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Monotonic Averages of Convex Functions

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We investigate the monotonicity of various averages of the values of a convex (or concave) function at n equally spaced points. For a convex function, averages without end points increase with n, while averages with end points decrease. Averages including one end point are treated as a special case of upper and lower Riemann sums, which are shown to decrease and increase, respectively. Corresponding results for mid-point Riemann sums and the trapezium estimate require convexity or concavity of the derivative as well as the function. Special cases include some known results and some new ones, unifying them in a more systematic theory. Further applications include results on series and power majorization.

1. INTRODUCTION

We consider the average of the values of a function at a sequence of n points equally spaced through an interval. All such averages give approximations of some kind to the integral of the function. Different averages are obtained depending on what is done with the end points: we may exclude them, include them, include one of them, or include them with half weighting (the trapezium estimate). For monotonic functions, the one-end-point expressions coincide with upper or lower Riemann sums, and in fact we consider such sums, regardless of whether the function is

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monotonic. A translation of the points leads to midpoint Riemann sums. Our objective is to find conditions under which the averages increase or decrease with n. In all cases, convexity (or concavity) of the function is the basic requirement. For such functions, the problem is well motivated because each of the averages is easily seen to be either greater or less than the first one in its sequence, or alternatively greater or less than the integral.

The quantities considered group naturally into three pairs: (1) averages with and without end points, (2) upper and lower Riemann sums, and (3) mid-point Riemann sums and the trapezium estimate. The results are essentially as expected, but in case (3), they require convexity or concavity of the derivative as well as the function.

Our results include as special cases a number of specific sequences that have been previously shown (by various methods) to be monotonic, and enable us to fill in missing cases to embed these results into a more systematic theory. A typical result is that if $\sigma_n = \sum_{r=1}^{n} r^p$, then σ_n / n^{p+1} decreases with n when $p \geq 0$ and increases when $p \leq 0$. An application to infinite series solves a problem from [11]: if $v_n = \sum_{j=1}^{\infty} 1/j^{\alpha}(j+n)^{\beta}$, where $0 \leq \alpha < 1$ and $\alpha + \beta > 1$, then $n^{\alpha+\beta-1} v_n$ increases with n. Another application is a simple solution of the "power majorization" problem (cf. [1, 5]).

2. AVERAGES WITH AND WITHOUT END POINTS

Define

$$A_n(f) = \frac{1}{n-1} \sum_{r=1}^{n-1} f\left(\frac{r}{n}\right) \qquad (n \geq 2),$$

$$B_n(f) = \frac{1}{n+1} \sum_{r=0}^{n} f\left(\frac{r}{n}\right) \qquad (n \geq 0).$$

These are, respectively, the averages excluding and including the end points. (Note that $A_1(f)$ is undefined; our notational convention is that the suffix n is used for quantities derived from intervals of length $1/n$.) The definitions, and the results below, adapt in the obvious way to intervals other than $[0, 1]$.

Some introductory remarks may help to show what might be expected. First, $A_2(f) = f(\frac{1}{2})$, while $B_1(f) = \frac{1}{2}[f(0) + f(1)]$. If f is convex, then $f(\frac{1}{2}) \leq \frac{1}{2}f(r/n) + \frac{1}{2}f[(n-r)/n]$, from which it is clear that $A_n(f) \geq A_2(f)$ for $n \geq 2$. Also, $f(r/n) \leq (1 - r/n)f(0) + (r/n)f(1)$, hence $f(r/n) + f[(n-r)/n] \leq f(0) + f(1)$, and $B_n(f) \leq B_1(f)$. In the same way,

$A_n(f) \le B_1(f)$. The relation with $A_n(f)$ is given by

$$B_n(f) = \frac{n-1}{n+1} A_n(f) + \frac{2}{n+1} B_1(f),$$

from which it is clear that $B_n(f) \ge A_n(f)$ for convex f.

These considerations make it plausible to suggest that when f is convex, $A_n(f)$ increases with n and $B_n(f)$ decreases. For $A_n(f)$, this statement was effectively proved both in [7, Lemma 8] and in [3], but in both cases disguised by being stated only for special choices of f. Our proof follows the pleasantly simple method of [3].

THEOREM 1. *Define $A_n(f)$ as above. If f is a convex function on the open interval* $(0, 1)$, *then $A_n(f)$ increases with n. If f is concave, $A_n(f)$ decreases with n.*

Proof. We prove the statement for convex f; the statement for concave f then follows by considering $-f$. For $1 \le r \le n-1$, the point r/n lies between $r/(n+1)$ and $(r+1)/(n+1)$. More exactly,

$$\frac{r}{n} = \frac{n-r}{n} \frac{r}{n+1} + \frac{r}{n} \frac{r+1}{n+1}.$$

Write $f[r/(n+1)] = a_r$. Since f is convex,

$$f\left(\frac{r}{n}\right) \le \frac{n-r}{n} a_r + \frac{r}{n} a_{r+1}.$$

Hence

$$\sum_{r=1}^{n-1} f\left(\frac{r}{n}\right) \le \frac{n-1}{n} a_1 + \frac{1}{n} a_2 + \frac{n-2}{n} a_2 + \cdots + \frac{1}{n} a_{n-1} + \frac{n-1}{n} a_n$$
$$= \frac{n-1}{n} \sum_{r=1}^{n} a_r,$$

which says that $A_n(f) \le A_{n+1}(f)$.

We now prove the companion result for $B_n(f)$. The authors are not aware of it having appeared previously, even in disguised form.

