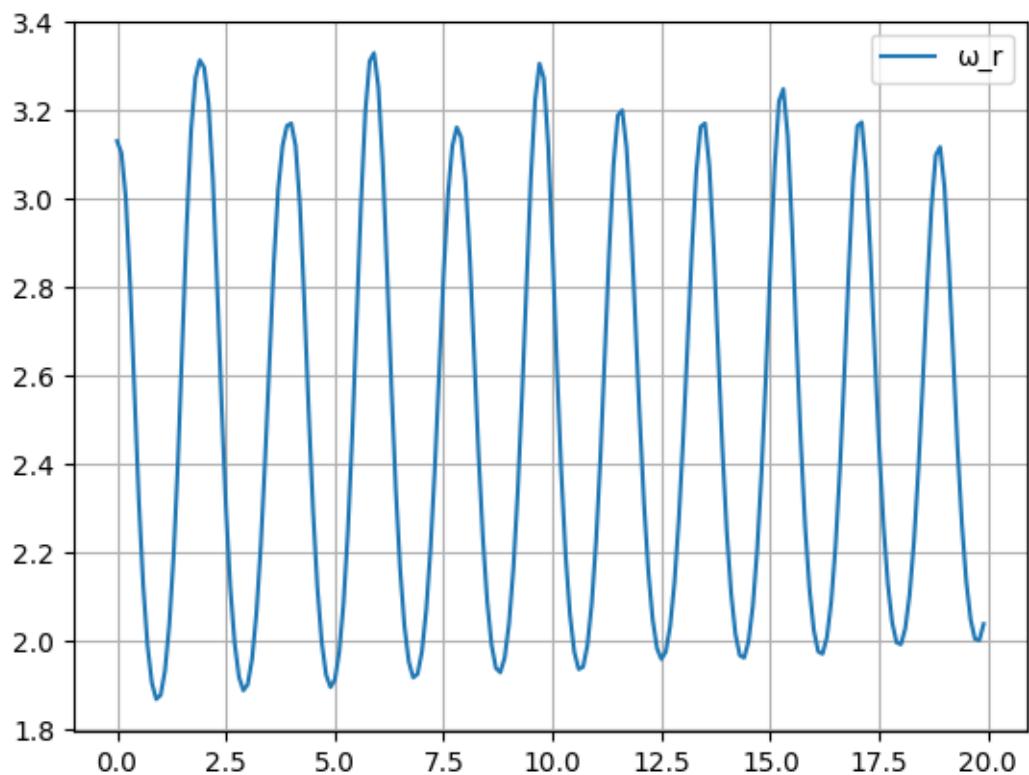


Figure B.27: Case 7 where  $m = 7 \text{ kg}$

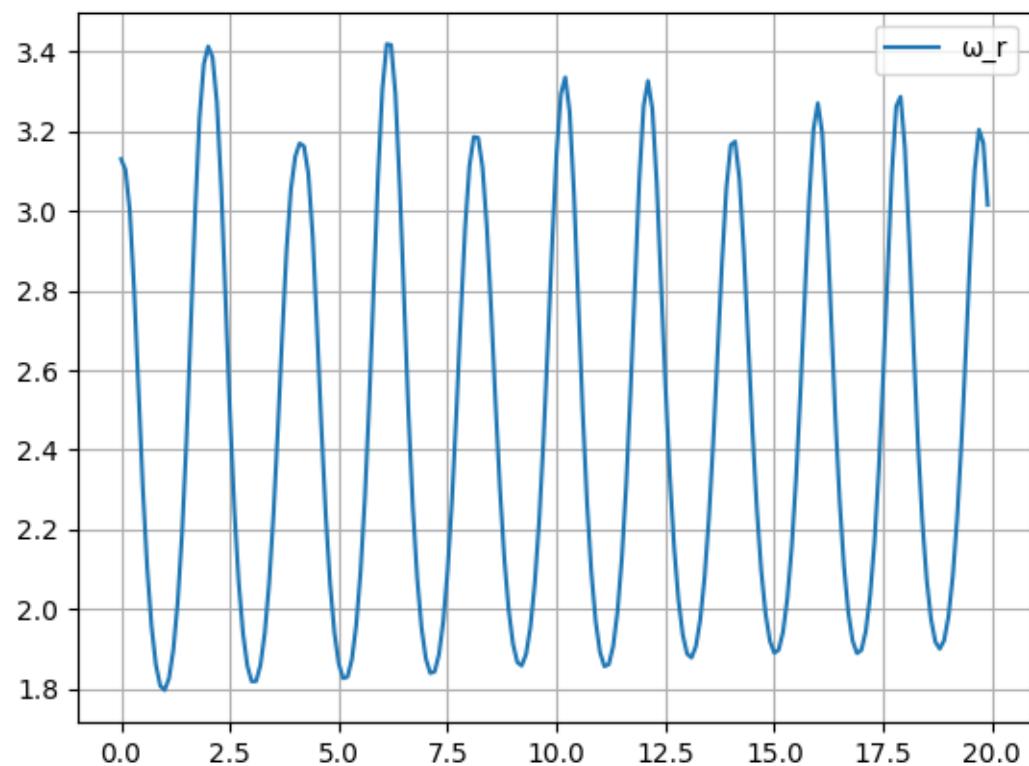


Figure B.28: *Case 8 where  $m = 8 \text{ kg}$*

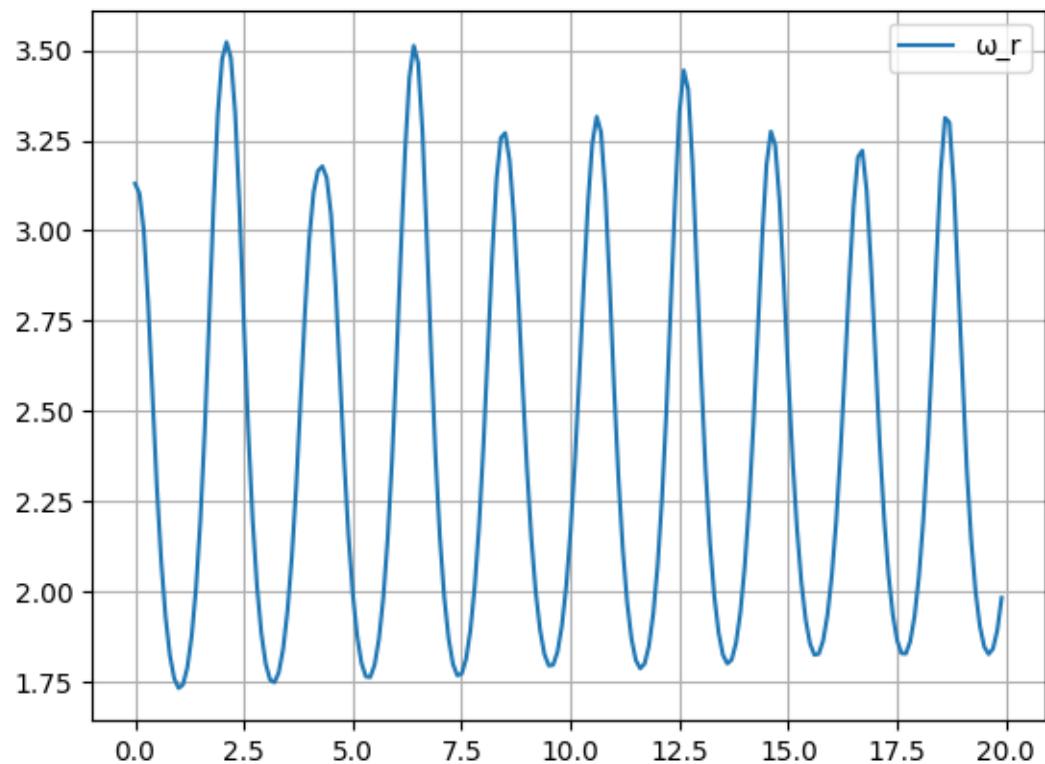


Figure B.29: Case 9 where  $m = 9 \text{ kg}$

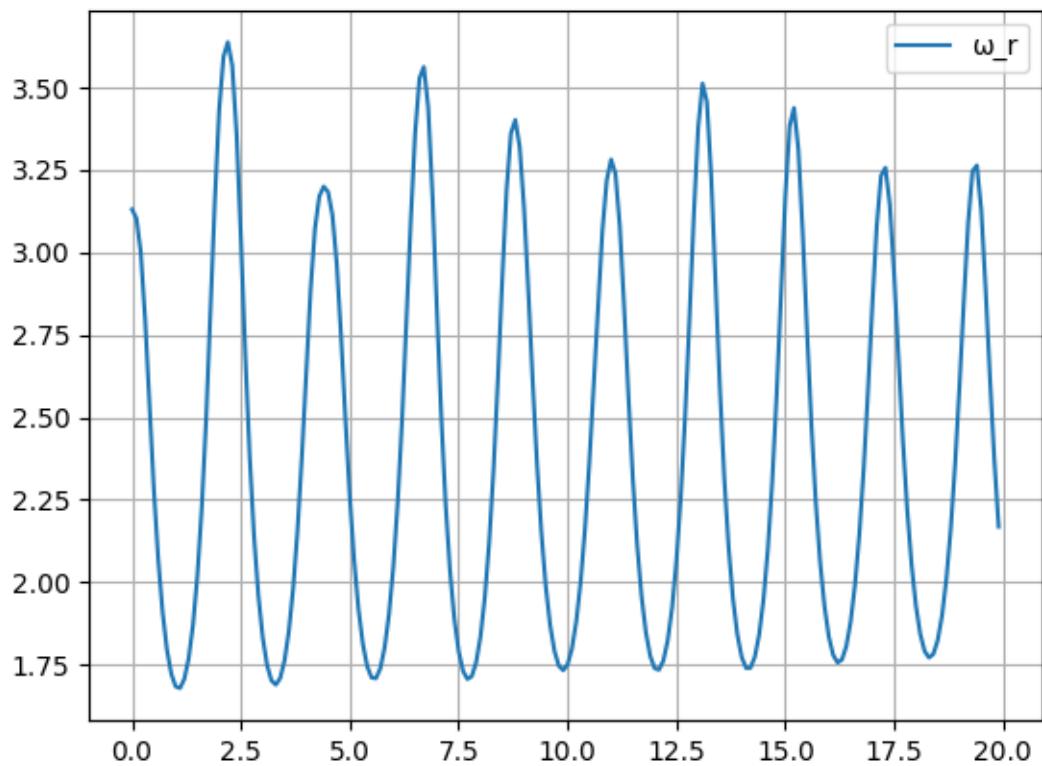


Figure B.30: *Case 10 where  $m = 10 \text{ kg}$*

## B.4 Absolute Frequency versus Time

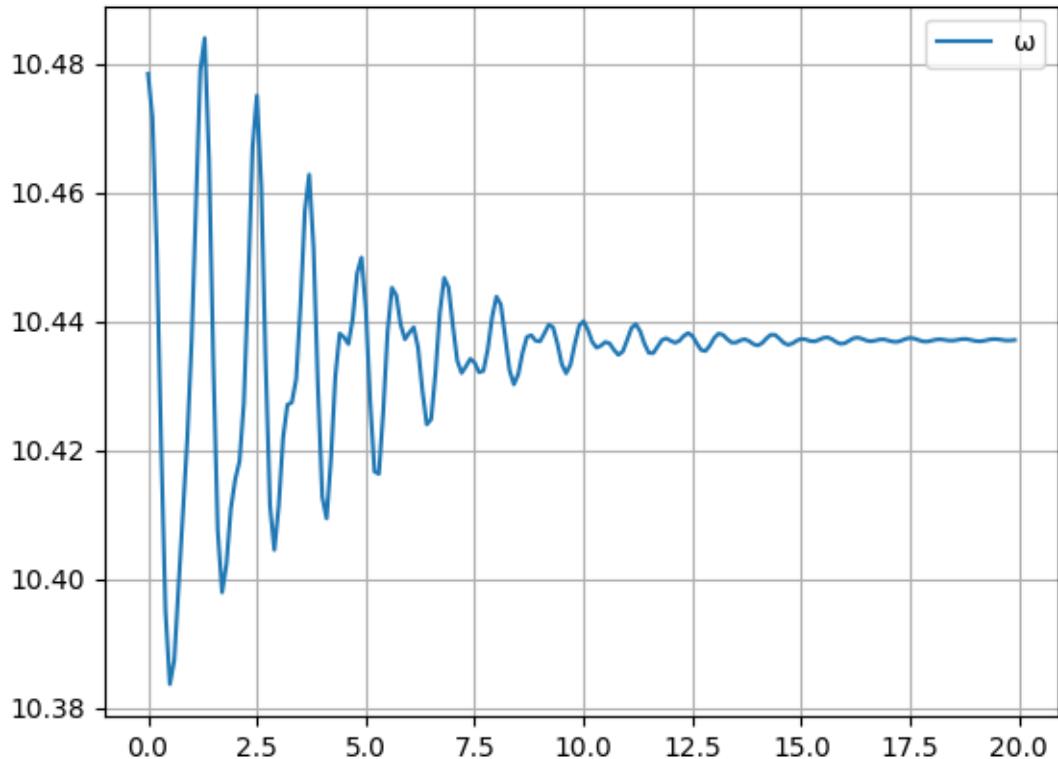


Figure B.31: Case 1 where  $m = 1 \text{ kg}$

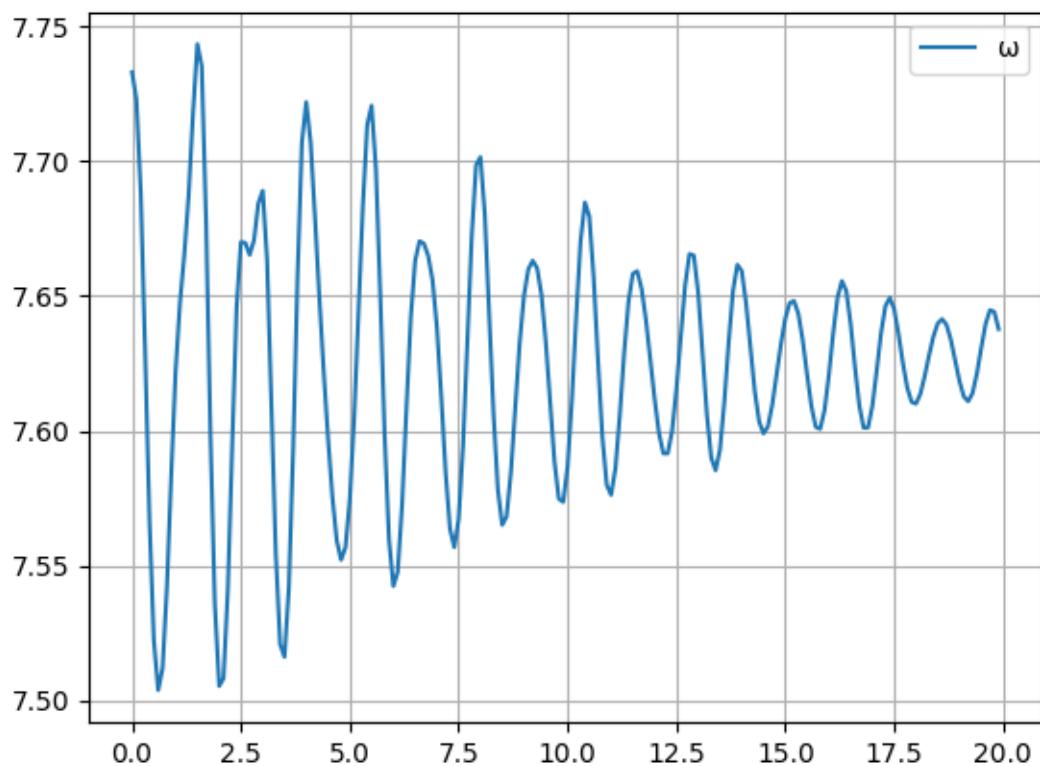


Figure B.32: *Case 2 where  $m = 2 \text{ kg}$*

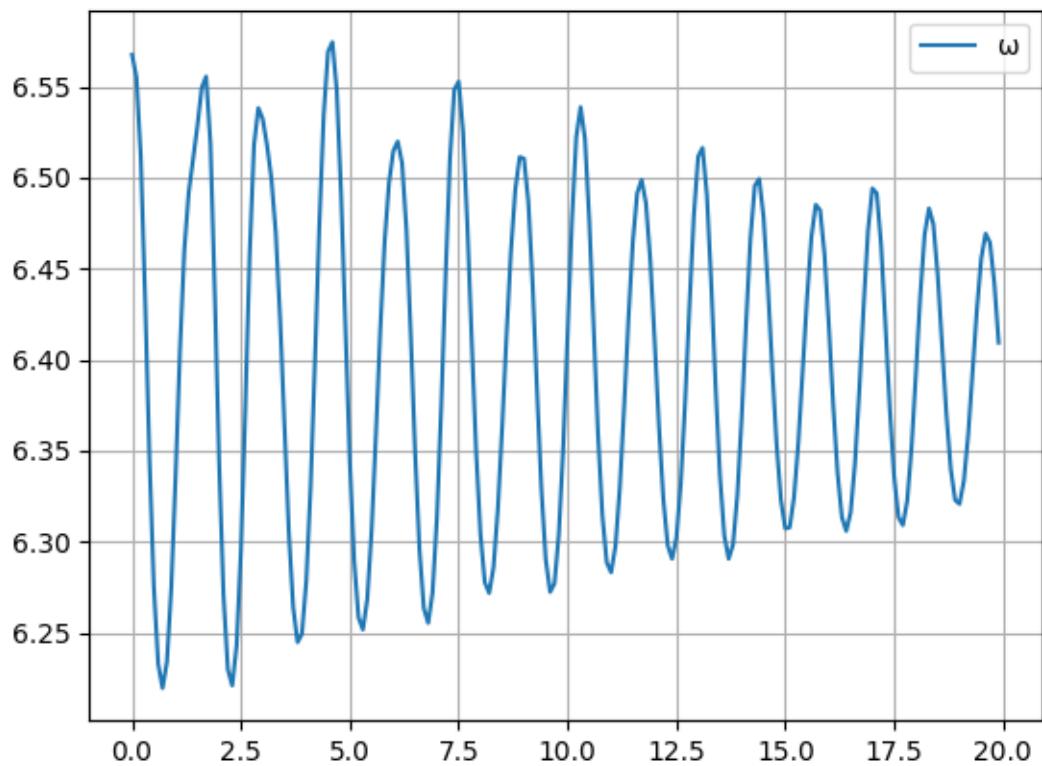


Figure B.33: Case 3 where  $m = 3 \text{ kg}$

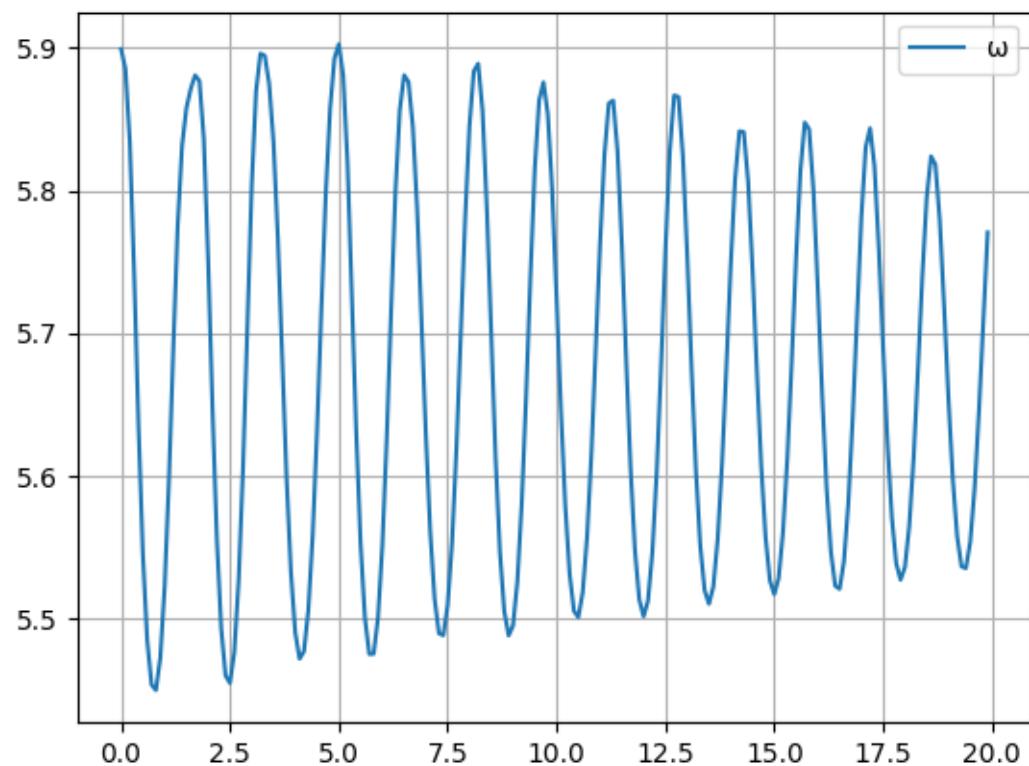


Figure B.34: Case 4 where  $m = 4 \text{ kg}$

