IB Mathematics AA HL

The Brachistochrone problem

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1 Introduction

The brachistochrone problem is a mathematical problem/challenge proposed by Bernoulli to fellow mathematicians in 1696. The brachistochrone curve is named after the simple words in Latin, brachistos (shortest) and chrone (time).

While exploring and researching for my Mathematics IA topic, I came across this topic, which at first sounded like a simple minimization problem whose solution can be found with high-school calculus. But after further reading and understanding the problem, I realized that the problem is quite complicated, though it seems simple.

The problem is about finding a curve between two points (where one point is higher than the other) on a two dimensional plane, such that a frictionless object travels in the shortest time possible. One might think it is a straight line, as I had initially thought, but this is the path for the shortest distance, not the shortest time.

I visualized this problem in such a way that if a bead is to be threaded on a string that can freely move between two points, namely A and B in the absence of drag and frictional forces

This question may appear to the person as a simple minimization problem at first glance. All calculus students well understand the potency of calculus in this aspect. When a function needs to be reduced, the derivative of that function equated to zero indicates the minimum and maximum points of the function.

Using this logic, we are to find a function that minimizes the travel time from point A to point B.

I aim to explore this problem from a high-school student perspective using high-school mathematics that I have learnt during my mathematics AA course and using some undergraduate mathematics that I have learnt in my free time.

The very idea that a straightforward mechanics problem is a reason behind great interesting developments in mathematics has caused me to explore this problem.

To solve this problem, I have to find the curve of fastest descent and to find that, I would need to find a function that optimizes time. This would be the first step to solve the problem.

2 Travel Time between two points

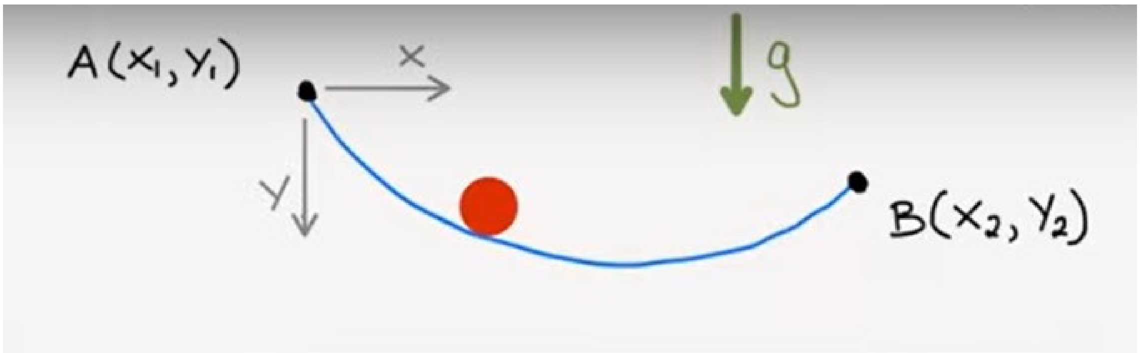

Figure 1: A graphial representation of the Brachistochrone problem

The travel time between two points is the sum of the infinitesimal changes in the time taken to cover a fixed infinitesimal distance from the path length. If I, define the total time taken to be T, then the mathematical relation is defined as,

$$T = \int dt$$

Because the phenomenon spans between in an finite time period, the above general integral can be written as,

$$T = \int_{t_1}^{t_2} dt$$

Where, the phenomenon spans between $t = t_1$ and $t = t_2$. So, the solution to the problem is obtained when the optimization (minimization) of the time integral above is computed, ie. Obtaining the curve on which the optimization (minimization) of $\boldsymbol{T}$ takes place.

If I define the velocity of the spherical mass in motion on the path (curve) to be,

$$v = \frac{ds}{dt}$$

Where ds is the incremental path length and dt is the incremental time.

