International Finance
Chapter 4: The Lucas Model

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The Lucas Model

One of the unattractive features of the Monetary Model is that the money demand function is imposed, and is not from decisions of optimizing agents.

$$m_t^d - p_t = \phi y_t - \lambda i_t + \epsilon_t$$

Lucas's model of exchange rate determination gives a rigorous theoretical framework for pricing foreign exchange and other assets in a flexible price environment.

The Lucas Model is a dynamic general equilibrium model of an endowment economy with complete markets where the fundamental determinants of the exchange rate are the same as those in the monetary model.

The Barter Economy Model Settings

Consider two countries each inhabited by a large number of individuals who have *identical utility functions and identical wealth*.

Because people are so similar you can normalize the constant populations of each country to 1 and model the people of each country by the actions of a single **representative agent**.

Firms in each country are pure endowment streams that generate a homogeneous **nonstorable** country-specific good using no labor or capital inputs.

Some people would like to think of these firms as fruit trees. Hence, this model is also called the Lucas Tree Model.

The Barter Economy Model Settings

Each firm issues one perfectly divisible share of common stock which is traded by shareholders in a competitive stock market.

The firms pay out all of their output as dividends to shareholders (i.e. representative agents).

Dividends from the firms are the sole source of support for representative agents.

x_t is the exogenous domestic output

y_t is the exogenous foreign output

The Barter Economy Model Settings

We treat x_t as the *numeraire good*, define q_t as the price of y_t in terms of x_t.

e_t is the ex-dividend market value of the domestic firm.

e_t^* is the ex-dividend market value of the foreign firm.

Domestic agent: consumes $c_{x,t}$ units of the home good, $c_{y,t}$ units of the foreign good and holds $w_{x,t}$ shares of the domestic firm, $w_{y,t}$ shares of the foreign firm.

Foreign agent: consumes $c_{x,t}^*$ units of the home good, $c_{y,t}^*$ units of the foreign good and holds $w_{x,t}^*$ shares of the domestic firm, $w_{y,t}^*$ shares of the foreign firm.

The Barter Economy Model

In period t, the domestic agent's wealth is:

$$W_t = w_{x,t-1}(x_t + e_t) + w_{y,t-1}(q_t y_t + e_t^*)$$

where $x_t + e_t$ and $q_t y_t + e_t^*$ are the with-dividend value of the home and foreign firms.

In period t, the domestic agent need to allocate current wealth toward new share purchases $e_t w_{x,t} + e_t^* w_{y,t}$ and consumption $c_{x,t} + q_t c_{y,t}$:

$$W_t = e_t w_{x,t} + e_t^* w_{y,t} + c_{x,t} + q_t c_{y,t}$$

Hence, in period t, the domestic agent's budget constraint is:

$$e_t w_{x,t} + e_t^* w_{y,t} + c_{x,t} + q_t c_{y,t} = w_{x,t-1}(x_t + e_t) + w_{y,t-1}(q_t y_t + e_t^*)$$

Domestic Agent Utility Maximization

Let $u(c_{x,t}, c_{y,t})$ be current period utility and $0 < \beta < 1$ be the subjective discount factor. Then the domestic agent's problem is to maximize expected lifetime utility under the consumption constraint.

$$\max_{c_{x,t}, c_{y,t}, w_{x,t}, w_{y,t}} E_t \left(\sum_{j=0}^{\infty} \beta^j u(c_{x,t+j}, c_{y,t+j}) \right)$$

$$s.t. \quad c_{x,t} + q_t c_{y,t} + e_t w_{x,t} + e_t^* w_{y,t} = w_{x,t-1}(x_t + e_t) + w_{y,t-1}(q_t y_t + e_t^*)$$

F.O.C (How?)

$$\begin{aligned}
c_{y,t} :& q_t u_1(c_{x,t}, c_{y,t}) = u_2(c_{x,t}, c_{y,t}) \\
w_{x,t} :& e_t u_1(c_{x,t}, c_{y,t}) = \beta E_t(u_1(c_{x,t+1}, c_{y,t+1}) \cdot (x_{t+1} + e_{t+1})) \\
w_{y,t} :& e_t^* u_1(c_{x,t}, c_{y,t}) = \beta E_t(u_1(c_{x,t+1}, c_{y,t+1}) \cdot (q_{t+1} y_{t+1} + e_{t+1}^*))
\end{aligned}$$

