The Olympiad Algebra Book (Vol. I) Complementary: 407 Polynomials & Trigonometry Review Problems

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Abstract

Here are 407 missing problems that did not fit into the pool of 1220 problems in the first volume of The Olympiad Algebra Book dedicated to Polynomials and Trigonometry. The majority of the questions are chosen from American competitions such as AIME (American Invitational Mathematics Examination), HMMT (Harvard-MIT Math Tournament), CHMMC (Caltech Harvey Mudd Math Competition), and PUMaC (Princeton University Math Competition). All of the AIME problems are copyright © Mathematical Association of America, and they can be found on the Contests page on the Art of Problem Solving website. In this document, the links to the problems posted on AoPS forums are embedded (if existent).

The document is divided into three sections. The first section attempts to review polynomials through the study of their roots in the complex numbers, and the second section is a collection of trigonometry problems in recent American competitions, along with miscellaneous geometric and combinatorial problems that are algebraic by heart. The third section contains partial but elegant solutions by AoPS users (with credits given to the authors).

Contents

1 Systems of Equations and Complex Numbers

1983 AIME, Problem 5 1. Suppose that the sum of the squares of two complex numbers x and y is 7 and the sum of the cubes is 10. What is the largest real value that $x+y$ can have?

1984 AIME, Problem 8 2. The equation z^6+z^3+1 has one complex root with argument θ between 90° and 180° in the complex plane. Determine the degree measure of θ.

1984 AIME, Problem 15 3. Determine $w^2+x^2+y^2+z^2$ if

$$\begin{aligned}
\frac{x^2}{2^2-1}+\frac{y^2}{2^2-3^2}+\frac{z^2}{2^2-5^2}+\frac{w^2}{2^2-7^2}&=1,\\
\frac{x^2}{4^2-1}+\frac{y^2}{4^2-3^2}+\frac{z^2}{4^2-5^2}+\frac{w^2}{4^2-7^2}&=1,\\
\frac{x^2}{6^2-1}+\frac{y^2}{6^2-3^2}+\frac{z^2}{6^2-5^2}+\frac{w^2}{6^2-7^2}&=1,\\
\frac{x^2}{8^2-1}+\frac{y^2}{8^2-3^2}+\frac{z^2}{8^2-5^2}+\frac{w^2}{8^2-7^2}&=1.
\end{aligned}$$

1986 AIME, Problem 11 4. The polynomial $1-x+x^2-x^3+\cdots+x^{16}-x^{17}$ may be written in the form $a_0+a_1y+a_2y^2+\cdots+a_{16}y^{16}+a_{17}y^{17}$, where $y=x+1$ and thet a_i's are constants. Find the value of a_2.

1988 AIME, Problem 11 5. Let $w_1,w_2,\dots,w_n$ be complex numbers. A line L in the complex plane is called a mean line for the points $w_1,w_2,\dots,w_n$ if L contains points (complex numbers) $z_1,z_2,\dots,z_n$ such that

$$\sum_{k=1}^{n}(z_k-w_k)=0.$$

For the numbers $w_1=32+170i$, $w_2=-7+64i$, $w_3=-9+200i$, $w_4=1+27i$, and $w_5=-14+43i$, there is a unique mean line with y-intercept 3. Find the slope of this mean line.

1988 AIME, Problem 13 6. Find a if a and b are integers such that x^2-x-1 is a factor of $ax^{17}+bx^{16}+1$.

1989 AIME, Problem 8 7. Assume that $x_1,x_2,\dots,x_7$ are real numbers such that

$$\begin{aligned}
x_1+4x_2+9x_3+16x_4+25x_5+36x_6+49x_7&=1,\\
4x_1+9x_2+16x_3+25x_4+36x_5+49x_6+64x_7&=12,\\
9x_1+16x_2+25x_3+36x_4+49x_5+64x_6+81x_7&=123.
\end{aligned}$$

Find the value of

$$16x_1+25x_2+36x_3+49x_4+64x_5+81x_6+100x_7.$$

1989 AIME, Problem 14 8. Given a positive integer n, it can be shown that every complex number of the form $r+si$, where r and s are integers, can be uniquely expressed in the base $-n+i$ using the integers $1,2,\dots,n^2$ as digits. That is, the equation

$$r+si=a_m(-n+i)^m+a_{m-1}(-n+i)^{m-1}+\cdots+a_1(-n+i)+a_0,$$

is true for a unique choice of non-negative integer m and digits $a_0,a_1,\dots,a_m$ chosen from the set $\{0,1,2,\dots,n^2\}$, with $a_m\neq 0$. We write

$$r+si=(a_ma_{m-1}\dots a_1a_0)_{-n+i},$$

to denote the base $-n+i$ expansion of $r+si$. There are only finitely many integers $k+0i$ that have four-digit expansions

$$k=(a_3a_2a_1a_0)_{-3+i}, \qquad a_3 \neq 0.$$

Find the sum of all such k.

1990 AIME, Problem 10 9. The sets $A=\{z : z^{18}=1\}$ and $B=\{w : w^{48}=1\}$ are both sets of complex roots of unity. The set $C=\{zw : z\in A \text{ and } w\in B\}$ is also a set of complex roots of unity. How many distinct elements are in C?

1990 AIME, Problem 15 10. Find ax^5+by^5 if the real numbers a, b, x, and y satisfy the equations

$$\begin{aligned} ax+by &= 3,\\ ax^2+by^2 &= 7,\\ ax^3+by^3 &= 16,\\ ax^4+by^4 &= 42. \end{aligned}$$

1991 AIME, Problem 7 11. Find A^2, where A is the sum of the absolute values of all roots of the following equation:

$$x = \sqrt{19}+\cfrac{91}{\sqrt{19}+\cfrac{91}{\sqrt{19}+\cfrac{91}{\sqrt{19}+\cfrac{91}{\sqrt{19}+\cfrac{91}{x}}}}}.$$

1992 AIME, Problem 8 12. For any sequence of real numbers $A=(a_1,a_2,a_3,\ldots)$, define ΔA to be the sequence $(a_2-a_1,a_3-a_2,a_4-a_3,\ldots)$, whose n^{th} term is $a_{n+1}-a_n$. Suppose that all of the terms of the sequence $\Delta(\Delta A)$ are 1, and that $a_{19}=a_{92}=0$. Find a_1.

1992 AIME, Problem 10 13. Consider the region A in the complex plane that consists of all points z such that both $z/40$ and $40/\overline{z}$ have real and imaginary parts between 0 and 1, inclusive. What is the integer that is nearest the area of A?

1993 AIME, Problem 5 14. Let $P_0(x)=x^3+313x^2-77x-8$. For integers $n\geq 1$, define $P_n(x)=P_{n-1}(x-n)$. What is the coefficient of x in $P_{20}(x)$?

1994 AIME, Problem 13 15. The equation

$$x^{10}+(13x-1)^{10}=0$$

has 10 complex roots $r_1,\overline{r_1},r_2,\overline{r_2},r_3,\overline{r_3},r_4,\overline{r_4},r_5,\overline{r_5}$, where the bar denotes complex conjugation. Find the value of

$$\frac{1}{r_1\overline{r_1}}+\frac{1}{r_2\overline{r_2}}+\frac{1}{r_3\overline{r_3}}+\frac{1}{r_4\overline{r_4}}+\frac{1}{r_5\overline{r_5}}.$$

1995 AIME, Problem 5 16. For certain real values of a, b, c, and d, the equation $x^4 + ax^3 + bx^2 + cx + d = 0$ has four non-real roots. The product of two of these roots is $13 + i$ and the sum of the other two roots is $3 + 4i$, where $i = \sqrt{-1}$. Find b.

1995 AIME, Problem 13 17. Let $f(n)$ be the integer closest to $\sqrt[4]{n}$. Find

$$\sum_{k=1}^{1995} \frac{1}{f(k)}.$$

1994 Turkey, Second Round, Problem 1 18. For a positive integer n, let a_n denote the closest integer to $\sqrt{n}$. Evaluate

$$\sum_{n=1}^{\infty} \frac{1}{a_n^3}.$$

1996 AIME, Problem 5 19. Suppose that the roots of $x^3 + 3x^2 + 4x - 11 = 0$ are a, b, and c, and that the roots of $x^3 + rx^2 + sx + t = 0$ are $a + b, b + c$, and $c + a$. Find t.

1996 AIME, Problem 11 20. Let P be the product of the roots of $z^6 + z^4 + z^3 + z^2 + 1 = 0$ that have positive imaginary part, and suppose that $P = r(\cos\theta^\circ + i\sin\theta^\circ)$, where $0 < r$ and $0 \le \theta < 360$. Find θ.

1997 AIME, Problem 14 21. Let v and w be distinct, randomly chosen roots of the equation $z^{1997} - 1 = 0$. Let m/n be the probability that $\sqrt{2 + \sqrt{3}} \le |v + w|$, where m and n are relatively prime positive integers. Find $m + n$.

1998 AIME, Problem 13 22. If $\{a_1, a_2, a_3, \ldots, a_n\}$ is a set of real numbers, indexed so that $a_1 < a_2 < a_3 < \cdots < a_n$, its complex power sum is defined to be

$$a_1 i + a_2 i^2 + a_3 i^3 + \cdots + a_n i^n,$$

where $i^2 = -1$. Let S_n be the sum of the complex power sums of all nonempty subsets of $\{1, 2, \ldots, n\}$. Given that $S_8 = -176 - 64i$ and $S_9 = p + qi$, were p and q are integers, find $|p| + |q|$.

1999 AIME, Problem 9 23. A function f is defined on the complex numbers by $f(z) = (a+bi)z$, where a and b are positive numbers. This function has the property that the image of each point in the complex plane is equidistant from that point and the origin. Given that $|a + bi| = 8$ and that $b^2 = m/n$, where m and n are relatively prime positive integers. Find $m + n$.

2000 AIME I, Problem 9 24. The system of equations

$$\begin{aligned} \log_{10}(2000xy) - (\log_{10} x)(\log_{10} y) &= 4, \\ \log_{10}(2yz) - (\log_{10} y)(\log_{10} z) &= 1, \\ \log_{10}(zx) - (\log_{10} z)(\log_{10} x) &= 0. \end{aligned}$$

has two solutions (x_1, y_1, z_1) and (x_2, y_2, z_2). Find $y_1 + y_2$.

2000 AIME II, Problem 1 25. The number

$$\frac{2}{\log_4 2000^6} + \frac{3}{\log_5 2000^6}$$

can be written as $\frac{m}{n}$ where m and n are relatively prime positive integers. Find $m+n$.

2000 AIME II, Problem 9 26. Given that z is a complex number such that

$$z + \frac{1}{z} = 2\cos 3^\circ,$$

find the least integer that is greater than

$$z^{2000} + \frac{1}{z^{2000}}.$$

2000 AIME II, Problem 13 27. The equation

$$2000x^6 + 100x^5 + 10x^3 + x - 2 = 0$$

has exactly two real roots, one of which is $(m+\sqrt{n})/r$, where m, n and r are integers, m and r are relatively prime, and $r > 0$. Find $m+n+r$.

2001 AIME I, Problem 3 28. Find the sum of the roots, real and non-real, of the equation

$$x^{2001} + \left(\frac{1}{2} - x\right)^{2001} = 0,$$

given that there are no multiple roots.

2001 AIME II, Problem 8 29. A certain function f has the properties that $f(3x) = 3f(x)$ for all positive real values of x, and that $f(x) = 1 - \mid x - 2 \mid$ for $1 \le x \le 3$. Find the smallest x for which $f(x) = f(2001)$.

2001 AIME II, Problem 14 30. There are $2n$ complex numbers that satisfy both

$$z^{28} - z^8 - 1 = 0 \qquad \text{and} \qquad |z| = 1.$$

These numbers have the form $z_m = \cos\theta_m + i\sin\theta_m$, where $0 \le \theta_1 < \theta_2 < \cdots < \theta_{2n} < 360$ and angles are measured in degrees. Find the value of $\theta_2 + \theta_4 + \cdots + \theta_{2n}$.

2002 AIME I, Problem 6 31. The solutions to the system of equations

$$\begin{aligned} \log_{225} x + \log_{64} y &= 4, \\ \log_x 225 - \log_y 64 &= 1, \end{aligned}$$

are (x_1, y_1) and (x_2, y_2). Find $\log_{30}(x_1 y_1 x_2 y_2)$.

2002 AIME I, Problem 7 32. The Binomial Expansion is valid for exponents that are not integers. That is, for all real numbers x, y, and r with $|x| > |y|$,

$$(x+y)^r = x^r + rx^{r-1}y + \frac{r(r-1)}{2}x^{r-2}y^2 + \frac{r(r-1)(r-2)}{3!}x^{r-3}y^3 + \cdots$$

What are the first three digits to the right of the decimal point in the decimal representation of $\left(10^{2002}+1\right)^{10/7}$?

2002 AIME I, Problem 12 33. Let

$$F(z) = \frac{z+i}{z-i},$$

for all complex numbers $z \neq i$, and let $z_n = F(z_{n-1})$ for all positive integers n. Given that

$$z_0 = \frac{1}{137} + i \qquad \text{and} \qquad z_{2002} = a + bi,$$

where a and b are real numbers, find $a + b$.

2002 AIME II, Problem 3 34. It is given that $\log_6 a + \log_6 b + \log_6 c = 6$, where a, b, and c are positive integers that form an increasing geometric sequence and $b - a$ is the square of an integer. Find $a + b + c$.

2002 AIME II, Problem 6 35. Find the integer that is closest to

$$1000 \sum_{n=3}^{10000} \frac{1}{n^2 - 4}.$$

2003 AIME I, Problem 4 36. Given that

$$\log_{10} \sin x + \log_{10} \cos x = -1,$$

and that

$$\log_{10}(\sin x + \cos x) = \frac{1}{2}(\log_{10} n - 1),$$

find n.

2003 AIME II, Problem 9 37. Consider the polynomials

$$P(x) = x^6 - x^5 - x^3 - x^2 - x,$$

and $Q(x) = x^4 - x^3 - x^2 - 1$. Given that z_1, z_2, z_3, and z_4 are the roots of $Q(x) = 0$, find

$$P(z_1) + P(z_2) + P(z_3) + P(z_4).$$

2003 AIME II, Problem 15 38. Let

$$P(x) = 24x^{24} + \sum_{j=1}^{23}(24-j)(x^{24-j} + x^{24+j}).$$

Let $z_1, z_2, \ldots, z_r$ be the distinct zeros of $P(x)$, and let $z_k^2 = a_k + b_k i$ for $k = 1, 2, \ldots, r$, where $i = \sqrt{-1}$, and a_k and b_k are real numbers. Let

$$\sum_{k=1}^{r} |b_k| = m + n\sqrt{p},$$

where m, n, and p are integers and p is not divisible by the square of any prime. Find $m + n + p$.

2004 AIME I, Problem 7 39. Let C be the coefficient of x^2 in the expansion of the product

$$(1-x)(1+2x)(1-3x)\cdots(1+14x)(1-15x).$$

Find $|C|$.

2004 AIME I, Problem 13 40. The polynomial

$$P(x) = (1 + x + x^2 + \cdots + x^{17})^2 - x^{17}$$

has 34 complex roots of the form $z_k = r_k[\cos(2\pi a_k) + i\sin(2\pi a_k)], k = 1, 2, 3, \ldots, 34$, with $0 < a_1 \le a_2 \le a_3 \le \cdots \le a_{34} < 1$ and $r_k > 0$. Given that $a_1 + a_2 + a_3 + a_4 + a_5 = m/n$, where m and n are relatively prime positive integers, find $m + n$.

2005 AIME I, Problem 6 41. Let P be the product of the non-real roots of $x^4 - 4x^3 + 6x^2 - 4x = 2005$. Find $\lfloor P \rfloor$.

2005 AIME I, Problem 8 42. The equation

$$2^{333x-2} + 2^{111x+2} = 2^{222x+1} + 1$$

has three real roots. Given that their sum is m/n where m and n are relatively prime positive integers, find $m + n$.

2005 AIME II, Problem 7 43. Let

$$x = \frac{4}{(\sqrt{5}+1)(\sqrt[4]{5}+1)(\sqrt[8]{5}+1)(\sqrt[16]{5}+1)}.$$

Find $(x+1)^{48}$.

2005 AIME II, Problem 9 44. For how many positive integers n less than or equal to 1000 is

$$(\sin t + i\cos t)^n = \sin nt + i\cos nt$$

true for all real t?

2005 AIME II, Problem 13 45. Let $P(x)$ be a polynomial with integer coefficients that satisfies $P(17) = 10$ and $P(24) = 17$. Given that $P(n) = n + 3$ has two distinct integer solutions n_1 and n_2, find the product $n_1 \cdot n_2$.

2006 AIME II, Problem 15 46. Given that x, y, and z are real numbers that satisfy:

$$\begin{aligned} x &= \sqrt{y^2 - \frac{1}{16}} + \sqrt{z^2 - \frac{1}{16}}, \\ y &= \sqrt{z^2 - \frac{1}{25}} + \sqrt{x^2 - \frac{1}{25}}, \\ z &= \sqrt{x^2 - \frac{1}{36}} + \sqrt{y^2 - \frac{1}{36}}, \end{aligned}$$

and that $x + y + z = m/\sqrt{n}$, where m and n are positive integers and n is not divisible by the square of any prime, find $m + n$.

2007 AIME I, Problem 3 47. The complex number z is equal to $9 + bi$, where b is a positive real number and $i^2 = -1$. Given that the imaginary parts of z^2 and z^3 are equal, find b.

2007 AIME I, Problem 8 48. The polynomial $P(x)$ is cubic. What is the largest value of k for which the polynomials

$$Q_1(x) = x^2 + (k - 29)x - k \qquad \text{and} \qquad Q_2(x) = 2x^2 + (2k - 43)x + k,$$

are both factors of $P(x)$?

2007 AIME II, Problem 7 49. Given a real number x, let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x. For a certain integer k, there are exactly 70 positive integers $n_1, n_2, \ldots, n_{70}$ such that

$$k = \lfloor \sqrt[3]{n_1} \rfloor = \lfloor \sqrt[3]{n_2} \rfloor = \cdots = \lfloor \sqrt[3]{n_{70}} \rfloor,$$

and k divides n_i for all i such that $1 \leq i \leq 70$. Find the maximum value of n_i/k for $1 \leq i \leq 70$.

2007 AIME II, Problem 14 50. Let $f(x)$ be a polynomial with real coefficients such that $f(0) = 1$, $f(2) + f(3) = 125$, and for all x, $f(x)f(2x^2) = f(2x^3 + x)$. Find $f(5)$.

2008 AIME I, Problem 13 51. Let

$$p(x, y) = a_0 + a_1x + a_2y + a_3x^2 + a_4xy + a_5y^2 + a_6x^3 + a_7x^2y + a_8xy^2 + a_9y^3.$$

Suppose that

$$\begin{aligned} p(0, 0) = p(1, 0) = p(-1, 0) = p(0, 1) = p(0, -1) \\ = p(1, 1) = p(1, -1) = p(2, 2) = 0. \end{aligned}$$

There is a point $(a/c, b/c)$ for which $p\,(a/c, b/c) = 0$ for all such polynomials, where a, b, and c are positive integers, a and c are relatively prime, and $c > 1$. Find $a + b + c$.

2008 AIME II, Problem 7 52. Let r, s, and t be the three roots of the equation

$$8x^3 + 1001x + 2008 = 0.$$

Find $(r + s)^3 + (s + t)^3 + (t + r)^3$.

2009 AIME I, Problem 2 53. There is a complex number z with imaginary part 164 and a positive integer n such that

$$\frac{z}{z+n} = 4i.$$

Find n.

2009 AIME II, Problem 2 54. Suppose that a, b, and c are positive real numbers such that $a^{\log_3 7} = 27$, $b^{\log_7 11} = 49$, and $c^{\log_{11} 25} = \sqrt{11}$. Find

$$a^{(\log_3 7)^2} + b^{(\log_7 11)^2} + c^{(\log_{11} 25)^2}.$$

2020 USAMTS, Year 32, Round 1, Problem 5 55. Find all pairs of rational numbers (a, b) such that $0 < a < b$ and

$$a^a = b^b.$$

2010 AIME I, Problem 3 56. Suppose that

$$y = \frac{3}{4}x \qquad \text{and} \qquad x^y = y^x.$$

The quantity $x + y$ can be expressed as a rational number r/s, where r and s are relatively prime positive integers. Find $r + s$.

2010 AIME I, Problem 6 57. Let $P(x)$ be a quadratic polynomial with real coefficients satisfying

$$x^2 - 2x + 2 \le P(x) \le 2x^2 - 4x + 3$$

for all real numbers x, and suppose $P(11) = 181$. Find $P(16)$.

2010 AIME I, Problem 8 58. For a real number a, let $\lfloor a \rfloor$ denominate the greatest integer less than or equal to a. Let $\mathcal{R}$ denote the region in the coordinate plane consisting of points (x, y) such that

$$\lfloor x \rfloor^2 + \lfloor y \rfloor^2 = 25.$$

The region $\mathcal{R}$ is completely contained in a disk of radius r (a disk is the union of a circle and its interior). The minimum value of r can be written as $\sqrt{m}/n$, where m and n are integers and m is not divisible by the square of any prime. Find $m + n$.

2010 AIME I, Problem 9 59. Let (a, b, c) be the real solution of the system of equations $x^3 - xyz = 2$, $y^3 - xyz = 6$, $z^3 - xyz = 20$. The greatest possible value of $a^3 + b^3 + c^3$ can be written in the form m/n, where m and n are relatively prime positive integers. Find $m + n$.

2010 AIME II, Problem 5 60. Positive numbers x, y, and z satisfy $xyz = 10^{81}$ and $(\log_{10} x)(\log_{10} yz) + (\log_{10} y)(\log_{10} z) = 468$. Find

$$\sqrt{(\log_{10} x)^2 + (\log_{10} y)^2 + (\log_{10} z)^2}.$$

2010 AIME II, Problem 6 61. Find the smallest positive integer n with the property that the polynomial $x^4 - nx + 63$ can be written as a product of two nonconstant polynomials with integer coefficients.

2010 AIME II, Problem 7 62. Let $P(z) = z^3 + az^2 + bz + c$, where a, b, and c are real. There exists a complex number w such that the three roots of $P(z)$ are $w + 3i$, $w + 9i$, and $2w - 4$, where $i^2 = -1$. Find $|a + b + c|$.

2010 AIME II, Problem 10 63. Find the number of second-degree polynomials $f(x)$ with integer coefficients and integer zeros for which $f(0) = 2010$.

2011 AIME I, Problem 6 64. Suppose that a parabola has vertex $(1/4, -9/8)$, and equation $y = ax^2 + bx + c$, where $a > 0$ and $a + b + c$ is an integer. The minimum possible value of a can be written as p/q, where p and q are relatively prime positive integers. Find $p + q$.

2011 AIME I, Problem 9 65. Suppose x is in the interval $[0, \pi/2]$ and

$$\log_{24 \sin x}(24 \cos x) = \frac{3}{2}.$$

Find $24 \cot^2 x$.

2011 AIME I, Problem 15 66. For some integer m, the polynomial $x^3 - 2011x + m$ has the three integer roots a, b, and c. Find $|a| + |b| + |c|$.

2011 AIME II, Problem 8 67. Let $z_1, z_2, z_3, \ldots, z_{12}$ be the 12 zeroes of the polynomial $z^{12} - 2^{36}$. For each j, let w_j be one of z_j or iz_j. Then the maximum possible value of the real part of $w_1 + w_2 + \cdots + w_{12}$ can be written as $m + \sqrt{n}$ where m and n are positive integers. Find $m + n$.

2011 AIME II, Problem 11 68. Let M_n be the $n \times n$ matrix with entries as follows: for $1 \le i \le n$, $m_{i,i} = 10$; for $1 \le i \le n - 1, m_{i+1,i} = m_{i,i+1} = 3$; all other entries in M_n are zero. Let D_n be the determinant of matrix M_n. Then,

$$\sum_{n=1}^{\infty} \frac{1}{8D_n + 1}$$

can be represented as p/q, where p and q are relatively prime positive integers. Find $p+q$. *Note*: The determinant of the 1×1 matrix $[a]$ is a, and the determinant of the 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$; for $n \ge 2$, the determinant of an $n \times n$ matrix with first row or first column $a_1\ a_2\ a_3 \ldots\ a_n$ is equal to

$$a_1C_1 - a_2C_2 + a_3C_3 - \cdots + (-1)^{n+1} a_n C_n,$$

where C_i is the determinant of the $(n - 1) \times (n - 1)$ matrix found by eliminating the row and column containing a_i.

2011 HMMT, Guts, Problem 17 69. Given positive real numbers x, y, and z that satisfy the following system of equations:

$$x^2 + y^2 + xy = 1,$$
$$y^2 + z^2 + yz = 4,$$
$$z^2 + x^2 + zx = 5,$$

find $x + y + z$.

2011 HMMT, Guts, Problem 27 70. Find the number of polynomials $p(x)$ with integer coefficients satisfying

$$p(x) \geq \min\{2x^4 - 6x^2 + 1, 4 - 5x^2\},$$

and

$$p(x) \leq \max\{2x^4 - 6x^2 + 1, 4 - 5x^2\},$$

for all real numbers x.

2011 HMMT, Algebra & Geometry, Problem 27 71. Let $f(x) = x^2 + 6x + c$ for all real numbers x, where c is some real number. For what values of c does $f(f(x))$ have exactly 3 distinct real roots?

2012 AIME I, Problem 6 72. The complex numbers z and w satisfy

$$z^{13} = w, w^{11} = z,$$

and the imaginary part of z is $\sin(m\pi/n)$ for relatively prime positive integers m and n with $m < n$. Find n.

2012 AIME I, Problem 9 73. Let x, y, and z be positive real numbers that satisfy

$$2\log_x(2y) = 2\log_{2x}(4z) = \log_{2x^4}(8yz) \neq 0.$$

The value of xy^5z can be expressed in the form $1/2^{p/q}$, where p and q are relatively prime integers. Find $p + q$.

2012 AIME I, Problem 14 74. Complex numbers a, b and c are the zeros of a polynomial $P(z) = z^3 + qz + r$, and $|a|^2 + |b|^2 + |c|^2 = 250$. The points corresponding to a, b, and c in the complex plane are the vertices of a right triangle with hypotenuse h. Find h^2.

2012 AIME II, Problem 6 75. Let $z = a + bi$ be the complex number with $|z| = 5$ and $b > 0$ such that the distance between $(1 + 2i)z^3$ and z^5 is maximized, and let $z^4 = c + di$.

2012 AIME II, Problem 8 76. The complex numbers z and w satisfy the system

$$z + \frac{20i}{w} = 5 + i, \text{ and}$$
$$w + \frac{12i}{z} = -4 + 10i.$$

Find the smallest possible value of $|zw|^2$.

2013 AIME I, Problem 5 77. The real root of $8x^3 - 3x^2 - 3x - 1 = 0$ can be written in the form

$$\frac{\sqrt[3]{a} + \sqrt[3]{b} + 1}{c},$$

where a, b, and c are positive integers. Find $a + b + c$.

2013 AIME I, Problem 10 78. There are nonzero integers a, b, r, and s such that the complex number $r+si$ is a zero of the polynomial $P(x)=x^3-ax^2+bx-65$. For each possible combination of a and b, let $p_{a,b}$ be the sum of the zeroes of $P(x)$. Find the sum of the $p_{a,b}$'s for all possible combinations of a and b.

2013 AIME II, Problem 2 79. Positive integers a and b satisfy the condition

$$\log_2(\log_{2^a}(\log_{2^b}(2^{1000})))=0.$$

Find the sum of all possible values of $a+b$.

2013 AIME II, Problem 12 80. Let S be the set of all polynomials of the form z^3+az^2+bz+c, where a, b, and c are integers. Find the number of polynomials in S such that each of its roots z satisfies either $|z|=20$ or $|z|=13$.