Implies,

$$dt = \frac{ds}{v}$$

Making a substitution for dt in the integral for $\boldsymbol{T}$ yields,

$$T = \int \frac{ds}{v}$$

Because this phenomenon consists of a finite path with a finite path length spanning between two points, namely x_1 and x_2, the above general integral can be written as,

$$T = \int_{x_1}^{x_2} \frac{ds}{v} \tag{1}$$

As I am considering y to be the vertical distance (height) at an instant from point x_1 to the spherical mass on the curve, by the conservation of energy we have that the kinetic energy of the spherical mass at a point in time on curve is the loss in potential energy from height y. Mathematically we have,

$$\frac{1}{2}mv^2 = mgy$$

Implies,

$$v = \sqrt{2gy}$$

If we look at the path (curve) closely and think about the infinitesimal path length ds, in terms of dx and dy, we have by Pythagorean theorem,

$$ds^2 = dx^2 + dy^2$$

Implies

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \sqrt{1 + (y')^2} dx$$

Making a substitution for ds and v in equation 1, we have,

$$T = \int_{x_1}^{x_2} \frac{ds}{v} = \int_{x_1}^{x_2} \frac{\sqrt{1+(y')^2}}{\sqrt{2gy}} dx = \int_{x_1}^{x_2} \sqrt{\frac{1+(y')^2}{2gy}} dx$$

Therefore we have,

$$T = \frac{1}{\sqrt{2g}} \int_{x_1}^{x_2} \sqrt{\frac{1+(y')^2}{y}} dx \tag{2}$$

Looking at the above equation, we can see that conventional calculus methods do not apply here. Instead of minimizing a specific point in a function, we are to minimize a family of curves (functions). This is because the function under the integral represents a category of special functions.

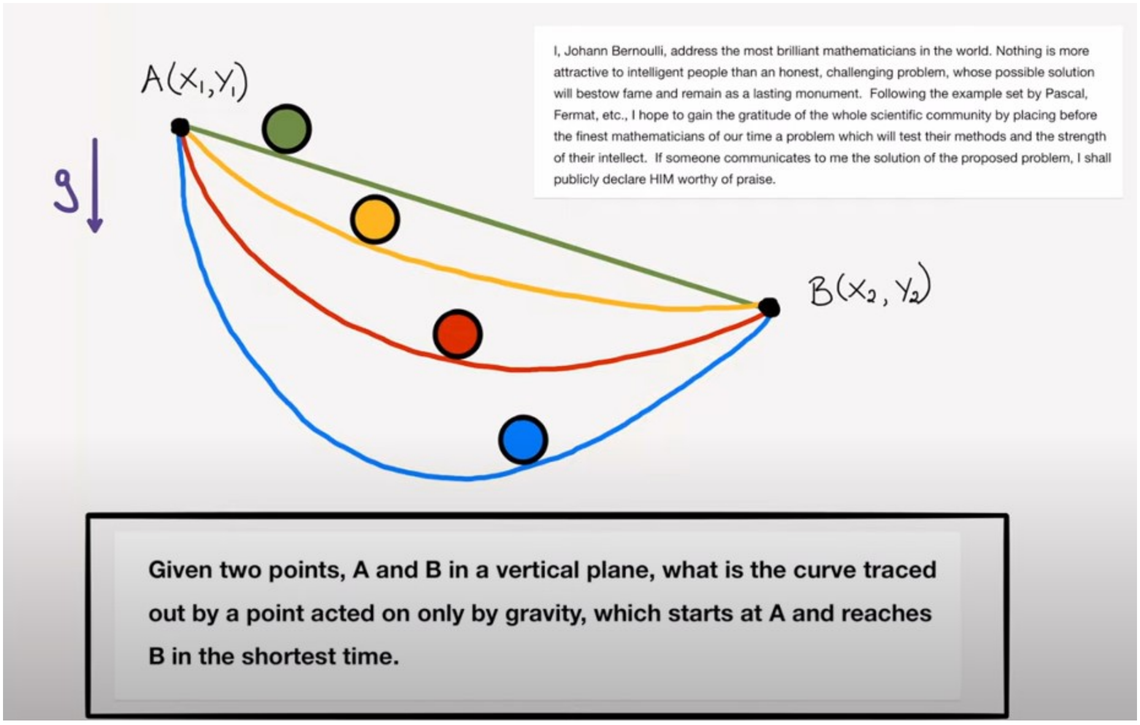

Figure 2: A graphial representation of some curves that could be a possible solution to the Brachistochrone problem

Upon observation we can evidently see that, $\sqrt{\frac{1+(y')^2}{y}}$ is a special function in y and y'. A function of this type (a function depending on a variable and their derivatives) is called a functional.

