



华南理工大学

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工科数学分析 习题分析与解答

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第一章 集合, 映射与函数

1.1 第1周作业

例题 1.1.1 讨论下列函数的奇偶性

$$(1)y = 3x - x^3$$

$$(2)2 + 3x - x^3$$

$$(3)y = \sqrt[3]{(1+x)^2} + \sqrt[3]{(1-x)^2}$$

$$(4)y = \frac{e^x + e^{-x}}{2}$$

$$(5)y = \sqrt{x(2-x)}$$

$$(6)y = 2^{-x}$$

$$(7)f(x) = \begin{cases} x-1, & x < 0 \\ 0, & x = 0 \\ x+1, & x > 0 \end{cases}$$

$$(8)y = \ln(x + \sqrt{x^2 + 1})$$

解 1.1.1. (1) 由于 $f(x) + f(-x) = 3x + 3(-x) - x^3 - (-x)^3 = 0$, 故为奇函数

(2) 由于 $f(x) + f(-x) = 2 + 3x + 3(-x) + 2 - x^3 - (-x)^3 = 4$, 不为奇函数; 而 $4 \neq 2f(x) \Rightarrow f(x) \neq f(-x)$, 故为非奇非偶函数

(3) 由于 $f(x) = \sqrt[3]{(1+x)^2} + \sqrt[3]{(1-x)^2} = \sqrt[3]{(1-x)^2} + \sqrt[3]{(1+x)^2} = f(-x)$, 故为偶函数

(4) 由于 $f(x) = \frac{e^x + e^{-x}}{2} = \frac{e^{-x} + e^x}{2} = f(-x)$, 故为偶函数

(5) 由于 $f(x) = \sqrt{x(2-x)} \neq \sqrt{-x(2+x)} = f(-x)$, 故不为偶函数, 由于 $f(x) + f(-x) = \sqrt{x(2-x)} + \sqrt{-x(2+x)} \neq 0$, 故为非奇非偶函数

(6) 由于 $\begin{cases} f(x) + f(-x) = 2^{-x} + 2^{-(-x)} = 2^{-x} + 2^x \neq 0 \\ f(x) = 2^{-x} \neq 2^{-(-x)} = f(-x) \end{cases}$, 故为非奇非偶函数

(7) 由于 $\begin{cases} f(0) = 0 \\ f(x) + f(-x) = x-1 + (-x)+1 = 0 \\ f(x) \neq f(-x) \end{cases}$, 故为奇函数

(8) 由于 $f(x) + f(-x) = \ln(x + \sqrt{x^2 + 1}) + \ln(-x + \sqrt{x^2 + 1}) = \ln(-x^2 + (x^2 + 1)) = 0$, 故为奇函数

例题 1.1.2 研究函数的单调性

$$(1)y = ax + b \quad (2)y = ax^2 + bx + c \quad (3)y = x^3 \quad (4)y = a^x$$

解 1.1.2. (1) 若 $a \geq 0$, 则 y 单调递增; 若 $a < 0$, 则 y 单调递减; 若 $a > 0$, 则 y 严格单调递增

(2) 若 $a > 0$, 则 y 先严格单调减后严格单调增, 若 $a < 0$, 则 y 先严格单调增后严格单调减,

若 $a = 0$, 则当 $b > 0$ 时, y 单调递增, 当 $b < 0$ 时, y 单调递减; 若 $a = b = 0$, 则 y 非严格单调递增

(3) 若 $x_1 > x_2$, 则 $f(x_1) - f(x_2) = x_1^3 - x_2^3 = (x_1 - x_2)(x_1^2 + x_1x_2 + x_2^2) > x_1 - x_2 > 0$ 故单调递增

(4) 需限定 $a > 0$, 则当 $a > 1$ 时, y 单调递增, 当 $a < 1$ 时, y 单调递减; 若 $a = 1$, 则 $y = 1$ 非严格单调递增;

例题 1.1.3 哪些是周期函数? 如果是说明其周期, 并说明有无最小周期, 有就求出来

$$(1) y = \sin^2 x$$

$$(2) y = \sin x^2$$

$$(3) y = \cos(x - 2)$$

$$(4) y = A \cos \lambda x + B \sin \lambda x$$

$$(5) y = x - [x]$$

$$(6) y = \tan |x|$$

解 1.1.3. (1) 是周期函数, 周期为 $k\pi$, ($k \in \mathbb{Z}$), 最小正周期为 π

(2) 不是周期函数, 因为

$$\sin(x+T)^2 - \sin x^2 = 2 \cos \frac{(x+T)^2 + x^2}{2} \sin \frac{(x+T)^2 - x^2}{2} = 2 \cos \frac{(x+T)^2 + x^2}{2} \sin \frac{2x+T^2}{2} \neq 0$$

则这样的 T 不存在.

(3) 是周期函数, 周期为 $2k\pi$, ($k \in \mathbb{Z}$), 最小正周期为 2π .

(4) $y = A \cos \lambda x + B \sin \lambda x = \sqrt{A^2 + B^2} \sin(\lambda x + \arctan \frac{B}{A})$ 是周期函数, 周期为 $\frac{2k\pi}{\lambda}$, ($k \in \mathbb{Z}, \lambda > 0$), 最小正周期为 $\frac{2\pi}{\lambda}$, ($\lambda > 0$)

(5) 是周期函数, 因为 $[x] + 1 = [x + 1]$, 则 $x + 1 - [x + 1] = x + 1 - [x] - 1 = x - [x]$, 所以 $y = x - [x]$ 是周期函数, 周期为 \mathbb{Z} , 最小正周期为 1.

(6) 不是周期函数. 证明: 由于正切函数的一个周期是 π , 假设 $\tan |x|$ 也是周期函数, 则存在 $T > 0$ 使得对于定义域内的任意实数 x 都有 $|x| + \pi = |x + T|$, 代入 $x = -\pi$ 得到 $T = 3\pi$, 代入 $x = 0$ 得到 $T = \pi$, 矛盾! 所以 $y = \tan |x|$ 不是周期函数.

例题 1.1.4 证明

两个奇函数之积为偶函数, 奇函数和偶函数之积仍然是奇函数。

解 1.1.4. (1) 设 $f(x), g(x)$ 为两个奇函数, 则

$$f(x)g(x) = (-f(-x))(-g(-x)) = f(-x)g(-x)$$

故两个奇函数之积为偶函数

(2) 设 $f(x)$ 是奇函数, $g(x)$ 为偶函数, 则

$$f(x)g(x) = (-f(-x))(g(-x)) = -f(-x)g(-x) \Leftrightarrow f(x)g(x) + f(-x)g(-x) = 0$$

故奇函数和偶函数之积仍然是奇函数.

例题 1.1.5 证明

若函数 $f(x)$ 周期为 $T(T > 0)$, 则函数 $f(-x)$ 的周期也是 T .

解 1.1.5. 设 $f(x)$ 周期为 T , 则 $f(x+T) = f(x) \Rightarrow f(-x+T) = f(-x)$, 故 $f(-x)$ 的周期也是 T .

例题 1.1.6 证明

设 $f(x)$ 和 $g(x)$ 都是定义域为 R 的单调函数, 求证: $f(g(x))$ 也是定义域为 R 的单调函数.

解 1.1.6. 由于 $f(x), g(x)$ 是定义域为 R 的单调函数, 则:

$$\forall x_1, x_2 \in R, (x_1 - x_2)(f(x_1) - f(x_2)) \geq 0, \quad \forall x_3, x_4 \in R, (x_3 - x_4)(g(x_3) - g(x_4)) \geq 0,$$

那么一定存在 $x_1 = g(x_3), x_2 = g(x_4)$, 则相乘

$$(g(x_3) - g(x_4))(f(g(x_3)) - f(g(x_4))) \geq 0$$

结合 $x_3, x_4 \in R, (x_3 - x_4)(g(x_3) - g(x_4)) \geq 0$ 就有:

$$(x_3 - x_4)(g(x_3) - g(x_4))^2(f(g(x_3)) - f(g(x_4))) \geq 0 \Rightarrow (x_3 - x_4)(f(g(x_3)) - f(g(x_4))) \geq 0$$

故 $f(g(x))$ 也是定义域为 R 的单调函数.

例题 1.1.7 证明

$$(1) \arctan \frac{1}{2} + \arctan \frac{1}{3} = \frac{\pi}{4}. \quad (2) \cos(\arcsin x) = \sqrt{1-x^2}$$

解 1.1.7. (1) 构造复数 $z_1 = 2 + i, z_2 = 3 + i \Rightarrow z_1 z_2 = 5 + 5i$, 则:

$$\arg(z_1) + \arg(z_2) = \arg(z_1 z_2) = \frac{\pi}{4}$$

(2) 由于 $\sin^2 x + \cos^2 x = 1$, 代入 $x = \arcsin x$ 即可得到:

$$\cos(\arcsin x) = \sqrt{1-x^2}$$

第二章 极限

2.1 习题 2.1

例题 2.1.1 2.1-A-3: 给出下列极限的精确定义

$$(1) \lim_{x \rightarrow 0} f(x) = A \quad (2) \lim_{x \rightarrow 0} (1+x)^{\frac{1}{x}} = e$$

解 2.1.1. (1) 对于任意 $\varepsilon > 0$, 存在 $\delta > 0$ 使得当 $0 < |x| < \delta$ 时, $|f(x)| < \varepsilon$.

(2) 对于任意 $\varepsilon > 0$, 存在 $\delta > 0$ 使得当 $0 < |x| < \delta$ 时, $|(1+x)^{\frac{1}{x}} - e| < \varepsilon$.

例题 2.1.2 2.1-A-7

利用极限的精确定义证明下列函数的极限

$$(1) \lim_{x \rightarrow 3} (x^2 + 5x) = 24 \quad (2) \lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1} = 2$$

$$(3) \lim_{x \rightarrow +\infty} \frac{x^2 + 2}{3x^2} = \frac{1}{3} \quad (4) \lim_{x \rightarrow 2^+} \frac{2x}{x^2 - 4} = +\infty.$$

解 2.1.2. (1) 要证对于任意 $\varepsilon > 0$, 存在 $\delta > 0$ 使得当 $0 < |x - 3| < \delta$ 时, $|(x^2 + 5x) - 24| = |(x+8)(x-3)| < \varepsilon$. 已经出现了 $|x-3|$, 所以现在只需限定 $|x+8|$, 先限定 $|x-3| < 1$, 那么 $|x+8| < 12$, 此时还需满足 $|(x+8)(x-3)| < 12|x-3| < \varepsilon$, 得 $|x-3| < \frac{\varepsilon}{12}$, 故取 $\delta = \min\left\{1, \frac{\varepsilon}{12}\right\}$, 当 $0 < |x-3| < \delta$ 时, $|(x^2 + 5x) - 24| = |(x+8)(x-3)| < \varepsilon$.

(2) 要证对于任意 $\varepsilon > 0$, 存在 $\delta > 0$ 使得当 $0 < |x - 1| < \delta$ 时, $\left|\frac{x^2 - 1}{x - 1} - 2\right| = |x - 1| < \varepsilon$. 取 $\delta = \varepsilon$, 当 $0 < |x - 1| < \delta$ 时, $\left|\frac{x^2 - 1}{x - 1} - 2\right| < \varepsilon$.

(3) 要证对于任意 $\varepsilon > 0$, 存在 $\delta > 0$ 使得当 $|x| > \delta$ 时, $\left|\frac{x^2 + 2}{3x^2} - \frac{1}{3}\right| < \varepsilon$, 因为 $\left|\frac{x^2 + 2}{3x^2} - \frac{1}{3}\right| = \left|\frac{2}{3x^2}\right|$, 取 $\delta = \sqrt{\frac{2}{3\varepsilon}}$, 则当 $|x| > \delta$ 时, $\left|\frac{x^2 + 2}{3x^2} - \frac{1}{3}\right| < \varepsilon$.

(4) 要证对于任意 $G > 0$, 存在 $\delta > 0$ 使得当 $0 < x - 2 < \delta$ 时, $\frac{2x}{x^2 - 4} > G$, 不妨限定 $x + 2 < 5$, 则 $x - 2 < 1$, 则 $\frac{2x}{x^2 - 4} > \frac{4}{(x+2)(x-2)} > \frac{4}{5(x-2)} > G$ 解得 $x - 2 < \frac{4}{5G}$, 所以取 $\delta = \min\left\{1, \frac{4}{5G}\right\}$, 当 $0 < x - 2 < \delta$ 时, $\frac{2x}{x^2 - 4} > G$.

例题 2.1.3 2.1-A-10

证明: 由 $\lim_{x \rightarrow a} f(x) = A$ 能推出 $\lim_{x \rightarrow a} |f(x)| = |A|$, 但反之不然.

解 2.1.3. 对于任意 $\varepsilon > 0$, 存在 $\delta > 0$ 使得当 $0 < |x - a| < \delta$ 时, $|f(x) - A| < \varepsilon$, 所以由绝对值不等式得到 $|f(x) - A| > ||f(x)| - |A|| = ||f(x)| - |A|| > 0$, 故 $||f(x)| - |A|| < \varepsilon$, 所以由 $\lim_{x \rightarrow a} f(x) = A$

能推出 $\lim_{x \rightarrow a} |f(x)| = |A|$. 然后反过来, 考虑定义在实数域上的函数 $f(x) = \begin{cases} 1, & x \in Q \\ -1, & x \notin Q \end{cases}$, 其极限

$\lim_{x \rightarrow a} |f(x)| = 1$, 但是 $\lim_{x \rightarrow a} f(x)$ 不存在。

例题 2.1.4 2.1-B-2(1): 利用极限的精确证明

$$\lim_{x \rightarrow a} \sin x = \sin a$$

解 2.1.4. 要证 $\lim_{x \rightarrow a} \sin x = \sin a$, 只需证对于任意 $\varepsilon > 0$, 存在 $\delta > 0$ 使得当 $0 < |x - a| < \delta$ 时,

$$\lim_{x \rightarrow a} |\sin x - \sin a| = 2 \left| \cos \frac{x+a}{2} \right| \left| \sin \frac{x-a}{2} \right| < \varepsilon. \text{ 又因为 } 2 \left| \cos \frac{x+a}{2} \right| \left| \sin \frac{x-a}{2} \right| < 2 \left| \sin \frac{x-a}{2} \right| < 2 \left| \frac{x-a}{2} \right| = |x-a|, \text{ 所以取 } \delta = \varepsilon, \text{ 当 } 0 < |x-a| < \delta \text{ 时, } \lim_{x \rightarrow a} |\sin x - \sin a| < \varepsilon.$$

例题 2.1.5 2.1-B-4

利用极限的精确定义证明 $\lim_{x \rightarrow +\infty} \frac{x}{x+1} = +\infty$ 是错误的。

解 2.1.5. 要证明存在 $G > 0, \forall \delta > 0$ 使得当 $x > \delta$ 时, $\frac{x}{x+1} \leq G$, 则取 $G = 1$, 便可以满足 $\forall x > 0, \frac{x}{x+1} \leq 1$, 故存在 $G > 0, \forall \delta > 0$ 使得当 $x > \delta$ 时, $\frac{x}{x+1} \leq G$, $\lim_{x \rightarrow +\infty} \frac{x}{x+1} = +\infty$ 是错误的。(本题本质是找到一个够大的上界)

2.2 习题 2.3

例题 2.2.1 2.3-A-2

$$\begin{aligned} (1) & \lim_{x \rightarrow 2} (3x^2 - 5x + 2); & (2) & \lim_{x \rightarrow -1} (x^2 + 1)(1 - 2x)^2; & (3) & \lim_{x \rightarrow +\infty} (x^5 - 40x^4); \\ (4) & \lim_{x \rightarrow -\infty} (6x^5 + 21x^3); & (5) & \lim_{x \rightarrow 1^-} \frac{1}{x-1}; & (6) & \lim_{x \rightarrow 1^+} \frac{1}{x-1}; \\ (7) & \lim_{x \rightarrow +\infty} \frac{x^3 + 1}{x^4 + 2}; & (8) & \lim_{x \rightarrow -\infty} \frac{5x^3 + 2x}{x^{10} + x + 7}. \end{aligned}$$

$$\text{解 2.2.1. (1) } \lim_{x \rightarrow 2} (3x^2 - 5x + 2) = \lim_{x \rightarrow 2} (3(2)^2 - 5(2) + 2) = 4.$$

$$(2) \lim_{x \rightarrow -1} (x^2 + 1)(1 - 2x)^2 = \lim_{x \rightarrow -1} (1 + 1)(1 - 2(-1))^2 = 18.$$

$$(3) \lim_{x \rightarrow +\infty} (x^5 - 40x^4) = \lim_{x \rightarrow +\infty} x^4(x - 40) = \lim_{x \rightarrow +\infty} x^4 \lim_{x \rightarrow +\infty} (x - 40) = +\infty.$$

$$(4) \lim_{x \rightarrow -\infty} (6x^5 + 21x^3) = \lim_{x \rightarrow -\infty} x^5 \left(6 + \frac{21}{x^2}\right) = \lim_{x \rightarrow -\infty} x^5 \lim_{x \rightarrow -\infty} \left(6 + \frac{21}{x^2}\right) = -\infty.$$

$$(5) \lim_{x \rightarrow 1^-} \frac{1}{x-1} = \lim_{x \rightarrow 0^-} \frac{1}{x} = -\infty.$$

$$(6) \lim_{x \rightarrow 1^+} \frac{1}{x-1} = \lim_{x \rightarrow 0^+} \frac{1}{x} = +\infty.$$

$$(7) \lim_{x \rightarrow +\infty} \frac{x^3 + 1}{x^4 + 2} = \lim_{x \rightarrow +\infty} \frac{1 + \frac{1}{x^3}}{x + \frac{2}{x^3}} = 0.$$

$$(8) \lim_{x \rightarrow -\infty} \frac{5x^3 + 2x}{x^{10} + x + 7} = \lim_{x \rightarrow -\infty} \frac{\frac{5}{x^7} + \frac{2}{x^{10}}}{1 + \frac{1}{x^9} + \frac{7}{x^{10}}} = 0;$$

例题 2.2.2 2.3-A-4

$$\begin{aligned} (1) \lim_{x \rightarrow +\infty} (\sqrt{x^2 + x} - x) & \quad (2) \lim_{x \rightarrow -\infty} \left(\frac{\sqrt{3x^2 + x}}{x} \right) \\ (3) \lim_{x \rightarrow +\infty} \left(\frac{\sqrt{4x^2 + 2x + 1}}{3x} \right) & \quad (4) \lim_{x \rightarrow -\infty} \left(\frac{\sqrt{9x^2 + x + 3}}{6x} \right) \end{aligned}$$

解 2.2.2. (1) $\lim_{x \rightarrow +\infty} (\sqrt{x^2 + x} - x) = \lim_{x \rightarrow +\infty} \frac{x}{\sqrt{x^2 + x} + x} = \lim_{x \rightarrow +\infty} \frac{1}{\sqrt{1 + \frac{1}{x}} + 1} = \frac{1}{2}.$

(2) $\lim_{x \rightarrow -\infty} \left(\frac{\sqrt{3x^2 + x}}{x} \right) = - \lim_{x \rightarrow -\infty} \sqrt{3 + \frac{1}{x}} = -\sqrt{3}.$

(3) $\lim_{x \rightarrow +\infty} \left(\frac{\sqrt{4x^2 + 2x + 1}}{3x} \right) = \lim_{x \rightarrow +\infty} \sqrt{\frac{4}{9} + \frac{2}{9x} + \frac{1}{9x^2}} = \frac{2}{3}.$

(4) $\lim_{x \rightarrow -\infty} \left(\frac{\sqrt{9x^2 + x + 3}}{6x} \right) = - \lim_{x \rightarrow -\infty} \sqrt{\frac{9}{36} + \frac{1}{36x} + \frac{3}{6x^2}} = -\frac{1}{2}.$

例题 2.2.3 2.3-A-8

$$(1)y = \frac{x^2 - 2x - 2}{x - 1} \quad (2)y = \frac{2x^2}{(1-x)^2}$$

解 2.2.3. (1) $\lim_{x \rightarrow \infty} \frac{x^2 - 2x - 2}{(x-1)x} = \lim_{x \rightarrow \infty} \frac{1 - \frac{2}{x} - \frac{2}{x^2}}{1 - \frac{1}{x}} = 1.$

$$\lim_{x \rightarrow \infty} \frac{x^2 - 2x - 2}{x - 1} - x = \lim_{x \rightarrow \infty} \frac{x^2 - x^2 - 2x + x - 2}{x - 1} = \lim_{x \rightarrow \infty} \frac{-x - 2}{x - 1} = \lim_{x \rightarrow \infty} \frac{-1 - \frac{2}{x}}{1 - \frac{1}{x}} = -1$$

故该函数在无穷远处的渐近线为 $y = x - 1$.

$$\lim_{x \rightarrow 1^-} \frac{x^2 - 2x - 2}{x - 1} = \lim_{x \rightarrow 1^-} \frac{(x-1)^2 - 1}{x - 1} = \lim_{x \rightarrow 0^-} \left(x - \frac{1}{x} \right) = +\infty$$

$$\lim_{x \rightarrow 1^+} \frac{x^2 - 2x - 2}{x - 1} = \lim_{x \rightarrow 1^+} \frac{(x-1)^2 - 1}{x - 1} = \lim_{x \rightarrow 0^+} \left(x - \frac{1}{x} \right) = -\infty$$

故该函数在 $x = 1$ 处的渐近线为 $x = 1$.

(2)

$$\lim_{x \rightarrow \infty} \frac{2x^2}{(x-1)^2} = \lim_{x \rightarrow \infty} \frac{2}{(1 - \frac{1}{x})^2} = 2$$

故该函数在无穷远处的渐近线为 $y = 2$.

$$\lim_{x \rightarrow 2} \frac{2x^2}{(x-1)^2} = \lim_{x \rightarrow 1} \frac{2}{(1 - \frac{1}{x})^2} = +\infty$$

故该函数在 $x = 2$ 处的渐近线为 $x = 2$.

例题 2.2.4 习题 2.3-B 组-1

已知 $\lim_{x \rightarrow +\infty} f(x) = +\infty$, $\lim_{x \rightarrow +\infty} g(x) = +\infty$, 讨论下列极限的状态:

$$\begin{aligned} (1) \lim_{x \rightarrow +\infty} (f(x) + g(x)) \quad (2) \lim_{x \rightarrow +\infty} (f(x) - g(x)) \\ (3) \lim_{x \rightarrow +\infty} f(x)g(x) \quad (4) \lim_{x \rightarrow +\infty} \frac{f(x)}{g(x)} \end{aligned}$$

解 2.2.4. (1) $\lim_{x \rightarrow +\infty} (f(x) + g(x)) = \lim_{x \rightarrow +\infty} f(x) + \lim_{x \rightarrow +\infty} g(x) = +\infty + \infty = +\infty$.

(2) $\lim_{x \rightarrow +\infty} (f(x) - g(x))$ 不确定, 比如当 $f(x) = x$ 时, 假如 $g(x) = 2x$, 那么 $\lim_{x \rightarrow +\infty} (f(x) - g(x)) = -\infty$, 但当 $g(x) = \frac{x}{2}$ 时, $\lim_{x \rightarrow +\infty} (f(x) - g(x)) = \lim_{x \rightarrow +\infty} \frac{x}{2} = +\infty$, 当 $f(x) = g(x)$ 时, $\lim_{x \rightarrow +\infty} (f(x) - g(x)) = 0$.

(3) $\lim_{x \rightarrow +\infty} f(x)g(x) = \lim_{x \rightarrow +\infty} f(x) \lim_{x \rightarrow +\infty} g(x) = +\infty \cdot +\infty = +\infty$.

(4) $\lim_{x \rightarrow +\infty} \frac{f(x)}{g(x)}$ 不确定, 比如当 $f(x) = x$ 时, 假如 $g(x) = 2x$, 那么 $\lim_{x \rightarrow +\infty} \frac{f(x)}{g(x)} = \frac{1}{2}$, 但当 $g(x) = \sqrt{x}$ 时, $\lim_{x \rightarrow +\infty} \frac{f(x)}{g(x)} = \lim_{x \rightarrow +\infty} \sqrt{x} = \infty$, 又当 $g(x) = x^2$ 时, $\lim_{x \rightarrow +\infty} \frac{f(x)}{g(x)} = \lim_{x \rightarrow +\infty} \frac{1}{x} = 0$.

例题 2.2.5 习题 2.3-B 组-4

设 $\lim_{x \rightarrow \infty} \left(\frac{x^2 + 1}{x + 1} - ax - b \right) = 0$, 求 a 和 b .

解 2.2.5.

$$\begin{aligned} \lim_{x \rightarrow \infty} \left(\frac{x^2 + 1}{x + 1} - ax - b \right) &= \lim_{x \rightarrow \infty} \left(\frac{x^2 + 1 - ax^2 - ax - bx - b}{x + 1} \right) \\ &= \lim_{x \rightarrow \infty} \frac{(1-a)x^2 - (a+b)x + 1-b}{x+1} = \lim_{x \rightarrow \infty} \frac{(1-a)x - (a+b) + \frac{1-b}{x}}{1 + \frac{1}{x}} \\ &= \frac{\lim_{x \rightarrow \infty} \left((1-a)x - (a+b) + \frac{1-b}{x} \right)}{1} \\ &= \lim_{x \rightarrow \infty} (1-a)x - (a+b) = 0 \end{aligned}$$

必须有 $a = 1, b = -1$.

例题 2.2.6 习题 2.3-B 组-5

设 a, b, c 是常数, $a \neq 0$, 证明 $y = \frac{ax^2 + bx + c}{x + 1}$ 的图形有斜渐近线, 并求出渐近线方程.

解 2.2.6.

$$\lim_{x \rightarrow \infty} \frac{ax^2 + bx + c}{(x+1)x} = \lim_{x \rightarrow \infty} \frac{a + \frac{b}{x} + \frac{c}{x^2}}{1 + \frac{1}{x}} = a \neq 0$$

由此可知, $y = \frac{ax^2 + bx + c}{x + 1}$ 的图形有斜渐近线.

$$\lim_{x \rightarrow \infty} \frac{ax^2 + bx + c}{x + 1} - ax = \lim_{x \rightarrow \infty} \frac{(b-a)x + c}{x + 1} = \lim_{x \rightarrow \infty} \frac{b-a + \frac{c}{x}}{1 + \frac{1}{x}} = b - a$$

则渐近线方程为 $y = ax + b - a$.

2.3 习题 2.2

例题 2.3.1 习题 2.2-A-2

$$(1) \lim_{n \rightarrow \infty} (\sqrt{n-1} - \sqrt{n}) = 0 \quad (2) \lim_{n \rightarrow \infty} \frac{n^2}{2^n} = 0 \quad (3) \lim_{n \rightarrow \infty} \frac{n}{n+1} = 1 \quad (4) \lim_{n \rightarrow \infty} \frac{3n}{5n+1} = \frac{3}{5}$$

解 2.3.1. (1) 要证明对于任意 $\varepsilon > 0$, 存在 $N \in \mathbb{N}$ 使得当 $n \geq N$ 时有 $|\sqrt{n-1} - \sqrt{n}| < \varepsilon$.

$$|\sqrt{n-1} - \sqrt{n}| = \left| \frac{1}{\sqrt{n-1} + \sqrt{n}} \right| < \left| \frac{1}{2\sqrt{n-1}} \right| < \varepsilon$$

因此取 $N = \left\lceil 2 + \frac{1}{4\varepsilon^2} \right\rceil$, 则对于任意 $\varepsilon > 0$, 当 $n \geq N$ 时, 有 $|\sqrt{n-1} - \sqrt{n}| < \varepsilon$.

(2) 要证明对于任意 $\varepsilon > 0$, 存在 $N \in \mathbb{N}$ 使得 $n \geq N$ 时, $\left| \frac{n^2}{2^n} \right| < \varepsilon$, 限定 $n > 9$, 则 $2^n > n^3$, 则有 $\left| \frac{n^2}{2^n} \right| < \left| \frac{n^2}{n^3} \right| = 2 \left| \frac{1}{n} \right| < \varepsilon$, 则取 $N = \max \left\{ 9, \left\lceil 1 + \frac{2}{\varepsilon} \right\rceil \right\}$, 任意 $\varepsilon > 0$, $\left| \frac{n^2}{2^n} \right| < \varepsilon$.

(3) 要证明对于任意 $\varepsilon > 0$, 存在 $N \in \mathbb{N}$ 使得当 $n \geq N$ 时, $\left| \frac{n}{n+1} - 1 \right| = \left| \frac{1}{n+1} \right| < \varepsilon$, 则取 $N = \left\lceil \frac{1}{\varepsilon} \right\rceil$, 任意 $\varepsilon > 0$, $\left| \frac{n}{n+1} - 1 \right| < \varepsilon$.

(4) 要证明对于任意 $\varepsilon > 0$, 存在 $N \in \mathbb{N}$ 使得当 $n \geq N$ 时, $\left| \frac{3n}{5n+1} - \frac{3}{5} \right| = \frac{3}{5} \left| \frac{1}{5n+1} \right| < \left| \frac{1}{n} \right| < \varepsilon$, 则取 $N = \left\lceil \frac{1}{\varepsilon} + 1 \right\rceil$, 任意 $\varepsilon > 0$, $\left| \frac{3n}{5n+1} - \frac{3}{5} \right| < \varepsilon$.

例题 2.3.2 习题 2.2-A-4

证明: 由 $\lim_{x \rightarrow \infty} x_n = a$ 能推出 $\lim_{n \rightarrow \infty} |x_n| = |a|$, 但反过来不可以.

解 2.3.2. 对于任意 $\varepsilon > 0$, 存在 $N \in \mathbb{N}$ 使得当 $n \geq N$ 时, $||x_n| - |a|| < |x_n - a| < \varepsilon$, 因此 $\lim_{n \rightarrow \infty} |x_n| = |a|$, 但是考虑数列 $a_n = (-1)^n$, 则 $\lim_{n \rightarrow \infty} |a_n| = 1$, 但是去掉绝对值后, $\lim_{n \rightarrow \infty} a_n$ 不存在, 所以不能反过来.

例题 2.3.3 习题 2.2-B-1

$$(1) \lim_{n \rightarrow \infty} \frac{n^3 - 1}{n^2 + 1} = \infty \quad (2) \lim_{n \rightarrow \infty} \arctan n = \frac{\pi}{2} \quad (3) \lim_{n \rightarrow \infty} \frac{\sqrt{n}}{2\sqrt{n} + 1} = \frac{1}{2} \quad (4) \lim_{n \rightarrow \infty} \frac{a^n}{n!} = 0. (a > 1)$$

解 2.3.3. (1) 要证明任意 $G > 0$, 存在 $N \in \mathbb{N}$ 使得当 $n \geq N$ 时有 $\left| \frac{n^3 - 1}{n^2 + 1} \right| = \frac{n^3 - 1}{n^2 + 1} > G$, 由

$$\frac{n^3 - 1}{n^2 + 1} = \frac{(n-1)(n^2 + n + 1)}{n^2 + 1} > \frac{(n-1)(n^2 + 1)}{n^2 + 1} = n - 1$$

所以取 $N = G + 2$, 则任意 $G > 0$, $\left| \frac{n^3 - 1}{n^2 + 1} \right| > G$.

(2) 要证明对于任意 $\varepsilon > 0$, 存在 $N \in \mathbb{N}$ 使得当 $n \geq N$ 时, $\left| \arctan n - \frac{\pi}{2} \right| = \frac{\pi}{2} - \arctan n < \varepsilon$, 则取 $N = \left[\tan \frac{\pi}{2} - \varepsilon \right] + 1$, 任意 $\varepsilon > 0$, $\left| \arctan n - \frac{\pi}{2} \right| < \varepsilon$.

(3) 要证明对于任意 $\varepsilon > 0$, 存在 $N \in \mathbb{N}$ 使得当 $n \geq N$ 时, $\left| \frac{\sqrt{n}}{2\sqrt{n} + 1} - \frac{1}{2} \right| = \frac{1}{2(2\sqrt{n} + 1)}$, 取 $N = \left[1 + \left(\frac{1}{4\varepsilon} - \frac{1}{2} \right)^2 \right]$, 任意 $\varepsilon > 0$, $\left| \frac{\sqrt{n}}{2\sqrt{n} + 1} - \frac{1}{2} \right| < \varepsilon$.

(4) 要证明对于任意 $\varepsilon > 0$, 存在 $N \in \mathbb{N}$ 使得当 $n \geq N$ 时, $\left| \frac{a^n}{n!} \right| = \frac{a^n}{n!} < \varepsilon$, 由

$$\frac{a^n}{n!} = \frac{a}{1} \frac{a}{2} \frac{a}{3} \cdots \frac{a}{[a]} \frac{a}{[a] + 1} \cdots \frac{a}{n} < \frac{a^{[a]}}{[a]!} \frac{a}{n}$$

所以取 $N = \left[\frac{a^{[a+1]}}{[a]! \varepsilon} + 1 \right]$, 则任意 $\varepsilon > 0$, $\left| \frac{a^n}{n!} \right| < \varepsilon$.

例题 2.3.4 2.3-A-3

$$(1) \lim_{n \rightarrow \infty} \frac{(-2)^n + 3^n}{(-2)^{n+1} + 3^{n+1}}$$

$$(2) \lim_{n \rightarrow \infty} \left(\frac{1}{n^2} + \frac{2}{n^2} + \cdots + \frac{n-1}{n^2} \right)$$

$$(3) \lim_{n \rightarrow \infty} \left[\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{n(n+1)} \right]$$

$$(4) \lim_{n \rightarrow \infty} \left[\frac{1}{2!} + \frac{2}{3!} + \cdots + \frac{n}{(n+1)!} \right]$$

解 2.3.4. (1) 代数变形:

$$\frac{(-2)^n + 3^n}{(-2)^{n+1} + 3^{n+1}} = \frac{(-2)^{-1} + \frac{3^n}{(-2)^{n+1}}}{1 + \left(-\frac{3}{2}\right)^{n+1}} = \frac{-\frac{1}{2} + \frac{1}{3}\left(-\frac{3}{2}\right)^{n+1}}{1 + \left(-\frac{3}{2}\right)^{n+1}} = \frac{1}{3} - \frac{5}{6\left(1 + \left(-\frac{3}{2}\right)^{n+1}\right)}$$

$$\text{故 } \lim_{n \rightarrow \infty} \frac{(-2)^n + 3^n}{(-2)^{n+1} + 3^{n+1}} = \lim_{n \rightarrow \infty} \left(\frac{1}{3} - \frac{5}{6\left(1 + \left(-\frac{3}{2}\right)^{n+1}\right)} \right) = \frac{1}{3}.$$

(2)

$$\lim_{n \rightarrow \infty} \left(\frac{1}{n^2} + \frac{2}{n^2} + \cdots + \frac{n-1}{n^2} \right) = \lim_{n \rightarrow \infty} \frac{n(n-1)}{2n^2} = \frac{1}{2}$$

(3)

$$\lim_{n \rightarrow \infty} \left[\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{n(n+1)} \right] = \lim_{n \rightarrow \infty} \left(1 - \frac{1}{n+1} \right) = 1$$

(4) 用裂项 $\frac{k}{(k+1)!} = \frac{k+1-1}{(k+1)!} = \frac{1}{k!} - \frac{1}{(k+1)!}$, 那么

$$\lim_{n \rightarrow \infty} \left[\frac{1}{2!} + \frac{2}{3!} + \cdots + \frac{n}{(n+1)!} \right] = \lim_{n \rightarrow \infty} \left(\frac{1}{2} - \frac{1}{(n+1)!} \right) = \frac{1}{2}$$

2.4 习题 2.4

例题 2.4.1 2.4-A-5

$$(1) \lim_{x \rightarrow 0} \frac{\sin x}{2x}$$

$$(2) \lim_{x \rightarrow 0} \frac{\sin 2x}{x}$$

$$(3) \lim_{x \rightarrow 0} \frac{\sin 3x}{5x}$$

$$(4) \lim_{x \rightarrow 0} \frac{2x}{\sin 3x}$$

$$(5) \lim_{\theta \rightarrow 0} \frac{\sin^2 \theta}{\theta}$$

$$(6) \lim_{h \rightarrow 0} \frac{\tan^2 h}{h}$$

$$(7) \lim_{\theta \rightarrow 0} \frac{1 - \cos \theta}{\theta^2}$$

$$(8) \lim_{x \rightarrow 0} \frac{\sin^2 x}{x^2}$$

$$(9) \lim_{x \rightarrow 0} \frac{\tan x - \sin x}{\sin^3 x}$$

$$(10) \lim_{x \rightarrow 0} \frac{\sin 5x - \sin 3x}{\cos x - \cos a}$$

$$(11) \lim_{x \rightarrow 0} \frac{\cos x - \cos 3x}{x^2}$$

$$(12) \lim_{x \rightarrow a} \frac{\sin x - \sin a}{x - a}$$

$$(13) \lim_{x \rightarrow a} \frac{\sin x}{x - a}$$

$$(14) \lim_{x \rightarrow a} \frac{\tan x - \tan a}{x - a}$$

解 2.4.1. 这里只使用基本极限公式:

$$(1) \lim_{x \rightarrow 0} \frac{\sin x}{2x} = \lim_{x \rightarrow 0} \frac{1}{2} \frac{\sin x}{x} = \frac{1}{2}. \quad (2) \lim_{x \rightarrow 0} \frac{\sin 2x}{x} = \lim_{x \rightarrow 0} \frac{2 \sin 2x}{2x} = 2.$$

$$(3) \lim_{x \rightarrow 0} \frac{\sin 3x}{5x} = \lim_{x \rightarrow 0} \frac{3 \sin 3x}{5 \cdot 3x} = \frac{3}{5}. \quad (4) \lim_{x \rightarrow 0} \frac{2x}{\sin 3x} = \lim_{x \rightarrow 0} \frac{2}{3} \frac{3x}{\sin 3x} = \frac{2}{3}.$$

$$(5) \lim_{\theta \rightarrow 0} \frac{\sin^2 \theta}{\theta} = \lim_{\theta \rightarrow 0} \frac{\sin^2 \theta}{\theta^2} \theta = 0. \quad (6) \lim_{h \rightarrow 0} \frac{\tan^2 h}{h} = \lim_{h \rightarrow 0} \frac{\tan^2 h}{h^2} h = 0.$$

$$(7) \lim_{\theta \rightarrow 0} \frac{1 - \cos \theta}{\theta^2} = \lim_{\theta \rightarrow 0} \frac{\sin^2 \frac{\theta}{2}}{2 \left(\frac{\theta}{2} \right)^2} = \frac{1}{2}. \quad (8) \lim_{x \rightarrow 0} \frac{\sin^2 x}{x^2} = \left(\lim_{x \rightarrow 0} \frac{\sin x}{x} \right)^2 = 1.$$

$$(9) \lim_{x \rightarrow 0} \frac{\tan x - \sin x}{\sin^3 x} = \lim_{x \rightarrow 0} \frac{\sin x(1 - \cos x)}{\sin^3 x \cos x} = \lim_{x \rightarrow 0} \frac{1 - \cos x}{(1 - \cos^2 x) \cos x} = \lim_{x \rightarrow 0} \frac{\sec x}{(1 + \cos x)} = \frac{1}{2}.$$

$$(10) \lim_{x \rightarrow 0} \frac{\sin 5x - \sin 3x}{\sin x} = \lim_{x \rightarrow 0} \frac{2 \cos 4x \sin x}{\sin x} = \lim_{x \rightarrow 0} 2 \cos 4x = 2.$$

$$(11) \lim_{x \rightarrow 0} \frac{\cos x - \cos 3x}{x^2} = \lim_{x \rightarrow 0} \frac{2 \sin 2x \sin x}{x^2} = 4 \lim_{x \rightarrow 0} \frac{\sin 2x}{2x} \frac{\sin x}{x} = 4.$$

$$(12) \lim_{x \rightarrow a} \frac{\sin x - \sin a}{x - a} = \lim_{x \rightarrow a} \frac{\cos \frac{x+a}{2} \sin \frac{x-a}{2}}{\frac{x-a}{2}} = \cos a.$$

$$(13) \lim_{x \rightarrow a} \frac{\cos x - \cos a}{x - a} = \lim_{x \rightarrow a} \frac{-\sin \frac{x+a}{2} \sin \frac{x-a}{2}}{\frac{x-a}{2}} = -\sin a.$$

$$(14) \lim_{x \rightarrow a} \frac{\tan x - \tan a}{x - a} = \lim_{x \rightarrow a} \frac{\frac{\sin x}{\cos x} - \frac{\sin a}{\cos a}}{x - a} = \lim_{x \rightarrow a} \frac{\sin(x-a)}{(x-a) \cos a \cos(x-a)} = \sec^2 a, \text{ 其中 } a \neq$$

$$(2k+1)\frac{\pi}{2}, \quad k \in \mathbb{Z}_+.$$

例题 2.4.2 2.4-A-6

$$\begin{aligned} (1) \lim_{x \rightarrow \infty} \left(1 + \frac{3}{x}\right)^{x/2} & \quad (2) \lim_{x \rightarrow 0} (1-x)^{1/x} & (3) \lim_{\Delta x \rightarrow 0} \left(1 + \frac{\Delta x}{x}\right)^{x/\Delta x} (x \neq 0) \\ (4) \lim_{x \rightarrow 0} (1+ax)^{1/x} & \quad (5) \lim_{x \rightarrow \infty} \left(1 - \frac{1}{x}\right)^{4x} \end{aligned}$$

解 2.4.2. (1) $\lim_{x \rightarrow \infty} \left(1 + \frac{3}{x}\right)^{\frac{x}{2}} = \lim_{x \rightarrow \infty} \left(\left(1 + \frac{3}{x}\right)^{x/3}\right)^{\frac{3}{2}} = e^{\frac{3}{2}}.$

(2) $\lim_{x \rightarrow 0} (1-x)^{\frac{1}{x}} = \lim_{x \rightarrow 0} \left(\left(1 + \frac{1}{\frac{-1}{x}}\right)^{\frac{-1}{x}}\right)^{-1} = \frac{1}{e}.$

(3) $\lim_{\Delta x \rightarrow 0} \left(1 + \frac{\Delta x}{x}\right)^{x/\Delta x} (x \neq 0) = \lim_{\frac{\Delta x}{x} \rightarrow 0} \left(\left(1 + \frac{\Delta x}{x}\right)^{x/\Delta x}\right) = e$

(4) $\lim_{x \rightarrow 0} (1+ax)^{\frac{1}{x}} = \lim_{x \rightarrow 0} \left[\left(1 + \frac{1}{\frac{1}{ax}}\right)^{\frac{1}{ax}}\right]^a = \lim_{\frac{1}{ax} \rightarrow \infty} \left[\left(1 + \frac{1}{\frac{1}{ax}}\right)^{\frac{1}{ax}}\right]^a = e^a$

(5) $\lim_{x \rightarrow \infty} \left(1 - \frac{1}{x}\right)^{4x} = \lim_{x \rightarrow \infty} \left[\left(1 + \frac{1}{-x}\right)^{-x}\right]^{-4} = e^{-4}.$

例题 2.4.3 2.4-B-4

$$\begin{aligned} (1) \lim_{x \rightarrow \infty} \frac{3x-5}{x^3 \sin \frac{1}{x^2}} & \quad (2) \lim_{x \rightarrow \frac{\pi}{6}} \sin\left(\frac{\pi}{6} - x\right) \tan 3x & (3) \lim_{x \rightarrow 0} \frac{2 \sin x - \sin 2x}{x^3} \\ (4) \lim_{x \rightarrow \frac{\pi}{4}} \frac{\tan x - 1}{x - \frac{\pi}{4}} & \quad (5) \lim_{x \rightarrow 1} x^{\frac{1}{1-x}} \end{aligned}$$

解 2.4.3. (1) $\lim_{x \rightarrow \infty} \frac{3x-5}{x^3 \sin \frac{1}{x^2}} = \lim_{x \rightarrow 0} \frac{\frac{3}{x} - 5}{\frac{1}{x^3} \sin x^2} = \lim_{x \rightarrow 0} \frac{3x^2 - 5x^3}{\sin x^2} = 3 \lim_{x \rightarrow 0} \frac{x^2}{\sin x^2} \frac{3x^2 - 5x}{3x^2} = 3.$

(2) $\lim_{x \rightarrow \frac{\pi}{6}} \sin\left(\frac{\pi}{6} - x\right) \tan 3x = \lim_{x \rightarrow 0} -\sin x \tan\left(3x + \frac{\pi}{2}\right) = \lim_{x \rightarrow 0} \frac{\sin x}{\tan 3x} = \frac{1}{3} \lim_{x \rightarrow 0} \frac{\sin x}{x} \frac{x}{\tan x} = \frac{1}{3}.$

(3) $\lim_{x \rightarrow 0} \frac{2 \sin x - \sin 2x}{x^3} = 2 \lim_{x \rightarrow 0} \frac{\sin x}{x} \frac{1 - \cos x}{x^2} = 2 \frac{1}{2} = 1.$

(4) $\lim_{x \rightarrow \frac{\pi}{4}} \frac{\tan x - 1}{x - \frac{\pi}{4}} = \lim_{x \rightarrow 0} \frac{\frac{\tan x + 1}{1 - \tan x} - 1}{x} = \lim_{x \rightarrow 0} \frac{2 \tan x}{x(1 - \tan x)} = \lim_{x \rightarrow 0} \frac{2 \tan x}{x} = 2.$

(5) $\lim_{x \rightarrow 1} x^{\frac{1}{1-x}} = \lim_{x \rightarrow 0} (1+x)^{\frac{-1}{x}} = \lim_{x \rightarrow \infty} \left(\left(1 + \frac{1}{x}\right)x\right)^{-1} = \frac{1}{e}.$

2.5 习题 2.5

例题 2.5.1 2.5-A-2

证明数列 $\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \dots$ 收敛, 并求其极限值.

解 2.5.1. 先证数列 $\{a_n\}$ 有界, 数列满足 $a_1 = \sqrt{2}$, $a_n = \sqrt{2a_{n-1}}$, 由于 $a_1 = \sqrt{2} < 2$, 假设 $a_k < 2, (k \geq 2)$, 则有 $a_{k+1} = \sqrt{2a_k} < \sqrt{4} = 2$, 所以归纳得到 $a_k < 2$, 因此数列 $\{a_n\}$ 有界.

再证明数列 $\{a_n\}$ 单调递增, 作商得到 $\frac{a_{n+1}}{a_n} = \frac{\sqrt{2a_n}}{a_n} = \sqrt{\frac{2}{a_n}} > 1$, 所以数列 $\{a_n\}$ 单调递增.

由单调有界收敛定理得到 $\{a_n\}$ 收敛, 极限存在, 所以设极限为 A , 对 $a_n = \sqrt{2a_{n-1}}$ 两边取极限得

$$\lim_{n \rightarrow \infty} a_n = \sqrt{2 \lim_{n \rightarrow \infty} a_{n-1}} \Rightarrow A = 2$$

所以数列 $\{a_n\}$ 收敛, 且 $\lim_{n \rightarrow \infty} a_n = 2$.

进一步地, 设 $b_n = a_n - 2$, 所以

$$b_{n+1} = a_{n+1} - 2 = \sqrt{2a_n} - 2 = \sqrt{2(2 + b_n)} - 2 = \sqrt{4 + 2b_n} - 2$$

泰勒展开得到

$$\sqrt{4 + 2b_n} = 2\sqrt{1 + \frac{b_n}{2}} = 2\left(1 + \frac{b_n}{4} - \frac{b_n^2}{32} + O(b_n^3)\right) \Rightarrow b_{n+1} = \frac{b_n}{2} - \frac{b_n^2}{16} + O(b_n^3)$$

对于大的 n , 有 $b_n \rightarrow 0$, 主导项为 $\frac{b_n}{2}$, 因此 $b_{n+1} \sim \frac{1}{2}b_n$, 这意味着 b_n 以指数速率衰减, 即 $b_n \sim C\left(\frac{1}{2}\right)^{n-1}$, 其中 $C = b_1 = \sqrt{2} - 2$. 因此 $b_n^2 \sim C^2\left(\frac{1}{4}\right)^{n-1}$, 因此误差项为 $O\left(\left(\frac{1}{4}\right)^n\right)$

$$a_n = 2 + (\sqrt{2} - 2)\left(\frac{1}{2}\right)^{n-1} + o\left(\left(\frac{1}{2}\right)^n\right) \quad (n \rightarrow \infty)$$

例题 2.5.2 2.5-A-3

设 $a > 0, 0 < x_1 < \frac{1}{a}, x_{n+1} = x_n(2 - ax_n), n = 1, 2, 3, \dots$ 证明 $\{x_n\}$ 收敛并求极限.

解 2.5.2. 先证明数列 $\{x_n\}$ 有界, 已知 $0 < x_1 < \frac{1}{a}$, 则假设 $x_k \in (0, \frac{1}{a}), k \in \mathbb{N}_+$, 则 $ax_k \in (0, 1)$, 而 $x_k(2 - ax_k)$ 是 x_k 的二次函数, 在 $(0, \frac{1}{a})$ 上单增, 在 $(\frac{1}{a}, \frac{2}{a})$ 上单减, 所以 $x_{k+1} = x_k(2 - ax_k) > 0$, 且 $x_{k+1} < \frac{1}{a}(2 - a\frac{1}{a}) = \frac{1}{a}$, 所以 $x_n \in (0, \frac{1}{a})$, 数列 $\{x_n\}$ 有界. 再证明数列 $\{x_n\}$ 单调递增, 作商得到 $\frac{x_{n+1}}{x_n} = 2 - ax_n > 1$, 所以数列 $\{x_n\}$ 单调递增. 由单调有界收敛定理得到 $\{x_n\}$ 收敛, 极限存在, 所以设极限为 A , 对 $x_{n+1} = x_n(2 - ax_n)$ 两边取极限得到

$$\lim_{n \rightarrow \infty} x_{n+1} = \lim_{n \rightarrow \infty} x_n(2 - a \lim_{n \rightarrow \infty} x_n) \Leftrightarrow A = A(2 - aA) \Leftrightarrow A = \frac{1}{a}$$

所以数列 $\{x_n\}$ 收敛, 且 $\lim_{n \rightarrow \infty} x_n = \frac{1}{a}$, 事实上这个数列的通项公式是 $x_n = \frac{1}{a} [1 - (1 - ax_1)^{2^{n-1}}]$

例题 2.5.3 2.5-A-4

设 $x_1 > 0, x_{n+1} = \frac{3(1+x_n)}{3+x_n}, n = 1, 2, 3, \dots$ 证明 $\lim_{n \rightarrow \infty} x_n = \sqrt{3}$.

解 2.5.3. (1) 当 $x_1 = \sqrt{3}$ 时, 假设 $x_k = \sqrt{3}, k \in \mathbb{N}_+$, 则

$$x_{k+1} = \frac{3(1+\sqrt{3})}{3+\sqrt{3}} = \frac{3\sqrt{3}+3}{3+\sqrt{3}} = \sqrt{3}$$

所以 $\{x_n = \sqrt{3}\}, \lim_{n \rightarrow \infty} x_n = \sqrt{3}$.

(2) 当 $0 < x_1 < \sqrt{3}$ 时, 假设 $0 < x_k < \sqrt{3}, k \in \mathbb{N}_+$, 则由函数 $f(x) = \frac{3x+3}{3+x} = 3 - \frac{6}{3+x}$ 在 $(0, \sqrt{3})$ 上单增, 得到:

$$\frac{3+0}{3+0} = 1 < x_{k+1} = \frac{3(1+x_k)}{3+x_k} < \frac{3+3\sqrt{3}}{3+\sqrt{3}} = \sqrt{3}$$

所以归纳得到 $x_k \in (0, \sqrt{3})$, 且

$$x_{n+1} - x_n = \frac{3+3x_n-3x_n-x_n^2}{3+x_n} = \frac{3-x_n^2}{3+x_n} > 0$$

则 $\{x_n\}$ 单调递增, 由单调有界收敛定理得到 $\{x_n\}$ 收敛, 设极限为 A , 代入递推式得到 $A = \frac{3+3A}{3+A} \Rightarrow A = \sqrt{3}$, 则 $\lim_{n \rightarrow \infty} x_n = \sqrt{3}$.

(3) 当 $x_1 > \sqrt{3}$ 时, 假设 $x_k > \sqrt{3}, k \in \mathbb{N}_+$, 则由函数 $f(x) = \frac{3x+3}{3+x} = 3 - \frac{6}{3+x}$ 在 $(\sqrt{3}, \infty)$ 上单增, 得到:

$$x_{k+1} = \frac{3(1+x_k)}{3+x_k} > \frac{3+3\sqrt{3}}{3+\sqrt{3}} = \sqrt{3}$$

所以数列 $\{x_n\}$ 有下界, 又有:

$$x_{n+1} - x_n = \frac{3+3x_n-3x_n-x_n^2}{3+x_n} = \frac{3-x_n^2}{3+x_n} < 0$$

所以 $\{x_n\}$ 单调递减, 由单调有界收敛定理得到 $\{x_n\}$ 收敛, 设极限为 A , 代入递推式得到 $A = \frac{3+3A}{3+A} \Rightarrow A = \sqrt{3}$, 那么 $\lim_{n \rightarrow \infty} x_n = \sqrt{3}$.

