Summer Kaywañan Algebra Competitions

A.K.A.

Summer KACY

KACY--I004:

Olympiad Pre-Algebra Contest 004

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Synopsis

The Olympiad Algebra Book comes in two volumes. The first volume, dedicated to Polynomials and Trigonometry, is a collection of lesson plans containing 1220 beautiful problems, around two-thirds of which are polynomial problems and one-third are trigonometry problems. The second volume of The Olympiad Algebra Book contains 1220 Problems on Functional Equations and Inequalities, and I hope to finish it before the end of Summer 2023. I hope I can finish collecting the FE and INEQ problems by June 29^{th}, as a reminder of the 1220 Number Theory Problems published as the first 1220 set of J29 Project. The current volumes has 843 Polynomial problems and 377 Trigonometry questions, the last 63 of which are bizarre spherical geometry problems!

The Olympiad Algebra Book is supposed to be a problem bank for Algebra, and it forms the resource for the first series of the KAYWAÑAN Algebra Contest. I suggest you start with Polynomials, and before you get bored or exhausted, also start solving Trigonometry problems. If you find these problems easy and not challenging enough, the Spherical Trigonometry lessons and problems are definitely going to be a must try!

This booklet contains problems and solutions of `KACY--I004` (Olympiad Pre–Algebra Contests), including the problems from the first book:

$$\text{KACY–I}\left\{5, 43, 44, 62, 79, 89, 90, 106, 116\right\}.$$

The numbers referred here are the question number out of the 1220 questions labeled from 1 to 1220. The competition's full title is "Kaywañan Olympiad Pre–Algebra Summer Contest 004," held on Saturday July 1^{st}, 2023.

Amir Parvardi,
Vancouver, British Columbia,
July 1, 2023

Contents

"Let No One Ignorant of Algebra Enter!"

KAYWAN

The rules of the KACY Competitions are simple:

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a) All problems whose titles contain **KACY–I** are questions of the Summer KACY Series, and all problems with a title containing **KACY–II** are questions of the Winter KACY Series.

b) This is the first volume of KACY, and it contains the SUMMER KACY questions. For the SUMMER KACY 2023 held weekly in Summer and Fall of 2023, only questions with title containing "**KACY–I**" are to be used in the actual KAYWAÑAN competitions.

This is because all the questions whose source does not contain **KACY–I** are either from a legit mathematical competition such as IMO, IMO Shortlist/Longlist, MAA Series (AMC, AIME, USAMO, USATST, USATSTST, USAMTS, etc.), National or Regional Olympiads (USA, APMC, Canada, etc.), or maybe from a book/paper I found and referenced in the question's title.

This assures that no famous problems are used in KACY, and that we actually identify and solve the non–KACY problems as exercises and examples in our journey of learning algebra during KAYWAÑAN Algebra Contest.

KACY–I004 Problems

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KACY–I 5. The quartic equation

$$x^4 - 5x^3 + 2x^2 + 20x - 24 = 0,$$

has four solutions $r_1 < r_2 \leq r_3 < r_4$. What is $r_1 + 2r_2 + 3r_3 + 4r_4$?

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KACY–I 43. The polynomial $(P(x)$ of degree six defined by

$$P(x) = x^6 + 2x^5 + 3x^4 + 24x^3 + 23x^2 + 22x + 21,$$

has been written as

$$P(x) = (x + a - 1)(x + a + 1)(x^2 + b)(x^2 - ax + 7b),$$

where a and b are integers. Find $a + b$.

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KACY–I 44. We know that the quartic polynomial

$$Q(x) = x^4 + 6x^2 + 18,$$

is expressed as the product of two monic quadratic polynomials. The absolute value of the coefficient of x in both quadratic factors of $Q(x)$ is $\sqrt{m(\sqrt{n} - p)}$, where $m > n > p$ are positive integers. What is mnp?

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KACY–I 62. In the factorization of $(x + y)^7 - x^7 - y^7$, one factor appears twice (it is the square of a polynomial in x and y). What is this factor?