Foreign Agent Utility Maximization

Similarly, the foreign agent's problem is to maximize expected lifetime utility under the consumption constraint.

$$\max_{c^*_{x,t}, c^*_{y,t}, w^*_{x,t}, w^*_{y,t}} E_t \left(\sum_{j=0}^{\infty} \beta^j u(c^*_{x,t+j}, c^*_{y,t+j}) \right)$$

$$s.t. \quad c^*_{x,t} + q_t c^*_{y,t} + e_t w^*_{x,t} + e^*_t w^*_{y,t} = w^*_{x,t-1}(x_t + e_t) + w^*_{y,t-1}(q_t y_t + e^*_t)$$

F.O.C

$$\begin{aligned}
c^*_{y,t} :& q_t u_1(c^*_{x,t}, c^*_{y,t}) = u_2(c^*_{x,t}, c^*_{y,t}) \\
w^*_{x,t} :& e_t u_1(c^*_{x,t}, c^*_{y,t}) = \beta E_t(u_1(c^*_{x,t+1}, c^*_{y,t+1}) \cdot (x_{t+1} + e_{t+1})) \\
w^*_{y,t} :& e^*_t u_1(c^*_{x,t}, c^*_{y,t}) = \beta E_t(u_1(c^*_{x,t+1}, c^*_{y,t+1}) \cdot (q_{t+1} y_{t+1} + e^*_{t+1}))
\end{aligned}$$

A Social Planner's Problem

In the early analysis, we focus on domestic agent and foreign agent their own utility maximization. Now suppose we have a social planner who cares about the whole world's utility.

Suppose the social planner attach a weight of ϕ to the domestic agent and $1-\phi$ to the foreign agent. Then the planner's problem is to allocate the x and y endowments optimally between the domestic and foreign individuals each period by maximizing:

$$E_t \sum_{j=0}^{\infty} \beta^j (\phi u(c_{x,t+j}, c_{y,t+j}) + (1-\phi) u(c^*_{x,t+j}, c^*_{y,t+j}))$$

Since the goods are not storable, the planner's problem can be reduced to the timeless problem of maximizing:

$$\phi u(c_{x,t}, c_{y,t}) + (1-\phi) u(c^*_{x,t}, c^*_{y,t})$$

A Social Planner's Problem

We can write the social planner's problem formally:

$$\begin{aligned} \max \quad & \phi u(c_{x,t}, c_{y,t}) + (1-\phi) u(c^*_{x,t}, c^*_{y,t}) \\ s.t. \quad & c_{x,t} + c^*_{x,t} = x_t \\ & c_{y,t} + c^*_{y,t} = y_t \end{aligned}$$

F.O.C (How?)

$$\phi u_1(c_{x,t}, c_{y,t}) = (1-\phi) u_1(c^*_{x,t}, c^*_{y,t})$$
$$\phi u_2(c_{x,t}, c_{y,t}) = (1-\phi) u_2(c^*_{x,t}, c^*_{y,t})$$

The above F.O.C are the optimal or efficient risk-sharing conditions. Risk-sharing is efficient when consumption is allocated so that the marginal utility of the domestic agent is proportional to the marginal utility of the marginal utility of the foreign agent.

A Social Planner's Problem

$$\phi u_1(c_{x,t}, c_{y,t}) = (1-\phi) u_1(c^*_{x,t}, c^*_{y,t})$$
$$\phi u_2(c_{x,t}, c_{y,t}) = (1-\phi) u_2(c^*_{x,t}, c^*_{y,t})$$

The weight ϕ can be interpreted as a measure of the size of the home country in the market version of the world economy. Since we have assumed that agents have equal wealth, we will let both agents be equally important to the planner, so $\phi = 1/2$.

Then the Pareto optimal allocation is to split the available output of x and y equally:

$$c_{x,t} = c^*_{x,t} = \frac{x_t}{2}, \quad \text{and} \quad c_{y,t} = c^*_{y,t} = \frac{y_t}{2}$$

Q: What if these two countries are not equal?

