1-1 已知矩阵 $[a_{ij}]$ 为

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{33} \\ a_{31} & a_{22} & a_{33} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 2 \\ 0 & 2 & 3 \end{pmatrix}$$

求　(1) a_{ii}；(2) $a_{ij}a_{ij}$；(3)当 $i=1,k=2$ 时 $a_{ij}a_{jk}$ 的值.

解　(1)
$$a_{ii} = \sum_{i=1}^{3} a_{ii} = a_{11} + a_{22} + a_{33} = 1 + 2 + 3 = 6$$

(2)
$$a_{ij}a_{ij} = a_{ij}a_{ij} = \sum_{i=1}^{3}\sum_{i=1}^{3} a_{ij}^2 = a_{11}^2 + a_{12}^2 + a_{13}^2 + a_{21}^2 + a_{22}^2 + a_{23}^2 + a_{31}^2 + a_{32}^2 + a_{33}^2$$
$$= 1^2 + 1^2 + 0^2 + 1^2 + 2^2 + 2^2 + 0^2 + 2^2 + 3^2 = 24$$

(3)当 $i=1,k=2$ 时，
$$a_{ij}a_{jk} = a_{1j}a_{j2} = \sum_{j=1}^{3} a_{1j}a_{j2} = a_{11}a_{12} + a_{12}a_{22} + a_{13}a_{32} = 1\times1 + 1\times2 + 0\times2 = 3$$

□

1-3 将下列各式按求和约定写成展开形式：

(1) $\dfrac{\partial\sigma_{ij}}{\partial x_j} + f_i = 0$；(2) $2E_{ij} = \dfrac{\partial u_i}{\partial x_j} + \dfrac{\partial u_j}{\partial x_i} + \dfrac{\partial u_k}{\partial x_i}\dfrac{\partial u_k}{\partial x_j}$.

解　(1)
$$\frac{\partial\sigma_{i1}}{\partial x_1} + \frac{\partial\sigma_{i2}}{\partial x_2} + \frac{\partial\sigma_{i3}}{\partial x_3} + f_i = 0 \quad (i = 1,2,3)$$

(2)
$$2E_{ij} = \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \frac{\partial u_1}{\partial x_i}\frac{\partial u_1}{\partial x_j} + \frac{\partial u_2}{\partial x_i}\frac{\partial u_2}{\partial x_j} + \frac{\partial u_3}{\partial x_i}\frac{\partial u_3}{\partial x_j} \quad (i,j = 1,2,3)$$
□

1-4 以指标符号表示 Hooke 定律

$$\begin{cases} \varepsilon_{11} = \dfrac{1}{E}[\sigma_{11} - \nu(\sigma_{22} + \sigma_{33})] \\ \varepsilon_{22} = \dfrac{1}{E}[\sigma_{22} - \nu(\sigma_{33} + \sigma_{11})] \\ \varepsilon_{33} = \dfrac{1}{E}[\sigma_{33} - \nu(\sigma_{11} + \sigma_{22})] \\ \varepsilon_{12} = \dfrac{1+\nu}{E}\sigma_{12} \\ \varepsilon_{23} = \dfrac{1+\nu}{E}\sigma_{23} \\ \varepsilon_{31} = \dfrac{1+\nu}{E}\sigma_{31} \end{cases}$$

解
$$\varepsilon_{ij} = \frac{1}{E}[(1+\nu)\sigma_{ij} - \delta_{ij}\nu\sigma_{kk}]$$

□

1-8 用 ε-δ 恒等式证明

$$(\boldsymbol{s}\times\boldsymbol{t})\cdot(\boldsymbol{u}\times\boldsymbol{v})=(\boldsymbol{s}\cdot\boldsymbol{u})(\boldsymbol{t}\cdot\boldsymbol{v})-(\boldsymbol{s}\cdot\boldsymbol{v})(\boldsymbol{t}\cdot\boldsymbol{u})$$

其中 $\boldsymbol{s},\boldsymbol{t},\boldsymbol{u},\boldsymbol{v}$ 是任意矢量.

证

$$\begin{aligned}
\text{左边} &= (\boldsymbol{s}\times\boldsymbol{t})\cdot(\boldsymbol{u}\times\boldsymbol{v})=(e_{ijk}s_jt_k\boldsymbol{e}_i)\cdot(e_{lmn}u_mv_n\boldsymbol{e}_l)=e_{ijk}e_{imn}s_jt_ku_mv_n\\
&=(\delta_{jm}\delta_{kn}-\delta_{km}\delta_{jn})s_jt_ku_mv_n=\delta_{jm}\delta_{kn}s_jt_ku_mv_n-\delta_{km}\delta_{jn}s_jt_ku_mv_n\\
&=s_jt_ku_jv_k-s_nt_mu_mv_n
\end{aligned}$$

$$\text{右边}=(\boldsymbol{s}\cdot\boldsymbol{u})(\boldsymbol{t}\cdot\boldsymbol{v})-(\boldsymbol{s}\cdot\boldsymbol{v})(\boldsymbol{t}\cdot\boldsymbol{u})=s_ou_ot_pv_p-s_qv_qt_ru_r=s_jt_ku_jv_k-s_nt_mu_mv_n$$

所以左边=右边,即 $(\boldsymbol{s}\times\boldsymbol{t})\cdot(\boldsymbol{u}\times\boldsymbol{v})=(\boldsymbol{s}\cdot\boldsymbol{u})(\boldsymbol{t}\cdot\boldsymbol{v})-(\boldsymbol{s}\cdot\boldsymbol{v})(\boldsymbol{t}\cdot\boldsymbol{u})$ 成立.

□

1-10 设 T_{ij} 是 2 阶张量,利用坐标转换关系证明 T_{ii} 是标量.

证 因为 T_{ij} 是 2 阶张量,由坐标转换关系知

$$T'_{ij}=\beta_{i'm}\beta_{j'n}T_{mn}\quad(i,j=1,2,3)$$

所以

$$T'_{ii}=\beta_{i'm}\beta_{i'n}T_{mn}=\delta_{mn}T_{mn}=T_{mm}=T_{ii}$$

即 T_{ii} 是标量. □

1-12 证明:若 $\boldsymbol{A}$ 为 2 阶对称张量,$\boldsymbol{B}$ 为 2 阶反对称张量,$\boldsymbol{a}$ 为矢量,则

$$\boldsymbol{A}\cdot\boldsymbol{a}=\boldsymbol{a}\cdot\boldsymbol{A},\quad \boldsymbol{B}\cdot\boldsymbol{a}=-\boldsymbol{a}\cdot\boldsymbol{B}$$

证

$$\boldsymbol{A}\cdot\boldsymbol{a}=A_{ij}\boldsymbol{e}_i\boldsymbol{e}_j\cdot a_k\boldsymbol{e}_k=A_{ij}a_j\boldsymbol{e}_i=A_{ji}a_j\boldsymbol{e}_i=a_jA_{ji}\boldsymbol{e}_i=a_k\boldsymbol{e}_k\cdot A_{ji}\boldsymbol{e}_j\boldsymbol{e}_i=\boldsymbol{a}\cdot\boldsymbol{A}$$

$$\boldsymbol{B}\cdot\boldsymbol{a}=B_{ij}\boldsymbol{e}_i\boldsymbol{e}_j\cdot a_k\boldsymbol{e}_k=B_{ij}a_j\boldsymbol{e}_i=-B_{ji}a_j\boldsymbol{e}_i=-a_jB_{ji}\boldsymbol{e}_i=-a_k\boldsymbol{e}_k\cdot B_{ji}\boldsymbol{e}_j\boldsymbol{e}_i=-\boldsymbol{a}\cdot\boldsymbol{B}$$

□

1-19 用指标符号证明

(1) $\nabla\cdot(\nabla\times\boldsymbol{A})=0$;(2) $\nabla\times(\nabla\phi)=\boldsymbol{0}$.

证 (1)

$$\begin{aligned}
\nabla\cdot(\nabla\times\boldsymbol{A}) &= \boldsymbol{e}_i\cdot\frac{\partial}{\partial x_i}\left(\boldsymbol{e}_j\times\frac{\partial\boldsymbol{A}}{\partial x_j}\right)=\boldsymbol{e}_i\cdot\frac{\partial}{\partial x_i}\left(\frac{\partial a_k}{\partial x_j}\boldsymbol{e}_j\times\boldsymbol{e}_k\right)=\boldsymbol{e}_i\cdot\frac{\partial}{\partial x_i}\left(\frac{\partial a_k}{\partial x_j}e_{ljk}\boldsymbol{e}_l\right)\\
&=\frac{\partial}{\partial x_i}\left(\frac{\partial a_k}{\partial x_j}e_{ijk}\right)=\frac{\partial}{\partial x_j}\left(\frac{\partial a_k}{\partial x_i}e_{ijk}\right)=\frac{\partial}{\partial x_i}\left(\frac{\partial a_k}{\partial x_j}e_{jik}\right)=-\frac{\partial}{\partial x_i}\left(\frac{\partial a_k}{\partial x_j}e_{ijk}\right)
\end{aligned}$$

由

$$\frac{\partial}{\partial x_i}\left(\frac{\partial a_k}{\partial x_j}e_{ijk}\right)=-\frac{\partial}{\partial x_i}\left(\frac{\partial a_k}{\partial x_j}e_{ijk}\right)$$

得

$$\nabla\cdot(\nabla\times\boldsymbol{A})=\frac{\partial}{\partial x_i}\left(\frac{\partial a_k}{\partial x_j}e_{ijk}\right)=0$$

⇨

(2)

$$\begin{aligned}
\nabla\times(\nabla\phi) &= \boldsymbol{e}_j\times\frac{\partial}{\partial x_j}\left(\frac{\partial\phi}{\partial x_k}\boldsymbol{e}_k\right)\\
&=\frac{\partial^2\phi}{\partial x_j\partial x_k}e_{ijk}\boldsymbol{e}_i=\frac{\partial^2\phi}{\partial x_k\partial x_j}e_{ijk}\boldsymbol{e}_i=\frac{\partial^2\phi}{\partial x_j\partial x_k}e_{ikj}\boldsymbol{e}_i=-\frac{\partial^2\phi}{\partial x_j\partial x_k}e_{ijk}\boldsymbol{e}_i
\end{aligned}$$

由

$$\frac{\partial^2 \phi}{\partial x_j \partial x_k} e_{ijk} \boldsymbol{e}_i = - \frac{\partial^2 \phi}{\partial x_j \partial x_k} e_{ijk} \boldsymbol{e}_i$$

得

$$\nabla \times (\nabla \phi) = \frac{\partial^2 \phi}{\partial x_j \partial x_k} e_{ijk} \boldsymbol{e}_i = \mathbf{0}$$

□

1-21 已知 ϕ 为标量,$\boldsymbol{u}$,$\boldsymbol{f}$,$\boldsymbol{\psi}$ 为矢量,写出下列张量方程在 Descartes 坐标系中的分量形式:
(1) $G\nabla \cdot \nabla \boldsymbol{u} + (\lambda + G) \nabla \nabla \cdot \boldsymbol{u} + \boldsymbol{f} = \mathbf{0}$;(2) $\boldsymbol{u} = \nabla \phi + \nabla \times \boldsymbol{\psi}$.

解 (1)由

$$G\nabla \cdot \nabla \boldsymbol{u} = G\boldsymbol{e}_i \cdot \frac{\partial}{\partial x_i}\left(\frac{\partial u_k}{\partial x_j} \boldsymbol{e}_j \boldsymbol{e}_k\right) = G \frac{\partial^2 u_k}{\partial x_i \partial x_i} \boldsymbol{e}_k = G \frac{\partial^2 u_i}{\partial x_j \partial x_j} \boldsymbol{e}_i$$

$$(\lambda + G) \nabla \nabla \cdot \boldsymbol{u} = (\lambda + G) \boldsymbol{e}_i \frac{\partial}{\partial x_i}\left(\boldsymbol{e}_j \cdot \frac{\partial u_k}{\partial x_j} \boldsymbol{e}_k\right) = (\lambda + G) \frac{\partial^2 u_j}{\partial x_i \partial x_j} \boldsymbol{e}_i$$

得原张量方程的分量形式为

$$G \frac{\partial^2 u_i}{\partial x_j \partial x_j} + (\lambda + G) \frac{\partial^2 u_j}{\partial x_i \partial x_j} + f_i = 0$$

⇨

(2)

$$u_i = \frac{\partial \phi}{\partial x_i} + e_{ijk} \frac{\partial \psi_k}{\partial x_j}$$

□

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2-1　在物体中一点 P 的应力张量为

$$\begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{33} \\ \sigma_{31} & \sigma_{22} & \sigma_{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 & -4 \\ 0 & 3 & 0 \\ -4 & 0 & 5 \end{pmatrix}$$

（1）　求过 P 点且外法线为 $\boldsymbol{\nu} = \frac{1}{2}\boldsymbol{e}_1 - \frac{1}{2}\boldsymbol{e}_2 + \frac{1}{\sqrt{2}}\boldsymbol{e}_3$ 的面上的应力矢量 $\boldsymbol{\sigma}_{(\nu)}$ ；

（2）　求应力矢量 $\boldsymbol{\sigma}_{(\nu)}$ 的大小；

（3）　求 $\boldsymbol{\sigma}_{(\nu)}$ 与 $\boldsymbol{\nu}$ 之间的夹角；

（4）　求 $\boldsymbol{\sigma}_{(\nu)}$ 的法向分量 σ_n ；

（5）　求 $\boldsymbol{\sigma}_{(\nu)}$ 的切向分量 τ .

解

（1）　由 Cauchy 公式得所求应力矢量为

$$\boldsymbol{\sigma}_{(\nu)} = \boldsymbol{\nu} \cdot \boldsymbol{\sigma} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 1 & 0 & -4 \\ 0 & 3 & 0 \\ -4 & 0 & 5 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} - 2\sqrt{2} & -\frac{3}{2} & \frac{5\sqrt{2}}{2} - 2 \end{pmatrix}$$

$$= \left(\frac{1}{2} - 2\sqrt{2}\right)\boldsymbol{e}_1 - \frac{3}{2}\boldsymbol{e}_2 + \left(\frac{5\sqrt{2}}{2} - 2\right)\boldsymbol{e}_3$$

（2）　应力矢量 $\boldsymbol{\sigma}_{(\nu)}$ 的大小为

$$\sigma_{(\nu)} = \sqrt{\left(\frac{1}{2} - 2\sqrt{2}\right)^2 + \left(-\frac{3}{2}\right)^2 + \left(\frac{5\sqrt{2}}{2} - 2\right)^2} = \sqrt{27 - 12\sqrt{2}}$$

（3）　$\boldsymbol{\sigma}_{(\nu)}$ 与 $\boldsymbol{\nu}$ 之间的夹角为

$$\langle \boldsymbol{\sigma}_{(\nu)}, \boldsymbol{\nu} \rangle = \arccos \frac{\boldsymbol{\sigma}_{(\nu)} \cdot \boldsymbol{\nu}}{|\boldsymbol{\sigma}_{(\nu)}||\boldsymbol{\nu}|} = \arccos \frac{\left(\frac{1}{2} - 2\sqrt{2}, -\frac{3}{2}, \frac{5\sqrt{2}}{2} - 2\right) \cdot \left(\frac{1}{2}, -\frac{1}{2}, \frac{1}{\sqrt{2}}\right)}{\sqrt{27 - 12\sqrt{2}}}$$

$$= \arccos \frac{7 - 4\sqrt{2}}{2\sqrt{27 - 12\sqrt{2}}}$$

（4）　$\boldsymbol{\sigma}_{(\nu)}$ 的法向分量 σ_n 为

$$\sigma_n = \boldsymbol{\sigma}_{(\nu)} \cdot \boldsymbol{\nu} = \left(\frac{1}{2} - 2\sqrt{2}, -\frac{3}{2}, \frac{5\sqrt{2}}{2} - 2\right) \cdot \left(\frac{1}{2}, -\frac{1}{2}, \frac{1}{\sqrt{2}}\right) = \frac{7 - 4\sqrt{2}}{2}$$

（5）　$\boldsymbol{\sigma}_{(\nu)}$ 的切向分量 τ 为

$$\tau = \sqrt{\sigma_{(\nu)}^2 - \sigma_n^2} = \sqrt{\left(\sqrt{27 - 12\sqrt{2}}\right)^2 - \left(\frac{7 - 4\sqrt{2}}{2}\right)^2} = \frac{\sqrt{27 + 8\sqrt{2}}}{2}$$

$\square$

2-3　如图 2-17 所示变宽度薄板，受轴向拉伸载荷 P. $\sigma_y = \tau_{yx} = \tau_{yz} = 0$. 试根据 Cauchy 公式确定薄板两侧外表面(法线为 $\boldsymbol{\nu} = \nu_1 \boldsymbol{e}_1 + \nu_3 \boldsymbol{e}_3$)处横截面正应力 σ_z 和材料力学中常被忽略的应力 σ_x，τ_{zx} 之间的关系.

解　由 Cauchy 公式得

$$\boldsymbol{\sigma}_{(\nu)} = \boldsymbol{\nu} \cdot \boldsymbol{\sigma} = (\nu_1 \quad 0 \quad \nu_3) \begin{pmatrix} \sigma_x & 0 & \tau_{xz} \\ 0 & 0 & 0 \\ \tau_{zx} & 0 & \sigma_z \end{pmatrix} = (\nu_1 \sigma_x + \nu_3 \tau_{zx} \quad 0 \quad \nu_1 \tau_{xz} + \nu_3 \sigma_z)$$

又变截面杆的侧面无载荷，即

$$\boldsymbol{\sigma}_{(\nu)} = (0 \quad 0 \quad 0)$$

所以

$$\begin{cases} \nu_1 \sigma_x + \nu_3 \tau_{zx} = 0 \\ \nu_1 \tau_{xz} + \nu_3 \sigma_z = 0 \end{cases}$$

设 $\boldsymbol{\nu}$ 与 x 轴正方向的夹角为 α，则

$$\nu_1 = \cos\alpha, \quad \nu_3 = \sin\alpha$$

所以

$$\sigma_z = -\tau_{xz}\cot\alpha = = -\tau_{zx}\cot\alpha = \sigma_x \cot^2\alpha$$

□

2-4　如图 2-18 所示三角形截面水坝，材料密度为 ρ，承受密度为 ρ_1 的液体压力. 已求得应力解为

$$\sigma_x = ax + by$$
$$\sigma_y = cx + dy - \rho g y$$
$$\tau_{xy} = -dx - ay$$

试根据直边与斜边上的边界条件确定常数 a, b, c, d.

解　由直边 ($x = 0$) 上的边界条件得

$$\begin{cases} \sigma_x = by = -\rho_1 g y \\ \tau_{xy} = \tau_{yx} = -ay = 0 \end{cases}$$

所以

$$a = 0, \quad b = -\rho_1$$

由斜边 ($x = y\tan\beta$) 上的边界条件得

$$(\cos\beta \quad -\sin\beta \quad 0) \begin{pmatrix} -\rho_1 y & -dx & 0 \\ -dx & cx + dy - \rho g y & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
$$= (-\rho_1 y\cos\beta + dy\tan\beta\sin\beta \quad -dy\tan\beta\cos\beta - (cy\tan\beta + dy - \rho g y)\sin\beta \quad 0)$$
$$= (0 \quad 0 \quad 0)$$

所以

$$\begin{cases} -\rho_1 \cos\beta + d\tan\beta\sin\beta = 0 \\ -d\tan\beta\cos\beta - (c\tan\beta + d - \rho g)\sin\beta = 0 \end{cases}$$

解得

$$c = \rho g \cot\beta, \quad d = \rho_1 \cot^2\beta$$

□

2-7 已知老坐标系 x, y, z 中的应力张量分量 σ_{ij}，将该坐标系绕 z 轴转 θ 角而得到新坐标系 x', y', z'. 求新坐标系中的应力张量分量 σ'_{ij}.

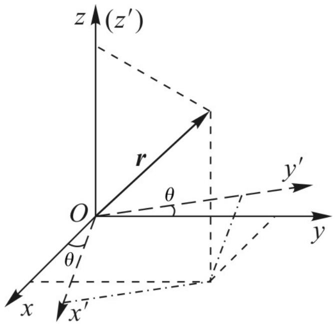


图 2.1 习题 2-7 图

解 在老坐标系 $Oxyz$ 中，

$$\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k} = (\boldsymbol{i} \quad \boldsymbol{j} \quad \boldsymbol{k})\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

在新坐标系 $Ox'y'z'$ 中，

$$\boldsymbol{r} = x'\boldsymbol{i}' + y'\boldsymbol{j}' + z'\boldsymbol{k}' = (\boldsymbol{i}' \quad \boldsymbol{j}' \quad \boldsymbol{k}')\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$$

又

$$\begin{cases} \boldsymbol{i}' = \boldsymbol{i}\cos\theta + \boldsymbol{j}\sin\theta \\ \boldsymbol{j}' = -\boldsymbol{i}\sin\theta + \boldsymbol{j}\cos\theta \\ \boldsymbol{k}' = \boldsymbol{k} \end{cases}$$

用矩阵形式表示为

$$(\boldsymbol{i}' \quad \boldsymbol{j}' \quad \boldsymbol{k}') = (\boldsymbol{i} \quad \boldsymbol{j} \quad \boldsymbol{k})\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

由矢量的不变性得

$$(\boldsymbol{i} \quad \boldsymbol{j} \quad \boldsymbol{k})\begin{pmatrix} x \\ y \\ z \end{pmatrix} = (\boldsymbol{i}' \quad \boldsymbol{j}' \quad \boldsymbol{k})\begin{pmatrix} x' \\ y' \\ z \end{pmatrix} = (\boldsymbol{i} \quad \boldsymbol{j} \quad \boldsymbol{k})\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x' \\ y' \\ z \end{pmatrix}$$

由 $\boldsymbol{r}$ 的任意性得

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

所以坐标转换系数矩阵为

$$\begin{pmatrix} \beta_{1'1} & \beta_{1'2} & \beta_{1'3} \\ \beta_{2'1} & \beta_{2'2} & \beta_{2'3} \\ \beta_{3'1} & \beta_{3'2} & \beta_{3'3} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

由

$$\sigma'_{ij} = \beta_{i'm}\beta_{j'm}\sigma_{mn}$$

即

$$\begin{pmatrix}\sigma'_{11} & \sigma'_{12} & \sigma'_{13}\\ \sigma'_{21} & \sigma'_{22} & \sigma'_{23}\\ \sigma'_{31} & \sigma'_{32} & \sigma'_{33}\end{pmatrix}=\begin{pmatrix}\beta_{1'1} & \beta_{1'2} & \beta_{1'3}\\ \beta_{2'1} & \beta_{2'2} & \beta_{2'3}\\ \beta_{3'1} & \beta_{3'2} & \beta_{3'3}\end{pmatrix}\begin{pmatrix}\sigma_{11} & \sigma_{12} & \sigma_{13}\\ \sigma_{21} & \sigma_{22} & \sigma_{33}\\ \sigma_{31} & \sigma_{22} & \sigma_{33}\end{pmatrix}\begin{pmatrix}\beta_{1'1} & \beta_{1'2} & \beta_{1'3}\\ \beta_{2'1} & \beta_{2'2} & \beta_{2'3}\\ \beta_{3'1} & \beta_{3'2} & \beta_{3'3}\end{pmatrix}^{\mathrm{T}}$$

$$=\begin{pmatrix}\cos\theta & \sin\theta & 0\\ -\sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\sigma_{11} & \sigma_{12} & \sigma_{13}\\ \sigma_{21} & \sigma_{22} & \sigma_{33}\\ \sigma_{31} & \sigma_{22} & \sigma_{33}\end{pmatrix}\begin{pmatrix}\cos\theta & \sin\theta & 0\\ -\sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{pmatrix}^{\mathrm{T}}$$

$$=\begin{pmatrix}\cos\theta & \sin\theta & 0\\ -\sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\sigma_{11} & \sigma_{12} & \sigma_{13}\\ \sigma_{21} & \sigma_{22} & \sigma_{33}\\ \sigma_{31} & \sigma_{22} & \sigma_{33}\end{pmatrix}\begin{pmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{pmatrix}$$

得新坐标系中的应力张量分量为

$$\begin{aligned}
\sigma'_{11} &=\sigma_{11}\cos^2\theta+\sigma_{22}\sin^2\theta+\sigma_{12}\cos\theta\sin\theta+\sigma_{21}\cos\theta\sin\theta\\
\sigma'_{12} &=-\sigma_{11}\cos\theta\sin\theta+\sigma_{22}\cos\theta\sin\theta+\sigma_{12}\cos^2\theta-\sigma_{21}\sin^2\theta\\
\sigma'_{13} &=\sigma_{23}\sin\theta+\sigma_{13}\cos\theta\\
\sigma'_{21} &=-\sigma_{11}\cos\theta\sin\theta+\sigma_{22}\cos\theta\sin\theta+\sigma_{21}\cos^2\theta-\sigma_{12}\sin^2\theta\\
\sigma'_{22} &=\sigma_{11}\sin^2\theta+\sigma_{22}\cos^2\theta-\sigma_{12}\cos\theta\sin\theta-\sigma_{21}\cos\theta\sin\theta\\
\sigma'_{23} &=\sigma_{23}\cos\theta-\sigma_{13}\sin\theta\\
\sigma'_{31} &=\sigma_{32}\sin\theta+\sigma_{31}\cos\theta\\
\sigma'_{32} &=\sigma_{32}\cos\theta-\sigma_{31}\sin\theta\\
\sigma'_{33} &=\sigma_{33}
\end{aligned}$$

□

2-9 若 I_1,I_2,I_3 为应力张量的第一,第二,第三不变量,I'_1,I'_2,I'_3,为应力偏量的第一,第二,第三不变量.证明

$$I'_2=-\left(\frac{1}{3}I_1^2-I_2\right),\quad I'_3=I_3-\frac{1}{3}I_1I_2+\frac{2}{27}I_1^3$$

