A Appendix

A.1 Notation Summary

Table 1: Summary of the main notation used throughout the paper.

Notation	**Description**
$A = \{1, \dots, \alpha\}$	The **set of players** in a coalitional game, or **set of algorithms** in a portfolio.
(A, v) or v	A **coalitional game** specifying the value $v(C)$ of every **coalition** $C \subseteq A$.
$\phi_i(A, v)$ or $\phi_i(v)$	The **Shapley value** of player i in the coalitional game (A, v); see Equation (1).
$\mathcal{T}(A) = \{T^1, \dots, T^q\}$	A partition of A into q **equivalence classes**; each class may represent all algorithms developed in the same year.
$Q(i)$	The **index of the equivalence class** of player (or algorithm) i.
$\succ$	A **precedence relation** between algorithms, where $\forall i, j \in A$, $i \succ j$ iff $Q(i) \succ Q(j)$.
$A^{\succ}$	A **strict partially ordered set** (i.e., poset) of the elements of A according to $\succ$.
$\mathcal{C}(A^{\succ})$	The set consisting of every **downward closed** set of algorithms, i.e., every $C \subseteq A$ such that, $\forall i \in C$, $i \succ j$ implies that $j \in C$.
$\Pi(A)$	The set of all **permutations** of A.
$\Pi^{\succ}(A)$	$\Pi^{\succ}(A) = \big\{\pi \in \Pi(A) : \forall 1 \leq i \leq \alpha \ \{\pi(1), \pi(2), \dots, \pi(i)\} \in \mathcal{C}(A^{\succ})\big\}$. In words, it is the set consisting of every **permutation for which every prefix is downward closed**.
$\pi(i)$	The i^{th} **element in permutation** π.
C_i^{π}	The coalition consisting of **all predecessors of player** i in permutation π.
$(A, \succ, v^{\succ})$ or $v^{\succ}$	A **temporal coalitional game** specifying the value $v^{\succ}(C)$ of every coalition $C \subseteq A$ that is **downward closed** according to $\succ$.
$\phi_i^{\succ}(A, \succ, v^{\succ})$ or $\phi_i^{\succ}$	The **temporal Shapley value** of player i in the temporal coalitional game $(A, \succ, v^{\succ})$; see Equations (2) and (3).

A.2 The Four Main Axioms of the Shapley Value

The following set of axioms are widely seen as desirable properties for a coalitional solution concept, ϕ, that captures each player's fair contribution to a coalition, C, in a coalitional game, (A, v), where $\phi_i(A, v)$ denotes the contribution of player i according to ϕ.

Axiom 1 (Additivity). *$\forall v, w \in \mathcal{V}(A)$, $\forall C \in 2^A$, let $[v+w](C)$ denote $v(C)+w(C)$. Then $\phi(A, v)+\phi(A, w) = \phi(A, [v+w])$.*

Axiom 2 (Efficiency). *The grand coalition's value is divided entirely among the players: $\forall v \in \mathcal{V}(A)$, $\sum_{i \in A} \phi_i(A, v) = v(A)$.*

Axiom 3 (Null Player). *A player who contributes nothing receives nothing: $\forall v \in \mathcal{V}(A), \forall i \in A$, if $v(C \cup \{i\}) - v(C) = 0$ for every $C \in 2^{A \setminus \{i\}}$, then $\phi_i(A, v) = 0$.*

Axiom 4 (Symmetry). *Payoffs do not depend on the players' names, i.e., for every $v \in \mathcal{V}(A)$ and every bijection $f : A \to A$, $\phi(A, f(v)) = f(\phi)(A, v)$.*

A celebrated theorem shows that these solution concepts can only be satisfied by a single solution concept, namely the Shapley value (see, e.g., Solan *et al.* [2013] for more details), which is defined as:

$$\phi_i(A, v) := \frac{1}{|A|!} \sum_{\pi \in \Pi(A)} v(C_i^{\pi} \cup \{i\}) - v(C_i^{\pi}).$$

A second, equivalent formulation of the Shapley value is as follows:

$$\phi_i(A, v) = \sum_{C \in 2^{A \setminus \{i\}}} \frac{(|A| - |C| - 1)!\, |C|!}{|A|!} \left(v(C \cup \{i\}) - v(C)\right). \tag{7}$$

A.3 The Proof of Lemma 1

Proof. Let us begin by rewriting Equation (2) for the temporal Shapley value as follows:

$$\phi_i^{\succ} = \frac{1}{\prod_{k=1}^{q} |T^k|!} \sum_{\pi \in \Pi^{\succ}(A)} v^{\succ}(C_i^{\pi} \cup \{i\}) - v^{\succ}(C_i^{\pi}),$$

which comes from the fact that there are exactly $\prod_{k=1}^{q} |T^k|!$ permutations in $\Pi^{\succ}(A)$.

