（天花板群测试三）设$x>0$，$y>0$，证明：$\sqrt{\frac{x^y}{y}}+\sqrt{\frac{y^x}{x}}\ge 2$.

证：等价于证$\frac{x}{x+y}\cdot x^{\frac{y-1}{2}}+\frac{y}{x+y}\cdot y^{\frac{x-1}{2}}\ge\frac{2\sqrt{xy}}{x+y}$，由琴生不等式有

$$\frac{x}{x+y}\cdot x^{\frac{y-1}{2}}+\frac{y}{x+y}\cdot y^{\frac{x-1}{2}}\ge \mathrm{e}^{\frac{(y-1)x\ln x+(x-1)y\ln y}{2x+2y}}$$

所以只需证

$$\frac{(y-1)x\ln x+(x-1)y\ln y}{2x+2y}\ge\ln\frac{2\sqrt{xy}}{x+y}$$

不妨设$x=ty$，$t\ge1$，即证

$$f(y)=(y-1)t\ln ty+(ty-1)\ln y+2(t+1)\ln\frac{t+1}{2\sqrt{t}}\ge 0$$

有$f'(y)=t\ln t+2t+2t\ln y-\frac{t+1}{y}$，易知$f'(y)$单调递增，

又$f'(1)=t\ln t+t-1\ge 0$，$f'\left(\frac{1}{t}\right)=t-t^2-t\ln t\le 0$，

故有$\frac{1}{t}\le y_0\le 1$，使得$f'(y_0)=0$，则可知

$$f(y)_{\min}=f(y_0)=t+1-2ty_0-t\ln t+2(t+1)\ln\frac{t+1}{2\sqrt{ty_0}}$$

又有$f'(y_0)=2t\left(1+\ln\sqrt{t}y_0-\frac{t+1}{2ty_0}\right)\ge 2t\left(2-\frac{1}{\sqrt{t}y_0}-\frac{t+1}{2ty_0}\right)$，

则$4ty_0\le t+2\sqrt{t}+1$，所以可知

$$f(y_0)\ge\frac{t+1-2\sqrt{t}}{2}-t\ln t+2(t+1)\ln\frac{t+1}{\sqrt{t}+1}$$

记$g(t)=\frac{t^2+1-2t}{2}-2t^2\ln t+2(t^2+1)\ln\frac{t^2+1}{t+1}$，$t\ge1$，只需证$g(t)\ge0$，

又有

$$g'(t)=4t\left[\frac{t^2+2t-3}{4t(t+1)}+\ln\frac{t^2+1}{t^2+t}\right]=4th(t)$$

又有 $h'(t)=\dfrac{(t-1)^2(3t^2+8t+3)}{4t^2(t^2+1)(t+1)^2}\geq 0$， $h(t)$ 单调递增，所以 $h(t)\geq h(1)=0$，

即有 $g'(t)\geq 0$，所以 $g(t)$ 单调递增，则有 $g(t)\geq g(1)=0$，得证.