2013 HMMT, Algebra, Problem 4 81. Determine all real values of A for which there exist distinct complex numbers x_1, x_2 such that the following three equations hold:

$$\begin{aligned} x_1(x_1+1)&=A,\\ x_2(x_2+1)&=A,\\ x_1^4+3x_1^3+5x_1&=x_2^4+3x_2^3+5x_2. \end{aligned}$$

2013 HMMT, Algebra, Problem 5 82. Let a and b be real numbers, and let r, s, and t be the roots of $f(x)=x^3+ax^2+bx-1$. Also, $g(x)=x^3+mx^2+nx+p$ has roots r^2, s^2, and t^2. If $g(-1)=-5$, find the maximum possible value of b.

2013 HMMT, Algebra, Problem 7 83. Compute

$$\sum_{a_1=0}^{\infty}\sum_{a_2=0}^{\infty}\cdots\sum_{a_7=0}^{\infty}\frac{a_1+a_2+\cdots+a_7}{3^{a_1+a_2+\cdots+a_7}}.$$

2013 HMMT, Algebra, Problem 8 84. Let x, y be complex numbers such that

$$\frac{x^2+y^2}{x+y}=4 \qquad \text{and} \qquad \frac{x^4+y^4}{x^3+y^3}=2.$$

Find all possible values of

$$\frac{x^6+y^6}{x^5+y^5}.$$

2013 HMMT, Algebra, Problem 9 85. Let z be a non-real complex number with $z^{23}=1$. Compute

$$\sum_{k=0}^{22}\frac{1}{1+z^k+z^{2k}}.$$

2013 HMMT, Guts, Problem 11 86. Compute the prime factorization of

$$1007021035035021007001.$$

You should write your answer in the form $p_1^{e_1} p_2^{e_2} \dots p_k^{e_k}$ where $p_1, \dots, p_k$ are distinct prime numbers and $e_1, \dots, e_k$ are positive integers.

2013 HMMT, Guts, Problem 20 87. The polynomial $f(x) = x^3 - 3x^2 - 4x + 4$ has three real roots r_1, r_2, and r_3. Let $g(x) = x^3 + ax^2 + bx + c$ be the polynomial which has roots s_1, s_2, and s_3, where

$$\begin{aligned} s_1 &= r_1 + r_2 z + r_3 z^2, \\ s_2 &= r_1 z + r_2 z^2 + r_3, \\ s_3 &= r_1 z^2 + r_2 + r_3 z, \end{aligned}$$

and $z = (-1 + i\sqrt{3})/2$. Find the real part of the sum of the coefficients of $g(x)$.

2013 HMMT, Guts, Problem 28 88. Let $z_0 + z_1 + z_2 + \cdots$ be an infinite complex geometric series such that $z_0 = 1$ and $z_{2013} = \dfrac{1}{2013^{2013}}$. Find the sum of all possible sums of this series.

2013 HMMT, Guts, Problem 33 89. Compute the value of $1^{25} + 2^{24} + 3^{23} + \ldots + 24^2 + 25^1$. If your answer is A and the correct answer is C, then your score on this problem (out of 25) will be

$$\left\lfloor 25 \min\left(\left(\frac{A}{C}\right)^2, \left(\frac{C}{A}\right)^2 \right) \right\rfloor.$$

2012 HMMT, Algebra, Problem 8 90. Let $x_1 = y_1 = x_2 = y_2 = 1$, then for $n \geq 3$ let

$$x_n = x_{n-1} y_{n-2} + x_{n-2} y_{n-1} \qquad \text{and} \qquad y_n = y_{n-1} y_{n-2} - x_{n-1} x_{n-2}.$$

What are the last two digits of $|x_{2012}|$?

2012 HMMT, Algebra, Problem 9 91. How many real triples (a, b, c) are there such that the polynomial

$$p(x) = x^4 + ax^3 + bx^2 + ax + c,$$

has exactly three distinct roots, which are equal to $\tan y$, $\tan 2y$, and $\tan 3y$ for some real number y?

2012 HMMT, Guts, Problem 18 92. Let x and y be positive real numbers such that $x^2 + y^2 = 1$ and

$$(3x - 4x^3)(3y - 4y^3) = -\frac{1}{2}.$$

Compute $x + y$.

2014 AIME I, Problem 6 93. The graphs of $y = 3(x - h)^2 + j$ and $y = 2(x - h)^2 + k$ have y-intercepts of 2013 and 2014, respectively, and each graph has two positive integer x-intercepts. Find h.

2014 AIME I, Problem 7 94. Let w and z be complex numbers such that $|w| = 1$ and $|z| = 10$. Let

$$\theta = \arg\left(\frac{w-z}{z}\right).$$

The maximum possible value of $\tan^2\theta$ can be written as p/q, where p and q are relatively prime positive integers. Find $p+q$.

Note that $\arg(w)$, for $w \neq 0$, denotes the measure of the angle that the ray from 0 to w makes with the positive real axis in the complex plane.

2014 AIME I, Problem 9 95. Let $x_1 < x_2 < x_3$ be three real roots of equation

$$\sqrt{2014}x^3 - 4029x^2 + 2 = 0.$$

Find $x_2(x_1 + x_3)$.

2014 AIME I, Problem 14 96. Let m be the largest real solution to the equation

$$\frac{3}{x-3} + \frac{5}{x-5} + \frac{17}{x-17} + \frac{19}{x-19} = x^2 - 11x - 4.$$

There are positive integers a, b, c such that $m = a + \sqrt{b + \sqrt{c}}$. Find $a+b+c$.

2014 AIME II, Problem 5 97. Real numbers r and s are roots of $p(x) = x^3 + ax + b$, and $r+4$ and $s-3$ are roots of $q(x) = x^3 + ax + b + 240$. Find the sum of all possible values of $|b|$.

2014 AIME II, Problem 7 98. Let

$$f(x) = (x^2 + 3x + 2)^{\cos(\pi x)}.$$

Find the sum of all positive integers n for which

$$\left|\sum_{k=1}^{n} \log_{10} f(k)\right| = 1.$$

2014 AIME II, Problem 10 99. Let z be a complex number with $|z| = 2014$. Let P be the polygon in the complex plane whose vertices are z and every w such that

$$\frac{1}{z+w} = \frac{1}{z} + \frac{1}{w}.$$

Then the area enclosed by P can be written in the form $n\sqrt{3}$, where n is an integer. Find the remainder when n is divided by 1000.

2014 HMMT, Algebra, Problem 3 100. Let

$$A = \frac{1}{6}((\log_2(3))^3 - (\log_2(6))^3 - (\log_2(12))^3 + (\log_2(24))^3).$$

Compute 2^A.

2014 HMMT, Algebra, Problem 4 101. Let b and c be real numbers and define the polynomial $P(x) = x^2 + bx + c$. Suppose that $P(P(1)) = P(P(2)) = 0$, and that $P(1) \neq P(2)$. Find $P(0)$.

2014 HMMT, Algebra, Problem 5 102. Find the sum of all real numbers x such that $5x^4 - 10x^3 + 10x^2 - 5x - 11 = 0$.

2014 HMMT, Algebra, Problem 6 103. Given w and z are complex numbers such that $|w + z| = 1$ and $|w^2 + z^2| = 14$, find the smallest possible value of $|w^3 + z^3|$. Here $|\cdot|$ denotes the absolute value of a complex number, given by $|a + bi| = \sqrt{a^2 + b^2}$ whenever a and b are real numbers.

2014 HMMT, Algebra, Problem 8 104. Find all real numbers k such that $r^4 + kr^3 + r^2 + 4kr + 16 = 0$ is true for exactly one real number r.

2014 HMMT, Algebra, Problem 105. Given a, b, and c are complex numbers satisfying

$$\begin{aligned} a^2 + ab + b^2 &= 1 + i, \\ b^2 + bc + c^2 &= -2, \\ c^2 + ca + a^2 &= 1, \end{aligned}$$

compute $(ab + bc + ca)^2$. Here, $i = \sqrt{-1}$ is the imaginary unit.

2014 HMMT, Algebra, Problem 10 106. For an integer n, let $f_9(n)$ denote the number of positive integers $d \leq 9$ dividing n. Suppose that m is a positive integer and $b_1, b_2, \ldots, b_m$ are real numbers such that

$$f_9(n) = \textstyle\sum_{j=1}^{m} b_j f_9(n - j),$$

for all $n > m$. Find the smallest possible value of m.

2014 HMMT, Guts, Problem 26 107. For $1 \leq j \leq 2014$, define

$$b_j = j^{2014} \prod_{i=1, i\neq j}^{2014} (i^{2014} - j^{2014}),$$

where the product is over all $i \in \{1, \ldots, 2014\}$ except $i = j$. Evaluate

$$\frac{1}{b_1} + \frac{1}{b_2} + \cdots + \frac{1}{b_{2014}}.$$

2014 HMMT, Guts, Problem 28 108. Let $f(n)$ and $g(n)$ be polynomials of degree 2014 such that $f(n) + (-1)^n g(n) = 2^n$ for $n = 1, 2, \ldots, 4030$. Find the coefficient of x^{2014} in $g(x)$.

2014 HMMT, Guts, Problem 32 109. Find all ordered pairs (a, b) of complex numbers with $a^2 + b^2 \neq 0$,

$$a + \frac{10b}{a^2 + b^2} = 5,$$

and

$$b + \frac{10a}{a^2 + b^2} = 4.$$

2014 HMMT, Team, Problem 5 110. Prove that there exists a nonzero complex number c and a real number d such that

$$\left|\left|\frac{1}{1+z+z^2}\right| - \left|\frac{1}{1+z+z^2} - c\right|\right| = d,$$

for all z with $|z| = 1$ and $1 + z + z^2 \neq 0$.

2014 HMIC, Problem 4 111. Let ω be a root of unity and f be a polynomial with integer coefficients. Show that if $|f(\omega)| = 1$, then $f(\omega)$ is also a root of unity.

2015 AIME I, Problem 10 112. Let $f(x)$ be a third-degree polynomial with real coefficients satisfying

$$|f(1)| = |f(2)| = |f(3)| = |f(5)| = |f(6)| = |f(7)| = 12.$$

Find $|f(0)|$.

2015 AIME II, Problem 6 113. Steve says to Jon, "I am thinking of a polynomial whose roots are all positive integers. The polynomial has the form $P(x) = 2x^3 - 2ax^2 + (a^2 - 81)x - c$ for some positive integers a and c. Can you tell me the values of a and c?" After some calculations, Jon says, "There is more than one such polynomial." Steve says, "You're right. Here is the value of a." He writes down a positive integer and asks, "Can you tell me the value of c?" Jon says, "There are still two possible values of c." Find the sum of the two possible values of c.

2015 AIME II, Problem 14 114. Let x and y be real numbers satisfying

$$x^4y^5 + y^4x^5 = 810 \qquad \text{and} \qquad x^3y^6 + y^3x^6 = 945.$$

Evaluate $2x^3 + (xy)^3 + 2y^3$.

2015 HMMT, Algebra, Problem 1 115. Let Q be a polynomial

$$Q(x) = a_0 + a_1x + \cdots + a_nx^n,$$

where $a_0, \ldots, a_n$ are non-negative integers. Given that $Q(1) = 4$ and $Q(5) = 152$, find $Q(6)$.

2015 HMMT, Algebra, Problem 7 116. Suppose (a_1, a_2, a_3, a_4) is a 4-term sequence of real numbers satisfying the following two conditions:

a) $a_3 = a_2 + a_1$ and $a_4 = a_3 + a_2$;

b) there exist real numbers a, b, c such that

$$an^2 + bn + c = \cos(a_n),$$

for all $n \in \{1, 2, 3, 4\}$.

Compute the maximum possible value of

$$\cos(a_1) - \cos(a_4),$$

over all such sequences (a_1, a_2, a_3, a_4).

2015 HMMT, Algebra, Problem 8 117. Find the number of ordered pairs of integers $(a, b) \in \{1, 2, \ldots, 35\}^2$ (not necessarily distinct) such that $ax + b$ is a "quadratic residue modulo $x^2 + 1$ and 35", i.e. there exists a polynomial $f(x)$ with integer coefficients such that either of the following *equivalent* conditions holds:

a) there exist polynomials P, Q with integer coefficients such that

$$f(x)^2 - (ax + b) = (x^2 + 1)P(x) + 35Q(x);$$

b) or more conceptually, the remainder when (the polynomial) $f(x)^2 - (ax + b)$ is divided by (the polynomial) $x^2 + 1$ is a polynomial with integer coefficients all divisible by 35.

2015 HMMT, Algebra, Problem 10 118. Find all ordered 4-tuples of integers (a, b, c, d) (not necessarily distinct) satisfying the following system of equations:

$$\begin{aligned} a^2 - b^2 - c^2 - d^2 &= c - b - 2 \\ 2ab &= a - d - 32 \\ 2ac &= 28 - a - d \\ 2ad &= b + c + 31. \end{aligned}$$

2015 HMMT, Team, Problem 3 119. Let $z = a + bi$ be a complex number with integer real and imaginary parts $a, b \in \mathbb{Z}$, where $i = \sqrt{-1}$ is the imaginary unit. If p is an odd prime number, show that the real part of $z^p - z$ is an integer divisible by p.

2015 HMMT, Team, Problem 9 120. Let

$$z = e^{2\pi i/101} \qquad \text{and} \qquad w = e^{2\pi i/10}.$$

Prove that

$$\prod_{a=0}^{9} \prod_{b=0}^{100} \prod_{c=0}^{100} (w^a + z^b + z^c)$$

is an integer and find (with proof) its remainder upon division by 101.

2015 HMMT, Guts, Problem 13 121. Let $P(x) = x^3 + ax^2 + bx + 2015$ be a polynomial all of whose roots are integers. Given that $P(x) \geq 0$ for all $x \geq 0$, find the sum of all possible values of $P(-1)$.

2015 HMMT, Guts, Problem 25 122. Let $r_1, \ldots, r_n$ be the distinct real zeroes of the equation

$$x^8 - 14x^4 - 8x^3 - x^2 + 1 = 0.$$

Evaluate $r_1^2 + \cdots + r_n^2$.

2015 HMMT, Guts, Problem 26 123. Let $a = \sqrt{17}$ and $b = i\sqrt{19}$, where $i = \sqrt{-1}$. Find the maximum possible value of the ratio $|a - z|/|b - z|$ over all complex numbers z of magnitude 1 (i.e. over the unit circle $|z| = 1$).

2015 HMMT, Guts, Problem 30 124. Find the sum of squares of all **distinct** complex numbers x satisfying the equation

$$0 = 4x^{10} - 7x^9 + 5x^8 - 8x^7 + 12x^6 - 12x^5 + 12x^4 - 8x^3 + 5x^2 - 7x + 4.$$

2015 HMIC, Problem 5 125. Let $\omega = e^{2\pi i/5}$ be a primitive fifth root of unity. Prove that there do not exist integers a, b, c, d, k with $k > 1$ such that

$$(a + b\omega + c\omega^2 + d\omega^3)^k = 1 + \omega.$$

2016 AIME I, Problem 7 126. For integers a and b consider the complex number

$$\frac{\sqrt{ab + 2016}}{ab + 100} - \left(\frac{\sqrt{|a + b|}}{ab + 100}\right) i.$$

Find the number of ordered pairs of integers (a, b) such that this complex number is a real number.

2016 AIME I, Problem 11 127. Let $P(x)$ be a nonzero polynomial such that

$$(x - 1)P(x + 1) = (x + 2)P(x),$$

for every real x, and $(P(2))^2 = P(3)$. Then $P(7/2) = m/n$, where m and n are relatively prime positive integers. Find $m + n$.

2016 AIME II, Problem 3 128. Let x, y and z be real numbers satisfying the system

$$\begin{aligned} \log_2(xyz - 3 + \log_5 x) &= 5, \\ \log_3(xyz - 3 + \log_5 y) &= 4, \\ \log_4(xyz - 3 + \log_5 z) &= 4. \end{aligned}$$

Find the value of $|\log_5 x| + |\log_5 y| + |\log_5 z|$.

2016 AIME II, Problem 6 129. For polynomial

$$P(x) = 1 - \frac{1}{3}x + \frac{1}{6}x^2,$$

define

$$Q(x) = P(x)P(x^3)P(x^5)P(x^7)P(x^9) = \sum_{i=0}^{50} a_i x^i.$$

Then,

$$\sum_{i=0}^{50} |a_i| = \frac{m}{n},$$

where m and n are relatively prime positive integers. Find $m + n$.

2016 HMMT, Algebra, Problem 10 130. Let a, b and c be real numbers such that

$$\begin{aligned} a^2+ab+b^2&=9,\\ b^2+bc+c^2&=52,\\ c^2+ca+a^2&=49. \end{aligned}$$

Compute the value of

$$\frac{49b^2+39bc+9c^2}{a^2}.$$

2016 HMMT, Guts, Problem 1 131. Let x and y be complex numbers such that $x+y=\sqrt{20}$ and $x^2+y^2=15$. Compute $|x-y|$.

2016 HMMT, Guts, Problem 23 132. Let $t=2016$ and $p=\ln 2$. Evaluate in closed form the sum

$$\sum_{k=1}^{\infty}\left(1-\sum_{n=0}^{k-1}\frac{e^{-t}t^n}{n!}\right)(1-p)^{k-1}p.$$

2016 HMMT, Team, Problem 7 133. Let $q(x)=q^1(x)=2x^2+2x-1$, and let $q^n(x)=q(q^{n-1}(x))$ for $n>1$. How many negative real roots does $q^{2016}(x)$ have?

2016 HMMT, November Theme, Problem 6 134. Let $P_1, P_2, \ldots, P_6$ be points in the complex plane, which are also roots of the equation $x^6+6x^3-216=0$. Given that $P_1P_2P_3P_4P_5P_6$ is a convex hexagon, determine the area of this hexagon.

2016 HMIC, Problem 4 135. Let P be an odd-degree integer-coefficient polynomial. Suppose that $xP(x)=yP(y)$ for infinitely many pairs x, y of integers with $x\neq y$. Prove that the equation $P(x)=0$ has an integer root.

2017 AIME I, Problem 10 136. Let $z_1=18+83i$, $z_2=18+39i$, and $z_3=78+99i$, where $i=\sqrt{-1}$. Let z be the unique complex number with the properties that

$$\frac{z_3-z_1}{z_2-z_1}\cdot\frac{z-z_2}{z-z_3},$$

is a real number and the imaginary part of z is the greatest possible. Find the real part of z.

2017 AIME I, Problem 14 137. Let $a>1$ and $x>1$ satisfy

$$\log_a(\log_a(\log_a 2)+\log_a 24-128)=128,$$

and $\log_a(\log_a x)=256$. Find the remainder when x is divided by 1000.

2017 AIME II, Problem 7 138. Find the number of integer values of k in the closed interval $[-500, 500]$ for which the equation $\log(kx)=2\log(x+2)$ has exactly one real solution.

2017 HMMT, Team, Problem 1 139. Let $P(x)$, $Q(x)$ be nonconstant polynomials with real number coefficients. Prove that if

$$\lfloor P(y)\rfloor=\lfloor Q(y)\rfloor,$$

for all real numbers y, then $P(x)=Q(x)$ for all real numbers x.

2017 HMMT, Team, Problem 2 140. Does there exist a two-variable polynomial $P(x, y)$ with real number coefficients such that $P(x, y)$ is positive exactly when x and y are both positive?

2017 HMMT, Algebra & Number Theory, Problem 1 141. Let $Q(x) = a_0 + a_1x + \cdots + a_nx^n$ be a polynomial with integer coefficients, and $0 \le a_i < 3$ for all $0 \le i \le n$.
Given that $Q(\sqrt{3}) = 20 + 17\sqrt{3}$, compute $Q(2)$.

2017 HMMT, Algebra & Number Theory, Problem 6 142. A polynomial P of degree 2015 satisfies the equation $P(n) = \frac{1}{n^2}$ for $n = 1, 2, \ldots, 2016$. Find

$$\lfloor 2017P(2017) \rfloor.$$

2017 HMMT, November General, Problem 5 143. Given that a, b, c are integers with $abc = 60$, and that complex number $\omega \neq 1$ satisfies $\omega^3 = 1$, find the minimum possible value of

$$|a + b\omega + c\omega^2|.$$

2017 HMMT, November Guts, Problem 11 144. Consider the graph in 3-space of

$$0 = xyz(x + y)(y + z)(z + x)(x - y)(y - z)(z - x).$$

This graph divides 3-space into N connected regions. What is N?

2017 HMMT, November Guts, Problem 16 145. Let a and b be complex numbers satisfying the two equations

$$a^3 - 3ab^2 = 36,$$
$$b^3 - 3ba^2 = 28i.$$

Let M be the maximum possible magnitude of a. Find all a such that $|a| = M$.

2017 HMMT, November Guts, Problem 25 146. Find all real numbers x satisfying the equation

$$x^3 - 8 = 16\sqrt[3]{x + 1}.$$

2018 AIME I, Problem 5 147. For each ordered pair of real numbers (x, y) satisfying

$$\log_2(2x + y) = \log_4(x^2 + xy + 7y^2),$$

there is a real number K such that

$$\log_3(3x + y) = \log_9(3x^2 + 4xy + Ky^2).$$

Find the product of all possible values of K.

2018 AIME I, Problem 6 148. Let N be the number of complex numbers z with the properties that $|z| = 1$ and $z^{6!} - z^{5!}$ is a real number. Find the remainder when N is divided by 1000.

2018 AIME II, Problem 5 149. Suppose that x, y, and z are complex numbers such that $xy = -80 - 320i$, $yz = 60$, and $zx = -96 + 24i$, where $i = \sqrt{-1}$. Then there are real numbers a and b such that $x + y + z = a + bi$. Find $a^2 + b^2$.

2018 AIME II, Problem 6 150. A real number a is chosen randomly and uniformly from the interval $[-20, 18]$. The probability that the roots of the polynomial

$$x^4 + 2ax^3 + (2a-2)x^2 + (-4a+3)x - 2,$$

are all real can be written in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

2019 AIME I, Problem 7 151. There are positive integers x and y that satisfy the system of equations

$$\log_{10} x + 2\log_{10}(\gcd(x,y)) = 60,$$
$$\log_{10} y + 2\log_{10}(\operatorname{lcm}(x,y)) = 570.$$

Let m be the number of (not necessarily distinct) prime factors in the prime factorization of x, and let n be the number of (not necessarily distinct) prime factors in the prime factorization of y. Find $3m+2n$.

2019 AIME I, Problem 10 152. For distinct complex numbers $z_1, z_2, \ldots, z_{673}$, the polynomial

$$(x-z_1)^3(x-z_2)^3\cdots(x-z_{673})^3,$$

can be expressed as $x^{2019} + 20x^{2018} + 19x^{2017} + g(x)$, where $g(x)$ is a polynomial with complex coefficients and with degree at most 2016. The value of

$$\left|\sum_{1\le j<k\le 673} z_j z_k\right|,$$

can be expressed in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

2019 AIME I, Problem 12 153. Given $f(z) = z^2 - 19z$, there are complex numbers z with the property that z, $f(z)$, and $f(f(z))$ are the vertices of a right triangle in the complex plane with a right angle at $f(z)$. There are positive integers m and n such that one such value of z is $m + \sqrt{n} + 11i$. Find $m+n$.

2019 AIME II, Problem 6 154. In a Martian civilization, all logarithms whose bases are not specified are assumed to be base b, for some fixed $b \ge 2$. A Martian student writes down

$$3\log(\sqrt{x}\log x) = 56,$$
$$\log_{\log(x)}(x) = 54,$$

and finds that this system of equations has a single real number solution $x > 1$. Find b.

2019 AIME II, Problem 8 155. The polynomial $f(z) = az^{2018} + bz^{2017} + cz^{2016}$ has real coefficients not exceeding 2019, and $f(\frac{1+\sqrt{3}i}{2}) = 2015 + 2019\sqrt{3}i$. Find the remainder when $f(1)$ is divided by 1000.

2019 PUMaC, Algebra, Problem 2 156. Let $f(x) = x^2 + 4x + 2$. Let r be the difference between the largest and smallest real solutions of the equation $f(f(f(f(x)))) = 0$. Then $r = a^{\frac{p}{q}}$ for some positive integers a, p, q so a is square-free and p, q are relatively prime positive integers. Compute $a+p+q$.

2019 PUMaC, Algebra, Problem 5 157. Let

$$\omega = e^{2\pi i/2017} \quad \text{and} \quad \zeta = e^{2\pi i/2019}.$$

Define

$$S = \{(a, b) \in \mathbb{Z} \,|\, 0 \le a \le 2016, 0 \le b \le 2018, (a, b) \ne (0, 0)\}.$$

Compute

$$\prod_{(a,b)\in S} (\omega^a - \zeta^b).$$

2019 PUMaC, Team Round, Problem 7 158. For all sets A of complex numbers, let $P(A)$ be the product of the elements of A. Let

$$S_z = \left\{1, 2, 9, 99, 999, \frac{1}{z}, \frac{1}{z^2}\right\},$$

and let T_z be the set of nonempty subsets of S_z (including S_z), and let

$$f(z) = 1 + \sum_{s\in T_z} P(s).$$

Suppose $f(z) = 6125000$ for some complex number z. Compute the product of all possible values of z.

2019 PUMaC, Team Round, Problem 13 159. Let $e_1, e_2, \ldots, e_{2019}$ be independently chosen from the set $\{0, 1, \ldots, 20\}$ uniformly at random. Let $\omega = e^{\frac{2\pi}{i}2019}$. Determine the expected value of

$$|e_1\omega + e_2\omega^2 + \cdots + e_{2019}\omega^{2019}|.$$

2019 HMMT, Team, Problem 3 160. Alan draws a convex 2020-gon

$$\mathcal{A} = A_1A_2 \cdots A_{2020},$$

with vertices in clockwise order and chooses 2020 angles $\theta_1, \theta_2, \ldots, \theta_{2020} \in (0, \pi)$ in radians with sum 1010π. He then constructs isosceles triangles $\triangle A_iB_iA_{i+1}$ on the exterior of $\mathcal{A}$ with $B_iA_i = B_iA_{i+1}$ and $\angle A_iB_iA_{i+1} = \theta_i$. (Here, $A_{2021} = A_1$.) Finally, he erases $\mathcal{A}$ and the point B_1. He then tells Jason the angles $\theta_1, \theta_2, \ldots, \theta_{2020}$ he chose. Show that Jason can determine where B_1 was from the remaining 2019 points, i.e. show that B_1 is uniquely determined by the information Jason has.

2019 HMMT, Team, Problem 9 161. Let $p > 2$ be a prime number. $\mathbb{F}_p[x]$ is defined as the set of polynomials in x with coefficients in $\mathbb{F}_p$ (the integers modulo p with usual addition and subtraction), so that two polynomials are equal if and only if the coefficients of x^k are equal in $\mathbb{F}_p$ for each non-negative integer k. For example,

$$(x + 2)(2x + 3) = 2x^2 + 2x + 1 \quad \text{in} \quad \mathbb{F}_5[x],$$

because the corresponding coefficients are equal modulo 5. Let $f, g \in \mathbb{F}_p[x]$. The pair (f, g) is called *compositional* if

$$f(g(x)) \equiv x^{p^2} - x \quad \text{in} \quad \mathbb{F}_p[x].$$

Find, with proof, the number of *compositional* pairs.

2019 HMMT, Team, Problem 10 162. Prove that for all positive integers n, all complex roots r of the polynomial

$$P(x) = (2n)x^{2n} + (2n-1)x^{2n-1} + \cdots + (n+1)x^{n+1} + nx^n + (n+1)x^{n-1} + \cdots + (2n-1)x + 2n$$

lie on the unit circle (i.e. $|r| = 1$).

2019 HMMT, Algebra & Number Theory, Problem 5 163. Let $a_1, a_2, \ldots$ be an arithmetic sequence and $b_1, b_2, \ldots$ be a geometric sequence. Suppose that $a_1b_1 = 20$, $a_2b_2 = 19$, and $a_3b_3 = 14$. Find the greatest possible value of a_4b_4.

2019 HMMT, Algebra & Number Theory, Problem 7 164. Find the value of

$$\sum_{a=1}^{\infty}\sum_{b=1}^{\infty}\sum_{c=1}^{\infty} \frac{ab(3a+c)}{4^{a+b+c}(a+b)(b+c)(c+a)}.$$

2018-2019 San Diego Power Contest, Winter, Problem 1 165. Let r_1, r_2, r_3 be the distinct real roots of $x^3 - 2019x^2 - 2020x + 2021 = 0$. Prove that $r_1^3 + r_2^3 + r_3^3$ is an integer multiple of 3.

