Mathematics IA Ver 1.01

Systems of Ordinary Differential Equations

On the Solutions of Non-linear Ordinary Differential Equations

Mohammed Sayeed Ahamad

Batch of 2022

Contents

Abstract

We will discuss the effects of change in the magnitude of ***mass*** *on the magnitude of the* ***radial position***, ***angular position*** *and the* ***integral frequencies*** *of the system of the damped elastic harmonic oscillator, by collecting raw data on the measurement of* ***radial displacement***, ***angular displacement***, ***angular frequency*** *and* ***absolute frequency*** *with respect to time under certain controlled spaces with* ***defined standard initial conditions***.

1 *Introduction*

I have chosen this research question because the fundamental relation between the mechanics of the damped elastic pendulum, its complex dynamics, its unstable chaotic behaviour and its excruciating difficulty in possibly modeling this behaviour has led me to explore this field, as this phenomenon is exciting, unique and wonderful, with ample scope to carry out research and collect data both quantitatively and qualitatively.

2 *Aim*

To ***collect experimental raw data*** *relating to the variation in* ***radial displacement****, the variation in* ***angular displacement****, the variation in* ***angular frequency*** *and the variation in* ***absolute frequency*** *in a lab controlled setting of a* ***damped elastic harmonic oscillator*** *and to compare it with* ***computer simulations*** *of the same and hence study its* ***different states*** *and further to study its* ***sensitivity*** *to* ***initial conditions*** *and calculate the value the* ***inconsistencies*** *in* ***accuracy*** *of the* ***mathematically modeled***

computer simulation *to that of the* ***actual experiment*** *and vice-versa.*

3 *Background Research*

Before beginning this investigation we must first know some important facts, formulae, laws and be familiar with the concepts that are to be incorporated in this investigation.

3.1 *The Lagrangian function*

Firstly, we must know that the fundamental concepts of Lagrangian mechanics and Lagrange's equations.

The Lagrangian is a function that is mathematically defined as,

$$L = T - V$$

Where, L is the Lagrangian, T is the kinetic energy of the system and V is the potential energy of the system.

The Lagrange's equation of second kind or the Lagrange-Euler Equation is mathematically defined as,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = Q_j$$

Where q_j *is the generalized coordinates and* Q_j *is the generalized forces.*

3.2 *Generalized coordinates*

According to our specific system that we are investigating, we have two generalized coordinates in polar coordinate representation, θ *and* x.

θ represents the angle formed by the spring component with the fictitious normal of the system at any given time and is a function of time (time-dependent).

x represents the extension in the spring component with the equilibrium length at any given time and is a function of time (time - dependent).

3.3 *Generalized forces*

According to our specific system that we are investigating, we only have one generalized forces (non-conservative force), the frictional force acting on the path of the system in the form of drag or fluid resistance.

Our frictional force can be embedded in Lagrange's equations in the form of the Q_j term aided with the utilisation of the Rayleigh dissipation function.

As there isn't any other non-conservative force, the net total sum of the Lagrange's equations with the incorporation of the Rayleigh dissipation function would equal zero.

3.4 *Air Drag/Fluid Resistance*

Air Drag/Fluid Resistance is the force acting opposite to the relative motion of any object moving in any fluid medium. Drag force is proportional to the square of velocity, as we are dealing with relatively high-speeds, which can be inferred from the huge Reynolds's number.

Drag forces decrease fluid velocity relative to the solid mass in the fluid's path.

The type of Drag in play in this system is that of an underdamped ($\zeta < 1$) oscillator with viscous drag.

***Note**: ζ here, in the context of Drag forces and resistive forces symbolizes the damping ratio.*

The general Drag equation is mathematically defined as,

$$F_D = \frac{1}{2}\rho v^2 C_D A$$

Where F_D is the Air/Fluid resistance between the mass and the fluid, ρ is the density of the fluid, v is the speed of the object relative to the fluid, C_D is *velocity decay constant (damping constant) and A is the cross sectional area.*

***Note**: The Drag force can be embedded and modeled into the Euler-Lagrange equation by the aid of the Rayleigh dissipation function and can be generally modeled using Stroke's Law, as we are using object masses that are spherical.*

3.5 *Rayleigh dissipation function*

Secondly, we must also know that,

*If the frictional force on a particle with velocity $\vec{v}$ can be written as $\vec{F_f} = -\vec{k} \cdot \vec{v}$, the Rayleigh dissipation function can be defined for a system of **n** particles as,*

$$R/D = \frac{1}{2} \sum_{i=0}^{n} C_D v_i^2 = \frac{1}{2} C_D \sum_{i=0}^{n} v_i^2$$

Where R or D is the Rayleigh dissipation function which is a function used to handle and model the effects of velocity-proportional frictional forces in Lagrangian mechanics, where C_D is the velocity decay constant (damping constant) and $\sum_{i=0}^{n} v_i^2$ is the sum of all velocities squared in all degrees of freedom pertaining to a mechanical system with n being the number of degrees of freedom in the system.

***Note**: An alternate representation could be also used to represent C_D in the form of a $n \times n$ matrix. Mathematically,*

$$C_D = \begin{bmatrix} k_i & 0 & 0 & \dots & 0 \\ 0 & k_j & 0 & 0 & 0 \\ 0 & 0 & k_k & 0 & 0 \\ \vdots & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & k_n \end{bmatrix}$$

3.6 *Stroke's Law*

We must know the Stroke's Law and the relation it defines to accurately model the change in Air Drag/Fluid Resistance with time.

Stroke's Law states a relation for the frictional force (drag force) exerted on spherical objects with very small Reynolds numbers in a viscous fluid. Mathematically,

$$F_D = 6\pi\mu R v_D$$

Note: *This relation is only applicable in this scenario because the mass that we are dealing with is spherical.*

Where F_D is the Air/Fluid resistance between the mass and the fluid, μ is the dynamic viscosity of the fluid, R is the radius of the mass and v_D is the flow velocity relative to the mass.

3.7 *Logarithmic Decrement*

The system exhibits an interesting feature, that of constant logarithmic decrements, that is,

$$\ln\frac{x_1}{x_2} = \ln\frac{x_2}{x_3} = \ln\frac{x_3}{x_4} = \cdots\cdots\cdots$$

Where x_n and x_{n+1} are the amplitudes of any two successive peaks ($n \in \mathbb{R}$).

Also interestingly that for any two successive peaks of a graphical representation of any parameter versus time employed in this investigation, if we define,

$$\delta = \ln\left(\frac{x_n}{x_{n+1}}\right)$$

For a graphical representation of any parameter employed in this investigation.

Then,

$$\zeta = \frac{\delta}{\sqrt{\delta^2 + (2\pi)^2}}$$

With the above findings and definitions, we can create an expression for percentage overshoot, that is,

$$Percentage\ Overshoot = 100 \cdot \exp\left(-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}\right)$$

Consequently we can find an expression for ζ in terms of ***percentage overshoot*** *(PO). That is,*

$$\zeta = \frac{-\ln\left(\frac{PO}{100}\right)}{\sqrt{\pi^2 + \ln^2\left(\frac{PO}{100}\right)}}$$

Note: *ζ here, in the context of Drag forces and resistive forces symbolizes the damping ratio.*

3.8 *Computer Simulation Software*

*To model the chaotic system of the Damped Harmonic Elastic Pendulum we would need the aid of **computational technology**. Software's such MATLAB, Mathematica or some simple, eccentric computer programming language code in a emulatable script format, that would compute and yield solutions for necessary simulations that are required.*

*For the purpose of this investigation we shall be using a **Python script**, the source code of which can be found in **appendix** **??**, to model and simulate the dynamic and chaotic system.*

4 Variables

Physical Quantity	*Symbol*
Spring constant	k
Equilibrium spring length	l_0
Extension length	x
Angular displacement/Polar Angle	θ
Radius of spherical mass	r

Table 4.1: *General physical quantities employed in this investigation*

Note: *In theory, the equilibrium spring length and the radius of the spherical mass incorporated in research could of any arbitrary value. For the purpose* *of this investigation, we shall specifically use springs of equilibrium spring length of 1 m and masses of radius* 5×10^{-2} *m.*

Independent Variable	*Dependent Variable*	*Controlled Variable*
Time	*Extension length*	*Temperature*
-	*Displacement*	*Fluid medium*
-	*Angular displacement*	*Reference point*
-	*Absolute frequency*	-

Table 4.2: *Segregation of employed variables as IV, DV or CV*

Note*: The spring constant, equilibrium length and the radius of the mass is no longer variable as we have defined a set value to it.*

5 Initial Conditions

We define certain initial conditions for this dynamic system and convert this system to an initial value problem (IVP) system.

$$x(0) = 0$$

Where $x(0)$ *is the initial extension length of the spring at time t = 0*

$$\theta(0) = \pi/2$$

Where $\theta(0)$ is the initial angular displacement of the system at time t = 0

6 *Equations*

6.1 *Equations of Motion*

We know that,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} + \frac{\partial R}{\partial q_j} = 0$$

Where L is the Lagrangian, q_i is the generalized coordinates, R is the Rayleigh dissipation function and "Q_j" is the generalized forces.

According to our specific case that we are investigating, we have two generalized coordinates, "θ" and "x".

Therefore the coupled set of differential equations that describe the motion of the damped elastic harmonic pendulum (Equations of Motion) with respect to the two generalized coordinates x and θ are,

$$m \cdot \ddot{x} - m \cdot (l_0 + x)\dot{\theta}^2 + kx - g \cdot m \cdot \cos\theta + C_D\dot{x} = 0$$

$$m \cdot (l_0 + x)^2\ddot{\theta} + 2m \cdot (l_0 + x)\dot{x}\dot{\theta} + g \cdot m \cdot (l_0 + x)\sin\theta + C_D\dot{\theta} = 0$$

With $\ddot{x}$ and $\ddot{\theta}$ isolated the equations are,

$$\ddot{x} = (l_0 + x)\dot{\theta}^2 - \frac{k}{m} \cdot x + g \cdot \cos\theta - \frac{C_D}{m}\dot{x}$$

$$\ddot{\theta} = -\frac{2}{(l_0 + x)}\dot{x}\dot{\theta} - \frac{g}{(l_0 + x)} \cdot \sin\theta - \frac{C_D}{m \cdot (l_0 + x)^2}\dot{\theta}$$

Where, m is the mass, l_0 is the rest length of the spring, x is the extension length of the spring, θ is the angle made between the spring and the normal, g is the gravitational acceleration, k is the spring stiffness constant and C_D damping coefficient.

***Note**: When damping is zero (Absolute Harmonic Motion), C_D is zero.*

*These equations will be directly inputted into the python source code from **appendix ??** to solve these equations numerically with an accuracy of $\pm 10^{-8}$ for each time step of the **Runge-Kutta 4th Order** (RK4) algorithm.*

***Note**: The numerical value of each time step involved in the simulation is 10^{-4} of a second.*

6.2 *Drag Equation*

We know that the general Drag equation is mathematically defined as,

$$F_D = \frac{1}{2}\rho v^2 C_D A$$

Where F_D is the Air/Fluid resistance between the mass and the fluid, ρ is the density of the fluid, v is the speed of the object relative to the fluid, C_D is velocity decay constant (damping constant) and A is the cross sectional area.

