Mathematics IA Ver 1.01

The Interrelation of various Transforms and Series

What is the relationship between the Laplace transform, the Fourier transform and the Fourier series of a function and how can it be used to compute the solutions of complex functions?

***Word Count**: 2000*

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Contents

Abstract

*We shall discuss the **Laplace Transform** extensively and the effects it has on functions when its is applied on them, and also on how an fundamental intuitive mathematical idea of **superposition** gave rise to one of the most important concepts and fields of study in **calculus**, **number theory**, **combinatorics** and how **information** is **transferred** and **processed** globally. We shall also discuss the **optimal methods** through which the transform can be applied to yield the most **accurate solutions** and **values** of a function.*

1 *Introduction*

*I have chosen this research question because I intend to study **Electrical Engineering**. The fundamental concepts of the **Laplace transform**, **Fourier transform** and the **Fourier series** form the **backbone** of a subfield in Electrical Engineering, that is "**Signal processing**", without which any sort of **communication**, through any **electrical device** would not be possible, which implies that **modern day computers** and **computational technology** that exists today would also **not be existent**.*

*The very idea that all of computational technology depends on the concepts laid out **150** to **200 years ago**, deeply interests me, and thus has led me to **study the complex relationships** within **various mathematical concepts**.*

2 *Aim*

*To study the various elements of the **Laplace** and **Fourier transforms** while also studying the **Fourier series** from different mathematical aspects so as to answer the statement posed at the beginning of paper, which is **to find if there exists any relationship** between the **Laplace transform**, **Fourier transform** and the **Fourier series**, if so what is its **mathematical significance** and how can it be optimally used to solve **Ordinary Differential Equations**. This paper aims to answer the above questions.*

3 Background Research

3.1 Generalized form of a Differential Equation

An Ordinary Differential Equation in one variable is generally mathematically defined as,

$$a_n\left(\overset{n}{\dot{y}}\right)^p + a_{n-1}\left(\overset{n-1}{\dot{y}}\right)^q + a_{n-2}\left(\overset{n-2}{\dot{y}}\right)^r + \cdots + a_2\left(\ddot{y}\right)^e + a_1\left(\dot{y}\right)^f + a_0\left(y\right)^g = c$$

Where y is a function of a base variable, ie. x and $\overset{n}{\dot{y}}$ *is the nth derivative of y, the principal function.*

Or in alternate matrix representation, this can be expressed as,

$$\begin{bmatrix} a_n & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & a_{n-1} & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & a_{n-2} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a_0 \end{bmatrix} \begin{bmatrix} \left(\overset{n}{\dot{y}}\right)^p \\ \left(\overset{n-1}{\dot{y}}\right)^q \\ \left(\overset{n-2}{\dot{y}}\right)^r \\ \vdots \\ \left(\ddot{y}\right)^e \\ \left(\dot{y}\right)^f \\ \left(y\right)^g \end{bmatrix} = c$$

Where n is the order of the differential equation, a_n, a_{n-1}, $a_{n-2}\cdots$ *are some arbitrary constants known as the coefficients of the differential equation and p, q, r* $\cdots$ *are also some arbitrary constants, of which the highest constant in value is the degree of the differential equation. It is these constants/functions differentiate differential equation with other forms of the differential equations. c is a fixed constant in the equation, usually zero. In theory these constants can be any function or a constant, but for every type of function that takes place, the values of p, q , r , etc. create some additional segregation and constraints in solving the differential equation.*

3.2 *Types of Ordinary Differential Equations*

There are various types of Ordinary Differential Equations based on the specific values/functions that replace the the values of the coefficients of the differential equation, order and degree of the equation.

3.2.1 *Linear Ordinary Differential Equations*

Linear Ordinary Differential Equations are differential equations where, the degree/exponent of the differential equation is 1 and the coefficients of the differential equation are some arbitrary differential functions in the base variable.

