# Week 3 : Introduction to vector spaces

## Lecture 1 : Introduction to vector spaces

### What are vector spaces?

### Recall : vectors in $R^2$

Consider vectors $(x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n)$ in $\mathbb{R}^n$ and $c \in \mathbb{R}$.

- Recall addition of these vectors is defined as :

  $$(x_1, x_2, \ldots, x_n) + (y_1, y_2, \ldots, y_n) = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$$

- and scalar multiplication is defined as :

  $$c(x_1, x_2, \ldots, x_n) = (cx_1, cx_2, \ldots, cx_n)$$

## Properties of addition and scalar multiplication

Let $v, w$ and $v'$ be vectors in $\mathbb{R}^n$ and $a, b \in \mathbb{R}$.

i) $v + w = w + v$.

ii) $(v + w) + v' = v + (w + v')$.

iii) The 0 vector satisfies that $v + 0 = 0 + v = v$.

iv) The vector $-v$ satisfies that $v + (-v) = 0$.

v) $1v = v$.

vi) $(ab)v = a(bv)$.

vii) $a(v + w) = av + aw$.

viii) $(a + b)v = av + bv$

A vector space is a set with two operations (called **addition** and **scalar multiplication** with the above properties (i)-(viii).

## Definition of a vector space

A vector space $V$ over $\mathbb{R}$ is a set along with two functions

$$+ : V \times V \to V \qquad \text{and} \qquad \cdot : \mathbb{R} \times V \to V$$

(i.e. for each pair of elements $v_1$ and $v_2$ in $V$, there is a unique element $v_1 + v_2$ in $V$, and for each $c \in \mathbb{R}$ and $v \in V$ there is a unique element $c \cdot v$ in $V$)

It is standard to suppress the $\cdot$ and only write $cv$ instead of $c.v$.

The functions $+$ and $\cdot$ are required to satisfy the following rules :

i) $v_1 + v_2 = v_2 + v_1$ for all $v_1, v_2 \in V$

ii) $(v_1 + v_2) + v_3 = v_1 + (v_2 + v_3)$ for all $v_1, v_2, v_3 \in V$

iii) There exists an element in $V$ denoted by 0 such that $v + 0 = v$ for all $v \in V$

iv) For each element $v \in V$ there exists an element $v' \in V$ such that $v + v' = 0$

v) For each element $v \in V$, $1v = v$

vi) For each pair of elements $a, b \in \mathbb{R}$ and each element $v \in V$, $(ab)v = a(bv)$

vii) For each element $a \in \mathbb{R}$ and each pair of elements $v_1$ and $v_2$, $a(v_1 + v_2) = av_1 + av_2$

viii) For each pair of elements $a, b \in \mathbb{R}$ and each element $v \in V$, $(a + b)v = av + bv$

## Example : Matrices

Let $M_{m \times n}(\mathbb{R})$ be the set of all $m \times n$ matrices with real numbers.

Recall that we have defined addition and scalar multiplication on $M_{m \times n}(\mathbb{R})$ as follows :

- $(A + B)_{ij} = A_{ij} + B_{ij}$
- $(cA)_{ij} = c(A)_{ij}$

where $A, B \in M_{m \times n}(\mathbb{R})$ and $c \in \mathbb{R}$.

Then $M_{m \times n}(\mathbb{R})$ along with addition and scalar multiplication forms a vector space.

## Example : Solutions of a homogeneous system

Consider the set of solutions $V$ of of a homogeneous system $Ax = 0$ where $A \in M_{m\times n}(\mathbb{R})$ (i.e. this is a homogeneous system of $m$ linear equations in $n$ variables).

Note that if $v, w \in V$ then

$$A(v + w) = Av + Aw = 0 + 0 = 0. \qquad \Longrightarrow v+w \in V$$

and if $c \in \mathbb{R}$ then

$$A(cv) = c(Av) = c(0) = 0. \qquad \Longrightarrow cv \in V$$

So addition and scalar multiplication on $\mathbb{R}^n$ restricts to the solution set. Hence it is a vector space.

This is an example of a subspace of a vector space.