Next, let us consider an arbitrary C_i^{π}. Recall that C_i^{π} is the coalition consisting of all the players that precede i in permutation $\pi \in \Pi^{\succ}(A)$. Let us rewrite C_i^{π} as follows:

$$C_i^{\pi} = \bigcup_{k=1}^{p-1} T^k \cup C^p,$$

where $p = Q(i)$. Note that $C^p \subseteq T^p \setminus \{i\}$ and that, as per Equation (2), $i \notin C_i^{\pi}$.

Observe that in temporal coalitional games, the order of players in each equivalence class does not matter. This means that, among all the permutations in $\Pi^{\succ}(A)$, there are exactly

$$\prod_{k=1}^{p-1} T^k!|C^p|!$$

orderings in which coalition $\bigcup_{k=1}^{p-1} T^k \cup C^p$ could be permuted. Furthermore, for each particular ordering of this coalition, there exist $(T^p \setminus C^p - 1)! \prod_{k=p+1}^{q} T^k!$ permutations in $\Pi^{\succ}(A)$ that contain this ordering, and where player i is exactly in place $|T^1 \cup T^2 \cup \ldots T^{p-1} \cup C^p| + 1$. In other words, there are exactly:

$$\prod_{k=1}^{p-1} T^k!|C^p|!(T^p \setminus C^p - 1)! \prod_{k=p+1}^{q} T^k!,$$

such permutations. Since, in each such permutation, player i contributes to coalition $\bigcup_{k=1}^{p-1} T^k \cup C^p$, we may write:

$$\phi_i^{\succ} = \sum_{C^p \subseteq T^p \setminus \{i\}} \frac{\prod_{k=1}^{p-1} |T^k|!|C^p|!(T^p \setminus C^p - 1)! \prod_{k=p+1}^{q} |T^k|!}{\prod_{k=1}^{q} |T^k|!} \left(v^{\succ} \left(\bigcup_{k=1}^{p-1} T^k \cup C^p \cup \{i\} \right) - v^{\succ} \left(\bigcup_{k=1}^{p-1} T^k \cup C^p \right) \right).$$

Now, reducing common divisors from the numerator and denominator, we obtain Equation (3). □

A.4 The Proof of Lemma 2

Proof. In the proof we will use the Möbius function—a unique function $\mu : \mathcal{C}(A^{\succ}) \times \mathcal{C}(A^{\succ}) \to \mathbb{Z}$ that satisfies the following system of equations:[2]

$$\mu(S, U) = 0 \quad \text{if } S \not\subseteq U, \tag{8}$$
$$\mu(S, U) = 1 \quad \text{if } S = U, \text{ and} \tag{9}$$
$$\sum_{S \subseteq W \subseteq U} \mu(S, W) = 0 \quad \text{if } S \subsetneq U. \tag{10}$$

In the first step, we will show that the Möbius function for temporal] coalitional games has the following form. For every $S, U \in \mathcal{C}(A^{\succ})$:

$$\mu^*(S, U) = \begin{cases} (-1)^{|U|-|S|} & \text{if } \forall_{i \in U \setminus S} \forall_{j \in A \setminus S}\; Q(i) \le Q(j), \\ 0 & \text{otherwise.} \end{cases} \tag{11}$$

Since the Möbius function is uniquely defined by (8)–(10), to prove (11), it suffices to show that μ^* satisfies all the three condition. As for (8) and (9), if $S \not\subseteq U$, then, from definition, $\mu^*(S, U) = 0$ and $\mu^*(S, S) = 1$ for every $S, U \in \mathcal{C}(A^{\succ})$, i.e., both these conditions are satisfied.

Let us now consider condition (10), i.e., let us assume that $S \subsetneq U$. Before proceeding, we need to introduce some additional notation. In particular, let us denote by L the set of players/solvers from $A \setminus S$ that belong to the lowest equivalence class, i.e., $L = \{i \in A \setminus S : \forall_{j \in A \setminus S} Q(i) \le Q(j)\}$.

Now, since $U \in \mathcal{C}(A^{\succ})$, the following two cases can be distinguished:

(a) either U contains only a part of L and no element from any higher equivalence class, i.e., $(U \setminus S) \subseteq L$; or

(b) U contains the whole L, i.e., $L \subseteq (U \setminus S)$.

