# Methods of Work and Energy

Work of a Force

In everyday life by the word 'work', we refer to a vast category of jobs. This meaning is not precise enough to be used as a physical quantity. It was the practical need of scientists and engineers of the late 18th century at the start of Industrial Revolution that made necessary to define work quantitatively as a physical quantity. Physical concept of work involves a force and displacement produced.

Work of a constant Force on a body in rectilinear motion

To understand concept of work, consider a block being pulled with the help of a string on frictionless horizontal ground. Let pull $\vec{F}$ of the string on the box is constant in magnitude as well as direction. The vertical component F_y of $\vec{F}$, the weight (mg) and the normal reaction N all act on the box in vertical direction but none of them can move it unless F_y becomes greater than the weight (mg). Consider that F_y is smaller than the weight of the box. Under this condition, the box moves along the plane only due to the horizontal component F_x of the force $\vec{F}$. The weight mg, the normal reaction N from the ground and vertical component F_y all are perpendicular to the displacement therefore have no contribution in its displacement. Therefore, work is done on the box only by the horizontal component F_x of the force $\vec{F}$.
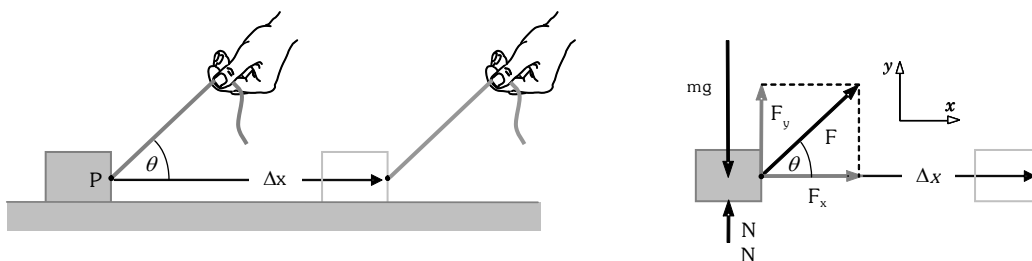
Here we must take care of one more point that is the box, which is a rigid body and undergoes translation motion therefore, displacement of every particle of the body including that on which the force is applied are equal. The particle of a body on which force acts is known as point of application of the force.

Now we observer that block is displaced & its speed is increased. And work W of the force $\vec{F}$ on the block is proportional to the product of its component in the direction of the displacement and the magnitude of the displacement Δx.

$$W \propto F_x \cdot \Delta x = F \cos\theta \cdot \Delta x$$

If we chose one unit of work as newton-meter, the constant of proportionality becomes unity and we have

$$W = F \cos\theta \cdot \Delta x = \vec{F} \cdot \Delta\vec{x}$$

The work W done by the force $\vec{F}$ is defined as scalar product of the force $\vec{F}$ and displacement $\Delta\vec{x}$ of point of application of the force.

Unit and Dimensions of Work of a Force

SI unit of work is "joule", named after famous scientist James Prescott Joule. It is abbreviated by letter J.

$$1 \text{ joule} = 1 \text{ newton} \quad 1 \text{ meter}$$

CGS unit of work is "erg". Its name is derived from the Greek ergon, meaning work.

$$1 \text{ erg} = 1 \text{ dyne} \quad 1 \text{ centimeter}$$
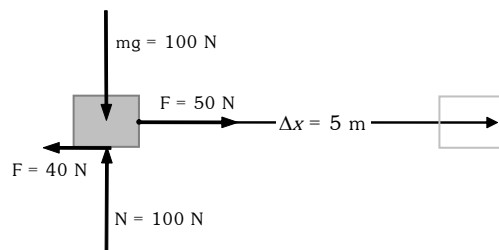
Dimensions of work are ML^2T^{-2}

Example

A 10 kg block placed on a rough horizontal floor is being pulled by a constant force 50 N. Coefficient of kinetic friction between the block and the floor is 0.4. Find work done by each individual force acting on the block over displacement of 5 m.
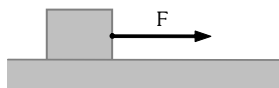
Solution.

Forces acting on the block are its weight (mg = 100 N),

normal reaction (N = 100 N) from the ground, force of

kinetic friction (f = 40 N) and the applied force (F = 50

N) and displacement of the block are shown in the given

figure.

All these force are constant force, therefore we use equation $\qquad W_{i \to f} = \vec{F} \cdot \Delta \vec{r}$.

Work done W_g by the gravity i.e. weight of the block $\qquad W_g = 0$ J $\qquad \left(\because m\vec{g} \perp \Delta \vec{x} \right)$

Work done W_N by the normal reaction $\qquad W_N = 0$ J $\qquad \left(\because \vec{N} \perp \Delta \vec{x} \right)$

Work done W_F by the applied force $\qquad W_F = 250$ J $\qquad \left(\because \vec{F} \parallel \Delta \vec{x} \right)$

Work done W_f by the force of kinetic friction $\qquad W_f = -200$ J $\qquad \left(\because \vec{f} \uparrow\downarrow \Delta \vec{x} \right)$

Example

A 10 kg block placed on a rough horizontal floor is being pulled by a constant force 100 N acting at angle 37 . Coefficient of kinetic friction between the block and the floor is 0.4. Find work done by each individual force acting on the block over displacement of 5 m.
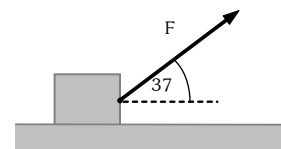
Solution.

Forces acting on the block are its weight (mg = 100 N), normal reaction (N = 40 N) from the ground, force of kinetic friction (f = 16 N) and the applied force (F = 100 N) and displacement of the block are shown in the given figure.
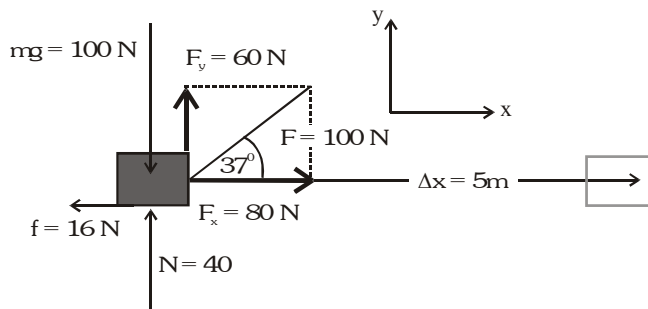
All these force are constant force, therefore we use equation $\qquad W_{i \to f} = \vec{F} \cdot \Delta \vec{r}$.

Work done W_g by the gravity i.e. weight of the block $\qquad W_g = 0$ J $\qquad \left(\because m\vec{g} \perp \Delta \vec{x} \right)$

Work done W_N by the normal reaction $\qquad W_N = 0$ J $\qquad \left(\because \vec{N} \perp \Delta \vec{x} \right)$
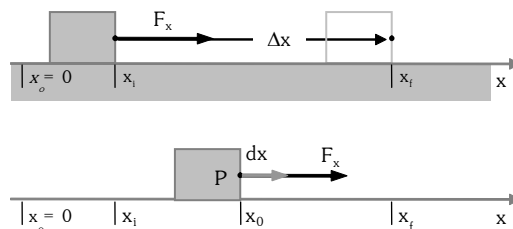
Work done W_F by the applied force $\qquad W_F = \vec{F} \cdot \Delta \vec{x} = F_x \Delta x = 400$ J

Work done W_f by the force of kinetic friction $\qquad W_f = -80$ J $\qquad \left(\because \vec{f} \uparrow\downarrow \Delta \vec{x} \right)$

## Work of a variable Force on a body in rectilinear motion

Usually a variable force does not vary appreciably during an infinitely small displacement of its point of application and therefore can be assumed constant in that infinitely small displacement.

Consider a box being pulled by a variable horizontal force F_x which is known as function of position x. We now calculate work done by this force in moving the box from position x_i to x_f. Over any infinitely small displacement dx the force does not vary appreciably and can be assumed constant. Therefore to calculate work done dW by the force F_x during infinitely small displacement dx is given by $dW = \vec{F}_x \cdot d\vec{x} = F_x dx$. Integrating dW from x_i to x_f we obtain work done by the force in moving the box from position x_i to x_f.

$$W_{i \to f} = \int_{x_i}^{x_f} F_x \cdot dx$$

The above equation also suggests that in rectilinear motion work done by a force equals to area under the force-position graph and the position axis.
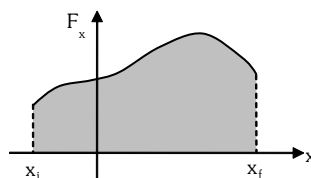
In the given figure is shown how a force F_x varies with position coordinate x. Work done by this force in moving its point of application from position x_i to x_f equals to area of the shaded portion.

Example

A force which varies with position coordinate x according to equation F_x = (4x+2) N. Here x is in meters. Calculate work done by this force in carrying a particle from position x_i = 1 m to x_f = 2 m.
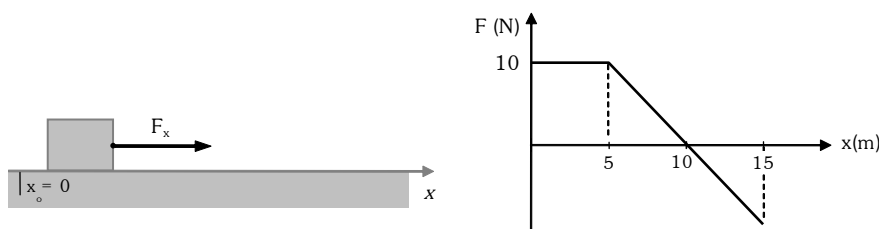
Solution.

Using the equation $W_{i \to f} = \int_{x_i}^{x_f} F_x \cdot dx$, we have $W_{i \to f} = \int_1^2 (4x + 2) \cdot dx = 8$ J

The above problem can also be solved by using graph

Example

A horizontal force F is used to pull a box placed on floor. Variation in the force with position coordinate x measured along the floor is shown in the graph.

(a) Calculate work done by the force in moving the box from x = 0 m to x = 10 m.

(b) Calculate work done by the force in moving the box from x = 10 m to x = 15 m.

(c) Calculate work done by the force in moving the box from x = 0 m to x = 15 m.

Solution.

In rectilinear motion work done by a force equals to area under the force-position graph and the position axis

(a) $W_{0\to10}$ = Area of trapazium OABC = 75 J

(b) $W_{10\to15}$ = −Area of triangle CDE = −25 J

(c) $W_{0\to15}$ = Area of trapazium OABC − Area of triangle CDE = 50 J
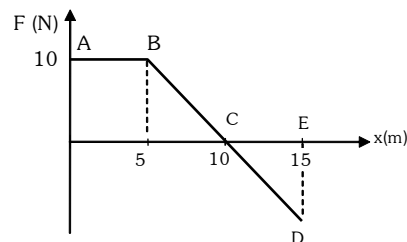
Example

A coiled spring with one end fixed has a realaxed length l_0 and a spring constant k. What amount of work must be done to stretch the spring by an amount s?