证 设应力张量为 σ_{ij},则应力偏量为 $\sigma'_{ij}=\sigma_{ij}-\sigma_0\delta_{ij}$,其中 $\sigma_0=\frac{1}{3}\sigma_{kk}=\frac{1}{3}I_1$.所以

$$I_1=\sigma_{kk}=\sigma_{11}+\sigma_{22}+\sigma_{33};$$

$$I_2=\frac{1}{2}(\sigma_{ii}\sigma_{jj}-\sigma_{ij}\sigma_{ij})=\frac{1}{2}(I_1^2-\sigma_{ij}\sigma_{ij})=\sigma_{11}\sigma_{22}+\sigma_{22}\sigma_{33}+\sigma_{33}\sigma_{11}-(\sigma_{12}^2+\sigma_{23}^2+\sigma_{31}^2);$$

$$\begin{aligned}
I_3 &=e_{ijk}\sigma_{1i}\sigma_{2j}\sigma_{3k}=\frac{1}{3}\sigma_{ij}\sigma_{jk}\sigma_{ki}+I_1\left(I_2-\frac{1}{3}I_1^2\right)\\
&=\sigma_{11}\sigma_{22}\sigma_{33}+2\sigma_{12}\sigma_{23}\sigma_{31}-\sigma_{11}\sigma_{23}^2-\sigma_{22}\sigma_{31}^2-\sigma_{33}\sigma_{12}^2;
\end{aligned}$$

$$I'_1=\sigma'_{kk}=\sigma'_{11}+\sigma'_{22}+\sigma'_{33}=0;$$

$$\begin{aligned}
I'_2 &=\frac{1}{2}(\sigma'_{ii}\sigma'_{jj}-\sigma'_{ij}\sigma'_{ij})=\frac{1}{2}(I_1'^2-\sigma'_{ij}\sigma'_{ij})=-\frac{1}{2}\sigma'_{ij}\sigma'_{ij}\\
&=-\frac{1}{2}(\sigma_{11}'^2+\sigma_{22}'^2+\sigma_{33}'^2+2\sigma_{12}^2+2\sigma_{23}^2+2\sigma_{31}^2)\\
&=-\frac{1}{2}[(\sigma_{11}-\sigma_0)^2+(\sigma_{22}-\sigma_0)^2+(\sigma_{33}-\sigma_0)^2+2\sigma_{12}^2+2\sigma_{23}^2+2\sigma_{31}^2]
\end{aligned}$$

$$=-\frac{1}{2}\left[\left(\frac{2}{3}\sigma_{11}-\frac{1}{3}\sigma_{22}-\frac{1}{3}\sigma_{33}\right)^2+\left(\frac{2}{3}\sigma_{22}-\frac{1}{3}\sigma_{33}-\frac{1}{3}\sigma_{11}\right)^2+\left(\frac{2}{3}\sigma_{33}-\frac{1}{3}\sigma_{11}-\frac{1}{3}\sigma_{22}\right)^2\right]$$
$$-(\sigma_{12}^2+\sigma_{23}^2+\sigma_{31}^2)$$
$$=-\frac{1}{2}\left[\frac{2}{3}(\sigma_{11}^2+\sigma_{22}^2+\sigma_{33}^2-\sigma_{11}\sigma_{22}-\sigma_{22}\sigma_{33}-\sigma_{33}\sigma_{11})\right]-(\sigma_{12}^2+\sigma_{23}^2+\sigma_{31}^2)$$
$$=-\frac{1}{3}(\sigma_{11}^2+\sigma_{22}^2+\sigma_{33}^2+2\sigma_{11}\sigma_{22}+2\sigma_{22}\sigma_{33}+2\sigma_{33}\sigma_{11})$$
$$+(\sigma_{11}\sigma_{22}+\sigma_{22}\sigma_{33}+\sigma_{33}\sigma_{11})-(\sigma_{12}^2+\sigma_{23}^2+\sigma_{31}^2)$$
$$=-\frac{1}{3}I_1^2+I_2$$
$$=-\left(\frac{1}{3}I_1^2-I_2\right);$$

$$I_3'=\frac{1}{3}\sigma_{ij}'\sigma_{jk}'\sigma_{ki}'+I_1'\left(I_2'-\frac{1}{3}I_1'^2\right)=\frac{1}{3}\sigma_{ij}'\sigma_{jk}'\sigma_{ki}'$$
$$=\frac{1}{3}(\sigma_{11}'^3+\sigma_{22}'^3+\sigma_{33}'^3)+\sigma_{11}\sigma_{12}^2+\sigma_{11}\sigma_{31}^2+\sigma_{22}\sigma_{12}^2+\sigma_{22}\sigma_{23}^2+\sigma_{33}\sigma_{23}^2+\sigma_{33}\sigma_{31}^2+2\sigma_{12}\sigma_{23}\sigma_{31}$$
$$=\frac{1}{3}\left[\left(\frac{2}{3}\sigma_{11}-\frac{1}{3}\sigma_{22}-\frac{1}{3}\sigma_{33}\right)^3+\left(\frac{2}{3}\sigma_{22}-\frac{1}{3}\sigma_{33}-\frac{1}{3}\sigma_{11}\right)^3+\left(\frac{2}{3}\sigma_{33}-\frac{1}{3}\sigma_{11}-\frac{1}{3}\sigma_{22}\right)^3\right]$$
$$+\left(\frac{2}{3}\sigma_{11}-\frac{1}{3}\sigma_{22}-\frac{1}{3}\sigma_{33}\right)\sigma_{12}^2+\left(\frac{2}{3}\sigma_{11}-\frac{1}{3}\sigma_{22}-\frac{1}{3}\sigma_{33}\right)\sigma_{31}^2$$
$$+\left(\frac{2}{3}\sigma_{22}-\frac{1}{3}\sigma_{33}-\frac{1}{3}\sigma_{11}\right)\sigma_{12}^2+\left(\frac{2}{3}\sigma_{22}-\frac{1}{3}\sigma_{33}-\frac{1}{3}\sigma_{11}\right)\sigma_{23}^2$$
$$+\left(\frac{2}{3}\sigma_{33}-\frac{1}{3}\sigma_{11}-\frac{1}{3}\sigma_{22}\right)\sigma_{23}^2+\left(\frac{2}{3}\sigma_{33}-\frac{1}{3}\sigma_{11}-\frac{1}{3}\sigma_{22}\right)\sigma_{31}^2$$
$$+2\sigma_{12}\sigma_{23}\sigma_{31}$$
$$=\frac{2}{27}\sigma_{11}^3+\frac{2}{27}\sigma_{22}^3+\frac{2}{27}\sigma_{33}^3+\frac{4}{9}\sigma_{11}\sigma_{22}\sigma_{33}$$
$$+\frac{1}{3}\sigma_{11}\sigma_{12}^2-\frac{1}{9}\sigma_{11}\sigma_{22}^2-\frac{2}{3}\sigma_{11}\sigma_{23}^2+\frac{1}{3}\sigma_{11}\sigma_{31}^2-\frac{1}{9}\sigma_{11}\sigma_{33}^2$$
$$-\frac{1}{9}\sigma_{11}^2\sigma_{22}+\frac{1}{3}\sigma_{12}^2\sigma_{22}+\frac{1}{3}\sigma_{22}\sigma_{23}^2-\frac{2}{3}\sigma_{22}\sigma_{31}^2-\frac{1}{9}\sigma_{22}\sigma_{33}^2$$
$$-\frac{1}{9}\sigma_{11}^2\sigma_{33}-\frac{2}{3}\sigma_{12}^2\sigma_{33}+\frac{1}{3}\sigma_{23}^2\sigma_{33}+\frac{1}{3}\sigma_{31}^2\sigma_{33}-\frac{1}{9}\sigma_{22}^2\sigma_{33}$$
$$+2\sigma_{12}\sigma_{23}\sigma_{31}$$
$$=\sigma_{11}\sigma_{22}\sigma_{33}+2\sigma_{12}\sigma_{23}\sigma_{31}-\sigma_{11}\sigma_{23}^2-\sigma_{22}\sigma_{31}^2-\sigma_{33}\sigma_{12}^2$$
$$-\frac{1}{3}(\sigma_{11}+\sigma_{22}+\sigma_{33})[\sigma_{11}\sigma_{22}+\sigma_{22}\sigma_{33}+\sigma_{33}\sigma_{11}-(\sigma_{12}^2+\sigma_{23}^2+\sigma_{31}^2)]+\frac{2}{27}(\sigma_{11}+\sigma_{22}+\sigma_{33})^3$$
$$=I_3-\frac{1}{3}I_1I_2+\frac{2}{27}I_1^3.$$

□

2-11　取主轴为参考轴，$\sigma_1,\sigma_2,\sigma_3$ 为主应力. 证明：在单位法向矢量为 $\boldsymbol{\nu}$ 的任何平面上剪应力大小为

$$\tau=\sqrt{\nu_1^2\nu_2^2(\sigma_1-\sigma_2)^2+\nu_2^2\nu_3^2(\sigma_2-\sigma_3)^2+\nu_3^2\nu_1^2(\sigma_3-\sigma_1)^2}=|\boldsymbol{\nu}\times(\boldsymbol{\sigma}\cdot\boldsymbol{\nu})|$$

证　因为

$$\nu_1^2+\nu_2^2+\nu_3^2=1$$

所以

$$\begin{aligned}\tau^2 &= \sigma_\nu^2 - \sigma_n^2 = \nu_i^2\sigma_i^2 - (\nu_i^2\sigma_i)^2 \\ &= (\nu_1^2\sigma_1^2 + \nu_2^2\sigma_2^2 + \nu_3^2\sigma_3^2) - (\nu_1^4\sigma_1^2 + \nu_2^4\sigma_2^2 + \nu_3^4\sigma_3^2 + 2\nu_1^2\sigma_1\nu_2^2\sigma_2 + 2\nu_2^2\sigma_2\nu_3^2\sigma_3 + 2\nu_3^2\sigma_3\nu_1^2\sigma_1) \\ &= \nu_1^2\sigma_1^2(1-\nu_1^2) + \nu_2^2\sigma_2^2(1-\nu_2^2) + \nu_3^2\sigma_3^2(1-\nu_3^2) - (2\nu_1^2\sigma_1\nu_2^2\sigma_2 + 2\nu_2^2\sigma_2\nu_3^2\sigma_3 + 2\nu_3^2\sigma_3\nu_1^2\sigma_1) \\ &= \nu_1^2\sigma_1^2(\nu_2^2+\nu_3^2) + \nu_2^2\sigma_2^2(\nu_3^2+\nu_1^2) + \nu_3^2\sigma_3^2(\nu_1^2+\nu_2^2) - (2\nu_1^2\sigma_1\nu_2^2\sigma_2 + 2\nu_2^2\sigma_2\nu_3^2\sigma_3 + 2\nu_3^2\sigma_3\nu_1^2\sigma_1) \\ &= \nu_1^2\nu_2^2(\sigma_1^2 - 2\sigma_1\sigma_2 + \sigma_2^2) + \nu_2^2\nu_3^2(\sigma_2^2 - 2\sigma_2\sigma_3 + \sigma_3^2) + \nu_3^2\nu_1^2(\sigma_3^2 - 2\sigma_3\sigma_1 + \sigma_1^2) \\ &= \nu_1^2\nu_2^2(\sigma_1-\sigma_2)^2 + \nu_2^2\nu_3^2(\sigma_2-\sigma_3)^2 + \nu_3^2\nu_1^2(\sigma_3-\sigma_1)^2\end{aligned}$$

又

$$\begin{aligned}|\boldsymbol{\nu}\times(\boldsymbol{\sigma}\cdot\boldsymbol{\nu})|^2 &= |(\nu_1,\nu_2,\nu_3)\times(\nu_1\sigma_1,\nu_2\sigma_2,\nu_3\sigma_3)|^2 \\ &= (\nu_2\nu_3\sigma_3 - \nu_3\nu_2\sigma_2)^2 + (\nu_3\nu_1\sigma_1 - \nu_1\nu_3\sigma_3)^2 + (\nu_1\nu_2\sigma_2 - \nu_2\nu_1\sigma_1)^2 \\ &= \nu_1^2\nu_2^2(\sigma_1-\sigma_2)^2 + \nu_2^2\nu_3^2(\sigma_2-\sigma_3)^2 + \nu_3^2\nu_1^2(\sigma_3-\sigma_1)^2\end{aligned}$$

所以

$$\tau = \sqrt{\nu_1^2\nu_2^2(\sigma_1-\sigma_2)^2 + \nu_2^2\nu_3^2(\sigma_2-\sigma_3)^2 + \nu_3^2\nu_1^2(\sigma_3-\sigma_1)^2} = |\boldsymbol{\nu}\times(\boldsymbol{\sigma}\cdot\boldsymbol{\nu})|$$

□

2-13　图 2-20 所示悬臂薄板，已知板内的应力分量为

$$\sigma_x = ax;\quad \sigma_y = a(2x+y-l-h);\quad \tau_{xy} = -ax$$

其中 a 为常数，其余应力分量为 0.

求此薄板所受的边界载荷及体力，并在图 2.2 上画出边界载荷.

解　边界 $x=0$ 上的载荷为

$$\sigma_x = 0,\quad \tau_{xy} = 0\quad (0\leqslant y\leqslant l)$$

边界 $y=l$ 上的载荷为

$$\sigma_y = a(2x-h),\quad \tau_{xy} = -ax\quad (0\leqslant x\leqslant h)$$

所以

$$F_x = \int_0^h \tau_{xy}\,\mathrm{d}x = \int_0^h(-ax)\,\mathrm{d}x = -\frac{1}{2}ah^2$$

$$F_y = \int_0^h \sigma_y\,\mathrm{d}x = \int_0^h a(2x-h)\,\mathrm{d}x = 0$$

$$M = \int_0^h \sigma_y x\,\mathrm{d}x = \int_0^h a(2x-h)x\,\mathrm{d}x = \frac{1}{6}ah^3$$

边界 $x+y=l+h$ 上的载荷为

$$\sigma_x = ax,\quad \sigma_y = ax,\quad \tau_{xy} = -ax\quad (l\leqslant x\leqslant h+l)$$

坐标变换得

$$\begin{aligned}\sigma &= \left(\frac{\sqrt{2}}{2}\right)^2\sigma_x + \left(\frac{\sqrt{2}}{2}\right)^2\sigma_y + 2\tau_{xy}\frac{\sqrt{2}}{2}\frac{\sqrt{2}}{2} \\ &= \frac{1}{2}ax + \frac{1}{2}ax - 2ax\frac{\sqrt{2}}{2}\frac{\sqrt{2}}{2} = 0\end{aligned}$$

$$\begin{aligned}\tau &= -\sigma_x\frac{\sqrt{2}}{2}\frac{\sqrt{2}}{2} + \sigma_y\frac{\sqrt{2}}{2}\frac{\sqrt{2}}{2} + \tau_{xy}\left(\frac{\sqrt{2}}{2}\right)^2 - \tau_{xy}\left(\frac{\sqrt{2}}{2}\right)^2 \\ &= -ax\frac{\sqrt{2}}{2}\frac{\sqrt{2}}{2} + ax\frac{\sqrt{2}}{2}\frac{\sqrt{2}}{2} - ax\left(\frac{\sqrt{2}}{2}\right)^2 + ax\left(\frac{\sqrt{2}}{2}\right)^2 = 0\end{aligned}$$

即边界 $x+y=l+h$ 上无外载荷.由平衡方程

$$\begin{cases}\dfrac{\partial\sigma_x}{\partial x}+\dfrac{\partial\tau_{yx}}{\partial y}+f_x=a+0+f_x=0\\[2ex]\dfrac{\partial\tau_{xy}}{\partial x}+\dfrac{\partial\sigma_y}{\partial y}+f_y=-a+a+f_y=0\end{cases}$$

得体力为

$$f_x=-a,\quad f_y=0$$

边界载荷如图 2.2 所示.

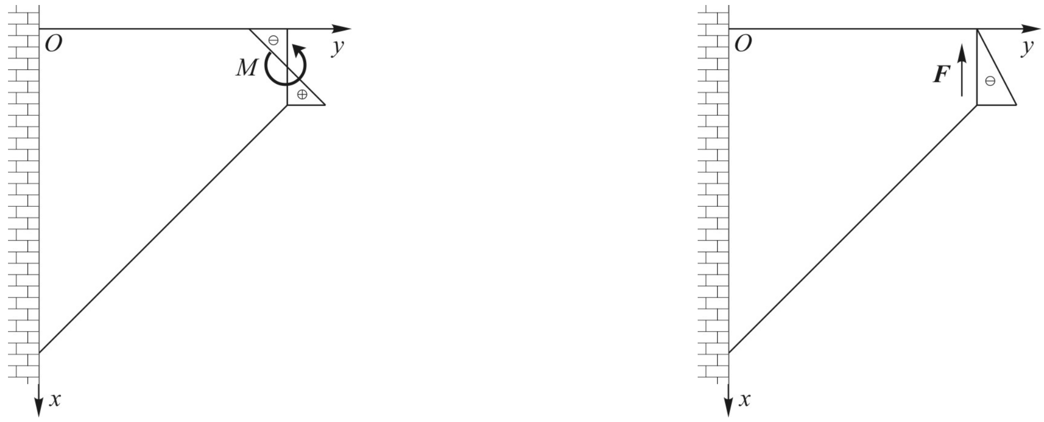


图 2.2　习题 2-13 图

□

2-17　已知应力场

$$(\sigma_{ij})=\begin{pmatrix}\sigma_{11}(x_1,x_2) & \sigma_{12}(x_1,x_2) & 0\\ \sigma_{21}(x_1,x_2) & \sigma_{22}(x_1,x_2) & 0\\ 0 & 0 & 0\end{pmatrix}$$

(1)　写出各应力分量间须满足的方程;

(2)　引入一标量函数 $\phi(x_1,x_2)$,使得

$$\sigma_{11}=\frac{\partial^2\phi}{\partial x_2^2};\quad \sigma_{22}=\frac{\partial^2\phi}{\partial x_1^2};\quad \sigma_{12}=-\frac{\partial^2\phi}{\partial x_1\partial x_2}$$

证明:以 ϕ 表示的上述应力分量将自动满足无体力的方程.

解

(1)　各应力分量间须满足的无体力的方程为

$$\begin{cases}\dfrac{\partial\sigma_{11}}{\partial x_1}+\dfrac{\partial\sigma_{21}}{\partial x_2}=0\\[2ex]\dfrac{\partial\sigma_{12}}{\partial x_1}+\dfrac{\partial\sigma_{22}}{\partial x_2}=0\end{cases}\tag{2.17.1}$$

⇨

(2)　$\sigma_{11}=\dfrac{\partial^2\phi}{\partial x_2^2}$;　$\sigma_{22}=\dfrac{\partial^2\phi}{\partial x_1^2}$;　$\sigma_{12}=-\dfrac{\partial^2\phi}{\partial x_1\partial x_2}$ 代入方程(2.17.1)左边得

$$\begin{cases}\text{左边}=\dfrac{\partial}{\partial x_1}\left(\dfrac{\partial^2\phi}{\partial x_2^2}\right)+\dfrac{\partial}{\partial x_2}\left(-\dfrac{\partial^2\phi}{\partial x_1\partial x_2}\right)=\dfrac{\partial^3\phi}{\partial x_1\partial x_2^2}-\dfrac{\partial^3\phi}{\partial x_2\partial x_1\partial x_2}=0=\text{右边}\\[2ex]\text{左边}=\dfrac{\partial}{\partial x_1}\left(-\dfrac{\partial^2\phi}{\partial x_1\partial x_2}\right)+\dfrac{\partial}{\partial x_2}\left(\dfrac{\partial^2\phi}{\partial x_1^2}\right)=-\dfrac{\partial^3\phi}{\partial x_1^2\partial x_2}+\dfrac{\partial^3\phi}{\partial x_2\partial x_1^2}=0=\text{右边}\end{cases}$$

即以 ϕ 表示的上述应力分量将自动满足无体力的方程.　□

3-1 初始时刻位于 (a_1, a_2, a_3) 的质点,在某一时刻 t 的位置为

$$x_1 = a_1 + ka_3;\quad x_2 = a_2 + ka_3;\quad x_3 = a_3$$

其中 $k = 1 \times 10^{-5}$. 求 Green 应变张量的分量.

解 Green 应变张量为

$$E_{ij} = \frac{1}{2}\left(\frac{\partial x_m}{\partial a_i}\frac{\partial x_m}{\partial a_j} - \delta_{ij}\right)$$

其分量为

$$E_{11} = \frac{1}{2}\left(\frac{\partial x_1}{\partial a_1}\frac{\partial x_1}{\partial a_1} + \frac{\partial x_2}{\partial a_1}\frac{\partial x_2}{\partial a_1} + \frac{\partial x_3}{\partial a_1}\frac{\partial x_3}{\partial a_1} - \delta_{11}\right) = \frac{1}{2}(1+0+0-1) = 0$$

$$E_{12} = \frac{1}{2}\left(\frac{\partial x_1}{\partial a_1}\frac{\partial x_1}{\partial a_2} + \frac{\partial x_2}{\partial a_1}\frac{\partial x_2}{\partial a_2} + \frac{\partial x_3}{\partial a_1}\frac{\partial x_3}{\partial a_2} - \delta_{12}\right) = \frac{1}{2}(0+0+0-0) = 0$$

$$E_{13} = \frac{1}{2}\left(\frac{\partial x_1}{\partial a_1}\frac{\partial x_1}{\partial a_3} + \frac{\partial x_2}{\partial a_1}\frac{\partial x_2}{\partial a_3} + \frac{\partial x_3}{\partial a_1}\frac{\partial x_3}{\partial a_3} - \delta_{13}\right) = \frac{1}{2}(k+0+0-0) = \frac{k}{2} = 5 \times 10^{-6}$$

$$E_{21} = \frac{1}{2}\left(\frac{\partial x_1}{\partial a_2}\frac{\partial x_1}{\partial a_1} + \frac{\partial x_2}{\partial a_2}\frac{\partial x_2}{\partial a_1} + \frac{\partial x_3}{\partial a_2}\frac{\partial x_3}{\partial a_1} - \delta_{21}\right) = \frac{1}{2}(0+0+0-0) = 0$$

$$E_{22} = \frac{1}{2}\left(\frac{\partial x_1}{\partial a_2}\frac{\partial x_1}{\partial a_2} + \frac{\partial x_2}{\partial a_2}\frac{\partial x_2}{\partial a_2} + \frac{\partial x_3}{\partial a_2}\frac{\partial x_3}{\partial a_2} - \delta_{22}\right) = \frac{1}{2}(0+1+0-1) = 0$$

$$E_{23} = \frac{1}{2}\left(\frac{\partial x_1}{\partial a_2}\frac{\partial x_1}{\partial a_3} + \frac{\partial x_2}{\partial a_2}\frac{\partial x_2}{\partial a_3} + \frac{\partial x_3}{\partial a_2}\frac{\partial x_3}{\partial a_3} - \delta_{23}\right) = \frac{1}{2}(0+k+0-0) = \frac{k}{2} = 5 \times 10^{-6}$$

$$E_{31} = \frac{1}{2}\left(\frac{\partial x_1}{\partial a_3}\frac{\partial x_1}{\partial a_1} + \frac{\partial x_2}{\partial a_3}\frac{\partial x_2}{\partial a_1} + \frac{\partial x_3}{\partial a_3}\frac{\partial x_3}{\partial a_1} - \delta_{31}\right) = \frac{1}{2}(k+0+0-0) = \frac{k}{2} = 5 \times 10^{-6}$$

$$E_{32} = \frac{1}{2}\left(\frac{\partial x_1}{\partial a_3}\frac{\partial x_1}{\partial a_2} + \frac{\partial x_2}{\partial a_3}\frac{\partial x_2}{\partial a_2} + \frac{\partial x_3}{\partial a_3}\frac{\partial x_3}{\partial a_2} - \delta_{32}\right) = \frac{1}{2}(0+k+0-0) = \frac{k}{2} = 5 \times 10^{-6}$$

$$E_{13} = \frac{1}{2}\left(\frac{\partial x_1}{\partial a_3}\frac{\partial x_1}{\partial a_3} + \frac{\partial x_2}{\partial a_3}\frac{\partial x_2}{\partial a_3} + \frac{\partial x_3}{\partial a_3}\frac{\partial x_3}{\partial a_3} - \delta_{33}\right) = \frac{1}{2}(k^2+k^2+1-1) = k^2 = 1 \times 10^{-10}$$

□

3-2 方板由图 3-14(a)均匀变形至(b), $x_3 = a_3$. 试求 Green 应变分量 E_{11}, E_{22} 和 E_{12}. 并求沿 a_1 方向($\boldsymbol{\nu} = \boldsymbol{e}_1$), a_2 方向($\boldsymbol{\nu} = \boldsymbol{e}_2$)及沿 45° 方向$\left(\boldsymbol{\nu} = \frac{1}{\sqrt{2}}\boldsymbol{e}_1 + \frac{1}{\sqrt{2}}\boldsymbol{e}_2\right)$上的线元变形长度比,及变形后的新方向.

