Summer Kaywañan Algebra Competitions

A.K.A.

Summer KACY

KACY--I005:

Olympiad Pre-Algebra Contest 005

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Synopsis

The Olympiad Algebra Book comes in two volumes. The first volume, dedicated to Polynomials and Trigonometry, is a collection of lesson plans containing 1220 beautiful problems, around two-thirds of which are polynomial problems and one-third are trigonometry problems. The second volume of The Olympiad Algebra Book contains 1220 Problems on Functional Equations and Inequalities, and I hope to finish it before the end of Summer 2023. I hope I can finish collecting the FE and INEQ problems by June 29^{th}, as a reminder of the 1220 Number Theory Problems published as the first 1220 set of J29 Project. The current volumes has 843 Polynomial problems and 377 Trigonometry questions, the last 63 of which are bizarre spherical geometry problems!

The Olympiad Algebra Book is supposed to be a problem bank for Algebra, and it forms the resource for the first series of the KAYWAÑAN Algebra Contest. I suggest you start with Polynomials, and before you get bored or exhausted, also start solving Trigonometry problems. If you find these problems easy and not challenging enough, the Spherical Trigonometry lessons and problems are definitely going to be a must try!

This booklet contains problems and solutions of `KACY--I005` (Olympiad Pre–Algebra Contests), including the problems from the first book:

$$\text{KACY–I}\left\{6, 45, 46, 63, 80, 91, 92, 107, 117\right\}.$$

The numbers referred here are the question number out of the 1220 questions labeled from 1 to 1220. The competition's full title is "Kaywañan Olympiad Pre–Algebra Summer Contest 005," held on Saturday July 8^{th}, 2023.

Amir Parvardi,
Vancouver, British Columbia,
July 8, 2023

Contents

"Let No One Ignorant of Algebra Enter!"

KAYWAN

The rules of the KACY Competitions are simple:

KACY Summer League

a) All problems whose titles contain **KACY–I** are questions of the Summer KACY Series, and all problems with a title containing **KACY–II** are questions of the Winter KACY Series.

b) This is the first volume of KACY, and it contains the SUMMER KACY questions. For the SUMMER KACY 2023 held weekly in Summer and Fall of 2023, only questions with title containing "**KACY–I**" are to be used in the actual KAYWAÑAN competitions.

This is because all the questions whose source does not contain **KACY–I** are either from a legit mathematical competition such as IMO, IMO Shortlist/Longlist, MAA Series (AMC, AIME, USAMO, USATST, USATSTST, USAMTS, etc.), National or Regional Olympiads (USA, APMC, Canada, etc.), or maybe from a book/paper I found and referenced in the question's title.

This assures that no famous problems are used in KACY, and that we actually identify and solve the non–KACY problems as exercises and examples in our journey of learning algebra during KAYWAÑAN Algebra Contest.

KACY–I005 Problems

KACY Summer League

KACY–I 6. Solve for x:

$$x^4 - 3x^3 - 8x^2 + 12x + 16 = 0.$$

KACY Summer League

KACY–I 45. If n is a positive integer, factorize

$$a^{5n} + a^n + 1.$$

KACY Summer League

KACY–I 46. Factorize

$$(x+1)(x+3)(x+5)(x+7) + 15.$$

KACY Summer League

KACY–I 63. Show that $(x+y)^n - x^n - y^n$ always has a factor of

$$nxy(x+y)(x^2+xy+y^2)^2,$$

if $n = 6k+1$ for some integer $k \geq 1$.

KACY Summer League

KACY–I 80. Factorize

$$(x-y)^5 + (y-z)^5 + (z-x)^5.$$

KACY Summer League

KACY–I 91. Factorize

$$(x^2+y^2)^3+(z^2-x^2)^3-(y^2+z^2)^3.$$

KACY Summer League

KACY–I 92. Factorize

$$x^3(y-z)+y^3(z-x)+z^3(x-y).$$

KACY Summer League

KACY–I 107. Assuming

$$f(x+1)=x^2-3x+2,$$

Find $f(x)$.

KACY Summer League

KACY–I 117.

