清华大学深圳研究生院

应用信息论

2018年春季学期

作业 1

YOUR NAME　　　　2025 年 8 月 31 日

1.1. 设X和Y是各有均值m_x, m_y,方差为σ_x^2, σ_y^2，且相互独立的高斯随机变量，已知$U = X + Y, V = X - Y$。试求$I(U;V)$。

解. U, V的联合分布是均值为$[\mu_x + \mu_y, \mu_x - \mu_y]$，协方差矩阵为

$$\Lambda_{U,V} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \sigma_x^2 & 0 \\ 0 & \sigma_x^2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^T = \begin{bmatrix} \sigma_x^2 + \sigma_y^2 & \sigma_x^2 - \sigma_y^2 \\ \sigma_x^2 - \sigma_y^2 & \sigma_x^2 + \sigma_y^2 \end{bmatrix}$$

由多元高斯分布微分熵的公式

$$h(U) = \frac{1}{2}\log((2\pi e)^2 |\Lambda_{U,V}|) = \frac{1}{2}\log(16\pi^2 e^2 \sigma_x^2 \sigma_y^2)$$

$U|V = v$也是高斯分布,方差为$\frac{4\sigma_x^2\sigma_y^2}{\sigma_x^2+\sigma_y^2}$，与$v$无关，因此

$$h(U|V) = \mathbb{E}_V[h(U|V = v)] = \frac{1}{2}\log(2\pi e \frac{4\sigma_x^2\sigma_y^2}{\sigma_x^2 + \sigma_y^2}) \Rightarrow$$

$$\begin{aligned} I(U;V) =& h(U) - h(U|V) \\ =& \frac{1}{2}\log(16\pi^2 e^2 \sigma_x^2 \sigma_y^2) - \frac{1}{2}\log(2\pi e \frac{4\sigma_x^2\sigma_y^2}{\sigma_x^2 + \sigma_y^2}) \\ =& \frac{1}{2}\log(2\pi e(\sigma_x^2 + \sigma_y^2)) \end{aligned}$$

1.2. 设有随机变量X, Y, Z均取值于$\{0, 1\}$,已知$I(X;Y) = 0, I(X;Y|Z) = 1$。求证$H(Z) = 1, H(X, Y, Z) = 2$

证明.
$I(X;Y|Z) = H(X|Z) - H(X|Y,Z) \leq H(X|Z) \leq H(X) \leq \log(2) = 1$
所以等号全都成立$\Rightarrow X \sim B(\frac{1}{2})$。同理可知$Y \sim B(\frac{1}{2})$。另外$H(Y|Z) = H(Y) \Rightarrow I(Y;Z) = 0 \Rightarrow H(Z|Y) = H(Z)$

$$\begin{aligned} & H(X|Y,Z) = 0 \\ \iff & H(X,Y,Z) = H(Y,Z) \\ \iff & H(X,Y) + H(Z|X,Y) = H(Y) + H(Z|Y) \\ \iff & 2 + H(Z|X,Y) = 1 + H(Z) \\ \iff & H(Z) = 1 + H(Z|X,Y) \end{aligned}$$

由上式推
出$H(Z) \geq 1$,又$H(Z) \leq 1 \Rightarrow H(Z) = 1 \Rightarrow H(X,Y,Z) = 2$ □

1.3. 设有信号X经过处理器A后获输出Y,Y再经处理器B后获输出Z。已知处理器A和B 分别独立处理X和Y。试证：$I(X;Z) \leq I(X;Y)$

证明. $I(X;Z) = H(Z) - H(Z|X) = H(Z); I(Y;Z) = H(Y)$因为$Z$是$Y$的函数$\Rightarrow H(Z) \leq H(Y) \Rightarrow I(X;Z) \leq I(X;Y)$ □

1.4. 已知随机变量X和Y的联合概率密度$p(a_k, b_j)$满足

$$p(a_1) = \frac{1}{2}, p(a_2) = p(a_3) = \frac{1}{4}, p(b_1) = \frac{2}{3}, p(b_2) = p(b_3) = \frac{1}{6}$$

试求能使$H(X,Y)$取得最大值的联合概率密度分布。

解. $H(X,Y) = H(X) + H(Y) - I(X;Y) \leq H(X) + H(Y) = \frac{7}{6} + \log 3$
等号成立当且仅当X, Y相互独立$\Rightarrow p(x,y) = p(x)p(y)$

1.5. 设随机变量X, Y, Z满足$p(x,y,z) = p(x)p(y|x)p(z|y)$。求证$I(X;Y) \geq I(X;Y|Z)$

证明. 因为$p(x,y,z) = p(x)p(y|x)p(z|y,x) \Rightarrow p(z|y,x) = p(z|x) \Rightarrow x$与$z$关于$y$条件独立$\Rightarrow I(X;Y|Z) = H(X|Z) - H(X|Y,Z) = H(X|Z) - H(X|Y) \leq H(X) - H(X|Y) = I(X;Y)$ □

1.6. 求证$I(X;Y;Z) = H(X,Y,Z) - H(X) - H(Y) - H(Z) + I(X;Y) + I(Y;Z) + I(Z;X)$,其中 $I(X;Y;Z) \triangleq I(X;Y) - I(X;Y|Z)$

