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SCHUR CONVEXITY AND CONCAVITY OF GNAN MEAN

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ABSTRACT. In this paper, the Schur convexity and Schur concavity of the Gnan mean and its dual form in two variables are discussed using strong mathematical induction by grouping of terms.

1. INTRODUCTION

For positive numbers a, b, let

$$I = I(a,b) = \begin{cases} \exp\left[\dfrac{b\ln b - a\ln a}{b-a} - 1\right], & a < b; \\ a, & a = b; \end{cases} \tag{1.1}$$

$$L = L(a,b) = \begin{cases} \dfrac{a-b}{\ln a - \ln b}, & a \neq b; \\ a, & a = b; \end{cases} \tag{1.2}$$

$$H = H(a,b) = \frac{a + \sqrt{ab} + b}{3}. \tag{1.3}$$

These are respectively called the Identric, Logarithmic and Heron means.

In [?, ?, ?], V. Lokesha et al. studied extensively and obtained some remarkable results on the weighted Heron mean, the weighted Heron dual mean and the weighted product type means and its monotonicities.

In [?, ?], Zhang et al. gave the generalizations of Heron mean, similar product type means and their dual forms. For two variables, the above means are as follows:
(1.4)

$$I(a,b;k) = \prod_{i=1}^{k}\left(\frac{(k+1-i)a + ib}{k+1}\right)^{\frac{1}{k}}, \quad I^*(a,b;k) = \prod_{i=0}^{k}\left(\frac{(k-i)a + ib}{k}\right)^{\frac{1}{k+1}},$$

and

$$H(a,b;k) = \frac{1}{k+1}\sum_{i=0}^{k} a^{\frac{k-i}{k}} b^{\frac{i}{k}}, \quad h(a,b;k) = \frac{1}{k}\sum_{i=1}^{k} a^{\frac{k+1-i}{k+1}} b^{\frac{i}{k+1}}, \tag{1.5}$$

where k is a natural number. Authors have proved that $H(a,b;k)$ and $I^*(a,b;k)$ are monotonic decreasing functions and $h(a,b;k)$ and $I(a,b;k)$ are monotonic increasing functions with k and also established the following limiting values of these means.

$$\lim_{k\to+\infty} I(a,b;k) = \lim_{k\to+\infty} I^*(a,b;k) = I(a,b),$$

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and

$$\lim_{k\to+\infty} H(a,b;k) = \lim_{k\to+\infty} h(a,b;k) = L(a,b).$$

In [?, ?] V. Lokesha et al. defined the Gnan mean and its dual form for two variables and then generalized for n variables. Also, they obtained some interesting properties, monotonic results and its limitations. The definitions of Gnan mean and its dual form are given in next section. The Schur convex function was introduced by I Schur, in 1923 and it has many important applications in analytic inequalities . In 2003 X.M. Zhang proposed the concept of "Schur-Harmonically convex function" which is an extension of "Schur-Convexity function". The detailed discussion on convexity and Schur convexity can be found in ([?]-[?]).

2. Definitions and Lemmas

In this section, we recall the definitions and lemmas which are essential to develop this paper.

Definition 2.1. [?] *Let $a, b \geqslant 0$, k be a positive integer, and α, β be any two real numbers. The Gnan mean $G(a,b;k,\alpha,\beta)$ and its dual form $g(a,b;k,\alpha,\beta)$ are defined as follows;*

$$\begin{aligned} G(a,b;k,\alpha,\beta) &= \left[\frac{1}{k}\sum_{i=1}^{k}\left(\frac{(k+1-i)a^{\alpha}+ib^{\alpha}}{k+1}\right)^{\frac{\beta}{\alpha}}\right]^{\frac{1}{\beta}}, \\ G(a,b;k,0,\beta) &= \left[\frac{1}{k}\sum_{i=1}^{k} a^{\frac{(k+1-i)\beta}{k+1}} b^{\frac{i\beta}{k+1}}\right]^{\frac{1}{\beta}}, \\ G(a,b;k,\alpha,0) &= \prod_{i=1}^{k}\left(\frac{(k+1-i)a^{\alpha}+ib^{\alpha}}{k+1}\right)^{\frac{1}{k\alpha}}, \\ G(a,b;k,0,0) &= \sqrt{ab}; \end{aligned} \tag{2.1}$$

and

$$\begin{aligned} g(a,b;k,\alpha,\beta) &= \left[\frac{1}{k+1}\sum_{i=0}^{k}\left(\frac{(k-i)a^{\alpha}+ib^{\alpha}}{k}\right)^{\frac{\beta}{\alpha}}\right]^{\frac{1}{\beta}}, \\ g(a,b;k,0,\beta) &= \left[\frac{1}{k+1}\sum_{i=0}^{k} a^{\frac{(k-i)\beta}{k}} b^{\frac{i\beta}{k}}\right]^{\frac{1}{\beta}}, \\ g(a,b;k,\alpha,0) &= \prod_{i=0}^{k}\left(\frac{(k-i)a^{\alpha}+ib^{\alpha}}{k}\right)^{\frac{1}{(k+1)\alpha}}, \\ g(a,b;k,0,0) &= \sqrt{ab}. \end{aligned} \tag{2.2}$$

Definition 2.2. [?, ?] *Let $x = (x_1, x_2, ..., x_n)$ and $y = (y_1, y_2, ..., y_n) \in R^n$*