THEOREM 2. *Define $B_n(f)$ as above. If f is convex on* $[0, 1]$, *then $B_n(f)$ decreases with n. If f is concave, $B_n(f)$ increases with n.*

Proof. Assume f is convex. For $0 \le r \le n$, we have

$$\frac{r}{n} = \frac{r}{n} \frac{r-1}{n-1} + \frac{n-r}{n} \frac{r}{n-1}.$$

Write $f[r/(n-1)] = b_r$. By convexity of f,

$$f\left(\frac{r}{n}\right) \le \frac{r}{n}b_{r-1} + \frac{n-r}{n}b_r.$$

Hence

$$\sum_{r=0}^{n} f\left(\frac{r}{n}\right) \le \frac{n}{n}b_0 + \frac{1}{n}b_0 + \frac{n-1}{n}b_1 + \frac{2}{n}b_1 + \cdots + \frac{1}{n}b_{n-1} + \frac{n}{n}b_{n-1}$$
$$= \frac{n+1}{n}\sum_{r=0}^{n-1} b_r,$$

which says that $B_n(f) \le B_{n-1}(f)$.

Though these two theorems are simple to prove, they are rich in applications. We start with a unified survey of results on expressions involving

$$\sigma_n = \sum_{r=1}^{n} r^p.$$

The results include, and in certain cases improve, some that have appeared sporadically elsewhere (with a variety of proofs). They also complement these by providing companion results that seem to be missing from the literature.

PROPOSITION 1 [7, Lemma 8]. *Let*

$$c_n = \frac{\sigma_n}{n(n+1)^p}.$$

Then c_n increases with n when $p \ge 1$ or $p \le 0$, and decreases with n when $0 \le p \le 1$.

Proof. This is a case of Theorem 1, since $c_n = A_{n+1}(f)$, with $f(x) = x^p$.

PROPOSITION 2. *Let*

$$d_n = \frac{\sigma_n}{n^p(n+1)}.$$

Then d_n decreases with n when $p \ge 1$ and increases with n when $p \le 1$.

Proof. For $p \ge 0$, we have $d_n = B_n(f)$, where $f(x) = x^p$ (note that the term $r = 0$ contributes zero), so the statement is a case of Theorem 2. For $p < 0$, we have $d_n = [(n+1)^{p-1}/n^{p-1}]c_n$, so the statement follows from Proposition 1.

Proposition 3. *Let* $u_n = \sigma_n / n^{p+1}$. *Then* u_n *decreases with* n *when* $p \geq 0$ *and increases with* n *when* $p \leq 0$.

Proof. With c_n as in Proposition 1, $u_n = [(n+1)^p / n^p] c_n$; the statements for $0 \leq p \leq 1$ and $p < 0$ follow. Also, $u_n = [(n+1)/n] d_n$, so Proposition 2 gives the result for $p \geq 1$.

The case $p \leq 0$ of Proposition 3 has appeared, with different methods, in [8, p. 89; 11, Proposition 5], and the case $p \geq 0$ follows from the theorem of [2]. Clearly, Proposition 3 is weaker (though simpler to state) than Propositions 1 and 2. Since σ_n can be regarded also as a sum of $n+1$ terms (starting with a zero), the following companion result is equally natural.

Proposition 4 . *Let* $v_n = \sigma_n / (n+1)^{p+1}$. *Then* v_n *is increasing for all* p.

Proof. For $p \geq 1$ and $p \leq 0$, the statement follows from the identity $v_n = [n/(n+1)] c_n$. For $0 \leq p \leq 1$, it follows from $v_n = [n^p / (n+1)^p] d_n$.

For the quantity x_n considered in the next proposition, it was shown in [2] that $x_n / x_{n+1} \geq n/(n+1)$ whenever $p > 0$. Our result improves this estimate (separately for $p < 1$ and $p > 1$), and is exact when $p = 1$.

Proposition 5. *For* $p \neq 0$, *define* $x_n = (\sigma_n / n)^{1/p}$. *For* $p \neq 1$, *let* $p^* = p/(p-1)$. *If* $0 < p < 1$, *then*

$$\frac{n+1}{n+2} \leq \frac{x_n}{x_{n+1}} \leq \left(\frac{n}{n+1} \right)^{1/p^*} \left(\frac{n+1}{n+2} \right)^{1/p}.$$

The opposite inequalities hold when $p > 1$. *The left-hand inequality holds when* $p < 0$.

Proof. First, $x_n = (n+1) c_n^{1/p}$. The comparisons with $(n+1)/(n+2)$ follow, by Proposition 1. Second, $x_n = n^{1/p^*} (n+1)^{1/p} d_n^{1/p}$; if z_n denotes the quantity on the right-hand side, the comparisons with z_n follow, by Proposition 2. (Note that when $p < 0$, z_n is not relevant because $z_n < (n+1)/(n+2)$.)

Again there is a companion result:

Proposition 6. *For* $p \neq 0$, *let* $y_n = [\sigma_n / (n+1)]^{1/p}$. *If* $0 < p < 1$, *then*

$$\left(\frac{n}{n+1} \right)^{1/p} \left(\frac{n+1}{n+2} \right)^{1/p^*} \leq \frac{y_n}{y_{n+1}} \leq \frac{n}{n+1}.$$

The opposite inequalities hold when $p > 1$. *Then left-hand inequality holds when* $p < 0$.

Proof. This is similar, using the identities $y_n = nd_n^{1/p}$ and $y_n = n^{1/p}(n+1)^{1/p^*}c_n$.

We now give an assortment of further applications of Theorems 1 and 2.

PROPOSITION 7 ([3; 6, Lemma 8] for $p > 0$). *For all real p, the expression*

$$\frac{1}{n-1}\sum_{r=1}^{n-1}\left(\frac{n-r}{r}\right)^p$$

increases with n.

