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# Inequalities for Sums of Powers

ZSOLT PÁLES*

*Department of Mathematics, L. Kossuth University, H-4010 Debrecen, pf. 12, Hungary*

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## 1. INTRODUCTION

In this paper we continue our investigations started in [6]. Applying the same methods, we shall discuss, among others, the following inequality

$$M_{a,b}(x, y) \leqslant M_{c,d}(x, y) \qquad (x, y > 0), \tag{1}$$

where the mean $M$ is defined by

$$\begin{aligned} M_{a,b}(x, y) &= \left(\frac{x^a + y^a}{x^b + y^b}\right)^{1/(a-b)} && \text{if} \quad a \neq b, \\ &= \exp\left(\frac{x^a \ln x + y^a \ln y}{x^a + y^a}\right) && \text{if} \quad a = b. \end{aligned} \tag{2}$$

This mean is the restriction of the following more general $n$-variable mean

$$\begin{aligned} M_{a,b}(x_1, ..., x_n) &= \left(\sum_{i=1}^{n} x_i^a \Big/ \sum_{i=1}^{n} x_i^b\right)^{1/(a-b)} && \text{if} \quad a \neq b, \\ &= \exp\left(\sum_{i=1}^{n} x_i^a \ln x_i \Big/ \sum_{i=1}^{n} x_i^a\right) && \text{if} \quad a = b, \end{aligned} \tag{3}$$

where $n$ is a positive integer and $x_1, ..., x_n$ are positive real numbers.

Concerning comparison of the means defined by (3) the following result is known (see Daróczy and Losonczi [2]):

THEOREM A. *Let $a$, $b$, $c$, $d$ be real numbers. Then in order that*

$$M_{a,b}(x_1, ..., x_n) \leqslant M_{c,d}(x_1, ..., x_n) \tag{4}$$

* This article was written while the author was a visitor at the Mathematatsches Institut I, Universität Karlsruhe, Karlsruhe, Federal Republic of Germany.



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*be valid for all* $n \in \mathbb{N}$ *and* $x_1, ..., x_n > 0$ *it is necessary and sufficient that*

$$\min(a, b) \leqslant \min(c, d) \quad \textit{and} \quad \max(a, b) \leqslant \max(c, d). \tag{5}$$

In [1] Brenner showed independently that (5) is a sufficient condition for (4). He also obtained a number of more general results.

There exist other generalizations of Theorem A (see for instance Losonczi [3], Pàles [5]) but now we show one way that can also be applied to the inequality (1).

Assume that $a \neq b$ and $c \neq d$ and then rearrange (4) to obtain

$$\begin{aligned} 1 \leqslant {} & (x_1^a + \cdots + x_n^a)^{1/(b-a)}(x_1^b + \cdots + x_n^b)^{1/(a-b)} \\ & \times (x_1^c + \cdots + x_n^c)^{1/(c-d)}(x_1^d + \cdots + x_n^d)^{1/(d-c)}. \end{aligned}$$

It seems to be natural to consider the following more general inequality

$$1 \leqslant (x_1^{a_1} + \cdots + x_n^{a_1})^{\alpha_1} \cdots (x_1^{a_k} + \cdots + x_n^{a_k})^{\alpha_k}, \tag{6}$$

where $k$ is a positive integer, $a_1, ..., a_k$, $\alpha_1, ..., \alpha_k$ are real values with

$$\alpha_1 + \cdots + \alpha_k = 0. \tag{7}$$

In [4] the author found the following result:

THEOREM B. *Let* $a_1, ..., a_k$, $\alpha_1, ..., \alpha_k$ *be real parameters with* (7). *Then* (6) *holds for all* $n \in \mathbb{N}$ *and* $x_1, ..., x_n > 0$ *if and only if*

$$0 \leqslant \alpha_1 |a_1 - a_i| + \cdots + \alpha_k |a_k - a_i| \tag{8}$$

*is valid for* $i = 1, ..., k$.

If $k = 4$ and if (6) is equivalent to (4) then it is a simple calculation to show that (8) is equivalent to (5).

In a similar way, the inequality

$$1 \leqslant (x^{a_1} + y^{a_1})^{\alpha_1} \cdots (x^{a_k} + y^{a_k})^{\alpha_k} \tag{9}$$

(where $a_1, ..., a_k$, $\alpha_1, ..., \alpha_k$ are real values with (7)) can be considered as a generalization of (1).