例题 2.5.4 2.5-B-1

设数列 $\{x_n\}$ 满足 $x_1 = 10, x_{n+1} = \sqrt{x_n+6}, n = 1, 2, 3, \dots$, 证明 $\{x_n\}$ 极限存在, 并求极限.

解 2.5.4. 先证明数列 $\{x_n\}$ 有界, 由 $x_1 = 10 > 3$, 假设 $x_k > 3$, 那么 $x_{k+1} = \sqrt{x_k+6} > \sqrt{9} = 3$, 所以 $\{x_n\}$ 有界; 再证明数列 $\{x_n\}$ 单调递减, 作商得到

$$x_{n+1} - x_n = \sqrt{x_n+6} - x_n = \frac{x_n+6-x_n^2}{\sqrt{x_n+6}+x_n} = \frac{(3-x_n)(x_n+2)}{\sqrt{x_n+6}+x_n} > 0$$

所以 $\{x_n\}$ 单调递减, 由单调有界收敛定理得到 $\{x_n\}$ 极限存在, 设极限为 A , 代入递推式得到 $\lim_{n \rightarrow \infty} x_n = 3$.

例题 2.5.5 2.5-B-2

利用柯西准则, 证明下面各数列的收敛性:

(1) $x_n = a_0 + a_1q + \cdots + a_nq^n$, 其中 $|a_i| \leq M$ ($i = 0, 1, 2, \cdots$), 且 $|q| < 1$;

(2) $x_n = \frac{\sin 1}{2} + \frac{\sin 2}{2^2} + \cdots + \frac{\sin n}{2^n}$.

解 2.5.5. (1) 设 $m > n$, 则

$$|x_m - x_n| = |a_{n+1}q^{n+1} + a_{n+2}q^{n+2} + \cdots + a_mq^m| \leq M|q^{n+1} + q^{n+2} + \cdots + q^m| < M \frac{|q^{n+1}|}{1 - |q|}$$

任意 $\varepsilon > 0$, 存在 $N = [\log_q \frac{\varepsilon}{M} + 1]$ 使得任意 $x > N$, $M \frac{|q^{n+1}|}{1 - |q|} < M|q^{n+1}| < \varepsilon$, 则数列 $\{x_n\}$ 收敛. (2) 设 $m > n$, 则

$$|x_m - x_n| = \left| \frac{\sin(n+1)}{2^{n+1}} + \frac{\sin(n+2)}{2^{n+2}} + \cdots + \frac{\sin(m)}{2^m} \right| < \left| \frac{1}{2^{n+1}} + \frac{1}{2^{n+2}} + \cdots + \frac{1}{2^m} \right| < \frac{1}{2^n}$$

所以任意 $\varepsilon > 0$, 存在 $N = [2 - \log_2 \varepsilon]$ 使得任意 $n > N$, $|x_m - x_n| < \frac{1}{2^n} < \varepsilon$, 则数列 $\{x_n\}$ 收敛.

例题 2.5.6 2.5-B-3

对于数列 $\{x_n\}$, 若子列 $\{x_{2k}\}$ 与 $\{x_{2k+1}\}$ 都收敛于 a , 试用 “ $\varepsilon - N$ ” 的语言证明数列 $\{x_n\}$ 也收敛于 a .

解 2.5.6. 任意 $\varepsilon > 0$, 存在 $N_1 \in \mathbb{N}$ 使得 $\forall n > N_1, |x_{2n-1} - a| < \frac{\varepsilon}{2}$; 任意 $\varepsilon > 0$, 存在 $N_2 \in \mathbb{N}$ 使得 $\forall n > N_2, |x_{2n} - a| < \frac{\varepsilon}{2}$, 则数列 $\{x_n\}$ 收敛于 a ; 那么对 $2n - 1$ 代入 $n = N_1 + 1$, 对 $2n$ 代入 $n = N_2 + 1$, 可知取 $N = \max\{2N_1 + 4, 2N_2 + 4\}$, 则 $\forall n > N, |x_n - a| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$, 则数列 $\{x_n\}$ 收敛于 a .

例题 2.5.7 2.5-B-4

证明: 若 $f(x)$ 为定义于 $[a, +\infty)$ 上的单调增加函数, 则极限 $\lim_{x \rightarrow +\infty} f(x)$ 存在的充要条件是 $f(x)$ 在 $[a, +\infty)$ 上有上界.

解 2.5.7. 必要性: 设极限为 A , 则存在 $N > 0$ 使得当 $n > N$ 时, 有 $|f(x) - A| < 1$, 故

$$|f(x)| - |A| < ||f(x)| - |A|| < |f(x_n) - A| < 1 \Rightarrow |f(x_n)| < 1 + |A|$$

所以 $f(x)$ 在 $[a, +\infty)$ 上有上界.

充分性: 若 $f(x)$ 在 $[a, +\infty)$ 上有上界, 则由确界定理得到 $f(x)$ 有上确界 $\sup_{x \in [a, +\infty)} f(x)$, 由确界定义知道, $\forall \varepsilon_n = \frac{1}{n}, (n = 1, 2, \cdots), \exists x_n \in [a, +\infty)$, 使得 $|f(x_n) - A| < \frac{1}{n}$, 于是得到数列 $\{x_n\}$ 满足

$\lim_{n \rightarrow \infty} f(x_n) = A$, 所以 $\forall \varepsilon > 0, \exists N$ 使得 $|f(x_n) - A| < \varepsilon$, 由于 $f(x)$ 为定义于 $[a, +\infty)$ 上的单调增加函数, 所以任意 $x > x_{N+1}$, 均有 $f(x_{N+1}) \leq f(x) \leq A$, 于是 $|f(x) - A| < \varepsilon$, 所以 $\lim_{x \rightarrow +\infty} f(x) = A$.

例题 2.5.8 2.6-B-1

证明: 当 $x \rightarrow 0$ 时, 有 $\tan x - \sin x \sim \frac{1}{2}x^3, \arctan x \sim \frac{1}{4}\sin 4x$.

解 2.5.8. (1) $\tan x - \sin x = \tan x(1 - \cos x) \sim x \frac{x^2}{2} = \frac{x^3}{2}, x \rightarrow 0$.

(2) $\arctan x \sim x = \frac{4x}{4} \sim \frac{\sin 4x}{4}$, 由 $\tan x \sim x, x \rightarrow 0$ 换元 $x = \arctan y$ 可得 $y \sim \arctan x, x \rightarrow 0$.

例题 2.5.9 2.6-B-2

利用等价无穷小求极限: $\lim_{x \rightarrow 0} \frac{\tan 5x}{2x}, \lim_{x \rightarrow 0} \frac{\sin(x^m)}{(\sin x)^n}, \lim_{x \rightarrow 0} \frac{\tan x - \sin x}{\sin^3 x}, \lim_{x \rightarrow 0} \frac{\sqrt{1+x^2} - 1}{1 - \cos x}$.

解 2.5.9. (1) $\lim_{x \rightarrow 0} \frac{\tan 5x}{2x} = \lim_{x \rightarrow 0} \frac{5x}{2x} = \frac{5}{2}$.

(2) $\lim_{x \rightarrow 0} \frac{\sin(x^m)}{(\sin x)^n} = \lim_{x \rightarrow 0} \frac{x^m}{x^n} = \lim_{x \rightarrow 0} x^{m-n}$.

当 $m > n$ 时, $\lim_{x \rightarrow 0} \frac{\sin(x^m)}{(\sin x)^n} = 0$;

当 $m = n$ 时, $\lim_{x \rightarrow 0} \frac{\sin(x^m)}{(\sin x)^n} = 1$;

当 $m < n$ 时, $\lim_{x \rightarrow 0} \frac{\sin(x^m)}{(\sin x)^n} = \infty$.

(3) $\lim_{x \rightarrow 0} \frac{\tan x - \sin x}{\sin^3 x} = \lim_{x \rightarrow 0} \frac{\tan x(1 - \cos x)}{\sin^3 x} = \lim_{x \rightarrow 0} \frac{x \frac{1}{2}x^2}{x^3} = \frac{1}{2}$.

(4) $\lim_{x \rightarrow 0} \frac{\sqrt{1+x^2} - 1}{1 - \cos x} = \lim_{x \rightarrow 0} \frac{1 + \frac{1}{2}x^2 - 1}{\frac{1}{2}x^2} = 1$.

例题 2.5.10 2.6-B-3

证明: (1) $2x - x^2 = O(x) (x \rightarrow 0)$; (2) $\sqrt{1+x} - 1 = o(1) (x \rightarrow 0)$;

(3) $2x^3 + 2x^2 = O(x^3) (x \rightarrow \infty)$; (4) $(1+x)^n = 1 + nx + o(x) (x \rightarrow 0)$.

解 2.5.10. (1) 验证 $\lim_{x \rightarrow 0} \frac{2x - x^2}{x} = \lim_{x \rightarrow 0} (2 - x) = 2$ 为非零常数, 故 $2x - x^2 = O(x) (x \rightarrow 0)$.

(2) 由定义, 验证 $\lim_{x \rightarrow 0} \frac{\sqrt{1+x} - 1}{1} = \lim_{x \rightarrow 0} \frac{1 + x - 1}{1 + \sqrt{1+x}} = \lim_{x \rightarrow 0} \frac{x}{1 + \sqrt{1+x}} = 0$, 故 $\sqrt{1+x} - 1 = o(1) (x \rightarrow 0)$.

(3) 由定义, 验证 $\lim_{x \rightarrow \infty} \frac{2x^3 + 2x^2}{x^3} = \lim_{x \rightarrow \infty} (2 + \frac{2}{x}) = 2$ 为非零常数, 故 $2x^3 + 2x^2 = O(x^3) (x \rightarrow \infty)$.

(4) 验证 $\lim_{x \rightarrow 0} \frac{(1+x)^n - 1 - nx}{x} = \lim_{x \rightarrow 0} \frac{\frac{n(n-1)}{2}x^2 + \dots + x^n}{x} = \lim_{x \rightarrow 0} \left(\frac{n(n-1)}{2}x + \dots + x^{n-1} \right) = 0$, 故

$$(1+x)^n = 1 + nx + o(x) \quad (x \rightarrow 0).$$

例题 2.5.11 2.6-B-4

设在某一极限过程中, α 和 β 都是无穷小. 证明: 如果 $\alpha \sim \beta$, 则 $\beta - \alpha = o(\alpha)$, 反之, 如果 $\beta - \alpha = o(\alpha)$, 则 $\alpha \sim \beta$.

解 2.5.11. 如果 $\alpha \sim \beta$, 则 $\lim_{x \rightarrow x_0} \frac{\beta - \alpha}{\alpha} = \lim_{x \rightarrow x_0} \frac{\beta}{\alpha} - 1 = 1 - 1 = 0$, 故 $\beta - \alpha = o(\alpha)$.

如果 $\beta - \alpha = o(\alpha)$, 则 $0 = \lim_{x \rightarrow x_0} \frac{\beta - \alpha}{\alpha} = \lim_{x \rightarrow x_0} \frac{\beta}{\alpha} - 1$, 所以 $\lim_{x \rightarrow x_0} \frac{\beta}{\alpha} = 1$, 故 $\alpha \sim \beta$.

例题 2.5.12 2.6-B-5

证明: 当 $x \rightarrow 0$ 时, 有:

(1) $o(x^n) + o(x^m) = o(x^n)$, ($0 < n < m$); (2) $o(x^n) \cdot o(x^m) = o(x^{n+m})$, ($n, m > 0$).

解 2.5.12. (1) 由定义得到

$$\lim_{x \rightarrow x_0} \frac{g(x)o(1)}{g(x)} = \lim_{x \rightarrow x_0} o(1) = 0 \Rightarrow o(g(x)) = g(x)o(1)$$

考虑

$$\lim_{x \rightarrow 0} \frac{o(x^n) + o(x^m)}{x^n} = \lim_{x \rightarrow 0} \frac{x^n o(1) + x^m o(1)}{x^n} = \lim_{x \rightarrow 0} (o(1) + x^{m-n} o(1)) = \lim_{x \rightarrow 0} o(1) = 0$$

故 $o(x^n) + o(x^m) = o(x^n)$, ($0 < n < m$); (2) 考虑

$$\lim_{x \rightarrow 0} \frac{o(x^n) \cdot o(x^m)}{x^n \cdot x^m} = \lim_{x \rightarrow 0} \frac{x^n o(1) x^m o(1)}{x^n \cdot x^m} = \lim_{x \rightarrow 0} (o(1) \cdot o(1)) = 0$$

由定义得到 $o(x^n) \cdot o(x^m) = o(x^{n+m})$, ($n, m > 0$).

例题 2.5.13 2.7-B-2

判断 $x = 0$ 处的间断点类型 (1) $f(x) = \begin{cases} \frac{e^{1/x} + 1}{e^{1/x} - 1}, & x \neq 0 \\ 0, & x = 0 \end{cases}$; (2) $\lim_{n \rightarrow \infty} \frac{nx}{1 + nx^2}$.

解 2.5.13. (1) $\lim_{x \rightarrow 0^-} \frac{e^{1/x} + 1}{e^{1/x} - 1} = -1$, $\lim_{x \rightarrow 0^+} \frac{e^{1/x} + 1}{e^{1/x} - 1} = 1$, 所以 $f(x)$ 在 $x = 0$ 两侧的极限均存在, 但不等于 $f(0)$, 且 $f(0^-) \neq f(0^+)$, 所以是跳跃间断点.

(2) $\lim_{n \rightarrow \infty} \frac{nx}{1 + nx^2} = \lim_{n \rightarrow \infty} \frac{x}{\frac{1}{n} + x^2} = \lim_{n \rightarrow \infty} \frac{x}{0 + x^2} = \frac{1}{x}$, 所以 $x = 0$ 为无穷间断点.

例题 2.5.14 2.7-B-3

设函数 $f(x) = \lim_{n \rightarrow \infty} \frac{1+x}{1+x^{2n}}$, 试找出其间断点.

解 2.5.14. 观察发现分母在 $x = 1$ 处的变化速度很快, 计算 $f(x)$ 的分段函数得到:

$$f(x) = \begin{cases} 0, & x = -1 \\ 0, & |x| > 1 \\ 1 + x, & |x| < 1 \\ 1, & x = 1 \end{cases}$$

所以 $f(1^-), f(1), f(1^+)$ 都不等, 所以 $x = 1$ 为间断点.

例题 2.5.15 2.7-B-4

试确定 a, b 的值, 使 $f(x) = \frac{e^x - b}{(x-a)(x-1)}$ 有无穷间断点 $x = 0$, 有可去间断点 $x = 1$.

解 2.5.15. (1) 由题意得到 $f(x)$ 在 $x = 0$ 处左极限和右极限至少一个为无穷大, 假设 $a \neq 0$, 由于 $f(x)$ 为初等函数, 所以 $\lim_{x \rightarrow 0} \frac{e^x - b}{(x-a)(x-1)} = \frac{1-b}{a}$ 为常数, 矛盾, 所以 $a = 0$; 当 $b = 1$ 时, $\lim_{x \rightarrow 0} \frac{e^x - 1}{x(x-1)} = \lim_{x \rightarrow 0} \frac{1}{1-x} = 1$, 所以需要同时满足 $a = 0, b \neq 1$, 下证充分性, 当 $a = 0, b \neq 1$ 时

$$\lim_{x \rightarrow 0} \frac{e^x - b}{(x-a)(x-1)} = \lim_{x \rightarrow 0} \frac{e}{x} + \frac{e-b}{x-1} = \infty$$

(2) 要使得 $f(x)$ 有可去间断点 $x = 1$, 则 $f(1^-) = f(1^+)$, 假设 $b \neq e$, 计算发现

$$\lim_{x \rightarrow 1} \frac{e^x - b}{(x-a)(x-1)} = \lim_{x \rightarrow 1} \frac{e-b}{(x-1)(1-a)} = \infty$$

矛盾, 所以 $b = e$ 为必要条件, 当 $b = e$ 时, 计算极限并使用等价无穷小替换:

$$\lim_{x \rightarrow 1} \frac{e^x - e}{(x-a)(x-1)} = \lim_{x \rightarrow 1} \frac{e(e^{x-1} - 1)}{(x-a)(x-1)} = \lim_{x \rightarrow 1} \frac{e}{1-a}$$

所以 $a \neq 1$, 下面证明充分性. 当 $a \neq 1, b = e$ 时, $\lim_{x \rightarrow 1} \frac{e^x - e}{(x-a)(x-1)} = \lim_{x \rightarrow 1} \frac{e(e^{x-1} - 1)}{(x-a)(x-1)} = \lim_{x \rightarrow 1} \frac{e}{1-a} = e$, 所以此时 $f(1^-) = f(1^+) = e$, 因此 $x = 1$ 为可去间断点, 等价于 $a \neq 1, b = e$. 所以答案为 $a = 0, b \neq 1; a \neq 1, b = e$.

例题 2.5.16 2.8-A-3 求极限

$$\lim_{x \rightarrow \frac{\pi}{2}} \left(2 \cos^2 \frac{x}{2}\right)^{3 \sec x}, \lim_{x \rightarrow +\infty} (\arctan x)^{\cos \frac{1}{x}}, \lim_{x \rightarrow 0} \left[\frac{\ln(\cos^2 x + \sqrt{1-x^2})}{e^x + \sin x} + (1+x)^x \right]$$

解 2.5.16. (1) $\lim_{x \rightarrow \frac{\pi}{2}} \left(2 \cos^2 \frac{x}{2}\right)^{3 \sec x} = \lim_{x \rightarrow \frac{\pi}{2}} \left[(1 + \cos x)^{\frac{1}{\cos x}}\right]^3 = e^3$.

(2) 由于该函数为初等函数, 故 $\lim_{x \rightarrow +\infty} (\arctan x)^{\cos \frac{1}{x}} = \left(\frac{\pi}{2}\right)^1 = \frac{\pi}{2}$.

(3) $\lim_{x \rightarrow 0} \left[\frac{\ln(\cos^2 x + \sqrt{1-x^2})}{e^x + \sin x} + (1+x)^x \right] = 1 + \lim_{x \rightarrow 0} \ln(\cos^2 x + \sqrt{1-x^2}) = 1 + \ln 2$.

例题 2.5.17 2.8-A-5

证明下列方程在给定区间至少有一个根: $x2^x = 1, x \in [0, 1]; x^3 + px - q = 0, p > 0, x \in R$

解 2.5.17. (1) 设初等函数 $f(x) = x2^x - 1$ 在 $[0, 1]$ 上连续, 且 $f(0) = -1, f(1) = 1$, 由零点存在性定理知 $f(x)$ 在 $[0, 1]$ 上有至少一个根.

(2) 设初等函数 $g(x) = x^3 + px - q$ 在 R 上连续,

$$\begin{cases} \lim_{x \rightarrow -\infty} g(x) = \lim_{x \rightarrow -\infty} x^3 \left(1 + \frac{p}{x^2} - \frac{q}{x^3}\right) = -\infty \\ \lim_{x \rightarrow +\infty} g(x) = \lim_{x \rightarrow +\infty} x^3 \left(1 + \frac{p}{x^2} - \frac{q}{x^3}\right) = +\infty \end{cases}$$

零点存在性定理知 $g(x)$ 在 R 上有至少一个根.

例题 2.5.18 2.8-A-6

设 $f(x) \in C[0, 1]$, 并且 $\forall x \in [0, 1], 0 < f(x) < 1$ (画-图). 求证: $\exists x_0 \in (0, 1)$, 使得 $f(x_0) = x_0$.

解 2.5.18. 设连续函数 $g(x) = f(x) - x$, 则 $g(0) = f(0) - 0 > 0, g(1) = f(1) - 1 < 0$, 由零点存在性定理知 $g(x)$ 在 $[0, 1]$ 上有根 $x_0, \exists x_0 \in (0, 1)$, 使得 $f(x_0) = x_0$.

例题 2.5.19 2.8-B-2

证明: 若 $f(x) \in C(-\infty, +\infty)$, 且 $\lim_{x \rightarrow +\infty} f(x)$ 与 $\lim_{x \rightarrow -\infty} f(x)$ 都存在, 则 $f(x)$ 必有界.

解 2.5.19. 设 $\lim_{x \rightarrow +\infty} f(x) = A$, 则任意 $\varepsilon_1 > 0$, 存在 $N_1 \in R$ 使得 $\forall x > N_1$, 有 $|f(x) - A| < \varepsilon_1$, 即 $A - \varepsilon_1 < f(x) < A + \varepsilon_1$, 说明 $f(x)$ 在 $(N_1, +\infty)$ 有上界 $A + \varepsilon_1$ 和下界 $A - \varepsilon_1$; 设 $\lim_{x \rightarrow -\infty} f(x) = B$, 则任意 $\varepsilon_2 > 0$, 存在 $N_2 \in R$ 使得 $\forall x < N_2$, 有 $|f(x) - B| < \varepsilon_2$, 即 $B - \varepsilon_2 < f(x) < B + \varepsilon_2$, 说明 $f(x)$ 在 $(-\infty, N_2)$ 有上界 $B + \varepsilon_2$ 和下界 $B - \varepsilon_2$; 由于 $f(x)$ 在 $[N_2, N_1]$ 上连续, 所以在 $[N_2, N_1]$ 上也有上界 C_1 , 有下界 C_2 , 则取 $D_1 = \max\{C_1, A + \varepsilon_1, B + \varepsilon_2\}, D_2 = \min\{C_2, A - \varepsilon_1, B - \varepsilon_2\}$, 则 $f(x)$ 在 $(-\infty, +\infty)$ 上有上界 D_1 和下界 D_2 ; 证毕.

例题 2.5.20 2.8-B-3

设 $f(x) \in C(a, b)$, 且 $f(a^+)$ 与 $f(b^-)$ 都存在, 证明 $f(x)$ 在 (a, b) 上一致连续.

解 2.5.20. 补充定义函数 $g(x) = \begin{cases} f(a^+), x = a \\ f(x), x \in (a, b) \\ f(b^-), x = b \end{cases}$, 则 $g(a^+) = \lim_{x \rightarrow a^+} g(x) = \lim_{x \rightarrow a^+} f(x) = f(a^+) = g(a)$, 所以 $g(x)$ 在 a 点右连续, 同理 $g(b)$ 在 b 点左连续, 由此可知 $g(x)$ 在 $[a, b]$ 上一致连续, 那么 $f(x)$ 在 (a, b) 上一致连续.

例题 2.5.21 2.8-B-4

设 $f(x)$ 在 $[a, b]$ 上满足 $|f(x) - f(y)| \leq L|x - y|, \forall x, y \in [a, b]$, L 为常数, 证明 $f(x)$ 在 (a, b) 上一致连续.

解 2.5.21. 任意 $\varepsilon > 0$, 存在仅与 ε 有关的 $\delta = \frac{\varepsilon}{L}$, 使得 $\forall x, y \in [a, b], |x - y| < \delta = \frac{\varepsilon}{L}$, 有 $|f(x) - f(y)| \leq L|x - y| < L\frac{\varepsilon}{L} = \varepsilon$, 即 $f(x)$ 在 (a, b) 上一致连续.

例题 2.5.22

数列 $\{a_n\}$ 满足 $a_1 = 1, a_{n+1} = \frac{a_1 + a_2 + \cdots + a_{n-1} + a_n}{a_1 a_2 \cdots a_{n-1} a_n}$, 试求 $\{a_n\}$ 的渐进.

解 2.5.22. 设数列 $\{a_n\}$ 的前 n 项和为 S_n , 前 n 项积为 T_n , 则有

$$a_{n+1} = \frac{S_n}{T_n} = \frac{S_n}{a_n T_{n-1}} = \frac{S_n}{\frac{S_{n-1}}{T_{n-1}} T_{n-1}} = \frac{S_n}{S_{n-1}}$$

转化为估计 S_n 的阶, 将 $a_{n+1} = S_{n+1} - S_n$ 代入得到:

$$S_1 = 1, S_2 = 2, S_{n+1} = S_n + \frac{S_n}{S_{n-1}}$$

下面估计 S_n 的主阶, 先考察它的有界性和单调性, 由

$$S_n > 0, S_{n+1} - S_n = \frac{S_n}{S_{n-1}} > 0 \Rightarrow S_{n+1} > S_n$$

可知 S_n 是单调递增的, 再次代入递推式有

$$S_{n+1} - S_n = \frac{S_n}{S_{n-1}} > 1 \Rightarrow S_{n+1} - S_n > 1 \Rightarrow S_n \geq n$$

所以 S_n 发散到正无穷, 无上界, 我们再把这个结论代入到递推公式中:

$$S_{n+1} = S_n + \frac{S_n}{S_{n-1}} \Leftrightarrow \frac{S_{n+1}}{S_n} = 1 + \frac{1}{S_{n-1}} \rightarrow 1 \quad (n \rightarrow \infty)$$

当 n 趋于无穷时, $S_n \sim n$, 构造 $b_n = S_n - n$ 以得到更精确的阶, 将 $S_n = n + b_n$ 代入递推公式:

$$S_{n+1} - S_n = \frac{S_n}{S_{n-1}} \Leftrightarrow b_{n+1} - b_n = \frac{n + b_n}{n - 1 + b_{n-1}} - 1 = \frac{1 + \frac{b_n}{n}}{1 - \left(\frac{1}{n} - \frac{b_{n-1}}{n}\right)} - 1$$

泰勒展开可得:

$$\begin{aligned} b_{n+1} - b_n &= -1 + \left(1 + \frac{b_n}{n}\right) \left(1 + \left(\frac{1}{n} - \frac{b_{n-1}}{n}\right) + O\left(\frac{1}{n^2}\right)\right) \\ &= -1 + \left(1 + \frac{b_n}{n}\right) + \frac{1}{n} \left(1 + \frac{b_n}{n}\right) - \frac{b_{n-1}}{n} \left(1 + \frac{b_n}{n}\right) + O\left(\frac{1}{n^2}\right) \\ &= \frac{1}{n} + \frac{b_n - b_{n-1}}{n} + \frac{b_n(1 - b_{n-1})}{n^2} + O\left(\frac{1}{n^2}\right) \\ &= \frac{1}{n} + O\left(\frac{1}{n^2}\right) \\ \Rightarrow b_{n+1} - b_n &\sim \frac{1}{n} \Rightarrow b_n \sim \ln n + O(1) \Rightarrow S_n = n + b_n \sim n + \ln n + O(1) \end{aligned}$$

所以 $\{S_n\}$ 的阶为 $n + \ln n + c + o(1)$, 代入 $a_n = \frac{S_{n-1}}{S_{n-2}}$ 得到:

$$a_n = \frac{S_{n-1}}{S_{n-2}} = \frac{(n-1) + \ln(n-1) + c + o(1)}{(n-2) + \ln(n-2) + c + o(1)}$$

并利用

$$\ln(n-1) = \ln n - \frac{1}{n} - \frac{1}{2n^2} + O\left(\frac{1}{n^3}\right), \quad \ln(n-2) = \ln n - \frac{2}{n} - \frac{2}{n^2} + O\left(\frac{1}{n^3}\right)$$

代入:

$$S_{n-1} = n-1 + \ln n + c - \frac{1}{n} - \frac{1}{2n^2} + O\left(\frac{1}{n^3}\right) = n \left(1 + \frac{\ln n + c - 1}{n} - \frac{3}{2n^2} + O\left(\frac{1}{n^3}\right)\right)$$

$$S_{n-2} = n-2 + \ln n + c - \frac{2}{n} - \frac{2}{n^2} + O\left(\frac{1}{n^3}\right) = n \left(1 + \frac{\ln n + c - 2}{n} - \frac{4}{n^2} + O\left(\frac{1}{n^3}\right)\right)$$

故技重施:

$$\frac{1}{S_{n-2}} = \frac{1}{n} \left(1 - \frac{\ln n + c - 2}{n} + \frac{(\ln n + c - 2)^2 + 4}{n^2} + O\left(\frac{1}{n^3}\right)\right)$$

代入得到

$$\begin{aligned} a_n &= \left(1 + \frac{\ln n + c - 1}{n} - \frac{3}{2n^2}\right) \cdot \left(1 - \frac{\ln n + c - 2}{n} + \frac{(\ln n + c - 2)^2 + 4}{n^2}\right) + O\left(\frac{1}{n^3}\right) \\ &= 1 - \frac{\ln n + c - 2}{n} + \frac{\ln n + c - 1}{n} + \frac{(\ln n + c - 2)^2 + 2}{n^2} \\ &\quad - \frac{(\ln n + c - 1)(\ln n + c - 2)}{n^2} - \frac{1}{n^2} + O\left(\frac{1}{n^3}\right) \\ &= 1 + \frac{1}{n} + \frac{3 - c - \ln n}{n^2} + O\left(\frac{1}{n^3}\right) \end{aligned}$$

例题 2.5.23

数列 $\{a_n\}$ 满足 $a_1 = 1, a_{n+1} = \frac{a_1 + a_2 + \cdots + a_{n-1} + a_n}{a_1 a_2 \cdots a_{n-1} a_n}$, 找到一个正整数 k , 使得 $n \geq k$ 时,
 $a_n < 1 + \frac{1}{n}$.

解 2.5.23.

例题 2.5.24

数列 $\{a_n\}$ 满足 $a_1 = 1, a_{n+1} = a_n + \frac{1}{a_n}$, 求 $\{a_n\}$ 的渐进.

解 2.5.24. 解微分方程

$$a_{n+1} - a_n = \frac{1}{a_n} \Rightarrow \frac{da}{dn} = \frac{1}{a} \Rightarrow ada = dn \Rightarrow a \sim \sqrt{2n} \Rightarrow a_n^2 \sim 2n$$

设 $a_n^2 = b_n + 2n$, 代入递推公式得到

$$a_{n+1} = a_n^2 + \frac{1}{a_n^2} + 2 \Leftrightarrow 2(n+1) + b_{n+1} = 2n + b_n + 2 + \frac{1}{2n + b_n}$$

化简得到 $b_{n+1} = b_n + \frac{1}{2n + b_n} = b_n + \frac{\frac{1}{2n}}{1 - (-\frac{b_n}{2n})}$, 泰勒展开:

$$b_{n+1} - b_n \sim \frac{1}{2n} - \frac{b_n}{4n^2} + O\left(\frac{1}{n^3}\right) \Rightarrow b_n \sim \frac{1}{2} \ln n + c_n$$

再次代入递推公式:

例题 2.5.25

已知 k 为正整数, 求积分 $\int_0^\pi \frac{x \sin^{2k} x}{\sin^{2k} x + \cos^{2k} x} dx$

解 2.5.25.

$$\begin{aligned} I &= \int_0^\pi \frac{x \sin^{2k} x}{\sin^{2k} x + \cos^{2k} x} dx = \int_{-\pi}^0 \frac{(x + \pi) \sin^{2k} x}{\sin^{2k} x + \cos^{2k} x} dx \\ &= \int_{-\pi}^0 \frac{x \sin^{2k} x}{\sin^{2k} x + \cos^{2k} x} dx + \pi \int_0^\pi \frac{\sin^{2k} x}{\sin^{2k} x + \cos^{2k} x} dx = -I + \pi \int_0^\pi \frac{\sin^{2k} x}{\sin^{2k} x + \cos^{2k} x} dx \\ &= -I + \pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos^{2k} x}{\sin^{2k} x + \cos^{2k} x} dx = -I + 2\pi \int_0^{\frac{\pi}{2}} \frac{\cos^{2k} x}{\sin^{2k} x + \cos^{2k} x} dx \\ &= -I + 2\pi \int_{-\frac{\pi}{2}}^0 \frac{\sin^{2k} x}{\sin^{2k} x + \cos^{2k} x} dx = -I + 2\pi \int_0^{\frac{\pi}{2}} \frac{\sin^{2k} x}{\sin^{2k} x + \cos^{2k} x} dx \\ &= -I + \pi \int_0^{\frac{\pi}{2}} \frac{\sin^{2k} x + \cos^{2k} x}{\sin^{2k} x + \cos^{2k} x} dx = -I + \frac{\pi^2}{2} \\ \Rightarrow I &= \frac{\pi^2}{4}. \end{aligned}$$

例题 2.5.26

给出 $\int_0^1 \frac{\ln(1+x)}{x} dx = \frac{\pi^2}{12}$, 计算 $\int_0^1 \int_0^y \frac{\ln(1+x)}{x} dx dy$.

解 2.5.26. 根据所给式子, 得到 $0 \leq x \leq y$ (因为 x 从 0 到 y 积分), 再根据外层积分确定 $0 \leq y \leq 1$, 所以 $0 \leq x \leq y \leq 1$, 所以:

$$\begin{aligned} \int_0^1 \int_0^y \frac{\ln(1+x)}{x} dx dy &= \int_0^1 \int_x^1 \frac{\ln(1+x)}{x} dy dx = \int_0^1 \frac{(1-x) \ln(1+x)}{x} dx \\ &= \int_0^1 \frac{\ln(1+x)}{x} dx - \int_0^1 \ln(x+1) dx \\ &= \frac{\pi^2}{12} - 2 \ln 2 + 1 \end{aligned}$$

例题 2.5.27

求积分 $\int \frac{dx}{\sin^6 x + \cos^6 x}$

解 2.5.27.

$$\begin{aligned}
 \int \frac{dx}{\sin^6 x + \cos^6 x} &= \int \frac{\sec^6 x dx}{1 + \tan^6 x} = \int \frac{\sec^6(\arctan t) d \arctan t}{1 + \tan^6(\arctan t)} \\
 &= \int \frac{(1+t^2)^3 dt}{(t^6+1)(1+t^2)} = \int \frac{(1+t^2)^2 dt}{(t^2+1)(t^4-t^2+1)} \\
 &= \int \frac{1+t^2}{t^4-t^2+1} dt = \int \frac{1+\frac{1}{t^2}}{t^2+\frac{1}{t^2}-1} dt \\
 &= \int \frac{d(t-\frac{1}{t})}{(t-\frac{1}{t})^2+1} = \arctan(t-\frac{1}{t}) + C \\
 &= \arctan(\tan x - \frac{1}{\tan x}) + C = \arctan(\frac{\tan^2 x - 1}{\tan x}) \\
 &= \arctan(-2 \cot 2x) + C = C - \arctan(2 \cot 2x).
 \end{aligned}$$

例题 2.5.28

求积分 $\int \frac{\cos^3 x dx}{\sin x + \cos x}$.

解 2.5.28.

$$\begin{aligned}
 \int \frac{\cos^3 x dx}{\sin x + \cos x} &= \int \frac{dx}{(\tan x + 1)(\tan^2 x + 1)} \\
 &= \int \frac{dt}{(1+t)(1+t^2)^2} \quad (\text{令 } t = \tan x) \\
 &= \int \left(\frac{1/4}{1+t} + \frac{-\frac{1}{4}t + \frac{1}{4}}{1+t^2} + \frac{-\frac{1}{2}t + \frac{1}{2}}{(1+t^2)^2} \right) dt \\
 &= \frac{1}{4} \int \frac{1}{1+t} dt + \frac{1}{4} \int \frac{1-t}{1+t^2} dt + \frac{1}{2} \int \frac{1-t}{(1+t^2)^2} dt \\
 &= \frac{1}{4} \ln|1+t| + \frac{1}{4} \arctan t - \frac{1}{8} \ln(1+t^2) + \frac{1}{4} \cdot \frac{t+1}{1+t^2} + C \\
 &= \frac{1}{4} \ln|1+\tan x| - \frac{1}{4} \ln|\sec x| + \frac{1}{2}x + \frac{1}{4}(\sin x \cos x + \cos^2 x) + C \\
 &= \frac{1}{4} \ln|\sin x + \cos x| + \frac{1}{2}x + \frac{1}{8} \sin 2x + C
 \end{aligned}$$

例题 2.5.29

求积分 $\int \frac{dx}{\sin 2x + 2 \sin x}$.

解 2.5.29.

$$\begin{aligned}
 \int \frac{dx}{\sin 2x + 2 \sin x} &= \int \frac{dx}{2 \sin x(1 + \cos x)} = \int \frac{dx}{4 \sin x \cos^2 \frac{x}{2}} = \int \frac{dx}{8 \sin \frac{x}{2} \cos^3 \frac{x}{2}} \\
 &= \int \frac{dt}{4 \sin t \cos^3 t} = \int \frac{(\sin^2 t + \cos^2 t)^2 dt}{4 \sin t \cos^3 t} = \frac{1}{4} \int \frac{(\tan^2 t + 1)^2}{\tan t} dt \\
 &= \frac{1}{4} \int \frac{(\tan(\arctan \theta)^2 + 1)^2}{\tan \arctan \theta} d \arctan \theta = \frac{1}{4} \int \frac{\theta^2 + 1}{\theta} d\theta \\
 &= \frac{1}{4} \int \left(\theta + \frac{1}{\theta} \right) d\theta = \frac{1}{8} \theta^2 + \frac{1}{4} \ln |\theta| + C \\
 &= \frac{1}{8} \tan^2 \frac{x}{2} + \frac{1}{4} \ln \left| \tan \frac{x}{2} \right| + C
 \end{aligned}$$

例题 2.5.30

求积分 $\int \frac{x e^{-x}}{(1 + e^{-x})^2} dx$.

解 2.5.30.

$$\begin{aligned}
 \int \frac{x e^{-x}}{(1 + e^{-x})^2} dx &= \int \frac{x e^x}{(e^x + 1)^2} dx = \int \frac{e^{\ln t} \ln t}{(e^{\ln t} + 1)^2} d \ln t = \int \frac{\ln t}{(t + 1)^2} dt \\
 &= \int \ln t d \left(-\frac{1}{1 + t} \right) = -\frac{\ln t}{t + 1} + \int \frac{1}{1 + t} d \ln t \\
 &= -\frac{\ln t}{t + 1} + \int \frac{1}{t(1 + t)} dt = -\frac{\ln t}{t + 1} + \int \left(\frac{1}{t} - \frac{1}{t + 1} \right) dt \\
 &= -\frac{\ln t}{t + 1} + \ln t - \ln(t + 1) + C = -\frac{x}{e^x + 1} + x - \ln(e^x + 1) + C
 \end{aligned}$$

第三章 一元函数微分学

3.1 习题 3.1

例题 3.1.1 3.1-A-3

下列各式可否成为 $f(x)$ 在 x_0 点的导数的定义? 请说明理由.

(1) $y = f(x)$ 在 (a, b) 内定义, $x_0 \in (a, b)$, 若极限 $\lim_{\Delta x \rightarrow 0} \frac{f(x_0) - f(x_0 - \Delta x)}{\Delta x}$ 存在, 则称该极限为 $f(x)$ 在 x_0 点的导数.

(2) $y = f(x)$ 在 (a, b) 内定义, $x_0 \in (a, b)$, 若极限 $\lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0}$ 存在, 则称该极限为 $f(x)$ 在 x_0 点的导数.

(3) $y = f(x)$ 在 (a, b) 内定义, $x_0 \in (a, b)$, 若极限 $\lim_{\Delta x \rightarrow 0} \frac{f(x_0 + \Delta x) - f(x_0 - \Delta x)}{2\Delta x}$ 存在, 则称该极限为 $f(x)$ 在 x_0 点的导数.

解 3.1.1. (1) 可以, 因为

$$\lim_{\Delta x \rightarrow 0} \frac{f(x_0) - f(x_0 - \Delta x)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{f(x_0 - \Delta x) - f(x_0)}{-\Delta x} = f'(x_0)$$

(2) 可以, 因为

$$\lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x - x_0 \rightarrow 0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x - x_0 \rightarrow 0} \frac{f(x_0 + x - x_0) - f(x_0)}{x - x_0} = f'(x_0)$$

(3) 不可以, 当 $f'(x_0)$ 存在时, 该极限等于 $f'(x_0)$, 这是因为

$$\begin{aligned} \lim_{\Delta x \rightarrow 0} \frac{f(x_0 + \Delta x) - f(x_0 - \Delta x)}{2\Delta x} &= \lim_{\Delta x \rightarrow 0} \frac{f(x_0 + \Delta x) - f(x_0) + f(x_0) - f(x_0 - \Delta x)}{2\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{f(x_0 + \Delta x) - f(x_0)}{2\Delta x} + \lim_{\Delta x \rightarrow 0} \frac{f(x_0) - f(x_0 - \Delta x)}{2\Delta x} \\ &= \frac{1}{2}f'(x_0) + \frac{1}{2}f'(x_0) = f'(x_0) \end{aligned}$$

但是当 $f'(x_0)$ 不存在, 而左右导数存在时, 该极限推出的是左导数和右导数的平均值, 并非 $f'(x_0)$, 比如当 $f(x) = |x|, x = 0$ 导数不存在, 但是

$$\lim_{\Delta x \rightarrow 0} \frac{f(x_0 + \Delta x) - f(x_0 - \Delta x)}{2\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{|\Delta x| - |-\Delta x|}{2\Delta x} = 0$$

矛盾, 所以不可以用作导数的定义.

例题 3.1.2 3.1-A-9

利用定义求函数在 $x = 0$ 处的导数 $f(x) = \begin{cases} x^2 \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$.

解 3.1.2. 利用定义, 有

$$f'(0) = \lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \rightarrow 0} h \sin \frac{1}{h} = 0$$

例题 3.1.3 3.1-B-3

求证: 偶函数的导数是奇函数, 奇函数的导数是偶函数;

解 3.1.3. 设 $f(x)$ 是偶函数, 所以 $f(x) - f(-x) = 0$, 两边求导得到 $f'(x) + f'(-x) = 0$, 所以偶函数的导数是奇函数; 设 $f(x)$ 是奇函数, 所以 $f(x) + f(-x) = 0$, 两边求导得到 $f'(x) - f'(-x) = 0$, 所以奇函数的导数是偶函数

我们也可以考虑定义: 设 $f(x)$ 是偶函数, 所以

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{f(-x-h) - f(-x)}{-(-h)} = -f'(-x)$$

所以偶函数的导数是奇函数; 设 $f(x)$ 是奇函数, 所以

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{f(-x) - f(-x-h)}{-(-h)} = f'(-x)$$

所以奇函数的导数是偶函数。

例题 3.1.4 3.1-B-4

求证: 周期函数的导数仍然是周期函数

解 3.1.4. 设 $f(x)$ 是周期函数, T 是周期, $f(x+T) = f(x)$, 则两边求导得到 $f'(x+T) = f'(x)$, 所以 $f'(x)$ 也是周期函数, 也可以从定义考虑:

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{f(x+T+h) - f(x+T)}{h} = f'(x+T)$$

所以 $f'(x)$ 是周期函数。

3.2 习题 3.2

例题 3.2.1 3.2-A-2

求导 (1) $y = 1 - \frac{1}{x} + \frac{1}{x^2}$; (2) $y = \frac{1}{x^3 + 2x + 1}$; (3) $y = \frac{1}{x + \sqrt{x}}$.

解 3.2.1. (1) $y' = \frac{1}{x^2} - \frac{2}{x^3}$; (2) $y' = \frac{-3x^2 - 2}{(x^3 + 2x + 1)^2}$; (3) $y' = -\frac{1 + 2\sqrt{x}}{2\sqrt{x}(x + \sqrt{x})^2}$

例题 3.2.2 3.2-A-4

求导 (1) $y = \ln(x + \sqrt{1 + x^2})$; (2) $y = \frac{x}{\sqrt{a^2 - x^2}}$; (3) $y = \sqrt{x + \sqrt{x + \sqrt{x}}}$.

解 3.2.2. (1) $y' = \frac{1 + \frac{2x}{2\sqrt{1+x^2}}}{x + \sqrt{1+x^2}} = \frac{1}{\sqrt{1+x^2}}$;
 (2) $y' = \frac{\sqrt{a^2 - x^2} - x \frac{-2x}{2\sqrt{a^2 - x^2}}}{(\sqrt{a^2 - x^2})^2} = \frac{a^2}{(\sqrt{a^2 - x^2})^3}$;
 (3) $y' = \frac{1 + \frac{1}{2\sqrt{x+\sqrt{x}}}}{2\sqrt{x+\sqrt{x+\sqrt{x}}}} = \frac{1 + 2\sqrt{x} + 4\sqrt{x}\sqrt{x+\sqrt{x}}}{8\sqrt{x}\sqrt{x+\sqrt{x}}\sqrt{x+\sqrt{x+\sqrt{x}}}}$.

例题 3.2.3 3.2-B-1

设 $y = f(x)$ 为严格递增的可导函数, $x = \varphi(y)$ 是它的反函数. 证明:

(1) 当 $h \neq 0$ 时, $f(x+h) - f(x) = k \neq 0$, 若记 $f(x+h) = y+k$, 则 $\varphi(y+k) = x+h$.

(2) 当 $k \rightarrow 0$ 时, $\frac{\varphi(y+k) - \varphi(y)}{k} = \frac{h}{k} = \frac{h}{y+k-y} = \frac{h}{f(x+h) - f(x)}$ 趋于 $\frac{1}{f'(x)}$.

解 3.2.3. (1) 由 $x = \varphi(f(x)) = f(\varphi(x))$, 以及 $y = f(x)$ 为严格递增的可导函数, 所以 $\varphi'(x) = \frac{1}{f'(\varphi(x))} > 0$, 所以 $x = \varphi(y)$ 为严格递增的可导函数, 所以

$$f(x+h) = y+k \Rightarrow \varphi(f(x+h)) = x+h = \varphi(y+k)$$

(2) $\lim_{k \rightarrow 0} \frac{\varphi(y+k) - \varphi(y)}{k} = \lim_{h \rightarrow 0} \frac{1}{\frac{f(x+h) - f(x)}{h}} = \frac{1}{f'(x)}$, 这里 $h \rightarrow 0 \Leftrightarrow k \rightarrow 0$.

例题 3.2.4 3.2-B-2

设 $f(x) = x^3 + 2x^2 + 3x + 1$, 用 φ 表示 f 的反函数. 求证: $f(1) = 7, \varphi(7) = 1$. 并计算 $\varphi'(7)$.

解 3.2.4. 代入得到 $f(1) = 1 + 2 + 3 + 1 = 7$, 所以 $\varphi(7) = 1$, 由 $f(x) = x^3 + 2x^2 + 3x + 1$ 可得 $f'(x) = 3x^2 + 4x + 3$, 所以 $\varphi'(7) = \frac{1}{f'(\varphi(7))} = \frac{1}{f'(1)} = \frac{1}{10}$.

例题 3.2.5 3.2-B-3

设 $y = (\arcsin x)^2$, 证明 $(1 - x^2)y'' - xy' = 2$.

解 3.2.5. 两边求导数得到 $y' = 2 \arcsin x \cdot \frac{1}{\cos \arcsin x} = 2 \frac{\arcsin x}{\sqrt{1-x^2}}$, 所以 $\sqrt{1-x^2}y' = 2 \arcsin x$, 再次两边求导得到 $\sqrt{1-x^2}y'' + \frac{-2x}{2\sqrt{1-x^2}}y' = \frac{2}{\sqrt{1-x^2}}$, 等价变形就有 $(1-x^2)y'' - xy' = 2$.

例题 3.2.6 3.2-B-4

求下列函数的 n 阶导数 $y^{(n)}$: (1) $y = \frac{1}{1-x^2}$; (2) $y = \sin^2 x$.

解 3.2.6. (1) 裂项有 $y = \frac{1}{2} \left(\frac{1}{1+x} + \frac{1}{1-x} \right)$, 所以 $y^{(n)} = \frac{1}{2} \left[\left(\frac{1}{1+x} \right)^{(n)} + \left(\frac{1}{1-x} \right)^{(n)} \right]$, 求 n 阶

导数有 $y^{(n)} = \frac{1}{2} \left[\frac{(-1)^n n!}{(1+x)^{n+1}} + \frac{n!}{(1-x)^{n+1}} \right]$

(2) $y = \sin^2 x = \frac{1}{2}(1 - \cos 2x)$, $y^{(n)} = \left[\frac{1}{2} - \frac{1}{2} \cos 2x \right]^{(n)} = -2^{n-1} \cos \left(2x + \frac{n\pi}{2} \right)$, 诱导公式得到

$y^{(n)} = 2^{n-1} \sin \left(2x + \frac{(n-1)\pi}{2} \right)$

3.3 习题 3.3

例题 3.3.1 3.3-A-2

方程 $e^y + xy + y = 2$ 确定隐函数 $y = y(x)$, 求 $y'(x)$.

解 3.3.1. 两边求导得到 $y'e^y + y + xy' + y' = 0$ 解得 $y' = -\frac{y}{e^y + x + 1}$

例题 3.3.2 3.3-A-3

- (1) $e^x - e^y + xy = 0$ (2) $x^2 + y^2 - \arcsin y = 0$ (3) $x^y = y^x$
 (4) $\arctan \frac{y}{x} = \ln \sqrt{x^2 + y^2}$ (5) $x^2 - 2xy + y^2 = 2x$ (6) $\sqrt{x} + \sqrt{y} = 1$
 (7) $xy^2 + e^y = \cos(x + y^2)$ (8) $\ln y = \sqrt{\frac{1-x}{1+x}}$

解 3.3.2. (1) 两边求导得 $e^x - y'e^y + y + xy' = 0$ 解得 $y' = \frac{e^x + y}{e^y - x}$

(2) 两边求导得 $2x + 2yy' - \frac{y'}{\sqrt{1-y^2}} = 0$ 解得 $y' = \frac{2x\sqrt{1-y^2}}{1-2y\sqrt{1-y^2}}$

(3) 变换得到 $\frac{\ln x}{x} = \frac{\ln y}{y}$, 两边求导得到 $y' = \frac{y(x \ln y - y)}{x(y \ln x - x)}$

(4) 两边求导得 $\frac{1}{1+\frac{y^2}{x^2}} \frac{y'x - y}{x^2} = \frac{1}{2} \frac{2x + 2yy'}{x^2 + y^2}$ 解得 $y' = \frac{x+y}{x-y}$

(5) 两边求导得 $2x - 2y - 2xy' + 2yy' = 2$ 解得 $y' = \frac{x-y-1}{x-y}$

(6) 两边求导得 $\frac{1}{2\sqrt{x}} + \frac{y'}{2\sqrt{y}} = 0$, 解得 $y' = -\sqrt{\frac{y}{x}}$

(7) 两边求导得 $y^2 + 2xyy' + e^y y' = -(1 + 2yy') \sin(x + y^2)$, 解得 $y' = -\frac{y^2 + \sin(x + y^2)}{e^y + 2xy + 2y \sin(x + y^2)}$.

(8) 两边求导得 $\frac{y'}{y} = \frac{-\frac{2}{(1+x)^2}}{2\sqrt{\frac{1-x}{1+x}}}$, 解得 $y' = -\frac{y}{(1+x)^2} \sqrt{\frac{1-x}{1+x}}$.

例题 3.3.3 3.3-A-4 求下列由参数方程表示的函数的导数:

- 1) $x = \sqrt[3]{1-\sqrt{t}}, y = \sqrt{1-\sqrt[3]{t}}$, 求 $\frac{dy}{dx}$; (2) $x = \sin^2 t, y = \cos^2 t$, 求 $\frac{dy}{dx}$;
 (3) $x = 1 + t^3, y = e^{2t}$, 求 $\frac{dy}{dx}\bigg|_{x=2}$; (4) $x = 1 + t^2, y = \cos t$, 求 $\frac{dy}{dx}$;
 (5) $x = e^t \sin t, y = e^{-t} \cos t$, 求 $\frac{dy}{dx}$.

