The Brachistochrone problem

1 Introduction

The brachistochrone problem, first proposed by Galileo and rediscovered by Johann Bernoulli in 1697, is one of mathematics' most intriguing solved problems. The brachistochrone curve is named after the basic words brachistos (shortest) and chrone (time). This problem is lovely not only because of the question's simplicity but also because of the numerous solutions it encourages. We may observe some of the finest brains in mathematics wrestle and struggle to create more information for all through this puzzle.

Simply put, the reader is asked to find a line connecting two points. According to Euclid's first postulate, a straight line segment can always be drawn connecting any two points. Between two locations on a Euclidian surface, the shortest path is automatically this line segment. What if we didn't want to determine the fastest path between these two sites but rather the shortest time?

Assume a string has a bead threaded on it, and the bead can freely move from point A to point B due to the absence of friction and drag forces. What curve should the string be in this situation, with a constant downward acceleration g, to reduce the bead's journey time?

This question may appear to the reader as a simple minimization problem at first glance. All calculus students well understand the potency of calculus in this aspect. When a function needs to be reduced, the derivative of that function equated to zero indicates the minimum and maximum points of the function.

Using this logic, we are to find a function that minimizes the travel time from point A to point B.

I have chosen this topic for my exploration as I am a physics student, physics fanatic and a deep admirer of the beauty of calculus. I intend to study and explore the mathematics behind this simple mechanics phenomenon from an high-school student perspective. This topic is the due reason for the major advancements in calculus, by Bernoulli, Euler and Lagrange.

The very idea that a very simple mechanics problem that is reason behind great interesting developments in mathematics has caused me to think and explore this problem and the mathematics underlying this problem using the mathematics I have learnt during my mathematics AA course and using some undergraduate mathematics that I have learnt in my free time as a matter of pure interest.

2 Travel Time between two points

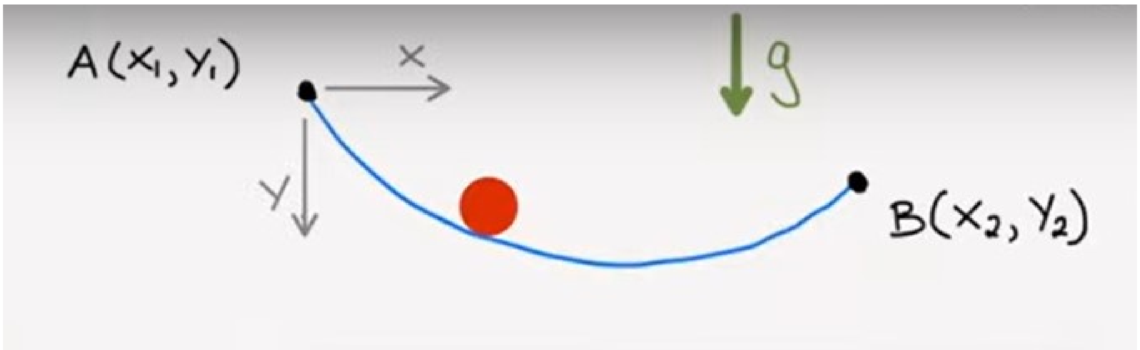

Figure 1: A graphial representation of the Brachistochrone problem

The travel time between two points is the sum of the infinitesimal changes in the time taken to cover a fixed infinitesimal distance from the path length. If we define the total time taken to be T, then the mathematical relation is defined as,

$$T = \int dt$$

Because the phenomenon spans between in an finite time period, the above general integral can be written as,

$$T = \int_0^t dt$$

So, the solution to the problem is the the optimization (minimization) of the time integral above, ie. The optimization (minimization) of $\boldsymbol{T}$.

If we define the velocity of the spherical mass in motion on the path (curve) to be,

$$v = \frac{ds}{dt}$$

Where ds is the incremental path length and dt is the incremental time.