解 设变形前方板内一点 A 的位置为 $\boldsymbol{a} = a_1\boldsymbol{e}_1 + a_2\boldsymbol{e}_2$, 变形后该点的位置为 $\boldsymbol{x} = x_1\boldsymbol{e}_1 + x_2\boldsymbol{e}_2$, 则

(1)

$$\boldsymbol{x} = x_1\boldsymbol{e}_1 + x_2\boldsymbol{e}_2 = a_1\boldsymbol{e}_1' + a_2\boldsymbol{e}_2' = a_1(1.4\boldsymbol{e}_1) + a_2(1.2\boldsymbol{e}_2) = 1.4a_1\boldsymbol{e}_1 + 1.2a_2\boldsymbol{e}_2$$

故

$$\begin{cases} x_1 = 1.4a_1 \\ x_2 = 1.2a_2 \end{cases}$$

又由已知

$$x_3 = a_3$$

所以

$$E_{11}=\frac{1}{2}\left(\frac{\partial x_1}{\partial a_1}\frac{\partial x_1}{\partial a_1}+\frac{\partial x_2}{\partial a_1}\frac{\partial x_2}{\partial a_1}+\frac{\partial x_3}{\partial a_1}\frac{\partial x_3}{\partial a_1}-\delta_{11}\right)=\frac{1}{2}(1.4^2+0+0-1)=\frac{12}{25}$$

$$E_{22}=\frac{1}{2}\left(\frac{\partial x_1}{\partial a_2}\frac{\partial x_1}{\partial a_2}+\frac{\partial x_2}{\partial a_2}\frac{\partial x_2}{\partial a_2}+\frac{\partial x_3}{\partial a_2}\frac{\partial x_3}{\partial a_2}-\delta_{22}\right)=\frac{1}{2}(0+1.2^2+0-1)=\frac{11}{50}$$

$$E_{12}=\frac{1}{2}\left(\frac{\partial x_1}{\partial a_1}\frac{\partial x_1}{\partial a_2}+\frac{\partial x_2}{\partial a_1}\frac{\partial x_2}{\partial a_2}+\frac{\partial x_3}{\partial a_1}\frac{\partial x_3}{\partial a_2}-\delta_{12}\right)=\frac{1}{2}(0+0+0-0)=0$$

由长度比

$$\lambda_\nu=\sqrt{1+2E_{ij}\nu_i\nu_j}$$

得 a_1 方向

$$\lambda_{a_1}=\sqrt{1+2E_{11}}=\sqrt{1+2\times\frac{12}{25}}=1.4$$

a_2 方向

$$\lambda_{a_2}=\sqrt{1+2E_{22}}=\sqrt{1+2\times\frac{11}{50}}=1.2$$

沿 45° 方向

$$\lambda_{45^\circ}=\sqrt{1+(E_{11}+E_{22}+2E_{12})}=\sqrt{1+\frac{12}{25}+\frac{11}{50}+2\times 0}=\frac{\sqrt{170}}{10}$$

(2)

$$\begin{aligned}\boldsymbol{x}&=x_1\boldsymbol{e}_1+x_2\boldsymbol{e}_2=a_1\boldsymbol{e}_1'+a_2\boldsymbol{e}_2'\\&=a_1(1.2\boldsymbol{e}_1)+a_2(0.5\boldsymbol{e}_1+1.2\boldsymbol{e}_2)\\&=(1.2a_1+0.5a_2)\boldsymbol{e}_1+1.2a_2\boldsymbol{e}_2\end{aligned}$$

故

$$\begin{cases}x_1=1.2a_1+0.5a_2\\x_2=1.2a_2\end{cases}$$

又由已知

$$x_3=a_3$$

所以

$$E_{11}=\frac{1}{2}\left(\frac{\partial x_1}{\partial a_1}\frac{\partial x_1}{\partial a_1}+\frac{\partial x_2}{\partial a_1}\frac{\partial x_2}{\partial a_1}+\frac{\partial x_3}{\partial a_1}\frac{\partial x_3}{\partial a_1}-\delta_{11}\right)=\frac{1}{2}(1.2^2+0+0-1)=\frac{11}{50}$$

$$E_{22}=\frac{1}{2}\left(\frac{\partial x_1}{\partial a_2}\frac{\partial x_1}{\partial a_2}+\frac{\partial x_2}{\partial a_2}\frac{\partial x_2}{\partial a_2}+\frac{\partial x_3}{\partial a_2}\frac{\partial x_3}{\partial a_2}-\delta_{22}\right)=\frac{1}{2}(0.5^2+1.2^2+0-1)=\frac{69}{200}$$

$$E_{12}=\frac{1}{2}\left(\frac{\partial x_1}{\partial a_1}\frac{\partial x_1}{\partial a_2}+\frac{\partial x_2}{\partial a_1}\frac{\partial x_2}{\partial a_2}+\frac{\partial x_3}{\partial a_1}\frac{\partial x_3}{\partial a_2}-\delta_{12}\right)=\frac{1}{2}(1.2\times 0.5+0+0-0)=\frac{3}{10}$$

由长度比

$$\lambda_\nu=\sqrt{1+2E_{ij}\nu_i\nu_j}$$

得 a_1 方向

$$\lambda_{a_1}=\sqrt{1+2E_{11}}=\sqrt{1+2\times\frac{11}{50}}=1.2$$

a_2 方向

$$\lambda_{a_2}=\sqrt{1+2E_{22}}=\sqrt{1+2\times\frac{69}{200}}=1.3$$

沿 45°方向

$$\lambda_{45^\circ} = \sqrt{1+(E_{11}+E_{22}+2E_{12})} = \sqrt{1+\frac{11}{50}+\frac{69}{200}+2\times\frac{3}{10}} = \frac{\sqrt{866}}{20}$$

□

3-3 试导出以 Almansi 应变张量 $\boldsymbol{e} = e_{ij}\boldsymbol{e}_i\boldsymbol{e}_j$ 表示的线元变形长度比 $\lambda = \dfrac{\mathrm{d}S}{\mathrm{d}S_0}$.

解 设变形前的任意线元 $\overrightarrow{PQ}$，其端点 $P(a_1, a_2, a_3)$ 及 $Q(a_1+\mathrm{d}a_1, a_2+\mathrm{d}a_2, a_3+\mathrm{d}a_3)$ 的矢径分别为

$$\overrightarrow{OP} = \boldsymbol{a} = a_i\boldsymbol{e}_i$$

$$\overrightarrow{OQ} = \boldsymbol{a}+\mathrm{d}\boldsymbol{a} = (a_i+\mathrm{d}a_i)\boldsymbol{e}_i$$

因而线元为

$$\overrightarrow{PQ} = \overrightarrow{OQ}-\overrightarrow{OP} = \mathrm{d}\boldsymbol{a} = \mathrm{d}a_i\boldsymbol{e}_i$$

变形后，P,Q 两点分别位移至 P',Q'，相应的矢径和线元为

$$\overrightarrow{OP'} = \boldsymbol{x} = x_i\boldsymbol{e}_i$$

$$\overrightarrow{OQ'} = \boldsymbol{x}+\mathrm{d}\boldsymbol{x} = (x_i+\mathrm{d}x_i)\boldsymbol{e}_i$$

$$\overrightarrow{P'Q'} = \overrightarrow{OQ'}-\overrightarrow{OP'} = \mathrm{d}\boldsymbol{x} = \mathrm{d}x_i\boldsymbol{e}_i$$

变形前后，线元 $\overrightarrow{PQ}$ 和$\overrightarrow{P'Q'}$ 的长度的平方为

$$(\mathrm{d}S_0)^2 = \mathrm{d}\boldsymbol{a}\cdot\mathrm{d}\boldsymbol{a} = \mathrm{d}a_m\mathrm{d}a_m \tag{3.1}$$

$$(\mathrm{d}S)^2 = \mathrm{d}\boldsymbol{x}\cdot\mathrm{d}\boldsymbol{x} = \mathrm{d}x_i\mathrm{d}x_i = \delta_{ij}\mathrm{d}x_i\mathrm{d}x_j \tag{3.2}$$

采用 Euler 描述法，$a_m = a_m(x_i)$，则

$$\mathrm{d}a_m = \frac{\partial a_m}{\partial x_i}\mathrm{d}x_i$$

代入式(3.1)得

$$(\mathrm{d}S_0)^2 = \frac{\partial a_m}{\partial x_i}\frac{\partial a_m}{\partial x_j}\mathrm{d}x_i\mathrm{d}x_j$$

式(3.2)减去式(3.1)可得变形后线元长度平方的变化

$$(\mathrm{d}S)^2-(\mathrm{d}S_0)^2 = 2e_{ij}\mathrm{d}x_i\mathrm{d}x_j$$

其中

$$e_{ij} = \frac{1}{2}\left(\delta_{ij}-\frac{\partial a_m}{\partial x_i}\frac{\partial a_m}{\partial x_j}\right)$$

为 Almansi 应变张量. 设变形后，线元 $\overrightarrow{P'Q'}$ 方向的单位矢量为

$$\boldsymbol{\nu}' = \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}S} = \frac{\mathrm{d}x_i}{\mathrm{d}S}\boldsymbol{e}_i = \nu'_i\boldsymbol{e}_i$$

其中

$$\nu'_i = \frac{\mathrm{d}x_i}{\mathrm{d}S}$$

为线元 $\overrightarrow{P'Q'}$ 的方向余弦. 则长度比为

$$\lambda = \frac{\mathrm{d}S}{\mathrm{d}S_0} = \sqrt{\frac{1}{1-2\dfrac{\mathrm{d}x_i}{\mathrm{d}S}e_{ij}\dfrac{\mathrm{d}x_j}{\mathrm{d}S}}} = \sqrt{\frac{1}{1-2\nu'_ie_{ij}\nu'_j}} = \sqrt{\frac{1}{1-2\boldsymbol{\nu}'\cdot\boldsymbol{e}\cdot\boldsymbol{\nu}'}}$$

□

3-4 位移场为坐标线性函数的变形称为均匀变形. 证明:对小应变情况,均匀变形 $u_i = A_{ij}a_j$(其中 A_{ij} 的各分量为常数)具有如下性质

(1) 直线在变形后仍然是直线;

(2) 相同方向的直线按同样的比例伸缩;

(3) 平行直线在变形后仍然平行;

(4) 平面在变形后仍为平面;

(5) 平行平面在变形后仍然平行.

证

由

$$x_i = u_i + a_i$$

得矩阵形式

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} + \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} A_{11}+1 & A_{12} & A_{13} \\ A_{21} & A_{22}+1 & A_{23} \\ A_{31} & A_{32} & A_{33}+1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

(1) 设直线方程为 $a_i = a_{0i} + b_i t$,则变形后方程为

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} A_{11}+1 & A_{12} & A_{13} \\ A_{21} & A_{22}+1 & A_{23} \\ A_{31} & A_{32} & A_{33}+1 \end{bmatrix} \begin{bmatrix} a_{01} \\ a_{02} \\ a_{03} \end{bmatrix} + \begin{bmatrix} A_{11}+1 & A_{12} & A_{13} \\ A_{21} & A_{22}+1 & A_{23} \\ A_{31} & A_{32} & A_{33}+1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} t = \begin{bmatrix} x_{01}+b'_1 t \\ x_{02}+b'_2 t \\ x_{03}+b'_3 t \end{bmatrix}$$

即直线 $a_i = a_{0i} + b_i t$ 在变形后为直线 $x_i = x_{0i} + b'_i t$. ⇨

(2) 设相同方向直线方程分别为 $a_i = a_{1i} + b_i t, a_i = a_{2i} + b_i t$,则变形后的方程分别为

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} A_{11}+1 & A_{12} & A_{13} \\ A_{21} & A_{22}+1 & A_{23} \\ A_{31} & A_{32} & A_{33}+1 \end{bmatrix} \begin{bmatrix} a_{11} \\ a_{12} \\ a_{13} \end{bmatrix} + \begin{bmatrix} A_{11}+1 & A_{12} & A_{13} \\ A_{21} & A_{22}+1 & A_{23} \\ A_{31} & A_{32} & A_{33}+1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} t = \begin{bmatrix} x_{11} \\ x_{12} \\ x_{13} \end{bmatrix} + \begin{bmatrix} b'_1 \\ b'_2 \\ b'_3 \end{bmatrix} t$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} A_{11}+1 & A_{12} & A_{13} \\ A_{21} & A_{22}+1 & A_{23} \\ A_{31} & A_{32} & A_{33}+1 \end{bmatrix} \begin{bmatrix} a_{21} \\ a_{22} \\ a_{23} \end{bmatrix} + \begin{bmatrix} A_{11}+1 & A_{12} & A_{13} \\ A_{21} & A_{22}+1 & A_{23} \\ A_{31} & A_{32} & A_{33}+1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} t = \begin{bmatrix} x_{21} \\ x_{22} \\ x_{23} \end{bmatrix} + \begin{bmatrix} b'_1 \\ b'_2 \\ b'_3 \end{bmatrix} t$$

在直线 $a_i = a_{1i} + b_i t$ 上取点 $A_1(a_{11}, a_{12}, a_{13}), A'_1(a_{11}+b_1 t, a_{12}+b_2 t, a_{13}+b_3 t)$,在直线 $a_i = a_{2i} + b_i t$ 上取点 $A_2(a_{21}, a_{22}, a_{23}), A'_2(a_{21}+b_1 t, a_{22}+b_2 t, a_{23}+b_3 t)$,则变形后, A_1, A'_1, A_2, A'_2 分别为

$$X_1 = (x_{11}, x_{12}, x_{13}), \quad X'_1 = (x_{11}+b'_1 t, x_{12}+b'_2 t, x_{13}+b'_3 t)$$
$$X_2 = (x_{21}, x_{22}, x_{23}), \quad X'_2 = (x_{21}+b'_1 t, x_{22}+b'_2 t, x_{23}+b'_3 t)$$

所以

$$\overrightarrow{A_1 A'_1} = (b_1 t, b_2 t, b_3 t) = \overrightarrow{A_2 A'_2}, \quad \overrightarrow{X_1 X'_1} = (b'_1 t, b'_2 t, b'_3 t) = \overrightarrow{X_2 X'_2}$$

故两直线的伸缩比例为

$$\lambda_1 = \frac{|\overrightarrow{X_1 X'_1}|}{|\overrightarrow{A_1 A'_1}|} = \frac{\sqrt{b_1'^2 + b_2'^2 + b_3'^2}}{\sqrt{b_1^2 + b_2^2 + b_3^2}}$$

$$\lambda_2 = \frac{|\overrightarrow{X_2 X'_2}|}{|\overrightarrow{A_2 A'_2}|} = \frac{\sqrt{b_1'^2 + b_2'^2 + b_3'^2}}{\sqrt{b_1^2 + b_2^2 + b_3^2}}$$

即 $\lambda_1 = \lambda_2$, 相同方向的直线按同样的比例伸缩. ⇨

(3) 设两平行直线方程分别为 $a_i = a_{1i} + b_i t$，$a_i = a_{2i} + b_i t$ ，则变形后的方程分别为

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} A_{11}+1 & A_{12} & A_{13} \\ A_{21} & A_{22}+1 & A_{23} \\ A_{31} & A_{32} & A_{33}+1 \end{bmatrix} \begin{bmatrix} a_{11} \\ a_{12} \\ a_{13} \end{bmatrix} + \begin{bmatrix} A_{11}+1 & A_{12} & A_{13} \\ A_{21} & A_{22}+1 & A_{23} \\ A_{31} & A_{32} & A_{33}+1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} t = \begin{bmatrix} x_{11} \\ x_{12} \\ x_{13} \end{bmatrix} + \begin{bmatrix} b_1' \\ b_2' \\ b_3' \end{bmatrix} t$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} A_{11}+1 & A_{12} & A_{13} \\ A_{21} & A_{22}+1 & A_{23} \\ A_{31} & A_{32} & A_{33}+1 \end{bmatrix} \begin{bmatrix} a_{21} \\ a_{22} \\ a_{23} \end{bmatrix} + \begin{bmatrix} A_{11}+1 & A_{12} & A_{13} \\ A_{21} & A_{22}+1 & A_{23} \\ A_{31} & A_{32} & A_{33}+1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} t = \begin{bmatrix} x_{21} \\ x_{22} \\ x_{23} \end{bmatrix} + \begin{bmatrix} b_1' \\ b_2' \\ b_3' \end{bmatrix} t$$

其中两直线的方向向量均为

$$\begin{bmatrix} b_1' \\ b_2' \\ b_3' \end{bmatrix} = \begin{bmatrix} A_{11}+1 & A_{12} & A_{13} \\ A_{21} & A_{22}+1 & A_{23} \\ A_{31} & A_{32} & A_{33}+1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

即平行直线在变形后仍然平行. ⇨

(4) 设平面方程为 $c_i a_i + c_0 = 0$，写成矩阵形式为

$$\begin{bmatrix} c_1 & c_2 & c_3 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} + c_0 = 0$$

则变形后方程为

$$\begin{bmatrix} c_1 & c_2 & c_3 \end{bmatrix} \begin{bmatrix} A_{11}+1 & A_{12} & A_{13} \\ A_{21} & A_{22}+1 & A_{23} \\ A_{31} & A_{32} & A_{33}+1 \end{bmatrix}^{-1} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + c_0 = 0$$

即

$$\begin{bmatrix} c_1' & c_2' & c_3' \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + c_0 = 0$$

所以平面在变形后仍为平面. ⇨

(4) 设两平行平面方程分别为 $c_i a_i + c_{01} = 0$，$c_i a_i + c_{02} = 0$ ，写成矩阵形式为

$$\begin{bmatrix} c_1 & c_2 & c_3 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} + c_{01} = 0$$

$$\begin{bmatrix} c_1 & c_2 & c_3 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} + c_{02} = 0$$

则变形后方程为

$$\begin{bmatrix} c_1 & c_2 & c_3 \end{bmatrix} \begin{bmatrix} A_{11}+1 & A_{12} & A_{13} \\ A_{21} & A_{22}+1 & A_{23} \\ A_{31} & A_{32} & A_{33}+1 \end{bmatrix}^{-1} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + c_{01} = 0$$

$$\begin{bmatrix} c_1 & c_2 & c_3 \end{bmatrix} \begin{bmatrix} A_{11}+1 & A_{12} & A_{13} \\ A_{21} & A_{22}+1 & A_{23} \\ A_{31} & A_{32} & A_{33}+1 \end{bmatrix}^{-1} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + c_{02} = 0$$

变形后两平面的法向量均为

$$[c_1' \quad c_2' \quad c_3'] = [c_1 \quad c_2 \quad c_3]\begin{bmatrix} A_{11}+1 & A_{12} & A_{13} \\ A_{21} & A_{22}+1 & A_{23} \\ A_{31} & A_{32} & A_{33}+1 \end{bmatrix}^{-1}$$

所以平行平面在变形后仍然平行.

□

3-7 将直角坐标系绕 x_3 轴转动 φ 角，求新旧坐标系间应变分量的转换关系.

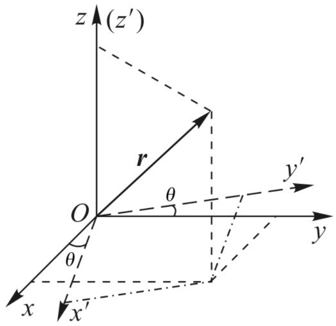


图 3.1 习题 3-7 图

解 在旧坐标系 $Ox_1x_2x_3$ 中，

$$\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k} = (\boldsymbol{i} \quad \boldsymbol{j} \quad \boldsymbol{k})\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

在新坐标系 $Ox_1'x_2'x_3'$ 中，

$$\boldsymbol{r} = x'\boldsymbol{i}' + y'\boldsymbol{j}' + z'\boldsymbol{k}' = (\boldsymbol{i}' \quad \boldsymbol{j}' \quad \boldsymbol{k}')\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$$

又

$$\begin{cases} \boldsymbol{i}' = \boldsymbol{i}\cos\varphi + \boldsymbol{j}\sin\varphi \\ \boldsymbol{j}' = -\boldsymbol{i}\sin\varphi + \boldsymbol{j}\cos\varphi \\ \boldsymbol{k}' = \boldsymbol{k} \end{cases}$$

用矩阵形式表示为

$$(\boldsymbol{i}' \quad \boldsymbol{j}' \quad \boldsymbol{k}') = (\boldsymbol{i} \quad \boldsymbol{j} \quad \boldsymbol{k})\begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

由矢量的不变性得

$$(\boldsymbol{i} \quad \boldsymbol{j} \quad \boldsymbol{k})\begin{pmatrix} x \\ y \\ z \end{pmatrix} = (\boldsymbol{i}' \quad \boldsymbol{j}' \quad \boldsymbol{k})\begin{pmatrix} x' \\ y' \\ z \end{pmatrix} = (\boldsymbol{i} \quad \boldsymbol{j} \quad \boldsymbol{k})\begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x' \\ y' \\ z \end{pmatrix}$$

由 $\boldsymbol{r}$ 的任意性得

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

所以坐标转换系数矩阵为

$$\begin{pmatrix} \beta_{1'1} & \beta_{1'2} & \beta_{1'3} \\ \beta_{2'1} & \beta_{2'2} & \beta_{2'3} \\ \beta_{3'1} & \beta_{3'2} & \beta_{3'3} \end{pmatrix} = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

由

$$\varepsilon'_{ij} = \beta_{i'm}\beta_{j'm}\varepsilon_{mn}$$

即

$$\begin{pmatrix} \varepsilon'_{11} & \varepsilon'_{12} & \varepsilon'_{13} \\ \varepsilon'_{21} & \varepsilon'_{22} & \varepsilon'_{23} \\ \varepsilon'_{31} & \varepsilon'_{32} & \varepsilon'_{33} \end{pmatrix} = \begin{pmatrix} \beta_{1'1} & \beta_{1'2} & \beta_{1'3} \\ \beta_{2'1} & \beta_{2'2} & \beta_{2'3} \\ \beta_{3'1} & \beta_{3'2} & \beta_{3'3} \end{pmatrix} \begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{33} \\ \varepsilon_{31} & \varepsilon_{22} & \varepsilon_{33} \end{pmatrix} \begin{pmatrix} \beta_{1'1} & \beta_{1'2} & \beta_{1'3} \\ \beta_{2'1} & \beta_{2'2} & \beta_{2'3} \\ \beta_{3'1} & \beta_{3'2} & \beta_{3'3} \end{pmatrix}^{\mathrm{T}}$$

$$= \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{33} \\ \varepsilon_{31} & \varepsilon_{22} & \varepsilon_{33} \end{pmatrix} \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}^{\mathrm{T}}$$

$$= \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{33} \\ \varepsilon_{31} & \varepsilon_{22} & \varepsilon_{33} \end{pmatrix} \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

得新坐标系中的应变张量分量为

$$\begin{aligned}
\varepsilon'_{11} &= \varepsilon_{11}\cos^2\varphi + \varepsilon_{22}\sin^2\varphi + \varepsilon_{12}\cos\varphi\sin\varphi + \varepsilon_{21}\cos\varphi\sin\varphi \\
\varepsilon'_{12} &= -\varepsilon_{11}\cos\varphi\sin\varphi + \varepsilon_{22}\cos\varphi\sin\varphi + \varepsilon_{12}\cos^2\varphi - \varepsilon_{21}\sin^2\varphi \\
\varepsilon'_{13} &= \varepsilon_{23}\sin\varphi + \varepsilon_{13}\cos\varphi \\
\varepsilon'_{21} &= -\varepsilon_{11}\cos\varphi\sin\varphi + \varepsilon_{22}\cos\varphi\sin\varphi + \varepsilon_{21}\cos^2\varphi - \varepsilon_{12}\sin^2\varphi \\
\varepsilon'_{22} &= \varepsilon_{11}\sin^2\varphi + \varepsilon_{22}\cos^2\varphi - \varepsilon_{12}\cos\varphi\sin\varphi - \varepsilon_{21}\cos\varphi\sin\varphi \\
\varepsilon'_{23} &= \varepsilon_{23}\cos\varphi - \varepsilon_{13}\sin\varphi \\
\varepsilon'_{31} &= \varepsilon_{32}\sin\varphi + \varepsilon_{31}\cos\varphi \\
\varepsilon'_{32} &= \varepsilon_{32}\cos\varphi - \varepsilon_{31}\sin\varphi \\
\varepsilon'_{33} &= \varepsilon_{33}
\end{aligned}$$

□

3-10 假设体积不可压缩,位移 $u_1(x_1,x_2)$ 与 $u_2(x_1,x_2)$ 很小, $u_3 \equiv 0$. 在一定区域内已知 $u_1 = (1 - x_2^2)(a + bx_1 + cx_1^2)$,$\varepsilon_{12} = d$, 其中 a,b,c,d 为常数,求 $u_2(x_1,x_2)$.

解 由体积不可压缩得

$$\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = 0$$

所以

$$\frac{\partial u_2}{\partial x_2} = -\frac{\partial u_1}{\partial x_1} - \frac{\partial u_3}{\partial x_3} = -(1 - x_2^2)(b + 2cx_1) - 0 = (x_2^2 - 1)(2cx_1 + b)$$

由

$$\varepsilon_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) = d$$

得

$$\frac{\partial u_2}{\partial x_1} = d - \frac{\partial u_1}{\partial x_2} = d - (-2x_2)(a + bx_1 + cx_1^2) = d + 2x_2(a + bx_1 + cx_1^2)$$

又由 u_2 的连续可微性知

$$\frac{\partial}{\partial x_2}\left(\frac{\partial u_2}{\partial x_1}\right)=2(a+bx_1+cx_1^2)=2c(x_2^2-1)=\frac{\partial}{\partial x_1}\left(\frac{\partial u_2}{\partial x_2}\right)$$

比较系数得

$$a=b=c=0$$

所以

$$\mathrm{d}u_2=\frac{\partial u_2}{\partial x_1}\mathrm{d}x_1+\frac{\partial u_2}{\partial x_2}\mathrm{d}x_2=[d+2x_2(a+bx_1+cx_1^2)]\mathrm{d}x_1+(x_2^2-1)(2cx_1+b)\mathrm{d}x_2=d\mathrm{d}x_1$$

故

$$u_2(x_1,x_2)=dx_1+C(\text{常数})$$

□

3-11 对于平面应变状态（$\varepsilon_3=\gamma_{13}=\gamma_{23}=0$），如果已知平面内 0°,45° 和 90° 方向的正应变,试求主应变的大小及方向.画出此状态的应变 Mohr 圆.

解 设平面内 0°,45° 和 90° 方向的正应变分别为 $\varepsilon_{0^\circ},\varepsilon_{45^\circ},\varepsilon_{90^\circ}$，则应变张量为

$$[\varepsilon_{ij}]=\begin{bmatrix}\varepsilon_{0^\circ} & \varepsilon_{12} & 0\\ \varepsilon_{21} & \varepsilon_{90^\circ} & 0\\ 0 & 0 & 0\end{bmatrix}$$

由 Cauchy 公式得

$$\varepsilon_{45^\circ}=\frac{1}{2}(\varepsilon_{0^\circ}+\varepsilon_{90^\circ})+\varepsilon_{12}$$

所以

$$\varepsilon_{12}=\varepsilon_{21}=\varepsilon_{45^\circ}-\frac{1}{2}(\varepsilon_{0^\circ}+\varepsilon_{90^\circ})$$

由

$$\begin{vmatrix}\varepsilon_{0^\circ}-\lambda & \varepsilon_{12} & 0\\ \varepsilon_{21} & \varepsilon_{90^\circ}-\lambda & 0\\ 0 & 0 & -\lambda\end{vmatrix}=0$$

得主应变的大小为

$$\varepsilon_1=\frac{\varepsilon_{0^\circ}+\varepsilon_{90^\circ}+\sqrt{2(\varepsilon_{0^\circ}^2-2\varepsilon_{0^\circ}\varepsilon_{45^\circ}+2\varepsilon_{45^\circ}^2-2\varepsilon_{45^\circ}\varepsilon_{90^\circ}+\varepsilon_{90^\circ}^2)}}{2}$$

$$\varepsilon_2=\frac{\varepsilon_{0^\circ}+\varepsilon_{90^\circ}-\sqrt{2(\varepsilon_{0^\circ}^2-2\varepsilon_{0^\circ}\varepsilon_{45^\circ}+2\varepsilon_{45^\circ}^2-2\varepsilon_{45^\circ}\varepsilon_{90^\circ}+\varepsilon_{90^\circ}^2)}}{2}$$

$$\varepsilon_3=0$$

由

$$\boldsymbol{v}\cdot\boldsymbol{\varepsilon}=\lambda\boldsymbol{v}$$

主应变的方向分别为

$$\boldsymbol{v}_1=k_1(-a+c+\sqrt{2(\varepsilon_{0^\circ}^2-2\varepsilon_{0^\circ}\varepsilon_{45^\circ}+2\varepsilon_{45^\circ}^2-2\varepsilon_{45^\circ}\varepsilon_{90^\circ}+\varepsilon_{90^\circ}^2)},a-2b+c,0)$$

$$\boldsymbol{v}_2=k_2(a+c-\sqrt{2(\varepsilon_{0^\circ}^2-2\varepsilon_{0^\circ}\varepsilon_{45^\circ}+2\varepsilon_{45^\circ}^2-2\varepsilon_{45^\circ}\varepsilon_{90^\circ}+\varepsilon_{90^\circ}^2)},a-2b+c,0)$$

$$\boldsymbol{v}_2=k_3(0,0,1)$$

此状态的应变 Mohr 圆方程为

$$\left(\varepsilon_{11}-\frac{\varepsilon_{0^\circ}+\varepsilon_{90^\circ}}{2}\right)^2+\varepsilon_{12}^2=\frac{\varepsilon_{0^\circ}^2-2\varepsilon_{0^\circ}\varepsilon_{45^\circ}+2\varepsilon_{45^\circ}^2-2\varepsilon_{45^\circ}\varepsilon_{90^\circ}+\varepsilon_{90^\circ}^2}{2}$$

应变 Mohr 圆如图 3.2 所示.