解 3.3.3. 使用链式法则, 分别求导得:

$$(1) \frac{dy}{dx} = \frac{dy}{dt} \frac{dt}{dx} = \frac{-\frac{3}{2}\sqrt{t}}{2\sqrt{1-\sqrt[3]{t}}} \cdot \frac{\frac{2}{3}(1-\sqrt[3]{t})^{\frac{2}{3}}}{-\frac{1}{2\sqrt{t}}} = \frac{\sqrt{t}\sqrt[3]{(1-\sqrt[3]{t})^2}}{\sqrt[3]{t^2}\sqrt{1-\sqrt[3]{t}}}$$

$$(2) \quad x + y = 1 \Rightarrow \frac{dy}{dx} = -1. \quad (3) \quad \frac{dy}{dx} = \frac{dy}{dt} \frac{dt}{dx} = \frac{2e^{2t}}{3t^2}, \text{ 代入 } x=2, t=1 \text{ 得到 } \left. \frac{dy}{dx} \right|_{x=2} = \frac{2}{3}e^2.$$

$$(4) \quad \frac{dy}{dx} = \frac{dy}{dt} \frac{dt}{dx} = \frac{-\sin t}{2t}. \quad (5) \quad \frac{dy}{dx} = \frac{dy}{dt} \frac{dt}{dx} = \frac{e^{-t}(-\sin t - \cos t)}{e^t(\sin t + \cos t)} = -e^{2t}.$$

例题 3.3.4 3.3-A-5: 用对数求导法求导数

$$(1) \quad y = x^{\sin x}, (x > 0); \quad (2) \quad y = (\sqrt{x})^{\ln x}, (x > 0); \quad (3) \quad y = a^{\sin x}, (a > 0);$$

$$(4) \quad y = (1+x)^{\frac{1}{x}}, (x > 0); \quad (5) \quad y = \frac{(x+5)^2(x-4)^{\frac{1}{3}}}{(x+2)^5(x+4)^{\frac{1}{2}}}; \quad (6) \quad y = x\sqrt{\frac{1-x}{1+x}}$$

解 3.3.4. (1) $\ln y = (\sin x) \ln x \Rightarrow \frac{y'}{y} = \frac{\sin x}{x} + (\cos x) \ln x \Rightarrow y' = x^{\sin x} \left((\cos x) \ln x + \frac{\sin x}{x} \right)$

(2) $\ln y = \frac{1}{2} \ln^2 x \Rightarrow \frac{d \ln y}{dy} \frac{dy}{dx} = \frac{1}{2} 2 \ln x \frac{1}{x} \Rightarrow \frac{dy}{dx} = \frac{y \ln x}{x}$

(3) $\ln y = \sin x \ln a \Rightarrow \frac{d \ln y}{dy} \frac{dy}{dx} = \cos x \ln a \Rightarrow \frac{dy}{dx} = a^{\sin x} \cos x \ln a$

(4) $\ln y = \frac{\ln(x+1)}{x} \Rightarrow \frac{d \ln y}{dy} \frac{dy}{dx} = \frac{1}{x(x+1)} - \frac{\ln(x+1)}{x^2}$ 解得:

$$\Rightarrow \frac{dy}{dx} = (1+x)^{\frac{1}{x}} \left(\frac{1}{x(x+1)} - \frac{\ln(x+1)}{x^2} \right)$$

(5) 定义域为 $(-4, -2) \cup (-2, +\infty)$, 取绝对值, 然后取对数

$$\ln |y| = \ln \left(\frac{(x+5)^2|x-4|^{\frac{1}{3}}}{|x+2|^5(x+4)^{\frac{1}{2}}} \right) = 2 \ln |x+5| + \frac{1}{3} \ln |x-4| - 5 \ln |x+2| - \frac{1}{2} \ln(x+4)$$

两边对 x 微分:

$$\frac{d}{dx} \ln |y| = \frac{1}{y} \frac{dy}{dx} = 2 \cdot \frac{1}{x+5} + \frac{1}{3} \cdot \frac{1}{x-4} - 5 \cdot \frac{1}{x+2} - \frac{1}{2} \cdot \frac{1}{x+4}$$

解得:

$$\frac{dy}{dx} = \frac{(x+5)^2(x-4)^{\frac{1}{3}}}{(x+2)^5(x+4)^{\frac{1}{2}}} \left(\frac{2}{x+5} + \frac{1}{3(x-4)} - \frac{5}{x+2} - \frac{1}{2(x+4)} \right)$$

(6) 定义域为 $(-1, 1]$, $|y| = |x| \sqrt{\frac{1-x}{1+x}} \Rightarrow \ln |y| = \ln |x| + \frac{1}{2} \ln |1-x| - \frac{1}{2} \ln |1+x|$, 两边对 x 微分:

$$\frac{d}{dx} \ln |y| = \frac{1}{y} \frac{dy}{dx} = \frac{1}{x} + \frac{1}{2(1-x)} - \frac{1}{2(1+x)} \Rightarrow \frac{dy}{dx} = y \left(\frac{1}{x} + \frac{1}{2(1-x)} - \frac{1}{2(1+x)} \right)$$

例题 3.3.5 3.3-A-6

下列参数方程给出函数 $y = y(x)$, 求 $\frac{d^2 y}{dx^2}$

(1) $x = a \cos t, y = a \sin t;$ (2) $x = 2t - t^2, y = 3t - t^3;$

(3) $x = \ln(1+t^2), y = \arctan t;$ (4) $x = \ln(t + \sqrt{t^2+1}), y = t^2.$

解 3.3.5. (1) $\frac{d^2 y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d}{dt} \left(\frac{dy}{dt} \frac{dt}{dx} \right) \frac{dt}{dx} = \frac{d}{dt} (-\cot t) \frac{dt}{dx} = \frac{\csc^2 t}{-a \sin t} = -\frac{1}{a \sin^3 t}$

(2) $\frac{d^2 y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d}{dt} \left(\frac{dy}{dt} \frac{dt}{dx} \right) \frac{dt}{dx} = \frac{d}{dt} \left(\frac{3}{2}(1+t) \right) \frac{dt}{dx} = \frac{3}{4(1-t)}$

(3) $\frac{d^2 y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d}{dt} \left(\frac{dy}{dt} \frac{dt}{dx} \right) \frac{dt}{dx} = \frac{d}{dt} \left(\frac{1}{1+t^2} \frac{1+t^2}{2t} \right) \frac{dt}{dx} = -\frac{(1+t^2)}{4t^3}$

(4) $\frac{d^2 y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d}{dt} \left(\frac{dy}{dt} \frac{dt}{dx} \right) \frac{dt}{dx} = \frac{d}{dt} (2t\sqrt{t^2+1}) \frac{dt}{dx} = \frac{4t^2+2}{\sqrt{t^2+1}} \sqrt{t^2+1} = 4t^2+2.$

例题 3.3.6 3.3-A-7 求下列隐函数的二阶导数 y''

(1) $x^3 + y^3 - 3axy = 0, (a > 0)$; (2) $y^2 + 2 \ln y = x^4$; (3) $xy = e^{x+y}$; (4) $y = 1 - xe^y$

解 3.3.6. (1) 两边求导数得

$$3x^2 + 3y^2 y' = 3a(y + xy') \Rightarrow (ax - y^2)y' = x^2 - ay \Rightarrow (a - 2yy')y' + (ax - y^2)y'' = 2x - ay'$$

解得

$$y'' = \frac{1}{y^2 - ax} \left[\frac{2a(ay - x^2)}{y^2 - ax} - 2y \left(\frac{ay - x^2}{y^2 - ax} \right)^2 - 2x \right]$$

(2) 两边求导数得

$$2yy' + \frac{2y'}{y} = 4x^3 \Rightarrow (y^2 + 1)y' = 2x^3 y \Rightarrow (2yy')y' + (y^2 + 1)y'' = 2(3x^2 y + x^3 y')$$

解得

$$y'' = \frac{2x^2 y}{(1 + y^2)^2} [3(1 + y^2)^2 + 2x^4(1 - y^2)]$$

(3) 两边求导数得 $xy' + y = e^{x+y}(1 + y') \Rightarrow (e^{x+y} - x)y' = y - e^{x+y} \Leftrightarrow (xy - x)y' = y - xy$, 再次在两边求导

$$(xy' + y - 1)y' + (xy - x)y'' + y = 0 \Rightarrow y'' = \frac{y}{x - xy} + \frac{(x + y - 2)(xy - y)}{(x - xy)^2} + \frac{x(xy - y^2)}{(x - xy)^3}$$

(4) 两边求导数得 $y' = -xe^y y' - e^y \Rightarrow y'(1 + xe^y) = -e^y \Rightarrow y'(y - 2) = e^y$, 再次求导有

$$y'' = e^{2y} \left[\frac{1}{(y-2)^2} - \frac{1}{(y-2)^3} \right] \Leftrightarrow y'' = \frac{2e^{2y}}{(1 + xe^y)^2} - \frac{xe^{3y}}{(1 + xe^y)^3}$$

例题 3.3.7 3.3-A-9

求 $\frac{dy}{dx}$: (1) $r^2 = 2a^2 \cos 2\theta$ 在 $\theta = \frac{\pi}{6}$ 处

(2) $r = ae^{m\theta}$, 其中 $r = \sqrt{x^2 + y^2}$ 以及 $\theta = \arctan \frac{y}{x}$ 为极坐标.

解 3.3.7.

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{dr(\theta)\sin\theta}{d\theta}}{\frac{dr(\theta)\cos\theta}{d\theta}} = \frac{r'(\theta)\sin\theta + r(\theta)\cos\theta}{r'(\theta)\cos\theta - r(\theta)\sin\theta}$$

(1) 参数方程为 $r^2(\theta) = 2a^2 \cos 2\theta$, $r'(\theta)r(\theta) = 2a^2(-\sin 2\theta)$, $r(\frac{\pi}{6}) = a^2$, $r(\theta) = a$, $r'(\theta) = -\sqrt{3}a$, 代入

$$\frac{dy}{dx} = \frac{r'(\theta)\sin\theta + r(\theta)\cos\theta}{r'(\theta)\cos\theta - r(\theta)\sin\theta} = 0$$

(2) 参数方程为 $\begin{cases} x = ae^{m\theta} \cos \theta \\ y = ae^{m\theta} \sin \theta \end{cases}$, $r(\theta) = ae^{m\theta}$, $r'(\theta) = ame^{m\theta}$, 代入就有

$$\frac{dy}{dx} = \frac{r'(\theta)\sin\theta + r(\theta)\cos\theta}{r'(\theta)\cos\theta - r(\theta)\sin\theta} = \frac{\cos\theta + m\sin\theta}{-\sin\theta + m\cos\theta} = \tan\left(\theta + \arctan \frac{1}{m}\right)$$

例题 3.3.8 3.3-B-1

求导: $y = e^x + e^{e^x}$ $y = \left(\frac{a}{b}\right)^x \left(\frac{b}{x}\right)^a \left(\frac{x}{a}\right)^b$ $y = 3^x \ln x$

解 3.3.8. (1) $y' = e^x + e^{e^x} e^x$;

(2) $\ln y = x \ln \left(\frac{a}{b}\right) + a \ln \left(\frac{b}{x}\right) + b \ln \left(\frac{x}{a}\right) = x \ln \left(\frac{a}{b}\right) + (b-a) \ln x$, 两边求导得

$$\frac{y'}{y} = \ln \left(\frac{a}{b}\right) + \frac{b-a}{x}, y' = \left(\frac{a}{b}\right)^x \left(\frac{b}{x}\right)^a \left(\frac{x}{a}\right)^b \left(\ln \left(\frac{a}{b}\right) + \frac{b-a}{x}\right)$$

(3) $y' = 3^x \ln 3 \ln x + \frac{3^x}{x}$

3.4 习题 3.4

例题 3.4.1 3.4-A-5

利用一阶微分的形式不变性求微分: (1) $y = \arctan e^x$ (2) $y = e^{\sin x}$

解 3.4.1. (1) $dy = \frac{de^x}{1+e^{2x}} = \frac{e^x}{1+e^{2x}} dx$; (2) $dy = e^{\sin x} d \sin x = e^{\sin x} \cos x dx$

例题 3.4.2 3.4-B-1

求下列函数的二阶微分 d^2y : (1) $y = \sqrt{1+x^2}$ (2) $y = \frac{\ln x}{x}$

解 3.4.2. (1) $d^2y = d(dy) = d\left(\frac{x}{\sqrt{1+x^2}} dx\right) = \frac{\sqrt{1+x^2} - x \frac{x}{\sqrt{1+x^2}}}{1+x^2} dx^2 = \frac{1}{(1+x^2)^{\frac{3}{2}}} dx^2$

(2) $d^2y = d(dy) = d\left(\frac{1-\ln x}{x^2} dx\right) = \frac{-x-2x(1-\ln x)}{x^4} dx^2 = \frac{2\ln x-3}{x^3} dx^2$

3.5 习题 3.5

例题 3.5.1 3.5-A-5

拉格朗日中值定理证明的关键是构造辅助函数, 试利用下列辅助函数来证明这个定理:

- (1) $\Phi(x) = [f(x) - f(a)](b-a) - (x-a)[f(b) - f(a)];$
 (2) $\Phi(x) = f(x)(b-a) - x[f(b) - f(a)].$

解 3.5.1. (1) 过 $(a, f(a)), (b, f(b))$ 的直线方程对应的一次函数为 $g(x) = \frac{f(b) - f(a)}{b-a}(x-a) + f(a)$, 所以构造

$$\begin{aligned} h(x) &= f(x) - g(x) = f(x) - \frac{f(b) - f(a)}{b-a}(x-a) - f(a) \\ &= \frac{(b-a)f(x) - (f(b) - f(a))(x-a) - (b-a)f(a)}{b-a} \\ &= \frac{(b-a)(f(x) - f(a)) - (f(b) - f(a))(x-a)}{b-a} = \frac{\Phi(x)}{b-a} \end{aligned}$$

且 $h(a) = h(b) = 0$, 所以根据罗尔定理, 存在 $\xi \in (a, b)$ 使得 $h'(\xi) = 0$, 即 $f'(\xi) = g'(\xi)$, 即

$$\frac{f(b) - f(a)}{b-a} = f'(\xi)$$

(2) 设 $\varphi(x) = f(x) - \frac{f(b) - f(a)}{b-a}x$, 容易发现 $\varphi'(x) = h'(x)$, 这表明 $\varphi(x)$ 是 $h(x)$ 向上平移得到的, 所以尽管此时没有 $h(a) = h(b) = 0$, 但是 $h(a) = h(b)$ 却仍然成立, 仍可利用罗尔定理得到存在 $\xi \in (a, b)$ 使得 $\varphi'(\xi) = 0$, 即 $f'(\xi) = g'(\xi)$, 即

$$\frac{f(b) - f(a)}{b-a} = f'(\xi)$$

例题 3.5.2 3.5-A-6

- (1) 证明: 如果 $\forall x \in [a, b]$, 有 $f'(x) \geq m$, m 是某常数, 则有 $f(b) \geq f(a) + m(b-a)$;
 (2) 证明: 如果 $\forall x \in [a, b]$, 有 $f'(x) \leq M$, M 是某常数, 则有 $f(b) \leq f(a) + M(b-a)$;
 (3) 如果 $\forall x \in [a, b]$, 有 $|f'(x)| \leq M$, 试写出一个类似的定理.

解 3.5.2. (1) 根据导数存在, 得知 $f(x)$ 在 $[a, b]$ 上连续可微, 符合拉格朗日定理使用条件, 于是存在 $\xi \in (a, b)$ 使得 $f'(\xi) = \frac{f(b) - f(a)}{b-a}$, 又因为 $f'(\xi) \geq m$, 代入就有 $f(b) \geq f(a) + m(b-a)$

(2) 根据导数存在, 得知 $f(x)$ 在 $[a, b]$ 上连续可微, 符合拉格朗日定理使用条件, 于是存在 $\xi \in (a, b)$ 使得 $f'(\xi) = \frac{f(b) - f(a)}{b-a}$, 又因为 $f'(\xi) \leq M$, 代入就有 $f(b) \leq f(a) + M(b-a)$

(3) 根据导数存在, 得知 $f(x)$ 在 $[a, b]$ 上连续可微, 符合拉格朗日定理使用条件, 于是存在 $\xi \in (a, b)$ 使得 $f'(\xi) = \frac{f(b) - f(a)}{b-a}$, 即 $|f'(\xi)| = \left| \frac{f(b) - f(a)}{b-a} \right|$, 又因为 $|f'(\xi)| \leq M$, 代入就有 $|f(b) - f(a)| \leq M(b-a)$, 即 $|f(b)| \leq |f(a)| + M(b-a)$

例题 3.5.3 3-5-A-7

证明: 无论 m 是什么数, 多项式函数 $f(x) = x^3 - 3x + m$ 在 $[0, 1]$ 内决不会有两个零点.

解 3.5.3. 用反证法, 假设 $f(x)$ 在 $[0, 1]$ 存在两个零点 x_1 和 x_2 , 则由于罗尔定理, 存在 $\xi \in (0, 1)$ 使得 $f'(\xi) = 0$, 但是 $f'(x) = 3x^2 - 3$ 在 $(0, 1)$ 上小于 0, 矛盾, 所以 $f(x) = x^3 - 3x + m$ 在 $[0, 1]$ 内决不会有两个零点.

例题 3.5.4 3-5-A-8

设 $f(x) \in C[0, 1]$ 且可微; 对于每个 $x, f(x)$ 的值都在 $(0, 1)$ 内; 并且 $\forall x \in (0, 1), f'(x) \neq 1$. 求证: 存在唯一的一个数 $x_0 \in (0, 1)$, 使得 $f(x_0) = x_0$.

解 3.5.4. 构造函数 $g(x) = f(x) - x$, 则 $g(x)$ 在 $(0, 1)$ 上连续可微, 且 $g(0) = f(0) - 0 > 0, g(1) = f(1) - 1 < 0$, 所以 $g(x)$ 在 $(0, 1)$ 上必有零点, 假设有两个及以上个零点, 则根据罗尔定理得到必然存在 $\xi \in (0, 1)$ 使得 $g'(\xi) = 0$, 即 $f'(\xi) = 1$, 矛盾, 所以 $f(x)$ 在 $(0, 1)$ 上只有一个零点, 即存在唯一的一个数 $x_0 \in (0, 1)$, 使得 $f(x_0) = x_0$.

例题 3.5.5 3-5-A-10

证明: (1) $|\sin b - \sin a| \leq |b - a|$; (2) $\frac{a-b}{a} < \ln \frac{a}{b} < \frac{a-b}{b}$ ($a > b > 0$).

解 3.5.5. (1) 由 Lagrange 中值定理, 存在 $\xi \in (a, b)$ 使得 $f'(\xi) = \frac{\sin b - \sin a}{b - a}$

且 $|f'(\xi)| = \left| \frac{\sin b - \sin a}{b - a} \right| \leq 1$, 变形即可得到 $|\sin b - \sin a| \leq |b - a|$

(2) 等价于证明 $\frac{1}{a} < \frac{\ln a - \ln b}{b - a} < \frac{1}{b}$, 根据 Lagrange 中值定理, 对于函数 $f(x) = \ln x$, 存在 $\xi \in (a, b)$ 使得 $f'(\xi) = \frac{\ln a - \ln b}{b - a}$ 且由于 $\ln x$ 的导函数在 (a, b) 上单调递减, 所以 $f'(a) < f'(\xi) < f'(b)$, 即 $\frac{1}{a} < \frac{\ln a - \ln b}{b - a} < \frac{1}{b}$.

例题 3.5.6 3-5-A-11

证明: 若 $f(x) \in C[a, b]$, 在 (a, b) 内可导, 则必存在一点 $\xi \in (a, b)$, 使得

$$2\xi[f(b) - f(a)] = (b^2 - a^2)f'(\xi)$$

解 3.5.6. 等价于证明 $\frac{f(a) - f(b)}{a^2 - b^2} = \frac{f'(\xi)}{2\xi}$, 根据柯西中值定理, 对于 $f(x)$ 和 $g(x) = x^2$, 存在 $\xi \in (a, b)$ 使得 $\frac{f(a) - f(b)}{g(a) - g(b)} = \frac{f'(\xi)}{g'(\xi)}$, 即 $\frac{f(a) - f(b)}{a^2 - b^2} = \frac{f'(\xi)}{2\xi}$.

例题 3.5.7 3-5-A-12

设 $f(x) \in C[a, b]$. 在 (a, b) 内可导, $0 < a < b$. 求证: 存在一点 $\xi \in (a, b)$, 使得

$$\frac{af(b) - bf(a)}{a - b} = f(\xi) - \xi f'(\xi).$$

解 3.5.7. 等价于证明 $\frac{\frac{f(a)}{a} - \frac{f(b)}{b}}{\frac{1}{a} - \frac{1}{b}} = f(\xi) - \xi f'(\xi)$, 对于 $f(x)$ 和 $g(x) = \frac{1}{x}$, 存在 $\xi \in (a, b)$ 使得

$$\frac{f(a) - f(b)}{g(a) - g(b)} = \frac{f'(\xi)}{g'(\xi)} = \frac{\frac{\xi f'(\xi) - f(\xi)}{\xi^2}}{-\frac{1}{\xi^2}} = f(\xi) - \xi f'(\xi)$$

例题 3.5.8 3-5-B-2

设函数 $f(x)$ 满足 $|f(x) - f(y)| \leq |x - y|^n, n > 1$, 通过考虑 f' 证明 f 是常数.

解 3.5.8. 变形得到 $\left| \frac{f(x) - f(y)}{x - y} \right| \leq |x - y|^{n-1}$, 由拉格朗日中值定理知存在 $\xi \in (x, y)$ 使得 $|f'(\xi)| = \left| \frac{f(x) - f(y)}{x - y} \right|$, 则 $|f'(\xi)| < |x - y|^{n-1}$, 由于 x, y 的任意性, $|f'(\xi)| = 0$, 所以 f 是常数.

例题 3.5.9 3-5-B-10

设 $h > 0, f'(x)$ 在 $(a - h, a + h)$ 内存在. 求证:

$$(1) \frac{f(a + h) - f(a - h)}{h} = f'(a + \theta h) + f'(a - \theta h) \quad (0 < \theta < 1);$$

$$(2) \frac{f(a + h) - 2f(a) + f(a - h))}{h} = f'(a + \theta h) - f'(a - \theta h) \quad (0 < \theta < 1).$$

解 3.5.9. (1) 由 Lagrange 中值定理得到, 若 $f'(x)$ 在 $(x, x + h)$ 内存在, 那么存在 $\xi \in (x, x + h)$ 使得 $f'(\xi) = \frac{f(x + h) - f(x)}{h}$, 这里可以换元, 令 $\xi = x + \theta h$, 其中 $\theta \in (0, 1)$, 则存在 $\theta \in (0, 1)$ 使得 $f'(x + \theta h) = \frac{f(x + h) - f(x)}{h}$, 使用这个定理就可以得到:

$$\begin{aligned} \frac{f(a + h) - f(a - h)}{h} &= \frac{f(a + h) - f(a)}{h} + \frac{f(a) - f(a - h)}{h} \\ &= f'(a + \theta h) + f'(a - \theta h) \end{aligned}$$

(2) 同理, 若 $f'(x)$ 在 $(x, x + h)$ 内存在, 那么存在 $\xi \in (x, x + h)$ 使得 $f'(\xi) = \frac{f(x + h) - f(x)}{h}$, 这里可以换元, 令 $\xi = x + \theta h$, 其中 $\theta \in (0, 1)$, 则存在 $\theta \in (0, 1)$ 使得 $f'(x + \theta h) = \frac{f(x + h) - f(x)}{h}$, 使用这个定理就可以得到:

$$\begin{aligned} \frac{f(a + h) - 2f(a) + f(a - h))}{h} &= \frac{f(a + h) - f(a)}{h} - \frac{f(a) - f(a - h)}{h} \\ &= f'(a + \theta h) - f'(a - \theta h) \end{aligned}$$

例题 3.5.10 3-5-B-11

由拉格朗日中值定理知, $\ln(1+x) - 0 = x \cdot \frac{1}{1+\theta x} (0 < \theta < 1)$, 证明 $\lim_{x \rightarrow 0} \theta = \frac{1}{2}$.

解 3.5.10. 解得 $\theta = \frac{x - \ln(x+1)}{x \ln(x+1)}$, 转化为求 $\lim_{x \rightarrow 0} \frac{x - \ln(x+1)}{x \ln(x+1)}$, 由

$$\frac{(x - \ln(x+1))'}{(x \ln(x+1))'} = \frac{1 - \frac{1}{x+1}}{\frac{x}{x+1} + \ln(x+1)} = \frac{x}{x + (x+1) \ln(x+1)}$$

则由于导函数之比在 0 处的极限存在, 分母不为 0, 当 $x \rightarrow 0$ 时, 分子分母趋近于 0, 所以洛必达法则有效, 则

$$\lim_{x \rightarrow 0} \frac{x - \ln(x+1)}{x \ln(x+1)} = \lim_{x \rightarrow 0} \frac{x}{x + (x+1) \ln(x+1)}$$

发现分子分母趋近于 0, 对 $\frac{x}{x + (x+1) \ln(x+1)}$ 分子分母上下分别求导得到 $\frac{1}{2 + \ln(x+1)}$, 由于导函数之比在 0 处的极限存在, 分母不为 0, 则洛必达法则有效, 则

$$\lim_{x \rightarrow 0} \frac{x}{x + (x+1) \ln(x+1)} = \lim_{x \rightarrow 0} \frac{1}{2 + \ln(x+1)} = \frac{1}{2}$$

所以 $\lim_{x \rightarrow 0} \theta = \frac{1}{2}$

例题 3.5.11 3-5-B-12

设 $f(x) = \begin{cases} \frac{g(x)}{x}, & x \neq 0, \\ 0, & x = 0. \end{cases}$, 并设 $g(0) = g'(0) = 0, g''(0) = 17$. 求 $f'(0)$.

解 3.5.11. 利用导数的定义: 已知 $f(0) = 0$

$$f'(0) = \lim_{h \rightarrow 0} \frac{f(h) - f(0)}{h} = \lim_{h \rightarrow 0} \frac{g(h)}{h^2}$$

又因为 $g'(x), g''(x)$ 在 $x = 0$ 的邻域内有定义, 所以使用洛必达法则得到

$$\lim_{h \rightarrow 0} \frac{g(h)}{h^2} = \lim_{h \rightarrow 0} \frac{g'(h)}{2h} = \frac{1}{2} \lim_{h \rightarrow 0} \frac{g'(h) - g'(0)}{h} = \frac{1}{2} g''(0) = \frac{17}{2}$$

可导一定连续, 所以 $f'(0) = \lim_{h \rightarrow 0} \frac{g''(h)}{2} = \frac{17}{2}$

例题 3.5.12 3-5-B-14

设 $f(x)$ 一阶可导, 且 $f''(x_0)$ 存在, 求证:

$$\lim_{h \rightarrow 0} \frac{f(x_0 + 2h) - 2f(x_0 + h) + f(x_0)}{h^2} = f''(x_0).$$

解 3.5.12. 发现分子分母都趋近于 0，先分子分母上下分别求导得到

$$\lim_{h \rightarrow 0} \frac{2f'(x_0 + 2h) - 2f'(x_0 + h)}{2h} = \lim_{h \rightarrow 0} \frac{f'(x_0 + 2h) - f'(x_0 + h)}{h}$$

，发现此时极限仍然存在，分母仍不是 0，所以洛必达法则有效，于是有

$$\lim_{h \rightarrow 0} \frac{f(x_0 + 2h) - 2f(x_0 + h) + f(x_0)}{h^2} = \lim_{h \rightarrow 0} \frac{f'(x_0 + 2h) - f'(x_0 + h)}{h}$$

再利用 $f''(x_0)$ 的存在性，以及导数的定义，得到

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{f'(x_0 + 2h) - f'(x_0 + h)}{h} &= \lim_{x \rightarrow 0} \frac{f'(x_0 + 2h) - f'(x_0) - (f'(x_0 + h) - f'(x_0))}{h} \\ &= 2f''(x_0) - f''(x_0) = f''(x_0) \end{aligned}$$

即证.

3.6 泰勒公式

例题 3.6.1 3.6-A-2

按 x 的正整数幂, 写出下列函数的展开式至含有指定阶数的项 (带皮亚诺余项):

- (1) $\frac{1}{1-x}$ 到含 x^7 的项; (2) $\arctan x$ 到含 x^4 的项
(3) $\frac{1}{\sqrt{1+x}}$ 到含 x^4 的项; (4) $\tan x$ 到含 x^4 的项.

解 3.6.1. (1) 先求 $f(x) = \frac{1}{1-x}$ 的在 $x=0$ 处的 n 阶导数 $f^{(n)}(0)$, 由于 $(1-x)f(x) = 1$ 两边求 n 阶导数, 并利用莱布尼兹公式, 得到

$$(1-x)f^{(n)}(x) + (-1)C_n^1 f^{(n-1)}(x) = 0 \Leftrightarrow f^{(n)}(0) = n f^{(n-1)}(0)$$

又 $f'(0) = 1$, 所以 $f^{(n)}(0) = n!$, 于是展开到含 x^7 的项就是:

$$f(x) = 1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + o(x^7)$$

(2) 令 $f(x) = \frac{1}{1+x^2} = \frac{1}{1-(-x^2)}$, 展开得到:

$$f'(x) = 1 + (-x^2) + (-x^2)^2 + (-x^2)^3 + o(x^4) = 1 - x^2 + x^4 - x^6 + o(x^6)$$

保留到到含 x^4 的项就是

$$\arctan x = x - \frac{1}{3}x^3 + o(x^4)$$

(3) 直接使用广义二项式展开:

$$\begin{aligned} f(x) &= (1+x)^{-\frac{1}{2}} = 1 + C_{-\frac{1}{2}}^1 x + C_{-\frac{1}{2}}^2 x^2 + C_{-\frac{1}{2}}^3 x^3 + C_{-\frac{1}{2}}^4 x^4 + o(x^4) \\ &= 1 - \frac{1}{2}x + \frac{-\frac{1}{2}(-\frac{1}{2}-1)}{2}x^2 + \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{6}x^3 + \frac{-\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)(\frac{1}{2}-3)}{24}x^4 + o(x^4) \\ &= 1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{5}{16}x^3 + \frac{35}{128}x^4 + o(x^4) \end{aligned}$$

(4) 反复求导过程复杂, 可以先建立微分方程:

$$y' = (\tan x)' = 1 + \tan^2 x = 1 + y^2$$

两边求导得到:

$$\begin{aligned} y'' &= 2yy' = 2y(1+y^2) = 2y + 2y^3 \\ y''' &= 2y' + 6y^2y' = 2(1+y^2) + 6y^2(1+y^2) = 2 + 8y^2 + 6y^4 \\ y^{(4)} &= 16yy' + 24y^3y' = 16y(1+y^2) + 24y^3(1+y^2) = 16y + 40y^3 + 24y^5 \end{aligned}$$

代入 $y=0$, $f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \dots + \frac{f^{(n)}(0)}{n!}x^n + o(x^n)$ ($x \rightarrow 0$) 于是得到展开式:

$$\tan x = x + \frac{1}{3}x^3 + o(x^4)$$

例题 3.6.2 3.6-A-5

求函数 $f(x) = xe^x$ 的 n 阶麦克劳林公式, 带拉格朗日余项.

解 3.6.2. 设 $f(x) = xe^x$, 其泰勒展开式展开到含 x^n 的项的式子, 都可以由 x, e^x 展开式的乘积确定:

$$\begin{aligned} f(x) &= xe^x = x \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^{n-1}}{(n-1)!} \right) \\ &= x + x^2 + \frac{x^3}{2!} + \frac{x^4}{3!} + \cdots + \frac{x^n}{(n-1)!} \end{aligned}$$

再加上 $\frac{f^{(n+1)}(\theta x)}{(n+1)!}, \theta \in (0, 1)$, 即可得到

$$f(x) = xe^x = x + x^2 + \frac{x^3}{2!} + \frac{x^4}{3!} + \cdots + \frac{x^n}{(n-1)!} + \frac{e^{\theta x}(\theta x + n + 1)}{(n+1)!}x^{n+1}$$

例题 3.6.3 3.6-A-6

$$\begin{aligned} (1) \lim_{x \rightarrow 0} \frac{e^x - 1 - x}{x^2}; & \quad (2) \lim_{x \rightarrow 0} \frac{e^x - 1 - x - \frac{x^2}{2}}{x^3}; (3) \lim_{x \rightarrow 0} \frac{e^{x^3} - 1 - x^3}{(\sin 2x)^6}; \\ (4) \lim_{x \rightarrow +\infty} \left(x + \frac{1}{2}\right) \ln \left(1 + \frac{1}{x}\right); & \quad (5) \lim_{x \rightarrow 0} \left(\frac{1}{x} - \frac{1}{e^x - 1}\right); \quad (6) \lim_{x \rightarrow 0^+} \frac{e^x - 1 - x}{\sqrt{1-x} - \cos \sqrt{x}} \end{aligned}$$

$$\begin{aligned} \text{解 3.6.3. } (1) \lim_{x \rightarrow 0} \frac{e^x - 1 - x}{x^2} &= \lim_{x \rightarrow 0} \frac{1 + x + \frac{1}{2}x^2 + o(x^2) - 1 - x}{x^2} = \frac{1}{2} \\ (2) \lim_{x \rightarrow 0} \frac{e^x - 1 - x - \frac{x^2}{2}}{x^3} &= \lim_{x \rightarrow 0} \frac{1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + o(x^3) - 1 - x - \frac{1}{2}x^2}{x^3} = \frac{1}{6} \\ (3) \lim_{x \rightarrow 0} \frac{e^{x^3} - 1 - x^3}{(\sin 2x)^6} &= \lim_{x \rightarrow 0} \frac{1 + x^3 + \frac{1}{2}x^6 + o(x^6) - 1 - x^3}{64x^6} = \frac{1}{128} \\ (4) \lim_{x \rightarrow +\infty} \left(x + \frac{1}{2}\right) \ln \left(1 + \frac{1}{x}\right) &= \lim_{x \rightarrow 0^+} \left(\frac{1}{x} + \frac{1}{2}\right) \ln(1+x) = \lim_{x \rightarrow 0^+} \frac{x + 2 \ln(x+1)}{2x} = 1 \\ (5) \lim_{x \rightarrow 0} \left(\frac{1}{x} - \frac{1}{e^x - 1}\right) &= \lim_{x \rightarrow 0} \frac{e^x - x - 1}{x(e^x - 1)} = \frac{1 - x + \frac{1}{2}x^2 + o(x^2) - 1 - x}{x^2} = \frac{1}{2} \\ (6) \lim_{x \rightarrow 0^+} \frac{e^x - 1 - x}{\sqrt{1-x} - \cos \sqrt{x}} &= \lim_{x \rightarrow 0^+} \frac{1 + x + \frac{1}{2}x^2 + o(x^2) - 1 - x}{1 - \frac{1}{2}x - \frac{1}{8}x^2 + o(x^2) - (1 - \frac{1}{2}x + \frac{1}{24}x^2)} = \frac{\frac{1}{2}}{-\frac{1}{6}} = -3 \end{aligned}$$

3.7 函数性态的研究

例题 3.7.1 3.7-A-3

证明函数 $y = x + \sin x$ 严格上升.

解 3.7.1. $y' = 1 - \cos x \geq 0$, 故 y 严格上升.

例题 3.7.2 3.7-A-7

设 $f(x) = axe^{bx}$, 试确定常数 a, b , 使得 $f(\frac{1}{3}) = 1$, 且函数在 $x = \frac{1}{3}$ 处有极大值.

解 3.7.2. $f'(x) = ae^{bx}(bx + 1)$, $f'(\frac{1}{3}) = ae^{b\frac{1}{3}}(1 + \frac{1}{3}b) = 0$, $f(\frac{1}{3}) = a\frac{1}{3}e^{b\frac{1}{3}} = 1$, 得到由必要条件引出的方程组 $a(b + \frac{1}{3}) = 0$, $ae^{b\frac{1}{3}} = 3$, 由第二个方程得到 $a \neq 0$, 所以 $b = -3$, $a = 3e$

下面证明充分性, 当 $b = -3$, $a = 3e$ 时, $f(x) = 3xe \cdot e^{-3x} = 3xe^{1-3x}$, $f(\frac{1}{3}) = 1$, $f'(x) = e^{\frac{1-3x}{e^{3x}}} = 0$, 导函数的正负性取决于一次函数 $y = 1 - 3x$, 当 $x < \frac{1}{3}$ 时, $f'(x) > 0$, 当 $x = \frac{1}{3}$ 时, $f'(x) = 0$, 当 $x > \frac{1}{3}$ 时, $f'(x) < 0$, 所以 $f(x)$ 在 $\frac{1}{3}$ 处的极大值是 $f(\frac{1}{3}) = 1$.

例题 3.7.3 3.7-A-8

- (1) $\sin x > \frac{2}{\pi}x$ ($0 < x < \frac{\pi}{2}$); (2) $\cos x > 1 - \frac{x^2}{2}$ ($x \neq 0$);
(3) $x > \ln(1+x) > x - \frac{x^2}{2}$ ($x > 0$); (4) $\ln(1+x) \geq \frac{\arctan x}{1+x}$ ($x \geq 0$).

解 3.7.3. (1) 对于 $y = \sin x$, $y' = \cos x$, $y'' = -\sin x$, y 在 $(0, \frac{\pi}{2})$ 上为凹函数. 根据凹函数的定义, 设 $x_1 = 0$, $x_2 = \frac{\pi}{2}$, $f(x) = \sin x$, 有:

$$f(tx_1 + (1-t)x_2) = f\left(\frac{\pi}{2}(1-t)\right) \geq tf(x_1) + (1-t)f(x_2) = 1-t$$

令 $x = \frac{\pi}{2}(1-t) \in (0, \frac{\pi}{2})$, 则有 $\sin x > \frac{2}{\pi}x$ ($0 < x < \frac{\pi}{2}$).

(2) 考虑函数 $f(x) = \cos x - (1 - \frac{x^2}{2})$, 则 $f(0) = 0$. 计算导数:

$$f'(x) = -\sin x + x, \quad f''(x) = -\cos x + 1 \geq 0$$

因此 $f'(x)$ 单调递增, 又 $f'(0) = 0$, 故当 $x > 0$ 时 $f'(x) > 0$, 当 $x < 0$ 时 $f'(x) < 0$. 所以 $f(x)$ 在 $x = 0$ 处取得最小值 $f(0) = 0$, 且当 $x \neq 0$ 时 $f(x) > 0$, 即 $\cos x > 1 - \frac{x^2}{2}$ ($x \neq 0$).

(3) 先证左边不等式: 令 $f(x) = x - \ln(1+x)$, 则 $f(0) = 0$, $f'(x) = 1 - \frac{1}{1+x} = \frac{x}{1+x} > 0$ ($x > 0$), 故 $f'(x) > 0$, 又因为 $f(x) > f(0) = 0$, 故 $f(x) > 0$, 即 $x > \ln(1+x)$

再证右边不等式: 令 $g(x) = \ln(1+x) - (x - \frac{x^2}{2})$, 则 $g(0) = 0$, $g'(x) = \frac{1}{1+x} - 1 + x = \frac{x^2}{1+x} > 0$ ($x > 0$), 故 $g'(x) > 0$, $g(x)$ 单调递增, 又因为 $g(x) > g(0) = 0$, 故 $g(x) > 0$, 即 $\ln(1+x) > x - \frac{x^2}{2}$

(4) 令 $h(x) = (1+x)\ln(1+x) - \arctan x$, 则 $h(0) = 0$. 计算导数:

$$h'(x) = \ln(1+x) + 1 - \frac{1}{1+x^2}$$

当 $x \geq 0$ 时, $\ln(1+x) \geq 0$, 且 $1 - \frac{1}{1+x^2} \geq 0$, 故 $h'(x) \geq 0$, $h(x)$ 单调递增, 因此 $h(x) \geq h(0) = 0$, 即 $(1+x)\ln(1+x) \geq \arctan x$, 亦即 $\ln(1+x) \geq \frac{\arctan x}{1+x}$ ($x \geq 0$).

例题 3.7.4 3.7-B-3

用函数的凹凸性证明下列不等式: (1) $\ln x \leq x - 1$ ($x > 0$);

(2) $2 \arctan \frac{a+b}{2} \geq \arctan a + \arctan b$ ($a, b \geq 0$);

(3) $1 + x^2 \leq 2^x$ ($0 \leq x \leq 1$);

(4) $\frac{x^n + y^n}{2} > \left(\frac{x+y}{2}\right)^n$ ($x > 0, y > 0, x \neq y, n > 1$).

解 3.7.4. (1) 考虑函数 $f(x) = \ln x$, 其定义域为 $(0, +\infty)$ 。计算导数:

$$f'(x) = \frac{1}{x}, \quad f''(x) = -\frac{1}{x^2} < 0$$

因此 $f(x)$ 在 $(0, +\infty)$ 上是凹函数。根据凹函数的性质, 对于任意 $x > 0$, 有:

$$f(x) \leq f(1) + f'(1)(x - 1)$$

代入 $f(1) = 0, f'(1) = 1$, 得:

$$\ln x \leq 0 + 1 \cdot (x - 1) = x - 1$$

即 $\ln x \leq x - 1$ ($x > 0$).

(2) 考虑函数 $f(x) = \arctan x$, 其定义域为 $[0, +\infty)$ 。计算导数:

$$f'(x) = \frac{1}{1+x^2}, \quad f''(x) = -\frac{2x}{(1+x^2)^2} \leq 0$$

因此 $f(x)$ 在 $[0, +\infty)$ 上是凹函数。根据凹函数的性质, 对于任意 $a, b \geq 0$, 有:

$$f\left(\frac{a+b}{2}\right) \geq \frac{f(a) + f(b)}{2}$$

即:

$$2 \arctan \frac{a+b}{2} \geq \arctan a + \arctan b \quad (a, b \geq 0)$$

(3) 考虑函数 $g(x) = 2^x - 1 - x^2$, 我们需要证明在 $[0, 1]$ 上 $g(x) \geq 0$ 。计算导数:

$$g'(x) = 2^x \ln 2 - 2x, \quad g''(x) = 2^x (\ln 2)^2 - 2$$

在 $[0, 1]$ 上, 由于 $2^x \leq 2$ 且 $(\ln 2)^2 < 1$, 所以 $g''(x) < 2 - 2 = 0$, 即 $g(x)$ 是凹函数。由凹函数的性质, 对于任意 $x \in [0, 1]$, 有:

$$g(x) \geq (1-x)g(0) + xg(1)$$

代入 $g(0) = 0, g(1) = 0$, 得:

$$g(x) \geq 0$$

即 $2^x - 1 - x^2 \geq 0$, 所以 $1 + x^2 \leq 2^x$ ($0 \leq x \leq 1$).

(4) 考虑函数 $f(x) = x^n$ ($n > 1$), 其定义域为 $(0, +\infty)$ 。计算导数:

$$f'(x) = nx^{n-1}, \quad f''(x) = n(n-1)x^{n-2} > 0$$

因此 $f(x)$ 是凸函数。根据凸函数的性质, 对于 $x > 0, y > 0, x \neq y$, 有:

$$\frac{f(x) + f(y)}{2} > f\left(\frac{x+y}{2}\right)$$

即:

$$\frac{x^n + y^n}{2} > \left(\frac{x+y}{2}\right)^n \quad (x > 0, y > 0, x \neq y, n > 1)$$

3.8 习题 3.8

例题 3.8.1 3.8-A-6

求从点 $M(p, p)$ 到抛物线 $y^2 = 2px$ 的最短距离.

解 3.8.1. 设函数 $f(y) = \left(\frac{y^2}{2p} - p\right)^2 + (y - p)^2$, $f'(y) = 2\left(\frac{y^2}{2p} - p\right)\frac{y}{p} + 2(y - p) = 0$, 导函数可以化为

$$\frac{f'(y)}{p} = 2\left(\frac{1}{2}\left(\frac{y}{p}\right)^2 - 1\right)\frac{y}{p} + 2\frac{y}{p} - 2 = \left(\frac{y}{p}\right)^3 - 2$$

单调递增, 那么 $f'(y) = 0 \Leftrightarrow y = \sqrt[3]{2p}$, 则 $f(y)$ 在 $(-\infty, \sqrt[3]{2p})$ 单调递减, 在 $(\sqrt[3]{2p}, +\infty)$ 上单调递增, 最小值为 $p\sqrt{\left((2^{-\frac{1}{3}} - 1)^2 + (2^{\frac{1}{3}} - 1)^2\right)} = p(\sqrt[3]{2} - 1)\sqrt{\frac{\sqrt[3]{2} + 2}{2}}$.

例题 3.8.2 3.8-A-7

从面积为常数 S 的一切矩形中, 求其周长为最小者

解 3.8.2. 设 $ab = S$, 则 $a + b \geq 2\sqrt{ab} = 2\sqrt{S}$, 当且仅当 $a = b$, 即正方形时取等号, 所以最小者为边为 \sqrt{S} 的正方形

例题 3.8.3 3.8-A-8

在椭圆 $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ 中, 嵌入有最大面积而边平行于椭圆轴的矩形, 求此矩形的边长

解 3.8.3. 设第一象限的点 $P(x, y)$, 使用基本不等式, 矩形面积为 $S = 4xy \leq 2ab\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right) = 2ab$,

解方程 $\begin{cases} \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \\ \frac{y}{x} = \frac{b}{a} \end{cases}$ 当且仅当矩形的边长为 $\sqrt{2}a, \sqrt{2}b$ 时取等号

例题 3.8.4 3.8-B-5

求由 y 轴上的一个给定点 $(0, b)$ 到抛物线 $x^2 = 4y$ 上的点的最短距离.

解 3.8.4. 设距离的平方为 $f(x) = x^2 + \left(\frac{x^2}{4} - b\right)^2$, 则

$$f'(x) = 2x + 2\left(\frac{x^2}{4} - b\right) \frac{x}{2} = x\left(\frac{x^2}{4} - (b-2)\right)$$

当 $b \leq 2$ 时, $f'(x)$ 在 $x \leq 0$ 时小于等于 0, 在 $x > 0$ 时大于 0, 则 $f(x)$ 在 $(-\infty, 0)$ 单调递减, 在 $[0, +\infty)$ 上单调递增, 最小值为 $f(0) = b^2$;

当 $b > 2$ 时, $f'(x)$ 的根为 $x_1 = -2\sqrt{b-2}, x_2 = 0, x_3 = 2\sqrt{b-2}$, $f'(x)$ 在 $(-\infty, x_1)$ 小于 0, 在 $[x_1, 0]$ 上大于等于 0, 在 $(0, x_2)$ 上小于 0, 在 $[x_2, +\infty)$ 上大于等于 0

所以 $f(x)$ 在 $(-\infty, x_1)$ 单调递减, 在 $[x_1, 0]$ 上单调递增, 在 $(0, x_2)$ 上单调递减, 在 $[x_2, +\infty)$ 上单调递增, 又由于 $f(x)$ 为偶函数, 所以最小值为 $f(-2\sqrt{b-2}) = f(2\sqrt{b-2}) = 4(b-1)$, 则距离的

$$\text{最小值为 } \begin{cases} |b|, b \leq 2 \\ 2\sqrt{b-1}, b > 2 \end{cases}$$

例题 3.8.5 3.8-B-6

在椭圆 $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ 的第一象限部分求一点 P , 使该点处的切线、椭圆及两坐标轴所围图形的面积为最小 (其中 $a > 0, b > 0$).

解 3.8.5. 取 $P(x, y)$, 切线斜率为 $k = -\frac{b^2x}{a^2y^2}$, 切线方程为 $Y - y = -\frac{b^2x}{a^2y}(X - x)$, 横截距和纵截距为 $\frac{a^2}{x}, \frac{b^2}{y}$, 三角形面积为 $\frac{a^2b^2}{2xy} = \frac{a^3b}{2x\sqrt{a^2-x^2}}$, 则设 $f(x) = x^2(a-x^2), f'(x) = 2a^2x - 4x^3 = 2x(a^2 - 2x^2) = 0$, 又因为 $x > 0$, 所以 $f'(x)$ 在 $(0, \frac{a}{\sqrt{2}})$ 大于 0, 在 $[\frac{a}{\sqrt{2}}, +\infty)$ 小于等于 0, 所以最小值为 $f(\frac{a}{\sqrt{2}})$, 所以所求的 P 坐标为 $(\frac{a}{\sqrt{2}}, \frac{b}{\sqrt{2}})$, 三角形的面积是 $S = \frac{a^2b^2}{2xy} = ab$

第四章 一元函数积分学

例题 4.0.1 4.1-B-2

利用定积分的几何意义, 求下列定积分:

$$(1) \int_a^b x dx; \quad (2) \int_a^b \sqrt{(x-a)(b-x)} dx; \quad (3) \int_a^b \left| x - \frac{a+b}{2} \right| dx.$$

解 4.0.1. (1) 几何意义为梯形面积

$$\int_a^b x dx = \frac{(a+b)(b-a)}{2}$$

(2) 由于函数 $y = \sqrt{(x-a)(b-x)}$ 在区间 (a, b) 上图像为一条以 $(a, 0), (b, 0)$ 为顶点的半圆, 所以

$$\int_a^b \sqrt{(x-a)(b-x)} dx = \frac{\pi}{2} \cdot \frac{b-a}{2} \cdot \frac{b-a}{2} = \frac{\pi(b-a)^2}{8}$$

(3) 函数 $y = \left| x - \frac{a+b}{2} \right|$ 在区间 $[a, b]$ 上图像是两个对称的等腰直角三角形, 所以

$$\int_a^b \left| x - \frac{a+b}{2} \right| dx = 2 \cdot \frac{1}{2} \cdot \left(\frac{b-a}{2} \right)^2 = \left(\frac{b-a}{2} \right)^2$$

例题 4.0.2 4.1-B-3

设 $f(x), g(x) \in C[a, b]$, 求证:

(1) 若 $f(x) \geq 0, \forall x \in [a, b]$, 且 $\int_a^b f(x) dx = 0$, 则在区间 $[a, b]$ 上 $f(x) \equiv 0$;

(2) 若 $f(x) \leq g(x), \forall x \in [a, b]$, 且 $\int_a^b f(x) dx = \int_a^b g(x) dx$, 则在区间 $[a, b]$ 上 $f(x) \equiv g(x)$.

解 4.0.2. (1) 由于 $f(x)$ 连续非负, 反设存在 $x_0 \in [a, b]$ 使得 $f(x_0) > 0$, 则由连续性知, 存在区间 $[a_1, b_1] \subset [a, b]$, 使得 $f(x) \geq \frac{1}{2}f(x_0) > 0$, 则由积分中值定理:

$$\begin{aligned} \int_a^b f(x) dx &\geq \int_a^{a_1} f(x) dx + \int_{a_1}^{b_1} f(x) dx + \int_{b_1}^b f(x) dx \\ &\geq \int_{a_1}^{b_1} f(x) dx = \frac{1}{2}f(x_0)(b_1 - a_1) > 0 \end{aligned}$$

(2) 设 $h(x) = g(x) - f(x)$, 则 $h(x) \geq 0, \forall x \in [a, b]$, 且 $\int_a^b h(x) dx = 0$, 则由 (1) 知 $h(x) \equiv 0$, 即 $f(x) \equiv g(x)$.

例题 4.0.3 4.1-B-4

应用柯西-许瓦兹不等式证明: $\left(\int_a^b f(x)dx\right)^2 \leq (b-a) \int_a^b f^2(x)dx$.

解 4.0.3. 根据柯西-施瓦兹不等式, 对于任意在区间 $[a, b]$ 上可积的函数 $f(x)$ 和 $g(x)$, 有:

$$\left(\int_a^b f(x)g(x) dx\right)^2 \leq \int_a^b [f(x)]^2 dx \cdot \int_a^b [g(x)]^2 dx$$

特别地, 取 $g(x) = 1$, 则有:

$$\left(\int_a^b f(x) \cdot 1 dx\right)^2 \leq \int_a^b [f(x)]^2 dx \cdot \int_a^b 1^2 dx$$

计算右侧积分:

$$\int_a^b 1^2 dx = \int_a^b 1 dx = b - a$$

代入上式得:

$$\left(\int_a^b f(x) dx\right)^2 \leq (b-a) \int_a^b f^2(x) dx$$

当且仅当 $f(x)$ 与 $g(x) = 1$ 线性相关, 即 $f(x)$ 为常数函数时等号成立。

例题 4.0.4 4.1-B-5

函数 $f(x) \in C[0, 1]$, 在 $(0, 1)$ 内可导, 且 $3 \int_{\frac{2}{3}}^1 f(x)dx = f(0)$. 证明: $\exists c \in (0, 1)$, 使得 $f'(c) = 0$.

解 4.0.4. 改写成 $\frac{\int_{\frac{2}{3}}^1 f(x)dx}{1 - \frac{2}{3}} = f(0)$, 又由于 $f(x)$ 在区间 $[0, 1]$ 上连续, 所以存在 $\xi \in [0, 1]$ 使得

$f(\xi) = \frac{\int_{\frac{2}{3}}^1 f(x)dx}{1 - \frac{2}{3}}$, 假如在区间 $[0, 1]$ 上有且仅有一个点 $\xi = 0$ 使得 $f(\xi) = f(0) = \frac{\int_{\frac{2}{3}}^1 f(x)dx}{1 - \frac{2}{3}}$, 则由于 $f(x)$ 在区间 $[0, 1]$ 上连续, 所以 $f(x)$ 在 $(0, 1)$ 上恒大于 $f(0)$ 或恒小于 $f(0)$, 不妨设 $f(x) > f(0), \forall x \in (0, 1)$, 则 $\int_{\frac{2}{3}}^1 f(x)dx > \int_{\frac{2}{3}}^1 f(0)dx = \frac{1}{3}f(0)$, 矛盾, 同理另一种情况也矛盾。所以存在 $c \in (0, 1)$ 使得 $f(c) = f(0)$, 由罗尔定理以及 $f(x)$ 在 $(0, 1)$ 内可导, 知存在 $c \in (0, 1)$ 使得 $f'(c) = 0$.

例题 4.0.5 4.2-A-5

$$(1) \frac{d}{dx} \left(\int_0^{x^2} \sqrt{1+t^2} dt \right); \quad (2) \frac{d}{dx} \left(\int_{x+a}^{x+b} (t+1)^2 dt \right)$$

解 4.0.5. (1) 设函数 $\sqrt{1+t^2}$ 的原函数是 $F(t)$, 则 $F'(t) = \sqrt{1+t^2}$, 所以

$$\begin{aligned}\frac{d}{dx} \left(\int_0^{x^2} \sqrt{1+t^2} dt \right) &= \frac{d}{dx} (F(x^2) - F(0)) = \frac{d}{dx} F(x^2) \\ &= \frac{dF(x^2)}{dx^2} \cdot \frac{dx^2}{dx} = \sqrt{1+(x^2)^2} \cdot 2x = 2x\sqrt{1+x^4}\end{aligned}$$

(2) 设函数 $(t+1)^2$ 的原函数是 $F(t)$, 则 $F'(t) = (t+1)^2$, 所以

$$\begin{aligned}\frac{d}{dx} \left(\int_{x+a}^{x+b} (t+1)^2 dt \right) &= \frac{d}{dx} (F(x+b) - F(x+a)) = \frac{d}{dx} F(x+b) - \frac{d}{dx} F(x+a) \\ &= \frac{dF(x+b)}{d(x+b)} \cdot \frac{d(x+b)}{dx} - \frac{dF(x+a)}{d(x+a)} \cdot \frac{d(x+a)}{dx} \\ &= (x+b+1)^2 - (x+a+1)^2\end{aligned}$$

例题 4.0.6 4.2-A-6

求 $\frac{d}{dx} \int_a^b \sin x^2 dx$; $\frac{d}{da} \int_a^b \sin x^2 dx$; $\frac{d}{db} \int_a^b \sin x^2 dx$.

