

考虑函数

$$g(x) = x^2 - x \left(\frac{x - \ln x + \ln(x - \ln x)}{2} + \sqrt{2 + \frac{(x - \ln x + \ln(x - \ln x))^2}{4}} \right).$$

令

$$A = x - \ln x + \ln(x - \ln x).$$

则

$$g(x) = x^2 - \frac{x}{2} (A + \sqrt{A^2 + 8}).$$

渐近展开： $x \rightarrow \infty$

首先展开 A 。记 $t = \frac{\ln x}{x}$ ，则 $t \rightarrow 0$ 。我们有

$$x - \ln x = x(1 - t), \quad \ln(x - \ln x) = \ln x + \ln(1 - t).$$

所以

$$A = x - \ln x + \ln x + \ln(1 - t) = x + \ln(1 - t).$$

对 $\ln(1 - t)$ 进行泰勒展开：

$$\ln(1 - t) = -t - \frac{t^2}{2} - \frac{t^3}{3} - \frac{t^4}{4} - \frac{t^5}{5} + O(t^6).$$

代入 $t = \frac{\ln x}{x}$ 得

$$A = x - \frac{\ln x}{x} - \frac{\ln^2 x}{2x^2} - \frac{\ln^3 x}{3x^3} - \frac{\ln^4 x}{4x^4} - \frac{\ln^5 x}{5x^5} + O\left(\frac{\ln^6 x}{x^5}\right).$$

其次，展开 $\sqrt{A^2 + 8}$ 。由于 $A \sim x$ ，有

$$\sqrt{A^2 + 8} = A \sqrt{1 + \frac{8}{A^2}} = A \left(1 + \frac{4}{A^2} - \frac{8}{A^4} + \frac{32}{A^6} + O\left(\frac{1}{A^8}\right) \right) = A + \frac{4}{A} - \frac{8}{A^3} + \frac{32}{A^5} + O\left(\frac{1}{A^7}\right).$$

因此

$$A + \sqrt{A^2 + 8} = 2A + \frac{4}{A} - \frac{8}{A^3} + \frac{32}{A^5} + O\left(\frac{1}{A^7}\right).$$

代入 $g(x)$ ：

$$g(x) = x^2 - \frac{x}{2} \left(2A + \frac{4}{A} - \frac{8}{A^3} + \frac{32}{A^5} + \cdots \right) = x^2 - xA - \frac{2x}{A} + \frac{4x}{A^3} - \frac{16x}{A^5} + \cdots.$$

现在逐项处理。

1. 计算 xA

$$xA = x \left(x - \frac{\ln x}{x} - \frac{\ln^2 x}{2x^2} - \frac{\ln^3 x}{3x^3} - \frac{\ln^4 x}{4x^4} - \frac{\ln^5 x}{5x^5} + \cdots \right) = x^2 - \ln x - \frac{\ln^2 x}{2x} - \frac{\ln^3 x}{3x^2} - \frac{\ln^4 x}{4x^3} - \frac{\ln^5 x}{5x^4} + O\left(\frac{\ln^6 x}{x^5}\right)$$

所以

$$x^2 - xA = \ln x + \frac{\ln^2 x}{2x} + \frac{\ln^3 x}{3x^2} + \frac{\ln^4 x}{4x^3} + \frac{\ln^5 x}{5x^4} + O\left(\frac{\ln^6 x}{x^5}\right).$$

2. 计算 $-\frac{2x}{A}$

令 $\delta = \frac{\ln x}{x} + \frac{\ln^2 x}{2x^2} + \frac{\ln^3 x}{3x^3} + \cdots$, 则 $A = x - \delta$ 。于是

$$\frac{1}{A} = \frac{1}{x - \delta} = \frac{1}{x} \cdot \frac{1}{1 - \delta/x} = \frac{1}{x} \left(1 + \frac{\delta}{x} + \frac{\delta^2}{x^2} + \frac{\delta^3}{x^3} + \cdots \right).$$

计算

$$\frac{\delta}{x} = \frac{\ln x}{x^2} + \frac{\ln^2 x}{2x^3} + \frac{\ln^3 x}{3x^4} + \cdots, \quad \frac{\delta^2}{x^2} = \frac{\ln^2 x}{x^4} + O\left(\frac{\ln^3 x}{x^5}\right).$$

因此

$$-\frac{2x}{A} = -2 \left(1 + \frac{\delta}{x} + \frac{\delta^2}{x^2} + \cdots \right) = -2 - \frac{2 \ln x}{x^2} + O\left(\frac{\ln^2 x}{x^3}\right).$$

3. 计算 $\frac{4x}{A^3}$

$$\frac{1}{A^3} = \frac{1}{x^3} \left(1 + 3\frac{\delta}{x} + 6\frac{\delta^2}{x^2} + \cdots \right),$$

所以

$$\frac{4x}{A^3} = \frac{4}{x^2} \left(1 + 3\frac{\delta}{x} + 6\frac{\delta^2}{x^2} + \cdots \right) = \frac{4}{x^2} + O\left(\frac{\ln x}{x^4}\right).$$

4. 更高阶项

项 $-\frac{16x}{A^5}$ 及更高阶项至少为 $O(x^{-4})$, 可忽略。

5. 合并

将以上结果代入：

$$\begin{aligned} g(x) &= \left(\ln x + \frac{\ln^2 x}{2x} + \frac{\ln^3 x}{3x^2} + \frac{\ln^4 x}{4x^3} + \cdots \right) \\ &\quad + \left(-2 - \frac{2 \ln x}{x^2} \right) + \left(\frac{4}{x^2} \right) + O\left(\frac{\ln^4 x}{x^3}\right) \\ &= \ln x - 2 + \frac{\ln^2 x}{2x} + \frac{\ln^3 x}{3x^2} - \frac{2 \ln x}{x^2} + \frac{4}{x^2} + O\left(\frac{\ln^4 x}{x^3}\right). \end{aligned}$$

整理 x^{-2} 项得

$$g(x) = \ln x - 2 + \frac{\ln^2 x}{2x} + \frac{1}{x^2} \left(\frac{\ln^3 x}{3} - 2 \ln x + 4 \right) + O\left(\frac{\ln^4 x}{x^3}\right).$$

因此，当 $x \rightarrow \infty$ 时， $g(x)$ 的渐近展开到 $O\left(\frac{\ln^3 x}{x^2}\right)$ 为

$$g(x) \sim \ln x - 2 + \frac{\ln^2 x}{2x} + \frac{\ln^3 x}{3x^2} - \frac{2 \ln x}{x^2} + \frac{4}{x^2}.$$

或等价地，

$$g(x) = \ln x - 2 + \frac{\ln^2 x}{2x} + \frac{1}{x^2} \left(\frac{\ln^3 x}{3} - 2 \ln x + 4 \right) + O\left(\frac{\ln^4 x}{x^3}\right).$$