If we define this functional as F, we have,

$$F[y, y'] = \sqrt{\frac{1 + (y')^2}{y}}$$

3 Euler-Lagrange Equation

Though Newton and Bernoulli's solutions were fabulous and stunning, they approached the problem in an geometric approach according different cases and phenomenon. It was Euler in collaboration with Lagrange that generalized these sets of problems on the optimization of problems that involved functionals.

Their works are now known as the "Calculus of Variations" as it embarks to employ the calculus of functions that are dependent on other functions (functionals).

In order to solve the Brachistochrone problem that involves the minimization of a functional, I shall use the Euler-Lagrange equation, as this was the primary equation that Euler used to solve the Brachistochrone problem. The equation states that,

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

As the functional "F", does not explicitly depend on x, the Euler-Lagrange equation reduces to what is known as the Beltrami identity. The Beltrami identity states that,

$$F - y'\frac{\partial F}{\partial y'} = C$$

3.1 Evaluating the Beltrami Identity for the Brachistochrone problem

Applying the Beltrami identity for the Brachistochrone problem yields,

$$F - y'\frac{\partial F}{\partial y'} = C = \sqrt{\frac{1 + (y')^2}{y}} - y'\frac{y'}{\sqrt{1 + y'}}$$

Implies,

$$\frac{1}{\sqrt{1+(y')^2}} = C$$

Therefore by squaring both sides and rearranging we get,

$$y\left(1+(y')^2\right) = \frac{1}{C^2} = k_1$$

Analyzing the above differential equation and rearranging it yields,

$$dx = \sqrt{\frac{y}{k_1 - y}}dy$$

Integrating both sides we get,

$$x + k_2 = \int \sqrt{\frac{y}{k_1 - y}}dy$$

To solve the above integral, I can make an trigonometric substitution for y, that is, $y = k_1 \sin^2\theta$ for some θ between 0 and $\pi/2$. Therefore we have,

$$dy = 2k_1 \sin\theta\cos\theta d\theta$$

Substituting for y and dy we have,

$$x = \int \sqrt{\frac{y}{k_1 - y}}dy = \int \sqrt{\frac{k_1 \sin^2\theta}{k_1 - k_1\sin^2\theta}} \cdot 2k_1 \sin\theta\cos\theta d\theta$$

Simplifying the above integral we have,

$$\int \sqrt{\frac{\sin^2\theta}{1-\sin^2\theta}} \cdot 2k_1 \sin\theta\cos\theta d\theta = \int \sqrt{\frac{\sin^2\theta}{\cos^2\theta}} \cdot 2k_1 \sin\theta\cos\theta d\theta = 2k_1 \int \sin^2\theta d\theta$$

If I am to make a trigonometric substitution for $\sin^2\theta = 1/2\,(1-\cos 2\theta)$, we have,

$$x = 2k_1 \int \sin^2 \theta d\theta = 2k_1 \int \frac{1}{2} (1 - \cos 2\theta) \, d\theta = k_1\theta - \frac{k_1}{2} \sin 2\theta$$

Therefore we have,

$$x = k_1\theta - \frac{k_1}{2} \sin 2\theta = \frac{k_1}{2} (2\theta - \sin 2\theta)$$

Also we have that,

$$y = k_1 \sin^2 \theta = \frac{k_1}{2} (1 - \cos 2\theta)$$

Therefore we have,

$$\boxed{\begin{array}{r} x = \frac{k_1}{2} (2\theta - \sin 2\theta) \\ y = \frac{k_1}{2} (1 - \cos 2\theta) \end{array}} \tag{3}$$

After solving the integral, we obtain parametric equations for x and y that represent a cycloid.

4 The Curve: Cycloid

In the process of using the Calculus of Variations, to minimize the time integral, we have come across the two defining parametric equations of a cycloid.

A cycloid is the curve formed when traced by a point on a circle as it rolls along a straight line without it slipping.

I know this for a fact, as the standard cycloid equations are of the form, $x = r \cos \theta$ and $y = r \sin \theta$.