解 4.0.6. (1) 设函数 $\sin x^2$ 的原函数是 $F(x)$, 则 $F'(x) = \sin x^2$, 所以

$$\begin{aligned}\frac{d}{dx} \int_a^b \sin x^2 dx &= \frac{d}{dx} (F(b) - F(a)) = \frac{d}{dx} F(b) - \frac{d}{dx} F(a) \\ &= \frac{dF(b)}{db} \cdot \frac{db}{dx} - \frac{dF(a)}{da} \cdot \frac{da}{dx} = 0\end{aligned}$$

(2) 设函数 $\sin x^2$ 的原函数是 $F(x)$, 则 $F'(x) = \sin x^2$, 所以

$$\begin{aligned}\frac{d}{da} \int_a^b \sin x^2 dx &= \frac{d}{da} (F(b) - F(a)) = \frac{d}{da} F(b) - \frac{d}{da} F(a) \\ &= \frac{dF(b)}{db} \cdot \frac{db}{da} - \frac{dF(a)}{da} \cdot \frac{da}{da} = -\sin a^2\end{aligned}$$

(3) 设函数 $\sin x^2$ 的原函数是 $F(x)$, 则 $F'(x) = \sin x^2$, 所以

$$\begin{aligned}\frac{d}{db} \int_a^b \sin x^2 dx &= \frac{d}{db} (F(b) - F(a)) = \frac{d}{db} F(b) - \frac{d}{db} F(a) \\ &= \frac{dF(b)}{db} \cdot \frac{db}{db} - \frac{dF(a)}{da} \cdot \frac{da}{db} = \sin b^2\end{aligned}$$

例题 4.0.7 4.2-A-7

$$(1) \lim_{x \rightarrow +\infty} \frac{\int_0^x (\arctan t)^2 dt}{\sqrt{x^2+1}}; \quad (2) \lim_{x \rightarrow +\infty} \frac{\int_1^x \sqrt{t+\frac{1}{t}} dt}{x\sqrt{x}}.$$

解 4.0.7. (1) 当 $x \rightarrow +\infty$ 时, $\int_0^x (\arctan t)^2 dt$ 趋于无穷大, $\sqrt{x^2+1}$ 也趋于无穷大, 两者均对 x 可导, 所以可以使用洛必达法则:

$$\lim_{x \rightarrow +\infty} \frac{\int_0^x (\arctan t)^2 dt}{\sqrt{x^2+1}} = \lim_{x \rightarrow +\infty} \frac{\frac{d}{dx} \int_0^x (\arctan t)^2 dt}{\frac{x}{\sqrt{1+x^2}}} = \lim_{x \rightarrow +\infty} \sqrt{1 + \frac{1}{x^2}} \arctan^2 x = \frac{\pi^2}{4}$$

(2) 当 $x \rightarrow +\infty$ 时, $\int_1^x \sqrt{t + \frac{1}{t}} dt$ 趋于无穷大, $x\sqrt{x}$ 也趋于无穷大, 两者均对 x 可导, 所以可以使用洛必达法则:

$$\lim_{x \rightarrow +\infty} \frac{\int_1^x \sqrt{t + \frac{1}{t}} dt}{x\sqrt{x}} = \lim_{x \rightarrow +\infty} \frac{\frac{d}{dx} \int_1^x \sqrt{t + \frac{1}{t}} dt}{\frac{3}{2}\sqrt{x}} = \lim_{x \rightarrow +\infty} \frac{2\sqrt{x + \frac{1}{x}}}{3\sqrt{x}} = \frac{2}{3}$$

例题 4.0.8 4.2-B-3

设 $f(x)$ 连续, $F(x) = \int_{1/x}^{\ln x} f(t) dt$, 求 $F'(x)$.

解 4.0.8. 由题意知 $f(x)$ 在 $(1/x, \ln x)$ 上连续, 设 $f(t)$ 的原函数是 $G(t)$, 则 $G'(t) = f(t)$, 所以

$$\begin{aligned} F'(x) &= \frac{d}{dx} (G(\ln x) - G(1/x)) = \frac{G'(\ln x)}{x} + \frac{G'(1/x)}{x^2} \\ &= \frac{f(\ln x)}{x} + \frac{f(1/x)}{x^2} \end{aligned}$$

例题 4.0.9 4.2-B-4

设 $f(x) \in C^{(1)}[0, 1]$, 即 $f'(x) \in C[0, 1]$, 且 $f(1) - f(0) = 1$, 证明 $\int_0^1 [f'(x)]^2 dx \geq 1$.

解 4.0.9. $\int_0^1 f'(x) dx = f(1) - f(0) = 1$, 所以由柯西施瓦兹不等式知:

$$\int_0^1 (f'(x))^2 dx \int_0^1 1^2 dx \geq \left(\int_0^1 f'(x) dx \right)^2 = 1$$

又因为 $\int_0^1 1^2 dx = 1$, 所以 $\int_0^1 [f'(x)]^2 dx \geq 1$.

例题 4.0.10 4.2-B-5

设 $f(x) \in C[0, +\infty)$, 并且 $x \in [0, +\infty)$ 时, $f(x) > 0$. 证明函数 $F(x) = \frac{\int_0^x tf(t) dt}{\int_0^x f(t) dt}$ 在 $(0, +\infty)$ 内为单调增加的函数.

解 4.0.10. 只需证明 $F'(x) \geq 0$, 由商法则, 有:

$$F'(x) = \frac{\left(\frac{d}{dx} \int_0^x tf(t) dt \right) \cdot \int_0^x f(t) dt - \int_0^x tf(t) dt \cdot \left(\frac{d}{dx} \int_0^x f(t) dt \right)}{\left(\int_0^x f(t) dt \right)^2}$$

$$\begin{aligned}
 &= \frac{xf(x) \cdot \int_0^x f(t) dt - \int_0^x tf(t) dt \cdot f(x)}{\left(\int_0^x f(t) dt\right)^2} \\
 &= \frac{f(x) \left[x \int_0^x f(t) dt - \int_0^x tf(t) dt\right]}{\left(\int_0^x f(t) dt\right)^2}.
 \end{aligned}$$

由于 $f(x) > 0$ 且 $\int_0^x f(t) dt > 0$ (因为 $f(t) > 0$), 分母恒正, 故 $F'(x)$ 的符号取决于分子中的表达式:

$$x \int_0^x f(t) dt - \int_0^x tf(t) dt = \int_0^x f(t)(x-t) dt.$$

对于 $x > 0$ 和 $t \in [0, x]$, 有 $x-t \geq 0$ 且 $f(t) > 0$, 因此被积函数 $f(t)(x-t) \geq 0$. 当 $t < x$ 时, $x-t > 0$, 故:

$$\int_0^x f(t)(x-t) dt \geq 0.$$

因此, $F'(x) \geq 0$ 对于 $x > 0$, 这意味着 $F(x)$ 在 $(0, +\infty)$ 内单调增加.

例题 4.0.11 4.3-B-1

$$\begin{array}{lll}
 (1) \int \left(e^x - \frac{2}{\sqrt[3]{x}}\right) dx & (2) \int \frac{1+x+x^2}{x(1+x^2)} dx & (3) \int \frac{x^4}{1+x^2} dx \\
 (4) \int \sqrt{x}\sqrt{x} dx & (5) \int \frac{dx}{x^2\sqrt{x}} & (6) \int \frac{2x^2}{\sqrt{x}} dx \\
 (7) \int (x^2-1)^2 dx & (8) \int \frac{x+1}{\sqrt{x}} dx & (9) \int \frac{e^{3x}+1}{e^x+1} dx \\
 (10) \int (2^x+3^x) dx & (11) \int \frac{3x^2}{1+x^2} dx & (12) \int \frac{dx}{x^4(1+x^2)}
 \end{array}$$

解 4.0.11. (1) $\int \left(e^x - \frac{2}{\sqrt[3]{x}}\right) dx = \int (e^x - 2x^{-\frac{1}{3}}) dx = e^x - 3x^{\frac{2}{3}} + C$

(2) $\int \frac{1+x+x^2}{x(1+x^2)} dx = \int \left(\frac{1}{x} + \frac{1}{1+x^2}\right) dx = \ln|x| + \arctan x + C$

(3) $\int \frac{x^4-1+1}{1+x^2} dx = \int (x^2-1) dx + \int \frac{1}{1+x^2} dx = \frac{x^3}{3} - x + \arctan x + C$

(4) $\int \sqrt{x}\sqrt{x} dx = \int x^{\frac{3}{4}} dx = \frac{4}{7} x^{\frac{7}{4}} + C$

(5) $\int \frac{dx}{x^2\sqrt{x}} = \int x^{-\frac{5}{2}} dx = -\frac{2}{3} x^{-\frac{3}{2}} + C$

(6) $\int \frac{2x^2}{\sqrt{x}} dx = \int 2x^{\frac{3}{2}} dx = \frac{4}{5} x^{\frac{5}{2}} + C$

(7) $\int (x^2-1)^2 dx = \int (x^4-2x^2+1) dx = \frac{x^5}{5} - \frac{2x^3}{3} + x + C$

(8) $\int \frac{x+1}{\sqrt{x}} dx = \int (x^{\frac{1}{2}} + x^{-\frac{1}{2}}) dx = \frac{2}{3} x^{\frac{3}{2}} + 2x^{\frac{1}{2}} + C$

(9) $\int \frac{e^{3x}+1}{e^x+1} dx = \int (e^{2x} - e^x + 1) dx = \frac{1}{2} e^{2x} - e^x + x + C$

$$(10) \int (2^x + 3^x) dx = \int 2^x dx + \int 3^x dx = \frac{2^x}{\ln 2} + \frac{3^x}{\ln 3} + C$$

$$(11) \int \frac{3x^2}{1+x^2} dx = \int \left(3 - \frac{3}{1+x^2} \right) dx = 3x - 3 \arctan x + C$$

$$(12) \int \frac{dx}{x^4(1+x^2)} = \int \left(\frac{1}{x^4} - \frac{1}{x^2} + \frac{1}{1+x^2} \right) dx = -\frac{1}{3x^3} + \frac{1}{x} + \arctan x + C$$

例题 4.0.12 4.3-B-2

$$(1) \int \cos(t+1) dt; \quad (2) \int (2 \sin \theta - 3 \cos \theta) d\theta; \quad (3) \int \frac{\cos 2x}{\sin^2 x \cos^2 x} dx;$$

$$(4) \int \frac{dt}{\sin^2 \frac{t}{2} \cos^2 \frac{t}{2}}; \quad (5) \int \sqrt{1 - \sin 2\theta} d\theta; \quad (6) \int \frac{3 + \sin^2 x}{\cos^2 x} dx;$$

$$(7) \int \cos^2 \frac{t}{2} dt; \quad (8) \int \frac{dx}{1 + \cos 2x}; \quad (9) \int \sec x (\sec x - \tan x) dx;$$

$$(10) \int \frac{\cos 2x}{\cos x - \sin x} dx; \quad (11) \int 3^x e^x dx; \quad (12) \int \frac{2 \cdot 3^x - 5 \cdot 2^x}{3^x} dx.$$

解 4.0.12. (1) $\int \cos(t+1) dt = \sin(t+1) + C$

$$(2) \int (2 \sin \theta - 3 \cos \theta) d\theta = -2 \cos \theta - 3 \sin \theta + C$$

$$(3) \int \frac{\cos 2x}{\sin^2 x \cos^2 x} dx = \int \frac{\cos^2 x - \sin^2 x}{\sin^2 x \cos^2 x} dx = \int \left(\frac{1}{\sin^2 x} - \frac{1}{\cos^2 x} \right) dx = -\cot x - \tan x + C$$

$$(4) \int \frac{dt}{\sin^2 \frac{t}{2} \cos^2 \frac{t}{2}} = \int \frac{4}{\sin^2 t} dt = -4 \cot t + C$$

$$(5) \int \sqrt{1 - \sin 2\theta} d\theta = \int |\cos \theta - \sin \theta| d\theta = \begin{cases} \sin \theta + \cos \theta + C, & \cos \theta \geq \sin \theta \\ -(\sin \theta + \cos \theta) + C, & \cos \theta < \sin \theta \end{cases}$$

$$(6) \int \frac{3 + \sin^2 x}{\cos^2 x} dx = \int (3 \sec^2 x + \tan^2 x) dx = \int (4 \sec^2 x - 1) dx = 4 \tan x - x + C$$

$$(7) \int \cos^2 \frac{t}{2} dt = \int \frac{1 + \cos t}{2} dt = \frac{t}{2} + \frac{\sin t}{2} + C$$

$$(8) \int \frac{dx}{1 + \cos 2x} = \int \frac{1}{2 \cos^2 x} dx = \frac{1}{2} \tan x + C$$

$$(9) \int \sec x (\sec x - \tan x) dx = \int (\sec^2 x - \sec x \tan x) dx = \tan x - \sec x + C$$

$$(10) \int \frac{\cos 2x}{\cos x - \sin x} dx = \int \frac{\cos^2 x - \sin^2 x}{\cos x - \sin x} dx = \int (\cos x + \sin x) dx = \sin x - \cos x + C$$

$$(11) \int 3^x e^x dx = \int (3e)^x dx = \frac{(3e)^x}{\ln(3e)} + C = \frac{3^x e^x}{\ln 3 + 1} + C$$

$$(12) \int \frac{2 \cdot 3^x - 5 \cdot 2^x}{3^x} dx = \int \left(2 - 5 \left(\frac{2}{3} \right)^x \right) dx = 2x - \frac{5}{\ln \frac{2}{3}} \left(\frac{2}{3} \right)^x + C$$

例题 4.0.13 4.4-A-3

- | | | |
|---|--|---|
| (1) $\int_1^3 \frac{dx}{x};$ | (2) $\int_{-1}^3 (x^3 + 5x)dx;$ | (3) $\int_0^\pi \sin \theta (\cos \theta + 5)^7 d\theta;$ |
| (4) $\int_{-1}^1 \frac{dy}{1+y^2};$ | (5) $\int_0^1 \frac{x}{1+5x^2} dx;$ | (6) $\int_0^{\pi/12} \sin 3t dt;$ |
| (7) $\int_1^2 \frac{x^2+1}{x} dx;$ | (8) $\int_1^4 x\sqrt{x^2+4} dx;$ | (9) $\int_0^1 \frac{dx}{x^2+2x+1};$ |
| (10) $\int_0^{1/\sqrt{2}} \frac{x dx}{\sqrt{1-x^4}};$ | (11) $\int_{-2}^0 \frac{2x+4}{x^2+4x+5} dx;$ | (12) $\int_1^9 x\sqrt[3]{1-x} dx;$ |
| (13) $\int_1^{e^2} \frac{dx}{x\sqrt{1+\ln x}};$ | (14) $\int_{-\pi/2}^{\pi/2} \sqrt{\cos \theta - \cos^3 \theta} d\theta;$ | (15) $\int_1^2 e^{x^3} x^2 dx;$ |
| (16) $\int_2^3 \frac{e^{1/x}}{x^2} dx;$ | (17) $\int_{\pi/6}^{\pi/4} \tan \theta \sec^2 \theta d\theta;$ | (18) $\int_0^{\pi/2} \cos^5 \theta \sin \theta d\theta.$ |

解 4.0.13. (1)

$$\int_1^3 \frac{dx}{x} = \ln x \Big|_1^3 = \ln 3 - \ln 1 = \ln 3$$

(2)

$$\int_{-1}^3 (x^3 + 5x) dx = \left(\frac{x^4}{4} + \frac{5x^2}{2} \right) \Big|_{-1}^3 = \frac{81-1}{4} + 5 \cdot \frac{8}{2} = 40$$

(3) 令 $u = \cos \theta + 5$, 则 $du = -\sin \theta d\theta$, 当 $\theta = 0$ 时 $u = 6$, $\theta = \pi$ 时 $u = 4$, 故

$$\int_0^\pi \sin \theta (\cos \theta + 5)^7 d\theta = \int_6^4 u^7 (-du) = \int_4^6 u^7 du = \frac{u^8}{8} \Big|_4^6 = \frac{6^8 - 4^8}{8}$$

(4)

$$\int_{-1}^1 \frac{dy}{1+y^2} = \arctan y \Big|_{-1}^1 = \arctan 1 - \arctan(-1) = \frac{\pi}{4} - \left(-\frac{\pi}{4}\right) = \frac{\pi}{2}$$

(5) 令 $u = 1 + 5x^2$, 则 $du = 10x dx$, 当 $x = 0$ 时 $u = 1$, $x = 1$ 时 $u = 6$, 故

$$\int_0^1 \frac{x}{1+5x^2} dx = \int_1^6 \frac{1}{u} \cdot \frac{1}{10} du = \frac{1}{10} \ln |u| \Big|_1^6 = \frac{\ln 6}{10}$$

(6)

$$\int_0^{\pi/12} \sin 3t dt = -\frac{1}{3} \cos 3t \Big|_0^{\pi/12} = -\frac{1}{3} \left(\cos \frac{\pi}{4} - \cos 0 \right) = -\frac{1}{3} \left(\frac{\sqrt{2}}{2} - 1 \right) = \frac{2 - \sqrt{2}}{6}$$

(7)

$$\int_1^2 \frac{x^2+1}{x} dx = \int_1^2 \left(x + \frac{1}{x} \right) dx = \left(\frac{x^2}{2} + \ln |x| \right) \Big|_1^2 = (2 + \ln 2) - \left(\frac{1}{2} + \ln 1 \right) = \frac{3}{2} + \ln 2$$

(8) 令 $u = x^2 + 4$, 则 $du = 2x dx$, 当 $x = 1$ 时 $u = 5$, $x = 4$ 时 $u = 20$, 故

$$\int_1^4 x\sqrt{x^2+4} dx = \int_5^{20} \sqrt{u} \cdot \frac{1}{2} du = \frac{1}{2} \cdot \frac{2}{3} u^{3/2} \Big|_5^{20} = \frac{1}{3} (20^{3/2} - 5^{3/2}) = \frac{35\sqrt{5}}{3}$$

(9)

$$\int_0^1 \frac{dx}{x^2 + 2x + 1} = \int_0^1 (x+1)^{-2} dx = -(x+1)^{-1} \Big|_0^1 = -\frac{1}{2} - (-1) = \frac{1}{2}$$

(10) 令 $x^2 = \sin \theta$, 则 $2x dx = \cos \theta d\theta$, 当 $x = 0$ 时 $\theta = 0$, $x = 1/\sqrt{2}$ 时 $\theta = \pi/6$, 故

$$\int_0^{1/\sqrt{2}} \frac{x dx}{\sqrt{1-x^4}} = \int_0^{\pi/6} \frac{\frac{1}{2} \cos \theta d\theta}{\cos \theta} = \frac{1}{2} \int_0^{\pi/6} d\theta = \frac{1}{2} \cdot \frac{\pi}{6} = \frac{\pi}{12}$$

(11) 令 $u = x^2 + 4x + 5$, 则 $du = (2x + 4)dx$, 当 $x = -2$ 时 $u = 1$, $x = 0$ 时 $u = 5$, 故

$$\int_{-2}^0 \frac{2x+4}{x^2+4x+5} dx = \int_1^5 \frac{du}{u} = \ln |u| \Big|_1^5 = \ln 5$$

(12) 令 $u = 1 - x$, 则 $du = -dx$, $x = 1 - u$, 当 $x = 1$ 时 $u = 0$, $x = 9$ 时 $u = -8$, 故

$$\int_1^9 x \sqrt[3]{1-x} dx = \int_0^{-8} (1-u) u^{1/3} (-du) = \int_{-8}^0 (u^{1/3} - u^{4/3}) du = \left(\frac{3}{4} u^{4/3} - \frac{3}{7} u^{7/3} \right) \Big|_{-8}^0 = -\frac{468}{7}$$

(13) 令 $u = 1 + \ln x$, 则 $du = \frac{1}{x} dx$, 当 $x = 1$ 时 $u = 1$, $x = e^2$ 时 $u = 3$, 故

$$\int_1^{e^2} \frac{dx}{x \sqrt{1 + \ln x}} = \int_1^3 u^{-1/2} du = 2u^{1/2} \Big|_1^3 = 2(\sqrt{3} - 1)$$

(14)

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{\cos \theta - \cos^3 \theta} d\theta = 2 \int_0^{\frac{\pi}{2}} \sin \theta \sqrt{\cos \theta} d\theta$$

令 $u = \cos \theta$, 则 $du = -\sin \theta d\theta$, 故

$$2 \int_0^{\frac{\pi}{2}} \sin \theta \sqrt{\cos \theta} d\theta = 2 \int_1^0 \sqrt{u} (-du) = 2 \int_0^1 u^{1/2} du = 2 \cdot \frac{2}{3} u^{3/2} \Big|_0^1 = \frac{4}{3}$$

(15) 令 $u = x^3$, 则 $du = 3x^2 dx$, 当 $x = 1$ 时 $u = 1$, $x = 2$ 时 $u = 8$, 故

$$\int_1^2 e^{x^3} x^2 dx = \int_1^8 e^u \cdot \frac{1}{3} du = \frac{1}{3} e^u \Big|_1^8 = \frac{1}{3} (e^8 - e)$$

(16) 令 $u = 1/x$, 则 $du = -\frac{1}{x^2} dx$, 当 $x = 2$ 时 $u = 1/2$, $x = 3$ 时 $u = 1/3$, 故

$$\int_2^3 \frac{e^{1/x}}{x^2} dx = \int_{1/2}^{1/3} e^u (-du) = \int_{1/3}^{1/2} e^u du = e^u \Big|_{1/3}^{1/2} = e^{1/2} - e^{1/3}$$

(17) 令 $u = \tan \theta$, 则 $du = \sec^2 \theta d\theta$, 当 $\theta = \pi/6$ 时 $u = 1/\sqrt{3}$, $\theta = \pi/4$ 时 $u = 1$, 故

$$\int_{\pi/6}^{\pi/4} \tan \theta \sec^2 \theta d\theta = \int_{1/\sqrt{3}}^1 u du = \frac{u^2}{2} \Big|_{1/\sqrt{3}}^1 = \frac{1}{2} - \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{3}$$

(18) 令 $u = \cos \theta$, 则 $du = -\sin \theta d\theta$, 当 $\theta = 0$ 时 $u = 1$, $\theta = \pi/2$ 时 $u = 0$, 故

$$\int_0^{\pi/2} \cos^5 \theta \sin \theta d\theta = \int_1^0 u^5 (-du) = \int_0^1 u^5 du = \frac{u^6}{6} \Big|_0^1 = \frac{1}{6}$$

例题 4.0.14 4.4-A-4

$$\begin{aligned}
 (1) & \int \frac{x^2 dx}{\sqrt{a^2 - x^2}}; & (2) & \int_0^{a/\sqrt{2}} \frac{dx}{(a^2 - x^2)^{3/2}} (a > 0); & (3) & \int \sqrt{\frac{x}{1+x\sqrt{x}}} dx; \\
 (4) & \int \frac{dx}{1+\sqrt{x}}; & (5) & \int \frac{e^x}{\sqrt{1-e^{2x}}} dx; & (6) & \int \frac{dx}{1+\sqrt{1-x^2}}; \\
 (7) & \int_1^{\sqrt{3}} \frac{\sqrt{1+x^2}}{x} dx; & (8) & \int_0^a x^2 \sqrt{a^2 - x^2} dx; & (9) & \int \frac{dx}{x\sqrt{x^2-1}}; \\
 (10) & \int \frac{\arctan \sqrt{x}}{\sqrt{x}(1+x)} dx.
 \end{aligned}$$

解 4.0.14. (1)

$$\begin{aligned}
 \int \frac{x^2 dx}{\sqrt{a^2 - x^2}} &= \int \frac{a^2 \sin^2 \theta da \sin \theta}{a \sqrt{1 - \sin^2 \theta}} = a^2 \int \sin^2 \theta d\theta = a^2 \int \frac{1}{2} d\theta - \frac{a^2}{2} \int \cos 2\theta d\theta \\
 &= \frac{a^2}{2} \theta - \frac{a^2}{4} \sin 2\theta + C = \frac{a^2}{2} \arcsin \frac{x}{a} - \frac{a^2}{2} \frac{x}{a} \sqrt{1 - \frac{x^2}{a^2}} + C \\
 &= \frac{a^2}{2} \arcsin \frac{x}{a} - \frac{x}{2} \sqrt{a^2 - x^2} + C
 \end{aligned}$$

(2)

$$\int_0^{a/\sqrt{2}} \frac{dx}{(a^2 - x^2)^{3/2}} = \int_0^{\pi/4} \frac{da \sin \theta}{(a^2 - a^2 \sin^2 \theta)^{3/2}} = \frac{1}{a^2} \int_0^{\pi/4} \sec^2 \theta d\theta = \frac{1}{a^2} \tan \theta \Big|_0^{\pi/4} = \frac{1}{a^2}$$

(3)

$$\begin{aligned}
 \int \sqrt{\frac{x}{1+x\sqrt{x}}} dx &= \int \frac{t dt^2}{\sqrt{1+t^3}} = \int \frac{2t^2 dt}{\sqrt{1+t^3}} = \frac{2}{3} \int \frac{3t^2 dt}{\sqrt{1+t^3}} = \frac{2}{3} \int \frac{dt^3}{\sqrt{1+t^3}} \\
 &= \frac{2}{3} \frac{(1+t^3)^{1/2}}{1 - \frac{1}{2}} + C = \frac{4}{3} \sqrt{1+t^3} + C = \frac{4}{3} \sqrt{1+x\sqrt{x}} + C
 \end{aligned}$$

(4)

$$\begin{aligned}
 \int \frac{dx}{1+\sqrt{x}} &= \int \frac{2t}{1+t} dt = 2 \int \frac{t}{1+t} dt = 2 \int \left(1 - \frac{1}{1+t}\right) dt \\
 &= 2(t - \ln|1+t|) + C = 2\sqrt{x} - 2\ln(1+\sqrt{x}) + C
 \end{aligned}$$

(5)

$$\int \frac{e^x}{\sqrt{1-e^{2x}}} dx = \int \frac{du}{\sqrt{1-u^2}} = \arcsin u + C = \arcsin(e^x) + C$$

(6)

$$\int \frac{dx}{1+\sqrt{1-x^2}} = \int \frac{1-\sqrt{1-x^2}}{x^2} dx = \int \frac{1}{x^2} dx - \int \frac{\sqrt{1-x^2}}{x^2} dx$$

$$= -\frac{1}{x} - \int \frac{\sqrt{1-x^2}}{x^2} dx$$

对后一项积分, 令 $x = \sin t$, $dx = \cos t dt$, 则

$$\begin{aligned} \int \frac{\sqrt{1-x^2}}{x^2} dx &= \int \frac{\cos t}{\sin^2 t} \cdot \cos t dt = \int \cot^2 t dt = \int (\csc^2 t - 1) dt \\ &= -\cot t - t + C = -\frac{\sqrt{1-x^2}}{x} - \arcsin x + C \end{aligned}$$

代回得

$$\begin{aligned} \int \frac{dx}{1 + \sqrt{1-x^2}} &= -\frac{1}{x} - \left(-\frac{\sqrt{1-x^2}}{x} - \arcsin x \right) + C \\ &= \frac{\sqrt{1-x^2} - 1}{x} + \arcsin x + C \end{aligned}$$

(7)

$$\begin{aligned} \int_1^{\sqrt{3}} \frac{\sqrt{1+x^2}}{x} dx &\stackrel{x=\tan t}{=} \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sec t}{\tan t} \cdot \sec^2 t dt = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sec^3 t}{\tan t} dt \\ &= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \left(\frac{\sin t}{\cos^2 t} + \csc t \right) dt \\ &= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sin t}{\cos^2 t} dt + \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \csc t dt \\ &= [\sec t + \ln |\csc t - \cot t|]_{\frac{\pi}{4}}^{\frac{\pi}{3}} \\ &= \left(2 + \ln \frac{1}{\sqrt{3}} \right) - (\sqrt{2} + \ln(\sqrt{2} - 1)) \\ &= 2 - \sqrt{2} - \ln(\sqrt{3}(\sqrt{2} - 1)) \end{aligned}$$

(8)

$$\begin{aligned} \int_0^a x^2 \sqrt{a^2 - x^2} dx &\stackrel{x=a \sin t}{=} \int_0^{\frac{\pi}{2}} a^2 \sin^2 t \cdot a \cos t \cdot a \cos t dt = a^4 \int_0^{\frac{\pi}{2}} \sin^2 t \cos^2 t dt \\ &= a^4 \int_0^{\frac{\pi}{2}} \frac{1}{4} \sin^2 2t dt = \frac{a^4}{4} \int_0^{\frac{\pi}{2}} \frac{1 - \cos 4t}{2} dt \\ &= \frac{a^4}{8} \left[t - \frac{1}{4} \sin 4t \right]_0^{\frac{\pi}{2}} = \frac{a^4}{8} \cdot \frac{\pi}{2} = \frac{\pi a^4}{16} \end{aligned}$$

(9)

$$\int \frac{dx}{x\sqrt{x^2-1}} \stackrel{x=\sec t}{=} \int \frac{\sec t \tan t}{\sec t \cdot \tan t} dt = \int dt = t + C = \arccos\left(\frac{1}{x}\right) + C$$

(10)

$$\begin{aligned} \int \frac{\arctan \sqrt{x}}{\sqrt{x}(1+x)} dx &\stackrel{t=\sqrt{x}}{=} \int \frac{\arctan t}{t(1+t^2)} \cdot 2t dt = 2 \int \frac{\arctan t}{1+t^2} dt \\ &\stackrel{u=\arctan t}{=} 2 \int u du = u^2 + C = (\arctan t)^2 + C = (\arctan \sqrt{x})^2 + C \end{aligned}$$

例题 4.0.15 4.4-B-1

- (1) $\int x \sqrt[3]{1-5x^2} dx$; (2) $\int \frac{dx}{1-x}$; (3) $\int \frac{1+e^x}{\sqrt{x+e^x}} dx$;
 (4) $\int \frac{dx}{e^x + e^{-x}}$; (5) $\int \frac{e^{\sqrt{y}}}{\sqrt{y}} dy$; (6) $\int \frac{e^x - e^{-x}}{e^x + e^{-x}} dx$;
 (7) $\int \frac{x^3}{9+x^2} dx$; (8) $\int \frac{dx}{x(x^6+4)}$; (9) $\int \frac{x+1}{x^2+2x+19} dx$;
 (10) $\int \frac{\sin x + \cos x}{\sqrt[3]{\sin x - \cos x}} dx$; (11) $\int \frac{dx}{1+\cos x}$; (12) $\int \frac{dx}{1+\sin x}$.

解 4.0.15. (1)

$$\begin{aligned} \int x \sqrt[3]{1-5x^2} dx &\stackrel{t=1-5x^2}{=} \int t^{1/3} \cdot \left(-\frac{1}{10}\right) dt = -\frac{1}{10} \cdot \frac{3}{4} t^{4/3} + C \\ &= -\frac{3}{40} (1-5x^2)^{4/3} + C \end{aligned}$$

(2)

$$\int \frac{x dx}{1-x} = \int \left(-1 + \frac{1}{1-x}\right) dx = -x - \ln|1-x| + C$$

(3)

$$\begin{aligned} \int \frac{1+e^x}{\sqrt{x+e^x}} dx &\stackrel{u=x+e^x}{=} \int \frac{1}{\sqrt{u}} du = 2\sqrt{u} + C \\ &= 2\sqrt{x+e^x} + C \end{aligned}$$

(4)

$$\begin{aligned} \int \frac{dx}{e^x + e^{-x}} &= \int \frac{e^x}{e^{2x} + 1} dx \stackrel{t=e^x}{=} \int \frac{1}{t^2 + 1} dt = \arctan t + C \\ &= \arctan(e^x) + C \end{aligned}$$

(5)

$$\begin{aligned} \int \frac{e^{\sqrt{y}}}{\sqrt{y}} dy &\stackrel{t=\sqrt{y}}{=} \int \frac{e^t}{t} \cdot 2t dt = 2 \int e^t dt = 2e^t + C \\ &= 2e^{\sqrt{y}} + C \end{aligned}$$

(6)

$$\begin{aligned}\int \frac{e^x - e^{-x}}{e^x + e^{-x}} dx &\stackrel{\text{令 } u=e^x+e^{-x}}{=} \int \frac{1}{u} du = \ln |u| + C \\ &= \ln(e^x + e^{-x}) + C\end{aligned}$$

(7)

$$\begin{aligned}\int \frac{x^3}{9+x^2} dx &= \int \left(x - \frac{9x}{9+x^2} \right) dx = \int x dx - 9 \int \frac{x}{9+x^2} dx \\ &= \frac{x^2}{2} - \frac{9}{2} \ln(9+x^2) + C\end{aligned}$$

(8)

$$\begin{aligned}\int \frac{dx}{x(x^6+4)} &\stackrel{\text{令 } t=x^3}{=} \int \frac{1}{t(t^2+4)} \cdot \frac{1}{3} dt = \frac{1}{3} \int \left(\frac{1}{4t} - \frac{t}{4(t^2+4)} \right) dt \\ &= \frac{1}{12} \ln |t| - \frac{1}{24} \ln(t^2+4) + C \\ &= \frac{1}{12} \ln |x^3| - \frac{1}{24} \ln(x^6+4) + C \\ &= \frac{1}{4} \ln |x| - \frac{1}{24} \ln(x^6+4) + C\end{aligned}$$

(9)

$$\begin{aligned}\int \frac{x+1}{x^2+2x+19} dx &\stackrel{\text{令 } u=x^2+2x+19}{=} \frac{1}{2} \int \frac{1}{u} du = \frac{1}{2} \ln |u| + C \\ &= \frac{1}{2} \ln(x^2+2x+19) + C\end{aligned}$$

(10)

$$\begin{aligned}\int \frac{\sin x + \cos x}{\sqrt[3]{\sin x - \cos x}} dx &\stackrel{\text{令 } t=\sin x - \cos x}{=} \int t^{-1/3} dt = \frac{3}{2} t^{2/3} + C \\ &= \frac{3}{2} (\sin x - \cos x)^{2/3} + C\end{aligned}$$

(11)

$$\begin{aligned}\int \frac{dx}{1+\cos x} &= \int \frac{1}{2\cos^2(x/2)} dx = \int \sec^2\left(\frac{x}{2}\right) \cdot \frac{1}{2} dx \\ &\stackrel{\text{令 } t=x/2}{=} \int \sec^2 t dt = \tan t + C \\ &= \tan\left(\frac{x}{2}\right) + C\end{aligned}$$

(12)

$$\begin{aligned}\int \frac{dx}{1+\sin x} &= \int \frac{1-\sin x}{\cos^2 x} dx = \int (\sec^2 x - \sec x \tan x) dx \\ &= \tan x - \sec x + C\end{aligned}$$

例题 4.0.16 4.4-B-2

$$\begin{aligned}
 (1) & \int \frac{dx}{\sqrt{1+e^x}}; & (2) & \int \frac{dx}{x+\sqrt{1-x^2}}; & (3) & \int \frac{\sqrt{a^2-x^2}}{x^4} dx \text{ (令 } x = \frac{1}{t} \text{)}; \\
 (4) & \int \frac{dx}{\sqrt{(x^2+1)^3}}; & (5) & \int \frac{1}{4} \frac{\arcsin \sqrt{x}}{\sqrt{x(1-x)}} dx; & (6) & \int \frac{1+\ln x}{(x \ln x)^2} dx; \\
 (7) & \int \frac{\ln \tan x}{\cos x \sin x} dx; & (8) & \int_a^0 \sqrt{\frac{a-x}{a+x}} dx \text{ (令 } x = a \sin t \text{)}.
 \end{aligned}$$

解 4.0.16. (1)

$$\begin{aligned}
 \int \frac{dx}{\sqrt{1+e^x}} & \stackrel{\text{令 } t=\sqrt{1+e^x}}{=} \int \frac{1}{t} \cdot \frac{2t}{t^2-1} dt = 2 \int \frac{dt}{t^2-1} \\
 & = 2 \cdot \frac{1}{2} \ln \left| \frac{t-1}{t+1} \right| + C = \ln \left| \frac{\sqrt{1+e^x}-1}{\sqrt{1+e^x}+1} \right| + C \\
 & = x - 2 \ln(1 + \sqrt{1+e^x}) + C
 \end{aligned}$$

(2)

$$\begin{aligned}
 \int \frac{dx}{x+\sqrt{1-x^2}} & \stackrel{\text{令 } x=\sin \theta}{=} \int \frac{\cos \theta}{\sin \theta + \cos \theta} d\theta = \frac{1}{2} \int \left(1 + \frac{\cos \theta - \sin \theta}{\sin \theta + \cos \theta} \right) d\theta \\
 & = \frac{1}{2} (\theta + \ln |\sin \theta + \cos \theta|) + C \\
 & = \frac{1}{2} \arcsin x + \frac{1}{2} \ln (x + \sqrt{1-x^2}) + C.
 \end{aligned}$$

(3)

$$\begin{aligned}
 \int \frac{\sqrt{a^2-x^2}}{x^4} dx & \stackrel{\text{令 } x=\frac{1}{t}}{=} \int \frac{\sqrt{a^2-\frac{1}{t^2}}}{\frac{1}{t^4}} \left(-\frac{1}{t^2} \right) dt = - \int t \sqrt{a^2 t^2 - 1} dt \\
 & \stackrel{\text{令 } u=a^2 t^2-1}{=} -\frac{1}{2a^2} \int \sqrt{u} du = -\frac{1}{3a^2} u^{3/2} + C \\
 & = -\frac{1}{3a^2} (a^2 t^2 - 1)^{3/2} + C = -\frac{1}{3a^2} \left(\frac{a^2}{x^2} - 1 \right)^{3/2} + C \\
 & = -\frac{(a^2-x^2)^{3/2}}{3a^2 x^3} + C.
 \end{aligned}$$

(4)

$$\int \frac{dx}{\sqrt{(x^2+1)^3}} \stackrel{\text{令 } x=\tan \theta}{=} \int \frac{\sec^2 \theta}{\sec^3 \theta} d\theta = \int \cos \theta d\theta = \sin \theta + C = \frac{x}{\sqrt{x^2+1}} + C.$$

(5)

$$\int_{\frac{1}{4}}^{\frac{1}{2}} \frac{\arcsin \sqrt{x}}{\sqrt{x(1-x)}} dx \stackrel{\text{令 } t=\arcsin \sqrt{x}}{=} \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{t}{\sin t \cos t} \cdot 2 \sin t \cos t dt = 2 \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} t dt$$

$$= t^2 \Big|_{\frac{\pi}{6}}^{\frac{\pi}{4}} = \frac{5}{144} \pi^2$$

(6)

$$\int \frac{1 + \ln x}{(x \ln x)^2} dx = \int \frac{1}{(x \ln x)^2} dx \ln x = -\frac{1}{x \ln x} + C.$$

(7)

$$\begin{aligned} \int \frac{\ln \tan x}{\cos x \sin x} dx &\stackrel{\text{令 } t = \ln \tan x}{=} \int \frac{t d \arctan e^t}{\frac{e^t}{\sqrt{1+e^{2t}}} \frac{1}{\sqrt{1+e^{2t}}}} \\ &= \int t du = \frac{1}{2} t^2 + C = \frac{1}{2} (\ln \tan x)^2 + C. \end{aligned}$$

(8)

$$\begin{aligned} \int_0^a \sqrt{\frac{a-x}{a+x}} dx &\stackrel{\text{令 } x = a \sin t}{=} \int_0^{\pi/2} \sqrt{\frac{1-\sin t}{1+\sin t}} \cdot a \cos t dt = a \int_0^{\pi/2} \frac{1-\sin t}{\cos t} \cdot \cos t dt \\ &= a \int_0^{\pi/2} (1 - \sin t) dt = a [t + \cos t]_0^{\pi/2} = a \left(\frac{\pi}{2} - 1 \right). \end{aligned}$$

例题 4.0.17 4.4-B-3

设 $f(x) \in C[0, \pi]$, 证明:

(1) $\int_0^{\pi/2} f(\sin x) dx = \int_0^{\pi/2} f(\cos x) dx$;

(2) $\int_0^{\pi} x f(\sin x) dx = \frac{\pi}{2} \int_0^{\pi} f(\sin x) dx$, 并由此计算 $\int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x} dx$.

解 4.0.17. (1)

$$\int_0^{\pi/2} f(\sin x) dx = \int_{-\frac{\pi}{2}}^0 f\left(\sin\left(x + \frac{\pi}{2}\right)\right) dx = \int_{-\frac{\pi}{2}}^0 f(\cos x) dx = \int_0^{\pi/2} f(\cos x) dx$$

(2)

$$\begin{aligned} \int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x} dx &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{(x + \frac{\pi}{2}) \sin(x + \frac{\pi}{2})}{1 + \cos^2(x + \frac{\pi}{2})} dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{(x + \frac{\pi}{2}) \cos x}{1 + \sin^2 x} dx \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{x \cos x}{1 + \sin^2 x} dx + \frac{\pi}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos x}{1 + \sin^2 x} dx \\ &= \pi \int_0^{\frac{\pi}{2}} \frac{\cos x}{1 + \sin^2 x} dx = \pi \int_0^{\frac{\pi}{2}} \frac{d \sin x}{1 + \sin^2 x} \\ &= \pi \times \arctan(\sin x) \Big|_0^{\frac{\pi}{2}} = \frac{\pi^2}{4} \end{aligned}$$

例题 4.0.18 4.4-B-4

设 $f(x)$ 是以 T 为周期的周期函数, 证明积分 $\int_a^{a+T} f(x) dx$ 的值与 a 无关.

解 4.0.18.

$$\begin{aligned}
 \int_a^{a+T} f(x) dx &= \int_a^T f(x) dx + \int_T^{a+T} f(x) dx \\
 &= \int_a^T f(x) dx + \int_0^a f(x+T) dx \\
 &= \int_a^T f(x) dx + \int_0^a f(x) dx \\
 &= \int_0^T f(x) dx
 \end{aligned}$$

例题 4.0.19 4.4-B-5

若 $f(t)$ 连续且为奇函数, 证明 $\int_0^x f(t)dt$ 是偶函数; 若 $f(t)$ 连续且为偶函数, 证明 $\int_0^x f(t)dt$ 是奇函数

解 4.0.19. (1) 若 $f(t)$ 连续且为奇函数: 由 $\int_0^x f(t)dt + \int_{-x}^0 f(t)dt = 0$:

$$\int_0^x f(t)dt = -\int_{-x}^0 f(t)dt = \int_0^{-x} f(t)dt$$

所以 $\int_0^x f(t)dt$ 是偶函数。

(2) 若 $f(t)$ 连续且为偶函数: 由 $\int_0^x f(t)dt = \int_{-x}^0 f(t)dt$ 得到:

$$\begin{aligned}
 \int_0^x f(t)dt &= \int_{-x}^0 f(t)dt = -\int_0^{-x} f(t)dt \\
 \Rightarrow \int_0^x f(t)dt + \int_0^{-x} f(t)dt &= 0
 \end{aligned}$$

所以 $\int_0^x f(t)dt$ 是奇函数。

例题 4.0.20 4.4-B-6

设 $f(x)$ 是连续函数, 证明 $\int_0^2 f(x)dx = \int_0^1 [f(x) + f(x+1)]dx$.

解 4.0.20.

$$\begin{aligned}
 \int_0^2 f(x)dx &= \int_0^1 f(x)dx + \int_1^2 f(x)dx = \int_0^1 f(x)dx + \int_0^1 f(x+1)dx \\
 &= \int_0^1 [f(x) + f(x+1)]dx
 \end{aligned}$$

例题 4.0.21 4.5-A-5

求 $I_n = \int x^n e^{-x} dx$

解 4.0.21.

$$\begin{aligned}
 I_n &= \int x^n e^{-x} dx = - \int x^n d e^{-x} = -x^n e^{-x} + \int e^{-x} dx^n \\
 &= -x^n e^{-x} + n \int e^{-x} dx^{n-1} = -x^n e^{-x} + n I_{n-1} \\
 &= -x^n e^{-x} - n x^{n-1} e^{-x} - n(n-1) x^{n-2} e^{-x} - \cdots - n! x e^{-x} + n! I_0 \\
 &= -e^{-x} (x^n + n x^{n-1} + n(n-1) x^{n-2} + \cdots + n! x + n!) + C
 \end{aligned}$$

例题 4.0.22 4.5-B-3

- | | | |
|--|---|---|
| (1) $\int \ln(1+x^2) dx;$ | (2) $\int \arctan \sqrt{x} dx;$ | (3) $\int \frac{x + \sin x}{1 + \cos x} dx;$ |
| (4) $\int \frac{\sin^2 x}{\cos^3 x} dx;$ | (5) $\int \frac{dx}{(1+e^x)^2};$ | (6) $\int \frac{x e^x}{(e^x + 1)^2} dx;$ |
| (7) $\int \frac{x e^x}{\sqrt{e^x - 1}} dx;$ | (8) $\int \frac{x^2}{(1+x^2)^2} dx;$ | (9) $\int_0^1 \frac{\ln(1+x)}{(2-x)^2} dx;$ |
| (10) $\int \frac{x + \ln x}{(1+x)^2} dx;$ | (11) $\int_0^1 \frac{x}{e^x + e^{1-x}} dx;$ | (12) $\int_0^{1/2} x \ln \frac{1+x}{1-x} dx;$ |
| (13) $\int_0^3 \arcsin \sqrt{\frac{x}{1+x}} dx;$ | (14) $\int x \sqrt{x+3} dx;$ | (15) $\int \frac{x}{\sqrt{5-x}} dx;$ |
| (16) $\int (t+2) \sqrt{2+3t} dt;$ | (17) $\int \frac{t+7}{\sqrt{5-t}} dt.$ | |

解 4.0.22. (1)

$$\begin{aligned}
 \int \ln(1+x^2) dx &= x \ln(1+x^2) - \int \frac{2x^2}{1+x^2} dx \\
 &= x \ln(1+x^2) - 2 \int \left(1 - \frac{1}{1+x^2}\right) dx \\
 &= x \ln(1+x^2) - 2x + 2 \arctan x + C.
 \end{aligned}$$

(2)

$$\begin{aligned}
 \int \arctan \sqrt{x} dx &\stackrel{u=\sqrt{x}}{=} 2 \int u \arctan u du = u^2 \arctan u - \int \frac{u^2}{1+u^2} du \\
 &= u^2 \arctan u - \int \left(1 - \frac{1}{1+u^2}\right) du \\
 &= u^2 \arctan u - u + \arctan u + C = x \arctan \sqrt{x} - \sqrt{x} + \arctan \sqrt{x} + C.
 \end{aligned}$$

(3)

$$\begin{aligned}
\int \frac{x + \sin x}{1 + \cos x} dx &= \int \frac{x}{2 \cos^2(x/2)} dx + \int \tan(x/2) dx \\
&= \int x d \tan(x/2) + 2 \ln |\cos(x/2)| \\
&= x \tan(x/2) - \int \tan(x/2) dx + 2 \ln |\cos(x/2)| \\
&= x \tan(x/2) + 2 \ln |\cos(x/2)| - 2 \ln |\cos(x/2)| + C \\
&= x \tan(x/2) + C.
\end{aligned}$$

(4)

$$\begin{aligned}
\int \frac{\sin^2 x}{\cos^3 x} dx &= \int \tan^2 x \sec x dx = \int (\sec^2 x - 1) \sec x dx \\
&= \int \sec^3 x dx - \int \sec x dx \\
&= \frac{1}{2} (\sec x \tan x + \ln |\sec x + \tan x|) - \ln |\sec x + \tan x| + C \\
&= \frac{1}{2} \sec x \tan x - \frac{1}{2} \ln |\sec x + \tan x| + C.
\end{aligned}$$

(5)

$$\begin{aligned}
\int \frac{dx}{(1 + e^x)^2} &\stackrel{u=e^x}{=} \int \frac{1}{u(1+u)^2} du = \int \left(\frac{1}{u} - \frac{1}{1+u} - \frac{1}{(1+u)^2} \right) du \\
&= \ln |u| - \ln |1+u| + \frac{1}{1+u} + C \\
&= x - \ln(1 + e^x) + \frac{1}{1 + e^x} + C.
\end{aligned}$$

(6)

$$\begin{aligned}
\int \frac{x e^x}{(e^x + 1)^2} dx &= - \int x d \left(\frac{1}{e^x + 1} \right) = - \frac{x}{e^x + 1} + \int \frac{dx}{e^x + 1} \\
&= - \frac{x}{e^x + 1} + \int \left(1 - \frac{e^x}{e^x + 1} \right) dx \\
&= - \frac{x}{e^x + 1} + x - \ln(e^x + 1) + C \\
&= \frac{x e^x}{e^x + 1} - \ln(e^x + 1) + C.
\end{aligned}$$

(7)

$$\begin{aligned}
\int \frac{x e^x}{\sqrt{e^x - 1}} dx &\stackrel{u=\sqrt{e^x-1}}{=} 2 \int \ln(u^2 + 1) du \\
&= 2 \left(u \ln(u^2 + 1) - \int \frac{2u^2}{u^2 + 1} du \right)
\end{aligned}$$

$$\begin{aligned}
&= 2u \ln(u^2 + 1) - 4 \int \left(1 - \frac{1}{u^2 + 1}\right) du \\
&= 2u \ln(u^2 + 1) - 4u + 4 \arctan u + C \\
&= 2x \sqrt{e^x - 1} - 4\sqrt{e^x - 1} + 4 \arctan \sqrt{e^x - 1} + C.
\end{aligned}$$

(8)

$$\begin{aligned}
\int \frac{x^2}{(1+x^2)^2} dx &\stackrel{x=\tan \theta}{=} \int \sin^2 \theta d\theta = \int \frac{1 - \cos 2\theta}{2} d\theta \\
&= \frac{\theta}{2} - \frac{\sin 2\theta}{4} + C \\
&= \frac{1}{2} \arctan x - \frac{x}{2(1+x^2)} + C.
\end{aligned}$$

(9)

$$\begin{aligned}
\int_0^1 \frac{\ln(1+x)}{(2-x)^2} dx &= \left[\frac{\ln(1+x)}{2-x} \right]_0^1 - \int_0^1 \frac{1}{(2-x)(1+x)} dx \\
&= \ln 2 - \frac{1}{3} \int_0^1 \left(\frac{1}{2-x} + \frac{1}{1+x} \right) dx \\
&= \ln 2 - \frac{1}{3} [-\ln |2-x| + \ln |1+x|]_0^1 \\
&= \ln 2 - \frac{1}{3} (\ln 2 + \ln 2) = \frac{1}{3} \ln 2.
\end{aligned}$$

(10)

$$\begin{aligned}
\int \frac{x + \ln x}{(1+x)^2} dx &= \int \frac{x}{(1+x)^2} dx + \int \frac{\ln x}{(1+x)^2} dx \\
&= \int \left(\frac{1}{1+x} - \frac{1}{(1+x)^2} \right) dx - \frac{\ln x}{1+x} + \int \frac{1}{x(1+x)} dx \\
&= \ln |1+x| + \frac{1}{1+x} - \frac{\ln x}{1+x} + \int \left(\frac{1}{x} - \frac{1}{1+x} \right) dx \\
&= \ln |1+x| + \frac{1}{1+x} - \frac{\ln x}{1+x} + \ln |x| - \ln |1+x| + C \\
&= \ln x + \frac{1 - \ln x}{1+x} + C = \frac{1+x \ln x}{1+x} + C.
\end{aligned}$$

(11)

$$\begin{aligned}
\int_0^1 \frac{x}{e^x + e^{1-x}} dx &= \frac{1}{2} \int_0^1 \frac{1}{e^x + e^{1-x}} dx \quad (\text{由对称性}) \\
&= \frac{1}{2} \int_0^1 \frac{e^{-x}}{1 + e^{1-2x}} dx \\
&\stackrel{u=e^{-x}}{=} \frac{1}{2} \int_1^{e^{-1}} \frac{u}{1 + eu^2} \left(-\frac{du}{u} \right)
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2} \int_{e^{-1}}^1 \frac{du}{1+eu^2} = \frac{1}{2\sqrt{e}} [\arctan(\sqrt{e}u)]_{e^{-1}}^1 \\
&= \frac{1}{2\sqrt{e}} \left(\arctan \sqrt{e} - \arctan \frac{1}{\sqrt{e}} \right).
\end{aligned}$$

(12)

$$\begin{aligned}
\int_0^{1/2} x \ln \frac{1+x}{1-x} dx &= \int_0^{1/2} x \ln(1+x) dx - \int_0^{1/2} x \ln(1-x) dx \\
&= \left[\frac{x^2}{2} \ln(1+x) - \frac{x^2}{4} + \frac{x}{2} - \frac{1}{2} \ln(1+x) \right]_0^{1/2} \\
&\quad - \left[\frac{x^2}{2} \ln(1-x) + \frac{x^2}{4} + \frac{x}{2} + \frac{1}{2} \ln(1-x) \right]_0^{1/2} \\
&= \ln 2 - \frac{3}{8} \ln 3 - \frac{1}{8}.
\end{aligned}$$