Implies,

$$dt = \frac{ds}{v}$$

Making a substitution for dt in the integral yields,

$$T = \int \frac{ds}{v}$$

Because this phenomenon consists of a finite path with a finite path length spanning between two points, namely A and B, the above general integral can be written as,

$$T = \int_A^B \frac{ds}{v} \tag{1}$$

As we are considering y to be the vertical distance (height) at an instant from point A to the spherical mass on the curve, by the conservation of energy we have that the kinetic energy of the spherical mass at a point in time on curve is the loss in potential energy from height y. Mathematically we have,

$$\frac{1}{2}mv^2 = mgy$$

Implies,

$$v = \sqrt{2gy}$$

If we look at the path (curve) closely and think about the infinitesimal path length ds, in terms of dx and dy, we have by Pythagorean theorem,

$$ds^2 = dx^2 + dy^2$$

Implies

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \sqrt{1 + (y')^2} dx$$

Making a substitution for ds and v in equation 1, we have,

$$T = \int_A^B \frac{ds}{v} = \int_{x_1}^{x_2} \frac{\sqrt{1 + (y')^2}}{\sqrt{2gy}} dx = \int_{x_1}^{x_2} \sqrt{\frac{1 + (y')^2}{2gy}} dx$$

Therefore we have,

$$T = \frac{1}{\sqrt{2g}} \int_{x_1}^{x_2} \sqrt{\frac{1 + (y')^2}{y}} dx \tag{2}$$

Looking at the above equation, we can see that conventional calculus methods do not apply here. Instead of minimizing a specific point in a function, we are to minimize a family of curves (functions).

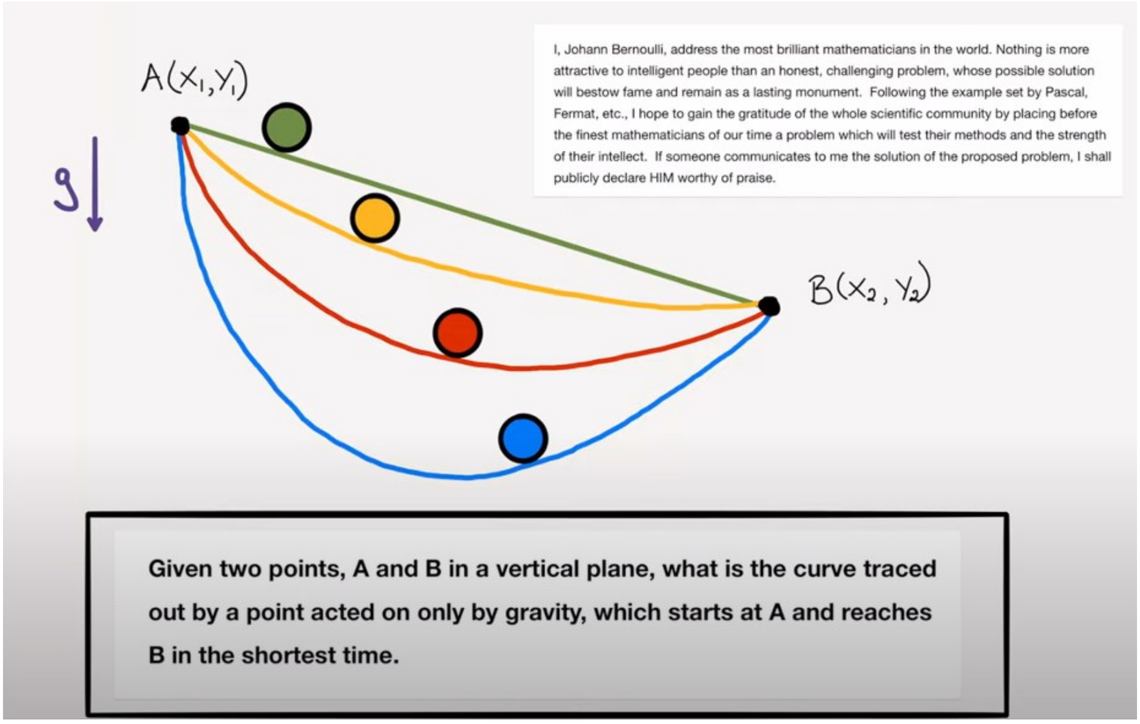

Figure 2: A graphial representation of some curves that could be a possible solution to the Brachistochrone problem

Upon observation we can evidently see that, $\sqrt{\frac{1+(y')^2}{y}}$ is functional in y and y'.