But if we are to allow the circle that forms the cycloid curve to rotate in a clockwise direction with an angle θ from the bottom of the circle, the above standard equations representing the cyloid curve must be corrected as follows,

$$x = -r \sin \theta, y = r \cos \theta$$

Also as the circle that forms the cyloid curve is to move in the positive x direction, we have to add this periodic motion mathematically into the parametric function of the cycloid. Therefore we have,

$$x = -r\sin\theta + \Delta x$$

$$\Delta x = 2\pi r \cdot \frac{\theta}{2\pi} = r\theta$$

Therefore we have,

$$x = -r\sin\theta + r\theta = r\left(\theta - \sin\theta\right)$$

However in the y direction, the only correction that needs to be done is the generalization, that the center of the circle that forms the cycloid is at (r, r) and not at $(0, 0)$, so as to ensure that t he bottom of the cycloid rest at the x axis. Therefore with a vertical translation of r units in the y axis we have,

$$y = r - r\cos\theta = r\left(1 - \cos\theta\right)$$

Therefore we have,

$$\boxed{\begin{aligned} x &= r\left(\theta - \sin\theta\right) \\ y &= r\left(1 - \cos\theta\right) \end{aligned}} \tag{4}$$

From the derivation above, I have confirmed that the parametric equations, that are a solution to the brachistorcone problem are indeed, that of a cycloid, as both of the curves have the same form.

On comparison with the parametric equations I have found as a solution to the brachistochrone problem with the improvised parametric equations of a cycloid, we observe that, $r = k_1/2$. In other words, k_1 is equal to the diameter of the circle which forms the cycloid

5 Differential Equations of the Cycloid

If I am to take the differential of the parametric equations from equations 4 with respect to θ, to find the rates of changes in the vertical and horizontal distances, we have,

$$\frac{dx}{d\theta} = \frac{d}{d\theta}\left(r\left(\theta - \sin\theta\right)\right) = r\left(1 - \cos\theta\right)$$

$$\frac{dy}{d\theta} = \frac{d}{d\theta}\left(r\left(1 - \cos\theta\right)\right) = r\sin\theta$$

Then by using the chain rule, I calculated the rate of change of the vertical distance with respect to the horizontal distance.

$$\frac{dy}{dx} = \frac{dy}{d\theta}\cdot\frac{d\theta}{dx} = \frac{r\sin\theta}{r\left(1-\cos\theta\right)} = \frac{\sin\theta}{1-\cos\theta}$$

Squaring on both sides we have,

$$\left(\frac{dy}{dx}\right)^2 = \frac{\sin^2\theta}{\left(1-\cos\theta\right)^2} = \frac{1-\cos^2\theta}{\left(1-\cos\theta\right)^2} = \frac{1+\cos\theta}{1-\cos\theta} \tag{5}$$

From equation 4, I had known that, $y = r\left(1-\cos\theta\right)$. Therefore with algebraic manipulation we have,

$$\cos\theta = 1 - \frac{y}{r} \tag{6}$$

Therefore by substituting equation 6 in equation 5, we have,

$$\left(\frac{dy}{dx}\right)^2 = \frac{2r-y}{y} \tag{7}$$

Therefore we have,

$$y\left(\left(y'\right)^2 + 1\right) = 2r$$

6 Solving for Travel Time Minimization

I have calculated and derived the parametric equations to the curve of fastest descent, I now would like to calculate the time it takes to travel on this path.

From equation 2 we have,

$$T = \frac{1}{\sqrt{2g}} \int_{x_1}^{x_2} \sqrt{\frac{1 + (y')^2}{y}} dx$$

And from equation 7 we have,

$$\left(\frac{dy}{dx}\right)^2 = \frac{2r - y}{y}$$

Implying,

$$dx = \sqrt{\frac{y}{2r - y}} dy$$

Substituting dx in equation 2 we have,

$$T = \frac{1}{\sqrt{2g}} \int_{y_1}^{y_2} \sqrt{\frac{1 + \left(\sqrt{\frac{2r-y}{y}}\right)^2}{y} \cdot \frac{y}{2r - y}} dy$$

