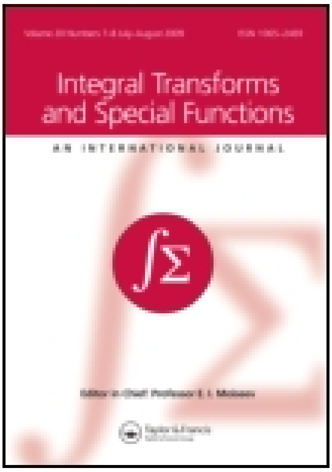

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A weighted and exponential generalization of Wilker's inequality and its applications

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A weighted and exponential generalization of Wilker's inequality and its applications

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In this paper, the authors first prove a weighted and exponential generalization of Wilker's inequality. The main result presented here is then applied with a view to deriving an improved version of the Sándor–Bencze conjectured inequality. Some other closely-related inequalities are also considered.

Keywords: Wilker's inequality; Huygens's inequality; Weighted generalization; Sándor–Bencze conjectured inequality; Weighted inequalities; Exponential generalization

AMS Subject Classification: Primary: 26D15; Secondary: 26D05; 33B20

1. Introduction

The following inequality is known in the literature as Wilker's inequality [1]:

$$\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} > 2 \quad \left(0 < x < \frac{\pi}{2}\right). \tag{1}$$

Wilker's inequality (1) has attracted remarkable interest of many mathematicians and has motivated a large number of research papers involving different proofs, various generalizations and improvements ([1–8]; see also the references therein). Recently, the following similar inequality (proved by Huygens [9]) was considered by Sándor and Bencze [10]:

$$2\left(\frac{\sin x}{x}\right) + \frac{\tan x}{x} > 3 \quad \left(0 < x < \frac{\pi}{2}\right). \tag{2}$$

Huygens's inequality (2) prompts us to ask a natural question: Does there exist an inequality which unifies (and possibly also extends) Wilker's inequality (1) and Huygens's inequality (2)? The following theorem gives an affirmative answer to this question.

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THEOREM 1 *Let*

$$0 < x < \frac{\pi}{2}, \quad \lambda > 0, \quad \mu > 0 \quad \textit{and} \quad p \leqq \frac{2q\mu}{\lambda}.$$

Then, for

$$q > 0 \quad \textit{or} \quad q \leqq \min\left\{-\frac{\lambda}{\mu}, -1\right\},$$

the following inequality holds true:

$$\frac{\lambda}{\mu+\lambda}\left(\frac{\sin x}{x}\right)^p + \frac{\mu}{\mu+\lambda}\left(\frac{\tan x}{x}\right)^q > 1. \tag{3}$$

2. A set of lemmas

In order to prove Theorem 1, we need the following lemmas.

LEMMA 1 (see [11, p. 17]). *If*

$$x_i > 0, \quad \lambda_i > 0 \quad (i = 1, \ldots, n) \quad \textit{and} \quad \sum_{i=1}^{n} \lambda_i = 1,$$

then

$$\sum_{i=1}^{n} \lambda_i x_i \geqq \prod_{i=1}^{n} x_i^{\lambda_i}. \tag{4}$$

LEMMA 2 (see [12, p. 238]). *The following two-sided trigonometric inequality holds true*:

$$\cos x < \left(\frac{\sin x}{x}\right)^3 < 1 \quad \left(0 < x < \frac{\pi}{2}\right). \tag{5}$$

LEMMA 3 *The following trigonometric inequality holds true*:

$$\left(\frac{x}{\sin x}\right)^2 + \frac{x}{\tan x} > 2 \quad \left(0 < x < \frac{\pi}{2}\right). \tag{6}$$