2018-2019 San Diego Power Contest, Winter, Problem 5 166. Prove that there exists a positive integer N such that for every polynomial $P(x)$ of degree 2019, there exist N linear polynomials $p_1, p_2, \ldots p_N$ such that $P(x) = p_1(x)^{2019} + p_2(x)^{2019} + \ldots + p_N(x)^{2019}$. (Assume all polynomials in this problem have real coefficients, and leading coefficients cannot be zero.)

2019-2020 San Diego Power Contest, Fall, Problem 3 167. Find all polynomials P with integer coefficients such that for all positive integers x, y,

$$\frac{P(x) - P(y)}{x^2 + y^2}$$

evaluates to an integer (in particular, it can be zero).

2020 AIME I, Problem 2 168. There is a unique positive real number x such that the three numbers $\log_8(2x)$, $\log_4 x$, and $\log_2 x$, in that order, form a geometric progression with positive common ratio. The number x can be written as m/n, where m and n are relatively prime positive integers. Find $m + n$.

2020 AIME I, Problem 11 169. For integers a, b, c, and d, let $f(x) = x^2 + ax + b$ and $g(x) = x^2 + cx + d$. Find the number of ordered triples (a, b, c) of integers with absolute values not exceeding 10 for which there is an integer d such that $g(f(2)) = g(f(4)) = 0$.

2020 AIME I, Problem 14 170. Let $P(x)$ be a quadratic polynomial with complex coefficients whose x^2 coefficient is 1. Suppose the equation $P(P(x)) = 0$ has four distinct solutions, $x = 3, 4, a, b$. Find the sum of all possible values of $(a + b)^2$.

2020 AIME II, Problem 3 171. The value of x that satisfies

$$\log_{2^x} 3^{20} = \log_{2^{x+3}} 3^{2020},$$

can be written as m/n, where m and n are relatively prime positive integers. Find $m + n$.

2020 AIME II, Problem 8 172. Define a sequence of functions recursively by $f_1(x) = |x - 1|$ and $f_n(x) = f_{n-1}(|x - n|)$ for integers $n > 1$. Find the least value of n such that the sum of the zeros of f_n exceeds 500,000.

2020 AIME II, Problem 11 173. Let $P(x) = x^2 - 3x - 7$, and let $Q(x)$ and $R(x)$ be two quadratic polynomials also with the coefficient of x^2 equal to 1. David computes each of the three sums $P + Q$, $P + R$, and $Q + R$ and is surprised to find that each pair of these sums has a common root, and these three common roots are distinct. If $Q(0) = 2$, then $R(0) = m/n$, where m and n are relatively prime positive integers. Find $m + n$.

2020 AIME II, Problem 14 174. For real number x let $\lfloor x \rfloor$ be the greatest integer less than or equal to x, and define $\{x\} = x - \lfloor x \rfloor$ to be the fractional part of x. For example, $\{3\} = 0$ and $\{4.56\} = 0.56$. Define $f(x) = x\{x\}$, and let N be the number of real-valued solutions to the equation $f(f(f(x))) = 17$ for $0 \leq x \leq 2020$. Find the remainder when N is divided by 1000.

2020 PUMaC, Algebra, Problem A4/B6 175. Let P be a 10-degree monic polynomial with roots $r_1, r_2, \ldots, r_{10} \neq 0$, and let Q be a 45-degree monic polynomial with roots

$$\frac{1}{r_i} + \frac{1}{r_j} - \frac{1}{r_i r_j},$$

where $i < j$ and $i, j \in \{1, \ldots, 10\}$. If $P(0) = Q(1) = 2$, then $\log_2(|P(1)|)$ can be written as a/b for relatively prime integers a, b. Find $a + b$.

2020 PUMaC, Algebra, Problem A6/B8 176. Given integer n, let W_n be the set of complex numbers of the form $re^{2qi\pi}$, where q is a rational number so that $q_n \in \mathbb{Z}$ and r is a real number. Suppose that p is a polynomial of degree ≥ 2 such that there exists a non-constant function $f : W_n \to C$ so that

$$p(f(x))p(f(y)) = f(xy),$$

for all $x, y \in W_n$. If p is the unique monic polynomial of lowest degree for which such an f exists for $n = 65$, find $p(10)$.

2020 PUMaC, Algebra, Problem A7 177. Suppose that p is the unique monic polynomial of minimal degree such that its coefficients are rational numbers and one of its roots is

$$\sin \frac{2\pi}{7} + \cos \frac{4\pi}{7}.$$

If $p(1) = a/b$, where a, b are relatively prime integers, find $|a + b|$.

2020 HMMT, Algebra & Number Theory, Problem 1 178. Let

$$P(x) = x^3 + x^2 - r^2 x - 2020,$$

be a polynomial with roots r, s, t. What is $P(1)$?

2020 HMMT, Algebra & Number Theory, Problem 3 179. Let $a = 256$. Find the unique real number $x > a^2$ such that

$$\log_a \log_a \log_a x = \log_{a^2} \log_{a^2} \log_{a^2} x.$$

2020 HMMT, Algebra & Number Theory, Problem 6 180. A polynomial $P(x)$ is a *base-n polynomial* if it is of the form $a_dx^d + a_{d-1}x^{d-1} + \cdots + a_1x + a_0$, where each a_i is an integer between 0 and $n-1$ inclusive and $a_d > 0$. Find the largest positive integer n such that for any real number c, there exists at most one base-n polynomial $P(x)$ for which $P(\sqrt{2} + \sqrt{3}) = c$.

2020 HMMT, Algebra & Number Theory, Problem 8 181. $P(x)$ is the unique polynomial of degree at most 2020 satisfying $P(k^2) = k$ for $k = 0, 1, 2, \ldots, 2020$. Compute $P(2021^2)$.

2020 HMMT, Algebra & Number Theory, Problem 9 182. Let

$$P(x) = x^{2020} + x + 2,$$

which has 2020 distinct roots. Let $Q(x)$ be the monic polynomial of degree $\binom{2020}{2}$ whose roots are the pairwise products of the roots of $P(x)$. Let α satisfy $P(\alpha) = 4$. Compute the sum of all possible values of $Q(\alpha^2)^2$.

2020 HMMT, Algebra & Number Theory, Problem 10 183. We define $\mathbb{F}_{101}[x]$ as the set of all polynomials in x with coefficients in $\mathbb{F}_{101}$ (the integers modulo 101 with usual addition and subtraction), so that two polynomials are equal if and only if the coefficients of x^k are equal in $\mathbb{F}_{101}$ for each non-negative integer k. For example,

$$(x+3)(100x+5) = 100x^2 + 2x + 15 \quad \text{in} \quad \mathbb{F}_{101}[x],$$

because the corresponding coefficients are equal modulo 101.

We say that $f(x) \in \mathbb{F}_{101}[x]$ is *lucky* if it has degree at most 1000 and there exist $g(x), h(x) \in \mathbb{F}_{101}[x]$ such that

$$f(x) = g(x)(x^{1001} - 1) + h(x)^{101} - h(x) \quad \text{in} \quad \mathbb{F}_{101}[x].$$

Find the number of *lucky* polynomials.

2020 HMMT, Team, Problem 7 184. Positive real numbers x and y satisfy

$$\left|\left|\cdots\left|\left||x|-y\right|-x\right|\cdots-y\right|-x\right| = \left|\left|\cdots\left|\left||y|-x\right|-y\right|\cdots-x\right|-y\right|,$$

where there are 2019 absolute value signs $|\cdot|$ on each side. Determine, with proof, all possible values of x/y.

2020 HMIC, Problem 4 185. Let

$$C_k = \frac{1}{k+1}\binom{2k}{k}, \qquad k = 1, 2, 3, \ldots$$

denote the k^{th} Catalan number and p be an odd prime. Prove that exactly half of the numbers in the set

$$\left\{ \sum_{k=1}^{p-1} C_k n^k \,\middle|\, n \in \{1, 2, \ldots, p-1\} \right\},$$

are divisible by p.

2020-2021 San Diego Power Contest, Winter, Day 1, Problem 4 186. Find all polynomials $P(x)$ with integer coefficients such that for all positive integers n, we have that $P(n)$ is not zero and $P(\overline{nn})/P(n)$ is an integer, where $\overline{nn}$ is the integer obtained upon concatenating n with itself.

2020-21 CHMMC Winter, Individual Round, Problem 4 187. Let $P(x) = x^3 - 6x^2 - 5x + 4$. Suppose that y and z are real numbers such that

$$zP(y) = P(y-n) + P(y+n),$$

for all reals n. Evaluate $P(y)$.

2020-21 CHMMC Winter, Individual Round, Problem 8 188. Define

$$S = \tan^{-1}(2020) + \sum_{j=0}^{2020} \tan^{-1}(j^2 - j + 1).$$

Then S can be written as $\frac{m\pi}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

2020-21 CHMMC Winter, Team Round, Problem 6 189. Suppose that

$$\prod_{n=1}^{\infty} \left(\frac{1 + i\cot\left(\frac{n\pi}{2n+1}\right)}{1 - i\cot\left(\frac{n\pi}{2n+1}\right)} \right)^{\frac{1}{n}} = \left(\frac{p}{q}\right)^{i\pi},$$

where p and q are relatively prime positive integers. Find $p+q$.

Note: for a complex number $z = re^{i\theta}$ for reals $r > 0, 0 \le \theta < 2\pi$, we define $z^n = r^n e^{i\theta n}$ for all positive reals n.

2020-21 CHMMC Winter, Team Round, Problem 10 190. Let ω be a non-real 47^{th} root of unity. Suppose that $\mathcal{S}$ is the set of polynomials of degree at most 46 and coefficients equal to either 0 or 1. Let N be the number of polynomials $Q \in \mathcal{S}$ such that

$$\sum_{j=0}^{46} \frac{Q(\omega^{2j}) - Q(\omega^j)}{\omega^{4j} + \omega^{3j} + \omega^{2j} + \omega^j + 1} = 47.$$

The prime factorization of N is $p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s}$ where $p_1, \dots, p_s$ are distinct primes and $\alpha_1, \alpha_2, \dots, \alpha_s$ are positive integers. Compute $\sum_{j=1}^{s} p_j \alpha_j$.

2020-21 CHMMC Winter, Tiebreaker Round, Problem 2 191. Find the sum of all positive integers $x < 241$ such that both $x^{24} + x^{18} + x^{12} + x^6 + 1$ and $x^{20} + x^{10} + 1$ are multiples of 241.

2020-21 CHMMC Winter, Tiebreaker Round, Problem 7 192. Consider the polynomial $x^3 - 3x^2 + 10$. Let a, b, c be its roots. Compute

$$a^2b^2c^2 + a^2b^2 + b^2c^2 + c^2a^2 + a^2 + b^2 + c^2.$$

2021 AIME I, Problem 8 193. Find the number of integers c such that the equation

$$\big||20|x| - x^2| - c\big| = 21,$$

has 12 distinct real solutions.

2021 AIME I, Problem 15 194. Let S be the set of positive integers k such that the two parabolas

$$y = x^2 - k \text{ and } x = 2(y-20)^2 - k$$

intersect in four distinct points, and these four points lie on a circle with radius at most 21. Find the sum of the least element of S and the greatest element of S.

2021 AIME II, Problem 4 195. There are real numbers $a, b, c,$ and d such that -20 is a root of $x^3 + ax + b$ and -21 is a root of $x^3 + cx^2 + d$. These two polynomials share a complex root $m + \sqrt{n} \cdot i$, where m and n are positive integers and $i = \sqrt{-1}$. Find $m + n$.

2021 AIME II, Problem 7 196. Let $a, b, c,$ and d be real numbers that satisfy the system of equations

$$\begin{aligned} a + b &= -3, \\ ab + bc + ca &= -4, \\ abc + bcd + cda + dab &= 14, \\ abcd &= 30. \end{aligned}$$

There exist relatively prime positive integers m and n such that

$$a^2 + b^2 + c^2 + d^2 = \frac{m}{n}.$$

Find $m + n$.

2021 PUMaC, Team Round, Problem 7 197. The roots of the polynomial $f(x) = x^8 + x^7 - x^5 - x^4 - x^3 + x + 1$ are all roots of unity. We say that a real number $r \in [0, 1)$ is nice if $e^{2i\pi r} = \cos 2\pi r + i \sin 2\pi r$ is a root of the polynomial f and if $e^{2i\pi r}$ has positive imaginary part. Let S be the sum of the values of nice real numbers r. If $S = \frac{p}{q}$ for relatively prime positive integers p, q, find $p + q$.

2021 PUMaC, Algebra, Problem A3/B5 198. Let $f(x) = 1 + 2x + 3x^2 + 4x^3 + 5x^4$ and let

$$\zeta = e^{2\pi i/5} = \cos\frac{2\pi}{5} + i \sin\frac{2\pi}{5}.$$

Find the value of the following expression:

$$f(\zeta)f(\zeta^2)f(\zeta^3)f(\zeta^4).$$

2021 PUMaC, Algebra, Problem A4/B6 199. The roots of a monic cubic polynomial p are positive real numbers forming a geometric sequence. Suppose that the sum of the roots is equal to 10. Under these conditions, the largest possible value of $|p(-1)|$ can be written as m/n, where m, n are relatively prime integers. Find $m + n$.

2021 PUMaC, Algebra, Problem A6/B8 200. Let f be a polynomial. We say that a complex number p is a double *attractor* if there exists a polynomial $h(x)$ so that $f(x) - f(p) = h(x)(x-p)^2$ for all $x \in \mathbb{R}$. Now, consider the polynomial

$$f(x) = 12x^5 - 15x^4 - 40x^3 + 540x^2 - 2160x + 1,$$

and suppose that its double *attractors* are $a_1, a_2, \ldots, a_n$. If the sum

$$\sum_{i=1}^{n} |a_i| = \sqrt{a} + \sqrt{b},$$

where a, b are positive integers, find $a + b$.

2021 PUMaC, Algebra, Problem B1 201. Let x, y be distinct positive real numbers satisfying

$$\frac{1}{\sqrt{x+y} - \sqrt{x-y}} + \frac{1}{\sqrt{x+y} + \sqrt{x-y}} = \frac{x}{\sqrt{y^3}}.$$

If $x/y = (a + \sqrt{b})/c$ for positive integers a, b, c with $gcd(a, c) = 1$, find $a + b + c$.

2021 HMMT, Algebra & Number Theory, Problem 2 202. Compute the number of ordered pairs of integers (a, b), with $2 \leq a, b \leq 2021$, that satisfy the equation

$$a^{\log_b\left(a^{-4}\right)} = b^{\log_a\left(ba^{-3}\right)}.$$

2021 HMMT, Algebra & Number Theory, Problem 3 203. Among all polynomials $P(x)$ with integer coefficients for which $P(-10) = 145$ and $P(9) = 164$, compute the smallest possible value of $|P(0)|$.

2021 HMMT, Algebra & Number Theory, Problem 4 204. Suppose that $P(x, y, z)$ is a homogeneous degree 4 polynomial in three variables such that $P(a, b, c) = P(b, c, a)$ and $P(a, a, b) = 0$ for all real a, b, and c. If $P(1, 2, 3) = 1$, compute $P(2, 4, 8)$.

Note: $P(x, y, z)$ is a homogeneous degree 4 polynomial if it satisfies

$$P(ka, kb, kc) = k^4 P(a, b, c),$$

for all real k, a, b, c.

2021 HMMT, Algebra & Number Theory, Problem 7 205. Suppose that x, y, and z are complex numbers of equal magnitude that satisfy

$$x + y + z = -\frac{\sqrt{3}}{2} - i\sqrt{5} \qquad \text{and} \qquad xyz = \sqrt{3} + i\sqrt{5}.$$

If $x = x_1 + ix_2$, $y = y_1 + iy_2$, and $z = z_1 + iz_2$ for real x_1, x_2, y_1, y_2, z_1 and z_2 then $(x_1x_2 + y_1y_2 + z_1z_2)^2$ can be written as a/b for relatively prime positive integers a and b. Compute $100a + b$.

2021 HMMT, Algebra & Number Theory, Problem 9 206. Let f be a monic cubic polynomial satisfying $f(x) + f(-x) = 0$ for all real numbers x. For all real numbers y, define $g(y)$ to be the number of distinct real solutions x to the equation $f(f(x)) = y$. Suppose that the set of possible values of $g(y)$ over all real numbers y is exactly $\{1, 5, 9\}$. Compute the sum of all possible values of $f(10)$.

2021 HMIC, Problem 3 207. Let A be a set of $n \geq 2$ positive integers, and let $f(x) = \sum_{a \in A} x^a$. Prove that there exists a complex number z with $|z| = 1$ and $|f(z)| = \sqrt{n-2}$.

2021-22 CHMMC Winter, Team Round, Problem 3 208. Suppose a, b, c are complex numbers with $a + b + c = 0$, $a^2 + b^2 + c^2 = 0$, and $|a|, |b|, |c| \leq 5$. Suppose further at least one of a, b, c have real and imaginary parts that are both integers. Find the number of possibilities for such ordered triples (a, b, c).

2021-22 CHMMC Winter, Team Round, Problem 6 209. There is a unique degree-10 monic polynomial with integer coefficients $f(x)$ such that

$$f\left(\sum_{j=0}^{9} \sqrt[10]{2021^j}\right) = 0.$$

Find the remainder when $f(1)$ is divided by 1000.

2021-22 CHMMC Winter, Proof Round, Problem 3 210. Let $F(x_1, \ldots, x_n)$ be a polynomial with real coefficients in$n > 1$ "indeterminate" variables $x_1, \ldots, x_n$. We say that F is n-alternating if for all integers $1 \leq i < j \leq n$,

$$F(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_n) = F(x_1, \ldots, x_j, \ldots, x_i, \ldots, x_n),$$

i.e., swapping the order of indeterminates x_i, x_j flips the sign of the polynomial. For example, $x_1^2 x_2 - x_2^2 x_1$ is 2-alternating, whereas $x_1 x_2 x_3 + 2x_2 x_3$ is not 3-alternating.

Note: two polynomials $P(x_1, \ldots, x_n)$ and $Q(x_1, \ldots, x_n)$ are considered equal if and only if each monomial constituent $ax_1^{k_1} \cdots x_n^{k_n}$ of P appears in Q with the same coefficient a, and vice versa. This is equivalent to saying that $P(x_1, \ldots, x_n) = 0$ if and only if every possible monomial constituent of P has coefficient 0.

a) Compute a 3-alternating polynomial of degree 3.

b) Prove that the degree of any nonzero n-alternating polynomial is at least $\binom{n}{2}$.

2022 AIME I, Problem 1 211. Quadratic polynomials $P(x)$ and $Q(x)$ have leading coefficients of 2 and -2, respectively. The graphs of both polynomials pass through the two points $(16, 54)$ and $(20, 53)$. Find $P(0) + Q(0)$.

2022 AIME I, Problem 4 212. Let

$$w = \frac{\sqrt{3} + i}{2} \qquad \text{and} \qquad z = \frac{-1 + i\sqrt{3}}{2},$$

where $i = \sqrt{-1}$. Find the number of ordered pairs (r, s) of positive integers not exceeding 100 that satisfy the equation $i \cdot w^r = z^s$.

2022 AIME I, Problem 15 213. Let x, y, and z be positive real numbers satisfying the system of equations

$$\sqrt{2x - xy} + \sqrt{2y - xy} = 1,$$
$$\sqrt{2y - yz} + \sqrt{2z - yz} = \sqrt{2},$$
$$\sqrt{2z - zx} + \sqrt{2x - zx} = \sqrt{3}.$$

Then $\left[(1-x)(1-y)(1-z)\right]^2$ can be written as m/n, where m and n are relatively prime positive integers. Find $m+n$.

2022 AIME II, Problem 4 214. There is a positive real number x not equal to either $1/20$ or $1/2$ such that

$$\log_{20x}(22x) = \log_{2x}(202x).$$

The value $\log_{20x}(22x)$ can be written as $\log_{10}(m/n)$, where m and n are relatively prime positive integers. Find $m+n$.

2022 AIME II, Problem 12 215. Let $a, b, x,$ and y be real numbers with $a > 4$ and $b > 1$ such that

$$\frac{x^2}{a^2} + \frac{y^2}{a^2-16} = \frac{(x-20)^2}{b^2-1} + \frac{(y-11)^2}{b^2} = 1.$$

Find the least possible value of $a+b$.

2022 AIME II, Problem 13 216. There is a polynomial $P(x)$ with integer coefficients such that

$$P(x) = \frac{(x^{2310}-1)^6}{(x^{105}-1)(x^{70}-1)(x^{42}-1)(x^{30}-1)},$$

holds for every $0 < x < 1$. Find the coefficient of x^{2022} in $P(x)$.

2022 HMMT, Algebra & Number Theory, Problem 1 217. Positive integers a, b, and c are all powers of k for some positive integer k. It is known that the equation $ax^2 - bx + c = 0$ has exactly one real solution r, and this value r is less than 100. Compute the maximum possible value of r.

2022 HMMT, Algebra & Number Theory, Problem 9 218. Suppose $P(x)$ is a monic polynomial of degree 2023 such that

$$P(k) = k^{2023} P\left(1 - \frac{1}{k}\right),$$

for every positive integer $1 \le k \le 2023$. Then $P(-1) = a/b$ where a and b are relatively prime integers. Compute the unique integer $0 \le n < 2027$ such that $bn - a$ is divisible by the prime 2027.

2022 HMMT, Team, Problem 6 219. Let $P(x) = x^4 + ax^3 + bx^2 + x$ be a polynomial with four distinct roots that lie on a circle in the complex plane. Prove that $ab \ne 9$.

2022 HMIC, Problem 1 220. Is

$$\prod_{k=0}^{\infty} \left(1 - \frac{1}{2022^{k!}}\right)$$

rational?

2022 Stanford Math Tournament, Team Round #5 221. Let $a, b,$ and c be the roots of the polynomial $x^3 - 3x^2 - 4x + 5$. Compute $\frac{a^4+b^4}{a+b} + \frac{b^4+c^4}{b+c} + \frac{c^4+a^4}{c+a}$.

2022 Stanford Math Tournament, Algebra #9 222. Let $P(x) = 8x^3 + ax + b + 1$ for $a, b \in \mathbb{Z}$. It is known that P has a root $x_0 = p + \sqrt{q} + \sqrt[3]{r}$, where $p, q, r \in \mathbb{Q}$, $q \ge 0$; however, P has no *rational* roots. Find the smallest possible value of $a+b$.

2022 Stanford Math Tournament, Algebra #10 223. Let $f^1(x) = x^3 - 3x$. Let $f^n(x) = f(f^{n-1}(x))$. Let $\mathcal{R}$ be the set of roots of $\frac{f^{2022}(x)}{x}$. If

$$\sum_{r\in\mathcal{R}} \frac{1}{r^2} = \frac{a^b - c}{d},$$

for positive integers a, b, c, d, where b is as large as possible and c and d are relatively prime, find $a + b + c + d$.

2022 Baltic Way, Problem 3 224. We call a two-variable polynomial $P(x, y)$ secretly one-variable, if there exist polynomials $Q(x)$ and $R(x, y)$ such that $\deg(Q) \geq 2$ and $P(x, y) = Q(R(x, y))$ (e.g., $x^2 + 1$ and $x^2y^2 + 1$ are secretly one-variable, but $xy + 1$ is not). Prove or disprove the following statement: If $P(x, y)$ is a polynomial such that both $P(x, y)$ and $P(x, y) + 1$ can be written as the product of two non-constant polynomials, then P is secretly one-variable.

Note: All polynomials are assumed to have real coefficients.

2022-23 CHMMC Winter, Individual Round, Problem 10 225. Find the number of pairs of positive integers (m, n) such that $n < m \leq 100$ and the polynomial $x^m + x^n + 1$ has a root on the unit circle.

2022-23 CHMMC Winter, Team Round, Problem 8 226. Suppose $a_3x^3 - x^2 + a_1x - 7 = 0$ is a cubic polynomial in x whose roots α, β, γ are positive real numbers satisfying

$$\frac{225\alpha^2}{\alpha^2 + 7} = \frac{144\beta^2}{\beta^2 + 7} = \frac{100\gamma^2}{\gamma^2 + 7}.$$

Find a_1.

2022-23 CHMMC Winter, Team Round, Problem 10 227. Suppose that $\xi \neq 1$ is a root of the polynomial $f(x) = x^{167} - 1$. Compute

$$\left| \sum_{0<a<b<167} \xi^{a^2+b^2} \right|.$$

2023 AIME I, Problem 2 228. If $\sqrt{\log_b n} = \log_b \sqrt{n}$ and $b \log_b n = \log_b bn$, then the value of n is equal to j/k, where j and k are relatively prime. What is $j + k$?

2023 AIME I, Problem 9 229. Find the number of cubic polynomials $p(x) = x^3 + ax^2 + bx + c$, where a, b, and c are integers in

$$\{-20, -19, -18, \ldots, 18, 19, 20\},$$

such that there is a unique integer $m \neq 2$ with $p(m) = p(2)$.

2023 AIME I, Problem 15 230. Find the largest prime number $p < 1000$ for which there exists a complex number z satisfying

a) the real and imaginary part of z are both integers;

b) $|z| = \sqrt{p}$, and

c) there exists a triangle whose three side lengths are p, the real part of z^3, and the imaginary part of z^3.

2023 AIME II, Problem 2 231. Let x, y, and z be real numbers satisfying the system of equations

$$\begin{aligned} xy + 4z &= 60, \\ yz + 4x &= 60, \\ zx + 4y &= 60. \end{aligned}$$

Let S be the set of possible values of x. Find the sum of the squares of the elements of S.

2023 AIME II, Problem 8 232. Let

$$\omega = \cos\frac{2\pi}{7} + i \cdot \sin\frac{2\pi}{7},$$

where $i = \sqrt{-1}$. Find

$$\prod_{k=0}^{6} (\omega^{3k} + \omega^k + 1).$$

2023 AIME II, Problem 13 233. Let A be an acute angle such that $\tan A = 2\cos A$. Find the number of positive integers n less than or equal to 1000 such that $\sec^n A + \tan^n A$ is a positive integer whose units digit is 9.

2023 Stanford Math Tournament, Algebra #10 234. Suppose that $p(x), q(x)$ are monic polynomials with non-negative integer coefficients such that

$$\frac{1}{5x} \geq \frac{1}{q(x)} - \frac{1}{p(x)} \geq \frac{1}{3x^2},$$

for all integers $x \geq 2$. Compute the minimum possible value of $p(1) \cdot q(1)$.

2023 Stanford Math Tournament, Algebra Tiebreaker #1 235. Compute the area of the polygon formed by connecting the roots of

$$x^{10} + x^9 + x^8 + x^6 + x^5 + x^4 + x^2 + x + 1,$$

graphed in the complex plane with line segments in counterclockwise order.

2023 Stanford Math Tournament, Algebra Tiebreaker #2 236. $f(x)$ is a non-constant polynomial. Given that $f(f(x)) + f(x) = f(x)^2$, compute $f(3)$.

2023 Stanford Math Tournament, Algebra Tiebreaker #3 237. Define

$$f(x) = x^3 - 6x^2 + \frac{25}{2}x - 7.$$

There is an interval $[a, b]$ such that for any real number x, the sequence $x, f(x), f(f(x)), \ldots$ is bounded (i.e., has a lower and upper bound) if and only if $x \in [a, b]$. Compute $(a - b)^2$.

2023 Bulgaria National Olympiad, Problem 3 238. Let $f(x)$ be a polynomial with positive integer coefficients. For every $n \in \mathbb{N}$, let $a_1^{(n)}, a_2^{(n)}, \ldots, a_n^{(n)}$ be fixed positive integers that give pairwise different residues modulo n and let

$$g(n) = \sum_{i=1}^{n} f(a_i^{(n)}) = f(a_1^{(n)}) + f(a_2^{(n)}) + \cdots + f(a_n^{(n)}).$$

Prove that there exists a constant M such that for all integers $m > M$ we have

$$\gcd(m, g(m)) > 2023^{2023}.$$

2023 Canada National Olympiad, Problem 4 239. Let $f(x)$ be a non-constant polynomial with integer coefficients such that $f(1) \neq 1$. For a positive integer n, define $\text{divs}(n)$ to be the set of positive divisors of n.