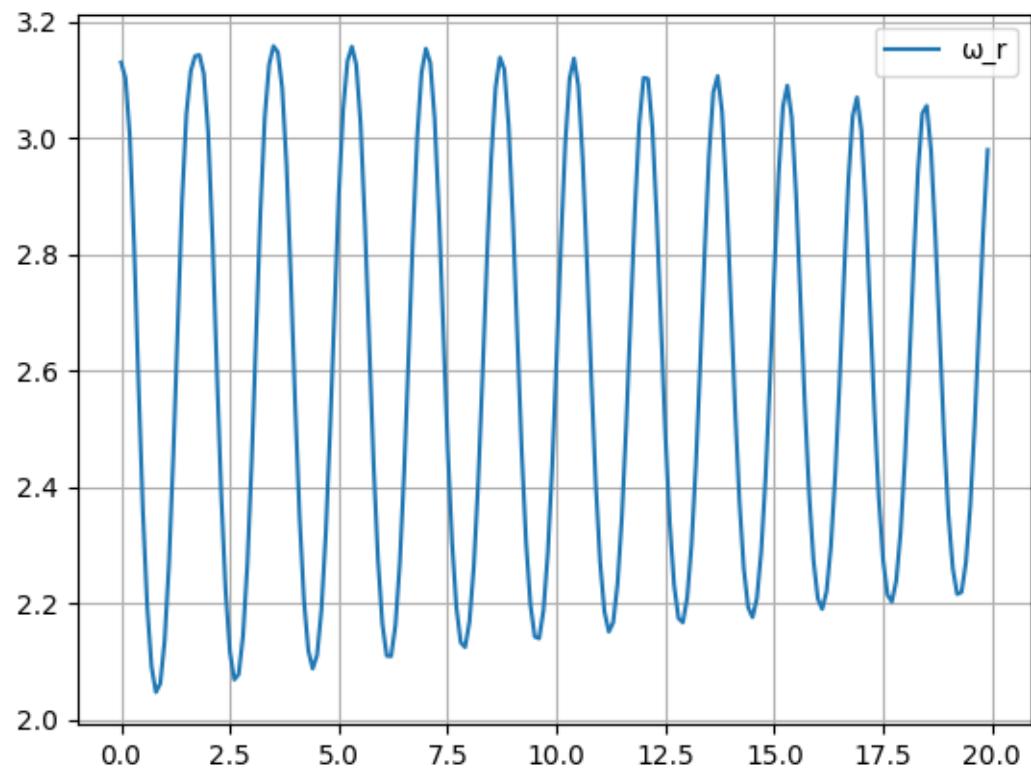


Figure B.35: Case 5 where  $m = 5 \text{ kg}$

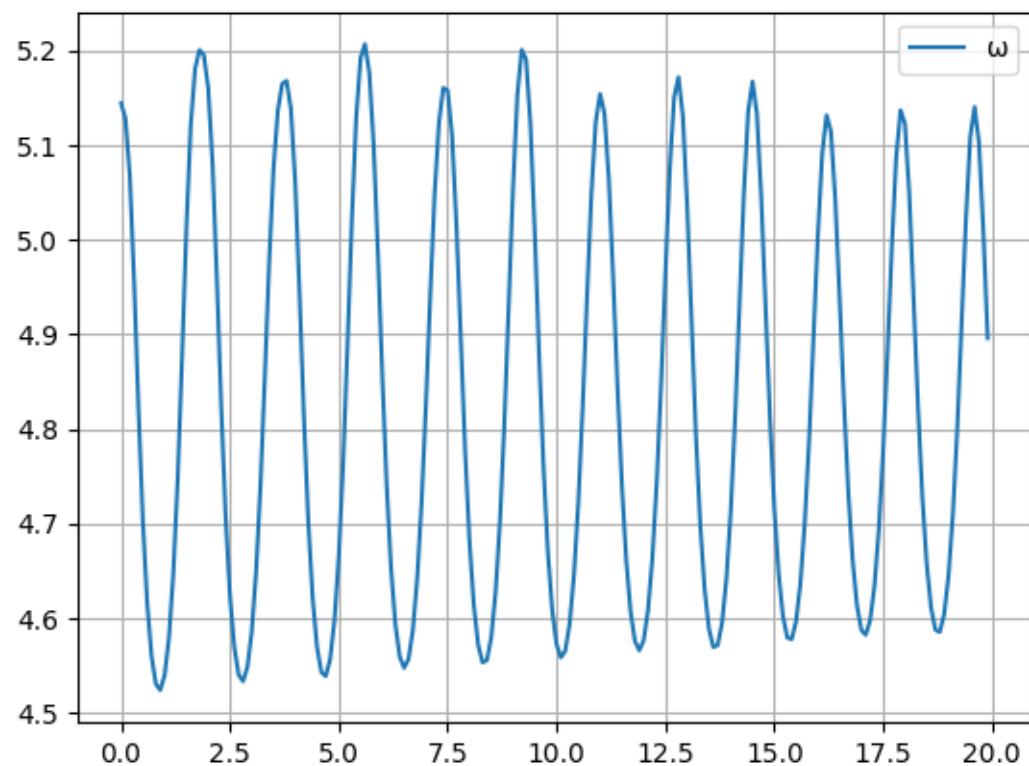


Figure B.36: *Case 6 where  $m = 6 \text{ kg}$*

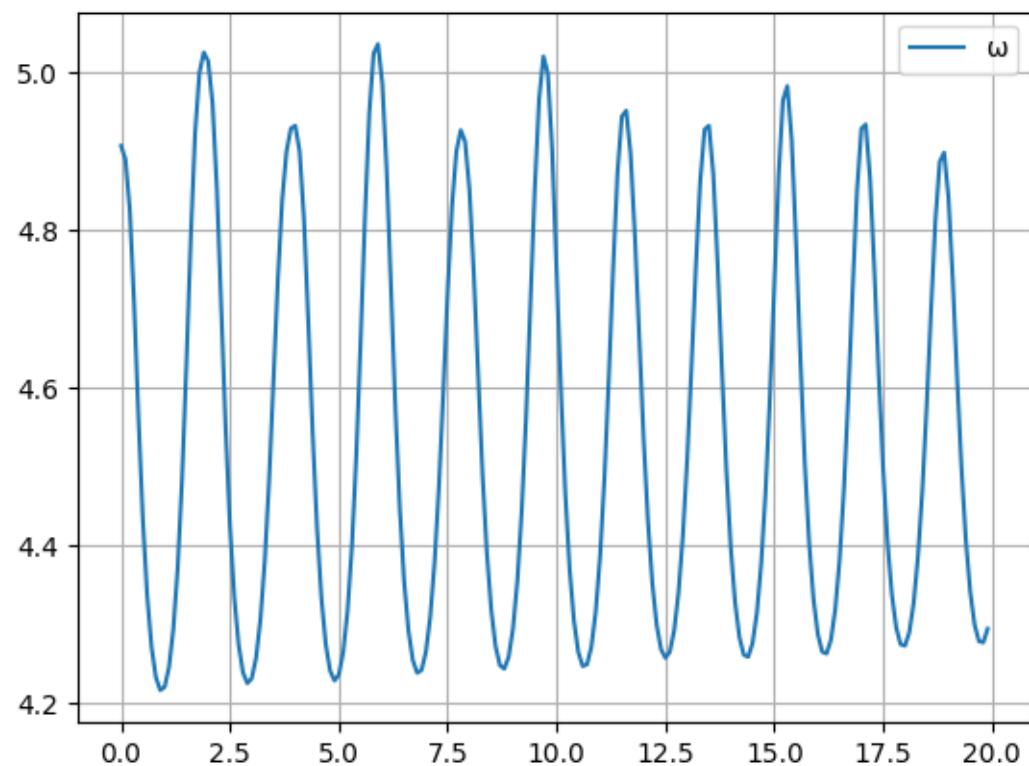


Figure B.37: Case 7 where  $m = 7\text{ kg}$

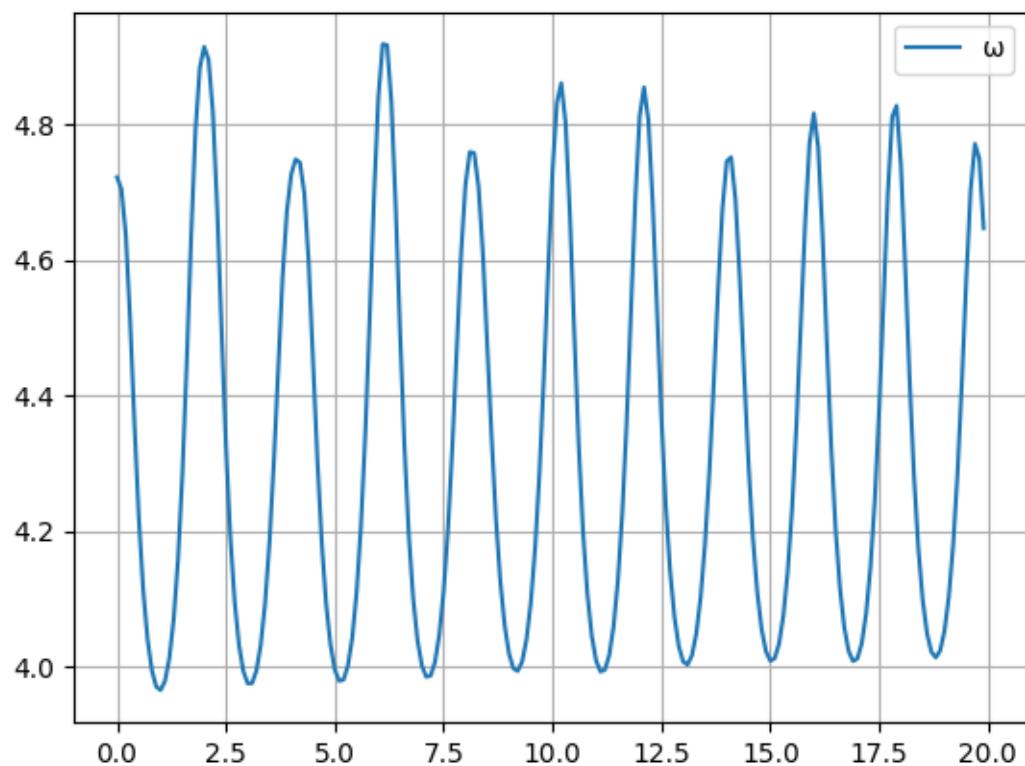


Figure B.38: Case 8 where  $m = 8 \text{ kg}$

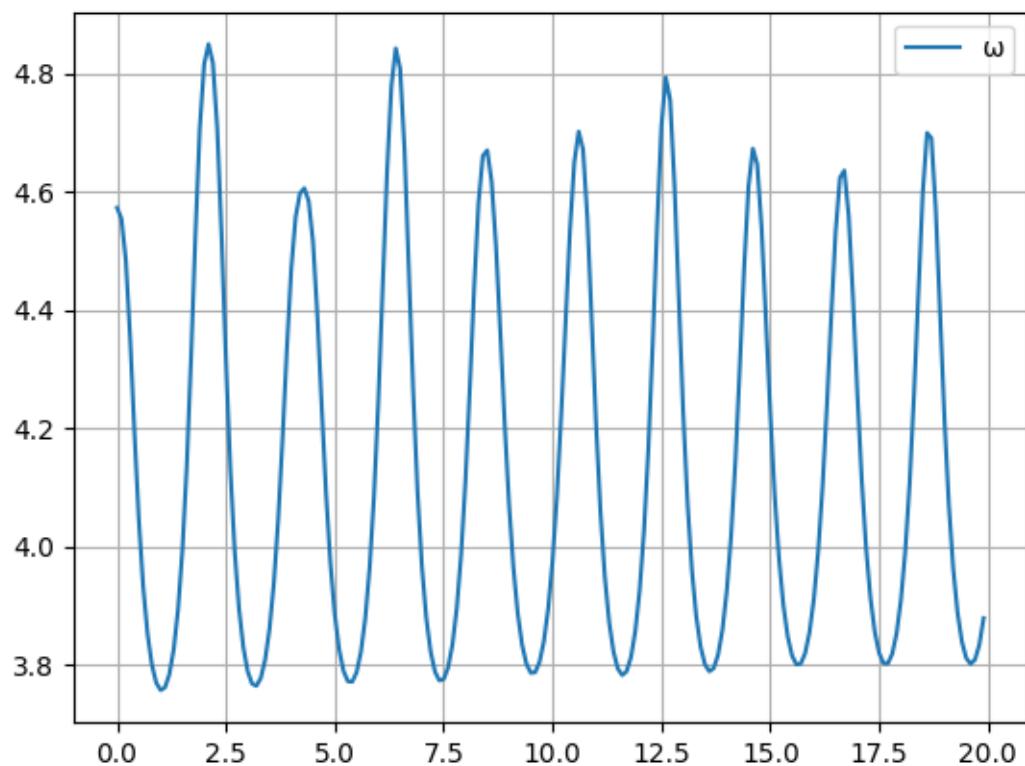


Figure B.39: Case 9 where  $m = 9 \text{ kg}$

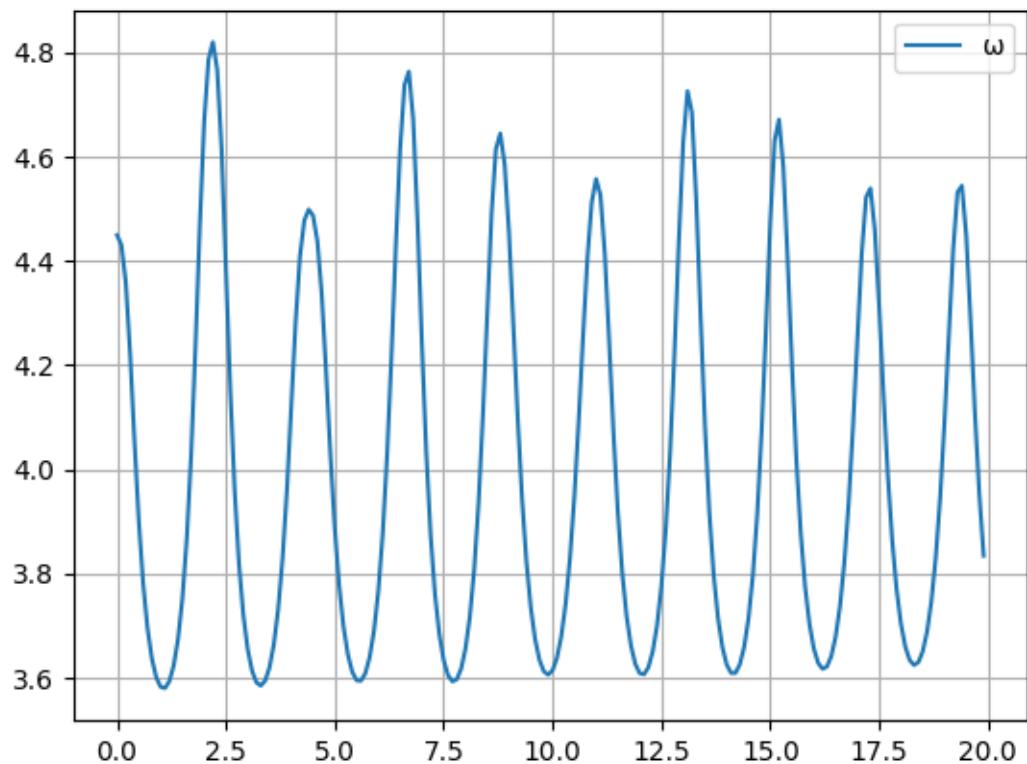


Figure B.40: Case 10 where  $m = 10 \text{ kg}$

## C *Raw Simulation Data*

### C.1 *Radial Displacement versus Time*

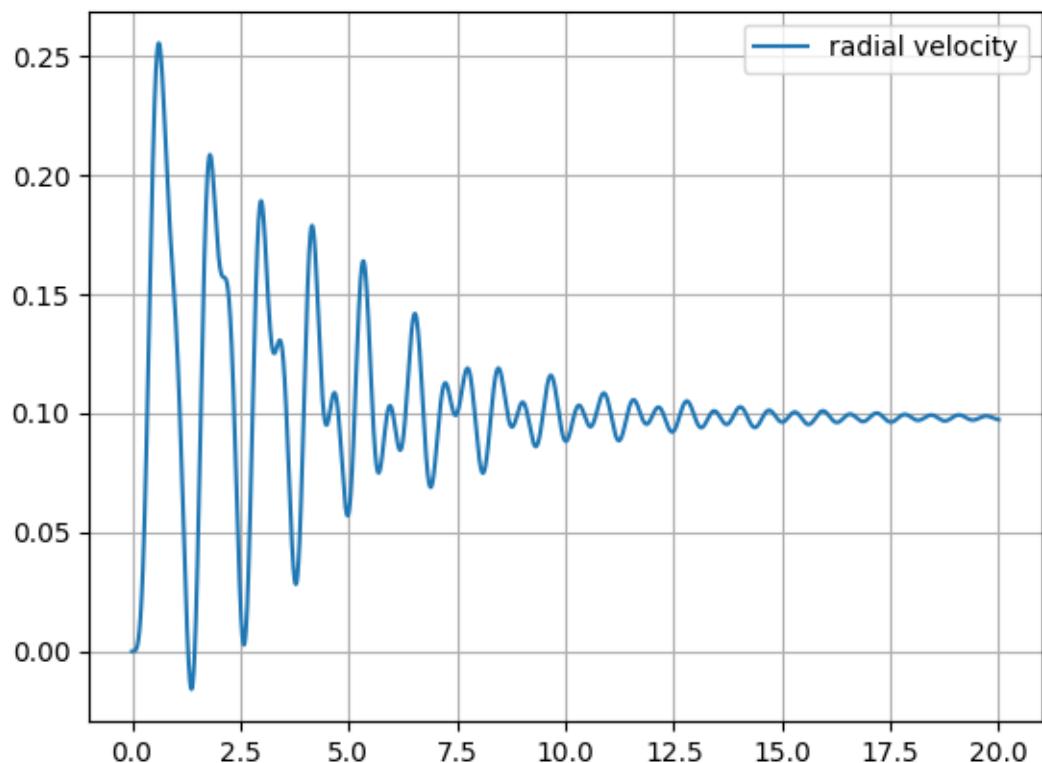


Figure C.1: *Case 1 where  $m = 1 \text{ kg}$*

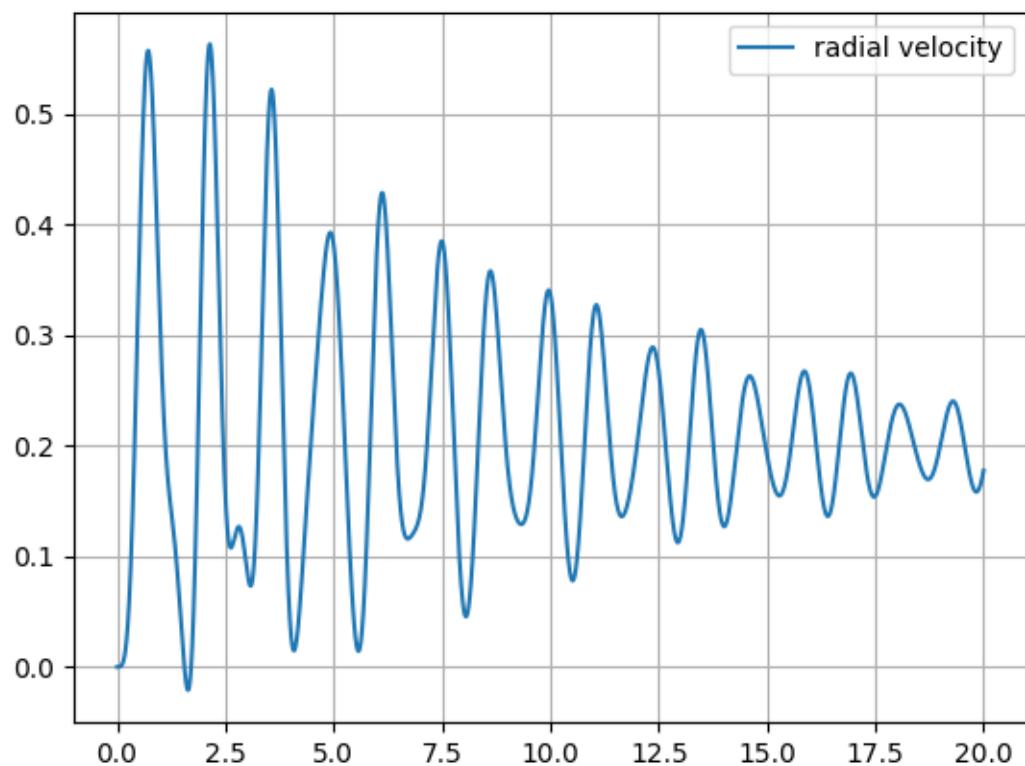


