Linear Combination of vectors:

Let V be a vector space and $v_1, v_2, \ldots, v_n \in V$. The linear combination of $v_1, v_2, \ldots, v_n$ with coefficients $a_1, a_2, \ldots, a_n \in \mathbb{R}$ is the vector $\sum_{i=1}^{n} a_i v_i \in V$.

A vector $v \in V$ is a linear combination of $v_1, v_2, \ldots, v_n$ if there exist some $a_1, a_2, \ldots, a_n \in \mathbb{R}$ so that $v = \sum_{i=1}^{n} a_i v_i$.

Example: Vector space $\mathbb{R}^2$.

$v = (5, 10)$

$$(5, 10) = 5(1, 0) + 10(0, 1)$$

$$(5, 10) = (5, -5) + (0, 15)$$

$$(5, 10) = 5(1, -1) + 15(0, 1)$$

* $(4,3) = a(1,2) + b(3,2)$
find 'a', 'b' ?

$$(4,3) = \tfrac{1}{4}(1,2) + \tfrac{5}{4}(3,2)$$
$$= (a, 2a) + (3b, 2b)$$
$$(4,3) = (a+3b,\ 2a+2b)$$

$$\left.\begin{aligned} a+3b &= 4 \\ 2a+2b &= 3. \end{aligned}\right\} \Rightarrow \text{2 eq}^n \text{ and 2 variable.}$$

$$\underbrace{\begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix}}_{\text{coefficient matrix}} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$$

$$\det\begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix} = -4 \neq 0.$$

$\left\{\begin{array}{l} \text{(1) Cremer's Rule} \\ \text{(2) Invertible matrix} \end{array}\right\}$ (3) Row echelon form

$$\left[\begin{array}{cc|c} 1 & 3 & 4 \\ 2 & 2 & 3 \end{array}\right] \xrightarrow{R_2 - 2R_1} \left[\begin{array}{cc|c} 1 & 3 & 4 \\ 0 & -4 & -5 \end{array}\right] \xrightarrow{\left(\frac{-1}{4}\right)R_2} \left[\begin{array}{cc|c} 1 & 3 & 4 \\ 0 & 1 & 5/4 \end{array}\right]$$

$$\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 4 \\ 5/4 \end{pmatrix}$$

$$\Rightarrow \quad \begin{aligned} a+3b &= 4 \\ b &= 5/4 \end{aligned} \quad \rightsquigarrow \quad a = 1/4.$$

* Linear dependence:

A set of vectors $V_1, V_2, \ldots, V_n$ from a vector space V is said to be linearly dependent, if there exist scalars $a_1, a_2, \ldots, a_n$, not all zero such that

$$a_1 V_1 + a_2 V_2 + \cdots + a_n V_n = 0.$$

Example: Find out if vectors are linearly dependent or not?

(1) $(4,5)$ and $(1/4, 1/5)$

Suppose $a, b \in \mathbb{R}$,

$$a(4,5) + b\left(\tfrac{1}{4}, \tfrac{1}{5}\right) = (0,0)$$

$$\begin{aligned} 5 \times \; & 4a + b/4 = 0 \\ 4 \times \; & 5a + b/5 = 0 \end{aligned}$$

$$20a + 5b/4 = 0$$
$$20a + 4b/5 = 0$$

$$\frac{5b}{4} - \frac{4b}{5} = 0$$

$$\frac{25b - 16b}{20} = \frac{9b}{20} = 0$$

$$\Rightarrow \boxed{b=0}$$

$$4a + b/4 = 0$$

$$4a = 0 \Rightarrow \boxed{a=0}$$

Not linearly dependent.

(2) $(3,-3)$ and $(2,-2)$? $\to$ linearly dependent.

$$a(3,-3)+b(2,-2)=(0,0)$$

$$\left.\begin{aligned} 3a+2b&=0\\ -3a-2b&=0 \end{aligned}\right\} \rightsquigarrow 3a+2b=0$$

$$a=\frac{-2b}{3}$$

$b\in\mathbb{R}$.

$$\left[\begin{array}{cc|c} 3 & 2 & 0\\ -3 & -2 & 0 \end{array}\right] \xrightarrow{R_2+R_1} \left[\begin{array}{cc|c} 3 & 2 & 0\\ 0 & 0 & 0 \end{array}\right]$$

$$\downarrow R_1/3$$

$$\left[\begin{array}{cc|c} 1 & 2/3 & 0\\ 0 & 0 & 0 \end{array}\right]$$

a - dependent

b - independent.

