Summer Kaywañan Algebra Competitions

A.K.A.

Summer KACY

KACY--I003:

Olympiad Pre-Algebra Contest 003

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Synopsis

The Olympiad Algebra Book comes in two volumes. The first volume, dedicated to Polynomials and Trigonometry, is a collection of lesson plans containing 1220 beautiful problems, around two-thirds of which are polynomial problems and one-third are trigonometry problems. The second volume of The Olympiad Algebra Book contains 1220 Problems on Functional Equations and Inequalities, and I hope to finish it before the end of Summer 2023. I hope I can finish collecting the FE and INEQ problems by June 29^{th}, as a reminder of the 1220 Number Theory Problems published as the first 1220 set of J29 Project. The current volumes has 843 Polynomial problems and 377 Trigonometry questions, the last 63 of which are bizarre spherical geometry problems!

The Olympiad Algebra Book is supposed to be a problem bank for Algebra, and it forms the resource for the first series of the KAYWAÑAN Algebra Contest. I suggest you start with Polynomials, and before you get bored or exhausted, also start solving Trigonometry problems. If you find these problems easy and not challenging enough, the Spherical Trigonometry lessons and problems are definitely going to be a must try!

This booklet contains problems and solutions of `KACY--I003` (Olympiad Pre–Algebra Contests), including the problems from the first book:

$$\text{KACY–I}\,\{4, 41, 42, 60, 61, 78, 88, 105, 115\}\,.$$

The numbers referred here are the question number out of the 1220 questions labeled from 1 to 1220. The competition's full title is "Kaywañan Olympiad Pre–Algebra Summer Contest 003," and it was held on Saturday June 24, 2023.

Amir Parvardi,
Vancouver, British Columbia,
June 24, 2023

Contents

"Let No One Ignorant of Algebra Enter!"

KAYWAN

The rules of the KACY Competitions are simple:

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a) All problems whose titles contain **KACY–I** are questions of the Summer KACY Series, and all problems with a title containing **KACY–II** are questions of the Winter KACY Series.

b) This is the first volume of KACY, and it contains the SUMMER KACY questions. For the SUMMER KACY 2023 held weekly in Summer and Fall of 2023, only questions with title containing "**KACY–I**" are to be used in the actual KAYWAÑAN competitions.

This is because all the questions whose source does not contain **KACY–I** are either from a legit mathematical competition such as IMO, IMO Shortlist/Longlist, MAA Series (AMC, AIME, USAMO, USATST, USATSTST, USAMTS, etc.), National or Regional Olympiads (USA, APMC, Canada, etc.), or maybe from a book/paper I found and referenced in the question's title.

This assures that no famous problems are used in KACY, and that we actually identify and solve the non–KACY problems as exercises and examples in our journey of learning algebra during KAYWAÑAN Algebra Contest.

KACY–I003 Problems

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KACY–I 4. Solve for x:

$$2x^4 + 3x^3 - 10x^2 - 2x + 3 = 0.$$

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KACY–I 41. Factorize $(1 + x + x^2 + \cdots + x^n)^2 - x^n$.

KACY Summer League

KACY–I 42. Factorize $2x^4 + x^3 + 3x^2 + x + 2$.

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KACY–I 60. Factorize $(x+y)^3 - x^3 - y^3$.

KACY Summer League

KACY–I 61. Factorize $(x+y)^5 - x^5 - y^5$.

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KACY–I 78. Factorize $(x-y)^3 + (y-z)^3 + (z-x)^3$.

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KACY–I 88. Factorize $8x^3(y+z) - y^3(z+2x) - z^3(2x-y)$.

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KACY–I 105. If $f(x) + f(y) = f(z)$, find z in terms of x and y so that:

a) $f(x) = ax$;

b) $f(x) = 1/x$; and

If you are familiar with inverse of trigonometric and exponential functions:

c) $f(x) = \arctan x$, where $|x| < 1$;

d) $f(x) = \log((1+x)/(1-x))$.

Lagrange Interpolation

KACY Summer League

KACY–I 115. Consider the function

$$f(x) = \frac{a(x-b)(x-c)}{(a-b)(a-c)} + \frac{b(x-c)(x-a)}{(b-c)(b-a)} + \frac{c(x-a)(x-b)}{(c-a)(c-b)}.$$

Find the roots of $f(x) - x = 0$ and conclude that $f(x) = x$ for all x.

KACY–I003 Answers

The problems and solutions of KACY–I003 are courtesy of Parviz Shahriari, and they are taken from his eternal two-volume Farsi contribution to mathematics:: "Methods of Algebra." May he rest in peace!

Solution 4. Simplifies to $(2x-1)(x+3)(x^2-x-1)=0$ which has rational solutions $x=1/2,-3$ and irrational solutions $(1\pm\sqrt{5})/2$.

Solution 41. Answer: $(x^{n+1}+x^n+\cdots+x+1)(x^{n-1}+x^{n-2}+\cdots+x+1)$.

Solution 42. Answer: $(x^2+x+1)(2x^2-x+2)$.

Solution 60. Answer: $3xy(x+y)$.

Solution 61. Answer: $5xy(x+y)(x^2+xy+y^2)$.

Solution 78. Answer: $3(x-y)(y-z)(z-x)$.

Solution 88. Answer: $(y+z)(2x-y)(2x+z)(2x+y-z)$.

Solution 105. Answer: (a) $z=x+y$; (b) $z=\frac{xy}{x+y}$; (c) $z=\frac{x+y}{1-xy}$; (d) $z=\frac{x+y}{1+xy}$.

Solution 115. It is easy to see that $f(a)=a, f(b)=b$, and $f(c)=c$, so that the equation $f(x)=x$ has at least three roots. However, the equation $f(x)-x=0$ is quadratic and having three roots implies that it is always zero, so that $f(x)=x$ for all x.