Summer Kaywañan Algebra Competitions

A.K.A.

Summer KACY

`KACY--I007:`

Olympiad Pre-Algebra Contest 007

Amir Parvardi

July 22, 2023

Synopsis

The Olympiad Algebra Book comes in two volumes. The first volume, dedicated to Polynomials and Trigonometry, is a collection of lesson plans containing 1220 beautiful problems, around two-thirds of which are polynomial problems and one-third are trigonometry problems. The second volume of The Olympiad Algebra Book contains 1220 Problems on Functional Equations and Inequalities, and I hope to finish it before the end of Summer 2023. The current volumes has 843 Polynomial problems and 377 Trigonometry questions, the last 63 of which are bizarre spherical geometry problems! I also added 407 complementary review problems to the first volume on July 16^{th}, 2023.

The Olympiad Algebra Book is supposed to be a problem bank for Algebra, and it forms the resource for the first series of the KAYWAÑAN Algebra Contest. I suggest you start with Polynomials, and before you get bored or exhausted, also start solving Trigonometry problems. If you find these problems easy and not challenging enough, the Spherical Trigonometry lessons and problems are definitely going to be a must try!

This booklet contains problems and solutions of `KACY--I007` (Olympiad Pre–Algebra Contests), including the problems from the first book:

$$\text{KACY–I}\left\{14, 48, 65, 66, 82, 96, 97, 98, 110, 119\right\}.$$

The numbers referred here are the question number out of the 1220 questions labeled from 1 to 1220. The competition's full title is "Kaywañan Olympiad Pre–Algebra Summer Contest 007," held on Saturday July 22^{nd}, 2023.

Amir Parvardi,
Vancouver, British Columbia,
July 22, 2023

Contents

> "Let No One Ignorant of Algebra Enter!"
>
> KAYWAN

The rules of the KACY Competitions are simple:

KACY Summer League

a) All problems whose titles contain **KACY–I** are questions of the Summer KACY Series, and all problems with a title containing **KACY–II** are questions of the Winter KACY Series.

b) This is the first volume of KACY, and it contains the SUMMER KACY questions. For the SUMMER KACY 2023 held weekly in Summer and Fall of 2023, only questions with title containing "**KACY–I**" are to be used in the actual KAYWAÑAN competitions.

This is because all the questions whose source does not contain **KACY–I** are either from a legit mathematical competition such as IMO, IMO Shortlist/Longlist, MAA Series (AMC, AIME, USAMO, USATST, USATSTST, USAMTS, etc.), National or Regional Olympiads (USA, APMC, Canada, etc.), or maybe from a book/paper I found and referenced in the question's title.

This assures that no famous problems are used in KACY, and that we actually identify and solve the non–KACY problems as exercises and examples in our journey of learning algebra during KAYWAÑAN Algebra Contest.

KACY–I007 Problems

KACY Summer League

KACY–I 14. If a and b are real numbers such that

$$\sqrt[3]{a}-\sqrt[3]{b}=12 \quad \text{and} \quad ab=\left(\frac{a+b+8}{6}\right)^3,$$

find the value of $a-b$.

KACY Summer League

KACY–I 48. Factorize $x^3+9x^2+11x-21$.

KACY Summer League

KACY–I 65. Factorize $x^2+y^2+z^2+2xy+2yz+2zx$.

KACY Summer League

KACY–I 66. Factorize

$$(xy^3+yz^3+zx^3)-(x^3y+y^3z+z^3x).$$

KACY Summer League

KACY–I 82. Factorize $(x^2+y^2+z^2)^2-2(x^4+y^4+z^4)$.

KACY Summer League

KACY–I 96. If $f(x)=x^2-2x$, find $f(2x+1)$.

KACY Summer League

KACY–I 97. If

$$f(x) = x + \frac{1}{x},$$

find $f(f(x))$.

KACY Summer League

KACY–I 98. If

$$f(x) = \frac{x-1}{x+1},$$

find $f(f(f(x))) \cdot f(x)$.

KACY Summer League

KACY–I 110. If

$$f\left(\frac{2x-1}{x+2}\right) = \frac{3x^2 - 3x + 7}{(x+2)^2},$$

Find $f(x)$.

KACY Summer League

KACY–I 119. For a real x and positive integer n, the function $F_n(x)$ is recursively defined by $F_1(x) = \cos x$ and

$$F_{n+1}(x) + F_n(x+1) = F_n(x).$$

Find $F_n(x)$ for different values of n modulo 4.

KACY–I007 Answers

The problems and solutions of KACY–I007 (except for the first one which is solved by pco), are courtesy of Parviz Shahriari, and they are taken from his eternal two-volume Farsi contribution to mathematics: "Methods of Algebra." May he rest in peace!

Solution 14. For easier writing, let $x = \sqrt[3]{a}$ and $y = \sqrt[3]{b}$. Then, we have $x - y = 12$ and $x^3 + y^3 = 6xy - 8$. We are looking for

$$x^3 - y^3 = (x - y)^3 + 3xy(x - y) = 1728 + 36xy,$$

and so we are looking for xy.

From

$$x^3 + y^3 = 6xy - 8 \qquad \text{and} \qquad x^3 - y^3 = 1728 + 36xy,$$

we get $x^3 = 21xy + 860$ and $y^3 = -15xy - 868$.

And so,

$$x^3y^3 = -(21xy + 860)(15xy + 868),$$

which is a cubic with a unique real root -35 (not very difficult to find by tests and trials ans rational root theorem). Therefore, $xy = -35$ and

$$a - b = x^3 - y^3 = 1728 + 36xy = \boxed{468}.$$

Solution 48. The answer is $(x - 1)(x + 3)(x + 7)$.

Solution 65. The answer is $(x + y + z)^2$.

Solution 66. The answer is $(x + y + z)(x - y)(y - z)(z - x)$.

Solution 82. The answer is

$$(x + y + z)(-x + y + z)(x - y + z)(x + y - z).$$

Solution 96. The answer is $f(2x + 1) = 4x^2 - 1$.

Solution 97. The answer is

$$f(f(x)) = \frac{x^4 + 3x^2 + 1}{x(x^2 + 1)}.$$

Solution 98. The answer is

$$f(f(f(x))) \cdot f(x) = -1.$$

Solution 110. The answer is $f(x) = x^2 - x + 1$.

Solution 119. By induction, we arrive at:

$$F_n(x) = \begin{cases} -2^{n-1}\left(\sin\frac{1}{2}\right)^{n-1} \cdot \sin\left(x + \frac{n-1}{2}\right), & \text{if } n = 4k, \\ 2^{n-1}\left(\sin\frac{1}{2}\right)^{n-1} \cdot \cos\left(x + \frac{n-1}{2}\right), & \text{if } n = 4k+1, \\ 2^{n-1}\left(\sin\frac{1}{2}\right)^{n-1} \cdot \sin\left(x + \frac{n-1}{2}\right), & \text{if } n = 4k+2, \\ -2^{n-1}\left(\sin\frac{1}{2}\right)^{n-1} \cdot \cos\left(x + \frac{n-1}{2}\right), & \text{if } n = 4k+3. \end{cases}$$