Solution.
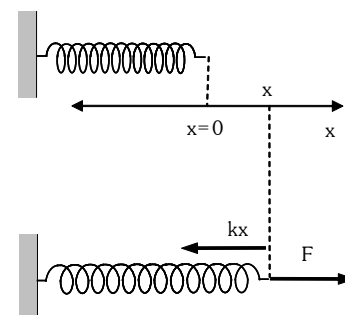
In order to stretch the free end of the spring to a point x, some agency must exert a force F, which must everywhere be equal to spring force.

$$F = kx$$

The applied force and the spring force are shown in the adjoining figure.

The work done W_F by the applied force in moving the free end of the spring from x = 0 to x = s be

$$W_F = \int_0^s \vec{F} \cdot d\vec{x} = \tfrac{1}{2}ks^2$$

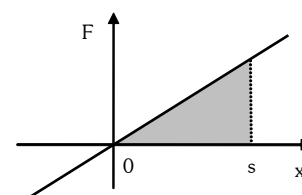**Ans.**

Use of Graph.

The variation in F with extension x in the spring is linear therefore area under the force extension graph can easily be calculated. This area equals to the work done by the applied force. The graph showing variation in F with x is shown in the adjoining figure.

$$W_F = \int_0^s \vec{F} \cdot d\vec{x} = \text{ Area of the shaded portion} = \tfrac{1}{2}ks^2$$

Variation in F with extension x.

Work of a variable Force on a body in curvilinear translation motion

Till now we have learnt how to calculate work of a force in rectilinear motion. We can extend this idea to calculate work of a variable force on any curvilinear path. To understand this let us consider a particle moving from point A to B. There may be several forces acting on it but here we show only that force whose work we want to calculate. This force may be constant or variable.

Let this force is denoted by $\vec{F}$. Consider an infinitely small path length PQ. Over this infinitely small path length, the force can be assumed constant. Work of this force $\vec{F}$ over this path length PQ is given by

$$dW = \vec{F} \cdot d\vec{r}$$

E
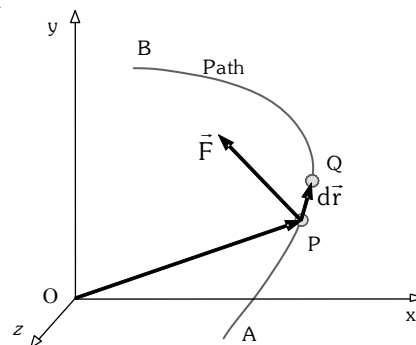
The whole path from A to B can be divided in several such infinitely small elements and work done by the force over the whole path from A to B is sum of work done over every such infinitely small element. This we can calculate by integration. Therefore, work done $W_{A \to B}$ by the force $\vec{F}$ is given by the following equation.

$$W_{A \to B} = \int_{\vec{r}_A}^{\vec{r}_B} \vec{F} \cdot d\vec{r}$$

Work of a Variable Force

For a generalization, let point A be the initial point and point B be the final point. Now we can express work $W_{i \to f}$ of a force $\vec{F}$ when its point of application moves from position vector $\vec{r}_i$ to $\vec{r}_f$ over a path by the following equation.

$$W_{i \to f} = \int_{\vec{r}_i}^{\vec{r}_f} \vec{F} \cdot d\vec{r}$$

The integration involved in the above equation must be carried over the path followed. Such kind of integration is known as path integrals.

Work of a constant Force

In simple situations where force $\vec{F}$ is constant, the above equation reduces to a simple form.

$$W_{i \to f} = \vec{F} \cdot \left(\int_{\vec{r}_i}^{\vec{r}_f} d\vec{r} \right) = \vec{F} \cdot \left(\vec{r}_f - \vec{r}_i \right) = \vec{F} \cdot \Delta\vec{r}$$

Example

Calculate work done by the force $\vec{F} = \left(3\tilde{i} - 2\tilde{j} + 4\tilde{k} \right) N$ in carrying a particle from point (–2 m, 1 m, 3 m) to (3 m, 6 m, –2 m).

Solution.

The force $\vec{F}$ is a constant force, therefore we can use equation $W_{i \to f} = \vec{F} \cdot \Delta\vec{r}$.

$W = \vec{F} \cdot \Delta\vec{r} = (3\hat{i} - 2\hat{j} + 4\hat{k}).(5\hat{i} + 5\hat{j} - 5\hat{k}) = -15$ J

Example

A particle is shifted form point (0, 0, 1 m) to (1 m, 1 m, 2 m), under simultaneous action of several forces. Two of the forces are $\vec{F}_1 = \left(2i + 3\tilde{j} - \tilde{k} \right) N$ and $\vec{F}_2 = \left(\tilde{i} - 2\tilde{j} + 2\tilde{k} \right) N$. Find work done by these two forces.

Solution.

Work done by a constant force equals to dot product of the force and displacement vectors.

$$W = \vec{F} \cdot \Delta\vec{r} \to W = \left(\vec{F}_1 + \vec{F}_2 \right) \cdot \Delta\vec{r}$$

Substituting given values, we have

$$W = \left(3\tilde{i} + \tilde{j} + \tilde{k} \right) \cdot \left(\tilde{i} + \tilde{j} + \tilde{k} \right) = 3 + 1 + 1 = 5 \text{ J}$$

Work of a force depends on frame of reference

A force does not depend on frame of reference and assumes same value in all frame of references, but displacement depends on frame of reference and may assume different values relative to different reference frames. Therefore, work of a force depends on choice of reference frame. For example, consider a man holding a suitcase stands in a lift that is moving up. In the reference frame attached with the lift, the man applies a force equal to weight of the bag but the displacement of the bag is zero, therefore work of this force on the bag is zero. However, in a reference frame attached with the ground the bag has displacement equal to that of the lift and the force applied by the man does a nonzero work.

Work and Energy

Suppose you have to push a heavy box on a rough horizontal floor. You apply a force on the box it moves and you do work. If you continue pushing, after some time you get tired and become unable to maintain your speed and eventually become unable to push the box further. You take rest and next day you can repeat the experiment and same thing happens. Why you get tired and eventually become unable to pull the box further? The explanation lies in fact that you have a capacity to do work, and when it is used up, you become unable to do work further. Next day you recollect this capacity and repeat the experiment. This capacity of doing work is known as energy. Here it comes from chemical reactions occurring with food in our body and is called chemical energy.

Consider another experiment in which we drop a block on a nail as shown in the figure. When set free, weight of the block accelerates it through the distance it falls and when it strikes the nail, its motion vanishes and what appear are the work that drives the nail, heat that increases temperature of the surrounding, and sound that causes air molecules to oscillate. If the block were placed on the nail and pressed hard, it would not have been so effective. Actually, the weight and the distance through which the hammer falls on the nail decide its effectiveness. We can explain these events by assuming that the block possesses energy due to its position at height against gravity.
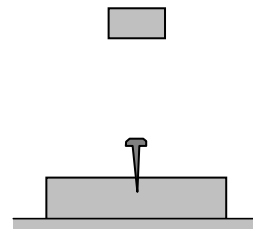
This energy is known as gravitational potential energy. When the block falls, this potential energy is converted into another form that is energy due to motion. This energy is known as kinetic energy. Moreover, when the block strikes the nail this kinetic energy is converted into work driving the nail, increasing temperature and producing sound.

Potential, Kinetic and Mechanical Energy

If a material-body is moved against a force like gravitational, electrostatic, or spring, a work must be done. In addition, if the force continues to acts even after the displacement, the work done can be recovered in form of energy, if the body is set loose. This recoverable stored energy by virtue of position in a force field is defined as potential energy, a name given by William Rankine.

All material bodies have energy due to their motion. This energy is known as kinetic energy, a name given by Lord Kelvin.

These two forms of energies - the kinetic energy and the potential energy are directly connected with motion of the body and force acting on the body respectively. They are collectively known as mechanical energy.

Other forms of Energy

Thermal energy, sound energy, chemical energy, electrical energy and nuclear energy are examples of some other forms of energy. Actually, in very fundamental way every form of energy is either kinetic or potential in nature. Thermal energy which is contribution of kinetic energy of chaotic motion of molecules in a body and potential energy due to intermolecular forces within the body. Sound energy is contribution kinetic energy of oscillating molecules and potential energy due intermolecular forces within the medium in which sound propagates. Chemical energy is contribution of potential energy due inter-atomic forces. Electric energy is

kinetic energy of moving charge carries in conductors. In addition, nuclear energy is contribution of electrostatic potential energy of nucleons.

In fact, every physical phenomenon involves in some way conversion of one form of energy into other. Whenever mechanical energy is converted into other forms or vice versa it always occurs through forces and displacements of their point of applications i.e. work. Therefore, we can say that work is measure of transfer of mechanical energy from one body to other. That is why the unit of energy is usually chosen equal to the unit of work.

Work-Kinetic Energy Theorem

Consider the situation described in the figure. The body shown is in translation motion on a curvilinear path with increasing speed. The net force acting on the body must have two components – the tangential component necessary to increase the speed and the normal component necessary to change the direction of motion. Applying Newton's laws of motion in an inertial frame, we have

$$\sum F_T = ma_T \text{ and } \sum F_N = ma_N$$
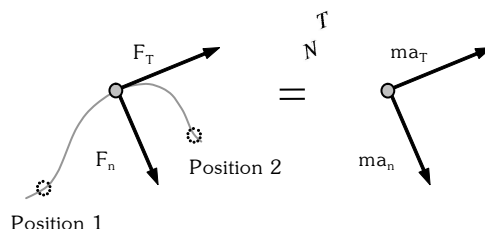
Let the body starts at position 1 with speed v_1 and reaches position 2 with speed v_2. If an infinitely small path increment is represented by vector $d\vec{s}$, the work done by the net force during the process is

$$W_{1\to 2} = \int_1^2 \vec{F} \cdot d\vec{s} = \int_1^2 \left(\vec{F}_T + \vec{F}_N\right) \cdot d\vec{s} = \int_1^2 F_T ds$$

$$W_{1\to 2} = \int_1^2 ma_T ds = \int_{v_1}^{v_2} mv dv = \tfrac{1}{2}mv_2^2 - \tfrac{1}{2}mv_1^2$$

The terms $\tfrac{1}{2}mv_1^2$ and $\tfrac{1}{2}mv_2^2$ represent the kinetic energies K_1 and K_2 of the particle at position-1 and 2 respectively. With this information the above equation reduces to $\qquad W_{1\to 2} = K_2 - K_1$

The above equation expresses that the work done by all external forces acting on a body in carrying it from one position to another equals to the change in the kinetic energy of the body between these positions. This statement is known as the work kinetic energy theorem.

How to apply work kinetic energy theorem

The work kinetic energy theorem is deduced here for a single body moving relative to an inertial frame, therefore it is recommended at present to use it for a single body in inertial frame. To use work kinetic energy theorem the following steps should be followed.

• Identify the initial and final positions as position 1 and 2 and write expressions for kinetic energies, whether known or unknown.

• Draw the free body diagram of the body at any intermediate stage between positions 1 and 2. The forces shown will help in deciding their work. Calculate work by each force and add them to obtain work done $W_{1\to 2}$ by all the forces.