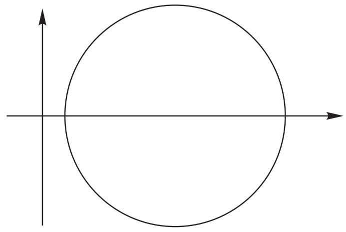


图 3.2 习题 3.11 图

□

3-14 设位移场为 $\boldsymbol{u}=a(x_1-x_3)^2\boldsymbol{e}_1+a(x_2+x_3)^2\boldsymbol{e}_2-ax_1x_2\boldsymbol{e}_3$，其中 a 为远小于 1 的常数. 确定在 $P(0,2,-1)$ 点的小应变张量分量、转动张量分量和转动矢量分量.

解 因为

$$\begin{bmatrix}\frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3}\\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3}\\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3}\end{bmatrix}=\begin{bmatrix}2a(x_1-x_3) & 0 & -2a(x_1-x_3)\\ 0 & 2a(x_2+x_3) & 2a(x_2+x_3)\\ ax_2 & ax_1 & 0\end{bmatrix}$$

所以在 $P(0,2,-1)$ 点的小应变张量为

$$[\varepsilon_{ij}]=\frac{1}{2}\left[\frac{\partial u_i}{\partial x_j}\right]+\frac{1}{2}\left[\frac{\partial u_j}{\partial x_i}\right]=\frac{1}{2}\begin{bmatrix}\frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3}\\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3}\\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3}\end{bmatrix}+\frac{1}{2}\begin{bmatrix}\frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3}\\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3}\\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3}\end{bmatrix}^{\mathrm{T}}$$

$$=\frac{1}{2}\begin{bmatrix}2a(x_1-x_3) & 0 & -2a(x_1-x_3)\\ 0 & 2a(x_2+x_3) & 2a(x_2+x_3)\\ ax_2 & ax_1 & 0\end{bmatrix}_{(x_1,x_2,x_3)=(0,2,-1)}$$

$$+\frac{1}{2}\begin{bmatrix}2a(x_1-x_3) & 0 & -2a(x_1-x_3)\\ 0 & 2a(x_2+x_3) & 2a(x_2+x_3)\\ ax_2 & ax_1 & 0\end{bmatrix}^{\mathrm{T}}_{(x_1,x_2,x_3)=(0,2,-1)}$$

$$=\begin{bmatrix}a & 0 & -a\\ 0 & a & a\\ a & 0 & 0\end{bmatrix}+\begin{bmatrix}a & 0 & a\\ 0 & a & 0\\ -a & a & 0\end{bmatrix}=\begin{bmatrix}2a & 0 & 0\\ 0 & 2a & a\\ 0 & a & 0\end{bmatrix}$$

转动张量为

$$[\Omega_{ij}]=\frac{1}{2}\left[\frac{\partial u_j}{\partial x_i}\right]-\frac{1}{2}\left[\frac{\partial u_i}{\partial x_j}\right]=\frac{1}{2}\begin{bmatrix}\frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3}\\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3}\\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3}\end{bmatrix}^{\mathrm{T}}-\frac{1}{2}\begin{bmatrix}\frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3}\\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3}\\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3}\end{bmatrix}$$

$$=\frac{1}{2}\begin{bmatrix}2a(x_1-x_3) & 0 & -2a(x_1-x_3)\\ 0 & 2a(x_2+x_3) & 2a(x_2+x_3)\\ ax_2 & ax_1 & 0\end{bmatrix}^{\mathrm{T}}_{(x_1,x_2,x_3)=(0,2,-1)}$$

$$-\frac{1}{2}\begin{bmatrix}2a(x_1-x_3) & 0 & -2a(x_1-x_3)\\ 0 & 2a(x_2+x_3) & 2a(x_2+x_3)\\ ax_2 & ax_1 & 0\end{bmatrix}_{(x_1,x_2,x_3)=(0,2,-1)}$$

$$=\begin{bmatrix}a & 0 & a\\ 0 & a & 0\\ -a & a & 0\end{bmatrix}-\begin{bmatrix}a & 0 & -a\\ 0 & a & a\\ a & 0 & 0\end{bmatrix}=\begin{bmatrix}0 & 0 & 2a\\ 0 & 0 & -a\\ -2a & a & 0\end{bmatrix}$$

转动矢量分量为

$$\omega_1=\Omega_{23}=-a,\quad \omega_2=\Omega_{31}=-2a,\quad \omega_3=\Omega_{12}=0$$

□

3-17 证明:由下式确定的应变

$$\varepsilon_{ij}=\frac{1}{2}(u_{i,j}+u_{j,i})$$

恒满足变形协调方程

$$e_{mjk}e_{nil}\varepsilon_{ij,kl}=0$$

证

$$左边=e_{mjk}e_{nil}\varepsilon_{ij,kl}=\begin{vmatrix}\delta_{mn} & \delta_{mi} & \delta_{ml}\\ \delta_{jn} & \delta_{ji} & \delta_{jl}\\ \delta_{kn} & \delta_{ki} & \delta_{kl}\end{vmatrix}\frac{\partial^2}{\partial x_k\partial x_l}\left[\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)\right]$$

$$=\frac{1}{2}(\delta_{mn}\delta_{ji}\delta_{kl}+\delta_{mi}\delta_{jl}\delta_{kn}+\delta_{ml}\delta_{jn}\delta_{ki}-\delta_{ml}\delta_{ji}\delta_{kn}-\delta_{mi}\delta_{jn}\delta_{kl}-\delta_{mn}\delta_{jl}\delta_{ki})\frac{\partial^3 u_i}{\partial x_k\partial x_l\partial x_j}$$

$$+\frac{1}{2}(\delta_{mn}\delta_{ji}\delta_{kl}+\delta_{mi}\delta_{jl}\delta_{kn}+\delta_{ml}\delta_{jn}\delta_{ki}-\delta_{ml}\delta_{ji}\delta_{kn}-\delta_{mi}\delta_{jn}\delta_{kl}-\delta_{mn}\delta_{jl}\delta_{ki})\frac{\partial^3 u_j}{\partial x_k\partial x_l\partial x_i}$$

$$=\frac{1}{2}\Big(\underbrace{\delta_{mn}\frac{\partial^3 u_i}{\partial x_k\partial x_k\partial x_i}}_{①}+\underbrace{\delta_{mi}\delta_{kn}\frac{\partial^3 u_i}{\partial x_k\partial x_j\partial x_j}}_{②}+\underbrace{\delta_{ml}\delta_{jn}\frac{\partial^3 u_i}{\partial x_i\partial x_l\partial x_j}}_{③}\Big)$$

$$-\frac{1}{2}\Big(\underbrace{\delta_{ml}\delta_{kn}\frac{\partial^3 u_i}{\partial x_k\partial x_l\partial x_i}}_{③}+\underbrace{\delta_{mi}\delta_{jn}\frac{\partial^3 u_i}{\partial x_k\partial x_k\partial x_j}}_{②}+\underbrace{\delta_{mn}\frac{\partial^3 u_i}{\partial x_i\partial x_j\partial x_j}}_{①}\Big)$$

$$+\frac{1}{2}\Big(\underbrace{\delta_{mn}\frac{\partial^3 u_i}{\partial x_k\partial x_k\partial x_i}}_{④}+\underbrace{\delta_{mi}\delta_{kn}\frac{\partial^3 u_j}{\partial x_k\partial x_j\partial x_i}}_{⑤}+\underbrace{\delta_{ml}\delta_{jn}\frac{\partial^3 u_j}{\partial x_i\partial x_l\partial x_i}}_{⑥}\Big)$$

$$-\frac{1}{2}\Big(\underbrace{\delta_{ml}\delta_{kn}\frac{\partial^3 u_i}{\partial x_k\partial x_l\partial x_i}}_{⑤}+\underbrace{\delta_{mi}\delta_{jn}\frac{\partial^3 u_j}{\partial x_k\partial x_k\partial x_i}}_{⑥}+\underbrace{\delta_{mn}\frac{\partial^3 u_j}{\partial x_i\partial x_j\partial x_i}}_{④}\Big)$$

$$=0+0=0=右边$$

□

4-2 证明:对各向同性弹性体,若主应力 $\sigma_1 \geqslant \sigma_2 \geqslant \sigma_3$,则相应的主应变 $\varepsilon_1 \geqslant \varepsilon_2 \geqslant \varepsilon_3$.

证 由广义 Hooke 定律得

$$\varepsilon_1 = \frac{1+\nu}{E}\sigma_1 - \frac{\nu}{E}(\sigma_1+\sigma_2+\sigma_3)$$

$$\varepsilon_2 = \frac{1+\nu}{E}\sigma_2 - \frac{\nu}{E}(\sigma_1+\sigma_2+\sigma_3)$$

$$\varepsilon_3 = \frac{1+\nu}{E}\sigma_3 - \frac{\nu}{E}(\sigma_1+\sigma_2+\sigma_3)$$

因为

$$\sigma_1 \geqslant \sigma_2 \geqslant \sigma_3$$

所以

$$\varepsilon_1 \geqslant \varepsilon_2 \geqslant \varepsilon_3$$

□

4-4 各向同性弹性体承受单向拉伸($\sigma_1 > 0, \sigma_2 = \sigma_3 = 0$),试确定只产生剪应变的截面位置,并求该截面上的正应力(取 $\nu = 0.3$).

解 以主方向为参考坐标. 由广义 Hooke 定律

$$\varepsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij}$$

得小应变张量为

$$[\varepsilon_{ij}] = \begin{bmatrix} \frac{1}{E}\sigma_1 & 0 & 0 \\ 0 & -\frac{\nu}{E}\sigma_1 & 0 \\ 0 & 0 & -\frac{\nu}{E}\sigma_1 \end{bmatrix}$$

设以单位向量 $\boldsymbol{\nu} = (\nu_1, \nu_2, \nu_3)$ 为法向的平面上只有剪应变,即正应变

$$\varepsilon_\nu = \varepsilon_{ij}\nu_i\nu_j = \varepsilon_{11}\nu_1^2 + \varepsilon_{22}\nu_2^2 + \varepsilon_{33}\nu_3^2 = \frac{\sigma_1}{E}\nu_1^2 - \frac{\nu\sigma_1}{E}\nu_2^2 - \frac{\nu\sigma_1}{E}\nu_3^2 = 0$$

由此得

$$\nu_1^2 - \nu\nu_2^2 - \nu\nu_3^2 = 0$$

又

$$\nu_1^2 + \nu_2^2 + \nu_3^2 = 1, \quad \nu = 0.3$$

所以

$$\nu_1^2 = \frac{3}{13}, \quad \nu_2^2 + \nu_3^2 = \frac{10}{13}$$

该截面上的正应力为

$$\sigma_n = \nu_i\nu_j\sigma_{ij} = \nu_1^2\sigma_1 = \frac{3}{13}\sigma_1$$

□

4-5 将弹性薄板粘接在两块刚性平板之间，受刚性平板压缩，压应力为 σ_z. 假设粘接完全阻止了板内的中面应变 ε_x 和 ε_y. 试求名义弹性模量 $\left(E_c=\dfrac{\sigma_z}{\varepsilon_z}\right)$. 并证明，当薄板近于不可压缩时，名义弹性模量 E_c 比 E 大得多.

解 由广义 Hooke 定律得

$$\begin{cases}\varepsilon_x=\dfrac{1+\nu}{E}\sigma_x-\dfrac{\nu}{E}(\sigma_x+\sigma_y+\sigma_z)=0\\ \varepsilon_y=\dfrac{1+\nu}{E}\sigma_y-\dfrac{\nu}{E}(\sigma_x+\sigma_y+\sigma_z)=0\\ \varepsilon_z=\dfrac{1+\nu}{E}\sigma_z-\dfrac{\nu}{E}(\sigma_x+\sigma_y+\sigma_z)\end{cases}$$

解得

$$\sigma_x=\sigma_y=\frac{\nu}{1-\nu}\sigma_z$$

所以名义弹性模量为

$$E_c=\frac{\sigma_z}{\varepsilon_z}=\frac{1-\nu}{1-\nu-2\nu^2}E$$

当薄板近于不可压缩时

$$\varepsilon_x+\varepsilon_y+\varepsilon_z=\varepsilon_z=\frac{1+\nu}{E}\sigma_z-\frac{\nu}{E}(\sigma_x+\sigma_y+\sigma_z)\to 0$$

所以

$$1-\nu-2\nu^2\to 0$$

此时

$$E_c=\frac{1-\nu}{1-\nu-2\nu^2}E\to+\infty$$

即名义弹性模量 E_c 比 E 大得多.

□

4-7 图 4-3(a)所示双向拉伸薄板，在变形过程中其应变 ε_x，ε_y 的关系如图 4-3(b)所示. 设材料为各向同性线弹性. 试沿路径积分求在 B 点时物体中单位体积的应变能.

解 由

$$\begin{cases}\varepsilon_x=\dfrac{1}{E}(\sigma_x-\nu\sigma_y)\\ \varepsilon_y=\dfrac{1}{E}(\sigma_y-\nu\sigma_x)\end{cases}$$

得

$$\sigma_x=\frac{E(\varepsilon_x+\nu\varepsilon_y)}{1-\nu^2},\quad \sigma_y=\frac{E(\varepsilon_y+\nu\varepsilon_x)}{1-\nu^2}$$

又在 OA 段，$\dfrac{\varepsilon_x}{\varepsilon_y}=\dfrac{\varepsilon_{x1}}{\varepsilon_{y1}}$，所以单位体积的应变能即应变能密度为

$$\begin{aligned}W&=\int_0^{\varepsilon_{x1}}\sigma_x\mathrm{d}\varepsilon_x+\int_{\varepsilon_{x1}}^{\varepsilon_{x2}}\sigma_x\mathrm{d}\varepsilon_x+\int_0^{\varepsilon_{y1}}\sigma_y\mathrm{d}\varepsilon_y\\&=\int_0^{\varepsilon_{x1}}\frac{E(\varepsilon_x+\nu\varepsilon_y)}{1-\nu^2}\mathrm{d}\varepsilon_x+\int_{\varepsilon_{x1}}^{\varepsilon_{x2}}\frac{E(\varepsilon_x+\nu\varepsilon_y)}{1-\nu^2}\mathrm{d}\varepsilon_x+\int_0^{\varepsilon_{y1}}\frac{E(\varepsilon_y+\nu\varepsilon_x)}{1-\nu^2}\mathrm{d}\varepsilon_y\end{aligned}$$

$$
=\int_0^{\varepsilon_{x1}} \frac{E\left(\varepsilon_x+\nu \dfrac{\varepsilon_{y1}}{\varepsilon_{x1}} \varepsilon_x\right)}{1-\nu^2} \mathrm{d} \varepsilon_x+\int_{\varepsilon_{x1}}^{\varepsilon_{x2}} \frac{E(\varepsilon_x+\nu \varepsilon_{y1})}{1-\nu^2} \mathrm{d} \varepsilon_x+\int_0^{\varepsilon_{y1}} \frac{E\left(\varepsilon_y+\nu \dfrac{\varepsilon_{x1}}{\varepsilon_{y1}} \varepsilon_y\right)}{1-\nu^2} \mathrm{d} \varepsilon_y
$$

$$
=\frac{E}{2(1-\nu^2)}(\varepsilon_{x2}^2+\varepsilon_{y1}^2+2\nu \varepsilon_{x2} \varepsilon_{y1})
$$

□

4-8 根据弹性理论的应变能公式(4.28),导出材料力学中杆件拉伸,弯曲及圆轴扭转的应变能公式

$$
U_{拉伸}=\frac{1}{2} \int_0^l \frac{N^2(x)}{EA} \mathrm{d} x=\frac{1}{2} \int_0^l EA\left(\frac{\mathrm{d} u}{\mathrm{d} x}\right)^2 \mathrm{d} x
$$

$$
U_{弯曲}=\frac{1}{2} \int_0^l \frac{M^2(x)}{EI} \mathrm{d} x=\frac{1}{2} \int_0^l EI\left(\frac{\mathrm{d}^2 w}{\mathrm{d} x^2}\right)^2 \mathrm{d} x
$$

$$
U_{扭转}=\frac{1}{2} \int_0^l \frac{M_t^2(z)}{GJ_\rho} \mathrm{d} z=\frac{1}{2} \int_0^l GJ_\rho\left(\frac{\mathrm{d} \phi}{\mathrm{d} z}\right)^2 \mathrm{d} z
$$

解 对杆件单向拉伸,有

$$
\sigma_x=\frac{N(x)}{A}, \quad \sigma_y=\sigma_z=\tau_{xy}=\tau_{yz}=\tau_{zx}=0, \quad \varepsilon_x=\frac{\sigma_x}{E}=\frac{\mathrm{d} u}{\mathrm{d} x}
$$

所以

$$
U_{拉伸}=\iiint_V W \mathrm{d} V=\int_0^l\left(\int_0^{\varepsilon_x} \sigma_x \mathrm{d} \varepsilon_x\right) A \mathrm{d} x=\int_0^l\left(\int_0^{\sigma_x} \frac{1}{E} \sigma_x \mathrm{d} \sigma_x\right) A \mathrm{d} x=\int_0^l \frac{\sigma_x^2}{2E} A \mathrm{d} x
$$

$$
=\int_0^l \frac{1}{2E}\left(\frac{N(x)}{A}\right)^2 A \mathrm{d} x=\frac{1}{2} \int_0^l \frac{N^2(x)}{EA} \mathrm{d} x
$$

$$
U_{拉伸}=\iiint_V W \mathrm{d} V=\int_0^l\left(\int_0^{\varepsilon_x} \sigma_x \mathrm{d} \varepsilon_x\right) A \mathrm{d} x=\int_0^l\left(\int_0^{\varepsilon_x} E \varepsilon_x \mathrm{d} \varepsilon_x\right) A \mathrm{d} x=\int_0^l \frac{\varepsilon_x^2}{2} EA \mathrm{d} x
$$

$$
=\int_0^l \frac{EA}{2}\left(\frac{\mathrm{d} u}{\mathrm{d} x}\right)^2 \mathrm{d} x=\frac{1}{2} \int_0^l EA\left(\frac{\mathrm{d} u}{\mathrm{d} x}\right)^2 \mathrm{d} x
$$

对杆件纯弯曲,有

$$
\sigma_x=\frac{M(x)}{I} z, \quad \sigma_y=\sigma_z=\tau_{xy}=\tau_{yz}=\tau_{zx}=0, \quad \varepsilon_x=\frac{\sigma_x}{E}=\frac{z}{\rho}=\frac{\mathrm{d}^2 w}{\mathrm{d} x^2} z
$$

所以

$$
U_{弯曲}=\iiint_V W \mathrm{d} V=\iint_A \int_0^l\left(\int_0^{\varepsilon_x} \sigma_x \mathrm{d} \varepsilon_x\right) \mathrm{d} x \mathrm{d} A=\iint_A \int_0^l\left(\int_0^{\sigma_x} \frac{1}{E} \sigma_x \mathrm{d} \sigma_x\right) \mathrm{d} x \mathrm{d} A=\iint_A \int_0^l \frac{\sigma_x^2}{2E} \mathrm{d} x \mathrm{d} A
$$

$$
=\iint_A \int_0^l \frac{1}{2E}\left(\frac{M(x)}{I} z\right)^2 \mathrm{d} x \mathrm{d} A=\int_0^l \frac{1}{2E}\left(\frac{M(x)}{I}\right)^2\left(\iint_A z^2 \mathrm{d} A\right) \mathrm{d} x=\frac{1}{2} \int_0^l \frac{M^2(x)}{EI} \mathrm{d} x
$$

$$
U_{弯曲}=\iiint_V W \mathrm{d} V=\iint_A \int_0^l\left(\int_0^{\varepsilon_x} \sigma_x \mathrm{d} \varepsilon_x\right) \mathrm{d} x \mathrm{d} A=\iint_A \int_0^l\left(\int_0^{\varepsilon_x} E \varepsilon_x \mathrm{d} \varepsilon_x\right) \mathrm{d} x \mathrm{d} A=\iint_A \int_0^l \frac{E \varepsilon_x^2}{2} \mathrm{d} x \mathrm{d} A
$$

$$
=\iint_A \int_0^l \frac{E}{2}\left(\frac{\mathrm{d}^2 w}{\mathrm{d} x^2} z\right)^2 \mathrm{d} x \mathrm{d} A=\int_0^l \frac{E}{2}\left(\frac{\mathrm{d}^2 w}{\mathrm{d} x^2}\right)^2\left(\iint_A z^2 \mathrm{d} A\right) \mathrm{d} x=\frac{1}{2} \int_0^l EI\left(\frac{\mathrm{d}^2 w}{\mathrm{d} x^2}\right)^2 \mathrm{d} x
$$

对圆轴扭转,有

$$
\tau_{z\theta}=\frac{M_t(z)}{J_\rho} \rho, \quad \sigma_\rho=\sigma_\theta=\sigma_z=\tau_{\rho\theta}=\tau_{z\rho}=0, \quad \varepsilon_x=\frac{\tau_{z\theta}}{G}=\rho \frac{\mathrm{d} \phi}{\mathrm{d} z}
$$

所以

$$
U_{扭转}=\iiint_V W \mathrm{d} V=\iint_A \int_0^l\left(\int_0^{\varepsilon_{z\theta}} \tau_{z\theta} \mathrm{d} \varepsilon_{z\theta}\right) \mathrm{d} z \mathrm{d} A=\iint_A \int_0^l\left(\int_0^{\tau_{z\theta}} \frac{1}{G} \tau_{z\theta} \mathrm{d} \tau_{z\theta}\right) \mathrm{d} z \mathrm{d} A=\iint_A \int_0^l \frac{\tau_{z\theta}^2}{2G} \mathrm{d} z \mathrm{d} A
$$

$$
=\iint_A \int_0^l \frac{1}{2G}\left(\frac{M_t(z)}{J_\rho}\rho\right)^2 \mathrm{d}z\mathrm{d}A = \int_0^l \frac{1}{2G}\left(\frac{M_t(z)}{J_\rho}\right)^2 \left(\iint_A \rho^2 \mathrm{d}A\right)\mathrm{d}z = \frac{1}{2}\int_0^l \frac{M_t^2(z)}{GJ_\rho}\mathrm{d}z
$$

$$
U_{扭转} = \iiint_V W\mathrm{d}V = \iint_A \int_0^l \left(\int_0^{\varepsilon_{z\theta}} \tau_{z\theta}\mathrm{d}\varepsilon_{z\theta}\right)\mathrm{d}z\mathrm{d}A = \iint_A \int_0^l \left(\int_0^{\varepsilon_{z\theta}} G\varepsilon_{z\theta}\mathrm{d}\varepsilon_{z\theta}\right)\mathrm{d}z\mathrm{d}A = \iint_A \int_0^l \frac{G\varepsilon_{z\theta}^2}{2}\mathrm{d}z\mathrm{d}A
$$

$$
=\iint_A \int_0^l \frac{G}{2}\left(\rho\frac{\mathrm{d}\phi}{\mathrm{d}z}\right)^2 \mathrm{d}z\mathrm{d}A = \int_0^l \frac{G}{2}\left(\frac{\mathrm{d}\phi}{\mathrm{d}z}\right)^2 \left(\iint_A \rho^2 \mathrm{d}A\right)\mathrm{d}z = \frac{1}{2}\int_0^l GJ_\rho\left(\frac{\mathrm{d}\phi}{\mathrm{d}z}\right)^2 \mathrm{d}z
$$

□

5-2 验证:如果 $\nabla^2\nabla^2 F_i = 0$,位移

$$u_i = \frac{\lambda+2G}{\nu(\lambda+G)}F_{i,jj} - \frac{1}{\nu}F_{j,ji}$$

是齐次 Lamé-Navier 方程的解.

证 把位移表达式代入齐次 Lamé-Navier 方程

$$Gu_{i,jj} + (\lambda+G)u_{j,ji} = 0$$

再由

$$\nabla^2\nabla^2 F_i = F_{i,jjkk} = 0$$

得

$$\begin{aligned}
\text{左边} &= G\left[\frac{\lambda+2G}{\nu(\lambda+G)}F_{i,kkjj} - \frac{1}{\nu}F_{j,kkji}\right] + (\lambda+G)\left[\frac{\lambda+2G}{\nu(\lambda+G)}F_{j,jikk} - \frac{1}{\nu}F_{k,jikj}\right]\\
&= 0 - \frac{G}{\nu}F_{j,kkji} + \frac{\lambda+2G}{\nu}F_{j,jikk} - \frac{\lambda+G}{\nu}F_{k,jikj}\\
&= -\frac{G}{\nu}F_{j,kkji} + \frac{\lambda+2G}{\nu}F_{j,kkji} - \frac{\lambda+G}{\nu}F_{k,jjki}\\
&= 0 = \text{右边}
\end{aligned}$$

所以位移

$$u_i = \frac{\lambda+2G}{\nu(\lambda+G)}F_{i,jj} - \frac{1}{\nu}F_{j,ji}$$

是齐次 Lamé-Navier 方程的解.

□

5-5 将橡皮方块放在与它同样尺寸的铁盒腔内,在铁盖上作用均匀压力 p(图 5-6).假设铁盒与铁盖可以视为刚体,在橡皮与铁之间没有摩擦.试用位移法求橡皮块中的位移、应变与应力.

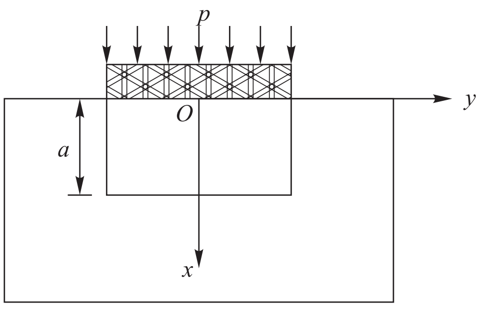


图 5.1　习题 5-5 图

解 建立如图 5.1 所示的空间直角坐标系 $Oxyz$.设不为零的位移分量只有 u,且 $u = u(x)$.则无体积力项的位移解法定解方程

$$Gu_{i,jj} + (\lambda+G)u_{j,ji} = 0$$

可化为

$$\frac{\partial^2 u}{\partial x^2} = 0$$

边界条件为

$$u(a) = 0,\quad -\sigma_x(0) = -(\lambda+2G)\frac{\partial u}{\partial x}(0) = p$$

解得

$$u(x)=\frac{p}{\lambda+2G}(a-x)$$

□

5-6 图 5-7 所示半空间体，密度为 ρ，在水平边界面上受均匀压力 q 作用. 已知该半空间体的水平位移 $u=v=0$，假设在 $z=h$ 处 $w=0$. 试用位移法求半空间体中的位移及应力.

