Mathematics IA Ver 1.01

The Interrelation of various Transforms and Series

What is the relationship between the Laplace transform, the Fourier transform and the Fourier series of a function and how can it be used to compute the solutions of complex functions?

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Contents

Abstract

We shall discuss the ***Laplace Transform*** *extensively and the effects it has on functions when its is applied on them, and also on how an fundamental intuitive mathematical idea of* ***superposition*** *gave rise to one of the most important concepts and fields of study in* ***calculus****,* ***number theory****,* ***combinatorics*** *and how* ***information*** *is* ***transferred*** *and* ***processed*** *globally. We shall also discuss the* ***optimal methods*** *through which the transform can be applied to yield the most* ***accurate solutions*** *and* ***values*** *of a function.*

1 *Introduction*

I have chosen this research question because I intend to study ***Electrical Engineering****. The fundamental concepts of the* ***Laplace transform****,* ***Fourier transform*** *and the* ***Fourier series*** *form the* ***backbone*** *of a subfield in Electrical Engineering, that is "****Signal processing****", without which any sort of* ***communication****, through any* ***electrical device*** *would not be possible, which implies that* ***modern day computers*** *and* ***computational technology*** *that exists today would also* ***not be existent****.*

The very idea that all of computational technology depends on the concepts laid out ***150*** *to* ***200 years ago****, deeply interests me, and thus has led me to* ***study the complex relationships*** *within* ***various mathematical concepts****.*

2 *Aim*

To study the various elements of the ***Laplace*** *and* ***Fourier transforms*** *while also studying the* ***Fourier series*** *from different mathematical aspects so as to answer the statement posed at the beginning of paper, which is* ***to find if there exists any relationship*** *between the* ***Laplace transform****,* ***Fourier transform*** *and the* ***Fourier series****, if so what is its* ***mathematical significance*** *and how can it be optimally used to solve* ***Ordinary Differential Equations****. This paper aims to answer the above questions.*

3 *Background Research*

3.1 *Generalized form of a Differential Equation*

An Ordinary Differential Equation in one variable is generally mathematically defined as,

$$a_n\left(\overset{n}{\dot{y}}\right)^p + a_{n-1}\left(\overset{n-1}{\dot{y}}\right)^q + a_{n-2}\left(\overset{n-2}{\dot{y}}\right)^r + \cdots + a_2\left(\ddot{y}\right)^e + a_1\left(\dot{y}\right)^f + a_0\left(y\right)^g = c$$

Where y is a function of a base variable, ie. x and $\overset{n}{\dot{y}}$ *is the nth derivative of y, the principal function.*

Or in alternate matrix representation, this can be expressed as,

$$\begin{bmatrix} a_n & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & a_{n-1} & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & a_{n-2} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a_0 \end{bmatrix} \begin{bmatrix} \left(\overset{n}{\dot{y}}\right)^p \\ \left(\overset{n-1}{\dot{y}}\right)^q \\ \left(\overset{n-2}{\dot{y}}\right)^r \\ \vdots \\ \left(\ddot{y}\right)^e \\ \left(\dot{y}\right)^f \\ \left(y\right)^g \end{bmatrix} = c$$

Where n is the order of the differential equation, a_n, a_{n-1}, $a_{n-2}\cdots$ *are some arbitrary constants known as the coefficients of the differential equation and p, q, r* $\cdots$ *are also some arbitrary constants, of which the highest constant in value is the degree of the differential equation. It is these constants/functions differentiate differential equation with other forms of the differential equations. c is a fixed constant in the equation, usually zero. In theory these constants can be any function or a constant, but for every type of function that takes place, the values of p, q , r , etc. create some additional segregation and constraints in solving the differential equation.*

3.2 *Types of Ordinary Differential Equations*

There are various types of Ordinary Differential Equations based on the specific values/functions that replace the the values of the coefficients of the differential equation, order and degree of the equation.