Proof. The function

$$f(x) = \left(\frac{1-x}{x}\right)^p + \left(\frac{x}{1-x}\right)^p$$

is convex on $(0,1)$. Also $f(r/n) = f[(n-r)/n]$, so the given expression is $2A_n(f)$. (If $p \geq 1$, it is sufficient to take $f(x) = [(1-x)/x]^p$.)

The Dirichlet kernel

$$D_n(x) = 1 + 2\sum_{r=1}^{n}\cos rx$$

is a sum of the type we are considering. From the identity $D_n(x) = \sin(n+\frac{1}{2})x/\sin\frac{1}{2}x$, it is easily seen that $(1/2n)D_n(a/n) \to (\sin a)/a$ as $n \to \infty$. Our theorems give:

PROPOSITION 8. *Let $0 < a \leq \pi/2$. Then*

$$\frac{1}{2n-1}D_{n-1}\left(\frac{a}{n}\right) \text{ decreases with } n,$$

$$\frac{1}{2n+1}D_n\left(\frac{a}{n}\right) \text{ increases with } n.$$

Proof. The function $f(x) = \cos x$ is concave on $[-a, a]$. The two expressions given are $A_{2n}(f)$ and $B_{2n}(f)$ for this interval.

Note. We have $D_{n-1}(\pi/2n) = D_n(\pi/2n) = \cot(\pi/4n)$. By considering $\sin x$ on $[0,\pi]$, one finds that $(n-1)\tan(\pi/2n)$ increases with n and $(n+1)\tan(\pi/2n)$ decreases; however, it is easy to prove by elementary calculus that $n\tan(\pi/2n)$ decreases.

We finish this section with some more general consequences of Theorems 1 and 2. First, a result on weighted averages:

PROPOSITION 9. *Suppose that the function $xf(x)$ is convex on* $[0,1]$. *Let*

$$\alpha_n = \frac{1}{n(n+1)}\sum_{r=1}^{n} rf\left(\frac{r}{n+1}\right), \qquad \beta_n = \frac{1}{n(n+1)}\sum_{r=1}^{n} rf\left(\frac{r}{n}\right).$$

Then (α_n) *is increasing and* (β_n) *is decreasing. The opposite applies if* $xf(x)$ *is concave.*

Proof. Write $xf(x) = g(x)$. Then $\alpha_n = A_{n+1}(g)$ and $\beta_n = B_n(g)$.

For any continuous function f on $[0,1]$, it is clear that $A_n(f)$ and $B_n(f)$ tend to $\int_0^1 f$ as $n \to \infty$, since the end point Riemann sums $(1/n)\sum_{r=1}^n f(r/n)$ tend to the integral. Hence we have:

PROPOSITION 10. *If f is convex on* $[0,1]$, *then* $A_n(f) \le \int_0^1 f \le B_n(f)$. *The opposite inequalities hold if f is concave.*

It is easy to give a direct proof for $B_n(f)$, as follows. Let $T_n(f)$ be the trapezium estimate for the integral (the behaviour of $T_n(f)$ itself is considered below). For convex f, it is obvious that $T_n(f) \ge \int_0^1 f$. One verifies easily that

$$T_n(f) = \left(1 + \frac{1}{n}\right)B_n(f) - \frac{1}{n}B_1(f).$$

Since $B_n(f) \le B_1(f)$, we have $B_n(f) \ge T_n(f)$.

3. AVERAGES WITH ONE END POINT AND RIEMANN SUMS

In the light of Theorems 1 and 2, it is natural to consider averages including *one* end point. For monotonic functions, these averages coincide with upper or lower Riemann sums. Indeed, let $S_n(f)$ and $s_n(f)$ be the upper and lower Riemann sums for the integral $\int_0^1 f$ resulting from division into n equal subintervals (formed by taking the supremum and infimum of $f(x)$ on each subinterval). Clearly, if f is increasing, then

$$s_n(f) = \frac{1}{n}\sum_{r=0}^{n-1} f\left(\frac{r}{n}\right), \qquad S_n(f) = \frac{1}{n}\sum_{r=1}^{n} f\left(\frac{r}{n}\right)$$

(and vice versa for decreasing f). On the principle that the sums should give closer approximations to the integral as n increases, one might expect $s_n(f)$ to increase and $S_n(f)$ to decrease. When m divides into n, it is clear that $s_m(f) \le s_n(f)$ and $S_m(f) \ge S_n(f)$, since each subinterval for m is divided into an integer number of subintervals for n, with smaller suprema and larger infima. However, a simple example is enough to show that our supposition is not true in general, even for monotonic functions.

EXAMPLE 1. Let f be the increasing, piecewise linear function defined by

$$f(0) = f\left(\tfrac{1}{2}\right) = 0, \qquad f\left(\tfrac{2}{3}\right) = f(1) = 1.$$

Then

$$S_2(f) = \tfrac{1}{2}\big[f\big(\tfrac{1}{2}\big) + f(1)\big] = \tfrac{1}{2},$$
$$S_3(f) = \tfrac{1}{3}\big[f\big(\tfrac{1}{3}\big) + f\big(\tfrac{2}{3}\big) + f(1)\big] = \tfrac{2}{3}.$$

We will show that the expected statements hold when f is either convex or concave. When f is also monotonic, this follows very simply from Theorems 1 and 2, so we state this case first as a provisional form of the theorem.