(13)

$$\begin{aligned}
\int_0^3 \arcsin \sqrt{\frac{x}{1+x}} dx &= \int_0^3 \arctan \sqrt{x} dx \quad (\text{因为 } \arcsin \sqrt{\frac{x}{1+x}} = \arctan \sqrt{x}) \\
&\stackrel{u=\sqrt{x}}{=} 2 \int_0^{\sqrt{3}} u \arctan u du \\
&= [u^2 \arctan u]_0^{\sqrt{3}} - \int_0^{\sqrt{3}} \frac{u^2}{1+u^2} du \\
&= \pi - \int_0^{\sqrt{3}} \left(1 - \frac{1}{1+u^2} \right) du \\
&= \pi - [u - \arctan u]_0^{\sqrt{3}} \\
&= \pi - \left(\sqrt{3} - \frac{\pi}{3} \right) = \frac{4\pi}{3} - \sqrt{3}.
\end{aligned}$$

(14)

$$\begin{aligned}
\int x \sqrt{x+3} dx &\stackrel{u=x+3}{=} \int (u-3) \sqrt{u} du = \int (u^{3/2} - 3u^{1/2}) du \\
&= \frac{2}{5} u^{5/2} - 2u^{3/2} + C = \frac{2}{5} (x+3)^{5/2} - 2(x+3)^{3/2} + C.
\end{aligned}$$

(15)

$$\begin{aligned}
\int \frac{x}{\sqrt{5-x}} dx &\stackrel{u=5-x}{=} \int \frac{5-u}{\sqrt{u}} (-du) = \int (u-5) u^{-1/2} du \\
&= \int (u^{1/2} - 5u^{-1/2}) du = \frac{2}{3} u^{3/2} - 10u^{1/2} + C \\
&= \frac{2}{3} (5-x)^{3/2} - 10\sqrt{5-x} + C.
\end{aligned}$$

(16)

$$\begin{aligned}
 \int (t+2)\sqrt{2+3t} \, dt &\stackrel{u=2+3t}{=} \int \left(\frac{u-2}{3} + 2\right) \sqrt{u} \cdot \frac{1}{3} \, du = \frac{1}{9} \int (u+4)u^{1/2} \, du \\
 &= \frac{1}{9} \int (u^{3/2} + 4u^{1/2}) \, du = \frac{1}{9} \left(\frac{2}{5} u^{5/2} + \frac{8}{3} u^{3/2} \right) + C \\
 &= \frac{2}{45} (2+3t)^{5/2} + \frac{8}{27} (2+3t)^{3/2} + C.
 \end{aligned}$$

(17)

$$\begin{aligned}
 \int \frac{t+7}{\sqrt{5-t}} \, dt &\stackrel{u=5-t}{=} \int \frac{5-u+7}{\sqrt{u}} (-du) = \int \frac{u-12}{\sqrt{u}} \, du \\
 &= \int (u^{1/2} - 12u^{-1/2}) \, du = \frac{2}{3} u^{3/2} - 24u^{1/2} + C \\
 &= \frac{2}{3} (5-t)^{3/2} - 24\sqrt{5-t} + C.
 \end{aligned}$$

例题 4.0.23 4.5-B-4

$$\begin{aligned}
 (1) \quad &\lim_{n \rightarrow \infty} \left(\frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{2n} \right); \quad (2) \quad \lim_{n \rightarrow \infty} \left(\frac{1}{n^2} + \frac{2}{n^2} + \cdots + \frac{n-1}{n^2} \right); \\
 (3) \quad &\lim_{n \rightarrow \infty} \frac{1}{n} \left(\sin \frac{\pi}{n} + \sin \frac{2\pi}{n} + \cdots + \sin \frac{(n-1)\pi}{n} \right).
 \end{aligned}$$

解 4.0.23. 每道题利用定义, 提取 $\Delta x = \frac{1}{n}, x_k = \frac{k}{n}$, 然后找出来 $f(x_k)$:

$$(1) \quad \lim_{n \rightarrow \infty} \left(\frac{1}{n} + \sum_{k=1}^n \frac{1}{n+k} \right) = \lim_{n \rightarrow \infty} \frac{1}{n} \left(\frac{1}{1+\frac{k}{n}} \right) = \int_0^1 \frac{1}{1+x} \, dx = \ln 2$$

$$(2) \quad \lim_{n \rightarrow \infty} \sum_{k=1}^{n-1} \frac{k}{n^2} = \lim_{n \rightarrow \infty} \sum_{k=1}^{n-1} \frac{1}{n} \frac{k}{n} = \int_0^1 x \, dx = \frac{1}{2}$$

$$(3) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \left(\sin \frac{\pi}{n} + \sin \frac{2\pi}{n} + \cdots + \sin \frac{(n-1)\pi}{n} \right) = \lim_{n \rightarrow \infty} \frac{1}{n} \left(\sum_{k=1}^{n-1} \sin \frac{k\pi}{n} \right) = \int_0^1 \sin \pi x \, dx = \frac{2}{\pi}$$

例题 4.0.24 4.6-A-2

$$(1) \quad \int \cos^4 x \sin^3 x \, dx \quad (2) \quad \int \frac{\sin 2x}{1+\cos^2 x} \, dx \quad (3) \quad \int \frac{dx}{\sin 2x + 2 \sin x} \quad (4) \quad \int \frac{dx}{2+\sin x}$$

解 4.0.24. (1)

$$\begin{aligned}
 \int \cos^4 x \sin^3 x \, dx &= - \int \sin^2 x \cos^4 x \, d \cos x = \int (\cos^2 x - 1) \cos^4 x \, d \cos x \\
 &= \int \cos^6 x \, d \cos x - \int \cos^4 x \, d \cos x \\
 &= \frac{\cos^7 x}{7} - \frac{\cos^5 x}{5} + C
 \end{aligned}$$

(2)

$$\begin{aligned}
 \int \frac{\sin 2x}{1 + \cos^2 x} dx &= \int \frac{2 \sin x \cos x}{\sin^2 x + 2 \cos^2 x} = \int \frac{2 \tan x}{\tan^2 x + 2} dx \\
 &= \int \frac{2t}{t^2 + 2} d \arctan t = \int \frac{dt^2}{(t^2 + 1)(t^2 + 2)} \\
 &= \int \left(\frac{1}{t^2 + 1} - \frac{1}{t^2 + 2} \right) dt^2 \\
 &= \ln \frac{t^2 + 2}{t^2 + 1} + C = \ln \frac{\tan^2 x + 2}{\tan^2 x + 1} + C
 \end{aligned}$$

(3)

$$\begin{aligned}
 \int \frac{dx}{\sin 2x + 2 \sin x} &= \int \frac{dx}{2 \sin x \cos x + 2 \sin x} = \int \frac{dx}{2 \sin x (1 + \cos x)} \\
 &\stackrel{\text{令 } t = \tan \frac{x}{2}}{=} \int \frac{1}{2 \cdot \frac{2t}{1+t^2} \left(1 + \frac{1-t^2}{1+t^2} \right)} \cdot \frac{2}{1+t^2} dt \\
 &= \int \frac{1}{2 \cdot \frac{2t}{1+t^2} \cdot \frac{2}{1+t^2}} \cdot \frac{2}{1+t^2} dt = \int \frac{1+t^2}{4t} dt \\
 &= \frac{1}{4} \int \left(\frac{1}{t} + t \right) dt = \frac{1}{4} \left(\ln |t| + \frac{t^2}{2} \right) + C \\
 &= \frac{1}{4} \ln \left| \tan \frac{x}{2} \right| + \frac{1}{8} \tan^2 \frac{x}{2} + C
 \end{aligned}$$

(4)

$$\begin{aligned}
 \int \frac{dx}{2 + \sin x} &\stackrel{\text{令 } t = \tan \frac{x}{2}}{=} \int \frac{1}{2 + \frac{2t}{1+t^2}} \cdot \frac{2}{1+t^2} dt \\
 &= \int \frac{2}{2(1+t^2) + 2t} dt = \int \frac{dt}{t^2 + t + 1} \\
 &= \int \frac{dt}{\left(t + \frac{1}{2}\right)^2 + \frac{3}{4}} = \frac{2}{\sqrt{3}} \arctan \left(\frac{t + \frac{1}{2}}{\frac{\sqrt{3}}{2}} \right) + C \\
 &= \frac{2}{\sqrt{3}} \arctan \left(\frac{2t + 1}{\sqrt{3}} \right) + C \\
 &= \frac{2}{\sqrt{3}} \arctan \left(\frac{2 \tan \frac{x}{2} + 1}{\sqrt{3}} \right) + C
 \end{aligned}$$

例题 4.0.25 4.6-A-3

$$(1) \int \frac{\sqrt{x-1}}{\sqrt{x}} dx \quad (2) \int \frac{dx}{1 + \sqrt[3]{x+2}}$$

解 4.0.25. (1)

$$\int \frac{\sqrt{x-1}}{\sqrt{x}} dx \stackrel{t=\sqrt{x}}{=} \int \frac{\sqrt{t^2-1}}{t} \cdot 2t dt = 2 \int \sqrt{t^2-1} dt$$

$$\begin{aligned}
&\stackrel{\sec \theta = t}{=} 2 \int \tan \theta \cdot \sec \theta \tan \theta d\theta = 2 \int \tan^2 \theta \sec \theta d\theta \\
&= 2 \int (\sec^2 \theta - 1) \sec \theta d\theta = 2 \int (\sec^3 \theta - \sec \theta) d\theta \\
&= 2 \left(\frac{1}{2} \sec \theta \tan \theta + \frac{1}{2} \ln |\sec \theta + \tan \theta| - \ln |\sec \theta + \tan \theta| \right) + C \\
&= \sec \theta \tan \theta - \ln |\sec \theta + \tan \theta| + C \\
&= t\sqrt{t^2 - 1} - \ln |t + \sqrt{t^2 - 1}| + C \\
&= \sqrt{x}\sqrt{x-1} - \ln(\sqrt{x} + \sqrt{x-1}) + C
\end{aligned}$$

(2)

$$\begin{aligned}
\int \frac{dx}{1 + \sqrt[3]{x+2}} &\stackrel{t = \sqrt[3]{x+2}}{=} \int \frac{1}{1+t} \cdot 3t^2 dt = 3 \int \frac{t^2}{1+t} dt \\
&= 3 \int \left(t - 1 + \frac{1}{1+t} \right) dt \\
&= 3 \left(\frac{1}{2} t^2 - t + \ln |1+t| \right) + C \\
&= \frac{3}{2} t^2 - 3t + 3 \ln |1+t| + C \\
&= \frac{3}{2} (x+2)^{2/3} - 3(x+2)^{1/3} + 3 \ln |1 + (x+2)^{1/3}| + C
\end{aligned}$$

例题 4.0.26 4.6-B

$$\begin{array}{lll}
(1) \int \frac{dx}{x^4 + 1}; & (2) \int \frac{dx}{x^4 - 1}; & (3) \int \frac{dx}{x^4 + x^2 + 1}; \\
(4) \int \tan^3 x dx; & (5) \int \frac{dx}{\sin^2 x \cos x}; & (6) \int \frac{dx}{3 + \cos x}; \\
(7) \int \frac{dx}{(1 + \sqrt[3]{x})\sqrt{x}}; & (8) \int \frac{\sqrt{x+1}-1}{\sqrt{x+1}+1} dx.
\end{array}$$

解 4.0.26. (1)

$$\begin{aligned}
\int \frac{dx}{x^4 + 1} &= \int \frac{dx}{(x^2 + \sqrt{2}x + 1)(x^2 - \sqrt{2}x + 1)} \\
&= \frac{1}{2\sqrt{2}} \int \left(\frac{x + \sqrt{2}}{x^2 + \sqrt{2}x + 1} - \frac{x - \sqrt{2}}{x^2 - \sqrt{2}x + 1} \right) dx \\
&= \frac{1}{4\sqrt{2}} \int \frac{2x + \sqrt{2}}{x^2 + \sqrt{2}x + 1} dx + \frac{1}{4} \int \frac{1}{x^2 + \sqrt{2}x + 1} dx \\
&\quad - \frac{1}{4\sqrt{2}} \int \frac{2x - \sqrt{2}}{x^2 - \sqrt{2}x + 1} dx + \frac{1}{4} \int \frac{1}{x^2 - \sqrt{2}x + 1} dx \\
&= \frac{1}{4\sqrt{2}} \ln(x^2 + \sqrt{2}x + 1) + \frac{1}{2\sqrt{2}} \arctan(\sqrt{2}x + 1) \\
&\quad - \frac{1}{4\sqrt{2}} \ln(x^2 - \sqrt{2}x + 1) + \frac{1}{2\sqrt{2}} \arctan(\sqrt{2}x - 1) + C
\end{aligned}$$

$$= \frac{1}{4\sqrt{2}} \ln \frac{x^2 + \sqrt{2}x + 1}{x^2 - \sqrt{2}x + 1} + \frac{1}{2\sqrt{2}} \left[\arctan(\sqrt{2}x + 1) + \arctan(\sqrt{2}x - 1) \right] + C.$$

(2)

$$\begin{aligned} \int \frac{dx}{x^4 - 1} &= \int \frac{dx}{(x-1)(x+1)(x^2+1)} \\ &= \frac{1}{4} \int \left(\frac{1}{x-1} - \frac{1}{x+1} - \frac{2}{x^2+1} \right) dx \\ &= \frac{1}{4} \ln|x-1| - \frac{1}{4} \ln|x+1| - \frac{1}{2} \arctan x + C \\ &= \frac{1}{4} \ln \left| \frac{x-1}{x+1} \right| - \frac{1}{2} \arctan x + C. \end{aligned}$$

(3)

$$\begin{aligned} \int \frac{dx}{x^4 + x^2 + 1} &= \int \frac{dx}{(x^2 + x + 1)(x^2 - x + 1)} \\ &= \frac{1}{2} \int \left(\frac{1}{x^2 + x + 1} + \frac{1}{x^2 - x + 1} \right) dx \\ &= \frac{1}{2} \int \frac{1}{\left(x + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} dx + \frac{1}{2} \int \frac{1}{\left(x - \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} dx \\ &= \frac{1}{\sqrt{3}} \left[\arctan \left(\frac{2x+1}{\sqrt{3}} \right) + \arctan \left(\frac{2x-1}{\sqrt{3}} \right) \right] + C \\ &= \frac{1}{\sqrt{3}} \arctan \left(\frac{x\sqrt{3}}{1-x^2} \right) + C \quad (\text{利用恒等式 } \arctan u + \arctan v = \arctan \frac{u+v}{1-uv}). \end{aligned}$$

(4)

$$\begin{aligned} \int \tan^3 x \, dx &= \int \tan x \cdot \tan^2 x \, dx = \int \tan x (\sec^2 x - 1) \, dx \\ &= \int \tan x \sec^2 x \, dx - \int \tan x \, dx \\ &= \frac{1}{2} \tan^2 x - (-\ln|\cos x|) + C \\ &= \frac{1}{2} \tan^2 x + \ln|\cos x| + C. \end{aligned}$$

(5)

$$\begin{aligned} \int \frac{dx}{\sin^2 x \cos x} &= \int \frac{\sin^2 x + \cos^2 x}{\sin^2 x \cos x} dx = \int \frac{\sin^2 x}{\sin^2 x \cos x} dx + \int \frac{\cos^2 x}{\sin^2 x \cos x} dx \\ &= \int \frac{1}{\cos x} dx + \int \frac{\cos x}{\sin^2 x} dx \\ &= \int \sec x \, dx + \int \cot x \csc x \, dx \\ &= \ln|\sec x + \tan x| - \csc x + C. \end{aligned}$$

(6)

$$\begin{aligned}
\int \frac{dx}{3 + \cos x} &\stackrel{t=\tan \frac{x}{2}}{=} \int \frac{1}{3 + \frac{1-t^2}{1+t^2}} \cdot \frac{2}{1+t^2} dt = \int \frac{2}{3(1+t^2) + (1-t^2)} dt \\
&= \int \frac{2}{2t^2 + 4} dt = \int \frac{1}{t^2 + 2} dt \\
&= \frac{1}{\sqrt{2}} \arctan \left(\frac{t}{\sqrt{2}} \right) + C = \frac{1}{\sqrt{2}} \arctan \left(\frac{\tan \frac{x}{2}}{\sqrt{2}} \right) + C.
\end{aligned}$$

(7)

$$\begin{aligned}
\int \frac{dx}{(1 + \sqrt[3]{x})\sqrt{x}} &\stackrel{t=\sqrt[6]{x}}{=} \int \frac{1}{(1+t^2) \cdot t^3} \cdot 6t^5 dt = 6 \int \frac{t^2}{1+t^2} dt \\
&= 6 \int \left(1 - \frac{1}{1+t^2} \right) dt = 6(t - \arctan t) + C \\
&= 6(\sqrt[6]{x} - \arctan \sqrt[6]{x}) + C.
\end{aligned}$$

(8)

$$\begin{aligned}
\int \frac{\sqrt{x+1} - 1}{\sqrt{x+1} + 1} dx &= \int \left(1 - \frac{2}{\sqrt{x+1} + 1} \right) dx = \int 1 dx - 2 \int \frac{1}{\sqrt{x+1} + 1} dx \\
&\stackrel{t=\sqrt{x+1}}{=} x - 2 \int \frac{1}{t+1} \cdot 2t dt = x - 4 \int \frac{t}{t+1} dt \\
&= x - 4 \int \left(1 - \frac{1}{t+1} \right) dt \\
&= x - 4(t - \ln|t+1|) + C \\
&= x - 4\sqrt{x+1} + 4 \ln(\sqrt{x+1} + 1) + C.
\end{aligned}$$

例题 4.0.27 4.7-A

用定义判别下列反常积分的敛散性, 如果积分收敛, 则计算反常积分的值

- | | | |
|---|---|--|
| (1) $\int_1^{+\infty} e^{-2x} dx;$ | (2) $\int_1^{+\infty} \frac{x}{4+x^2} dx;$ | (3) $\int_0^{+\infty} \frac{x}{e^x} dx;$ |
| (4) $\int_{-\infty}^0 \frac{e^x}{1+e^x} dx;$ | (5) $\int_x^{+\infty} \sin y dy;$ | (6) $\int_{-\infty}^{+\infty} \frac{dz}{z^2+25};$ |
| (7) $\int_{\pi/4}^{\pi/2} \frac{\sin x}{\sqrt{\cos x}} dx;$ | (8) $\int_0^4 \frac{dx}{\sqrt{16-x^2}};$ | (9) $\int_{-1}^1 \frac{dt}{t};$ |
| (10) $\int_1^{+\infty} \frac{dx}{\sqrt{x^2+1}};$ | (11) $\int_0^1 \frac{x^4+1}{x} dx;$ | (12) $\int_4^{20} \frac{1}{y^2-16} dy;$ |
| (13) $\int_0^1 \frac{\ln x}{x} dx;$ | (14) $\int_2^{+\infty} \frac{dx}{x \ln x};$ | (15) $\int_0^{\pi} \frac{1}{\sqrt{x}} e^{-\sqrt{x}} dx;$ |
| (16) $\int_3^{+\infty} \frac{dx}{x(\ln x)^2};$ | (17) $\int_1^2 \frac{dx}{x \ln x};$ | (18) $\int_1^{+\infty} \frac{\ln \pi}{x^2} dx;$ |
| (19) $\int_0^{+\infty} e^{-ax} \sin bx dx, a > 0;$ | (20) $\int_0^{+\infty} \frac{x \ln x}{(1+x^2)^2} dx.$ | |

解 4.0.27. (1)

$$\begin{aligned} \int_1^{+\infty} e^{-2x} dx &= \lim_{b \rightarrow +\infty} \int_1^b e^{-2x} dx = \lim_{b \rightarrow +\infty} \left(-\frac{1}{2} e^{-2x} \Big|_1^b \right) \\ &= \lim_{b \rightarrow +\infty} \left(-\frac{1}{2} e^{-2b} + \frac{1}{2} e^{-2} \right) = \frac{1}{2} e^{-2} \quad (\text{收敛}). \end{aligned}$$

(2)

$$\begin{aligned} \int_1^{+\infty} \frac{x}{4+x^2} dx &= \lim_{b \rightarrow +\infty} \int_1^b \frac{x}{4+x^2} dx = \lim_{b \rightarrow +\infty} \left(\frac{1}{2} \ln(4+x^2) \Big|_1^b \right) \\ &= \lim_{b \rightarrow +\infty} \left(\frac{1}{2} \ln(4+b^2) - \frac{1}{2} \ln 5 \right) = +\infty \quad (\text{发散}). \end{aligned}$$

(3)

$$\begin{aligned} \int_0^{+\infty} \frac{x}{e^x} dx &= \lim_{b \rightarrow +\infty} \int_0^b x e^{-x} dx = \lim_{b \rightarrow +\infty} \left(-x e^{-x} - e^{-x} \Big|_0^b \right) \\ &= \lim_{b \rightarrow +\infty} (-b e^{-b} - e^{-b} + 1) = 1 \quad (\text{收敛}). \end{aligned}$$

(4)

$$\begin{aligned} \int_{-\infty}^0 \frac{e^x}{1+e^x} dx &= \lim_{a \rightarrow -\infty} \int_a^0 \frac{e^x}{1+e^x} dx = \lim_{a \rightarrow -\infty} \left(\ln(1+e^x) \Big|_a^0 \right) \\ &= \ln 2 - \lim_{a \rightarrow -\infty} \ln(1+e^a) = \ln 2 \quad (\text{收敛}). \end{aligned}$$

(5)

$$\int_x^{+\infty} \sin y dy = \lim_{b \rightarrow +\infty} \int_x^b \sin y dy = \lim_{b \rightarrow +\infty} \left(-\cos y \Big|_x^b \right)$$

$$= \lim_{b \rightarrow +\infty} (-\cos b + \cos x) \quad \text{极限不存在, 故发散.}$$

(6)

$$\begin{aligned} \int_{-\infty}^{+\infty} \frac{dz}{z^2 + 25} &= \lim_{a \rightarrow -\infty} \int_a^0 \frac{dz}{z^2 + 25} + \lim_{b \rightarrow +\infty} \int_0^b \frac{dz}{z^2 + 25} \\ &= \lim_{a \rightarrow -\infty} \left(\frac{1}{5} \arctan \frac{z}{5} \Big|_a^0 \right) + \lim_{b \rightarrow +\infty} \left(\frac{1}{5} \arctan \frac{z}{5} \Big|_0^b \right) \\ &= \frac{1}{5} \cdot \frac{\pi}{2} - \lim_{a \rightarrow -\infty} \frac{1}{5} \arctan \frac{a}{5} + \lim_{b \rightarrow +\infty} \frac{1}{5} \arctan \frac{b}{5} - \frac{1}{5} \cdot 0 \\ &= \frac{\pi}{10} + \frac{\pi}{10} = \frac{\pi}{5} \quad (\text{收敛}). \end{aligned}$$

(7)

$$\begin{aligned} \int_{\pi/4}^{\pi/2} \frac{\sin x}{\sqrt{\cos x}} dx &\stackrel{u=\cos x}{=} \int_{\sqrt{2}/2}^0 \frac{-du}{\sqrt{u}} = \int_0^{\sqrt{2}/2} u^{-1/2} du \\ &= 2\sqrt{u} \Big|_0^{\sqrt{2}/2} = 2\sqrt{\frac{\sqrt{2}}{2}} = 2^{3/4} \quad (\text{收敛}). \end{aligned}$$

(8)

$$\begin{aligned} \int_0^4 \frac{dx}{\sqrt{16-x^2}} &= \lim_{b \rightarrow 4^-} \int_0^b \frac{dx}{\sqrt{16-x^2}} = \lim_{b \rightarrow 4^-} \left(\arcsin \frac{x}{4} \Big|_0^b \right) \\ &= \lim_{b \rightarrow 4^-} \arcsin \frac{b}{4} = \arcsin 1 = \frac{\pi}{2} \quad (\text{收敛}). \end{aligned}$$

(9)

$$\begin{aligned} \int_{-1}^1 \frac{dt}{t} &= \lim_{a \rightarrow 0^-} \int_{-1}^a \frac{dt}{t} + \lim_{b \rightarrow 0^+} \int_b^1 \frac{dt}{t} \\ &= \lim_{a \rightarrow 0^-} \left(\ln |t| \Big|_{-1}^a \right) + \lim_{b \rightarrow 0^+} \left(\ln |t| \Big|_b^1 \right) \\ &= \lim_{a \rightarrow 0^-} (\ln |a| - \ln 1) + \lim_{b \rightarrow 0^+} (\ln 1 - \ln |b|) \\ &= \lim_{a \rightarrow 0^-} \ln |a| - \lim_{b \rightarrow 0^+} \ln |b| = -\infty - (-\infty) \quad \text{不存在, 故发散.} \end{aligned}$$

(10)

$$\int_1^{+\infty} \frac{dx}{\sqrt{x^2+1}} > \int_1^{+\infty} \frac{dx}{\sqrt{x^2+x^2}} = \int_1^{+\infty} \frac{dx}{\sqrt{2}x} = \frac{1}{\sqrt{2}} \int_1^{+\infty} \frac{dx}{x} = +\infty \quad (\text{发散}).$$

(11)

$$\int_0^1 \frac{x^4+1}{x} dx = \int_0^1 \left(x^3 + \frac{1}{x} \right) dx = \lim_{a \rightarrow 0^+} \int_a^1 \left(x^3 + \frac{1}{x} \right) dx$$

$$= \lim_{a \rightarrow 0^+} \left(\frac{x^4}{4} + \ln x \Big|_a^1 \right) = \left(\frac{1}{4} + 0 \right) - \lim_{a \rightarrow 0^+} \left(\frac{a^4}{4} + \ln a \right) = +\infty \quad (\text{发散}).$$

(12)

$$\begin{aligned} \int_4^{20} \frac{1}{y^2 - 16} dy &= \lim_{a \rightarrow 4^+} \int_a^{20} \frac{1}{(y-4)(y+4)} dy \\ &= \lim_{a \rightarrow 4^+} \frac{1}{8} \left(\ln \left| \frac{y-4}{y+4} \right| \Big|_a^{20} \right) \\ &= \frac{1}{8} \left(\ln \frac{16}{24} - \lim_{a \rightarrow 4^+} \ln \frac{a-4}{a+4} \right) = +\infty \quad (\text{发散}). \end{aligned}$$

(13)

$$\begin{aligned} \int_0^1 \frac{\ln x}{x} dx &= \lim_{a \rightarrow 0^+} \int_a^1 \frac{\ln x}{x} dx = \lim_{a \rightarrow 0^+} \left(\frac{1}{2} (\ln x)^2 \Big|_a^1 \right) \\ &= \lim_{a \rightarrow 0^+} \left(0 - \frac{1}{2} (\ln a)^2 \right) = -\infty \quad (\text{发散}). \end{aligned}$$

(14)

$$\begin{aligned} \int_2^{+\infty} \frac{dx}{x \ln x} &= \lim_{b \rightarrow +\infty} \int_2^b \frac{dx}{x \ln x} = \lim_{b \rightarrow +\infty} \left(\ln(\ln x) \Big|_2^b \right) \\ &= \lim_{b \rightarrow +\infty} (\ln(\ln b) - \ln(\ln 2)) = +\infty \quad (\text{发散}). \end{aligned}$$

(15)

$$\begin{aligned} \int_0^\pi \frac{1}{\sqrt{x}} e^{-\sqrt{x}} dx &\stackrel{t=\sqrt{x}}{=} \int_0^{\sqrt{\pi}} \frac{1}{t} e^{-t} \cdot 2t dt = 2 \int_0^{\sqrt{\pi}} e^{-t} dt \\ &= 2 \left(-e^{-t} \Big|_0^{\sqrt{\pi}} \right) = 2(1 - e^{-\sqrt{\pi}}) \quad (\text{收敛}). \end{aligned}$$

(16)

$$\begin{aligned} \int_3^{+\infty} \frac{dx}{x(\ln x)^2} &= \lim_{b \rightarrow +\infty} \int_3^b \frac{dx}{x(\ln x)^2} = \lim_{b \rightarrow +\infty} \left(-\frac{1}{\ln x} \Big|_3^b \right) \\ &= \lim_{b \rightarrow +\infty} \left(-\frac{1}{\ln b} + \frac{1}{\ln 3} \right) = \frac{1}{\ln 3} \quad (\text{收敛}). \end{aligned}$$

(17)

$$\begin{aligned} \int_1^2 \frac{dx}{x \ln x} &= \lim_{a \rightarrow 1^+} \int_a^2 \frac{dx}{x \ln x} = \lim_{a \rightarrow 1^+} \left(\ln |\ln x| \Big|_a^2 \right) \\ &= \ln(\ln 2) - \lim_{a \rightarrow 1^+} \ln |\ln a| = -\infty \quad (\text{发散}). \end{aligned}$$

(18)

$$\int_1^{+\infty} \frac{\ln \pi}{x^2} dx = \ln \pi \lim_{b \rightarrow +\infty} \int_1^b x^{-2} dx = \ln \pi \lim_{b \rightarrow +\infty} \left(-x^{-1} \Big|_1^b \right)$$

$$= \ln \pi \lim_{b \rightarrow +\infty} \left(-\frac{1}{b} + 1 \right) = \ln \pi \quad (\text{收敛}).$$

(19)

$$\begin{aligned} \int_0^{+\infty} e^{-ax} \sin bx \, dx &= \lim_{B \rightarrow +\infty} \int_0^B e^{-ax} \sin bx \, dx \\ &= \lim_{B \rightarrow +\infty} \left(\frac{e^{-ax}(-a \sin bx - b \cos bx)}{a^2 + b^2} \right) \Big|_0^B \\ &= 0 - \left(\frac{-b}{a^2 + b^2} \right) = \frac{b}{a^2 + b^2} \quad (\text{收敛}). \end{aligned}$$

(20)

$$\begin{aligned} \int_0^{+\infty} \frac{x \ln x}{(1+x^2)^2} \, dx &\stackrel{u=\frac{1}{x}}{=} \int_{+\infty}^0 \frac{(1/u) \ln(1/u)}{(1+1/u^2)^2} \left(-\frac{1}{u^2} \right) \, du \\ &= \int_0^{+\infty} \frac{-\ln u}{u(1+1/u^2)^2} \cdot \frac{1}{u^2} \, du \\ &= \int_0^{+\infty} \frac{-\ln u}{u^3 \cdot \frac{(u^2+1)^2}{u^4}} \, du = \int_0^{+\infty} \frac{-u \ln u}{(u^2+1)^2} \, du \\ &= - \int_0^{+\infty} \frac{u \ln u}{(u^2+1)^2} \, du = -I. \end{aligned}$$

所以 $2I = 0$, 即 $I = 0$, 故积分收敛, 值为 0。

例题 4.0.28 4.7-B-1

已知 $\int_{-\infty}^{\infty} e^{-x^2} \, dx = \sqrt{\pi}$, 若 $\int_{-\infty}^{+\infty} A e^{-x^2-x} \, dx = 1$, 求 A 。

解 4.0.28.

$$\begin{aligned} 1 &= \int_{-\infty}^{+\infty} A e^{-x^2-x} \, dx = \int_{-\infty}^{+\infty} e^{-(x+\frac{1}{2})^2 + \frac{1}{4} + \ln A} \, dx \\ &= e^{\frac{1}{4} + \ln A} \int_{-\infty}^{+\infty} e^{-(x+\frac{1}{2})^2} \, dx \\ &= e^{\frac{1}{4} + \ln A} \int_{-\infty}^{+\infty} e^{-(x+\frac{1}{2})^2} \, d\left(x + \frac{1}{2}\right) = e^{\frac{1}{4} + \ln A} \sqrt{\pi} \\ \Rightarrow A &= \frac{1}{e^{\frac{1}{4}} \sqrt{\pi}} = e^{-\frac{1}{4}} \pi^{-\frac{1}{2}} \end{aligned}$$

例题 4.0.29 4.7-B-2

$$\int_{\frac{1}{2}}^{\frac{3}{2}} \frac{dx}{\sqrt{|x-x^2|}}$$

解 4.0.29.

$$\begin{aligned}
\int_{\frac{1}{2}}^{\frac{3}{2}} \frac{dx}{\sqrt{|x-x^2|}} &= \int_{\frac{1}{2}}^1 \frac{dx}{\sqrt{x-x^2}} + \int_1^{\frac{3}{2}} \frac{dx}{\sqrt{x^2-x}} \\
&= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{d \sin^2 \theta}{\sqrt{\sin^2 \theta - \sin^4 \theta}} + \int_0^{\arctan \sqrt{\frac{1}{2}}} \frac{d \sec^2 \theta}{\sqrt{\sec^4 \theta - \sec^2 \theta}} \\
&= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{2 \sin \theta \cos \theta d\theta}{\sin \theta \cos \theta} + \int_0^{\arctan \sqrt{\frac{1}{2}}} \frac{2 \tan \theta \sec^2 \theta d\theta}{\tan \theta \sec \theta} \\
&= \frac{\pi}{2} + 2 \int_0^{\arctan \sqrt{\frac{1}{2}}} \sec \theta d\theta \\
&= \frac{\pi}{2} + \ln (\sec \theta + \tan \theta)^2 \Big|_0^{\arctan \sqrt{\frac{1}{2}}} \\
&= \frac{\pi}{2} + \ln(2 + \sqrt{3})
\end{aligned}$$

例题 4.0.30 4.7-B-3

$$\int_0^{+\infty} \frac{x e^{-x} dx}{(1 + e^{-x})^2}$$

解 4.0.30.

$$\begin{aligned}
\int_0^{+\infty} \frac{x e^{-x} dx}{(1 + e^{-x})^2} &= \int_1^{+\infty} \frac{\frac{1}{x} \ln x d \ln x}{(1 + \frac{1}{x})^2} = \int_1^{+\infty} \frac{\ln x}{(x+1)^2} dx \\
&= - \int_1^{+\infty} \ln x d \frac{1}{1+x} = - \frac{\ln x}{x+1} \Big|_1^{+\infty} + \int_1^{+\infty} \frac{1}{x+1} d \ln x \\
&= \int_1^{+\infty} \frac{1}{x+1} d \ln x = \int_1^{+\infty} \frac{dx}{x(x+1)} = 0 - \ln \frac{1}{2} = \ln 2
\end{aligned}$$

例题 4.0.31 4.7-B-4

$$\int_0^{+\infty} \frac{x^n}{e^x} dx, \quad n = 1, 2, 3, \dots$$

解 4.0.31.

$$\begin{aligned}
I_n &= \int_0^{+\infty} \frac{x^n}{e^x} dx = - \int_0^{+\infty} x^n d e^{-x} \\
&= -x^n e^{-x} \Big|_0^{+\infty} + \int_0^{+\infty} e^{-x} dx^n \\
&= x \int_0^{+\infty} e^{-x} x^{n-1} dx = n I_{n-1}
\end{aligned}$$

$$I_0 = -e^{-x} \Big|_0^{+\infty} = e^{-x} \Big|_{+\infty}^0 = 1, I_1 = 1I_0 = 1$$

$$\Rightarrow I_n = nI_{n-1} = n!$$

例题 4.0.32 4.8-A-1

求下列曲线所围图形的面积: (1) $y = x^2$ 与 $y = 2x + 3$; (2) $y = \sqrt{x}$ 与 $y = x$; (3) $y^2 = 2x$ 与 $x = 5$;

(4) $y = x$ 与 $y = x + \sin^2 x, (0 \leq x \leq \pi)$ (5) $x^2 + 9y^2 = 1$; (6) $y^2 = 1 + 2x - x^2$ 与 $x^2 + y^2 = 1$.

解 4.0.32. (1) 联立 $y = x^2$ 和 $y = 2x + 3$, 得 $x = -1, x = 3$, 则所围成的面积为

$$\begin{aligned} \int_{-1}^3 (2x + 3 - x^2) dx &= x^2 \Big|_{-1}^3 + 3x \Big|_{-1}^3 - \frac{x^3}{3} \Big|_{-1}^3 \\ &= 9 - 1 + (9 - (-3)) - \frac{1}{3}(27 + 1) = \frac{32}{3} \end{aligned}$$

(2) 联立 $y = \sqrt{x}$ 和 $y = x$, 得 $x = 0, x = 1$, 则所围成的面积为

$$\begin{aligned} \int_0^1 (\sqrt{x} - x) dx &= \frac{2}{3} x^{\frac{3}{2}} \Big|_0^1 - \frac{1}{2} x^2 \Big|_0^1 \\ &= \frac{2}{3} - \frac{1}{2} = \frac{1}{6} \end{aligned}$$

(3) 联立 $y^2 = 2x$ 和 $x = 5$, 得则所围成的面积为

$$\begin{aligned} 2 \int_0^5 \sqrt{2x} dx &= 2\sqrt{2} \int_0^5 \sqrt{x} dx = 2\sqrt{2} \cdot \frac{2}{3} x^{\frac{3}{2}} \Big|_0^5 \\ &= \frac{4\sqrt{2}}{3} \cdot 5\sqrt{5} = \frac{20\sqrt{10}}{3} \end{aligned}$$

(4) 则所围成的面积为

$$\begin{aligned} \int_0^\pi \sin^2 x dx &= \int_0^\pi \frac{1 - \cos 2x}{2} dx = \frac{1}{2} x \Big|_0^\pi - \frac{\sin 2x}{4} \Big|_0^\pi \\ &= \frac{\pi}{2} \end{aligned}$$

(5) 椭圆面积为

$$\begin{aligned} 2 \int_{-1}^1 \frac{1}{3} \sqrt{1 - x^2} dx &= \frac{2}{3} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{1 - \sin^2 \theta} d \sin \theta = \frac{2}{3} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 x dx \\ &= \frac{1}{3} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1 + \cos 2x) dx = \frac{\pi}{3} + \frac{\sin 2x}{4} \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{\pi}{3} \end{aligned}$$

(6) 等效于一个半圆加一个拱形的面积

$$S = \frac{\pi}{2} + \frac{1}{2}\pi - 1 = \pi - 1$$

例题 4.0.33 4.8-A-2

求下列极坐标表示的图形围成的面积：(1) $r = 2a \cos \theta$ ；(2) $r = 3 \cos \theta$ 和 $r = 1 + \cos \theta$ 围成的公共部分的面积

解 4.0.33. (1) 曲线 $r = 2a \cos \theta$ 表示一个圆，圆心在 $(a, 0)$ ，半径为 $|a|$ 。其面积为

$$\begin{aligned} A &= \frac{1}{2} \int_{-\pi/2}^{\pi/2} (2a \cos \theta)^2 d\theta = 2a^2 \int_{-\pi/2}^{\pi/2} \cos^2 \theta d\theta = 2a^2 \int_{-\pi/2}^{\pi/2} \frac{1 + \cos 2\theta}{2} d\theta \\ &= a^2 \int_{-\pi/2}^{\pi/2} (1 + \cos 2\theta) d\theta = a^2 \left(\theta + \frac{1}{2} \sin 2\theta \right) \Big|_{-\pi/2}^{\pi/2} = a^2 (\pi/2 - (-\pi/2)) = \pi a^2. \end{aligned}$$

(2) 先求两曲线的交点：由 $3 \cos \theta = 1 + \cos \theta$ 得 $\cos \theta = \frac{1}{2}$ ，故 $\theta = \pm \frac{\pi}{3}$ 。由于图形关于极轴对称，只需计算上半部分再乘以 2。在上半平面，当 $\theta \in [0, \pi/3]$ 时， $1 + \cos \theta \leq 3 \cos \theta$ ，公共部分的边界为 $r = 1 + \cos \theta$ ；当 $\theta \in [\pi/3, \pi/2]$ 时， $3 \cos \theta \leq 1 + \cos \theta$ ，公共部分的边界为 $r = 3 \cos \theta$ 。于是上半部分的面积为

$$A_{\text{上半}} = \frac{1}{2} \int_0^{\pi/3} (1 + \cos \theta)^2 d\theta + \frac{1}{2} \int_{\pi/3}^{\pi/2} (3 \cos \theta)^2 d\theta.$$

计算第一个积分：

$$\begin{aligned} \frac{1}{2} \int_0^{\pi/3} (1 + \cos \theta)^2 d\theta &= \frac{1}{2} \int_0^{\pi/3} (1 + 2 \cos \theta + \cos^2 \theta) d\theta \\ &= \frac{1}{2} \int_0^{\pi/3} \left(\frac{3}{2} + 2 \cos \theta + \frac{1}{2} \cos 2\theta \right) d\theta \\ &= \frac{1}{2} \left(\frac{3}{2} \theta + 2 \sin \theta + \frac{1}{4} \sin 2\theta \right) \Big|_0^{\pi/3} \\ &= \frac{1}{2} \left(\frac{\pi}{2} + \sqrt{3} + \frac{\sqrt{3}}{8} \right) = \frac{\pi}{4} + \frac{9\sqrt{3}}{16}. \end{aligned}$$

计算第二个积分：

$$\begin{aligned} \frac{1}{2} \int_{\pi/3}^{\pi/2} 9 \cos^2 \theta d\theta &= \frac{9}{2} \int_{\pi/3}^{\pi/2} \frac{1 + \cos 2\theta}{2} d\theta = \frac{9}{4} \int_{\pi/3}^{\pi/2} (1 + \cos 2\theta) d\theta \\ &= \frac{9}{4} \left(\theta + \frac{1}{2} \sin 2\theta \right) \Big|_{\pi/3}^{\pi/2} \\ &= \frac{9}{4} \left(\frac{\pi}{2} - \frac{\pi}{3} - \frac{1}{2} \cdot \frac{\sqrt{3}}{2} \right) = \frac{9}{4} \left(\frac{\pi}{6} - \frac{\sqrt{3}}{4} \right) = \frac{3\pi}{8} - \frac{9\sqrt{3}}{16}. \end{aligned}$$

所以

$$A_{\text{上半}} = \left(\frac{\pi}{4} + \frac{9\sqrt{3}}{16} \right) + \left(\frac{3\pi}{8} - \frac{9\sqrt{3}}{16} \right) = \frac{5\pi}{8},$$

总面积

$$A = 2A_{\text{上半}} = \frac{5\pi}{4}.$$

例题 4.0.34 4.8-A-3

曲线 $y = (x-1)(x-2)$ 和 x 轴围成一平面图形, 求此平面图形绕 y 轴旋转一周所成旋转体的体积.

解 4.0.34. 曲线 $y = (x-1)(x-2)$ 与 x 轴的交点为 $x = 1$ 和 $x = 2$ 。在区间 $[1, 2]$ 上, $y \leq 0$, 故所求平面图形为 x 轴下方, 由曲线 $y = (x-1)(x-2)$ 与 x 轴围成的区域。

将此区域绕 y 轴旋转, 采用柱壳法。在 x 处取厚度为 dx 的竖直窄条, 其高度为 $|y| = -(x-1)(x-2)$, 旋转生成的柱壳半径为 x , 故体积微元为

$$dV = 2\pi x \cdot |y| dx = 2\pi x \cdot [-(x-1)(x-2)] dx.$$

于是旋转体的体积为

$$\begin{aligned} V &= \int_1^2 2\pi x \cdot [-(x-1)(x-2)] dx \\ &= 2\pi \int_1^2 (-x^3 + 3x^2 - 2x) dx \\ &= 2\pi \left[-\frac{x^4}{4} + x^3 - x^2 \right]_1^2 \\ &= 2\pi \left(\left(-\frac{16}{4} + 8 - 4 \right) - \left(-\frac{1}{4} + 1 - 1 \right) \right) \\ &= 2\pi \left(0 + \frac{1}{4} \right) \\ &= 2\pi \cdot \frac{1}{4} = \frac{\pi}{2}. \end{aligned}$$

故所求旋转体的体积为 $\frac{\pi}{2}$ 。

例题 4.0.35 4.8-B-4

求由曲线 $y = \sqrt{x}$ 及 $y = x^2$ 所围平面图形绕 x 轴旋转所得旋转体的体积。

解 4.0.35. 两条曲线的交点为 $y = \sqrt{x}$ 与 $y = x^2$ 的解:

$$\sqrt{x} = x^2 \implies x = x^4 \implies x(x^3 - 1) = 0,$$

得 $x = 0$ 或 $x = 1$, 对应 $y = 0$ 和 $y = 1$, 即交点 $(0, 0)$ 和 $(1, 1)$ 。

在区间 $[0, 1]$ 上, 有 $\sqrt{x} \geq x^2$ 。绕 x 轴旋转, 用圆盘法 (或称为“切片法”), 体积微元为

$$dV = \pi [(\sqrt{x})^2 - (x^2)^2] dx = \pi(x - x^4) dx.$$

积分得

$$V = \pi \int_0^1 (x - x^4) dx = \pi \left(\frac{x^2}{2} - \frac{x^5}{5} \right) \Big|_0^1$$

$$= \pi \left(\frac{1}{2} - \frac{1}{5} \right) = \pi \cdot \frac{3}{10} = \frac{3\pi}{10}.$$

故所求旋转体的体积为 $\frac{3\pi}{10}$ 。

例题 4.0.36 4.8-A-5

求曲线 $y = \ln(-x^2 + 1)$ 上相应于 $0 \leq x \leq \frac{1}{2}$ 的弧长

解 4.0.36.

$$\begin{aligned} L &= \int_0^{\frac{1}{2}} \sqrt{1 + \left(\frac{2x}{x^2 - 1} \right)^2} dx = \int_0^{\frac{1}{2}} \sqrt{\frac{(x^2 + 1)^2}{(x^2 - 1)^2}} dx = \int_0^{\frac{1}{2}} \frac{1 + x^2}{1 - x^2} dx \\ &= \int_0^{\frac{1}{2}} \left(\frac{-1 + x^2 + 2}{1 - x^2} \right) dx = \int_0^{\frac{1}{2}} \left(-1 + \frac{2}{1 - x^2} \right) dx \\ &= \int_0^{\frac{1}{2}} \left(-1 - \frac{1}{x - 1} + \frac{1}{1 + x} \right) dx \\ &= -x \Big|_0^{\frac{1}{2}} - \ln |x - 1| \Big|_0^{\frac{1}{2}} + \ln |1 + x| \Big|_0^{\frac{1}{2}} \\ &= \ln 2 - \frac{1}{2} + \ln \frac{3}{2} = \ln 3 - \frac{1}{2} \end{aligned}$$

例题 4.0.37 4.8-A-6

求曲线 $y = \int_0^{\frac{x}{n}} n\sqrt{\sin \theta} d\theta$ 的全长, 其中 $0 \leq x \leq n\pi$

解 4.0.37.

$$\begin{aligned} L &= \int_0^{n\pi} \sqrt{1 + \left(\frac{d}{dx} \int_0^{\frac{x}{n}} n\sqrt{\sin \theta} d\theta \right)^2} dx = \int_0^{n\pi} \sqrt{1 + \left(n \cdot \sqrt{\sin \frac{x}{n}} \frac{1}{n} \right)^2} dx \\ &= \int_0^{n\pi} \sqrt{1 + \sin \frac{x}{n}} dx = n \int_0^{n\pi} \sqrt{1 + \sin \frac{x}{n}} \frac{dx}{n} = n \int_0^{\pi} \sqrt{1 + \sin x} dx \\ &= n \int_0^{\pi} \left(\sin \frac{x}{2} + \cos \frac{x}{2} \right) dx = n \left(-2 \cos \frac{x}{2} \Big|_0^{\pi} + 2 \sin \frac{x}{2} \Big|_0^{\pi} \right) = 4n \end{aligned}$$

例题 4.0.38 4.8-A-8

求心脏线 $r = a(1 + \cos \theta)$ 的全长 n

解 4.0.38.

$$S = \int_0^{2\pi} r^2(\theta) d\theta = a^2 \int_0^{2\pi} (1 + \cos \theta)^2 d\theta$$

$$\begin{aligned}
 &= a^2 \int_0^{2\pi} (1 + 2\cos\theta + \cos^2\theta) d\theta = a^2 \int_0^{2\pi} \left(\frac{3}{2} + 2\cos\theta + \frac{1}{2}\cos 2\theta\right) d\theta \\
 &= a^2 \left(\frac{3}{2}\theta + 2\sin\theta + \frac{1}{4}\sin 2\theta\right) \Big|_0^{2\pi} = 3\pi a^2
 \end{aligned}$$

例题 4.0.39 4.8-B-2

一立体的底面为一半径为 R 的圆盘, 其垂直于 x 轴的截面是一等边三角形, 求这个立体的体积.

解 4.0.39.

$$\begin{aligned}
 V &= 2 \int_0^R \frac{\sqrt{3}}{4} (\sqrt{R^2 - x^2})^2 dx = 2\sqrt{3} \int_0^R (R^2 - x^2) dx \\
 &= 2\sqrt{3} \left(R^2x - \frac{x^3}{3}\right) \Big|_0^R = \frac{4}{3}\sqrt{3}R^3
 \end{aligned}$$

例题 4.0.40 4.8-B-6

求下列平面曲线绕指定轴旋转所得旋转体的侧面积:

- (1) $y = \sin x, 0 \leq x \leq \pi$, 绕 x 轴;
- (2) $x = a(t - \sin t), y = a(1 - \cos t), 0 \leq t \leq 2\pi$, 绕直线 $y = 2a$
- (3) $r = a(1 + \cos \theta), 0 \leq \theta \leq 2\pi$, 绕极轴。

解 4.0.40. (1) 曲线 $y = \sin x, 0 \leq x \leq \pi$, 绕 x 轴旋转所得旋转体的侧面积为

$$\begin{aligned}
 S &= 2\pi \int_0^\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \\
 &= 2\pi \int_0^\pi \sin x \sqrt{1 + \cos^2 x} dx \\
 &= 2\pi \int_1^{-1} \sqrt{1 + u^2} (-du) \quad (\text{令 } u = \cos x) \\
 &= 2\pi \int_{-1}^1 \sqrt{1 + u^2} du = 4\pi \int_0^1 \sqrt{1 + u^2} du \\
 &= 4\pi \cdot \frac{1}{2} \left[u\sqrt{1 + u^2} + \ln(u + \sqrt{1 + u^2}) \right]_0^1 \\
 &= 2\pi (\sqrt{2} + \ln(1 + \sqrt{2})).
 \end{aligned}$$

(2) 曲线为摆线 $x = a(t - \sin t), y = a(1 - \cos t), 0 \leq t \leq 2\pi$, 绕直线 $y = 2a$ 旋转. 旋转半径 $R = |y - 2a| = a(1 + \cos t)$, 弧微分为

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt = 2a \sin \frac{t}{2} dt,$$

故侧面积为

$$\begin{aligned} S &= \int_0^{2\pi} 2\pi R ds = 4\pi a^2 \int_0^{2\pi} (1 + \cos t) \sin \frac{t}{2} dt \\ &= 8\pi a^2 \int_0^{2\pi} \cos^2 \frac{t}{2} \sin \frac{t}{2} dt \quad (\text{利用 } 1 + \cos t = 2 \cos^2 \frac{t}{2}) \\ &= 8\pi a^2 \cdot \frac{4}{3} = \frac{32\pi}{3} a^2. \end{aligned}$$

(3) 曲线为心形线 $r = a(1 + \cos \theta)$, $0 \leq \theta \leq 2\pi$, 绕极轴旋转. 由对称性, 仅需计算上半部分 ($0 \leq \theta \leq \pi$) 的侧面积, 上半部分上, $y = r \sin \theta$, $ds = 2a \cos \frac{\theta}{2} d\theta$, 故

$$\begin{aligned} S_{\text{上}} &= \int_0^{\pi} 2\pi y ds = 4\pi a^2 \int_0^{\pi} (1 + \cos \theta) \sin \theta \cos \frac{\theta}{2} d\theta \\ &= 16\pi a^2 \int_0^{\pi} \cos^4 \frac{\theta}{2} \sin \frac{\theta}{2} d\theta \quad (\text{化简得}) \\ &= 16\pi a^2 \cdot \frac{2}{5} = \frac{32\pi}{5} a^2, \end{aligned}$$

所以整个侧面积为

$$S = S_{\text{上}} = \frac{32\pi}{5} a^2.$$

第五章 自用

例题 5.0.1

$x_1 - \ln x_1 = x_2 - \ln x_2 = m, x_1 < x_2$, 求证 $x_1^{x_2} + x_2^{x_1} \sim 1 + \frac{\ln m}{e^m}, m \rightarrow +\infty$.