## Non-example

Let us define addition and scalar multiplication in $\mathbb{R}^2$ as follows:

- $(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 - y_2)$
- $c(x_1, x_2) = (cx_1, cx_2)$

Check that (i), (ii) and (viii) fail to hold.

(i) fails :

$(x_1, x_2) + (y_1, y_2) = (x_1 + y_1 , x_2 - y_2)$
$\neq$
$(y_1, y_2) + (x_1, x_2) = (y_1 + x_1 , y_2 - x_2)$

$(0,0) + (1,1) = (1, -1)$
$\neq$
$(1,1) + (0,0) = (1, 1)$

Let us define addition and scalar multiplication in $\mathbb{R}^2$ as follows:

- $(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, 0)$
- $c(x_1, x_2) = (cx_1, 0)$

Check that (iii), (iv) and (v) fail to hold.

**QN : 3,4,5,7,8,9,10**

# Lecture 2 : Some properties of vector spaces

## Cancellation law of vector addition

If $v_1, v_2, v_3 \in V$ such that $v_1 + v_3 = v_2 + v_3$, then $v_1 = v_2$.

$(x_1, x_2, x_3) + (y_1, y_2, y_3) = (z_1, z_2, z_3) + (y_1, y_2, y_3)$ } in $\mathbb{R}^3$ or gen. in $\mathbb{R}^n$

$\Rightarrow x_i + y_i = z_i + y_i \quad , i = 1, 2, 3$

Subtract $y_i$ from both sides $\Rightarrow x_i = z_i \quad , i = 1, 2, 3.$

$v_1 + v_3 = v_2 + v_3 \Rightarrow (v_1 + v_3) + v_3' = (v_2 + v_3) + v_3'$ $\quad \exists v_3'$ s.t. $v_3 + v_3' = 0$

$\Rightarrow v_1 + (v_3 + v_3') = v_2 + (v_3 + v_3')$

$\Rightarrow v_1 + 0 = v_2 + 0$

$\Rightarrow v_1 = v_2.$

## Corollaries:

- The vector 0 described in (iii) is unique.

  Suppose $\exists w \in V$ s.t. $v + w = v \ \forall v \in V.$

  $\cancel{v} + w = \cancel{v} + 0 \Rightarrow w = 0.$

- The vector $v'$ described in (iv) is unique and it is standard to refer to it as $-v$.

  $v + v' = 0$ Suppose $v''$ also satisfies this.

  Then $v + v' = 0 = v + v''$. Cancel $v$. $\therefore v' = v''.$

## Some more important properties

In any vector space $V$ the following statements are true.

- $0v = 0$ for each $v \in V$.

$(0+0)v = 0v + 0v$

$0v = 0v + 0v \Rightarrow 0v + 0 = 0v + 0v$

Cancel $0v \Rightarrow 0 = 0v$.

- $(-c)v = -(cv) = c(-v)$ for each $c \in \mathbb{R}$ and for each $v \in V$.

$\underbrace{(c + (-c))v}_{0v = 0} = cv + (-c)v \Rightarrow cv + (-c)v = 0.$

$\Rightarrow (-c)v = -(cv).$

- $c0 = 0$ for each $c \in \mathbb{R}$.

Check this!

# Example : Stock Taking

## Earlier Example : Stock taking

**Stock taking in a grocery shop :**

| Items | In stock | Buyer A | Buyer B | Buyer C | New stock |
|---|---|---|---|---|---|
| Rice in kg | 150 | 8 | 12 | 3 | 100 |
| Dal in kg | 50 | 8 | 5 | 2 | 75 |
| Oil in Litres | 35 | 4 | 7 | 5 | 30 |
| Biscuits in packets | 70 | 10 | 10 | 5 | 80 |
| Soap Bars | 25 | 4 | 2 | 1 | 30 |

Vector space : { (Quantity of rice in kg, quantity of dal in kg, the
quantity of oil in litres, the no. of biscuit packets,
no. of soap bars) }

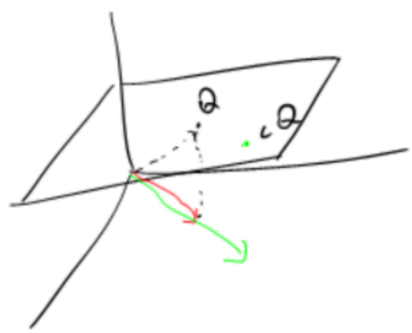
IS Like $\mathbb{R}^5$.