3.2.1 *Linear Ordinary Differential Equations*

Linear Ordinary Differential Equations are differential equations where, the degree/exponent of the differential equation is 1 and the coefficients of the differential equation are some arbitrary differential functions in the base variable.

3.2.2 *Non-Linear Ordinary Differential Equations*

An Ordinary Differential equation is said to be non-linear when the degree/exponent of the differential equation is anything but 1 and the co-efficient of the the differential equation is a non-differentiable function of in the principal function, ie. $\sin y$ *as* a_k *or* $\left(\overset{k}{\dot{y}}\right)^2$ *for some value k.*

3.2.3 *Homogeneous Ordinary Differential Equations*

An Ordinary Differential equation is said to be homogeneous, when with the nth derivative of the principal function, all other derivatives of lower order's are sequentially present, ie. $a_1\left(\dddot{y}\right)^{k_1} + a_2\left(\ddot{y}\right)^{k_2} + a_3\left(\dot{y}\right)^{k_3} + a_4\left(y\right)^{k_4} = c$

3.2.4 *Heterogeneous Ordinary Differential Equations*

An Ordinary Differential equation is said to be non-homogeneous, when with the nth derivative of the principal function, other derivatives of lower order's are not present, ie. $a_1\left(\dddot{y}\right)^{k_1} + a_2\left(\dot{y}\right)^{k_2} + a_3\left(y\right)^{k_3} = c$

3.3 *Types of Domains*

3.3.1 *Time Domain (Real Domain)*

The time domain, or the real domain is the domain in which we classically deal with functions, usually the x plane. In practical applications, every physical quantity is measured relatively against time, therefore the name as, the time domain.

3.3.2 *Frequency Domain (Complex Domain)*

The frequency domain, or the complex domain is the domain which is usually extended mathematically from the real domain, with the aid of imaginary numbers and complex analysis, usually dealt in the z plane.

In practical applications when a function or otherwise known as a signal (in Engineering terminologies) is transformed mathematically, it aids the observer to study with the concept of super-positioning to understand how much of each signal as a part summed up with other parts make up the original function.

In terms of physical quantities, this is done by studying the frequencies of each counterpart of a signal, thus its name as, the frequency domain.

3.4 *The Laplace Transform*

The Laplace Transform is a integral transform that transforms a function fro the time domain to the complex frequency-domain. The transform is mathematically defined as,

$$\mathcal{L}\{f\}(s) = F(s) = \int_0^\infty f(t)e^{-st}dt = \int_0^\infty f(t)e^{-\sigma t}e^{-iwt}dt$$

Where $s = \sigma + iw$. *Therefore the transform is defined as,*

$$\mathcal{L}\{f\}(s) = F(s) = \int_0^\infty f(t)e^{-\sigma t}e^{-iwt}dt$$

The Laplace Transform is a linear operator, therefore it is only applicable and can only be applied to linear polynomials and functions, which implies that this transform can be used to compute the solutions of differential equations of any order, provided that they are linear.

Note*: A table consisting some brief Laplacian conversions can be found in the appendix.*

3.5 *The Fourier Transform*

The Fourier Transform is also an integral transform that transforms a function from the time domain to the complex frequency-domain. The Fourier transform differs from the Laplace transform because the Fourier transform is a slice of *the Laplace transform, a component of the Laplace transform. The transform represents for one single value of the Laplace transform.*

The transform is mathematically defined as,

$$\mathcal{F}\{f\}(s) = F(s) = \int_{-\infty}^{+\infty} f(t)e^{-iwt}dt$$

The Fourier Transform is also a linear operator, therefore it is only applicable and can only be applied to linear polynomials and functions, which implies that this transform can be used to compute the solutions of differential equations of any order, provided that they are linear.

3.6 *Trigonometric Series*

A trigonometric series is a series that is of the form,

$$\sum_{n=0}^{\infty} (a_n \cdot \cos(nx) + b_n \cdot \sin(nx))$$

Where, a_n and b_n are some arbitrary functions or constants.