Figure C.2: Case 2 where  $m = 2 \text{ kg}$

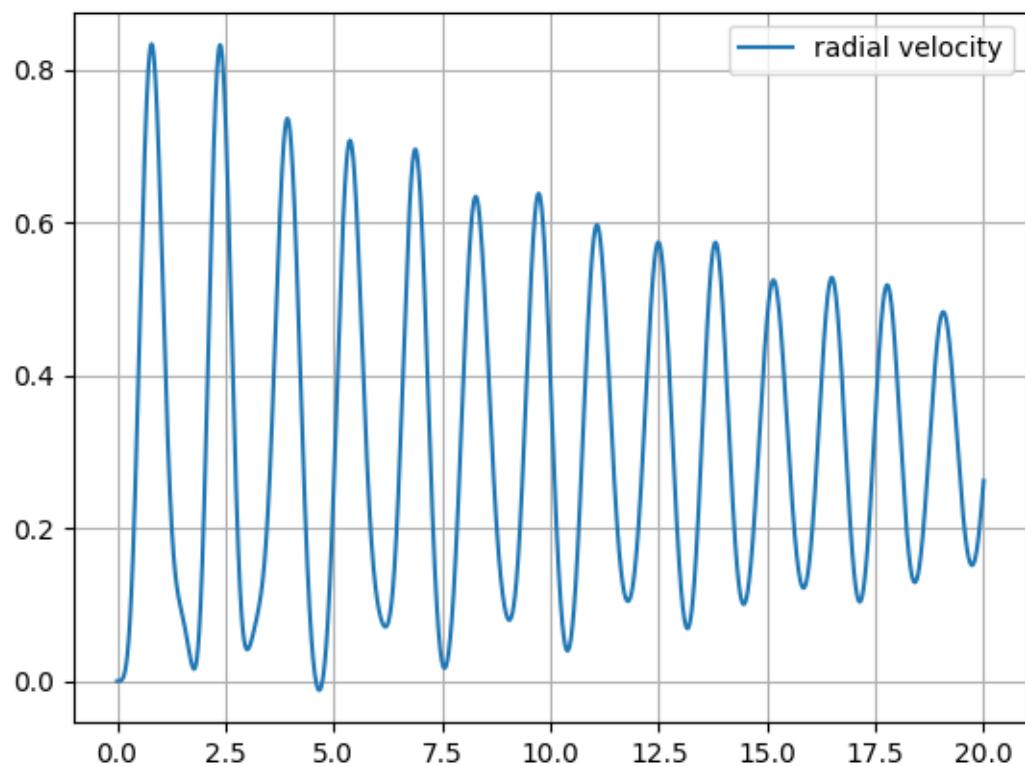


Figure C.3: *Case 3 where  $m = 3 \text{ kg}$*

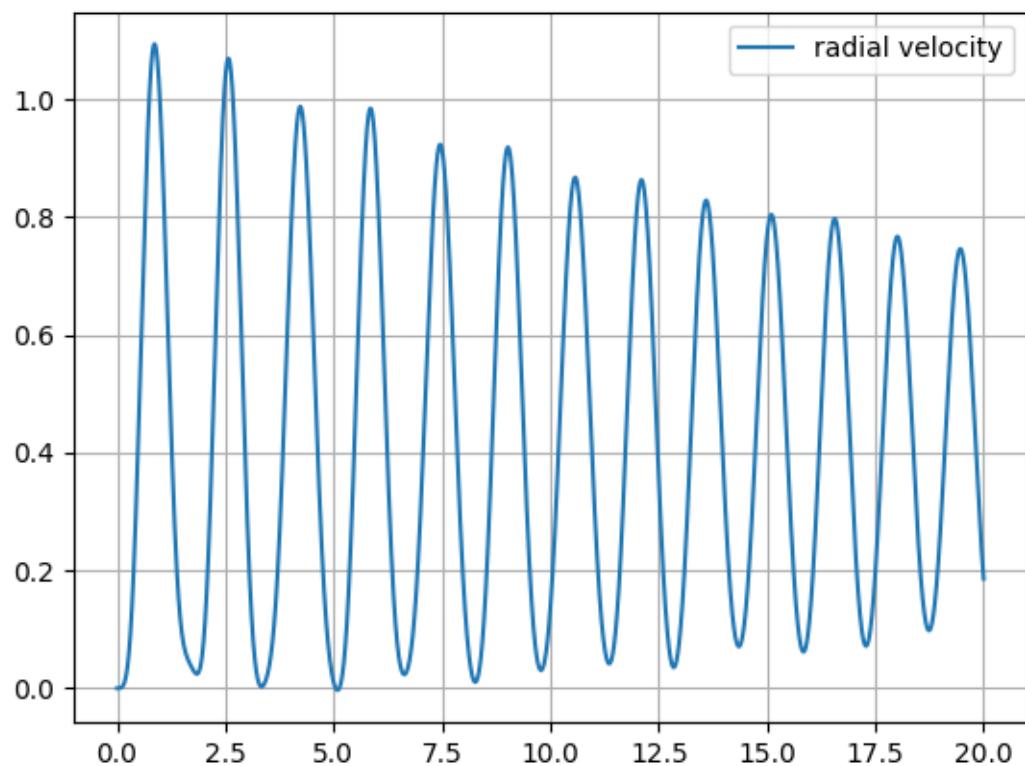


Figure C.4: Case 4 where  $m = 4 \text{ kg}$

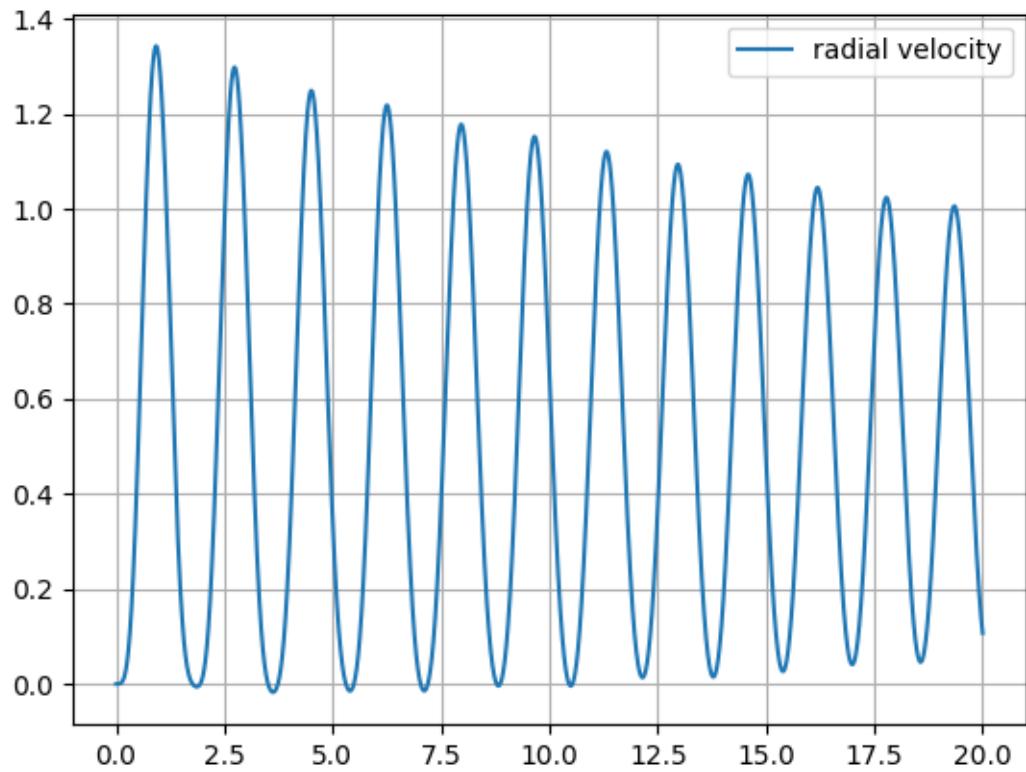


Figure C.5: Case 5 where  $m = 5 \text{ kg}$

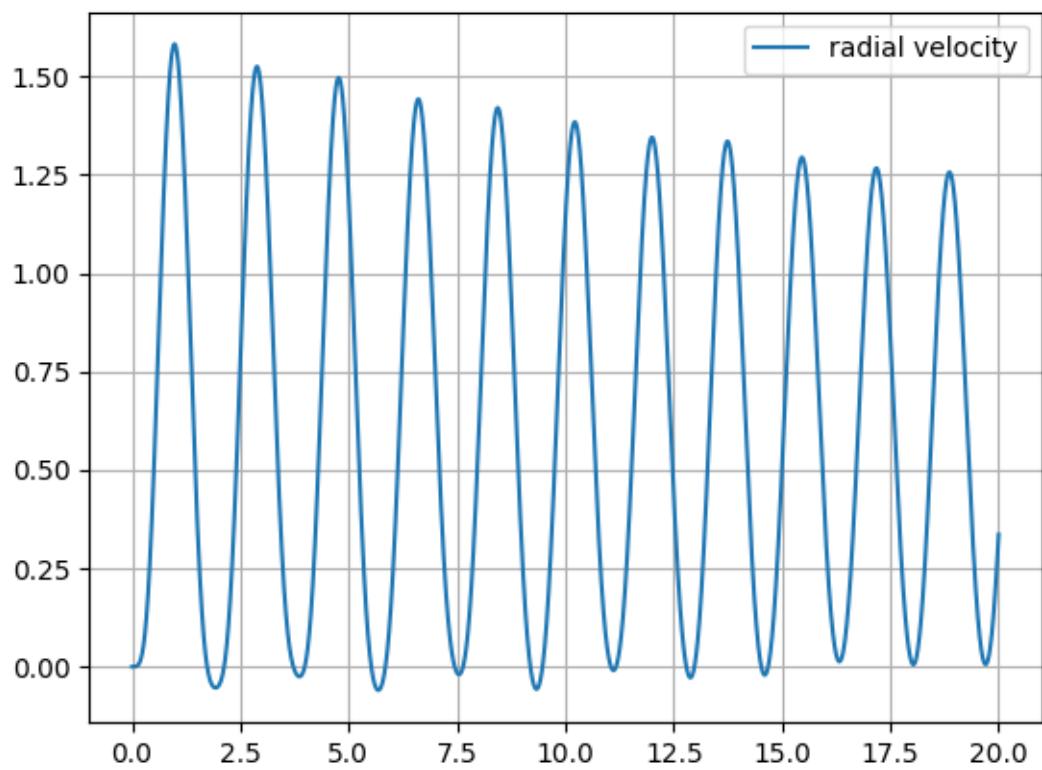


Figure C.6: Case 6 where  $m = 6 \text{ kg}$

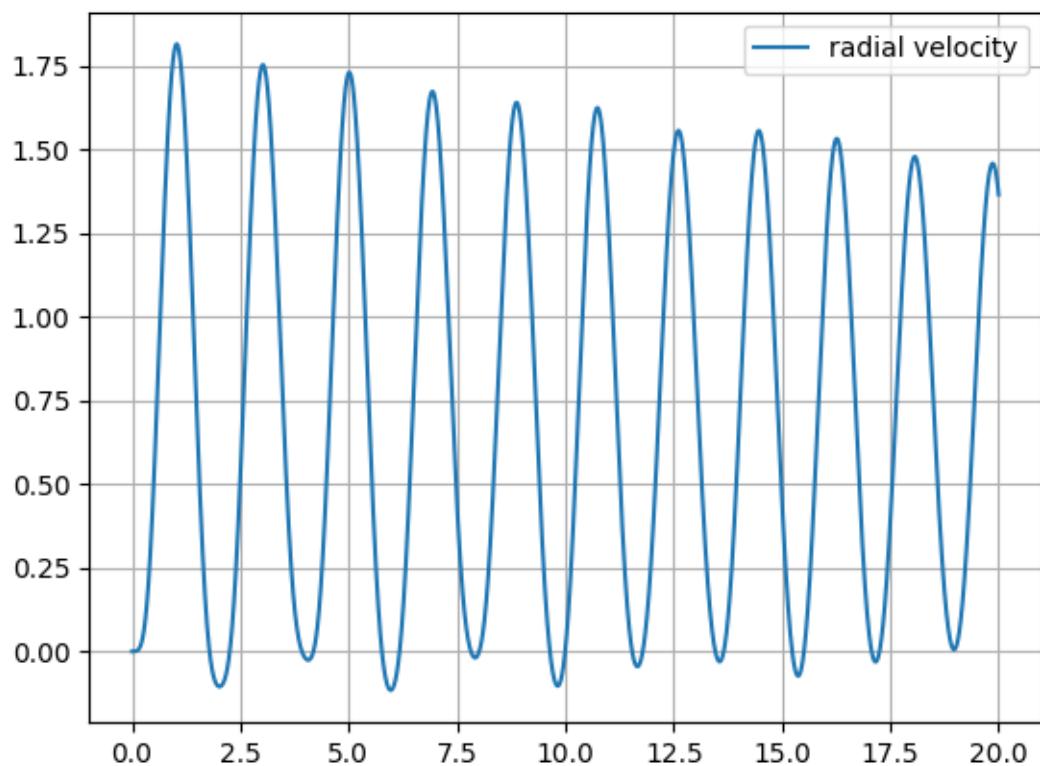


Figure C.7: *Case 7 where  $m = 7 \text{ kg}$*

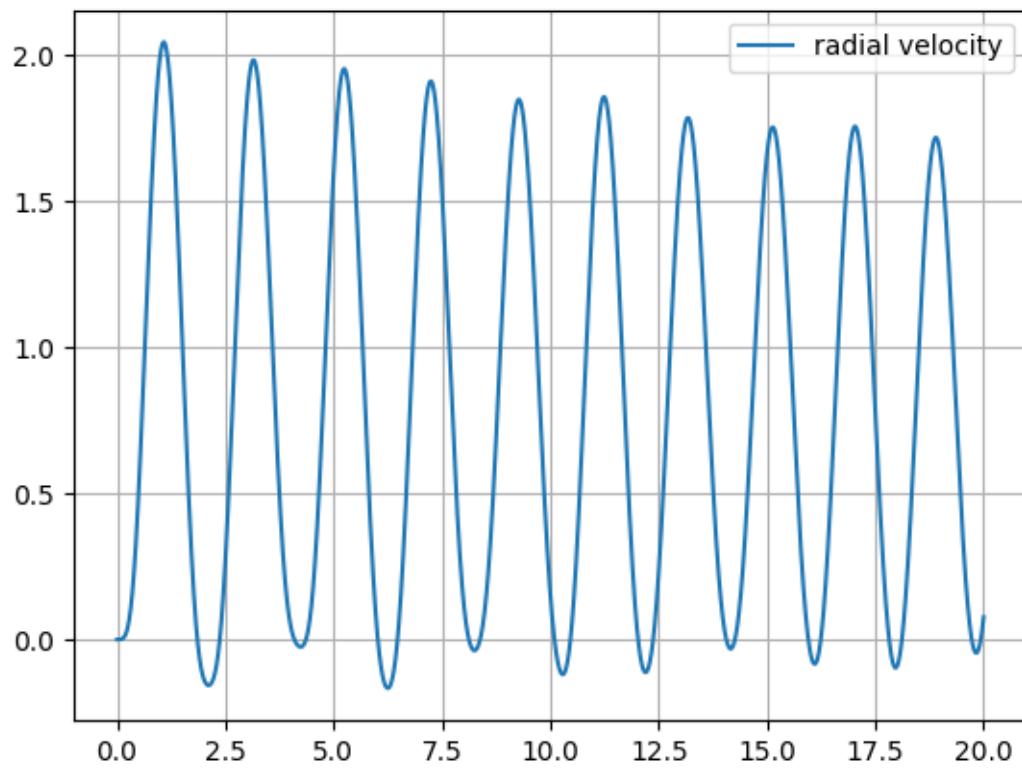


Figure C.8: *Case 8 where  $m = 8 \text{ kg}$*

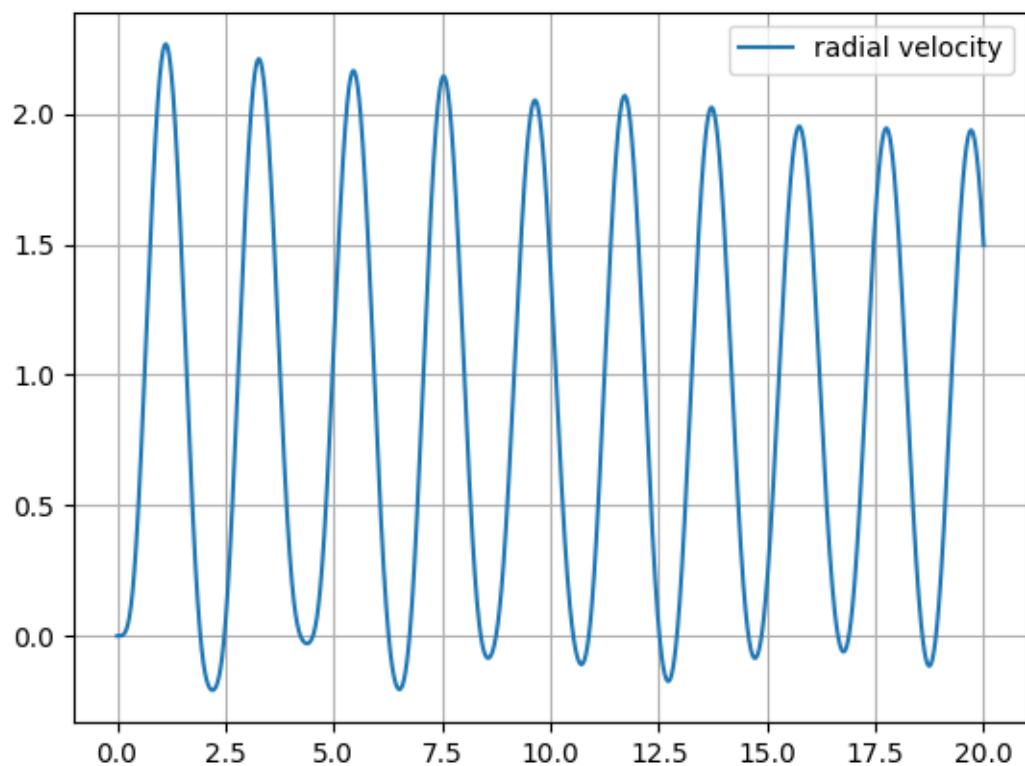


Figure C.9: *Case 9 where  $m = 9 \text{ kg}$*

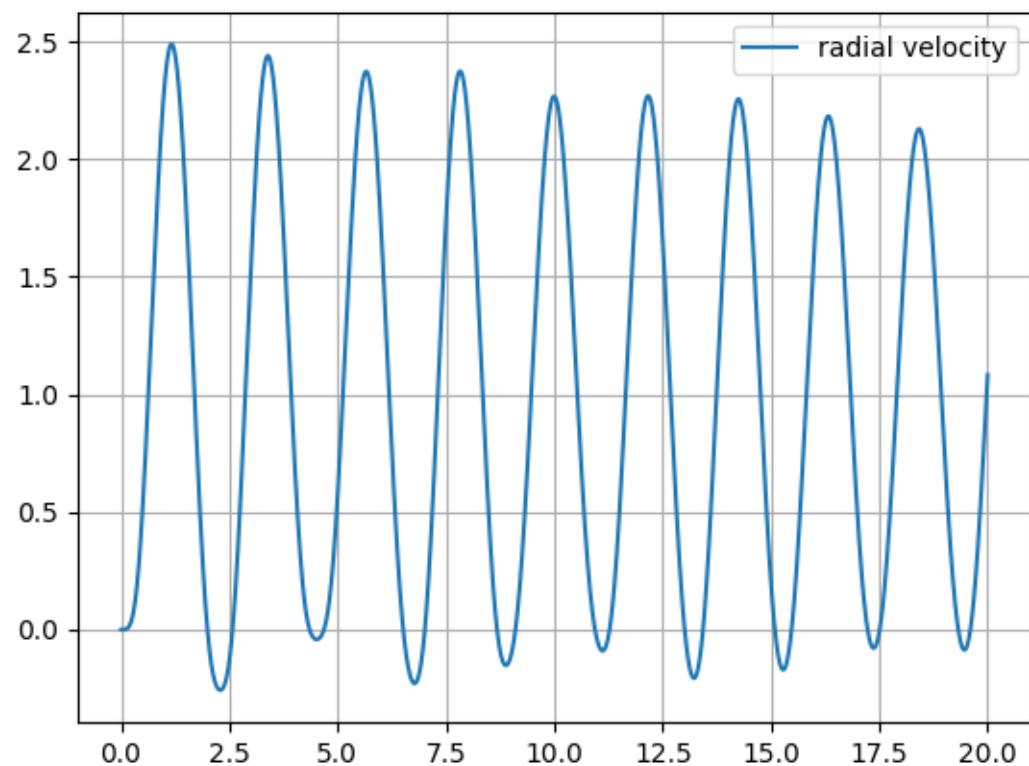


Figure C.10: *Case 10 where  $m = 10 \text{ kg}$*