THEOREM 3A. *Suppose that f is monotonic and either convex or concave on* $[0,1]$. *Then, with the above notation, $s_n(f)$ increases with n, and $S_n(f)$ decreases.*

Proof. We prove the statements for convex functions; those for concave functions then follow by considering $-f$. Further, we suppose that f is increasing: the decreasing case is then deduced by considering $f(1-x)$. Define $A_n(f)$ and $B_n(f)$ as above. If a constant is added to f, then the same constant is added to each Riemann sum. Hence we may assume that $f(0) = 0$. Then $A_n(f) \geq 0$ for $n \geq 2$, and

$$s_n(f) = \frac{1}{n}\sum_{r=0}^{n-1} f\left(\frac{r}{n}\right) = \frac{1}{n}\sum_{r=1}^{n-1} f\left(\frac{r}{n}\right) = \left(1 - \frac{1}{n}\right)A_n(f).$$

By Theorem 1, $A_n(f)$ is increasing. Hence $s_n(f)$ is increasing for $n \geq 2$ (strictly, unless f is constant). As remarked above, it is trivial that $s_1(f) \leq s_2(f)$. Further,

$$S_n(f) = \frac{1}{n}\sum_{r=1}^{n} f\left(\frac{r}{n}\right) = \frac{1}{n}\sum_{r=0}^{n} f\left(\frac{r}{n}\right) = \left(1 + \frac{1}{n}\right)B_n(f),$$

which is decreasing, since $B_n(f)$ is decreasing by Theorem 2.

Remark. A convex function may tend to infinity (but not $-\infty$) at 0 or 1, and $s_n(f)$ is still meaningful. Note that the above proof for $s_n(f)$ did not involve $f(1)$, so it remains valid in this case.

Note that Proposition 3 and Proposition 4 (for $p > 0$) are immediate consequences. Proposition 3 can be extended as follows:

PROPOSITION 11. *Fix $k \geq 1$, and let $E_{k,n} = \{j : (k-1)n < j \leq kn\}$. Then the expression*

$$\frac{1}{n^{p+1}} \sum_{r \in E_{k,n}} r^p$$

decreases with n when $p \geq 0$ and increases when $p < 0$.

Proof. Let $f(x) = x^p$ on the interval $[k-1, k]$. The given expression is $\frac{1}{n}\sum_{r \in E_{k,n}} f(r/n)$, which equals $S_n(f)$ when f is increasing, $s_n(f)$ when f is decreasing.

We now establish the full form of Theorem 3, dropping the condition that f is monotonic. The case of $s_n(f)$ with f convex is stated without proof in [8, Lemma 16.22].

THEOREM 3. *Suppose that f is either convex or concave on* $[0,1]$. *Then $s_n(f)$ increases with n, and $S_n(f)$ decreases.*

Proof. Again it is sufficient to prove the statements for convex f. We consider $s_n(f)$ first: a slight modification of the proof of Theorem 3A is enough to deal with this case. Let f be a convex function on $[0,1]$ that is not monotonic. Then f attains its least value on $[0,1]$ at some point c and is decreasing on $[0,c]$, increasing on $[c,1]$. By adding a constant, we may assume that $f(c) = 0$. Write $I_r = [(r-1)/n, r/n]$. Let k be such that $c \in I_k$. The least value of $f(x)$ on I_r is $f(r/n)$ for $r \le k-1$, 0 for $r = k$, and $f[(r-1)/n]$ for $r \ge k+1$. Hence

$$s_n(f) = \frac{1}{n}\sum_{r=1}^{n-1} f\left(\frac{r}{n}\right) = \left(1 - \frac{1}{n}\right) A_n(f),$$

as in Theorem 3A.

The proof for $S_n(f)$ is a good deal harder. Write $f(r/n) = a_r$. Since the maximum of a convex function on an interval occurs at an end point, we have

$$S_n(f) = \frac{1}{n}\sum_{r=0}^{n-1} \max(a_r, a_{r+1}).$$

Now $2\max(a,b) = (a+b) + |a-b|$ for any a, b, hence

$$2nS_n(f) = \sum_{r=0}^{n-1} (a_r + a_{r+1} + |a_r - a_{r+1}|).$$

As in Theorem 2, we have $f[r/(n+1)] \le u_r$, where

$$u_r = \frac{r}{n+1}a_{r-1} + \frac{n-r+1}{n+1}a_r = \frac{r}{n+1}(a_{r-1} - a_r) + a_r$$

(also, $u_0 = a_0$ and $u_{n+1} = a_n$). Hence $S_{n+1}(f) \le S'_{n+1}(f)$, where

$$2(n+1)S'_{n+1}(f) = \sum_{r=0}^{n} (u_r + u_{r+1} + |u_r - u_{r+1}|).$$

We will show that $S'_{n+1}(f) \leq S_n(f)$. Now

$$(n+1)\sum_{r=1}^{n} u_r = \sum_{r=0}^{n-1} a_r - na_n + (n+1)\sum_{r=1}^{n} a_r$$
$$= (a_0 + a_n) + (n+2)\sum_{r=1}^{n-1} a_r.$$

Hence

$$(n+1)\sum_{r=0}^{n} (u_r + u_{r+1}) = (n+1)(u_0 + u_{n+1}) + 2(n+1)\sum_{r=1}^{n} u_r$$
$$= (n+3)(a_0 + a_n) + 2(n+2)\sum_{r=1}^{n-1} a_r.$$

Also, for $0 \leq r \leq n$,

$$(n+1)(u_r - u_{r+1}) = r(a_{r-1} - a_r) + (n-r)(a_r - a_{r+1}),$$

from which we deduce that

$$\sum_{r=0}^{n} |u_r - u_{r+1}| \leq \sum_{r=0}^{n-1} |a_r - a_{r+1}|.$$

Hence

$$2(n+1)^2 S'_{n+1}(f) \leq (n+3)(a_0 + a_n) + 2(n+2)\sum_{r=1}^{n-1} a_r$$
$$+ (n+1)\sum_{r=0}^{n-1} |a_r - a_{r+1}|,$$

so we have

$$2n(n+1)^2 [S_n(f) - S'_{n+1}(f)] \geq B - A,$$

where

$$A = (n-1)(a_0 + a_n) - 2\sum_{r=1}^{n-1} a_r,$$
$$B = (n+1)\sum_{r=0}^{n-1} |a_r - a_{r+1}|.$$

But

$$A = (n-1)(a_0 - a_1) + (n-3)(a_1 - a_2) + \cdots - (n-1)(a_{n-1} - a_n),$$

so $A \leq B$, as required.