A positive integer m is f-cool if there exists a positive integer n for which

$$f[\text{divs}(m)] = \text{divs}(n).$$

Prove that for any such f, there are finitely many f-cool integers.

(The notation $f[S]$ for some set S denotes the set $\{f(s) : s \in S\}$.)

2023 Romanian District Olympiad, Problem 10.3 240. Let $n \geqslant 2$ be an integer. Determine all complex numbers z which satisfy

$$|z^{n+1} - z^n| \geqslant |z^{n+1} - 1| + |z^{n+1} - z|.$$

2023 Greece National Olympiad, Problem 1 241. Find all quadruplets (x, y, z, w) of positive real numbers that satisfy the following system:

$$\begin{cases} \dfrac{xyz+1}{x+1} = \dfrac{yzw+1}{y+1} & = \dfrac{zwx+1}{z+1} = \dfrac{wxy+1}{w+1}, \\ \\ x+y+z+w & = 48. \end{cases}$$

2023 India National Olympiad, Problem 2 242. Suppose $a_0, \ldots, a_{100}$ are positive reals. Consider the following polynomial for each k in $\{0, 1, \ldots, 100\}$:

$$a_{100+k}x^{100} + 100a_{99+k}x^{99} + a_{98+k}x^{98} + a_{97+k}x^{97} + \cdots + a_{2+k}x^2 + a_{1+k}x + a_k,$$

where indices are taken modulo 101, i.e., $a_{100+i} = a_{i-1}$ for any i in $\{1, 2, \ldots, 100\}$. Show that it is impossible that each of these 101 polynomials has all its roots real.

2023 Romanian Masters in Mathematics, Problem 3 243. Let $n \geq 2$ be an integer and let f be a $4n$-variable polynomial with real coefficients. Assume that, for any $2n$ points $(x_1, y_1), \ldots, (x_{2n}, y_{2n})$ in the Cartesian plane, $f(x_1, y_1, \ldots, x_{2n}, y_{2n}) = 0$ if and only if the points form the vertices of a regular $2n$-gon in some order, or are all equal.

Determine the smallest possible degree of f. Note, for example, that the degree of the polynomial

$$g(x, y) = 4x^3y^4 + yx + x - 2$$

is 7 because $7 = 3 + 4$.

2023 Macedonian Team Selection Test, Problem 5 244. Let $Q(x) = a_{2023}x^{2023} + a_{2022}x^{2022} + \cdots + a_1x + a_0 \in \mathbb{Z}[x]$ be a polynomial with integer coefficients. For an odd prime number p we define the polynomial $Q_p(x) = a_{2023}^{p-2}x^{2023} + a_{2022}^{p-2}x^{2022} + \cdots + a_1^{p-2}x + a_0^{p-2}$. Assume that there exist infinitely primes p such that

$$\frac{Q_p(x) - Q(x)}{p}$$

is an integer for all $x \in \mathbb{Z}$. Determine the largest possible value of $Q(2023)$ over all such polynomials Q.

2023 German TST 4, Problem 3 245. Let $f(x)$ be a monic polynomial of degree 2023 with integer coefficients. Show that for any sufficiently large integer N and any prime number $p > 2023N$, the product

$$f(1)f(2)\dots f(N)$$

is at most $\binom{2023}{2}$ times divisible by p.

2023 Purple Comet, Problem 16 246. The polynomial $P(x) = x^4 - ax^2 + 2023$ has roots

$$\alpha, -\alpha, \alpha\sqrt{\alpha^2 - 10}, -\alpha\sqrt{\alpha^2 - 10},$$

for some positive real number α. Find the value of a.

2023 British Math Olympiad, Problem 3 247. For each positive integer n, denote by $\omega(n)$ the number of distinct prime divisors of n (for example, $\omega(1) = 0$ and $\omega(12) = 2$). Find all polynomials $P(x)$ with integer coefficients, such that whenever n is a positive integer satisfying $\omega(n) > 2023^{2023}$, then $P(n)$ is also a positive integer with

$$\omega(n) \geq \omega(P(n)).$$

2023 Thailand Mock IMO, Problem 4 248. Find all polynomials $P(x)$ with integer coefficients for which there exists an integer M such that $P(n)$ divides $(n + 2023)!$ for all positive integers $n > M$.

2023 Pan African Math Olympiad, Problem 5 249. Let a, b be reals with $a \neq 0$ and let

$$P(x) = ax^4 - 4ax^3 + (5a + b)x^2 - 4bx + b.$$

Show that all roots of $P(x)$ are real and positive if and only if $a = b$.

2 Trigonometric and Exponential Functions; Algebra versus Geometry and Combinatorics

1984 AIME, Problem 13 250. Find the value of

$$10\cot(\cot^{-1}3+\cot^{-1}7+\cot^{-1}13+\cot^{-1}21).$$

1989 AIME, Problem 10 251. Let a, b, c be the three sides of a triangle, and let α, β, γ, be the angles opposite them. If $a^2+b^2=1989c^2$, find

$$\frac{\cot\gamma}{\cot\alpha+\cot\beta}.$$

1991 AIME, Problem 4 252. How many real numbers x satisfy the equation

$$\frac{1}{5}\log_2 x=\sin(5\pi x)?$$

1991 AIME, Problem 9 253. Suppose that

$$\sec x+\tan x=\frac{22}{7},$$

and that

$$\csc x+\cot x=\frac{m}{n},$$

where m/n is in lowest terms. Find $m+n$.

1993 AIME, Problem 14 254. A rectangle that is inscribed in a larger rectangle (with one vertex on each side) is called unstuck if it is possible to rotate (however slightly) the smaller rectangle about its center within the confines of the larger. Of all the rectangles that can be inscribed unstuck in a 6 by 8 rectangle, the smallest perimeter has the form $\sqrt{N}$, for a positive integer N. Find N.

1994 AIME, Problem 4 255. Find the positive integer n for which

$$\lfloor\log_2 1\rfloor+\lfloor\log_2 2\rfloor+\lfloor\log_2 3\rfloor+\cdots+\lfloor\log_2 n\rfloor=1994.$$

(For real x, $\lfloor x\rfloor$ is the greatest integer $\le x$.)

1994 AIME, Problem 7 256. For certain ordered pairs (a,b) of real numbers, the system of equations

$$\begin{aligned}ax+by&=1,\\ x^2+y^2&=50,\end{aligned}$$

has at least one solution, and each solution is an ordered pair (x,y) of integers. How many such ordered pairs (a,b) are there?

1995 AIME, Problem 2 257. Find the last three digits of the product of the positive roots of

$$\sqrt{1995}x^{\log_{1995} x} = x^2.$$

1995 AIME, Problem 7 258. Given that $(1+\sin t)(1+\cos t) = 5/4$ and

$$(1-\sin t)(1-\cos t) = \frac{m}{n} - \sqrt{k},$$

where k, m, and n are positive integers with m and n relatively prime, find $k+m+n$.

1995 AIME, Problem 12 259. Pyramid $OABCD$ has square base $ABCD$, congruent edges $\overline{OA}, \overline{OB}, \overline{OC}$, and $\overline{OD}$, and $\angle AOB = 45^\circ$. Let θ be the measure of the dihedral angle formed by faces OAB and OBC. Given that $\cos\theta = m + \sqrt{n}$, where m and n are integers, find $m+n$.

1996 AIME, Problem 10 260. Find the smallest positive integer solution to

$$\tan 19x^\circ = \frac{\cos 96^\circ + \sin 96^\circ}{\cos 96^\circ - \sin 96^\circ}.$$

1997 AIME, Problem 11 261. Let

$$x = \frac{\sum\limits_{n=1}^{44} \cos n^\circ}{\sum\limits_{n=1}^{44} \sin n^\circ}.$$

What is the greatest integer that does not exceed $100x$?

1997 AIME, Problem 13 262. Let S be the set of points in the Cartesian plane that satisfy

$$\Big|\big||x|-2\big|-1\Big| + \Big|\big||y|-2\big|-1\Big| = 1.$$

If a model of S were built from wire of negligible thickness, then the total length of wire required would be $a\sqrt{b}$, where a and b are positive integers and b is not divisible by the square of any prime number. Find $a+b$.

1998 AIME, Problem 3 263. The graph of $y^2 + 2xy + 40|x| = 400$ partitions the plane into several regions. What is the area of the bounded region?

1998 AIME, Problem 5 264. Given that

$$A_k = \frac{k(k-1)}{2} \cos \frac{k(k-1)\pi}{2},$$

find

$$|A_{19} + A_{20} + \cdots + A_{98}|.$$

1999 AIME, Problem 6 265. A transformation of the first quadrant of the coordinate plane maps each point (x, y) to the point $(\sqrt{x}, \sqrt{y})$. The vertices of quadrilateral $ABCD$ are $A = (900, 300), B = (1800, 600), C = (600, 1800)$, and $D = (300, 900)$. Let k be the area of the region enclosed by the image of quadrilateral $ABCD$. Find the greatest integer that does not exceed k.

1999 AIME, Problem 11 266. Given that

$$\sum_{k=1}^{35} \sin 5k = \tan \frac{m}{n},$$

where angles are measured in degrees, and m and n are relatively prime positive integers that satisfy $m < 90n$, find $m + n$.

2000 AIME II, Problem 15 267. Find the least positive integer n such that

$$\frac{1}{\sin 45^\circ \sin 46^\circ} + \frac{1}{\sin 47^\circ \sin 48^\circ} + \cdots + \frac{1}{\sin 133^\circ \sin 134^\circ} = \frac{1}{\sin n^\circ}.$$

2001 AIME I, Problem 5 268. An equilateral triangle is inscribed in the ellipse whose equation is $x^2 + 4y^2 = 4$. One vertex of the triangle is $(0, 1)$, one altitude is contained in the y-axis, and the length of each side is $\sqrt{m/n}$, where m and n are relatively prime positive integers. Find $m + n$.

2001 AIME I, Problem 9 269. In triangle ABC, $AB = 13$, $BC = 15$ and $CA = 17$. Point D is on $\overline{AB}$, E is on $\overline{BC}$, and F is on $\overline{CA}$. Let $AD = p \cdot AB$, $BE = q \cdot BC$, and $CF = r \cdot CA$, where p, q, and r are positive and satisfy $p + q + r = 2/3$ and $p^2 + q^2 + r^2 = 2/5$. The ratio of the area of triangle DEF to the area of triangle ABC can be written in the form m/n, where m and n are relatively prime positive integers. Find $m + n$.

2001 AIME II, Problem 4 270. Let $R = (8, 6)$. The lines whose equations are $8y = 15x$ and $10y = 3x$ contain points P and Q, respectively, such that R is the midpoint of $\overline{PQ}$. The length of PQ equals m/n, where m and n are relatively prime positive integers. Find $m + n$.

2001 AIME II, Problem 6 271. Square $ABCD$ is inscribed in a circle. Square $EFGH$ has vertices E and F on $\overline{CD}$ and vertices G and H on the circle. The ratio of the area of square $EFGH$ to the area of square $ABCD$ can be expressed as m/n where m and n are relatively prime positive integers and $m < n$. Find $10n + m$.

2001 AIME II, Problem 12 272. Given a triangle, its midpoint triangle is obtained by joining the midpoints of its sides. A sequence of polyhedra P_i is defined recursively as follows: P_0 is a regular tetrahedron whose volume is 1. To obtain P_{i+1}, replace the midpoint triangle of every face of P_i by an outward-pointing regular tetrahedron that has the midpoint triangle as a face. The volume of P_3 is m/n, where m and n are relatively prime positive integers. Find $m + n$.

2002 AIME I, Problem 5 273. Let $A_1, A_2, A_3, \ldots, A_{12}$ be the vertices of a regular dodecagon. How many distinct squares in the plane of the dodecagon have at least two vertices in the set $\{A_1, A_2, A_3, \ldots, A_{12}\}$?

2002 AIME I, Problem 274. Let $ABCD$ and $BCFG$ be two faces of a cube with $AB = 12$. A beam of light emanates from vertex A and reflects off face $BCFG$ at point P, which is 7 units from $\overline{BG}$ and 5 units from $\overline{BC}$. The beam continues to be reflected off the faces of the cube. The length of the light path from the time it leaves point A until it next reaches a vertex of the cube is given by $m\sqrt{n}$, where m and n are integers and n is not divisible by the square of any prime. Find $m+n$.

2002 AIME I, Problem 13 275. In triangle ABC the medians $\overline{AD}$ and $\overline{CE}$ have lengths 18 and 27, respectively, and $AB = 24$. Extend $\overline{CE}$ to intersect the circumcircle of ABC at F. The area of triangle AFB is $m\sqrt{n}$, where m and n are positive integers and n is not divisible by the square of any prime. Find $m+n$.

2002 AIME I, Problem 15 276. Polyhedron $ABCDEFG$ has six faces. Face $ABCD$ is a square with $AB = 12$; face $ABFG$ is a trapezoid with $\overline{AB}$ parallel to $\overline{GF}$, $BF = AG = 8$, and $GF = 6$; and face CDE has $CE = DE = 14$. The other three faces are $ADEG, BCEF$, and EFG. The distance from E to face $ABCD$ is 12. Given that $EG^2 = p - q\sqrt{r}$, where p, q, and r are positive integers and r is not divisible by the square of any prime, find $p+q+r$.

2002 AIME II, Problem 2 277. Three vertices of a cube are $P = (7, 12, 10)$, $Q = (8, 8, 1)$, and $R = (11, 3, 9)$. What is the surface area of the cube?

2002 AIME II, Problem 10 278. While finding the sine of a certain angle, an absent-minded professor failed to notice that his calculator was not in the correct angular mode. He was lucky to get the right answer. The two least positive real values of x for which the sine of x degrees is the same as the sine of x radians are $(m\pi)/(n-\pi)$ and $(p\pi)/(q+\pi)$, where m, n, p and q are positive integers. Find $m+n+p+q$.

2002 AIME II, Problem 15 279. Circles $\mathcal{C}_1$ and $\mathcal{C}_2$ intersect at two points, one of which is $(9, 6)$, and the product of the radii is 68. The x-axis and the line $y = mx$, where $m > 0$, are tangent to both circles. It is given that m can be written in the form $a\sqrt{b}/c$, where a, b, and c are positive integers, b is not divisible by the square of any prime, and a and c are relatively prime. Find $a+b+c$.

2003 AIME I, Problem 6 280. The sum of the areas of all triangles whose vertices are also vertices of a $1\times1\times1$ cube is $m+\sqrt{n}+\sqrt{p}$, where m, n, and p are integers. Find $m+n+p$.

2003 AIME I, Problem 11 281. An angle x is chosen at random from the interval $0^\circ < x < 90^\circ$. Let p be the probability that the numbers $\sin^2 x$, $\cos^2 x$, and $\sin x \cos x$ are not the lengths of the sides of a triangle. Given that $p = d/n$, where d is the number of degrees in $\arctan m$ and m and n are positive integers with $m+n < 1000$, find $m+n$.

2003 AIME II, Problem 4 282. In a regular tetrahedron the centers of the four faces are the vertices of a smaller tetrahedron. The ratio of the volume of the smaller tetrahedron to that of the larger is m/n, where m and n are relatively prime positive integers. Find $m+n$.

2003 AIME II, Problem 11 283. Triangle ABC is a right triangle with $AC = 7$, $BC = 24$, and right angle at C. Point M is the midpoint of AB, and D is on the same side of line AB as C so that $AD = BD = 15$. Given that the area of triangle CDM may be expressed as $(m\sqrt{n})/p$, where m, n, and p are positive integers, m and p are relatively prime, and n is not divisible by the square of any prime, find $m+n+p$.

2003 AIME II, Problem 13 284. A bug starts at a vertex of an equilateral triangle. On each move, it randomly selects one of the two vertices where it is not currently located, and crawls along a side of the triangle to that vertex. Given that the probability that the bug moves to its starting vertex on its tenth move is m/n, where m and n are relatively prime positive integers, find $m+n$.

2004 AIME I, Problem 11 285. A solid in the shape of a right circular cone is 4 inches tall and its base has a 3-inch radius. The entire surface of the cone, including its base, is painted. A plane parallel to the base of the cone divides the cone into two solids, a smaller cone-shaped solid C and a frustum-shaped solid F, in such a way that the ratio between the areas of the painted surfaces of C and F and the ratio between the volumes of C and F are both equal to k. Given that $k = m/n$, where m and n are relatively prime positive integers, find $m + n$.

2004 AIME I, Problem 12 286. Let S be the set of ordered pairs (x, y) such that $0 < x \leq 1$, $0 < y \leq 1$, and

$$\left[\log_2\left(\frac{1}{x}\right)\right] \qquad \text{and} \qquad \left[\log_5\left(\frac{1}{y}\right)\right]$$

are both even. Given that the area of the graph of S is m/n, where m and n are relatively prime positive integers, find $m+n$. The notation $[z]$ denotes the greatest integer that is less than or equal to z.

2004 AIME II, Problem 1 287. A chord of a circle is perpendicular to a radius at the midpoint of the radius. The ratio of the area of the larger of the two regions into which the chord divides the circle to the smaller can be expressed in the form $(a\pi + b\sqrt{c})/(d\pi - e\sqrt{f})$, where a, b, c, d, e, and f are positive integers, a and e are relatively prime, and neither c nor f is divisible by the square of any prime. Find the remainder when the product $abcdef$ is divided by 1000.

2004 AIME II, Problem 12 288. Let $ABCD$ be an isosceles trapezoid, whose dimensions are $AB = 6$, $BC = 5 = DA$, and $CD = 4$. Draw circles of radius 3 centered at A and B, and circles of radius 2 centered at C and D. A circle contained within the trapezoid is tangent to all four of these circles. Its radius is $(-k + m\sqrt{n})/p$, where k, m, n, and p are positive integers, n is not divisible by the square of any prime, and k and p are relatively prime. Find $k + m + n + p$.

2005 AIME I, Problem 7 289. In quadrilateral $ABCD$, $BC = 8$, $CD = 12$, $AD = 10$, and $m\angle A = m\angle B = 60^\circ$. Given that $AB = p + \sqrt{q}$, where p and q are positive integers, find $p + q$.

2005 AIME I, Problem 10 290. Triangle ABC lies in the Cartesian Plane and has an area of 70. The coordinates of B and C are $(12, 19)$ and $(23, 20)$, respectively, and the coordinates of A are (p, q). The line containing the median to side BC has slope -5. Find the largest possible value of $p + q$.

2005 AIME I, Problem 11 291. A semicircle with diameter d is contained in a square whose sides have length 8. Given the maximum value of d is $m - \sqrt{n}$, find $m + n$.

2005 AIME I, Problem 14 292. Consider the points $A(0, 12)$, $B(10, 9)$, $C(8, 0)$, and $D(-4, 7)$. There is a unique square S such that each of the four points is on a different side of S. Let K be the area of S. Find the remainder when $10K$ is divided by 1000.

2005 AIME I, Problem 15 293. Triangle ABC has $BC = 20$. The incircle of the triangle evenly trisects the median AD. If the area of the triangle is $m\sqrt{n}$ where m and n are integers and n is not divisible by the square of a prime, find $m + n$.

2005 AIME II, Problem 8 294. Circles C_1 and C_2 are externally tangent, and they are both internally tangent to circle C_3. The radii of C_1 and C_2 are 4 and 10, respectively, and the centers of the three circles are all collinear. A chord of C_3 is also a common external tangent of C_1 and C_2. Given that the length of the chord is $(m\sqrt{n})/p$ where $m, n,$ and p are positive integers, m and p are relatively prime, and n is not divisible by the square of any prime, find $m + n + p$.

2005 AIME II, Problem 10 295. Given that O is a regular octahedron, that C is the cube whose vertices are the centers of the faces of O, and that the ratio of the volume of O to that of C is m/n, where m and n are relatively prime integers, find $m+n$.

2005 AIME II, Problem 14 296. In triangle ABC, $AB = 13$, $BC = 15$, and $CA = 14$. Point D is on $\overline{BC}$ with $CD = 6$. Point E is on $\overline{BC}$ such that $\angle BAE \cong \angle CAD$. Given that $BE = p/q$ where p and q are relatively prime positive integers, find q.

2005 AIME II, Problem 15 297. Let w_1 and w_2 denote the circles $x^2+y^2+10x-24y-87 = 0$ and $x^2+y^2-10x-24y+153 = 0$, respectively. Let m be the smallest positive value of a for which the line $y = ax$ contains the center of a circle that is externally tangent to w_2 and internally tangent to w_1. Given that $m^2 = p/q$, where p and q are relatively prime integers, find $p+q$.

2006 AIME I, Problem 12 298. Find the sum of the values of x such that

$$\cos^3 3x + \cos^3 5x = 8\cos^3 4x \cos^3 x,$$

where x is measured in degrees and $100 < x < 200$.

2006 AIME I, Problem 14 299. A tripod has three legs each of length 5 feet. When the tripod is set up, the angle between any pair of legs is equal to the angle between any other pair, and the top of the tripod is 4 feet from the ground. In setting up the tripod, the lower 1 foot of one leg breaks off. Let h be the height in feet of the top of the tripod from the ground when the broken tripod is set up. Then h can be written in the form $m/\sqrt{n}$, where m and n are positive integers and n is not divisible by the square of any prime. Find $\lfloor m+\sqrt{n}\rfloor$. (The notation $\lfloor x\rfloor$ denotes the greatest integer that is less than or equal to x.)

2006 AIME II, Problem 12 300. Equilateral $\triangle ABC$ is inscribed in a circle of radius 2. Extend $\overline{AB}$ through B to point D so that $AD = 13$, and extend $\overline{AC}$ through C to point E so that $AE = 11$. Through D, draw a line l_1 parallel to $\overline{AE}$, and through E, draw a line l_2 parallel to $\overline{AD}$. Let F be the intersection of l_1 and l_2. Let G be the point on the circle that is collinear with A and F and distinct from A. Given that the area of $\triangle CBG$ can be expressed in the form $(p\sqrt{q})/r$, where p, q, and r are positive integers, p and r are relatively prime, and q is not divisible by the square of any prime, find $p+q+r$.

2007 AIME II, Problem 11 301. Two long cylindrical tubes of the same length but different diameters lie parallel to each other on a flat surface. The larger tube has radius 72 and rolls along the surface toward the smaller tube, which has radius 24. It rolls over the smaller tube and continues rolling along the flat surface until it comes to rest on the same point of its circumference as it started, having made one complete revolution. If the smaller tube never moves, and the rolling occurs with no slipping, the larger tube ends up a distance x from where it starts. The distance x can be expressed in the form $a\pi + b\sqrt{c}$, where a, b, and c are integers and c is not divisible by the square of any prime. Find $a+b+c$.

2007 AIME II, Problem 15 302. Four circles ω, ω_A, ω_B, and ω_C with the same radius are drawn in the interior of triangle ABC such that ω_A is tangent to sides AB and AC, ω_B to BC and BA, ω_C to CA and CB, and ω is externally tangent to ω_A, ω_B, and ω_C. If the sides of triangle ABC are 13, 14, and 15, the radius of ω can be represented in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

2008 AIME I, Problem 8 303. Find the positive integer n such that

$$\arctan\frac{1}{3}+\arctan\frac{1}{4}+\arctan\frac{1}{5}+\arctan\frac{1}{n}=\frac{\pi}{4}.$$

2008 AIME I, Problem 14 304. Let $\overline{AB}$ be a diameter of circle ω. Extend $\overline{AB}$ through A to C. Point T lies on ω so that line CT is tangent to ω. Point P is the foot of the perpendicular from A to line CT. Suppose $AB=18$, and let m denote the maximum possible length of segment BP. Find m^2.

2008 AIME I, Problem 15 305. A square piece of paper has sides of length 100. From each corner a wedge is cut in the following manner: at each corner, the two cuts for the wedge each start at distance $\sqrt{17}$ from the corner, and they meet on the diagonal at an angle of 60° (see the figure below). The paper is then folded up along the lines joining the vertices of adjacent cuts. When the two edges of a cut meet, they are taped together. The result is a paper tray whose sides are not at right angles to the base. The height of the tray, that is, the perpendicular distance between the plane of the base and the plane formed by the upper edges, can be written in the form $\sqrt[n]{m}$, where m and n are positive integers, $m<1000$, and m is not divisible by the nth power of any prime. Find $m+n$.

2008 AIME II, Problem 8 306. Let $a=\pi/2008$. Find the smallest positive integer n such that

$$2[\cos(a)\sin(a)+\cos(4a)\sin(2a)+\cos(9a)\sin(3a)+\cdots+\cos(n^2a)\sin(na)],$$

is an integer.

2008 AIME II, Problem 9 307. A particle is located on the coordinate plane at $(5,0)$. Define a move for the particle as a counterclockwise rotation of $\pi/4$ radians about the origin followed by a translation of 10 units in the positive x-direction. Given that the particle's position after 150 moves is (p,q), find the greatest integer less than or equal to $|p|+|q|$.

2008 AIME II, Problem 13 308. A regular hexagon with center at the origin in the complex plane has opposite pairs of sides one unit apart. One pair of sides is parallel to the imaginary axis. Let R be the region outside the hexagon, and let

$$S=\left\{\frac{1}{z}\mid z\in R\right\}.$$

Then the area of S has the form $a\pi+\sqrt{b}$, where a and b are positive integers. Find $a+b$.

2009 AIME I, Problem 11 309. Consider the set of all triangles OPQ where O is the origin and P and Q are distinct points in the plane with nonnegative integer coordinates (x,y) such that $41x+y=2009$. Find the number of such distinct triangles whose area is a positive integer.

2009 AIME I, Problem 12 310. In right $\triangle ABC$ with hypotenuse $\overline{AB}$, $AC=12$, $BC=35$, and $\overline{CD}$ is the altitude to $\overline{AB}$. Let ω be the circle having $\overline{CD}$ as a diameter. Let I be a point outside $\triangle ABC$ such that $\overline{AI}$ and $\overline{BI}$ are both tangent to circle ω. The ratio of the perimeter of $\triangle ABI$ to the length AB can be expressed in the form m/n, where m and n are relatively prime positive integers. Find $m+n$.

2009 AIME II, Problem 10 311. Four lighthouses are located at points A, B, C, and D. The lighthouse at A is 5 kilometers from the lighthouse at B, the lighthouse at B is 12 kilometers from the lighthouse at C, and the lighthouse at A is 13 kilometers from the lighthouse at C. To an observer at A, the angle determined by the lights at B and D and the angle determined by the lights at C and D are equal. To an observer at C, the angle determined by the lights at A and B and the angle determined by the lights at D and B are equal. The number of kilometers from A to D is given by $p\sqrt{r}/q$, where p, q, and r are relatively prime positive integers, and r is not divisible by the square of any prime. Find $p+q+r$.

2009 AIME II, Problem 13 312. Let A and B be the endpoints of a semicircular arc of radius 2. The arc is divided into seven congruent arcs by six equally spaced points $C_1, C_2, \dots, C_6$. All chords of the form $\overline{AC_i}$ or $\overline{BC_i}$ are drawn. Let n be the product of the lengths of these twelve chords. Find the remainder when n is divided by 1000.

2009 AIME II, Problem 15 313. Let $\overline{MN}$ be a diameter of a circle with diameter 1. Let A and B be points on one of the semicircular arcs determined by $\overline{MN}$ such that A is the midpoint of the semicircle and $MB = 3/5$. Point C lies on the other semicircular arc. Let d be the length of the line segment whose endpoints are the intersections of diameter $\overline{MN}$ with the chords $\overline{AC}$ and $\overline{BC}$. The largest possible value of d can be written in the form $r - s\sqrt{t}$, where r, s, and t are positive integers and t is not divisible by the square of any prime. Find $r+s+t$.

2010 AIME I, Problem 13 314. Rectangle $ABCD$ and a semicircle with diameter AB are coplanar and have nonoverlapping interiors. Let $\mathcal{R}$ denote the region enclosed by the semicircle and the rectangle. Line ℓ meets the semicircle, segment AB, and segment CD at distinct points N, U, and T, respectively. Line ℓ divides region $\mathcal{R}$ into two regions with areas in the ratio $1:2$. Suppose that $AU = 84$, $AN = 126$, and $UB = 168$. Then DA can be represented as $m\sqrt{n}$, where m and n are positive integers and n is not divisible by the square of any prime. Find $m+n$.