—ve corresponds to demand
+ve " to supply.

## Example : Affine flats

Let $V$ be a plane parallel to the $XY$-plane. We will define an "addition" and "scalar multiplication" of points on $V$.

Scalar multiplication : Let $Q \in V$ and $c \in \mathbb{R}$. Project $Q$ onto the $XY$-plane, scale the resulting vector by $c$ and project the result back to $V$. Define $cQ$ to be the tip of the obtained arrow.




Addition :

Check that this is a vector space both geometrically (visualize!) and by writing down the algebra.

**QN : 7,8,9,10**

# Lecture 3 : Linear dependence

## Linear dependence

## Vector addition (Recall)

Vector addition is a binary operation $+ : V \times V \to V$, which takes any two vectors $v$ and $w$ of $V$ and assigns to them a third vector denoted by $v + w$.

In $\mathbb{R}^n$ vector addition is defined by co-ordinate wise addition:

$$(x_1, x_2, \ldots, x_n) + (y_1, y_2, \ldots, y_n) = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$$

e.g.

$$\begin{aligned}(1.5, -3.3, 7.2, \frac{1}{2}, 1) + (-4, 5.8, 10, 5\frac{1}{2}, -3.4) \\ = (-2.5, 2.5, 17.2, 6, -2.4).\end{aligned}$$

## Scalar multiplication (Recall)

Scalar multiplication is a function $\cdot : \mathbb{R} \times V \to V$, which take any element $c \in \mathbb{R}$ and $v \in V$ and assigns a new vector denoted by $c \cdot v$.

It is standard to suppress the $\cdot$ and instead of $c \cdot v$, the notation $cv$ is used.

In $\mathbb{R}^n$, scalar multiplication is defined as follows:

$$c(x_1, x_2, \ldots, x_n) = (cx_1, cx_2, \ldots, cx_n)$$

e.g.