• Use the work obtained in step 2 and kinetic energies in step 1 into $W_{1\to 2} = K_2 - K_1$.

Example

A 5kg ball when falls through a height of 20 m acquires a speed of 10 m/s. Find the work done by air resistance.

Solution.

The ball starts falling from position 1, where its speed is zero; hence, kinetic energy is also zero.

$$K_1 = 0 \text{ J} \qquad \dots(i)$$

During downwards motion of the ball constant gravitational force mg acts downwards and air resistance R of unknown magnitude acts upwards as shown in the free body diagram. The ball reaches position 20 m below the position-1 with a speed v = 10 m/s, so the kinetic energy of the ball at position 2 is

$$K_2 = \tfrac{1}{2}mv^2 = 250 \text{ J} \qquad \dots(ii)$$

The work done by gravity

$$W_{g,1\to2} = mgh = 1000 \text{ J} \qquad \dots(iii)$$
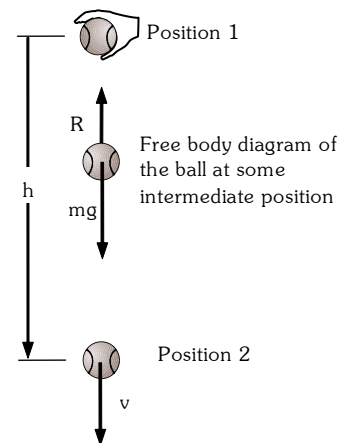
Denoting the work done by the air resistance $W_{R,1\to2}$ and making use of eq. 1, 2, and 3 in work kinetic energy theorem, we have $\qquad W_{1\to2} = K_2 - K_1 \to W_{g,1\to2} + W_{R,1\to2} = K_2 - K_1 \Rightarrow W_{R,1\to2} = -750J$
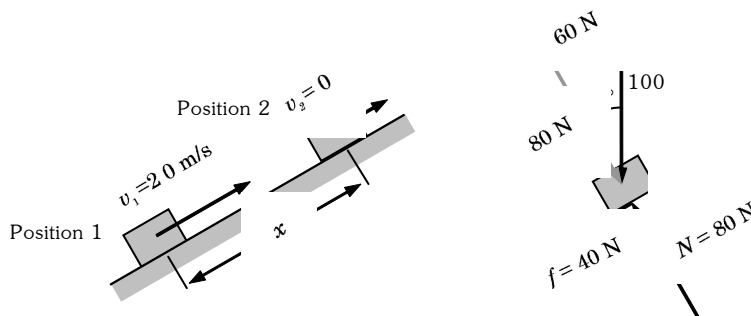
Example

A box of mass m = 10 kg is projected up an inclined plane from its foot with a speed of 20 m/s as shown in the figure. The coefficient of friction μ between the box and the plane is 0.5. Find the distance traveled by the box on the plane before it stops first time.
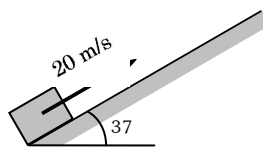
Solution.

The box starts from position 1 with speed v_1 = 20 m/s and stops at position 2.

Kinetic energy at position 1: $\quad K_1 = \tfrac{1}{2}mv_1^2 = 2000 \text{ J}$

Kinetic energy at position 2: $\quad K_2 = 0$ J

Work done by external forces as the box moves from position 1 to position 2:

$$W_{1\to2} = W_{g,1\to2} + W_{f,1\to2} = -60x - 40x = -100x \text{ J}$$

Applying work energy theorem for the motion of the box from position 1 to position 2, we have

$$W_{1\to2} = k_2 - k_1 \Rightarrow -100 x = 0 - 2000 \Rightarrow x = 20 \text{ m}$$

Example

A box of mass m is attached to one end of a coiled spring of force constant k. The other end of the spring is fixed and the box can slide on a rough horizontal surface, where the coefficient of friction is μ. The box is held against the spring force compressing the spring by a distance x_o. The spring force in this position is more than force of limiting friction. Find the speed of the box when it passes the equilibrium position, when released.
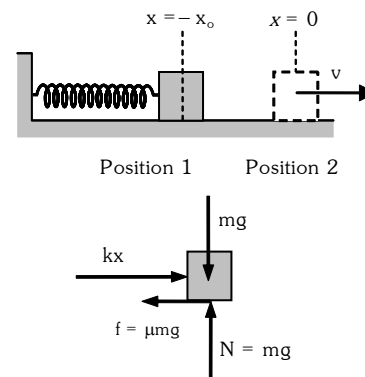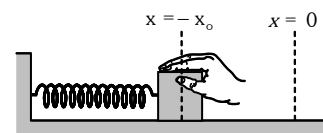
Solution.

Before the equilibrium position, when the box passes the position coordinate $-x$, forces acting on it are its weight mg, normal reaction N from the horizontal surface, the force of kinetic friction f, and spring force F = kx as shown in the free body diagram. Let the box passes the equilibrium position with a speed v_o.

Applying work energy theorem on the box when it moves from position 1 $(-x_o)$ to position 2 (x = 0), we have

$$W_{1\to2} = K_2 - K_1 \Rightarrow W_{F,1\to2} + W_{f,1\to2} = K_2 - K_1$$

$$\tfrac{1}{2}kx_o^2 - \mu mgx_o = \tfrac{1}{2}mv_o^2 - 0 \Rightarrow v_o = \sqrt{\left(kx_o^2 - 2\mu mgx_o\right)}$$

Example

A block of mass m is suspended from a spring of force constant k. it is held to keep the spring in its relaxed length as shown in the figure.

(a) The applied force is decreased gradually so that the block moves downward at negligible speed. How far below the initial position will the block stop?

(b) The applied force is removed suddenly. How far below the initial position, will the block come to an instantaneous rest?
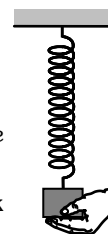
Solution.

(a) As the applied force (F) is decreased gradually, everywhere in its downward motion the block remains in the state of translational equilibrium and moves with negligible speed. Its weight (mg) is balanced by the upward spring force (kx) and the applied force. When the applied force becomes zero the spring force becomes equal to the weight and the block stops below a distance x_o from the initial position. The initial and final positions and free body diagram of the block at any intermediate position are shown in the adjoining figure. Applying the conditions of equilibrium, we have

$$x_o = \frac{mg}{k}$$

(b) In the previous situation the applied force was decreased gradually keeping the block everywhere in equilibrium. If the applied force is removed suddenly, the block will accelerate downwards. As the block moves, the increase in spring extension increases the upward force, due to which acceleration decreases until extension becomes x_o. At this extension, the block will acquire its maximum speed and it will move further downward. When extension becomes more than x_o spring force becomes more than the weight (mg) and the block decelerates and ultimately stops at a distance x_m below the initial position. The initial position-1, the final position-2, and the free body diagram of the block at some intermediate position when spring extension is x are shown in the adjoining figure.
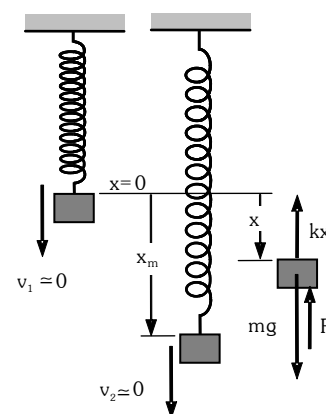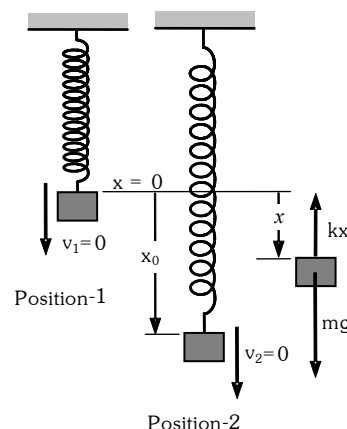
Kinetic energy in position-1 is $\quad\quad K_1 = 0$

Kinetic energy in position-2 is $\quad\quad K_2 = 0$

Work done $W_{1 \to 2}$ by gravity and the spring force is $\quad W_{1 \to 2} = W_{g,1 \to 2} + W_{spring,1 \to 2} = mgx_m - \frac{1}{2}kx_m^2$

Using above values in the work energy theorem, we have $\quad W_{1 \to 2} = K_2 - K_1 \to mgx_m - \frac{1}{2}kx_m^2 = 0$

$$x_m = 2mg/k$$

Example

A block of mass m = 0.5 kg slides from the point A on a horizontal track with an initial sped of v_1 = 3 m/s towards a weightless horizontal spring of length 1 m and force constant k = 2 N/m. The part AB of the track is frictionless and the part BD has the coefficients of static and kinetic friction as 0.22 and 0.2 respectively. If the distance AB and BC are 2 m and 2.14 m respectively, find the total distance through which the block moves before it comes to rest completely. g = 10 m/s².
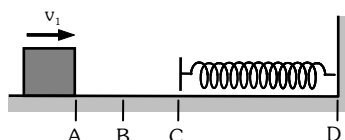
Solution

Since portion AB of the track is smooth, the block reaches B with velocity v_1. Afterward force of kinetic friction starts opposing its motion. As the block passes the point C the spring force also starts opposing its motion in addition to the force of kinetic friction. The work done by these forces decrease the kinetic energy of the block and stop the block momentarily at a distance x_m after the point C.
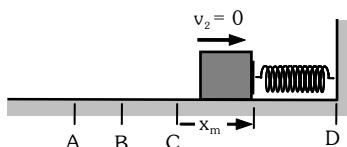
Kinetic energy of the block at position-1 is $\quad\quad\quad\quad\quad\quad K_1 = \frac{1}{2}mv_1^2 = 2.25$ J.

Kinetic energy of the block at position-2 is $\quad\quad\quad\quad\quad\quad K_2 = \frac{1}{2}mv_2^2 = 0$ J.

Work $W_{f,1 \to 2}$ done by the frictional force before the block stops is $\quad W_{f,1 \to 2} = \mu mg(BC + x_m) = 2.14 + x_m$

Work $W_{s,1 \to 2}$ done by the spring force before the block stops is $\quad W_{s,1 \to 2} = \int_{x=0}^{x_m} kxdx = \frac{1}{2}kx_m^2 = x_m^2$

Using above information and the work energy principle, we have

$$W_{1 \to 2} = K_2 - K_1 \Rightarrow 2.14 + x_m + x_m^2 = 2.25 \Rightarrow x_m = 0.1 \text{ m.}$$

The motion of block after it stops momentarily at position-2 depends upon the condition whether the spring force is more than or less than the force of limiting friction. If the spring force in position-2 is more than the force of limiting friction the block will move back and if the spring force in position-2 is less than the force of static friction the block will not move back and stop permanently.

Spring force F_s at position-2 is $\quad\quad\quad\quad\quad\quad F_s = kx_m = 0.2$ N.

The force of limiting friction f_m is $\quad\quad\quad\quad\quad\quad f_m = \mu_s mg = 1.1$ N.

The force of limiting friction is more than the spring force therefore the block will stop at position-2 permanently. The total distance traveled by the block = AB + BC + x_m =4.24 m.