2010 AIME I, Problem 15 315. In $\triangle ABC$ with $AB = 12$, $BC = 13$, and $AC = 15$, let M be a point on $\overline{AC}$ such that the incircles of $\triangle ABM$ and $\triangle BCM$ have equal radii. Let p and q be positive relatively prime integers such that

$$\frac{AM}{CM} = \frac{p}{q}.$$

Find $p+q$.

2010 AIME II, Problem 12 316. Two noncongruent integer-sided isosceles triangles have the same perimeter and the same area. The ratio of the lengths of the bases of the two triangles is $8:7$. Find the minimum possible value of their common perimeter.

2010 AIME II, Problem 14 317. In right triangle ABC with right angle at C, we know that $\angle BAC < 45^\circ$ and $AB = 4$. Point P on AB is chosen such that $\angle APC = 2\angle ACP$ and $CP = 1$. The ratio AP/BP can be represented in the form $p + q\sqrt{r}$, where p, q, r are positive integers and r is not divisible by the square of any prime. Find $p+q+r$.

2010 AIME II, Problem 15 318. In triangle ABC, $AC = 13, BC = 14$, and $AB = 15$. Points M and D lie on AC with $AM = MC$ and $\angle ABD = \angle DBC$. Points N and E lie on AB with $AN = NB$ and $\angle ACE = \angle ECB$. Let P be the point, other than A, of intersection of the circumcircles of $\triangle AMN$ and $\triangle ADE$. Ray AP meets BC at Q. The ratio BQ/CQ can be written in the form m/n, where m and n are relatively prime positive integers. Find $m-n$.

2011 AIME I, Problem 2 319. In rectangle $ABCD$, $AB = 12$ and $BC = 10$. Points E and F lie inside rectangle $ABCD$ so that $BE = 9$, $DF = 8$, $\overline{BE} \parallel \overline{DF}$, $\overline{EF} \parallel \overline{AB}$, and line BE intersects segment $\overline{AD}$. The length EF can be expressed in the form $m\sqrt{n} - p$, where m, n, and p are positive integers and n is not divisible by the square of any prime. Find $m + n + p$.

2011 AIME I, Problem 3 320. Let L be the line with slope $5/12$ that contains the point $A = (24, -1)$, and let M be the line perpendicular to line L that contains the point $B = (5, 6)$. The original coordinate axes are erased, and line L is made the x-axis, and line M the y-axis. In the new coordinate system, point A is on the positive x-axis, and point B is on the positive y-axis. The point P with coordinates $(-14, 27)$ in the original system has coordinates (α, β) in the new coordinate system. Find $\alpha + \beta$.

2011 AIME I, Problem 4 321. In triangle ABC, $AB = 125, AC = 117$, and $BC = 120$. The angle bisector of angle A intersects $\overline{BC}$ at point L, and the angle bisector of angle B intersects $\overline{AC}$ at point K. Let M and N be the feet of the perpendiculars from C to $\overline{BK}$ and $\overline{AL}$, respectively. Find MN.

2011 AIME I, Problem 13 322. A cube with side length 10 is suspended above a plane. The vertex closest to the plane is labelled A. The three vertices adjacent to vertex A are at heights 10, 11, and 12 above the plane. The distance from vertex A to the plane can be expressed as $(r - \sqrt{s})/t$, where r, s, and t are positive integers, and $r + s + t < 1000$. Find $r + s + t$.

2011 AIME I, Problem 14 323. Let $A_1A_2A_3A_4A_5A_6A_7A_8$ be a regular octagon. Let M_1, M_3, M_5, and M_7 be the midpoints of sides $\overline{A_1A_2}$, $\overline{A_3A_4}$, $\overline{A_5A_6}$, and $\overline{A_7A_8}$, respectively. For $i = 1, 3, 5, 7$, ray R_i is constructed from M_i towards the interior of the octagon such that $R_1 \perp R_3$, $R_3 \perp R_5$, $R_5 \perp R_7$, and $R_7 \perp R_1$. Pairs of rays R_1 and R_3, R_3 and R_5, R_5 and R_7, and R_7 and R_1 meet at B_1, B_3, B_5, B_7 respectively. If $B_1B_3 = A_1A_2$, then $\cos 2\angle A_3M_3B_1$ can be written in the form $m - \sqrt{n}$, where m and n are positive integers. Find $m + n$.

2011 AIME II, Problem 4 324. In triangle ABC, $11AB = 20AC$. The angle bisector of $\angle A$ intersects BC at point D, and point M is the midpoint of AD. Let P be the point of the intersection of AC and BM. The ratio of CP to PA can be expressed in the form m/n, where m and n are relatively prime positive integers. Find $m + n$.

2011 AIME II, Problem 10 325. A circle with center O has radius 25. Chord $\overline{AB}$ of length 30 and chord $\overline{CD}$ of length 14 intersect at point P. The distance between the midpoints of the two chords is 12. The quantity OP^2 can be represented as m/n, where m and n are relatively prime positive integers. Find the remainder where $m + n$ is divided by 1000.

2011 AIME II, Problem 13 326. Point P lies on the diagonal AC of square $ABCD$ with $AP > CP$. Let O_1 and O_2 be the circumcenters of triangles ABP and CDP respectively. Given that $AB = 12$ and $\angle O_1PO_2 = 120°$, then $AP = \sqrt{a} + \sqrt{b}$ where a and b are positive integers. Find $a + b$.

2011 AIME II, Problem 15 327. Let $P(x) = x^2 - 3x - 9$. A real number x is chosen at random from the interval $5 \le x \le 15$. The probability that

$$\lfloor \sqrt{P(x)} \rfloor = \sqrt{P(\lfloor x \rfloor)},$$

is equal to

$$\frac{\sqrt{a} + \sqrt{b} + \sqrt{c} - d}{e},$$

where a, b, c, d and e are positive integers and none of a, b, or c is divisible by the square of a prime. Find $a+b+c+d+e$.

2011 HMMT, Algebra, Problem 8 328. Let $z = \cos\frac{2\pi}{2011} + i\sin\frac{2\pi}{2011}$, and let

$$P(x) = x^{2008} + 3x^{2007} + 6x^{2006} + \cdots + \frac{2008 \cdot 2009}{2}x + \frac{2009 \cdot 2010}{2}$$

for all complex numbers x. Evaluate $P(z)P(z^2)P(z^3)\cdots P(z^{2010})$.

2012 AIME I, Problem 13 329. Three concentric circles have radii 3, 4, and 5. An equilateral triangle with one vertex on each circle has side length s. The largest possible area of the triangle can be written as

$$a + \frac{b}{c}\sqrt{d},$$

where a, b, c and d are positive integers, b and c are relatively prime, and d is not divisible by the square of any prime. Find $a+b+c+d$.

2012 AIME II, Problem 9 330. Let x and y be real numbers such that

$$\frac{\sin x}{\sin y} = 3 \qquad \text{and} \qquad \frac{\cos x}{\cos y} = \frac{1}{2}.$$

The value of

$$\frac{\sin 2x}{\sin 2y} + \frac{\cos 2x}{\cos 2y}$$

can be expressed in the form p/q, where p and q are relatively prime positive integers. Find $p+q$.

2012 AIME II, Problem 13 331. Equilateral $\triangle ABC$ has side length $\sqrt{111}$. There are four distinct triangles AD_1E_1, AD_1E_2, AD_2E_3, and AD_2E_4, each congruent to $\triangle ABC$, with $BD_1 = BD_2 = \sqrt{11}$. Find

$$\sum_{k=1}^{4}(CE_k)^2.$$

2012 AIME II, Problem 15 332. Triangle ABC is inscribed in circle ω with $AB = 5$, $BC = 7$, and $AC = 3$. The bisector of angle A meets side BC at D and circle ω at a second point E. Let γ be the circle with diameter DE. Circles ω and γ meet at E and a second point F. Then $AF^2 = m/n$, where m and n are relatively prime positive integers. Find $m+n$.

2012 HMMT, Algebra, Problem 4 333. During the weekends, Eli delivers milk in the complex plane. On Saturday, he begins at z and delivers milk to houses located at $z^3, z^5, z^7, \ldots, z^{2013}$ in that order; on Sunday, he begins at 1 and delivers milk to houses located at $z^2, z^4, z^6, \ldots, z^{2012}$ in that order. Eli always walks directly (in a straight line) between two houses. If the distance he must travel from his starting point to the last house is $\sqrt{2012}$ on both days, find the real part of z^2.

2013 AIME I, Problem 3 334. Let $ABCD$ be a square, and let E and F be points on $\overline{AB}$ and $\overline{BC}$, respectively. The line through E parallel to $\overline{BC}$ and the line through F parallel to $\overline{AB}$ divide $ABCD$ into two squares and two non square rectangles. The sum of the areas of the two squares is 9/10 of the area of square $ABCD$. Find

$$\frac{AE}{EB} + \frac{EB}{AE}.$$

2013 AIME I, Problem 7 335. A rectangular box has width 12 inches, length 16 inches, and height m/n inches, where m and n are relatively prime positive integers. Three faces of the box meet at a corner of the box. The center points of those three faces are the vertices of a triangle with an area of 30 square inches. Find $m + n$.

2013 AIME I, Problem 8 336. The domain of the function $f(x) = \arcsin(\log_m(nx))$ is a closed interval of length $1/2013$, where m and n are positive integers and $m > 1$. Find the remainder when the smallest possible sum $m + n$ is divided by 1000.

2013 AIME I, Problem 9 337. A paper equilateral triangle ABC has side length 12. The paper triangle is folded so that vertex A touches a point on side $\overline{BC}$ a distance 9 from point B. The length of the line segment along which the triangle is folded can be written as $(m\sqrt{p})/n$, where m, n, and p are positive integers, m and n are relatively prime, and p is not divisible by the square of any prime. Find $m + n + p$.

2013 AIME I, Problem 13 338. Triangle AB_0C_0 has side lengths $AB_0 = 12$, $B_0C_0 = 17$, and $C_0A = 25$. For each positive integer n, points B_n and C_n are located on $\overline{AB_{n-1}}$ and $\overline{AC_{n-1}}$, respectively, creating three similar triangles $\triangle AB_nC_n \sim \triangle B_{n-1}C_nC_{n-1} \sim \triangle AB_{n-1}C_{n-1}$. The area of the union of all triangles $B_{n-1}C_nB_n$ for $n \geq 1$ can be expressed as $\frac{p}{q}$, where p and q are relatively prime positive integers. Find q.

2013 AIME I, Problem 14 339. For $\pi \leq \theta < 2\pi$, let

$$P = \frac{1}{2}\cos\theta - \frac{1}{4}\sin 2\theta - \frac{1}{8}\cos 3\theta + \frac{1}{16}\sin 4\theta + \frac{1}{32}\cos 5\theta - \frac{1}{64}\sin 6\theta - \frac{1}{128}\cos 7\theta + \cdots,$$

and

$$Q = 1 - \frac{1}{2}\sin\theta - \frac{1}{4}\cos 2\theta + \frac{1}{8}\sin 3\theta + \frac{1}{16}\cos 4\theta - \frac{1}{32}\sin 5\theta - \frac{1}{64}\cos 6\theta + \frac{1}{128}\sin 7\theta + \cdots,$$

so that $P/Q = (2\sqrt{2})/7$. Then, $\sin\theta = -m/n$ where m and n are relatively prime positive integers. Find $m + n$.

2013 AIME II, Problem 4 340. In the Cartesian plane let $A = (1, 0)$ and $B = \left(2, 2\sqrt{3}\right)$. Equilateral triangle ABC is constructed so that C lies in the first quadrant. Let $P = (x, y)$ be the center of $\triangle ABC$. Then $x \cdot y$ can be written as $(p\sqrt{q})/r$, where p and r are relatively prime positive integers and q is an integer that is not divisible by the square of any prime. Find $p + q + r$.

2013 AIME II, Problem 5 341. In equilateral $\triangle ABC$ let points D and E trisect $\overline{BC}$. Then $\sin(\angle DAE)$ can be expressed in the form $(a\sqrt{b})/c$, where a and c are relatively prime positive integers, and b is an integer that is not divisible by the square of any prime. Find $a + b + c$.

2013 AIME II, Problem 13 342. In $\triangle ABC$, $AC = BC$, and point D is on $\overline{BC}$ so that $CD = 3 \cdot BD$. Let E be the midpoint of $\overline{AD}$. Given that $CE = \sqrt{7}$ and $BE = 3$, the area of $\triangle ABC$ can be expressed in the form $m\sqrt{n}$, where m and n are positive integers and n is not divisible by the square of any prime. Find $m + n$.

2013 AIME II, Problem 15 343. Let A, B, C be angles of an acute triangle with

$$\cos^2 A + \cos^2 B + 2\sin A\sin B\cos C = \frac{15}{8}, \text{ and}$$
$$\cos^2 B + \cos^2 C + 2\sin B\sin C\cos A = \frac{14}{9}.$$

There are positive integers p, q, r, and s for which

$$\cos^2 C + \cos^2 A + 2\sin C \sin A \cos B = \frac{p - q\sqrt{r}}{s},$$

where $p+q$ and s are relatively prime and r is not divisible by the square of any prime. Find $p+q+r+s$. *Note*: due to an oversight by the exam-setters, there is no acute triangle satisfying these conditions. You should instead assume ABC is obtuse with $\angle B > 90°$.

2013 HMIC, Problem 5 344. This problem has two parts:

a) Given a set X of points in the plane, let $f_X(n)$ be the largest possible area of a polygon with at most n vertices, all of which are points of X. Prove that if m, n are integers with $m \geq n > 2$ then $f_X(m) + f_X(n) \geq f_X(m+1) + f_X(n-1)$.

b) Let P_0 be a 1×2 rectangle (including its interior) and inductively define the polygon P_i to be the result of folding P_{i-1} over some line that cuts P_{i-1} into two connected parts. The diameter of a polygon P_i is the maximum distance between two points of P_i. Determine the smallest possible diameter of P_{2013}.

2013 HMMT, Guts, Problem 34 345. For how many unordered sets $\{a, b, c, d\}$ of positive integers, none of which exceed 168, do there exist integers w, x, y, z such that $(-1)^w a + (-1)^x b + (-1)^y c + (-1)^z d = 168$? If your answer is A and the correct answer is C, then your score (out of 25) on this problem will be

$$\left\lfloor 25e^{-3\frac{|C-A|}{C}} \right\rfloor.$$

2014 AIME I, Problem 10 346. A disk with radius 1 is externally tangent to a disk with radius 5. Let A be the point where the disks are tangent, C be the center of the smaller disk, and E be the center of the larger disk. While the larger disk remains fixed, the smaller disk is allowed to roll along the outside of the larger disk until the smaller disk has turned through an angle of $360°$. That is, if the center of the smaller disk has moved to the point D, and the point on the smaller disk that began at A has now moved to point B, then $\overline{AC}$ is parallel to $\overline{BD}$. Then $\sin^2(\angle BEA) = m/n$, where m and n are relatively prime positive integers. Find $m+n$.

2014 AIME II, Problem 11 347. In $\triangle RED, RD = 1, \angle DRE = 75°$ and $\angle RED = 45°$. Let M be the midpoint of segment $\overline{RD}$. Point C lies on side $\overline{ED}$ such that $\overline{RC} \perp \overline{EM}$. Extend segment $\overline{DE}$ through E to point A such that $CA = AR$. Then $AE = (a - \sqrt{b})/c$, where a and c are relatively prime positive integers, and b is a positive integer. Find $a+b+c$.

2014 AIME II, Problem 12 348. Suppose that the angles of $\triangle ABC$ satisfy $\cos(3A)+\cos(3B)+\cos(3C) = 1$. Two sides of the triangle have lengths 10 and 13. There is a positive integer m so that the maximum possible length for the remaining side of $\triangle ABC$ is $\sqrt{m}$. Find m.

2014 AIME II, Problem 14 349. In $\triangle ABC$, $AB = 10$, $\angle A = 30°$, and $\angle C = 45°$. Let H, D, and M be points on line $\overline{BC}$ such that $\overline{AH} \perp \overline{BC}$, $\angle BAD = \angle CAD$, and $BM = CM$. Point N is the midpoint of segment $\overline{HM}$, and point P is on ray AD such that $\overline{PN} \perp \overline{BC}$. Then $AP^2 = m/n$, where m and n are relatively prime positive integers. Find $m+n$.

2014 HMMT, Guts, Problem 31 350. Compute

$$\sum_{k=1}^{1007}\left(\cos\left(\frac{\pi k}{1007}\right)\right)^{2014}.$$

2014 HMMT, Team, Problem 10 351. Fix a positive real number $c > 1$ and positive integer n. Initially, a blackboard contains the numbers $1, c, \ldots, c^{n-1}$. Every minute, Bob chooses two numbers a, b on the board and replaces them with $ca + c^2b$. Prove that after $n-1$ minutes, the blackboard contains a single number no less than

$$\left(\frac{c^{n/L}-1}{c^{1/L}-1}\right)^L,$$

where $\phi = \frac{1+\sqrt{5}}{2}$ and $L = 1 + \log_\phi(c)$.

2015 AIME I, Problem 11 352. Triangle ABC has positive integer side lengths with $AB = AC$. Let I be the intersection of the bisectors of $\angle B$ and $\angle C$. Suppose $BI = 8$. Find the smallest possible perimeter of $\triangle ABC$.

2015 AIME I, Problem 13 353. With all angles measured in degrees, the product

$$\prod_{k=1}^{45}\csc^2(2k-1)^\circ = m^n,$$

where m and n are integers greater than 1. Find $m + n$.

2015 AIME II, Problem 4 354. In an isosceles trapezoid, the parallel bases have lengths $\log 3$ and $\log 192$, and the altitude to these bases has length $\log 16$. The perimeter of the trapezoid can be written in the form $\log 2^p 3^q$, where p and q are positive integers. Find $p + q$.

2015 AIME II, Problem 7 355. Triangle ABC has side lengths $AB = 12$, $BC = 25$, and $CA = 17$. Rectangle $PQRS$ has vertex P on $\overline{AB}$, vertex Q on $\overline{AC}$, and vertices R and S on $\overline{BC}$. In terms of the side length $PQ = w$, the area of $PQRS$ can be expressed as the quadratic polynomial

$$\text{Area}(PQRS) = \alpha w - \beta \cdot w^2$$

Then the coefficient $\beta = m/n$, where m and n are relatively prime positive integers. Find $m + n$.

2015 AIME II, Problem 13 356. Define the sequence $a_1, a_2, a_3, \ldots$ by

$$a_n = \sum_{k=1}^{n}\sin(k),$$

where k represents radian measure. Find the index of the 100th term for which $a_n < 0$.

2015 HMMT, Geometry, Problem 10 357. Let $\mathcal{G}$ be the set of all points (x, y) in the Cartesian plane such that $0 \le y \le 8$ and

$$(x-3)^2 + 31 = (y-4)^2 + 8\sqrt{y(8-y)}.$$

There exists a unique line ℓ of negative slope tangent to $\mathcal{G}$ and passing through the point $(0, 4)$. Suppose ℓ is tangent to $\mathcal{G}$ at a unique point P. Find the coordinates (α, β) of P.

2015 HMMT, Guts, Problem 4 358. Consider the function $z(x, y)$ describing the paraboloid

$$z = (2x - y)^2 - 2y^2 - 3y.$$

Archimedes and Brahmagupta are playing a game. Archimedes first chooses x. Afterwards, Brahmagupta chooses y. Archimedes wishes to minimize z while Brahmagupta wishes to maximize z. Assuming that Brahmagupta will play optimally, what value of x should Archimedes choose?

2015 HMMT, Guts, Problem 20 359. What is the largest real number θ less than π such that

$$\prod_{k=0}^{10} \cos\left(2^k\theta\right),$$

and

$$\prod_{k=0}^{10} \left(1 + \frac{1}{\cos(2^k\theta)}\right)?$$

2015 HMMT, Guts, Problem 28 360. Let w, x, y, and z be positive real numbers such that

$$\begin{aligned} 0 &\neq \cos w \cos x \cos y \cos z, \\ 2\pi &= w + x + y + z, \\ 3\tan w &= k(1 + \sec w), \\ 4\tan x &= k(1 + \sec x), \\ 5\tan y &= k(1 + \sec y), \\ 6\tan z &= k(1 + \sec z). \end{aligned}$$

Here, $\sec t$ denotes $1/\cos t$ when $\cos t \neq 0$. Find k.

2016 AIME I, Problem 9 361. Triangle ABC has $AB = 40$, $AC = 31$, and $\sin A = 1/5$. This triangle is inscribed in rectangle $AQRS$ with B on $\overline{QR}$ and C on $\overline{RS}$. Find the maximum possible area of $AQRS$.

2016 AIME II, Problem 5 362. Triangle ABC_0 has a right angle at C_0. Its side lengths are pairwise relatively prime positive integers, and its perimeter is p. Let C_1 be the foot of the altitude to $\overline{AB}$, and for $n \geq 2$, let C_n be the foot of the altitude to $\overline{C_{n-2}B}$ in $\triangle C_{n-2}C_{n-1}B$. Given the sum

$$\sum_{n=1}^{\infty} C_{n-1}C_n = 6p,$$

find p.

2016 HMMT, Guts, Problem 15 363. Compute

$$\tan\left(\frac{\pi}{7}\right)\tan\left(\frac{2\pi}{7}\right)\tan\left(\frac{3\pi}{7}\right).$$

2016 HMMT, November Theme, Problem 7 364. Seven lattice points form a convex heptagon with all sides having distinct lengths. Find the minimum possible value of the sum of the squares of the sides of the heptagon.

2017 AIME I, Problem 4 365. A pyramid has a triangular base with side lengths 20, 20, and 24. The three edges of the pyramid from the three corners of the base to the fourth vertex of the pyramid all have length 25. The volume of the pyramid is $m\sqrt{n}$, where m and n are positive integers, and n is not divisible by the square of any prime. Find $m+n$.

2017 AIME II, Problem 13 366. For each integer $n \geq 3$, let $f(n)$ be the number of 3-element subsets of the vertices of a regular n-gon that are the vertices of an isosceles triangle (including equilateral triangles). Find the sum of all values of n such that

$$f(n+1) = f(n) + 78.$$

2017 AIME II, Problem 15 367. Tetrahedron $ABCD$ has $AD = BC = 28$, $AC = BD = 44$, and $AB = CD = 52$. For any point X in space, define

$$f(X) = AX + BX + CX + DX.$$

The least possible value of $f(X)$ can be expressed as $m\sqrt{n}$, where m and n are positive integers, and n is not divisible by the square of any prime. Find $m+n$.

2017 HMMT, November Team, Problem 10 368. Denote $\phi = \frac{\sqrt{5}+1}{2}$ and consider the set of all finite binary strings without leading zeroes. Each string S has a "base-ϕ" value $p(S)$. For example, $p(1101) = \phi^3 + \phi^2 + 1$. For any positive integer n, let $f(n)$ be the number of such strings S that satisfy $p(S) = \frac{\phi^{48n}-1}{\phi^{48}-1}$. The sequence of fractions $\frac{f(n+1)}{f(n)}$ approaches a real number c as n goes to infinity. Determine the value of c.

2017 HMMT, November Guts, Problem 26 369. Kelvin the Frog is hopping on a number line (extending to infinity in both directions). Kelvin starts at 0. Every minute, he has a $\frac{1}{3}$ chance of moving 1 unit left, a $\frac{1}{3}$ chance of moving 1 unit right, and $\frac{1}{3}$ chance of getting eaten. Find the expected number of times Kelvin returns to 0 (not including the start) before he gets eaten.

2017 HMIC, Problem 3 370. Let $v_1, v_2, \ldots, v_m$ be vectors in $\mathbb{R}^n$, such that each has a strictly positive first coordinate. Consider the following process. Start with the zero vector $w = (0, 0, \ldots, 0) \in \mathbb{R}^n$. Every round, choose an i such that $1 \leq i \leq m$ and $w \cdot v_i \leq 0$, and then replace w with $w + v_i$.

Show that there exists a constant C such that regardless of your choice of i at each step, the process is guaranteed to terminate in (at most) C rounds. The constant C may depend on the vectors $v_1, \ldots, v_m$.

2018 AIME I, Problem 13 371. Let $\triangle ABC$ have side lengths $AB = 30$, $BC = 32$, and $AC = 34$. Point X lies in the interior of $\overline{BC}$, and points I_1 and I_2 are the incenters of $\triangle ABX$ and $\triangle ACX$, respectively. Find the minimum possible area of $\triangle AI_1I_2$ as X varies along $\overline{BC}$.

2018 AIME I, Problem 15 372. David found four sticks of different lengths that can be used to form three non-congruent convex cyclic quadrilaterals, A, B, C, which can each be inscribed in a circle with radius 1. Let φ_A denote the measure of the acute angle made by the diagonals of quadrilateral A, and define φ_B and φ_C similarly. Suppose that $\sin\varphi_A = 2/3$, $\sin\varphi_B = 3/5$, and $\sin\varphi_C = 6/7$. All three quadrilaterals have the same area K, which can be written in the form m/n, where m and n are relatively prime positive integers. Find $m+n$.

2018 AIME II, Problem 12 373. Let $ABCD$ be a convex quadrilateral with $AB = CD = 10$, $BC = 14$, and $AD = 2\sqrt{65}$. Assume that the diagonals of $ABCD$ intersect at point P, and that the sum of the areas of $\triangle APB$ and $\triangle CPD$ equals the sum of the areas of $\triangle BPC$ and $\triangle APD$. Find the area of quadrilateral $ABCD$.

2018 HMMT, Algebra & Number Theory, Problem 6 374. Let α, β, and γ be three real numbers. Suppose that

$$\cos\alpha + \cos\beta + \cos\gamma = 1$$

$$\sin\alpha + \sin\beta + \sin\gamma = 1.$$

Find the smallest possible value of $\cos\alpha$.

2018 HMMT, Algebra & Number Theory, Problem 10 375. Let S be a randomly chosen 6-element subset of the set $\{0, 1, 2, \ldots, n\}$. Consider the polynomial

$$P(x) = \sum_{i \in S} x^i.$$

Let X_n be the probability that $P(x)$ is divisible by some non-constant polynomial $Q(x)$ of degree at most 3 with integer coefficients satisfying $Q(0) \neq 0$. Find the limit of X_n as n goes to infinity.

2018 HMMT, Team, Problem 5 376. Is it possible for the projection of the set of points (x, y, z) with $0 \leq x, y, z \leq 1$ onto some two-dimensional plane to be a simple convex pentagon?

2018 HMMT, November General, Problem 10 377. Real numbers x, y, and z are chosen from the interval $[-1, 1]$ independently and uniformly at random. What is the probability that

$$|x| + |y| + |z| + |x + y + z| = |x + y| + |y + z| + |z + x|?$$

2019 AIME I, Problem 8 378. Let x be a real number such that

$$\sin^{10} x + \cos^{10} x = \frac{11}{36}.$$

Then,

$$\sin^{12} x + \cos^{12} x = \frac{m}{n},$$

where m and n are relatively prime positive integers. Find $m + n$.

2019 AIME I, Problem 11 379. In $\triangle ABC$, the sides have integers lengths and $AB = AC$. Circle ω has its center at the incenter of $\triangle ABC$. An excircle of $\triangle ABC$ is a circle in the exterior of $\triangle ABC$ that is tangent to one side of the triangle and tangent to the extensions of the other two sides. Suppose that the excircle tangent to $\overline{BC}$ is internally tangent to ω, and the other two excircles are both externally tangent to ω. Find the minimum possible value of the perimeter of $\triangle ABC$.

2019 AIME I, Problem 15 380. Let $\overline{AB}$ be a chord of a circle ω, and let P be a point on the chord $\overline{AB}$. Circle ω_1 passes through A and P and is internally tangent to ω. Circle ω_2 passes through B and P and is internally tangent to ω. Circles ω_1 and ω_2 intersect at points P and Q. Line PQ intersects ω at X and Y. Assume that $AP = 5$, $PB = 3$, $XY = 11$, and $PQ^2 = m/n$, where m and n are relatively prime positive integers. Find $m + n$.