Further Remarks on $s_n(f)$ and $S_n(f)$. These quantities differ from the others considered here in being non-linear functions of f, and in failing to equal the integral for functions of the form $ax + b$. From the expression for $s_n(f)$ given in the proof of Theorem 3, it is clear that $s_n(f) \le A_n(f)$ for convex functions. For convex f, it is also easily seen that

$$S_n(f) = \frac{1}{n}\sum_{r=0}^{n} f\left(\frac{r}{n}\right) - \frac{1}{n} f\left(\frac{p}{n}\right),$$

where $f(p/n)$ is the smallest of the numbers $f(r/n)$. It follows that $S_n(f) \ge B_n(f)$ for such f. An alternative proof of Theorem 3 (for $S_n(f)$) is possible along the following lines: construct a convex function h_n whose values at the points $r/(n-1)$ (for $0 \le r \le n-1$) are the values $f(r/n)$ excluding $f(p/n)$. Verify that $h_{n+1} \le h_n$ and apply Theorem 2. However, the details are quite lengthy.

4. MID-POINT RIEMANN SUMS AND THE TRAPEZIUM ESTIMATE

When $[0,1]$ is divided into n equal subintervals, the mid-point Riemann sum for $\int_0^1 f$ is

$$M_n(f) = \frac{1}{n}\sum_{r=1}^{n} f\left(\frac{2r-1}{2n}\right),$$

and the trapezium estimate is

$$T_n(f) = \frac{1}{n}\left[\tfrac{1}{2}f(0) + \sum_{r=1}^{n-1} f\left(\frac{r}{n}\right) + \tfrac{1}{2}f(1)\right].$$

The quantity $M_n(f)$ describes the area formed by taking tangents to the curve at the points $(r - \frac{1}{2})/n$, while $T_n(f)$ is the area formed by taking straight-line approximations between the points r/n. As approximations to the integral, both estimates are exact for linear functions, and dramatically better than upper or lower Riemann sums for "well-behaved" functions generally. For a *convex* function f, it is geometrically obvious that $M_n(f) \le \int_0^1 f \le T_n(f)$, and one might expect $M_n(f)$ to increase and $T_n(f)$ to decrease with n. It is again easy to see that this is true for pairs m, n such that m divides into n. As before, a simple example shows that it is not true in general.

EXAMPLE 2. Consider the convex function $f(x) = |x|$ on $[-1, 1]$. Clearly,

$$M_2(f) = T_2(f) = 1,$$

while

$$M_3(f) = 2 \times \tfrac{2}{3} \times \tfrac{2}{3} = \tfrac{8}{9},$$

$$T_3(f) = \tfrac{2}{3}\left(\tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{3} + \tfrac{1}{2}\right) = \tfrac{10}{9}.$$

Note that the derivative of $|x|$ fails, in the most extreme way possible, to be either convex or concave. We will show that the expected conclusions hold when f', as well as f, is convex or concave. The proof will be achieved by expressing $M_n(f)$ and $T_n(f)$ as integrals of f' and using the following lemma on convex functions, which is roughly an integral form of convexity.

LEMMA 1. *Suppose that $a < b < c < d$ and that p, q, r are non-negative numbers. Suppose further that*

$$b - a \le d - c \qquad \text{and} \qquad p \le r, \tag{1}$$

$$q(c - b) = p(b - a) + r(d - c). \tag{2}$$

Let g be a convex function on $[a, d]$, with $g(c) \ge g(b)$. Then

$$q\int_b^c g \le p\int_a^b g + r\int_c^d g.$$

Note. The point of condition (2) is to give the correct weighting to the intervals: it implies equality in the case when g is constant.

Proof. Let h be the linear function agreeing with g at b and c. Since g is convex, we have

$$h(x) \ge g(x) \qquad \text{for } b \le x \le c,$$

$$h(x) \le g(x) \qquad \text{for } a \le b \text{ and } c \le x \le d.$$

Hence it is sufficient to prove the statement for h. Now $h(x) = mx + n$, where $m \ge 0$, since $g(c) \ge g(b)$. By the note above, it is sufficient to consider the case $h(x) = x$. Substituting for $q(c - b)$ by (2), we have

$$\begin{aligned}
&2p\int_a^b x\,dx - 2q\int_b^c x\,dx + 2r\int_c^d x\,dx \\
&\quad = p(b^2 - a^2) - q(c^2 - b^2) + r(d^2 - c^2) \\
&\quad = p(b^2 - a^2) - (c + b)[p(b - a) + r(d - c)] + r(d^2 - c^2) \\
&\quad = r(d - c)(d - b) - p(b - a)(c - a) \\
&\quad \ge 0,
\end{aligned}$$

since $r \ge p$, $d - c \ge b - a$ and $d - b \ge c - a$ (clearly, these conditions could be relaxed: what we really need is the final inequality in the proof).

Throughout the following, f will be a function having a derivative that is either convex or concave. We consider $M_n(f)$ first.