We will now prove that, in both cases, (11) satisfies condition (10). In particular, from binomial theorem, for case (a) we get:

$$\sum_{S \subseteq W \subseteq U} \mu^*(S, W) = \sum_{S \subseteq W \subseteq U} (-1)^{|W|-|S|} = (1-1)^{|U|-|S|} = 0,$$

[2] See the book by Bona [2011] for a detailed introduction to the Möbius function.

and for case (b), we get:

$$\sum_{S\subseteq W\subseteq U}\mu^*(S,W)=\sum_{S\subseteq W\subseteq S\cup L}\mu^*(S,W)=(1-1)^{|L|}=0.$$

Hence, we proved the Möbius function for temporal coalitional games is given by formula (11).

Following Lemma 2 from Faigle and Kern[Faigle and Kern, 1992, Lemma 2], we may write:

$$v^{\succ}=\sum_{U\in\mathcal{C}(A^{\succ})}\left(\sum_{S\in\mathcal{C}(A^{\succ})}\mu^*(S,U)v^{\succ}(S)\right)\sigma_U.$$

In our setting, by representing U as $\bigcup_{r=1}^{p-1}T^r\cup C$ for some $p\in\{1,\ldots,q\}$ and $C\subseteq T^p$, we get equivalently

$$v^{\succ}=\sum_{p=1}^{q}\sum_{\substack{C\subseteq T^p\\ C\neq\emptyset}}\left(\sum_{S\in\mathcal{C}(A^{\succ})}\mu^*(S,\bigcup_{r=1}^{p-1}T^r\cup C)v^{\succ}(S)\right)\sigma_{\bigcup_{r=1}^{p-1}T^r\cup C}.$$

Finally, using (11) we get that $\mu^*(S,\bigcup_{r=1}^{p-1}T^r\cup C)$ is non-zero if and only if $S=\bigcup_{r=1}^{p-1}T^r\cup D$ for some $D\subseteq C$. Thus,

$$v^{\succ}=\sum_{p=1}^{q}\sum_{\substack{C\subseteq T^p\\ C\neq\emptyset}}\left(\sum_{D\subseteq C}(-1)^{|C|-|D|}v^{\succ}(\bigcup_{r=1}^{p-1}T^r\cup D)\right)\sigma_{\bigcup_{r=1}^{p-1}T^r\cup C}.$$

This concludes the proof of Lemma 2. □

A.5 Computing the Temporal Shapley Value in Polynomial Time for the Characteristic Function (6)

In general, if we interpret Equation (3) as an algorithm—which is less demanding than Equation (2) —computing the temporal Shapley value of every solver takes time $O(\sum_{p=1}^{q}2^{T^p})$, i.e., the computation time is exponential in the sizes of equivalence classes. However, the specific form of the characteristic function (6) was already shown to enable polynomial computations of the standard Shapley value [Fréchette *et al.*, 2016] for standard coalitional games. To this end, Fréchette *et al.* [2016] used marginal contribution networks (MC-nets)—a well-known compact representation for coalitional games that admits polynomial-time computation of the Shapley value [Ieong and Shoham, 2005; Chalkiadakis *et al.*, 2011] and that had been already generalized [Elkind *et al.*, 2009; Aadithya *et al.*, 2011] and extended in various ways. In this appendix we extend the result by Fréchette *et al.* [2016] to the temporal Shapley value and temporal coalitional games.

With this scheme, a game is represented by a set of rules, $\mathcal{R}$, each of which is of the form $\mathcal{F}\to V$, where $\mathcal{F}$ is a propositional formula over A and V is a real number. A coalition C is said to *meet* a given formula $\mathcal{F}$ if and only if $\mathcal{F}$ evaluates to `true` when all Boolean variables corresponding to the players in C are set to `true`, and all Boolean variables corresponding to players outside C are set to `false`. We write $C\models\mathcal{F}$ to denote that C meets $\mathcal{F}$. In MC-nets, if coalition C does not meet any rule then its value is 0. Otherwise, the value of C is the sum of V from every rule in which $\mathcal{F}$ is met by C. More formally:

$$v(C)=\sum_{\mathcal{F}\to V\in\mathcal{R}:\ C\models\mathcal{F}}V. \tag{12}$$

For example, the MC-net where $\mathcal{R}=\{2\to 3, 1\wedge 2\to 5\}$ corresponds to the game $G=(\{1,2\},v)$ where $v(\{1\})=0$, $v(\{2\})=2$ and $v(\{1,2\})=8$. Intuitively, in this example, the rules mean that whenever 2 is present in a coalition, the value of that coalition increases by 3, and whenever 1 and 2 are present together in a coalition, its value increases by 5.