## C.2 *Angular Displacement versus Time*

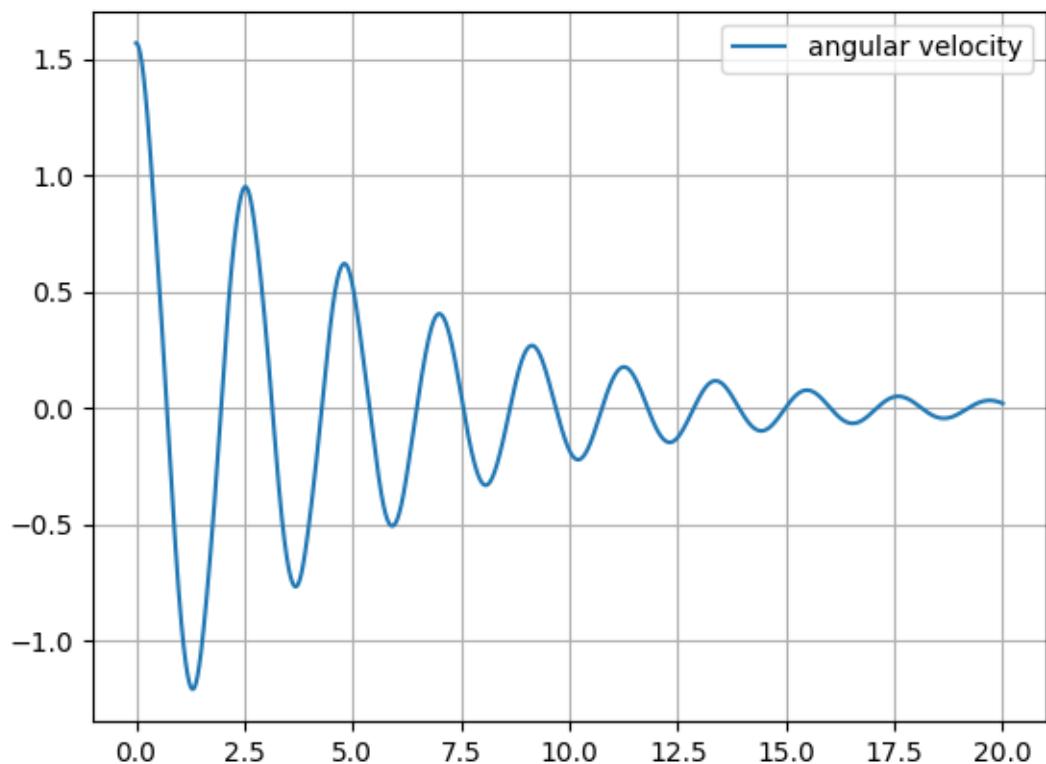


Figure C.11: *Case 1 where  $m = 1 \text{ kg}$*

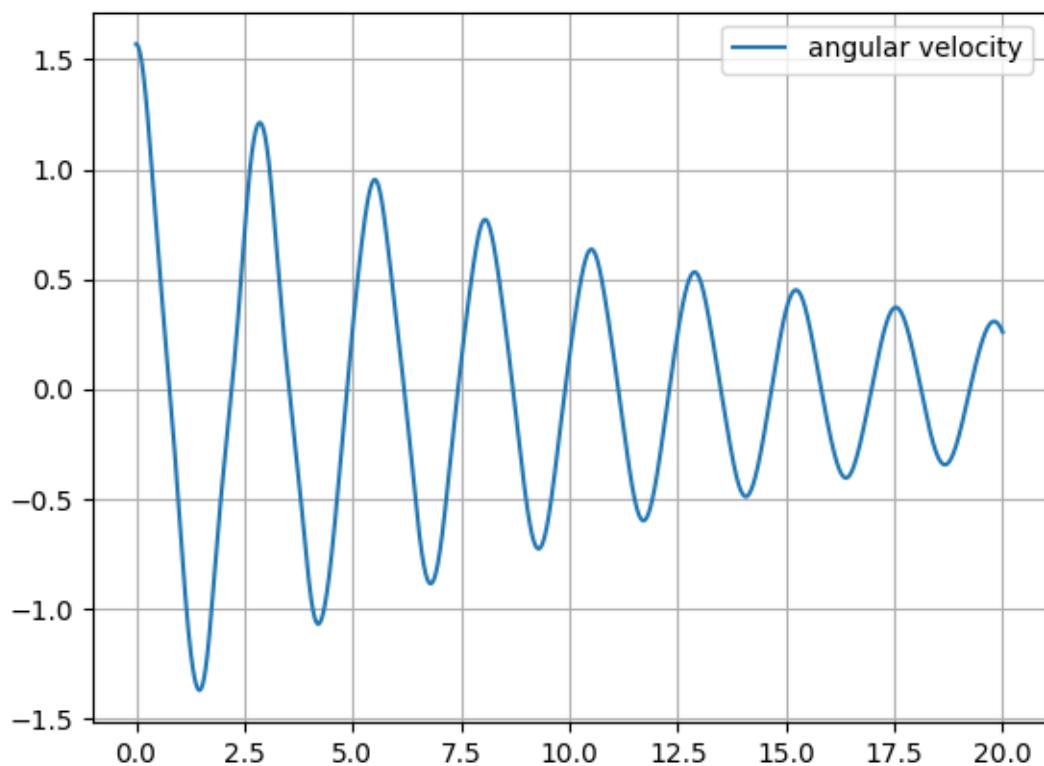


Figure C.12: Case 2 where  $m = 2 \text{ kg}$

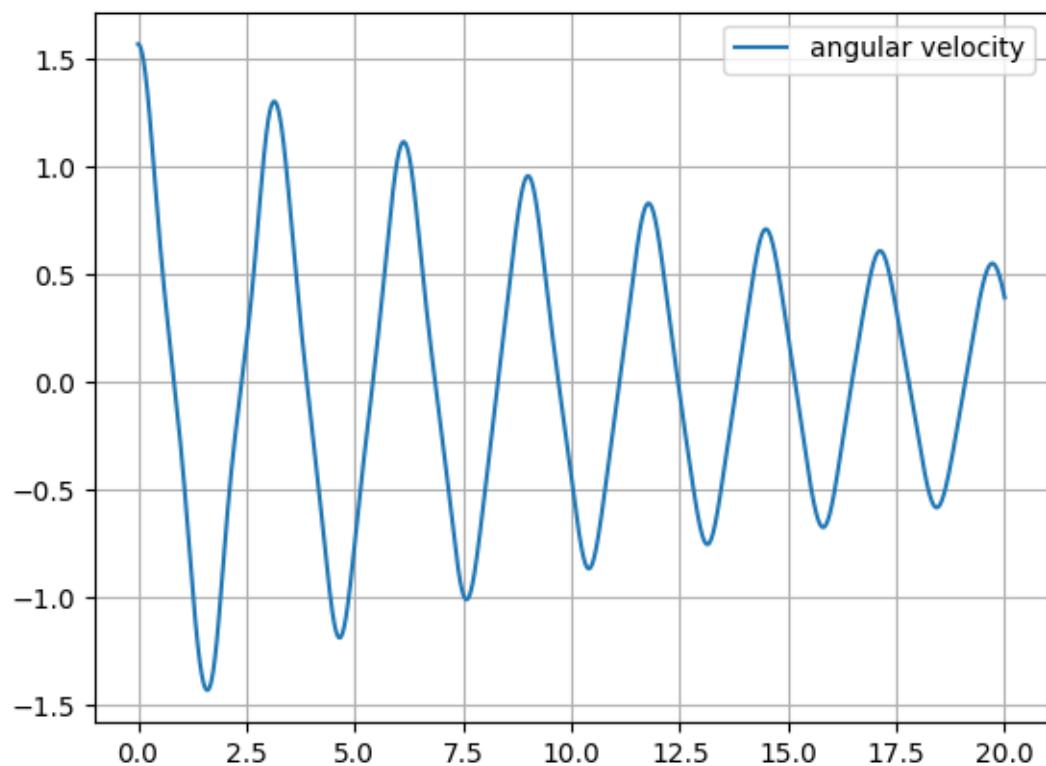


Figure C.13: Case 3 where  $m = 3 \text{ kg}$

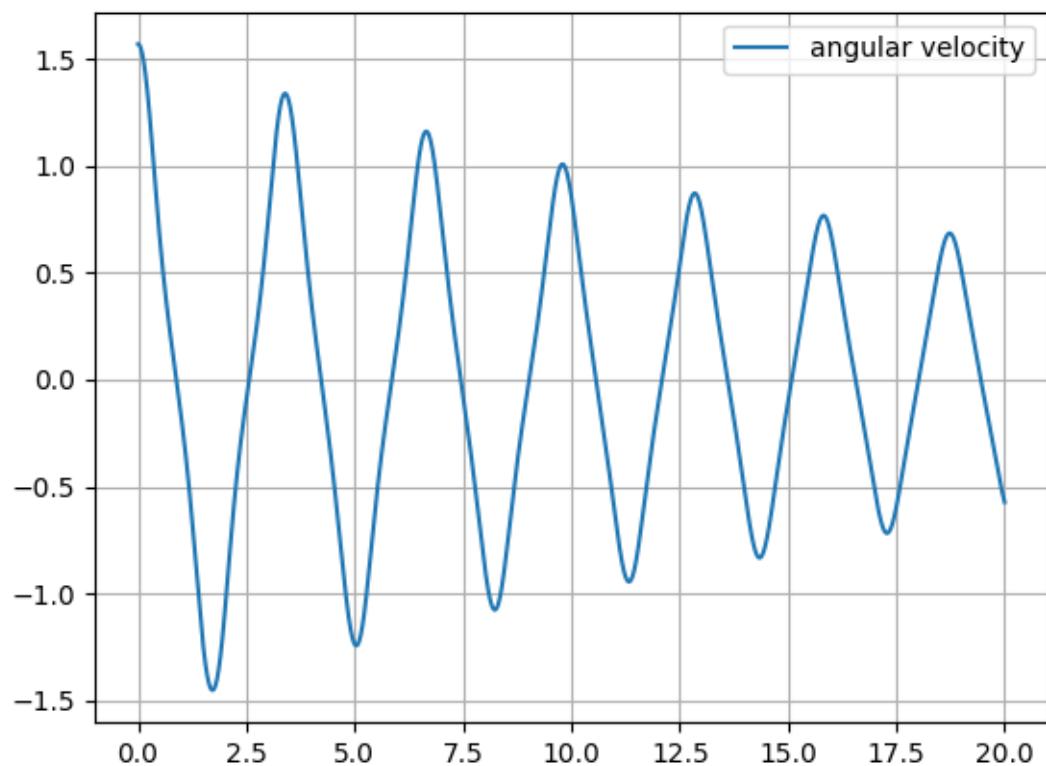


Figure C.14: *Case 4 where  $m = 4 \text{ kg}$*

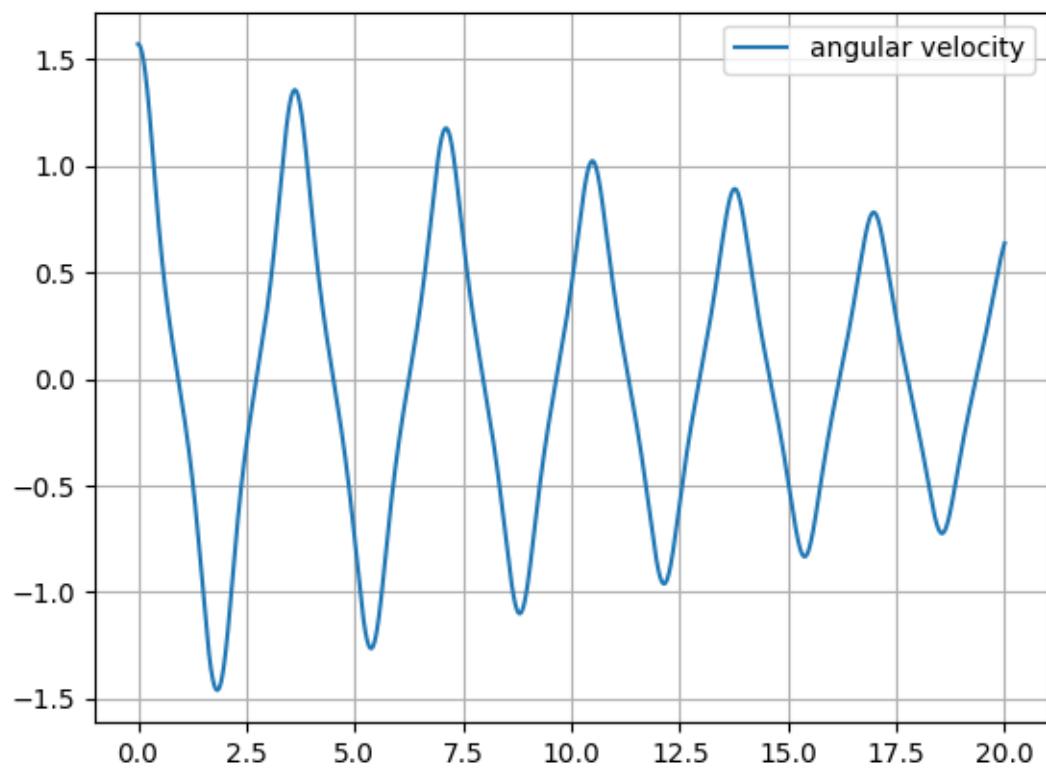


Figure C.15: Case 5 where  $m = 5 \text{ kg}$

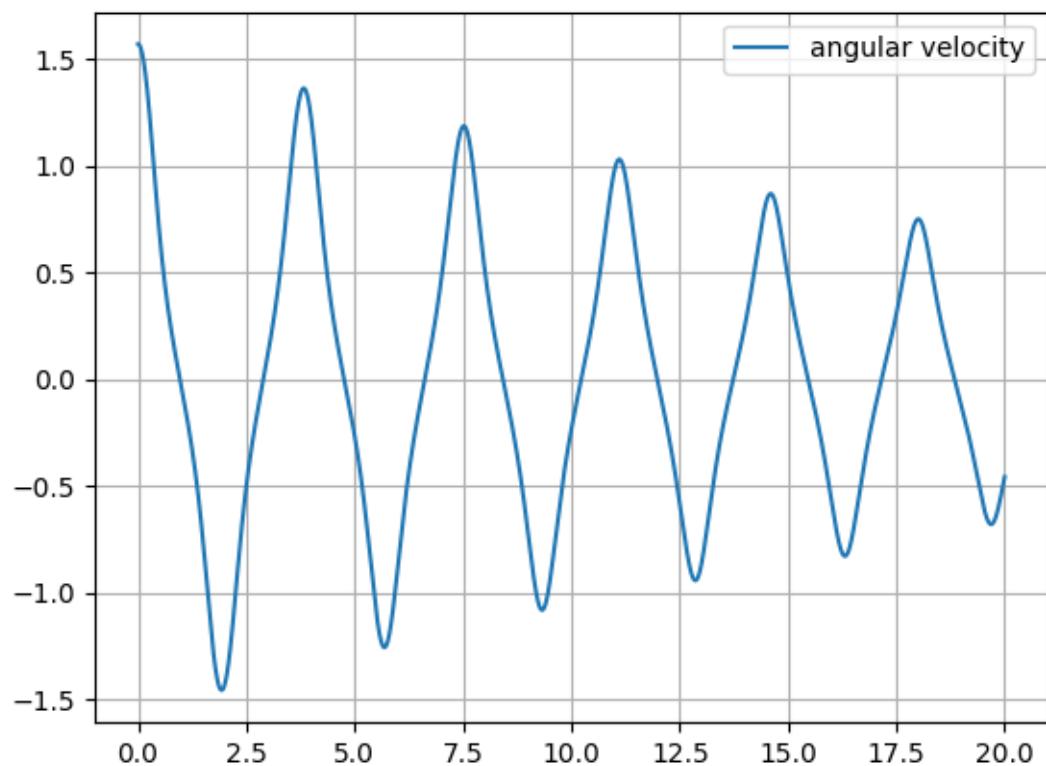


Figure C.16: Case 6 where  $m = 6 \text{ kg}$

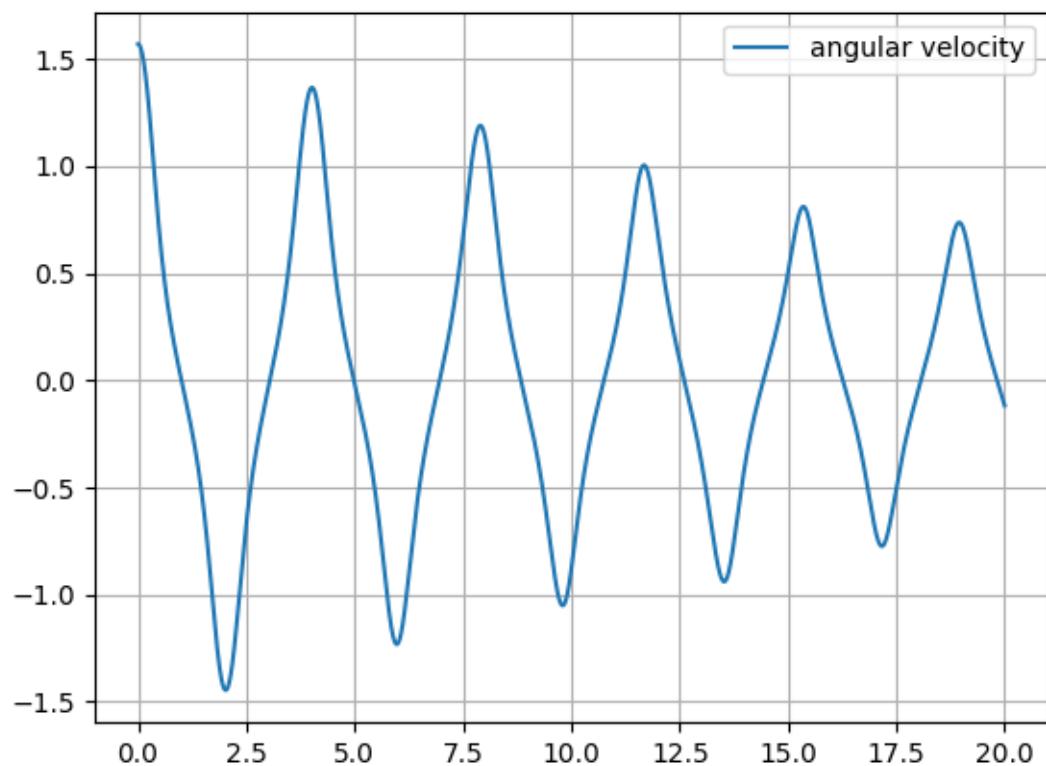


Figure C.17: Case 7 where  $m = 7 \text{ kg}$

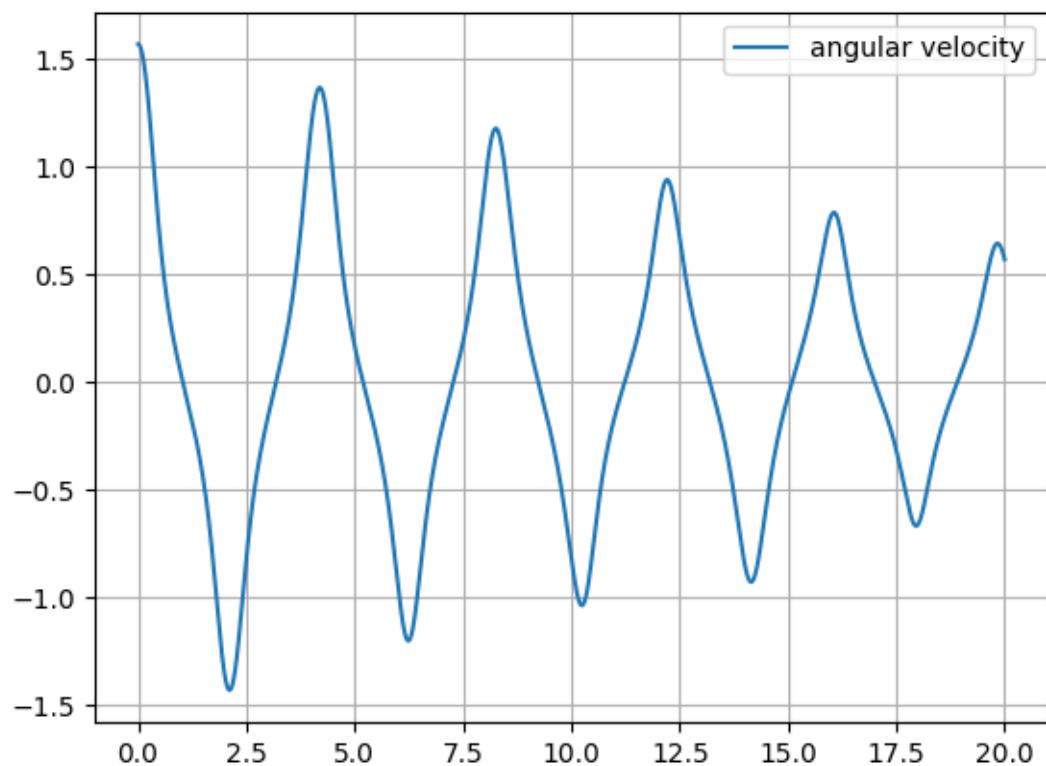