证明.

$$\begin{aligned}
I(X;Y;Z) =& I(X;Y) - I(X;Y|Z) \\
=& H(X) + H(Y) - H(X,Y) - (H(X|Z) - H(X|Y,Z)) \\
=& H(X) + H(Y) - H(X,Y) - (H(X,Z) - H(Z)) + H(X,Y,Z) - H(Y,Z) \\
=& H(X,Y,Z) - H(X) - H(Y) - H(Z) + (H(X) + H(Y) - H(X,Y)) \\
&+ (H(Y) + H(Z) - H(Y,Z)) + (H(Z) + H(X) - H(X,Z)) \\
=& H(X,Y,Z) - H(X) - H(Y) - H(Z) + I(X;Y) + I(Y;Z) + I(Z;X)
\end{aligned}$$

□

1.7. 令$p=(p_1,p_2,\dots,p_a)$是一个概率分布，满足$p_1\geq p_2\geq \dots p_a$,假设$\epsilon>0$使得$p_1-\epsilon\geq p_2+\epsilon$成立，证明：
$H(p_1,p_2,\dots,p_a)\leq H(p_1-\epsilon,p_2+\epsilon,p_3,\dots,p_a)$

证明. 设$f(\epsilon)=(p_1-\epsilon)\log(p_1-\epsilon)+(p_2+\epsilon)\log(p_2+\epsilon)$ 由已知$0\leq\epsilon\frac{p_2-p_1}{2}$ $f'(\epsilon)=\log\frac{p_2+\epsilon}{p_1-\epsilon}\leq 0$
$\Rightarrow f(\epsilon)\leq f(0)\Rightarrow H(p_1,p_2,\dots,p_a)\leq H(p_1-\epsilon,p_2+\epsilon,p_3,\dots,p_a)$ □

1.8. 设$p_i(x)\sim N(\mu_i,\sigma_i^2)$,试求相对熵$D(p_2||p_1)$

解.

$$
\begin{aligned}
D(p_2||p_1)=&\int_{\mathbb{R}}p_2(x)\log\frac{p_2(x)}{p_1(x)}dx\\
=&\int_{\mathbb{R}}p_2(x)\left(\log\frac{\sigma_1^2}{\sigma_2^2}+\frac{1}{2}((x-\mu_1)^2-(x-\mu_2)^2)\log e\right)dx\\
=&2\log\frac{\sigma_1}{\sigma_2}+\frac{1}{2}(\mu_1^2-\mu_2^2)\log e+(\mu_2-\mu_1)\mu_2\log e\\
=&2\log\frac{\sigma_1}{\sigma_2}+\frac{1}{2}(\mu_1-\mu_2)^2\log e
\end{aligned}
$$

1.9. 若$f(x)$分别是区间$(0,0.01),(0,0.5),(0,1),(0,2),(0,5)$上均匀分布的分布函数，计算$f(x)$的微分熵。

解. 设U_t是$(0,t)$上的均匀分布，则$h(U_t)=\log t$

- $h(U_{0.01})=\log 0.01$
- $h(U_{0.5})=-1$
- $h(U_1)=0$
- $h(U_2)=1$
- $h(U_5)=\log 5$

1.10. 设

$$
\begin{aligned}
p_1(x,y)=&\frac{1}{2\pi\sigma_x\sigma_y}\exp[-\frac{1}{2}(\frac{x^2}{\sigma_x^2}+\frac{y^2}{\sigma_y^2})]\\
p_2(x,y)=&\frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}}\exp[-\frac{1}{2(1-\rho^2)}(\frac{x^2}{\sigma_x^2}-2\rho\frac{xy}{\sigma_x\sigma_y}+\frac{y^2}{\sigma_y^2})]
\end{aligned}
$$

试求$D(p_2||p_1)$和$I(X;Y)$，其中$X,Y\sim p_2$

解.

$$
\begin{aligned}
D(p_2||p_1) = & \iint_{\mathbb{R}^2} p_2(x,y) \log \frac{p_2(x,y)}{p_1(x,y)} dxdy \\
& -\frac{1}{2} \log(1-\rho^2) \\
& -\frac{1}{2}(\log e) \iint_{\mathbb{R}^2} p_2(x,y) \left[\frac{\rho^2 x^2}{\sigma_x^2(1-\rho^2)} + \frac{\rho^2 y^2}{\sigma_y^2(1-\rho^2)} - \frac{2\rho xy}{(1-\rho^2)\sigma_x\sigma_y} \right] dxdy \\
& = -\frac{1}{2} \log(1-\rho^2)
\end{aligned}
$$

$X|Y=y$服从高斯分布，方差为$(1-\rho^2)\sigma_x^2$

$$
\begin{aligned}
I(X;Y) = & h(X) - h(X|Y) \\
= & \frac{1}{2} \log(2\pi e \sigma_x^2) - \frac{1}{2} \log(2\pi e \sigma_x^2 (1-\rho^2)) \\
= & \frac{1}{2} \log(\frac{2\pi e}{1-\rho^2})
\end{aligned}
$$