If we define this functional as F, we have,

$$F[y, y'] = \sqrt{\frac{1+(y')^2}{y}}$$

3 Euler-Lagrange Equation

Though Newton and Bernoulli's solutions were fabulous and stunning, they approached the problem in an geometric approach according different cases and phenomenon. It was Euler in collaboration with Lagrange that generalized these sets of problems on the optimization of problems that involved functionals.

Their works are now known as the "Calculus of Variations" as it embarks to employ the calculus of functions that are dependent on other functions (functionals).

In order to solve the Brachistochrone problem that involves the minimization of a functional, we shall use the Euler-Lagrange equation, as this was the primary equation that Euler used to solve the Brachistochrone problem. The equation states that,

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

As the functional "F", does not explicitly depend on x, the Euler-Lagrange equation reduces to what is known as the Beltrami identity. The Beltrami identity states that,

$$F - y'\frac{\partial F}{\partial y'} = C$$

3.1 Evaluating the Beltrami Identity for the Brachistochrone problem

Applying the Beltrami identity for the Brachistochrone problem yields,

$$F - y'\frac{\partial F}{\partial y'} = C = \sqrt{\frac{1+(y')^2}{y}} - y'\frac{y'}{\sqrt{1+y'}}$$

Implies,

$$\frac{1}{\sqrt{1+(y')^2}} = C$$

Therefore by squaring both sides and rearranging we get,

$$y\left(1+(y')^2\right) = \frac{1}{C^2} = k_1$$

Analyzing the above differential equation and rearranging it yields,

$$dx = \sqrt{\frac{y}{k_1 - y}}dy$$

Integrating both sides we get,

$$x + k_2 = \int \sqrt{\frac{y}{k_1 - y}} dy$$

To solve the above integral, we can make an trigonometric substitution for y, that is, $y = k_1 \sin^2 \theta$ for some θ between 0 and $\pi/2$. Therefore we have,

$$dy = 2k_1 \sin\theta \cos\theta d\theta$$

Substituting for y and dy we have,

$$x = \int \sqrt{\frac{y}{k_1 - y}} dy = \int \sqrt{\frac{k_1 \sin^2\theta}{k_1 - k_1 \sin^2\theta}} \cdot 2k_1 \sin\theta \cos\theta d\theta$$

Simplifying the above integral we have,

$$\int \sqrt{\frac{\sin^2\theta}{1 - \sin^2\theta}} \cdot 2k_1 \sin\theta \cos\theta d\theta = \int \sqrt{\frac{\sin^2\theta}{\cos^2\theta}} \cdot 2k_1 \sin\theta \cos\theta d\theta = 2k_1 \int \sin^2\theta d\theta$$

If we make a trigonometric substitution for $\sin^2\theta = 1/2\,(1 - \cos 2\theta)$, we have,

$$x = 2k_1 \int \sin^2\theta d\theta = 2k_1 \int \frac{1}{2}(1 - \cos 2\theta)\, d\theta = k_1\theta - \frac{k_1}{2}\sin 2\theta$$

Therefore we have,

$$x = k_1\theta - \frac{k_1}{2}\sin 2\theta = \frac{k_1}{2}(2\theta - \sin 2\theta)$$

Also we have that,

$$y = k_1 \sin^2\theta = \frac{k_1}{2}(1 - \cos 2\theta)$$

Therefore we have,

$$\boxed{\begin{array}{c} x = \frac{k_1}{2}\left(2\theta - \sin 2\theta\right) \\ y = \frac{k_1}{2}\left(1 - \cos 2\theta\right) \end{array}} \qquad (3)$$

After solving the integral, we obtain parametric equations for x and y that represent a cycloid.

4 The Curve: Cycloid

In the process of using the Calculus of Variations, to minimize the time integral, we have come across the two defining parametric equations of a cycloid.

A cycloid is the curve formed when traced by a point on a circle as it rolls along a straight line without it slipping.