LEMMA 2. *Write*

$$F_{0,n} = \left[0, \frac{1}{2n}\right], \qquad F_{r,n} = \left[\frac{2r-1}{2n}, \frac{2r+1}{2n}\right] \quad \text{for } 1 \le r \le n-1,$$

$$J_{r,n} = \int_{F_{r,n}} f'.$$

Then

$$M_n(f) - f(0) = \sum_{r=0}^{n-1} \left(1 - \frac{r}{n}\right) J_{r,n}.$$

Proof. We have

$$\begin{aligned} M_n(f) - f(0) &= \frac{1}{n} \sum_{r=1}^{n} \left[f\left(\frac{2r-1}{2n}\right) - f(0) \right] \\ &= \frac{1}{n} \sum_{r=1}^{n} (J_{0,n} + J_{1,n} + \cdots + J_{r-1,n}) \\ &= \frac{1}{n} \sum_{r=0}^{n-1} (n-r) J_{r,n}. \end{aligned}$$

THEOREM 4. *Suppose that f' is either convex or concave on* $[0,1]$. *If f is convex, then $M_n(f)$ increases with n; if f is concave, then $M_n(f)$ decreases with n.*

Proof. We prove the statements for convex f: those for concave f then follow by considering $-f$. Further, once the result for convex f' is known, we can derive the result for concave f' by considering $g(x) = f(1-x)$, for then $M_n(g) = M_n(f)$ and $g'(x) = -f'(1-x)$, so that g' is convex. So we suppose henceforth that f and f' are convex.

Fix n. Divide $F_{r,n}$ into two subintervals as follows:

$$C_r = F_{r,n} \cap F_{r,n+1} = \left[\frac{2r-1}{2n}, \frac{2r+1}{2(n+1)}\right] \quad \text{for } r \ge 1,$$

$$C_0 = \left[0, \frac{1}{2(n+1)}\right],$$

$$D_r = F_{r,n} \cap F_{r+1,n+1} = \left[\frac{2r+1}{2(n+1)}, \frac{2r+1}{2n}\right].$$

Denote by $\ell(A)$ the length of an interval A and write $n(n+1) = N$. Then for $r \geq 1$,

$$\ell(C_r) = \frac{1}{2N}[n(2r+1) - (n+1)(2r-1)] = \frac{2n-2r+1}{2N},$$

$$\ell(D_r) = \frac{2r+1}{2N}.$$

By Lemma 2,

$$M_{n+1}(f) - M_n(f) = \sum_{r=0}^{n-1} c_r \int_{C_r} f' + \sum_{r=0}^{n-1} d_r \int_{D_r} f',$$

where

$$c_r = \left(1 - \frac{r}{n+1}\right) - \left(1 - \frac{r}{n}\right) = \frac{r}{N},$$

$$d_r = \left(1 - \frac{r+1}{n+1}\right) - \left(1 - \frac{r}{n}\right) = -\frac{n-r}{N}.$$

In order to apply Lemma 1, we partition C_r into two intervals C'_r and C''_r with respective lengths $(n-r+1)/2N$ and $(n-r)/2N$ (the left-hand subinterval is C'_r). Also, take $C''_0 = C_0$. Clearly,

$$M_{n+1}(f) - M_n(f) = \frac{1}{N} \sum_{r=0}^{n-1} V_r(f'),$$

where for each $r \geq 0$,

$$V_r(f') = r \int_{C''_r} f' - (n-r) \int_{D_r} f' + (r+1) \int_{C'_{r+1}} f'.$$

The conditions of Lemma 1 are satisfied by $V_r(f')$, since f' is increasing and

$$\ell(C''_r) = \ell(C'_{r+1}) = \frac{n-r}{2N},$$

so that

$$r\ell(C''_r) + (r+1)\ell(C'_{r+1}) = \frac{1}{2N}(n-r)(2r+1) = (n-r)\ell(D_r).$$

Hence $V_r(f') \geq 0$ for each r, so $M_{n+1}(f) \geq M_n(f)$.

The choice $f(x) = x^p$ gives the following statement, which will be applied again to solve the power majorization problem.

PROPOSITION 12. *The expression*

$$\frac{1}{n^{p+1}} \sum_{r=1}^{n} (2r-1)^p$$

increases with n if $p \geq 1$ *or* $p \leq 0$, *and decreases with n if* $0 \leq p \leq 1$.

Proof. The function f is convex when $p \geq 1$ or $p \leq 0$, concave otherwise. Also, f' is either convex or concave in all cases.

We now turn to $T_n(f)$. The strategy is similar, but the details are different.

LEMMA 3. *Write*

$$E_{r,n} = \left[\frac{r-1}{n}, \frac{r}{n}\right], \qquad I_{r,n} = \int_{E_{r,n}} f'.$$

Then

$$T_n(f) - f(0) = \sum_{r=1}^{n} \left(1 - \frac{2r-1}{2n}\right) I_{r,n}.$$

Proof. Since

$$f\left(\frac{r}{n}\right) - f(0) = \int_0^{r/n} f' = I_{1,n} + \cdots + I_{r,n},$$

we have

$$\begin{aligned} T_n(f) - f(0) &= \frac{1}{n} \sum_{r=1}^{n-1} \left[f\left(\frac{r}{n}\right) - f(0) \right] + \frac{1}{2n} [f(1) - f(0)] \\ &= \frac{1}{n} \sum_{r=1}^{n-1} (I_{1,n} + \cdots + I_{r,n}) + \frac{1}{2n} (I_{1,n} + \cdots + I_{n,n}) \\ &= \frac{1}{n} \sum_{r=1}^{n} \left(n - r + \tfrac{1}{2}\right) I_{r,n}. \end{aligned}$$

THEOREM 5. *Suppose that* f' *is either convex or concave on* $[0,1]$. *If* f *is convex, then* $T_n(f)$ *decreases with n; if* f *is concave, then* $T_n(f)$ *increases with n.*