Solution Under CRRA Utility Function

CRRA function VS. CARA function

Now, let us adopt a particular function form for the utility function to get explicit solutions. Suppose the utility function is a CRRA function:

$$u(c_x, c_y) = \frac{C_t^{1-\gamma}}{1-\gamma}, \quad \text{where} \quad C_t = c_{x,t}^{\theta} c_{y,t}^{1-\theta}$$

Then based on equation (4.6)-(4.8), we can get (How?):

$$\begin{aligned}
q_t &= \frac{1-\theta}{\theta} \cdot \frac{x_t}{y_t} \\
\frac{e_t}{x_t} &= \beta E_t\left(\left(\frac{C_{t+1}}{C_t}\right)^{1-\gamma}\left(1 + \frac{e_{t+1}}{x_{t+1}}\right)\right) \\
\frac{e_t^*}{q_t y_t} &= \beta E_t\left(\left(\frac{C_{t+1}}{C_t}\right)^{1-\gamma}\left(1 + \frac{e_{t+1}^*}{q_{t+1} y_{t+1}}\right)\right)
\end{aligned}$$

Solution Under CRRA Utility Function

$$q_t = \frac{1-\theta}{\theta} \cdot \frac{x_t}{y_t}$$
$$\frac{e_t}{x_t} = \beta E_t((\frac{C_{t+1}}{C_t})^{1-\gamma}(1 + \frac{e_{t+1}}{x_{t+1}}))$$
$$\frac{e_t^*}{q_t y_t} = \beta E_t((\frac{C_{t+1}}{C_t})^{1-\gamma}(1 + \frac{e_{t+1}^*}{q_{t+1} y_{t+1}}))$$

The real exchange rate q_t is determined by relative output levels.

$\frac{e_t}{x_t}$ and $\frac{e_t^*}{q_t y_t}$ can be called “price-dividend“ ratios. The equity price-dividend ratio can be expressed as the present discounted value of future consumption growth raised to the power $1-\gamma$.

One-Money Monetary Economy

In the barter economy model, we do not have the concept of *money*. One fo the difficulties in getting money into the model is that the people in the barter economy get along just fine without money.

To get around this problem, Lucas prohibits barter in the monetary economy and imposed a *cash-in-advance* constraint that requires people to use money to buy goods. Hence, we have to add some assumptions:

- The total money supply M_t, evolves according to $M_t = \lambda_t M_{t-1}$. In each period, each agent receives a lump-sum money transfer $\frac{\Delta M_t}{2} = (\lambda_t - 1)\frac{M_{t-1}}{2}$
- Each household is split into *worker-shopper* (or *seller-buyer*) pairs. Part of them need to buy x-goods and y-goods, part of them need to sell x-goods or y-goods endowments in the mall.

Model Settings

Current-period wealth is comprised of (1) dividends from last period's goods sales, (2) the market value of ex-dividend equity shares, and (3) the lump-sum monetary transfer.

$$W_t = \underbrace{\frac{P_{t-1}(w_{x,t-1}x_{t-1} + w_{y,t-1}q_{t-1}y_{t-1})}{P_t}}_{\text{Dividends}}$$

$$+ \underbrace{w_{x,t-1}e_t + w_{y,t-1}e_t^*}_{\text{Ex-dividend share values}} + \underbrace{\frac{\Delta M_t}{2P_t}}_{\text{Money transfer}}$$

NOTE: compared with (4.1), we now use x_{t-1}, q_{t-1} and y_{t-1} instead of x_t, q_t and y_t. Why?

