对$x\geq 0,y>0,a<1$证明

$$\frac{(1+x)^a-1}{(1+y)^a-1}\geq\left(\frac{x}{y}\right)^{\frac{ax(1+x)^{a-1}}{(1+x)^a-1}}$$

设$f(t)=(1+t)^a-1$，即证

$$\frac{f(x)}{f(y)}\geq\left(\frac{x}{y}\right)^{\frac{xf'(x)}{f(x)}}\Leftrightarrow\ln f(x)-\ln f(y)\geq\frac{xf'(x)}{f(x)}\big(\ln x-\ln y\big)$$

不妨设$x>y$，由Cauchy中值定理知存在$\xi\in(y,x)$使得

$$\frac{\ln f(x)-\ln f(y)}{\ln x-\ln y}=\frac{(\ln f(t))'}{(\ln t)'}\bigg|_{t=\xi}=\frac{\xi f'(\xi)}{f(\xi)}$$

设$g(t)=t(\ln f(t))'=\dfrac{tf'(t)}{f(t)}$，则只需证$g(\xi)\geq g(x)$，我们有

$$\begin{aligned}g(t)&=\frac{at}{1+t}\bigg(1+\frac{1}{(1+t)^a-1}\bigg)\\g'(t)&=a\bigg[\frac{1}{(1+t)^2}\bigg(1+\frac{1}{(1+t)^a-1}\bigg)-\frac{t}{1+t}\cdot\frac{a(1+t)^{a-1}}{\big((1+t)^a-1\big)^2}\bigg]\\&=\frac{a(1+t)^a}{(1+t)^2\big((1+t)^a-1\big)}\bigg(1-\frac{at}{(1+t)^a-1}\bigg)\end{aligned}$$

由Bernoulli不等式得$g'(t)<0$，故$g(\xi)\geq g(x)$成立.