(1) *x is majorized by y (in symbol $x \prec y$) If $\sum_{i=1}^{k} x_{[i]} \leq \sum_{i=1}^{k} y_{[i]}$, and $\sum_{i=1}^{n} x_{[i]} = \sum_{i=1}^{n} y_{[i]}$, where $x_{[1]} \geq, ..., \geq x_{[n]}$ and $y_{[1]} \geq, ..., \geq y_{[n]}$ are rearrangements of x and y in descending order.*

(2) $x \geq y$ means $x_i \geq y_i$ for all $i = 1, 2, ..., n$. Let $\Omega \in R^n (n \geq 2)$. The function $\varphi : \Omega \to R$ is said to be increasing if $x \geq y$ implies $\varphi(x) \geq \varphi(y)$. φ is said to be decreasing if and only if $-\varphi$ is increasing.
(3) $\Omega \subseteq R^n$ is called a convex set if $(\alpha x_1 + \beta y_1, ..., \alpha x_n + \beta y_n) \in \Omega$ for every x and $y \in \Omega$ where $\alpha, \beta \in [0, 1]$ with $\alpha + \beta = 1$.
(4) Let $\Omega \subseteq R^n$. The function $\varphi : \Omega \to R$ be said to be a schur convex function on Ω if $x \leq y$ on Ω implies $\varphi(x) \leq \varphi(y)$. φ is said to be a schur concave function on Ω if and only if $-\varphi$ is schur convex.

Definition 2.3. [?] *Let* $x = (x_1, x_2, ..., x_n)$ *and* $y = (y_1, y_2, ..., y_n) \in R^n_+$
$\Omega \subseteq R^n$ *is called Harmonically convex set if* $(x_1^\alpha y_1^\beta, ..., x_1^\alpha y_1^\beta) \in \Omega$ *for all* x *and* $y \in \Omega$ *where* $\alpha, \beta \in [0, 1]$ *with* $\alpha + \beta = 1$
Let $\Omega \subseteq R^n_+$. *The function* $\varphi : \Omega \to R_+$ *is said to be schur Harmonically convex function on* Ω *if* $(lnx_1, ..., ln_n) \prec (lny_1, ..., lny_n)$ *on* Ω *implies* $\varphi(x) \leq \varphi(y)$. φ *is said to be a schur Harmonically concave function on* Ω *if and only if* $-\varphi$ *is schur Harmonically convex.*

Definition 2.4. [?], [?] $\Omega \subseteq R^n$ *is called symmetric set if* $x \in \Omega$ *implies* $Px \in \Omega$ *for every* $n \times n$ *permutation matrix* P.
The function $\varphi : \Omega \to R$ *is called symmetric if for every permutation matrix* P, $\varphi(Px) = \varphi(x)$ *for all* $x \in \Omega$.

Lemma 2.1. [?] *Let* $\Omega \subseteq R^n$ *be symmetric with non empty interior set* Ω^0 *and let* $\varphi : \Omega \to R$ *be continuous on* Ω *and differentiable in* Ω^0. *Then* φ *is Schur convex(Schur concave) on* Ω *if and only if* φ *is symmetric on* Ω *and*

$$(x_1 - x_2)\left(\frac{\partial\varphi}{\partial x_1} - \frac{\partial\varphi}{\partial x_2}\right) \geq 0 (\leq 0). \tag{2.3}$$

holds for any $x = (x_1, x_2, ..., x_n) \in \Omega^0$.

3. Main Result

In this section, the Schur convexity and concavity of Gnan mean and it's dual for two variables are established by the method of strong mathematical induction[?] with grouping of terms.

Theorem 3.1. *Let* α, β, a *and* b *be real numbers, with* $a \geq b$, *and* k *be non- negative integer. Then*

(1) For $\alpha, \beta \neq 0$, the Gnan mean $G(a, b, k; \alpha, \beta)$ and its dual $g(a, b, k; \alpha, \beta)$ are Schur convex in a and b for $1 \leq \alpha \leq \beta$.
(2) For $\alpha = 0, \beta \neq 0$, the Gnan mean $G(a, b, k; 0, \beta)$ and its dual $g(a, b, k; 0, \beta)$ are Schur concave in a and b for $\beta \leq 0$.
(3) For $\alpha \neq 0, \beta = 0$, the Gnan mean $G(a, b, k; \alpha, 0)$ and its dual $g(a, b, k; \alpha, 0)$ are Schur concave in a and b for $\alpha \leq 1/2$.
(4) For $\alpha = 0$, $\beta = 0$, the Gnan mean $G(a, b, k; 0, 0)$ and its dual $g(a, b, k; 0, 0)$ are Schur concave in a and b.

Proof: The proof of (1) is established by discussing the following two cases.

Case(i). For $\alpha, \beta \neq 0$, $a \geq b$, non negative integer k, we have the Gnan mean,

$$G = G(a,b;k,\alpha,\beta) = \left[\frac{1}{k}\sum_{i=1}^{k}\left(\frac{(k+1-i)a^{\alpha}+ib^{\alpha}}{k+1}\right)^{\frac{\beta}{\alpha}}\right]^{\frac{1}{\beta}}. \tag{3.1}$$

Taking logarithm on both sides, differentiating partially w.r.t a and rearranging lead to

$$\frac{\partial G}{\partial a} = \frac{G^{1-\beta}}{k}\sum_{i=1}^{k}\left(\frac{(k+1-i)a^{\alpha}+ib^{\alpha}}{k+1}\right)^{\frac{\beta}{\alpha}-1}\frac{k+1-i}{k+1}a^{\alpha-1}. \tag{3.2}$$