When a_n and b_n are of the form below, they are known as the ***Fourier series****.*

$$a_n = \frac{1}{L} \cdot \int_L f(x) \cdot \cos(nx)\, dx$$

$$a_n = \frac{1}{L} \cdot \int_L f(x) \cdot \sin(nx)\, dx$$

3.7 *The Fourier Series*

The Fourier Series is a infinite summation representation of sinusoidal functions of a periodic function. Mathematically,

$$f(x) = \sum_{n=0}^{\infty} (a_n \cdot \cos(nx) + b_n \cdot \sin(nx))$$

Alternatively,

$$f(x) = \sum_{n=0}^{\infty} a_n \cdot \cos(nx) + \sum_{n=0}^{\infty} b_n \cdot \sin(nx)$$

Or,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cdot \cos(nx) + \sum_{n=1}^{\infty} b_n \cdot \sin(nx)$$

Where,

$$a_n = \frac{2}{P} \cdot \int_P f(x) \cdot \cos\left(\frac{2\pi}{P} \cdot nx\right) dx$$

$$b_n = \frac{2}{P} \cdot \int_P f(x) \cdot \sin\left(\frac{2\pi}{P} \cdot nx\right) dx$$

With exceptions to the coefficient when $n = 0$*, also where* $f(x)$ *is a periodic function.*

Usually of the form,

$$a_n = \frac{1}{\pi} \cdot \int_{-\pi}^{+\pi} f(x) \cdot \cos(nx)\, dx$$

$$b_n = \frac{1}{\pi} \cdot \int_{-\pi}^{+\pi} f(x) \cdot \cos(nx)\, dx$$

By utilizing Euler's formula, we can convert the series from the real from to complex-exponential form. Therefore we have,

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \cdot e^{inx}$$

Where,

$$c_n = \frac{1}{2\pi} \cdot \int_L f(x) \cdot e^{-inx} dx$$

3.8 *Parseval's Theorem*

Parseval's Theorem states is a result of the Fourier Transform. It states that, the integral of the square of a function is equal to the sum of the square of its transform. Mathematically,

$$\frac{1}{\pi} \cdot \int_{-\pi}^{+\pi} (f(x))^2 \, dx = \sum_{n=0}^{\infty} \left(a_n^2 + b_n^2\right)$$

Conversely, could be generalized as,

$$\frac{1}{L} \cdot \int_L f^2(x)dx = \sum_{n=0}^{\infty} \left(a_n^2 + b_n^2\right)$$

3.9 *Maclaurin Series*

The Maclaurin Series of a function is an infinite sum of terms that are expressed in terms of the function's derivatives at 0. The Maclaurin series is mathematically defined as,

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} \cdot x^n$$

The Maclaurin series aids us to decompose any function directly/indirectly into, an infinite series, which then can be approximated accordingly. The real hidden use could be to decompose a term that obstructs a solution to an ODE, first into a Fourier series and then into a Maclaurin series.

The Maclaurin series of the base trigonometric functions are,

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \cdot x^{2n+1}$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \cdot x^{2n}$$

4 *Representation of the transforms and series*

4.1 *Analytical Representation*

When analyzing and investigating the Laplace transform, which was previously defined as,

$$\mathcal{L}\{f\}(s) = F(s) = \int_0^\infty f(t)e^{-\sigma t}e^{-iwt}dt$$

When $\sigma = 0$*, the transform reduces to,*

$$\mathcal{L}\{f\}(s) = F(s) = \int_0^\infty f(t)e^{-iwt}dt = \mathcal{F}\{f\}(s)|_0^\infty$$

Therefore, the Fourier transform is the Laplace transform when the complex variable, s has no real part. ie. $\sigma = 0 \implies s = iw$*.*

Visually, the Fourier transform is the Laplace transform at $\sigma = 0$*. It is a slice of the Laplace transform over the whole domain.*

4.2 *Graphical Representation*

To effectively demonstrate visually, on what the transforms and the series means and what it does to functions, we shall specifically consider the function $f(t) = e^{-t} \cdot \sin(2t)$*, as this function contains both an exponential component and a sinusoidal component.*