Figure C.18: *Case 8 where  $m = 8 \text{ kg}$*

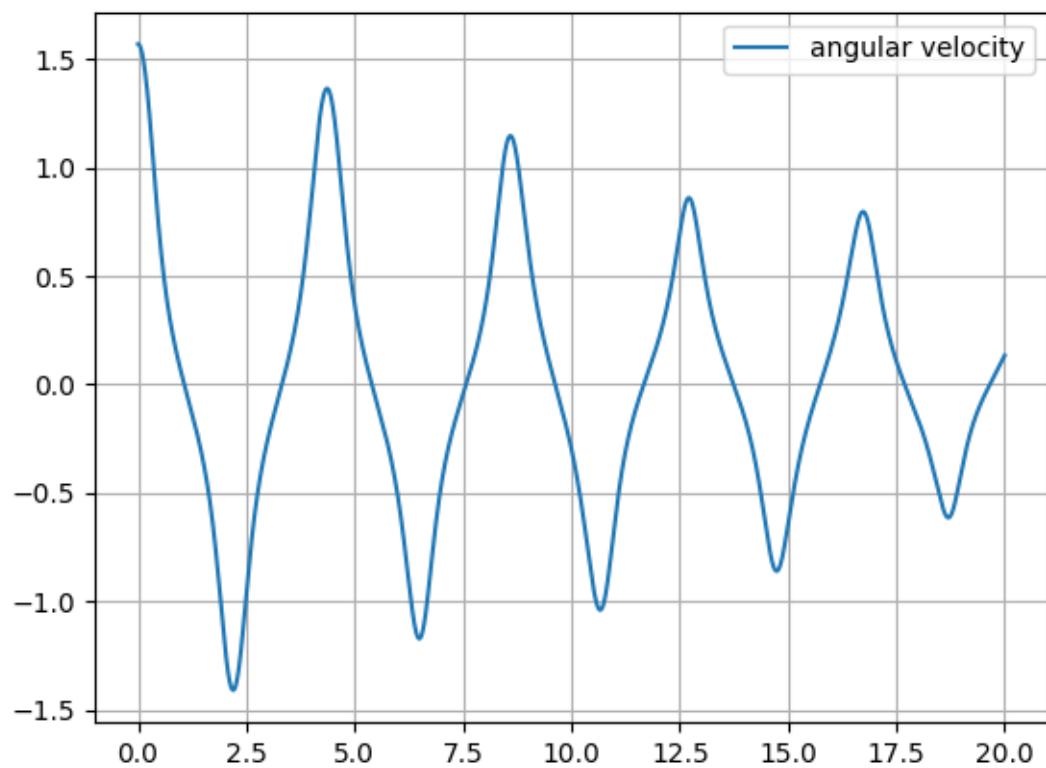


Figure C.19: *Case 9 where  $m = 9 \text{ kg}$*

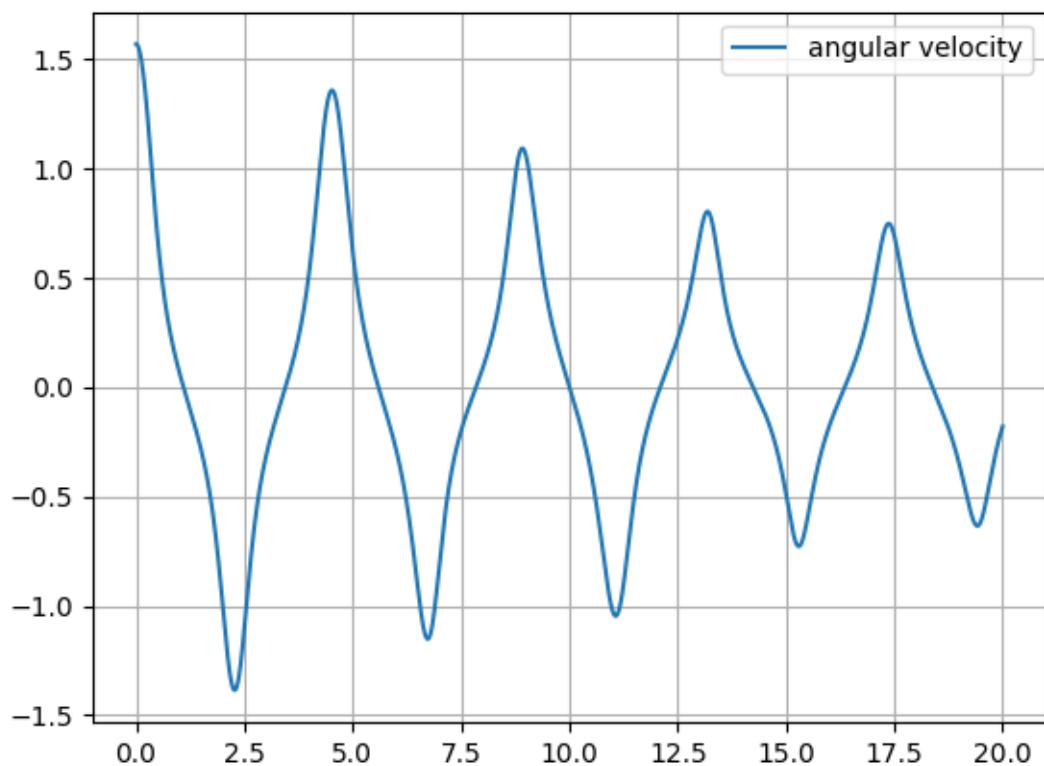


Figure C.20: *Case 10 where  $m = 10 \text{ kg}$*

### C.3 Angular Frequency versus Time

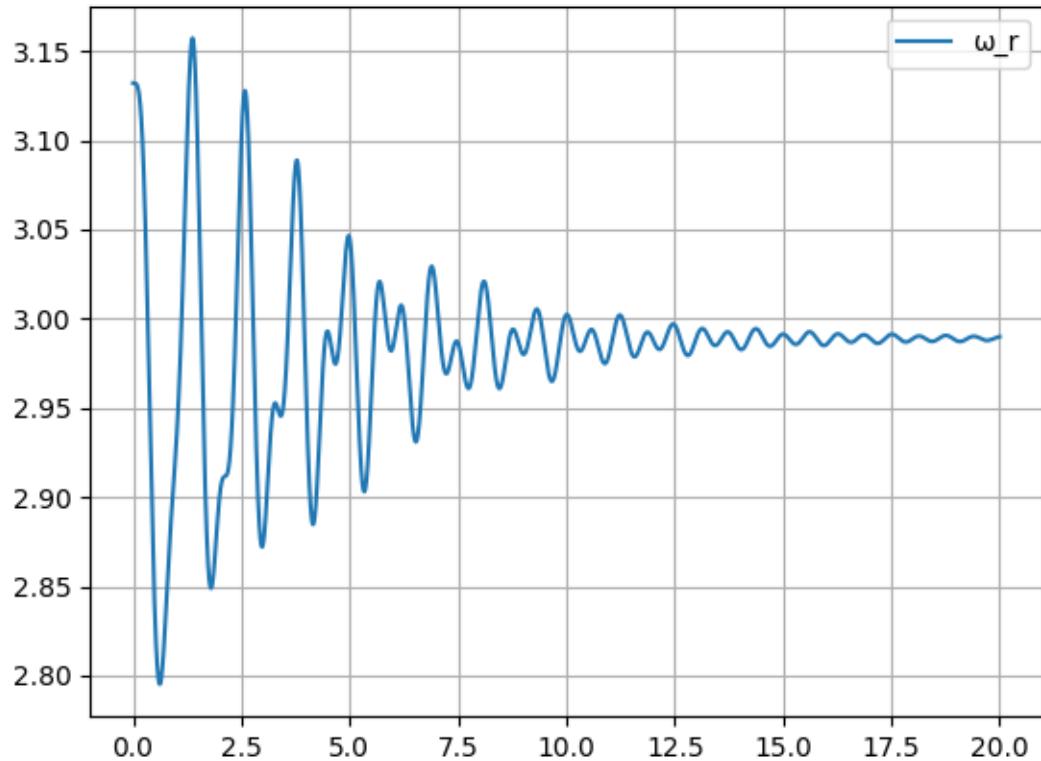


Figure C.21: Case 1 where  $m = 1 \text{ kg}$

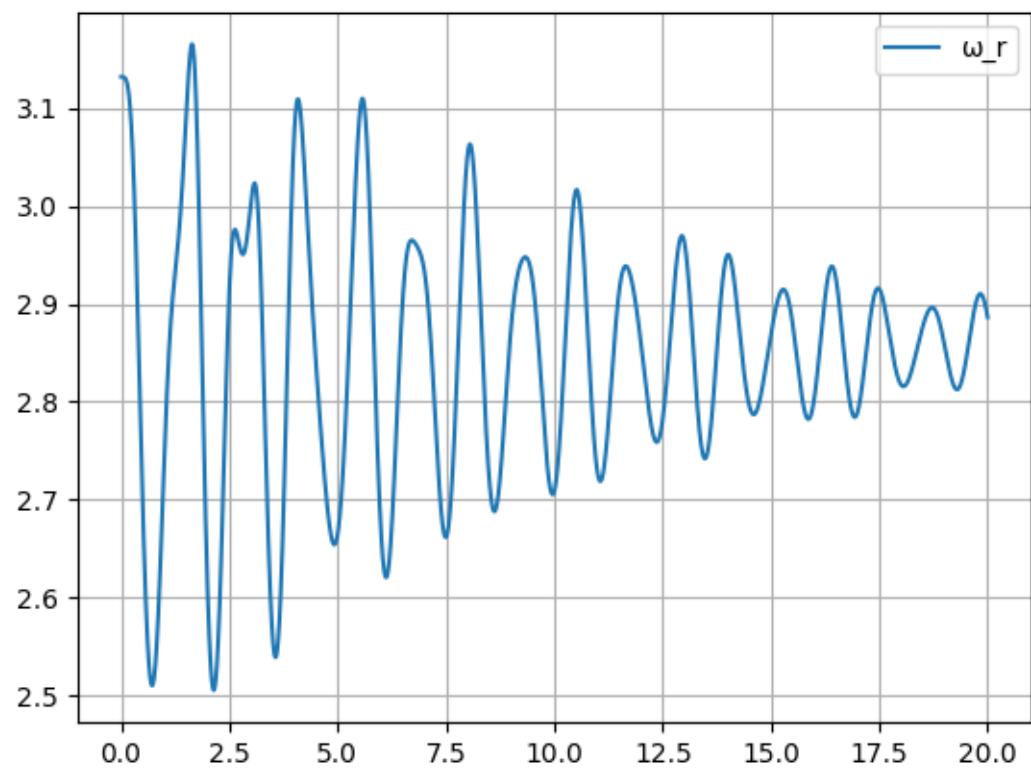


Figure C.22: Case 2 where  $m = 2 \text{ kg}$

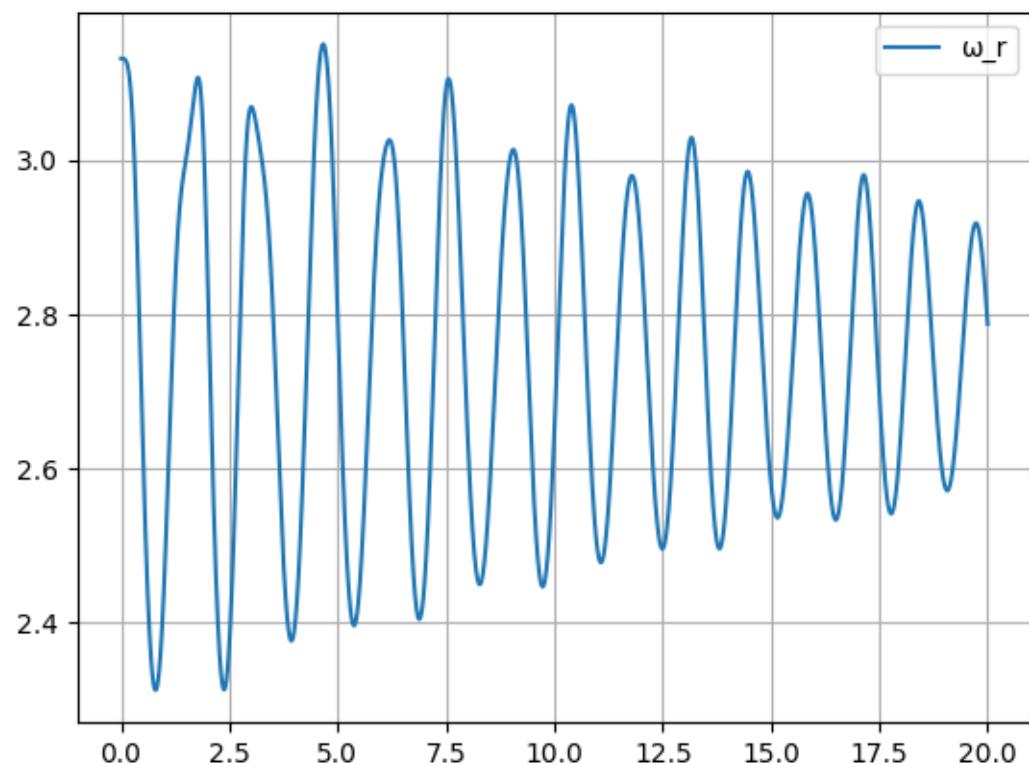


Figure C.23: Case 3 where  $m = 3 \text{ kg}$

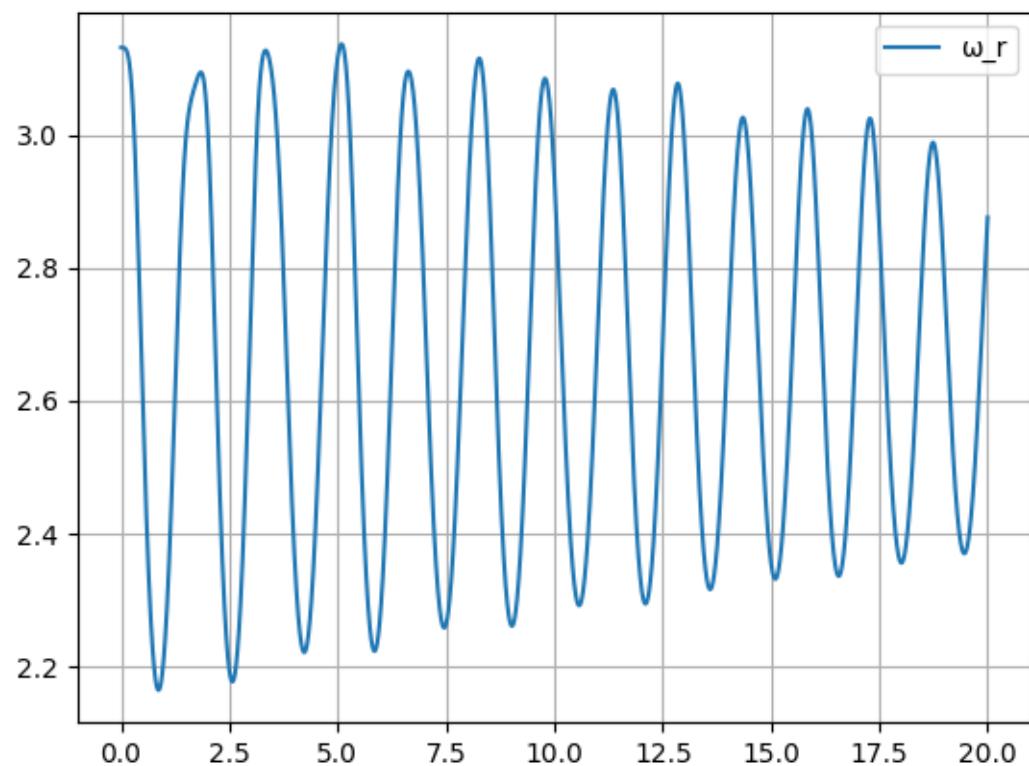


Figure C.24: *Case 4 where  $m = 4 \text{ kg}$*

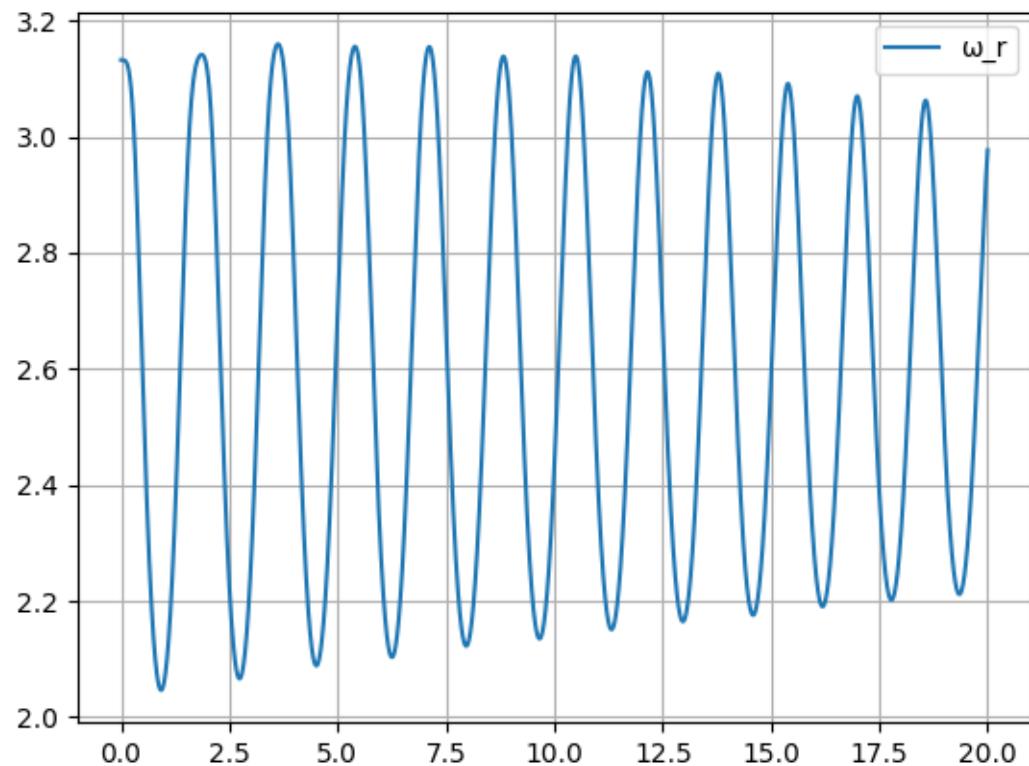


Figure C.25: Case 5 where  $m = 5 \text{ kg}$

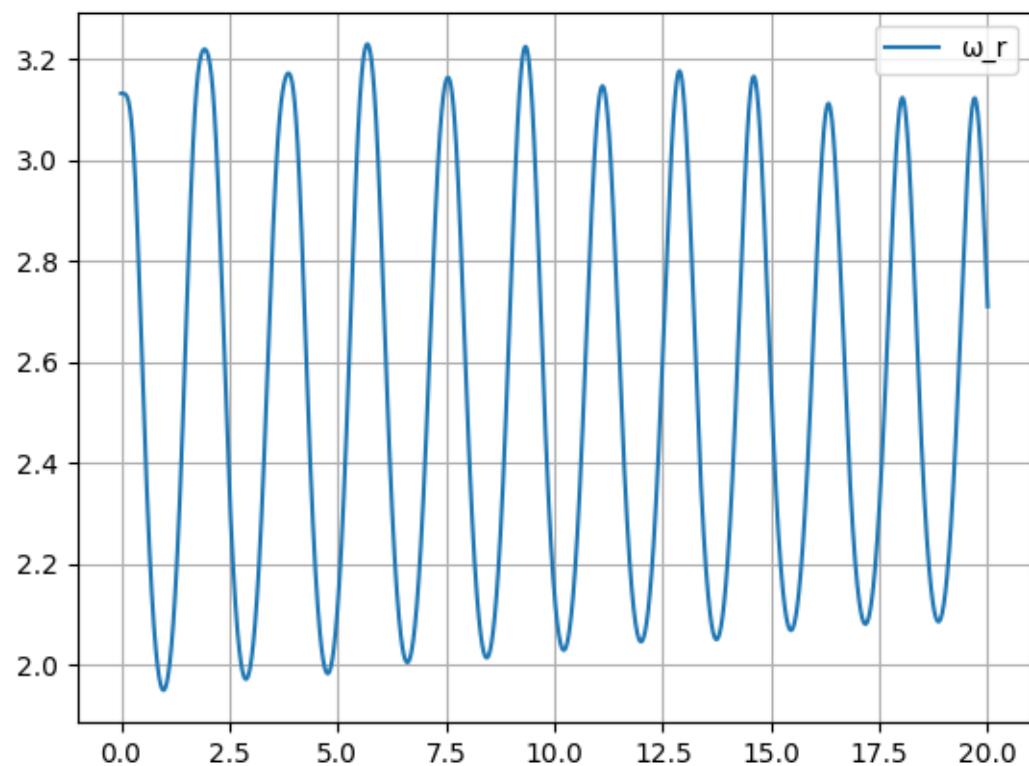