Model Settings

Then the domestic agent needs to allocates W_t towards cash m_t to finance shopping plans and to equities to secure next period dividends:

$$W_t = \frac{m_t}{P_t} + w_{x,t}e_t + w_{y,t}e_t^*$$

Meanwhile, the domestic agent knows that the amount of cash required to finance the current period consumption plan is:

$$m_t = P_t(c_{x,t} + q_t c_{y,t})$$

Combine the above three equations together, we can get the budge constraint:

$$c_{x,t} + q_t c_{y,t} + w_{x,t}e_t + w_{y,t}e_t^* =$$
$$\frac{P_{t-1}}{P_t}(w_{x,t-1}x_{t-1} + w_{y,t-1}q_{t-1}y_{t-1}) + \frac{\Delta M_t}{2P_t} + w_{x,t-1}e_t + w_{y,t-1}e_t^*$$

One-Money Monetary Economy Model

The domestic agent's maximization problem can be written:

$$\max_{c^*_{x,t},c^*_{y,t},w^*_{x,t},w^*_{y,t}} E_t\left(\sum_{j=0}^{\infty}\beta^j u(c^*_{x,t+j}, c^*_{y,t+j})\right)$$

$$s.t. \quad c_{x,t} + q_t c_{y,t} + w_{x,t}e_t + w_{y,t}e^*_t =$$

$$\frac{P_{t-1}}{P_t}(w_{x,t-1}x_{t-1} + w_{y,t-1}q_{t-1}y_{t-1}) + \frac{\Delta M_t}{2P_t} + w_{x,t-1}e_t + w_{y,t-1}e^*_t$$

Same as in section 4.1, the terms that matter in period t are:

$$u(c_{x,t}, c(y,t)) + \beta E_t u(c_{x,t+1}, c_{y,t+1})$$

Solve the maximization problem we can get the Euler equations (4.31) $\sim$ (4.33) characterizing optimal household behaviors.

One-Money Monetary Economy Model

After introducing money, we have another constraint related to the total money supply:

$$M_t = P_t(x_t + q_t y_t)$$

We still assume the utility function is $u(c_x, c_y) = \frac{C_t^{1-\gamma}}{1-\gamma}$ and $C_t = c_{x,t}^{\theta} c_{y,t}^{1-\theta}$. Combine this constraint with (4.23) $q_t = \frac{1-\theta}{\theta} \cdot \frac{x_t}{y_t}$, we can get:

$$\frac{P_t}{P_{t+1}} = \frac{M_t}{M_{t+1}} \frac{x_{t+1}}{x_t}$$

Combine the above equation with the explicit utility function, we can rewrite (4.32) and (4.33) as (How?):

$$\frac{e_t}{x_t} = \beta E_t \left[(\frac{C_{t+1}}{C_t})^{1-\gamma} (\frac{M_t}{M_{t+1}} + \frac{e_{t+1}}{x_{t+1}}) \right]$$

$$\frac{e_t^*}{q_t y_t} = \beta E_t \left[(\frac{C_{t+1}}{C_t})^{1-\gamma} (\frac{M_t}{M_{t+1}} + \frac{e_{t+1}^*}{q_{t+1} y_{t+1}}) \right]$$

Pricing Nominal Bonds

Suppose you are looking for the shadow price of a hypothetical nominal bond such that the public willingly keeps it in zero net supply. Let b_t be the nominal price of a bond that pays one dollar at the end of the period.

Then the utility cost of buying this bond is

$$\frac{u_1(c_{x,t}, c_{y,t})b_t}{P_t x_t}, \quad \text{where } u(.) \text{ is a CRRA function} \tag{1}$$

The discounted expected marginal utility of this one-dollar payoff one period later:

$$\beta E_t\left[\frac{u_1(c_{x,t+1}, c_{y,t+1})}{P_{t+1}x_{t+1}}\right] \tag{2}$$

$$(1) = (2) \Rightarrow b_t = \beta E_t\left[\left(\frac{C_{t+1}}{C_t}\right)^{(1-\gamma)}\frac{M_t}{M_{t+1}}\right], \quad \text{where} \quad C_t = c_{x,t}^{\theta}c_{y,t}^{(1-\theta)}$$

Two-Money Monetary Economy Model Settings

We now require that the home country's x-goods can only be purchased with dollars and the foreign country's y-goods can only be purchased with euros.

x-dividends are paid out in dollars and y-dividends are paid out in euros.

Agents can acquire the foreign currency required to finance consumption plans during securities market trading.

M_t is the outstanding stock of dollars, N_t is the outstanding stock of euros.

$$M_t = \lambda_t M_{t-1}, \quad \text{and} \quad N_t = \lambda_t^* N_{t-1}$$

where (λ_t, λ_t^*) are exogenous random gross rates of change in M and N.