Similarly,

$$\frac{\partial G}{\partial b} = \frac{G^{1-\beta}}{k}\sum_{i=1}^{k}\left(\frac{(k+1-i)a^{\alpha}+ib^{\alpha}}{k+1}\right)^{\frac{\beta}{\alpha}-1}\frac{i}{k+1}b^{\alpha-1}. \tag{3.3}$$

Consider,
(3.4)

$$(a-b)\left(\frac{\partial G}{\partial a}-\frac{\partial G}{\partial b}\right) = \frac{(a-b)}{k}G^{1-\beta}\sum_{i=1}^{k}\left(\frac{(k+1-i)a^{\alpha}+ib^{\alpha}}{k+1}\right)^{\frac{\beta}{\alpha}}\frac{(k+1-i)a^{\alpha-1}-ib^{\alpha-1}}{(k+1-i)a^{\alpha}+ib^{\alpha}}$$

$$(a-b)\left(\frac{\partial G}{\partial a}-\frac{\partial G}{\partial b}\right) = [\Delta]\,[\Theta]. \tag{3.5}$$

Where,

$$\Delta = \frac{(a-b)}{k}G^{1-\beta} \quad \text{and} \quad \Theta = \sum_{i=1}^{k}\left(\frac{(k+1-i)a^{\alpha}+ib^{\alpha}}{k+1}\right)^{\frac{\beta}{\alpha}}\frac{(k+1-i)a^{\alpha-1}-ib^{\alpha-1}}{(k+1-i)a^{\alpha}+ib^{\alpha}}.$$

Clearly $\Delta \geq 0$.

Now, we shall prove that $\Theta \geq 0$ for $1 \leq \alpha \leq \beta$, for all positive integral values of k, by strong mathematical induction.
For $k=1$,

$$\Theta = \left(\frac{a^{\alpha}+b^{\alpha}}{2}\right)^{\frac{\beta}{\alpha}-1}\left(\frac{a^{\alpha-1}-b^{\alpha-1}}{2}\right) \geq 0 \quad for \quad \alpha \geq 1.$$

For $k=2$,

$$\Theta = \left(\frac{2a^{\alpha}+b^{\alpha}}{3}\right)^{\frac{\beta}{\alpha}}\frac{2a^{\alpha-1}-b^{\alpha-1}}{2a^{\alpha}+b^{\alpha}} + \left(\frac{a^{\alpha}+2b^{\alpha}}{3}\right)^{\frac{\beta}{\alpha}}\frac{a^{\alpha-1}-2b^{\alpha-1}}{a^{\alpha}+2b^{\alpha}}$$

$$\Theta = \left(\frac{1}{3}\right)^{\frac{\beta}{\alpha}}\left[(2a^{\alpha}+b^{\alpha})^{\frac{\beta}{\alpha}-1}(2a^{\alpha-1}-b^{\alpha-1}) + (a^{\alpha}+2b^{\alpha})^{\frac{\beta}{\alpha}-1}(a^{\alpha-1}-2b^{\alpha-1})\right] \geq 0,$$

if,

$$\left(\frac{2a^{\alpha}+b^{\alpha}}{a^{\alpha}+2b^{\alpha}}\right)^{\frac{\beta}{\alpha}-1}\frac{2a^{\alpha-1}-b^{\alpha-1}}{2b^{\alpha-1}-a^{\alpha-1}} \geq 1,$$

that is, if

$$\left(\frac{2a^{\alpha}+b^{\alpha}}{a^{\alpha}+2b^{\alpha}}\right)^{\frac{\beta}{\alpha}-1} \geq 1, \quad \frac{2a^{\alpha-1}-b^{\alpha-1}}{2b^{\alpha-1}-a^{\alpha-1}} \geq 1,$$

that is, if

$$\beta \geq \alpha \quad and \quad \alpha \geq 1.$$

Hence,

$$\Theta \geq 0 \quad \text{for} \quad 1 \leq \alpha \leq \beta.$$

For $k = 3$,

$$\Theta = \left(\frac{3a^\alpha + b^\alpha}{4}\right)^{\frac{\beta}{\alpha}} \frac{3a^{\alpha-1} - b^{\alpha-1}}{3a^\alpha + b^\alpha} + \left(\frac{2a^\alpha + 2b^\alpha}{4}\right)^{\frac{\beta}{\alpha}} \frac{2a^{\alpha-1} - 2b^{\alpha-1}}{2a^\alpha + 2b^\alpha} + \left(\frac{a^\alpha + 3b^\alpha}{4}\right)^{\frac{\beta}{\alpha}} \frac{a^{\alpha-1} - 3b^{\alpha-1}}{a^\alpha + 3b^\alpha}.$$

Grouping first and last term, we have

$$\Theta = \left(\frac{1}{4}\right)^{\frac{\beta}{\alpha}} \left[(3a^\alpha + b^\alpha)^{\frac{\beta}{\alpha}-1} (3a^{\alpha-1} - b^{\alpha-1}) + (a^\alpha + 3b^\alpha)^{\frac{\beta}{\alpha}-1} (a^{\alpha-1} - 3b^{\alpha-1})\right] +$$

$$\left(\frac{2a^\alpha + 2b^\alpha}{4}\right)^{\frac{\beta}{\alpha}} \frac{a^{\alpha-1} - b^{\alpha-1}}{a^\alpha + b^\alpha} \geq 0,$$