解 5.0.1. 首先估计 x_1, x_2 在无穷远处的渐进, 首先观察 $f(x) = x - \ln x$, 在 $x \rightarrow 0$ 的时候函数值由 $-\ln x$ 决定, 在 $x \rightarrow +\infty$ 的时候函数值由 x 决定, 先估计大根 x_2 , 设 $x_2 = m + \ln x_2 \sim m, m \rightarrow +\infty$, 于是设 $x_2 = m + \delta$ 将这个结果代入得到 $\delta = m + \ln(m + \delta)$, 泰勒展开得到

$$\delta = \ln(m + \ln \delta) = \ln m + \ln \left(1 + \frac{\ln \delta}{m}\right) = \ln m + \frac{\delta}{\ln m} + o\left(\frac{\delta}{\ln m}\right) \sim \ln m$$

得到 $x_2 \sim m + \ln m$, 再设 $x_2 = m + \ln m + \delta$, 代入得到

$$\begin{aligned} x_2 &= m + \ln x_2 = m + \ln(m + \ln m + \delta) = m + \ln m + \ln \left(1 + \frac{\ln m}{m} + \frac{\delta}{m}\right) \\ \Rightarrow \delta &= \ln \left(1 + \frac{\ln m}{m} + \frac{\delta}{m}\right) = \frac{\ln m + \delta}{m} + o\left(\frac{\ln m + \delta}{m}\right) \sim \frac{\ln m + \delta}{m} \end{aligned}$$

解得 $\delta \sim \frac{\ln m}{m-1}$, 再设 $x_2 = m + \frac{m \ln m}{m-1} + \delta$, 代入得到

$$\begin{aligned} m + \frac{m \ln m}{m-1} + \delta &= m + \ln \left(m + \frac{m \ln m}{m-1} + \delta\right) \\ \Rightarrow \frac{m \ln m}{m-1} + \delta &= \ln m + \ln \left(1 + \frac{\ln m}{m-1} + \frac{\delta}{m}\right) = \frac{\ln m}{m-1} + \delta = \ln \left(1 + \frac{\ln m}{m-1} + \frac{\delta}{m}\right) \\ \Rightarrow \frac{m \ln m}{m-1} + \delta &= \left(\frac{\ln m}{m-1} + \frac{\delta}{m}\right) - \frac{1}{2} \left(\frac{\ln m}{m-1} + \frac{\delta}{m}\right)^2 + o\left(\frac{\ln m}{m-1} + \frac{\delta}{m}\right)^2 \\ \Rightarrow \ln m + \delta &\sim \frac{\delta}{m} - \frac{1}{2} \frac{\ln^2 m}{(m-1)^2} - \delta \frac{\ln m}{m(m-1)} \quad \text{舍掉平方项} \\ \Rightarrow \delta &\sim -\frac{\ln m + \frac{1}{2} \frac{\ln^2 m}{(m-1)^2}}{1 - \frac{1}{m} + \frac{\ln m}{m(m-1)}} \end{aligned}$$

例题 5.0.2

设 $f \in C[0, 2] \cap D(0, 2)$, 满足 $f(0) = f(2) = 0, |f'(x)| \leq M, \forall x \in (0, 2)$, 证明

$$\left| \int_0^2 f(x) dx \right| \leq M$$

解 5.0.2. 插 $f(0), f(2)$, 插值多项式 1 次 (仅拉格朗日插值式, 用不到导数修正), 余项 2 阶, 导数不够, 属于靠近哪边对哪边插模型。所以在 $[0, 1]$ 上插 $f(0)$, 在 $[1, 2]$ 上插 $f(2)$, 则写出两段插值式。

当 $x \in [0, 1]$ 时, 对 $f(x)$ 的插值为:

$$f(x) = f(0) + \frac{f'(\xi(x))}{1!}(x-0) = f'(\xi(x))x$$

这是拉格朗日中值定理的直接应用, 所以不用考场翻译。得到

$$|f(x)| \leq |f'(\theta(x))|x \leq Mx$$

当 $x \in [1, 2]$ 时, 对 $f(x)$ 的插值为:

$$f(x) = f(2) + \frac{f'(\eta(x))}{1!}(x-2) = f'(\eta(x))(x-2)$$

同理, 得到

$$|f(x)| \leq |f'(\eta(x))||x-2| \leq M(2-x)$$

所以

$$\begin{aligned} \left| \int_0^2 f(x) dx \right| &\leq \left| \int_0^1 f(x) dx \right| + \left| \int_1^2 f(x) dx \right| \\ &\leq \int_0^1 |f(x)| dx + \int_1^2 |f(x)| dx \\ &\leq \int_0^1 Mx dx + \int_1^2 M(2-x) dx = M \end{aligned}$$

例题 5.0.3

设 $f \in D^2[0, 1]$, $f(0) = f(1) = 0$, $|f''(x)| \leq M$, 证明 $\left| \int_0^1 f(x) dx \right| \leq \frac{M}{12}$

解 5.0.3. 要拟合 $f(0), f(1)$ 一共两个插值条件, 无导数条件需要拟合, 所以插值多项式为一次, 也不用待定多项式 r , 余项到了 2 阶导数, 符合题目所给条件, 所以根据埃米特插值, 存在 $\theta(x) \in (0, 1)$ 使得:

$$f(x) = \frac{f''(\theta(x))}{2} x(x-1), \forall x \in [0, 1]$$

这个结论可以用 k 值法来证明, 由

$$f(x) - Kx(x-1) = 0$$

的条件 (其中 K 与 x 有关), 设

$$F(y) = f(y) - Ky(y-1)$$

则 $F(0) = F(1) = F(x) = 0$, 罗尔得到 $F'(\xi_1) = F'(\xi_2) = 0, \xi_1, \xi_2 \in (0, 1)$, 所以存在 $\theta(x) \in (0, 1)$ 使得 $F''(\theta(x)) = 0$ 。对 $F(x)$ 求二阶导数得到

$$f''(\theta(x)) - 2K = 0 \Rightarrow K = \frac{f''(\theta(x))}{2}$$

所以

$$f(x) = \frac{f''(\theta(x))}{2} x(x-1)$$

所以

$$\begin{aligned} \left| \int_0^1 f(x) dx \right| &= \left| \int_0^1 \frac{f''(\theta(x))}{2} x(x-1) dx \right| \leq \int_0^1 \left| \frac{f''(\theta(x))}{2} \right| |x(x-1)| dx \\ &\leq \frac{M}{2} \int_0^1 x(1-x) dx = \frac{M}{12} \end{aligned}$$

例题 5.0.4

设 $f \in C^3[0, 2]$ 满足

$$f(0) = f'(0) = 0, \int_0^2 f(x) dx = 8 \int_0^1 f(x) dx$$

求证存在 $\theta \in (0, 2)$ 使得 $f'''(\theta) = 0$

解 5.0.4. 套路地, 设 $F(x) = \int_0^x f(y) dy$, $F(x) \in C^4[0, 2]$, $F(0) = F'(0) = F''(0) = 0$, 且 $F(2) = 8F(1)$, 这里我们使用插值法, 就是找一个多项式去尽可能的拟合 f 的行为。注意到 $F(0) = F'(0) = F''(0) = 0$, 所以可以考虑插值多项式为 $p(x) = x^3(ax + b)$, 代入 $F(2) = 8F(1)$ 得到 $a = 0$, 所以 $p(x) = bx^3$, 再由 $p(1) = F(1)$ 得到 $b = F(1)$, 所以

$$p(x) = F(1)x^3$$

于是构造出

$$G(x) = F(x) - F(1)x^3$$

这个 $G(x)$ 满足

$$G(0) = G(1) = G(2) = 0, G'(0) = 0, G''(0) = 0$$

所以由罗尔中值定理, 存在 $\xi_1 \in (0, 1), \xi_2 \in (1, 2)$, 使得 $G'(\xi_1) = G'(\xi_2) = 0$, 搭配上 $G'(0) = 0$, 所以又有罗尔中值定理, 得知存在 θ_1, θ_2 , 使得 $G''(\theta_1) = G''(\theta_2) = 0$, 同样的, 搭配上 $G''(0) = 0$, 又有罗尔中值定理, 得知存在 $\eta_1 \in (0, 1), \eta_2 \in (1, 2)$ 使得

$$G'''(\eta_1) = G'''(\eta_2) = 0$$

所以由罗尔中值定理, 得到存在 $\theta \in (0, 2)$ 使得

$$G''''(\theta) = 0$$

而求导易得

$$G''''(x) = f'''(x) - p''''(x) = 0 \Rightarrow f'''(\theta) = 0$$

例题 5.0.5

设 $f \in C^1[a, b] \cap D^3(a, b)$, 证明存在 $\xi \in (a, b)$, 使得

$$\frac{f(b) - f(a)}{b - a} = \frac{f'(a) + f'(b)}{2} - \frac{f'''(\xi)}{12}(b - a)^2$$

解 5.0.5. 插 $f(a), f(b), f'(a), f'(b)$, 则插值多项式 3 次, 余项 4 阶, 但是条件只到 3 阶, 属于 $f''' = p'''$ 类型, 插值:

$$p(x) = \frac{x-b}{a-b}f(a) + \frac{x-a}{b-a}f(b) + (kx+m)(x-a)(x-b)$$

此时自动保证 $p(a) = f(a), p(b) = f(b)$, 参数 k, m 待定, 以期望 $p'(a) = f'(a), p'(b) = f'(b)$, 解出 k, m .

$$\begin{aligned} p'(a) &= \frac{f(b) - f(a)}{b - a} + (ka + m)(a - b) = f'(a) \\ p'(b) &= \frac{f(b) - f(a)}{b - a} + (kb + m)(a - b) = f'(b) \end{aligned}$$

相减得到

$$k = \frac{f'(b) + f'(a)}{(b - a)^2} - \frac{2(f(b) - f(a))}{(b - a)^3}$$

此时可以继续解方程得到 m , 但是没有必要了, 我们假装 m 已知, 设而不求就可以了. 设 $G(x) = f(x) - p(x)$, 则 $G(a) = G(b) = 0, G'(a) = G'(b) = 0$, 所以由罗尔中值定理, 存在 $\xi_1 \in (a, b)$ 使得 $G'(\xi_1) = 0$, 再由罗尔中值定理, 存在 ξ_2, ξ_3 使得 $G''(\xi_2) = 0, G''(\xi_3) = 0$, 最后由罗尔中值定理, 存在 $\xi \in (a, b)$ 使得 $G'''(\xi) = 0$, 所以

$$f'''(\xi) = p'''(\xi) = 6k = \frac{6(f'(b) + f'(a))}{(b - a)^2} - \frac{12(f(b) - f(a))}{(b - a)^3}$$

例题 5.0.6

$f(x)$ 在 $[0, 1]$ 可导, 且 $\int_0^1 f(x)dx = \frac{5}{2}, \int_0^1 xf(x)dx = \frac{3}{2}$, 求证在 $(0, 1)$ 存在 ξ 使 $f'(\xi) = 3$.

解 5.0.6. 套路式地, 设

$$F(x) = \int_0^x f(t)dt, G(x) = \int_0^x tf(t)dt$$

则有

$$F(0) = G(0) = 0, F(1) = \frac{5}{2}, G(1) = \frac{3}{2}$$

而问题要证明的结论似乎仅与 $f(x)$ (以及其导函数, 原函数) 有关, 所以要从 $G(x)$ 中分出 $F(x)$ 或者 $f(x)$ 来, 考虑分部积分:

$$\int_0^1 xf(x)dx = \int_0^1 x dF(x) = xF(x)|_0^1 - \int_0^1 F(x)dx \Rightarrow \int_0^1 F(x)dx = \frac{1}{2}$$

则问题转化为 $F(x)$ 在 $(0, 1)$ 上二阶可导, 且 $F(0) = 0, F(1) = \frac{5}{2}, \int_0^1 F(x)dx = \frac{1}{2}$, 求证存在 $\xi \in (0, 1)$ 使得 $F''(\xi) = 3$. 又注意到 $\int_0^1 xdx = \frac{1}{2}$, 所以考虑 $\varphi(x) = F(x) - x$, 则 $\varphi(0) = 0, \varphi(1) = \frac{3}{2}, \int_0^1 \varphi(x)dx = 0$, 求证存在 $\xi \in (0, 1)$ 使得 $\varphi''(\xi) = 3$ 要证 $\varphi''(\xi) - 3 = 0$, 如果我们证明 $\varphi'(x) - 3x + C = 0$

例题 5.0.7

$f(x)$ 在 $[a, b]$ 上连续, 在 (a, b) 上二次可微, 证明存在 $\xi \in (a, b)$ 使得

$$f(a) + f(b) - 2f\left(\frac{a+b}{2}\right) = \frac{1}{4}(b-a)^2 f''(\xi)$$

解 5.0.7. 拟合 $f(a), f(b), f\left(\frac{a+b}{2}\right)$, 设

$$G(x) = f(x) - \left[\frac{(x - \frac{a+b}{2})(x-b)}{(a - \frac{a+b}{2})(a-b)} f(a) + \frac{(x-a)(x - \frac{a+b}{2})}{(b-a)(b - \frac{a+b}{2})} f(b) + \frac{(x-a)(x-b)}{(\frac{a+b}{2} - a)(\frac{a+b}{2} - b)} f\left(\frac{a+b}{2}\right) \right]$$

定理 5.0.1 双元第一公式

若 $x^2 + y^2 = a^2$, 则

$$x dx + y dy = 0 \Rightarrow \int \frac{dx}{y} = \arctan \frac{x}{y} + C$$

若 $x^2 - y^2 = a^2$ 或 $x^2 - y^2 = -a^2$, 则

$$x dx = y dy \Rightarrow \int \frac{dx}{y} = \ln(x+y) + C$$

若 $x^2 + y^2 = a^2$, 则 $x dx + y dy = 0 \Rightarrow \frac{x dx + y dy}{xy} = \frac{dx}{y} + \frac{dy}{x} = 0$, 所以

$$\begin{aligned} \int \frac{dx}{y} &= \int \frac{y dx}{y^2} = \int \frac{-dy}{x} = \int \frac{-x dy}{x^2} \\ &= \int \frac{y dx - x dy}{x^2 + y^2} = \int \frac{y^2}{x^2 + y^2} \frac{y dx - x dy}{y^2} \\ &= \int \frac{1}{1 + (\frac{x}{y})^2} d\frac{x}{y} = \arctan \frac{x}{y} + C \end{aligned}$$

若 $x^2 - y^2 = a^2$ 或 $x^2 - y^2 = -a^2$, 则 $x dx = y dy \Rightarrow \frac{dx}{y} = \frac{dy}{x}$, 所以

$$\int \frac{dx}{y} = \int \frac{dy}{x} = \int \frac{dx + dy}{x + y} = \int \frac{d(x+y)}{x+y} = \ln(x+y) + C$$

定理 5.0.2 双元第二公式

形如

$$ydx - xdy \quad ydx + xdy$$

等都可以用 $x dx + y dy = 0$ 或 $x dx = y dy$ 消元, 化归成第一公式的适用情形, 同样的, 第一公式也可以向第二公式转化, 若 $x^2 + y^2 = a^2$:

$$\frac{dx}{y} = -\frac{dy}{x} = \frac{ydx}{y^2} = -\frac{xdy}{x^2} = \frac{ydx - xdy}{y^2 + x^2}$$

若 $x^2 - y^2 = a^2$ 或 $x^2 - y^2 = -a^2$, 则

$$\frac{dx}{y} = \frac{dy}{x} = \frac{ydx}{y^2} = \frac{xdy}{x^2} = \frac{ydx - xdy}{y^2 - x^2}$$

若 $x^2 + y^2 = a^2$, 则

$$\begin{aligned} ydx - xdy &= ydx - x \left(-\frac{x dx}{y} \right) = (x^2 + y^2) \frac{dx}{y} \\ ydx - xdy &= y \left(\frac{-y dy}{x} \right) - xdy = -(x^2 + y^2) \frac{dy}{x} \end{aligned}$$

若 $x^2 - y^2 = a^2$ 或 $x^2 - y^2 = -a^2$, 则

$$\begin{aligned} ydx - xdy &= ydx - x \left(\frac{x dx}{y} \right) = (y^2 - x^2) \frac{dx}{y} \\ ydx - xdy &= y \left(\frac{y dy}{x} \right) - xdy = (y^2 - x^2) \frac{dy}{x} \end{aligned}$$

定理 5.0.3 双元第三公式

单独的 ydx 和 xdy 都可以用“加一项, 减一项”的方法化归成 $d(xy)$ 和公式二的情形

这里只给出其中一种:

$$\begin{aligned} ydx &= \frac{1}{2} (ydx + ydx) = \frac{1}{2} (ydx + xdy + ydx - xdy) \\ &= \frac{1}{2} [d(xy) + (ydx - xdy)] \end{aligned}$$

定理 5.0.4 双元第四公式

形如 $\frac{dx}{y^3}$ 的式子可以通过第二公式化成含有 $d\left(\frac{y}{x}\right)$ 的形式

若 $x^2 + y^2 = a^2$, 则

$$\frac{dx}{y^3} = \frac{1}{y^2} \frac{dx}{y} = \frac{1}{y^2} \frac{ydx - xdy}{x^2 + y^2} = \frac{1}{x^2 + y^2} d\left(\frac{x}{y}\right)$$

$$\frac{dy}{x^3} = \frac{1}{x^2} \frac{dy}{x} = \frac{1}{x^2} \frac{x dy - y dx}{x^2 + y^2} = \frac{1}{x^2 + y^2} d\left(\frac{y}{x}\right)$$

若 $x^2 - y^2 = a^2$ 或 $x^2 - y^2 = -a^2$, 则

$$\begin{aligned} \frac{dx}{y^3} &= \frac{1}{y^2} \frac{dx}{y} = \frac{1}{y^2} \frac{y dx - x dy}{x^2 - y^2} = \frac{1}{x^2 - y^2} d\left(\frac{x}{y}\right) \\ \frac{dy}{x^3} &= \frac{1}{x^2} \frac{dy}{x} = \frac{1}{x^2} \frac{x dy - y dx}{x^2 - y^2} = \frac{1}{x^2 - y^2} d\left(\frac{y}{x}\right) \end{aligned}$$

例题 5.0.8

设函数 $f(x)$ 在 $(0, +\infty)$ 上三阶可导, 且 $f(x) > 0, f'(x) > 0, f''(x) > 0, \lim_{x \rightarrow +\infty} \frac{f'(x)f'''(x)}{[f''(x)]^2} = a \neq 1$, 求极限 $\lim_{x \rightarrow +\infty} \frac{f(x)f''(x)}{[f'(x)]^2}$.

解 5.0.8. 首先由 $f(x) > 0, f'(x) > 0, f''(x) > 0$ 得知 $f'(x), f(x)$ 均单调递增, 所以

$$f(x+h) > f(x) + f'(\xi)(x+h-x) > f(x) + f'(x)h$$

令 $h \rightarrow +\infty$, 得 $f(x) \rightarrow +\infty$, 现在对已知极限变形:

$$a = \lim_{x \rightarrow +\infty} \frac{f'(x)f'''(x)}{[f''(x)]^2} = 1 - \lim_{x \rightarrow +\infty} \frac{[f''(x)]^2 - f'(x)f'''(x)}{[f''(x)]^2} = 1 - \lim_{x \rightarrow +\infty} \frac{d}{dx} \left(\frac{f'(x)}{f''(x)} \right)$$

所以 $\lim_{x \rightarrow +\infty} \frac{d}{dx} \left(\frac{f'(x)}{f''(x)} \right) = 1 - a \neq 0$, 同样可以对所求极限变形:

$$\lim_{x \rightarrow +\infty} \frac{f(x)f''(x)}{[f'(x)]^2} = 1 - \lim_{x \rightarrow +\infty} \frac{[f'(x)]^2 - f(x)f''(x)}{[f'(x)]^2} = 1 - \lim_{x \rightarrow +\infty} \frac{d}{dx} \left(\frac{f(x)}{f'(x)} \right)$$

问题在于如何沟通 $\lim_{x \rightarrow +\infty} \frac{d}{dx} \left(\frac{f'(x)}{f''(x)} \right) = 1 - a$ 和 $\lim_{x \rightarrow +\infty} \frac{d}{dx} \left(\frac{f(x)}{f'(x)} \right)$, 观察到它们是微分形式, 所以不妨反方向利用洛必达法则. 须知, 洛必达法则在 $\frac{*}{\infty}$ 的情形中是适用的, 即若 $f \rightarrow \infty, f, g$ 在 a 的某个去心邻域内可导, 且 $\lim_{x \rightarrow a} \frac{g'}{f'}$ 存在 (或为无穷大, 但分母不等于 0), 那么 $\lim_{x \rightarrow a} \frac{g}{f} = \lim_{x \rightarrow a} \frac{g'}{f'}$; 相关证明可以参阅数分教材 (如陈纪修的, 卓里奇的, 等等).

$$\begin{aligned} \lim_{x \rightarrow +\infty} \frac{\frac{f(x)}{f'(x)}}{x} &= \lim_{x \rightarrow +\infty} \frac{d}{dx} \left(\frac{f(x)}{f'(x)} \right) = \lim_{x \rightarrow +\infty} \frac{f(x)}{xf'(x)} \\ \lim_{x \rightarrow +\infty} \frac{\frac{f'(x)}{f''(x)}}{x} &= \lim_{x \rightarrow +\infty} \frac{d}{dx} \left(\frac{f'(x)}{f''(x)} \right) = \lim_{x \rightarrow +\infty} \frac{f'(x)}{xf''(x)} \end{aligned}$$

设 $\lim_{x \rightarrow +\infty} \frac{f(x)f''(x)}{[f'(x)]^2} = A$, $\lim_{x \rightarrow +\infty} \frac{d}{dx} \left(\frac{f(x)}{f'(x)} \right) = 1 - A$, $\lim_{x \rightarrow +\infty} \frac{d}{dx} \left(\frac{f'(x)}{f''(x)} \right) = \lim_{x \rightarrow +\infty} \frac{f'(x)}{xf''(x)} = 1 - a$

$$\lim_{x \rightarrow +\infty} \frac{xf''(x)}{f'(x)} \lim_{x \rightarrow +\infty} \frac{f(x)}{xf'(x)} = \lim_{x \rightarrow +\infty} \frac{f(x)f''(x)}{[f'(x)]^2}$$

由于 $\lim_{x \rightarrow +\infty} \frac{f'(x)}{xf''(x)} = 1 - a \Rightarrow \lim_{x \rightarrow +\infty} \frac{xf''(x)}{f'(x)} = \frac{1}{1-a}$ 即 $\frac{1-A}{1-a} = A$, 解得

$$A = \lim_{x \rightarrow +\infty} \frac{f(x)f''(x)}{[f'(x)]^2} = \frac{1}{2-a}$$

例题 5.0.9 求极限

$$\lim_{n \rightarrow +\infty} n^x \left[\left(1 + \frac{1}{n+1}\right)^{n+1} - \left(1 + \frac{1}{n}\right)^n \right]$$

解 5.0.9. 根据做差的形式不难想到拉格朗日中值定理, 但是如果直接凑分母 $1 = n+1 - n$, 那就会导致 $\xi \in (n, n+1)$ 的东西出现, 而没有很好的利用上

$$\left(1 + \frac{1}{n+1}\right)^{n+1} - \left(1 + \frac{1}{n}\right)^n \rightarrow 0, n \rightarrow +\infty$$

所以先化为 e 指数, 用取对数之后的差分形式作为分母 $e^f - e^g = e^\xi(f-g), \xi \in (f, g)$:

$$\begin{aligned} \lim_{n \rightarrow +\infty} n^x \left[\left(1 + \frac{1}{n+1}\right)^{n+1} - \left(1 + \frac{1}{n}\right)^n \right] &= \lim_{n \rightarrow +\infty} n^x \left[e^{(n+1)\ln(1+\frac{1}{n+1})} - e^{n\ln(1+\frac{1}{n})} \right] \\ &= \lim_{n \rightarrow +\infty} n^x \left[\frac{e^{(n+1)\ln(1+\frac{1}{n+1})} - e^{n\ln(1+\frac{1}{n})}}{(n+1)\ln(1+\frac{1}{n+1}) - n\ln(1+\frac{1}{n})} \right] \left[(n+1)\ln(1+\frac{1}{n+1}) - n\ln(1+\frac{1}{n}) \right] \\ &= \lim_{n \rightarrow +\infty} n^x e^\xi \left[(n+1)\ln(1+\frac{1}{n+1}) - n\ln(1+\frac{1}{n}) \right] \end{aligned}$$

其中 $e^\xi \in \left(\left(1 + \frac{1}{n}\right)^n, \left(1 + \frac{1}{n+1}\right)^{n+1} \right)$, 当 $n \rightarrow +\infty$ 时, $\xi \rightarrow 1$, 所以:

$$\begin{aligned} &= \lim_{n \rightarrow +\infty} n^x \cdot e \left[(n+1)\ln(1+\frac{1}{n+1}) - n\ln(1+\frac{1}{n}) \right] \\ &= \lim_{n \rightarrow +\infty} n^x \cdot e \left[\left(1 - \frac{1}{2(n+1)} + \frac{1}{3(n+1)^2} + o\left(\frac{1}{(n+1)^2}\right)\right) - \left(1 - \frac{1}{2n} + \frac{1}{3n^2} + o\left(\frac{1}{n^2}\right)\right) \right] \\ &= \lim_{n \rightarrow +\infty} n^x \cdot e \left[-\frac{1}{2(n+1)} + \frac{1}{2n} + \frac{1}{3(n+1)^2} - \frac{1}{3n^2} + o\left(\frac{1}{n^2}\right) \right] \\ &= \lim_{n \rightarrow +\infty} n^x \cdot e \left[\frac{1}{2n(n+1)} + \frac{1}{3} \cdot \frac{n^2 - (n+1)^2}{n^2(n+1)^2} + o\left(\frac{1}{n^2}\right) \right] \\ &= \lim_{n \rightarrow +\infty} n^x \cdot e \left[\frac{1}{2n(n+1)} - \frac{2n+1}{3n^2(n+1)^2} + o\left(\frac{1}{n^2}\right) \right] \\ &= \lim_{n \rightarrow +\infty} n^x \cdot e \left[\frac{1}{2n^2} + o\left(\frac{1}{n^2}\right) \right] \\ &= \frac{e}{2} \lim_{n \rightarrow +\infty} n^{x-2} \end{aligned}$$

因此, 极限为:

$$\lim_{n \rightarrow +\infty} n^x \left[\left(1 + \frac{1}{n+1}\right)^{n+1} - \left(1 + \frac{1}{n}\right)^n \right] = \begin{cases} 0 & \text{如果 } x < 2 \\ \frac{e}{2} & \text{如果 } x = 2 \\ +\infty \text{ 或不存在} & \text{如果 } x > 2 \end{cases}$$

例题 5.0.10

设 $f(x) = \left(\tan \frac{\pi x}{4} - 1\right) \left(\tan \frac{\pi x^2}{4} - 1\right) \left(\tan \frac{\pi x^3}{4} - 1\right) \cdots \left(\tan \frac{\pi x^n}{4} - 1\right)$, 求 $f^{(n)}(x)$

解 5.0.10. 等价于给定函数 $f(x) = \prod_{k=1}^n \left(\tan \frac{\pi x^k}{4} - 1\right)$, 求 $f^{(n)}(1)$ 。由于当 $x = 1$ 时, 每个因子 $\tan \frac{\pi x^k}{4} - 1 = 0$, 因此 $x = 1$ 是 $f(x)$ 的 n 重零点。由泰勒公式, 在 $x = 1$ 处展开:

$$f(x) = \frac{f^{(n)}(1)}{n!} (x-1)^n + O((x-1)^{n+1}).$$

计算 $h(1) = \lim_{x \rightarrow 1} \frac{f(x)}{(x-1)^n}$ 。由于 $f(x)$ 是乘积形式, 我们将每个因子除以 $x-1$ 有

$$\frac{f(x)}{(x-1)^n} = \prod_{k=1}^n \frac{\tan \frac{\pi x^k}{4} - 1}{x-1}.$$

取极限 $x \rightarrow 1$, 每个因子的极限为导数:

$$\lim_{x \rightarrow 1} \frac{\tan \frac{\pi x^k}{4} - 1}{x-1} = \frac{d}{dx} \left(\tan \frac{\pi x^k}{4} \right) \Big|_{x=1} = \frac{\pi}{4} \cdot k x^{k-1} \sec^2 \frac{\pi x^k}{4} \Big|_{x=1} = \frac{\pi}{4} \cdot k \cdot 1 \cdot 2 = \frac{\pi k}{2}.$$

所以

$$h(1) = \prod_{k=1}^n \frac{\pi k}{2} = \frac{\pi^n}{2^n} \prod_{k=1}^n k = \frac{\pi^n}{2^n} n! \quad f^{(n)}(1) = n! \cdot h(1) = \left(\frac{\pi}{2}\right)^n (n!)^2.$$

当然本题目还可以先代入泰勒展开, 再相乘, 结果是一样的, 对于第 k 个因子 $\tan \left(\frac{\pi x^k}{4}\right) - 1$ 直接泰勒展开得:

$$\tan \left(\frac{\pi x^k}{4}\right) - 1 = \frac{\pi k}{2} (x-1) + O((x-1)^2)$$

将各因子相乘, 注意每个因子都包含 $(x-1)$ 的一阶项:

$$f(x) = \prod_{k=1}^n \left(\tan \left(\frac{\pi x^k}{4}\right) - 1 \right) = \prod_{k=1}^n \left[\frac{\pi k}{2} (x-1) + O((x-1)^2) \right] = \frac{\pi^n}{2^n} \prod_{k=1}^n [k(x-1) + O((x-1)^2)]$$

实际上我们只关心 $(x-1)^n$ 项的系数 (因为高阶项对 $f^{(n)}(1)$ 无贡献), 可得:

$$f(x) = \left(\frac{\pi}{2}\right)^n \left(\prod_{k=1}^n k \right) (x-1)^n + O((x-1)^{n+1}) = \left(\frac{\pi}{2}\right)^n n! (x-1)^n + O((x-1)^{n+1})$$

由泰勒公式:

$$f(x) = \frac{f^{(n)}(1)}{n!} (x-1)^n + O((x-1)^{n+1})$$

比较 $(x-1)^n$ 的系数:

$$\frac{f^{(n)}(1)}{n!} = \left(\frac{\pi}{2}\right)^n n! \quad \Rightarrow \quad f^{(n)}(1) = \left(\frac{\pi}{2}\right)^n (n!)^2$$

例题 5.0.11

$f(x) = (x-1)(\sqrt{x}-1)(\sqrt[3]{x}-1)\cdots(\sqrt[n]{x}-1)$, 求 $f^{(n)}(1)$ 。

解 5.0.11. 由于当 $x=1$ 时, 每个因子 $x^{1/k}-1=0$, 因此 $x=1$ 是 $f(x)$ 的 n 重零点。在 $x=1$ 处展开:

$$f(x) = \frac{f^{(n)}(1)}{n!}(x-1)^n + O((x-1)^{n+1}).$$

现在计算 $g(1) = \lim_{x \rightarrow 1} \frac{f(x)}{(x-1)^n}$ 。由于 $f(x)$ 是乘积, 我们有

$$\frac{f(x)}{(x-1)^n} = \prod_{k=1}^n \frac{x^{1/k} - 1}{x - 1}.$$

取极限 $x \rightarrow 1$, 每个因子的极限为导数:

$$\lim_{x \rightarrow 1} \frac{x^{1/k} - 1}{x - 1} = \frac{d}{dx} (x^{1/k}) \Big|_{x=1} = \frac{1}{k} \cdot x^{1/k-1} \Big|_{x=1} = \frac{1}{k}.$$

因此

$$g(1) = \prod_{k=1}^n \frac{1}{k} = \frac{1}{1 \cdot 2 \cdots n} = \frac{1}{n!} \quad f^{(n)}(1) = n! \cdot \frac{1}{n!} = 1.$$

例题 5.0.12

函数 $f(x)$ 在区间 $[0, 1]$ 上有连续导数, 且 $\int_0^1 f(x)dx = 0$, 证明对于任意的 $\xi \in (0, 1)$:

$$\left| \int_0^\xi f(x)dx \right| \leq \frac{1}{8} \max_{x \in [0, 1]} |f'(x)|$$

解 5.0.12. 由于 $f(x)$ 在 $[0, 1]$ 上有连续导数, 根据牛顿-莱布尼茨公式, 对于任意 $\xi \in (0, 1)$, 考虑积分

$$\int_0^\xi f(x)dx = \int_0^\xi \left(f(0) + \int_0^x f'(t)dt \right) dx = f(0)\xi + \int_0^\xi \int_0^x f'(t)dt dx.$$

交换积分次序,

$$\int_0^\xi \int_0^x f'(t)dt dx = \int_0^\xi \int_t^\xi f'(t)dx dt = \int_0^\xi (\xi - t)f'(t)dt.$$

所以

$$\int_0^\xi f(x)dx = f(0)\xi + \int_0^\xi (\xi - t)f'(t)dt.$$

代入 $\xi = 1$, $\int_0^1 f(x)dx = 0 \Rightarrow f(0) = -\int_0^1 (1 - t)f'(t)dt$, 所以代入并分区间讨论, 利用绝对值不等式, 有

$$\left| \int_0^\xi f(x)dx \right| = \begin{cases} \left| \int_0^\xi (\xi - 1)t f'(t)dt \right| \leq \max_{x \in [0, 1]} |f'(x)| \left| \int_0^\xi (\xi - 1)t dt \right|, & 0 \leq t \leq \xi \\ \left| \int_\xi^1 \xi(t - 1)f'(t)dt \right| \leq \max_{x \in [0, 1]} |f'(x)| \left| \int_\xi^1 \xi(t - 1)dt \right|, & \xi \leq t \leq 1 \end{cases}$$

去绝对值, 只需分别计算:

$$\int_0^\xi (1 - \xi)t dt = (1 - \xi) \cdot \frac{1}{2}\xi^2 = \frac{1}{2}(1 - \xi)\xi^2 \quad \int_\xi^1 \xi(1 - t)dt = \xi \cdot \frac{1}{2}(1 - \xi)^2 = \frac{1}{2}\xi(1 - \xi)^2.$$

利用 $\frac{\left| \int_0^\xi f(x)dx \right|}{\max_{x \in [0, 1]} |f'(x)|} \leq \begin{cases} \left| \int_0^\xi (\xi - 1)t dt \right|, & 0 \leq t \leq \xi \\ \left| \int_\xi^1 \xi(t - 1)dt \right|, & \xi \leq t \leq 1 \end{cases}$ 相加得

$$\frac{\left| \int_0^1 f(x)dx \right|}{\max_{x \in [0, 1]} |f'(x)|} \leq \frac{1}{2}(1 - \xi)\xi^2 + \frac{1}{2}\xi(1 - \xi)^2 = \frac{1}{2}\xi(1 - \xi)(\xi + (1 - \xi)) = \frac{1}{2}\xi(1 - \xi) \leq \frac{1}{8}.$$

这就完成了证明。

例题 5.0.13

$$a_1 = 5, a_{n+1} = \frac{(1+a_n)^3 - 5}{3}, \text{ 计算 } \sum_{n=1}^{\infty} \frac{a_n - 1}{a_n^2 + a_n + 1}$$

解 5.0.13. 关键在于不动点方程的多重根, 因为对于 $a_{n+1} = f(a_n)$, 若方程 $f(x) - x = 0$ 有二重根 (甚至多重根) r , 那么设 $g(x) = f(x) - x$, 则其可以被分解出一个因式 $(x-r)^2$, 此时不妨设

$$a_{n+1} - a_n = f(a_n) - a_n = (a_n - r)^2 h(a_n)$$

然后就是等式两边减去 r , 此时:

$$a_{n+1} - r = f(a_n) - r = (a_n - r)^2 h(a_n) + a_n - r = (a_n - r)(1 + (a_n - r)h(a_n))$$

在尝试裂项求和时, 常常考虑 $\frac{1}{a_n - r}$ 的差分形式:

$$\begin{aligned} \frac{1}{a_n - r} - \frac{1}{a_{n+1} - r} &= \frac{a_{n+1} - a_n}{(a_n - r)(a_{n+1} - r)} = \frac{(a_n - r)^2 h(a_n)}{(a_n - r)(a_n - r)(1 + (a_n - r)h(a_n))} \\ &= \frac{h(a_n)}{1 + (a_n - r)h(a_n)} \end{aligned}$$

此时分母的次数的确一定低于 $f(a_n)$ 的次数, 反观如果是一重零点, 就变成

$$\begin{aligned} \frac{1}{a_n - r} - \frac{1}{a_{n+1} - r} &= \frac{a_{n+1} - a_n}{(a_n - r)(a_{n+1} - r)} = \frac{(a_n - r)h(a_n)}{(a_n - r)^2(1 + h(a_n))} \\ &= \frac{h(a_n)}{(a_n - r)(1 + h(a_n))} \end{aligned}$$

此时分母次数一定等于 $f(a_n)$ 的次数, 不符合题干结构。所以关注多重根, 不妨通过做差来解:

$$\begin{aligned} a_{n+1} - a_n &= \frac{(1+a_n)^3 - 3a_n - 5}{3} = \frac{a_n^3 + 3a_n^2 - 4}{3} = \frac{(a_n + 2)^2(a_n - 1)}{3} \\ \Leftrightarrow \frac{a_{n+1} - a_n}{(a_n + 2)(a_{n+1} + 2)} &= \frac{(a_n + 2)^2(a_n - 1)}{3(a_n + 2)(a_{n+1} + 2)} = \frac{a_n + 2}{a_{n+1} + 2} \cdot \frac{a_n - 1}{3} \\ \Leftrightarrow \frac{1}{a_n + 2} - \frac{1}{a_{n+1} + 2} &= \frac{a_n + 2}{\frac{(1+a_n)^3 - 5}{3} + 2} \cdot \frac{a_n - 1}{3} = (a_n - 1) \frac{a_n + 2}{(1+a_n)^3 + 1} \\ &= \frac{(a_n + 2)(a_n - 1)}{(a_n + 1 + 1)((a_n + 1)^2 - (a_n + 1) + 1)} = \frac{a_n - 1}{a_n^2 + a_n + 1} \\ \Rightarrow \sum_{n=1}^{\infty} \frac{a_n - 1}{a_n^2 + a_n + 1} &= \sum_{n=1}^{\infty} \left(\frac{1}{a_n + 2} - \frac{1}{a_{n+1} + 2} \right) = \frac{1}{a_1 + 2} = \frac{1}{7} \end{aligned}$$

第一行找出来了 -2 是多重根, 之后直接除以 $(a_n + 2)(a_{n+1} + 2)$ 强行凑裂项, 之后就出现了 $\frac{a_n - r}{a_{n+1} - r}$ 结构, 直接代入递推公式就可以, 这就是通法了。

定理 5.0.5

设 $\lambda > 1, a_0 \neq 0, a_1 = \frac{\lambda + \lambda^{-1}}{2}a_0$, 递推关系 $a_{n+2} - (\lambda + \lambda^{-1})a_{n+1} + a_n = 0$, 则

$$a_n = \frac{a_0}{2}(\lambda^n + \lambda^{-n})$$

进一步的

$$\sum_{n=0}^{\infty} \frac{4^n}{a_{2^n}^2} = \frac{4}{a_0^2} \frac{1}{(\lambda - \lambda^{-1})^2}$$

注意到恒等式

$$\begin{aligned} \frac{1}{(a + a^{-1})^2} &= \frac{(a - a^{-1})^2}{(a^2 - a^{-2})^2} = \frac{(a + a^{-1})^2 - 4}{(a^2 - a^{-2})^2} \\ &= \frac{(a + a^{-1})^2}{(a^2 - a^{-2})^2} - \frac{4}{(a^2 - a^{-2})^2} \\ &= \frac{1}{(a - a^{-1})^2} - \frac{4}{(a^2 - a^{-2})^2} \end{aligned}$$

代入 $a = \lambda^{2^n}$ 得:

$$\begin{aligned} \frac{1}{(\lambda^{2^n} + \lambda^{-2^n})^2} &= \frac{1}{(\lambda^{2^n} - \lambda^{-2^n})^2} - \frac{4}{(\lambda^{2^{n+1}} - \lambda^{-2^{n+1}})^2} \\ \Leftrightarrow \frac{4^n}{(\lambda^{2^n} + \lambda^{-2^n})^2} &= \frac{4^n}{(\lambda^{2^n} - \lambda^{-2^n})^2} - \frac{4^{n+1}}{(\lambda^{2^{n+1}} - \lambda^{-2^{n+1}})^2} \\ \Leftrightarrow \sum_{n=0}^{\infty} \frac{4^n}{a_{2^n}^2} &= \frac{4}{a_0^2} \sum_{n=0}^{\infty} \left(\frac{4^n}{(\lambda^{2^n} - \lambda^{-2^n})^2} - \frac{4^{n+1}}{(\lambda^{2^{n+1}} - \lambda^{-2^{n+1}})^2} \right) = \frac{4}{a_0^2} \frac{1}{(\lambda - \lambda^{-1})^2} \end{aligned}$$

定理 5.0.6

设 a_0 为已知量, 数列 $\{a_n\}$ 满足通项公式 $a_n = C(\lambda^n - \lambda^{-n})$, 则

$$\sum_{n=0}^{\infty} \frac{1}{a_{2^n}} = \frac{1}{a_1} + \frac{1}{a_k} \left(\frac{1}{\lambda^k} - \frac{a_{1-k}}{a_1} \right)$$

注意到对任意整数 k , 成立恒等式:

$$\begin{aligned} \frac{\lambda^k - \lambda^{-k}}{\lambda^{2n} - \lambda^{-2n}} &= \frac{\lambda^n \lambda^{-n} \lambda^k - \lambda^n \lambda^{-n} \lambda^{-k}}{\lambda^{2n} - \lambda^{-2n}} \\ &= \frac{\lambda^n \lambda^{-n} \lambda^k - \lambda^n \lambda^{-n} \lambda^{-k} - (\lambda^{2n} \lambda^{-k} - \lambda^{-2n} \lambda^k) + (\lambda^{2n} \lambda^{-k} - \lambda^{-2n} \lambda^k)}{\lambda^{2n} - \lambda^{-2n}} \\ &= \frac{\lambda^n \lambda^{-n} \lambda^k + \lambda^{-2n} \lambda^k - \lambda^n \lambda^{-n} \lambda^{-k} - \lambda^{2n} \lambda^{-k} + (\lambda^{2n} \lambda^{-k} - \lambda^{-2n} \lambda^k)}{\lambda^{2n} - \lambda^{-2n}} \\ &= \frac{(\lambda^n + \lambda^{-n})(\lambda^{-n} \lambda^k) - \lambda^n \lambda^{-k} (\lambda^n + \lambda^{-n}) + (\lambda^{2n} \lambda^{-k} - \lambda^{-2n} \lambda^k)}{\lambda^{2n} - \lambda^{-2n}} \\ &= \frac{(\lambda^n + \lambda^{-n})(\lambda^{-n} \lambda^k - \lambda^n \lambda^{-k}) + (\lambda^{2n} \lambda^{-k} - \lambda^{-2n} \lambda^k)}{\lambda^{2n} - \lambda^{-2n}} \\ &= -\frac{\lambda^n \lambda^{-k} - \lambda^{-n} \lambda^k}{\lambda^n - \lambda^{-n}} + \frac{\lambda^{2n} \lambda^{-k} - \lambda^{-2n} \lambda^k}{\lambda^{2n} - \lambda^{-2n}} \\ &= \frac{\lambda^{2n-k} - \lambda^{-(2n-k)}}{\lambda^{2n} - \lambda^{-2n}} - \frac{\lambda^{n-k} - \lambda^{-(n-k)}}{\lambda^n - \lambda^{-n}} \end{aligned}$$

赋值 $n = 2^{n-1}$, 保留 k , 那么

$$\frac{\lambda^k - \lambda^{-k}}{\lambda^{2^n} - \lambda^{-2^n}} = \frac{\lambda^{2^{n-1}-k} - \lambda^{-(2^{n-1}-k)}}{\lambda^{2^n} - \lambda^{-2^n}} - \frac{\lambda^{2^{n-1}-k} - \lambda^{-(2^{n-1}-k)}}{\lambda^{2^{n-1}} - \lambda^{-2^{n-1}}}$$

所以当数列的通项是 $a_n = C(\lambda^n - \lambda^{-n})$ 时, 即:

$$\begin{aligned} \frac{a_k}{a_{2^n}} &= \frac{a_{2^n-k}}{a_{2^n}} - \frac{a_{2^{n-1}-k}}{a_{2^{n-1}}} \Leftrightarrow \frac{1}{a_{2^n}} = \frac{1}{a_k} \left(\frac{a_{2^n-k}}{a_{2^n}} - \frac{a_{2^{n-1}-k}}{a_{2^{n-1}}} \right) \\ &\Rightarrow \sum_{n=0}^{\infty} \frac{1}{a_{2^n}} = \frac{1}{a_1} + \frac{1}{a_k} \sum_{n=1}^{\infty} \left(\frac{a_{2^n-k}}{a_{2^n}} - \frac{a_{2^{n-1}-k}}{a_{2^{n-1}}} \right) \\ &\Leftrightarrow \sum_{n=0}^{\infty} \frac{1}{a_{2^n}} = \frac{1}{a_1} + \frac{1}{a_k} \left(\frac{1}{\lambda^k} - \frac{a_{1-k}}{a_1} \right) \end{aligned}$$

此时需将 a_n 的定义域延拓至整数集合, 相应地, 若 $a_n = C(\lambda^n - (-\lambda^{-1})^n)$, 则只需取 k 为偶数 $2k$, 这样就回到了本题的情形:

定理 5.0.7

设 a_0 为已知量, 数列 $\{a_n\}$ 满足通项公式 $a_n = C(\lambda^n - (-\lambda^{-1})^n)$, 则

$$\sum_{n=0}^{\infty} \frac{1}{a_{2^n}} = \frac{1}{a_1} + \frac{1}{a_{2k}} \left(\frac{1}{\lambda^{2k}} - \frac{a_{1-2k}}{a_1} \right)$$

例题 5.0.14

$a_1 = a_2 = 1, a_{n+2} = a_{n+1} + a_n$, 求

例题 5.0.15

$f(x)$ 在区间 $[0, 1]$ 上连续, 证明存在 $\xi \in (0, 1)$ 使得

$$\int_{\xi}^1 f(x) dx = \xi f(\xi)$$

解 5.0.14. 设 $f(x)$ 的原函数是 $F(x)$, 则要证明的结论等价于:

$$\begin{aligned} \int_{\xi}^1 f(x) dx = \xi F(\xi) &\Leftrightarrow F(1) - F(\xi) = \xi F'(\xi) \\ &\Leftrightarrow \xi F'(\xi) + F(\xi) = F(1) \\ &\Leftrightarrow (\xi F(\xi))' = F(1) \end{aligned}$$

所以辅助函数就是 $g(x) = xF(x)$, 它在 $[0, 1]$ 内也连续, 要证明存在 $\xi \in (0, 1)$ 使得 $g'(\xi) = g(1)$, 即:

$$g'(\xi) = \frac{g(1) - 0}{1 - 0} = \frac{g(1) - g(0)}{1 - 0}$$

这正是拉格朗日中值定理的应用。

例题 5.0.16

设 $f(x)$ 在区间 $[0, 1]$ 上可导, 且 $f(1) = 0$, $\int_0^1 x f'(x) dx = 1$, 证明: 至少存在 $\xi \in (0, 1)$ 使得 $f'(\xi) = 2$ 。

解 5.0.15.

$$1 = \int_0^1 x f'(x) dx = \int_0^1 x df(x) = x f(x) \Big|_0^1 - \int_0^1 f(x) dx \Rightarrow \int_0^1 f(x) dx = -1$$

所以设 $f(x)$ 的原函数为 $F(x)$, 则有 $F(1) - F(0) = -1$, 使用泰勒公式, 存在 $\xi \in (0, 1)$ 使得:

$$F(0) = F(1) + F'(1)(0 - 1) + \frac{1}{2} F''(\xi) \Rightarrow F''(\xi) = f'(\xi) = 2$$

例题 5.0.17

$$\int_0^1 \frac{x^2 - 1}{\ln x} dx$$

解 5.0.16.

$$\begin{aligned}\int_0^1 \frac{x^2 - 1}{\ln x} dx &= \int_0^1 \frac{x^2 - x^0}{\ln x} dx = \int_0^1 \int_0^2 x^y dy dx \\&= \int_0^2 \int_0^1 x^y dx dy = \int_0^2 \frac{dy}{y+1} = \ln |y+1| \Big|_0^2 \\&= \ln 3\end{aligned}$$

例题 5.0.18

$$\int_0^{\frac{\pi}{2}} \frac{x \sin x}{1 + \cos^2 x} dx$$

解 5.0.17.

$$\begin{aligned} \int_0^{\frac{\pi}{2}} \frac{x \sin x}{1 + \cos^2 x} dx &= - \int_0^{\frac{\pi}{2}} \frac{x d \cos x}{1 + \cos^2 x} = \int_0^{\frac{\pi}{2}} x d \arctan(\cos x) \\ &= -x \arctan \cos x \Big|_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \arctan \cos x dx \\ &= \int_0^{\frac{\pi}{2}} \arctan \cos x dx = \int_0^{\frac{\pi}{2}} \arctan \sin x dx \end{aligned}$$

令 $I(a) = \int_0^{\frac{\pi}{2}} \arctan a \sin x dx$, 则

$$\begin{aligned} I'(a) &= \frac{d}{da} \left(\int_0^{\frac{\pi}{2}} \arctan a \sin x dx \right) = \int_0^{\frac{\pi}{2}} \left(\frac{d}{da} \arctan a \sin x \right) dx \\ &= \int_0^{\frac{\pi}{2}} \frac{\sin x}{1 + a^2 \sin^2 x} dx = \int_0^{\frac{\pi}{2}} \frac{\sin x}{(a^2 + 1) - a^2 \cos^2 x} dx \\ &= - \int_0^{\frac{\pi}{2}} \frac{d \cos x}{(a^2 + 1) - a^2 \cos^2 x} = \int_0^1 \frac{dt}{(a^2 + 1) - a^2 t^2} = \frac{1}{a^2} \int_0^1 \frac{dt}{\frac{a^2+1}{a^2} - t^2} \\ &= \frac{1}{2a\sqrt{a^2+1}} \ln \left| \frac{\frac{\sqrt{a^2+1}}{a} + t}{\frac{\sqrt{a^2+1}}{a} - t} \right| \Big|_0^1 = \frac{\ln(\sqrt{a^2+1} + a)}{a\sqrt{a^2+1}} \end{aligned}$$

于是

$$\begin{aligned} I(1) &= \int_0^1 I'(a) da \\ &= \int_0^1 \frac{\ln(\sqrt{a^2+1} + a)}{a\sqrt{a^2+1}} da \\ &= -\ln(\sqrt{1+a^2} + a) \ln \left(\frac{\sqrt{1+a^2} + 1}{a} \right) \Big|_0^1 + \int_0^1 \frac{\ln \left(\frac{\sqrt{1+a^2} + 1}{a} \right)}{\sqrt{1+a^2}} da \\ &= -\ln^2(1 + \sqrt{2}) + \int_0^1 \frac{\ln \left(\frac{\sqrt{1+a^2} + 1}{a} \right)}{\sqrt{1+a^2}} da \\ &= -\ln^2(1 + \sqrt{2}) + \int_1^\infty \frac{\ln(\sqrt{a^2+1} + a)}{a\sqrt{1+a^2}} da \\ &= -\ln^2(1 + \sqrt{2}) + I(\infty) - I(1) \end{aligned}$$

于是

$$I(1) = -\frac{1}{2} \ln^2(1 + \sqrt{2}) + \frac{I(\infty)}{2} = -\frac{1}{2} \ln^2(1 + \sqrt{2}) + \frac{\pi^2}{8}$$

例题 5.0.19

$$\int_0^{\frac{\pi}{2}} \frac{x \sin x}{1 + \cos^2 x} dx$$

解 5.0.18.

$$\begin{aligned}
 \int_0^{\frac{\pi}{2}} \frac{x \sin x}{1 + \cos^2 x} dx &= - \int_0^{\frac{\pi}{2}} \frac{x d \cos x}{1 + \cos^2 x} = \int_0^{\frac{\pi}{2}} x d \arctan(\cos x) \\
 &= -x \arctan \cos x \Big|_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \arctan \cos x dx = \int_0^{\frac{\pi}{2}} \arctan \cos x dx \\
 &= \int_0^{\frac{\pi}{2}} \int_0^1 \frac{\cos x}{1 + y^2 \cos^2 x} dy dx = \int_0^1 \int_0^{\frac{\pi}{2}} \frac{\cos x}{1 + y^2 \cos^2 x} dx dy \\
 &= \int_0^1 \int_0^1 \frac{d \sin x}{1 + y^2 (1 - \sin^2 x)} dy = \int_0^1 \int_0^1 \frac{d \sin x}{(\sqrt{1 + y^2})^2 - (y \sin x)^2} dy \\
 &= \frac{1}{2y\sqrt{1 + y^2}} \int_0^1 \int_0^1 \left(\frac{1}{\sqrt{1 + y^2} + y \sin x} + \frac{1}{\sqrt{1 + y^2} - y \sin x} \right) d \sin x dy \\
 &= \int_0^1 \frac{\ln(y + \sqrt{1 + y^2})}{y\sqrt{1 + y^2}} dy \stackrel{y = \frac{t^2 - 1}{2t} \text{ (万能代换)}}{=} 2 \int_1^{1 + \sqrt{2}} \frac{\ln t}{t^2 - 1} dt \\
 2 \int_1^{1 + \sqrt{2}} \frac{\ln t}{t^2 - 1} dt &= \int_1^{1 + \sqrt{2}} \ln t \left(\frac{1}{t - 1} - \frac{1}{t + 1} \right) dt = \int_1^{1 + \sqrt{2}} \ln t d \ln \frac{t - 1}{t + 1} \\
 &= -\ln^2(1 + \sqrt{2}) - \int_1^{1 + \sqrt{2}} \frac{1}{t} \ln \frac{t - 1}{t + 1} dt \quad \text{令 } \frac{t - 1}{t + 1} = m \\
 &= -\ln^2(1 + \sqrt{2}) - \int_0^{\sqrt{2} - 1} \frac{1 - m}{1 + m} \ln \frac{\frac{1 + m}{1 - m} - 1}{\frac{1 + m}{1 - m} + 1} d \frac{1 + m}{1 - m} \\
 &= -\ln^2(1 + \sqrt{2}) - 2 \int_0^{\sqrt{2} - 1} \frac{\ln m}{1 - m^2} dm = -\ln^2(1 + \sqrt{2}) - 2 \int_{+\infty}^{\sqrt{2} + 1} \frac{\ln \frac{1}{u}}{1 - \frac{1}{u^2}} d \frac{1}{u} \\
 &= -\ln^2(1 + \sqrt{2}) - 2 \int_{+\infty}^{\sqrt{2} + 1} \frac{\ln u}{u^2 - 1} du = -\ln^2(1 + \sqrt{2}) + 2 \int_{\sqrt{2} + 1}^{+\infty} \frac{\ln u}{u^2 - 1} du \\
 &= -\ln^2(1 + \sqrt{2}) + 2 \int_1^{+\infty} \frac{\ln t}{t^2 - 1} dt - 2 \int_1^{1 + \sqrt{2}} \frac{\ln t}{t^2 - 1} dt \\
 2 \int_1^{1 + \sqrt{2}} \frac{\ln t}{t^2 - 1} dt &= -\frac{1}{2} \ln^2(1 + \sqrt{2}) + \int_1^{+\infty} \frac{\ln t}{t^2 - 1} dt \stackrel{t \leftarrow \frac{1}{t}}{=} -\frac{1}{2} \ln^2(1 + \sqrt{2}) - \int_0^1 \frac{\ln t}{1 - t^2} dt \\
 &= -\frac{1}{2} \ln^2(1 + \sqrt{2}) - \int_0^1 \sum_{n=0}^{\infty} t^{2n} \ln t dt = -\frac{1}{2} \ln^2(1 + \sqrt{2}) + \sum_{n=0}^{\infty} \frac{1}{(2n + 1)^2} \\
 &= \frac{\pi^2}{8} - \frac{1}{2} \ln^2(1 + \sqrt{2})
 \end{aligned}$$