$$0.5(1.5, -3.3, 7.2, \frac{1}{2}, 1) = (0.75, -1.65, 3.6, \frac{1}{4}, 0.5).$$

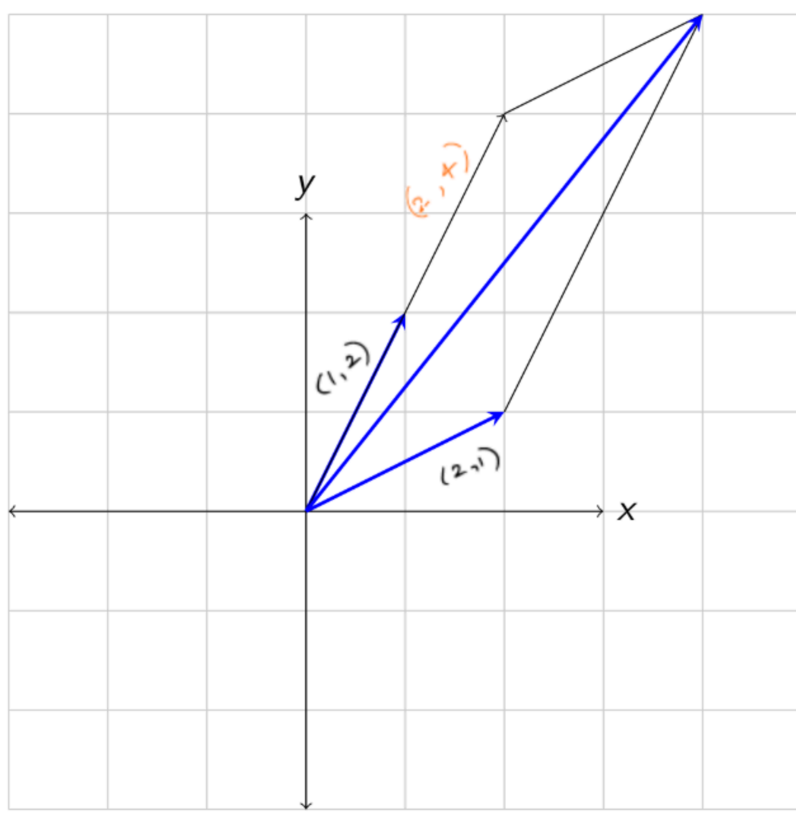
## Linear combination of vectors

Let $V$ be a vector space and $v_1, v_2, \ldots, v_n \in V$. The linear combination of $v_1, v_2, \ldots, v_n$ with coefficients $a_1, a_2, \ldots, a_n \in \mathbb{R}$ is the vector $\sum_{i=1}^{n} a_i v_i \in V$.

A vector $v \in V$ is a linear combination of $v_1, v_2, \ldots, v_n$ if there exist some $a_1, a_2, \ldots, a_n \in \mathbb{R}$ so that $v = \sum_{i=1}^{n} a_i v_i$.

## Example in $R^2$

Example in $\mathbb{R}^2$ : $\quad 2(1,2) + (2,1) = (4,5)$



In the previous example we see that $(4,5)$ is a linear combination of vectors $(1,2)$ and $(2,1)$, as follows:

$$2(1,2)+(2,1)=(2,4)+(2,1)=(4,5)$$

Moreover, each of the vectors in the expression is a linear combination of the other two vectors.

$$\frac{1}{2}(4,5)-\frac{1}{2}(2,1)=(1,2)$$
$$(4,5)-2(1,2)=(2,1)$$

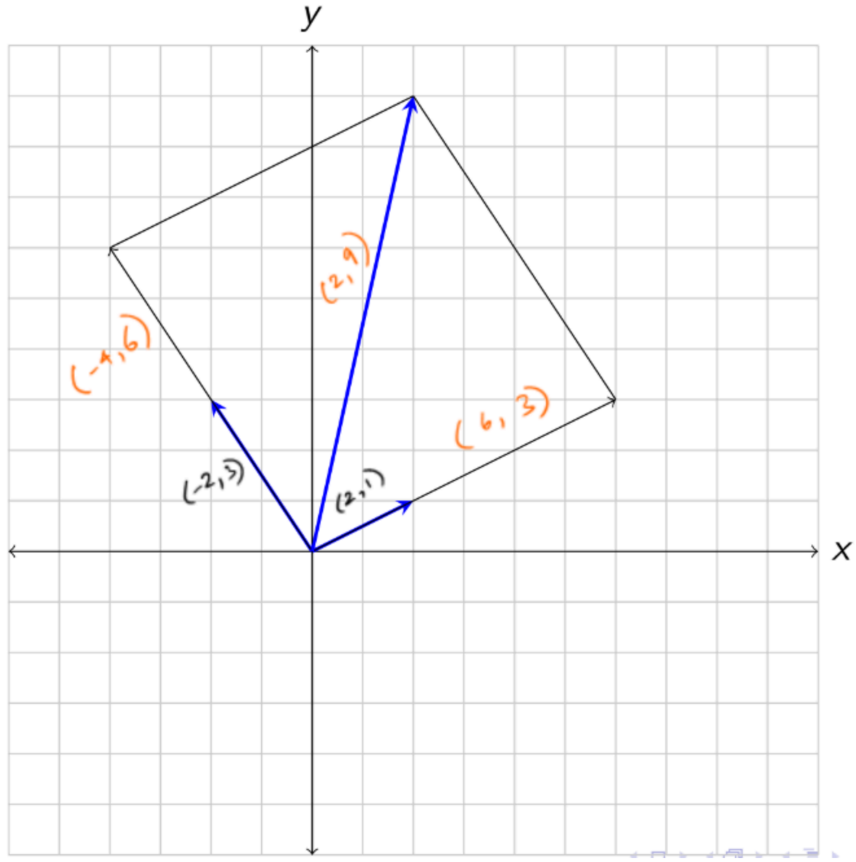
Note further that we can re-write these expressions as follows :

$$2(1,2)+(2,1)-(4,5)=(0,0)$$

Observe : the 0 vector is a linear combination of $(1,2),(2,1),(4,5)$ with non-zero coefficients.

