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Wilker-type inequalities for hyperbolic functions

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ABSTRACT

Using classical analytic techniques, the Wilker–Anglesio inequality and parameterized Wilker inequality for hyperbolic functions are proved. The main result is then applied to deriving a hyperbolic analogue of the Sándor–Bencze inequality.

1. Introduction

Wilker [1] proposed the following open problem:

(a) Prove that, if $0 < x < \frac{\pi}{2}$, then

$$\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} > 2. \tag{1}$$

(b) Under the same assumption as in part (a) above, find the largest congest constant c such that

$$\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} > 2 + cx^3 \tan x. \tag{2}$$

Different proofs of Wilker's inequality (1) are given by Sumner et al. [2], Guo et al. [3], Zhang and Zhu [4], and Zhu [5]. It is worth noting that, in [2], Anglesio showed the validity of the following sharp inequality, which gives a solution to part (b) of Wilker's problem.

$$\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} > 2 + \frac{16}{\pi^4}x^3 \tan x \quad \left(0 < x < \frac{\pi}{2}\right), \tag{3}$$

where the constant $(16/\pi^4)$ is best possible; that is, it cannot be replaced by a larger number.

During the past few years, considerable attention has been given to the Wilker inequality (1) and the Wilker–Anglesio inequality (3). Various interesting generalizations and improvements and variants of these inequalities have appeared in the literature; see [6–9] the references cited therein.

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Recently, Wu and Srivastava [10] gave a generalization of Wilker's inequality involving parameters of exponent and weight, as follows:

$$\frac{\lambda}{\mu+\lambda}\left(\frac{\sin x}{x}\right)^p+\frac{\mu}{\mu+\lambda}\left(\frac{\tan x}{x}\right)^q>1, \tag{4}$$

where $0<x<\pi/2$, $\lambda>0$, $\mu>0$, $p\le 2q\mu/\lambda$, $q>0$ *or* $q\le\min\{-\lambda/\mu,-1\}$.

The purpose of this paper is to establish the hyperbolic analogue of the Wilker–Anglesio inequality (3) and the parameterized Wilker inequality (4). We next provide an application to the Sándor–Bencze inequality (1).

2. Some lemmas

In order to prove the main results in Sections 3 and 4, we first introduce the following lemmas.

Lemma 1 (*See Hardy et al. [11]*)**.** *If $x_i>0$, $\lambda_i>0$ $(i=1,2,\ldots,n)$ and $\sum_{i=1}^n\lambda_i=1$, then*

$$\sum_{i=1}^n\lambda_i x_i\ge\prod_{i=1}^n x_i^{\lambda_i}. \tag{5}$$

Lemma 2 (*See Mitrinović and Vasić [12]*)**.** *For all nonzero real numbers x, the following inequality holds:*

$$1<\cosh x<\left(\frac{\sinh x}{x}\right)^3. \tag{6}$$

Lemma 3. *For all nonzero real numbers x, the following inequality holds:*

$$\left(\frac{x}{\sinh x}\right)^2+\frac{x}{\tanh x}>2. \tag{7}$$

Proof. Case (I): $x>0$.

Define a function $f:\mathbb{R}^+\longrightarrow\mathbb{R}$ by

$$f(x)=\left(\frac{x}{\sinh x}\right)^2+\frac{x}{\tanh x}.$$

Differentiating $f(x)$ with respect to x gives

$$f'(x)=\frac{1}{\sinh^3 x}\left(\sinh^2 x\cosh x+x\sinh x-2x^2\cosh x\right).$$

By making use of the arithmetic–geometric means inequality and inequality (6), we obtain for $x\in(0,+\infty)$ the following inequality:

$$\begin{aligned} f'(x)&\ge\frac{1}{\sinh^3 x}\left(2\sqrt{x\sinh^3 x\cosh x}-2x^2\cosh x\right)\\ &=\frac{2x^2}{\sinh^3 x}\sqrt{\cosh x}\left(\sqrt{\left(\frac{\sinh x}{x}\right)^3}-\sqrt{\cosh x}\right)>0.\end{aligned}$$

This means that function f is strictly increasing on $(0,+\infty)$, and we thus deduce from $\lim_{x\to0^+}f(x)=2$ that $f(x)>2$ for $(0,+\infty)$, which leads to the desired inequality (7).

Case (II): with $x<0$, so $-x>0$.