Proof Define a function

$$f : \left(0, \frac{\pi}{2}\right) \longrightarrow \mathbb{R}$$

by

$$f(x) = \left(\frac{x}{\sin x}\right)^2 + \frac{x}{\tan x}.$$

Then, upon differentiating $f(x)$ with respect to x, we get

$$f'(x) = \frac{1}{\sin^3 x}(\sin^2 x \cos x - 2x^2 \cos x + x \sin x).$$

Next, by applying Lemma 2 followed by a simple calculation, we find that

$$
\begin{aligned}
f'(x) &= \frac{x^2}{\sin^3 x}\left[\cos x\left(\left(\frac{\sin x}{x}\right)^2 - 2\right) + \frac{\sin x}{x}\right] \\
&= \frac{x^2}{\sin^3 x}\left[\left(\cos x - \left(\frac{\sin x}{x}\right)^3\right)\left(\left(\frac{\sin x}{x}\right)^2 - 2\right)\right. \\
&\qquad \left. + \left(\frac{\sin x}{x}\right)^3\left(\left(\frac{\sin x}{x}\right)^2 - 2\right) + \frac{\sin x}{x}\right] \\
&= \frac{x^2}{\sin^3 x}\left[\left(\cos x - \left(\frac{\sin x}{x}\right)^3\right)\left(\left(\frac{\sin x}{x}\right)^2 - 2\right)\right. \\
&\qquad \left. + \left(\frac{\sin x}{x}\right)\left(\left(\frac{\sin x}{x}\right)^2 - 1\right)^2\right] \\
&> 0 \quad \left(0 < x < \frac{\pi}{2}\right).
\end{aligned}
$$

This means that $f(x)$ is *strictly increasing* on the open interval $(0, \pi/2)$. Consequently, we can deduce from the following observation:

$$\lim_{x \to 0+} f(x) = 2$$

that

$$f(x) > 2 \quad \left(0 < x < \frac{\pi}{2}\right),$$

which leads us to the inequality (6) asserted by Lemma 3. ■

3. Proof of the main result (Theorem 1)

In our proof of Theorem 1, we consider the following two cases.

Case 1 Let

$$\lambda > 0, \quad \mu > 0, \quad p \leqq \frac{2q\mu}{\lambda} \quad \text{and} \quad q > 0.$$

Then, by applying Lemma 1 and Lemma 2, we obtain

$$
\begin{aligned}
&\frac{\lambda}{\mu+\lambda}\left(\frac{\sin x}{x}\right)^p + \frac{\mu}{\mu+\lambda}\left(\frac{\tan x}{x}\right)^q \\
&\geqq \left(\frac{\sin x}{x}\right)^{p\lambda/(\mu+\lambda)}\left(\frac{\tan x}{x}\right)^{q\mu/(\mu+\lambda)} \\
&= \left(\frac{\sin x}{x}\right)^{p\lambda/(\mu+\lambda)}\left(\frac{\sin x}{x}\right)^{q\mu/(\mu+\lambda)}\left(\frac{1}{\cos x}\right)^{q\mu/(\mu+\lambda)}
\end{aligned}
$$

$$> \left(\frac{\sin x}{x}\right)^{p\lambda/(\mu+\lambda)} \left(\frac{\sin x}{x}\right)^{q\mu/(\mu+\lambda)} \left(\frac{\sin x}{x}\right)^{-3q\mu/(\mu+\lambda)}$$

$$= \left(\frac{\sin x}{x}\right)^{(p\lambda-2q\mu)/(\mu+\lambda)}$$

$$\geqq 1 \quad \left(0 < x < \frac{\pi}{2}\right),$$

which is the desired inequality (3).

Case 2 Let

$$\lambda > 0, \quad \mu > 0, \quad p \leqq \frac{2q\mu}{\lambda} \quad \text{and} \quad q \leqq \min\left\{-\frac{\lambda}{\mu}, -1\right\}.$$

Then it follows from the hypothesis of Theorem 1 that

$$\frac{\lambda}{\mu+\lambda}\left(\frac{\sin x}{x}\right)^p + \frac{\mu}{\mu+\lambda}\left(\frac{\tan x}{x}\right)^q$$

$$\geqq \frac{\lambda}{\mu+\lambda}\left(\frac{\sin x}{x}\right)^{2q\mu/\lambda} + \frac{\mu}{\mu+\lambda}\left(\frac{\tan x}{x}\right)^q$$

$$= \frac{\lambda}{\mu+\lambda}\left(\frac{x}{\sin x}\right)^{-2q\mu/\lambda} + \frac{\mu}{\mu+\lambda}\left(\frac{x}{\tan x}\right)^{-q} \quad \left(-\frac{q\mu}{\lambda} \geqq 1; -q \geqq 1\right). \tag{7}$$