In Section 2 we derive necessary conditions for (9) and in Section 3 we show that these conditions are also sufficient if we assume several additional assumptions involving $a_1, ..., a_k$ and $k$. In the final section, applying our results, we give necessary and sufficient conditions for (1) to hold.

## 2. Necessary Conditions

THEOREM 1. *Let $a_1, ..., a_k, \alpha_1, ..., \alpha_k$ be real numbers satisfying* (7). *Then in order that* (9) *be valid for all positive $x$ and $y$, it is necessary that the following three conditions be fulfilled*:

(i) $0 = \alpha_1 a_1 + \cdots + \alpha_k a_k$,

(ii) $0 \leqslant \alpha_1 a_1^2 + \cdots + \alpha_k a_k^2$,

(iii) $0 \leqslant \alpha_1 f(a_1) + \cdots + \alpha_k f(a_k)$,

*where*

$$f(x) = 1 \quad \textit{for} \quad x = \min |a_i|,$$
$$= 0 \quad \textit{for} \quad x \neq \min |a_i|$$

*if either* $0 \leqslant \min a_i$ *or* $\max a_i \leqslant 0$, *and*

$$f(x) = |x| \quad \textit{for} \quad x \in \mathbb{R}$$

*if* $\min a_i < 0 < \max a_i$.

*Proof.* Let $x = y$ in (9). Then using (7), we have

$$1 \leqslant x^{\alpha_1 a_1 + \cdots + \alpha_k a_k}$$

for all positive $x$. Thus the necessity of (i) is obvious.

To prove (ii), put $x = e^s$ and $y = e^{-s}$ into (9). Then we obtain the inequality

$$0 \leqslant \alpha_1 g(a_1 s) + \cdots + \alpha_k g(a_k s), \tag{10}$$

where

$$g(x) = \ln(\cosh x) = x^2/2 - x^4/12 + \cdots.$$

Multiplying (10) by $2/s^2$ and taking the limit $s \to 0$, we get (ii).

In the proof of the necessity of (iii) we distinguish two cases.

*Case* I. Either $0 \leqslant \min a_i$ or $\max a_i \leqslant 0$. We deal only with the case $0 \leqslant \min a_i$, the proof of the other case is completely similar.

Let $y = 1$ in (9). After a simple calculation we get

$$0 \leqslant \sum_{i=1}^{k} \alpha_i (x^{a_i}/\min(x^{a_1}, ..., x^{a_k})) \ln(1 + x^{a_i})^{x^{-a_i}}. \tag{11}$$

Since

$$\lim_{x \to 0} \ln(1 + x^{a_i})^{x^{-a_i}} = 1 \qquad \text{if} \quad a_i > 0,$$

$$= \ln 2 \qquad \text{if} \quad a_i = 0,$$

$$\lim_{x \to 0} x^{a_i}/\min(x^{a_1}, ..., x^{a_k}) = 0 \qquad \text{if} \quad a_i > \min a_j,$$

$$= 1 \qquad \text{if} \quad a_i = \min a_j,$$

therefore (iii) follows from (11) if we take the limit $x \to 0$.

*Case* II. $\min a_i < 0 < \max a_i$. As we have seen, (10) is a consequence of (9). On the other hand, by L'Hospital's rule

$$\lim_{s \to \infty} g(a_i s)/s = \lim_{s \to \infty} a_i \tanh(a_i s) = |a_i|,$$

therefore, multiplying (10) by $1/s$ and taking the limit $s \to \infty$, we obtain

$$0 \leqslant \alpha_1 |a_1| + \cdots + \alpha_k |a_k|,$$

which completes the proof of the theorem.

## 3. Sufficient Conditions

We shall need the following

Lemma. *Let $a_1 \leqslant a_2 \leqslant a_3 \leqslant a_4$ be arbitrary with $0 \leqslant a_1 + a_4$ and $0 \leqslant a_2 + a_3$. Then there exist a, b real and c, d positive constants such that*

$$g(a_i) = a + ba_i + ca_i^2 + df(a_i) \tag{12}$$

*for $i = 1, 2, 3, 4$, where f and g are defined in condition* (iii) *and in the proof of Theorem* 1, *respecitvely.*

*Proof.* Without loss of generality we may suppose that $a_1 < a_2 < a_3 < a_4$. We shall distinguish four cases.

