设函数列$\{f_n\}$满足$f_1(x)=x\text{-}\ln x$, $f_{n+1}(x)=f_1(f_n(x))$,

函数列$\{g_n\}$满足$g_n(x)=\frac{e^{1-f_n(x)}}{x}$,

函数列$\{h_n\}$满足$h_n(x)=g_n(g_n(x))\text{-}x$.

证明：对于任意正整数$n\geq 2, h_n(x)$在(1，+∞)单调递增.

证明：由归纳假设易证, 对于任意正整数n,

$f_n(x)$在(0，1) 单调递减，在(1，+∞)单调递增，

$f_n(x)\geq f_n(1)=1$,当且仅当$x=1$ 取等

$h_n(x)=g_n(g_n(x))\text{-}x=\frac{e^{1-f_n(g_n(x))}}{g_n(x)}-x=x\frac{e^{1-f_n(g_n(x))}}{e^{1-f_n(x)}}-x$

设$p_n(x)=f_n(x)-f_n(g_n(x))$,

则$h_n(x)=xe^{p_n(x)}\text{-}x$

$h_n{}'(x)=e^{p_n(x)}(1+xp_n{}'(x))\text{-}1$

要证对于任意正整数$n\geq 2, h_n(x)$在(1，+∞)单调递增,即$h_n{}'(x)>0(x>1)$

只要证对于任意正整数$n\geq 2, p_n{}'(x)>0$且$p_n(x)>0(x>1)$

又易证对于任意正整数n, $f_n(1)=g_n(1)=1$, $p_n(1)=0$

故只要证对于任意正整数$n\geq 2, p_n{}'(x)>0(x>1)$

对于任意正整数$n\geq 2$,$p_n(x)=f_n(x)-f_n(g_n(x))$

$=f_1(f_{n-1}(x))\text{-}f_1(f_{n-1}(g_n(x)))$

$=f_{n-1}(x)\text{-}\ln f_{n-1}(x)\text{-}f_{n-1}(g_n(x))+\ln f_{n-1}(g_n(x))$,

$p_n{}'(x)=f_{n-1}{}'(x)\text{-}(f_{n-1}(g_n(x)))'\text{-}\frac{f_{n-1}{}'(x)}{f_{n-1}(x)}+\frac{(f_{n-1}(g_n(x)))'}{f_{n-1}(g_n(x))}$

对于任意正整数$n\geq 2$,设$q_n(x)=f_{n-1}(x)-f_{n-1}(g_n(x))$,

若证得对于任意正整数$n\geq 2$,$q_n{}'(x)>0$,即$q_n(x)$在(1，+∞)单调递增,

则对于任意正整数$n\geq 2$ 与$x>1$,

$q_n(x)>q_n(1)=0$, $f_{n-1}(x)>f_{n-1}(g_n(x))$

$p_n{}'(x)=f_{n-1}{}'(x)\text{-}(f_{n-1}(g_n(x)))'\text{-}\frac{f_{n-1}{}'(x)}{f_{n-1}(x)}+\frac{(f_{n-1}(g_n(x)))'}{f_{n-1}(g_n(x))}$

$>f_{n-1}{}'(x)\text{-}(f_{n-1}(g_n(x)))'\text{-}\frac{f_{n-1}{}'(x)}{f_{n-1}(x)}+\frac{(f_{n-1}(g_n(x)))'}{f_{n-1}(x)}=q_n{}'(x)(1\text{-}\frac{1}{f_{n-1}(x)})>0$

故只要证对于任意正整数$n\geq 2$, $q_n{}'(x)>0(x>1)$

以此类推, 设$r_n(x)=f_1(x)-f_1(g_n(x))$,

则只要证对于任意正整数$n\geq 2$, $r_n{}'(x)>0(x>1)$

对于任意$x>1$,

当$n=2$, $r_n(x)=r_2(x)=f_1(x)-f_1(g_2(x))=1\text{-}\ln x\text{-}\frac{x-\ln x}{e^{x-1}}+\ln(x\text{-}\ln x)$

$r_n{}'(x)=\text{-}\frac{1}{x}+\frac{x-\ln x-1+\frac{1}{x}}{e^{x-1}}+\frac{1-\frac{1}{x}}{x-\ln x}>\text{-}\frac{1}{x}+\frac{\frac{1}{x}}{e^{x-1}}+\frac{1-\frac{1}{x}}{x-\ln x}=\frac{e^{1-x}+\frac{\ln x-1}{x-\ln x}}{x}$

当 $1<x<e$, $r_n{}'(x)>\frac{e^{1-x}+\frac{\ln x-1}{x-\ln x}}{x}>\frac{e^{1-x}+\ln x-1}{x}$

设 $s(x)=e^{1-x}+\ln x-1$

$s'(x)=\frac{1}{x}\text{-}\frac{1}{e^{x-1}}\geq 0$

故 $s(x)$在(1，e)单调递增

当 1< $x<e$，$r_n{}'(x)$> $\frac{e^{1-x}+\ln x-1}{x}>\frac{s\ (1)}{x}$=0

当$x\geq e$，$r_n{}'(x)>\frac{e^{1-x}+\frac{\ln x-1}{x-\ln x}}{x}$>0

故$r_n{}'(x)\ >0$

假设当$n=k-1(k\in N^{+})$,

$r_n{}'(x)=\ r'_{k-1}(x)$> 0,即$f_1'(x)>(f_1(g_{k-1}(x)))'(x>1)$ ，

$r_{k-1}(x)\ >r_{k-1}(1)$=0,即 $f_1(x)>f_1(g_{k-1}(x))(x>1)$

则当$n=k$，$r_n(x)$= $r_k(x)$= $f_1(x)-f_1(g_k(x))$

$r_n{}'(x)\ =f_1{}'(x)-(f_1(g_k(x)))'>(f_1(g_{k-1}(x)))'-(f_1(g_k(x)))'$

=$(f_1(g_{k-1}(x))-f_1(g_{k-1}(x)f_{k-1}(x)))'$

=$(e^{1-f_{k-1}(x)-\ln x}(1-f_{k-1}(x))$+ $\ln f_{k-1}(x))'$

>$-e^{1-f_{k-1}(x)-\ln x}f_{k-1}{}'(x)$+$\frac{f_{k-1}{}'(x)}{f_{k-1}(x)}$

>$f_{k-1}{}'(x)(\frac{1}{f_{k-1}(x)}-\frac{1}{e^{f_{k-1}(x)-1}})$>0

故对于任意正整数$n\geq 2$，$r_n{}'(x)\ >0(x>1)$

证毕！