4.2.1 *The Laplace Transform*

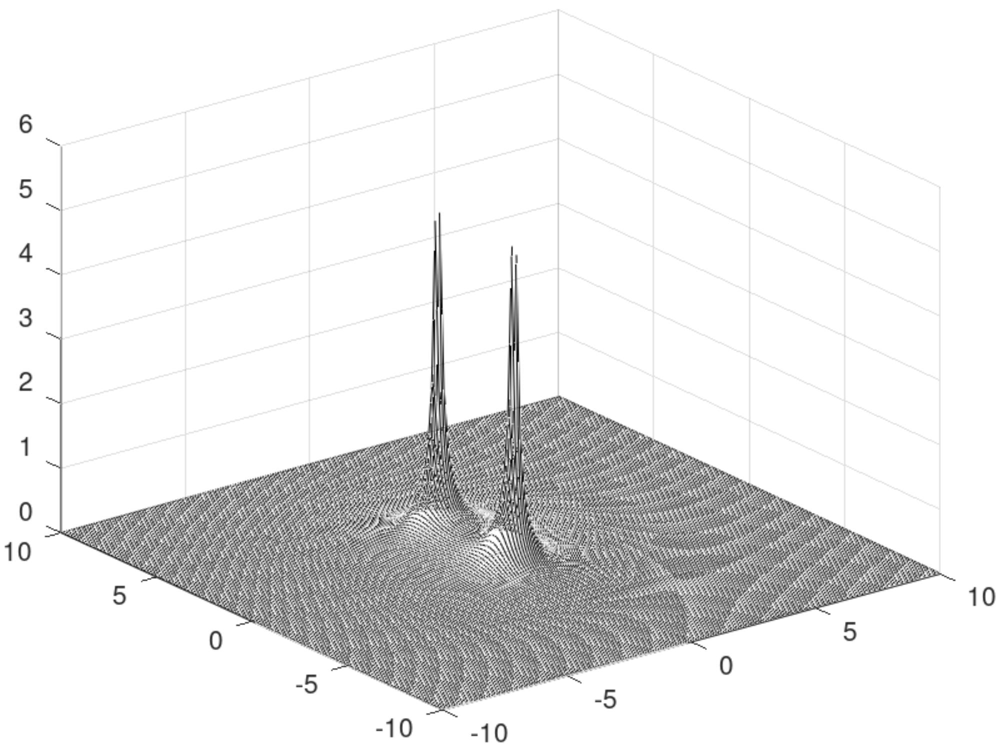

Figure 1: *The Laplace transform of* $e^{-x} \cdot \sin{(2x)}$

It is evident form observing the graph of the Laplace transform of $e^{-t} \cdot \sin{(2t)}$*, that the transform completely changes the function, all over the complex domain.*