Figure C.26: Case 6 where  $m = 6 \text{ kg}$

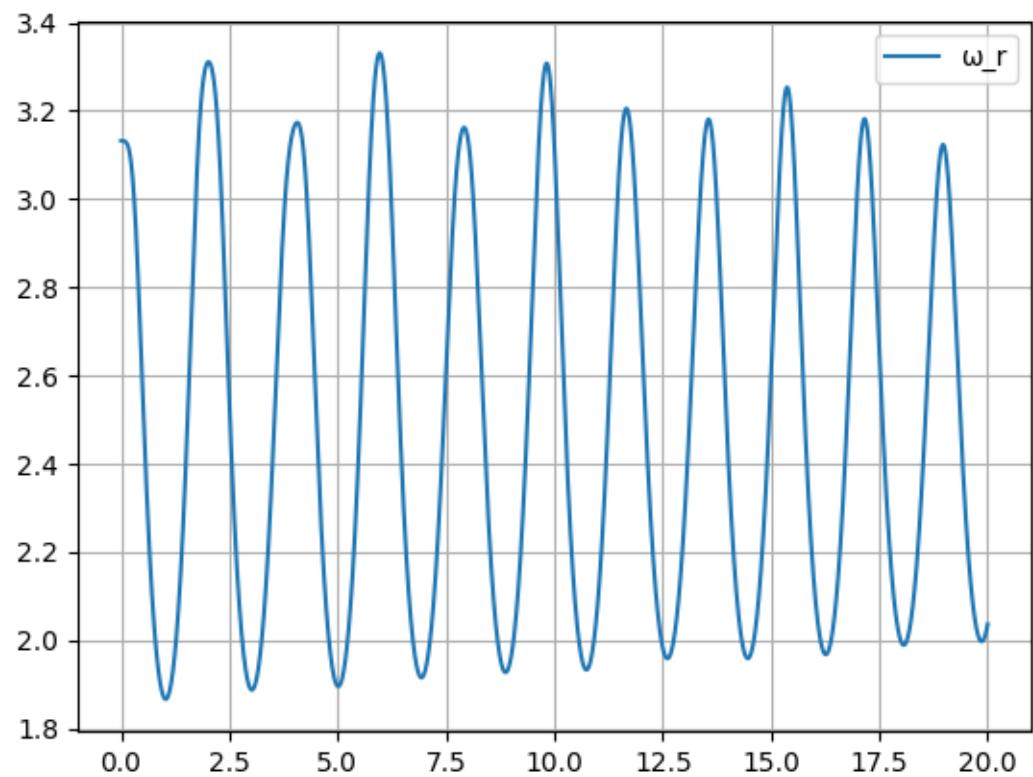


Figure C.27: Case 7 where  $m = 7 \text{ kg}$

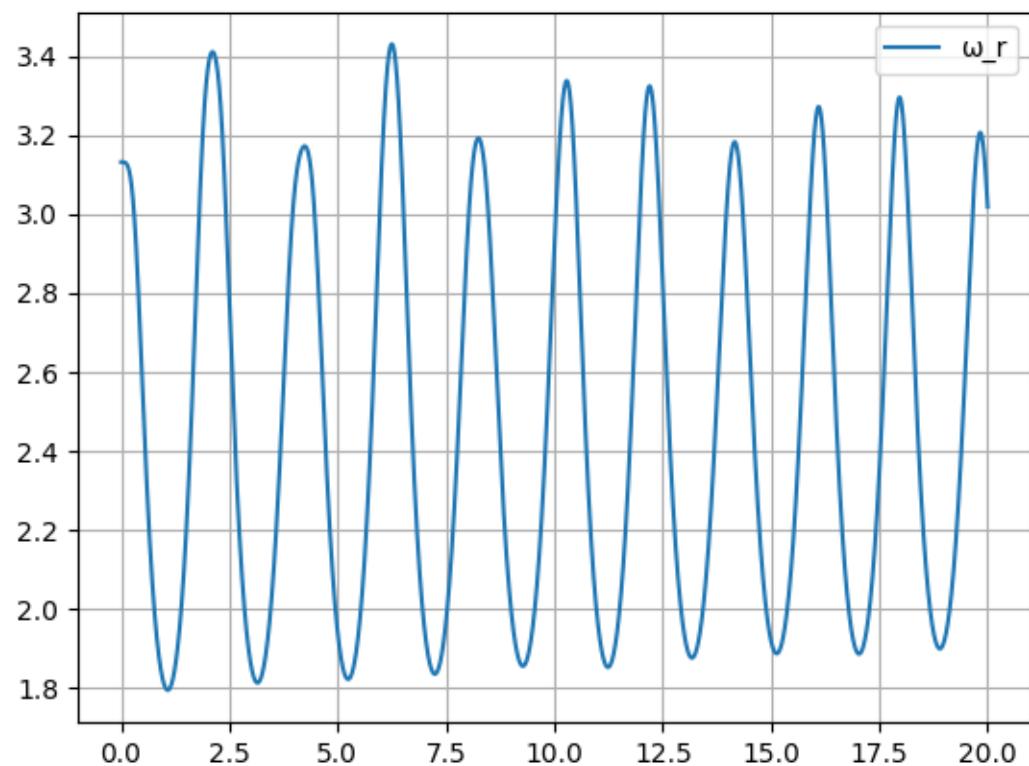


Figure C.28: Case 8 where  $m = 8 \text{ kg}$

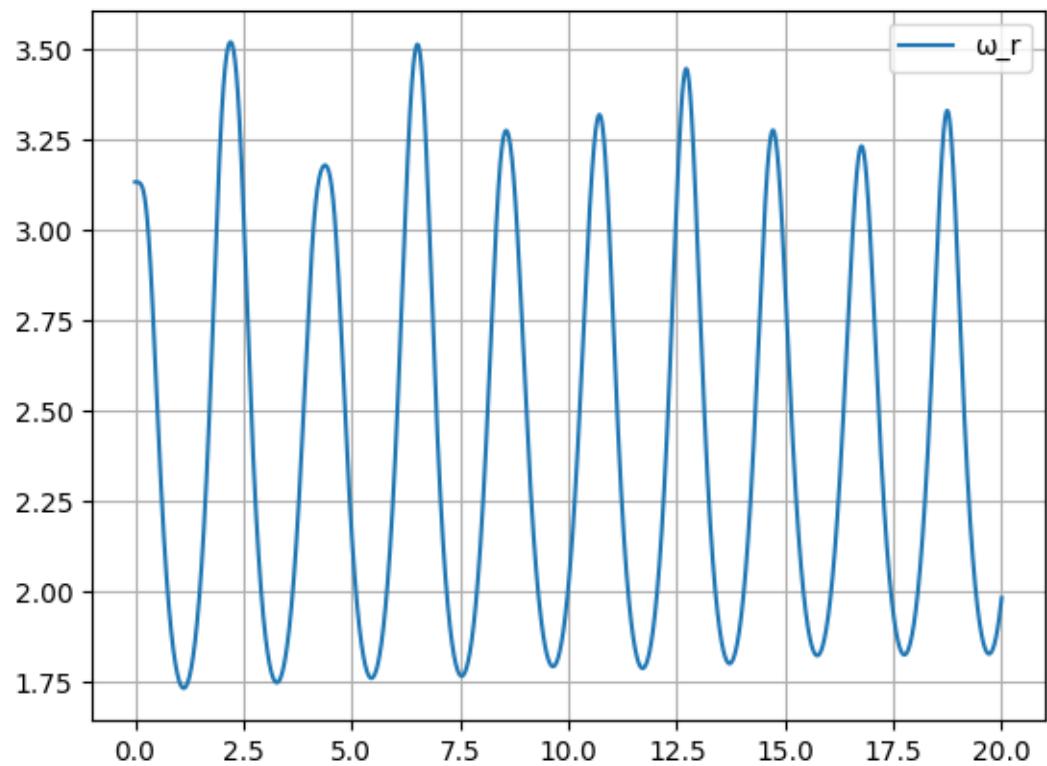


Figure C.29: *Case 9 where  $m = 9 \text{ kg}$*

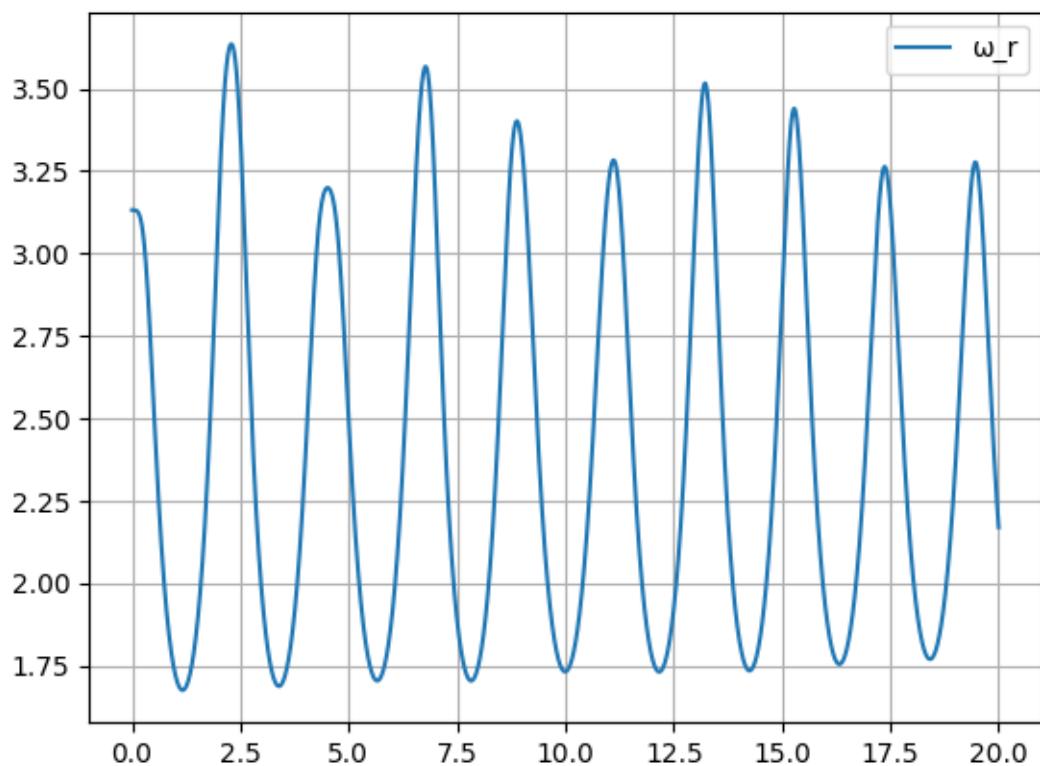


Figure C.30: *Case 10 where  $m = 10 \text{ kg}$*

#### C.4 *Absolute Frequency versus Time*

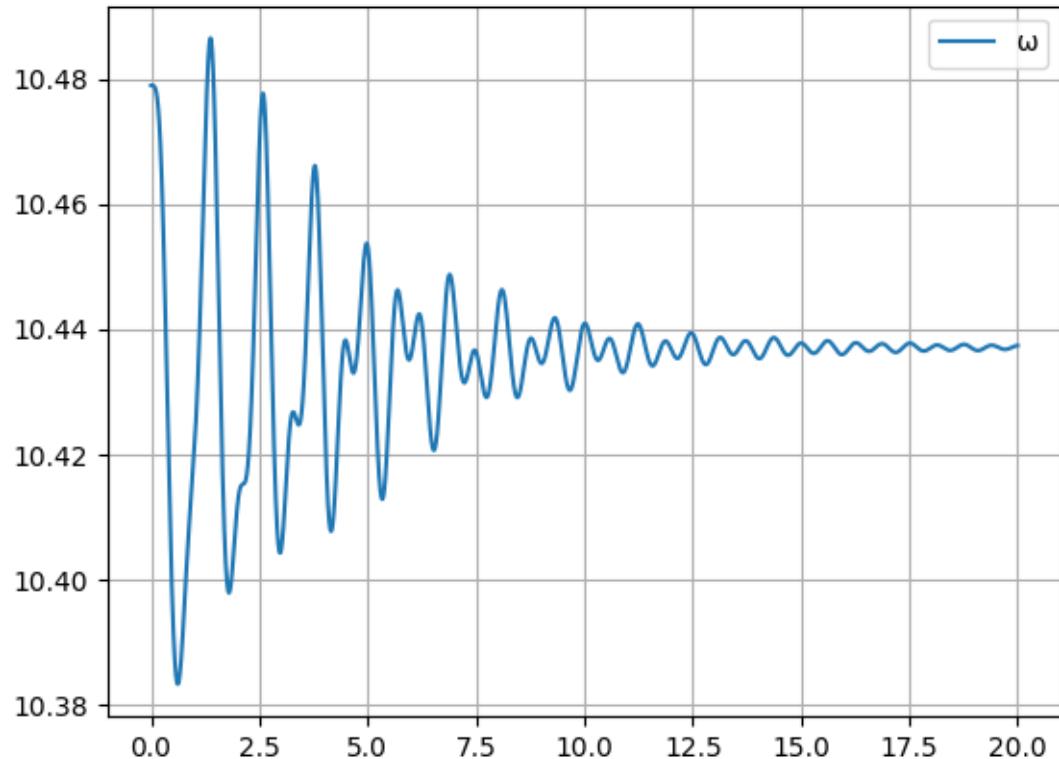


Figure C.31: *Case 1 where  $m = 1 \text{ kg}$*

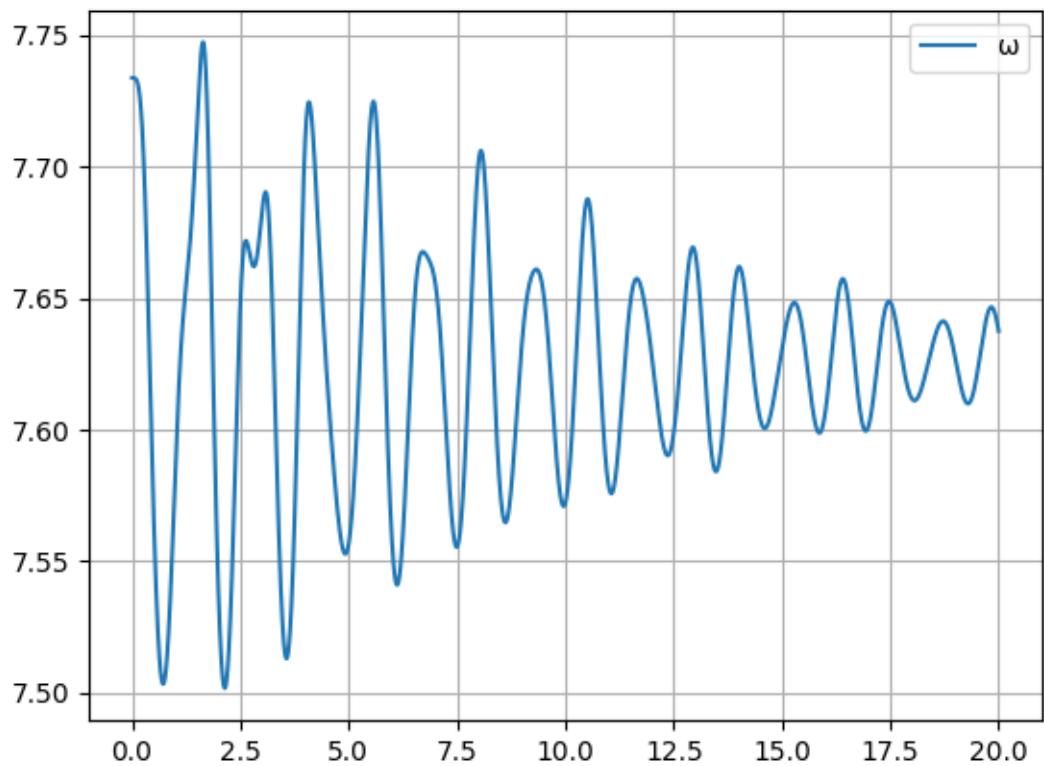


Figure C.32: *Case 2 where  $m = 2 \text{ kg}$*

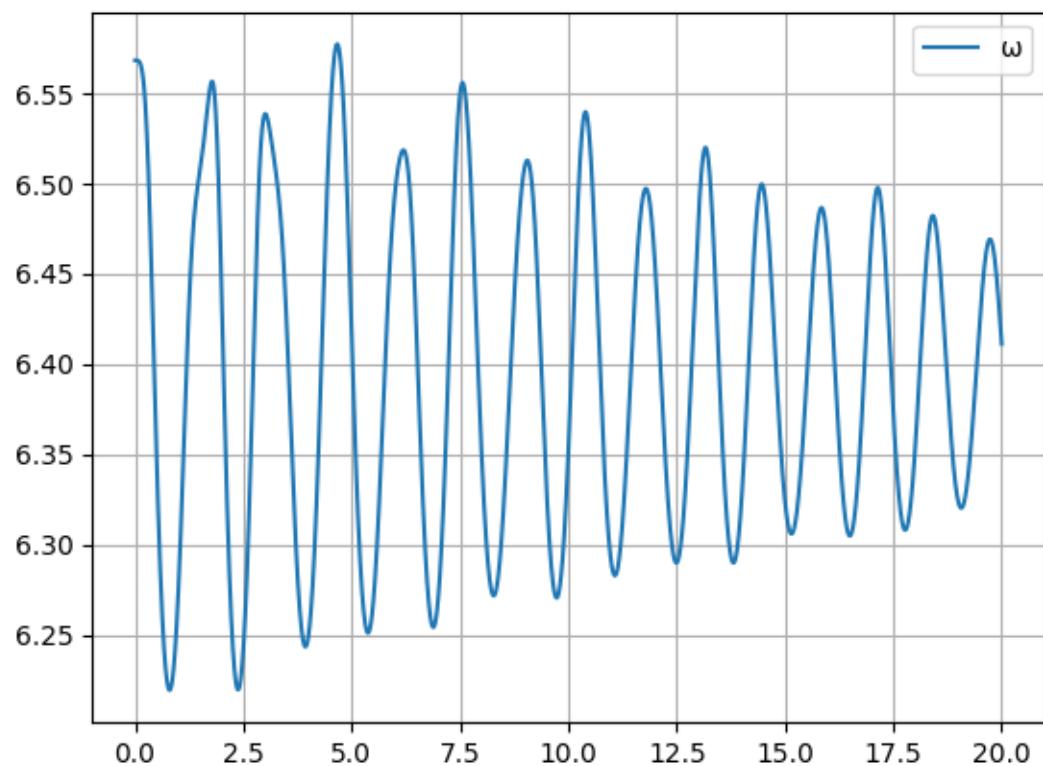


Figure C.33: Case 3 where  $m = 3 \text{ kg}$

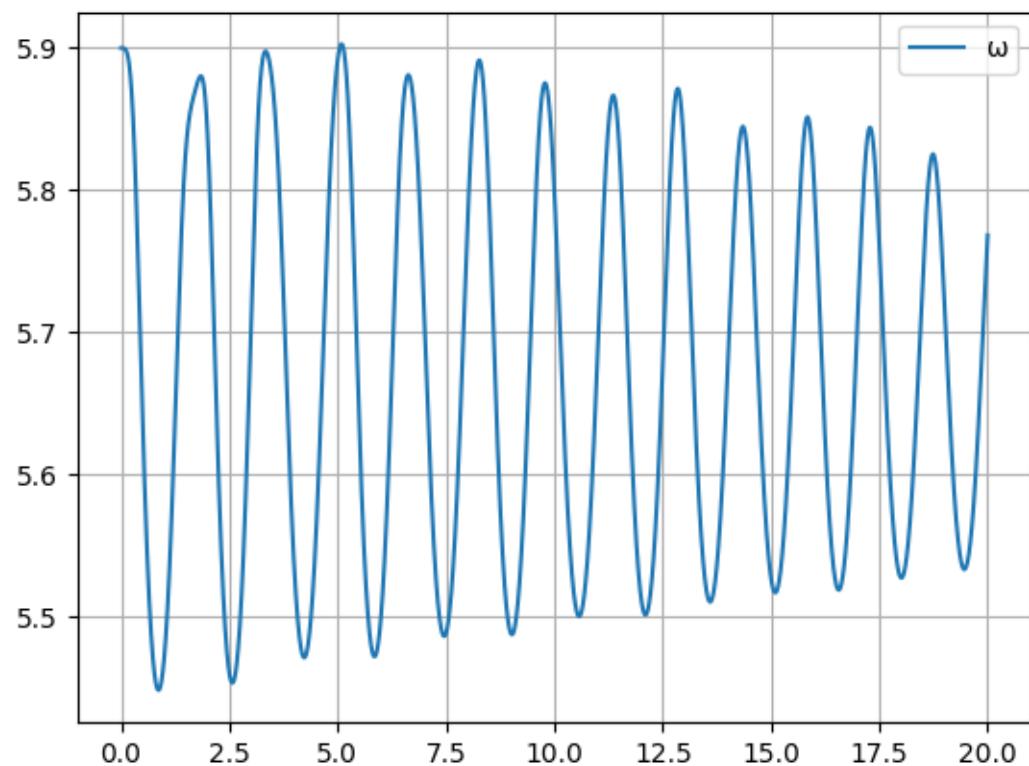