But as the fluid medium that we are dealing with is simply atmospheric air at $25°$ and at 1 atm pressure, $\rho = 1.1839 kg/m^3$. Also that, the physical nature of the mass that we are dealing with is spherical, therefore $C_D = 0.47$ and $A = \pi r^2$.

Therefore, we have,

$$F_D = \pi \cdot \frac{432165}{10^6} \cdot r^2 v^2$$

6.3 *Strokes's Law Equation*

We know that,

$$F_D = 6\pi\mu R v_D$$

Where F_D is the Air/Fluid resistance between the mass and the fluid, μ is the dynamic viscosity of the fluid, R is the radius of the mass and v_D is the flow velocity relative to the mass.

But as the fluid medium that we are dealing with is simply atmospheric air at $25°$ and at 1 atm pressure, $\mu = 18.6\mu P \cdot s = 1.86 \times 10^{-5} P \cdot s$.

Also with close observation it can be inferred that the magnitude of flow velocity relative to the mass is the same as the magnitude of the absolute velocity of the system. Therefore using Newtonian vector definition, we define,

$$v = \sqrt{(\ddot{x})^2 + (\ddot{\theta})^2}$$

Therefore we can infer that,

$$v_D = \sqrt{(\ddot{x})^2 + (\ddot{\theta})^2}$$

Therefore, we finally have,

$$F_D = \frac{11.16}{10^5} \cdot \pi \sqrt{(\ddot{x})^2 + (\ddot{\theta})^2} \cdot R$$

6.4 *Frequency Equations*

The frequency of the system can be broken down into two two types of frequencies if we consider it from the perspective of the two generalized coordinates. They are radial and angular frequencies

6.4.1 *Radial frequency*

Radial frequency is defined as the number of oscillations of the radial movement (Spring component of the system) per second. Physically and mathematically defined as,

$$\omega_r = \sqrt{\frac{k}{m}}$$

With observation, it can be inferred that, radial frequency remains constant whatsoever in the whole system.

6.4.2 *Angular frequency*

Angular frequency is defined as the number of oscillations of the angular movement (Pendulum component of the system) per second. Physically and mathematically defined as,

$$\omega_\theta = \sqrt{\frac{g}{l_0 + x}}$$

With observation, it can be inferred that, angular frequency is a function of radial position and will be a variable all throughout in the whole system.

6.4.3 *Absolute frequency*

If we are to talk about absolute frequency, we can define absolute frequency using Newtonian vector definition as,

$$f = \sqrt{(\omega_r)^2 + (\omega_\theta)^2}$$

Therefore we have,

$$f = \sqrt{\frac{k}{m} + \frac{g}{l_0 + x}}$$

6.5 *Fundamental Derived Equations*

The fundamental derived equations we shall be using in this investigation are:

$$\ddot{x} = (l_0 + x)\dot{\theta}^2 - \frac{k}{m} \cdot x + g \cdot \cos\theta - \frac{C_D}{m}\dot{x} \tag{6.1}$$

$$\ddot{\theta} = -\frac{2}{(l_0 + x)}\dot{x}\dot{\theta} - \frac{g}{(l_0 + x)} \cdot \sin\theta - \frac{C_D}{m \cdot (l_0 + x)^2}\dot{\theta} \tag{6.2}$$

Note*: The coefficient of damping (C_D) from now on will be equal to the numeric value of 0.47.*

$$F_D = \pi \cdot \frac{432165}{10^6} \cdot r^2 v^2 \tag{6.3}$$

$$F_D = \frac{11.16}{10^5} \cdot \pi\sqrt{(\ddot{x})^2 + (\ddot{\theta})^2} \cdot R \tag{6.4}$$

***Note**: R and r represent the same physical quantity, i.e., the radius of the mass used in a specific case.*

$$\omega_r = \sqrt{\frac{k}{m}} \tag{6.5}$$

$$\omega_\theta = \sqrt{\frac{g}{l_0 + x}} \tag{6.6}$$

$$f = \sqrt{\frac{k}{m} + \frac{g}{l_0 + x}} \tag{6.7}$$

***Note**: All the equations mentioned above shall be inputted algorithmically in an computer program/software and shall be numerically estimated to about an accuracy of approximately $\pm 10^{-8}$ of each time step interval.*

7 Observations

Let ζ_e, τ_e, χ_e and ψ_e be equal to the sum of the experimental observations from $t = 0$ to $t = 20$ seconds divided by the number of observations, for radial displacement, angular displacement, angular frequency and absolute frequency respectively for each and every case that we are investigating.

Let ζ_s, τ_s, χ_s and ψ_s be equal to the sum of the simulation observations from $t = 0$ to $t = 20$ seconds divided by the number of observations, for radial displacement, angular displacement, angular frequency and absolute frequency respectively for each and every case that we are investigating.

We further define, $\zeta_{0_n} = \frac{\zeta_e}{\zeta_s}$, $\tau_{0_n} = \frac{\tau_e}{\tau_s}$, $\chi_{0_n} = \frac{\chi_e}{\chi_s}$ and $\psi_{0_n} = \frac{\psi_e}{\psi_s}$ for each and every case.

Here, upon close observation we infer that, $100 \cdot \zeta_{0_n}$, $100 \cdot \tau_{0_n}$, $100 \cdot \chi_{0_n}$ *and* $100 \cdot \psi_{0_n}$ *are the percentage accuracy of the experimental values of radial displacement, angular displacement, angular frequency and absolute frequency respectively for each and every case that we are investigating.*