Making reference to the result proved in Case (I), we obtain

$$\left(\frac{x}{\sinh x}\right)^2+\frac{x}{\tanh x}=\left(\frac{(-x)}{\sinh(-x)}\right)^2+\frac{(-x)}{\tanh(-x)}>2.$$

The proof of Lemma 3 is complete. $\square$

3. Wilker–Anglesio's inequality for hyperbolic functions

Theorem 1. *Let x be nonzero real numbers; then the following inequality holds:*

$$\left(\frac{\sinh x}{x}\right)^2 + \frac{\tanh x}{x} > 2 + \frac{8}{45}x^3 \tanh x, \tag{8}$$

where the constant $(8/45)$ is best possible; that is, it cannot be replaced by a larger number.

Proof. Case (I): $x > 0$.

Define a function $g : \mathbb{R}^+ \longrightarrow \mathbb{R}$ by

$$g(x) = \frac{\left(\frac{\sinh x}{x}\right)^2 + \frac{\tanh x}{x} - 2}{x^3 \tanh x}.$$

Differentiating $g(x)$ with respect to x gives

$$\begin{aligned} g'(x) &= \frac{24x^2 \sinh 2x + 10 \sinh 2x - 5 \sinh 4x + 18x + 16x^3 + 2x \cosh 4x - 20x \cosh 2x}{2x^6 \sinh^2 x} \\ &= \frac{g_1(x)}{2x^6 \sinh^2 x}, \end{aligned}$$

where $g_1(x)$ is the numerator of $g'(x)$.

Now, computing the derivative of $g_1(x)$ gives

$$\begin{aligned} g_1'(x) &= 48x^2(\cosh 2x + 1) + 8x(\sinh 2x + \sinh 4x) - 18 \cosh 4x + 18 \\ &= (16x^2 \cosh^2 x)[6x^2 + \sinh x(4 \cosh x - \cosh^{-1} x)x - 9 \sinh^2 x] \\ &= (16x^2 \cosh^2 x) g_2(x). \end{aligned}$$

Similarly, we have

$$\begin{aligned} g_2'(x) &= \frac{-14 \sinh x \cosh^3 x - \sinh x \cosh x - x + 8x \cosh^2 x + 8x \cosh^4 x}{\cosh^2 x} \\ &= \frac{g_3(x)}{\cosh^2 x}, \\ g_3'(x) &= (4 \sinh 2x)(4x + 2x \cosh 2x - 3 \sinh 2x) \\ &= (4 \sinh 2x) g_4(x), \\ g_4'(x) &= (4 \sinh 2x)(x - \tanh x) \\ &= (4 \sinh 2x) g_5(x), \end{aligned}$$

and

$$g_5'(x) = \tanh^2 x.$$

From $g_5'(x) > 0$ ($x \in (0, +\infty)$) and $g_5(0) = 0$, we conclude that the function g_5 is increasing on $(0, +\infty)$, and $g_5(x) > 0$ for $x \in (0, +\infty)$. Similarly, by using the functional relationships stated above together with $g_4(0) = g_3(0) = g_2(0) = g_1(0) = 0$, we deduce that each of the functions g_4, g_3, g_2, g_1 is increasing and positive on $(0, +\infty)$. We hence conclude that g is increasing on $(0, +\infty)$.

Thus, we use

$$\lim_{x \to 0^+} g(x) = \frac{8}{45},$$

to deduce the following inequality

$$\frac{\left(\frac{\sinh x}{x}\right)^2 + \frac{\tanh x}{x} - 2}{x^3 \tanh x} > \frac{8}{45}.$$

This implies the desired inequality (8) and shows that the constant $(8/45)$ in (8) is best possible.

Case (II): with $x < 0$, so $-x > 0$.

Making reference to the result proved in Case (I), we have

$$\left(\frac{\sinh x}{x}\right)^2 + \frac{\tanh x}{x} = \left(\frac{\sinh(-x)}{(-x)}\right)^2 + \frac{\tanh(-x)}{(-x)} > 2 + \frac{8}{45}(-x)^3 \tanh(-x) = \left(2 + \frac{8}{45}x^3 \tanh x\right).$$

This completes the proof of Theorem 1. $\square$

4. Parameterized Wilker's inequality for hyperbolic functions

Theorem 2. *Let $x \neq 0$, $\lambda > 0$, $\mu > 0$ and $p \geq 2q\mu/\lambda$. Then, for $q > 0$ or $q \leq \min\{-\lambda/\mu, -1\}$, the following inequality holds:*

$$\frac{\lambda}{\mu+\lambda}\left(\frac{\sinh x}{x}\right)^p + \frac{\mu}{\mu+\lambda}\left(\frac{\tanh x}{x}\right)^q > 1. \tag{9}$$