Conservative and Non-conservative Forces

Gravitational, electrostatic, and restoring force of a spring are some of the natural forces with a property in common that work done by them in moving a particle from one point to another depends solely on the locations of the initial and final points and not on the path followed irrespective of pair of points selected. On the other hand, there are forces such as friction, whose work depends on path followed. Accordingly, forces are divided into two categories – one whose work is path independent and other whose work is path dependant. The forces of the former category are known as conservative forces and of the later one as non-conservative forces.

A force, whose finite non-zero work $W_{1\to2}$ expended in moving a particle from a position 1 to another position 2 is independent of the path followed, is defined as a conservative force.

Consider a particle moving from position 1 to position 2 along different paths A, B, and C under the action of a conservative force F as shown in figure. If work done by the force along path A, B, and C are $W_{1\to2,A}$, $W_{1\to2,B}$ and $W_{1\to2,C}$ respectively, we have

$$W_{1\to2,A} = W_{1\to2,B} = W_{1\to2,C}$$

If these works are positive, the work done $W_{2\to1,D}$ by the same force in moving the particle from position 2 to 1 by any other path say D will have the same magnitude but negative sign. Hence, we have

$$W_{1\to2,A} = W_{1\to2,B} = W_{1\to2,C} = -W_{2\to1,D}$$

$$W_{1\to2,A} + W_{2\to1,D} = W_{1\to2,B} + W_{2\to1,D} = W_{1\to2,C} + W_{2\to1,D} = 0$$

The above equations are true for any path between any pair of points-1 and 2.

Representing the existing conservative force by $\vec{F}$, an infinitely path increment by $d\vec{r}$ and integration over a closed path by $\oint(\ )\cdot d\vec{r}$, the above equation can be represented in an alternative form as $\oint \vec{F}\cdot d\vec{r} = 0$

The equation () shows that the total work done by a conservative force in moving a particle from position 1 to another position 2 and moving it back to position 1 i.e. around a closed path is zero. It is used as a fundamental property of a conservative force.
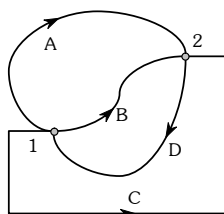
• A conservative force must be function only of position not of velocity or time.

• All uniform and constant forces are conservative forces. Here the term uniform means same magnitude and direction everywhere in the space and the term constant means same magnitude and direction at all instants of time.

• All central forces are conservative forces. A central force at any point acts always towards or away from a fixed point and its magnitude depends on the distance from the fixed point.

• All forces, whose magnitude or direction depends on the velocity, are non-conservative. Sliding friction, which acts in opposite direction to that of velocity, and viscous drag of fluid depends in magnitude of velocity and acts in opposite direction to that of velocity are non-conservative.

Potential Energy

Consider a ball of mass m placed on the ground and someone moves it at negligible speed through a height h above the ground as shown in figure. The ball remains in the state of equilibrium therefore the upward force F applied on it everywhere equals to the weight (mg) of the ball. The work $W_{a \to b}$ done by the applied force on the ball is $W_{a \to b} = \vec{F} \cdot \vec{h} = mgh$.
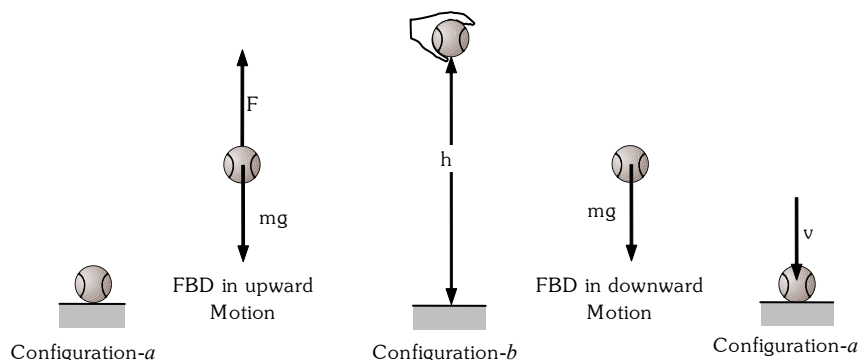

Now if the ball is dropped from the height h it starts moving downwards due to its weight and strikes the ground with speed v. The work $W_{b \to a}$ done by its weight during its downward motion imparts it a kinetic energy K_c, which is obtained by using work energy principle and the above equation as

$$W_{b \to a} = K_a - K_b \Rightarrow mgh = \tfrac{1}{2}mv^2 - 0 \Rightarrow K_a = \tfrac{1}{2}mv^2 = mgh$$

Instead of raising the ball to height h, if it were thrown upwards with a speed v it would have reached the height h and returned to the ground with the same speed. Now if we assume a new form of energy that depends on the separation between the ball and the ground, the above phenomena can be explained. This new form of energy is known as potential energy of the earth-ball system. When ball moves up, irrespective of the path or method how the ball has been moved, potential energy of the earth-ball system increases. This increase equals to work done by applied force F in moving the ball to height h or negative of work done by gravity. When the ball descends, potential energy of the earth ball system decreases; and is recovered as the kinetic energy of the ball when separation vanishes. During descend of the ball gravity does positive work, which equals to decrease in potential energy.

Potential energy of the earth ball system is due to gravitation force and therefore is call gravitation potential energy. Change in gravitational potential energy equals to negative of work done by gravitational force. It is denoted by ΔU.

In fact, when the ball is released both the ball and the earth move towards each other and acquire momenta of equal magnitude but the mass of the earth is infinitely large as compared to that of the ball, the earth acquires negligible kinetic energy. It is the ball, that acquires almost all the kinetic energy and therefore sometimes the potential energy is erroneously assigned with the ball and called the potential energy of the ball. Nevertheless, it must be kept in mind that the potential energy belongs to the entire system.

E
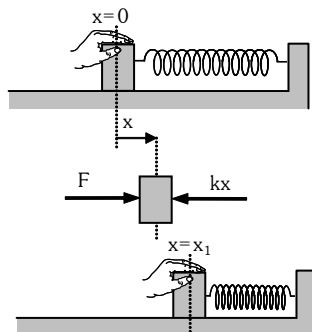
As another instance, consider a block of mass m placed on a smooth horizontal plane and connected to one end of a spring of force constant k, whose other end is connected to a fixed support. Initially, when the spring is relaxed, no net force acts on the block and it is in equilibrium at position x = 0. If the block is pushed gradually against the spring force and moves at negligible speed without acceleration, at every position x, the applied force F balances the spring force kx. The work done $W_{0\to1}$ by this force in moving the block from position x =0 to   x = x_1 is

$$W_{0\to1} = \int_{x=0}^{x=x_1} \vec{F}\cdot d\vec{x} = \tfrac{1}{2}kx_1^2$$

If the applied force is removed, the block moves back and reaches its initial position with a kinetic energy K_0, which is obtained by applying work energy theorem together with the above equation.

$$W_{1\to0} = K_0 - K_1 \Rightarrow K_0 = \tfrac{1}{2}mv^2 = \tfrac{1}{2}kx_1^2$$
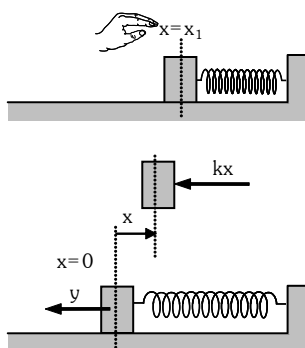
The above equation shows that the work done on the block by the applied force in moving it from x = 0 to x = x_1 is stored in the spring block system as increase in potential energy and when the block returns to its initial position x = 0 this stored potential energy decreases and is recovered as the kinetic energy of the block. The same result would have been obtained if the block were pulled elongating the spring and then released. The change in potential energy of the spring block system when the spring length is increased or decreased by x equals to negative of work done by the spring force.

 In both the above cases forces involved were conservative. In fact, work done against all conservative forces is recoverable. With every conservative force, we can associate a potential energy, whose change equals to negative of work done by the conservative force. For an infinitely small change in configuration, change in potential energy dU equals to the negative of work done dW_C by conservative forces.

$$dW = dU = - dW_C$$

Since a force is the interaction between two bodies, on very fundamental level potential energy is defined for every pair of bodies interacting with conservative forces. The potential energy of a system consisting of a large number of bodies thus will be sum of potential energies of all possible pairs of bodies constituting the system.

Because only change in potential energy has significance, we can chose potential energy of any configuration as reference value.

Gravitational potential energy for uniform gravitational force

Near the earth surface for heights small compared to the radius of the earth, the variation in the gravitational force between a body of mass m and the ground can be neglected. For such a system, change in gravitational potential energy in any vertically upward displacement h of mass m is given by $\Delta U = mgh$ and in vertical downward displacement h is given by $\Delta U = - mgh$.

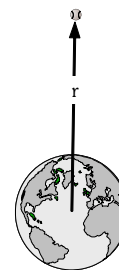## Gravitational potential energy for non-uniform gravitational force

When motion of a body of mass m involves distances from the earth surface large enough, the variation in the gravitational force between the body and the earth cannot be neglected. For such physical situations the configuration, when the body is at infinitely large distance from the earth center is taken as the reference configuration and potential energy of this configuration is arbitrarily assumed zero ($U_\infty = 0$).

If the body is brought at negligible speed to a distance r from infinitely large distance from the earth center, the work done W_g by the gravitational force is given by the following equation.

The ball at a distance *r* from the center of the earth.

$$W_g = \int_\infty^r \vec{F}_g \cdot d\vec{r} = \left[\frac{GMm}{r} \right]_\infty^r$$

Negative of this work done equals to change in potential energy of the system. Denoting potential energies in configuration of separation r and ∞ by U_r and U_∞, we have

$$U_r - U_\infty = -W_g \rightarrow U_r = -\frac{GMm}{r}$$

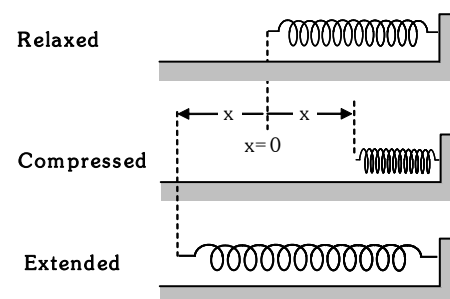## Potential energy associated with spring force

The potential energy associated with a spring force of an ideal spring when compressed or elongated by a distance x from its natural length is defined by the following equation
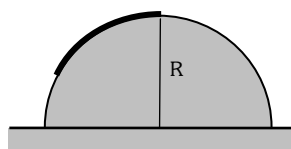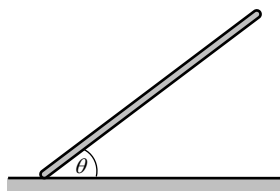
$$U = \tfrac{1}{2}kx^2$$

Relaxed

Compressed

Extended

Example

Find the gravitational potential energies in the following physical situations. Assume the ground as the reference potential energy level.

(a)   A thin rod of mass m and length L kept at angle θ with one of its end touching the ground.

(b)   A flexible rope of mass m and length L placed on a smooth hemisphere of radius R and one of the ends of the rope is fixed at the top of the hemisphere.