We know this for a fact, as the standard cycloid equations are of the form, $x = r\cos\theta$ and $y = r\sin\theta$.

But if we are to allow the circle that forms the cycloid curve to rotate in a clockwise direction with an angle t from the bottom of the circle, the above standard equations representing the cyloid curve must be corrected as follows,

$$x = -r\sin\theta, y = r\cos\theta$$

Also as the circle that forms the cyloid curve is to move in the positive x direction, we have to add this periodic motion mathematically into the parametric function of the cycloid. Therefore we have,

$$x = -r\sin\theta + \Delta x$$

$$\Delta x = 2\pi r \cdot \frac{\theta}{2\pi} = r\theta$$

Therefore we have,

$$x = -r\sin\theta + r\theta = r\left(\theta - \sin\theta\right)$$

However in the y direction, the only correction that needs to be done is the generalization, that the center of the circle that forms the cycloid is at (r, r)

and not at $(0,0)$, so as to ensure that t he bottom of the cycloid rest at the x axis. Therefore with a vertical translation of r units in the y axis we have,

$$y = r - r\cos\theta = r\,(1 - \cos\theta)$$

Therefore we have,

$$\boxed{\begin{aligned} x &= r\,(\theta - \sin\theta) \\ y &= r\,(1 - \cos\theta) \end{aligned}} \tag{4}$$

From the derivation above, we have confirmed that the parametric equations, that are a solution to the brachistorcone problem are indeed, that of a cycloid, as both of the curves have the same form.

On comparison with the parametric equations we have found as a solution to the brachistochrone problem with the improvised parametric equations of a cycloid, we observe that, $r = k_1/2$. In other words, k_1 is equal to the diameter of the circle which forms the cycloid

5 Differential Equations of the Cycloid

If we are to take the differential of the parametric equations from equations 4 with respect to θ, we have,

$$\frac{dx}{d\theta} = \frac{d}{d\theta}\left(r\,(\theta - \sin\theta)\right) = r\,(1 - \cos\theta)$$

$$\frac{dy}{d\theta} = \frac{d}{d\theta}\left(r\,(1 - \cos\theta)\right) = r\sin\theta$$

Then by using the chain rule we have,

$$\frac{dy}{dx} = \frac{dy}{d\theta} \cdot \frac{d\theta}{dx} = \frac{r\sin\theta}{r\,(1 - \cos\theta)} = \frac{\sin\theta}{1 - \cos\theta}$$

Squaring on both sides we have,

$$\left(\frac{dy}{dx}\right)^2 = \frac{\sin^2\theta}{(1 - \cos\theta)^2} = \frac{1 - \cos^2\theta}{(1 - \cos\theta)^2} = \frac{1 + \cos\theta}{1 - \cos\theta} \tag{5}$$

From equation 4 we know that, $y = r\left(1 - \cos\theta\right)$. Therefore with algebraic manipulation we have,

$$\cos\theta = 1 - \frac{y}{r} \tag{6}$$

Therefore by substituting equation 6 in equation 5, we have,

$$\left(\frac{dy}{dx}\right)^2 = \frac{2r - y}{y} \tag{7}$$

Therefore we have,

$$y\left(\left(y'\right)^2 + 1\right) = 2r$$

6 Solving for Travel Time Minimization

We have calculated and derived the parametric equations to the curve of fastest descent, we now would like to calculate the time it takes to travel on this path.

From equation 2 we have,

$$T = \frac{1}{\sqrt{2g}} \int_{x_1}^{x_2} \sqrt{\frac{1 + \left(y'\right)^2}{y}} dx$$

And from equation 7 we have,

$$\left(\frac{dy}{dx}\right)^2 = \frac{2r - y}{y}$$

Implying,

$$dx = \sqrt{\frac{y}{2r - y}} dy$$

Substituting dx in equation 2 we have,

$$T = \frac{1}{\sqrt{2g}} \int_{y_1}^{y_2} \sqrt{\frac{1 + \left(\sqrt{\frac{2r-y}{y}}\right)^2}{y} \cdot \frac{y}{2r-y}} dy$$