With some algebraic manipulation, the above equation reduces to,

$$T = \frac{1}{\sqrt{2g}} \int_{y_1}^{y_2} \sqrt{\frac{2r}{y\,(2r - y)}} dy$$

From equation 4, we know that,

$$y = r\,(1 - \cos\theta)$$

Therefore the time minimization integral can be reduced to,

$$T = \frac{1}{\sqrt{2g}} \int_{y_1}^{y_2} \sqrt{\frac{2r}{r\,(1 - \cos\theta)\,(2r - r\,(1 - \cos\theta))}} dy$$

If we look at the denominator of the fraction, which therein lies under the integral, we have,

$$r(1-\cos\theta)(2r - r(1-\cos\theta)) = -\left[(r-y)^2 - r^2 = r^2\cos^2\theta - r^2\right] = -\left[r^2\left(\cos^2\theta - 1\right)\right]$$

But,

$$-\left[r^2\left(\cos^2\theta - 1\right)\right] = r^2\sin^2\theta$$

Therefore we have,

$$T = \frac{1}{\sqrt{2g}}\int_{y_1}^{y_2}\sqrt{\frac{2r}{r^2\sin^2\theta}}dy = \frac{1}{\sqrt{2g}}\int_{y_1}^{y_2}\frac{\sqrt{2r}}{r\sin\theta}dy$$

Substituting $dy = r\sin\theta d\theta$, we have,

$$T = \frac{1}{\sqrt{2g}}\int_{\theta_1}^{\theta_2}\frac{\sqrt{2r}}{r\sin\theta}\cdot r\sin\theta d\theta = \frac{1}{\sqrt{2g}}\int_{\theta_1}^{\theta_2}\sqrt{2r}d\theta$$

Therefore we finally have,

$$T = \sqrt{\frac{r}{g}}\cdot\theta \tag{8}$$

For any values of the coordinates for which the time and path are to be minimized, the variables, r and θ can be calculated according to the case as they are variables in the parametric equations that define the cycloid.

7 Calculation of Sample Path

Up till now, all I have done is to try to understand the nature of the brachistiochrone problem and its solution. I have explored various parametric equations that describe the curve of fastest descent, but I now would like to apply it to a real problem with real coordinates to define a real curve from real points to investigate the curve and the time it takes for a spherical mass to follow this curve-path.

I shall investigate the brachistochrone curve between the points $A(x_1, y_1)$ and $B(x_2, y_2)$, where $x_1 = 0, y_1 = 0, x_2 = 4$ and $y_2 = -3$. Before evaluating the problem, we would like that the beginning point of the curve be placed at the peaks of the cycloid as this ensures that the mass sliding through the curve would attain the highest velocity (as the tangent at the initial point would be a vertical line), However this requires that the parametric equations to be of $\theta \in 2\pi n$, where $n \in \mathbb{Z}^+$.