By using the above definitions we have,

7.1 *Observation from Case 1 where m = 1 kg*

$\zeta_{e_1} = 0.075444$ *and* $\zeta_{s_1} = 0.072167$

$\tau_{e_1} = 0.296533$ *and* $\tau_{s_1} = 0.293811$

$\chi_{e_1} = 3.02011$ *and* $\chi_{s_1} = 3.01622$

$\psi_{e_1} = 10.4463$ *and* $\psi_{s_1} = 10.4477$

Therefore,

$\zeta_{0_1} = 1.04542$

$\tau_{0_1} = 1.00927$

$\chi_{0_1} = 1.00129$

$\psi_{0_1} = 0.999872$

7.2 *Observation from Case 2 where m = 2 kg*

$\zeta_{e_2} = 0.204333$ *and* $\zeta_{s_2} = 0.227944$

$\tau_{e_2} = 0.547644$ *and* $\tau_{s_2} = 0.441155$

$\chi_{e_2} = 2.86233$ *and* $\chi_{s_2} = 2.83269$

$\psi_{e_2} = 7.62984$ *and* $\psi_{s_2} = 7.6204$

Therefore,

$\zeta_{0_2} = 0.896417$

$\tau_{0_2} = 1.24139$

$\chi_{0_2} = 1.01047$

$\psi_{0_2} = 1.00124$

7.3 Observation from Case 3 where $m = 3$ kg

$\zeta_{e_3} = 0.339667$ *and* $\zeta_{s_3} = 0.342178$

$\tau_{e_3} = 0.025422$ *and* $\tau_{s_3} = 0.053266$

$\chi_{e_3} = 2.64633$ *and* $\chi_{s_3} = 2.73407$

$\psi_{e_3} = 6.39122$ *and* $\psi_{s_3} = 6.39197$

Therefore,

$\zeta_{0_3} = 0.992661$

$\tau_{0_3} = 0.477259$

$\chi_{0_3} = 0.967911$

$\psi_{0_3} = 0.999884$

7.4 Observation from Case 4 where $m = 4$ kg

$\zeta_{e_4} = 0.405456$ *and* $\zeta_{s_4} = 0.401744$

$\tau_{e_4} = 0.056311$ *and* $\tau_{s_4} = 0.096322$

$\chi_{e_4} = 2.70911$ *and* $\chi_{s_4} = 2.71456$

$\psi_{e_4} = 5.69371$ *and* $\psi_{s_4} = 5.69726$

Therefore,

$\zeta_{0_4} = 1.00924$

$\tau_{0_4} = 0.584612$

$\chi_{0_4} = 0.997992$

$\psi_{0_4} = 0.999377$

7.5 Observation from Case 5 where m = 5 kg

$\zeta_{e_5} = 0.470556$ *and* $\zeta_{s_5} = 0.4483$

$\tau_{e_5} = 0.149422$ *and* $\tau_{s_5} = 0.163$

$\chi_{e_5} = 2.64211$ *and* $\chi_{s_5} = 2.64337$

$\psi_{e_5} = 5.20113$ *and* $\psi_{s_5} = 5.20013$

Therefore,

$\zeta_{0_5} = 1.04965$

$\tau_{0_5} = 0.916699$

$\chi_{0_5} = 0.999523$

$\psi_{0_5} = 1.00019$

7.6 Observation from Case 6 where m = 6 kg

$\zeta_{e_6} = 0.587$ *and* $\zeta_{s_6} = 0.4685$

$\tau_{e_6} = 0.2162$ *and* $\tau_{s_6} = 0.277866$

$\chi_{e_6} = 2.56878$ *and* $\chi_{s_6} = 2.56469$

$\psi_{e_6} = 4.83438$ *and* $\psi_{s_6} = 4.83146$

Therefore,

$\zeta_{0_6} = 1.25293$

$\tau_{0_6} = 0.778073$

$\chi_{0_6} = 1.00159$

$\psi_{0_6} = 1.0006$

7.7 Observation from Case 7 where m = 7 kg

$\zeta_{e_7} = 0.744222$ *and* $\zeta_{s_7} = 0.712233$

$\tau_{e_7} = 0.132977$ *and* $\tau_{s_7} = 0.073288$

$\chi_{e_7} = 2.49967$ *and* $\chi_{s_7} = 2.5086$

$\psi_{e_7} = 4.53948$ *and* $\psi_{s_7} = 4.55099$

Therefore,

$\zeta_{0_7} = 1.04491$

$\tau_{0_7} = 1.81444$

$\chi_{0_7} = 0.99644$

$\psi_{0_7} = 0.997471$

7.8 *Observation from Case 8 where m = 8 kg*

$\zeta_{e_8} = 0.722889$ *and* $\zeta_{s_8} = 0.700322$

$\tau_{e_8} = 0.113444$ *and* $\tau_{s_8} = 0.128066$