Proof. In view of the fact that

$$\frac{\lambda}{\mu+\lambda}\left(\frac{\sinh(-x)}{(-x)}\right)^p + \frac{\mu}{\mu+\lambda}\left(\frac{\tanh(-x)}{(-x)}\right)^q = \frac{\lambda}{\mu+\lambda}\left(\frac{\sinh x}{x}\right)^p + \frac{\mu}{\mu+\lambda}\left(\frac{\tanh x}{x}\right)^q,$$

in order to prove Theorem 2, it is enough to prove that inequality (9) holds for $x > 0$.

Case (I): $x > 0$, $\lambda > 0$, $\mu > 0$, $p \geq 2q\mu/\lambda$ and $q > 0$.

Using Lemmas 1 and 2 gives

$$\begin{aligned}\frac{\lambda}{\mu+\lambda}\left(\frac{\sinh x}{x}\right)^p + \frac{\mu}{\mu+\lambda}\left(\frac{\tanh x}{x}\right)^q &\geq \left(\frac{\sinh x}{x}\right)^{p\lambda/(\mu+\lambda)}\left(\frac{\tanh x}{x}\right)^{q\mu/(\mu+\lambda)}\\ &= \left(\frac{\sinh x}{x}\right)^{p\lambda/(\mu+\lambda)}\left(\frac{\sinh x}{x}\right)^{q\mu/(\mu+\lambda)}\left(\frac{1}{\cosh x}\right)^{q\mu/(\mu+\lambda)}\\ &> \left(\frac{\sinh x}{x}\right)^{p\lambda/(\mu+\lambda)}\left(\frac{\sinh x}{x}\right)^{q\mu/(\mu+\lambda)}\left(\frac{\sinh x}{x}\right)^{-3q\mu/(\mu+\lambda)}\\ &= \left(\frac{\sinh x}{x}\right)^{(p\lambda-2q\mu)/(\mu+\lambda)}\\ &\geq 1,\end{aligned}$$

which is the desired inequality (9).

Case (II): $x > 0$, $\lambda > 0$, $\mu > 0$, $p \geq 2q\mu/\lambda$ and $q \leq \min\{-\lambda/\mu, -1\}$.

It follows from the hypotheses of Theorem 2 and Lemma 2 that

$$\begin{aligned}\frac{\lambda}{\mu+\lambda}\left(\frac{\sinh x}{x}\right)^p + \frac{\mu}{\mu+\lambda}\left(\frac{\tanh x}{x}\right)^q &\geq \frac{\lambda}{\mu+\lambda}\left(\frac{\sinh x}{x}\right)^{2q\mu/\lambda} + \frac{\mu}{\mu+\lambda}\left(\frac{\tanh x}{x}\right)^q\\ &= \frac{\lambda}{\mu+\lambda}\left(\frac{x}{\sinh x}\right)^{-2q\mu/\lambda} + \frac{\mu}{\mu+\lambda}\left(\frac{x}{\tanh x}\right)^{-q},\end{aligned} \tag{10}$$

where $-q\mu/\lambda \geq 1$, $-q \geq 1$.

On the other hand, it follows from Lemma 3 that

$$\frac{x}{\tanh x} > 2 - \left(\frac{x}{\sinh x}\right)^2 > 0.$$

Combining inequality (10) and the above inequality (9), we obtain

$$\frac{\lambda}{\mu+\lambda}\left(\frac{\sinh x}{x}\right)^p + \frac{\mu}{\mu+\lambda}\left(\frac{\tanh x}{x}\right)^q > \frac{\lambda}{\mu+\lambda}\left(\frac{x}{\sinh x}\right)^{-2q\mu/\lambda} + \frac{\mu}{\mu+\lambda}\left(2 - \left(\frac{x}{\sinh x}\right)^2\right)^{-q}. \tag{11}$$

Define a function $f : (0, 1) \longrightarrow \mathbb{R}$ by

$$f(t) = \frac{\lambda}{\mu+\lambda}t^{-q\mu/\lambda} + \frac{\mu}{\mu+\lambda}(2-t)^{-q}.$$

Differentiating with respect to t gives

$$f'(t) = \frac{q\mu}{\mu+\lambda}\left[(2-t)^{-q-1} - t^{-q\mu/\lambda-1}\right].$$

From $-q\mu/\lambda \geq 1$, $-q \geq 1$, $0 < t < 1$, it is easy to verify that $(2-t)^{-q-1} \geq 1 \geq t^{-q\mu/\lambda-1}$, and we thus conclude that $f'(t) \leq 0$ for $0 < t < 1$, which implies that f is decreasing on $(0, 1)$. Hence, we have $f(t) \geq f(1) = 1$ for $0 < t < 1$.