$$\left.\begin{aligned} b&=1\\ a&=-\frac{2}{3} \end{aligned}\right\}$$

(3.) $(1,3,4)$ & $(4,3,1)$?

$$a(1,3,4)+b(4,3,1)=(0,0,0)$$

$$\left.\begin{aligned} a+4b&=0\\ 3a+3b&=0\\ 4a+b&=0 \end{aligned}\right\}$$

2 variable
3 - eqn.

$$\left[\begin{array}{cc|c} 1 & 4 & 0\\ 3 & 3 & 0\\ 4 & 1 & 0 \end{array}\right] \longrightarrow \text{Exercise !!!}$$

(4) $(1,2,3)$, $(-1,0,1)$ and $(2,2,2)$ in $\mathbb{R}^3$.

$$a_1(1,2,3) + a_2(-1,0,1) + a_3(2,2,2) = (0,0,0)$$

Exercise!

<u>Ans.</u> $a_1 = -1$, $a_2 = 1$, $a_3 = 1$

$$-(1,2,3) + (-1,0,1) + (2,2,2) = (0,0,0).$$

$$\Updownarrow$$

$$(-1,0,1) + (2,2,2) = (1,2,3)$$

(*) If three vectors v_1, v_2, v_3 are dependent in $\mathbb{R}^3$ then one vector is a linear combination of other two vectors.

What about superset

$S = \{(1,2,3), (-1,0,1), (2,2,2), (4,3,1)\}$?

* $(-1)(1,2,3) + 1(-1,0,1) + 1(2,2,2) + 0(4,3,1)$
$$= (0,0,0)$$

Remark:

If a set is linearly dependent, then so is every superset of it.

LINEARLY INDEPENDENCE:

A set of vectors $v_1, v_2, \ldots, v_n$ from a vector space V is said to be <u>linearly independent</u> if $v_1, v_2, \ldots, v_n$ are not linearly dependent.

Equivalently,

A set of vectors $v_1, v_2, \ldots, v_n$ from a vector space is std. <u>linearly independent</u> if

$$a_1 v_1 + a_2 v_2 + \cdots + a_n v_n = 0$$

can only be satisfied when $a_i = 0 \ \forall i$.

Example:

(1) $(1,0), (3,1)$ in $\mathbb{R}^2$.

$$a(1,0) + b(3,1) = (0,0)$$

$$a + 3b = 0$$

$$3b = 0 \Rightarrow b = 0 \Rightarrow a = 0.$$

Linearly independent.

(2) $(3,-3)$ and $(2,1)$ in $\mathbb{R}^2$.

Exercise to check if they are linearly independent.

Equivalent notion:

$v_1, \dots, v_n \in V$ are std. L. I.

if the only linear combination of $v_1, \dots v_n$ which equal to 0 is the linear combination with all coefficient zero.

(1) Two vectors v_1, v_2 are linearly dependent $\Leftrightarrow$ one vector is a multiple of other vector.

E.g.: $(3,-3)$ and $(2,-2)$ $\rightarrow$ linearly dependent.

$$a(3,-3) + b(2,-2) = (0,0)$$

$$b = 1$$

$$a = -\frac{2}{3}$$

$$\left(\frac{-2}{3}\right)(3,-3) + (2,-2) = (0,0)$$

$$(2,-2) = \left(\frac{2}{3}\right)(3,-3)$$

In general,

$v_1, v_2 \in V$ are linearly dependent

then $\quad a v_1 + b v_2 = 0$

Suppose $a \neq 0$, then

$$v_1 + \frac{b}{a} v_2 = 0 \Rightarrow \boxed{v_1 = -\frac{b}{a} v_2}$$

(2) Three vectors are linear dependent iff one vector is linear combination of other two vectors.