if,

$$\left(\frac{3a^\alpha + b^\alpha}{a^\alpha + 3b^\alpha}\right)^{\frac{\beta}{\alpha}-1} \geq 1, \frac{3a^{\alpha-1} - b^{\alpha-1}}{3b^{\alpha-1} - a^{\alpha-1}} \geq 1,$$

that is, if

$$\beta \geq \alpha \quad and \quad \alpha \geq 1.$$

Hence,

$$\Theta \geq 0 \quad \text{for} \quad 1 \leq \alpha \leq \beta.$$

Let $\Theta \geq 0 \quad \text{for} \quad 1 \leq \alpha \leq \beta$, be true for all $k - 1$ values.
Now

$$\Theta(k) = \Theta(k-1) + \left(\frac{a^\alpha + kb^\alpha}{k+1}\right)^{\frac{\beta}{\alpha}} \left(\frac{a^{\alpha-1} - kb^{\alpha-1}}{a^\alpha + kb^\alpha}\right) \tag{3.6}$$

$$= \Theta(k-2) + \left(\frac{ka^\alpha + b^\alpha}{k+1}\right)^{\frac{\beta}{\alpha}} \left(\frac{ka^{\alpha-1} - b^{\alpha-1}}{ka^\alpha + b^\alpha}\right) + \left(\frac{a^\alpha + kb^\alpha}{k+1}\right)^{\frac{\beta}{\alpha}} \left(\frac{a^{\alpha-1} - kb^{\alpha-1}}{a^\alpha + kb^\alpha}\right).$$

$$\Theta(k-2) \geq 0.$$

It is easy to see that
(3.7)

$$\left(\frac{ka^\alpha + b^\alpha}{k+1}\right)^{\frac{\beta}{\alpha}} \left(\frac{ka^{\alpha-1} - b^{\alpha-1}}{ka^\alpha + b^\alpha}\right) + \left(\frac{a^\alpha + kb^\alpha}{k+1}\right)^{\frac{\beta}{\alpha}} \left(\frac{a^{\alpha-1} - kb^{\alpha-1}}{a^\alpha + kb^\alpha}\right) \geq 0 \quad \text{for} \quad 1 \leq \alpha \leq \beta.$$

Therefore, $\Theta(k) \geq 0 \quad \text{for} \quad 1 \leq \alpha \leq \beta$.
Hence by induction $\Theta \geq 0 \quad \text{for} \quad 1 \leq \alpha \leq \beta$, for all positive integral values of k.
Thus, on grouping first and last term, second and second to last term; and so on in Θ, we get

$$(a-b)\left(\frac{\partial G}{\partial a} - \frac{\partial G}{\partial b}\right) = [\Delta]\,[\Theta] \geq 0 \quad \text{for} \quad 1 \leq \alpha \leq \beta. \tag{3.8}$$

Hence $G(a, b, k; \alpha, \beta)$ is Schur convex for $1 \leq \alpha \leq \beta$.

Case(ii). For $\alpha, \beta \neq 0$, $a \geq b$, non negative integer k, we have the dual Gnan mean

$$g = g(a, b; k, \alpha, \beta) = \left[\frac{1}{k+1}\sum_{i=0}^{k}\left(\frac{(k-i)a^\alpha + ib^\alpha}{k}\right)^{\frac{\beta}{\alpha}}\right]^{\frac{1}{\beta}}. \tag{3.9}$$

Taking logarithm on both sides, differentiating partially w.r.t a and rearranging leads to

$$\frac{\partial g}{\partial a} = \frac{g}{(k+1)g^\beta}\sum_{i=0}^{k}\left(\frac{(k-i)a^\alpha + ib^\alpha}{k}\right)^{\frac{\beta}{\alpha}-1}\frac{k-i}{k}a^{\alpha-1}. \tag{3.10}$$

Similarly,

$$\frac{\partial g}{\partial b} = \frac{g}{(k+1)g^\beta}\sum_{i=0}^{k}\left(\frac{(k-i)a^\alpha + ib^\alpha}{k}\right)^{\frac{\beta}{\alpha}-1}\frac{i}{k}b^{\alpha-1}. \tag{3.11}$$

Consider
(3.12)

$$(a-b)\left(\frac{\partial g}{\partial a} - \frac{\partial g}{\partial b}\right) = \frac{(a-b)}{k+1}g^{1-\beta}\sum_{i=0}^{k}\left(\frac{(k-i)a^\alpha + ib^\alpha}{k}\right)^{\frac{\beta}{\alpha}}\frac{(k-i)a^{\alpha-1} - ib^{\alpha-1}}{(k-i)a^\alpha + ib^\alpha}$$

$$(a-b)\left(\frac{\partial g}{\partial a} - \frac{\partial g}{\partial b}\right) = [\Delta]\,[\Theta].$$

Where

$$\Delta = \frac{(a-b)}{k+1}g^{1-\beta} \quad \text{and} \quad \Theta = \sum_{i=0}^{k}\left(\frac{(k-i)a^\alpha + ib^\alpha}{k}\right)^{\frac{\beta}{\alpha}}\frac{(k-i)a^{\alpha-1} - ib^{\alpha-1}}{(k-i)a^\alpha + ib^\alpha}.$$