2019 AIME II, Problem 10 381. There is a unique angle θ between $0°$ and $90°$ such that for non-negative integers n, the value of $\tan(2^n\theta)$ is positive when n is a multiple of 3, and negative otherwise. The degree measure of θ is p/q, where p and q are relatively prime integers. Find $p+q$.

2019 AIME II, Problem 15 382. In acute triangle ABC points P and Q are the feet of the perpendiculars from C to $\overline{AB}$ and from B to $\overline{AC}$, respectively. Line PQ intersects the circumcircle of $\triangle ABC$ in two distinct points, X and Y. Suppose $XP=10$, $PQ=25$, and $QY=15$. The value of $AB\cdot AC$ can be written in the form $m\sqrt{n}$ where m and n are positive integers, and n is not divisible by the square of any prime. Find $m+n$.

2019 HMMT, Combinatorics, Problem 10 383. Fred the Four-Dimensional Fluffy Sheep is walking in 4-dimensional space. He starts at the origin. Each minute, he walks from his current position (a_1,a_2,a_3,a_4) to some position (x_1,x_2,x_3,x_4) with integer coordinates satisfying

$$(x_1-a_1)^2+(x_2-a_2)^2+(x_3-a_3)^2+(x_4-a_4)^2=4 \quad \text{and}$$
$$|(x_1+x_2+x_3+x_4)-(a_1+a_2+a_3+a_4)|=2.$$

In how many ways can Fred reach $(10,10,10,10)$ after exactly 40 minutes, if he is allowed to pass through this point during his walk?

2019 HMIC, Problem 3 384. Do there exist four points $P_i=(x_i,y_i)\in\mathbb{R}^2$ $(1\le i\le 4)$ on the plane such that:

a) for all $i=1,2,3,4$, the inequality $x_i^4+y_i^4\le x_i^3+y_i^3$ holds, and
b) for all $i\ne j$, the distance between P_i and P_j is greater than 1?

2020 AIME II, Problem 13 385. Convex pentagon $ABCDE$ has side lengths $AB=5$, $BC=CD=DE=6$, and $EA=7$. Moreover, the pentagon has an inscribed circle (a circle tangent to each side of the pentagon). Find the area of $ABCDE$.

2020 AIME II, Problem 15 386. Let $\triangle ABC$ be an acute scalene triangle with circumcircle ω. The tangents to ω at B and C intersect at T. Let X and Y be the projections of T onto lines AB and AC, respectively. Suppose $BT=CT=16$, $BC=22$, and

$$TX^2+TY^2+XY^2=1143.$$

Find XY^2.

2020 HMMT, Geometry, Problem 7 387. Let Γ be a circle, and ω_1 and ω_2 be two non-intersecting circles inside Γ that are internally tangent to Γ at X_1 and X_2, respectively. Let one of the common internal tangents of ω_1 and ω_2 touch ω_1 and ω_2 at T_1 and T_2, respectively, while intersecting Γ at two points A and B. Given that $2X_1T_1=X_2T_2$ and that ω_1, ω_2, and Γ have radii 2, 3, and 12, respectively, compute the length of AB.

2020 HMMT, Team, Problem 4 388. Alan draws a convex 2020-gon

$$\mathcal{A}=A_1A_2\cdots A_{2020}$$

with vertices in clockwise order and chooses 2020 angles $\theta_1,\theta_2,\dots,\theta_{2020}\in(0,\pi)$ in radians with sum 1010π. He then constructs isosceles triangles $\triangle A_iB_iA_{i+1}$ on the exterior of $\mathcal{A}$ with $B_iA_i=B_iA_{i+1}$ and $\angle A_iB_iA_{i+1}=\theta_i$. (Here, $A_{2021}=A_1$.) Finally, he erases $\mathcal{A}$ and the point B_1. He then tells

Jason the angles $\theta_1, \theta_2, \dots, \theta_{2020}$ he chose. Show that Jason can determine where B_1 was from the remaining 2019 points, i.e. show that B_1 is uniquely determined by the information Jason has.

2021 AIME I, Problem 6 389. Segments $\overline{AB}, \overline{AC}$, and $\overline{AD}$ are edges of a cube and $\overline{AG}$ is a diagonal through the center of the cube. Point P satisfies $BP = 60\sqrt{10}$, $CP = 60\sqrt{5}$, $DP = 120\sqrt{2}$, and $GP = 36\sqrt{7}$. Find AP.

2021 AIME I, Problem 7 390. Find the number of pairs (m, n) of positive integers with $1 \leq m < n \leq 30$ such that there exists a real number x satisfying

$$\sin(mx) + \sin(nx) = 2.$$

2021 AIME II, Problem 5 391. For positive real numbers s, let $\tau(s)$ denote the set of all obtuse triangles that have area s and two sides with lengths 4 and 10. The set of all s for which $\tau(s)$ is nonempty, but all triangles in $\tau(s)$ are congruent, is an interval $[a, b)$. Find $a^2 + b^2$.

2021 PUMaC, Team Round, Problem 5 392. Given a real number t with $0 < t < 1$, define the real-valued function

$$f(t, \theta) = \sum_{n=-\infty}^{\infty} t^{|n|}\omega^n,$$

where $\omega = e^{i\theta} = \cos\theta + i\sin\theta$. For $\theta \in [0, 2\pi)$, the polar curve $r(\theta) = f(t, \theta)$ traces out an ellipse E_t with a horizontal major axis whose left focus is at the origin. Let $A(t)$ be the area of the ellipse E_t. Let

$$A\left(\frac{1}{2}\right) = \frac{a\pi}{b},$$

where a, b are relatively prime positive integers. Find $100a + b$.

2021 PUMaC, Algebra, Problem 5 393. Consider the sum

$$S = \sum_{j=1}^{2021} \left|\sin\frac{2\pi j}{2021}\right|.$$

The value of S can be written as $\tan(c\pi/d)$ for some relatively prime positive integers c, d, satisfying $2c < d$. Find the value of $c + d$.

2021 PUMaC, Algebra, Problem 7 394. Consider the following expression

$$S = \log_2\left(\sum_{k=1}^{2019}\sum_{j=2}^{2020} \log_{2^{1/k}}(j)\log_{j^2}\left(\sin\frac{\pi k}{2020}\right)\right).$$

Find the smallest integer n which is bigger than S (i.e. find $\lceil S \rceil$).

2021 HMIC, Problem 4 395. Let $A_1A_2A_3A_4$, $B_1B_2B_3B_4$, and $C_1C_2C_3C_4$ be three regular tetrahedra in 3-dimensional space, no two of which are congruent. Suppose that, for each $i \in \{1, 2, 3, 4\}$, C_i is the midpoint of the line segment A_iB_i. Determine whether the four lines A_1B_1, A_2B_2, A_3B_3, and A_4B_4 must concur.

2022 AIME II, Problem 11 396. Let $ABCD$ be a convex quadrilateral with $AB = 2, AD = 7,$ and $CD = 3$ such that the bisectors of acute angles $\angle DAB$ and $\angle ADC$ intersect at the midpoint of $\overline{BC}$. Find the square of the area of $ABCD$.

2022 HMMT, Algebra & Number Theory, Problem 7 397. Let (x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4), and (x_5, y_5) be the vertices of a regular pentagon centered at $(0, 0)$. Compute the product of all positive integers k such that the equality

$$x_1^k + x_2^k + x_3^k + x_4^k + x_5^k = y_1^k + y_2^k + y_3^k + y_4^k + y_5^k,$$

must hold for all possible choices of the pentagon.

2022 Stanford Math Tournament, Algebra #6 398. Compute

$$\cot\left(\sum_{n=1}^{23} \cot^{-1}\left(1 + \sum_{k=1}^{n} 2k\right)\right).$$

2022 Stanford Math Tournament, Algebra Tiebreaker #2 399. What is the area of the region in the complex plane consisting of all points z satisfying both

$$\left|\frac{1}{z} - 1\right| < 1$$

and $|z - 1| < 1$? ($|z|$ denotes the magnitude of a complex number, i.e., $|a + bi| = \sqrt{a^2 + b^2}$.)

2023 AIME I, Problem 5 400. Let P be a point on the circumcircle of square $ABCD$ such that $PA \cdot PC = 56$ and $PB \cdot PD = 90$. What is the area of square $ABCD$?

2023 AIME I, Problem 8 401. Rhombus $ABCD$ has $\angle BAD < 90^\circ$. There is a point P on the incircle of the rhombus such that the distances from P to lines DA, AB, and BC are 9, 5, and 16, respectively. Find the perimeter of $ABCD$.

2023 AIME I, Problem 13 402. Each face of two noncongruent parallelepipeds is a rhombus whose diagonals have lengths $\sqrt{21}$ and $\sqrt{31}$. The ratio of the volume of the larger of the two polyhedra to the volume of the smaller is m/n, where m and n are relatively prime positive integers. Find $m + n$.

2023 Bulgaria National Olympiad, Problem 5 403. For every positive integer n determine the least possible value of the expression

$$|x_1| + |x_1 - x_2| + |x_1 + x_2 - x_3| + \cdots + |x_1 + x_2 + \cdots + x_{n-1} - x_n|,$$

given that $x_1, x_2, \ldots, x_n$ are real numbers satisfying $|x_1| + |x_2| + \cdots + |x_n| = 1$.

2023 British Mathematical Olympiad, Problem 4 404. Find the greatest integer $k \leq 2023$ for which the following holds: whenever Alice colours exactly k numbers of the set $\{1, 2, \ldots, 2023\}$ in red, Bob can colour some of the remaining uncoloured numbers in blue, such that the sum of the red numbers is the same as the sum of the blue numbers.

2023 Benelux, Problem 2 405. Determine all integers $k \geqslant 1$ with the following property: given k different colours, if each integer is coloured in one of these k colours, then there must exist integers $a_1 < a_2 < \cdots < a_{2023}$ of the same colour such that the differences $a_2 - a_1, a_3 - a_2, \ldots, a_{2023} - a_{2022}$ are all powers of 2.

2023 EGMO, Problem 2 406. We are given an acute triangle ABC. Let D be the point on its circumcircle such that AD is a diameter. Suppose that points K and L lie on segments AB and AC, respectively, and that DK and DL are tangent to circle AKL. Show that line KL passes through the orthocenter of triangle ABC.

2023 Italy Math Olympiad, Problem 6 407. Dedalo buys a finite number of binary strings, each of finite length and made up of the binary digits 0 and 1. For each string, he pays $(\frac{1}{2})^L$ drachmas, where L is the length of the string. The Minotaur is able to escape the labyrinth if he can find an infinite sequence of binary digits that does not contain any of the strings Dedalo bought. Dedalo's aim is to trap the Minotaur. For instance, if Dedalo buys the strings 00 and 11 for a total of half a drachma, the Minotaur is able to escape using the infinite string $01010101\ldots$. On the other hand, Dedalo can trap the Minotaur by spending 75 cents of a drachma: he could for example buy the strings 0 and 11, or the strings $00, 11, 01$. Determine all positive integers c such that Dedalo can trap the Minotaur with an expense of at most c cents of a drachma.

3 Partial Solutions

Solution by peace09 1. Let $s = x + y$ and $p = xy$, so that $x^2 + y^2 = s^2 - 2p = 7$ and $x^3 + y^3 = s(s^2 - 3p) = 10$, or, equivalently, $s^2 - 3p = \frac{10}{s}$. Seeking to create elimination in p, we multiply the two equations by 3 and 2 in that order to obtain $3s^2 - 6p = 21$ and $2s^2 - 6p = \frac{20}{s}$, which after subtracting yields $s^2 = 21 - \frac{20}{s}$ or $s^3 - 21s + 20 = 0$. Evidently the LHS is divisible by $s - 1$, and dividing by $s - 1$ gives $s^2 + s - 20$, which factors as $(s + 5)(s - 4)$. Therefore, the solutions to the equation are $s = -5$, $s = 1$, and $s = 4$, of which the largest real value is $\boxed{4}$, the requested answer.

Solution by S. Zhu 2. We desire the roots of $P(z) = z^6 + z^3 + 1$. Notice that

$$(z^3 - 1)P(z) = z^9 - 1,$$

which just has the ninth roots of unity. However, in multiplying by $(z^3 - 1)$, the extraneous third roots of unity have also been included. Thus, the roots of $P(z)$ are just the ninth roots of unity which are not third roots of unity. The ninth roots of unity are:

$$\cos(\theta) + i \cdot \sin(\theta),$$

for $\theta = (360°/9)\,(n)$ for $n = 0, 1, 2, 3, 4, 5, 6, 7, 8$ and $i = \sqrt{-1}$. The third roots of unity are

$$\cos(\phi) + i \cdot \sin(\phi),$$

for $\phi = (360°/3)\,(m)$ for $m = 0, 1, 2$. The only ninth root of unity which has argument $90° \leq \theta \leq 180°$ which is not a third root of unity is $\cos(160°) + i \cdot \sin(160°)$. Hence, the desired answer is $\boxed{160}$.

Solution by OlympusHero 5. This means $z_1 + z_2 + z_3 + z_4 + z_5 = 3 + 504i$. Write as points on the real plane, for simplicity. From the y intercept condition, the points are

$$(a, pa + 3), (b, pb + 3), (c, pc + 3), (d, pd + 3), (e, pe + 3).$$

Then $p(a + b + c + d + e) + 3 \cdot 5 = 504$ and $a + b + c + d + e = 3$, so the answer is

$$p = \frac{504 - 3 \cdot 5}{3} = \boxed{163}.$$

Solution by bobthegod78 6. For simplicity, let $f(x) = ax^{17} + bx^{16} + 1$ and $g(x) = x^2 - x - 1$. Notice that the roots of $g(x)$ are also roots of $f(x)$. Let these roots be u, v. We get the system

$$au^{17} + bu^{16} + 1 = 0,$$
$$av^{17} + bv^{16} + 1 = 0.$$

If we multiply the first equation by v^{16} and the second by u^{16} we get

$$u^{17}v^{16}a + u^{16}v^{16}b + v^{16} = 0,$$
$$u^{16}v^{17}a + u^{16}v^{16}b + u^{16} = 0.$$

Now subtracting, we get

$$a(u^{17}v^{16} - u^{16}v^{17}) = u^{16} - v^{16} \implies a = \frac{u^{16} - v^{16}}{u^{17}v^{16} - u^{16}v^{17}}.$$

By Vieta's, $uv = -1$ so the denominator becomes $u - v$. By difference of squares and dividing out $u - v$ we get

$$a = (u^8 + v^8)(u^4 + v^4)(u^2 + v^2)(u + v).$$

A simple exercise of Vieta's gets us $a = \boxed{987}$.

Solution by joml88 8. From $k = (a_3a_2a_1a_0)_{-3+i}$ we have

$$\begin{aligned}
k &= a_3(-3+i)^3 + a_2(-3+i)^2 + a_1(-3+i) + a_0 \\
&= a_3\left[(-3)^3 + 3(-3)^2 i + 3(-3)i^2 + i^3\right] + a_2\left[(-3)^2 + 2(-3)i + i^2\right] + a_1(-3+i) + a_0 \\
&= (-18 + 26i)a_3 + (8 - 6i)a_2 + (-3 + i)a_1 + a_0 \\
&= (-18a_3 + 8a_2 - 3a_1 + a_0) + (26a_3 - 6a_2 + a_1)i
\end{aligned}$$

The imaginary part is 0, hence

$$26a_3 - 6a_2 + a_1 = 0 \Leftrightarrow 26a_3 = 6a_2 - a_1.$$

Now note that $a_i \in \{0, 1, 2, \ldots, 9\}$. So the RHS of $26a_3 = 6a_2 - a_1$ is at most $6 \cdot 9 = 54$ meaning that a_3 can only be 1 or 2.

If $a_3 = 2$, then $a_2 = 9$ and $a_1 = 2$. So then

$$k = -18(2) + 8(9) - 3(2) + a_0 = 30 + a_0.$$

But a_0 can be anything from 0 to 9. We therefore sum up

$$(30 + 0) + (30 + 1) + \cdots + (30 + 9) = 345,$$

since we're interested in the sum of all k and also to account for every possibility of k.

On the other hand, if $a_3 = 1$ then $a_2 = 5$ and $a_1 = 4$. This gives $k = -18(1)+8(5)-3(4)+a_0 = 10 + a_0$. So we again take the sum

$$(10 + 0) + (10 + 1) + \cdots + (10 + 9) = 145.$$

To find the total sum of all possible k, we just take $345 + 145 = \boxed{490}$.

Solution by cobbler 9. Clearly $z = e^{\frac{2ai\pi}{18}}$ and $w = e^{\frac{2bi\pi}{48}}$, with $a \in \{1, 2, ..., 18\}$ and $b \in \{1, 2, \dots, 48\}$. So,

$$zw = e^{\frac{2\pi i(8a+3b)}{144}},$$

meaning each zw is a 144^{th} root of unity. Furthermore, by Chicken McNugget, every integer greater than $8 \cdot 3 - 8 - 3 = 13$ can be achieved by $8a+3b$. Now since adding a multiple of 2π to the numerator doesn't change the value of zw (i.e., roots of unity are periodic with period 2π), and since $8a + 3b$ can achieve $\equiv \{1, 2, \dots, 144\} \pmod{144}$ (because $8a + 3b$ can be any value > 13 from above), it follows that zw can be any distinct 144^{th} root of unity. So, the answer is just $\boxed{144}$.

Solution by 4everwise 12. For starters, we see that $(a_3 - a_2) - (a_2 - a_1) = a_3 - 2a_2 + a_1 = 1$. Similarly, $a_{n+2} - 2a_{n+1} + a_n = 1$, so that $a_{n+2} = 1 + 2a_{n+1} - a_n$. Now, we let $a = a_1$ and $b = a_2$, so that we're trying to solve for a. Notice that the next few terms are the sequence are $a_3 = 1+2b-a$, $a_4 = 3 + 3b - 2a$, etc., where the nth term a_n is

$$a_n = \frac{(n-1)(n-2)}{2} + (n-1)b - (n-2)a.$$

Because $a_{19} = a_{92} = 0$, we get two equations involving a and b. $17a - 18b = 153$ and $90a - 91b = 4095$. Solving, $a = a_1 = \boxed{819}$

Solution by OlympusHero 16. Let the first two roots be a_1, b_1 and let the other two be c_1, d_1. We have $a_1b_1 = 13 + i, c_1 + d_1 = 3 + 4i$, so $c_1d_1 = 13 - i, a_1 + b_1 = 3 - 4i$. From Vieta's formulas,

$$\begin{aligned} b &= a_1b_1 + a_1c_1 + a_1d_1 + b_1c_1 + b_1d_1 + c_1d_1 \\ &= 26 + (a_1c_1 + a_1d_1 + b_1c_1 + b_1d_1) \\ &= 26 + (a_1 + b_1)(c_1 + d_1) \\ &= 26 + 25 = \boxed{51}. \end{aligned}$$

Solution by Rust and Deedlit 18.

$$\sum_{n=1}^{\infty} \frac{1}{a_n^3} = \sum_{k=1}^{\infty} \sum_{n=k^2-k+1}^{k^2+k} \frac{1}{a_n^3} = \sum_{k=1}^{\infty} \frac{2k}{k^3} = \frac{\pi^2}{3}.$$

Solution by joml88 21. The solutions of the equation $z^{1997} = 1$ are the 1997th roots of unity and are equal to

$$\cos\left(\frac{2\pi k}{1997}\right) + i \sin\left(\frac{2\pi k}{1997}\right), \qquad \text{for } k = 0, 1, \dots, 1996.$$

They are also located at the vertices of a regular 1997–gon that is centered at the origin in the complex plane.

WLOG, let $v = 1$. Then,

$$\begin{aligned}
|v+w|^2 &= |\cos\left(\frac{2\pi k}{1997}\right) + i\sin\left(\frac{2\pi k}{1997}\right) + 1|^2 \\
&= \left|\left[\cos\left(\frac{2\pi k}{1997}\right) + 1\right] + i\sin\left(\frac{2\pi k}{1997}\right)\right|^2 \\
&= \cos^2\left(\frac{2\pi k}{1997}\right) + 2\cos\left(\frac{2\pi k}{1997}\right) + 1 + \sin^2\left(\frac{2\pi k}{1997}\right) \\
&= 2 + 2\cos\left(\frac{2\pi k}{1997}\right).
\end{aligned}$$

We want $|v+w|^2 \geq 2+\sqrt{3}$. From what we just obtained, this is equivalent to

$$\cos\left(\frac{2\pi k}{1997}\right) \geq \frac{\sqrt{3}}{2}.$$

This occurs when

$$\frac{\pi}{6} \geq \frac{2\pi k}{1997} \geq -\frac{\pi}{6},$$

which is satisfied by $k = 166, 165, \ldots, -165, -166$ (we don't include 0 because that corresponds to v). So, out of the 1996 possible k, 332 work. Thus, $m/n = 332/1996 = 83/499$. So our answer is $83 + 499 = \boxed{582}$.

Solution by joml88 31. Let $a = \log_{225} x$ and $b = \log_{64} y$. We know that $\log_x 225 = \frac{1}{\log_{225} x} = \frac{1}{a}$ and $\log_y 64 = \frac{1}{\log_{64} y} = \frac{1}{b}$. The system is thus

$$\begin{aligned}
a+b &= 4 \\
\frac{1}{a} - \frac{1}{b} &= 1.
\end{aligned}$$

We can solve for a in the first equation, substitute in into the second equation, and get b to be $3 \pm \sqrt{5}$. Using the same procedure again, we find that a is $1 \pm \sqrt{5}$. Therefore

$$x_1y_1x_2y_2 = \left(225^{3+\sqrt{5}}\right)\left(64^{1+\sqrt{5}}\right)\left(225^{3-\sqrt{5}}\right)\left(64^{1-\sqrt{5}}\right) = 30^{12}.$$

Therefore, our answer is $\log_{30} 30^{12} = \boxed{012}$.

Solution by OlympusHero 32. Note that if we have a negative exponent, we are multiplying by something extremely close to zero, so the result is negligible. This means we only need to consider $10^{2860} + \frac{10}{7} \cdot 10^{858}$. The first of these does not have a decimal, so it does not contribute anything. The second of these has 428571 after the decimal, so the answer is $\boxed{428}$.

Solution by paladin8 33. After a brilliant mass of algebra, we find that $F(F(F(z))) = z$. Then $z_i = z_j$ if and only if $i \equiv j \pmod 3$, so

$$z_{2002} = z_1 = F(z_0) = \frac{\frac{1}{137} + i + i}{\frac{1}{137} + i - i} = 1 + 274i,$$

so that $a + b = 1 + 274 = \boxed{275}$.

Solution by DottedCalculator 51. The graph of $p(x,y)=0$ is a curve of degree 3. Consider the equations $p(x,y)=x(x-1)(2x-3y+2)$ and $p(x,y)=(x-y)(x^2+y^2+xy-1)$. These two equations satisfy the conditions of the problem. Therefore, the point $\left(\frac{a}{c},\frac{b}{c}\right)$ must be a zero of both equations. Therefore, this point is the intersection of $2x-3y+2=0$ and $x^2+y^2+xy-1=0$ other than $(-1,0)$. From the first equation, we get $x=\frac{3}{2}y-1$, so substituting into the second equation, we get

$$\begin{aligned}\left(\frac{3}{2}y-1\right)^2+y^2+\left(\frac{3}{2}y-1\right)y-1&=0\\ \frac{9}{4}y^2-3y+1+y^2+\frac{3}{2}y^2-y-1&=0\\ \frac{19}{4}y^2-4y&=0.\end{aligned}$$

Since $y\neq 0$, we must have $y=\frac{16}{19}$. Then, $x=\frac{5}{19}$. Now, suppose that the graph of $p(x,y)=0$ passes through the eight points given in the problem. Then, by Cayley-Bacharach, it must pass through $\left(\frac{5}{19},\frac{16}{19}\right)$. Therefore, this point works, so $a=5$, $b=16$, and $c=19$, so $a+b+c=\boxed{040}$.

Solution 69. The answer is $\boxed{\sqrt{5+2\sqrt{3}}}$.

Solution 70. The answer is $\boxed{4}$. We have $p(x)=(2x^2-3)q(x)-7/2$, where $q(x)$ is either a constant polynomial (two solutions) or a quadratic polynomial (two other solutions).

Solution 71. The answer is $\boxed{(11-\sqrt{13})/2}$.

Solution by forthegreatergood 86. Notice the 'main' (non-zero) digits:

$$1-7-21-35-35-21-7-1.$$

These are from pascal's triangle! Since there are 2 zeros in between each pairs of numbers we have 1001^7 as the answer. It is a well known fact that $1001=7\cdot 11\cdot 13$ so we have

$$1007021035035021007001=\boxed{7^7\cdot 11^7\cdot 13^7}.$$

Solution 89. The answer is $\boxed{66071772829247409}$.

Solution 90. The answer is $\boxed{84}$.

Solution 91. The answer is $\boxed{18}$.

Solution 92. Let $x=\cos\theta$ and $y=\sin\theta$ and the equations are $\sin^2\theta+\cos^2\theta=1$ and $\sin 3\theta\cos 3\theta=1/2$, or simply $\sin 6\theta=1$, or $\theta=\pi/12$. Therefore,

$$x+y=\cos\frac{\pi}{12}+\sin\frac{\pi}{12}=\boxed{\frac{\sqrt{6}}{2}}.$$

Solution by Nyash 106. We want to find the minimal degree m such that there exists a polynomial $p(x)$ of degree m, that when multiplied to the generating function

$$F(x) = \sum_{n=0}^{\infty} f_9(n)x^n = \sum_{n=1}^{9} \frac{1}{1-x^n},$$

yields a polynomial.

Obviously, the denominator of $F(x)$ is at most $\prod_{n=1}^{9} \Phi_n(x)$, where $\Phi_n(x)$ denotes the nth cyclotomic polynomial.

By explicitly writing out the numerator as a product of cyclotomics, we can see that it does not have a root at any of the nth primitive roots of unity, where n is an integer between 1 and 9 (there should probably be a better way to do this).

Since m is at least the degree of the denominator, the minimum value is just

$$\varphi(1) + \varphi(2) + \cdots + \varphi(9) = 8 + (9-1) + (5-1) + (7-1) + 2 = \boxed{28}.$$

Solution 116. The answer is $-9 + 3\sqrt{3}$.

Solution by pi37 135. We prove the statement for all odd-degree polynomials P.

First, note that if $x = 0$, $y \neq 0$ is a pair of distinct integers satisfying $xP(x) = yP(y)$, then $P(y) = 0$, so we're done. Otherwise, assume that there are infinitely many pairs of distinct nonzero integers satisfying the problem statement.