Two-Money Monetary Economy Model Settings

Now individuals need to secure claims to future dollar and euro transfers:

- r_t is the price of a claim to all future dollar transfers in terms of x
- r_t^* is the price of a claim to all future euro transfers in terms of x
- $\psi_{M,t}$ is the claims held by domestic agent on the dollar streams
- $\psi_{N,t}$ is the claims held by domestic agent on the euro streams

Initially, the home agent is endowed with $\psi_{M,0} = 1$, $\psi_{N,0} = 0$ and the foreign agent has $\psi_{M,0} = 0$, $\psi_{N,0} = 1$ which they are free to trade.

Two-Money Monetary Economy Model

Now the domestic household's current-period wealth consists of

- nominal dividends paid from equity ownership carried over from last period
- current period monetary transfers
- the market value of equity and monetary transfer claims

$$W_t = \underbrace{\frac{P_{t-1}}{P_t} w_{x,t-1} x_{t-1} + \frac{S_t P^*_{t-1}}{P_t} w_{y,t-1} y_{t-1}}_{\text{Dividends}} + \underbrace{\frac{\psi_{M,t-1} \Delta M_t}{P_t} + \frac{\psi_{N,t-1} S_t \Delta N_t}{P_t}}_{\text{Monetary Transfers}}$$
$$+ \underbrace{w_{x,t-1} e_t + w_{y,t-1} e^*_t + \psi_{M,t-1} r_t + \psi_{N,t-1} r^*_t}_{\text{Market Value of Securities}}$$

Two-Money Monetary Economy Model

The domestic household needs to allocate the current-period wealth to

- equity
- claims to future monetary transfers
- dollars and euros for shopping

$$W_t = \underbrace{\frac{m_t}{P_t} + \frac{n_t S_t}{P_t}}_{\text{Goods}} + \underbrace{w_{x,t} e_t + w_{y,t} e_t^*}_{\text{Equity}} + \underbrace{\psi_{M,t} r_t + \psi_{N,t} r_t^*}_{\text{Monetary Transfers}}$$

In equilibrium, we have the binding cash-in-advance constraints: $m_t = P_t c_{x,t}$ and $n_t = P_t^* c_{y,t}$, hence the equation can be rewritten as:

$$W_t = \underbrace{c_{x,t} + \frac{S_t P_t^*}{P_t} c_{y,t}}_{\text{Goods}} + \underbrace{w_{x,t} e_t + w_{y,t} e_t^*}_{\text{Equity}} + \underbrace{\psi_{M,t} r_t + \psi_{N,t} r_t^*}_{\text{Monetary Transfers}}$$

Two-Money Monetary Economy Model

Combine the previous two slides, we can get the budget constraint fo the home individual. So the domestic household's problem is to maximize

$$\max_{\{c_{x,t}, c_{y,t}, w_{x,t}, w_{y,t}, \psi_{M,t}, \psi_{N,t}\}} E_t \left[\sum_{j=0}^{\infty} \beta^j u(c_{x,t+j}, c_{y,t+j}) \right]$$

$$\begin{aligned}
s.t. \quad & c_{x,t} + \frac{S_t P_t^*}{P_t} c_{y,t} + w_{x,t} e_t + w_{y,t} e_t^* + \psi_{M,t} r_t + \psi_{N,t} r_t^* = \\
& \frac{P_{t-1}}{P_t} w_{x,t-1} x_{t-1} + \frac{S_t P_{t-1}^*}{P_t} w_{y,t-1} y_{t-1} + \frac{\psi_{M,t-1} \Delta M_t}{P_t} \\
& + \frac{\psi_{N,t-1} \Delta N_t}{P_t} + w_{x,t-1} e_t + w_{y,t-1} e_t^* + \psi_{x,t-1} r_t + \psi_{y,t-1} r_t^*
\end{aligned}$$

Two-Money Monetary Economy Model

Following the same procedure in the previous two models, we can get the F.O.C (How?)