## Another Example in $R^2$

## Another example in $\mathbb{R}^2$ : $\quad 3(2,1) + 2(-2,3) = (2,9)$




In the previous example we see that $(2,9)$ is a linear combination of vectors $(2,1)$ and $(-2,3)$, as follows:

$$3(2,1) + 2(-2,3) = (6,3) + (-4,6) = (2,9)$$

Moreover, each of the vectors in the expression is a linear combination of the other two vectors.

$$\tfrac{1}{3}(2,9) - \tfrac{2}{3}(-2,3) = (2,1)$$
$$\tfrac{1}{2}(2,9) - \tfrac{3}{2}(2,1) = (-2,3)$$

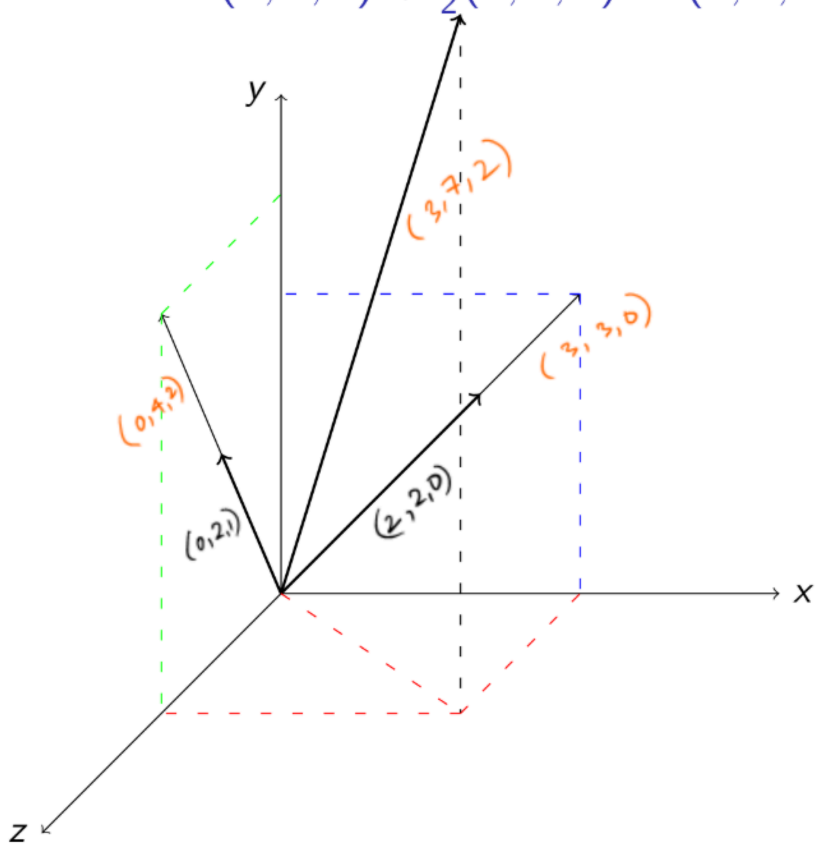
Note further that we can re-write these expressions as follows :

$$3(2,1) + 2(-2,3) - (2,9) = (0,0)$$

Observe : the 0 vector is a linear combination of $(2,1), (-2,3), (2,9)$ with non-zero coefficients.