解 半空间体 z 方向的位移分量 w 只与坐标 z 有关. 位移解法定解方程可化为

$$(\lambda+2G)\frac{\partial^2 w}{\partial z^2}+\rho g=0$$

边界条件为

$$-\sigma_z(0)=-(\lambda+2G)\frac{\partial u}{\partial x}(0)=q,\quad w(h)=0$$

解得半空间体中的位移为

$$w(z)=-\frac{\rho g}{2(\lambda+2G)}z^2-\frac{q}{\lambda+2G}z+\frac{h(\rho gh+2q)}{2(\lambda+2G)}$$

□

5-9 图 5-8 所示矩形薄板，一对边均匀受拉，另一对边均匀受压. 由叠加原理求板的应力和位移.

解 (1) 当板只均匀受拉时，位移解法定解方程为

$$\begin{cases} G\left(\dfrac{\partial^2 u_1}{\partial x^2}+\dfrac{\partial^2 u_1}{\partial y^2}\right)+(\lambda+G)\left(\dfrac{\partial^2 u_1}{\partial x^2}+\dfrac{\partial^2 v_1}{\partial x\partial y}\right)=0 \\ G\left(\dfrac{\partial^2 v_1}{\partial x^2}+\dfrac{\partial^2 v_1}{\partial y^2}\right)+(\lambda+G)\left(\dfrac{\partial^2 v_1}{\partial y^2}+\dfrac{\partial^2 u_1}{\partial x\partial y}\right)=0 \end{cases}$$

边界条件为

$$u(0,y)=0,\quad \sigma_x(a,y)=2G\frac{\partial u_1}{\partial x}+\lambda\left(\frac{\partial u_1}{\partial x}+\frac{\partial v_1}{\partial y}\right)=q_1$$

$$v(x,0)=0,\quad \sigma_y(x,b)=2G\frac{\partial v_1}{\partial y}+\lambda\left(\frac{\partial u_1}{\partial x}+\frac{\partial v_1}{\partial y}\right)=0$$

解得

$$u_1(x,y)=\frac{q_1(\lambda+2G)}{4G(\lambda+G)}x,\quad v_1(x,y)=-\frac{\lambda q_1}{4G(\lambda+G)}y$$

⇨

(2) 当板只均匀受压时，位移解法定解方程为

$$\begin{cases} G\left(\dfrac{\partial^2 u_2}{\partial x^2}+\dfrac{\partial^2 u_2}{\partial y^2}\right)+(\lambda+G)\left(\dfrac{\partial^2 u_2}{\partial x^2}+\dfrac{\partial^2 v_2}{\partial x\partial y}\right)=0 \\ G\left(\dfrac{\partial^2 v_2}{\partial x^2}+\dfrac{\partial^2 v_2}{\partial y^2}\right)+(\lambda+G)\left(\dfrac{\partial^2 v_2}{\partial y^2}+\dfrac{\partial^2 u_2}{\partial x\partial y}\right)=0 \end{cases}$$

边界条件为

$$u(0,y)=0,\quad \sigma_x(a,y)=2G\frac{\partial u_2}{\partial x}+\lambda\left(\frac{\partial u_2}{\partial x}+\frac{\partial v_2}{\partial y}\right)=0$$

$$v(x,0)=0,\quad \sigma_y(x,b)=2G\frac{\partial v_2}{\partial y}+\lambda\left(\frac{\partial u_2}{\partial x}+\frac{\partial v_2}{\partial y}\right)=-q_2$$

解得

$$u_2(x,y) = \frac{\lambda q_2}{4G(\lambda+G)}x, \quad v_2(x,y) = -\frac{q_2(\lambda+2G)}{4G(\lambda+G)}y$$

综合(1)(2)并由叠加原理得板的位移为

$$u(x,y) = u_1(x,y) + u_2(x,y) = \frac{q_1(\lambda+2G)+\lambda q_2}{4G(\lambda+G)}x$$

$$v(x,y) = v_1(x,y) + v_2(x,y) = -\frac{\lambda q_1 + q_2(\lambda+2G)}{4G(\lambda+G)}y$$

应力为

$$\sigma_x = 2G\frac{\partial u}{\partial x} + \lambda\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right) = q_1$$

$$\sigma_y = 2G\frac{\partial v}{\partial y} + \lambda\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right) = -q_2$$

$$\tau_{xy} = G\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right) = 0$$

□

5-10 矩形截面杆件受沿轴向的简单拉伸及绕 x,y 轴的弯矩的作用,如图 5-9 所示.不计体力.6 个应力分量为

$$\sigma_z \neq 0; \quad \sigma_x = \sigma_y = \tau_{yz} = \tau_{zx} = \tau_{xy} = 0$$

试用平衡方程和 B-M 方程求 σ_z 的函数形式.并利用端面力边界条件

$$\iint_A \tau_{yz}\,\mathrm{d}A = \iint_A \tau_{zx}\,\mathrm{d}A = \iint_A (x\tau_{yz} - y\tau_{zx})\,\mathrm{d}A = 0$$

$$\iint_A \sigma_z\,\mathrm{d}A = P_z, \quad \iint_A y\sigma_z\,\mathrm{d}A = M_x, \quad \iint_A x\sigma_z\,\mathrm{d}A = -M_y$$

和坐标原点处无平移和转动的条件确定积分常数.(A 为端部横截面面积,x,y 轴分别为截面的对称轴.截面对 x,y 轴的惯性矩分别为 I_x,I_y.)

解 在题设条件下,无体积力项 B-M 方程

$$\nabla^2\sigma_{ij} + \frac{1}{1+\nu}\sigma_{kk,ij} = 0$$

化为

$$\frac{\partial^2\sigma_z}{\partial x^2} = \frac{\partial^2\sigma_z}{\partial y^2} = \nabla^2\sigma_z + \frac{1}{1+\nu}\frac{\partial^2\sigma_z}{\partial z^2} = \frac{\partial^2\sigma_z}{\partial x\partial y} = \frac{\partial^2\sigma_z}{\partial y\partial z} = \frac{\partial^2\sigma_z}{\partial z\partial x} = 0$$

无体积力项平衡方程

$$\sigma_{ji,j} = 0$$

化为

$$\frac{\partial\sigma_z}{\partial z} = 0$$

所以 σ_z 的函数形式为

$$\sigma_z = ax + by + c$$

由端面力边界条件及对称条件可得

$$\iint_A \sigma_z\,\mathrm{d}A = \iint_A (ax+by+c)\,\mathrm{d}A = cA = P_z$$

$$\iint_A y\sigma_z\,\mathrm{d}A = \iint_A y(ax+by+c)\,\mathrm{d}A = bI_x = M_x$$

$$\iint_A x\sigma_z \mathrm{d}A = \iint_A x(ax + by + c)\mathrm{d}A = aI_y = -M_y$$

所以积分常数为

$$a = -\frac{M_y}{I_y}, \quad b = \frac{M_x}{I_x}, \quad c = \frac{P_z}{A}$$

故

$$\sigma_z = -\frac{M_y}{I_y}x + \frac{M_x}{I_x}y + \frac{P_z}{A}$$

□

6-5 证明翘曲函数

$$\psi(x,y)=\frac{b^2-a^2}{b^2+a^2}xy \qquad ①$$

能用来求解椭圆杆 $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ 的扭转问题,并且扭矩为

$$M=\frac{G\alpha\pi a^3b^3}{a^2+b^2}$$

证（1）翘曲函数①满足调和方程

$$\nabla^2\psi=\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}=0+0=0$$

（2）翘曲函数①满足边界条件

$$\frac{\partial\psi}{\partial\nu}=\nabla\psi\cdot\boldsymbol{\nu}=\left(\frac{b^2-a^2}{b^2+a^2}y,\frac{b^2-a^2}{b^2+a^2}x\right)\cdot\frac{1}{\sqrt{\left(\frac{2x}{a^2}\right)^2+\left(\frac{2y}{b^2}\right)^2}}\left(\frac{2x}{a^2},\frac{2y}{b^2}\right)$$

$$\overset{\substack{x=a\cos\theta\\ y=b\sin\theta}}{=\!=\!=\!=}\underbrace{\frac{b\cos\theta}{\sqrt{b^2\cos^2\theta+a^2\sin^2\theta}}}_{\nu_1}\underbrace{b\sin\theta}_{y}-\underbrace{\frac{a\sin\theta}{\sqrt{b^2\cos^2\theta+a^2\sin^2\theta}}}_{\nu_2}\underbrace{a\cos\theta}_{x}=\nu_1y-\nu_2x$$

综合(1)(2)知,翘曲函数能用来求解椭圆杆 $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ 的扭转问题. 扭矩为

$$\begin{aligned}
M_z&=\alpha G\iint_A\left[\left(\frac{\partial\psi}{\partial y}x-\frac{\partial\psi}{\partial x}y\right)+(x^2+y^2)\right]\mathrm{d}A\\
&=\alpha G\iint_A\left[\left(\frac{b^2-a^2}{b^2+a^2}x^2-\frac{b^2-a^2}{b^2+a^2}y^2\right)+(x^2+y^2)\right]\mathrm{d}A\\
&=2\alpha G\iint_A\frac{b^2x^2+a^2y^2}{b^2+a^2}\mathrm{d}x\mathrm{d}y\\
&=2\alpha G\iint_D\frac{b^2(ar\cos\theta)^2+a^2(br\sin\theta)^2}{b^2+a^2}\left|\det\begin{pmatrix}a\cos\theta & -ar\sin\theta\\ b\sin\theta & br\cos\theta\end{pmatrix}\right|\mathrm{d}r\mathrm{d}\theta\\
&=\frac{8G\alpha a^3b^3}{a^2+b^2}\int_0^{\frac{\pi}{2}}\int_0^1 r^3\mathrm{d}r\mathrm{d}\theta=\frac{8G\alpha a^3b^3}{a^2+b^2}\cdot\frac{\pi}{2}\cdot\frac{1}{4}=\frac{G\alpha\pi a^3b^3}{a^2+b^2}
\end{aligned}$$

□

6-6 试比较边长为 a 的正方形截面杆与面积相等的圆截面杆,承受同样大小扭矩时所产生的最大剪应力及抗扭刚度.

解 设两杆均承受扭矩 M_t. 则边长为 a 的正方形截面杆的最大剪应力为

$$\tau_{\max,1}=\frac{M_t}{0.208a^3}$$

抗扭刚度为

$$D_{t,1}=0.141Ga^4$$

面积为 a^2 的圆的半径为

$$r=\sqrt{\frac{a^2}{\pi}}=\frac{a}{\sqrt{\pi}}$$

最大切应力为

$$\tau_{\max,2}=\frac{M_t}{\frac{\pi r^3}{2}}=\frac{2\sqrt{\pi}M_t}{a^3}$$

抗扭刚度为

$$D_{t,2}=G\frac{\pi r^4}{2}=\frac{Ga^4}{2\pi}$$

所以

$$\frac{\tau_{\max,1}}{\tau_{\max,2}}=\frac{\frac{M_t}{0.208a^3}}{\frac{2\sqrt{\pi}M_t}{a^3}}=1.356$$

$$\frac{D_{t,1}}{D_{t,2}}=\frac{0.141Ga^4}{\frac{Ga^4}{2\pi}}=0.886$$

即两杆承受同样大小扭矩时,正方形截面杆所产生的最大剪应力较大,抗扭刚度较小.

□

6-7 边长为 a 的正方形截面杆承受扭转,坐标如图 6-28 所示,扭矩为 M_z. $ABCD$ 为与横截面成 $\frac{\pi}{4}$ 角的斜截面, E,F 分别为 AD 及 AB 的中点. 已知在扭矩 M_z 作用下,正方形横截面各边中点的剪应力为

$$\tau=\frac{M_z}{0.208a^3}$$

求斜截面 $ABCD$ 上点 E 与点 F 处的正应力与剪应力.

解 点 E 处的应力张量为

$$\boldsymbol{\sigma}_E=\begin{bmatrix}\sigma_x & \tau_{xy} & \tau_{xz}\\ \tau_{yx} & \sigma_y & \tau_{yz}\\ \tau_{zx} & \tau_{zy} & \sigma_z\end{bmatrix}=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & \tau\\ 0 & \tau & 0\end{bmatrix}$$

点 F 处的应力张量为

$$\boldsymbol{\sigma}_F=\begin{bmatrix}\sigma_x & \tau_{xy} & \tau_{xz}\\ \tau_{yx} & \sigma_y & \tau_{yz}\\ \tau_{zx} & \tau_{zy} & \sigma_z\end{bmatrix}=\begin{bmatrix}0 & 0 & \tau\\ 0 & 0 & 0\\ \tau & 0 & 0\end{bmatrix}$$

截面 $ABCD$ 的单位法向量为

$$\boldsymbol{\nu}=\left(\frac{\sqrt{2}}{2},0,\frac{\sqrt{2}}{2}\right)$$

由 Cauchy 公式得截面 $ABCD$ 上点 E 与点 F 处的应力矢量分别为

$$\boldsymbol{\sigma}_E=\boldsymbol{\nu}\cdot\boldsymbol{\sigma}_E=\begin{bmatrix}\frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2}\end{bmatrix}\cdot\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & \tau\\ 0 & \tau & 0\end{bmatrix}=\begin{bmatrix}0 & \frac{\sqrt{2}}{2}\tau & 0\end{bmatrix}$$

$$\boldsymbol{\sigma}_F = \boldsymbol{\nu} \cdot \boldsymbol{\sigma}_F = \begin{bmatrix} \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & \tau \\ 0 & 0 & 0 \\ \tau & 0 & 0 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2}\tau & 0 & \frac{\sqrt{2}}{2}\tau \end{bmatrix}$$

点 E 与点 F 处的正应力分别为

$$\sigma_{n,E} = \boldsymbol{\nu} \cdot \boldsymbol{\sigma}_E = \begin{bmatrix} \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \end{bmatrix} \cdot \begin{bmatrix} 0 & \frac{\sqrt{2}}{2}\tau & 0 \end{bmatrix} = 0$$

$$\sigma_{n,F} = \boldsymbol{\nu} \cdot \boldsymbol{\sigma}_F = \begin{bmatrix} \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \end{bmatrix} \cdot \begin{bmatrix} \frac{\sqrt{2}}{2}\tau & 0 & \frac{\sqrt{2}}{2}\tau \end{bmatrix} = \tau = \frac{M_z}{0.208a^3} = 4.808\frac{M_z}{a^3}$$

点 E 与点 F 处的剪应力分别为

$$\tau_E = \sqrt{\sigma_E^2 - \sigma_{n,E}^2} = \sqrt{\left(\frac{\sqrt{2}}{2}\tau\right)^2 - 0^2} = \frac{\sqrt{2}}{2}\tau = \frac{\sqrt{2}}{2}\frac{M_z}{0.208a^3} = 3.400\frac{M_z}{a^3}$$

$$\tau_F = \sqrt{\sigma_F^2 - \sigma_{n,F}^2} = \sqrt{\left(\frac{\sqrt{2}}{2}\tau\right)^2 + \left(\frac{\sqrt{2}}{2}\tau\right)^2 - \tau^2} = 0$$

□

6-9 求图 6-29 所示薄壁构件的扭转刚度. 材料剪切模量为 G.

解 如图设薄壁构件的总扭转刚度等于各部分的扭转刚度之和,即

$$D_{\mathrm{t}} = \frac{G}{3}\sum_{i=1}^{4} a_i\delta_i^3 + \frac{4GA^2\delta}{s} = \frac{4G}{3}b\delta^3 + \frac{4G[(a+\delta)(a-\delta)]^2\delta}{2(a+\delta)+2(a-\delta)} = \frac{4Gb\delta^3}{3} + \frac{G\delta(a^2-\delta^2)^2}{a}$$

□

6-10 图 6-30 所示均匀厚度的薄壁管,承受扭矩 M_z 作用,试求管壁中的剪应力及管的单位长度扭转角.

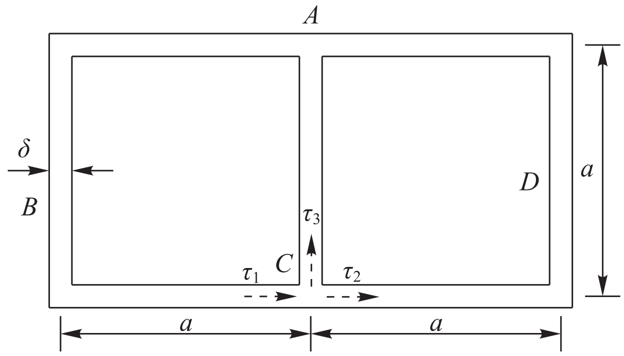


图 6.1　习题 6-10 图

解 如图 6.1 所示,设左侧管壁内切应力为 τ_1,右侧管壁内切应力为 τ_2,隔板内切应力为 τ_3. 则通过各截面段的切应力流为

$$\tau_1\delta = h_1, \quad \tau_2\delta = h_2, \quad \tau_3\delta = h_3 = h_1 - h_2 = \tau_1\delta - \tau_2\delta$$

扭矩为

$$M_{\mathrm{t}} = 2V = 2(A_1h_1 + A_2h_2) = 2A_1\delta\tau_1 + 2A_2\delta\tau_2$$

其中 A_1, A_2 为两孔(壁厚中心线包围的)面积. 所以环量为

$$\oint_{ABCA}\tau\,\mathrm{d}s = \tau_1 s_1 + \tau_3 s_3 = 2G\alpha A_1$$

$$\oint_{CDAC}\tau\,\mathrm{d}s = \tau_2 s_2 - \tau_3 s_3 = 2G\alpha A_2$$

其中 s_1, s_2, s_3 分别为中心线 $\overparen{ABC}, \overparen{CDA}, \overline{CA}$ 的长度.

解得管壁边框内的剪应力为

$$\tau_1 = \frac{M_z}{2[2\delta^2 \cdot 3a \cdot (a^2)^2 + \delta^2 a(2a^2)^2]}(\delta a \cdot 2a^2 + \delta \cdot 3a \cdot a^2) = \frac{M_z}{4\delta a^2}$$

隔板内的剪应力为

$$\tau_2 = \frac{M_z}{2[2\delta^2 \cdot 3a \cdot (a^2)^2 + \delta^2 a(2a^2)^2]}(\delta \cdot 3a \cdot a^2 - \delta \cdot 3a \cdot a^2) = 0$$

管的单位长度扭转角为

$$\alpha = \frac{1}{2Ga^2}(\tau_1 \cdot 3a + \tau_2 \cdot a) = \frac{3M_z}{8G\delta a^3}$$

□

7-3 一矩形截面悬臂薄板,在自由端受集中力 P 作用,如图 7-40 所示. 由材料力学知

$$\sigma_x = \frac{P(l-x)}{I}y$$

设梁处于平面应力状态. 试根据平衡方程及边界条件写出 σ_y 及 τ_{xy} 的表达式,并验证它们是否表示正确的解答.

解

平衡方程为

$$\begin{cases} \dfrac{\partial \sigma_x}{\partial x} + \dfrac{\partial \tau_{xy}}{\partial y} = 0 \\ \dfrac{\partial \tau_{xy}}{\partial x} + \dfrac{\partial \sigma_y}{\partial y} = 0 \end{cases}$$

边界条件为

$$\begin{cases} y = \pm \dfrac{h}{2} & \sigma_y = 0, \quad \tau_{xy} = 0 \\ x = l & \iint_A \sigma_x \mathrm{d}A = 0, \quad \iint_A \tau_{xy} \mathrm{d}A = -P \end{cases}$$

把 $\sigma_y = \dfrac{P(l-x)}{I}y$ 代入平衡方程得

$$\frac{\partial \tau_{xy}}{\partial y} = -\frac{\partial \sigma_x}{\partial x} = \frac{P}{I}y$$

所以

$$\tau_{xy} = \frac{P}{2I}y^2 + f(x), \quad \frac{\partial \sigma_y}{\partial y} = -\frac{\partial \tau_{xy}}{\partial x} = -f'(x), \quad \sigma_y = -f'(x)y + g(x)$$

由侧面边界条件得

$$\tau_{xy} = \frac{P}{2I}\left(\pm \frac{h}{2}\right)^2 + f(x) = 0, \quad \sigma_y = \mp f'(x)\frac{h}{2} + g(x) = 0$$

所以

$$f(x) = -\frac{Ph^2}{8I}, \quad g(x) = 0$$

故

$$\tau_{xy} = \frac{P}{8I}(4y^2 - h^2), \quad \sigma_y = 0$$

验证代入无体力 B-M 方程得

$$\text{左边} = \nabla^2(\sigma_x + \sigma_y) = \nabla^2\left[\frac{P(l-x)}{I}y + 0\right] = 0 = \text{右边}$$

及端面边界条件

$$\text{左边} = \iint_A \frac{P(l-l)}{I}y\,\mathrm{d}A = 0 = \text{右边}$$

$$\text{左边} = \iint_A \frac{P}{8I}(4y^2 - h^2)\mathrm{d}A = \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{P}{8I}(4y^2 - h^2)\mathrm{d}y = -\frac{2}{3}\frac{P}{8I}h^3 = -\frac{2}{3}\frac{P}{\frac{8}{12}h^3}h^3 = -P = \text{右边}$$

都成立.又

$$\Theta=\sigma_x+\sigma_y=\frac{P(l-x)}{I}y+0\neq ax+by+c$$

所以该应力状态不是平面应力状态,为广义平面应力状态.

□

7-4 如图 7-41 所示矩形薄板,厚度为 1.验证 $\phi=axy^3$ 能否作为应力函数?若能,写出应力分量表达式(不计体力).并利用边界条件画出边界上的作用力.

解

验证无体力应力协调方程

$$\text{左边}=\nabla^2\nabla^2\phi=\frac{\partial^4\phi}{\partial x^4}+2\frac{\partial^4\phi}{\partial x^2\partial y^2}+\frac{\partial^4\phi}{\partial y^4}=0+0+0=0=\text{右边}$$

成立,所以 $\phi=axy^3$ 能作为应力函数.应力分量表达式为

$$\sigma_x=\frac{\partial^2\phi}{\partial y^2}=6axy,\quad \sigma_y=\frac{\partial^2\phi}{\partial x^2}=0,\quad \tau_{xy}=\frac{\partial^2\phi}{\partial x\partial y}=3ay^2$$

边界条件为

❶ 左侧边界 $x=0$

以 O 为参考点,因该点 $\phi=axy^3=0,\dfrac{\partial\phi}{\partial y}=3axy^2=0,\dfrac{\partial\phi}{\partial x}=ay^3=0$. 所以

$$\frac{\partial\phi}{\partial y}=0=\int_0^y\overline{X}(-\mathrm{d}\tilde{y})=R_x$$

$$\frac{\partial\phi}{\partial x}=ay^3=-\int_0^y\overline{Y}(-\mathrm{d}\tilde{y})=-R_y$$

解得

$$\overline{X}=0,\quad \overline{Y}=3ay^2$$

❷ 右侧边界 $x=l$

以右侧边界中点 $O'(l,0)$ 为参考点,因该点 $\phi=axy^3=0,\dfrac{\partial\phi}{\partial y}=3axy^2=0,\dfrac{\partial\phi}{\partial x}=ay^3=0$.

所以

$$\frac{\partial\phi}{\partial y}=3aly^2=\int_0^y\overline{X}\mathrm{d}\tilde{y}=R_x$$

$$\frac{\partial\phi}{\partial x}=ay^3=-\int_0^y\overline{Y}\mathrm{d}\tilde{y}=-R_y$$

解得

$$\overline{X}=6aly,\quad \overline{Y}=-3ay^2$$

❸ 上侧边界 $y=\dfrac{h}{2}$

以右侧边界中点 $O'(l,0)$ 为参考点,因该点 $\phi=axy^3=0,\dfrac{\partial\phi}{\partial y}=3axy^2=0,\dfrac{\partial\phi}{\partial x}=ay^3=0$.

所以

$$\left.\frac{\partial\phi}{\partial y}\right|_B=3ax\left(\frac{h}{2}\right)^2=\int_{O'}^B\overline{X}\,\mathrm{d}s=\int_0^{\frac{h}{2}}\overline{X}\mathrm{d}y+\int_l^x\overline{X}(-\mathrm{d}\tilde{x})=\int_0^{\frac{h}{2}}6aly\mathrm{d}y+\int_l^x\overline{X}(-\mathrm{d}\tilde{x})=R_x$$

$$\left.\frac{\partial\phi}{\partial x}\right|_B = a\left(\frac{h}{2}\right)^3 = -\int_{O'}^{B}\bar{Y}\,\mathrm{d}s = -\left[\int_0^{\frac{h}{2}}\bar{Y}\,\mathrm{d}y + \int_l^x \bar{Y}(-\,\mathrm{d}\tilde{x})\right] = -\left[-\int_0^{\frac{h}{2}}3ay^2\,\mathrm{d}y - \int_l^x\bar{Y}\,\mathrm{d}\tilde{x}\right] = -R_y$$

解得

$$\bar{X} = -\frac{3}{4}ah^2,\quad \bar{Y} = 0$$

❹ 下侧边界 $y = -\frac{h}{2}$

以 O 为参考点,因该点 $\phi = axy^3 = 0$,$\frac{\partial\phi}{\partial y} = 3axy^2 = 0$,$\frac{\partial\phi}{\partial x} = ay^3 = 0$. 所以

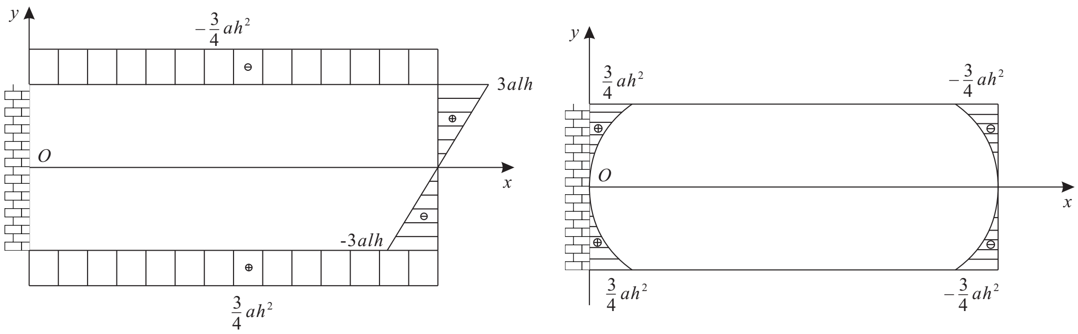
$$\left.\frac{\partial\phi}{\partial y}\right|_D = 3ax\left(-\frac{h}{2}\right)^2 = \int_O^D\bar{X}\,\mathrm{d}s = \int_0^{-\frac{h}{2}}\bar{X}\,\mathrm{d}y + \int_0^x\bar{X}\,\mathrm{d}\tilde{x} = \int_0^{-\frac{h}{2}}0\,\mathrm{d}y + \int_x^0\bar{X}\,\mathrm{d}\tilde{x} = R_x$$

$$\left.\frac{\partial\phi}{\partial x}\right|_D = a\left(-\frac{h}{2}\right)^3 = -\int_O^D\bar{Y}\,\mathrm{d}s = -\left(\int_0^{-\frac{h}{2}}\bar{Y}\,\mathrm{d}y + \int_0^x\bar{Y}\,\mathrm{d}\tilde{x}\right) = -\left(\int_0^{-\frac{h}{2}}3ay^2\,\mathrm{d}y + \int_0^x\bar{Y}\,\mathrm{d}\tilde{x}\right) = -R_y$$

解得

$$\bar{X} = \frac{3}{4}ah^2,\quad \bar{Y} = 0$$

画出各边界上的作用力如图 7.1 所示.