$\chi_{e_8} = 2.528$ *and* $\chi_{s_8} = 2.53933$

$\psi_{e_8} = 4.36351$ *and* $\psi_{s_8} = 4.37021$

Therefore,

$\zeta_{0_8} = 1.03222$

$\tau_{0_8} = 0.885824$

$\chi_{0_8} = 0.995538$

$\psi_{0_8} = 0.998467$

7.9 *Observation from Case 9 where m = 9 kg*

$\zeta_{e_9} = 0.951$ *and* $\zeta_{s_9} = 0.835733$

$\tau_{e_9} = 0.132755$ *and* $\tau_{s_9} = 0.103266$

$\chi_{e_9} = 2.40522$ *and* $\chi_{s_9} = 2.44351$

$\psi_{e_9} = 4.13234$ *and* $\psi_{s_9} = 4.15612$

Therefore,

$\zeta_{0_9} = 1.13792$

$\tau_{0_9} = 1.28556$

$\chi_{0_9} = 0.98433$

$\psi_{0_9} = 0.994278$

7.10 *Observation from Case 10 where $m = 10\ kg$*

$\zeta_{e_{10}} = 0.874667$ *and* $\zeta_{s_{10}} = 0.822289$

$\tau_{e_{10}} = 0.124977$ *and* $\tau_{s_{10}} = 0.127677$

$\chi_{e_{10}} = 2.538$ *and* $\chi_{s_{10}} = 2.52378$

$\psi_{e_{10}} = 4.03814$ *and* $\psi_{s_{10}} = 4.07456$

Therefore,

$\zeta_{0_{10}} = 1.0637$

$\tau_{0_{10}} = 0.978853$

$\chi_{0_{10}} = 1.00563$

$\psi_{0_{10}} = 0.991062$

8 Analysis

8.1 Graphical Analysis

8.1.1 Graphs in relation to the Experimental values versus time

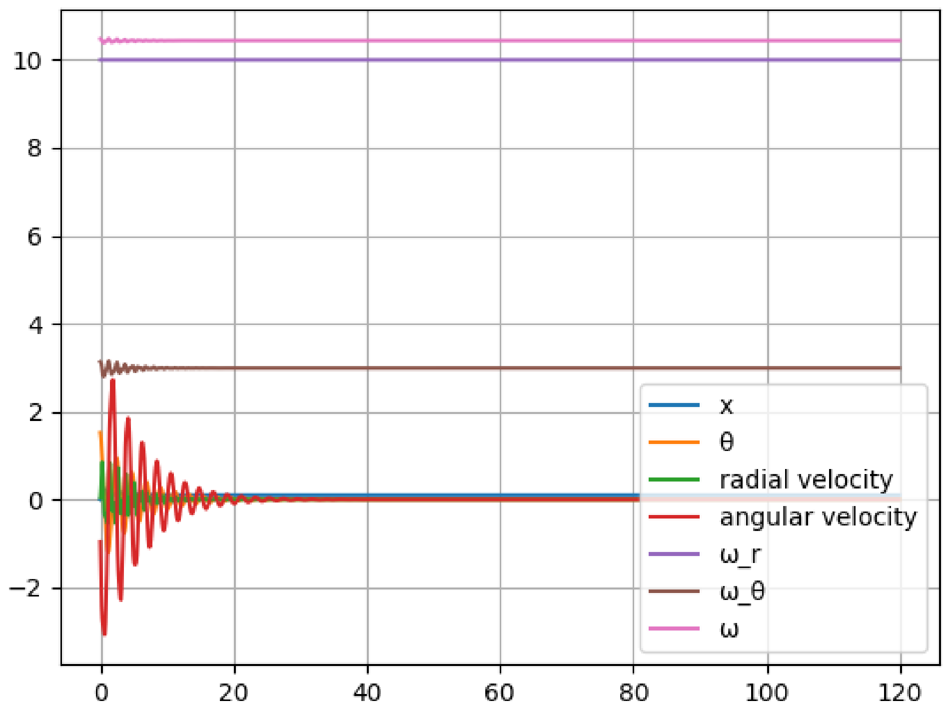

Figure 8.1: *Case 1 where m = 1 kg*

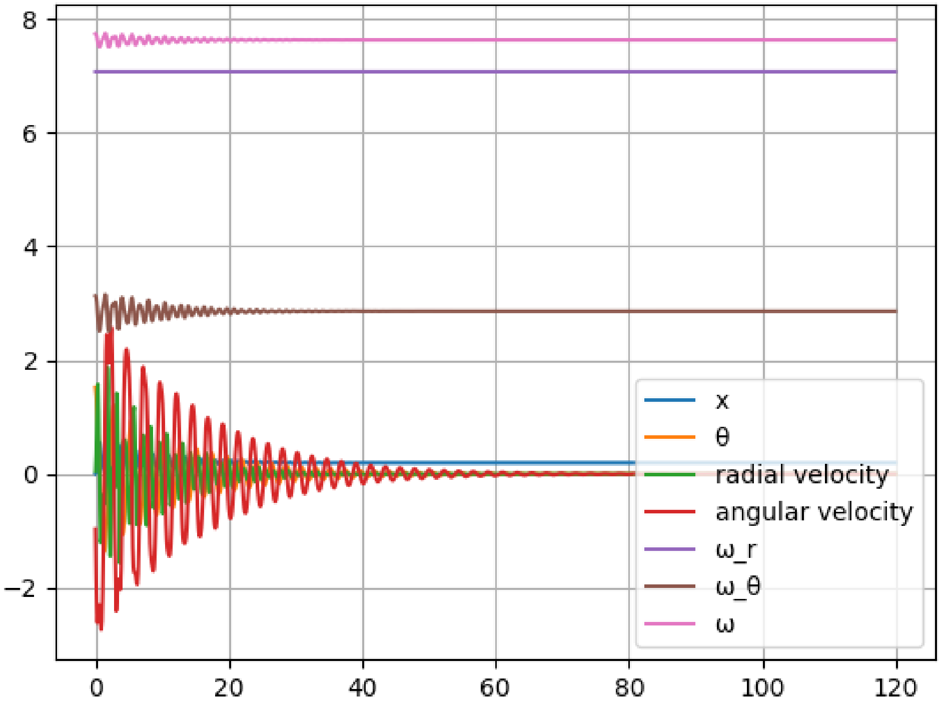

Figure 8.2: *Case 2 where m = 2 kg*

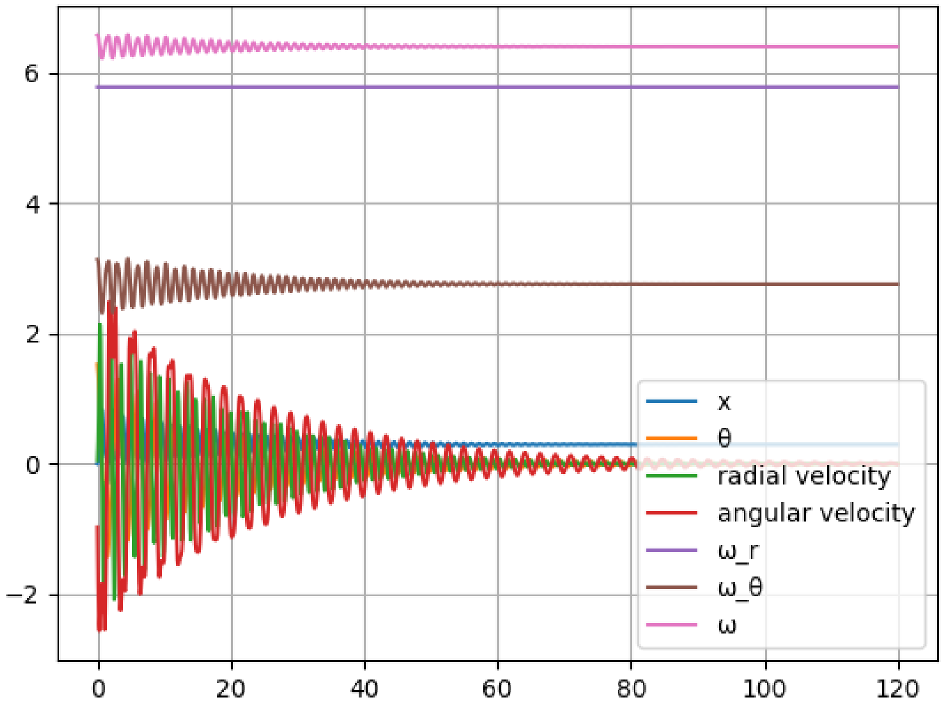

Figure 8.3: *Case 3 where m = 3 kg*

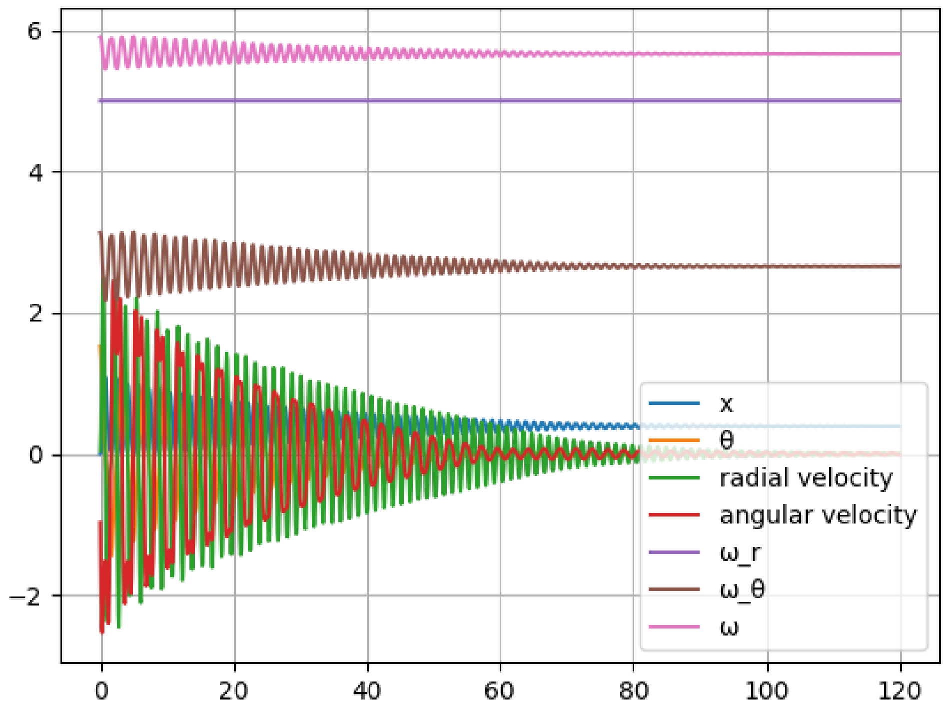

Figure 8.4: *Case 4 where* $m = 4$ *kg*

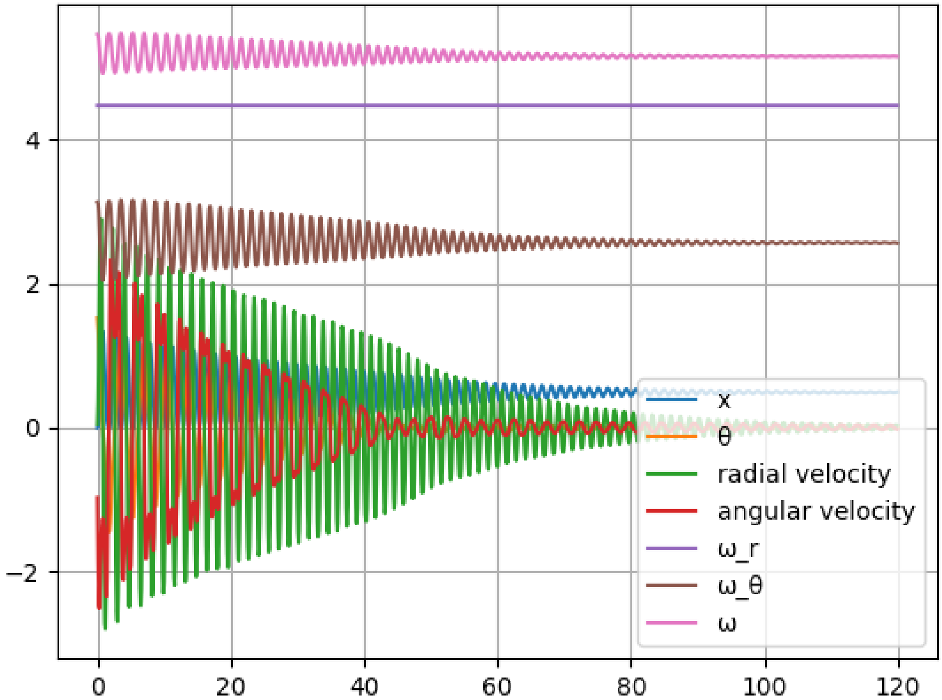

Figure 8.5: *Case 5 where $m = 5$ kg*

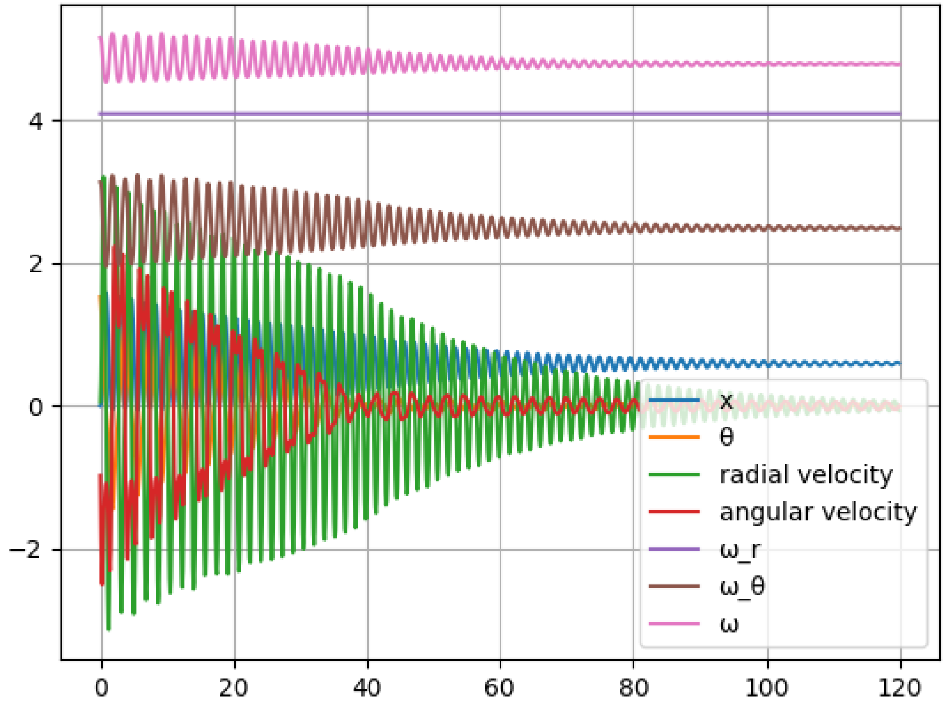

Figure 8.6: *Case 6 where* $m = 6\ kg$

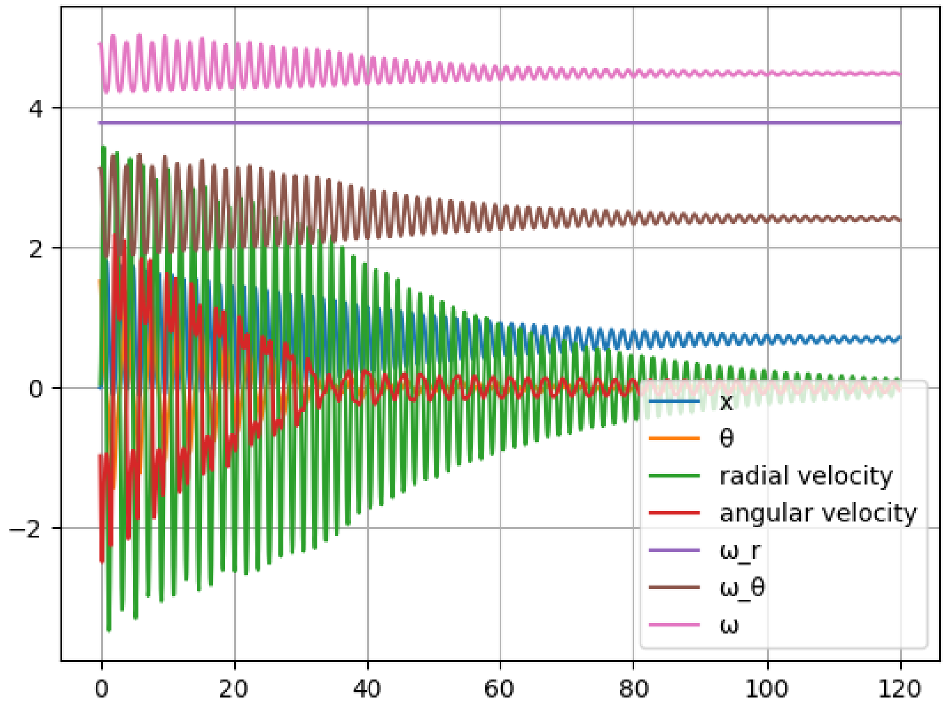

Figure 8.7: *Case 7 where* $m = 7\ kg$

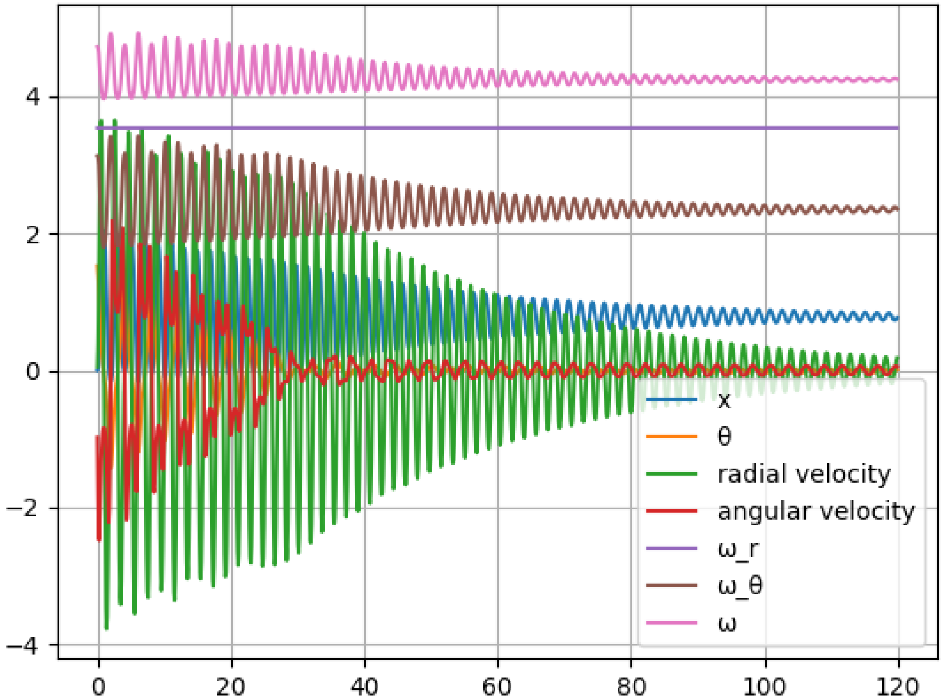

Figure 8.8: *Case 8 where m = 8 kg*

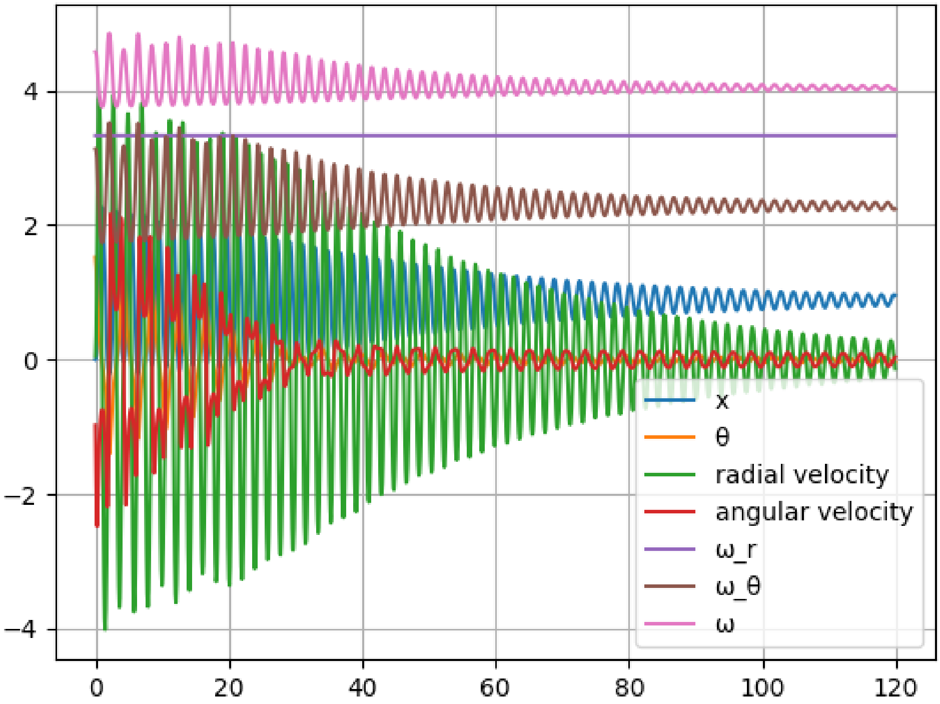

Figure 8.9: *Case 9 where m = 9 kg*

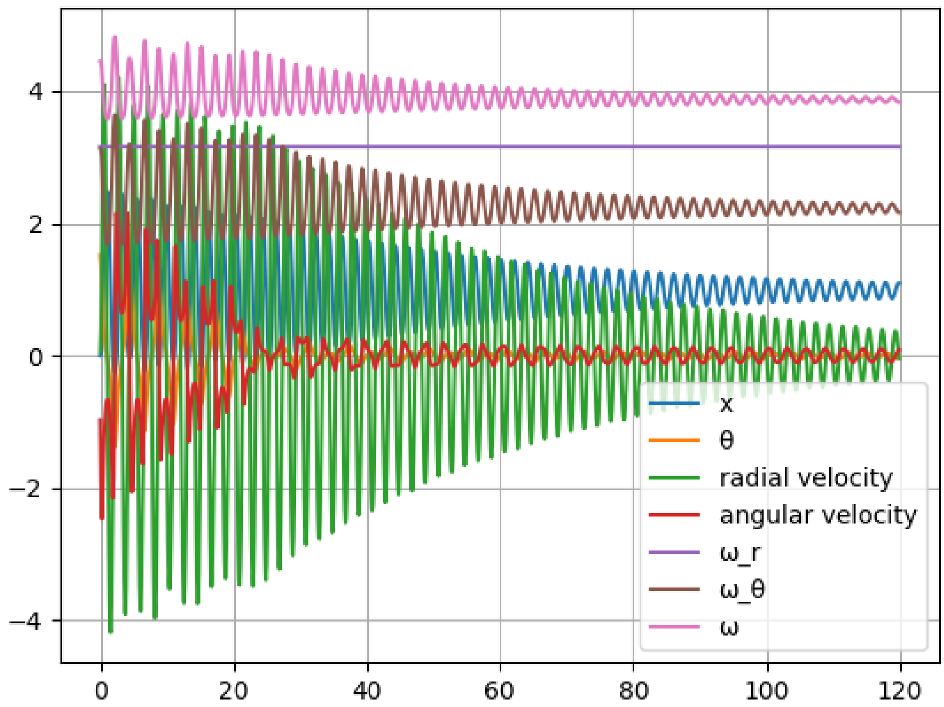

Figure 8.10: *Case 10 where m = 10 kg*

8.1.2 *Graphs in relation to the Simulation values versus time*

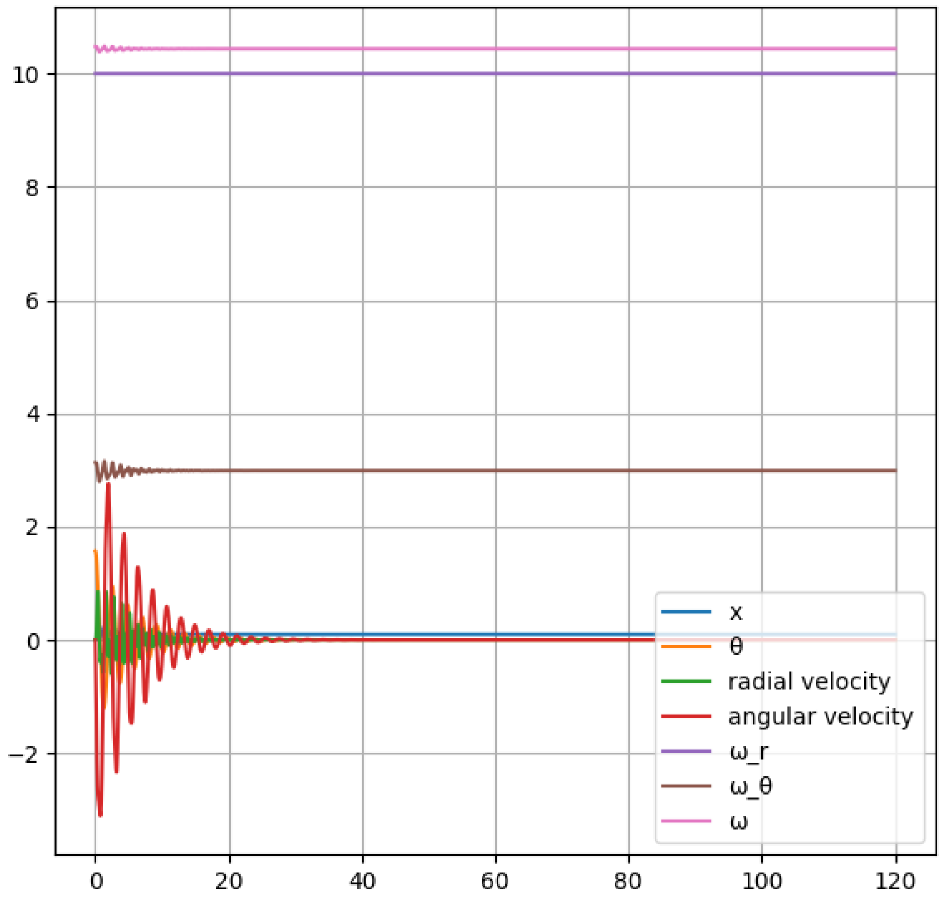

Figure 8.11: *Case 1 where m = 1 kg*

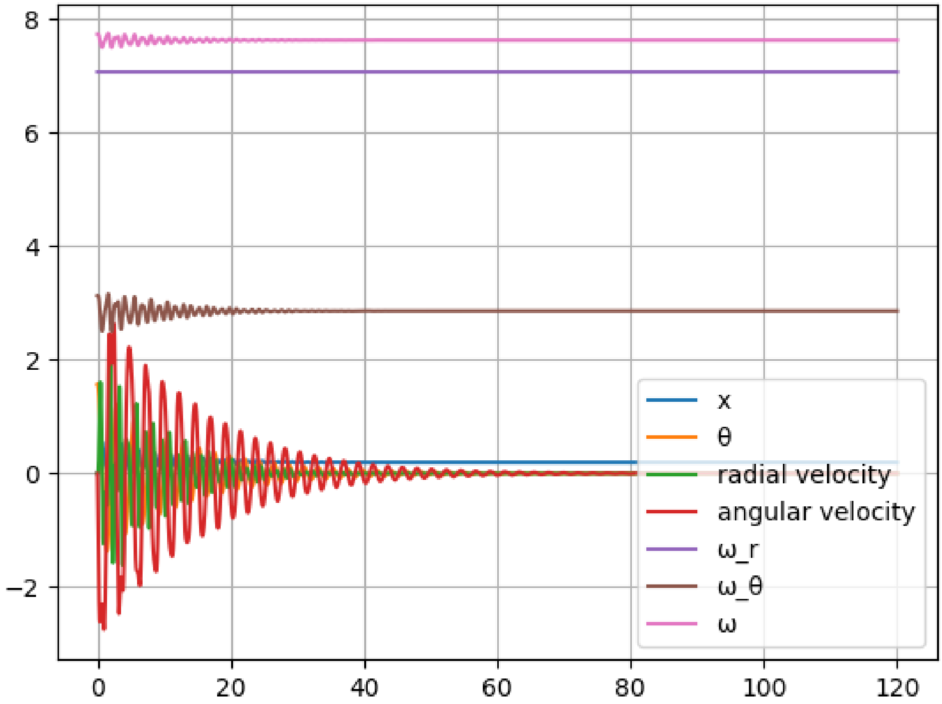

Figure 8.12: *Case 2 where m = 2 kg*

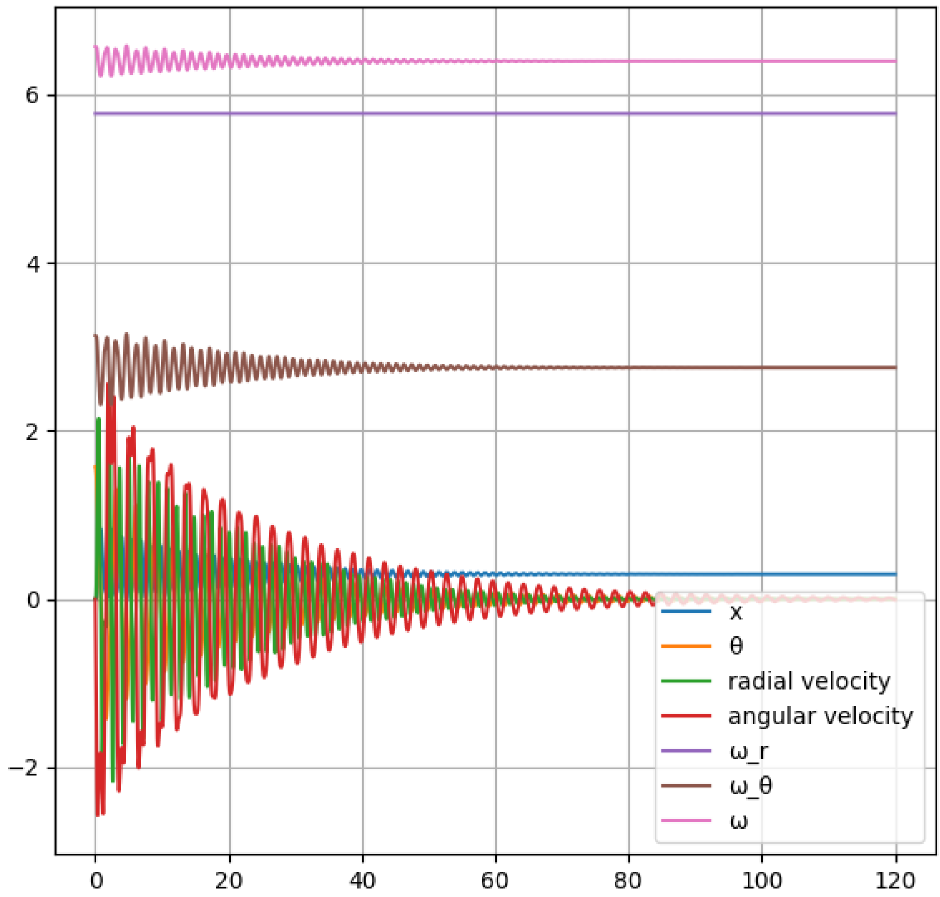

Figure 8.13: *Case 3 where m = 3 kg*

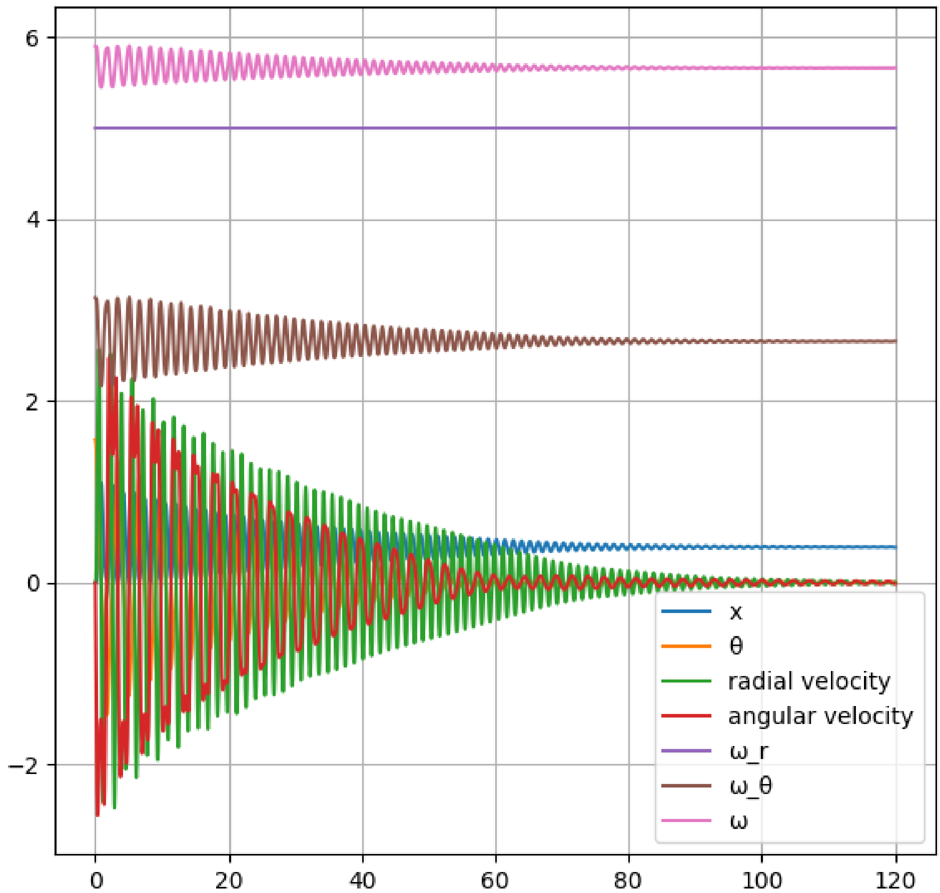

Figure 8.14: *Case 4 where m = 4 kg*

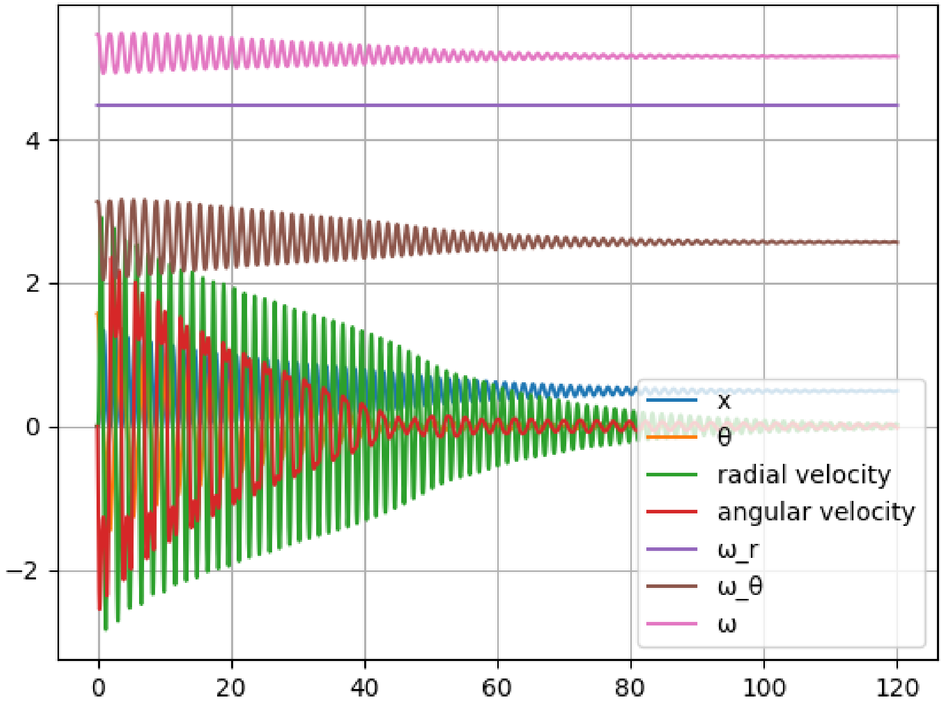

Figure 8.15: *Case 5 where m = 5 kg*

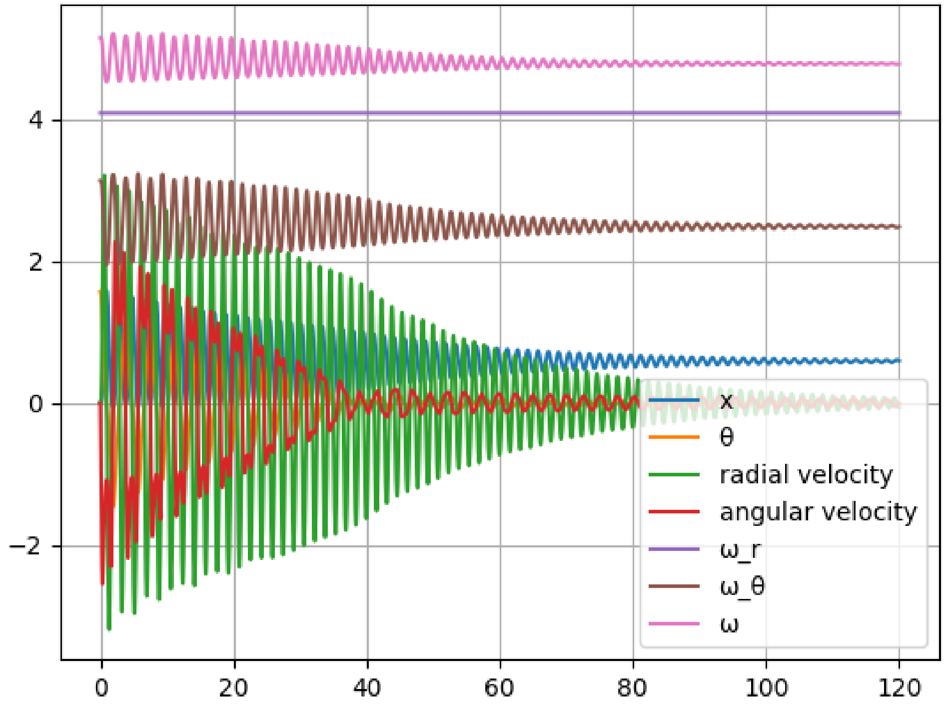

Figure 8.16: *Case 6 where m = 6 kg*

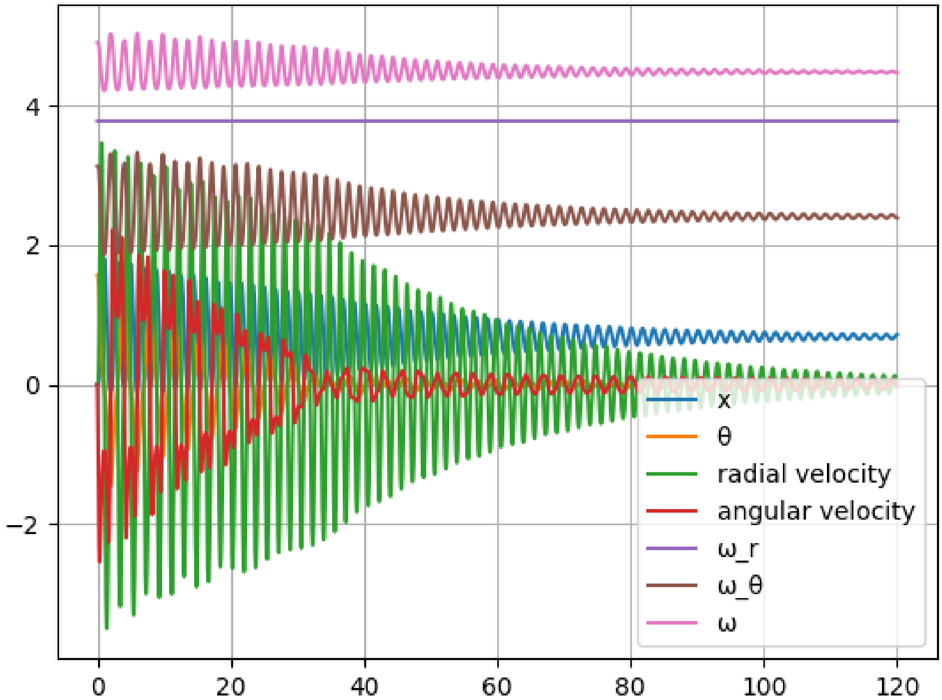

Figure 8.17: *Case 7 where $m = 7\ kg$*

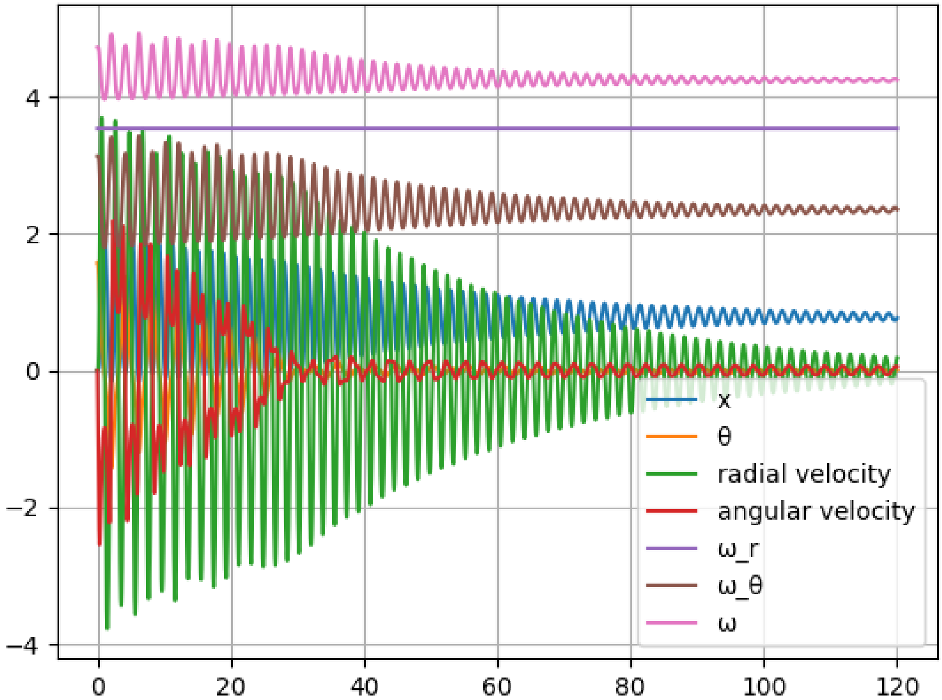

Figure 8.18: *Case 8 where* m *= 8 kg*

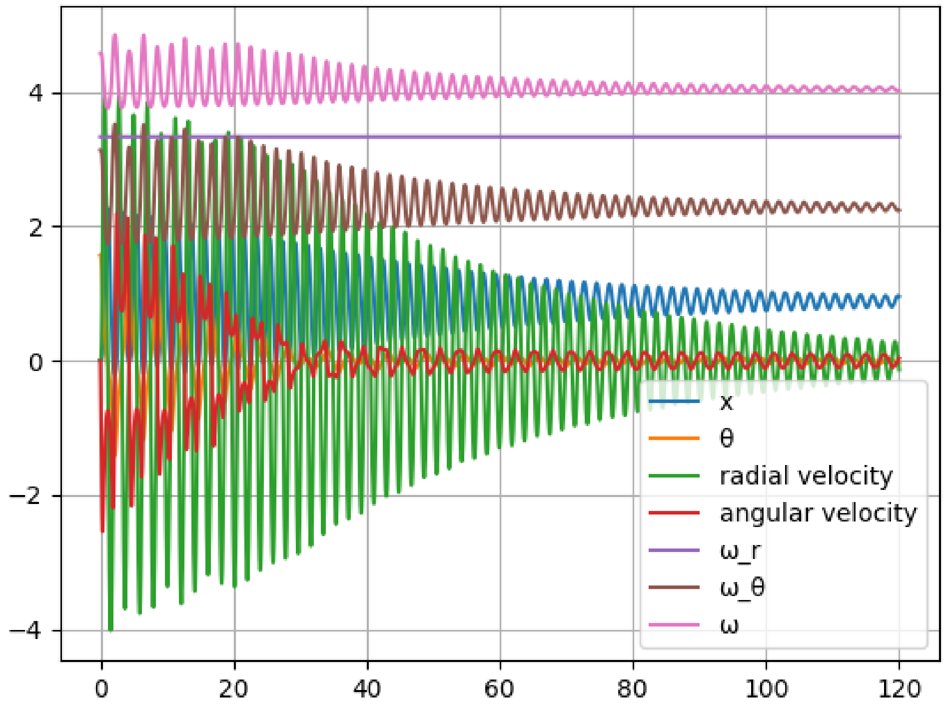

Figure 8.19: *Case 9 where m = 9 kg*

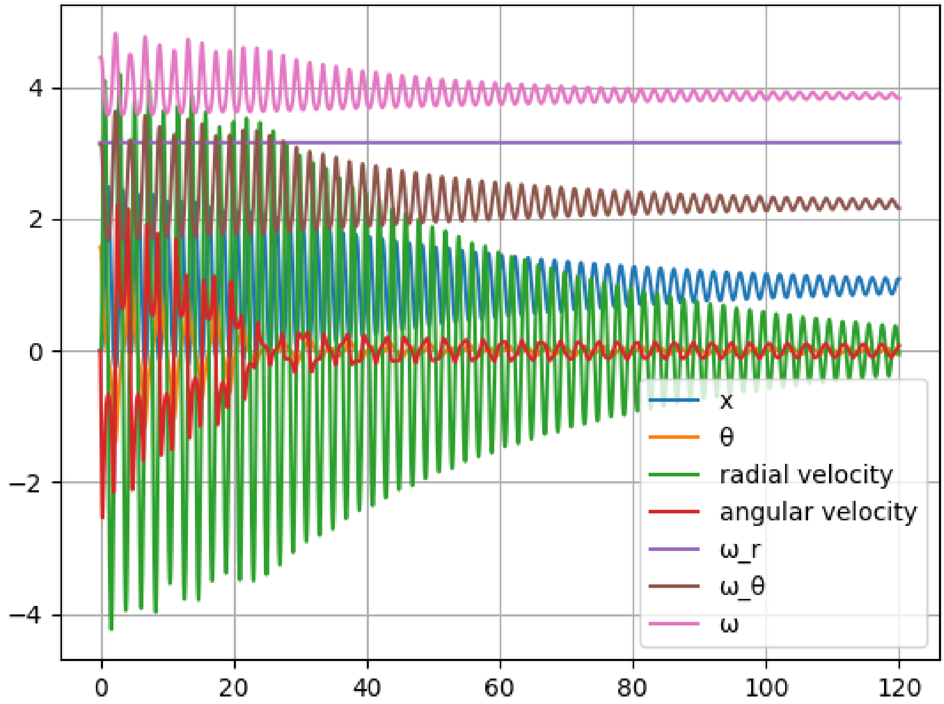

Figure 8.20: *Case 10 where m = 10 kg*