$$
\begin{aligned}
c_{y,t}: &\quad \frac{S_t P_t^*}{P_t} u_1(c_{x,t}, c_{y,t}) = u_2(c_{x,t}, c_{y,t}) \\
w_{x,t}: &\quad e_t u_1(c_{x,t}, c_{y,t}) = \beta E_t \left[u_1(c_{x,t+1}, c_{y,t+1}) \left(\frac{P_t}{P_{t+1}} x_t + e_{t+1} \right) \right] \\
w_{y,t}: &\quad e_t^* u_1(c_{x,t}, c_{y,t}) = \beta E_t \left[u_1(c_{x,t+1}, c_{y,t+1}) \left(\frac{S_{t+1} P_t^*}{P_{t+1}} y_t + e_{t+1}^* \right) \right] \\
\psi_{M,t}: &\quad r_t u_1(c_{x,t}, c_{y,t}) = \beta E_t \left[u_1(c_{x,t+1}, c_{y,t+1}) \left(\frac{\Delta M_{t+1}}{P_{t+1}} + r_{t+1} \right) \right] \\
\psi_{N,t}: &\quad r_t^* u_1(c_{x,t}, c_{y,t}) = \beta E_t \left[u_1(c_{x,t+1}, c_{y,t+1}) \left(\frac{\Delta N_{t+1} S_{t+1}}{P_{t+1}} + r_{t+1}^* \right) \right]
\end{aligned}
$$

Two-Money Monetary Economy Model

In this Two-Money Monetary Economy Model, we have two adding-up constraints:

$$M_t = P_t x_t$$
$$N_t = P_t^* y_t$$

Based on the F.O.C for $c_{y,t}$ we can know that the real exchange rate is:

$$\frac{S_t P_t^*}{P_t} = \frac{u_2(c_{x,t}, c_{y,t})}{u_1(c_{x,t}, c_{y,t})}$$

The nominal exchange rate is:

$$S_t = \frac{u_2(c_{x,t}, c_{y,t})}{u_1(c_{x,t}, c_{y,t})} \cdot \frac{P_t}{P_t^*} = \frac{u_2(c_{x,t}, c_{y,t})}{u_1(c_{x,t}, c_{y,t})} \frac{M_t}{N_t} \frac{y_t}{x_t}$$

Determinants of the Nominal Exchange Rate

In the Two-Money Monetary Economy Model:

$$S_t = \frac{u_2(c_{x,t}, c_{y,t})}{u_1(c_{x,t}, c_{y,t})} \frac{M_t}{N_t} \frac{y_t}{x_t}$$

In the Monetary Model:

$$s_t = \gamma \sum_{j=0}^{\infty} (\psi)^j E_t f_{t+j}, \quad \text{where} \quad f_t \equiv (m_t - m_t^*) - \phi(y_t - y_t^*)$$

There are two major difference:

- Besides relative money supplies, and relative GDP, in the Lucas model, the exchange rate also depends on preferences (utility).
- In the Lucas model, the exchange rate does not depend explicitly on expectations of the future.

One-period Ahead Forward Exchange Rate

Similar to the setting in the One-Money Monetary Economy Model:

- b_t is the date t dollar price of a 1-period nominal discount bond that pays one dollar at period $t+1$
- b_t^* is the date t euro price of a 1-period nominal discount bond that pays one euro at period $t+1$

By covered interest parity, the one-period ahead forward exchange rate is

$$F_t = S_t \frac{b_t^*}{b_t}$$

The equilibrium bond prices are

$$b_t = \beta E_t \left[(\frac{C_{t+1}}{C_t})^{(1-\gamma)} \frac{M_t}{M_{t+1}} \right]$$

$$b_t^* = \beta E_t \left[(\frac{C_{t+1}}{C_t})^{(1-\gamma)} \frac{N_t}{N_{t+1}} \right]$$

Introduction to the Calibration Method

Cooley and Prescott set out the calibration proceeds as follows:

- Obtain a set of measurements from real-world data that we want to explain. These measurements are typically a set of sample moments.
- Solve and calibrate the candidate model. Assign values to the *deep parameters* of utility functions and production functions that make sense.
- Run/simulate the model by computing and generating time-series of the variables that we want to explain
- Decide whether the generated data “look like “ the observations that you want to explain.