Ieong and Shoham [2005] focused on a particular version of their representation, called *basic MC-nets*, where $\mathcal{F}$ is made only of conjunctions of positive and/or negative literals, i.e., it has the form

$$p_{i_1}\wedge\ldots\wedge p_{i_k}\wedge\neg n_{j_1}\wedge\ldots\wedge\neg n_{j_l}\to V. \tag{13}$$

Let us write such a basic rule as $\mathcal{F}(\mathcal{P},\mathcal{N})\to V$, where $\mathcal{P}$ ($\mathcal{N}$) is the set of positive (negative) literals. Ieong and Shoham showed that if a coalitional game is represented by a set of such basic rules:

$$\mathcal{R}=\left\{\mathcal{F}\left(\mathcal{P}_1,\mathcal{N}_1\right)\to V_1,\ldots,\mathcal{F}\left(\mathcal{P}_{|\mathcal{R}|},\mathcal{N}_{|\mathcal{R}|}\right)\to V_{|\mathcal{R}|}\right\},$$

then the Shapley value can be computed in time $O(|A|\cdot|\mathcal{R}|)$.

We will now develop a method which, given the temporal coalitional game $(A,\succ,v^{\succ})$, where $v^{\succ}$ is given by the characteristic function (6), allows for computing the temporal Shapley value in polynomial time.

In particular, we will show that, for every temporal coalitional game with the characteristic function (6), there exists a standard coalitional game:

- that can be represented with the set of basic MC-net rules that is of size polynomial in the number of solvers, $|A|$, and instances, $|X|$, and
- the Shapley value of this standard coalitional game equals to the temporal Shapley value of the temporal coalitional game.

In the first step, for each instance $x \in X$, and for each equivalence class $T^p \in \mathcal{T}$, let us sort solvers $i \in A$ in the *ascending order* with respect to their individual performance on x. Given $x \in X$, we will denote the sequence of such orderings for $p = 1, \ldots, q$ by $\overrightarrow{s}_x$. Formally, $\overrightarrow{s}_x$ is a function $A \rightarrow \{1, \ldots, |A|\}$ and we will denote by $\overrightarrow{s}_x^{-1}(i)$ the position of solver i in $\overrightarrow{s}_x$.

The following holds:

Theorem 2. *Let $(A, \succ, v^{\succ})$ be the temporal coalitional game, where the characteristic function $v^{\succ}$ is given by* (6)*. Furthermore, let (A, v) be a standard coalitional that can be represented as the following set of basic MC-net rules:*

$$\mathcal{R} = \bigcup_{x \in X} \left\{ \begin{array}{l} \overrightarrow{s}_x(1) \bigwedge_{k=2}^{|T^1|} \neg \overrightarrow{s}_x(k) \rightarrow \mathsf{score}_x(\overrightarrow{s}_x(1)) \\ \ldots \\ \overrightarrow{s}_x(|T^1|) \rightarrow \mathsf{score}_x(\overrightarrow{s}_x(|T^1|)) \\ \overrightarrow{s}_x(|T^1|+1) \bigwedge_{k=2}^{|T^2|} \neg \overrightarrow{s}_x(k) \rightarrow \mathsf{score}_x(\overrightarrow{s}_x(|T^1|+1)) - \mathsf{score}_x(\overrightarrow{s}_x(|T^1|)) \\ \ldots \\ \overrightarrow{s}_x(|T^1 \cup T^2|) \rightarrow \mathsf{score}_x(\overrightarrow{s}_x(|T^1 \cup T^2|)) - \mathsf{score}_x(\overrightarrow{s}_x(|T^1|)) \\ \ldots \\ \overrightarrow{s}_x(|\bigcup_{k=1}^{q-1} T^k|+1) \bigwedge_{k=2}^{|A|} \neg \overrightarrow{s}_x(k) \rightarrow \mathsf{score}_x(\overrightarrow{s}_x(|\bigcup_{k=1}^{q-1} T^k|+1) - \mathsf{score}_x(\overrightarrow{s}_x(|\bigcup_{k=1}^{q-1} T^k|)) \\ \ldots \\ \overrightarrow{s}_x(|A|) \rightarrow \mathsf{score}_x(\overrightarrow{s}_x(|A|) - \mathsf{score}_x(\overrightarrow{s}_x(|\bigcup_{k=1}^{q-1} T^k|)) \end{array} \right\} \tag{14}$$

the size of which is $|X| \times |A|$. Then, for all $i \in A$ it holds that: $\phi_i(A, v) = \phi_i^{\succ}(A, \succ, v^{\succ})$.