Upon close visual observation, we see that the difference in the plotted values of that of the simulation and experimental values of the radial displacement, angular displacement, angular frequency and absolute frequency from the ***x-t*** *graph are very minute to the extent that it would be right to say and consider that the experimental values are both accurate and precise in relation to that of the literature/theoretical/simulation values.*

We also observe that, with increase in mass, the ***amplitude*** *of the parameters employed in this investigation also increases while the* ***frequency*** *of the parameters employed stays approximately constant.*

8.2 *Numerical Analysis*

8.2.1 *Radial Displacement versus Time*

We initially defined and conditionalized the initial value of the radial displacement versus time to be equal to 0.

Due to viscous underdamped motion, it gains energy and momentum and preserves it, with gradual but steady loss int its energy state and is in motion for more than 300 seconds before coming to a complete halt. ie. no motion.

8.2.2 *Angular Displacement versus Time*

We initially defined and conditionalized the initial value of the angular displacement versus time to be equal to $\pi/2$.

Due to viscous underdamped motion, it gains energy and momentum and preserves it, with gradual but steady loss int its energy state and is in motion for more than 120 seconds before coming to a complete halt. ie. no motion.

8.2.3 *Angular Frequency versus Time*

We previously defined angular frequency (ω_θ) *to be as equation 6.6.*

The initial state ($\omega_\theta(0)$) *that follows is,*

$$\omega_\theta(0) = \sqrt{\frac{g}{l_0 + x(0)}} = \sqrt{\frac{g}{l_0}}$$

But we had previously mentioned that the magnitude of rest length of the spring would be 1 meter ($l_0 = 1m$)*. Therefore we have,*

$$\omega_\theta(0) = \sqrt{g} \approx 3.13209195$$

This parameter keeps fluctuating through the time domain as this is a function of radial displacement which in turn is a function of time.

This initial state defined and derived above is uniform irrespective of the mass employed

8.2.4 *Absolute Frequency versus Time*

We previously defined absolute frequency (ω_r) to be as equation 6.5.

The initial state ($\omega_r(0)$) that follows is,

$$\omega_r(0) = \sqrt{\frac{k}{m}}$$

This parameter does not fluctuate through the time domain as this is not a function of any arbitrary variable that would be in turn is a function of time.

This initial state defined and derived above is constant for any particular case with a steady constant mass m, throught a case

9 *Evaluation*

We further define,

$$\zeta_0 = \overline{\zeta_{0_n}} = \frac{\sum_{n=1}^{n} \zeta_{0_n}}{n} = \frac{\zeta_{0_1} + \zeta_{0_2} + \zeta_{0_3} + \zeta_{0_4} + \zeta_{0_5} + \zeta_{0_6} + \zeta_{0_7} + \zeta_{0_8} + \zeta_{0_9} + \zeta_{0_{10}}}{10}$$