Eg.: $(1,2,3)$, $(-1,0,1)$ and $(2,2,2)$ in $\mathbb{R}^3$.

$$(-1,0,1) + (2,2,2) = (1,2,3)$$

In general,

Suppose v_1, v_2 and v_3 are three vectors in V such that they are dependent. Then,

$a_1 v_1 + a_2 v_2 + a_3 v_3 = 0$ where let a_1 is nonzero.

$$v_1 = \left(-\frac{a_2}{a_1}\right) v_2 + \left(\frac{-a_3}{a_1}\right) v_3.$$

(3) $v_1, v_2, \dots, v_n \in \mathbb{R}^m$ linear independency.

$$a_1 v_1 + a_2 v_2 + \dots + a_n v_n = 0$$

In particular, $v_i = (v_{1i}, v_{2i}, \dots, v_{mi}) \in \mathbb{R}^m$

$$a_1 v_{11} + a_2 v_{12} + \dots + a_n v_{1n} = 0$$

$$\vdots$$

$$a_1 v_{m1} + a_2 v_{m2} + \dots + a_n v_{mn} = 0$$

Homogenous System of linear Eq^n's.

If above system has a unique sol^n, i.e. $(0, 0, \dots, 0) \in \mathbb{R}^m$, then vectors $v_1, \dots, v_n$ are linearly independent.

Example:

- $(1,0,2)$ and $(3,3,5)$ in $\mathbb{R}^3$

$$a_1(1,0,2) + a_2(3,3,5) = (0,0,0)$$
$$(a_1 + 3a_2,\ 3a_2,\ 2a_1 + 5a_2) = (0,0,0)$$

$$(1,\underline{0},2) = k(3,\underline{3},5)$$
$$3k = 0 \Rightarrow k=0$$
$$(1,0,2) \neq (0,0,0)$$
$\Rightarrow$ $(1,0,2)$ and $(3,3,5)$ not linearly dependent

$\Rightarrow$ vectors are linearly independent.

- $(1,2)$, $(1,3)$ and $(3,3)$ in $\mathbb{R}^2$

$$a_1(1,2) + a_2(1,3) + a_3(3,3) = (0,0)$$

$$a_1 + a_2 + 3a_3 = 0$$
$$2a_1 + 3a_2 + 3a_3 = 0$$

$m=2$, $n=3$. $\quad n > m$

$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 2 & 3 & 3 & 0 \end{array}\right] \rightarrow \text{Infinite sol}^n$$

Whenever # of eq^n < # of variables then there will always be an <u>independent</u> variable.

i.e. there are infinitely Sol^n to the homogenous system of eq^n.

Conclusion: More than 2 vectors in $\mathbb{R}^2$ are always linearly dependent.

Similarly,
More than 'n' vectors in $\mathbb{R}^n$ are linearly dependent.

Using Determinants:

Recall: Consider n-variables & n-eq^n system:

$A\bar{x} = \bar{0}$ has unique solution

$\Longleftrightarrow$ $\det A \neq 0$. (i.e. A is invertible).

Thus,

To check linear independence of 2 vectors (1,3) and (4,5) in $\mathbb{R}^2$

$$\Longleftrightarrow \det\begin{pmatrix} 1 & 4 \\ 3 & 5 \end{pmatrix} = -7 \neq 0$$

Example: (2×2)

Example (3x3):

$(1,3,4), (0,1,0), (1,2,0)$ in $\mathbb{R}^3$

$\because \det\begin{pmatrix} 1 & 3 & 4 \\ 0 & 1 & 0 \\ 1 & 2 & 0 \end{pmatrix} = -4.$

$\therefore$ vectors are linearly independent.

* To check the linear independency of $v_1, \ldots, v_n \in \mathbb{R}^n$

$v_1, v_2, \ldots, v_n \in \mathbb{R}^n$ are linear independent if and only if $\det\begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix} \neq 0.$

$\square$

AQ 3.1: $V = \{(1,x) \mid x \in \mathbb{R}\} \subseteq \mathbb{R}^2$
with usual addition &
scalar multiplication as in $\mathbb{R}^2$.

Is V closed under addition:

$(1,2), (1,3) \in V, \quad (2,5) \notin V.$

$3(1,x) = (3, 3x) \notin V$

Suppose $(a,b) \in V$ s.t. $(a,b) + (1,x) = (1,x)$

$\Rightarrow \quad (a+1, b+x) = (1,x)$

$\Rightarrow \quad a+1 = 1, \quad b+x = x$

$\Rightarrow \quad a=0, \quad b=0$

$(0,0) \notin V$?

$(a+b)v = av + bv, \quad a,b \in \mathbb{R}, v \in V.$

L.H.S. $(a+b)(1,x) = ((a+b),(a+b)x) = (a+b, ax+bx)$

R.H.S. $a(1,x) + b(1,x) = (a+b, ax+bx)$