3.2.2 *Non-Linear Ordinary Differential Equations*

An Ordinary Differential equation is said to be non-linear when the degree/exponent of the differential equation is anything but 1 and the co-efficient of the the differential equation is a non-differentiable function of in the principal function, ie. $\sin y$ *as* a_k *or* $\left(\overset{k}{\dot{y}}\right)^2$ *for some value* k*.*

3.2.3 *Homogeneous Ordinary Differential Equations*

An Ordinary Differential equation is said to be homogeneous, when with the nth derivative of the principal function, all other derivatives of lower order's are sequentially present, ie. $a_1\left(\dddot{y}\right)^{k_1} + a_2\left(\ddot{y}\right)^{k_2} + a_3\left(\dot{y}\right)^{k_3} + a_4\left(y\right)^{k_4} = c$

3.2.4 *Heterogeneous Ordinary Differential Equations*

An Ordinary Differential equation is said to be non-homogeneous, when with the nth derivative of the principal function, other derivatives of lower order's are not present, ie. $a_1\left(\dddot{y}\right)^{k_1} + a_2\left(\dot{y}\right)^{k_2} + a_3\left(y\right)^{k_3} = c$

3.3 *Types of Domains*

3.3.1 *Time Domain (Real Domain)*

The time domain, or the real domain is the domain in which we classically deal with functions, usually the x *plane. In practical applications, every physical quantity is measured relatively against time, therefore the name as, the time domain.*

3.3.2 *Frequency Domain (Complex Domain)*

The frequency domain, or the complex domain is the domain which is usually extended mathematically from the real domain, with the aid of imaginary numbers and complex analysis, usually dealt in the z plane.

In practical applications when a function or otherwise known as a signal (in Engineering terminologies) is transformed mathematically, it aids the observer to study with the concept of super-positioning to understand how much of each signal as a part summed up with other parts make up the original function.

In terms of physical quantities, this is done by studying the frequencies of each counterpart of a signal, thus its name as, the frequency domain.

3.4 *The Laplace Transform*

The Laplace Transform is a integral transform that transforms a function fro the time domain to the complex frequency-domain. The transform is mathematically defined as,

$$\mathcal{L}\{f\}(s) = F(s) = \int_0^\infty f(t)e^{-st}dt = \int_0^\infty f(t)e^{-\sigma t}e^{-iwt}dt$$

Where $s = \sigma + iw$. Therefore the transform is defined as,

$$\mathcal{L}\{f\}(s) = F(s) = \int_0^\infty f(t)e^{-\sigma t}e^{-iwt}dt$$

The Laplace Transform is a linear operator, therefore it is only applicable and can only be applied to linear polynomials and functions, which implies that this transform can be used to compute the solutions of differential equations of any order, provided that they are linear.

3.5 *The Fourier Transform*

The Fourier Transform is also an integral transform that transforms a function from the time domain to the complex frequency-domain. The Fourier transform differs from the Laplace transform because the Fourier transform is a slice of the Laplace transform, a component of the Laplace transform. The transform represents for one single value of the Laplace transform.

The transform is mathematically defined as,

$$\mathcal{F}\{f\}(s) = F(s) = \int_0^\infty f(t)e^{-iwt}dt$$

The Fourier Transform is also a linear operator, therefore it is only applicable and can only be applied to linear polynomials and functions, which implies that this transform can be used to compute the solutions of differential equations of any order, provided that they are linear.

3.6 *Trigonometric Series*

A trigonometric series is a series that is of the form,

$$\sum_{n=0}^{\infty} (a_n \cdot \cos(nx) + b_n \cdot \sin(nx))$$

Where, a_n and b_n are some arbitrary functions or constants.

*When a_n and b_n are of the form below, they are known as the **Fourier series**.*

$$a_n = \frac{1}{L} \cdot \int_L f(x) \cdot \cos(nx)\, dx$$

$$a_n = \frac{1}{L} \cdot \int_L f(x) \cdot \sin(nx)\, dx$$

3.7 *The Fourier Series*

The Fourier Series is a infinite summation representation of sinusoidal functions of a periodic function. Mathematically,

$$f(x) = \sum_{n=0}^{\infty} (a_n \cdot \cos(nx) + b_n \cdot \sin(nx))$$

Alternatively,

$$f(x) = \sum_{n=0}^{\infty} a_n \cdot \cos(nx) + \sum_{n=0}^{\infty} b_n \cdot \sin(nx)$$

Or,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cdot \cos(nx) + \sum_{n=1}^{\infty} b_n \cdot \sin(nx)$$