## Another Example in $R^3$

Example in $\mathbb{R}^3$ :  $2(0,2,1) + \frac{3}{2}(2,2,0) = (3,7,2)$



In the previous example we see that $(3,7,2)$ is a linear combination of vectors $(0,2,1)$ and $(2,2,0)$, as follows:

$$2(0,2,1)+\frac{3}{2}(2,2,0)=(0,4,2)+(3,3,0)=(3,7,2)$$

Moreover, each of the vectors in the expression is a linear combination of the other two vectors.

$$\frac{1}{2}(3,7,2)-\frac{3}{4}(2,2,0)=(0,2,1)$$
$$\frac{2}{3}(3,7,2)-\frac{4}{3}(0,2,1)=(2,2,0)$$

Note further that we can re-write these expressions as follows :

$$2(0,2,1)+\frac{3}{2}(2,2,0)-(3,7,2)=(0,0,0)$$

Observe : the 0 vector is a linear combination of $(0,2,1),(2,2,0),(3,7,2)$ with non-zero coefficients.

The plane of the two vectors $(0,2,1)$ and $(2,2,0)$ can be expressed by the equation $2x-2y+4z=0$.

Let us choose a vector which is not on the plane, say $(1,2,0)$.
We claim that, $(1,2,0)$ cannot be written as a linear combination of $(0,2,1)$ and $(2,2,0)$.

If possible let us assume we can write $(1, 2, 0)$ as a linear combination of the other two vectors as follows,

$$a(0, 2, 1) + b(2, 2, 0) = (1, 2, 0)$$

which implies, $2b = 1$, $2a + 2b = 2$ and $a = 0$. Clearly these three equations simultaneously cannot have a solution. Hence our assumption was false.

We can use the discussion above to conclude that

$$a(0, 2, 1) + b(2, 2, 0) + c(1, 2, 0) = (0, 0, 0) \text{ if and only if } a = b = c = 0.$$

i.e. the only way the 0 vector is a linear combination of $(0, 2, 1), (2, 2, 0), (1, 2, 0)$ is if the coefficients are 0.

## Definition of Linear dependence

A set of vectors $v_1, v_2, \ldots, v_n$ from a vector space $V$ is said to be linearly dependent, if there exist scalars $a_1, a_2, \ldots, a_n$, not all zero, such that

$$a_1 v_1 + a_2 v_2 + \ldots + a_n v_n = 0$$

Equivalently, the 0 vector is a linear combination of $v_1, v_2, \ldots, v_n$ with non-zero coefficients.

## More examples

## Example : 1

Consider the following two vectors in $\mathbb{R}^3$,

$$(2,3,7) \text{ and } (\tfrac{5}{3}, \tfrac{5}{2}, \tfrac{35}{6}).$$

It is easy to check that

$$5(2,3,7) - 6(\tfrac{5}{3}, \tfrac{5}{2}, \tfrac{35}{6}) = (0,0,0)$$

Hence these two vectors are linearly dependent. Also observe that one is a scalar multiple of the other.

## Example : 2

Consider the following three vectors in $\mathbb{R}^3$,

$$(2,1,2), (3,0,1) \text{ and } (10,-4,-2)$$

It is easy to check that

$$2(2,1,2) - 3(3,0,1) + \tfrac{1}{2}(10,-4,-2) = (0,0,0)$$

$$(4,2,4) - (9,0,3) + (5,-2,-1)$$

Hence these three vectors are linearly dependent.

## Example : 3

Add one more vector $(2,3,7)$ to the set of vectors in the previous slide. Hence we have the following set of vectors in $\mathbb{R}^3$.

$$\{(2,1,2),\ (3,0,1),\ (10,-4,-2),\ (2,3,7)\}$$

It is easy to check that

$$2(2,1,2) - 3(3,0,1) + \tfrac{1}{2}(10,-4,-2) + 0(2,3,7) = (0,0,0)$$

It still satisfies the definition of linear dependence as all the scalars are not zero. Hence these four vectors are also linearly dependent.

## Important remark

The previous example points to the following fact :

If a set is linearly dependent, then so is every superset of it.

**QN :1, 2,6,7,9,10**

# Lecture 4 : Linear independence - Part 1

## Linear independence

## Linear dependence (recall)

A set of vectors $v_1, v_2, \ldots, v_n$ from a vector space $V$ is said to be linearly dependent if there exists scalars $a_1, a_2, \ldots, a_n$, not all zero, such that

$$a_1 v_1 + a_2 v_2 + \ldots + a_n v_n = 0$$

Equivalently : $v_1, v_2, \ldots, v_n$ are linearly dependent if the 0 vector can be expressed as a linear combination of $v_1, v_2, \ldots, v_n$ with non-zero coefficients (i.e. at least one coefficient is non-zero).