*Case* I. $0 \leqslant a_1$. Then $f(a_1) = 1$ and $f(a_i) = 0$ for $i = 2, 3, 4$. Thus (12) reduces to the following system of equations

$$g(a_1) = a + ba_1 + ca_1^2 + d, \tag{13}$$

$$g(a_i) = a + ba_i + ca_i^2 \qquad (i = 2, 3, 4). \tag{14}$$

It is obvious that there exists a unique solution of the system (13) and (14). We have only to show that $c$ and $d$ are nonnegative. Let

$$h(x) = g(x) - a - bx - cx^2.$$

Then $h(a_i) = 0$ for $i = 2, 3, 4$; therefore, by the Rolle's theorem, $h''(x)$ vanishes at a point $x = x_1 > 0$; that is,

$$g''(x_1) = 2c.$$

But $g''(x) = 1/\cosh^2 x > 0$, thus $c > 0$.

To prove $d \geqslant 0$, let $P(x) = p_0(x - a_2)(x - a_3)(x - a_4)$ and choose $p_0$ so that $P(a_1) = 1$. Since $a_1 < a_i$, hence we get very easily that $p_0 < 0$. With the help of $P$, (13) and (14) can be rewritten as

$$g(a_i) = a + ba_i + ca_i^2 + dP(a_i) \qquad (i = 1, 2, 3, 4).$$

Now let

$$h(x) = g(x) - a - bx - cx^2 - dP(x).$$

Then $h(a_i) = 0$ for $i = 1, 2, 3, 4$; therefore the Rollé theorem implies the existence of a value $x = x_2 > 0$ where $h_2'''$ is zero, i.e.,

$$g'''(x_2) = 4p_0 d.$$

Since $g'''(x) = -2 \sinh x/\cosh^3 x < 0$ and $p_0 < 0$, hence we obtain $d > 0$.

In the proof of the lemma we have still the following cases:

*Case* II. $a_1 < a \leqslant a_2$.

*Case* III. $a_2 < 0 \leqslant a_3$ and $a_1 a_2 < a_3 a_4$.

*Case* IV. $a_1 + a_4 = a_2 + a_3 = 0$.

We omit the proof of these cases since the proof of Case III, Case IV, and Case V of Lemma 2 in [6] can be repeated almost word for word here. Of course, the meaning of the function $g$ is different but in [6] the only properties of $g$ used in the argument are $g''(x) > 0$ and $g'''(x) < 0$ for $x > 0$, and this property also holds in our case.

THEOREM 2. *Let $k = 4$ and let $a_1, a_2, a_3, a_4, \alpha_1, \alpha_2, \alpha_3, \alpha_4$ be real numbers satisfying $a_1 \leqslant a_2 \leqslant a_3 \leqslant a_4$, $(a_1 + a_4)(a_2 + a_3) \geqslant 0$, and* (7). *Then in order that* (9) *be valid for all positive $x$ and $y$ it is necessary and sufficient that conditions* (i), (ii), *and* (iii) *of Theorem* 1 *be satisfied.*

We omit the proof of this theorem because it is completely similar to the one of Theorem 2 in [6].

## 4. Application

Using Theorem 2, we can solve the problem of comparison of the means defined by (2).

THEOREM 3. *Let $a$, $b$, $c$, $d$ be arbitrary real numbers with $a \neq b$ and $c \neq d$. Then* (1) *is satisfied for all positive $x$ and $y$ if and only if*

$$a+b \leqslant c+d$$

*and*

$$m(a, b) \leqslant m(c, d),$$

*where*

$$\begin{aligned} m(x, y) &= \min(x, y) && \textit{if}\ \ 0 \leqslant \min(a, b, c, d), \\ &= (|x| - |y|)/(x-y) && \textit{if}\ \ \min(a, b, c, d) < 0 < \max(a, b, c, d), \\ &= \max(x, y) && \textit{if}\ \ \max(a, b, c, d) \leqslant 0. \end{aligned}$$

The proof of this theorem is similar to the proof of the corollary in [6], so we omit it.

We remark that Theorem 3 can be extended to the case $(a-b)(c-d)=0$ very easily.

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