4.2.2 *The Fourier Transform*

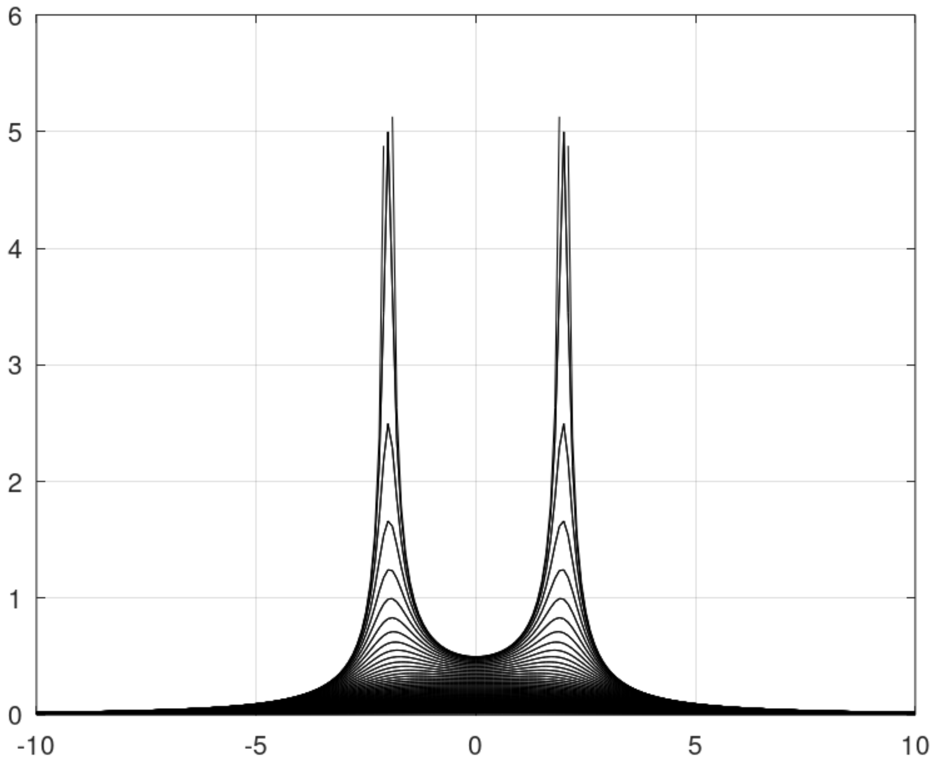

Figure 2: *The Fourier transform of* $e^{-x} \cdot \sin(2x)$

By observation and graphical comparison, it is evident that, the Fourier transform is a slice of the Laplace transform, over the whole domain. In other words it is the Laplace transform at a specific coordinate value of the Laplace transform, ie. when the real part is zero.

5 *Solving ODE's using transforms and series*

5.1 *Linear ODE*

When dealing with linear ODE's, applying integral transforms and/or series to decompose terms in the ODE into sum of infinitesimal sines and cosines is made possible with ease.

This is because the Laplace and the Fourier transforms are linear operators, and consequently can be only applied when an ODE is linear, as for the Fourier series, it can be applied on linear or non-linear ODE's, whereas a linearized ODE simplifies the decomposition process of terms in an ODE.

To further explain this concept, I shall be utilizing an linear ODE as an example. Namely,

$$a_3\dddot{y} + a_2\ddot{y} + a_1\dot{y} + a_0 y = c$$

Where, a_1*,* a_2*,* a_3 *and* c *are some arbitrary constants or functions of some arbitrary variable, other than the principal function,* y*.*

5.1.1 *The Laplace Transform*

Applying the Laplace transform on the ODE (assuming that a_n*'s and* c *are arbitrary constants) we have,*

$$\mathcal{L}\{a_3\dddot{y} + a_2\ddot{y} + a_1\dot{y} + a_0 y\} = \mathcal{L}\{c\}$$

Because the Laplace transform is an linear operator, it can be said that the Laplace transform of the whole expression is the sum of the Laplace transform of individual terms, also that the Laplace transform of any constant/function multiplied by the function of the principal function is the Laplace transform of the function of the principal function times the constant/function.

Therefore, we have

$$a_3 \cdot \mathcal{L}\{\dddot{y}\} + a_2 \cdot \mathcal{L}\{\ddot{y}\} + a_1 \cdot \mathcal{L}\{\dot{y}\} + a_0 \cdot \mathcal{L}\{y\} = \mathcal{L}\{c\}$$

When transformed, this expression equals,

$$\begin{aligned} a_3 \cdot &\left[s^3 Y(s) - s^2 y(0) - s y'(0) - y''(0)\right] \\ &+ a_2 \cdot \left[s^2 Y(s) - s y(0) - y'(0)\right] + a_1 \cdot [sY(s) - y(0)] + a_0 \cdot Y(s) = \frac{c}{s} \end{aligned}$$

Rearranging and regrouping the terms we get,

$$Y(s)\cdot\left[a_3s^3+a_2s^2+a_1s+a_0\right]-y(0)\cdot\left[a_3s^2+a_2s+a_1\right]$$
$$-\,y'(0)\cdot[a_3s+a_2]-a_3\cdot y''(0)=\frac{c}{s}$$