(a) x 分量　　(b) y 分量

图 7.1　各边界上的作用力

7-1 利用坐标变化的方法,将笛卡尔坐标系中的平衡方程

$$\begin{cases}\dfrac{\partial\sigma_x}{\partial x} + \dfrac{\partial\tau_{xy}}{\partial y} + f_x = 0\\[2ex]\dfrac{\partial\tau_{xy}}{\partial x} + \dfrac{\partial\sigma_y}{\partial y} + f_y = 0\end{cases}$$

转换到极坐标中表示的形式

$$\begin{cases}\dfrac{\partial\sigma_r}{\partial r} + \dfrac{1}{r}\dfrac{\partial\tau_{r\theta}}{\partial\theta} + \dfrac{\sigma_r - \sigma_\theta}{r} + f_r = 0\\[2ex]\dfrac{\partial\tau_{r\theta}}{\partial r} + \dfrac{1}{r}\dfrac{\partial\sigma_\theta}{\partial\theta} + 2\dfrac{\tau_{r\theta}}{r} + f_\theta = 0\end{cases}$$

解

由

$$r = \sqrt{x^2 + y^2},\quad \theta = \arctan\frac{y}{x}$$

得

$$\frac{\partial r}{\partial x}=\frac{x}{r}=\cos\theta,\quad \frac{\partial r}{\partial y}=\frac{y}{r}=\sin\theta,\quad \frac{\partial\theta}{\partial x}=-\frac{y}{r^2}=-\frac{\sin\theta}{r},\quad \frac{\partial\theta}{\partial y}=\frac{x}{r^2}=\frac{\cos\theta}{r}$$

又

$$\begin{bmatrix}\sigma_x & \tau_{xy}\\ \tau_{xy} & \sigma_y\end{bmatrix}=\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}\sigma_r & \tau_{r\theta}\\ \tau_{r\theta} & \sigma_\theta\end{bmatrix}\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}$$

$$\begin{bmatrix}f_x\\ f_y\end{bmatrix}=\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}f_r\\ f_\theta\end{bmatrix}$$

所以

$$\begin{aligned}
\sigma_x &= \sigma_r\cos^2\theta+\sigma_\theta\sin^2\theta-2\tau_{r\theta}\sin\theta\cos\theta\\
\sigma_y &= \sigma_r\sin^2\theta+\sigma_\theta\cos^2\theta+2\tau_{r\theta}\sin\theta\cos\theta\\
\tau_{xy} &= \tau_{r\theta}(\cos^2\theta-\sin^2\theta)+(\sigma_r-\sigma_\theta)\sin\theta\cos\theta\\
f_x &= f_r\cos\theta-f_\theta\sin\theta\\
f_y &= f_r\sin\theta+f_\theta\cos\theta
\end{aligned}$$

且

$$\begin{aligned}
\frac{\partial\sigma_x}{\partial x}=&\frac{\partial(\sigma_r\cos^2\theta+\sigma_\theta\sin^2\theta-2\tau_{r\theta}\sin\theta\cos\theta)}{\partial r}\frac{\partial r}{\partial x}+\frac{\partial(\sigma_r\cos^2\theta+\sigma_\theta\sin^2\theta-2\tau_{r\theta}\sin\theta\cos\theta)}{\partial\theta}\frac{\partial\theta}{\partial x}\\
=&\frac{\partial\sigma_r}{\partial r}\cos^3\theta+\frac{\partial\sigma_\theta}{\partial r}\cos\theta\sin^2\theta-2\frac{\partial\tau_{r\theta}}{\partial r}\cos^2\theta\sin\theta-\frac{\partial\sigma_r}{\partial\theta}\frac{\cos^2\theta\sin\theta}{r}-\frac{\partial\sigma_\theta}{\partial\theta}\frac{\sin^3\theta}{r}+2\frac{\partial\tau_{r\theta}}{\partial\theta}\frac{\sin^2\theta\cos\theta}{r}\\
&+2\sigma_r\frac{\sin^2\theta\cos\theta}{r}-2\sigma_\theta\frac{\sin^2\theta\cos\theta}{r}+2\tau_{r\theta}\frac{\sin\theta\cos^2\theta}{r}-2\tau_{r\theta}\frac{\sin^3\theta}{r}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial\tau_{xy}}{\partial y}=&\frac{\partial[\tau_{r\theta}(\cos^2\theta-\sin^2\theta)+(\sigma_r-\sigma_\theta)\sin\theta\cos\theta]}{\partial r}\frac{\partial r}{\partial y}+\frac{\partial[\tau_{r\theta}(\cos^2\theta-\sin^2\theta)+(\sigma_r-\sigma_\theta)\sin\theta\cos\theta]}{\partial\theta}\frac{\partial\theta}{\partial y}\\
=&\frac{\partial\tau_{r\theta}}{\partial r}\cos^2\theta\sin\theta-\frac{\partial\tau_{r\theta}}{\partial r}\sin^3\theta+\frac{\partial\sigma_r}{\partial r}\cos\theta\sin^2\theta-\frac{\partial\sigma_\theta}{\partial r}\cos\theta\sin^2\theta\\
&+\frac{\partial\tau_{r\theta}}{\partial\theta}\frac{\cos^3\theta}{r}-\frac{\partial\tau_{r\theta}}{\partial\theta}\frac{\sin^2\theta\cos\theta}{r}+\frac{\partial\sigma_r}{\partial\theta}\frac{\sin\theta\cos^2\theta}{r}-\frac{\partial\sigma_\theta}{\partial\theta}\frac{\sin\theta\cos^2\theta}{r}\\
&-4\tau_{r\theta}\frac{\cos^2\theta\sin\theta}{r}+(\sigma_r-\sigma_\theta)\frac{\cos^3\theta-\sin^2\theta\cos\theta}{r}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial\tau_{xy}}{\partial x}=&\frac{\partial[\tau_{r\theta}(\cos^2\theta-\sin^2\theta)+(\sigma_r-\sigma_\theta)\sin\theta\cos\theta]}{\partial r}\frac{\partial r}{\partial x}+\frac{\partial[\tau_{r\theta}(\cos^2\theta-\sin^2\theta)+(\sigma_r-\sigma_\theta)\sin\theta\cos\theta]}{\partial\theta}\frac{\partial\theta}{\partial x}\\
=&\frac{\partial\tau_{r\theta}}{\partial r}\cos^3\theta-\frac{\partial\tau_{r\theta}}{\partial r}\sin^2\theta\cos\theta+\frac{\partial\sigma_r}{\partial r}\sin\theta\cos^2\theta-\frac{\partial\sigma_\theta}{\partial r}\sin\theta\cos^2\theta\\
&-\frac{\partial\tau_{r\theta}}{\partial\theta}\frac{\cos^2\theta\sin\theta}{r}+\frac{\partial\tau_{r\theta}}{\partial\theta}\frac{\sin^3\theta}{r}-\frac{\partial\sigma_r}{\partial\theta}\frac{\cos\theta\sin^2\theta}{r}+\frac{\partial\sigma_\theta}{\partial\theta}\frac{\cos\theta\sin^2\theta}{r}\\
&+4\tau_{r\theta}\frac{\cos\theta\sin^2\theta}{r}-(\sigma_r-\sigma_\theta)\frac{\cos^2\theta\sin\theta-\sin^3\theta}{r}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial\sigma_y}{\partial y}=&\frac{\partial(\sigma_r\sin^2\theta+\sigma_\theta\cos^2\theta+2\tau_{r\theta}\sin\theta\cos\theta)}{\partial r}\frac{\partial r}{\partial y}+\frac{\partial(\sigma_r\sin^2\theta+\sigma_\theta\cos^2\theta+2\tau_{r\theta}\sin\theta\cos\theta)}{\partial\theta}\frac{\partial\theta}{\partial y}\\
=&\frac{\partial\sigma_r}{\partial r}\sin^3\theta+\frac{\partial\sigma_\theta}{\partial r}\cos^2\theta\sin\theta+2\frac{\partial\tau_{r\theta}}{\partial r}\sin^2\theta\cos\theta+\frac{\partial\sigma_r}{\partial\theta}\frac{\sin^2\theta\cos\theta}{r}+\frac{\partial\sigma_\theta}{\partial\theta}\frac{\cos^3\theta}{r}+2\frac{\partial\tau_{r\theta}}{\partial\theta}\frac{\sin\theta\cos^2\theta}{r}\\
&+2\sigma_r\frac{\sin\theta\cos^2\theta}{r}-2\sigma_\theta\frac{\sin\theta\cos^2\theta}{r}+2\tau_{r\theta}\frac{\cos^3\theta}{r}-2\tau_{r\theta}\frac{\sin^2\theta\cos\theta}{r}
\end{aligned}$$

代入直角坐标平衡方程得

$$
\begin{aligned}
&\frac{\partial\sigma_x}{\partial x}+\frac{\partial\tau_{xy}}{\partial y}+f_x\\
=&\frac{\partial\sigma_r}{\partial r}\cos^3\theta+\frac{\partial\sigma_\theta}{\partial r}\cos\theta\sin^2\theta-2\frac{\partial\tau_{r\theta}}{\partial r}\cos^2\theta\sin\theta-\frac{\partial\sigma_r}{\partial\theta}\frac{\cos^2\theta\sin\theta}{r}-\frac{\partial\sigma_\theta}{\partial\theta}\frac{\sin^3\theta}{r}+2\frac{\partial\tau_{r\theta}}{\partial\theta}\frac{\sin^2\theta\cos\theta}{r}\\
&+2\sigma_r\frac{\sin^2\theta\cos\theta}{r}-2\sigma_\theta\frac{\sin^2\theta\cos\theta}{r}+2\tau_{r\theta}\frac{\sin\theta\cos^2\theta}{r}-2\tau_{r\theta}\frac{\sin^3\theta}{r}\\
&+\frac{\partial\tau_{r\theta}}{\partial r}\cos^2\theta\sin\theta-\frac{\partial\tau_{r\theta}}{\partial r}\sin^3\theta+\frac{\partial\sigma_r}{\partial r}\cos\theta\sin^2\theta-\frac{\partial\sigma_\theta}{\partial r}\cos\theta\sin^2\theta\\
&+\frac{\partial\tau_{r\theta}}{\partial\theta}\frac{\cos^3\theta}{r}-\frac{\partial\tau_{r\theta}}{\partial\theta}\frac{\sin^2\theta\cos\theta}{r}+\frac{\partial\sigma_r}{\partial\theta}\frac{\sin\theta\cos^2\theta}{r}-\frac{\partial\sigma_\theta}{\partial\theta}\frac{\sin\theta\cos^2\theta}{r}\\
&-4\tau_{r\theta}\frac{\cos^2\theta\sin\theta}{r}+(\sigma_r-\sigma_\theta)\frac{\cos^3\theta-\sin^2\theta\cos\theta}{r}\\
=&\frac{\partial\sigma_r}{\partial r}\cos\theta-\frac{\partial\tau_{r\theta}}{\partial r}\sin\theta-\frac{\partial\sigma_\theta}{\partial\theta}\frac{\sin\theta}{r}+\frac{\partial\tau_{r\theta}}{\partial\theta}\frac{\cos\theta}{r}+(\sigma_r-\sigma_\theta)\frac{\cos\theta}{r}-2\tau_{r\theta}\frac{\sin\theta}{r}+f_r\cos\theta-f_\theta\sin\theta\\
=&\left(\frac{\partial\sigma_r}{\partial r}+\frac{1}{r}\frac{\partial\tau_{r\theta}}{\partial\theta}+\frac{\sigma_r-\sigma_\theta}{r}+f_r\right)\cos\theta-\left(\frac{\partial\tau_{r\theta}}{\partial r}+\frac{1}{r}\frac{\partial\sigma_\theta}{\partial\theta}+2\frac{\tau_{r\theta}}{r}+f_\theta\right)\sin\theta=0 \qquad ❶
\end{aligned}
$$

$$
\begin{aligned}
&\frac{\partial\tau_{xy}}{\partial x}+\frac{\partial\sigma_y}{\partial y}+f_y\\
=&\frac{\partial\tau_{r\theta}}{\partial r}\cos^3\theta-\frac{\partial\tau_{r\theta}}{\partial r}\sin^2\theta\cos\theta+\frac{\partial\sigma_r}{\partial r}\sin\theta\cos^2\theta-\frac{\partial\sigma_\theta}{\partial r}\sin\theta\cos^2\theta\\
&-\frac{\partial\tau_{r\theta}}{\partial\theta}\frac{\cos^2\theta\sin\theta}{r}+\frac{\partial\tau_{r\theta}}{\partial\theta}\frac{\sin^3\theta}{r}-\frac{\partial\sigma_r}{\partial\theta}\frac{\cos\theta\sin^2\theta}{r}+\frac{\partial\sigma_\theta}{\partial\theta}\frac{\cos\theta\sin^2\theta}{r}\\
&+4\tau_{r\theta}\frac{\cos\theta\sin^2\theta}{r}-(\sigma_r-\sigma_\theta)\frac{\cos^2\theta\sin\theta-\sin^3\theta}{r}\\
&+\frac{\partial\sigma_r}{\partial r}\sin^3\theta+\frac{\partial\sigma_\theta}{\partial r}\cos^2\theta\sin\theta+2\frac{\partial\tau_{r\theta}}{\partial r}\sin^2\theta\cos\theta+\frac{\partial\sigma_r}{\partial\theta}\frac{\sin^2\theta\cos\theta}{r}+\frac{\partial\sigma_\theta}{\partial\theta}\frac{\cos^3\theta}{r}+2\frac{\partial\tau_{r\theta}}{\partial\theta}\frac{\sin\theta\cos^2\theta}{r}\\
&+2\sigma_r\frac{\sin\theta\cos^2\theta}{r}-2\sigma_\theta\frac{\sin\theta\cos^2\theta}{r}+2\tau_{r\theta}\frac{\cos^3\theta}{r}-2\tau_{r\theta}\frac{\sin^2\theta\cos\theta}{r}+f_r\sin\theta+f_\theta\cos\theta\\
=&\frac{\partial\sigma_r}{\partial r}\sin\theta+\frac{\partial\tau_{r\theta}}{\partial r}\cos\theta+\frac{\partial\sigma_\theta}{\partial\theta}\frac{\cos\theta}{r}+\frac{\partial\tau_{r\theta}}{\partial\theta}\frac{\sin\theta}{r}+(\sigma_r-\sigma_\theta)\frac{\sin\theta}{r}+2\tau_{r\theta}\frac{\cos\theta}{r}+f_r\sin\theta+f_\theta\cos\theta\\
=&\left(\frac{\partial\sigma_r}{\partial r}+\frac{1}{r}\frac{\partial\tau_{r\theta}}{\partial\theta}+\frac{\sigma_r-\sigma_\theta}{r}+f_r\right)\cos\theta+\left(\frac{\partial\tau_{r\theta}}{\partial r}+\frac{1}{r}\frac{\partial\sigma_\theta}{\partial\theta}+2\frac{\tau_{r\theta}}{r}+f_\theta\right)\sin\theta=0 \qquad ❷
\end{aligned}
$$

由❶❷得

$$
\begin{cases}
\left(\dfrac{\partial\sigma_r}{\partial r}+\dfrac{1}{r}\dfrac{\partial\tau_{r\theta}}{\partial\theta}+\dfrac{\sigma_r-\sigma_\theta}{r}+f_r\right)\cos\theta=0\\
\left(\dfrac{\partial\tau_{r\theta}}{\partial r}+\dfrac{1}{r}\dfrac{\partial\sigma_\theta}{\partial\theta}+2\dfrac{\tau_{r\theta}}{r}+f_\theta\right)\sin\theta=0
\end{cases}
$$

对任意 θ 都成立，所以

$$
\begin{cases}
\dfrac{\partial\sigma_r}{\partial r}+\dfrac{1}{r}\dfrac{\partial\tau_{r\theta}}{\partial\theta}+\dfrac{\sigma_r-\sigma_\theta}{r}+f_r=0\\
\dfrac{\partial\tau_{r\theta}}{\partial r}+\dfrac{1}{r}\dfrac{\partial\sigma_\theta}{\partial\theta}+2\dfrac{\tau_{r\theta}}{r}+f_\theta=0
\end{cases}
$$

此即极坐标系中的平衡方程. □

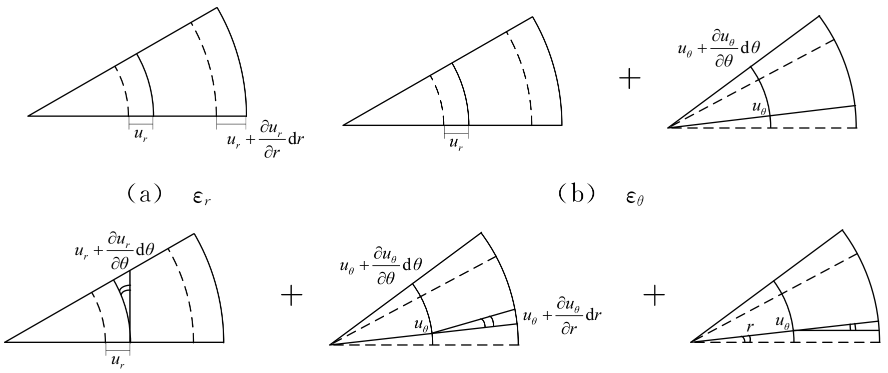
7-2 利用微元的方法，将几何关系

$$
\begin{cases}
\varepsilon_x = \dfrac{\partial u}{\partial x} \\[2ex]
\varepsilon_y = \dfrac{\partial v}{\partial y} \\[2ex]
\gamma_{xy} = \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x}
\end{cases}
$$

转换到极坐标中表示的形式

$$
\begin{cases}
\varepsilon_r = \dfrac{\partial u_r}{\partial r} \\[2ex]
\varepsilon_\theta = \dfrac{1}{r}\dfrac{\partial v_\theta}{\partial \theta} + \dfrac{u_r}{r} \\[2ex]
\gamma_{r\theta} = \dfrac{1}{r}\dfrac{\partial u_r}{\partial \theta} + \dfrac{\partial v_\theta}{\partial r} - \dfrac{v_\theta}{r}
\end{cases}
$$

解



（a） ε_r （b） ε_θ

（c） $\gamma_{r\theta}$

图 7.3 习题 7-2 图

由图 7.3 知

$$
\varepsilon_r = \frac{u_r + \dfrac{\partial u_r}{\partial r}\mathrm{d}r - u_r}{\mathrm{d}r} = \frac{\partial u_r}{\partial r}
$$

$$
\varepsilon_\theta = \frac{u_\theta + \dfrac{\partial u_\theta}{\partial \theta}\mathrm{d}\theta - u_\theta}{r\mathrm{d}\theta} + \frac{(r+u_r)\mathrm{d}\theta - r\mathrm{d}\theta}{r\mathrm{d}\theta} = \frac{\partial u_\theta}{r\partial \theta} + \frac{u_r}{r}
$$

$$
\gamma_{r\theta} = \frac{u_r + \dfrac{\partial u_\theta}{\partial \theta}\mathrm{d}\theta - u_r}{r\mathrm{d}\theta} + \frac{u_\theta + \dfrac{\partial u_\theta}{\partial r}\mathrm{d}r - u_\theta}{\mathrm{d}r} - \frac{u_\theta}{r} = \frac{\partial u_\theta}{r\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}
$$

□

7-7 图 7-43 所示单位厚度的正方形板,顶角受一对 P 力作用.试求边界 AB,BC,CD,DA 上的 ϕ,$\dfrac{\partial \phi}{\partial n}$,$\dfrac{\partial \phi}{\partial s}$ 值.

解 由正方形板的对称性,将 A 点处的载荷 P 分为作用于 A 点附近的两个载荷 $\dfrac{P}{2}$,如图 8.1 所示.对 B 处的载荷 P 作同样的处理.

设 $\phi|_A = 0, \left.\dfrac{\partial \phi}{\partial n}\right|_A = 0, \left.\dfrac{\partial \phi}{\partial s}\right|_A = 0$.

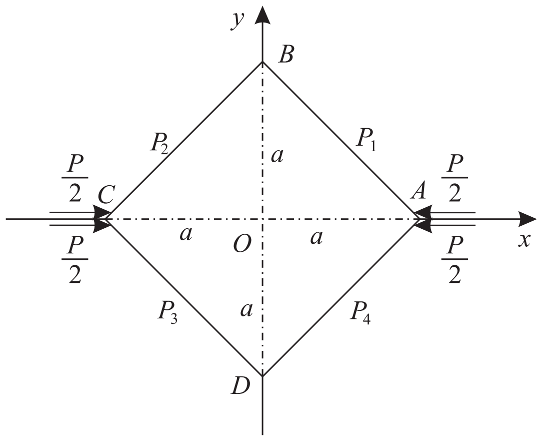


图 8.1　习题 7-7 图

在 $AB(x+y=a, 0<x<a)$ 上任取一点 $P_1(x,y)$,则

$$\left.\frac{\partial \phi}{\partial n}\right|_{P_1} = -R_s = -\frac{P}{2}\cos 45^\circ = -\frac{\sqrt{2}P}{4},\quad \left.\frac{\partial \phi}{\partial s}\right|_{P_1} = R_n = -\frac{P}{2}\sin 45^\circ = -\frac{\sqrt{2}P}{4}$$

$$\phi|_{P_1} = -\frac{P}{2}y = \frac{P}{2}(x-a)$$

在 $BC(-x+y=a, -a<x<0)$ 上任取一点 $P_2(x,y)$,则

$$\left.\frac{\partial \phi}{\partial n}\right|_{P_2} = -R_s = -\frac{P}{2}\cos 45^\circ = -\frac{\sqrt{2}P}{4},\quad \left.\frac{\partial \phi}{\partial s}\right|_{P_2} = R_n = \frac{P}{2}\sin 45^\circ = \frac{\sqrt{2}P}{4}$$

$$\phi|_{P_2} = -\frac{P}{2}y = -\frac{P}{2}(x+a)$$

在 $CD(-x-y=a, -a<x<0)$ 上任取一点 $P_3(x,y)$,则

$$\left.\frac{\partial \phi}{\partial n}\right|_{P_3} = -R_s = -\frac{P}{2}\cos 45^\circ = -\frac{\sqrt{2}P}{4},\quad \left.\frac{\partial \phi}{\partial s}\right|_{P_3} = R_n = -\frac{P}{2}\sin 45^\circ = -\frac{\sqrt{2}P}{4}$$

$$\phi|_{P_3} = -\frac{P}{2}y + Py = \frac{P}{2}y = -\frac{P}{2}(x+a)$$

在 $DA(x-y=a, 0<x<a)$ 上任取一点 $P_4(x,y)$,则

$$\left.\frac{\partial \phi}{\partial n}\right|_{P_4} = -R_s = -\frac{P}{2}\cos 45^\circ = -\frac{\sqrt{2}P}{4},\quad \left.\frac{\partial \phi}{\partial s}\right|_{P_4} = R_n = \frac{P}{2}\sin 45^\circ = \frac{\sqrt{2}P}{4}$$

$$\phi|_{P_4} = -\frac{P}{2}y + Py = \frac{P}{2}y = \frac{P}{2}(x-a)$$

□

7-9 矩形截面的柱的厚度为 1，$h \ll l$，在一边侧面上受均布剪力 q 作用(图 7-45). 试求柱中的应力分量.

解 由 $h \ll l$ 且左边界 $x=0$ 及右边界 $x=l$ 上 $\sigma_x=0$，可设整个应力场中 $\sigma_x(x,y)=0$，即 $\dfrac{\partial^2\phi(x,y)}{\partial y^2}=0$，所以

$$\phi(x,y)=yf(x)+g(x)$$

$$\sigma_y(x,y)=\frac{\partial^2\phi(x,y)}{\partial x^2}=yf''(x)+g''(x)$$

$$\tau_{xy}(x,y)=-\frac{\partial^2\phi(x,y)}{\partial x\partial y}=-f'(x)$$

控制方程为

$$\nabla^2\nabla^2\phi(x,y)=yf^{(4)}(x)+g^{(4)}(x)=0$$

由于上式对任何 y 值都应满足,所以 y 的各次幂的系数都应为 0,即

$$f^{(4)}(x)=0,\quad g^{(4)}(x)=0$$

所以

$$f(x)=Ax^3+Bx^2+Cx,\quad g(x)=Dx^3+Ex^2$$

主要边界条件

$$x=0\quad \tau_{xy}(0,y)=-f'(0)=-C=0$$
$$x=h\quad \tau_{xy}(h,y)=-f'(h)=-(3Ah^2+2Bh+C)=q$$

次要边界条件

$$y=0\quad\begin{cases}\sigma_y(x,0)=g''(x)=Dx^3+Ex^2=0\\ \displaystyle\int_0^h\tau_{xy}(x,0)\mathrm{d}x=\int_0^h-f'(x)\mathrm{d}x=f(0)-f(h)=-(Ah^3+Bh^2+Ch)=0\end{cases}$$

$$y=l\quad\begin{cases}\displaystyle\int_0^h\sigma_y(x,l)\mathrm{d}x=\int_0^h[lf''(x)+g''(x)]\mathrm{d}x=l[f'(h)-f'(0)]+g'(h)-g'(0)\\ \qquad\qquad\qquad=l(3Ah^2+2Bh+C)+3Dh^2+2Eh=-ql\\ \displaystyle\int_0^h\tau_{xy}(x,l)\mathrm{d}x=\int_0^h-f'(x)\mathrm{d}x=f(0)-f(h)=-(Ah^3+Bh^2+Ch)=0\end{cases}$$

解得

$$A=-\frac{q}{h^2},\quad B=\frac{q}{h},\quad C=0,\quad D=0,\quad E=0$$

所以

$$\sigma_x(x,y)=0,\quad \sigma_y(x,y)=-6\frac{q}{h^2}xy+2\frac{q}{h}y,\quad \tau_{xy}(x,y)=3\frac{q}{h^2}x^2-2\frac{q}{h}x$$

□

7-11 三角悬臂梁只受重力作用,梁的密度为 ρ(图 7-47). 试用纯三次的应力函数求梁中应力.