Clearly $\Delta \geq 0$.
Now, we shall prove that $\Theta \geq 0$ for $1 \leq \alpha \leq \beta$, for all positive integral values of k, by strong mathematical induction.
For $k = 1$,

$$\Theta = (a^\beta)\frac{1}{a} + b^\beta\left(-\frac{1}{b}\right)$$
$$= a^{\beta-1} - b^{\beta-1} \geq 0$$

if

$$a^{\beta-1} \geq b^{\beta-1}$$

that is if

$$\beta \geq 1.$$

Hence

$$\Theta \geq 0 \quad \text{for} \quad \beta \geq 1. \tag{3.13}$$

For $k = 2$,

$$\Theta = (a^\alpha)^{\frac{\beta}{\alpha}}\left(\frac{1}{a}\right) + \left(\frac{a^\alpha + b^\alpha}{2}\right)^{\frac{\beta}{\alpha}}\left(\frac{a^{\alpha-1} - b^{\alpha-1}}{a^\alpha + b^\alpha}\right) + (b^\alpha)^{\frac{\beta}{\alpha}}(-\frac{1}{b}).$$

Grouping first and third term, we have

$$\Theta = (a^\alpha)^{\frac{\beta}{\alpha}}\left(\frac{1}{a}\right) + (b^\alpha)^{\frac{\beta}{\alpha}}(-\frac{1}{b}) + \left(\frac{a^\alpha + b^\alpha}{2}\right)^{\frac{\beta}{\alpha}}\left(\frac{a^{\alpha-1} - b^{\alpha-1}}{a^\alpha + b^\alpha}\right) \geq 0$$

if

$$a^{\beta-1} - b^{\beta-1} \geq 0, \left(\frac{a^{\alpha-1} - b^{\alpha-1}}{a^\alpha + b^\alpha}\right) \geq 0,$$

that is if

$$\beta \geq 1, \alpha \geq 1.$$

Hence

$$\Theta \geq 0, \text{for} \quad \beta \geq 1, \alpha \geq 1. \tag{3.14}$$

For $k = 3$, grouping first and last term, second and second to last term, we get

$$\Theta = \left(a^\beta\right)\left(\frac{1}{a}\right) - \left(b^\beta\right)\left(\frac{1}{b}\right) + \left(\frac{2a^\alpha + b^\alpha}{3}\right)^{\frac{\beta}{\alpha}-1}\left(\frac{2a^{\alpha-1} - b^{\alpha-1}}{3}\right) + \left(\frac{a^\alpha + 2b^\alpha}{3}\right)^{\frac{\beta}{\alpha}-1}\left(\frac{a^{\alpha-1} - 2b^{\alpha-1}}{3}\right) \geq 0$$

if

$$a^{\beta-1} - b^{\beta-1} \geq 0, \left(\frac{2a^\alpha + b^\alpha}{a^\alpha + 2b^\alpha}\right)^{\frac{\beta}{\alpha}-1}\left(\frac{2a^{\alpha-1} - b^{\alpha-1}}{2b^{\alpha-1} - a^{\alpha-1}}\right) \geq 1$$

that is if

$$\beta \geq 1, \beta \geq \alpha, \alpha \geq 1.$$

Hence

$$\Theta \geq 0 \quad \text{for} \quad \alpha, \beta \geq 1, \alpha \leq \beta.$$

That is

$$\Theta \geq 0 \quad \text{for} \quad 1 \leq \alpha \leq \beta.$$

Let $\Theta \geq 0$ for $1 \leq \alpha \leq \beta$, be true for all $k-1$ values.
Now

$$\Theta(k) = \Theta(k-1) + \left(\frac{a^\alpha + (k-1)b^\alpha}{k}\right)^{\frac{\beta}{\alpha}-1}\left(\frac{a^{\alpha-1} - (k-1)b^{\alpha-1}}{k}\right) \tag{3.15}$$

$$= \Theta(k-2) + \left(\frac{(k-1)a^\alpha + b^\alpha}{k}\right)^{\frac{\beta}{\alpha}-1}\left(\frac{(k-1)a^{\alpha-1} - b^{\alpha-1}}{k}\right) + \left(\frac{a^\alpha + (k-1)b^\alpha}{k}\right)^{\frac{\beta}{\alpha}-1}\left(\frac{a^{\alpha-1} - (k-1)b^{\alpha-1}}{k}\right). \tag{3.16}$$

$$\Theta(k-2) \geq 0.$$

It is easy to see that

$$\left(\frac{(k-1)a^\alpha + b^\alpha}{k}\right)^{\frac{\beta}{\alpha}-1}\left(\frac{(k-1)a^{\alpha-1} - b^{\alpha-1}}{k}\right) + \left(\frac{a^\alpha + (k-1)b^\alpha}{k}\right)^{\frac{\beta}{\alpha}-1}\left(\frac{a^{\alpha-1} - (k-1)b^{\alpha-1}}{k}\right) \geq 0$$

for $1 \leq \alpha \leq \beta$.
Therefore, $\Theta(k) \geq 0$ for $1 \leq \alpha \leq \beta$.
Hence by induction $\Theta \geq 0$ for $1 \leq \alpha \leq \beta$, for all positive integral values of k.

Thus, on grouping first and last term, second and second to last term, and so on in Θ, we get

$$(3.17) \qquad (a-b)\left(\frac{\partial g}{\partial a}-\frac{\partial g}{\partial b}\right)=[\Delta]\,[\Theta]\geq 0 \quad \text{for} \quad 1\leq \alpha\leq \beta.$$

Hence $g(a,b,k;\alpha,\beta)$ is Schur convex for $1\leq\alpha\leq\beta$.
This completes the proof of (1).

The proof of (2) is established by discussing the following two cases.