第六章 积分表

6.1 含有 $ax + b$ 的积分

例题 6.1.1

$$\int \frac{\mathrm{d}x}{ax + b}$$

$$\int \frac{\mathrm{d}x}{ax + b} = \frac{1}{a} \int \frac{\mathrm{d}(ax + b)}{ax + b} = \frac{1}{a} \ln |ax + b| + C.$$

例题 6.1.2

$$\int (ax + b)^n \mathrm{d}x$$

$$\int (ax + b)^n \mathrm{d}x = \frac{1}{a} \int (ax + b)^n \mathrm{d}(ax + b) = \frac{1}{a} \cdot \frac{(ax + b)^{n+1}}{n+1} + C \quad (n \neq -1).$$

例题 6.1.3

$$\int \frac{x}{ax + b} \mathrm{d}x$$

$$\begin{aligned} \int \frac{x}{ax + b} \mathrm{d}x &= \int \frac{1}{a} \left(1 - \frac{b}{ax + b} \right) \mathrm{d}x \\ &= \frac{1}{a} \int \mathrm{d}x - \frac{b}{a} \int \frac{\mathrm{d}x}{ax + b} \\ &= \frac{x}{a} - \frac{b}{a^2} \ln |ax + b| + C. \end{aligned}$$

例题 6.1.4

$$\int \frac{x^2}{ax + b} \mathrm{d}x$$

$$\int \frac{x^2}{ax + b} \mathrm{d}x = \frac{1}{a} \int \frac{ax^2}{ax + b} \mathrm{d}x = \frac{1}{a} \int \frac{ax^2 + bx - bx}{ax + b} \mathrm{d}x$$

$$\begin{aligned}
 &= \frac{1}{a} \int \left(x - \frac{bx}{ax+b} \right) dx = \frac{1}{a} \int \left(x - \frac{bx + \frac{b^2}{a} - \frac{b^2}{a}}{ax+b} \right) dx \\
 &= \frac{1}{a} \int \left(x - \frac{b}{a} + \frac{b^2}{a} \frac{1}{ax+b} \right) dx \\
 &= \frac{1}{a} \int x dx - \frac{b}{a^2} \int dx + \frac{b^2}{a^2} \int \frac{dx}{ax+b} \\
 &= \frac{x^2}{2a} - \frac{bx}{a^2} + \frac{b^2}{a^3} \ln|ax+b| + C.
 \end{aligned}$$

例题 6.1.5

$$\int \frac{dx}{x(ax+b)}$$

$$\begin{aligned}
 \int \frac{dx}{x(ax+b)} &= \frac{1}{b} \int \left(\frac{1}{x} - \frac{a}{ax+b} \right) dx \\
 &= \frac{1}{b} (\ln|x| - \ln|ax+b|) + C = \frac{1}{b} \ln \left| \frac{x}{ax+b} \right| + C.
 \end{aligned}$$

例题 6.1.6

$$\int \frac{dx}{x^2(ax+b)}$$

$$\begin{aligned}
 \int \frac{dx}{x^2(ax+b)} &= \int \left(\frac{-\frac{a}{b^2}x + \frac{1}{b}}{x^2} + \frac{\frac{a^2}{b^2}}{ax+b} \right) dx \\
 &= \frac{1}{b} \int \left(\frac{a}{x} - \frac{a^2}{ax+b} - \frac{1}{x^2} \right) dx \\
 &= \frac{1}{b} \left(a \ln|x| - a \ln|ax+b| + \frac{1}{x} \right) + C \\
 &= \frac{a}{b} \ln \left| \frac{x}{ax+b} \right| + \frac{1}{bx} + C.
 \end{aligned}$$

例题 6.1.7

$$\int \frac{x dx}{(ax+b)^2}$$

$$\begin{aligned}
 \int \frac{x dx}{(ax+b)^2} &= \int \frac{x + \frac{b}{a} - \frac{b}{a}}{(ax+b)^2} dx \\
 &= \int \frac{1}{a} \left(\frac{1}{ax+b} - \frac{b}{(ax+b)^2} \right) dx
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{a} \int \frac{dx}{ax+b} - \frac{b}{a} \int \frac{dx}{(ax+b)^2} \\
 &= \frac{1}{a^2} \ln|ax+b| + \frac{b}{a^2} \cdot \frac{1}{ax+b} + C.
 \end{aligned}$$

例题 6.1.8

$$\int \frac{x^2 dx}{(ax+b)^2}$$

$$\begin{aligned}
 \int \frac{x^2 dx}{(ax+b)^2} &= \frac{1}{a^2} \int \frac{a^2 x^2 dx}{(ax+b)^2} = \frac{1}{a^2} \int \frac{a^2 x^2 + 2abx + b^2 - 2abx - b^2}{(ax+b)^2} dx \\
 &= \frac{1}{a^2} \int \left(1 - \frac{2abx + b^2}{(ax+b)^2} \right) dx = \frac{1}{a^2} \int \left(1 - \frac{2abx + 2b^2 - b^2}{(ax+b)^2} \right) dx \\
 &= \frac{1}{a^2} \int \left(1 - \frac{2b}{ax+b} + \frac{b^2}{(ax+b)^2} \right) dx \\
 &= \frac{x}{a^2} - \frac{2b}{a^3} \ln|ax+b| - \frac{b^2}{a^3} \cdot \frac{1}{ax+b} + C.
 \end{aligned}$$

例题 6.1.9

$$\int \frac{dx}{x(ax+b)^2}$$

$$\begin{aligned}
 \int \frac{dx}{x(ax+b)^2} &= \frac{1}{b^2} \int \frac{b^2}{x(ax+b)^2} dx = \frac{1}{b^2} \int \left(\frac{1}{x} - \frac{a^2 x + 2ab}{(ax+b)^2} \right) dx \\
 &= \frac{1}{b^2} \int \left(\frac{1}{x} - \frac{a^2 x + ab + ab}{(ax+b)^2} \right) dx = \frac{1}{b^2} \int \left(\frac{1}{x} - \frac{a}{ax+b} - \frac{ab}{(ax+b)^2} \right) dx \\
 &= \frac{1}{b^2} \left(\ln|x| - \ln|ax+b| + \frac{b}{ax+b} \right) + C \\
 &= \frac{1}{b^2} \ln \left| \frac{x}{ax+b} \right| + \frac{1}{b(ax+b)} + C.
 \end{aligned}$$

例题 6.1.10

$$\int \frac{dx}{(ax+b)^2}$$

$$\int \frac{dx}{(ax+b)^2} = \frac{1}{a} \int \frac{d(ax+b)}{(ax+b)^2} = -\frac{1}{a} \cdot \frac{1}{ax+b} + C.$$

例题 6.1.11

$$\int \frac{dx}{(ax+b)^3}$$

$$\int \frac{dx}{(ax+b)^3} = \frac{1}{a} \int \frac{d(ax+b)}{(ax+b)^3} = -\frac{1}{2a} \cdot \frac{1}{(ax+b)^2} + C.$$

6.2 含有 $\sqrt{ax+b}$ 的积分

例题 6.2.1

$$\int \sqrt{ax+b} dx$$

$$\int \sqrt{ax+b} dx = \frac{1}{a} \int (ax+b)^{1/2} d(ax+b) = \frac{2}{3a} (ax+b)^{3/2} + C.$$

例题 6.2.2

$$\int x\sqrt{ax+b} dx$$

$$\begin{aligned} \int x\sqrt{ax+b} dx &= \frac{1}{a^2} \int ax\sqrt{ax+b} d(ax+b) \\ &= \frac{1}{a^2} \int (ax+b-b)\sqrt{ax+b} d(ax+b) \\ &= \frac{1}{a^2} \int ((ax+b)^{3/2} - b(ax+b)^{1/2}) d(ax+b) \\ &= \frac{1}{a^2} \left(\frac{2}{5} (ax+b)^{5/2} - \frac{2b}{3} (ax+b)^{3/2} \right) + C \\ &= \frac{2(ax+b)^{3/2}(3ax-2b)}{15a^2} + C. \end{aligned}$$

例题 6.2.3

$$\int x^2\sqrt{ax+b} dx$$

$$\begin{aligned} \int x^2\sqrt{ax+b} dx &= \frac{1}{a} \int x^2\sqrt{ax+b} d(ax+b) \\ &= \int \left(\frac{ax+b-b}{a} \right)^2 \sqrt{ax+b} d(ax+b) \\ &= \frac{1}{a} \int \left(\frac{(ax+b)^2 - 2b(ax+b) + b^2}{a^2} \right) \sqrt{ax+b} d(ax+b) \\ &= \frac{1}{a^3} \int ((ax+b)^{5/2} - 2b(ax+b)^{3/2} + b^2(ax+b)^{1/2}) d(ax+b) \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{a^3} \left(\frac{2}{7}(ax+b)^{7/2} - \frac{4b}{5}(ax+b)^{5/2} + \frac{2b^2}{3}(ax+b)^{3/2} \right) + C \\
 &= \frac{2(ax+b)^{3/2}(15a^2x^2 - 12abx + 8b^2)}{105a^3} + C.
 \end{aligned}$$

例题 6.2.4

$$\int \frac{\sqrt{ax+b}}{x} dx$$

$$\begin{aligned}
 \int \frac{\sqrt{ax+b}}{x} dx &= \int \frac{ax+b}{x\sqrt{ax+b}} dx = \int \frac{adx}{\sqrt{ax+b}} + \int \frac{b dx}{x\sqrt{ax+b}} \\
 &= 2\sqrt{ax+b} + b \int \frac{dx}{x\sqrt{ax+b}} \\
 &= 2\sqrt{ax+b} + b \int \left(\frac{\sqrt{ax+b}}{x} - \frac{a}{\sqrt{ax+b}} \right) dx
 \end{aligned}$$

对于后一项积分, 需分情况讨论:

当 $b > 0$ 时, 令 $\sqrt{ax+b} = t$, 则 $x = \frac{t^2-b}{a}$, $dx = \frac{2t}{a} dt$, 有

$$\begin{aligned}
 \int \frac{dx}{x\sqrt{ax+b}} &= \int \frac{1}{\frac{t^2-b}{a}} \cdot \frac{2t}{a} \cdot \frac{1}{t} dt = 2 \int \frac{dt}{t^2-b} \\
 &= \begin{cases} \frac{1}{\sqrt{b}} \ln \left| \frac{t-\sqrt{b}}{t+\sqrt{b}} \right| + C, & b > 0, \\ -\frac{2}{\sqrt{-b}} \arctan \frac{t}{\sqrt{-b}} + C, & b < 0. \end{cases}
 \end{aligned}$$

因此原积分为

$$\int \frac{\sqrt{ax+b}}{x} dx = 2\sqrt{ax+b} + b \cdot \begin{cases} \frac{1}{\sqrt{b}} \ln \left| \frac{\sqrt{ax+b}-\sqrt{b}}{\sqrt{ax+b}+\sqrt{b}} \right| + C_1, & b > 0, \\ -\frac{2}{\sqrt{-b}} \arctan \frac{\sqrt{ax+b}}{\sqrt{-b}} + C_1, & b < 0. \end{cases}$$

若 $b = 0$, 则积分简化为 $\int \frac{\sqrt{a}}{\sqrt{x}} dx = 2\sqrt{a}\sqrt{x} + C$.

例题 6.2.5

$$\int \frac{\sqrt{ax+b}}{x^2} dx$$

$$\int \frac{\sqrt{ax+b}}{x^2} dx = -\frac{\sqrt{ax+b}}{x} + \frac{a}{2} \int \frac{dx}{x\sqrt{ax+b}} + C.$$

后一项积分同上讨论。

例题 6.2.6

$$\int \frac{dx}{\sqrt{ax+b}}$$

$$\int \frac{dx}{\sqrt{ax+b}} = \frac{1}{a} \int (ax+b)^{-1/2} d(ax+b) = \frac{2}{a} \sqrt{ax+b} + C.$$

例题 6.2.7

$$\int \frac{x dx}{\sqrt{ax+b}}$$

$$\begin{aligned} \int \frac{x dx}{\sqrt{ax+b}} &= \frac{1}{a} \int \frac{x}{\sqrt{ax+b}} d(ax+b) \\ &= \frac{1}{a} \int \frac{\frac{ax+b-b}{a}}{\sqrt{ax+b}} d(ax+b) \\ &= \frac{1}{a^2} \int \left(\sqrt{ax+b} - \frac{b}{\sqrt{ax+b}} \right) d(ax+b) \\ &= \frac{1}{a^2} \left(\frac{2}{3} (ax+b)^{3/2} - 2b \sqrt{ax+b} \right) + C \\ &= \frac{2(ax-2b)\sqrt{ax+b}}{3a^2} + C. \end{aligned}$$

例题 6.2.8

$$\int \frac{x^2 dx}{\sqrt{ax+b}}$$

$$\begin{aligned} \int \frac{x^2 dx}{\sqrt{ax+b}} &= \frac{1}{a} \int \frac{x^2}{\sqrt{ax+b}} d(ax+b) \\ &= \frac{1}{a} \int \frac{\frac{(ax+b)^2 - 2b(ax+b) + b^2}{a^2}}{\sqrt{ax+b}} d(ax+b) \\ &= \frac{1}{a^3} \int \left((ax+b)^{3/2} - 2b \sqrt{ax+b} + b^2 (ax+b)^{-1/2} \right) d(ax+b) \\ &= \frac{1}{a^3} \left(\frac{2}{5} (ax+b)^{5/2} - \frac{4b}{3} (ax+b)^{3/2} + 2b^2 \sqrt{ax+b} \right) + C \\ &= \frac{2(3a^2 x^2 - 4abx + 8b^2)\sqrt{ax+b}}{15a^3} + C. \end{aligned}$$

例题 6.2.9

$$\int \frac{dx}{x\sqrt{ax+b}}$$

$$\int \frac{dx}{x\sqrt{ax+b}} = \begin{cases} \frac{1}{\sqrt{b}} \ln \left| \frac{\sqrt{ax+b} - \sqrt{b}}{\sqrt{ax+b} + \sqrt{b}} \right| + C, & b > 0, \\ \frac{2}{\sqrt{-b}} \arctan \sqrt{\frac{ax+b}{-b}} + C, & b < 0. \end{cases}$$

注：当 $b > 0$ 时，令 $t = \sqrt{ax+b}$ ，则 $x = \frac{t^2-b}{a}$ ， $dx = \frac{2t}{a}dt$ ，代入得

$$\int \frac{dx}{x\sqrt{ax+b}} = \int \frac{1}{\frac{t^2-b}{a}} \cdot \frac{2t}{a} \cdot \frac{1}{t} dt = 2 \int \frac{dt}{t^2-b} = \frac{1}{\sqrt{b}} \ln \left| \frac{t-\sqrt{b}}{t+\sqrt{b}} \right| + C.$$

当 $b < 0$ 时，类似可得反正切形式。

例题 6.2.10

$$\int \frac{dx}{x^2\sqrt{ax+b}}$$

$$\int \frac{dx}{x^2\sqrt{ax+b}} = -\frac{\sqrt{ax+b}}{bx} - \frac{a}{2b} \int \frac{dx}{x\sqrt{ax+b}} + C.$$

其中后一项积分如上所述。

6.3 含有 $x^2 + a^2$ 的积分

例题 6.3.1

$$\int \frac{dx}{x^2 + a^2}$$

$$\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \arctan \frac{x}{a} + C \quad (a > 0).$$

例题 6.3.2

$$\int \frac{x dx}{x^2 + a^2}$$

$$\int \frac{x dx}{x^2 + a^2} = \frac{1}{2} \ln(x^2 + a^2) + C.$$

例题 6.3.3

$$\int \frac{dx}{x(x^2 + a^2)}$$

$$\begin{aligned}
 \int \frac{\mathrm{d}x}{x(x^2 + a^2)} &= \frac{1}{a^2} \int \left(\frac{1}{x} - \frac{x}{x^2 + a^2} \right) \mathrm{d}x \\
 &= \frac{1}{a^2} \left(\ln|x| - \frac{1}{2} \ln(x^2 + a^2) \right) + C \\
 &= \frac{1}{a^2} \ln \left| \frac{x}{\sqrt{x^2 + a^2}} \right| + C.
 \end{aligned}$$

例题 6.3.4

$$\int \frac{\mathrm{d}x}{x^2(x^2 + a^2)}$$

$$\begin{aligned}
 \int \frac{\mathrm{d}x}{x^2(x^2 + a^2)} &= \frac{1}{a^2} \int \left(\frac{1}{x^2} - \frac{1}{x^2 + a^2} \right) \mathrm{d}x \\
 &= \frac{1}{a^2} \left(-\frac{1}{x} - \frac{1}{a} \arctan \frac{x}{a} \right) + C \\
 &= -\frac{1}{a^2 x} - \frac{1}{a^3} \arctan \frac{x}{a} + C.
 \end{aligned}$$

例题 6.3.5

$$\int \frac{\mathrm{d}x}{x^2 - a^2}$$

$$\int \frac{\mathrm{d}x}{x^2 - a^2} = \frac{1}{2a} \ln \left| \frac{x-a}{x+a} \right| + C \quad (a > 0, |x| \neq a).$$

例题 6.3.6

$$\int \frac{x \mathrm{d}x}{x^2 - a^2}$$

$$\int \frac{x \mathrm{d}x}{x^2 - a^2} = \frac{1}{2} \ln|x^2 - a^2| + C.$$

例题 6.3.7

$$\int \frac{\mathrm{d}x}{x(x^2 - a^2)}$$

$$\int \frac{\mathrm{d}x}{x(x^2 - a^2)} = \frac{1}{a^2} \int \left(\frac{1}{x} - \frac{x}{x^2 - a^2} \right) \mathrm{d}x$$

$$\begin{aligned}
 &= \frac{1}{a^2} \left(\ln|x| - \frac{1}{2} \ln|x^2 - a^2| \right) + C \\
 &= \frac{1}{a^2} \ln \left| \frac{x}{\sqrt{|x^2 - a^2|}} \right| + C.
 \end{aligned}$$

例题 6.3.8

$$\int \frac{dx}{x^2(x^2 - a^2)}$$

$$\begin{aligned}
 \int \frac{dx}{x^2(x^2 - a^2)} &= \frac{1}{a^2} \int \left(-\frac{1}{x^2} - \frac{1}{x^2 - a^2} \right) dx \\
 &= \frac{1}{a^2} \left(\frac{1}{x} - \frac{1}{2a} \ln \left| \frac{x-a}{x+a} \right| \right) + C \\
 &= \frac{1}{a^2 x} - \frac{1}{2a^3} \ln \left| \frac{x-a}{x+a} \right| + C.
 \end{aligned}$$

例题 6.3.9

$$\int \frac{dx}{x^3(x^2 - a^2)}$$

$$\begin{aligned}
 \int \frac{dx}{x^3(x^2 - a^2)} &= \frac{1}{a^2} \int \left(-\frac{1}{x^3} - \frac{1}{a^2 x} + \frac{x}{a^2(x^2 - a^2)} \right) dx \\
 &= \frac{1}{a^2} \left(\frac{1}{2x^2} - \frac{1}{a^2} \ln|x| + \frac{1}{2a^2} \ln|x^2 - a^2| \right) + C \\
 &= \frac{1}{2a^2 x^2} - \frac{1}{a^4} \ln|x| + \frac{1}{2a^4} \ln|x^2 - a^2| + C.
 \end{aligned}$$

例题 6.3.10

$$\int \frac{dx}{(x^2 + a^2)^n}$$

$$I_n = \int \frac{dx}{(x^2 + a^2)^n} \quad (n \in \mathbb{N}^*).$$

利用递推公式:

$$I_n = \frac{x}{2a^2(n-1)(x^2 + a^2)^{n-1}} + \frac{2n-3}{2a^2(n-1)} I_{n-1}, \quad n \geq 2,$$

其中

$$I_1 = \int \frac{dx}{x^2 + a^2} = \frac{1}{a} \arctan \frac{x}{a} + C.$$

例题 6.3.11

$$\int \frac{dx}{(x^2 - a^2)^n}$$

$$J_n = \int \frac{dx}{(x^2 - a^2)^n} \quad (n \in \mathbb{N}^*).$$

利用递推公式:

$$J_n = -\frac{x}{2a^2(n-1)(x^2 - a^2)^{n-1}} + \frac{2n-3}{2a^2(n-1)}J_{n-1}, \quad n \geq 2,$$

其中

$$J_1 = \int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \ln \left| \frac{x-a}{x+a} \right| + C.$$

6.4 含有 $ax^2 + bx + c$ 的积分

例题 6.4.1

$$\int \frac{dx}{ax^2 + bx + c}$$

$$\begin{aligned} \int \frac{dx}{ax^2 + bx + c} &= \frac{1}{a} \int \frac{dx}{x^2 + \frac{b}{a}x + \frac{c}{a}} \\ &= \frac{1}{a} \int \frac{dx}{\left(x + \frac{b}{2a}\right)^2 + \frac{4ac-b^2}{4a^2}}. \end{aligned}$$

令 $t = x + \frac{b}{2a}$, $\Delta = b^2 - 4ac$, 则

$$\int \frac{dx}{ax^2 + bx + c} = \frac{1}{a} \int \frac{dt}{t^2 + \frac{4ac-b^2}{4a^2}}.$$

分情况讨论:

1. 若 $\Delta < 0$, 则 $4ac - b^2 > 0$, 记 $k^2 = \frac{4ac-b^2}{4a^2}$,

$$\int \frac{dx}{ax^2 + bx + c} = \frac{1}{a} \cdot \frac{1}{k} \arctan \frac{t}{k} + C = \frac{2}{\sqrt{4ac-b^2}} \arctan \frac{2ax+b}{\sqrt{4ac-b^2}} + C.$$

2. 若 $\Delta = 0$, 则

$$\int \frac{dx}{ax^2 + bx + c} = \frac{1}{a} \int \frac{dt}{t^2} = -\frac{1}{at} + C = -\frac{2}{2ax+b} + C.$$

3. 若 $\Delta > 0$, 则分母可分解为两个不同实根 x_1, x_2 ,

$$\int \frac{dx}{ax^2 + bx + c} = \frac{1}{a(x_1 - x_2)} \ln \left| \frac{x - x_1}{x - x_2} \right| + C = \frac{1}{\sqrt{\Delta}} \ln \left| \frac{2ax + b - \sqrt{\Delta}}{2ax + b + \sqrt{\Delta}} \right| + C.$$

例题 6.4.2

$$\int \frac{x dx}{ax^2 + bx + c}$$

$$\begin{aligned} \int \frac{x dx}{ax^2 + bx + c} &= \frac{1}{2a} \int \frac{2ax + b - b}{ax^2 + bx + c} dx \\ &= \frac{1}{2a} \int \frac{2ax + b}{ax^2 + bx + c} dx - \frac{b}{2a} \int \frac{dx}{ax^2 + bx + c}. \end{aligned}$$

其中

$$\int \frac{2ax + b}{ax^2 + bx + c} dx = \ln |ax^2 + bx + c| + C_1,$$

而 $\int \frac{dx}{ax^2 + bx + c}$ 的结果如上所述。因此

$$\int \frac{x dx}{ax^2 + bx + c} = \frac{1}{2a} \ln |ax^2 + bx + c| - \frac{b}{2a} \int \frac{dx}{ax^2 + bx + c} + C.$$

具体形式根据判别式代入即可。

6.5 含有 $\sqrt{x^2 + a^2}$ 的积分

例题 6.5.1

$$\int \frac{dx}{\sqrt{x^2 + a^2}}$$

$$\begin{aligned} \int \frac{dx}{\sqrt{x^2 + a^2}} &\stackrel{x=a \tan t}{=} \int \frac{a \sec^2 t}{\sqrt{a^2 \tan^2 t + a^2}} dt = \int \frac{a \sec^2 t}{a \sec t} dt \\ &= \int \sec t dt = \ln |\sec t + \tan t| + C \\ &\stackrel{t=\arctan(x/a)}{=} \ln \left| \frac{\sqrt{x^2 + a^2}}{a} + \frac{x}{a} \right| + C \\ &= \ln |x + \sqrt{x^2 + a^2}| + C' \quad (C' = C - \ln a). \end{aligned}$$

例题 6.5.2

$$\int \frac{x dx}{\sqrt{x^2 + a^2}}$$

$$\int \frac{x \, dx}{\sqrt{x^2 + a^2}} \stackrel{u=x^2+a^2}{=} \frac{1}{2} \int \frac{du}{\sqrt{u}} = \sqrt{u} + C = \sqrt{x^2 + a^2} + C.$$

例题 6.5.3

$$\int \frac{x^2 dx}{\sqrt{x^2 + a^2}}$$

$$\begin{aligned} \int \frac{x^2 dx}{\sqrt{x^2 + a^2}} &\stackrel{x=a \tan t}{=} \int \frac{a^2 \tan^2 t \cdot a \sec^2 t}{a \sec t} dt = a^2 \int \tan^2 t \sec t \, dt \\ &= a^2 \int (\sec^2 t - 1) \sec t \, dt = a^2 \left(\int \sec^3 t \, dt - \int \sec t \, dt \right) \\ &= a^2 \left(\frac{1}{2} \sec t \tan t + \frac{1}{2} \ln |\sec t + \tan t| - \ln |\sec t + \tan t| \right) + C \\ &= \frac{a^2}{2} (\sec t \tan t - \ln |\sec t + \tan t|) + C \\ &\stackrel{t=\arctan(x/a)}{=} \frac{a^2}{2} \left(\frac{\sqrt{x^2 + a^2}}{a} \cdot \frac{x}{a} - \ln \left| \frac{\sqrt{x^2 + a^2}}{a} + \frac{x}{a} \right| \right) + C \\ &= \frac{x}{2} \sqrt{x^2 + a^2} - \frac{a^2}{2} \ln |x + \sqrt{x^2 + a^2}| + C'. \end{aligned}$$

例题 6.5.4

$$\int \frac{dx}{x\sqrt{x^2 + a^2}}$$

$$\begin{aligned} \int \frac{dx}{x\sqrt{x^2 + a^2}} &\stackrel{x=a \tan t}{=} \int \frac{a \sec^2 t}{a \tan t \cdot a \sec t} dt = \frac{1}{a} \int \frac{\sec t}{\tan t} dt \\ &= \frac{1}{a} \int \csc t \, dt = \frac{1}{a} \ln |\csc t - \cot t| + C \\ &\stackrel{t=\arctan(x/a)}{=} \frac{1}{a} \ln \left| \frac{\sqrt{x^2 + a^2}}{x} - \frac{a}{x} \right| + C \\ &= \frac{1}{a} \ln \left| \frac{\sqrt{x^2 + a^2} - a}{x} \right| + C. \end{aligned}$$

例题 6.5.5

$$\int \frac{dx}{x^2 \sqrt{x^2 + a^2}}$$

$$\int \frac{dx}{x^2 \sqrt{x^2 + a^2}} \stackrel{x=a \tan t}{=} \int \frac{a \sec^2 t}{a^2 \tan^2 t \cdot a \sec t} dt = \frac{1}{a^2} \int \frac{\sec t}{\tan^2 t} dt$$

$$\begin{aligned}
 &= \frac{1}{a^2} \int \frac{\cos t}{\sin^2 t} dt = \frac{1}{a^2} \int \csc t \cot t dt = -\frac{1}{a^2} \csc t + C \\
 &\stackrel{t=\arctan(x/a)}{=} -\frac{1}{a^2} \cdot \frac{\sqrt{x^2 + a^2}}{x} + C = -\frac{\sqrt{x^2 + a^2}}{a^2 x} + C.
 \end{aligned}$$

例题 6.5.6

$$\int \frac{dx}{\sqrt{(x^2 + a^2)^3}}$$

$$\begin{aligned}
 \int \frac{dx}{\sqrt{(x^2 + a^2)^3}} &\stackrel{x=a \tan t}{=} \int \frac{a \sec^2 t}{(a^3 \sec^3 t)} dt = \frac{1}{a^2} \int \cos t dt \\
 &= \frac{1}{a^2} \sin t + C \stackrel{t=\arctan(x/a)}{=} \frac{1}{a^2} \cdot \frac{x}{\sqrt{x^2 + a^2}} + C = \frac{x}{a^2 \sqrt{x^2 + a^2}} + C.
 \end{aligned}$$

例题 6.5.7

$$\int \frac{x dx}{\sqrt{(x^2 + a^2)^3}}$$

$$\begin{aligned}
 \int \frac{x dx}{\sqrt{(x^2 + a^2)^3}} &\stackrel{u=x^2+a^2}{=} \frac{1}{2} \int u^{-3/2} du = \frac{1}{2} \cdot (-2) u^{-1/2} + C = -\frac{1}{\sqrt{u}} + C \\
 &= -\frac{1}{\sqrt{x^2 + a^2}} + C.
 \end{aligned}$$

例题 6.5.8

$$\int \frac{x^2 dx}{\sqrt{(x^2 + a^2)^3}}$$

$$\begin{aligned}
 \int \frac{x^2 dx}{\sqrt{(x^2 + a^2)^3}} &\stackrel{x=a \tan t}{=} \int \frac{a^2 \tan^2 t \cdot a \sec^2 t}{(a^3 \sec^3 t)} dt = \frac{1}{a} \int \frac{\tan^2 t}{\sec t} dt \\
 &= \frac{1}{a} \int \sin t \tan t dt = \frac{1}{a} \int \frac{\sin^2 t}{\cos t} dt = \frac{1}{a} \int \frac{1 - \cos^2 t}{\cos t} dt \\
 &= \frac{1}{a} \int (\sec t - \cos t) dt = \frac{1}{a} (\ln |\sec t + \tan t| - \sin t) + C \\
 &\stackrel{t=\arctan(x/a)}{=} \frac{1}{a} \left(\ln \left| \frac{\sqrt{x^2 + a^2}}{a} + \frac{x}{a} \right| - \frac{x}{\sqrt{x^2 + a^2}} \right) + C \\
 &= \frac{1}{a} \ln |x + \sqrt{x^2 + a^2}| - \frac{x}{a \sqrt{x^2 + a^2}} + C'.
 \end{aligned}$$

例题 6.5.9

$$\int \sqrt{x^2 + a^2} \, dx$$

$$\begin{aligned} \int \sqrt{x^2 + a^2} \, dx &\stackrel{x=a \tan t}{=} \int a \sec t \cdot a \sec^2 t \, dt = a^2 \int \sec^3 t \, dt \\ &= \frac{a^2}{2} (\sec t \tan t + \ln |\sec t + \tan t|) + C \\ &\stackrel{t=\arctan(x/a)}{=} \frac{a^2}{2} \left(\frac{\sqrt{x^2 + a^2}}{a} \cdot \frac{x}{a} + \ln \left| \frac{\sqrt{x^2 + a^2}}{a} + \frac{x}{a} \right| \right) + C \\ &= \frac{x}{2} \sqrt{x^2 + a^2} + \frac{a^2}{2} \ln |x + \sqrt{x^2 + a^2}| + C'. \end{aligned}$$

例题 6.5.10

$$\int x \sqrt{x^2 + a^2} \, dx$$

$$\int x \sqrt{x^2 + a^2} \, dx \stackrel{u=x^2+a^2}{=} \frac{1}{2} \int \sqrt{u} \, du = \frac{1}{2} \cdot \frac{2}{3} u^{3/2} + C = \frac{1}{3} (x^2 + a^2)^{3/2} + C.$$

例题 6.5.11

$$\int x^2 \sqrt{x^2 + a^2} \, dx$$

$$\begin{aligned} \int x^2 \sqrt{x^2 + a^2} \, dx &\stackrel{x=a \tan t}{=} \int a^2 \tan^2 t \cdot a \sec t \cdot a \sec^2 t \, dt = a^4 \int \tan^2 t \sec^3 t \, dt \\ &= a^4 \int (\sec^2 t - 1) \sec^3 t \, dt = a^4 \left(\int \sec^5 t \, dt - \int \sec^3 t \, dt \right). \end{aligned}$$

利用递推公式或分部积分，最终可得：

$$\int x^2 \sqrt{x^2 + a^2} \, dx = \frac{x}{8} (2x^2 + a^2) \sqrt{x^2 + a^2} - \frac{a^4}{8} \ln |x + \sqrt{x^2 + a^2}| + C.$$

例题 6.5.12

$$\int \frac{\sqrt{x^2 + a^2}}{x} \, dx$$

$$\int \frac{\sqrt{x^2 + a^2}}{x} \, dx \stackrel{x=a \tan t}{=} \int \frac{a \sec t}{a \tan t} \cdot a \sec^2 t \, dt = a \int \frac{\sec^3 t}{\tan t} \, dt$$

$$\begin{aligned}
&= a \int \frac{1}{\cos^3 t} \cdot \frac{\cos t}{\sin t} dt = a \int \frac{1}{\cos^2 t \sin t} dt \\
&= a \int \frac{\sin^2 t + \cos^2 t}{\cos^2 t \sin t} dt = a \int \left(\frac{\sin t}{\cos^2 t} + \frac{\cos t}{\sin t} \right) dt \\
&= a \left(\int \frac{\sin t}{\cos^2 t} dt + \int \cot t dt \right) \\
&= a (\sec t + \ln |\sin t|) + C \\
&\stackrel{t=\arctan(x/a)}{=} a \left(\frac{\sqrt{x^2 + a^2}}{a} + \ln \left| \frac{x}{\sqrt{x^2 + a^2}} \right| \right) + C \\
&= \sqrt{x^2 + a^2} + a \ln \left| \frac{x}{\sqrt{x^2 + a^2}} \right| + C.
\end{aligned}$$

例题 6.5.13

$$\int \frac{\sqrt{x^2 + a^2}}{x^2} dx$$

$$\begin{aligned}
\int \frac{\sqrt{x^2 + a^2}}{x^2} dx &\stackrel{x=a \tan t}{=} \int \frac{a \sec t}{a^2 \tan^2 t} \cdot a \sec^2 t dt = \frac{1}{a} \int \frac{\sec^3 t}{\tan^2 t} dt \\
&= \frac{1}{a} \int \frac{1}{\cos^3 t} \cdot \frac{\cos^2 t}{\sin^2 t} dt = \frac{1}{a} \int \frac{1}{\cos t \sin^2 t} dt \\
&= \frac{1}{a} \int \frac{\sin^2 t + \cos^2 t}{\cos t \sin^2 t} dt = \frac{1}{a} \int \left(\frac{\sin t}{\cos t} + \frac{\cos t}{\sin^2 t} \right) dt \\
&= \frac{1}{a} \left(\int \tan t dt + \int \cot t \csc t dt \right) \\
&= \frac{1}{a} (-\ln |\cos t| - \csc t) + C \\
&\stackrel{t=\arctan(x/a)}{=} \frac{1}{a} \left(-\ln \left| \frac{a}{\sqrt{x^2 + a^2}} \right| - \frac{\sqrt{x^2 + a^2}}{x} \right) + C \\
&= -\frac{\sqrt{x^2 + a^2}}{ax} + \frac{1}{a} \ln \left| \frac{\sqrt{x^2 + a^2}}{a} \right| + C.
\end{aligned}$$

例题 6.5.14

$$\int \sqrt{(x^2 + a^2)^3} dx$$

$$\begin{aligned}
\int \sqrt{(x^2 + a^2)^3} dx &= \int (x^2 + a^2)^{3/2} dx \stackrel{x=a \tan t}{=} \int (a^2 \sec^2 t)^{3/2} \cdot a \sec^2 t dt \\
&= a^4 \int \sec^5 t dt.
\end{aligned}$$

利用递推公式:

$$\int \sec^n t \, dt = \frac{\sec^{n-2} t \tan t}{n-1} + \frac{n-2}{n-1} \int \sec^{n-2} t \, dt,$$

可得:

$$\begin{aligned} \int \sec^5 t \, dt &= \frac{\sec^3 t \tan t}{4} + \frac{3}{4} \int \sec^3 t \, dt \\ &= \frac{\sec^3 t \tan t}{4} + \frac{3}{4} \left(\frac{\sec t \tan t}{2} + \frac{1}{2} \ln |\sec t + \tan t| \right) + C. \end{aligned}$$

代回并整理得:

$$\int \sqrt{(x^2 + a^2)^3} \, dx = \frac{x}{8}(2x^2 + 5a^2)\sqrt{x^2 + a^2} + \frac{3a^4}{8} \ln |x + \sqrt{x^2 + a^2}| + C.$$

例题 6.5.15

$$\int \frac{\sqrt{(x^2 + a^2)^3}}{x} \, dx$$

$$\int \frac{\sqrt{(x^2 + a^2)^3}}{x} \, dx = \int \frac{(x^2 + a^2)^{3/2}}{x} \, dx \xrightarrow{u=x^2+a^2} \frac{1}{2} \int \frac{u^{3/2}}{u-a^2} \, du.$$

然后进行有理化代换或分部积分, 最终可得:

$$\int \frac{\sqrt{(x^2 + a^2)^3}}{x} \, dx = \frac{1}{3}(x^2 + a^2)^{3/2} + a^2 \sqrt{x^2 + a^2} - a^3 \ln \left| \frac{a + \sqrt{x^2 + a^2}}{x} \right| + C.$$

例题 6.5.16

$$\int \frac{\sqrt{(x^2 + a^2)^3}}{x^2} \, dx$$

$$\begin{aligned} \int \frac{\sqrt{(x^2 + a^2)^3}}{x^2} \, dx &\xrightarrow{x=a \tan t} \int \frac{a^3 \sec^3 t}{a^2 \tan^2 t} \cdot a \sec^2 t \, dt = a^2 \int \frac{\sec^5 t}{\tan^2 t} \, dt \\ &= a^2 \int \frac{\sec^3 t}{\sin^2 t} \, dt = a^2 \int \frac{1}{\cos^3 t \sin^2 t} \, dt. \end{aligned}$$

该积分较为复杂, 最终表达式为:

$$\int \frac{\sqrt{(x^2 + a^2)^3}}{x^2} \, dx = -\frac{\sqrt{(x^2 + a^2)^3}}{x} + \frac{3}{2}x\sqrt{x^2 + a^2} + \frac{3}{2}a^2 \ln |x + \sqrt{x^2 + a^2}| + C.$$

例题 6.5.17

$$\int x \sqrt{(x^2 + a^2)^3} \, dx$$

$$\int x \sqrt{(x^2 + a^2)^3} dx \stackrel{u=x^2+a^2}{=} \frac{1}{2} \int u^{3/2} du = \frac{1}{2} \cdot \frac{2}{5} u^{5/2} + C = \frac{1}{5} (x^2 + a^2)^{5/2} + C.$$

例题 6.5.18

$$\int x^2 \sqrt{(x^2 + a^2)^3} dx$$

$$\begin{aligned} \int x^2 \sqrt{(x^2 + a^2)^3} dx &= \int x^2 (x^2 + a^2)^{3/2} dx \stackrel{x=a \tan t}{=} \int a^2 \tan^2 t \cdot a^3 \sec^3 t \cdot a \sec^2 t dt \\ &= a^6 \int \tan^2 t \sec^5 t dt = a^6 \int (\sec^2 t - 1) \sec^5 t dt \\ &= a^6 \left(\int \sec^7 t dt - \int \sec^5 t dt \right). \end{aligned}$$

利用递推公式，最终可得：

$$\int x^2 \sqrt{(x^2 + a^2)^3} dx = \frac{x}{48} (8x^4 + 26a^2 x^2 + 33a^4) \sqrt{x^2 + a^2} + \frac{5a^6}{16} \ln |x + \sqrt{x^2 + a^2}| + C.$$

6.6 含有 $\sqrt{x^2 - a^2}$ 的积分

例题 6.6.1

$$\int \frac{dx}{\sqrt{x^2 - a^2}}$$

$$\begin{aligned} \int \frac{dx}{\sqrt{x^2 - a^2}} &\stackrel{x=a \sec t}{=} \int \frac{a \sec t \tan t dt}{\sqrt{a^2 \sec^2 t - a^2}} = \int \frac{a \sec t \tan t}{a \tan t} dt \\ &= \int \sec t dt = \ln |\sec t + \tan t| + C \\ &\stackrel{t=\arccos(a/x)}{=} \ln \left| \frac{x}{a} + \frac{\sqrt{x^2 - a^2}}{a} \right| + C \\ &= \ln |x + \sqrt{x^2 - a^2}| + C' \quad (C' = C - \ln a). \end{aligned}$$

注意：通常写作 $\int \frac{dx}{\sqrt{x^2 - a^2}} = \ln |x + \sqrt{x^2 - a^2}| + C.$

例题 6.6.2

$$\int \frac{x dx}{\sqrt{x^2 - a^2}}$$

$$\int \frac{x dx}{\sqrt{x^2 - a^2}} \stackrel{u=x^2-a^2}{=} \frac{1}{2} \int \frac{du}{\sqrt{u}} = \frac{1}{2} \cdot 2\sqrt{u} + C = \sqrt{x^2 - a^2} + C.$$

例题 6.6.3

$$\int \frac{x^2 dx}{\sqrt{x^2 - a^2}}$$

$$\begin{aligned} \int \frac{x^2 dx}{\sqrt{x^2 - a^2}} &\stackrel{x=a \sec t}{=} \int \frac{a^2 \sec^2 t \cdot a \sec t \tan t dt}{\sqrt{a^2 \sec^2 t - a^2}} = \int \frac{a^3 \sec^3 t \tan t}{a \tan t} dt \\ &= a^2 \int \sec^3 t dt = a^2 \left(\frac{1}{2} \sec t \tan t + \frac{1}{2} \ln |\sec t + \tan t| \right) + C \\ &\stackrel{t=\arccos(a/x)}{=} a^2 \left(\frac{1}{2} \cdot \frac{x}{a} \cdot \frac{\sqrt{x^2 - a^2}}{a} + \frac{1}{2} \ln \left| \frac{x}{a} + \frac{\sqrt{x^2 - a^2}}{a} \right| \right) + C \\ &= \frac{x}{2} \sqrt{x^2 - a^2} + \frac{a^2}{2} \ln |x + \sqrt{x^2 - a^2}| + C'. \end{aligned}$$

例题 6.6.4

$$\int \frac{dx}{x\sqrt{x^2 - a^2}}$$

$$\begin{aligned} \int \frac{dx}{x\sqrt{x^2 - a^2}} &\stackrel{x=a \sec t}{=} \int \frac{a \sec t \tan t dt}{a \sec t \cdot a \tan t} = \frac{1}{a} \int dt = \frac{t}{a} + C \\ &\stackrel{t=\arccos(a/x)}{=} \frac{1}{a} \arccos \frac{a}{x} + C. \end{aligned}$$

也可写作: $\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a} \sec^{-1} \left| \frac{x}{a} \right| + C.$

例题 6.6.5

$$\int \frac{dx}{x^2 \sqrt{x^2 - a^2}}$$

$$\begin{aligned} \int \frac{dx}{x^2 \sqrt{x^2 - a^2}} &\stackrel{x=a \sec t}{=} \int \frac{a \sec t \tan t dt}{a^2 \sec^2 t \cdot a \tan t} = \frac{1}{a^2} \int \cos t dt = \frac{1}{a^2} \sin t + C \\ &\stackrel{t=\arccos(a/x)}{=} \frac{1}{a^2} \cdot \frac{\sqrt{x^2 - a^2}}{x} + C = \frac{\sqrt{x^2 - a^2}}{a^2 x} + C. \end{aligned}$$

例题 6.6.6

$$\int \frac{dx}{\sqrt{(x^2 - a^2)^3}}$$

$$\int \frac{dx}{\sqrt{(x^2 - a^2)^3}} \stackrel{x=a \sec t}{=} \int \frac{a \sec t \tan t dt}{(a^3 \tan^3 t)} = \frac{1}{a^2} \int \frac{\sec t}{\tan^2 t} dt = \frac{1}{a^2} \int \frac{\cos t}{\sin^2 t} dt$$

$$= \frac{1}{a^2} \int \csc t \cot t \, dt = -\frac{1}{a^2} \csc t + C$$

$$\stackrel{t=\arccos(a/x)}{=} -\frac{1}{a^2} \cdot \frac{x}{\sqrt{x^2 - a^2}} + C = -\frac{x}{a^2 \sqrt{x^2 - a^2}} + C.$$

例题 6.6.7

$$\int \frac{x \, dx}{\sqrt{(x^2 - a^2)^3}}$$

$$\int \frac{x \, dx}{\sqrt{(x^2 - a^2)^3}} \stackrel{u=x^2-a^2}{=} \frac{1}{2} \int u^{-3/2} \, du = \frac{1}{2} \cdot (-2)u^{-1/2} + C = -\frac{1}{\sqrt{u}} + C$$

$$= -\frac{1}{\sqrt{x^2 - a^2}} + C.$$

例题 6.6.8

$$\int \frac{x^2 \, dx}{\sqrt{(x^2 - a^2)^3}}$$

$$\int \frac{x^2 \, dx}{\sqrt{(x^2 - a^2)^3}} \stackrel{x=a \sec t}{=} \int \frac{a^2 \sec^2 t \cdot a \sec t \tan t \, dt}{(a^3 \tan^3 t)} = \frac{1}{a} \int \frac{\sec^3 t}{\tan^2 t} \, dt = \frac{1}{a} \int \frac{1}{\cos^3 t} \cdot \frac{\cos^2 t}{\sin^2 t} \, dt$$

$$= \frac{1}{a} \int \frac{1}{\cos t \sin^2 t} \, dt = \frac{1}{a} \int \frac{\sin^2 t + \cos^2 t}{\cos t \sin^2 t} \, dt$$

$$= \frac{1}{a} \int \left(\frac{\sin t}{\cos t} + \frac{\cos t}{\sin^2 t} \right) \, dt = \frac{1}{a} \left(\int \tan t \, dt + \int \cot t \csc t \, dt \right)$$

$$= \frac{1}{a} (-\ln |\cos t| - \csc t) + C$$

$$\stackrel{t=\arccos(a/x)}{=} \frac{1}{a} \left(-\ln \left| \frac{a}{x} \right| - \frac{x}{\sqrt{x^2 - a^2}} \right) + C$$

$$= \frac{1}{a} \ln \left| \frac{x}{a} \right| - \frac{x}{a\sqrt{x^2 - a^2}} + C$$

$$= \frac{1}{a} \ln |x| - \frac{x}{a\sqrt{x^2 - a^2}} + C'.$$

例题 6.6.9

$$\int \sqrt{x^2 - a^2} \, dx$$

$$\int \sqrt{x^2 - a^2} \, dx \stackrel{x=a \sec t}{=} \int \sqrt{a^2 \sec^2 t - a^2} \cdot a \sec t \tan t \, dt = \int a \tan t \cdot a \sec t \tan t \, dt$$

$$= a^2 \int \sec t \tan^2 t \, dt = a^2 \int \sec t (\sec^2 t - 1) \, dt$$

$$= a^2 \left(\int \sec^3 t \, dt - \int \sec t \, dt \right).$$

利用 $\int \sec^3 t \, dt = \frac{1}{2}(\sec t \tan t + \ln |\sec t + \tan t|) + C$ 和 $\int \sec t \, dt = \ln |\sec t + \tan t| + C$, 得

$$\int \sqrt{x^2 - a^2} \, dx = a^2 \left(\frac{1}{2} \sec t \tan t - \frac{1}{2} \ln |\sec t + \tan t| \right) + C.$$

代回 $t = \arccos(a/x)$, $\sec t = x/a$, $\tan t = \sqrt{x^2 - a^2}/a$:

$$\begin{aligned} \int \sqrt{x^2 - a^2} \, dx &= a^2 \left(\frac{1}{2} \cdot \frac{x}{a} \cdot \frac{\sqrt{x^2 - a^2}}{a} - \frac{1}{2} \ln \left| \frac{x}{a} + \frac{\sqrt{x^2 - a^2}}{a} \right| \right) + C \\ &= \frac{x}{2} \sqrt{x^2 - a^2} - \frac{a^2}{2} \ln |x + \sqrt{x^2 - a^2}| + C'. \end{aligned}$$

例题 6.6.10

$$\int x \sqrt{x^2 - a^2} \, dx$$

$$\int x \sqrt{x^2 - a^2} \, dx \stackrel{u=x^2-a^2}{=} \frac{1}{2} \int \sqrt{u} \, du = \frac{1}{2} \cdot \frac{2}{3} u^{3/2} + C = \frac{1}{3} (x^2 - a^2)^{3/2} + C.$$

例题 6.6.11

$$\int x^2 \sqrt{x^2 - a^2} \, dx$$

$$\begin{aligned} \int x^2 \sqrt{x^2 - a^2} \, dx &\stackrel{x=a \sec t}{=} \int a^2 \sec^2 t \cdot a \tan t \cdot a \sec t \tan t \, dt = a^4 \int \sec^3 t \tan^2 t \, dt \\ &= a^4 \int \sec^3 t (\sec^2 t - 1) \, dt = a^4 \left(\int \sec^5 t \, dt - \int \sec^3 t \, dt \right). \end{aligned}$$

利用递推公式计算, 最终可得:

$$\int x^2 \sqrt{x^2 - a^2} \, dx = \frac{x}{8} (2x^2 - a^2) \sqrt{x^2 - a^2} - \frac{a^4}{8} \ln |x + \sqrt{x^2 - a^2}| + C.$$

例题 6.6.12

$$\int \frac{\sqrt{x^2 - a^2}}{x} \, dx$$

$$\begin{aligned} \int \frac{\sqrt{x^2 - a^2}}{x} \, dx &\stackrel{x=a \sec t}{=} \int \frac{a \tan t}{a \sec t} \cdot a \sec t \tan t \, dt = a \int \tan^2 t \, dt = a \int (\sec^2 t - 1) \, dt \\ &= a(\tan t - t) + C \end{aligned}$$

$$\begin{aligned} & \frac{t=\arccos(a/x)}{=} a \left(\frac{\sqrt{x^2 - a^2}}{a} - \arccos \frac{a}{x} \right) + C \\ & = \sqrt{x^2 - a^2} - a \arccos \frac{a}{x} + C. \end{aligned}$$

也可写作: $\int \frac{\sqrt{x^2 - a^2}}{x} dx = \sqrt{x^2 - a^2} - a \sec^{-1} \left| \frac{x}{a} \right| + C.$

例题 6.6.13

$$\int \frac{\sqrt{x^2 - a^2}}{x^2} dx$$

$$\begin{aligned} \int \frac{\sqrt{x^2 - a^2}}{x^2} dx & \stackrel{x=a \sec t}{=} \int \frac{a \tan t}{a^2 \sec^2 t} \cdot a \sec t \tan t dt = \frac{1}{a} \int \frac{\tan^2 t}{\sec t} dt = \frac{1}{a} \int \frac{\sin^2 t}{\cos t} dt \\ & = \frac{1}{a} \int \frac{1 - \cos^2 t}{\cos t} dt = \frac{1}{a} \int (\sec t - \cos t) dt \\ & = \frac{1}{a} (\ln |\sec t + \tan t| - \sin t) + C \\ & \stackrel{t=\arccos(a/x)}{=} \frac{1}{a} \left(\ln \left| \frac{x}{a} + \frac{\sqrt{x^2 - a^2}}{a} \right| - \frac{\sqrt{x^2 - a^2}}{x} \right) + C \\ & = \frac{1}{a} \ln |x + \sqrt{x^2 - a^2}| - \frac{\sqrt{x^2 - a^2}}{ax} + C'. \end{aligned}$$

例题 6.6.14

$$\int \sqrt{(x^2 - a^2)^3} dx$$

$$\begin{aligned} \int \sqrt{(x^2 - a^2)^3} dx & = \int (x^2 - a^2)^{3/2} dx \stackrel{x=a \sec t}{=} \int (a^2 \tan^2 t)^{3/2} \cdot a \sec t \tan t dt \\ & = \int a^3 |\tan^3 t| \cdot a \sec t \tan t dt \quad (\text{假设 } x > a > 0, \text{ 则 } \tan t > 0) \\ & = a^4 \int \tan^4 t \sec t dt = a^4 \int \tan^2 t \cdot \tan^2 t \sec t dt \\ & = a^4 \int (\sec^2 t - 1) \sec t \tan^2 t dt = a^4 \left(\int \sec^3 t \tan^2 t dt - \int \sec t \tan^2 t dt \right). \end{aligned}$$

利用递推, 最终可得:

$$\int \sqrt{(x^2 - a^2)^3} dx = \frac{x}{8} (2x^2 - 5a^2) \sqrt{x^2 - a^2} + \frac{3a^4}{8} \ln |x + \sqrt{x^2 - a^2}| + C.$$

例题 6.6.15

$$\int \frac{\sqrt{(x^2 - a^2)^3}}{x} dx$$

$$\begin{aligned}\int \frac{\sqrt{(x^2 - a^2)^3}}{x} dx &= \int \frac{(x^2 - a^2)^{3/2}}{x} dx \stackrel{x=a \sec t}{=} \int \frac{a^3 \tan^3 t}{a \sec t} \cdot a \sec t \tan t dt \\ &= a^3 \int \tan^4 t dt = a^3 \int (\sec^2 t - 1)^2 dt = a^3 \int (\sec^4 t - 2 \sec^2 t + 1) dt.\end{aligned}$$

计算各项:

$$\begin{aligned}\int \sec^4 t dt &= \int \sec^2 t \sec^2 t dt = \int (1 + \tan^2 t) d(\tan t) = \tan t + \frac{1}{3} \tan^3 t + C, \\ \int \sec^2 t dt &= \tan t + C, \quad \int 1 dt = t + C.\end{aligned}$$

所以

$$\int \frac{\sqrt{(x^2 - a^2)^3}}{x} dx = a^3 \left(\tan t + \frac{1}{3} \tan^3 t - 2 \tan t + t \right) + C = a^3 \left(t - \tan t + \frac{1}{3} \tan^3 t \right) + C.$$

代回 $t = \arccos(a/x)$, $\tan t = \sqrt{x^2 - a^2}/a$:

$$\begin{aligned}\int \frac{\sqrt{(x^2 - a^2)^3}}{x} dx &= a^3 \left(\arccos \frac{a}{x} - \frac{\sqrt{x^2 - a^2}}{a} + \frac{1}{3} \left(\frac{\sqrt{x^2 - a^2}}{a} \right)^3 \right) + C \\ &= a^3 \arccos \frac{a}{x} - a^2 \sqrt{x^2 - a^2} + \frac{1}{3} (x^2 - a^2)^{3/2} + C.\end{aligned}$$

例题 6.6.16

$$\int \frac{\sqrt{(x^2 - a^2)^3}}{x^2} dx$$

$$\int \frac{\sqrt{(x^2 - a^2)^3}}{x^2} dx = -\frac{\sqrt{(x^2 - a^2)^3}}{x} + \frac{3}{2} x \sqrt{x^2 - a^2} - \frac{3}{2} a^2 \ln |x + \sqrt{x^2 - a^2}| + C.$$

例题 6.6.17

$$\int x \sqrt{(x^2 - a^2)^3} dx$$

$$\int x \sqrt{(x^2 - a^2)^3} dx \stackrel{u=x^2-a^2}{=} \frac{1}{2} \int u^{3/2} du = \frac{1}{2} \cdot \frac{2}{5} u^{5/2} + C = \frac{1}{5} (x^2 - a^2)^{5/2} + C.$$

例题 6.6.18

$$\int x^2 \sqrt{(x^2 - a^2)^3} dx$$

$$\begin{aligned}
 \int x^2 \sqrt{(x^2 - a^2)^3} dx &= \int x^2 (x^2 - a^2)^{3/2} dx \\
 &\stackrel{x=a \sec t}{=} \int a^2 \sec^2 t \cdot a^3 \tan^3 t \cdot a \sec t \tan t dt \\
 &= a^6 \int \sec^3 t \tan^4 t dt = a^6 \int \sec^3 t (\sec^2 t - 1)^2 dt \\
 &= a^6 \int \sec^3 t (\sec^4 t - 2 \sec^2 t + 1) dt \\
 &= a^6 \left(\int \sec^7 t dt - 2 \int \sec^5 t dt + \int \sec^3 t dt \right).
 \end{aligned}$$

利用递推公式, 可得最终表达式, 但较冗长。通常用分部积分或递推, 此处略去详细步骤。

例题 6.6.19

$$\int \frac{dx}{\sqrt{a^2 - x^2}}$$

$$\begin{aligned}
 \int \frac{dx}{\sqrt{a^2 - x^2}} &\stackrel{x=a \sin t}{=} \int \frac{a \cos t dt}{\sqrt{a^2 - a^2 \sin^2 t}} = \int \frac{a \cos t}{a \cos t} dt = \int dt = t + C \\
 &\stackrel{t=\arcsin(x/a)}{=} \arcsin \frac{x}{a} + C \quad (|x| < a).
 \end{aligned}$$

也可以表示为: $\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin \frac{x}{a} + C$.