解 由题意设 $\phi(x,y)=Ax^3+Bx^2y+Cxy^2+Dy^3$，则

$$\sigma_x(x,y)=\frac{\partial^2\phi(x,y)}{\partial y^2}+V=2Cx+6Dy-\rho gy$$

$$\sigma_y(x,y)=\frac{\partial^2\phi(x,y)}{\partial x^2}+V=6Ax+2By-\rho gy$$

$$\tau_{xy}(x,y)=-\frac{\partial^2\phi(x,y)}{\partial x\partial y}=-(2Bx+2Cy)$$

主要边界条件

$$y=0\quad\begin{cases}\sigma_y(x,0)=6Ax=0\\ \tau_{xy}(x,0)=-2Bx=0\end{cases}$$

$$y=x\tan\alpha\quad\begin{aligned}&[-\sin\alpha\quad\cos\alpha]\begin{bmatrix}2Cx+6Dx\tan\alpha-\rho gx\tan\alpha & -(2Bx+2Cx\tan\alpha)\\ -(2Bx+2Cx\tan\alpha) & 6Ax+2Bx\tan\alpha-\rho gx\tan\alpha\end{bmatrix}\\ &=\left[4Cx\sin\alpha-6Dx\frac{\sin^2\alpha}{\cos\alpha}+\rho gx\frac{\sin^2\alpha}{\cos\alpha}\quad 2Cx\frac{\sin^2\alpha}{\cos\alpha}-\rho gx\sin\alpha\right]=[0\quad 0]\end{aligned}$$

解得

$$A=0,\quad B=0,\quad C=\frac{1}{2}\rho g\cot\alpha,\quad D=\frac{1}{6}\rho g(1-2\cot^2\alpha)$$

所以

$$\sigma_x(x,y)=2Cx+6Dy-\rho gy=\rho gx\cot\alpha+\rho gy(1-2\cot^2\alpha)-\rho gy=\rho g(x\cot\alpha-2y\cot^2\alpha)$$

$$\sigma_y(x,y)=6Ax+2By-\rho gy=-\rho gy$$

$$\tau_{xy}(x,y)=-(2Bx+2Cy)=-\rho gy\cot\alpha$$

□

7-18 设在厚壁管外套以绝对刚性的外套，使管不能发生轴向位移. 厚壁管受均匀内压力 q(图7-50)，试求厚壁管中的应力及位移.

解 由管不能发生轴向位移可知，此问题为平面应变问题. 边界条件为

$$r=b\quad u_r=\frac{1+\nu}{E}\left[\frac{(1-2\nu)(a^2q-b^2p_0)}{b^2-a^2}b+\frac{a^2b^2(q-p_0)}{b^2-a^2}\frac{1}{b}\right]=0$$

解得

$$p_0=\frac{2a^2q(1-\nu)}{a^2+b^2(1-2\nu)}$$

由 Lamé 公式得应力分量为

$$\sigma_r=\frac{a^2}{b^2-a^2}\left(1-\frac{b^2}{r^2}\right)q-\frac{b^2}{b^2-a^2}\left(1-\frac{a^2}{r^2}\right)\frac{2a^2q(1-\nu)}{a^2+b^2(1-2\nu)}=-\frac{a^2[b^2(1-2\nu)+r^2]}{r^2[b^2(1-2\nu)+a^2]}q$$

$$\sigma_\theta=\frac{a^2}{b^2-a^2}\left(1+\frac{b^2}{r^2}\right)q-\frac{b^2}{b^2-a^2}\left(1+\frac{a^2}{r^2}\right)\frac{2a^2q(1-\nu)}{a^2+b^2(1-2\nu)}=\frac{a^2[b^2(1-2\nu)-r^2]}{r^2[b^2(1-2\nu)+a^2]}q$$

$$\tau_{r\theta}=0$$

$$\sigma_z=\nu(\sigma_r+\sigma_\theta)=-\frac{2\nu a^2}{a^2+b^2(1-2\nu)}q$$

$$\tau_{rz}=0$$

$$\tau_{\theta z}=0$$

位移分量为

$$u_r=\frac{1+\nu}{E}\left[\frac{(1-2\nu)(a^2q-b^2p_0)}{b^2-a^2}r+\frac{a^2b^2(q-p_0)}{b^2-a^2}\frac{1}{r}\right]=\frac{a^2q(b^2-r^2)(1-2\nu)}{r[a^2+b^2(1-2\nu)]}$$

$$v_\theta=0$$

$$w_z=0$$

□

7-19 图 7-51 所示薄圆环，在 $r=a$ 处固定，在 $r=b$ 处受均匀分布的剪力 τ. 以位移法及应力函数法示圆环中的应力和位移.

解 ❶ **位移解法**

采用半逆法.由轴对称性可设

$$u_r = u_(r), \quad v_\theta = v_\theta(r)$$

代入几何方程得

$$\varepsilon_r = \frac{\mathrm{d}u_r}{\mathrm{d}r}, \quad \varepsilon_\theta = \frac{u_r}{r}, \quad \gamma_{r\theta} = \frac{\mathrm{d}v_\theta}{\mathrm{d}r} - \frac{v_\theta}{r}$$

由本构方程得

$$\sigma_r = \frac{E}{1-\nu^2}\left(\frac{\mathrm{d}u_r}{\mathrm{d}r} + \nu\frac{u_r}{r}\right), \quad \sigma_\theta = \frac{E}{1-\nu^2}\left(\frac{u_r}{r} + \nu\frac{\mathrm{d}u_r}{\mathrm{d}r}\right), \quad \tau_{r\theta} = G\left(\frac{\mathrm{d}v_\theta}{\mathrm{d}r} - \frac{v_\theta}{r}\right)$$

在轴对称问题中,平衡方程简化为

$$\frac{\mathrm{d}\sigma_r}{\mathrm{d}r} + \frac{\sigma_r - \sigma_\theta}{r} + f_r = 0$$

$$\frac{\mathrm{d}\tau_{r\theta}}{\mathrm{d}r} + 2\frac{\tau_{r\theta}}{r} + f_\theta = 0$$

把本构方程代入上式,并考虑无体力情况,则用位移表示的平衡方程为

$$\frac{\mathrm{d}^2 u_r}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}u_r}{\mathrm{d}r} - \frac{u_r}{r^2} = 0$$

$$\frac{\mathrm{d}^2 v_\theta}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}v_\theta}{\mathrm{d}r} - \frac{v_\theta}{r^2} = 0$$

其通解为

$$u_r = C_1 r + \frac{C_2}{r}, \quad v_\theta = C_3 r + \frac{C_4}{r}$$

边界条件为

$$r = a \quad u_r = C_1 a + \frac{C_2}{a} = 0, \quad v_\theta = C_3 a + \frac{C_4}{a} = 0$$

$$r = b \quad \begin{cases} \sigma_r = \dfrac{E}{1-\nu^2}\left(\dfrac{\mathrm{d}u_r}{\mathrm{d}r} + \nu\dfrac{u_r}{r}\right) = \dfrac{E}{1-\nu^2}\left[C_1(1+\nu) - \dfrac{C_2}{b^2}(1-\nu)\right] = 0 \\ \tau_{r\theta} = G\left(\dfrac{\mathrm{d}v_\theta}{\mathrm{d}r} - \dfrac{v_\theta}{r}\right) = -\dfrac{2GC_4}{b^2} = \tau \end{cases}$$

解得

$$C_1 = 0, \quad C_2 = 0, \quad C_3 = \frac{\tau b^2}{2Ga^2}, \quad C_4 = -\frac{\tau b^2}{2G}$$

所以圆环中的位移分量为

$$u_r = 0, \quad v_\theta = \frac{\tau b^2}{2Ga^2}r - \frac{\tau b^2}{2Gr} = \frac{\tau b^2}{2G}\left(\frac{r}{a^2} - \frac{1}{r}\right)$$

应力分量为

$$\sigma_r = 0, \quad \sigma_\theta = 0, \quad \tau_{r\theta} = \frac{\tau b^2}{r^2}$$

⇨

❷ **应力函数解法**

由该问题关于直径的反对称性知,对称应力分量 $\sigma_r = 0$,$\sigma_\theta = 0$,即

$$\sigma_r = \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} = 0 \qquad ①$$

$$\sigma_\theta = \frac{\partial^2 \phi}{\partial r^2} = 0 \tag{②}$$

由②可设应力分量

$$\phi = r f(\theta) + g(\theta)$$

代入①得

$$\frac{1}{r} f''(\theta) + \frac{1}{r^2} g''(\theta) + \frac{1}{r} f(\theta) = 0$$

由于上式对任意半径 r 都成立，所以

$$f''(\theta) = 0, \quad g''(\theta) = 0, \quad f(\theta) = 0$$

故

$$\phi = g(\theta) = A\theta + B$$

所以

$$\tau_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \phi}{\partial \theta}\right) = \frac{A}{r^2}$$

由 $r = b$ 处边界条件得

$$\tau_{r\theta} = \frac{A}{b^2} = \tau$$

得

$$A = \tau b^2$$

所以圆环中的应力分量为

$$\sigma_r = 0, \quad \sigma_\theta = 0, \quad \tau_{r\theta} = \frac{\tau b^2}{r^2}$$

由反对称性知，对称位移分量 $u_r = 0$. 应变分量为

$$\varepsilon_r = \frac{1}{E}(\sigma_r - \nu\sigma_\theta) = 0 = \frac{\partial u_r}{\partial r}$$

$$\varepsilon_\theta = \frac{1}{E}(\sigma_\theta - \nu\sigma_r) = 0 = \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{u_r}{r}$$

$$\gamma_{r\theta} = \frac{1}{G}\tau_{r\theta} = \frac{\tau b^2}{G r^2} = \frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r}$$

解得

$$u_r = 0, \quad v_\theta = Br - \frac{\tau b^2}{2Gr}$$

由 $r = a$ 处的边界条件

$$v_\theta = Ba - \frac{\tau b^2}{2Ga} = 0$$

得

$$B = \frac{\tau b^2}{2Ga^2}$$

所以位移分量为

$$u_r = 0$$

$$v_\theta = \frac{\tau b^2}{2Ga^2} r - \frac{\tau b^2}{2Gr} = \frac{\tau b^2}{2G}\left(\frac{r}{a^2} - \frac{1}{r}\right)$$

□

7-22 矩形薄板受纯剪力作用，剪力强度为 q. 设距板边缘较远处有一小圆孔，孔的半径为 a. 试求孔边的最大和最小正应力.

解 如图 8.2 所示矩形薄板受纯剪力 q 作用的情况，边界上的应力为

$$\sigma_x = 0,\quad \sigma_y = 0,\quad \tau_{xy} = q \tag{①}$$

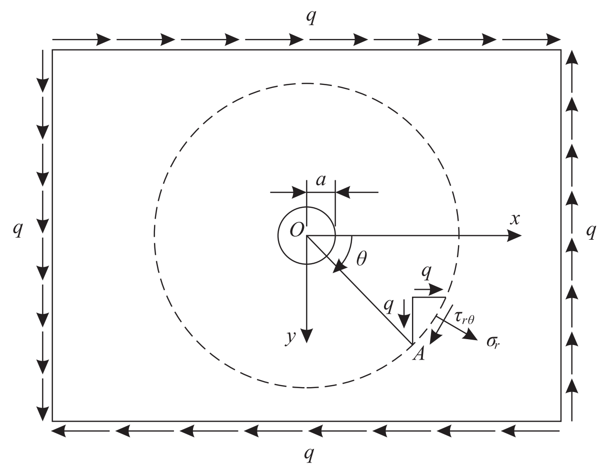


图 8.2 习题 7-22 图

选用极坐标，板的矩形边界用半径为 b 的同心圆来代替. 当 b 足够大时，局部应力已完全衰减，所以为求外圆边界 $r = b$ 上的应力，把①代入转轴公式得

$$\sigma_r\big|_{r=b} = q\sin 2\theta,\quad \sigma_\theta\big|_{r=b} = -q\sin 2\theta,\quad \tau_{r\theta}\big|_{r=b} = q\cos 2\theta \tag{②}$$

在内孔处的边界条件为

$$\sigma_r\big|_{r=a} = 0,\quad \tau_{r\theta}\big|_{r=a} = 0 \tag{③}$$

②表明，σ_r 的环向分布规律为 $\sin 2\theta$. 由

$$\sigma_r = \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + \frac{1}{r}\frac{\partial\phi}{\partial r}$$

知 σ_r 与 $\dfrac{\partial^2\phi}{\partial\theta^2}$ 与 $\dfrac{1}{r}\dfrac{\partial\phi}{\partial r}$ 有关. 所以应力函数也按 $\sin 2\theta$ 变化，应设为

$$\phi = f(r)\sin 2\theta \tag{④}$$

代入控制方程

$$\nabla^2\nabla^2\phi(r,\theta) = 0$$

得

$$\sin 2\theta\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r} - \frac{4}{r^2}\right)\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r} - \frac{4}{r^2}\right)f = 0$$

消去因子 $\sin 2\theta$ 得 Euler 方程，其特征方程为

$$[(k-2)(k-3) + (k-2) - 4][k(k-1) + k - 4] = 0$$

即

$$k(k-4)(k+2)(k-2) = 0$$

故其通解为

$$f(r) = Ar^4 + Br^2 + C + \frac{D}{r^2}$$

代入④并由

$$\sigma_r = \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + \frac{1}{r}\frac{\partial\phi}{\partial r},\quad \sigma_\theta = \frac{\partial^2\phi}{\partial r^2},\quad \tau_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right)$$

得应力分量为

$$\sigma_r=-\sin 2\theta\left(2B+\frac{4C}{r^2}+\frac{6D}{r^4}\right)$$

$$\sigma_\theta=\sin 2\theta\left(12Ar^2+2B+\frac{6D}{r^4}\right)$$

$$\tau_{r\theta}=-\cos 2\theta\left(6Ar^2+2B-\frac{2C}{r^2}-\frac{6D}{r^4}\right)$$

利用边界条件②③定出积分常数

$$A=\frac{q}{Nb^2}\beta^2(1-\beta^2),\quad B=-\frac{q}{2N}(1+3\beta^4-6\beta^6),\quad C=\frac{qa^2}{N}(1-\beta^6),\quad D=-\frac{qa^4}{2N}(1-\beta^4)$$

对无限大板小圆孔情况，$\beta\to 0, N\to 1$，各常数化为

$$A=0,\quad B=-\frac{q}{2},\quad C=qa^2,\quad D=-\frac{qa^4}{2}$$

所以无限大板中小圆孔附近的应力为

$$\sigma_r=q\left(1-\frac{a^2}{r^2}\right)\left(1-3\frac{a^2}{r^2}\right)\sin 2\theta$$

$$\sigma_\theta=-q\left(1+3\frac{a^4}{r^4}\right)\sin 2\theta$$

$$\tau_{r\theta}=q\left(1-\frac{a^2}{r^2}\right)\left(1+3\frac{a^2}{r^2}\right)\cos 2\theta$$

在孔边 $r=a$ 处，

$$\sigma_\theta=-4q\sin 2\theta$$

当 $\theta=\frac{\pi}{4}$ 时，σ_θ 取最小值 $-4q$，当 $\theta=\frac{3\pi}{4}$ 时，σ_θ 取最大值 $4q$. 所以孔边的最大和最小正应力分别为

$$\sigma_{\theta,\max}=4q,\quad \sigma_{\theta,\min}=-4q$$

□

7-25 图 7-56 所示楔形体的两侧面受均布剪力 q 作用，试求应力分量.

解 如图 8.3 所示，选 A 为起始参考点，沿逆时针方向计算应力函数 ϕ 及其导数的边界值.

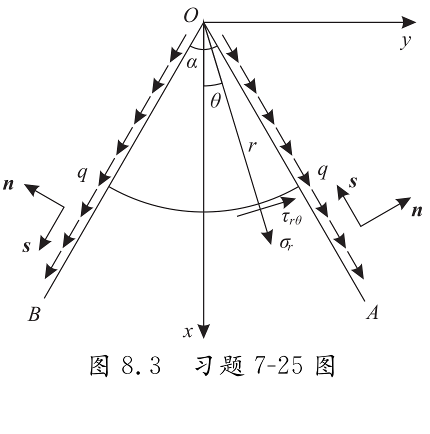


图 8.3 习题 7-25 图

在 AO 边 $(\theta=\alpha)$ 上

$$\phi=0,\quad \frac{\partial\phi}{\partial r}=-\frac{\partial\phi}{\partial s}=-R_n=0,\quad \frac{\partial\phi}{r\partial\theta}=\frac{\partial\phi}{\partial n}=-R_s=-q(a-r) \qquad ①$$

在 OB 边 $(\theta=-\alpha)$ 上

$$\phi=-qar\sin 2\alpha,\ \frac{\partial\phi}{\partial r}=-\frac{\partial\phi}{\partial s}=-R_n=qa\sin 2\alpha,\ \frac{\partial\phi}{r\partial\theta}=\frac{\partial\phi}{\partial n}=-R_s=-q(a\cos 2\alpha+r)$$

②

由①②可知，ϕ 与 r^2 成正比，所以设

$$\phi = r^2 f(\theta) \tag{③}$$

把③代入协调方程

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\phi = 0$$

得

$$\frac{1}{r^2}\left[4f''(\theta)+f^{(4)}(\theta)\right]=0$$

其通解为

$$f(\theta)=A+B\theta+C\cos 2\theta+D\sin 2\theta$$

代回③式有

$$\phi = r^2(A+B\theta+C\cos 2\theta+D\sin 2\theta)$$

代入应力公式

$$\sigma_r=\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}+\frac{1}{r}\frac{\partial\phi}{\partial r},\quad \sigma_\theta=\frac{\partial^2\phi}{\partial r^2},\quad \tau_{r\theta}=-\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right)$$

得

$$\begin{aligned}\sigma_r &= 2(A+B\theta-C\cos 2\theta-D\sin 2\theta)\\ \sigma_\theta &= 2(A+B\theta+C\cos 2\theta+D\sin 2\theta)\\ \tau_{r\theta} &= -B+2C\sin 2\theta-2D\cos 2\theta\end{aligned}$$

下面检验边界条件. 在两个侧面（$\theta=\pm\alpha$）上有

$$\sigma_\theta = 2(A+B\alpha+C\cos 2\alpha+D\sin 2\alpha)=0,\quad \tau_{r\theta}=-B+2C\sin 2\alpha-2D\cos 2\alpha=q$$

$$\sigma_\theta = 2(A-B\alpha+C\cos 2\alpha-D\sin 2\alpha)=0,\quad \tau_{r\theta}=-B-2C\sin 2\alpha-2D\cos 2\alpha=-q$$

解得

$$A=-\frac{q}{2}\cot 2\alpha,\quad B=0,\quad C=\frac{q}{2\sin 2\alpha},\quad D=0$$

所以应力分量为

$$\sigma_r=-\frac{\cos 2\alpha+\cos 2\theta}{\sin 2\alpha}q,\quad \sigma_\theta=\frac{\cos 2\theta-\cos 2\alpha}{\sin 2\alpha}q,\quad \tau_{r\theta}=q\frac{\cos 2\theta}{\sin 2\alpha}$$

□

7-27 图 7-58 所示半平面在边界上受两集中力作用，利用叠加原理求 A 点的应力.

解 由叠加原理知，A 点的应力分量为

$$\sigma_x=-\frac{2P}{\pi}\frac{b^3}{(b^2+a^2)^2}-\frac{2(-P)}{\pi}\frac{b^3}{[b^2+(-a)^2]^2}=0$$

$$\sigma_y=-\frac{2P}{\pi}\frac{ba^2}{(b^2+a^2)^2}-\frac{2(-P)}{\pi}\frac{b(-a)^2}{[b^2+(-a)^2]^2}=0$$

$$\tau_{xy}=-\frac{2P}{\pi}\frac{b^2a}{(b^2+a^2)^2}-\frac{2(-P)}{\pi}\frac{b^2(-a)}{[b^2+(-a)^2]^2}=-\frac{4P}{\pi}\frac{b^2a}{(b^2+a^2)^2}$$

习题 7-26 中，应力函数解法的边界条件为

$$\begin{cases}\theta=\pm\alpha\quad \sigma_\theta=\tau_{r\theta}=0 & ①\\ r=a\quad \displaystyle\int_{-\alpha}^{\alpha}(\sigma_r\cos\theta-\tau_{r\theta}\sin\theta)r\mathrm{d}\theta=0,\quad \int_{-\alpha}^{\alpha}(\sigma_r\sin\theta+\tau_{r\theta}\cos\theta)r\mathrm{d}\theta=0 & \\ \displaystyle\int_{-\alpha}^{\alpha}\tau_{r\theta}r^2\mathrm{d}\theta=-M & ②\end{cases}$$

设应力函数的形式为 $\phi = f(r)$，由控制方程

$$\nabla^2 \nabla^2 \phi = 0$$

得 Euler 方程

$$4f''(\theta) + f^{(4)}(\theta) = 0$$

其通解为

$$f(\theta) = A + B\theta + C\cos 2\theta + D\sin 2\theta$$

由

$$\sigma_r = \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} + \frac{1}{r}\frac{\partial \phi}{\partial r}, \quad \sigma_\theta = \frac{\partial^2 \phi}{\partial r^2}, \quad \tau_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \phi}{\partial \theta}\right)$$

得应力分量为

$$\sigma_r = -\frac{4}{r^2}(C\cos 2\theta + D\sin 2\theta), \quad \sigma_\theta = 0, \quad \tau_{r\theta} = \frac{1}{r^2}(B - 2C\sin 2\theta + 2D\cos 2\theta)$$

由①得

$$B - 2C\sin 2\alpha + 2D\cos 2\alpha = 0, \quad B + 2C\sin 2\alpha + 2D\cos 2\alpha$$

由②得

$$2B\alpha + 2D\sin 2\alpha = -M$$

解得

$$B = \frac{M\cos 2\alpha}{\sin 2\alpha - \alpha\cos 2\alpha}, \quad C = 0, \quad D = \frac{-M}{2(\sin 2\alpha - 2\alpha\cos 2\alpha)}$$

所以

$$\phi = \frac{M\theta\cos 2\alpha}{\sin 2\alpha - \alpha\cos 2\alpha} - \frac{M\sin 2\theta}{2(\sin 2\alpha - 2\alpha\cos 2\alpha)}$$

$$\sigma_r = \frac{2}{r^2}\frac{M\sin 2\theta}{\sin 2\alpha - 2\alpha\cos 2\alpha}$$

$$\sigma_\theta = 0$$

$$\tau_{r\theta} = \frac{M}{\sin 2\alpha - 2\alpha\cos 2\alpha}\frac{\cos 2\alpha - \cos 2\theta}{r^2}$$

取 $\alpha = \frac{\pi}{2}$，x 轴上相应点 $\theta = 0$，$r = b$ 的应力为

$$\sigma_r = 0, \quad \sigma_\theta = 0, \quad \tau_{r\theta} = -\frac{2M}{\pi b^2}$$

习题 7-27 中，当 $b \gg a$ 时

$$\tau_{xy} = -\frac{4P}{\pi}\frac{b^2 a}{(b^2 + a^2)^2} = -\frac{4Pa}{b^2\pi}\left[1 + \left(\frac{a}{b}\right)^2\right]^{-2} \approx -\frac{4Pa}{b^2\pi} = -\frac{2M}{\pi b^2}$$

其中 $M = 2Pa$. 所以当 $b \gg a$ 时，两静力等效的作用效果相同，即把作用在物体局部表面上的外力，用另一组与它静力等效的力系来代替，这种等效处理对物体内部应力状态的影响将随远离该局部作用区的距离增加而迅速衰减，此即 Saint-Venant 原理.

□

2010 年 12 月 22 日　第 15 周　星期三　**弹性力学**　第 10 章　第 120,121,122,123 页

10-1 任一弹性体,受一对共线集中力 P 作用,见图 10-19.不计体力,用功的互等定理求由此而产生的体积变化 ΔV.

解 设弹性体表面各点受均匀压强 q 作用,l 的收缩量为 Δl. 此时弹性体处于三向均匀压缩状态,任意微元的应力张量为

$$\boldsymbol{\sigma}=\begin{bmatrix}-q & & \\ & -q & \\ & & -q\end{bmatrix}$$

所以

$$\Delta l=-\frac{l}{E}[-q-\nu(-q-q)]=\frac{1-2\nu}{E}ql$$

由外力功互等定理

$$q\Delta V=P\Delta l$$

得

$$\Delta V=\frac{1-2\nu}{E}Pl$$

□

10-2 图 10-20 所示结构,各杆铰结. $AB=BC=CD=DA=a$,$\angle ABC=90^\circ$. 材料常数及载荷如图所示,结构发生小变形.忽略变形的高阶小量,用功的互等定理求 AC 杆的伸长.

解 在 A,C 两点分别作用水平向左、向右,大小均为 Q 的拉力,AB,BC,CD,DA 杆不受力,则 AC 杆的伸长量为

$$\Delta l_{AC}=\frac{Q}{F_3E_3}\sqrt{2}a$$

由几何关系知,B,D 点之间的距离缩短

$$\Delta l_{BD}=\Delta l_{AC}=\frac{Q}{F_3E_3}\sqrt{2}a$$

由功互等定理

$$Q\Delta l=P\Delta l_{BD}=P\frac{Q}{F_3E_3}\sqrt{2}a$$

所以 AC 杆的伸长为

$$\Delta l=\frac{P}{F_3E_3}\sqrt{2}a$$

□

10-3 用虚功原理计算在图 10-21 中的结点 B 处,引起位移 u_1,u_2 所需的作用力 P_1,P_2. 假定杆件截面积均为 F,弹性模量为 E.