Where,

$$a_n = \frac{2}{P} \cdot \int_P f(x) \cdot \cos\left(\frac{2\pi}{P} \cdot nx\right) dx$$

$$b_n = \frac{2}{P} \cdot \int_P f(x) \cdot \sin\left(\frac{2\pi}{P} \cdot nx\right) dx$$

With exceptions to the coefficient when $n = 0$*, also where* $f(x)$ *is a periodic function.*

Usually of the form,

$$a_n = \frac{1}{\pi} \cdot \int_{-\pi}^{+\pi} f(x) \cdot \cos(nx)\, dx$$

$$b_n = \frac{1}{\pi} \cdot \int_{-\pi}^{+\pi} f(x) \cdot \cos(nx)\, dx$$

By utilizing Euler's formula, we can convert the series from the real from to complex-exponential form. Therefore we have,

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \cdot e^{inx}$$

Where,

$$c_n = \frac{1}{2\pi} \cdot \int_L f(x) \cdot e^{-inx} dx$$

3.8 *Parseval's Theorem*

Parseval's Theorem states is a result of the Fourier Transform. It states that, the integral of the square of a function is equal to the sum of the square of its transform. Mathematically,

$$\frac{1}{\pi} \cdot \int_{-\pi}^{+\pi} (f(x))^2 \, dx = \sum_{n=0}^{\infty} \left(a_n^2 + b_n^2\right)$$

Conversely, could be generalized as,

$$\frac{1}{L} \cdot \int_L f^2(x) dx = \sum_{n=0}^{\infty} \left(a_n^2 + b_n^2\right)$$

4 *Representation of the transforms and series*

4.1 *Analytical Representation*

When analyzing and investigating the Laplace transform, which was previously defined as,

$$\mathcal{L}\{f\}(s) = F(s) = \int_0^{\infty} f(t) e^{-\sigma t} e^{-iwt} dt$$

When $\sigma = 0$, the transform reduces to,

$$\mathcal{L}\{f\}(s) = F(s) = \int_0^{\infty} f(t) e^{-iwt} dt = \mathcal{F}\{f\}(s)$$

Therefore, the Fourier transform is the Laplace transform when the complex variable, s has no real part. ie. $\sigma = 0 \implies s = iw$.

Visually, the Fourier transform is the Laplace transform at $\sigma = 0$. It is a slice of the Laplace transform over the whole domain.

4.2 *Graphical Representation*

To effectively demonstrate visually, on what the transforms and the series means and what it does to functions, we shall specifically consider the function $f(t) = e^{-t} \cdot \sin(2t)$, as this function contains both an exponential component and a sinusoidal component.