Now, from inequality (11) and a variation of inequality (6), $0 < \frac{x}{\sinh x} < 1$ $(x > 0)$, we deduce that

$$\frac{\lambda}{\mu+\lambda}\left(\frac{\sinh x}{x}\right)^p + \frac{\mu}{\mu+\lambda}\left(\frac{\tanh x}{x}\right)^q > f\left(\left(\frac{x}{\sinh x}\right)^2\right) \geq 1.$$

Inequality (9) is proved. This completes the proof of Theorem 2. □

By setting $(p, q) = (2, 1)$ and $(\lambda, \mu) = (2, 1)$ in Theorem 2, we obtain the following weighted and exponential Wilker-type inequalities for hyperbolic functions, respectively.

Corollary 1. *Let $x \neq 0, \lambda \geq \mu > 0$. Then the following inequality holds:*

$$\frac{\lambda}{\mu+\lambda}\left(\frac{\sinh x}{x}\right)^2 + \frac{\mu}{\mu+\lambda}\left(\frac{\tanh x}{x}\right) > 1. \tag{12}$$

Corollary 2. *Let $x \neq 0, p \geq q$. Then for $q > 0$ or $q \leq -2$, the following inequality also holds:*

$$2\left(\frac{\sinh x}{x}\right)^p + \left(\frac{\tanh x}{x}\right)^q > 3. \tag{13}$$

We next provide an interesting application of Theorem 2.

Recently, Sándor and Bencze [13] proposed the following open problem: Prove that, for all $x \in (0, \frac{\pi}{2})$ and $\alpha \in (0, +\infty)$, the following inequality holds:

$$\left(\frac{\sin x}{x}\right)^\alpha > \frac{\cos^\alpha x}{1+\cos^\alpha x}. \tag{14}$$

Inequality (14) was proved by Wu and Srivastava [10]; we give here a hyperbolic analogue of Sándor–Bencze's inequality.

Corollary 3. *Let $x \neq 0$. Then, for $\alpha > 0$ or $\alpha \leq -1$, the following inequality holds:*

$$\left(\frac{\sinh x}{x}\right)^\alpha > \frac{\cosh^\alpha x}{1+\cosh^\alpha x}. \tag{15}$$

Proof. Putting $\lambda = \mu = 1, p = 2\alpha$ and $q = \alpha$ in Theorem 2 gives

$$\left(\frac{\sinh x}{x}\right)^{2\alpha} + \cosh^{-\alpha} x \left(\frac{\sinh x}{x}\right)^\alpha - 2 > 0 \quad (x \neq 0,\ \alpha > 0 \text{ or } \alpha \leq -1),$$

which is equivalent to the following inequality:

$$\left[\left(\frac{\sinh x}{x}\right)^\alpha + \frac{\cosh^{-\alpha} x + \sqrt{\cosh^{-2\alpha} x + 8}}{2}\right]\left[\left(\frac{\sinh x}{x}\right)^\alpha + \frac{\cosh^{-\alpha} x - \sqrt{\cosh^{-2\alpha} x + 8}}{2}\right] > 0.$$

We can now deduce from the above inequality that

$$\left(\frac{\sinh x}{x}\right)^\alpha + \frac{\cosh^{-\alpha} x - \sqrt{\cosh^{-2\alpha} x + 8}}{2} > 0,$$

and we thus have

$$\begin{aligned}\left(\frac{\sinh x}{x}\right)^\alpha &> \frac{-\cosh^{-\alpha} x + \sqrt{\cosh^{-2\alpha} x + 8}}{2}\\ &= \frac{4\cosh^\alpha x}{1+\sqrt{1+8\cosh^{2\alpha} x}}\\ &> \frac{4\cosh^\alpha x}{1+(1+2\sqrt{2}\cosh^\alpha x)}\\ &> \frac{\cosh^\alpha x}{1+\cosh^\alpha x}.\end{aligned}$$

Thus, inequality (15) is proved. □

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