Summer Kaywañan Algebra Competitions

A.K.A.

Summer KACY

KACY--I009:

Olympiad Pre-Algebra Contest 009

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Synopsis

The Olympiad Algebra Book comes in two volumes. The first volume, dedicated to Polynomials and Trigonometry, is a collection of lesson plans containing 1220 beautiful problems, around two-thirds of which are polynomial problems and one-third are trigonometry problems. The second volume of The Olympiad Algebra Book contains 1220 Problems on Functional Equations and Inequalities, and I hope to finish it before the end of Summer 2023. The current volumes has 843 Polynomial problems and 377 Trigonometry questions, the last 63 of which are bizarre spherical geometry problems! I also added 407 complementary review problems to the first volume on July 16^{th}, 2023.

The Olympiad Algebra Book is supposed to be a problem bank for Algebra, and it forms the resource for the first series of the KAYWAÑAN Algebra Contest. I suggest you start with Polynomials, and before you get bored or exhausted, also start solving Trigonometry problems. If you find these problems easy and not challenging enough, the Spherical Trigonometry lessons and problems are definitely going to be a must try!

This booklet contains problems and solutions of `KACY--I009` (Olympiad Pre–Algebra Contests), including the problems from the first book:

$$\text{KACY–I}\,\{33, 55, 57, 72, 73, 85, 101, 102, 112, 121\}\,.$$

The numbers referred here are the question number out of the 1220 questions labeled from 1 to 1220. The competition's full title is "Kaywañan Olympiad Pre–Algebra Summer Contest 009," held on Saturday August 5^{th}, 2023.

Amir Parvardi,
Vancouver, British Columbia,
August 5, 2023

Contents

> "Let No One Ignorant of Algebra Enter!"
>
> KAYWAN

The rules of the KACY Competitions are simple:

KACY Summer League

a) All problems whose titles contain **KACY–I** are questions of the Summer KACY Series, and all problems with a title containing **KACY–II** are questions of the Winter KACY Series.

b) This is the first volume of KACY, and it contains the SUMMER KACY questions. For the SUMMER KACY 2023 held weekly in Summer and Fall of 2023, only questions with title containing "**KACY–I**" are to be used in the actual KAYWAÑAN competitions.

This is because all the questions whose source does not contain **KACY–I** are either from a legit mathematical competition such as IMO, IMO Shortlist/Longlist, MAA Series (AMC, AIME, USAMO, USATST, USATSTST, USAMTS, etc.), National or Regional Olympiads (USA, APMC, Canada, etc.), or maybe from a book/paper I found and referenced in the question's title.

This assures that no famous problems are used in KACY, and that we actually identify and solve the non–KACY problems as exercises and examples in our journey of learning algebra during KAYWAÑAN Algebra Contest.

KACY–I009 Problems

KACY Summer League

KACY–I 33. How many numbers in the 100^{th} row of the Pascal triangle (the one starting with $1, 100, \ldots$) are not divisible by 3?

KACY Summer League

KACY–I 55. Factorize $x^{2^k} - y^{2^k}$ for all k.

KACY Summer League

Sophie Parker Identity 57. Factorize $x^4 + x^2y^2 + y^4$.

KACY Summer League

KACY–I 72. Factorize $a^4 + b^4 + c^4 - 2a^2b^2 - 2a^2c^2 - 2b^2c^2$.

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KACY–I 73. Factorize $x^2y^2z^2 + (x^2 + yz)(y^2 + zx)(z^2 + xy)$.

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KACY–I 85. Factorize $(x + y + z)^5 - x^5 - y^5 - z^5$.

KACY Summer League

KACY–I 101. If $f(x) = ax^2 + bx + c$, what is the value of the following expression?

$$g(x) = f(x + 3) - 3f(x + 2) + 3f(x + 1) - f(x).$$

KACY Summer League

KACY–I 102. Find a function in the form of $f(x) = a + bc^x$ such that

$$f(0) = 15, f(2) = 30, f(4) = 90.$$

KACY Summer League

KACY–I 112. The function $f(x)$ is defined for $x > 1$ as

$$f(x) = \log(x + \sqrt{x^2 - 1}).$$

Find $f(2x^2 - 1)$ and $f(4x^3 - 3x)$ in terms of $f(x)$.

KACY Summer League

KACY–I 121. What is the sum of coefficients of the following polynomial after expansion?

$$p(x) = (12x^3 - 54x^2 + 19x + 22)^{71}.$$

KACY–I009 Answers

The last problems and solutions of KACY–I009 are courtesy of Parviz Shahriari, and they are taken from his eternal two-volume Farsi contribution to mathematics: "Methods of Algebra." May he rest in peace!