4.2.1 *The Laplace Transform*

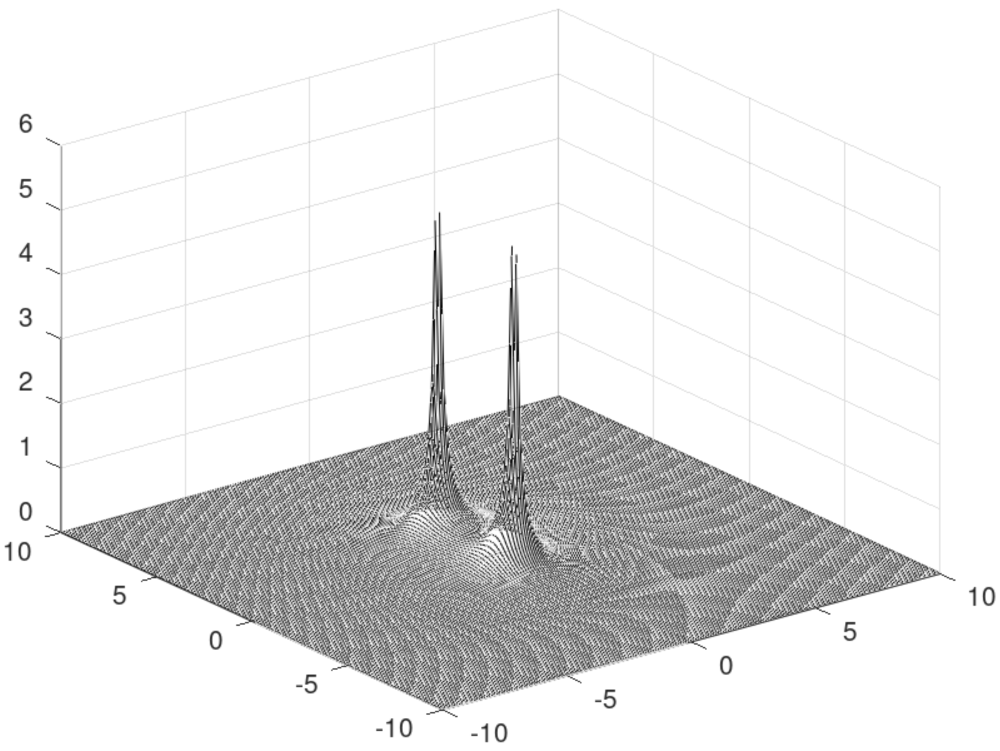

Figure 1: *The Laplace transform of* $e^{-x} \cdot \sin{(2x)}$

4.2.2 *The Fourier Transform*

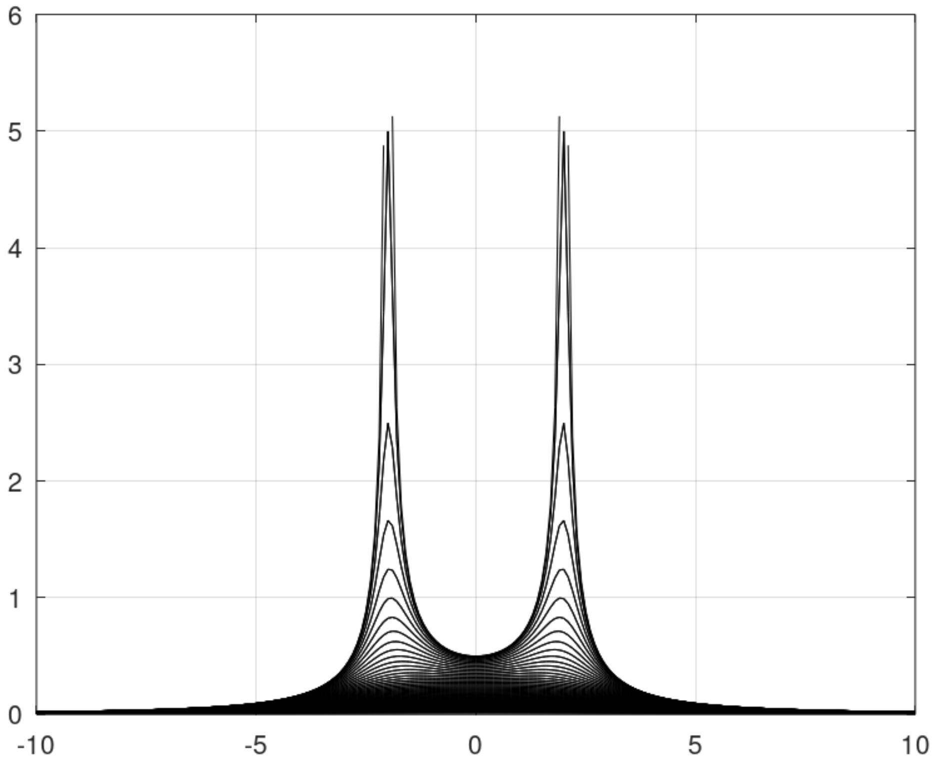

Figure 2: *The Fourier transform of* $e^{-x} \cdot \sin(2x)$

4.2.3 *The Fourier Series*

Figure 3: *The Fourier series of* $e^{-x} \cdot \sin(2x)$

5 *Solving ODE's using transforms and series*

5.1 *Linear ODE*

When dealing with linear ODE's, applying integral transforms and or series to decompose terms in the ODE into sum of infinitesimal sines and cosines is made possible with ease.

This is because the Laplace and the Fourier transforms are linear operators, and consequently can be only applied when an ODE is linear, as for the Fourier series, it can be applied on linear or non-linear ODE's, whereas a linearized ODE simplifies the decomposition process of terms in an ODE.

To further explain this concept, I shall be utilizing an linear ODE as an example. Namely,

$$a_3 \dddot{y} + a_2\ddot{y} + a_1\dot{y} + a_0 y = c$$

Where, a_1, a_2, a_3 *and* c *are some arbitrary constants or functions of some arbitrary variable, other than the principal function,* y.

