If a and b are nonnegative real numbers such that $a+b=2$, then

$$a^{2\sqrt{b}}+b^{2\sqrt{a}}\le 2.$$

Proof. Assume that $a\ge b$. For $a=2$ and $b=0$, the inequality is obvious. Otherwise, using the substitution $a=1+x$ and $b=1-x$, $0\le x<1$, we can write the desired inequality as $f(x)\le 2$, where

$$f(x)=e^{2\sqrt{1-x}\ln(1+x)}+e^{2\sqrt{1+x}\ln(1-x)}.$$

Case $\frac{13}{20}\le x<1$. If $f'(x)\le 0$, then $f(x)$ is decreasing, and hence $f(x)\le f(0)=2$. Since

$$f'(x)=[\frac{2\sqrt{1-x}}{1+x}-\frac{\ln(1+x)}{\sqrt{1-x}}]e^{2\sqrt{1-x}\ln(1+x)}+[\frac{-2\sqrt{1+x}}{1-x}+\frac{\ln(1-x)}{\sqrt{1+x}}]e^{2\sqrt{1+x}\ln(1-x)}$$

$$<[\frac{2\sqrt{1-x}}{1+x}-\frac{\ln(1+x)}{\sqrt{1-x}}]e^{2\sqrt{1-x}\ln(1+x)},$$

we have $f'(x)<0$ if $g(x)\le 0$, where $g(x)=\frac{2(1-x)}{1+x}-\ln(1+x)$. Since $g(x)$ is strictly decreasing and $g(\frac{13}{20})=\frac{14}{33}-\ln\frac{33}{20}<0$, it follows that $g(x)\le 0$ for $\frac{13}{20}\le x<1$.

Case $0\le x\le\frac{13}{20}$. Applying Lemma below, it suffices to show that $f(x)\le 2$, where

$$f(x)=e^{2x-2x^2+\frac{11}{12}x^3-\frac{1}{2}x^4}+e^{-(2x+2x^2+\frac{11}{12}x^3+\frac{1}{2}x^4)}.$$

If $f'(x)\le 0$, then $f(x)$ is decreasing, and hence $f(x)\le f(0)=2$. Since

$$f'(x)=(2-4x+\frac{11}{4}x^2-2x^3)e^{2x-2x^2+\frac{11}{12}x^3-\frac{1}{2}x^4}-(2+4x+\frac{11}{4}x^2+2x^3)e^{-(2x+2x^2+\frac{11}{12}x^3+\frac{1}{2}x^4)},$$

the inequality $f'(x)\le 0$ is equivalent to

$$e^{-4x-\frac{11}{6}x^3}\ge\frac{8-16x+11x^2-8x^3}{8+16x+11x^2+8x^3}.$$

For the nontrivial case $8-16x+11x^2-8x^3>0$, we rewrite this inequality as $g(x)\ge 0$, where

$$g(x)=-4x-\frac{11}{6}x^3-\ln(8-16x+11x^2-8x^3)+\ln(8+16x+11x^2+8x^3).$$

If $g'(x)\ge 0$, then $g(x)$ is increasing, and hence $g(x)\ge g(0)=0$. From

$$g'(x) = -4 - \frac{11}{2}x^2 + \frac{(16+24x^2)-22x}{8+11x^2-(16x+8x^3)} + \frac{(16+24x^2)+22x}{8+11x^2+(16x+8x^3)},$$

it follows that $g'(x) \ge 0$ is equivalent to

$$4[(16+24x^2)(8+11x^2)-22x(16x+8x^3)] \ge (8+11x^2)[(8+11x^2)^2-(16x+8x^3)^2].$$

Since

$$(8+11x^2)^2-(16x+8x^3)^2 \le (8+11x^2)^2 - 256x^2 - 256x^4 \le 16(4-5x^2),$$

it suffices to show that

$$(4+6x^2)(8+11x^2)-11x(8x+4x^3) \ge (8+11x^2)(4-5x^2).$$

This reduces to $77x^4 \ge 0$, which is clearly true. This completes the proof. Equality holds for $a = b = 1$.

Lemma. *If* $-1 < t \le \frac{13}{20}$, *then*

$$\sqrt{1-t}\,\ln(1+t) \le t - t^2 + \frac{11}{24}t^3 - \frac{1}{4}t^4.$$

Proof. We have two cases to consider.