With some algebraic manipulation, the above equation reduces to,

$$T = \frac{1}{\sqrt{2g}} \int_{y_1}^{y_2} \sqrt{\frac{2r}{y\left(2r-y\right)}} dy$$

From equation 4, we know that,

$$y = r\left(1 - \cos\theta\right)$$

Therefore the time minimization integral can be reduced to,

$$T = \frac{1}{\sqrt{2g}} \int_{y_1}^{y_2} \sqrt{\frac{2r}{r\left(1-\cos\theta\right)\left(2r - r\left(1-\cos\theta\right)\right)}} dy$$

If we look at the denominator of the fraction, which therein lies under the integral, we have,

$$r\left(1-\cos\theta\right)\left(2r - r\left(1-\cos\theta\right)\right) = -\left[\left(r-y\right)^2 - r^2 = r^2\cos^2\theta - r^2\right] = -\left[r^2\left(\cos^2\theta - 1\right)\right]$$

But,

$$-\left[r^2\left(\cos^2\theta - 1\right)\right] = r^2\sin^2\theta$$

Therefore we have,

$$T = \frac{1}{\sqrt{2g}} \int_{y_1}^{y_2} \sqrt{\frac{2r}{r^2\sin^2\theta}} dy = \frac{1}{\sqrt{2g}} \int_{y_1}^{y_2} \frac{\sqrt{2r}}{r\sin\theta} dy$$

Substituting $dy = r\sin\theta d\theta$, we have,

$$T = \frac{1}{\sqrt{2g}} \int_{\theta_1}^{\theta_2} \frac{\sqrt{2r}}{r\sin\theta} \cdot r\sin\theta d\theta = \frac{1}{\sqrt{2g}} \int_{\theta_1}^{\theta_2} \sqrt{2r} d\theta$$

Therefore we finally have,

$$T = \sqrt{\frac{r}{g}} \cdot \theta \tag{8}$$

For any values of the coordinates for which the time and path are to be minimized, the variables, r and θ can be calculated according to the case as they are variables in the parametric equations that define the cycloid.

7 Calculation of Sample Path

Up till now, all we have done is to try to understand the nature of the brachistiochrone problem and its solution. We have explored various parametric equations that describe the curve of fastest descent, but we now would like to apply it to a real problem with real coordinates to define a real curve from real points to investigate the curve and the time it takes for a spherical mass to follow this curve-path.

Let us investigate the brachistochrone curve between the points $A(x_1, y_1)$ and $B(x_2, y_2)$, where $x_1 = 0, y_1 = 0, x_2 = 4$ and $y_2 = -3$. Before evaluating the problem, we would like that the beginning point of the curve be placed at the peaks of the cycloid as this ensures that the mass sliding through the curve would attain the highest velocity (as the tangent at the initial point would be a vertical line), However this requires that the parametric equations to be of $\theta \in 2\pi n$, where $n \in \mathbb{Z}^+$.