Moreover, we find from Lemma 3 that

$$\left(\frac{x}{\sin x}\right)^2 > 2 - \frac{x}{\tan x} > 0 \quad \left(0 < x < \frac{\pi}{2}\right).$$

By combining the inequality (7) with the above trigonometric inequality, we obtain

$$\frac{\lambda}{\mu+\lambda}\left(\frac{\sin x}{x}\right)^p + \frac{\mu}{\mu+\lambda}\left(\frac{\tan x}{x}\right)^q$$

$$> \frac{\lambda}{\mu+\lambda}\left(2 - \frac{x}{\tan x}\right)^{-q\mu/\lambda}$$

$$+ \frac{\mu}{\mu+\lambda}\left(\frac{x}{\tan x}\right)^{-q} \quad \left(0 < x < \frac{\pi}{2}\right). \tag{8}$$

We now define a function

$$f : (0, 1) \longrightarrow \mathbb{R}$$

by

$$f(x) = \frac{\lambda}{\mu+\lambda}(2-x)^{-q\mu/\lambda} + \frac{\mu}{\mu+\lambda}x^{-q},$$

which, upon differentiating with respect to x, yields

$$f'(x) = \frac{q\mu}{\mu+\lambda}\left[(2-x)^{-(q\mu/\lambda)-1} - x^{-q-1}\right].$$

For

$$-\frac{q\mu}{\lambda} \geqq 1, \quad -q \geqq 1 \quad \text{and} \quad 0 < x < 1,$$

it is easy to verify that

$$(2-x)^{-(q\mu/\lambda)-1} \geqq 1 \geqq x^{-q-1}.$$

We thus conclude that

$$f'(x) \leqq 0 \quad (0 < x < 1),$$

which immediately implies that $f(x)$ is *decreasing* on the open interval $(0, 1)$. Hence we have

$$f(x) \geqq f(1) = 1 \quad (0 < x < 1).$$

Now, making use of the inequality (8) together with the following well-known trigonometric inequality:

$$0 < \frac{x}{\tan x} < 1 \quad \left(0 < x < \frac{\pi}{2}\right),$$

we deduce that

$$\frac{\lambda}{\mu+\lambda}\left(\frac{\sin x}{x}\right)^p + \frac{\mu}{\mu+\lambda}\left(\frac{\tan x}{x}\right)^q > f\left(\frac{x}{\tan x}\right) \geqq 1,$$

which proves the inequality (3). This completes the proof of Theorem 1.

By setting

$$(p, q) = (2, 1) \quad \text{and} \quad (\lambda, \mu) = (2, 1)$$

in Theorem 1, we obtain a weighted generalization of Wilker's inequality (1) and an exponential generalization of Huygens's inequality (2), given by Corollary 1 and Corollary 2, respectively.

COROLLARY 1 *Let*

$$0 < x < \frac{\pi}{2} \quad \textit{and} \quad 0 < \lambda \leqq \mu.$$

Then

$$\frac{\lambda}{\mu+\lambda}\left(\frac{\sin x}{x}\right)^2 + \frac{\mu}{\mu+\lambda}\left(\frac{\tan x}{x}\right) > 1. \tag{9}$$

COROLLARY 2 *Let*

$$0 < x < \frac{\pi}{2} \quad \textit{and} \quad p \leqq q.$$

Then, for

$$q > 0 \quad \textit{or} \quad q \leqq -2,$$

the following inequality holds true:

$$2\left(\frac{\sin x}{x}\right)^p + \left(\frac{\tan x}{x}\right)^q > 3. \tag{10}$$

Remark 1 It is obvious that Wilker's inequality (1) would follow as a special case of the inequality (9) when

$$\lambda = \mu = 1.$$

Furthermore, in its special case when

$$p = q = 1,$$

the inequality (10) reduces to Huygens's inequality (2).