Case(i). For $\alpha=0$, $\beta\neq 0$, and $a\geq b$, we have the Gnan mean,

$$(3.18) \qquad G=G(a,b;k,0,\beta)=\left[\frac{1}{k}\sum_{i=1}^{k}a^{\frac{k+1-i}{k+1}\beta}b^{\frac{i}{k+1}\beta}\right]^{\frac{1}{\beta}}$$

then,
(3.19)

$$(a-b)\left(\frac{\partial G}{\partial a}-\frac{\partial G}{\partial b}\right)=\frac{(a-b)}{k(k+1)ab}G^{1-\beta}\left[\sum_{i=1}^{k}a^{\frac{k+1-i}{k+1}\beta}b^{\frac{i}{k+1}\beta}(b(k+1-i)-ai)\right].$$

$$(3.20) \qquad (a-b)\left(\frac{\partial G}{\partial a}-\frac{\partial G}{\partial b}\right)=[\Delta]\,[\Theta].$$

Where,

$$\Delta=\frac{(a-b)}{k(k+1)ab}G^{1-\beta}, \quad \text{and} \quad \Theta=\sum_{i=1}^{k}a^{\frac{k+1-i}{k+1}\beta}b^{\frac{i}{k+1}\beta}(b(k+1-i)-ai).$$

Clearly $\Delta\geq 0$.
Now, we shall prove that $\Theta\leq 0$ for $\beta\leq 0$, for all positive integral values of k, by strong mathematical induction.
For $k=1$,

$$\Theta=a^{\frac{\beta}{2}}b^{\frac{\beta}{2}}\,(b-a)\leq 0.$$

For $k=2$,

$$\Theta=\left(a^{\frac{2\beta}{3}}b^{\frac{\beta}{3}}\right)(2b-a)+\left(a^{\frac{\beta}{3}}b^{\frac{2\beta}{3}}\right)(b-2a)\leq 0, \text{for} \quad \beta\leq 0$$

For $k=3$, grouping first and last term, we have

$$\Theta=\left(a^{\frac{3\beta}{4}}b^{\frac{\beta}{4}}\right)(3b-a)+\left(a^{\frac{\beta}{4}}b^{\frac{3\beta}{4}}\right)(b-3a)+\left(a^{\frac{\beta}{2}}b^{\frac{\beta}{2}}\right)(2b-2a)\leq 0, \text{for} \quad \beta\leq 0$$

For $k=4$, grouping first and last term, second and second to last term, we have

$$\Theta=\left(a^{\frac{4\beta}{5}}b^{\frac{\beta}{5}}\right)(4b-a)+\left(a^{\frac{\beta}{5}}b^{\frac{4\beta}{5}}\right)(b-4a)+\left(a^{\frac{3\beta}{5}}b^{\frac{2\beta}{5}}\right)(3b-2a)+\left(a^{\frac{2\beta}{5}}b^{\frac{3\beta}{5}}\right)(2b-3a)\leq 0, \text{for} \quad \beta\leq 0$$

Let $\Theta\leq 0$ for $\beta\leq 0$, be true for all $k-1$ values.
Now

$$(3.21) \qquad \Theta(k)=\Theta(k-1)+a^{\frac{\beta}{k+1}}b^{\frac{k\beta}{k+1}}(b-ka)$$
$$=\Theta(k-2)+a^{\frac{k\beta}{k+1}}b^{\frac{\beta}{k+1}}(kb-a)+a^{\frac{\beta}{k+1}}b^{\frac{k\beta}{k+1}}(b-ka)$$
$$\Theta(k-2)\leq 0.$$

It is easy to see that

$$a^{\frac{k\beta}{k+1}}b^{\frac{\beta}{k+1}}(kb-a)+a^{\frac{\beta}{k+1}}b^{\frac{k\beta}{k+1}}(b-ka)\leq 0 \quad \text{for} \quad \beta\leq 0.$$

Therefore, $\Theta(k)\leq 0$ for $\beta\leq 0$
Hence by induction $\Theta\leq 0$ for $\beta\leq 0$, for all positive integral values of k.

Thus, on grouping first and last term, second and second to last term, and so on in Θ, we get

$$(3.22)\qquad (a-b)\left(\frac{\partial G}{\partial a}-\frac{\partial G}{\partial b}\right)=[\Delta]\,[\Theta]\leq 0 \quad \text{for} \quad \beta\leq 0.$$

Hence $G(a,b,k;0,\beta)$ is Schur concave for $\beta\leq 0$.

Case(ii) For $\alpha=0$, $\beta\neq 0$, $a\geq b$, We have the the dual Gnan mean

$$(3.23)\qquad g=g(a,b;k,0,\beta)=\left[\frac{1}{k+1}\sum_{i=0}^{k}a^{\frac{k-i}{k}\beta}b^{\frac{i}{k}\beta}\right]^{\frac{1}{\beta}}$$

then,

$$(3.24)\qquad (a-b)\left(\frac{\partial g}{\partial a}-\frac{\partial g}{\partial b}\right)=\frac{(a-b)}{k(k+1)ab}g^{1-\beta}\left[\sum_{i=0}^{k}a^{\frac{k-i}{k}\beta}b^{\frac{i}{k}\beta}(b(k-i)-ai)\right].$$

$$(3.25)\qquad (a-b)\left(\frac{\partial g}{\partial a}-\frac{\partial g}{\partial b}\right)=[\Delta]\,[\Theta].$$

Where

$$\Delta=\frac{(a-b)}{k(k+1)ab}g^{1-\beta} \quad \text{and} \quad \Theta=\left[\sum_{i=0}^{k}a^{\frac{k-i}{k}\beta}b^{\frac{i}{k}\beta}(b(k-i)-ai)\right].$$