$$\tau_0 = \overline{\tau_{0_n}} = \frac{\sum_{n=1}^{n} \tau_{0_n}}{n} = \frac{\tau_{0_1} + \tau_{0_2} + \tau_{0_3} + \tau_{0_4} + \tau_{0_5} + \tau_{0_6} + \tau_{0_7} + \tau_{0_8} + \tau_{0_9} + \tau_{0_{10}}}{10}$$

$$\chi_0 = \overline{\chi_{0_n}} = \frac{\sum_{n=1}^{n} \chi_{0_n}}{n} = \frac{\chi_{0_1} + \chi_{0_2} + \chi_{0_3} + \chi_{0_4} + \chi_{0_5} + \chi_{0_6} + \chi_{0_7} + \chi_{0_8} + \chi_{0_9} + \chi_{0_{10}}}{10}$$

$$\psi_0 = \overline{\psi_{0_n}} = \frac{\sum_{n=1}^{n} \psi_{0_n}}{n} = \frac{\psi_{0_1} + \psi_{0_2} + \psi_{0_3} + \psi_{0_4} + \psi_{0_5} + \psi_{0_6} + \psi_{0_7} + \psi_{0_8} + \psi_{0_9} + \psi_{0_{10}}}{10}$$

Therefore we have,

$\zeta_0 = 1.0525068$

$\tau_0 = 0.997198$

$\chi_0 = 0.9960714$

$\psi_0 = 0.9982441$

Upon observation, we see that the value of ζ_0*,* τ_0*,* χ_0 *and* ψ *we have found is not equal to 1, but is relatively very close, so we can say that we have some errors in calculating the experimental values of radial displacement, angular displacement, angular frequency and absolute frequency respectively versus time.*

Percentage uncertainty *in measurement of radial displacement versus time is* $(1 - \zeta_0) \cdot 100\% =$ ***5.25068%***

Percentage uncertainty *in measurement of angular displacement versus time is* $(1 - \tau_0) \cdot 100\% =$ ***0.2802%***

Percentage uncertainty *in measurement of angular frequency versus time is* $(1 - \chi_0) \cdot 100\% =$ ***0.39286%***

Percentage uncertainty *in measurement of absolute frequency versus time is* $(1 - \psi_0) \cdot 100\% =$ ***0.17559%***

10 Limitations of Study

There are various limitations in this study/investigation as we have placed forth, certain strict initial conditions and other conditions that make this system so constrained and disables us to expand our researching capability of this chaotic phenomenon. Conditions such as,

- *Restricting the value of* $\theta(0)$ *to* $\pi/2$
- *Restricting the value of* $x(0)$ *to* 0
- *Restricting the value of* l_0 *to 1 m*

- *Restricting the value of the time domain from 0 to 20 seconds*
- *Restricting the value of the radius of the mass employed to* $5 \times 10^{-2}\ m$
- *Restricting the value of the spring stiffness constant to 100 N/m*

11 Sources of Error

There are various ways through which errors might have crept into our raw and ordered data, some of the possible sources of errors are:

1. *Raw experimental data presented here in the investigation report is collected through lab experimentation, and there are chances that the data collected may have slight discrepancy in it*
2. *Insignificant random human errors by the observer, ie. parallax errors*
3. *Uncertainties that cannot be minimized due to lack of highly sophisticated equipment and materials used in the experiments in this investigation*