Therefore by inputting the values of x_2 and y_2 we have,

$$4 = r\,(\theta - \sin\theta) \tag{9}$$

$$-3 = r\,(1 - \cos\theta) \tag{10}$$

Implies,

$$r = \frac{4}{\theta - \sin\theta}$$

Substituting the value for r in 10, we have,

$$-3 = \left(\frac{4}{\theta - \sin\theta}\right)(1 - \cos\theta)$$

Implies,

$$-3\,(\theta - \sin\theta) = 4\,(1 - \cos\theta)$$

Solving for θ using a GDC, we have, $\theta = -2.87995$. Therefore we have,

$$r = \frac{4}{-2.87995 - \sin -2.87995} \approx -1.52597$$

Sustituting this value of r in equations 4, we have,

$$x = -1.53\,(\theta - \sin\theta)$$

$$y = 1.53\,(1 - \cos\theta)$$

Plotting the above parametric equations on Desmos we get,

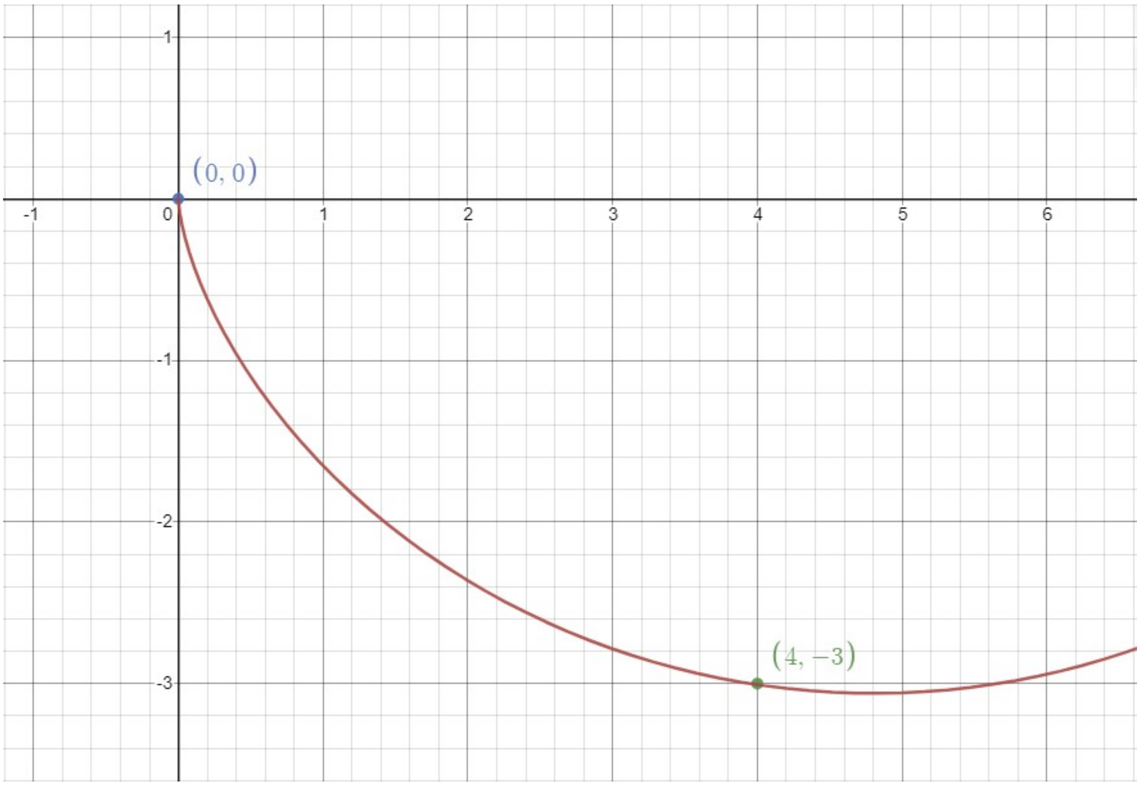

Figure 3: Plot of the parametric equations that plot the cycloid

As we can see, the above curve passes through the points $A(0,0)$ and $B(3,-4)$

Using equation 8, we can find the time it takes for an object/mass to slide from the initial point, to the final point, ie. the the points $A(0,0)$ and $B(3,-4)$. Therefore the time it takes to reach from the initial to the final point is,

If we consider each point to be in the units of meters, then we have,

$$T = \left| \sqrt{\frac{-1.52597}{-9.81}} \cdot -2.87995 \right|$$

Therefore,

$$T \approx 1.13586s$$

This time is relatively much shorter than if an object were to slide down a ramp from point $A(0,0)$ to $B(3,-4)$.

8 Conclusion

A small problem can lead to complicated and often unfathomable notions and solutions, which is the wonder of mathematics. The Brachistochrone curve, which began as a simple task to infer the shortest path between two objects in space aided by gravity, evolved into the Calculus of Variations, an entire field of mathematics.

I discovered various Mathematical Association of America research papers through my "basic" investigation, which I discovered while surfing Wikipedia late one night, Articles, lecture notes, books, physics applications.

Not only did I get my first taste of the calculus of variations, a field I'd never heard of before.

Generally taught at the college level; however, I now understand partial derivatives and how to use them.

The branch between analytic calculus and the realm of geometry is known as the parametric equation. Surprisingly, I discovered my shortcomings in Conventional Mathematics due to this investigation.