Clearly $\Delta\geq 0$.
Now, we shall prove that $\Theta\leq 0$ for $\beta\leq 0$, for all positive integral values of k, by strong mathematical induction.
For $k=1$,

$$\Theta=a^{\beta-1}-b^{\beta-1}\leq 0. \quad \text{for} \quad \beta\leq 1.$$

For $k=2$,

$$\Theta=a^{\beta}(2b)+a^{\frac{\beta}{2}}b^{\frac{\beta}{2}}(b-a)+b^{\beta}(-2a)\leq 0, \quad \text{for} \quad \beta\leq 1.$$

For $k=3$, grouping 1first and last term, second and second to last term, we have

$$\Theta=a^{\beta}(3b)+b^{\beta}(-3a)+a^{\frac{2\beta}{3}}b^{\frac{\beta}{3}}(2b-a)+a^{\frac{\beta}{3}}b^{\frac{2\beta}{3}}(b-2a)\leq 0, \quad \text{for} \quad \beta\leq 0.$$

For $k=4$, grouping first and last term, second and second to last term, we have

$$\Theta=a^{\beta}(4b)+b^{\beta}(-4a)+a^{\frac{3\beta}{4}}b^{\frac{\beta}{4}}(3b-a)+a^{\frac{\beta}{4}}b^{\frac{3\beta}{4}}(b-3a)+a^{\frac{\beta}{2}}b^{\frac{\beta}{2}}(2b-2a)\leq 0, \quad \text{for} \quad \beta\leq 0.$$

Let $\Theta\leq 0$ for $\beta\leq 0$, be true for all $k-1$ values.
Now

$$(3.26)\qquad \Theta(k)=\Theta(k-2)+a^{\frac{(k-1)\beta}{k}}b^{\frac{\beta}{k}}((k-1)b-a)+a^{\frac{\beta}{k}}b^{\frac{(k-1)\beta}{k}}(b-(k-1)a)$$

$$\Theta(k-2)\leq 0.$$

It is easy to see that

$$a^{\frac{(k-1)\beta}{k}}b^{\frac{\beta}{k}}((k-1)b-a)+a^{\frac{\beta}{k}}b^{\frac{(k-1)\beta}{k}}(b-(k-1)a)\leq 0 \quad \text{for} \quad \beta\leq 0.$$

Therefore, $\Theta(k)\leq 0$ for $\beta\leq 0$.
Hence by induction $\Theta\leq 0$ for $\beta\leq 0$, for all positive integral values of k.
Thus, on grouping first and last term, second and second to last term, and so on in Θ, we get

$$(a-b)\left(\frac{\partial g}{\partial a}-\frac{\partial g}{\partial b}\right)=[\Delta]\,[\Theta]\leq 0 \quad \text{for} \quad \beta\leq 0. \tag{3.27}$$

Hence $g(a,b,k;0,\beta)$ is Schur concave for $\beta\leq 0$.
This completes the proof of (2).

The proof of (3) is established by discussing the following two cases.

Case(i). For $\beta=0$, $\alpha\neq 0$, $a\geq b$, We have the Gnan mean,

$$G=G(a,b;k,\alpha,0)=\prod_{i=1}^{k}\left(\frac{(k+1-i)a^{\alpha}+ib^{\alpha}}{k+1}\right)^{\frac{1}{k\alpha}}. \tag{3.28}$$

Let,

$$\Theta=(a-b)\left(\frac{\partial G}{\partial a}-\frac{\partial G}{\partial b}\right). \tag{3.29}$$

Now, we shall prove that $\Theta\leq 0$ for $\alpha\leq 1/2$, for all positive integral values of k, by strong mathematical induction.
For $k=1$,

$$\Theta=(a-b)G\left[\frac{a^{\alpha-1}-b^{\alpha-1}}{a^{\alpha}+b^{\alpha}}\right]\leq 0 \quad \text{for} \quad \alpha\leq 1. \tag{3.30}$$

For $k=2$, grouping first and last term, second and second to last term, we have
(3.31)

$$\Theta=\frac{(a-b)}{2}G\left[\frac{2a^{\alpha-1}}{2a^{\alpha}+b^{\alpha}}-\frac{2b^{\alpha-1}}{a^{\alpha}+2b^{\alpha}}+\frac{a^{\alpha-1}}{a^{\alpha}+2b^{\alpha}}-\frac{b^{\alpha-1}}{2a^{\alpha}+b^{\alpha}}\right]\leq 0 \quad \text{for} \quad \alpha\leq 1/2.$$

For $k=3$, grouping first and last term, second and second to last term, and so on ,we have

$$\Theta=\frac{(a-b)}{3}G[\frac{3a^{\alpha-1}}{3a^{\alpha}+b^{\alpha}}-\frac{3b^{\alpha-1}}{a^{\alpha}+3b^{\alpha}}+\frac{2a^{\alpha-1}}{2a^{\alpha}+2b^{\alpha}}-\frac{2b^{\alpha-1}}{2a^{\alpha}+2b^{\alpha}}+ \tag{3.32}$$
$$\frac{a^{\alpha-1}}{a^{\alpha}+3b^{\alpha}}-\frac{b^{\alpha-1}}{3a^{\alpha}+b^{\alpha}}]\leq 0 \quad \text{for} \quad \alpha\leq 1/2l.$$