4. Applications of Theorem 1

As an *open problem*, Sándor and Bencze [10] asked to prove that, for all

$$x \in \left(0, \frac{\pi}{2}\right) \quad \text{and} \quad \alpha \in (0, \infty),$$

the following inequality holds true:

$$\left(\frac{\sin x}{x}\right)^{\alpha} > \frac{\cos^{\alpha} x}{1 + \cos^{\alpha} x}. \tag{11}$$

The Sándor–Bencze conjectured inequality (11) provides a good opportunity to illustrate the application of the foregoing results. Based upon the improved Wilker inequality (3) asserted by Theorem 1, we give here a sharp and generalized version of the Sándor–Bencze conjectured inequality (11).

THEOREM 2 *Let*

$$0 < x < \frac{\pi}{2}.$$

Then, for

$$\alpha > 0 \quad \text{or} \quad \alpha \leqq -1,$$

the following inequality holds true:

$$\left(\frac{\sin x}{x}\right)^{\alpha} > \frac{4\cos^{\alpha} x}{1 + \sqrt{1 + 8\cos^{2\alpha} x}}. \tag{12}$$

Proof Putting

$$\lambda = \mu = 1, \quad p = 2\alpha \quad \text{and} \quad q = \alpha$$

in Theorem 1, we get

$$\left(\frac{\sin x}{x}\right)^{2\alpha} + \cos^{-\alpha} x \left(\frac{\sin x}{x}\right)^{\alpha} - 2 > 0 \quad \left(0 < x < \frac{\pi}{2};\ \alpha > 0 \quad \text{or} \quad \alpha \leqq -1\right),$$

which is equivalent to the following inequality:

$$\left[\left(\frac{\sin x}{x}\right)^{\alpha} + \frac{\cos^{-\alpha} x + \sqrt{\cos^{-2\alpha} x + 8}}{2}\right] \cdot \left[\left(\frac{\sin x}{x}\right)^{\alpha} + \frac{\cos^{-\alpha} x - \sqrt{\cos^{-2\alpha} x + 8}}{2}\right] > 0.$$

We can now deduce from the above inequality that

$$\left(\frac{\sin x}{x}\right)^{\alpha} + \frac{\cos^{-\alpha} x - \sqrt{\cos^{-2\alpha} x + 8}}{2} > 0,$$

this is, that

$$\left(\frac{\sin x}{x}\right)^{\alpha} > \frac{-\cos^{-\alpha} x + \sqrt{\cos^{-2\alpha} x + 8}}{2} = \frac{4\cos^{\alpha} x}{1 + \sqrt{1 + 8\cos^{2\alpha} x}}. \tag{13}$$

The proof of Theorem 2 is thus completed. ■

As a consequence of Theorem 2, we immediately obtain the following refinement of the Sándor–Bencze conjectured inequality (11):

COROLLARY 3 *If*

$$0 < x < \frac{\pi}{2} \quad \textit{and} \quad \alpha > 0,$$

then

$$\left(\frac{\sin x}{x}\right)^{\alpha} > \frac{4\cos^{\alpha} x}{1 + \sqrt{1 + 8\cos^{2\alpha} x}} > \frac{2\cos^{\alpha} x}{1 + \cos^{\alpha} x} > \frac{\cos^{\alpha} x}{1 + \cos^{\alpha} x}. \tag{14}$$

In addition, upon replacing α by $-\alpha$ in Theorem 2, a *reversed* version of the Sándor–Bencze conjectured inequality (11) is derived as follows.

COROLLARY 4 *If*

$$0 < x < \frac{\pi}{2} \quad \textit{and} \quad \alpha \geqq 1,$$

then

$$\left(\frac{\sin x}{x}\right)^{\alpha} < \frac{\cos^{\alpha} x + \sqrt{\cos^{2\alpha} x + 8}}{4}. \tag{15}$$

Remark 2 Corollary 3 and Corollary 4 show that the inequality (12) is sharper and more general than the Sándor–Bencze conjectured inequality (11).

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