4. *Assumptions and certain conditions put forth on the the system to model its chaotic behavior*

12 Conclusion

In this paper, I have shown the effects of changes in the ***mass*** *in an* ***elastic pendulum*** *oscillating in a fluid medium (Air(Damped motion)) on the* ***radial displacement****,* ***angular displacement****,* ***angular frequency*** *and* ***absolute frequency*** *by collecting raw data relating to the above parameters, under certain controlled conditions as so to completely study the motion/dynamics* *of the elastic pendula motion. I have also shown the possible* ***uncertainties in measurement*** *of the* ***radial displacement****,* ***angular displacement****,* ***angular frequency*** *and* ***absolute frequency****.*

Bibliography

[1] Girgin, Z., and E. Demir. Investigation of Elastic Pendulum Oscillations by Simulation Technique*. Journal of Engineering Sciences, 2009,* `jag.journalagent.com/pajes/pdfs/PAJES_15_1_81_86.pdf`*.*

[2] Digilov, Rafael M., et al. "Damping in a Variable Mass on a Spring Pendulum." American Journal of Physics*, vol. 73, no. 10, 2005, pp. 901–05.* Crossref*, doi:10.1119/1.1979498.*

[3] John Deyst, and Jonathan How. 16.61 Aerospace Dynamics*. Spring 2003. Massachusetts Institute of Technology: MIT OpenCourseWare,* `https://ocw.mit.edu`*. License: Creative Commons BY-NC-SA.*

[4] Minguzzi, E. "Rayleigh's Dissipation Function at Work." European Journal of Physics*, vol. 36, no. 3, 2015, p. 035014.* Crossref*, doi:10.1088/0143-0807/36/3/035014.*

[5] Bayly, P. V., and L. N. Virgin. "An Empirical Study of the Stability of Periodic Motion in the Forced Spring-Pendulum." Proceedings: Mathematical and Physical Sciences*, vol. 443, no. 1918, 1993, pp. 391–408. JSTOR,* `www.jstor.org/stable/52258`*. Accessed 2 Aug. 2021.*

[6] Bellomo, Nicola, et al. "Mechanics and Dynamical Systems With Mathematica®." Modeling and Simulation in Science, Engineering and Technology*, 2000, pp. 341–408.* Crossref*, doi:10.1007/978-1-4612-1338-3.*

[7] "1. History of Dynamics; Motion in Moving Reference Frames." YouTube*, uploaded by MIT OpenCourseWare, 3 Sept. 2013,* `www.youtube.com/watch?v=GUvoVvXwoOQ`*.*

[8] Zammit, Corey, et al. The Dynamics Of The Elastic Pendulum*. Tucson, University of Arizona, 2018.*

[9] *"22. Finding Natural Frequencies Mode Shapes of a 2 DOF System."* YouTube*, uploaded by MIT OpenCourseWare, 4 Sept. 2013,* *www.youtube.com/watch?v=9CPA6WG6mRo*.

[10] *Adomian, G., and R. Rach. "On the Solution of Nonlinear Differential Equations With Convolution Product Nonlinearities."* Journal of Mathematical Analysis and Applications*, vol. 114, no. 1, 1986, pp. 171–75. Crossref, doi:10.1016/0022-247x(86)90074-0.*

[11] *"Advance Mechanics: Lagrangian Example 11."* YouTub*, uploaded by Adnan Iqbal, 9 Aug. 2020,* *www.youtube.com/watch?v=zQhaOmmNgsA*.