Proof. As before, it is sufficient to consider the case when f and f' are convex. Fix n. Divide $E_{r,n}$ into two subintervals, as follows. Let

$$A_r = E_{r,n} \cap E_{r,n+1} = \left[\frac{r-1}{n}, \frac{r}{n+1}\right],$$

$$B_r = E_{r,n} \cap E_{r+1,n=1} = \left[\frac{r}{n+1}, \frac{r}{n}\right].$$

Write $n(n+1) = N$. Then

$$\ell(A_r) = \frac{1}{N}[rn - (r-1)(n+1)] = \frac{n-r+1}{N},$$

$$\ell(B_r) = \frac{r}{N}.$$

By Lemma 3,

$$T_n(f) - T_{n+1}(f) = \sum_{r=1}^{n} a_r \int_{A_r} f' + \sum_{r=1}^{n} b_r \int_{B_r} f',$$

where

$$a_r = \left(1 - \frac{2r-1}{2n}\right) - \left(1 - \frac{2r-1}{2(n+1)}\right) = -\frac{2r-1}{2N},$$

$$b_r = \left(1 - \frac{2r-1}{2n}\right) - \left(1 - \frac{2r+1}{2(n+1)}\right) = \frac{2n-2r+1}{2N}.$$

This time, we partition the functions instead of the intervals. We claim that

$$T_n(f) - T_{n+1}(f) = \frac{1}{2N} \sum_{r=1}^{n} W_r(f'),$$

where

$$W_r(f') = (n-r+1)\int_{B_{r-1}} f' - (2r-1)\int_{A_r} f' + (n-r+1)\int_{B_r} f'$$

(the left-hand term in $W_1(f')$ is taken to be 0). Indeed, when $W_{r+1}(f')$ is added to $W_r(f')$ (where $1 \le r \le n-1$), we obtain the correct multiple $(2n-2r+1)f'$ on B_r, while $W_n(f')$ gives the required multiple $1.f'$ on B_n.

Condition (2) of Lemma 1 is satisfied by $W_r(f')$, since

$$(n-r+1)\left[\ell(B_{r-1})+\ell(B_r)\right] = (n-r+1)\frac{2r-1}{N}$$
$$= (2r-1)\ell(A_r).$$

(This is also valid for $r = 1$, with $\ell(B_0) = 0$.) Hence $W_r(f') \geq 0$ for each r, so that $T_n(f) \geq T_{n+1}(f)$.

In general, $T_n(f)$ leads to awkward expressions because of the half values at the end points. However, these disappear if $f(0) = f(1) = 0$. For such a function, write $S_n = \sum_{r=1}^{n-1} f(r/n)$. If f is concave, then by Theorems 1 and 2, $S_n/(n-1)$ is decreasing and $S_n/(n+1)$ is increasing. Theorem 5 improves the second statement to S_n/n is increasing. An example of this situation is given in the next result.

PROPOSITION 13. *Let* $0 < p \leq 1$. *Then*

$$\frac{1}{n^{p+2}} \sum_{r=1}^{n-1} r^p(n-r)$$

increases with n.

Proof. Let $f(x) = x^p - x^{p+1}$. Then f is concave, since x^p is concave and x^{p+1} is convex. Similarly, f' is convex. Also,

$$T_n(f) = \frac{1}{n}\sum_{r=1}^{n-1} \frac{r^p}{n^p}\,\frac{n-r}{n} = \frac{1}{n^{p+2}} \sum_{r=1}^{n-1} r^p(n-r).$$

Remark. When $p \leq 0$, the statement in Proposition 13 (in fact, a stronger statement) follows from Theorem 1. With S_n defined as above, the expression in Proposition 13 is S_n/n. It is easily checked that f is convex on $(0, 1)$, so Theorem 1 shows that $S_n/(n-1)$ is increasing. When $p > 1$, f fails to be either convex or concave on $[0, 1]$, and we leave it as an open question whether the expression in Proposition 13 is always increasing (it is, for example, when $p = 2$).

When applied to the Dirichlet kernel $D_n(x)$ (see Proposition 8), Theorem 5 shows that $n\tan(\pi/2n)$ is decreasing; however, this is easy to show by elementary means.

Further Notes. The following easily checked identities relate $T_n(f)$ to $A_n(f)$ and $B_n(f)$:

$$T_n(f) = \left(1+\frac{1}{n}\right)B_n(f) - \frac{1}{n}B_1(f) = \frac{n-1}{2n}A_n(f) + \frac{n+1}{2n}B_n(f).$$

Since $A_n(f)$ and $B_n(f)$ pull in opposite directions, it is not surprising that the results of this section need a more delicate analysis than Theorems 1 and 2. Some expressions for $M_n(f)$ in terms of the other quantities are

$$M_n(f) = 2T_{2n}(f) - T_n(f) = \frac{2n+1}{n}B_{2n}(f) - \frac{n+1}{n}B_n(f).$$

5. APPLICATION TO AN INFINITE SERIES

We show how Theorem 3A applies to an infinite series considered in [11]. Fix α, β such that $0 \leq \alpha < 1$ and $\alpha + \beta > 1$. Let

$$v_n(\alpha, \beta) = \sum_{j=1}^{\infty} \frac{1}{j^{\alpha}(j+n)^{\beta}}.$$

Note that $v_n(0, \beta) = \sum_{k=n+1}^{\infty} 1/k^{\beta}$, the tail of the zeta function. We have

$$v_n(\alpha, \beta) \leq \int_0^{\infty} \frac{1}{x^{\alpha}(x+n)^{\beta}}\, dx = \frac{1}{n^{\alpha+\beta-1}} I,$$

where

$$I = \int_0^{\infty} \frac{1}{x^{\alpha}(x+1)^{\beta}}\, dx = B(1-\alpha, \alpha+\beta-1)$$
$$= \frac{1}{\Gamma(\beta)}\Gamma(1-\alpha)\Gamma(\alpha+\beta-1).$$