Figure C.34: Case 4 where  $m = 4 \text{ kg}$

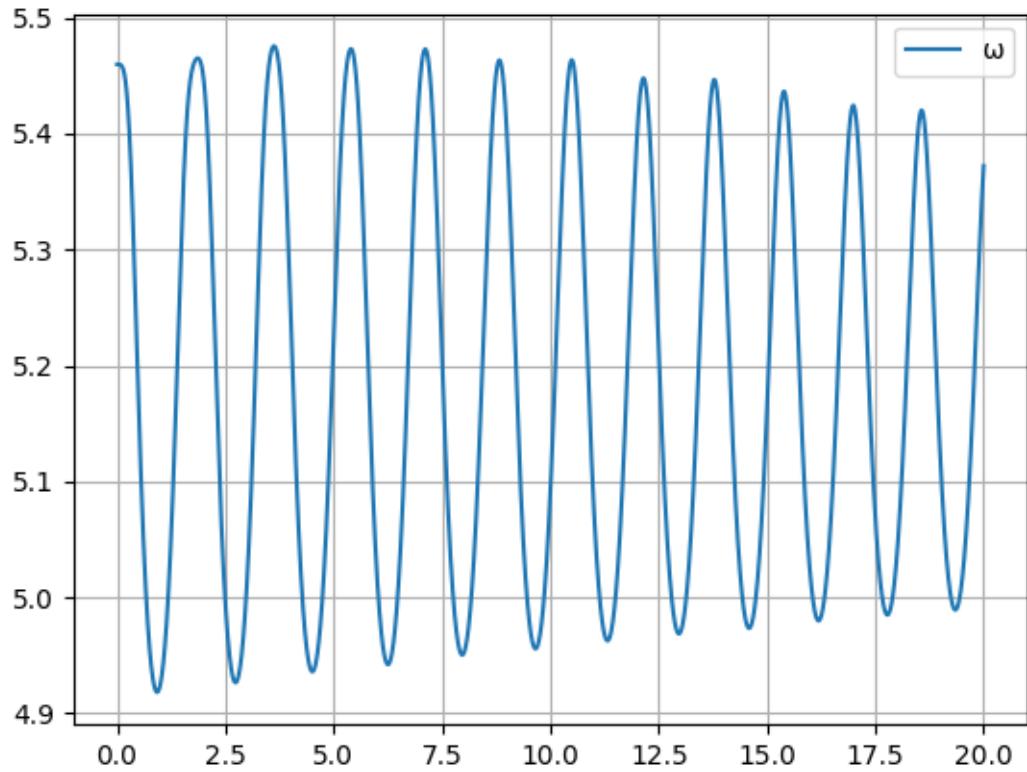


Figure C.35: Case 5 where  $m = 5 \text{ kg}$

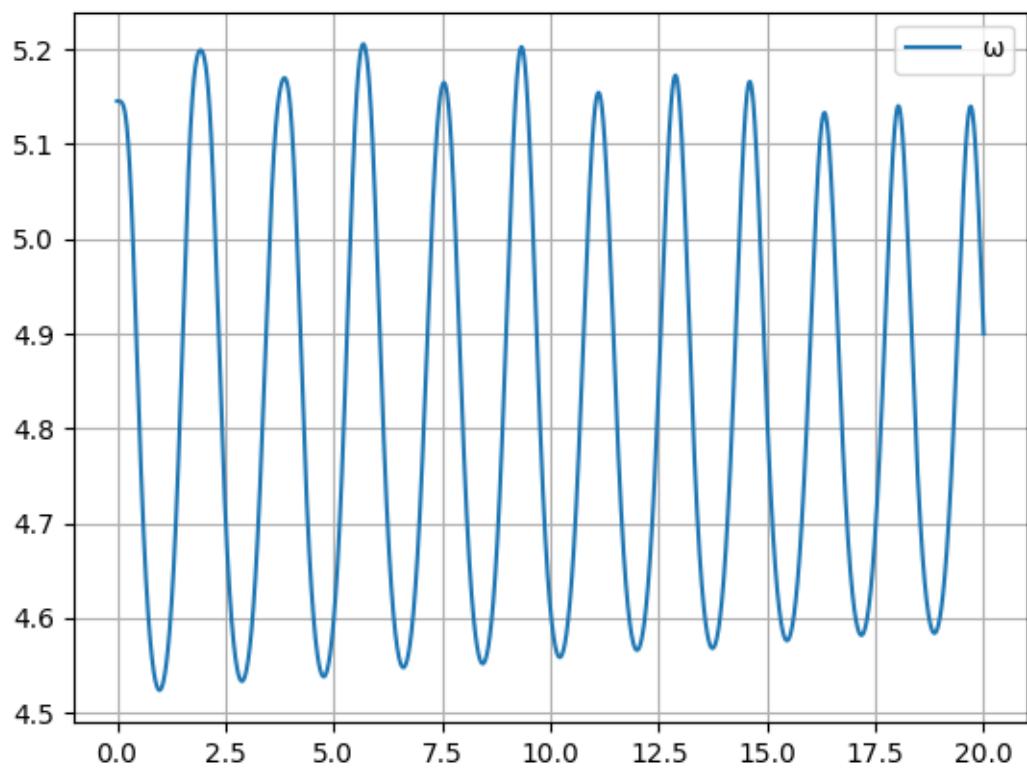


Figure C.36: Case 6 where  $m = 6 \text{ kg}$

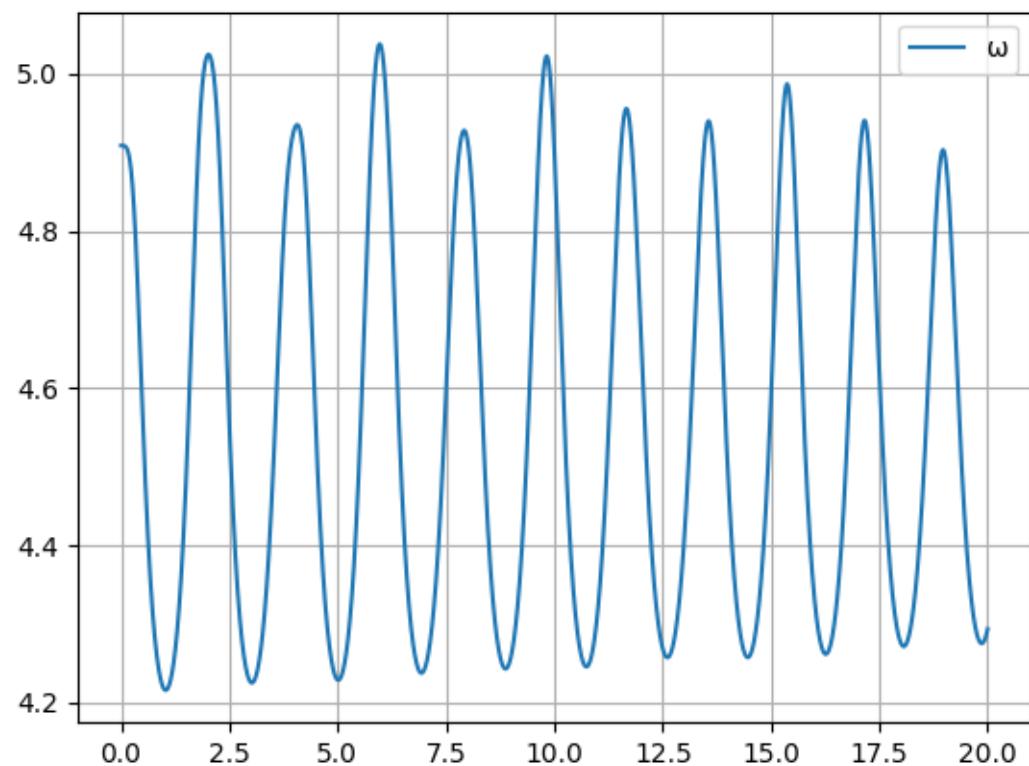


Figure C.37: Case 7 where  $m = 7 \text{ kg}$

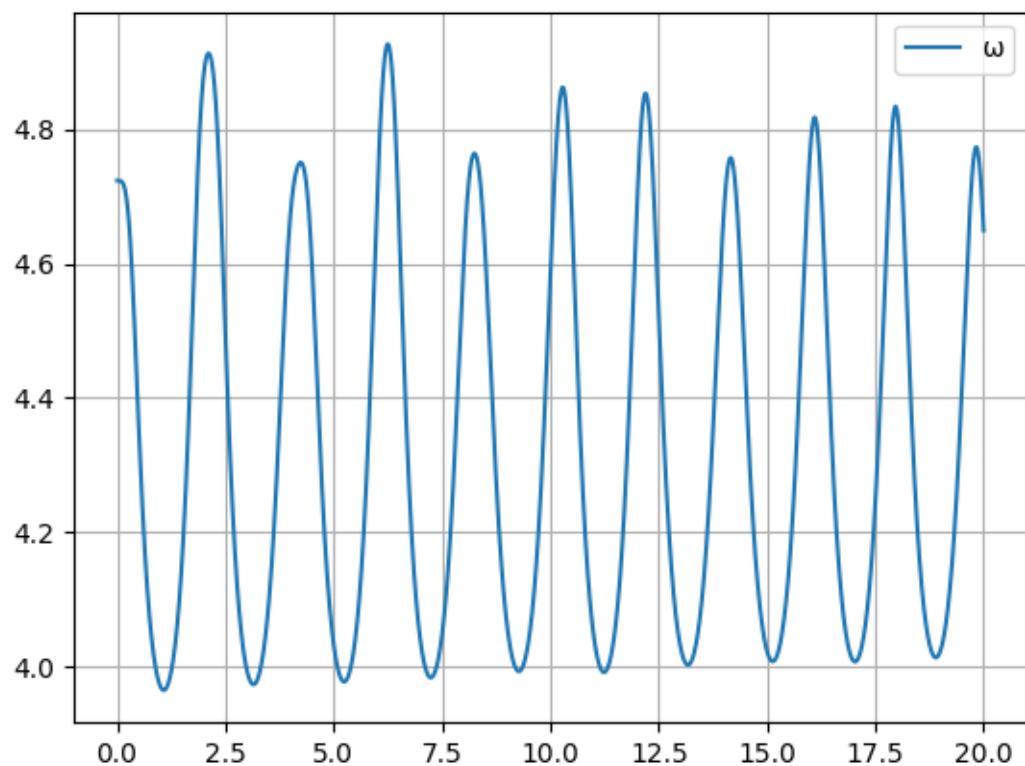


Figure C.38: Case 8 where  $m = 8 \text{ kg}$

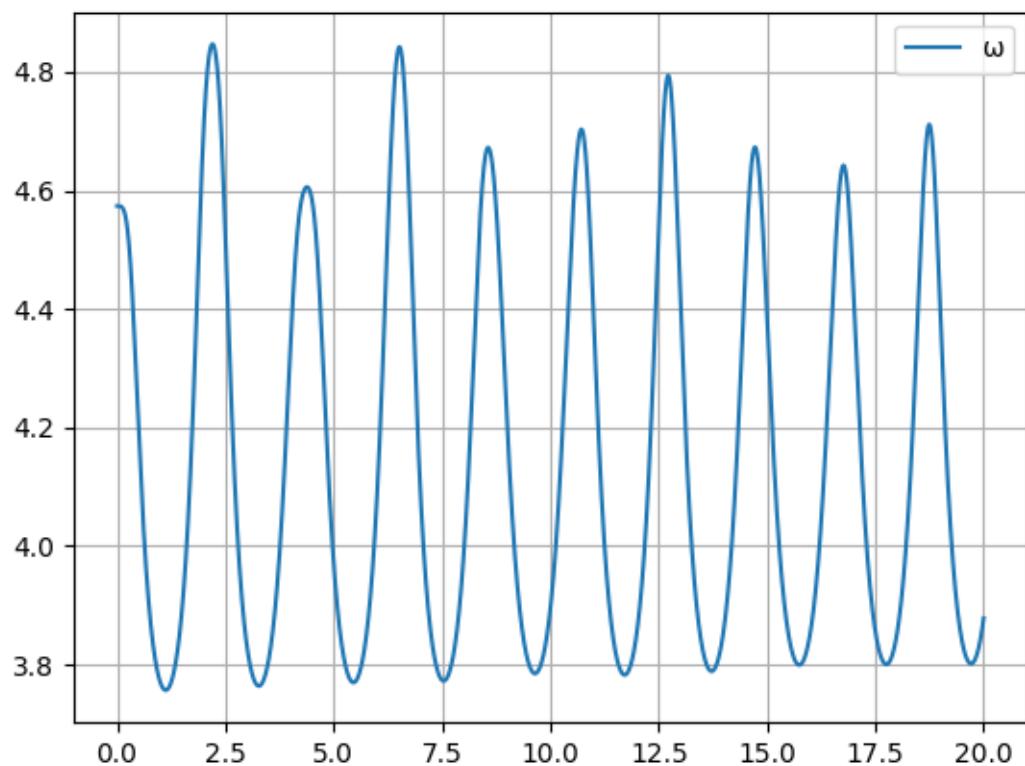


Figure C.39: Case 9 where  $m = 9 \text{ kg}$

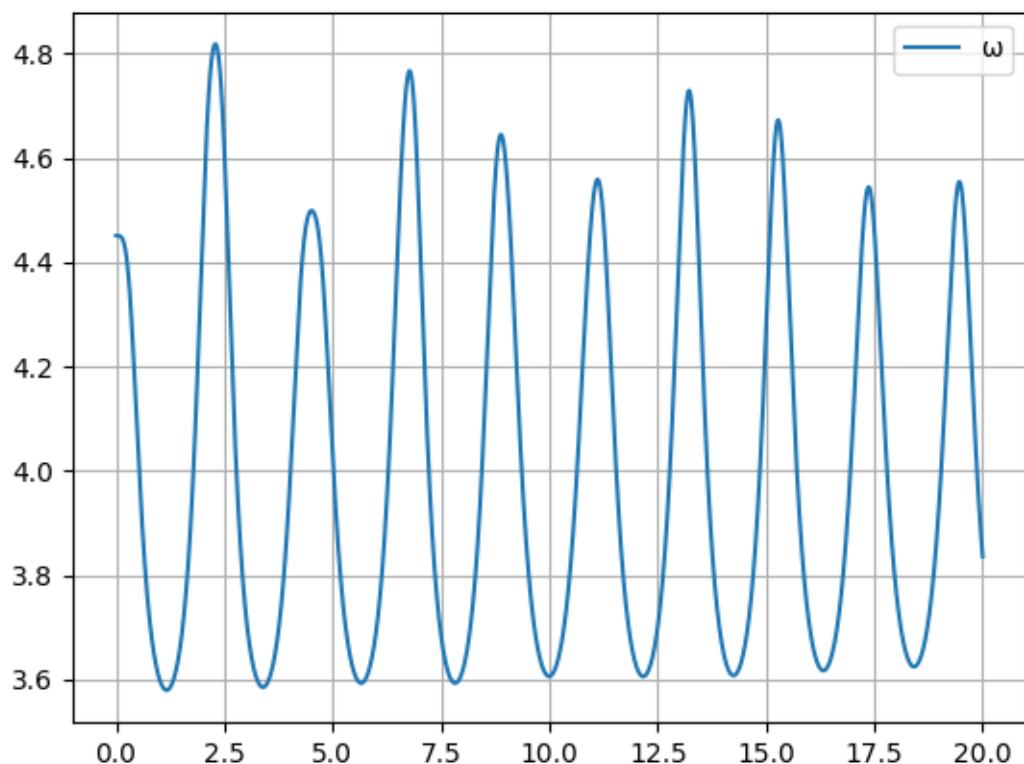


Figure C.40: Case 10 where  $m = 10 \text{ kg}$

## D *Derived Data*

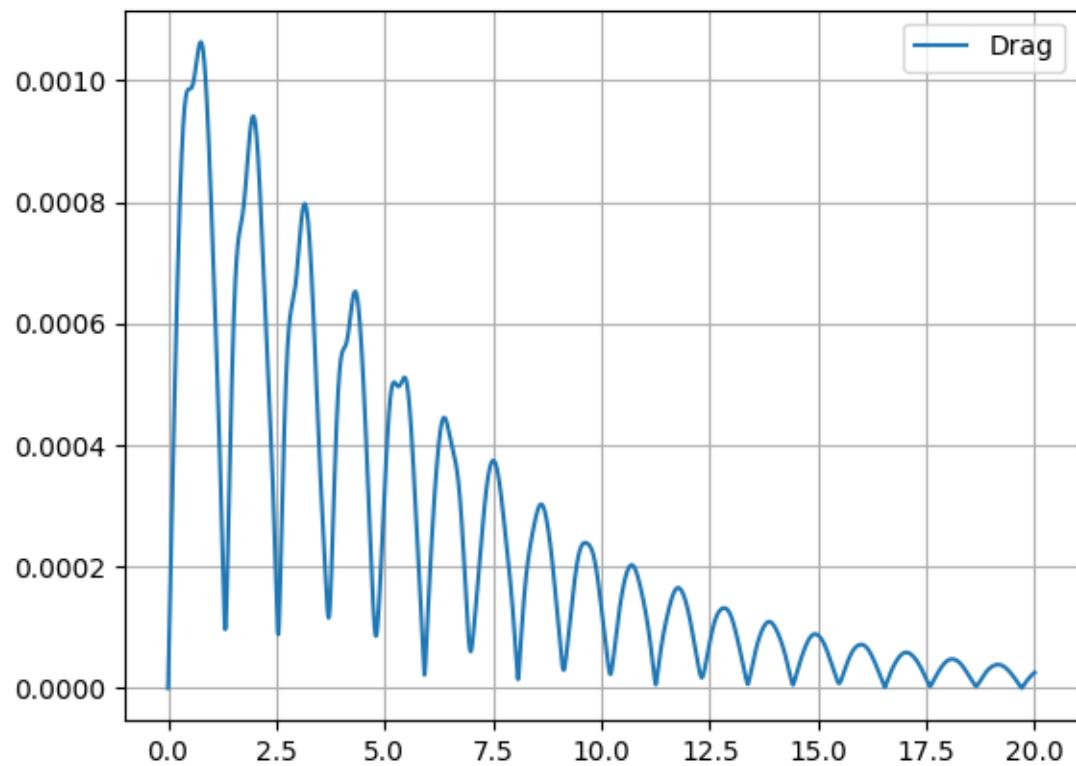


Figure D.1: *Case 1 where  $m = 1 \text{ kg}$*

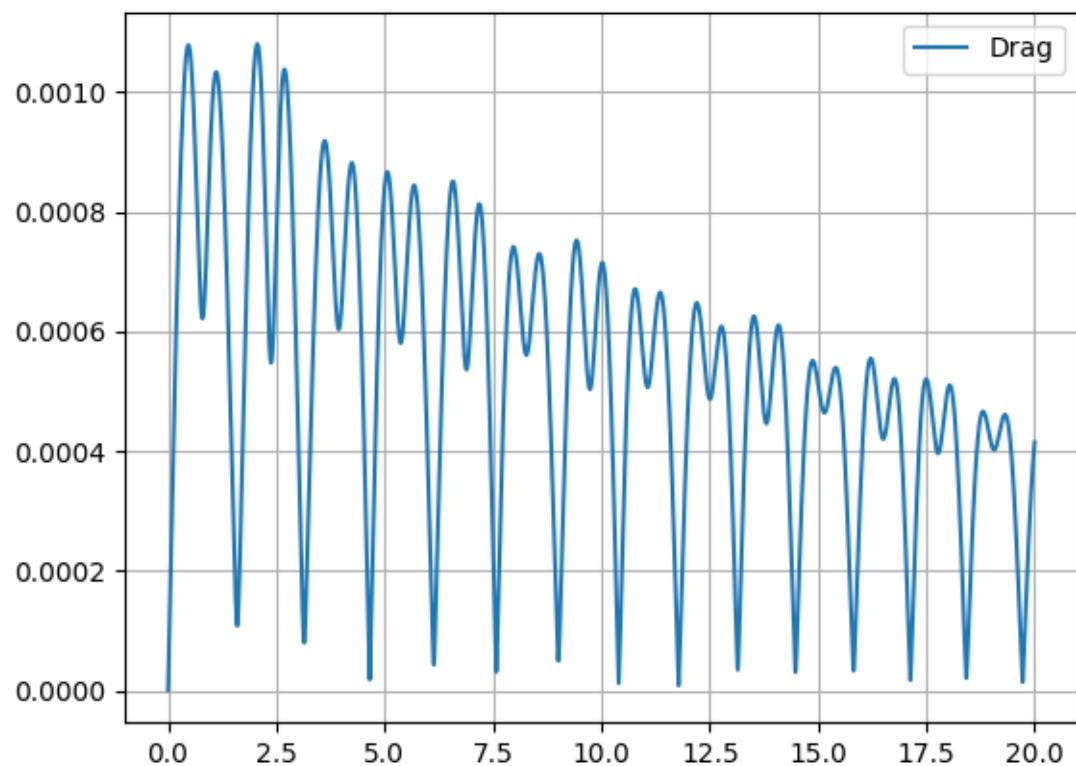