We see that, in the above equation, everything is an arbitrary constant, except $Y(s)$. Also we need not worry about s, which is the complex frequency variable, we call later transform it back to the time variable/real variable. Isolating $Y(s)$, we have,

$$Y(s)=\frac{y(0)\cdot[a_3s^3+a_2s^2+a_1s]+y'(0)\cdot[a_3s^2+a_2s]+a_3\cdot sy''(0)+c}{a_3s^4+a_2s^3+a_1s^2+a_0s}$$

Therefore, we have,

$$\mathcal{L}^{-1}\{Y(s)\}=y(x)=y=$$
$$\mathcal{L}^{-1}\left\{\frac{y(0)\cdot[a_3s^3+a_2s^2+a_1s]+y'(0)\cdot[a_3s^2+a_2s]+a_3\cdot sy''(0)+c}{a_3s^4+a_2s^3+a_1s^2+a_0s}\right\}$$

We can infer that all that is left in the expression on the R.H.S that forms the inverse Laplace transform equals the principal function y, can be decomposed into simpler partial fractions and the inverse transform can be applied to each individual partial fraction, as it consists of only constants/function not in y and the complex frequency variable s.

5.1.2 *The Fourier Transform*

The Fourier transform, is of not much use, as it is the same as the Laplace transform except that it is mathematically restrictive in terms of its domain to a set of values. The Fourier transform would also yield a function in the complex frequency domain but without the real part of the complex variable.

Applying the transform, simplifying the terms in the ODE, algebraically ordering and then applying the inverse Fourier transform would yield the same results as when done with the Laplace transform.

The real difference would be be when we compute the transform of Partial Differential Equations (PDE's). When computing the transform of PDE's, *the Laplacian does little help, when compared to the Fourierian, as its distinct approach to annihilate the real part of the complex frequency makes it increasingly easy to study the non-linearity of PDE's for example.*

5.1.3 *The Fourier Series*

The Fourier series breaks down, a term that is an obstacle to an eloquent solution, by decomposing a specific term and/or a coefficient, into a sum of infinite sines and cosines. When the ODE is linear, this process is simplified extremely, such that it is easier to compute any equation that has sines and cosines and linear ODE as its part.

The Fourier series can be extensively used as an appropriate approximation method (before any numerical integration algorithm is applied, ie. RK4) to compute solutions to equations that may be linear/non-linear but still cannot be solved analytically.

5.2 *Non-Linear ODE*

When dealing with non-linear ODE's, applying integral transforms and/or series to decompose terms in the ODE into sum of infinitesimal sines and cosines is excruciatingly difficult.

A we discussed above, this is because the Laplace and the Fourier transforms are linear operators, and consequently can be only applied when an ODE is linear, as for the Fourier series, it can be applied on linear or non-linear ODE's, whereas a linearized ODE simplifies the decomposition process of terms in an ODE.

But this does not the limit the scope of the use of integral transforms and/or series as we fundamentally know that, any and all non-linear ODE's can be decomposed into a linear system with some clever change of variables and substitution.

Once we have accomplished to convert a non-linear system/ODE into a linear system/ODE then the process what follows for the application of the Laplace/Fourier transforms and/or Fourier series is pretty much the same as the original linear ODE case we have begun with, it might be even simpler.

***Note**: Linearizing an non-linear ODE is an additional step to solve and compute the transforms/series of non-linear systems.*

5.2.1 *Linearization of non-linear ODE*

If we have a non-linear system comprising of a non-linear ODE, then in order to apply the Laplace and Fourier transform, we can convert the non-linear ODE into a linear ODE with some clever change of variables and variable substitution.