Let $\Theta\leq 0$ for $\beta\leq 0$, be true for all $k-1$ values.
Now

$$\Theta(k)=\Theta(k-1)+\frac{(a-b)G}{k}\left[\frac{ka^{\alpha-1}}{ka^{\alpha}+b^{\alpha}}-\frac{kb^{\alpha-1}}{a^{\alpha}+kb^{\alpha}}\right]. \tag{3.33}$$

$$\Theta(k-1)\leq 0.$$

It is easy to see that

$$\frac{(a-b)G}{k}\left[\frac{ka^{\alpha-1}}{ka^\alpha+b^\alpha}-\frac{kb^{\alpha-1}}{a^\alpha+kb^\alpha}\right]\le 0 \quad \text{for} \quad \alpha\le 1/2.$$

Therefore, $\Theta(k)\le 0$ for $\alpha\le 1/2$.
Hence by induction $\Theta\le 0$ for $\alpha\le 1/2$, for all positive integral values of k.

Thus, on grouping first and last term, second and second to last term, and so on in Θ, we get

$$(a-b)\left(\frac{\partial G}{\partial a}-\frac{\partial G}{\partial b}\right)\le 0 \quad \text{for} \quad \alpha\le 1/2. \tag{3.34}$$

Hence $G(a,b,k;\alpha,0)$ is Schur concave for $\alpha\le 1/2$.

Case(ii). For $\beta=0$, $\alpha\ne 0$, $a\ge b$, We have the dual Gnan mean

$$g=g(a,b;k,\alpha,0)=\prod_{i=0}^{k}\left(\frac{(k-i)a^\alpha+ib^\alpha}{k}\right)^{\frac{1}{(k+1)\alpha}}. \tag{3.35}$$

Let,

$$\Theta=(a-b)\left(\frac{\partial g}{\partial a}-\frac{\partial g}{\partial b}\right). \tag{3.36}$$

Now, we shall prove that $\Theta\le 0$ for $\alpha\le 1/2$, for all positive integral values of k, by strong mathematical induction.
For $k=1$,

$$\Theta=\frac{(a-b)(b-a)}{2\sqrt{ab}}\le 0. \tag{3.37}$$

For $k=2$, grouping first and last term, second and second to last term, we have

$$\Theta=\frac{(a-b)}{3}g\left[\frac{1}{a}-\frac{1}{b}+\frac{a^{\alpha-1}}{a^\alpha+b^\alpha}-\frac{b^{\alpha-1}}{a^\alpha+b^\alpha}\right]\le 0 \quad \text{for} \quad \alpha\le 1. \tag{3.38}$$

For $k=3$, grouping first and last term, second and second to last term, we have

$$\begin{aligned}\Theta=\frac{(a-b)}{4}g[\frac{1}{a}-\frac{1}{b}+\frac{2a^{\alpha-1}}{2a^\alpha+b^\alpha}-\frac{2b^{\alpha-1}}{a^\alpha+2b^\alpha}+\\ \frac{a^{\alpha-1}}{a^\alpha+2b^\alpha}-\frac{b^{\alpha-1}}{2a^\alpha+b^\alpha}]\le 0 \quad \text{for} \quad \alpha\le 1/2\end{aligned} \tag{3.39}$$

Let $\Theta\le 0$ for $\alpha\le 1/2$, be true for all $k-1$ values.
Now

$$\begin{aligned}\Theta(k)=\Theta(k-1)+\frac{(a-b)g}{k+1}\left[\frac{(k-1)a^{\alpha-1}}{(k-1)a^\alpha+b^\alpha}-\frac{(k-1)b^{\alpha-1}}{a^\alpha+(k-1)b^\alpha}\right]\\ \Theta(k-1)\le 0.\end{aligned} \tag{3.40}$$

It is easy to see that

$$\frac{(a-b)G}{k}\left[\frac{(k-1)a^{\alpha-1}}{(k-1)a^\alpha+b^\alpha}-\frac{(k-1)b^{\alpha-1}}{a^\alpha+(k-1)b^\alpha}\right]\le 0 \quad \text{for} \quad \alpha\le 1/2.$$

Therefore, $\Theta(k) \leq 0$ for $\alpha \leq 1/2$.
Hence by induction $\Theta \leq 0$ for $\alpha \leq 1/2$, for all positive integral values of k.

Thus, on grouping first and last term, second and second to last term, and so on in Θ, we get

$$(3.41) \qquad (a-b)\left(\frac{\partial g}{\partial a} - \frac{\partial g}{\partial b}\right) \leq 0 \quad \text{for} \quad \alpha \leq 1/2.$$

Hence $g(a,b,k;\alpha,0)$ is Schur concave for $\alpha \leq 1/2$.
This completes the proof of (3).

Proof of(4):
For $\alpha = 0$, $\beta = 0$, and $a \geq b$, we have the Gnan mean = dual Gnan mean, given by

$$G(a,b;k,0,0) = g(a,b;k,0,0) = \sqrt{ab}.$$

Differentiating partially with respect a and b, we have

$$\frac{\partial G}{\partial a} = \frac{b}{2\sqrt{ab}}, \quad \frac{\partial G}{\partial b} = \frac{a}{2\sqrt{ab}}.$$

Now,

$$(3.42) \qquad (a-b)\left(\frac{\partial G}{\partial a} - \frac{\partial G}{\partial b}\right) = \frac{(a-b)(b-a)}{2\sqrt{ab}} \leq 0.$$

Therefore, for $\alpha = 0$, $\beta = 0$, $G(a,b,k;0,0) = g(a,b,k;0,0) = \sqrt{ab}$ is schur concave.

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