Proof. We will continue to use the notation from the proof of Lemma 1, i.e., we will decompose any $C \in \mathcal{C}(A)$ into $C = \bigcup_{k=1}^{p-1} C^p$, where $C^p \cup T^p \neq \emptyset$ and $C^p \cap T^{p+1} = \emptyset$.

We begin by comparing the formula for the temporal Shapley value obtained in Lemma 1, i.e.:

$$\phi_i^{\succ}(A, \succ, v^{\succ}) = \sum_{C^p \subseteq T^p \setminus \{i\}} \frac{|C^p|!(|T^p \setminus C^p| - 1)!}{|T^p|!} \left(v^{\succ}\left(\bigcup_{k=1}^{p-1} T^k \cup C^p \cup \{i\}\right) - v^{\succ}\left(\bigcup_{k=1}^{p-1} T^k \cup C^p\right)\right), \tag{15}$$

where $p = Q(i)$, to the corresponding formula for the standard Shapley value, i.e.:

$$\phi_i(A, v) = \sum_{C \in 2^{A \setminus \{i\}}} \frac{(|A| - |C| - 1)!\, |C|!}{|A|!} \left(v(C \cup \{i\}) - v(C)\right). \tag{16}$$

We observe that both formulas are, in principle, the same. In particular, the coefficient $\frac{|C^p|!(|T^p \setminus C^p|-1)!}{|T^p|!}$ is exactly the same as $\frac{(|A|-|C|-1)!\,|C|!}{|A|!}$, if we consider T^p to be the set of players. Theoretically this is not the case because in each coalition that contains any player from T^p, by definition, there must be all the players from the previous equivalence classes $T^1, \ldots, T^{p-1}$. However, all such players from the previous equivalence classes have no direct bearing on the temporal Shapley value as the sum in the formula (15) from Lemma 1 only cycles over coalitions from T^p. In other words, they should be considered as constant.

Furthermore, we observe that the element:

$$\left(v^{\succ}\left(\bigcup_{k=1}^{p-1} T^k \cup C^p \cup \{i\}\right) - v^{\succ}\left(\bigcup_{k=1}^{p-1} T^k \cup C^p\right)\right)$$

in formula (15) and the element:

$$(v(C \cup \{i\}) - v(C))$$

in formula (16) are both marginal contributions.

We have just established that the formula for the temporal Shapley value is the same as the formula for the standard Shapley value, where the equivalence class T^p, $p = Q^{-1}(i)$, is the set of players A, and all the players from previous equivalence classes should be considered as constants. This means that we can use the result from Fréchette *et al.* [2016] for computing the standard Shapley value for the characteristic function (6) using simple MC-nets. In particular, assuming for clarity that there is only one problem instance, i.e., $X = \{x\}$, we have the following set of rules:

$$\mathcal{R}_x = \left\{ \begin{array}{l} \overrightarrow{s}_x(1) \bigwedge_{k=2}^{|A|} \neg \overrightarrow{s}_x(k) \rightarrow \mathsf{score}_x(\overrightarrow{s}_x(1)) \\ \overrightarrow{s}_x(2) \bigwedge_{k=3}^{|A|} \neg \overrightarrow{s}_x(k) \rightarrow \mathsf{score}_x(\overrightarrow{s}_x(2) \\ \ldots \\ \overrightarrow{s}_x(|A|) \rightarrow \mathsf{score}_x(\overrightarrow{s}_x(|A|)) \end{array} \right\}. \tag{17}$$

Here, $A = T^p$, where $p = Q^{-1}(i)$. Furthermore, we observe that the value of each rule $\mathsf{score}_x(\overrightarrow{s}_x(.)$ shoudl be modified. This is because, the interpretation of MC-nets in the context of standard Shapley value is that the value of each coalition is initially zero and then it is increased by the value of each rule satisfied by the coalitions, i.e., the value of the rule is the marginal contribution of the players in the rule to each coalition that satisfies the rule. Conversely, in the context of temporal Shapley valeu and the MC-net rules over the set of players T^p, we have to take into account that the value of each coalition is not zero initially but:

$$v^{\succ}\left(\bigcup_{k=1}^{p-1} T^k\right).$$

That is why, we need to modfiy the value of each rule by $-\mathsf{score}_x(\overrightarrow{s}_x(|\bigcup_{k=1}^{p-1} T^k|))$ for all $p = 2, \ldots, q$.

This concludes the proof of Theorem 2. $\square$