To elucidate this in detail, I shall be utilizing the non-linear ODE as an example. Namely,

$$a_3 \left(\dddot{y} \right)^2 + a_2 \ddot{y} + a_1 \left(\dot{y} \right)^2 + a_0 y = c$$

If we make a substitution for $y = u^4$, for some arbitrary variable u, then we have,

$$\dot{y} = 4u^3$$

$$\ddot{y} = 12u^2$$

$$\dddot{y} = 24u$$

Therefore, we have,

$$a_3 \left(24u\right)^2 + a_2 \left(12u^2\right) + a_1 \left(4u^3\right)^2 + a_0 \left(u^4\right) = c$$

Implies,

$$\left(576a_3\right) u^2 + \left(12a_2\right) u^2 + \left(16a_1\right) u^6 + \left(a_0\right) u^4 = c$$

Now what the above expression appears to be is a nasty algebraic equation in variable u, but this can be further depressed, if we make a substitution for $v = u^2$, for some arbitrary variable v, then we have,

$$\left(576a_3\right) v + \left(12a_2\right) v + \left(16a_1\right) v^3 + \left(a_0\right) v^2 = c$$

Regrouping and rearranging the terms in the equation, we have,

$$(16a_1)\, v^3 + (a_0)\, v^2 + (576a_3 + 12a_2)\, v = c$$

As it is evident from the above equation, we have cleverly manipulated variables and derivatives to depress the original non-linear ODE into a cubic polynomial in variable v, if the a_n's and c are some arbitrary constants or not functions in y, then this can be solved directly with some algebraic manipulation. The necessary aid of transforms truly comes when, the a_n's and c are functions in y.

To elucidate on the solution of non-linear ODE when the a_n's and c are functions in a better manner, I shall be assign $a_0 = 1$, $a_1 = \sin y$, $a_2 = 1$, $a_3 = \cos x$ and $c = 0$.

Therefore we have,

$$(16 \sin y)v^3 + v^2 + (576 \cos y + 12)v = 0$$

But, $y = u^4 = (u^2)^2 = v^2$, therefore we have,

$$16 \sin\left(v^2\right) v^3 + v^2 + (576 \cos\left(v^2\right) + 12)v = 0$$

5.2.2 *Fourier Series*

The Fourier series would be of use when, a_n's and c are non-trig functions of y, then we an yield the Fourier series and continue with the Maclaurin series, as the Fourier series would decompose any function into a series of sines and cosines. But as in this case, we already have all functions in trigonometric function form, therefore we can continue with applying the Fourier series.

5.2.3 *Maclaurin Series*

To approximately solve the above ODE we can use the Maclaurin series of sine and cosine, which is,

$$\sin y = \sin\left(v^2\right) \approx v^2 - \frac{(v^2)^3}{3!} + \frac{(v^2)^5}{5!} - \frac{(v^2)^7}{7!} + \frac{(v^2)^9}{9!}$$

$$\cos y = \cos\left(v^2\right) \approx 1 - \frac{(v^2)^2}{2!} + \frac{(v^2)^4}{4!} - \frac{(v^2)^6}{6!} + \frac{(v^2)^8}{8!}$$

We can make the above substitution in the non-linear ODE with trig functions in the principal function, and either solve the equation with algebraic manipulation or by applying the Transforms used to solve linear ODE's as after the substitution, the system would become linear.

***Note**: The above method on using the Fourier and Maclaurin series just yeilds an approximate solution. The accuracy of the solution depends on the number of terms from the infinite expansion taken into account, for substitution in the ODE.*

6 *Conclusion*

In this paper, I have answered the question posed at the beginning of the paper. Namely, to find any existent relationship that exists between the Laplace/Fourier Transforms with various infinite series and how can they be optimally utilized to solve ODE's considering many cases.

I have in the due process of this exploration discovered an unique relationship between various transforms, series and the many ways with the aid of mathematical conversion and systematic approach to optimally solve various ODE's in an systematic approach.