Solution

In both the above situations, mass is distributed over a range of position coordinates. In such situations calculate potential energy of an infinitely small portion of the body and integrate the expression obtained over the entire range of position coordinates covered by the body.

E

(a) Assume a small portion of length $\delta\ell$ of the rod at distance ℓ from the bottom end and height of the midpoint of this portion from the ground is h. Mass of this portion is δm. When $\delta\ell$ approaches to zero, the gravitational potential energy dU of the assumed portion becomes

$$dU = \frac{m}{L}ghd\ell = \frac{m}{L}g\ell\sin\theta d\ell$$
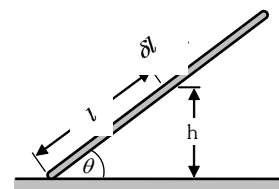
The gravitational potential energy U of the rod is obtained by carrying integration of the above equation over the entire length of the rod.

$$U = \int_{\ell=0}^{L} \frac{m}{L}g\ell\sin\theta d\ell = \tfrac{1}{2}mgL\sin\theta$$

(b) The gravitational potential energy dU of a small portion of length $\delta\ell$ shown in the adjoining figure, when $\delta\ell$ approaches to zero is

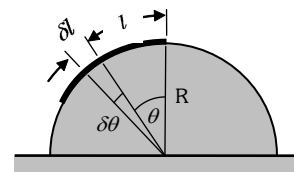$$dU = \frac{m}{L}gR^2\sin\theta d\theta$$

The gravitational potential energy U of the rope is obtained by carrying integration of the above equation over the entire length of the rope.

$$U = \int_{\theta=0}^{\theta=\frac{L}{R}} \frac{m}{L}gR^2\sin\theta d\theta = \frac{m}{L}gR^2\left\{1 - \cos\left(\frac{L}{R}\right)\right\}$$

Conservation of mechanical energy

The total potential energy of the system and the total kinetic energy of all the constituent bodies together are known as the mechanical energy of the system. If E, K, and U respectively denote the total mechanical energy, total kinetic energy, and the total potential energy of a system in any configuration, we have

E = K + U

Consider a system on which no external force acts and all the internal forces are conservative. If we apply work-kinetic energy $\left(W_{1\to2} = K_2 - K_1\right)$ theorem, the work $W_{1\to2}$ will be the work done by internal conservative forces, negative of which equals change in potential energy. Rearranging the kinetic energy and potential energy terms, we have

$$E = K_1 + U_1 = K_2 + U_2$$

The above equation takes the following forms

E = K + U = constant

$\Delta E = 0 \Rightarrow \Delta K + \Delta U = 0$

Above equations, express the principle of conservation of mechanical energy.

If there is no net work done by any external force or any internal non-conservative force, the total mechanical energy of a system is conserved.

The principle of conservation of mechanical energy is developed from the work energy principle for systems where change in configuration takes place under internal conservative forces only. Therefore, in physical situations, where external forces or non-conservative internal forces are involved, the use of work energy principle should be preferred.

In systems, where external forces or internal nonconservative forces do work, the net work done by these forces becomes equal to change in the mechanical energy of the system.

Potential energy and the associated conservative force

We know how to find potential energy associated with a conservative force. Now we learn how to obtain the conservative force if potential energy function is known. Consider work done dW by a conservative force in moving a particle through an infinitely small path length $d\vec{s}$ as shown in the figure.
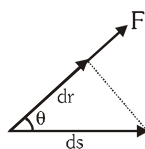
$$dU = -dW = -\vec{F} \cdot d\vec{s} = -Fds\cos\theta$$

From the above equation, the magnitude F of the conservative force can be expressed.

$$F = -\frac{dU}{ds\cos\theta} = -\frac{dU}{dr}$$

If we assume an infinitely small displacement $d\vec{r}$ in the direction of the force, magnitude of the force is given by the following equation.

$$F = -\frac{dU}{dr}$$

Here minus sign suggest that the force acts in the direction of decreasing potential energy. Therefore if we assume unit vector $\tilde{e}_r$ in the direction of $d\vec{r}$, force vector $\vec{F}$ is given by the following equation.

$$\vec{F} = -\frac{dU}{dr}\tilde{e}_r$$

Example

Force between the atoms of a diatomic molecule has its origin in the interactions between the electrons and the nuclei present in each atom. This force is conservative and associated potential energy U(r) is, to a good approximation, represented by the Lennard – Jones potential.

$$U(r) = U_o\left\{\left(\frac{a}{r}\right)^{12} - \left(\frac{a}{r}\right)^{6}\right\}$$

Here r is the distance between the two atoms and U_o and a are positive constants. Develop expression for the associated force and find the equilibrium separation between the atoms.

Solution.

Using equation $F = -\dfrac{dU}{dr}$, we obtain the expression for the force

$$F = \frac{6U_0}{a}\left\{2\left(\frac{a}{r}\right)^{13} - \left(\frac{a}{r}\right)^{7}\right\}$$

At equilibrium the force must be zero. Therefore the equilibrium separation r_o is

$$r_o = 2^{\frac{1}{6}}a$$

Potential energy and nature of equilibrium

The above equation suggests that on every location where the potential energy function assumes a minimum or a maximum value or in every region where the potential energy function assumes a constant value, the associated conservative force becomes zero and a body under the action of only this conservative force must be in the state of equilibrium. Different status of potential energy function in the state of equilibrium suggests us to define three different types of equilibriums – the stable, unstable and neutral equilibrium.

The state of stable and unstable equilibrium is associated with a point location, where the potential energy function assumes a minimum and maximum value respectively, and the neutral equilibrium is associated with region of space, where the potential energy function assumes a constant value.

For the sake of simplicity, consider a one dimensional potential energy function U of a central force F. Here r is the radial coordinate of a particle. The central force F experienced by the particle equals to the negative of the slope of the potential energy function. Variation in the force with r is also shown in the figure.

At locations $r = r_1$, $r = r_2$, and in the region $r \geq r_3$, where potential energy function assumes a minimum, a maximum, and a constant value respectively, the force becomes zero and the particle is in the state of equilibrium.
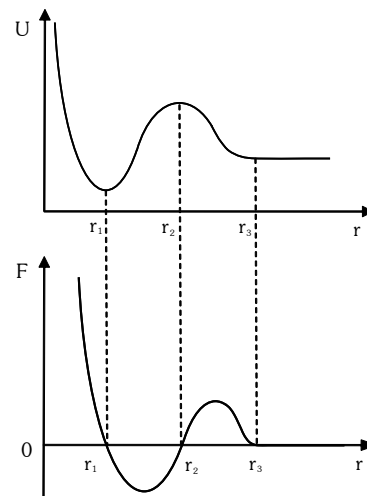
Force is negative of the slope of the potential energy function.

Stable Equilibrium.

At $r = r_1$ the potential energy function is a minima and the force on either side acts towards the point $r = r_1$. If the particle is displaced on either side and released, the force tries to restore it at $r = r_1$. At this location the particle is in the state of stable equilibrium. The dip in the potential energy curve at the location of stable equilibrium is known as potential well. A particle when disturbed from the state of stable within the potential well starts oscillations about the location of stable equilibrium. At the locations of stable equilibrium we have

$$F(r) = -\frac{\partial U}{\partial r} = 0 \; ; \; \text{and} \; \frac{\partial F}{\partial r} < 0 \; ; \; \text{and} \; \frac{\partial^2 U}{\partial r^2} > 0$$

Unstable Equilibrium.

At $r = r_2$ the potential energy function is a maxima, the force acts away from the point $r = r_2$. If the particle is displaces slightly on either side, it will not return to the location $r = r_2$. At this location, the particle is in the state of unstable equilibrium. At the locations of unstable equilibrium we have

$$F(r) = -\frac{\partial U}{\partial r} = 0 \; \text{therefore} \; \frac{\partial F}{\partial r} > 0 \; ; \; \text{and} \; \frac{\partial^2 U}{\partial r^2} < 0$$

Neutral Equilibrium.

In the region $r \geq r_3$, the potential energy function is constant and the force is zero everywhere. In this region, the particle is in the state of neutral equilibrium. At the locations of neutral equilibrium we have

$$F(r) = -\frac{\partial U}{\partial r} = 0 \; \text{therefore} \; \frac{\partial F}{\partial r} = 0 \; \text{and} \; \frac{\partial^2 U}{\partial r^2} = 0$$

POWER

When we purchase a car or jeep we are interested in the horsepower of its engine. We know that usually an engine with large horsepower is most effective in accelerating the automobile.

In many cases it is useful to know not just the total amount of work being done, but how fast the work is done. We define power as the rate at which work is being done.

$$\text{Average Power} = \frac{\text{Work done}}{\text{Time taken to do work}} = \frac{\text{Total change in kinetic energy}}{\text{Total change in time}}$$

If ΔW is the amount of work done in the time interval Δt. Then $P = \dfrac{\Delta W}{\Delta t} = \dfrac{W_2 - W_1}{t_2 - t_1}$

When work is measured in joules and t is in seconds, the unit for power is the joule per second, which is called watt. For motors and engines, power is usually measured in horsepower, where horsepower is 1 hp = 746 W. The definition of power is applicable to all types of work like mechanical, electrical, thermal.

$$\text{Instanteneous power} \quad P = \frac{dW}{dt} = \frac{\vec{F}.d\vec{r}}{dt} = \vec{F}.\vec{v}$$

Where v is the instantaneous velocity of the particle and dot product is used as only that component of force will contribute to power which is acting in the direction of instantaneous velocity.

- Power is a scalar quantity with dimension $M^1L^2T^{-3}$

- SI unit of power is J/s or watt

- 1 horsepower = 746 watt

Example

A vehicle of mass m starts moving such that its speed v varies with distance traveled s according to the law $v = k\sqrt{s}$, where k is a positive constant. Deduce a relation to express the instantaneous power delivered by its engine.