(If $\alpha = 0$, then $I = 1/(\beta - 1)$, and if $\beta = 1$, then $I = \pi/\sin \alpha\pi$.) Also,

$$v_n(\alpha, \beta) \geq \int_1^{\infty} \frac{1}{x^{\alpha}(x+n)^{\beta}}\, dx$$

and

$$\int_0^1 \frac{1}{x^{\alpha}(x+n)^{\beta}}\, dx \leq \frac{1}{(1-\alpha)n^{\beta}},$$

from which it is clear that $n^{\alpha+\beta-1}(\alpha, \beta) \to I$ as $n \to \infty$. In [11], the problem was to find the infimum and supremum of $n^{\alpha+\beta-1}v_n(\alpha, \beta)$ for the cases $\alpha = 0$ and $\beta = 1$; these determine the norm and "lower bound" (in a certain sense) of the Cesaro and Hilbert operators on the Lorentz sequence space $d(w, 1)$, where $w_n = 1/n^{\alpha}$. In the case $\alpha = 0$, it was shown

(Proposition 6) that the sequence is increasing (cf. also [8, Remark 4.10]). For the case $\beta = 1$, without showing the sequence to be increasing, it was established that the infimum is $v_1(\alpha, \beta)$ and the supremum is I.

PROPOSITION 14. *With $v_n(\alpha, \beta)$ defined as above, $n^{\alpha+\beta-1}v_n(\alpha, \beta)$ increases with n.*

Proof. Note first that f is convex on $\mathbb{R}^+$, where

$$f(x) = \frac{1}{x^\alpha(1+x)^\beta},$$

since

$$f'(x) = -\frac{\alpha}{x^{\alpha+1}(1+x)^\beta} - \frac{\beta}{x^\alpha(1+x)^{\beta+1}}.$$

Let $E_{k,n} = \{j \in \mathbb{Z} : (k-1)n < j \le kn\}$. Then $n^{\alpha+\beta-1}v_n(\alpha, \beta) = \sum_{k=1}^\infty a_{k,n}$, where

$$\begin{aligned} a_{k,n} &= \sum_{j \in E_{k,n}} \frac{n^{\alpha+\beta-1}}{j^\alpha(j+n)^\beta} \\ &= \frac{1}{n} \sum_{j \in E_{k,n}} \frac{1}{(j/n)^\alpha(1+j/n)^\beta} \\ &= \frac{1}{n} \sum_{j \in E_{k,n}} f\left(\frac{j}{n}\right) \\ &= \frac{1}{n} \sum_{i=1}^n f\left(k-1+\frac{i}{n}\right) \end{aligned}$$

(substitute $i = j-(k-1)n$). Now f is decreasing, so by Theorem 3A, $a_{k,n}$ increases with n for fixed k. The statement follows.

Note. When $\alpha \ge 1$, the problem is trivial, because $n^{\alpha+\beta-1}/(j+n)^\beta$ increases with n for each fixed j.

6. APPLICATION TO THE POWER MAJORIZATION PROBLEM

Let $x = (x_1, x_2, \dots, x_n)$ and $y = (y_1, y_2, \dots, y_n)$ be finite sequences. Then x is said to be *majorized* by y if for all convex functions f, we have

$$\sum_{j=1}^n f(x_j) \le \sum_{j=1}^n f(y_j).$$

We write $x \leq_{maj} y$ if this occurs. The majorization principle, also known as Karamata's inequality, states that if (x_j) and (y_j) are *decreasing*, then $x \leq_{maj} y$ is equivalent to

$$x_1 + \cdots + x_j \leq y_1 + \cdots + y_j \qquad \text{for } 1 \leq j \leq n-1,$$
$$x_1 + \cdots + x_n = y_1 + \cdots + y_n.$$

(For a simple proof, see [4, Sect. 1.30]; cf. also [10, 12].)

The sequence (x_j) is said to be *power majorized* by (y_j) if the above holds for all functions of the form $f(x) = x^p$, that is,

$$\sum_{j=1}^{n} x_j^p \leq \sum_{j=1}^{n} y_j^p \qquad \text{for } p \geq 1 \text{ and } p \leq 0,$$

$$\sum_{j=1}^{n} x_j^p \geq \sum_{j=1}^{n} y_j^p \qquad \text{for } 0 < p < 1.$$

Write $x \leq_{\pi} y$ if this occurs. In answer to a question of A. Clausing [9], examples have been given in [1, 5] to show that power majorization does not imply majorization (although it does so for vectors of length not more than 3). However, these examples are specially constructed and rather elaborate. Our Theorem 4 provides a "natural" example of the type required.

PROPOSITION 15. *Let*

$$x = (9,9,9,3,3,3), \qquad y = (10,10,6,6,2,2).$$

Then $x \leq_{\pi} y$, but x is not majorized by y.

Proof. The third partial sum shows that x is not majorized by y. By Proposition 12, for $p \geq 1$ and $p \leq 0$,

$$\frac{1}{2.2^p}(1^p + 3^p) \leq \frac{1}{3.3^p}(1^p + 3^p + 5^p),$$

hence

$$3(9^p + 3^p) \leq 2(10^p + 6^p + 2^p),$$

with the reverse inequality holding for $0 < p < 1$. In other words, $x \leq_{\pi} y$.

Note. Using Theorem 5 instead of Theorem 4, one obtains another such example:

$$x = (10,10,10,10,8,8,8,8), \qquad y = (12,9,9,9,9,9,9,6).$$

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