[12] *"Applications of Second-Order Differential Equations."* Index of /Data/CALCULUS Concepts and Contexts/Upfiles, *www.stewartcalculus.com/data/CALCULUS%20Concepts%20and%20Contexts/upfiles. Accessed 2 Aug. 2021.*

[13] *"Classical Mechanics — Lecture 3."* YouTube*, uploaded by Stanford, 16 Dec. 2011,* *www.youtube.com/watch?v=3apIZCpmdls*.

[14] *Craig, K. "Spring-Pendulum Dynamic System."* NYU Engineering, *engineering.nyu.edu/mechatronics/Control_Lab/Criag/Craig_RPI/200. Accessed 2 Aug. 2021.*

[15] *Denis, Byakatonda.* An Overview of Numerical and Analytical Methods for Solving Ordinary Differential Equations*. arXiv, 2020. arXiv,* *arxiv.org/ftp/arxiv/papers/2012/2012.07558.pdf*.

[16] *"Differential Equations Notes."* Chapter 2 - Second Order Differential Equations*,* *people.uncw.edu/hermanr/mat361/ODEBook. Accessed 2 Aug. 2021.*

[17] *"Equation of Motion for the Simple Pendulum (SDOF)."* YouTube*, uploaded by Good Vibrations with Freeball, 29 Dec. 2016,* *www.youtube.com/watch?v=8VJ1CJ55Np0*.

[18] *"Equations of Motion for the Elastic Pendulum (2DOF) Using Lagrange's Equations."* YouTube*, uploaded by Good Vibrations with Freeball, 10 July 2020,* *www.youtube.com/watch?v=iULa9AOOJpA*.

[19] *"Equations of Motion for the Multi Degree of Freedom (MDOF) Problem Using LaGrange's Equations."* YouTube*, uploaded by Good Vibrations with Freeball, 16 Feb. 2017,* *www.youtube.com/watch?v=uAKD5CGZuSs*.

[20] *"Equations of Motion for the Nonlinear Oscillator (2DOF)."* YouTube, *uploaded by Good Vibrations with Freeball, 18 July 2020,* www.youtube.com/watch?v=l4XQtlkyC2o.

[21] *"Euler-Lagrange Equation Explained Intuitively - Lagrangian Mechanics."* YouTube, *uploaded by Physics Videos by Eugene Khutoryansky, 18 Nov. 2018,* www.youtube.com/watch?v=EceVJJGAFFI.

[22] *"State Space Equation for Spring Pendulum."* Physics Stack Exchange, *Physics Stack Exchange, 4 Dec. 2014,* physics.stackexchange.com/questions/150463/state-space-equation-for-spring-pendulum.

[23] *Goldstein, Herbert, and C. P. Poole.* Classical Mechanics. *2nd ed., Zaltbommel, Netherlands, Van Haren Publishing, 1980.*

[24] *"Hamilton's Principle and Euler-Lagrange Equations."* YouTube, *uploaded by Rangeet Bhattacharyya, 15 Sept. 2020,* www.youtube.com/watch?v=FBcS9RHzHRw.

[25] *Han, Yün. "ECE 486 Control Systems."* ECE 486 Control Systems - Lecture 2, *University of Illinois, 18 Jan. 2018,* courses.engr.illinois.edu/ece486/sp2021/documentation/handbook/lec02.html.

[26] *"Introduction to Lagrangian Mechanics."* YouTube, *uploaded by Physics Explained, 18 Oct. 2020,* www.youtube.com/watch?v=8UtnDaGHpq0.

[27] *Ismail, A. I. "New Vertically Planed Pendulum Motion."* Mathematical Problems in Engineering, *edited by El-Sayed Abo-Dahab, vol. 2020, 2020, pp. 1–6.* Crossref, *doi:10.1155/2020/8861738.*

[28] *"Lagrangian Mechanics Part 8: Spring Pendulum."* YouTube, *uploaded by Neil Shah Physics, 9 July 2020,* www.youtube.com/watch?v=ET3IlPgC3dI.

[29] *Libretexts. "10.4: Rayleigh's Dissipation Function."* Physics LibreTexts, *28 Feb. 2021,* phys.libretexts.org/Bookshelves/Classical_Mechanics/\Variational_Principles_in_Classical_Mechanics_(Cline)/\10%3A_Nonconservative_Systems/10.04%3A_Rayleighs_Dissipatio_Function.

[30] *"M2. Differential Equations in Physics."* YouTube, *uploaded by doctorphys, 7 Oct. 2011,* www.youtube.com/watch?v=Hl65dG6g7fg.

[31] *"MM61: Simple Pendulum and Elliptic Integrals."* YouTube*, uploaded by Vivek Narayanan, 26 Oct. 2020,* www.youtube.com/watch?v=wSBUvJINEuo*.*

[32] *"Modified Hamilton's Principle."* YouTube*, uploaded by Rangeet Bhattacharyya, 25 Sept. 2020,* www.youtube.com/watch?v=pzkwl0L2N9M*.*

[33] *Morin, David.* Chapter 6 - The Lagrangian Method*. Massachusetts, Harvard University, 2007.*

[34] *"Oscillations Demo: Mass Spring System."* YouTube*, uploaded by Physics Demos, 20 Aug. 2016,* www.youtube.com/watch?v=FJBPNJR2QJU*.*

[35] *Paulsson, Oskar.* The Rayleigh Dissipation Function - Theory and Application*. Oskar Paulsson, 2014.*

[36] *"Physics - Adv. Mechanics: Lagrangian Mech. (1 of 25) What Is Lagrangian Mechanics?"* YouTube*, uploaded by Michel van Biezen, 30 Mar. 2016,* www.youtube.com/watch?v=4uJaKJASKnY*.*

[37] *"Rayleigh Dissipation Function — Classical Mechanics — LetThereBeMath —."* YouTube*, uploaded by Let there be math, 24 Oct. 2017,* www.youtube.com/watch?v=AX_XjLxTXWU*.*

[38] *Scaife, B. K. P. "On the Rayleigh Dissipation Function for Dielectric Media."* Journal of Molecular Liquids*, vol. 43, 1989, pp. 101–07. Crossref, doi:10.1016/0167-7322(89)80010-8.*

[39] *"Showcase of Lagrangian Mechanics Using an Elastic Pendulum."* YouTube*, uploaded by Benedict Wesson, 24 May 2015,* www.youtube.com/watch?v=CA34i245meQ*.*

[40] *"Simple Pendulum with Air Resistance — Classical Mechanics — LetThereBeMath —."* YouTube*, uploaded by LetThereBeMath, 25 Oct. 2017,* www.youtube.com/watch?v=erveOJD_qv4*.*

[41] *Sivasrinivaskolukula. "Spring Pendulum."* Sivasrinivaskolukula*,* sites.google.com/site/kolukulasivasrinivas/mechanics/spring-pendulum*. Accessed 2 Aug. 2021.*

[42] *Stanovnik, Aleš, and Borut Jurcic-Zlobec. "Numerical Study of the Elastic Pendulum on the Rotating Earth."* ISRN Mathematical Physics*, vol. 2012, 2012, pp. 1–7.* Crossref*, doi:10.5402/2012/806231.*

[43] *The Open University.* MST224 - Mathematical Methods - Second-Order Differential Equations. *Amsterdam, Netherlands, Reed Business Education, 2014.*

[44] *"Two Degree of Freedom (2DOF) Problem With Damping - Equations of Motion (EOMs)."* YouTube, *uploaded by Good Vibrations with Freeball, 14 Sept. 2019,* www.youtube.com/watch?v=NV7cd-7Rz-I.

[45] *"Two Degree of Freedom (2DOF) Problem With Damping - Equations of Motion (EOMs)"* YouTube, *uploaded by Good Vibrations with Freeball, 14 Sept. 2019,* www.youtube.com/watch?v=NV7cd-7Rz-I.

[46] *Vazquez-Leal, Hector, et al. "Modified Taylor Series Method for Solving Nonlinear Differential Equations With Mixed Boundary Conditions Defined on Finite Intervals."* SpringerPlus, *vol. 3, no. 1, 2014.* Crossref, *doi:10.1186/2193-1801-3-160.*

[47] *"What Does the Laplace Transform Really Tell Us? A Visual Explanation (plus Applications)."* YouTube, *uploaded by Zach Star, 3 Nov. 2019,* www.youtube.com/watch?v=n2y7n6jw5d0.

[48] *Wikipedia contributors. "Elastic Pendulum."* Wikipedia, *7 May 2021,* en.wikipedia.org/wiki/Elastic_pendulum.

[49] *Pokorny, P. "Stability Condition for Vertical Oscillation of 3-Dim Heavy Spring Elastic Pendulum."* Regular and Chaotic Dynamics, *vol. 13, no. 3, 2008, pp. 155–65.* Crossref, *doi:10.1134/s1560354708030027.*

[50] *Pokorny, P. "Continuation of Periodic Solutions of Dissipative and Conservative Systems: Application to Elastic Pendulum."* Mathematical Problems in Engineering, *vol. 2009, 2009, pp. 1–15. Crossref, doi:10.1155/2009/104547.*

[51] *Anurag, et al. "Understanding the Order-Chaos-Order Transition in the Planar Elastic Pendulum."* Physica D: Nonlinear Phenomena, *vol. 402, 2020, p. 132256. Crossref, doi:10.1016/j.physd.2019.132256.*

[52] *Simionescu, P.* Computer-Aided Graphing and Simulation Tools for AutoCAD Users. *Abingdon, United Kingdom, Taylor Francis, 2014.*

[53] *Kreyszig, Erwin.* Advanced Engineering Mathematics. *Hoboken, NJ, United States, Wiley, 1972.*

[54] *Hayek, Sabih I. "Mechanical Vibration and Damping."* Digital Encyclopedia of Applied Physics, *2003.* Crossref, *doi:10.1002/3527600434.eap231.*

[55] *Fowles, Grant R., and George L. Cassiday.* Analytic Mechanics*. 5th ed., Fort Worth, Saunders College, 1986.*

[56] *Serway, Raymond, and John Jewett.* Physics for Scientists and Engineers*. Thomson-Brooks/Cole, 2004.*

[57] *Tipler, Paul.* Physics for Scientists and Engineers*. W. H. Freeman, 1998.*

[58] *Wylie, Clarence Raymond.* Advanced Engineering Mathematics*. 4th ed., McGraw-Hill, 1975.*

[59] *Rowell, Derek. "State-Space Representation of LTI Systems."* 2.14 Analysis and Design of Feedback Control Systems*, D. Rowell, 2002, pp. 1–18.*

[60] *Malham, Simon.* Differential Equations and Linear Algebra - Lecture Notes*. Heriot-Watt University, 2016.*

[61] *Mauti, Eddah, and Ali Raza.* Spring Pendulum System*. 2018.*

[62] *Goldstein, Herbert.* Classical Mechanics (Addison-Wesley Series in Physics*. 2nd ed., Addison-Wesley, 1980.*

[63] *J. Vandiver, and David Gossard. 2.003SC Engineering Dynamics. Fall 2011. Massachusetts Institute of Technology: MIT OpenCourseWare, https://ocw.mit.edu. License: Creative Commons BY-NC-SA.*