设函数列$\{f_n\}$满足$f_1(x)=\ x\text{-}\ln x$，$f_{n+1}(x)=\ f_1(f_n(x))$,

函数列$\{g_n\}$满足$g_n(x)=\frac{e^{1-f_n(x)}}{x}$,

函数列$\{h_n\}$满足$h_n(x)=\ g_n(g_n(x))\text{-}\ x$.

证明：对于任意正整数$n\geq 2, h_n(x)$在(1，$+\infty$)单调递增.

证明：由归纳假设易证，对于任意正整数n,

$f_n(x)$在(0，1) 单调递减，在(1，$+\infty$)单调递增，

$f_n(x)\ \geq f_n(1)$=1,当且仅当$x=1$ 取等

$h_n(x)=\ g_n(g_n(x))\text{-}\ x\ =\ \frac{e^{1-f_n(g_n(x))}}{g_n(x)}-x=x\frac{e^{1-f_n(g_n(x))}}{e^{1-f_n(x)}}-x$

设$p_n(x)=\ f_n(x)\ -f_n(g_n(x))$,

则$h_n(x)\ =\ xe^{p_n(x)}\text{-}\ x$

$h_n{}'(x)=\ e^{p_n(x)}(1+xp_n{}'(x))\text{-}1$

要证对于任意正整数$n\geq 2, h_n(x)$在(1，$+\infty$)单调递增,即$h_n{}'(x)>0(x>1)$

只要证对于任意正整数$n\geq 2, p_n{}'(x)\ >0$且$p_n(x)\ >0(x>1)$

又易证对于任意正整数n，$f_n(1)=g_n(1)=1$，$p_n(1)=0$

故只要证对于任意正整数$n\geq 2, p_n{}'(x)\ >0(x>1)$

对于任意正整数$n\geq 2$,$p_n(x)=\ f_n(x)\ -f_n(g_n(x))$

$=\ f_1(f_{n-1}(x))\text{-}\ f_1(f_{n-1}(g_n(x)))$

$=f_{n-1}(x)\text{-}\ \ln f_{n-1}(x)\text{-}f_{n-1}(g_n(x))+\ \ln f_{n-1}(g_n(x))$,

$p_n{}'(x)=f_{n-1}{}'(x)\text{-}(f_{n-1}(g_n(x)))'\text{-}\frac{f_{n-1}{}'(x)}{f_{n-1}(x)}+\frac{(f_{n-1}(g_n(x)))'}{f_{n-1}(g_n(x))}$

对于任意正整数$n\geq 2$,设$q_n(x)=\ f_{n-1}(x)-f_{n-1}(g_n(x))$,

若证得对于任意正整数$n\geq 2$,$q_n{}'(x)>0$,即$q_n(x)$在(1，$+\infty$)单调递增,

则对于任意正整数$n\geq 2$ 与$x>1$,

$q_n(x)\ >q_n(1)$=0，$f_{n-1}(x)>f_{n-1}(g_n(x))$

$p_n{}'(x)=f_{n-1}{}'(x)\text{-}(f_{n-1}(g_n(x)))'\text{-}\frac{f_{n-1}{}'(x)}{f_{n-1}(x)}+\frac{(f_{n-1}(g_n(x)))'}{f_{n-1}(g_n(x))}$

$>f_{n-1}{}'(x)\text{-}(f_{n-1}(g_n(x)))'\text{-}\frac{f_{n-1}{}'(x)}{f_{n-1}(x)}+\frac{(f_{n-1}(g_n(x)))'}{f_{n-1}(x)}=q_n{}'(x)(1\text{-}\frac{1}{f_{n-1}(x)})>0$

故只要证对于任意正整数$n\geq 2$，$q_n{}'(x)\ >0(x>1)$

以此类推，设$r_n(x)=\ f_1(x)-f_1(g_n(x))$,

则只要证对于任意正整数$n\geq 2$，$r_n{}'(x)\ >0(x>1)$

对于任意$x>1$,

当$n=2$，$r_n(x)=\ r_2(x)=f_1(x)-f_1(g_2(x))$=1-$\ln x$-$\frac{x-\ln x}{e^{x-1}}$+$\ln(x\text{-}\ln x)$

$r_n{}'(x)=\text{-}\frac{1}{x}+\frac{x-\ln x-1+\frac{1}{x}}{e^{x-1}}+\frac{1-\frac{1}{x}}{x-\ln x}>\text{-}\frac{1}{x}+\frac{\frac{1}{x}}{e^{x-1}}+\frac{1-\frac{1}{x}}{x-\ln x}=\frac{e^{1-x}+\frac{\ln x-1}{x-\ln x}}{x}$

当 1$<x<e$，$r_n{}'(x)>\frac{e^{1-x}+\frac{\ln x-1}{x-\ln x}}{x}>\frac{e^{1-x}+\ln x-1}{x}$

设 $s(x)=\ e^{1-x}+\ln x-1$

$s'(x)=\frac{1}{x}\text{-}\frac{1}{e^{x-1}}\geq 0$

故 $s(x)$在$(1，e)$单调递增

当 $1< x < e$，$r_n{}'(x)> \frac{e^{1-x}+\ln x-1}{x} > \frac{s\,(1)}{x}=0$

当$x \geq e$，$r_n{}'(x) > \frac{e^{1-x}+\frac{\ln x-1}{x-\ln x}}{x}>0$

故$r_n{}'(x)\ >0$

假设当$n = k - 1(k \in N^+)$,

$r_n{}'(x) = r'_{k-1}(x) > 0$,即$f_1'(x) > (f_1(g_{k-1}(x)))'(x > 1)$ ，

$r_{k-1}(x)\ > r_{k-1}(1)=0$,即 $f_1(x) > f_1(g_{k-1}(x))(x > 1)$

则当$n = k$，$r_n(x)= r_k(x)= f_1(x) - f_1(g_k(x))$

$r_n{}'(x)\ = f_1{}'(x) - (f_1(g_k(x)))' > (f_1(g_{k-1}(x)))' - (f_1(g_k(x)))'$

$=(f_1(g_{k-1}(x)) - f_1(g_{k-1}(x)f_{k-1}(x)))'$

$=(e^{1-f_{k-1}(x)-\ln x}(1 - f_{k-1}(x))+ \ln f_{k-1}(x))'$

$>-e^{1-f_{k-1}(x)-\ln x}f_{k-1}{}'(x)+\frac{f_{k-1}{}'(x)}{f_{k-1}(x)}$

$>f_{k-1}{}'(x)(\frac{1}{f_{k-1}(x)} - \frac{1}{e^{f_{k-1}(x)-1}}>0$

故对于任意正整数$n \geq 2$，$r_n{}'(x)\ > 0(x > 1)$

证毕！