Summer Kaywañan Algebra Competitions

A.K.A.

Summer KACY

KACY--I006:

Olympiad Pre-Algebra Contest 006

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Synopsis

The Olympiad Algebra Book comes in two volumes. The first volume, dedicated to Polynomials and Trigonometry, is a collection of lesson plans containing 1220 beautiful problems, around two-thirds of which are polynomial problems and one-third are trigonometry problems. The second volume of The Olympiad Algebra Book contains 1220 Problems on Functional Equations and Inequalities, and I hope to finish it before the end of Summer 2023. I hope I can finish collecting the FE and INEQ problems by June 29^{th}, as a reminder of the 1220 Number Theory Problems published as the first 1220 set of J29 Project. The current volumes has 843 Polynomial problems and 377 Trigonometry questions, the last 63 of which are bizarre spherical geometry problems!

The Olympiad Algebra Book is supposed to be a problem bank for Algebra, and it forms the resource for the first series of the KAYWAÑAN Algebra Contest. I suggest you start with Polynomials, and before you get bored or exhausted, also start solving Trigonometry problems. If you find these problems easy and not challenging enough, the Spherical Trigonometry lessons and problems are definitely going to be a must try!

This booklet contains problems and solutions of `KACY--I006` (Olympiad Pre–Algebra Contests), including the problems from the first book:

$$\text{KACY–I}\,\{13, 47, 64, 81, 93, 95, 108, 109, 118\}\,.$$

The numbers referred here are the question number out of the 1220 questions labeled from 1 to 1220. The competition's full title is "Kaywañan Olympiad Pre–Algebra Summer Contest 006," held on Saturday July 15^{th}, 2023.

Amir Parvardi,
Vancouver, British Columbia,
July 15, 2023

Contents

"Let No One Ignorant of Algebra Enter!"

KAYWAN

The rules of the KACY Competitions are simple:

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a) All problems whose titles contain **KACY–I** are questions of the Summer KACY Series, and all problems with a title containing **KACY–II** are questions of the Winter KACY Series.

b) This is the first volume of KACY, and it contains the SUMMER KACY questions. For the SUMMER KACY 2023 held weekly in Summer and Fall of 2023, only questions with title containing "**KACY–I**" are to be used in the actual KAYWAÑAN competitions.

This is because all the questions whose source does not contain **KACY–I** are either from a legit mathematical competition such as IMO, IMO Shortlist/Longlist, MAA Series (AMC, AIME, USAMO, USATST, USATSTST, USAMTS, etc.), National or Regional Olympiads (USA, APMC, Canada, etc.), or maybe from a book/paper I found and referenced in the question's title.

This assures that no famous problems are used in KACY, and that we actually identify and solve the non–KACY problems as exercises and examples in our journey of learning algebra during KAYWAÑAN Algebra Contest.

KACY–I006 Problems

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KACY–I 13. Let p be a prime number. Prove that the polynomial

$$P(x) = x^{p-1} + 2x^{p-2} + \cdots + (p-1)x + p,$$

is irreducible over $\mathbb{Z}[x]$.

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KACY–I 47. Factorize $x^3 + 5x^2 + 3x - 9$.

KACY Summer League

KACY–I 64. Factorize $4(x^2+xy+y^2)^3-27x^2y^2(x+y)^2$.

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KACY–I 81. Factorize $(x-y)^7+(y-z)^7+(z-x)^7$.

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KACY–I 93. Factorize $x^3(z-y^2)+y^3(x-z^2)+z^3(y-x^2)+xyz(xyz-1)$.

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KACY–I 95. Factorize

$$[(x^2+y^2)(a^2+b^2)+4abxy]^2-4[xy(a^2+b^2)+ab(x^2+y^2)]^2.$$

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KACY–I 108. Assuming

$$f\left(x+\frac{1}{x}\right)=x^2+\frac{1}{x^2},$$

Find $f(x)$ for $|x|\geq 2$.

KACY Summer League

KACY–I 118. Find $f(x,y)$ given that

$$f\left(x+y,\frac{y}{x}\right)=x^2-y^2.$$

KACY–I006 Answers

The problems and solutions of KACY–I006 (except for the first one) are courtesy of Parviz Shahriari, and they are taken from his eternal two-volume Farsi contribution to mathematics: "Methods of Algebra." May he rest in peace!