Solution

Let the particle is moving on a curvilinear path. When it has traveled a distance s, the force F acting on it and its speed v are shown in the adjoining figure.
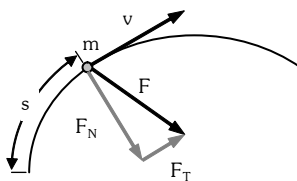
Instantaneous power delivered by the engine: $\qquad P = \vec{F}.\vec{v} = (\vec{F}_T + \vec{F}_N)\cdot\vec{v} = F_T v = ma_T v$

Tangential acceleration of the vehicle: $\qquad a_T = v\dfrac{dv}{ds} = \dfrac{k^2}{2}$

From above equations, we have $\qquad P = \dfrac{mk^3}{2}\sqrt{s}$

CIRCULAR MOTION IN VERTICAL PLANE

Suppose a particle of mass m is attached to an inextensible light string of length R. The particle is moving in a vertical circle of radius R about a fixed point O. It is imparted a velocity u in horizontal direction at lowest point A. Let v be its velocity at point P of the circle as shown in figure. Here,

$$h = R(1 - \cos\theta) \qquad \qquad ...(i)$$

From conservation of mechanical energy

$$\frac{1}{2}m(u^2 - v^2) = mgh \quad \Rightarrow \quad v^2 = u^2 - 2gh \qquad ....(ii)$$
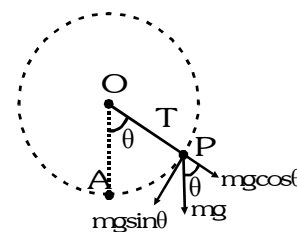
The necessary centripetal force is provided by the resultant of tension T and mg cosθ

$$T - mg\cos\theta = \frac{mv^2}{R} \qquad ...(iii)$$

Since speed of the particle decreases with height, hence tension is maximum at the bottom, where $\cos\theta = 1$ (as $\theta = 0^0$)

$$\Rightarrow \quad T_{max} = \frac{mv^2}{R} + mg; \quad T_{min} = \frac{mv'^2}{R} - mg \text{ at the top. Here, } v' = \text{ speed of the particle at the top.}$$

Condition of Looping the Loop $\left(u \geq \sqrt{5gR}\right)$

The particle will complete the circle if the string does not slack even at the highest point $(\theta = \pi)$. Thus, tension in the string should be greater than or equal to zero (T ≥ 0) at $\theta = \pi$. In critical case substituting T=0 and $\theta = \pi$

in Eq. (iii), we get mg= $\frac{mv_{min}^2}{R}$ $\Rightarrow$ $v_{min} = \sqrt{gR}$ (at highest point)

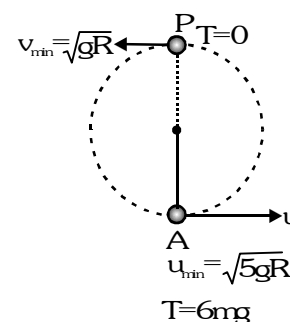Substituting $\theta = \pi$ in Eq. (i), Therefore, from Eq. (ii)

$$u_{min}^2 = v_{min}^2 + 2gh = gR + 2g(2R) = 5gR \Rightarrow u_{min} = \sqrt{5gR}$$

Thus, if $u \geq \sqrt{5gR}$, the particle will complete the circle. At u = $\sqrt{5gR}$,

velocity at highest point is v = $\sqrt{gR}$ and tension in the string is zero.

Substituting $\theta = 0^0$ and v= $\sqrt{5gR}$ in Eq. (iii), we get T =6 mg or in the critical condition tension in the string at lowest position is 6 mg. This is shown in figure. If u < $\sqrt{5gR}$, following two cases are possible.

Condition of Leaving the Circle $\left(\sqrt{2gR} < u < \sqrt{5gR}\right)$

If u < $\sqrt{5gR}$, the tension in the string will become zero before reaching the highest point. From Eq. (iii), tension in the string becomes zero (T=0) where, $\cos\theta = \frac{-v^2}{Rg}$ $\Rightarrow$ $\cos\theta = \frac{2gh - u^2}{Rg}$

Substituting, this value of $\cos\theta$ in Eq. (i), we get $\frac{2gh - u^2}{Rg} = 1 - \frac{h}{R}$ $\Rightarrow$ $h = \frac{u^2 + Rg}{3g} = h_1$ (say) .....(iv)

or we can say that at height h_1 tension in the string becomes zero. Further, if $u < \sqrt{5gR}$, velocity of the particle

becomes zero when $0 = u^2 - 2gh \Rightarrow h = \dfrac{u^2}{2g} = h_2(\text{say})...(v)$ i.e., at height h_2 velocity of particle becomes zero.

Now, the particle will leave the circle if tension in the string becomes zero but velocity is not zero. or $T = 0$ but $v \neq 0$. This is possible only when $h_1 < h_2$
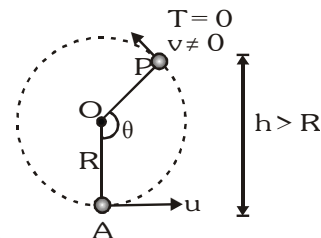
$$\Rightarrow \frac{u^2 + Rg}{3g} < \frac{u^2}{2g} \Rightarrow 2u^2 + 2Rg < 3u^2 \Rightarrow u^2 > 2Rg \Rightarrow u > \sqrt{2Rg}$$

Therefore, if $\sqrt{2gR} < u < \sqrt{5gR}$, the particle leaves the circle.

From Eq. (iv), we can see that $h > R$ if $u^2 > 2gR$. Thus, the particle, will leave

the circle when $h > R$ or $90^0 < \theta < 180^0$. This situation is shown in the figure

$$\sqrt{2gR} < u < \sqrt{5gR} \quad \text{or} \quad 90^0 < \theta < 180^0$$

Note : After leaving the circle, the particle will follow a parabolic path.

### Condition of Oscillation $\left(0 < u \leq \sqrt{2gR}\right)$

The particle will oscillate if velocity of the particle becomes zero but tension in the string is not zero or $v = 0$, but $T \neq 0$. This is possible when $h_2 < h_1$

$$\Rightarrow \frac{u^2}{2g} < \frac{u^2 + Rg}{3g} \Rightarrow 3u^2 < 2u^2 + 2Rg \Rightarrow u^2 < 2Rg \Rightarrow u < \sqrt{2Rg}$$

Moreover, if $h_1 = h_2$, $u = \sqrt{2Rg}$ and tension and velocity both becomes zero

simultaneously. Further, from Eq. (iv), we can see that $h \leq R$ if $u \leq \sqrt{2Rg}$.

Thus, for $0 < u \leq \sqrt{2gR}$, particle oscillates in lower half of the circle $(0^0 < \theta \leq 90)$

This situation is shown in the figure. $\quad 0 < u \leq \sqrt{2gR} \quad$ or $\quad 0 < \theta \leq 90$
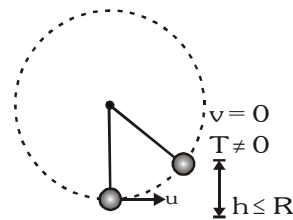
Example

Calculate following for shown situation :–
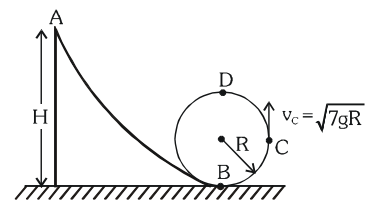
(a) Speed at D   (b) Normal reaction at D   (c) Height H

Solution

(a) $v_D^2 = v_C^2 - 2gR = 5gR \Rightarrow v_D = \sqrt{5gR}$

(b) $mg + N_D = \dfrac{mv_D^2}{R} \Rightarrow N_D = \dfrac{m(5gR)}{R} - mg = 4mg$

(c) by energy conservation between point A & C

$$mgH = \frac{1}{2}mv_C^2 + mgR = \frac{1}{2}mv_D^2 + mg2R = \frac{1}{2}m(5gR) + mg2R = \frac{9}{2}mgR \Rightarrow H = \frac{9}{2}R$$
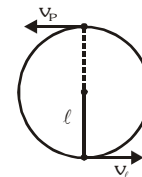
E

Example

A stone of mass 1 kg tied to a light string of length $\ell = \dfrac{10}{3}$ m is whirling in a circular path in vertical plane. If the ratio of the maximum to minimum tension in the string is 4, find the speed of the stone at the lowest and highest points

Solution

$$\therefore \ \frac{T_{max}}{T_{min}} = 4 \ \therefore \ \frac{\dfrac{mv_\ell^2}{\ell} + mg}{\dfrac{mv_p^2}{\ell} - mg} = 4 \ \Rightarrow \ \frac{v_\ell^2 + g\ell}{v_p^2 - g\ell} = 4$$

We know $v_\ell^2 = v_p^2 + 4g\ell \ \Rightarrow \ \dfrac{v_P^2 + 5g\ell}{v_p^2 - g\ell} = 4 \ \Rightarrow \ 3v_p^2 = 9g\ell$

$$\Rightarrow v_p = \sqrt{3g\ell} = \sqrt{3 \times 10 \times \frac{10}{3}} = 10 \text{ ms}^{-1} \Rightarrow v_\ell = \sqrt{7g\ell} = \sqrt{7 \times 10 \times \frac{10}{3}} = 15.2 \text{ ms}^{-1}$$

$$\Rightarrow T = (mg + ma) + 2m(g+a)(1-\cos\theta) = m(g+a)(3 - 2\cos\theta)$$

Example

A heavy particle hanging from a fixed point by a light inextensible string of length ℓ, is projected horizontally with speed $\sqrt{(g\ell)}$. Find the speed of the particle and the inclination of the string to the vertical at the instant of the motion when the tension in the string is equal to the weight of the particle.
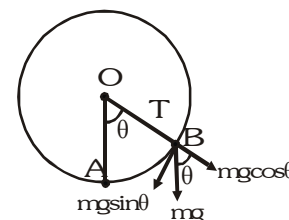
Solution

Let tension in the string becomes equal to the weight of the particle when particle reaches the point B and deflection of the string from vertical is θ. Resolving mg along the string and perpendicular to the string, we get net radial force on the particle at B i.e.

$$F_R = T - mg\cos\theta \qquad ....(i)$$

If v_B be the speed of the particle at B, then

$$F_R = \frac{mv_B^2}{\ell} \qquad ....(ii)$$

From (i) and (ii), we get, $T - mg\cos\theta = \dfrac{mv_B^2}{\ell}$ ....(iii)

Since at B, T = mg $\Rightarrow$ mg $(1-\cos\theta) = \dfrac{mv_B^2}{\ell} \Rightarrow v_B^2 = g\ell(1-\cos\theta)$ ...(iv)

Applying conservation of mechanical energy of the particle at point A and B, we have

$$\frac{1}{2}mv_A^2 = mg\ell(1-\cos\theta) + \frac{1}{2}mv_B^2 ; \text{ where } v_A = \sqrt{g\ell} \text{ and } v_B = \sqrt{g\ell(1-\cos\theta)}$$

$$\Rightarrow g\ell = 2g\ell(1-\cos\theta) + g\ell(1-\cos\theta) \Rightarrow \cos\theta = \frac{2}{3} \Rightarrow \theta = \cos^{-1}\left(\frac{2}{3}\right)$$

Putting the value of $\cos\theta$ in equation (iv), we get : $v_B = \sqrt{\dfrac{g\ell}{3}}$

SOME WORKED OUT EXAMPLES

Example #1

When a conservative force does positive work on a body, then

(A) its potential energy must increase.   (B) its potential energy must decrease.

(C) its kinetic energy must increase.   (D) its total energy must decrease.

Solution   **Ans. (B)**

Work done by conservative force = $-\Delta U$ = positive $\Rightarrow \Delta U \downarrow$
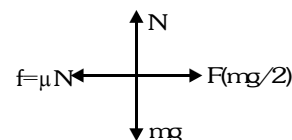
Example #2

A box of mass m is initially at rest on a horizontal surface. A constant horizontal force of mg/2 is applied to the box directed to the right. The coefficient of friction of the surface changes with the distance pushed as $\mu = \mu_0 x$ where x is the distance from the initial location. For what distance is the box pushed until it comes to rest again?