解 由几何关系知,BC 杆的长度为 l,AB 杆的长度为 $\frac{5}{4}l$. 则

$$\varepsilon_{BC}=\frac{u_1}{l},\quad \sigma_{BC}=E\varepsilon_{BC}=E\frac{u_1}{l}$$

$$\varepsilon_{AB}=\frac{\frac{4}{5}u_1-\frac{3}{5}u_2}{\frac{5}{4}l}=\frac{16u_1-12u_2}{25l},\quad \sigma_{AB}=E\varepsilon_{AB}=\frac{E(16u_1-12u_2)}{25l}$$

所以外力虚功为

$$P_1\delta u_1+P_2\delta u_2$$

内力虚功为

$$\sigma_{BC}\delta\varepsilon_{BC}Fl+\sigma_{AB}\delta\varepsilon_{AB}F\ \frac{5}{4}l=E\frac{u_1}{l}\ \frac{\delta u_1}{l}Fl+\frac{E(16u_1-12u_2)}{25l}\ \frac{16\delta u_1-12\delta u_2}{25l}F\ \frac{5}{4}l$$

由虚位移原理得

$$P_1\delta u_1+P_2\delta u_2=E\frac{u_1}{l}\ \frac{\delta u_1}{l}Fl+\frac{E(16u_1-12u_2)}{25l}\ \frac{16\delta u_1-12\delta u_2}{25l}F\ \frac{5}{4}l$$

所以

$$P_1=E\frac{u_1}{l}F+\frac{16E(16u_1-12u_2)}{25^2l}F\ \frac{5}{4}=\frac{EF}{l}\left(\frac{189}{125}u_1-\frac{48}{125}u_2\right)$$

$$P_2=-\frac{12E(16u_1-12u_2)}{25^2l}F\ \frac{5}{4}=\frac{EF}{l}\left(-\frac{48}{125}u_1+\frac{36}{125}u_2\right)$$

□

10-4 图 10-22 中的两杆，长度均为 l，弹性模量及截面积分别为 E_1,F_1 及 E_2,F_2. 用虚功原理求在 P 力作用下两杆铰接处的位移.

解 设 P 作用点水平向右的位移分量为 u，竖直向下的位移分量为 v. 所以

$$\varepsilon_1=\frac{\sqrt{3}u+v}{2l},\quad \sigma_1=E_1\varepsilon_1=E_1\frac{\sqrt{3}u+v}{2l}$$

$$\varepsilon_2=\frac{v-\sqrt{3}u}{2l},\quad \sigma_2=E_2\varepsilon_2=E_2\frac{v-\sqrt{3}u}{2l}$$

所以外力虚功为

$$P\delta v$$

内力虚功为

$$\sigma_1\delta\varepsilon_1F_1l+\sigma_2\delta\varepsilon_2F_2l=E_1\varepsilon_1=E_1\frac{\sqrt{3}u+v}{2l}\ \frac{\sqrt{3}\delta u+\delta v}{2l}F_1l+E_2\ \frac{v-\sqrt{3}u}{2l}\ \frac{\delta v-\sqrt{3}\delta u}{2l}F_2l$$

由虚位移原理得

$$P\delta v=E_1\frac{\sqrt{3}u+v}{2l}\ \frac{\sqrt{3}\delta u+\delta v}{2l}F_1l+E_2\ \frac{v-\sqrt{3}u}{2l}\ \frac{\delta v-\sqrt{3}\delta u}{2l}F_2l$$

解得

$$u=\frac{Pl}{\sqrt{3}}\left(\frac{1}{F_1E_1}-\frac{1}{F_2E_2}\right)$$

$$v=Pl\left(\frac{1}{F_1E_1}+\frac{1}{F_2E_2}\right)$$

□

10-5 试用虚功原理求图 10-23 中所示梁的挠度曲线，并求当 $a=\frac{l}{2}$ 时中点的挠度值.

解 由题设边界条件，设梁的挠度方程为 $w=\sum_{n=1}^{+\infty}a_n\sin\frac{n\pi x}{l}$. 令第 m 个位移参数 a_m 产生虚增

量 $\delta a_m \neq 0$，其他位移参数的虚增量均为 0，则相应虚挠度为

$$\delta w = \delta a_m \sin\frac{m\pi x}{l}, \quad \delta w\,|_{x=a} = \delta a_m \sin\frac{m\pi a}{l}$$

求导得

$$\delta w' = \frac{m\pi}{l}\delta a_m \cos\frac{m\pi x}{l}, \quad \delta w'' = -\left(\frac{m\pi}{l}\right)^2 \delta a_m \sin\frac{m\pi x}{l}$$

所以梁内的变形可能弯矩为

$$M = EIw'' = -EI\sum_{n=1}^{+\infty}\left(\frac{n\pi}{l}\right)^2 a_n \sin\frac{n\pi x}{l}$$

由虚位移原理得

$$\int_0^l EI\left(\frac{m\pi}{l}\right)^2 \delta a_m \sin\frac{m\pi x}{l}\sum_{n=1}^{+\infty}\left(\frac{n\pi}{l}\right)^2 a_n \sin\frac{n\pi x}{l}\mathrm{d}x = \int_0^l M\delta w''\mathrm{d}x = P\delta w\,|_{x=a} = P\delta a_m \sin\frac{m\pi a}{l}$$

解得

$$a_m = \frac{2Pl^3}{EI\pi^4 m^4}\sin\frac{m\pi a}{l}$$

所以梁的挠度曲线方程为

$$w = \sum_{n=1}^{+\infty}\frac{2Pl^3}{EI\pi^4 n^4}\sin\frac{n\pi a}{l}\sin\frac{n\pi x}{l}$$

当 $a = \dfrac{l}{2}$ 时中点的挠度值为

$$w\left(\frac{l}{2}\right) = \sum_{n=1}^{+\infty}\frac{2Pl^3}{EI\pi^4 n^4}\sin\frac{n\pi}{2}\sin\frac{n\pi}{2} = \frac{2Pl^3}{EI\pi^4}\sum_{n=1}^{+\infty}\frac{1}{n^4}\sin^2\frac{n\pi}{2} = \frac{2Pl^3}{EI\pi^4}\frac{\pi^4}{96} = \frac{Pl^3}{48EI}$$

□

10-6 用虚功原理计算图 10-24 中半圆曲梁中点 B 处向上的铅直位移.

解 设 B 点有竖直向上的虚力 δF. 由平衡条件知，在角度为 θ 的截面上，弯矩为

$$M(\theta) = M - NR(1-\cos\theta)$$

$M(\theta)$ 引起的转角为

$$\mathrm{d}\varphi = \frac{M(\theta)}{EI}R\,\mathrm{d}\theta$$

虚力 δF 引起的虚弯矩为

$$\delta M = -R\delta F\cos\theta$$

所以可能变形在虚应力上做的余虚功为

$$\int_{\frac{\pi}{2}}^{\pi}\delta M\frac{\mathrm{d}\varphi}{\mathrm{d}\theta}\mathrm{d}\theta = \int_{\frac{\pi}{2}}^{\pi} -\delta FR\cos\theta\frac{M(\theta)R}{EI}\mathrm{d}\theta = \int_{\frac{\pi}{2}}^{\pi} -\delta FR\cos\theta[M - NR(1-\cos\theta)]\frac{R}{EI}\mathrm{d}\theta$$

$$= \frac{R^2}{EI}\left[M - N\left(1+\frac{\pi}{4}\right)R\right]\delta F$$

给定位移 Δ 在虚反力 δF 所做的余虚功为 $\Delta\delta F$. 由余虚功原理知

$$\Delta\delta F = \frac{R^2}{EI}\left[M - N\left(1+\frac{\pi}{4}\right)R\right]\delta F$$

所以

$$\Delta = \frac{R^2}{EI}\left[M - N\left(1+\frac{\pi}{4}\right)R\right]$$

□

10-8 矩形截面悬臂梁受均布载荷 q 作用. 梁截面宽为 b，高为 h，长为 l. 设材料为非线性弹性，其单向应力应变关系为 $\sigma_x = E\sqrt{|\varepsilon_x|}\operatorname{sgn}\varepsilon_x$. 其中 E 为材料常数. 用余虚功原理求梁自由端的垂直位移.

解 设自由端有虚力 δF，其引起的虚弯矩为 $\delta M = -(l-x)\delta F$. 所以

$$\delta\sigma_x = \frac{(l-x)y}{I}\delta F$$

均布载荷 q 引起的弯矩为

$$M = -\frac{1}{2}qx^2 + qlx - \frac{1}{2}ql^2$$

应力为

$$\sigma_x = -\frac{M}{I}y$$

应变为

$$\varepsilon_x = \left(\frac{\sigma_x}{E}\right)^2 \operatorname{sgn}\sigma_x = \left(\frac{\sigma_x}{E}\right)^2 \operatorname{sgn} y$$

由余虚功原理知

$$\Delta\delta F = \iint_S \varepsilon_x \delta\sigma_x b\,\mathrm{d}x\mathrm{d}y = \int_0^l \int_{-\frac{h}{2}}^{\frac{h}{2}} \left(-\frac{1}{2}qx^2 + qlx - \frac{1}{2}ql^2\right)^2 \operatorname{sgn} y \,\frac{(l-x)y^3}{E^3 I^3} b\delta F\mathrm{d}y\mathrm{d}x$$

所以

$$\Delta = \frac{bh^4 l^6 q^2}{768E^2 I^3}$$

把 $I = \dfrac{bh^3}{12}$ 代入上式得梁自由端的垂直位移为

$$\Delta = \frac{9l^6 q^2}{4E^2 b^2 h^5}$$

□

10-9 图 10-25 所示梁，一端固定一端弹性支承. 梁的抗弯刚度 EI 为常数，弹簧刚度为 k. 梁上作用有分布载荷 $q(x)$，梁的作用有集中力 P，梁端作用力偶 M. 试用最小势能原理导出平衡方程、边界条件及中点处的连接条件.

解 设梁的挠度方程为 $w = w(x)$，则由图 10-25 得边界条件为

$$w(0) = 0, w'(0) = 0;\quad w_-\left(\frac{l}{2}\right) = w_+\left(\frac{l}{2}\right), w'_-\left(\frac{l}{2}\right) = w'_+\left(\frac{l}{2}\right)$$

系统的势能为

$$\Pi = \int_0^l \frac{1}{2}EI[w''(x)]^2\mathrm{d}x - Pw\left(\frac{l}{2}\right) - \int_0^l q(x)w(x)\mathrm{d}x + \frac{1}{2}kw^2(l) + Mw'(l)$$

由最小势能原理知

$$\begin{aligned}\delta\Pi &= \int_0^l EIw''(x)\delta w''(x)\mathrm{d}x - P\delta w\left(\frac{l}{2}\right) - \int_0^l q(x)\delta w(x)\mathrm{d}x + kw(l)\delta w(l) + M\delta w'(l)\\
&= EIw''(x)\delta w'(x)\,|_0^l - \int_0^l EIw'''(x)\delta w'(x)\mathrm{d}x - P\delta w\left(\frac{l}{2}\right) - \int_0^l q(x)\delta w(x)\mathrm{d}x + kw(l)\delta w(l) + M\delta w'(l)\\
&= EIw''(l)\delta w'(l) - EIw'''(x)\delta w(x)\,|_0^l + \int_0^l EIw^{(4)}(x)\delta w(x)\mathrm{d}x - P\delta w\left(\frac{l}{2}\right) - \int_0^l q(x)\delta w(x)\mathrm{d}x + kw(l)\delta w(l) + M\delta w'(l)\end{aligned}$$

$$=[EIw''(l)+M]\delta w'(l)-\left[EIw'''(l)\delta w(l)-EIw'''_{+}\left(\frac{l}{2}\right)\delta w\left(\frac{l}{2}\right)+EIw'''_{-}\left(\frac{l}{2}\right)\delta w\left(\frac{l}{2}\right)\right]$$

$$+\int_0^l[EIw^{(4)}(x)-q(x)]\delta w(x)\mathrm{d}x-P\delta w\left(\frac{l}{2}\right)+kw(l)\delta w(l)$$

$$=[EIw''(l)+M]\delta w'(l)+[kw(l)-EIw'''(l)]\delta w(l)+\left[EIw'''_{+}\left(\frac{l}{2}\right)-EIw'''_{-}\left(\frac{l}{2}\right)-P\right]\delta w\left(\frac{l}{2}\right)+\int_0^l[EIw^{(4)}(x)-q(x)]\delta w(x)\mathrm{d}x$$

$$=0$$

所以可得平衡方程为

$$EIw^{(4)}(x)-q(x)=0$$

边界条件为

$$EIw''(l)+M=0,\quad kw(l)-EIw'''(l)=0$$

中点处的连续条件为

$$EIw'''_{+}\left(\frac{l}{2}\right)-EIw'''_{-}\left(\frac{l}{2}\right)-P=0$$

□

10-11 等直杆长为 l，端部受扭矩 M 作用. 设位移为 $u=-\alpha zy$，$v=\alpha zx$，$w=\alpha\psi(x,y)$，其中 ψ 为翘曲函数. 用最小势能原理导出以 ψ 表示的基本方程及力边界条件.

解 由应变分量

$$\gamma_{zx}=\alpha\left(\frac{\partial\psi}{\partial x}-y\right),\quad \gamma_{zy}=\alpha\left(\frac{\partial\psi}{\partial y}+x\right)$$

得系统总势能为

$$\Pi=\iiint_V\frac{1}{2}(\tau_{zx}\gamma_{zx}+\tau_{zy}\gamma_{zy})\mathrm{d}V-M\alpha l$$

$$=\iint_S\frac{Gl}{2}\left\{\left[\alpha\left(\frac{\partial\psi}{\partial x}-y\right)\right]^2+\left[\alpha\left(\frac{\partial\psi}{\partial y}+x\right)\right]^2\right\}\mathrm{d}x\mathrm{d}y-M\alpha l$$

由最小势能原理得

$$\delta\Pi=Gl\alpha^2\iint_S\left[\left(\frac{\partial\psi}{\partial x}-y\right)\delta\left(\frac{\partial\psi}{\partial x}\right)+\left(\frac{\partial\psi}{\partial y}+x\right)\delta\left(\frac{\partial\psi}{\partial y}\right)\right]\mathrm{d}x\mathrm{d}y$$

$$=Gl\alpha^2\iint_S\left[\left(\frac{\partial\psi}{\partial x}-y\right)\delta\left(\frac{\partial\psi}{\partial x}\right)+\left(\frac{\partial\psi}{\partial y}+x\right)\delta\left(\frac{\partial\psi}{\partial y}\right)\right]\mathrm{d}x\mathrm{d}y$$

$$=-Gl\alpha^2\iint_S\left(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}\right)\delta\psi\mathrm{d}x\mathrm{d}y+Gl\alpha^2\int_\Gamma\left[\left(\frac{\partial\psi}{\partial x}-y\right)\nu_1+\left(\frac{\partial\psi}{\partial y}+x\right)\nu_2\right]\delta\psi\mathrm{d}s$$

$$=0$$

所以控制方程为

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}=0$$

边界条件为

$$\left(\frac{\partial\psi}{\partial x}-y\right)\nu_1+\left(\frac{\partial\psi}{\partial y}+x\right)\nu_2=0$$

□

10-15 简支梁长为 l，抗弯刚度为 EI. 试用 Галёркин 法求解在均布载荷 q 作用下，梁弯曲的挠度曲线. 提示：选取试验函数 $w=\sum_{n=1}^{+\infty}a_n\sin\frac{n\pi x}{l}$.

解 选挠度为 w 为自变函数,梁弯曲问题的 Галёркин 求解方程为

$$\int_0^l (EIw^{(4)} - q)w_n \mathrm{d}x = 0 \quad (n = 1,2,\cdots,N)$$

把试验函数 $w = \sum\limits_{n=1}^{+\infty} a_n \sin \dfrac{n\pi x}{l}$ 代入上式得

$$\int_0^l \left[EI \sum_{i=1}^{+\infty} \left(\frac{i\pi}{l}\right)^4 a_i \sin \frac{i\pi x}{l} - q\right] a_n \sin \frac{n\pi x}{l} \mathrm{d}x = EI \frac{n^4\pi^4}{2l^3} a_n^2 - q a_n l \frac{1-(-1)^n}{n\pi} = 0$$

解得

$$a_n = \frac{2ql^4[1-(-1)^n]}{EIn^5\pi^5}$$

所以梁弯曲的挠度曲线为

$$w = \sum_{n=1}^{+\infty} \frac{2ql^4[1-(-1)^n]}{EIn^5\pi^5} \sin \frac{n\pi x}{l} = \sum_{k=1}^{+\infty} \frac{4ql^4}{EI(2k-1)^5\pi^5} \sin \frac{(2k-1)\pi x}{l}$$

□

10-17 图 10-30 所示简支梁长为 l,抗弯刚度为 EI,中点受 P 力作用,支座之间有弹性介质支承.其弹性系数为 k(即每单位长介质对单位挠度提供的反力).设 $w = \sum\limits_{n=1}^{+\infty} a_n \sin \dfrac{n\pi x}{l}$,试用 Ritz 法求梁中点的挠度.

解 系统的总势能为

$$\Pi = \int_0^l \frac{1}{2} EI[w''(x)]^2 \mathrm{d}x - Pw\left(\frac{l}{2}\right) + \int_0^l \frac{1}{2} kw^2(x) \mathrm{d}x$$

由最小势能原理

$$\begin{aligned}
\delta\Pi &= EI\int_0^l w''(x)\delta w''(x)\mathrm{d}x - P\delta w\left(\frac{l}{2}\right) + k\int_0^l w(x)\delta w(x)\mathrm{d}x \\
&= EI\int_0^l \left[\sum_{n=1}^{+\infty}\left(\frac{n\pi}{l}\right)^2 a_n \sin\frac{n\pi x}{l}\right]\left[\sum_{n=1}^{+\infty}\left(\frac{n\pi}{l}\right)^2 \sin\frac{n\pi x}{l}\right]\delta a_n \mathrm{d}x - P\sum_{n=1}^{+\infty}\sin\frac{n\pi}{2}\delta a_n \\
&\quad + k\int_0^l \left[\sum_{n=1}^{+\infty} a_n \sin\frac{n\pi x}{l}\right]\left[\sum_{n=1}^{+\infty}\sin\frac{n\pi x}{l}\right]\delta a_n \mathrm{d}x \\
&= EI\sum_{n=1}^{+\infty}\frac{n^4\pi^4}{l^4} a_n \int_0^l \sin^2\frac{n\pi x}{l}\mathrm{d}x\delta a_n - P\sum_{n=1}^{+\infty}\sin\frac{n\pi}{2}\delta a_n + k\sum_{n=1}^{+\infty} a_n \int_0^l \sin^2\frac{n\pi x}{l}\mathrm{d}x\delta a_n \\
&= EI\sum_{n=1}^{+\infty}\frac{n^4\pi^4}{2l^3} a_n\delta a_n - P\sum_{n=1}^{+\infty}\sin\frac{n\pi}{2}\delta a_n + k\sum_{n=1}^{+\infty} a_n \frac{l}{2}\delta a_n = 0
\end{aligned}$$

所以

$$a_n = \frac{2Pl^3}{\pi^4 EIn^4 + kl^4} \sin \frac{n\pi}{2}$$

故梁的挠度方程为

$$w = \sum_{n=1}^{+\infty} \frac{2Pl^3}{\pi^4 EIn^4 + kl^4} \sin \frac{n\pi}{2} \sin \frac{n\pi x}{l}$$

且梁中点的挠度为

$$w\left(\frac{l}{2}\right) = \sum_{n=1}^{+\infty} \frac{2Pl^3}{\pi^4 EIn^4 + kl^4} \sin^2 \frac{n\pi}{2} = \sum_{m=1}^{+\infty} \frac{2Pl^3}{\pi^4 EI(2m-1)^4 + kl^4}$$

□

10-19 图 10-32 中一端固支的悬臂梁，B 点有一拉杆作用以减低梁中的弯曲应力. 试用最小余能原理求拉杆的内力.

解 设拉杆内的内力为 T. 由静力平衡条件知梁内轴力

$$F_{\mathrm{N}}(x)=\begin{cases}0 & (0\leqslant x\leqslant l)\\ -T\sin 60^\circ=-\dfrac{\sqrt{3}}{2}T & (l\leqslant x\leqslant 2l)\end{cases}$$

梁内弯矩为

$$M(x)=\begin{cases}-\dfrac{1}{2}qx^2 & (0\leqslant x\leqslant l)\\ T(x-l)\cos 60^\circ-\dfrac{1}{2}qx^2=\dfrac{T}{2}(x-l)-\dfrac{1}{2}qx^2 & (l\leqslant x\leqslant 2l)\end{cases}$$

由于固支及铰支端没有位移，所以余势 $V_{\mathrm{C}}=0$. 故系统的总余能为

$$\varPi_{\mathrm{C}}=\int_0^{\frac{2\sqrt{3}}{3}l}\frac{T^2}{2E_CF_C}\mathrm{d}x+\int_0^{2l}\frac{F_{\mathrm{N}}^2}{2EF}\mathrm{d}x+\int_0^{2l}\frac{M^2}{2EI}\mathrm{d}x$$

由最小余能原理知

$$\begin{aligned}\delta\varPi_{\mathrm{C}}&=\int_0^{\frac{2\sqrt{3}}{3}l}\frac{T}{E_CF_C}\delta T\mathrm{d}x+\int_0^{2l}\frac{F_{\mathrm{N}}}{EF}\delta F_{\mathrm{N}}\mathrm{d}x+\int_0^{2l}\frac{M}{EI}\delta M\mathrm{d}x\\&=\int_0^{\frac{2\sqrt{3}}{3}l}\frac{T}{E_CF_C}\delta T\mathrm{d}x+\int_l^{2l}\frac{3T}{4EF}\delta T\mathrm{d}x+\int_l^{2l}\frac{T(x-l)-qx^2}{4EI}(x-l)\delta T\mathrm{d}x\\&=\frac{T}{E_CF_C}\frac{2\sqrt{3}}{3}l\delta T+\frac{3T}{4EF}l\delta T+\left(\frac{Tl^3}{12EI}-\frac{17ql^4}{48EI}\right)\delta T\\&=0\end{aligned}$$

所以拉杆内的内力为

$$T=\frac{\dfrac{17ql^3}{48EI}}{\dfrac{1}{E_CF_C}\dfrac{2\sqrt{3}}{3}+\dfrac{3}{4EF}+\dfrac{l^2}{12EI}}$$

□

12-3 设坝体内有半径为 a 的圆形孔道，孔道附近的变温可近似地表示为 $T=-T_a\dfrac{a}{r}$，其中 T_a 为孔边的温度，r 为距孔道中心的距离. 求坝体中的温度应力.

解 该问题为平面应变问题. 由边界条件得

$$\begin{cases}\sigma_r\mid_{r=a}=\dfrac{E_1}{1-\nu_1^2}\left[C_1(1+\nu_1)-C_2(1-\nu_1)\dfrac{1}{a^2}\right]=0\\ \sigma_\theta\mid_{r=a}=-\alpha_1E_1T+\dfrac{E_1}{1-\nu_1^2}\left[C_1(1+\nu_1)+C_2(1-\nu_1)\dfrac{1}{a^2}\right]=0\end{cases}$$

解得

$$C_1=\frac{\alpha(\nu-1)}{2}T_a,\quad C_2=-\frac{\alpha(\nu+1)}{2}a^2T_a$$

所以坝体中的温度应力为

$$\sigma_r=\alpha_1E_1\frac{T_aa(r-a)}{r^2}+\frac{E_1}{1-\nu_1^2}\left[\frac{\alpha(\nu-1)}{2}T_a(1+\nu_1)+\frac{\alpha(\nu+1)}{2}a^2T_a(1-\nu_1)\frac{1}{r^2}\right]$$

$$\sigma_\theta=\alpha_1E_1\frac{T_aa(a-r)}{r^2}-\alpha_1E_1T+\frac{E_1}{1-\nu_1^2}\left[\frac{\alpha(\nu-1)}{2}T_a(1+\nu_1)-\frac{\alpha(\nu+1)}{2}a^2T_a(1-\nu_1)\frac{1}{r^2}\right]$$

其中

$$E_1=\frac{E}{1-\nu^2},\quad \nu_1=\frac{\nu}{1-\nu},\quad \alpha_1=(1+\nu)\alpha$$

□

12-4 图 12-6 所示薄圆环，内半径为 a，外半径为 b. 内孔边自由，温度升高 T. 圆环外边界受约束，$u_r=0$，且温度保持不变. 内部无热源，温度沿板厚均匀，求圆环内的应力.

解 该问题为平面应力问题. 由题意知，圆环内的温度分布为

$$T(r)=T-T\frac{\ln\dfrac{r}{a}}{\ln\dfrac{b}{a}}=T\frac{\ln\dfrac{b}{r}}{\ln\dfrac{b}{a}}$$

由边界条件得

$$\begin{cases}u_r\mid_{r=b}=(1+\nu_1)\alpha_1\dfrac{1}{b}\displaystyle\int_a^bT(\rho)\rho\mathrm{d}\rho+C_1b+\dfrac{C_2}{b}=0\\ \sigma_r\mid_{r=a}=\dfrac{E_1}{1-\nu_1^2}\left[C_1(1+\nu_1)-C_2(1-\nu_1)\dfrac{1}{a^2}\right]=0\end{cases}$$

把 $T(r)$ 代入上式，解得

$$C_1=\frac{T\alpha(1-\nu)(1+\nu)\left(a^2-b^2+2a^2\ln\dfrac{b}{a}\right)}{4[a^2(1+\nu)+b^2(1-\nu)]\ln\dfrac{b}{a}}$$

$$C_2=\frac{a^2T\alpha(1+\nu)^2\left(a^2-b^2+2a^2\ln\dfrac{b}{a}\right)}{4[a^2(1+\nu)+b^2(1-\nu)]\ln\dfrac{b}{a}}$$

所以圆环内的应力为

$$\sigma_r = -\alpha_1 E_1 \frac{1}{r^2}\int_a^r T(\rho)\rho\mathrm{d}\rho + \frac{E_1}{1-\nu_1^2}\left[C_1(1+\nu_1) - C_2(1-\nu_1)\frac{1}{r^2}\right]$$

$$\sigma_\theta = \alpha_1 E_1 \frac{1}{r^2}\int_a^r T(\rho)\rho\mathrm{d}\rho - \alpha_1 E_1 T(\rho) + \frac{E_1}{1-\nu_1^2}\left[C_1(1+\nu_1) + C_2(1-\nu_1)\frac{1}{r^2}\right]$$

其中

$$\int_a^r T(\rho)\rho\mathrm{d}\rho = -\frac{a^2 T}{2} - \frac{T\left(a^2 - r^2 - 2r^2\ln\dfrac{b}{r}\right)}{4\ln\dfrac{b}{a}}$$

$$E_1 = E, \quad \nu_1 = \nu, \quad \alpha_1 = \alpha$$

□

12-5 周边自由的矩形板(图 12-7)中发生变温 $T = T_0\dfrac{y^3}{b^3}$，假定 a 远大于 b，求板中的温度应力.

解 由于板很薄，厚度方向的应力忽略，则只有纵向应力 σ_x 产生.

由第 170 页例 2 知，板中的热应力为

$$\sigma_x = E\alpha T_0\frac{y^3}{b^3} + \frac{E\alpha}{2b}\int_{-b}^{b} T_0\frac{y^3}{b^3}\mathrm{d}y + \frac{3yE\alpha}{2b^3}\int_{-b}^{b} T_0\frac{y^3}{b^3}y\mathrm{d}y = E\alpha T_0\frac{y^3}{b^3} + \frac{3T_0E\alpha}{5b}y$$

□