(a) Find two roots for the following equation:

$$f(x)=f\left(\frac{x+8}{x-1}\right).$$

(b) If $f(x)=x^2-12x+3$, find all the roots of the equation given in (a).

KACY–I005 Answers

The problems and solutions of KACY–I005 are courtesy of Parviz Shahriari, and they are taken from his eternal two-volume Farsi contribution to mathematics:: "Methods of Algebra." May he rest in peace!

Solution 6. Simplifies to $(x-4)(x-2)(x+1)(x+2)=0$ which has solutions

$$x=-1,\pm 2,4.$$

Solution 45. Answer: $(a^{2n}+a^n+1)(a^{3n}-a^{2n}+1)$.

Solution 46. Answer: $(x+2)(x+6)(x+4+\sqrt{6})(x+4-\sqrt{6})$.

Solution 63. It is easy using the Binomial Theorem to prove that for all positive integers n, $(x+y)^n-x^n-y^n$ has a factor of n. Define

$$f(x,y)=(x+y)^n-x^n-y^n.$$

Since $f(0,y)=f(x,0)=0$, we find that $f(x,y)$ is always divisible by xy. Furthermore, when n is odd, we have $f(x,-x)=0$, which means that $f(x,y)$ is divisible by $x+y$ when n is odd. Finally, assume that x^2+xy+y^2 is a quadratic in x and solve it using the third roots of unity ω and ω^2 (these are the roots of $\omega^3=1$):

$$x^2+xy+y^2=0 \implies x_{1,2}=y\cdot\frac{-1\pm i\sqrt{3}}{2},$$

which gives us $x_1=y\omega$ and $x_2=y\omega^2$. Now calculate $f(y\omega,y)$:

$$f(y\omega,y)=(y\omega+y)^n-(y\omega)^n-y^n=y^n(\omega+1)^n-y^n(\omega^n+1).$$

Since $\omega^2+\omega+1=0$, we can replace $\omega+1$ with $-\omega^2$ and rewrite the above equation as

$$f(y\omega,y)=-y^n(\omega^{2n}+\omega^n+1).$$

As a result, since $\omega^{2n}+\omega^n+1$ equals zero if and only if $n\equiv 1 \pmod 3$, we find that $f(x,y)$ is divisible by x^2+xy+y^2. So far, we have proved that $f(x,y)$ is divisible by $xy(x+y)(x^2+xy+y^2)$ whenever $n=6k+1$. To finish, we need to prove that $f(x,y)$ is divisible by $(x^2+xy+y^2)^2$, which can be done by proving that the derivative of $f(x,y)$ (with respect to either x or y) is divisible by (x^2+xy+y^2). If we let $\varphi(x)=(x+y)^n-x^n-y^n$, then

$$\varphi'(x)=n(x+y)^{n-1}-nx^{n-1}.$$

Now, calculate either $\varphi'(y\omega)$ or $\varphi'(y\omega^2)$:

$$\varphi'(y\omega)=ny^{n-1}\left((\omega+1)^{n-1}-\omega^{n-1}\right).$$

Since $n-1$ is even, $(\omega+1)^{n-1}=(-\omega^2)^{n-1}=\omega^{2(n-1)}$, and if we replace this in the previous equation (and let $n=6k+1$):

$$\varphi'(y\omega)=ny^{n-1}\left(\omega^{12k}-\omega^{6k}\right)=0.$$

Since both $\varphi)(x)$ and $\varphi'(x)$ are divisible by x^2+xy+y^2, it means that $\varphi(x)$ is divisible by $(x^2+xy+y^2)^2$ and we are done.

Solution 80. Answer: $5(x-y)(y-z)(z-x)(x^2+y^2+z^2-xy-yz-zx)$.

Solution 91. Answer: $3(y^2+z^2)(x^2+y^2)(x-z)(x+z)$.

Solution 92. Answer: $-(x+y+z)(x-y)(y-z)(z-x)$.

Solution 107. Answer: $f(x)=x^2-5x+7$.

Solution 117. Answer: (a) $x=-2,4$, (b) $x=-2,2,4,10$.