Figure D.2: Case 2 where  $m = 2 \text{ kg}$

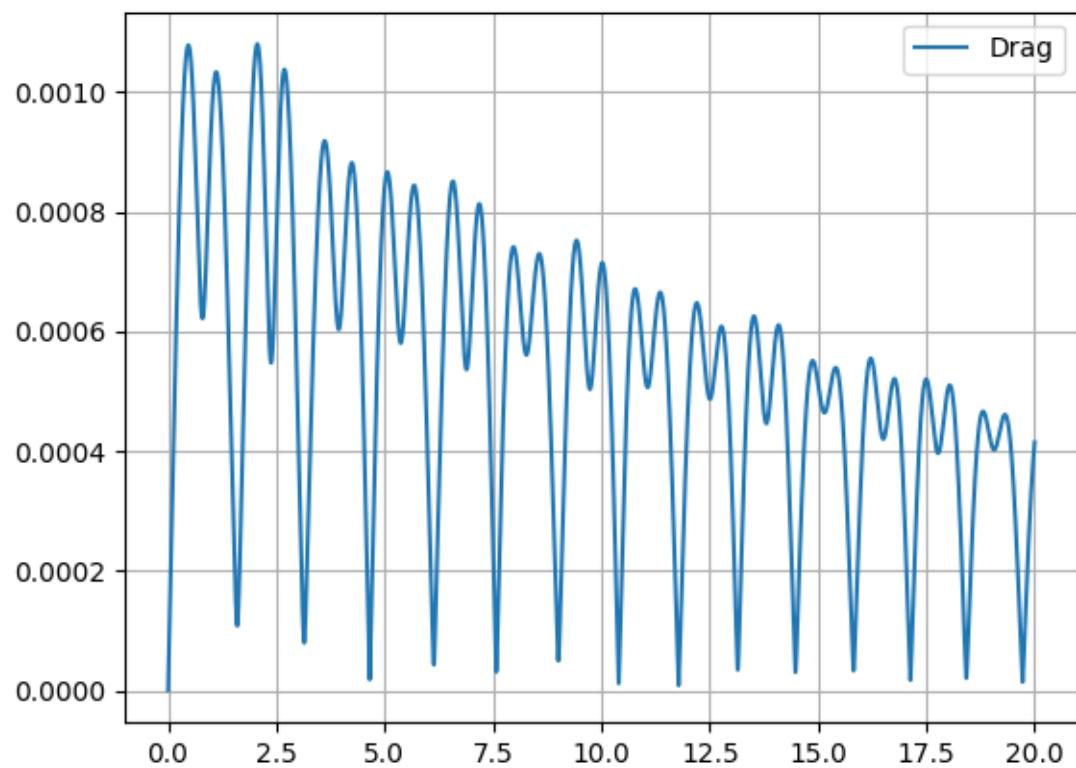


Figure D.3: *Case 3 where  $m = 3 \text{ kg}$*

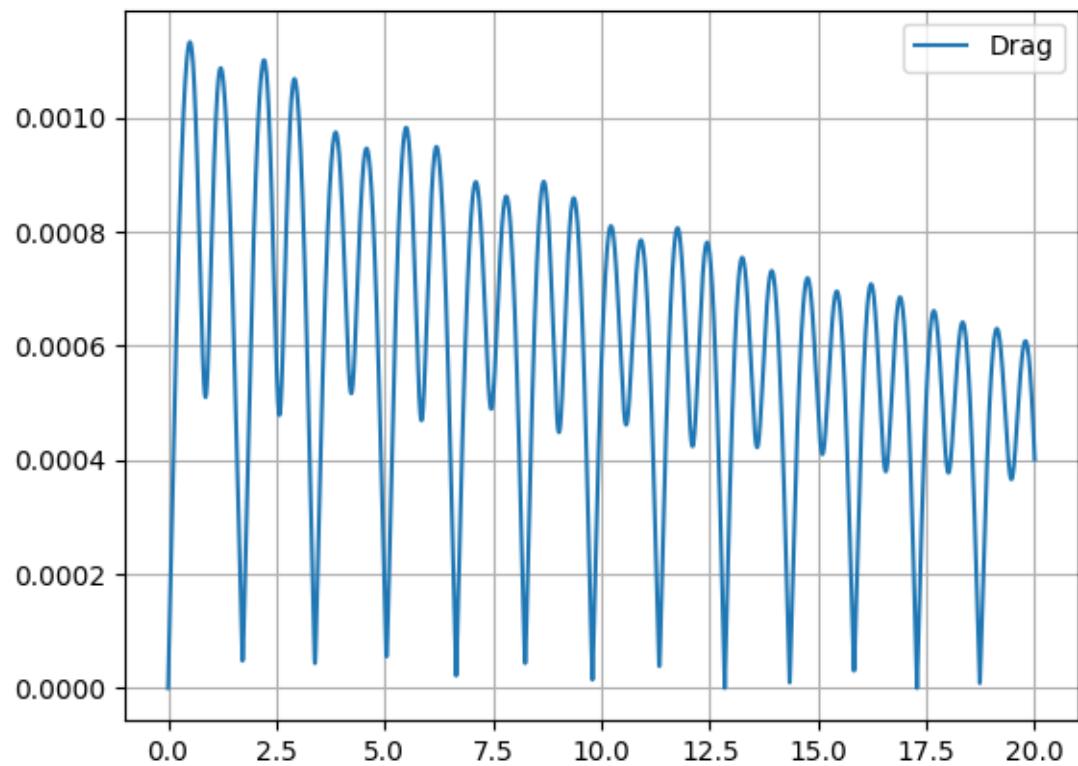


Figure D.4: Case 4 where  $m = 4 \text{ kg}$

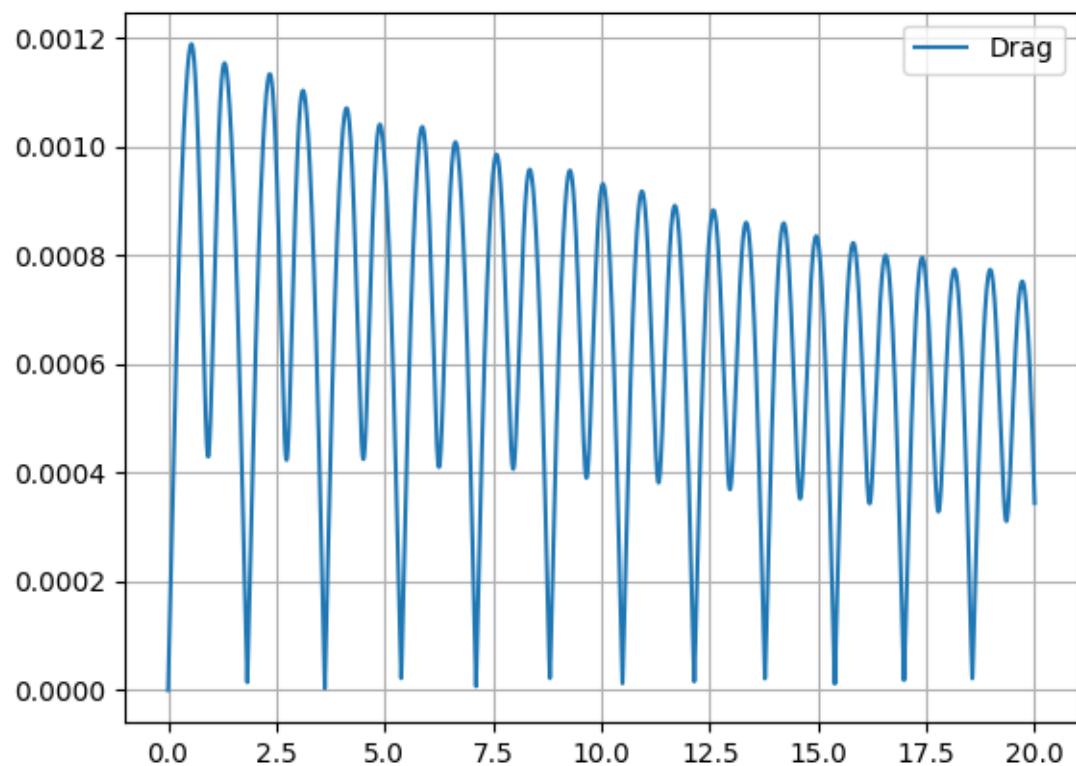


Figure D.5: Case 5 where  $m = 5 \text{ kg}$

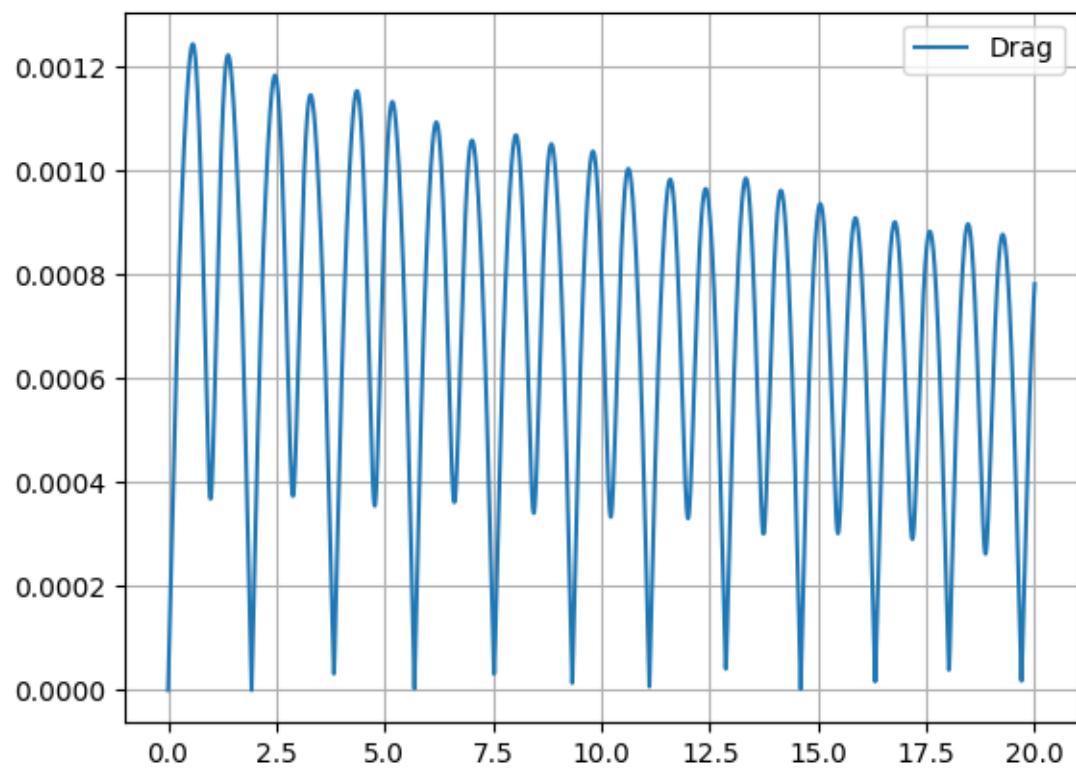


Figure D.6: Case 6 where  $m = 6 \text{ kg}$

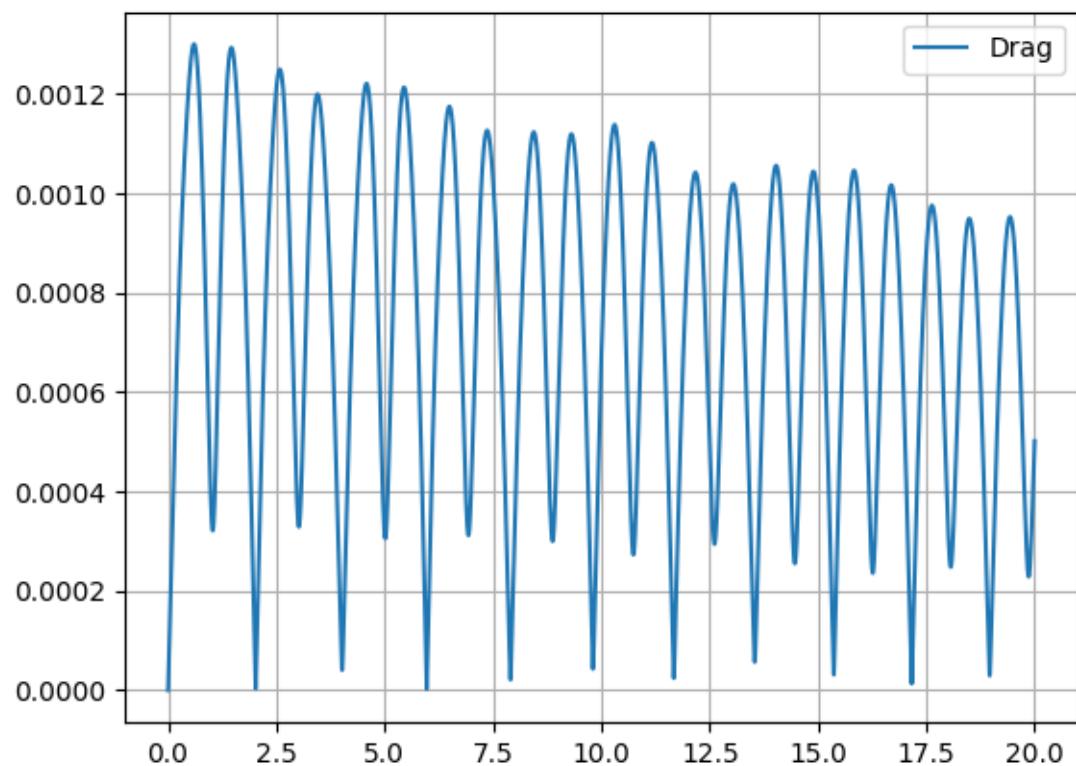


Figure D.7: Case 7 where  $m = 7 \text{ kg}$

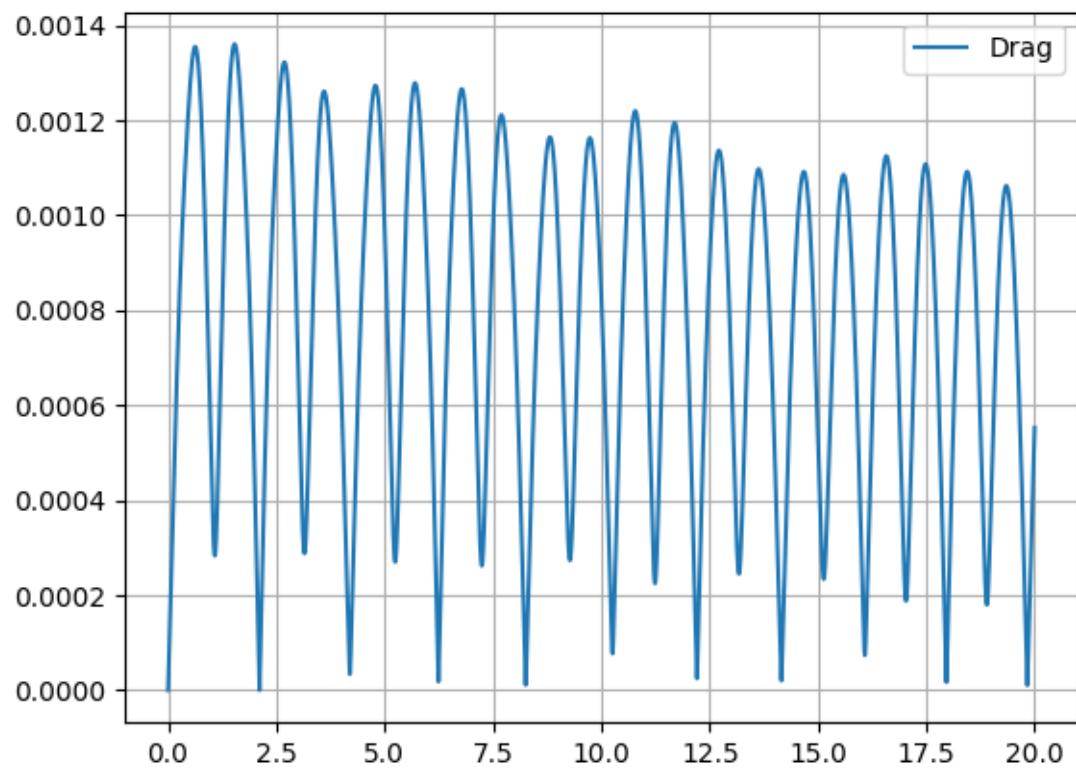


Figure D.8: *Case 8 where  $m = 8 \text{ kg}$*

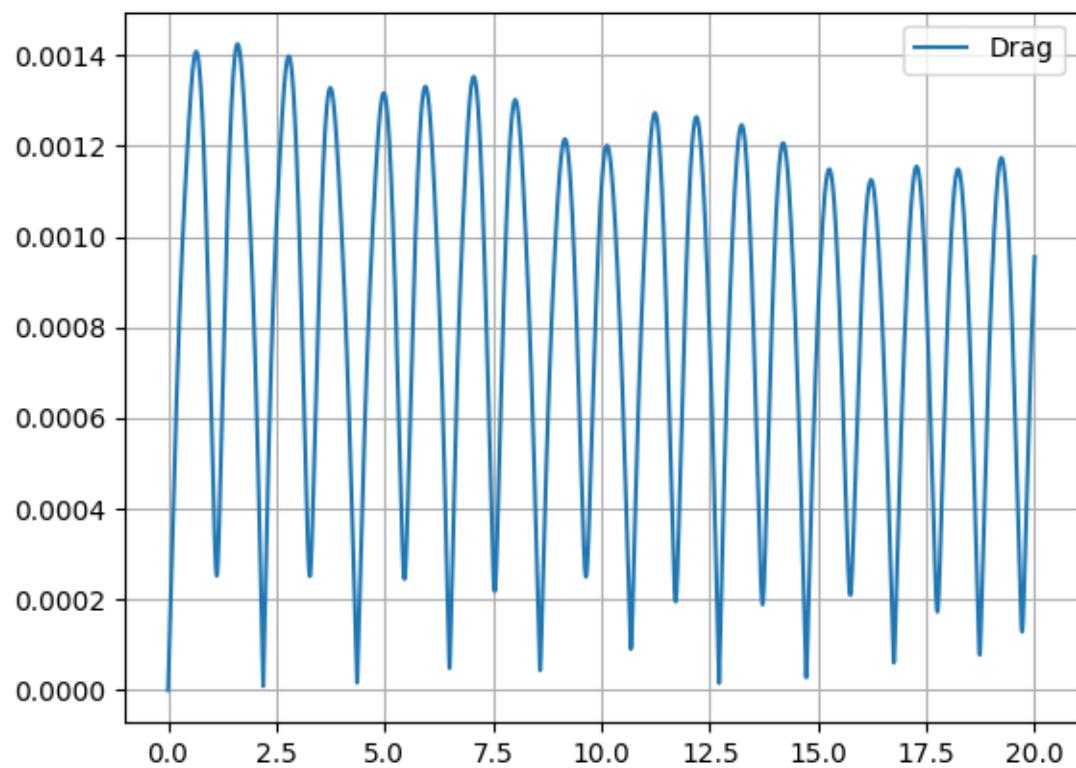


Figure D.9: *Case 9 where  $m = 9 \text{ kg}$*

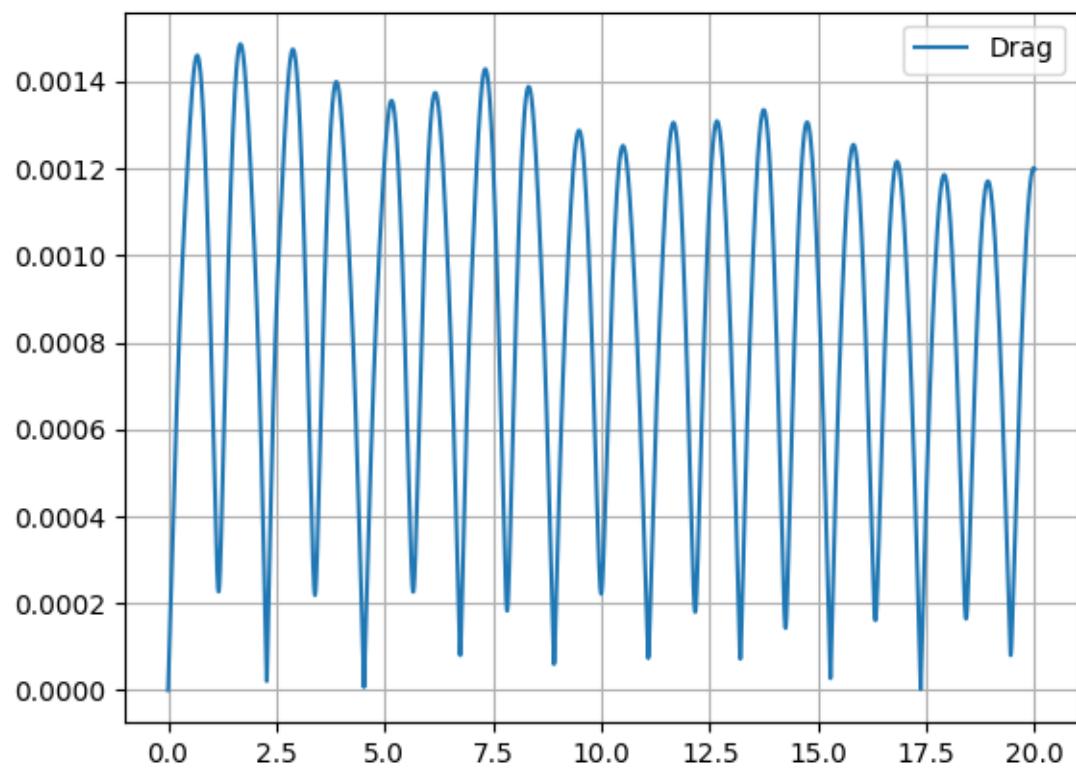


Figure D.10: *Case 10 where  $m = 10 \text{ kg}$*