例题 6.6.20

$$\int \frac{x dx}{\sqrt{a^2 - x^2}}$$

$$\int \frac{x dx}{\sqrt{a^2 - x^2}} \stackrel{u=a^2-x^2}{=} \frac{1}{2} \int \frac{-du}{\sqrt{u}} = -\frac{1}{2} \int u^{-1/2} du = -\sqrt{u} + C = -\sqrt{a^2 - x^2} + C.$$

例题 6.6.21

$$\int \frac{x^2 dx}{\sqrt{a^2 - x^2}}$$

$$\begin{aligned}
 \int \frac{x^2 dx}{\sqrt{a^2 - x^2}} &\stackrel{x=a \sin t}{=} \int \frac{a^2 \sin^2 t \cdot a \cos t dt}{\sqrt{a^2 - a^2 \sin^2 t}} = \int \frac{a^3 \sin^2 t \cos t}{a \cos t} dt \\
 &= a^2 \int \sin^2 t dt = a^2 \int \frac{1 - \cos 2t}{2} dt = \frac{a^2}{2} \left(t - \frac{\sin 2t}{2} \right) + C
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{a^2}{2} (t - \sin t \cos t) + C \\
 &\stackrel{t=\arcsin(x/a)}{=} \frac{a^2}{2} \left(\arcsin \frac{x}{a} - \frac{x}{a} \cdot \frac{\sqrt{a^2 - x^2}}{a} \right) + C \\
 &= \frac{a^2}{2} \arcsin \frac{x}{a} - \frac{x}{2} \sqrt{a^2 - x^2} + C.
 \end{aligned}$$

例题 6.6.22

$$\int \frac{dx}{x\sqrt{a^2 - x^2}}$$

$$\begin{aligned}
 \int \frac{dx}{x\sqrt{a^2 - x^2}} &\stackrel{x=a \sin t}{=} \int \frac{a \cos t dt}{a \sin t \cdot a \cos t} = \frac{1}{a} \int \csc t dt = \frac{1}{a} \ln |\csc t - \cot t| + C \\
 &\stackrel{t=\arcsin(x/a)}{=} \frac{1}{a} \ln \left| \frac{a}{x} - \frac{\sqrt{a^2 - x^2}}{x} \right| + C = \frac{1}{a} \ln \left| \frac{a - \sqrt{a^2 - x^2}}{x} \right| + C.
 \end{aligned}$$

也可写作: $\int \frac{dx}{x\sqrt{a^2 - x^2}} = -\frac{1}{a} \ln \left| \frac{a + \sqrt{a^2 - x^2}}{x} \right| + C.$

例题 6.6.23

$$\int \frac{dx}{x^2\sqrt{a^2 - x^2}}$$

$$\begin{aligned}
 \int \frac{dx}{x^2\sqrt{a^2 - x^2}} &\stackrel{x=a \sin t}{=} \int \frac{a \cos t dt}{a^2 \sin^2 t \cdot a \cos t} = \frac{1}{a^2} \int \csc^2 t dt = -\frac{1}{a^2} \cot t + C \\
 &\stackrel{t=\arcsin(x/a)}{=} -\frac{1}{a^2} \cdot \frac{\sqrt{a^2 - x^2}}{x} + C = -\frac{\sqrt{a^2 - x^2}}{a^2 x} + C.
 \end{aligned}$$

例题 6.6.24

$$\int \frac{dx}{\sqrt{(a^2 - x^2)^3}}$$

$$\begin{aligned}
 \int \frac{dx}{\sqrt{(a^2 - x^2)^3}} &\stackrel{x=a \sin t}{=} \int \frac{a \cos t dt}{(a^3 \cos^3 t)} = \frac{1}{a^2} \int \sec^2 t dt = \frac{1}{a^2} \tan t + C \\
 &\stackrel{t=\arcsin(x/a)}{=} \frac{1}{a^2} \cdot \frac{x}{\sqrt{a^2 - x^2}} + C = \frac{x}{a^2 \sqrt{a^2 - x^2}} + C.
 \end{aligned}$$

例题 6.6.25

$$\int \frac{x dx}{\sqrt{(a^2 - x^2)^3}}$$

$$\begin{aligned} \int \frac{x \, dx}{\sqrt{(a^2 - x^2)^3}} &\stackrel{u=a^2-x^2}{=} \frac{1}{2} \int \frac{-du}{u^{3/2}} = -\frac{1}{2} \int u^{-3/2} \, du = -\frac{1}{2} \cdot (-2) u^{-1/2} + C \\ &= \frac{1}{\sqrt{u}} + C = \frac{1}{\sqrt{a^2 - x^2}} + C. \end{aligned}$$

例题 6.6.26

$$\int \frac{x^2 dx}{\sqrt{(a^2 - x^2)^3}}$$

$$\begin{aligned} \int \frac{x^2 dx}{\sqrt{(a^2 - x^2)^3}} &\stackrel{x=a \sin t}{=} \int \frac{a^2 \sin^2 t \cdot a \cos t \, dt}{(a^3 \cos^3 t)} = \frac{1}{a} \int \frac{\sin^2 t}{\cos^2 t} \, dt = \frac{1}{a} \int \tan^2 t \, dt \\ &= \frac{1}{a} \int (\sec^2 t - 1) \, dt = \frac{1}{a} (\tan t - t) + C \\ &\stackrel{t=\arcsin(x/a)}{=} \frac{1}{a} \left(\frac{x}{\sqrt{a^2 - x^2}} - \arcsin \frac{x}{a} \right) + C. \end{aligned}$$

例题 6.6.27

$$\int \sqrt{a^2 - x^2} \, dx$$

$$\begin{aligned} \int \sqrt{a^2 - x^2} \, dx &\stackrel{x=a \sin t}{=} \int \sqrt{a^2 - a^2 \sin^2 t} \cdot a \cos t \, dt = \int a \cos t \cdot a \cos t \, dt \\ &= a^2 \int \cos^2 t \, dt = a^2 \int \frac{1 + \cos 2t}{2} \, dt = \frac{a^2}{2} \left(t + \frac{\sin 2t}{2} \right) + C \\ &= \frac{a^2}{2} (t + \sin t \cos t) + C \\ &\stackrel{t=\arcsin(x/a)}{=} \frac{a^2}{2} \left(\arcsin \frac{x}{a} + \frac{x}{a} \cdot \frac{\sqrt{a^2 - x^2}}{a} \right) + C \\ &= \frac{a^2}{2} \arcsin \frac{x}{a} + \frac{x}{2} \sqrt{a^2 - x^2} + C. \end{aligned}$$

例题 6.6.28

$$\int x \sqrt{a^2 - x^2} \, dx$$

$$\begin{aligned} \int x \sqrt{a^2 - x^2} \, dx &\stackrel{u=a^2-x^2}{=} \frac{1}{2} \int \sqrt{u} \, (-du) = -\frac{1}{2} \int u^{1/2} \, du = -\frac{1}{2} \cdot \frac{2}{3} u^{3/2} + C \\ &= -\frac{1}{3} (a^2 - x^2)^{3/2} + C. \end{aligned}$$

例题 6.6.29

$$\int x^2 \sqrt{a^2 - x^2} dx$$

$$\begin{aligned} \int x^2 \sqrt{a^2 - x^2} dx &\stackrel{x=a \sin t}{=} \int a^2 \sin^2 t \cdot a \cos t \cdot a \cos t dt = a^4 \int \sin^2 t \cos^2 t dt \\ &= a^4 \int \frac{1}{4} \sin^2 2t dt = \frac{a^4}{4} \int \frac{1 - \cos 4t}{2} dt = \frac{a^4}{8} \left(t - \frac{\sin 4t}{4} \right) + C \\ &= \frac{a^4}{8} \left(t - \frac{1}{4} \cdot 2 \sin 2t \cos 2t \right) + C = \frac{a^4}{8} \left(t - \frac{1}{2} \sin 2t \cos 2t \right) + C. \end{aligned}$$

利用 $\sin 2t = 2 \sin t \cos t = 2 \cdot \frac{x}{a} \cdot \frac{\sqrt{a^2 - x^2}}{a} = \frac{2x\sqrt{a^2 - x^2}}{a^2}$, $\cos 2t = 1 - 2 \sin^2 t = 1 - 2 \frac{x^2}{a^2}$, 代入化简得:

$$\begin{aligned} \int x^2 \sqrt{a^2 - x^2} dx &= \frac{a^4}{8} \left(\arcsin \frac{x}{a} - \frac{1}{2} \cdot \frac{2x\sqrt{a^2 - x^2}}{a^2} \cdot \left(1 - 2 \frac{x^2}{a^2} \right) \right) + C \\ &= \frac{a^4}{8} \arcsin \frac{x}{a} - \frac{x}{8} \sqrt{a^2 - x^2} (a^2 - 2x^2) + C. \end{aligned}$$

也可整理为:

$$\int x^2 \sqrt{a^2 - x^2} dx = \frac{x}{8} (2x^2 - a^2) \sqrt{a^2 - x^2} + \frac{a^4}{8} \arcsin \frac{x}{a} + C.$$

例题 6.6.30

$$\int \frac{\sqrt{a^2 - x^2}}{x} dx$$

$$\begin{aligned} \int \frac{\sqrt{a^2 - x^2}}{x} dx &\stackrel{x=a \sin t}{=} \int \frac{a \cos t}{a \sin t} \cdot a \cos t dt = a \int \frac{\cos^2 t}{\sin t} dt \\ &= a \int \frac{1 - \sin^2 t}{\sin t} dt = a \int (\csc t - \sin t) dt \\ &= a (\ln |\csc t - \cot t| + \cos t) + C \\ &\stackrel{t=\arcsin(x/a)}{=} a \left(\ln \left| \frac{a}{x} - \frac{\sqrt{a^2 - x^2}}{x} \right| + \frac{\sqrt{a^2 - x^2}}{a} \right) + C \\ &= a \ln \left| \frac{a - \sqrt{a^2 - x^2}}{x} \right| + \sqrt{a^2 - x^2} + C. \end{aligned}$$

例题 6.6.31

$$\int \frac{\sqrt{a^2 - x^2}}{x^2} dx$$

$$\begin{aligned}
 \int \frac{\sqrt{a^2 - x^2}}{x^2} dx &\stackrel{x=a \sin t}{=} \int \frac{a \cos t}{a^2 \sin^2 t} \cdot a \cos t dt = \frac{1}{a} \int \frac{\cos^2 t}{\sin^2 t} dt \\
 &= \frac{1}{a} \int \cot^2 t dt = \frac{1}{a} \int (\csc^2 t - 1) dt \\
 &= \frac{1}{a} (-\cot t - t) + C \\
 &\stackrel{t=\arcsin(x/a)}{=} -\frac{1}{a} \left(\frac{\sqrt{a^2 - x^2}}{x} + \arcsin \frac{x}{a} \right) + C \\
 &= -\frac{\sqrt{a^2 - x^2}}{ax} - \frac{1}{a} \arcsin \frac{x}{a} + C.
 \end{aligned}$$

例题 6.6.32

$$\int \sqrt{(a^2 - x^2)^3} dx$$

$$\begin{aligned}
 \int \sqrt{(a^2 - x^2)^3} dx &= \int (a^2 - x^2)^{3/2} dx \stackrel{x=a \sin t}{=} \int (a^2 \cos^2 t)^{3/2} \cdot a \cos t dt \\
 &= \int a^3 \cos^3 t \cdot a \cos t dt = a^4 \int \cos^4 t dt.
 \end{aligned}$$

利用 $\cos^4 t = \left(\frac{1+\cos 2t}{2}\right)^2 = \frac{1}{4}(1 + 2\cos 2t + \cos^2 2t) = \frac{1}{4}\left(1 + 2\cos 2t + \frac{1+\cos 4t}{2}\right) = \frac{1}{8}(3 + 4\cos 2t + \cos 4t)$, 所以

$$\int \cos^4 t dt = \frac{1}{8} \int (3 + 4\cos 2t + \cos 4t) dt = \frac{1}{8} \left(3t + 2\sin 2t + \frac{1}{4}\sin 4t \right) + C.$$

又 $\sin 2t = 2\sin t \cos t = 2 \cdot \frac{x}{a} \cdot \frac{\sqrt{a^2 - x^2}}{a} = \frac{2x\sqrt{a^2 - x^2}}{a^2}$, $\sin 4t = 2\sin 2t \cos 2t = 2 \cdot \frac{2x\sqrt{a^2 - x^2}}{a^2} \cdot \left(1 - 2\frac{x^2}{a^2}\right)$, 代入并整理得:

$$\begin{aligned}
 \int \sqrt{(a^2 - x^2)^3} dx &= \frac{a^4}{8} \left(3\arcsin \frac{x}{a} + \frac{2x\sqrt{a^2 - x^2}}{a^2} \left(5 - \frac{2x^2}{a^2} \right) \right) + C \\
 &= \frac{3a^4}{8} \arcsin \frac{x}{a} + \frac{x}{8} \sqrt{a^2 - x^2} (5a^2 - 2x^2) + C.
 \end{aligned}$$

也可写作:

$$\int \sqrt{(a^2 - x^2)^3} dx = \frac{x}{8} (5a^2 - 2x^2) \sqrt{a^2 - x^2} + \frac{3a^4}{8} \arcsin \frac{x}{a} + C.$$

例题 6.6.33

$$\int \frac{\sqrt{(a^2 - x^2)^3}}{x} dx$$

$$\begin{aligned}
 \int \frac{\sqrt{(a^2 - x^2)^3}}{x} dx &= \int \frac{(a^2 - x^2)^{3/2}}{x} dx \stackrel{x=a \sin t}{=} \int \frac{a^3 \cos^3 t}{a \sin t} \cdot a \cos t dt \\
 &= a^3 \int \frac{\cos^4 t}{\sin t} dt = a^3 \int \frac{(1 - \sin^2 t)^2}{\sin t} dt \\
 &= a^3 \int (\csc t - 2 \sin t + \sin^3 t) dt.
 \end{aligned}$$

计算各项:

$$\begin{aligned}
 \int \csc t dt &= \ln |\csc t - \cot t|, \\
 \int \sin t dt &= -\cos t, \\
 \int \sin^3 t dt &= \int (1 - \cos^2 t) \sin t dt = -\cos t + \frac{\cos^3 t}{3}.
 \end{aligned}$$

所以

$$\begin{aligned}
 \int \frac{\sqrt{(a^2 - x^2)^3}}{x} dx &= a^3 \left(\ln |\csc t - \cot t| + 2 \cos t - \cos t + \frac{\cos^3 t}{3} \right) + C \\
 &= a^3 \left(\ln |\csc t - \cot t| + \cos t + \frac{\cos^3 t}{3} \right) + C.
 \end{aligned}$$

代回 $t = \arcsin(x/a)$, $\cos t = \sqrt{1 - \frac{x^2}{a^2}} = \frac{\sqrt{a^2 - x^2}}{a}$, $\csc t = \frac{a}{x}$, $\cot t = \frac{\sqrt{a^2 - x^2}}{x}$:

$$\begin{aligned}
 \int \frac{\sqrt{(a^2 - x^2)^3}}{x} dx &= a^3 \left(\ln \left| \frac{a}{x} - \frac{\sqrt{a^2 - x^2}}{x} \right| + \frac{\sqrt{a^2 - x^2}}{a} + \frac{1}{3} \left(\frac{\sqrt{a^2 - x^2}}{a} \right)^3 \right) + C \\
 &= a^3 \ln \left| \frac{a - \sqrt{a^2 - x^2}}{x} \right| + a^2 \sqrt{a^2 - x^2} + \frac{1}{3} (a^2 - x^2)^{3/2} + C.
 \end{aligned}$$

整理得:

$$\int \frac{\sqrt{(a^2 - x^2)^3}}{x} dx = \frac{1}{3} (a^2 - x^2)^{3/2} + a^2 \sqrt{a^2 - x^2} + a^3 \ln \left| \frac{a - \sqrt{a^2 - x^2}}{x} \right| + C.$$

例题 6.6.34

$$\int \frac{\sqrt{(a^2 - x^2)^3}}{x^2} dx$$

$$\begin{aligned}
 \int \frac{\sqrt{(a^2 - x^2)^3}}{x^2} dx &\stackrel{x=a \sin t}{=} \int \frac{a^3 \cos^3 t}{a^2 \sin^2 t} \cdot a \cos t dt = a^2 \int \frac{\cos^4 t}{\sin^2 t} dt \\
 &= a^2 \int \frac{(1 - \sin^2 t)^2}{\sin^2 t} dt = a^2 \int (\csc^2 t - 2 + \sin^2 t) dt
 \end{aligned}$$

$$= a^2 \left(-\cot t - 2t + \int \sin^2 t \, dt \right).$$

而 $\int \sin^2 t \, dt = \frac{t}{2} - \frac{\sin 2t}{4} = \frac{t}{2} - \frac{\sin t \cos t}{2}$, 所以

$$\begin{aligned} \int \frac{\sqrt{(a^2 - x^2)^3}}{x^2} \, dx &= a^2 \left(-\cot t - 2t + \frac{t}{2} - \frac{\sin t \cos t}{2} \right) + C \\ &= a^2 \left(-\cot t - \frac{3t}{2} - \frac{\sin t \cos t}{2} \right) + C. \end{aligned}$$

代回 $t = \arcsin(x/a)$, $\cot t = \frac{\sqrt{a^2 - x^2}}{x}$, $\sin t = \frac{x}{a}$, $\cos t = \frac{\sqrt{a^2 - x^2}}{a}$:

$$\begin{aligned} \int \frac{\sqrt{(a^2 - x^2)^3}}{x^2} \, dx &= a^2 \left(-\frac{\sqrt{a^2 - x^2}}{x} - \frac{3}{2} \arcsin \frac{x}{a} - \frac{1}{2} \cdot \frac{x}{a} \cdot \frac{\sqrt{a^2 - x^2}}{a} \right) + C \\ &= -\frac{a^2 \sqrt{a^2 - x^2}}{x} - \frac{3a^2}{2} \arcsin \frac{x}{a} - \frac{x}{2} \sqrt{a^2 - x^2} + C. \end{aligned}$$

整理得:

$$\int \frac{\sqrt{(a^2 - x^2)^3}}{x^2} \, dx = -\frac{\sqrt{a^2 - x^2}}{x} \left(a^2 + \frac{x^2}{2} \right) - \frac{3a^2}{2} \arcsin \frac{x}{a} + C.$$

例题 6.6.35

$$\int x \sqrt{(a^2 - x^2)^3} \, dx$$

$$\begin{aligned} \int x \sqrt{(a^2 - x^2)^3} \, dx &\stackrel{u=a^2-x^2}{=} \frac{1}{2} \int u^{3/2} (-du) = -\frac{1}{2} \int u^{3/2} \, du = -\frac{1}{2} \cdot \frac{2}{5} u^{5/2} + C \\ &= -\frac{1}{5} (a^2 - x^2)^{5/2} + C. \end{aligned}$$

例题 6.6.36

$$\int x^2 \sqrt{(a^2 - x^2)^3} \, dx$$

$$\begin{aligned} \int x^2 \sqrt{(a^2 - x^2)^3} \, dx &= \int x^2 (a^2 - x^2)^{3/2} \, dx \stackrel{x=a \sin t}{=} \int a^2 \sin^2 t \cdot a^3 \cos^3 t \cdot a \cos t \, dt \\ &= a^6 \int \sin^2 t \cos^4 t \, dt = a^6 \int \sin^2 t (1 - \sin^2 t)^2 \, dt \\ &= a^6 \int (\sin^2 t - 2 \sin^4 t + \sin^6 t) \, dt. \end{aligned}$$

这些积分可以通过递推公式或利用三角恒等式计算, 但过程较长。最终结果为:

$$\int x^2 \sqrt{(a^2 - x^2)^3} \, dx$$

$$= -\frac{x}{8}(2x^2 - 5a^2)\sqrt{(a^2 - x^2)^3} + \frac{3a^6}{16} \left(\arcsin \frac{x}{a} - \frac{x}{a^2}\sqrt{a^2 - x^2}(2x^2 - 3a^2) \right) + C.$$

为简洁, 通常用递推公式或分部积分得出, 代入前面 $\int \sqrt{(a^2 - x^2)^3} dx$ 的结果可得完整表达式。

$$\int x^2 \sqrt{(a^2 - x^2)^3} dx = -\frac{x}{6}(a^2 - x^2)^{5/2} + \frac{a^2}{6} \int \sqrt{(a^2 - x^2)^3} dx + C.$$

6.7 含有 $\sqrt{ax^2 + bx + c}$ 和的积分

例题 6.7.1

$$\int \sqrt{ax^2 + bx + c} dx$$

$$\begin{aligned} \int \sqrt{ax^2 + bx + c} dx &= \int \sqrt{a \left(x^2 + \frac{b}{a}x + \frac{c}{a} \right)} dx \\ &= \sqrt{a} \int \sqrt{\left(x + \frac{b}{2a} \right)^2 + \frac{4ac - b^2}{4a^2}} dx \quad (\text{假设 } a > 0) \\ &\stackrel{u=x+\frac{b}{2a}, \Delta=4ac-b^2}{=} \sqrt{a} \int \sqrt{u^2 + \frac{\Delta}{4a^2}} du. \end{aligned}$$

记 $k = \frac{\sqrt{|\Delta|}}{2|a|}$ 。分情况讨论:

1. 若 $\Delta > 0$, 则 $\sqrt{a} \int \sqrt{u^2 + k^2} du$, 其中 $k = \frac{\sqrt{\Delta}}{2a}$ 。利用公式:

$$\int \sqrt{u^2 + k^2} du = \frac{u}{2} \sqrt{u^2 + k^2} + \frac{k^2}{2} \ln |u + \sqrt{u^2 + k^2}| + C.$$

2. 若 $\Delta = 0$, 则积分简化为 $\sqrt{a} \int |u| du = \frac{\sqrt{a}}{2} u|u| + C$, 但注意 $u = x + b/(2a)$, 原被积函数为 $\sqrt{a}|u|$ 。

3. 若 $\Delta < 0$, 则 $\sqrt{a} \int \sqrt{u^2 - k^2} du$, 其中 $k = \frac{\sqrt{-\Delta}}{2a}$ 。利用公式:

$$\int \sqrt{u^2 - k^2} du = \frac{u}{2} \sqrt{u^2 - k^2} - \frac{k^2}{2} \ln |u + \sqrt{u^2 - k^2}| + C.$$

最终结果用 x 表示, 这里不展开全部。若 $a < 0$, 则被开方数为负, 积分可能仅在定义域内实数, 此时通常提取 $-a$ 并处理为 $\sqrt{-a} \sqrt{-(x^2 + \frac{b}{a}x + \frac{c}{a})}$, 转化为第二种类型。

例题 6.7.2

$$\int \sqrt{-ax^2 + bx + c} dx$$

$$\begin{aligned}
 \int \sqrt{-ax^2 + bx + c} \, dx &= \int \sqrt{-a \left(x^2 - \frac{b}{a}x - \frac{c}{a} \right)} \, dx \quad (\text{假设 } a > 0) \\
 &= \sqrt{a} \int \sqrt{- \left(x^2 - \frac{b}{a}x - \frac{c}{a} \right)} \, dx \\
 &= \sqrt{a} \int \sqrt{k^2 - \left(x - \frac{b}{2a} \right)^2} \, dx,
 \end{aligned}$$

其中完成平方后, 设 $k^2 = \frac{b^2+4ac}{4a^2}$ (需保证被开方数非负)。令 $u = x - \frac{b}{2a}$, 则积分化为

$$\sqrt{a} \int \sqrt{k^2 - u^2} \, du = \sqrt{a} \left(\frac{u}{2} \sqrt{k^2 - u^2} + \frac{k^2}{2} \arcsin \frac{u}{k} \right) + C,$$

其中 $|u| \leq k$ 。代回 u 和 k 即得结果。

例题 6.7.3

$$\int x \sqrt{ax^2 + bx + c} \, dx$$

$$\begin{aligned}
 \int x \sqrt{ax^2 + bx + c} \, dx &= \frac{1}{2a} \int (2ax + b - b) \sqrt{ax^2 + bx + c} \, dx \\
 &= \frac{1}{2a} \int (2ax + b) \sqrt{ax^2 + bx + c} \, dx - \frac{b}{2a} \int \sqrt{ax^2 + bx + c} \, dx.
 \end{aligned}$$

对于第一项, 令 $u = ax^2 + bx + c$, 则 $du = (2ax + b)dx$, 所以

$$\int (2ax + b) \sqrt{ax^2 + bx + c} \, dx = \int \sqrt{u} \, du = \frac{2}{3} u^{3/2} = \frac{2}{3} (ax^2 + bx + c)^{3/2}.$$

第二项即为上一个积分的结果。因此

$$\int x \sqrt{ax^2 + bx + c} \, dx = \frac{1}{3a} (ax^2 + bx + c)^{3/2} - \frac{b}{2a} \int \sqrt{ax^2 + bx + c} \, dx + C.$$

例题 6.7.4

$$\int x \sqrt{-ax^2 + bx + c} \, dx$$

$$\begin{aligned}
 \int x \sqrt{-ax^2 + bx + c} \, dx &= \frac{1}{-2a} \int (-2ax + b - b) \sqrt{-ax^2 + bx + c} \, dx \\
 &= -\frac{1}{2a} \int (-2ax + b) \sqrt{-ax^2 + bx + c} \, dx + \frac{b}{2a} \int \sqrt{-ax^2 + bx + c} \, dx.
 \end{aligned}$$

对于第一项, 令 $u = -ax^2 + bx + c$, 则 $du = (-2ax + b)dx$, 所以

$$\int (-2ax + b)\sqrt{-ax^2 + bx + c} dx = \int \sqrt{u} du = \frac{2}{3}u^{3/2}.$$

因此

$$\int x\sqrt{-ax^2 + bx + c} dx = -\frac{1}{3a}(-ax^2 + bx + c)^{3/2} + \frac{b}{2a} \int \sqrt{-ax^2 + bx + c} dx + C.$$

例题 6.7.5

$$\int \frac{dx}{\sqrt{ax^2 + bx + c}}$$

$$\begin{aligned} \int \frac{dx}{\sqrt{ax^2 + bx + c}} &= \frac{1}{\sqrt{a}} \int \frac{dx}{\sqrt{\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a^2}}} \quad (\text{假设 } a > 0) \\ &\stackrel{u=x+\frac{b}{2a}, \Delta=4ac-b^2}{=} \frac{1}{\sqrt{a}} \int \frac{du}{\sqrt{u^2 + \frac{\Delta}{4a^2}}}. \end{aligned}$$

分情况:

1. 若 $\Delta > 0$, 记 $k = \frac{\sqrt{\Delta}}{2a} > 0$, 则

$$\int \frac{du}{\sqrt{u^2 + k^2}} = \ln|u + \sqrt{u^2 + k^2}| + C.$$

2. 若 $\Delta = 0$, 则积分简化为 $\frac{1}{\sqrt{a}} \int \frac{du}{|u|} = \frac{1}{\sqrt{a}} \ln|u| + C$.

3. 若 $\Delta < 0$, 记 $k = \frac{\sqrt{-\Delta}}{2a} > 0$, 则

$$\int \frac{du}{\sqrt{u^2 - k^2}} = \ln|u + \sqrt{u^2 - k^2}| + C.$$

统一形式:

$$\int \frac{dx}{\sqrt{ax^2 + bx + c}} = \frac{1}{\sqrt{a}} \ln|2ax + b + 2\sqrt{a}\sqrt{ax^2 + bx + c}| + C \quad (a > 0).$$

若 $a < 0$, 则需用反正弦函数等形式。

例题 6.7.6

$$\int \frac{x dx}{\sqrt{ax^2 + bx + c}}$$

$$\begin{aligned}\int \frac{x \, dx}{\sqrt{ax^2 + bx + c}} &= \frac{1}{2a} \int \frac{2ax + b - b}{\sqrt{ax^2 + bx + c}} \, dx \\ &= \frac{1}{2a} \int \frac{2ax + b}{\sqrt{ax^2 + bx + c}} \, dx - \frac{b}{2a} \int \frac{dx}{\sqrt{ax^2 + bx + c}}.\end{aligned}$$

第一项：令 $u = ax^2 + bx + c$ ，则 $du = (2ax + b)dx$ ，所以

$$\int \frac{2ax + b}{\sqrt{ax^2 + bx + c}} \, dx = \int \frac{du}{\sqrt{u}} = 2\sqrt{u} = 2\sqrt{ax^2 + bx + c}.$$

因此

$$\int \frac{x \, dx}{\sqrt{ax^2 + bx + c}} = \frac{1}{a} \sqrt{ax^2 + bx + c} - \frac{b}{2a} \int \frac{dx}{\sqrt{ax^2 + bx + c}} + C.$$

例题 6.7.7

$$\int \frac{x^2 dx}{\sqrt{ax^2 + bx + c}}$$

$$\begin{aligned}\int \frac{x^2 dx}{\sqrt{ax^2 + bx + c}} &= \frac{1}{a} \int \frac{ax^2}{\sqrt{ax^2 + bx + c}} \, dx \\ &= \frac{1}{a} \int \frac{ax^2 + bx + c - bx - c}{\sqrt{ax^2 + bx + c}} \, dx \\ &= \frac{1}{a} \int \sqrt{ax^2 + bx + c} \, dx - \frac{b}{a} \int \frac{x \, dx}{\sqrt{ax^2 + bx + c}} - \frac{c}{a} \int \frac{dx}{\sqrt{ax^2 + bx + c}}.\end{aligned}$$

前两个积分已知，代入即可。

例题 6.7.8

$$\int \frac{dx}{\sqrt{-ax^2 + bx + c}}$$

$$\begin{aligned}\int \frac{dx}{\sqrt{-ax^2 + bx + c}} &= \frac{1}{\sqrt{a}} \int \frac{dx}{\sqrt{\frac{b^2 + 4ac}{4a^2} - \left(x - \frac{b}{2a}\right)^2}} \quad (\text{假设 } a > 0) \\ &\stackrel{u = x - \frac{b}{2a}, k^2 = \frac{b^2 + 4ac}{4a^2}}{=} \frac{1}{\sqrt{a}} \int \frac{du}{\sqrt{k^2 - u^2}} \\ &= \frac{1}{\sqrt{a}} \arcsin \frac{u}{k} + C = \frac{1}{\sqrt{a}} \arcsin \frac{2ax - b}{\sqrt{b^2 + 4ac}} + C.\end{aligned}$$

例题 6.7.9

$$\int \frac{x \, dx}{\sqrt{-ax^2 + bx + c}}$$

$$\begin{aligned}\int \frac{x \, dx}{\sqrt{-ax^2 + bx + c}} &= \frac{1}{-2a} \int \frac{-2ax + b - b}{\sqrt{-ax^2 + bx + c}} \, dx \\ &= -\frac{1}{2a} \int \frac{-2ax + b}{\sqrt{-ax^2 + bx + c}} \, dx + \frac{b}{2a} \int \frac{dx}{\sqrt{-ax^2 + bx + c}}.\end{aligned}$$

第一项：令 $u = -ax^2 + bx + c$ ，则 $du = (-2ax + b)dx$ ，所以

$$\int \frac{-2ax + b}{\sqrt{-ax^2 + bx + c}} \, dx = \int \frac{du}{\sqrt{u}} = 2\sqrt{u} = 2\sqrt{-ax^2 + bx + c}.$$

因此

$$\int \frac{x \, dx}{\sqrt{-ax^2 + bx + c}} = -\frac{1}{a} \sqrt{-ax^2 + bx + c} + \frac{b}{2a} \int \frac{dx}{\sqrt{-ax^2 + bx + c}} + C.$$

例题 6.7.10

$$\int \frac{x^2 dx}{\sqrt{-ax^2 + bx + c}}$$

$$\begin{aligned}\int \frac{x^2 dx}{\sqrt{-ax^2 + bx + c}} &= \frac{1}{-a} \int \frac{-ax^2}{\sqrt{-ax^2 + bx + c}} \, dx \\ &= -\frac{1}{a} \int \frac{-ax^2 + bx + c - bx - c}{\sqrt{-ax^2 + bx + c}} \, dx \\ &= -\frac{1}{a} \int \sqrt{-ax^2 + bx + c} \, dx + \frac{b}{a} \int \frac{x \, dx}{\sqrt{-ax^2 + bx + c}} + \frac{c}{a} \int \frac{dx}{\sqrt{-ax^2 + bx + c}}.\end{aligned}$$

6.8 含有 $\sqrt{\pm \frac{x-a}{x-b}}, \sqrt{(x-a)(b-x)}$ 的积分

例题 6.8.1

$$\int \sqrt{\frac{x-a}{x-b}} \, dx$$

解 6.8.1. 设 $t = \sqrt{\frac{x-a}{x-b}}$ ，则 $t^2 = \frac{x-a}{x-b}$ ，解得 $x = \frac{a-bt^2}{1-t^2}$ 。微分得

$$dx = \frac{2(b-a)t}{(1-t^2)^2} \, dt.$$

于是

$$\int \sqrt{\frac{x-a}{x-b}} \, dx = \int t \cdot \frac{2(b-a)t}{(1-t^2)^2} \, dt = 2(b-a) \int \frac{t^2}{(1-t^2)^2} \, dt.$$

将被积函数分解为部分分式:

$$\frac{t^2}{(1-t^2)^2} = \frac{1}{4} \left(\frac{1}{1-t} + \frac{1}{1+t} \right) + \frac{1}{2} \left(\frac{1}{(1-t)^2} + \frac{1}{(1+t)^2} \right).$$

积分得

$$\begin{aligned} \int \frac{t^2}{(1-t^2)^2} dt &= \frac{1}{4} (-\ln|1-t| + \ln|1+t|) + \frac{1}{2} \left(\frac{1}{1-t} - \frac{1}{1+t} \right) + C \\ &= \frac{1}{4} \ln \left| \frac{1+t}{1-t} \right| + \frac{t}{1-t^2} + C. \end{aligned}$$

代回 $t = \sqrt{\frac{x-a}{x-b}}$, 并利用 $1-t^2 = \frac{b-a}{x-b}$, $t/(1-t^2) = \frac{\sqrt{(x-a)(x-b)}}{b-a}$, 以及

$$\frac{1+t}{1-t} = \frac{\sqrt{x-b} + \sqrt{x-a}}{\sqrt{x-b} - \sqrt{x-a}},$$

可得

$$\begin{aligned} \int \sqrt{\frac{x-a}{x-b}} dx &= 2(b-a) \left(\frac{1}{4} \ln \left| \frac{\sqrt{x-b} + \sqrt{x-a}}{\sqrt{x-b} - \sqrt{x-a}} \right| + \frac{\sqrt{(x-a)(x-b)}}{b-a} \right) + C \\ &= \sqrt{(x-a)(x-b)} + \frac{b-a}{2} \ln \left| \frac{\sqrt{x-b} + \sqrt{x-a}}{\sqrt{x-b} - \sqrt{x-a}} \right| + C. \end{aligned}$$

利用恒等式 $\ln \left| \frac{\sqrt{x-b} + \sqrt{x-a}}{\sqrt{x-b} - \sqrt{x-a}} \right| = 2 \ln (\sqrt{x-a} + \sqrt{x-b}) + \text{常数}$, 可进一步写为

$$\int \sqrt{\frac{x-a}{x-b}} dx = \sqrt{(x-a)(x-b)} + (b-a) \ln (\sqrt{x-a} + \sqrt{x-b}) + C',$$

其中 C' 为常数。通常假设 $x > b > a$, 故可略去绝对值。

例题 6.8.2

$$\int \sqrt{\frac{x-a}{b-x}} dx$$

解 6.8.2. 设 $t = \sqrt{\frac{x-a}{b-x}}$, 则 $t^2 = \frac{x-a}{b-x}$, 解得 $x = \frac{a+bt^2}{1+t^2}$ 。微分得

$$dx = \frac{2(b-a)t}{(1+t^2)^2} dt.$$

于是

$$\int \sqrt{\frac{x-a}{b-x}} dx = \int t \cdot \frac{2(b-a)t}{(1+t^2)^2} dt = 2(b-a) \int \frac{t^2}{(1+t^2)^2} dt.$$

计算 $\int \frac{t^2}{(1+t^2)^2} dt$, 令 $t = \tan u$, 则 $dt = \sec^2 u du$, 且

$$\int \frac{t^2}{(1+t^2)^2} dt = \int \frac{\tan^2 u}{\sec^4 u} \sec^2 u du = \int \sin^2 u du = \frac{u}{2} - \frac{\sin 2u}{4} + C = \frac{1}{2} \arctan t - \frac{t}{2(1+t^2)} + C.$$

因此

$$\int \sqrt{\frac{x-a}{b-x}} dx = 2(b-a) \left(\frac{1}{2} \arctan t - \frac{t}{2(1+t^2)} \right) + C = (b-a) \left(\arctan t - \frac{t}{1+t^2} \right) + C.$$

代回 $t = \sqrt{\frac{x-a}{b-x}}$, 注意 $\frac{t}{1+t^2} = \frac{\sqrt{(x-a)(b-x)}}{b-a}$, 且 $\arctan t = \arcsin \sqrt{\frac{x-a}{b-a}}$ (因为当 $t \geq 0$ 时, $\arctan t = \arcsin \frac{t}{\sqrt{1+t^2}} = \arcsin \sqrt{\frac{x-a}{b-a}}$). 所以

$$\int \sqrt{\frac{x-a}{b-x}} dx = (b-a) \arcsin \sqrt{\frac{x-a}{b-a}} - \sqrt{(x-a)(b-x)} + C.$$

也可写为

$$\int \sqrt{\frac{x-a}{b-x}} dx = \sqrt{(x-a)(b-x)} + (b-a) \arcsin \sqrt{\frac{x-a}{b-a}} + C,$$

其中符号差异可并入常数。通常采用后一种形式。

例题 6.8.3

$$\int \frac{dx}{\sqrt{(x-a)(b-x)}}$$

解 6.8.3. 注意到 $(x-a)(b-x) = -(x-a)(x-b) = \left(\frac{b-a}{2}\right)^2 - \left(x - \frac{a+b}{2}\right)^2$. 令 $u = x - \frac{a+b}{2}$, 则

$$\int \frac{dx}{\sqrt{(x-a)(b-x)}} = \int \frac{du}{\sqrt{\left(\frac{b-a}{2}\right)^2 - u^2}} = \arcsin \frac{2u}{b-a} + C = \arcsin \frac{2x-a-b}{b-a} + C.$$

另一种常见形式: 令 $t = \sqrt{\frac{x-a}{b-x}}$, 可得

$$\int \frac{dx}{\sqrt{(x-a)(b-x)}} = 2 \arcsin \sqrt{\frac{x-a}{b-a}} + C.$$

两种形式等价, 因为 $\arcsin \frac{2x-a-b}{b-a} = 2 \arcsin \sqrt{\frac{x-a}{b-a}} - \frac{\pi}{2}$, 相差常数。

例题 6.8.4

$$\int \sqrt{(x-a)(b-x)} dx$$

解 6.8.4. 完成平方:

$$(x-a)(b-x) = -[x^2 - (a+b)x + ab] = \left(\frac{b-a}{2}\right)^2 - \left(x - \frac{a+b}{2}\right)^2.$$

令 $u = x - \frac{a+b}{2}$, 则

$$\int \sqrt{(x-a)(b-x)} \, dx = \int \sqrt{\left(\frac{b-a}{2}\right)^2 - u^2} \, du.$$

利用公式 $\int \sqrt{k^2 - u^2} \, du = \frac{u}{2} \sqrt{k^2 - u^2} + \frac{k^2}{2} \arcsin \frac{u}{k} + C$, 其中 $k = \frac{b-a}{2}$, 得

$$\begin{aligned} \int \sqrt{(x-a)(b-x)} \, dx &= \frac{u}{2} \sqrt{\left(\frac{b-a}{2}\right)^2 - u^2} + \frac{1}{2} \left(\frac{b-a}{2}\right)^2 \arcsin \frac{u}{(b-a)/2} + C \\ &= \frac{2x-a-b}{4} \sqrt{(x-a)(b-x)} + \frac{(b-a)^2}{8} \arcsin \frac{2x-a-b}{b-a} + C. \end{aligned}$$

也可写为

$$\int \sqrt{(x-a)(b-x)} \, dx = \frac{1}{2} \left(x - \frac{a+b}{2}\right) \sqrt{(x-a)(b-x)} + \frac{(b-a)^2}{8} \arcsin \frac{2x-a-b}{b-a} + C.$$

6.9 含有三角函数的积分

例题 6.9.1

$$\int \tan x \, dx$$

$$\int \tan x \, dx = -\ln |\cos x| + C = \ln |\sec x| + C.$$

例题 6.9.2

$$\int \cot x \, dx$$

$$\int \cot x \, dx = \ln |\sin x| + C.$$

例题 6.9.3

$$\int \sec x \, dx$$

$$\int \sec x \, dx = \ln |\sec x + \tan x| + C.$$

例题 6.9.4

$$\int \csc x \, dx$$

$$\int \csc x \, dx = \ln |\csc x - \cot x| + C = \ln \left| \tan \frac{x}{2} \right| + C.$$

例题 6.9.5

$$\int \tan^2 x \, dx$$

$$\int \tan^2 x \, dx = \int (\sec^2 x - 1) \, dx = \tan x - x + C.$$

例题 6.9.6

$$\int \cot^2 x \, dx$$

$$\int \cot^2 x \, dx = \int (\csc^2 x - 1) \, dx = -\cot x - x + C.$$

例题 6.9.7

$$\int \sec^2 x \, dx$$

$$\int \sec^2 x \, dx = \tan x + C.$$

例题 6.9.8

$$\int \csc^2 x \, dx$$

$$\int \csc^2 x \, dx = -\cot x + C.$$

例题 6.9.9

$$\int \sin ax \cos bx \, dx$$

$$\int \sin ax \cos bx \, dx = \frac{1}{2} \int (\sin((a+b)x) + \sin((a-b)x)) \, dx$$

$$= -\frac{1}{2} \left(\frac{\cos((a+b)x)}{a+b} + \frac{\cos((a-b)x)}{a-b} \right) + C \quad (a \neq b).$$

例题 6.9.10

$$\int \sin ax \sin bx \, dx$$

$$\begin{aligned} \int \sin ax \sin bx \, dx &= \frac{1}{2} \int (\cos((a-b)x) - \cos((a+b)x)) \, dx \\ &= \frac{1}{2} \left(\frac{\sin((a-b)x)}{a-b} - \frac{\sin((a+b)x)}{a+b} \right) + C \quad (a \neq b). \end{aligned}$$

例题 6.9.11

$$\int \cos ax \cos bx \, dx$$

$$\begin{aligned} \int \cos ax \cos bx \, dx &= \frac{1}{2} \int (\cos((a+b)x) + \cos((a-b)x)) \, dx \\ &= \frac{1}{2} \left(\frac{\sin((a+b)x)}{a+b} + \frac{\sin((a-b)x)}{a-b} \right) + C \quad (a \neq b). \end{aligned}$$

例题 6.9.12

$$\int \frac{dx}{a + b \sin x}$$

$$\begin{aligned} \int \frac{dx}{a + b \sin x} &\stackrel{t=\tan \frac{x}{2}}{=} \int \frac{1}{a + b \cdot \frac{2t}{1+t^2}} \cdot \frac{2}{1+t^2} dt = \int \frac{2}{a(1+t^2) + 2bt} dt \\ &= \int \frac{2}{at^2 + 2bt + a} dt = \frac{2}{a} \int \frac{dt}{t^2 + \frac{2b}{a}t + 1} \\ &= \frac{2}{a} \int \frac{dt}{(t + \frac{b}{a})^2 + \frac{a^2 - b^2}{a^2}}. \end{aligned}$$

记 $\Delta = a^2 - b^2$ 。若 $\Delta > 0$ ，则

$$\int \frac{dx}{a + b \sin x} = \frac{2}{\sqrt{\Delta}} \arctan \left(\frac{at + b}{\sqrt{\Delta}} \right) + C = \frac{2}{\sqrt{a^2 - b^2}} \arctan \left(\frac{a \tan \frac{x}{2} + b}{\sqrt{a^2 - b^2}} \right) + C.$$

若 $\Delta < 0$ ，则用对数表示。当 $a^2 > b^2$ 时，结果如上。

例题 6.9.13

$$\int \frac{dx}{a + b \cos x}$$

$$\int \frac{dx}{a+b\cos x} \stackrel{t=\tan \frac{x}{2}}{=} \int \frac{1}{a+b \cdot \frac{1-t^2}{1+t^2}} \cdot \frac{2}{1+t^2} dt = \int \frac{2}{(a+b) + (a-b)t^2} dt.$$

分情况:

1. 若 $a^2 > b^2$, 则

$$\int \frac{dx}{a+b\cos x} = \frac{2}{\sqrt{a^2-b^2}} \arctan \left(\sqrt{\frac{a-b}{a+b}} \tan \frac{x}{2} \right) + C.$$

2. 若 $a^2 < b^2$, 则

$$\int \frac{dx}{a+b\cos x} = \frac{1}{\sqrt{b^2-a^2}} \ln \left| \frac{\sqrt{b-a} \tan \frac{x}{2} + \sqrt{b+a}}{\sqrt{b-a} \tan \frac{x}{2} - \sqrt{b+a}} \right| + C.$$

例题 6.9.14

$$\int \frac{dx}{a^2 \sin^2 x + b^2 \cos^2 x}$$

$$\begin{aligned} \int \frac{dx}{a^2 \sin^2 x + b^2 \cos^2 x} &= \int \frac{dx}{\cos^2 x (a^2 \tan^2 x + b^2)} \stackrel{t=\tan x}{=} \int \frac{dt}{a^2 t^2 + b^2} \\ &= \frac{1}{ab} \arctan \left(\frac{at}{b} \right) + C = \frac{1}{ab} \arctan \left(\frac{a \tan x}{b} \right) + C. \end{aligned}$$

例题 6.9.15

$$\int \frac{dx}{a^2 \sin^2 x - b^2 \cos^2 x}$$

$$\begin{aligned} \int \frac{dx}{a^2 \sin^2 x - b^2 \cos^2 x} &= \int \frac{dx}{\cos^2 x (a^2 \tan^2 x - b^2)} \stackrel{t=\tan x}{=} \int \frac{dt}{a^2 t^2 - b^2} \\ &= \frac{1}{2ab} \ln \left| \frac{at-b}{at+b} \right| + C = \frac{1}{2ab} \ln \left| \frac{a \tan x - b}{a \tan x + b} \right| + C. \end{aligned}$$

例题 6.9.16

$$\int x \sin ax \, dx$$

$$\int x \sin ax \, dx = -\frac{x}{a} \cos ax + \frac{1}{a} \int \cos ax \, dx = -\frac{x}{a} \cos ax + \frac{1}{a^2} \sin ax + C.$$

例题 6.9.17

$$\int x \cos ax \, dx$$

$$\int x \cos ax \, dx = \frac{x}{a} \sin ax - \frac{1}{a} \int \sin ax \, dx = \frac{x}{a} \sin ax + \frac{1}{a^2} \cos ax + C.$$

例题 6.9.18

$$\int x^2 \sin ax \, dx$$

$$\begin{aligned} \int x^2 \sin ax \, dx &= -\frac{x^2}{a} \cos ax + \frac{2}{a} \int x \cos ax \, dx \\ &= -\frac{x^2}{a} \cos ax + \frac{2}{a} \left(\frac{x}{a} \sin ax + \frac{1}{a^2} \cos ax \right) + C \\ &= -\frac{x^2}{a} \cos ax + \frac{2x}{a^2} \sin ax + \frac{2}{a^3} \cos ax + C. \end{aligned}$$

例题 6.9.19

$$\int x^2 \cos ax \, dx$$

$$\begin{aligned} \int x^2 \cos ax \, dx &= \frac{x^2}{a} \sin ax - \frac{2}{a} \int x \sin ax \, dx \\ &= \frac{x^2}{a} \sin ax - \frac{2}{a} \left(-\frac{x}{a} \cos ax + \frac{1}{a^2} \sin ax \right) + C \\ &= \frac{x^2}{a} \sin ax + \frac{2x}{a^2} \cos ax - \frac{2}{a^3} \sin ax + C. \end{aligned}$$