Solution by Batominovski 13. If $p = 2$, the problem is trivial. If $p = 3$, we get $P(x) = x^2 + 2x + 3$, which is also irreducible. So, we assume that $p \geq 5$.

Note that $(x-1)P(x) = x\left(x^{p-1} + x^{p-2} + \cdots + 1\right) - p$. Therefore,

$$(x-1)^2 P(x) = x\left(x^p - 1\right) - p(x-1)\,.$$

Let $Q(x) := P(x+1)$. We see that

$$x^2 Q(x) = (x+1)\left((x+1)^p - 1\right) - px = (x+1)^{p+1} - (p+1)x - 1\,.$$

Therefore,

$$Q(x) = x^{p-1} + \binom{p+1}{p}x^{p-2} + \binom{p+1}{p-1}x^{p-3} + \cdots + \binom{p+1}{3}x + \binom{p+1}{2}\,.$$

Note that p divides all $\binom{p+1}{2}$, $\binom{p+1}{3}$, $\ldots$, $\binom{p+1}{p-1}$. Hence, reducing modulo p, we see that

$$Q(x) \equiv x^{p-1} + x^{p-2} \pmod{p}\,.$$

Suppose contrary that P is reducible (and, hence, so is Q). Write $Q(x) = R(x)S(x)$ for some non-constant $R(x), S(x) \in \mathbb{Z}[x]$. From

$$R(x)S(x) = Q(x) \equiv x^{p-2}(x+1) \pmod{p}\,,$$

if both R and S have degrees at least 2, we then see that x divides both $R(x)$ and $S(x)$ in $\mathbb{F}_p[x]$. Thus, p divides the constant terms of both R and S. This is a contradiction as the constant term of $Q(x)$ is not divisible by p^2. Thus, $\deg(R) = 1$ or $\deg(S) = 1$. Without loss of generality, $\deg(R) = 1$, or $R(x) = x + a$. It is easy to see that $a \equiv 1 \pmod{p}$. Also,

$$S(x) = x^{p-2} + s_{p-3}x^{p-3} + \cdots + s_1 x + s_0\,,$$

where p divides all s_0, s_1, $\ldots$, s_{p-3}. Since $Q(x) = (x+a)S(x)$, equating the constant term, we get $s_0 a = p\binom{p+1}{2}$, or

$$a \mid a\left(\frac{s_0}{p}\right) = \frac{p+1}{2}\,.$$

Since $a \equiv 1 \pmod{p}$, we see that

$$a = 1 \text{ or } |a| \geq (p-1) > \frac{p+1}{2}\,.$$

As a divides $\frac{p+1}{2}$, we must have $a = 1$. Hence, $Q(x)$ is divisible by $x+1$. Therefore, $P(x) = Q(x-1)$ is divisible by x, which is, again, absurd. Therefore, $P(x)$ is irreducible, as desired.

Solution by Parviz Shahriari 47. Answer: $(x-1)(x+3)^2$.

Solution by Parviz Shahriari 64. The expression is symmetric with respect to x and y, and it becomes zero by plugging $x=y$ and $x=-2y$, so the factorization is

$$4(x^2+xy+y^2)^3-27x^2y^2(x+y)^2=[(x+2y)(2x+y)(x-y)]^2.$$

Solution by Parviz Shahriari 81. This is a special case of a problem in KACY–I005 with $n=7$, so the factorization is

$$(x-y)^7+(y-z)^7+(z-x)^7=7(x-y)(y-z)(z-x)(x^2+y^2+z^2-xy-yz-zx)^2.$$

Solution by Parviz Shahriari 93. Answer: $(x^2-y)(y^2-z)(z^2-x)$.

Solution by Parviz Shahriari 95. The term in the first bracket simplifies to $(ax+by)^2+(ay+bx)^2$, and the term in the second bracket is $(ay+bx)(ax+by)$. Use the difference of squares to see that the given expression factorizes into

$$[(x^2+y^2)(a^2+b^2)+4abxy]^2-4[xy(a^2+b^2)+ab(x^2+y^2)]^2=(a-b)^2(a+b)^2(x-y)^2(x+y)^2.$$

Solution by Parviz Shahriari 108. Answer: $f(x)=x^2-2$.

Solution by Parviz Shahriari 118. Answer:

$$f(x,y)=x^2\cdot\frac{1-y}{1+y}.$$