Solution by Boris 33. The answer is 12. To find the numbers in the 100^{th} row of the Pascal triangle (the one starting with $1, 100, \dots$) that are not divisible by 3, we need to find the number of coefficients in the polynomial

$$(1+x)^{100} = 1 + \binom{100}{1}x + \binom{100}{2}x^2 + \cdots + x^{100},$$

which are not equal to 0 modulo 3. Note that by Binomial Theorem, and taking modulo 3, one has,

$$(1+x)^3 = 1 + 3x + 3x^2 + x^3 \equiv 1 + x^3 \pmod 3.$$

and so also

$$(1+x)^9 \equiv (1+x^3)^3 \equiv 1 + x^9 \pmod 3,$$

and so on, for any power of 3. Now, $100 = 81 + 2 \cdot 9 + 1$. Therefore, modulo 3 one has

$$(1+x)^{100} = (1+x)^{81}\left((1+x)^9\right)^2(1+x) = (1+x^{81})(1+2x^9+x^{18})(1+x).$$

In this product all $2 \cdot 3 \cdot 2 = 12$ powers of x are different (because every integer can be written in base 3 in a unique way), and the coefficients are all nonzero modulo 3. So, the answer is 12.

Solution by Amir Parvardi 55. Since $x = y$ yields $x^{2^k} - y^{2^k} = 0$, we know that $(x-y)$ is a factor of $x^{2^k} - y^{2^k}$. We also get the quotient as in the n^{th} Negative Double–Variable Identity:

$$\frac{x^{2^k} - y^{2^k}}{x-y} = x^{2^k-1} + x^{2^k-2}y + \cdots + xy^{2^k-2} + y^{2^k-1}.$$

We see that the degree of x in the quotient is $2^k - 1$, which happens to be equal to $1 + 2 + 2^2 + \cdots + 2^{k-1}$, meaning that the leading term in the quotient, x^{2^k-1}, is in fact a product of k terms $x \cdot x^2 \cdot x^{2^2} \cdots x^{2^{k-1}}$, and there must be an identity in this form:

$$x^{2^k-1} + x^{2^k-2}y + \cdots + xy^{2^k-2} + y^{2^k-1} = (x+\dots)(x^2+\dots)(x^{2^2}+\dots)\cdots(x^{2^{k-1}}+\dots),$$

and the same technique could be applied on y since everything is symmetric, and the missing terms are easily found:

$$\frac{x^{2^k} - y^{2^k}}{x-y} = (x+y)(x^2+y^2)(x^{2^2}+y^{2^2})\cdots(x^{2^{k-1}}+x^{2^{k-1}}),$$

giving us the magical $2^{k^{th}}$ Negative Double–Variable Identity:

$$x^{2^k} - y^{2^k} = (x-y)(x+y)(x^2+y^2)(x^{2^2}+y^{2^2})\cdots(x^{2^{k-1}}+y^{2^{k-1}}).$$

Solution by Sophie Parker 57.

- **Difference of Squares**: It is easy to see that adding x^2y^2 to the given expression completes the square, making it $(x^2+y^2)^2 = x^4+2x^2y^2+y^4$. The Difference of Squares Identity yields the final factorization:

$$\begin{aligned} x^4+x^2y^2+y^4 &= (x^2+y^2)^2-(xy)^2 \\ &= (x^2+y^2-xy)(x^2+y^2+xy). \end{aligned}$$

- **Difference of Squares and Cubes**: What if we begin with x^6-y^6? If I apply The Difference of Squares on this expression, I would have on one hand $x^6-y^6 = (x^3-y^3)(x^3+y^3)$, and on the other hand I can apply the n^{th} Negative Double–Variable Identity for $n=3$ on $x^6-y^6 = (x^2-y^2)(x^4+x^2y^2+y^4)$.

$$\begin{aligned} x^6-y^6 &= (x^3-y^3)(x^3+y^3) \\ &= (x-y)(x^2+xy+y^2)\cdot(x+y)(x^2-xy+y^2) \\ x^6-y^6 &= (x^2-y^2)(x^4+x^2y^2+y^4) \\ &= (x-y)(x+y)(x^4+x^2y^2+y^4). \end{aligned}$$

Therefore,

$$(x-y)(x^2+xy+y^2)\cdot(x+y)(x^2-xy+y^2) = (x-y)(x+y)(x^4+x^2y^2+y^4).$$

Assuming $x \neq \pm y$, we can cancel the terms $x-y$ and $x+y$ from both sides of the equation and obtain the factorization of $x^4+x^2y^2+y^4$ as a consequence:

$$(x^2+xy+y^2)\cdot(x^2-xy+y^2) = x^4+x^2y^2+y^4.$$

Solution 72. Answer: $(a+b+c)(a-b-c)(a+b-c)(a-b+c)$.

Solution 73. Answer: $(xy^2+yz^2+zx^2)(x^2y+y^2x+z^2x)$.

Solution 85. Answer: $5(x+y)(y+z)(z+x)(x^2+y^2+z^2+xy+yz+zx)$.

Solution 101. Answer: $g(x) \equiv 0$.

Solution 102. Answer: $f(x) = 10+5\cdot 2^x$.

Solution 112. Answer: $f(2x^2-1) = 2f(x)$ and $f(4x^3-3x) = 3f(x)$.

Solution 121. Answer: $p(1) = -1$.