5.1.1 *The Laplace Transform*

Applying the Laplace transform on the ODE (assuming that a_n*'s and* c *are arbitrary constants) we have,*

$$\mathcal{L}\{a_3 \dddot{y} + a_2\ddot{y} + a_1\dot{y} + a_0 y\} = \mathcal{L}\{c\}$$

Because the Laplace transform is an linear operator, it can be said that the Laplace transform of the whole expression is the sum of the Laplace transform of individual terms, also that the Laplace transform of any constant/function multiplied by the function of the principal function is the Laplace transform of the function of the principal function times the constant/function.

Therefore, we have

$$a_3 \cdot \mathcal{L}\{\dddot{y}\} + a_2 \cdot \mathcal{L}\{\ddot{y}\} + a_1 \cdot \mathcal{L}\{\dot{y}\} + a_0 \cdot \mathcal{L}\{y\} = \mathcal{L}\{c\}$$

When transformed, this expression equals,

$$\begin{aligned} a_3 \cdot &\left[s^3Y(s) - s^2y(0) - sy'(0) - y''(0)\right] \\ &+ a_2 \cdot \left[s^2Y(s) - sy(0) - y'(0)\right] + a_1 \cdot \left[sY(s) - y(0)\right] + a_0 \cdot Y(s) = \frac{c}{s} \end{aligned}$$

Rearranging and regrouping the terms we get,

$$Y(s) \cdot \left[a_3 s^3 + a_2 s^2 + a_1 s + a_0\right] - y(0) \cdot \left[a_3 s^2 + a_2 s + a_1\right] - y'(0) \cdot [a_3 s + a_2] - a_3 \cdot y''(0) = \frac{c}{s}$$

We see that, in the above equation, everything is an arbitrary constant, except $Y(s)$. Also we need not worry about s, which is the complex frequency variable, we call later transform it back to the time variable/real variable. Isolating $Y(s)$, we have,

$$Y(s) = \frac{y(0) \cdot [a_3 s^3 + a_2 s^2 + a_1 s] + y'(0) \cdot [a_3 s^2 + a_2 s] + a_3 \cdot s y''(0) + c}{a_3 s^4 + a_2 s^3 + a_1 s^2 + a_0 s}$$

Therefore, we have,

$$\mathcal{L}^{-1}\{Y(s)\} = y(x) = y = \mathcal{L}^{-1}\left\{\frac{y(0) \cdot [a_3 s^3 + a_2 s^2 + a_1 s] + y'(0) \cdot [a_3 s^2 + a_2 s] + a_3 \cdot s y''(0) + c}{a_3 s^4 + a_2 s^3 + a_1 s^2 + a_0 s}\right\}$$

5.1.2 *The Fourier Transform*

5.1.3 *The Fourier Series*

5.2 *Non-Linear ODE*

*

5.2.1 *Linearization of non-linear ODE*

5.2.2 *

5.2.3 *

6 *Limitations of Study*

There are various limitations in this study/investigation as we have placed forth, certain strict initial conditions and other conditions that make this system so constrained and disables us to expand our researching capability of this chaotic phenomenon. Conditions such as,

- *Restricting the value of* $\theta(0)$ *to* $\pi/2$
- *Restricting the value of* $x(0)$ *to* 0
- *Restricting the value of* l_0 *to 1 m*
- *Restricting the value of the time domain from 0 to 20 seconds*
- *Restricting the value of the radius of the mass employed to* 5×10^{-2} *m*
- *Restricting the value of the spring stiffness constant to 100 N/m*

7 Conclusion

In this paper, I have shown the effects of changes in the ***mass*** *in an* ***elastic pendulum*** *oscillating in a fluid medium (Air(Damped motion)) on the* ***radial displacement****,* ***angular displacement****,* ***angular frequency*** *and* ***absolute frequency*** *by collecting raw data relating to the above parameters, under certain controlled conditions as so to completely study the motion/dynamics of the elastic pendula motion. I have also shown the possible* ***uncertainties in measurement*** *of the* ***radial displacement****,* ***angular displacement****,* ***angular frequency*** *and* ***absolute frequency****.*