The crucial claim is that for any x, y satisfying $xP(x) = yP(y)$, $|x + y|$ is bounded above. Let

$$P(x) = a_{2n-1}x^{2n-1} + a_{2n-2}x^{2n-2} + \cdots + a_0$$

for $a_i \in \mathbb{Z}$, $a_{2n-1} \neq 0$, $n \geq 1$. Then

$$xP(x) - yP(y) = a_{2n-1}(x^{2n} - y^{2n}) + a_{2n-2}(x^{2n-1} - y^{2n-1}) + \cdots + a_0(x - y),$$

which gives

$$\begin{aligned} xP(x) - yP(y) = (x-y)(a_{2n-1}(x+y)(x^{2n-2} + x^{2n-4}y^2 + \cdots y^{2n-2}) \\ + a_{2n-2}(x^{2n-2} + x^{2n-3}y + \cdots y^{2n-2}) + \cdots + a_0). \end{aligned}$$

So if $x \neq y$, then

$$a_{2n-1}(x+y)(x^{2n-2} + x^{2n-4}y^2 + \cdots y^{2n-2}) = -(a_{2n-2}(x^{2n-2} + x^{2n-3}y + \cdots y^{2n-2}) + \cdots + a_0)$$

Now since $a_{2n-1} \neq 0$ and $a_{2n-1} \in \mathbb{Z}$, we have $|a_{2n-1}| \geq 1$. Furthermore, $x^{2k}y^{2\ell} \geq 0$ for all $k, \ell \geq 0$, so

$$|x^{2n-2} + x^{2n-4}y^2 + \cdots y^{2n-2}| \geq |x^{2n-2}|$$

Now assume WLOG $|x| \geq |y|$. Noting $|x^i| \geq |x^j|$ for $i \geq j$ and using the triangle inequality, we get

$$\begin{aligned} |x+y||x^{2n-2}| &\leq |a_{2n-1}(x+y)(x^{2n-2} + x^{2n-4}y^2 + \cdots + y^{2n-2}| \\ &= |a_{2n-2}(x^{2n-2} + x^{2n-3}y + \cdots y^{2n-2}) + \cdots + a_0| \\ &\leq |a_{2n-2}(2n-1)x^{2n-2}| + |a_{2n-3}(2n-2)x^{2n-2}| + \cdots + |a_0(1)x^{2n-2}| \\ &\leq ((2n-1)|a_{2n-2}| + (2n-2)|a_{2n-3}| + \cdots + |a_0|)|x^{2n-2}|. \end{aligned}$$

But since $x \neq 0$,
$$|x+y| \leq (2n-1)|a_{2n-2}| + (2n-2)|a_{2n-3}| + \cdots + |a_0|.$$
Thus by the infinite Pigeonhole Principle, there exists an integer c such that there are infinitely many pairs x, y with $x+y=c$ and $xP(x)=yP(y)$. In other words, $xP(x)=(c-x)P(c-x)$ holds for infinitely many x, so it holds identically. If $c \neq 0$, plugging in $x=0$ yields $cP(c)=0$, so $P(c)=0$, as desired. If $c=0$, then $xP(x)=-xP(-x)$, so $P(x)=-P(-x)$, and 0 is a root of P.

Solution by freeman66 165. Using the identity
$$(r_1+r_2+r_3)^3-(r_1^3+r_2^3+r_3^3)=3(r_1+r_2)(r_2+r_3)(r_3+r_1),$$
and $r_1+r_2+r_3=2019$ (by Vieta's), we get
$$\begin{aligned}2019^3-(r_1^3+r_2^3+r_3^3) &= 3(2019-r_1)(2019-r_2)(2019-r_3)\\ &= 3(2019^3-2019^2(r_1+r_2+r_3)+2019(r_1r_2+r_2r_3+r_3r_1)-r_1r_2r_3).\end{aligned}$$
Taking mod 3 easily gives us the desired answer, since
$$2019^3-2019^2(r_1+r_2+r_3)+2019(r_1r_2+r_2r_3+r_3r_1)-r_1r_2r_3,$$
is an integer.

Solution by stroller 186. We claim that $P(x)=x^m$ for some $m \in \mathbb{N}_0$. It's easy to see that such P are solutions.

Now suppose P satisfies the problem conditions. For each $q|P(n)$ for some n let $S_q := \{m : q|P(m), m \in \mathbb{Z}\}$, then
$$m \in S_q \iff m-q \in S_q.$$
For each q such that $S_q \neq \emptyset, S_q \cap \mathbb{Z} \neq \mathbb{Z}$ and $q > 10^{\deg P}$, we contend that S_q consists of only multiples of q. Note that
$$n \in S_q \cap \mathbb{N} \implies \overline{nn} = (10^{s(n)}+1)n \in S_q \cap \mathbb{N},$$
where $s(n)$ is the number of digits of n. However note that
$$n \in S_q \implies n+kq \in S_q \forall k \implies (10^{s(n+kq)}+1)(n+kq) \in S_q \implies (10^{s(n+kq)}+1)n \in S_q.$$
Since $\{s(n+kq) : k \in \mathbb{N}\}$ contains all sufficiently large integers we have from Fermat's Little Theorem that $n(10^t+1) \in S_q \forall t \in \mathbb{N}$. Now if $n \neq 0$ then 10^t attains at least $\log(q+1) > \log q \geq \deg P$ values mod q, hence P has $\deg P+1$ roots $\pmod q$ (namely $n(10^t+1), t=0,1,\ldots,\deg P$), so P must be identically zero $\pmod q$, again a contradiction. Thus $n \in S_q \implies n=0$, as desired.

Since P cannot be identically zero there exists K such that for all q such that $q > K$, S_q consists of only multiples of q or is empty. Call this $(*)$.

Write $P(x)=x^aQ(x)$ with $Q(0) \neq 0$. Then Q satisfies $(*)$ as well and $Q(0) \neq 0$. However note that $0 \in S_q$ if $S_q \neq$ it follows that the number of primes $q > K$ dividing some value of P is bounded, so P has only finitely many prime divisors, contradiction.

Solution by Assassino9931 210. By fixing any $n-1$ of the variables and considering the result as a polynomial on a single variable, we see that $x_i - x_j$ divides F for all $i<j$, hence the result. Equality for $n=3$ holds, e.g., for $(x_1-x_2)(x_2-x_3)(x_1-x_3)$.

Solution by Tintarn 224. Someone raised the question of whether there is a polynomial P such that both $P(x)$ and $P(x)+1$ are reducible. Quickly, we found $P(x)=x^2-1$ as a solution. We then asked whether we can classify all the solutions, but this turns out to be quite complicated. We then went on to ask what happens if we ask for polynomials in several variables. Soon we realized that this question was ill-posed: If we have a solution to the one-variable problem, then we can always still just take $P(x,y)=x^2-1$ as a solution (i.e. just let P not depend on the second variable). But even if we force $P(x,y)$ to actually depend on both variables, there are a lot of trivial solutions like $P(x,y)=x^2y^2-1$ or more generally $P(x,y)=Q(R(x,y))$ where $Q(x)$ is a solution to the one-variable problem and $R(x,y)$ is arbitrary. We then asked: Do all solutions for the two-variable problem arise in this way from the one-variable problem? And thus, the notion of "secretly one-variable" was born. The way we went on to solve the problem originally was like this: We realized that solutions to the problem correspond to matrices with polynomial entries and determinant 1 since $\det\begin{pmatrix} a & b \\ c & d\end{pmatrix}=ad-bc$ and so $ad-bc=1$ implies that we can take $p=bc$ and $p+1=ad$. For instance, the solution $P(x)=x^2-1$ corresponds to the matrix $\begin{pmatrix} x & x-1 \\ x+1 & x\end{pmatrix}$. (We are thus asking, in fancy terms, for the structure of the group $SL_2(\mathbb{R}[x,y])$.) But from this point of view, there is a natural way to produce new examples: Take two matrices and multiply them. Starting from the "secretly one-variable" solutions, we can therefore construct the new example

$$\begin{pmatrix} x & x-1 \\ x+1 & x\end{pmatrix}\cdot\begin{pmatrix} y & y-1 \\ y+1 & y\end{pmatrix}=\begin{pmatrix} 2xy+y-x-1 & 2xy-x-y \\ 2xy+x+y & 2xy+x-y-1\end{pmatrix}.$$

And so we get the solution

$$P(x,y)=4x^2y^2-x^2-y^2-2xy=(2xy+x+y)(2xy-x-y),$$

with

$$P(x,y)+1=4x^2y^2-x^2-y^2-2xy+1=(2xy+x-y-1)(2xy+y-x-1),$$

and all that remains to be checked is that $P(x,y)$ is indeed not secretly one-variable which is a rather boring but not very difficult thing, given that there are not so many possibilities for the degree of Q and R.

Solution 225. The answer is $\boxed{1260}$.

Solution 226. The answer is $\boxed{77/15}$.

Solution 227. The answer is $\boxed{3\sqrt{798}}$.

Solution by peace09 234. The latter half of the inequality rewrites as

$$3x^2(p(x)-q(x))\geq p(x)q(x),\ (*)$$

which, by checking degree and sign, forces $\deg p\geq\deg q$ and $\deg q\leq 2$. Before we perform casework on $\deg q$, rearrange the former half of the inequality:

$$5x(p(x)-q(x))\leq p(x)q(x),\ (\dagger)$$

which implies that $\deg q\geq 1$.

If $\deg q = 1$, $\deg p = 2$ gives $5x^3 + \cdots \le x^3 + \ldots$ in (†), impossible. So $p(x) = x + a$ and $q(x) = x + b$; then, the original inequality becomes

$$\frac{1}{5x} \ge \frac{1}{x+a} - \frac{1}{x+b} \ge \frac{1}{3x^2}.$$

We can quickly verify that $a, b \neq 0$, which means that $p(1)q(1) \ge 4$ which rules out this case in favor of the next:

If $\deg q = 2$, $\deg p = 2$ gives $3cx^2 + \cdots \ge x^4 + \ldots$ in $(*)$, impossible; so $\deg p \ge 3$. Now, the key is that it is possible to have p or q contain a single term x^n, which beats the established bound of $p(1)q(1) \ge 4$ otherwise.

First $q(x) = x^2$ fails, because $\frac{1}{p(x)} - \frac{1}{q(x)} \ge \frac{1}{4} - \frac{1}{8} > \frac{1}{10}$ in that case.

Next $p(x) = x^n$ gives $n = 5$ and $q(x) = x^2 + 2x$ after eliminating cases $n = 3, 4$. So $p(1)q(1) \ge \boxed{3}$, and the aforementioned equality case can easily be verified directly.

Solution by gracemoon124 235. Notice that it may be factored as

$$x^8(x^2+x+1) + x^4(x^2+x+1) + (x^2+x+1) = (x^2+x+1)(x^8+x^4+1).$$

$x^8 + x^4 + 1$ may be broken down as $(x^4+1)^2 - x^4 = (x^4+x^2+1)(x^4-x^2+1) = (x^2+x+1)(x^2-x+1)(x^4-x^2+1)$. It only remains to factor $x^4 - x^2 + 1$, but this may be done using the same method:

$$x^4 - x^2 + 1 = (x^2+1)^2 - 3x^2 = (x^2 + x\sqrt{3} + 1)(x^2 - x\sqrt{3} + 1).$$

The roots are therefore $e^{\pi i/6}$, $e^{\pi i/3}$, $e^{2\pi i/3}$, $e^{5\pi i/6}$, and their conjugates. This forms an octagon on the unit circle. Split into a rectangle and two trapezoids, to obtain the area as

$$\sqrt{3} + \left(\frac{\sqrt{3}-1}{2}\right)(\sqrt{3}+1) = \boxed{\sqrt{3}+1}.$$

Solution by David Altizio 240. The Triangle Inequality applied to the right hand side yields

$$\begin{aligned} |z|^n|z-1| &\geqslant |z^{n+1}-1| + |z^{n+1}-z| \\ &= |z^{n+1}-1| + |z-z^{n+1}| \geqslant |z-1|. \qquad (\dagger) \end{aligned}$$

This inequality is true when either $|z-1| = 0$ or $|z|^n \geqslant 1$; either way, $|z| \geqslant 1$. Analogously, the Triangle Inequality applied to the left hand side yields

$$\begin{aligned} |z^{n+1}-1| + |z^{n+1}-z| &\leqslant |z^{n+1}-z^n| \\ &\leqslant |z^{n+1}-1| + |z^n-1|, \end{aligned}$$

so $|z||z^n-1| \leqslant |z^n-1|$. This inequality is true when either $|z^n-1| = 0$ or $|z| \leqslant 1$; either way, $|z| \leqslant 1$. Combining both inequalities together yields $|z| = 1$.

In particular, the inequality (†) must be an equality, so 1, z, and z^{n+1} are collinear. But all three complex numbers have magnitude 1, so this is only possible when two of them are equal to each other.

It follows that z is either an n^{th} root of unity or an $(n+1)^{\text{st}}$ root of unity. These solutions work when plugged back in.

Solution by joml88 250. Note that

$$\begin{aligned}\cot(\alpha+\beta) &= \frac{1}{\tan(\alpha+\beta)} \\ &= \frac{1-\tan\alpha\tan\beta}{\tan\alpha+\tan\beta} \\ &= \frac{\cot\alpha\cot\beta-1}{\cot\alpha+\cot\beta}.\end{aligned}$$

Let $x=\cot^{-1}3+\cot^{-1}7$ and $y=\cot^{-1}13+\cot^{-1}21$. Thus

$$\begin{aligned}\cot x &= \frac{\cot(\cot^{-1}3)\cot(\cot^{-1}7)-1}{\cot(\cot^{-1}3)+\cot(\cot^{-1}7)} \\ &= \frac{3\cdot 7-1}{3+7} \\ &= 2,\end{aligned}$$

and

$$\begin{aligned}\cot y &= \frac{\cot(\cot^{-1}13)\cot(\cot^{-1}21)-1}{\cot(\cot^{-1}13)+\cot(\cot^{-1}21)} \\ &= \frac{13\cdot 21-1}{13+21} \\ &= 8\end{aligned}$$

So then,

$$\begin{aligned}\cot(x+y) &= \frac{\cot x\cot y-1}{\cot x+\cot y} \\ &= \frac{2\cdot 8-1}{2+8} \\ &= \frac{3}{2}\end{aligned}$$

So the answer is therefore $\boxed{15}$.

Solution by OlympusHero 251. Rewrite as

$$\frac{\frac{\cos\gamma}{\sin\gamma}}{\frac{\cos\alpha}{\sin\alpha}+\frac{\cos\beta}{\sin\beta}},$$

or

$$\frac{\frac{\cos\gamma}{\sin\gamma}}{\frac{\sin(\alpha+\beta)}{\sin\alpha\sin\beta}} = \frac{\cos\gamma\sin\alpha\sin\beta}{\sin(\alpha+\beta)\sin\gamma},$$

or $\cos\gamma\sin\alpha\sin\beta/\sin^2\gamma$. From Law of Cosines on our original triangle, we have

$$1989c^2-2ab\cos\gamma=c^2,$$

or $ab\cos\gamma = 994c^2$. But
$$\frac{\sin\alpha\sin\beta}{\sin^2\gamma} = \frac{ab}{c^2} = \frac{994}{\cos\gamma},$$
so the answer is $\cos\gamma\cdot\frac{994}{\cos\gamma} = \boxed{994}$.

Solution by chess64 252. The range of $\sin(x)$ is $[-1,1]$. Hence, we only need to consider $\left|\frac{1}{5}\log_2 x\right| \le 1$. This is satisfied for $\frac{1}{32} \le x \le 32$. First let's consider $\frac{1}{32} \le x < 1$. In this interval, the left hand side is negative while the right hand side is negative only in $[1/5, 2/5]$ and $[3/5, 4/5]$, so there are 4 solutions.

When $1 < x \le 32$, the left hand side is positive, and the right hand side is positive only in
$$[6/5, 7/5], [8/5, 9/5], \ldots, [158/5, 169/5].$$
There are then 2 points of intersection in each of these intervals, and there are 77 intervals.

When $x = 1$ both sides of the equation are 0, so in all we have $4 + 77\cdot 2 + 1 = \boxed{159}$ solutions.

Solution by chess64 253. Recall that $\sec^2 x - \tan^2 x = 1$, from which we find that $\sec x - \tan x = 7/22$. Adding the equations
$$\begin{aligned}\sec x + \tan x &= 22/7\\ \sec x - \tan x &= 7/22\end{aligned}$$
together and dividing by 2 gives $\sec x = 533/308$, and subtracting the equations and dividing by 2 gives $\tan x = 435/308$. Hence, $\cos x = 308/533$ and $\sin x = \tan x\cos x = (435/308)(308/533) = 435/533$. Thus, $\csc x = 533/435$ and $\cot x = 308/435$. Finally,
$$\csc x + \cot x = \frac{841}{435} = \frac{29}{15},$$
so, $m + n = \boxed{44}$.

Solution by Qiaochu Yuan 256. Interpreted geometrically, we have a line $ax + by = 1$ intersecting a circle of radius $\sqrt{50}$ centered at the origin. It can either intersect tangent to a lattice point or secant to two lattice points, and the lattice points are $(x, y) = (\pm5, \pm5), (\pm1 \pm 7), (\pm7, \pm1)$ for a total of 12 lattice points.

There are 12 tangent solutions, one for each point. A secant solution can be found by choosing any one of the twelve and drawing the line connecting that point and any of the other 11 points except the point diametrically opposite. The line cannot pass through the origin (I didn't realize this before) because then $a(0) + b(0) = 1$, which does not have real solutions.

Hence, the number of solutions really is $12 + \frac{12(10)}{2} = \boxed{72}$.

Solution by joml88 257. Take the log base 1995 of both sides
$$\begin{aligned}\log_{1995}\left(\sqrt{1995}x^{\log_{1995} x}\right) &= \log_{1995}(x^2)\\ \log_{1995} 1995^{1/2} + (\log_{1995} x)^2 &= 2\log_{1995} x\\ 2(\log_{1995} x)^2 - 4\log_{1995} x + 1 &= 0,\end{aligned}$$
so that
$$\log_{1995} x = \frac{2\pm\sqrt{2}}{2}.$$

Both $(2+\sqrt{2})/2$ and $(2-\sqrt{2})/2$ are positive, so they both produce a solution. So then $x_1 = 1995^{(2+\sqrt{2})/2}$ and $x_2 = 1995^{(2-\sqrt{2})/2}$ where x_1, x_2 are solutions to the original equation. The product x_1x_2 is therefore equal to 1995^2. We only want the last three digits, which is the same as taking it mod 1000. So, our answer is

$$1995^2 \equiv (-5)^2 \equiv \boxed{025} \pmod{1000}.$$

Solution by joml88 263. When $x \geq 0$, $|x| = x$. This gives $y^2 + 2xy + 40x = 400 \Rightarrow 2x(y+20) = (20+y)(20-y)$. Thus either $y + 20 = 0 \Rightarrow y = -20$ or $2x = 20 - y \Rightarrow y = -2x + 20$. This produces two rays.

When $x < 0$, $|x| = -x$. This gives $y^2 + 2xy - 40x = 400 \Rightarrow 2x(20-y) = -(20+y)(20-y)$. So either $y = 20$ or $y = -2x - 20$. This produces two more rays.

Graphing these four rays in the coordinate plane makes a parallelogram with base 20, height 40, and, therefore, area 800.

Solution by JesusFreak197 274. It moves 5 to the side and 7 up, and we must end up at another vertex. Every time it reflects, it will move 5 units to the side and 7 units either up or down. If it hits a face partway through a reflection, it will continue the remaining number of units in the opposite direction. We first have that both $5x$ and $7x$ are multiples of 12 for $x = 12$. The length AP is $\sqrt{5^2 + 7^2 + 12^2} = \sqrt{218}$, so we multiply that by 12, and we have $12\sqrt{218} \Rightarrow 12 + 218 = \boxed{230}$.

Solution by RminusQ 318. Warning: This solution is brute force and ugly.

Set this triangle to the plane, $A = (0,0), B = (15,0)$. It can be seen $C = (33/5, 56/5)$. Then $M = (33/10, 28/5), N = (15/2, 0)$, $D = (99/29, 168/29), E = (65/9, 0)$, the last two by angle bisector theorem.

Now, we know nothing about circumcircles, so we set about finding X, the center of AMN, and Y, the center of ADE. Then let $X = (p,q)$, and $Y = (r,s)$. We can find $p = 15/4$ by inspection of A and N; similarly $r = 65/18$ by looking at A and E. Now,

$$\left(\frac{15}{4}\right)^2 + q^2 = \left(\frac{9}{20}\right)^2 + \left(\frac{28}{5} - q\right)^2,$$

which leads to $14.0625 = .2025 + 31.36 - 11.2q$ and

$$q = \frac{17.5}{11.2} = \frac{25}{16}.$$

Similarly,

$$\left(\frac{65}{18}\right)^2 + s^2 = \left(\frac{99}{29} - \frac{65}{18}\right)^2 + \left(\frac{168}{29} - s\right)^2,$$

which leads to

$$s = \frac{1235}{696},$$

(no that doesn't reduce).

So our circles are $(x-p)^2 + (y-q)^2 = p^2 + q^2$ and $(x-r)^2 + (y-s)^2 = r^2 + s^2$ which reduce to $x^2 + y^2 = 2px + 2qy$ and $x^2 + y^2 = 2rx + 2sy$. Setting these equal, we get $px + qy = rx + sy$. Plugging in and multiplying through by 2088 to clear fractions, we arrive at

$$15660x + 6525y = 15080x + 7410y,$$

or $580x = 885y$. Thus, whatever point P is, ray AP is the first quadrant part of the line

$$y = \frac{580}{885}x = \frac{116}{177}x.$$

Solving for the equation of that ray and BC, we get the y coordinate of that intersection is

$$\frac{145}{22} = \frac{725}{110}.$$

That's the vertical distance between Q and B; the vertical distance between Q and C is thus

$$\frac{507}{110},$$

and our desired ratio is $725/507$, giving an answer of $725 - 507 = \boxed{218}$.

Solution 328. The answer is $\boxed{2011^{2008}(1005^{2011} - 1004^{2011})}$.

Solution by jaymuro 333. The sum of the Saturday distances is

$$|z^3 - z| + |z^5 - z^3| + \cdots + |z^{2013} - z^{2011}| = \sqrt{2012}.$$

The sum of the Sunday distances is

$$|z^2 - 1| + |z^4 - z^2| + \cdots + |z^{2012} - z^{2010}| = \sqrt{2012}.$$

Note that

$$|z^3 - z| + |z^5 - z^3| + \cdots + |z^{2013} - z^{2011}| = |z|(|z^2 - 1| + |z^4 - z^2| + \cdots + |z^{2012} - z^{2010}|),$$

so $|z| = 1$. Then

$$\begin{aligned} |z^2 - 1| + |z^4 - z^2| + \cdots + |z^{2012} - z^{2010}| &= |z^2 - 1| + |z^2||z^2 - 1| + \cdots + |z^{2010}||z^2 - 1| \\ &= |z^2 - 1| + |z|^2|z^2 - 1| + \cdots + |z|^{2010}|z^2 - 1| \\ &= 1006|z^2 - 1|, \end{aligned}$$

so,

$$|z^2 - 1| = \frac{\sqrt{2012}}{1006}.$$

We have that $|z^2| = |z|^2 = 1$. Let $z^2 = a + bi$, where a and b are real numbers, so $a^2 + b^2 = 1$. From the equation $|z^2 - 1| = \sqrt{2012}/1006$,

$$(a - 1)^2 + b^2 = \frac{2012}{1006^2} = \frac{1}{503}.$$

Subtracting these equations, we get

$$2a - 1 = 1 - \frac{1}{503} = \frac{502}{503},$$

so

$$a = \boxed{\frac{1005}{1006}}.$$

Solution by peelybonehead 342. Let $BD = x$ and $AE = ED = y$. Thus, $CD = 3x$ and $AC = 4x$. Noting that $\cos\angle ADC = -\cos\angle ADB$, applying Law of Cosines to angles $\angle ADC$ and $\angle ADB$ gives

$$7 = 9x^2 + y^2 - 6xy\cos\angle ADC$$
$$9 = x^2 + y^2 + 2xy\cos\angle ADC.$$

Multiplying both sides of the second equation by 3 and adding both equations together gives $34x^2 = 12 + 4y^2$. Now, noting that $\cos\angle CED = -\cos\angle AEC$, applying Law of Cosines to angles $\angle CED$ and $\angle AEC$ gives

$$9x^2 = 7 + y^2 - 2\sqrt{7}y\cos\angle CED$$
$$16x^2 = 7 + y^2 + 2\sqrt{7}y\cos\angle CED.$$

Adding the two equations, we get $25x^2 = 14 + 2y^2$. Next, subtracting $34x^2 = 12 + 4y^2$ and $50x^2 = 28 + 4y^2$ and solving for the positive value of x gives $x = 1$. Since we know that $x = 1$, $CD = 4$ and $AC = 4$. Applying Stewart's Theorem to triangle ACD with median $\overline{CE}$, we get

$$2y^2 + 14y = 16y + 9y$$

which can be solved to get $y = \frac{\sqrt{22}}{2}$. We now apply Law of Cosines on angles $\angle ADB$ and $\angle ADC$, which gives

$$16 = 22 + 9 - 6\sqrt{22}\cos\angle ADC$$
$$AB^2 = 22 + 1 + 2\sqrt{22}\cos\angle ADC$$

Multiplying both sides of the second equation by 3 and adding both equations together gives $3AB^2 = 84 \implies AB = 2\sqrt{7}$. Since $AC = BC$, $\triangle ABC$ is an isosceles triangle and thus $h_c = \sqrt{4^2 - 7} = 3$. Therefore,

$$[ABC] = \frac{3 \cdot 2\sqrt{7}}{2} = 3\sqrt{7}.$$

Thus, the answer is $\boxed{10}$.

Solution by peelybonehead 343. We use the identity

$$\cos^2 A + \cos^2 B + \cos^2 C + 2\cos A\cos B\cos C = 1.$$

Note that

$$\cos A\cos B - \sin A\sin B = \cos(A+B) = \cos(180^\circ - C) = -\cos C.$$

To get rid of the $2\sin A\sin B\cos C$ term on the left hand side and apply the identity mentioned above, we add $\cos^2 C + 2\cos A\cos B\cos C - 2\sin A\sin B\cos C$ to both sides of

$$\cos^2 A + \cos^2 B + 2\sin A\sin B\cos C = \frac{15}{8}.$$

Since $\cos^2 C + 2\cos A\cos B\cos C - 2\sin A\sin B\cos C$ is equal to

$$\cos^2 C + 2\cos C(\cos A\cos B\cos C - \sin A\sin) = \cos^2 C - 2\cos^2 C,$$

adding it to both sides gives

$$1 = \frac{15}{8} - \cos^2 C.$$

Thus, $\cos C = \sqrt{7/8}$, and therefore $\sin C = \sqrt{\frac{1}{8}}$. Similarly, we add

$$\cos^2 A + 2\cos A\cos B\cos C - 2\sin B\sin C\cos A$$

to both sides of

$$\cos^2 B + \cos^2 C + 2\sin B\sin C\cos A = \frac{14}{9}$$

to get

$$1 = \frac{14}{9} - \cos^2 C.$$

Thus, $\cos A = \sqrt{5/9}$ and $\sin A = 2/3$. Let our desired value be x. Adding

$$\cos^2 B + 2\cos A\cos B\cos C - 2\sin C\sin A\cos B$$

to both sides of

$$\cos^2 C + \cos^2 A + 2\sin C\sin A\cos B = x,$$

we get

$$1 = x - \cos^2 B.$$

We can now solve for x using our findings from above:

$$\begin{aligned}
x &= \cos^2 B + 1 \\
&= \cos^2(A + C) + 1 \\
&= (\cos A\cos C - \sin A\sin C)^2 + 1 \\
&= \left(\sqrt{\frac{5}{9}} \cdot \sqrt{\frac{7}{8}} - \frac{2}{3} \cdot \sqrt{\frac{7}{8}}\right)^2 + 1 \\
&= \left(\frac{\sqrt{35} - 2}{\sqrt{72}}\right)^2 + 1 \\
&= \frac{35 + 4 - 2\sqrt{140}}{72} + 1 \\
&= \frac{111 - 4\sqrt{35}}{72}.
\end{aligned}$$

Therefore,

$$x = \frac{111 - 4\sqrt{35}}{72},$$

and the answer is $p + q + r + s = \boxed{222}$.

Solution 345. The answer is equal to the coefficient of x^{168} in the generating function

$$(x^{-168} + x^{-167} + \cdots + x^{167} + x^{168})^4 = \frac{(x^{169} - x^{-169})^4}{(x - 1)^4},$$

which is $\boxed{761474}$.

Solution by TheUltimate123 351. Let N be the final number; it is sufficient to show

$$N^{1/L} \overset{?}{\geq} \frac{c^{n/L}-1}{c^{1/L}-1} = 1^{1/L} + c^{1/L} + \left(c^2\right)^{1/L} + \cdots + \left(c^{n-1}\right)^{1/L}.$$

Thus we only need the following monovariant:

Claim. For any a, b, we have $\left(ca+c^2b\right)^{1/L} \geq a^{1/L}+b^{1/L}$.