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KACY–I 79. The two polynomials $f(x,y,z)$ and $g(x,y,z)$ are defined by

$$f(x,y,z) = x^2 + y^2 + z^2 - xy - yz - zx,$$
$$g(x,y,z) = (x-y)^4 + (y-z)^4 + (z-x)^4.$$

Find the quotient of the division of $g(x,y,z)$ by $(f(x,y,z))^2$.

KACY Summer League

KACY–I 89. Factorize $x^2y + xy^2 + x^2z + xz^2 + y^2z + yz^2 + 2xyz$.

KACY Summer League

KACY–I 90. Factorize $x^2y + xy^2 + x^2z + xz^2 + y^2z + yz^2 + 3xyz$.

KACY Summer League

KACY–I 106. Assuming

$$f(x) = \frac{1}{1-x},$$

Which one is $f(f(f(x)))$?

(A) $\frac{1}{x}$ **(B)** $\frac{1}{x-1}$ **(C)** x **(D)** $x-1$ **(E)** $\frac{x-1}{x}$

KACY Summer League

KACY–I 116. If n is an odd integer, $a^2 \neq 1$, and $f(x)$ is defined for all x by the equation

$$af(x^n) + f(-x^n) = bx,$$

the result of the division $f(x)/\sqrt[n]{x}$ would be the fraction m/n. Find $m+n$ in terms of a and b.

KACY–I004 Answers

The problems and solutions of KACY–I004 are courtesy of Parviz Shahriari, and they are taken from his eternal two-volume Farsi contribution to mathematics:: "Methods of Algebra." May he rest in peace!

Solution 5. the answer is 20. The equation simplifies to $(x-2)^2(x+2)(x-3)=0$ which has solutions $x=\pm 2, 3$. Therefore, $r_1=-2<r_2=r_3=2<r_4=3$, and

$$r_1+2r_2+3r_3+4r_4=-2+2\cdot 2+3\cdot 2+4\cdot 3=\boxed{20}.$$

Solution 43. The answer is 3. Begin with calculating the square root of the expression, equal to x^3+x^2+x+11 with a remainder of -100, then use difference of squares to arrive at $(x^3+x^2+x+1)(x^3+x^2+x+21)$. The final answer is $(x+1)(x+3)(x^2+1)(x^2-2x+7)$. Therefore, $a=2$ and $b=1$, so that $a+b=2+1=\boxed{3}$.

Solution 44. The answer is 12. We can complete the square to factorize $Q(x)$ as

$$Q(x)=\left(x^2+x\sqrt{6(\sqrt{2}-1)}+3\sqrt{2}\right)\left(x^2-x\sqrt{6(\sqrt{2}-1)}+3\sqrt{2}\right),$$

so that $(m,n,p)=(6,2,1)$ and finally $mnp=6\cdot 2\cdot 1=\boxed{12}$.

Solution 62. The answer is x^2+xy+y^2. The given expression factorizes into

$$(x+y)^7-x^7-y^7=7xy(x+y)(x^2+xy+y^2)^2.$$

The only factor that appears twice is $\boxed{x^2+xy+y^2}$.

Solution 79. The answer is 2. Expanding $g(x,y,z)=(x-y)^4+(y-z)^4+(z-x)^4$,

$$\begin{aligned}g(x,y,z)=(x-y)^4+(y-z)^4+(z-x)^4&=2(x^2+y^2+z^2-xy-yz-zx)^2\\&=2(f(x,y,z))^2,\end{aligned}$$

which yields $g(x,y,z)/(f(x,y,z))^2=\boxed{2}$.

Solution 89. Answer: $\boxed{(x+y)(y+z)(z+x)}$.

Solution 90. Answer: $\boxed{(x+y+z)(xy+yz+zx)}$.

Solution 106. The answer is $f(f(f(x)))=x$. It is easy to see that $f(f(x))=\frac{x-1}{x}$, and applying f one more time results in $f(f(f(x)))=\boxed{x\ \textbf{(C)}}$.

Solution 116. The answer is $a+b-1$. Change x to $-x$ in the given equation to easily arrive at

$$f(x)=\frac{b}{a-1}\sqrt[n]{x}.$$

Therefore,

$$\frac{f(x)}{\sqrt[n]{x}}=\frac{b}{a-1},$$

and the answer would be $\boxed{b+a-1}$.