Therefore by inputting the values of x_2 and y_2 we have,

$$4 = r\left(\theta - \sin\theta\right) \tag{9}$$

$$-3 = r\left(1 - \cos\theta\right) \tag{10}$$

Implies,

$$r = \frac{4}{\theta - \sin\theta}$$

Substituting the value for r in 10, we have,

$$-3 = \left(\frac{4}{\theta - \sin\theta}\right)\left(1 - \cos\theta\right)$$

Implies,

$$-3\left(\theta - \sin\theta\right) = 4\left(1 - \cos\theta\right)$$

Solving for θ using a GDC, we have, $\theta = -2.87995$. Therefore we have,

$$r = \frac{4}{-2.87995 - \sin -2.87995} \approx -1.52597$$

Sustituting this value of r in equations 4, we have,

$$x = -1.53\left(\theta - \sin\theta\right)$$

$$y = 1.53\left(1 - \cos\theta\right)$$

Plotting the above parametric equations on Desmos we get,

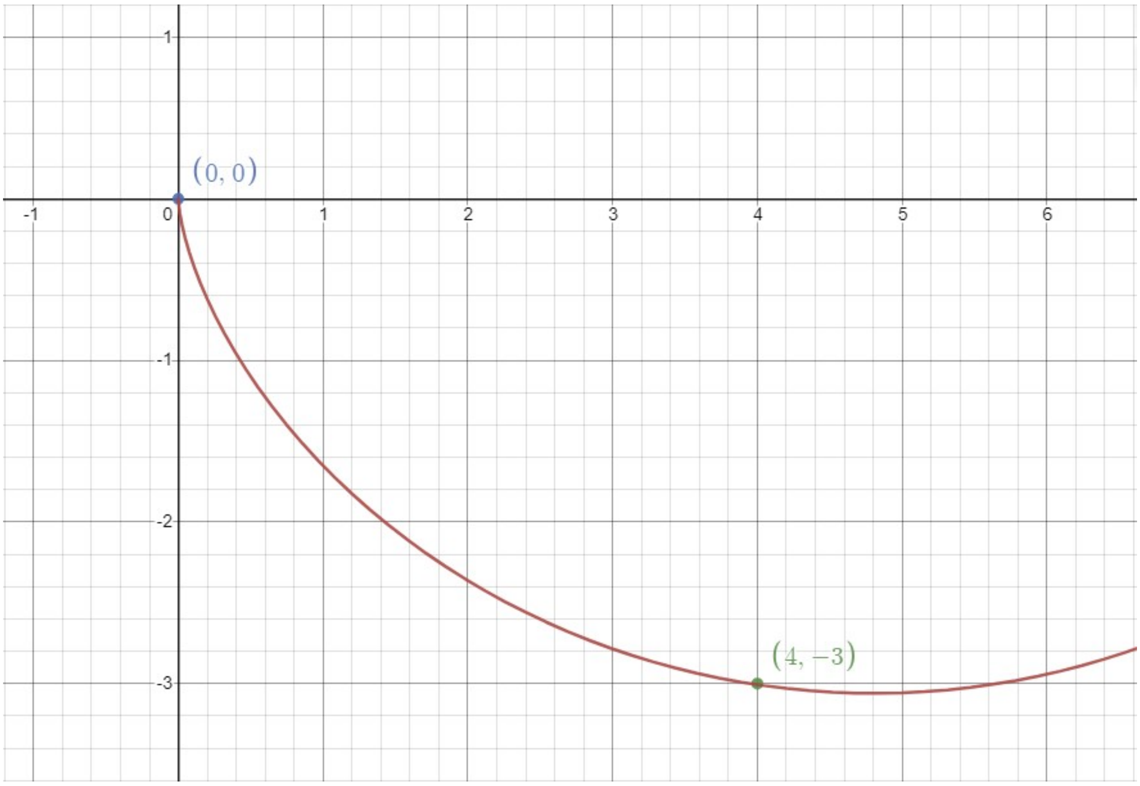

Figure 3: Plot of the parametric equations that plot the cycloid

As we can see, the above curve passes through the points $A(0, 0)$ and $B(3, -4)$

Using equation 8, we can find the time it takes for an object/mass to slide from the initial point, to the final point, ie. the the points $A(0, 0)$ and $B(3, -4)$. Therefore the time it takes to reach from the initial to the final point is,

If I am to consider each point to be in the units of meters, then we have,

$$T = \left| \sqrt{\frac{-1.52597}{-9.81}} \cdot -2.87995 \right|$$

Therefore,

$$T \approx 1.13586s$$

This time is relatively much shorter than if an object were to slide down a ramp from point $A(0,0)$ to $B(3,-4)$.

8 Conclusion

A small problem can lead to complicated and often unfathomable notions and solutions, which is the wonder of mathematics. The Brachistochrone curve, which began as a simple task to infer the shortest path between two objects in space aided by gravity, evolved into the Calculus of Variations, an entire field of mathematics.

The very fact that this simple-sounding problem was the eventual reason for significant further advancement in the fields of mathematics and physics drew me to explore this problem and study it thoroughly.

I discovered and read various Mathematical Association of America research papers through my "basic" investigation, which I found while surfing Wikipedia late one night, Articles, lecture notes, books, physics applications.

In the beginning, it was pretty complicated to understand the mathematics behind the analytical solutions placed by Euler and Lagrange. Still, after learning more about certain topics, I finally learned and understood them.

This exploration enabled me to study the most basic yet important parts of the Calculus of Variations, a field foreign to me.

Cetain topics used in this investigation are generally taught at an undergraduate level, but with time, I learnt and understood them with the use of mathematical intuition and rigour. Topics such as partial derivatives and functionals.

This exploration enabled me to explore more mathematics on my own, and explore and fix my shortcomings and conceptual errors in my errors on mathematical understanding and ideas.