Case $0 \le t \le \frac{13}{20}$. It suffices to show that

$$\sqrt{1-t} \le 1 - \frac{t}{2} - \frac{t^2}{8} - \frac{t^3}{16},$$

$$\ln(1+t) \le t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \frac{t^5}{5}$$

and

$$(1 - \frac{t}{2} - \frac{t^2}{8} - \frac{t^3}{16})(t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \frac{t^5}{5}) \le t - t^2 + \frac{11}{24}t^3 - \frac{1}{4}t^4.$$

The first inequality is equivalent to $f(t) \ge 0$, where

$$f(t) = \ln(1 - \frac{t}{2} - \frac{t^2}{8} - \frac{t^3}{16}) - \frac{1}{2}\ln(1-t).$$

From

$$f'(t) = \frac{-(\frac{1}{2} + \frac{t}{4} + \frac{3t^2}{16})}{1 - \frac{t}{2} - \frac{t^2}{8} - \frac{t^3}{16}} + \frac{1}{2(1-t)} = \frac{1}{2(1-t)} - \frac{8+4t+3t^2}{16-8t-2t^2-t^3}$$

$$=\frac{5t^3}{2(1-t)(16-8t-2t^2-t^3)}\ge 0,$$

it follows that $f(t)$ is increasing; therefore, $f(t)\ge f(0)=0$.

The second inequality is equivalent to $f(t)\ge 0$, where

$$f(t)=t-\frac{t^2}{2}+\frac{t^3}{3}-\frac{t^4}{4}+\frac{t^5}{5}-\ln(1+t).$$

From

$$f'(t)=1-t+t^2-t^3+t^4-\frac{1}{1+t}=\frac{t^5}{1+t},$$

it follows that $f(t)$ is increasing; therefore, $f(t)\ge f(0)=0$.

The third inequality is true if $f(t)\ge 0$, where

$$f(t)=t-t^2+\frac{11}{24}t^3-\frac{1}{4}t^4-(1-\frac{t}{2}-\frac{t^2}{8}-\frac{t^3}{16})(t-\frac{t^2}{2}+\frac{t^3}{3}-\frac{t^4}{4}+\frac{t^5}{5})$$

$$=t^4(\frac{1}{6}-\frac{151t}{480}+\frac{43t^2}{480}+\frac{3t^3}{320}+\frac{t^4}{80})=\frac{t^4(160-302t+86t^2+9t^3+12t^4)}{960}.$$

For $0\le t\le\frac{13}{20}$, we have

$$f(t)>\frac{2t^4(80-151t+43t^2)}{960}>0.$$

Case $-1<t\le 0$. Write the inequality as

$$-\sqrt{1-t}\,\ln(1+t)\ge -t+t^2-\frac{11}{24}t^3+\frac{1}{4}t^4.$$

It suffices to show that

$$\sqrt{1-t}\ge 1-\frac{t}{2}-\frac{t^2}{8},$$

$$-\ln(1+t)\ge -t+\frac{t^2}{2}-\frac{t^3}{3}+\frac{t^4}{4}$$

and

$$(1-\frac{t}{2}-\frac{t^2}{8})(-t+\frac{t^2}{2}-\frac{t^3}{3}+\frac{t^4}{4})\ge -t+t^2-\frac{11}{24}t^3+\frac{1}{4}t^4.$$

The first inequality is equivalent to $f(t)\ge 0$, where

$$f(t)=\frac{1}{2}\ln(1-t)-\ln(1-\frac{t}{2}-\frac{t^2}{8}).$$

From

$$f'(t)=\frac{-1}{2(1-t)}+\frac{\frac{1}{2}+\frac{t}{4}}{1-\frac{t}{2}-\frac{t^2}{8}}=\frac{-1}{2(1-t)}+\frac{2(t+2)}{8-4t-t^2}=\frac{-3t^2}{2(1-t)(8-4t-t^2)}\le 0\,,$$

it follows that $f(t)$ is decreasing; therefore, $f(t)\ge f(0)=0$.

The second inequality is equivalent to $f(t)\ge 0$, where

$$f(t)=t-\frac{t^2}{2}+\frac{t^3}{3}-\frac{t^4}{4}-\ln(1+t)\,.$$

From

$$f'(t)=1-t+t^2-t^3-\frac{1}{1+t}=\frac{-t^4}{1+t}\le 0\,,$$

it follows that $f(t)$ is decreasing; therefore, $f(t)\ge f(0)=0$.

The third inequality is true since

$$f(t)=(1-\frac{t}{2}-\frac{t^2}{8})(-t+\frac{t^2}{2}-\frac{t^3}{3}+\frac{t^4}{4})+t-t^2+\frac{11t^3}{24}-\frac{t^4}{4}=\frac{t^4(10-8t-3t^2)}{960}>0\,.$$