By Hölder's inequality,

$$\left[\left(c^{1/L}a^{1/L}\right)^L + \left(c^{2/L}b^{1/L}\right)^L\right]^{1/L}\left[\left(c^{-1/L}\right)^{L/(L-1)}\left(c^{-2/L}\right)^{L/(L-1)}\right]^{(L-1)/L} \geq a^{1/L}+b^{1/L}.$$

However $c = \phi^{L-1}$ and $1 = \phi^{-1}+\phi^{-2}$, so

$$\left(c^{-1/L}\right)^{L/(L-1)} + \left(c^{-2/L}\right)^{L/(L-1)} = c^{-1/(L-1)} + c^{-2/(L-1)} = 1,$$

and we are done.

Solution by shinichiman 363. First, we prove that

$$\cos\frac{\pi}{7}\cos\frac{2\pi}{7}\cos\frac{3\pi}{7} = \frac{1}{8}.$$

We have

$$\begin{aligned}
-\sin\frac{\pi}{7} = \sin\frac{8\pi}{7} &= 2\cos\frac{4\pi}{7}\sin\frac{4\pi}{7}, \\
&= 4\cos\frac{4\pi}{7}\cos\frac{2\pi}{7}\sin\frac{2\pi}{7}, \\
&= 8\cos\frac{4\pi}{7}\cos\frac{2\pi}{7}\cos\frac{\pi}{7}\sin\frac{\pi}{7}.
\end{aligned}$$

Therefore,

$$\cos\frac{4\pi}{7}\cos\frac{2\pi}{7}\cos\frac{\pi}{7} = \frac{-1}{8}.$$

Since

$$\cos\frac{4\pi}{7} = -\cos\frac{3\pi}{7},$$

so

$$\cos\frac{\pi}{7}\cos\frac{2\pi}{7}\cos\frac{3\pi}{7} = \frac{1}{8},$$

as desired.

Next, we will prove that

$$\sin\frac{\pi}{7}\sin\frac{2\pi}{7}\sin\frac{3\pi}{7} = \frac{\sqrt{7}}{8}.$$

Indeed, let $z = e^{\pi i/7}$ then we have $z^7 = -1$ and $z^{14} = 1$. Thus,

$$(1-z)(1-z^6) = 1 + z^7 - z - z^6 = \frac{1}{z} - z,$$

and

$$(1-z^{13})(1-z^8) = \left(1-\frac{1}{z}\right)\left(1-\frac{1}{z^6}\right) = -\frac{1}{z} - \frac{1}{z^6} = z - \frac{1}{z}.$$

Thus,

$$\sin^2\frac{\pi}{7}=\left[\frac{1}{2i}\left(z-\frac{1}{z}\right)\right]^2=\frac{1}{4}(1-z)(1-z^6)(1-z^8)(1-z^{13}).$$

Similarly, we obtain

$$\begin{aligned}\sin^2\frac{5\pi}{7} &= \frac{1}{4}(1-z^2)(1-z^5)(1-z^{12})(1-z^9),\\ \sin^2\frac{3\pi}{7} &= \frac{1}{4}(1-z^3)(1-z^4)(1-z^{10})(1-z^{11}).\end{aligned}$$

Thus, by multiplying all of the above together and with $1-z^7=2$ we obtain

$$\left(\sin\frac{\pi}{7}\sin\frac{2\pi}{7}\sin\frac{3\pi}{7}\right)^2=\frac{1}{4^3\cdot 2}\prod_{i=1}^{13}(1-z^i).$$

Note that $\prod_{i=1}^{13}(1-z^i)=f(1)=14$, where $f(x)=x^{13}+x^{12}+\cdots+x+1=(x^{14}-1)/(x-1)$ that has 13 roots $z,z^2,\cdots,z^{13}$. Thus,

$$\left(\sin\frac{\pi}{7}\sin\frac{2\pi}{7}\sin\frac{3\pi}{7}\right)^2=\frac{7}{8^2}$$

or

$$\sin\frac{\pi}{7}\sin\frac{2\pi}{7}\sin\frac{3\pi}{7}=\frac{\sqrt{7}}{8}.$$

Combining these two identities we obtain

$$\tan\frac{\pi}{7}\tan\frac{2\pi}{7}\tan\frac{3\pi}{7}=\sqrt{7}.$$

Solution by Shaddoll 369. We consider the probability that the frog ever returns to 0 if the frog is currently at 1(by symmetry, this probability is the same when the frog is at -1), let it be p. Now, note that p is also the probability that the frog can ever get back to 1 when it's at 2, by translation. Thus,

$$p=\frac{1}{3}+\frac{1}{3}p^2,$$

(either he gets back to 0 with probability 1/3, or he goes to 2 with probability 1/3 and has to pass by 1 on his way to 0 with probability p he'll ever do it, and another factor of p to get back to 0, or he gets eaten and obviously can't return to 0), and solving, we have

$$p=\frac{3-\sqrt{5}}{2},$$

(the other root is greater than 1, so it clearly can't work). Thus, the chance that he ever returns to 0 when he's currently at 0 is

$$\frac{2p}{3}=\frac{3-\sqrt{5}}{3},$$

denote this by q(since he avoids getting eaten off the start, then returns to 0), so the expected number of times he returns to 0 is

$$q+q^2+q^3+\cdots=\frac{1}{\frac{\sqrt{5}}{3-\sqrt{5}}}=\frac{3-\sqrt{5}}{\sqrt{5}}=\frac{3\sqrt{5}-5}{5}.$$

Solution by Michael Tang 370. Denote the first coordinate of a vector v by v'. Let $\ell = \max(|v_1|, |v_2|, \ldots, |v_m|)$ and $t = \min(v_1', v_2', \ldots, v_m') > 0$. We will show that $C = (\ell/t)^2$ works.

Lemma: After R rounds ($R \geq 1$), we have $|w| \leq \ell\sqrt{R}$, regardless of our choices for i.

Proof of Lemma: Induct on R. For $R = 1$, we have $w \in \{v_1, \ldots, v_m\}$, so by the definition of ℓ, we have $|w| \leq \ell = \ell\sqrt{1}$, as desired. Now suppose that, for some $R \geq 1$, we have $|w| \leq \ell\sqrt{R}$ after R rounds, regardless of our choices for i. Let v_j be the chosen vector in round $R+1$. Then

$$|w + v_j|^2 = |w|^2 + |v_j|^2 + 2(w \cdot v_j) \leq (\ell\sqrt{R})^2 + \ell^2 + 0 = \ell^2(R+1)$$

so $|w + v_j| \leq \ell\sqrt{R+1}$, completing the induction. ■

Now suppose the process lasts R rounds ($R \geq 1$). Since in each round we add some v_i to w, after R rounds we must have $w' \geq R \cdot t$. Thus $|w| \geq w' \geq R \cdot t$. But by the Lemma, we also have $|w| \leq \ell\sqrt{R}$, hence

$$R \cdot t \leq |w| \leq \ell\sqrt{R} \implies R \leq \left(\frac{\ell}{t}\right)^2 \leq C,$$

as claimed.

Solution by David Altizio 371. First note that

$$\angle I_1AI_2 = \angle I_1AX + \angle XAI_2 = \frac{\angle BAX}{2} + \frac{\angle CAX}{2} = \frac{\angle A}{2},$$

is a constant not depending on X, so by

$$[AI_1I_2] = \tfrac{1}{2}(AI_1)(AI_2)\sin\angle I_1AI_2,$$

it suffices to minimize $(AI_1)(AI_2)$. Let $a = BC$, $b = AC$, $c = AB$, and $\alpha = \angle AXB$. Remark that

$$\angle AI_1B = 180^\circ - (\angle I_1AB + \angle I_1BA) = 180^\circ - \tfrac{1}{2}(180^\circ - \alpha) = 90^\circ + \tfrac{\alpha}{2}.$$

Applying the Law of Sines to $\triangle ABI_1$ gives

$$\frac{AI_1}{AB} = \frac{\sin\angle ABI_1}{\sin\angle AI_1B} \qquad \Rightarrow \qquad AI_1 = \frac{c\sin\frac{B}{2}}{\cos\frac{\alpha}{2}}.$$

Analogously one can derive $AI_2 = \frac{b\sin\frac{C}{2}}{\sin\frac{\alpha}{2}}$, and so

$$[AI_1I_2] = \frac{bc\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}{2\cos\frac{\alpha}{2}\sin\frac{\alpha}{2}} = \frac{bc\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}{\sin\alpha} \geq bc\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2},$$

with equality when $\alpha = 90^\circ$, that is, when X is the foot of the perpendicular from A to $\overline{BC}$. In this case the desired area is $bc\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$. To make this feasible to compute, note that

$$\sin\frac{A}{2} = \sqrt{\frac{1-\cos A}{2}} = \sqrt{\frac{1 - \frac{b^2+c^2-a^2}{2bc}}{2}} = \sqrt{\frac{(a-b+c)(a+b-c)}{4bc}}.$$

Applying similar logic to $\sin\frac{B}{2}$ and $\sin\frac{C}{2}$ and simplifying yields a final answer of

$$\begin{aligned} bc\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} &= bc \cdot \frac{(a-b+c)(b-c+a)(c-a+b)}{8abc} \\ &= \frac{(30-32+34)(32-34+30)(34-30+32)}{8 \cdot 32} = \boxed{126}. \end{aligned}$$

Solution by David Altizio 372. We claim that in general the answer is

$$K = 2R^2 \sin\varphi_A \sin\varphi_B \sin\varphi_C,$$

where R is the common circumradius. Note that this generalizes the $K = 2R^2 \sin A \sin B \sin C$ formula for the area of a triangle. Further note that since $\sin t = \sin(\pi - t)$ for all $t \in \mathbb{R}$, the acute versus obtuse distinction on the φ_Xs is irrelevant. The key observation is that all the information given in the problem can be condensed into a single quadrilateral.

Consider points W, X, Y, and Z in this order about a circle ω such that $WXYZ$ is quadrilateral A. Now let X' be a point on minor arc $\widehat{WY}$ such that $XX' \parallel WY$. Then $WXX'Y$ is an isosceles trapezoid, so $WX'YZ$ is another quadrilateral which can be formed by the four given sticks; WLOG let it be quadrilateral B. Now a bit of angle chasing reveals that

$$\varphi_B = \angle(WY, ZX') = \angle ZX'X = \angle ZYX,$$

and so we deduce that one of the interior angles of quadrilateral A has angular measure φ_B. Similarly, one of the other two interior angles (i.e. neither $\angle ZYX$ nor $\angle ZWX$) has angular measure φ_C.

The problem is practically finished from here, as we deduce

$$\begin{aligned} K = \tfrac{1}{2}(WY)(ZX)\sin\varphi_A = \tfrac{1}{2}(2R\sin\varphi_B)(2R\sin\varphi_C)\sin\varphi_A \\ = 2R^2 \sin\varphi_A \sin\varphi_B \sin\varphi_C, \end{aligned}$$

which is what we wanted. Applying this to the particular case at hand, a quick computation gives $K = \frac{24}{35}$ for an answer of $\boxed{059}$.

(For anyone curious, the specifications in the problem are satisfied if the lengths of the sticks are approximately 0.32, 0.91, 1.06, and 1.82. I actually forgot to check this when I submitted the problem - thanks CAMC for making sure the configuration is valid!)

Solution by HrishiP 374. Allow $a = \cos\alpha + i\sin\alpha$, and define b, c similarly. Note that on the complex plane, we require a, b, c lie on the unit circle (set of points containing complex numbers z such that $|z| = 1$). Since $a + b + c = 1 + i$, we are motivated to isolate a and find $1 + i - a = b + c$. The only thing we know is that $|b + c| \le 2$. So, we have

$$|b + c|^2 = |2\cos\beta \cdot \cos\gamma + 2\sin\beta + 2\sin\gamma + 2| = |2\cos(\beta - \gamma) + 2|.$$

It is optimal to have equality, because we are trying to minimize the real part of a, and if $\cos\beta, \cos\gamma$ are as large as possible then $\cos\alpha$ is small. Since $|b + c|^2 \le 2^2$ we need $\cos(\beta - \gamma) = 1 \implies \beta = \gamma$. Now, we solve the system

$$\begin{aligned} \cos\alpha + 2\cos\beta = 1, \\ \sin\alpha + 2\sin\beta = 1. \end{aligned}$$

Now we solve this system. If we square the equations and add them, we have

$$\begin{aligned} (\cos^2\alpha + \sin^2\alpha) + 4(\cos\alpha\cos\beta + \sin\alpha\sin\beta) + 4(\cos^2\beta + \sin^2\beta) = 2, \\ 5 + 4(\cos\alpha\cos\beta + \sin\alpha\sin\beta) = 2, \\ \cos\alpha\cos\beta + \sin\alpha\sin\beta = -\tfrac{3}{4}. \end{aligned}$$

Note that

$$\cos\beta = \frac{1 - \cos\alpha}{2} \text{ and } \sin\beta = \frac{1 - \sin\alpha}{2}.$$

If we substitute these into the derived equation, and rearrange (also using Pythagorean identities) we have

$$\begin{aligned}\cos\alpha + \sin\alpha &= -\tfrac{1}{2},\\ \sqrt{1-\cos^2\alpha} &= -\left(\tfrac{1}{2}+\cos\alpha\right),\\ 2\cos^2\alpha + \cos\alpha - \tfrac{3}{4} &= 0.\end{aligned}$$

Solving the quadratic for $\cos\alpha$ gives $\cos\alpha = \frac{-1\pm\sqrt{7}}{4}$. Since we want the smaller solution, the answer is

$$\boxed{\tfrac{-1-\sqrt{7}}{4}}.$$

Solution by MathStudent2002 375. Assume Q is irreducible and fix Q. Now, let Y_Q be the limit of the probability that $Q \mid P$. We note that we only care about the Q so that $Y_Q \neq 0$. For size reasons this means Q's roots all must have modulus 1 (since for a root z and big n the i's are usually spread apart enough that z is just never a root), which means Q's roots are roots of unity, thus Q must be cyclotomic, but $\deg Q \leq 3$ so Q is either $X-1, X+1, X^2+X+1$, or X^2+1. Clearly the first one also has $Y_Q = 0$.

Let $Y_1 = Y_{X+1}, Y_2 = Y_{X^2+X+1}, Y_3 = Y_{X^2+1}$, and let $Y_4 = Y_1 + Y_3$. Note that if S modulo 4 has a, b, c, d of $0, 1, 2, 3$, respectively, then Y_1 occurs if and only if $a + c = b + d$, and Y_3 occurs if and only if $a = c, b = d$. These two never coincide, so Y_4 is the probability that P is divisible by one of $X + 1, X^2 + 1$. Furthermore,

$$Y_1 = \frac{\binom{6}{3}}{2^6} = \frac{5}{16},$$

while

$$Y_3 = \frac{5}{16}\cdot\frac{5}{16},$$

so that

$$Y_4 = \frac{105}{256}.$$

Finally,

$$Y_3 = \frac{\binom{6}{2,2,2}}{3^6} = \frac{10}{81}.$$

So, since divisibility by X^2+X+1 depends on $S \pmod 3$ while divisibility by one of $X+1, X^2+1$ depends on $S \pmod 4$, by Chinese Theorem these are essentially independent for big n. So, the probability that one of the divisibilities occurs is

$$1-(1-Y_3)(1-Y_4) = 1-\frac{71}{81}\cdot\frac{151}{256} = 1-\frac{10721}{20736} = \boxed{\frac{10015}{20736}}.$$

Solution by CantonMathGuy 376. No. The image must be centrally symmetric about the image of $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, so it cannot be a polygon with an odd number of vertices.

Solution by iNomOnCountdown 377. If x, y, z are all positive, this always works. If one of them is negative, WLOG, let z be negative and x, y be positive. Casework shows that the equation holds if and only if $|z| > x + y$, which happens $\frac{1}{6}$ of the time in this case. Additionally, note that multiplying all the variables by -1 does not change the result. So, our answer is:

$$\frac{\frac{1}{8} + \frac{3}{8} \cdot \frac{1}{6}}{\frac{1}{2}} = \boxed{\frac{3}{8}}.$$

Solution by Michael Tang 382. Let R denote the foot of the A-altitude. Recall that

$$\triangle APQ \sim \triangle RPB \sim \triangle RCQ.$$

Thus, we have $\frac{AP}{PQ} = \frac{RP}{PB}$, which yields

$$PR = \frac{AP \cdot PB}{PQ} = \frac{XP \cdot PY}{PQ} = \frac{10 \cdot 40}{25} = 16.$$

Analogous reasoning yields $QR = 21$.

Now recall that $\triangle ABC$ is the excentral triangle of $\triangle PQR$; this follows since e.g. $BQ \perp AC$ and BQ bisects $\angle PQR$. Thus, the problem is reduced to the following: if $\triangle ABC$ satisfies $BC = 25$, $AB = 21$, and $AC = 16$, and I_A, I_B, and I_C are the excenters of $\triangle ABC$, then what is $I_AI_B \cdot I_AI_C$?

Denote by A' and B' the projections of I_A and I_B onto BC and AC respectively. Compute $CA' = s - b$ and $CB' = s - a$, and so armed with the fact that $C \in \overline{I_AI_B}$ we obtain

$$I_AI_B = I_AC + CI_B = \frac{s-a}{\sin\frac{C}{2}} + \frac{s-b}{\sin\frac{C}{2}} = \frac{c}{\sin\frac{C}{2}}.$$

Similarly, $I_AI_C = b/\sin\frac{B}{2}$, so

$$I_AI_B \cdot I_AI_C = \frac{bc}{\sin\frac{B}{2}\sin\frac{C}{2}}.$$

Two applications of the Law of Cosines give $\cos B = 27/35$ and $\cos C = 11/20$. Thus,

$$\sin\frac{B}{2} = \sqrt{\frac{1-\cos B}{2}} = \sqrt{\frac{4}{35}},$$

and

$$\sin\frac{C}{2} = \sqrt{\frac{9}{40}},$$

so, finally,

$$\frac{bc}{\sin\frac{B}{2}\sin\frac{C}{2}} = \frac{21 \cdot 16}{\sqrt{\frac{4}{35} \cdot \frac{9}{40}}} = 560\sqrt{14}.$$

The requested answer is $560 + 14 = \boxed{574}$.

Solution by CantonMathGuy 383. The allowed displacements are permutations of $(\pm 2, 0, 0, 0)$ and $(\pm 1, \pm 1, \pm 1, \mp 1)$, corresponding to the factor

$$\sum_{\text{cyc}} (a^2 + a^{-2}) + \sum_{\text{cyc}} (a^{-1}bcd + ab^{-1}c^{-1}d^{-1}).$$

Miraculously, this factors as

$$(abcd)^{-2}(abcd+1)(ab+cd)(ac+bd)(ad+bc).$$

It follows that the desired quantity is the coefficient of $(abcd)^{90}$ in the expression

$$(abcd+1)^{40}(ab+cd)^{40}(ac+bd)^{40}(ad+bc)^{40}.$$

Now we may check that the only way to obtain $(abcd)^{90}$ is to choose $abcd$ 30 times, 1 10 times, and each of ab, cd, ac, bd, ad, and bc, 20 times. Thus, the answer is

$$\tbinom{40}{10}\tbinom{40}{20}^3.$$

Solution by pieater314159 384.

$$(0,0), (1,1), \left(-\frac{1}{4}, \frac{31}{32}\right), \left(\frac{31}{32}, -\frac{1}{4}\right).$$

Solution by HamstPan38825 387. Set O_1 and O_2 to be the centers of ω_1 and ω_2, respectively. Let $\theta_1 = \angle X_1O_1T_1$ and $\theta_2 = \angle T_2O_2X_2$. Then, the condition $2X_1T_1 = X_2T_2$ yields

$$2\sin\frac{\theta_1}{2} = \frac{1}{2}\left(3\sin\frac{\theta_2}{2}\right) \iff \sin\frac{\theta_2}{2} = \frac{4}{3}\sin\frac{\theta_1}{2}.$$

Now, we compute the distance from O to $\overline{T_1T_2}$ in two ways. Let ℓ be the line through O perpendicular to $\overline{T_1T_2}$.

First, drawing the line through O_1 parallel to $\overline{T_1T_2}$, the distance from O to this line is $10\cos\theta_1$, so the altitude has length $10\cos\theta_1 + 2$. Next, let $P = \overline{OX_2} \cap \overline{T_1T_2}$. As $\theta_2 > 90^\circ$, we can compute $O_2P = -\frac{3}{\cos\theta_2}$, so $OX = 9 + \frac{3}{\cos\theta_2}$ and as $\measuredangle(OX_2, \ell) = -\theta_2$, we have that the distance also equals

$$d = -\cos\theta_2\left(9 + \frac{3}{\cos\theta_2}\right) = -9\cos\theta_2 - 3.$$

Setting these equal and using double angle,

$$10\left(1 - 2\sin^2\frac{\theta_1}{2}\right) + 5 + 9\left(1 - 2\sin^2\frac{\theta_2}{2}\right) = 0,$$

so this yields $\sin\frac{\theta_1}{2} = \sqrt{\frac{6}{13}}$. Then, the altitude has length $\frac{36}{13}$, so

$$AB = 2\sqrt{12^2 - \left(\frac{36}{13}\right)^2} = \boxed{\frac{96\sqrt{10}}{13}}.$$

Solution 398. The answer is $\boxed{(n+2)/n}$.

Solution 399. The answer is $\boxed{\frac{2\pi}{3}+\frac{\sqrt{3}}{4}}$.

Solution by naman12 402. Note that the volume of a parallelopiped is simply

$$V = (\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}$$

where $\mathbf{a},\mathbf{b},\mathbf{c}$ are the three sides. Note that this is the triple product, so

$$V = \det\mathbf{M} = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}.$$

Thus, if α,β,γ are the angles $\angle(\mathbf{b},\mathbf{c}),\angle(\mathbf{c},\mathbf{a}),\angle(\mathbf{a},\mathbf{b})$,

$$\begin{aligned}
V^2 &= \det(\mathbf{M}\cdot\mathbf{M}^\intercal)\\
&= \det\begin{vmatrix} a_x^2+a_y^2+a_z^2 & a_xb_x+a_yb_y+a_zb_z & a_xc_x+a_yc_y+a_zc_z \\ b_xa_x+b_ya_y+b_za_z & b_x^2+b_y^2+b_z^2 & b_xc_x+b_yc_y+b_zc_z \\ c_xa_x+c_ya_y+c_za_z & c_xb_x+c_yb_y+c_zb_z & c_x^2+c_y^2+c_z^2 \end{vmatrix}\\
&= \det\begin{vmatrix} \mathbf{a}\cdot\mathbf{a} & \mathbf{a}\cdot\mathbf{b} & \mathbf{a}\cdot\mathbf{c} \\ \mathbf{b}\cdot\mathbf{a} & \mathbf{b}\cdot\mathbf{b} & \mathbf{b}\cdot\mathbf{c} \\ \mathbf{c}\cdot\mathbf{a} & \mathbf{c}\cdot\mathbf{b} & \mathbf{c}\cdot\mathbf{c} \end{vmatrix}\\
&= \ a^2(b^2c^2-b^2c^2\cos^2\alpha) - ab\cos\gamma(abc^2\cos\gamma - abc^2\cos\alpha\cos\beta)\\
&\quad + ac\cos\beta(ab^2c\cos\gamma\cos\alpha - ab^2c\cos\beta)\\
&= a^2b^2c^2(1+2\cos\alpha\cos\beta\cos\gamma-\cos^2\alpha-\cos^2\beta-\cos^2\gamma).
\end{aligned}$$

Note that the side length of the rhombus is

$$\frac{1}{2}\sqrt{\sqrt{21}^2+\sqrt{31}^2} = \sqrt{13}.$$

Let θ be the acute angle of the rhombus. Note

$$\sqrt{\frac{1-\cos\theta}{2}} = \sin\frac{1}{2}\theta = \frac{\sqrt{21}}{2\sqrt{13}}.$$

Thus,

$$\frac{1-\cos\theta}{2} = \frac{21}{52} \implies \cos\theta = \frac{5}{26}.$$

Thus, the ratio of the volumes is

$$\frac{1+2(\frac{5}{26})^3-3(\frac{5}{26})^2}{1-2(\frac{5}{26})^3-3(\frac{5}{26})^2} = \frac{63}{62}.$$

Therefore, the answer is $63+62=\boxed{125}$.

Solution by Dragomir Grozev 404. Different approach. It needs additional work and calculations, but I hope it could be carried out. So, you have a set X of 1431 integers in the interval $[1, 2023]$. For any subset $A \subset X$ denote $S_A := \sum_{a \in A} a$. The main idea is to show that as A runs through all subsets of X, S_A covers all integers in $[L, S_X - L]$ for some not too large integer L. For example $L := 14000$ will do the job, but it can be optimized a lot! In some sense, this is a stronger claim. This claim almost solves the original problem, so let us sketch the proof.

Since $S_{X \setminus A} = S_X$-S_A, it is enough to prove S_A can cover the interval $[L, S_X/2]$. Consider the set $X' \subset X$ that consists of the last 715 largest numbers in X. Apparently, $S_{X'} \geq S_X/2$. Let $X'' := X \setminus X'$. Since $|X'| = 715$, it follows $X'' \subset [1..1310]$. (we can optimize the sets X'' and X' here, 715 is a very rough estimate).

1 For each $k \in [1200, 1403]$, there exists $A \subset X'', |A| = 2$ with $S_{A''} = k$. Indeed, assume the opposite and consider the the appropriate pairs $(k - i, i)$. If both numbers in a pair belong to X'' we are done. But it's impossible all of these pairs to have at most one number in X'' since the density of X'' prevents it.

2 For any $k_i \in [1200, 1403], i = 1, 2, \dots 10$ there exist distinct $x_i, y_i \in X'', i = 1, \dots, 10$ such that $x_i + y_i = k_i$. We can prove it using 1) subsequently for $i = 1, 2, \dots, 10$ and removing the pair x_i, y_i.

3 Thus, when the set $A \subset X'', |A| = 20$ varies, S_A covers all integers in $[10 \cdot 1400, 10 \cdot 1400 + 2030]$. Then, for any fixed $A' \subset X'$ the value $S_{A'} + S_A$ as $A \subset X''$ varies will contain interval with length 2030 and since each offset $x \in X'$ is less than 2030 it means we can cover the interval $[L, S_X/2]$ where $L = 14000$.

It remains some small cases. In case $S_X - 14000 \geq 2023 \cdot 1012 - S_X$ we are done. Consider now the case $2S_X < 2023 \cdot 1012 + 14000$. In this case X slightly differs from the first 1431 natural numbers, so it can be checked directly?

Solution by Matthew Kroesche 406. Here is a quick complex bash! Let $\triangle AKL$ be inscribed in the unit circle, and let O and H be the circumcenter and orthocenter of $\triangle ABC$ respectively, so that

$$\begin{aligned}
|a| = |k| = |\ell| &= 1, \\
d &= \frac{2k\ell}{k+\ell}, \\
o = \frac{a+d}{2} &= \frac{ak + a\ell + 2k\ell}{2(k+\ell)}, \\
b = k + o - ak\overline{o} &= \frac{k^2 + 3k\ell + a\ell - ak}{2(k+\ell)}, \\
c = \ell + o - a\ell\overline{o} &= \frac{\ell^2 + 3k\ell + ak - a\ell}{2(k+\ell)}, \\
h = a + b + c - 2o = \frac{k^2 + \ell^2 + 2k\ell}{2(k+\ell)} &= \frac{k+\ell}{2}.
\end{aligned}$$

So in fact the orthocenter of $\triangle ABC$ is the midpoint of $\overline{KL}$.

Solution by Gabriel Goh 407. Take a big n and use the strings with $n+1$ 1s and the string with $n+1$ 0s. These cost $1/2^n$ total. Now any possible valid string can only have at most n consecutive bits that are equal. Replace the binary string with an integer array depicting the length of consecutive equal blocks. For example 011000100... becomes 12312...

Choose a big m. We pick the binary string corresponding to every possible m length array. For example if $n=2$ and $m=3$ we would take 010, 0100, 0110, 01100, 0010, 00100, 00110, 001100 and their complementaries. We calculate the cost. It is equal to

$$2\left(\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2^n}\right)^m$$

by generating functions (the first factor of 2 is because it can start with 0 or 1). This is equal to

$$2\left(1-\frac{1}{2^n}\right)^m.$$

Hence we have found a construction with cost

$$\frac{1}{2^n}+2\left(1-\frac{1}{2^n}\right)^m,$$

and taking m, n sufficiently large finishes.