Calibrating the Lucas Model

Measurement

The measurements that we ask the Lucas model to match are the **volatility** (standard deviation) and first-order **auto-correlation** of:

(1) gross rate of depreciation, $\frac{S_{t+1}}{S_t}$

(2) forward premium, $\frac{F_t}{S_t}$

(3) realized forward profit $\frac{F_t - S_{t+1}}{S_t}$

and the slope coefficient from regressing $\frac{S_{t+1}}{S_t}$ on $\frac{F_t}{S_t}$

Calibrating the Lucas Model

Table 4.2: Measured and Implied Moments, US-Germany

		Volatility			Autocorrelation		
	Slope	$\frac{S_{t+1}}{S_t}$	$\frac{F_t}{S_t}$	$\frac{(F_t-S_{t+1})}{S_t}$	$\frac{S_{t+1}}{S_t}$	$\frac{F_t}{S_t}$	$\frac{(F_t-S_{t+1})}{S_t}$
Data	-0.293	0.060	0.008	0.061	0.007	0.888	0.026
Model	-1.444	0.014	0.006	0.029	0.105	0.006	0.628

Note: Model values generated with $\gamma = 10$, $\theta = 0.5$.

Forward Premium Puzzle: the forward premium predicts the future depreciation, but with a negative sign.

Recall that the uncovered interest parity condition implies that the forward premium should predict the future depreciation with a coefficient of 1.

Calibrating the Lucas Model

Calibration

The "technology" that underlies the model are the exogenous monetary growth rate $\bar{\lambda}$, $\bar{\lambda}^*$, and the exogenous output growth rate $\bar{g}$, $\bar{g}^*$. These exogenous variables need to be fed into the model, and we can estimate these variables using real-world data.

Let the state vector be ϕ=($\bar{\lambda}$, $\bar{\lambda}^*$,$\bar{g}$, $\bar{g}^*$). The process governing the state vector is a finite-state Markov chain with stationary probabilities: each element of the state vector is allowed to be in either one of two possible states-high ("1") and low ("2").

We denote the 16 x 16 probability transition matrix for the state by ***P***, where $p_{i,j} = P[\phi_{t+1} = \phi_j | \phi_t = \phi_j]$

Calibrate the Transition Matrix

Now how to estimate the probability transition matrix $\boldsymbol{P}$?

Step 1: we use real-world data (i.e. the US and Germany data in this case) to estimate the high and low state values of ϕ

Step 2: We classify the data into the ψ states according to whether the observations lie above or below the mean then set the transition probabilities $p_{j,k}$ equal to the relative frequency of transitions from state ϕ_j to ϕ_k found in the data.

$$
\begin{bmatrix}
.00 & .00 & .20 & .00 & .40 & .00 & .00 & .00 & .20 & .00 & .00 & .00 & .20 & .00 & .00 & .00 \\
.20 & .20 & .20 & .20 & .00 & .20 & .00 & .00 & .00 & .00 & .00 & .00 & .00 & .00 & .00 & .00 \\
.17 & .17 & .00 & .17 & .17 & .00 & .00 & .00 & .00 & .00 & .00 & .00 & .00 & .00 & .17 & .17 \\
.00 & .00 & .00 & .00 & .17 & .00 & .00 & .00 & .00 & .17 & .33 & .17 & .00 & .00 & .17 & .00 \\
.08 & .08 & .08 & .08 & .15 & .08 & .08 & .08 & .15 & .08 & .08 & .00 & .00 & .00 & .00 & .00 \\
.20 & .00 & .00 & .00 & .20 & .00 & .00 & .00 & .00 & .00 & .20 & .00 & .00 & .20 & .20 & .00 \\
.00 & .00 & .00 & .20 & .40 & .00 & .00 & .20 & .00 & .00 & .00 & .00 & .20 & .00 & .00 & .00 \\
.25 & .00 & .00 & .00 & .00 & .50 & .00 & .00 & .00 & .00 & .00 & .00 & .00 & .00 & .00 & .25 \\
.00 & .14 & .00 & .00 & .00 & .00 & .14 & .00 & .14 & .14 & .00 & .00 & .00 & .14 & .14 & .14 \\
.00 & .00 & .00 & .00 & .00 & .00 & .25 & .00 & .25 & .00 & .00 & .25 & .25 & .00 & .00 & .00 \\
.00 & .00 & .20 & .00 & .20 & .00 & .00 & .00 & .20 & .20 & .00 & .20 & .00 & .00 & .00 & .00 \\
.00 & .25 & .00 & .25 & .25 & .00 & .00 & .00 & .00 & .00 & .00 & .00 & .00 & .25 & .00 & .00 \\
.00 & .00 & .00 & .00 & .13 & .00 & .00 & .13 & .13 & .00 & .13 & .13 & .25 & .00 & .13 & .00 \\
.00 & .00 & .20 & .00 & .00 & .00 & .00 & .00 & .00 & .00 & .00 & .00 & .20 & .00 & .40 & .20 \\
.00 & .00 & .00 & .00 & .25 & .00 & .25 & .13 & .00 & .00 & .00 & .00 & .13 & .13 & .00 & .13 \\
.00 & .00 & .00 & .20 & .00 & .20 & .00 & .00 & .00 & .00 & .00 & .00 & .20 & .20 & .20 & .00
\end{bmatrix}
$$