(A) $\dfrac{2}{\mu_0}$ (B) $\dfrac{1}{\mu_0}$ (C) $\dfrac{1}{2\mu_0}$ (D) $\dfrac{1}{4\mu_0}$

Solution   **Ans. (B)**

Net change in kinetic energy = 0 $\Rightarrow$ net work W = 0

$$W = \int dW = \int F dx - \int \mu N dx = \frac{mg}{2}x - mg\mu_0 \int_0^x x\,dx = 0 \Rightarrow x = \frac{1}{\mu_0}$$
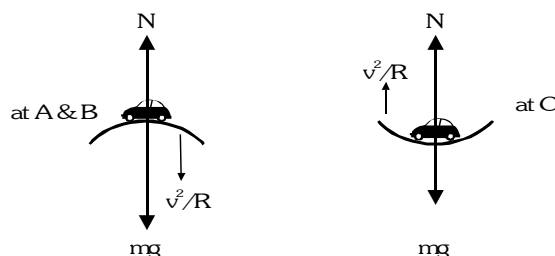

Example #3

A car is moving along a hilly road as shown (side view). The coefficient of static friction between the tyres and pavement is constant and the car maintains a steady speed. If, at one of the points shown the driver applies the brakes as hard as possible without making the tyres slip, the magnitude of the frictional force immediately after the brakes are applied will be maximum if the car was at
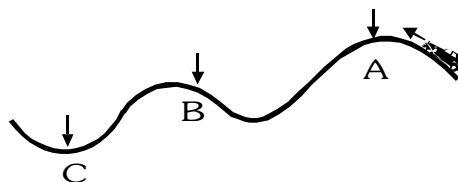
(A) point A   (B) point B   (C) point C   (D) friction force same for positions A, B and C
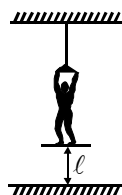
Solution   **Ans. (C)**

At A & B, N = mg$-$ mv^2/R & at C, N = mg + mv^2/R $\therefore$ $f_{max} = \mu_s N \rightarrow$ maximum for C

E

Example #4

One end of a light rope is tied directly to the ceiling. A man of mass M initially at rest on the ground starts climbing the rope hand over hand upto a height ℓ. From the time he starts at rest on the ground to the time he is hanging at rest at a height ℓ, how much work was done on the man by the rope?

(A) 0   (B) $Mg\ell$

(C) $- Mg\ell$ (D) It depends on how fast the man goes up.

Solution   **Ans. (B)**

Total work done on man = 0 $\Rightarrow$ Work done by string = – work done by gravity = –(– $Mg\ell$) = $Mg\ell$

Example #5

Consider a roller coaster with a circular loop. A roller coaster car starts from rest from the top of a hill which is 5 m higher than the top of the loop. It rolls down the hill and through the loop. What must the radius of the loop be so that the passengers of the car will feel at highest point, as if they have their normal weight?
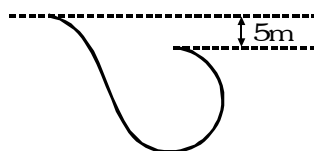
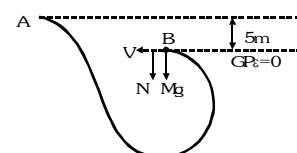(A) 5 m   (B) 10 m   (C) 15 m   (D) 20 m

Solution   **Ans. (A)**

According to mechanical energy conservation between A and B
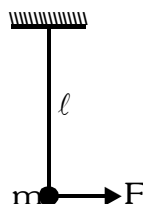
$$mg(5) = O + \frac{1}{2}mv^2 \Rightarrow v^2 = 10g \text{ ...(i)}$$

According to centripetal force equation

$$N + mg = \frac{mv^2}{r} \text{ for N = mg; } 2mg = \frac{mv^2}{r} \Rightarrow r = \frac{v^2}{2g} = \frac{10g}{2g} = 5m$$

Example #6

A pendulum bob of mass m is suspended at rest. A cosntant horiozntal force F = mg/2 starts acting on it. The maximum angular deflection of the string is

(A) 90   (B) 53   (C) 37   (D) 60

Solution   **Ans. (C)**

Let at angular deflection θ its velocity be v then by work energy theorem W= ΔKE
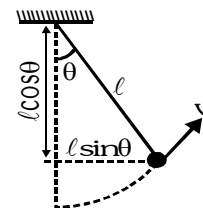
$$\frac{1}{2}mv^2 = -mg(\ell - \ell\cos\theta) + F\ell\sin\theta$$

At maximum angular deflection, v = 0

$$0 = -mg\ell\,(1-\cos\theta) + \frac{mg}{2}\sin\theta \Rightarrow 2-2\cos\theta = \sin\theta \Rightarrow 4 + 4\cos^2\theta - 8\cos\theta = \sin^2\theta = 1 - \cos^2\theta$$

$$\Rightarrow 5\cos^2\theta - 8\cos\theta + 3 = 0 \Rightarrow 5\cos^2\theta - 5\cos\theta - 3\cos\theta + 3 = 0$$

$$\Rightarrow 5\cos\theta(\cos\theta-1) - 3(\cos\theta-1) = 0 \Rightarrow (5\cos\theta-3)(\cos\theta-1) = 0$$

$$\Rightarrow \cos\theta = \frac{3}{5} \text{ or } \cos\theta = 1 \Rightarrow \theta = 37 \text{ or } \theta = 0$$

Example #7

The potential energy for the force $\vec{F} = yz\tilde{i} + xz\tilde{j} + xy\tilde{k}$, if the zero of the potential energy is to be chosen at the point (2, 2, 2), is

(A) 8 + xyz   (B) 8 – xyz   (C) 4–xyz   (D) 4 + xyz

Solution   **Ans. (B)**

$$\because \vec{F} = -\frac{\partial U}{\partial x}\tilde{i} - \frac{\partial U}{\partial y}\tilde{j} - \frac{\partial U}{\partial z}\tilde{k} \qquad \therefore \frac{\partial U}{\partial x} = -yz, \frac{\partial U}{\partial y} = -xz, \frac{\partial U}{\partial z} = -xy$$

Therefore U = – xyz + C where C = constant   As at (2, 2, 2), U =0 so C =8

OR

Objective question approach : Check that U =0 at (2, 2, 2)

Example #8

A particle is projected along the inner surface of a smooth vertical circle of radius R, its velocity at the lowest point being $\frac{1}{5}\sqrt{95Rg}$. It will leave the circle at an angular distance.... from the highest point

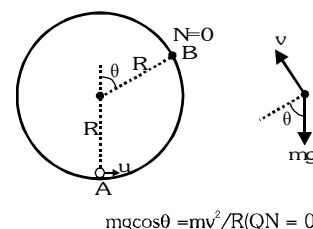(A) 37   (B) 53   (C) 60   (D) 30

Solution   **Ans. (B)**

By conservation of mechanical energy [between point A and B]

$$\frac{1}{2}mu^2 = mgR(1+\cos\theta) + \frac{1}{2}mv^2$$

$$\frac{1}{2}m\left(\frac{1}{5}\sqrt{95Rg}\right)^2 = mgR(1+\cos\theta) + \frac{1}{2}mgR\cos\theta$$

mgcosθ =mv²/R(QN = 0)

$$\Rightarrow \frac{95}{25} = 2 + 2\cos\theta + \cos\theta \Rightarrow 3\cos\theta = \frac{45}{25} \Rightarrow \cos\theta = \frac{15}{25} = \frac{3}{5} \Rightarrow \theta = 53°$$

Example #9

The upper half of an inclined plane with inclination θ is perfectly smooth while the lower half is rough. A body starting from rest at the top will again come to rest at the bottom if the coefficient of friction for the lower half is given by

(A) tanθ   (B) 2tanθ   (C) 2cosθ   (D) 2sinθ
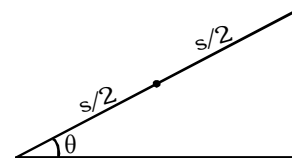
Solution   **Ans. (B)**

Refer to figure. In the journey over the upper half of incline, $v^2 - u^2 = 2$ as

$$v^2 - 0 = 2 \, (g\sin\theta) \, \frac{s}{2} = g\sin\theta.s$$

In the journey over the lower half of incline $v^2 - u^2 = 2$ as

$$0 - g\sin\theta.s = 2 \, g(\sin\theta - \mu\cos\theta)\frac{s}{2} \Rightarrow -\sin\theta = \sin\theta - \mu\cos\theta \Rightarrow \mu = \frac{2\sin\theta}{\cos\theta} = 2\tan\theta$$

Example #10

Simple pendulums P_1 and P_2 have lengths $\ell_1 = 80$ cm and $\ell_2 = 100$ cm respectively. The bobs are of masses m_1 and m_2. Initially both are at rest in equilibrium position. If each of the bobs is given a displacement of 2 cm, the work done is W_1 and W_2 respectively. Then,

(A) $W_1 > W_2$ if $m_1 = m_2$ (B) $W_1 < W_2$ if $m_1 = m_2$ (C) $W_1 = W_2$ if $\dfrac{m_1}{m_2} = \dfrac{5}{4}$ (D) $W_1 = W_2$ if $\dfrac{m_1}{m_2} = \dfrac{4}{5}$
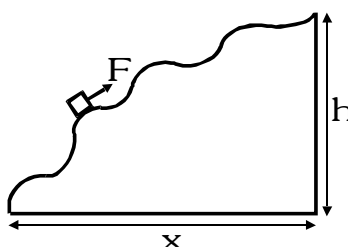
Solution   **Ans. (A,D)**

With usual notation, the height through which the bob falls is $h = \ell(1 - \cos\theta) = \ell\left(2\sin^2\dfrac{\theta}{2}\right) = 2\ell\left(\dfrac{\theta^2}{4}\right)$ since θ is

small. Therefore, we can write $h = \dfrac{\ell\theta^2}{2} = \dfrac{\ell}{2}\left(\dfrac{a}{\ell}\right)^2 = \dfrac{a^2}{2\ell}$. where a = amplitude

Thus, the work done W= P.E. = mgh = $\dfrac{mga^2}{2\ell} \Rightarrow W \propto \dfrac{1}{\ell}$

Example #11

A body of mass m is slowly halved up the rough hill by a force F at which each point is directed along a tangent to the hill.
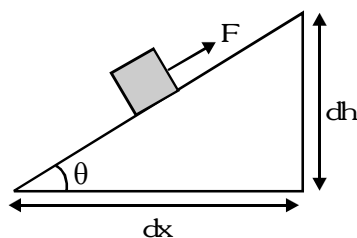

Work done by the force

(A) independent of shape of trajectory.   (B) depends upon x.

(C) depends upon h.   (D) depends upon coefficient of friction (μ)

Solution   **Ans. (ABCD)**

Work done by the force = Work done against gravity (W_g) + work done against friction (W_f)

$$= \ W_g = \int \left(mg\sin\theta\right)ds = mg\int \ ds\sin\theta = mg\int \ dh = mgh$$

and $\ W_f = \int \left(\mu mg\cos\theta\right)ds = \mu mg\int \ ds\cos\theta = \mu mg\int \ dx = \mu mgx$

Example #12

The kinetic energy of a particle continuously increase with time. Then

(A) the magnitude of its linear momentum is increasing continuously.