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Appendices

Table of Laplace Transforms

	$f(t)=\mathfrak{L}^{-1}\{F(s)\}$	$F(s)=\mathfrak{L}\{f(t)\}$		$f(t)=\mathfrak{L}^{-1}\{F(s)\}$	$F(s)=\mathfrak{L}\{f(t)\}$
1.	1	$\frac{1}{s}$	2.	$\mathbf{e}^{at}$	$\frac{1}{s-a}$
3.	$t^n, \quad n=1,2,3,\ldots$	$\frac{n!}{s^{n+1}}$	4.	$t^p, p>-1$	$\frac{\Gamma(p+1)}{s^{p+1}}$
5.	$\sqrt{t}$	$\frac{\sqrt{\pi}}{2s^{\frac{3}{2}}}$	6.	$t^{n-\frac{1}{2}}, \quad n=1,2,3,\ldots$	$\frac{1\cdot 3\cdot 5\cdots(2n-1)\sqrt{\pi}}{2^n s^{n+\frac{1}{2}}}$
7.	$\sin(at)$	$\frac{a}{s^2+a^2}$	8.	$\cos(at)$	$\frac{s}{s^2+a^2}$
9.	$t\sin(at)$	$\frac{2as}{(s^2+a^2)^2}$	10.	$t\cos(at)$	$\frac{s^2-a^2}{(s^2+a^2)^2}$
11.	$\sin(at)-at\cos(at)$	$\frac{2a^3}{(s^2+a^2)^2}$	12.	$\sin(at)+at\cos(at)$	$\frac{2as^2}{(s^2+a^2)^2}$
13.	$\cos(at)-at\sin(at)$	$\frac{s(s^2-a^2)}{(s^2+a^2)^2}$	14.	$\cos(at)+at\sin(at)$	$\frac{s(s^2+3a^2)}{(s^2+a^2)^2}$
15.	$\sin(at+b)$	$\frac{s\sin(b)+a\cos(b)}{s^2+a^2}$	16.	$\cos(at+b)$	$\frac{s\cos(b)-a\sin(b)}{s^2+a^2}$
17.	$\sinh(at)$	$\frac{a}{s^2-a^2}$	18.	$\cosh(at)$	$\frac{s}{s^2-a^2}$
19.	$\mathbf{e}^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}$	20.	$\mathbf{e}^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}$
21.	$\mathbf{e}^{at}\sinh(bt)$	$\frac{b}{(s-a)^2-b^2}$	22.	$\mathbf{e}^{at}\cosh(bt)$	$\frac{s-a}{(s-a)^2-b^2}$
23.	$t^n\mathbf{e}^{at}, \quad n=1,2,3,\ldots$	$\frac{n!}{(s-a)^{n+1}}$	24.	$f(ct)$	$\frac{1}{c}F\left(\frac{s}{c}\right)$
25.	$u_c(t)=u(t-c)$ Heaviside Function	$\frac{\mathbf{e}^{-cs}}{s}$	26.	$\delta(t-c)$ Dirac Delta Function	$\mathbf{e}^{-cs}$
27.	$u_c(t)f(t-c)$	$\mathbf{e}^{-cs}F(s)$	28.	$u_c(t)g(t)$	$\mathbf{e}^{-cs}\mathfrak{L}\{g(t+c)\}$
29.	$\mathbf{e}^{ct}f(t)$	$F(s-c)$	30.	$t^n f(t), \quad n=1,2,3,\ldots$	$(-1)^n F^{(n)}(s)$
31.	$\frac{1}{t}f(t)$	$\int_s^{\infty} F(u)\,du$	32.	$\int_0^t f(v)\,dv$	$\frac{F(s)}{s}$
33.	$\int_0^t f(t-\tau)g(\tau)\,d\tau$	$F(s)G(s)$	34.	$f(t+T)=f(t)$	$\frac{\int_0^T \mathbf{e}^{-st}f(t)\,dt}{1-\mathbf{e}^{-sT}}$
35.	$f'(t)$	$sF(s)-f(0)$	36.	$f''(t)$	$s^2F(s)-sf(0)-f'(0)$
37.	$f^{(n)}(t)$	$s^nF(s)-s^{n-1}f(0)-s^{n-2}f'(0)\cdots-sf^{(n-2)}(0)-f^{(n-1)}(0)$			

Figure 3: *Laplace Conversion Table of some functions*