Risk Premium

Based on the results in last section:

$$b_t = \beta E_t \left[(\frac{C_{t+1}}{C_t})^{(1-\gamma)} \frac{M_t}{M_{t+1}} \right]$$

$$b_t^* = \beta E_t \left[(\frac{C_{t+1}}{C_t})^{(1-\gamma)} \frac{N_t}{N_{t+1}} \right]$$

we can write the price of the bonds as:

$$b_t = \frac{\beta E_t \left[(g_{t+1}^{\theta} g_{t+1}^{*(1-\theta)})^{1-\gamma} \right]}{\lambda_t}$$

$$b_t^* = \frac{\beta E_t \left[(g_{t+1}^{\theta} g_{t+1}^{*(1-\theta)})^{1-\gamma} \right]}{\lambda_t^*}$$

How did we get this result?

Risk Premium

We define $d = \frac{\lambda}{\lambda^*}$ as the gross rate of depreciation of the home currency. And define $G = \frac{(g^{\theta} g^{*(1-\theta)})^{1-\gamma}}{\lambda}$ and $G^* = \frac{(g^{\theta} g^{*(1-\theta)})^{1-\gamma}}{\lambda^*}$, then

The domestic bond price is $b_k = \beta \sum_{i=1}^{16} p_{k,i} G_i$

The foreign bond price is $b_k^* = \beta \sum_{i=1}^{16} p_{k,i} G_i^*$

The expected gross change in the nominal exchagne rate is $\sum_{i=1}^{16} p_{k,i} d_i$

The state-k contingent risk premium is (How?)

$$rp_k = \sum_{i=1}^{16} p_{k,i} d_i - \frac{\sum_{i=1}^{16} p_{k,i} G_i^*}{\sum_{i=1}^{16} p_{k,i} G_i}$$

Simulation Results

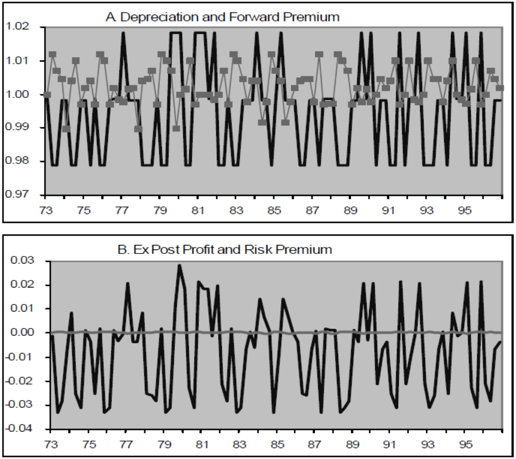

Figure 4.1: From the Lucas Model. A: Implied gross one-period ahead change in nominal exchange rate S_{t+1}/S_t and current forward premium F_t/S_t (in boxes). B. Implied ex post forward payoff $(S_{t+1} - F_t)/S_t$ (jagged line) and risk premium $\mathrm{E}_t(S_{t+1} - F_t)/S_t$ (smooth line).