(B) its height above the ground must continuously decrease.

(C) the work done by all forces acting on the particle must be positive.

(D) the resultant force on the particle must be parallel to the velocity at all times.

Solution   **Ans. (A, C)**

For (A) : $p = \sqrt{2mK}$ if K $\uparrow$ then p $\uparrow$

For (B) : Its height may $\uparrow$ or $\downarrow$

For (C) : W = ΔK if ΔK = positive then W = positive

For (D) : The resultant force on the particle must be at an angle less than 90 all times

Example #13

A particle moves in one dimensional field with total mechanical energy E. If potential energy of particle is U(x), then

(A) Particle has zero speed where U(x) = E   (B) Particle has zero acceleration where U(x) = E

(C) Particle has zero velocity where $\dfrac{dU(x)}{dx} = 0$ (D) Particle has zero acceleration where $\dfrac{dU(x)}{dx} = 0$

Solution   **Ans. (A,D)**

Mechanical energy = kinetic energy + potential energy   E = K + U(x) where K = $\dfrac{1}{2}$ mv²

If K = 0 then E = U(x)

If F = 0 then $F = -\dfrac{dU(x)}{dx} = 0 \Rightarrow \dfrac{dU(x)}{dx} = 0$

Example #14

A spring block system is placed on a rough horizontal surface having coefficient of friction μ. The spring is given

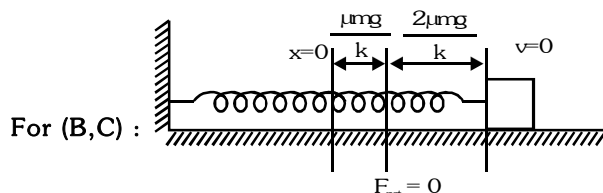initial elongation $\dfrac{3\mu mg}{k}$ and the block is released from rest. For the subsequent motion

(A) Initial acceleration of block is $2\mu g$.

(B) Maximum compression in spring is $\dfrac{\mu mg}{k}$.

(C) Minimum compression in spring is zero.

(D) Maximum speed of the block is $2\mu g\sqrt{\dfrac{m}{k}}$
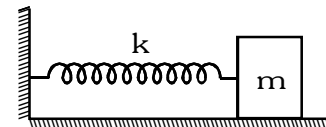
Solution   **Ans. (A,B,C,D)**

For (A) : Initial acceleration = $\dfrac{k\left(\dfrac{3\mu mg}{k}\right) - \mu mg}{m} = 2\mu g$

For (B,C) :

Therefore maximum compression = $\dfrac{2\mu mg}{k} - \dfrac{\mu mg}{k} = \dfrac{\mu mg}{k}$ and minimum compression = 0

For (D) : At maximum speed $F_{net} = 0$ so by using work energy theorem

$$\dfrac{1}{2}mv^2 = \dfrac{1}{2}k\left(\dfrac{3\mu mg}{k}\right)^2 - \dfrac{1}{2}k\left(\dfrac{\mu mg}{k}\right)^2 - \mu mg\left(\dfrac{2\mu mg}{k}\right) \Rightarrow v = 2\mu g\sqrt{m/k}$$

Example #15 to 17

A particle of mass m = 1 kg is moving along y-axis and a single conservative force F(y) acts on it. The potential energy of particle is given by $U(y) = (y^2 - 6y + 14)$ J where y is in meters. At y = 3 m the particle has kinetic energy of 15 J.

15. The total mechanical energy of the particle is

(A) 15 J   (B) 5 J   (C) 20 J   (D) can't be determined

16. The maximum speed of the particle is

(A) 5 m/s   (B) $\sqrt{30}$ m/s   (C) $\sqrt{40}$ m/s   (D) $\sqrt{10}$ m/s

17. The largest value of y (position of particle) is

(A) $3+\sqrt{5}$ (B) $3-\sqrt{5}$ (C) $3+\sqrt{15}$ (D) $6+\sqrt{15}$

Solution

15. **Ans. (C)**

Total mechanical energy = kinetic energy + potential energy = 15+ [3²–6(3)+14] = 15 +5 = 20 J

16. **Ans. (B)**

At maximum speed (i.e. maximum kinetic energy), potential energy is minimum U = $y^2 - 6y + 14 = 5 + (y-3)^2$
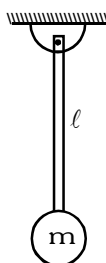
which is minimum at y=3 m so U_{min} = 5J

Therefore K_{max} = 20 – 5 = 15 J $\Rightarrow \dfrac{1}{2}mv_{max}^2 = 15 \Rightarrow v_{max} = \sqrt{30}$ m/s

17. **Ans. (C)**

For particle K ≥ 0 $\Rightarrow$ E – U ≥ 0 $\Rightarrow$ 20 – (5+(y–3)²] ≥ 0 $\Rightarrow$ (y–3)² ≤ 15 $\Rightarrow$ y–3 ≤ √15 $\Rightarrow$ y ≤ 3 + √15

Example #18 to 20

A rigid rod of length ℓ and negligible mass has a ball with mass m attached to one end and its other end fixed, to form a pendulum as shown in figure. The pendulum is inverted, with the rod straight up, and then released.

18. At the lowest point of trajectory, what is the ball's speed?

(A) $\sqrt{2g\ell}$　　　　(B) $\sqrt{4g\ell}$　　　　(C) $2\sqrt{2g\ell}$　　　　(D) $\sqrt{8g\ell}$

19. What is the tension in the rod at the lowest point of trajectory of ball?

(A) 6 mg　　　　(B) 3 mg　　　　(C) 4 mg　　　　(D) 5 mg

20. Now, if the pendulum is released from rest from a horizontal position. At what angle from the vertical does the tension in the rod equal to the weight of the ball?

(A) $\cos^{-1}\left(\dfrac{2}{3}\right)$　　　(B) $\cos^{-1}\left(\dfrac{1}{3}\right)$　　　(C) $\cos^{-1}\left(\dfrac{1}{2}\right)$　　　(D) $\cos^{-1}\left(\dfrac{1}{4}\right)$

Solution

18. **Ans. (B)**

From COME : $2mg\ell = \dfrac{1}{2}mv^2 \Rightarrow v = \sqrt{4g\ell} = 2\sqrt{g\ell}$
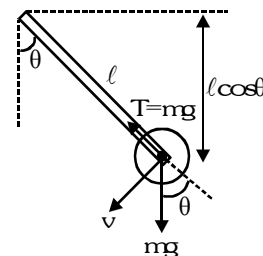
19. **Ans. (D)**

At the lowest point $T - mg = \dfrac{mv^2}{\ell} \Rightarrow T = mg + \dfrac{m}{\ell}(4g\ell) = 5mg$

E

20. **Ans. (B)**

Force equation $T - mg\cos\theta = \dfrac{mv^2}{\ell}$

Energy equation $mg\ell\cos\theta = \dfrac{1}{2}mv^2$

Therefore $mg - mg\cos\theta = 2mg\cos\theta \Rightarrow 3\cos\theta = 1 \Rightarrow \cos\theta = \dfrac{1}{3}$
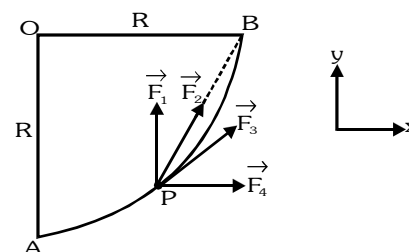
Example #21

AB is a quarter of a smooth horizontal circular track of radius R. A particle P of mass m moves along the track from A to B under the action of following forces :

$\vec{F}_1$ = F (always towards y-axis)

$\vec{F}_2$ = F (always towards point B)

$\vec{F}_3$ = F (always along the tangent to path AB)

$\vec{F}_4$ = F (always towards x-axis)

Column I		Column II	
(A)	Work done by $\vec{F}_1$	(P)	$\sqrt{2}$ FR
(B)	Work done by $\vec{F}_2$	(Q)	$\dfrac{1}{\sqrt{2}}$ FR
(C)	Work done by $\vec{F}_3$	(R)	FR
(D)	Work done by $\vec{F}_4$	(S)	$\dfrac{\pi FR}{2}$
		(T)	$\dfrac{2FR}{\pi}$

Solution   **Ans. (A) →(R); (B) →(P); (C) →(S); (D) →(R)**

For (A) : Work done by $\vec{F}_1 = FR$
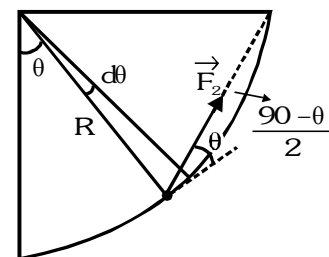
For (B) : $dW = \vec{F}.d\vec{s} = (FRd\theta)\cos\left(45 - \dfrac{\theta}{2}\right) = FR\left(45 - \dfrac{\theta}{2}\right)d\theta$

$W = \displaystyle\int_0^{\pi/4} FR\cos\left(45 - \dfrac{\theta}{2}\right) = -2FR\left(\sin 45° - \dfrac{\theta}{2}\right)_0^{\pi/2} = \sqrt{2}FR$

For (C) : $W = \int \vec{F}.d\vec{s} = F\left(\dfrac{\pi R}{2}\right) = \dfrac{\pi FR}{2}$

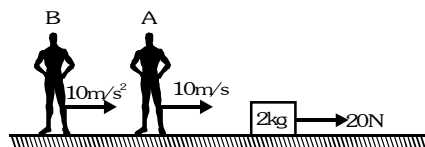**For (D) :** $W = \int \vec{F}.d\vec{s} = (F)(R) = FR$

Example #22

A block of mass 2 kg is dragged by a force of 20 N on a smooth horizontal surface. It is observed from three reference frames ground, observer A and observer B. Observer A is moving with constant velocity of 10 m/s and B is moving with constant acceleration of 10 m/s². The observer B and block starts simultaneously at t =0.

Column I	Column II
(A) Work energy theorem is applicable in	(P) 100 J
(B) Work done on block in 1s as observed by ground is	(Q) – 100 J
(C) Work done on block is 1 s as observed by observer A is	(R) zero
(D) Work done on block in 1 s as observed by observer B is	(S) only ground & A
	(T) all frames ground, A & B

Solution   **Ans. (A) →(T); (B) →(P); (C) →(Q); (D) →(R)**

For (A) :   Work energy theorem is applicable in all reference frames.

For (B) :   w.r.t. ground : At t =0, u =0 and t = 1 s, $v = at = \left(\dfrac{20}{2}\right)(1) = 10$ m/s

Work done = change in kinetic energy = $\dfrac{1}{2}(2)(10)^2 - \dfrac{1}{2}(2)(0)^2 = 100$ J

For (C) :   w.r.t. observer A : Initial velocity = 0 – 10 = – 10 m/s, Final velocity = 10 – 10 = 0

Work done = $\dfrac{1}{2}(2)(0)^2 - \dfrac{1}{2}(2)(-10)^2 = -100$ J

For (D) :   w.r.t. observer B : Initial velocity = 0 – 0 = 0

Final velocity = 10 – 10 = 0; Work done = 0