## Definition of linear independence

A set of vectors $v_1, v_2, \ldots, v_n$ from a vector space $V$ is said to be linearly independent if $v_1, v_2, \ldots, v_n$ are not linearly dependent.

Equivalently : A set of vectors $v_1, v_2, \ldots, v_n$ from a vector space $V$ is said to be linearly independent, if the equation

$$a_1 v_1 + a_2 v_2 + \ldots + a_n v_n = 0$$

can only be satisfied when $a_i = 0$ for all $i = 1, 2, \ldots, n$.

Equivalently : A set of vectors $v_1, v_2, \ldots, v_n$ from a vector space $V$ is said to be linearly independent if the only linear combination of $v_1, v_2, \ldots, v_n$ which equals 0 is the linear combination with all coefficients 0.

## Example in $R^2$

Consider the two vectors $(-1, 3)$ and $(2, 0)$ in $\mathbb{R}^2$.

Consider the following equation :

$$a(-1, 3) + b(2, 0) = (0, 0)$$

Hence we have the following system of linear equations:

$$-a + 2b = 0 \text{ and } 3a = 0.$$

Hence $a = 0,\ b = 0$ is the unique solution of the system of linear equations, which implies that the vectors $(-1, 3)$ and $(2, 0)$ are linearly independent.

## The 0 vector

Let $v_1, v_2, \ldots, v_n$ be a set of vectors containing the 0 vector.

Suppose $v_i = 0$. Then we can choose $a_i = 1$ and $a_j = 0$ for $j \neq i$.

Then the linear combination $a_1 v_1 + a_2 v_2 + \ldots + a_n v_n$ is 0 but not all coefficients are 0.

Hence, a set of vectors $v_1, v_2, \ldots, v_n$ containing the 0 vector is always a linearly dependent set.

## When are two non-zero vectors linearly independent?

Let $v_1$ and $v_2$ be two non-zero vectors.

Suppose that $v_1$ and $v_2$ are linearly dependent.

Then $a_1 v_1 + a_2 v_2 = 0$ for some coefficients $a_1$ and $a_2$. (where at least one of $a_1$ or $a_2$ is not 0)

Note that since the vectors are non-zero, both $a_1$ and $a_2$ must be non-zero.

Dividing by $a_1$ and putting $c = -a_2/a_1$, we get that $v_1 = cv_2$.

Hence $v_1$ and $v_2$ are multiples of each other.

We can reverse the implications above and conclude that if $v_1$ and $v_2$ are multiples of each other then they are linearly dependent.

**Conclusion : Two non-zero vectors are linearly independent precisely when they are not multiples of each other .**

## Linear independence of three vectors

Suppose $v_1$,$v_2$ and $v_3$ are linearly dependent.

Then $a_1v_1 + a_2v_2 + a_3v_3 = 0$ for some coefficients $a_1, a_2, a_3$ where at least one of the coefficients is non-zero.

If $a_1 \neq 0$, then we can write $v_1 = b_2v_2 + b_2v_3$ where $b_2 = -a_2/a_1$ and $b_3 = -a_3/a_1$. Hence, $v_1$ is a linear combination of the other two vectors.

Similarly if $a_2 \neq 0$, $v_2$ is a linear combination of the other two vectors and if $a_3 \neq 0$, $v_3$ is a linear combination of the other two vectors.

Since the implications are reversible, we obtain that $v_1$,$v_2$ and $v_3$ are linearly dependent exactly when one of the vectors is a linear combination of the others.

**Conclusion : If three vectors are linearly independent then none of these vectors is a linear combination of the other two.**

## Example in $R^3$

Let us consider three vectors $(1,1,2)$, $(1,2,0)$ and $(0,2,1)$ in $\mathbb{R}^3$ and also consider the following equation:

$$a(1,1,2) + b(1,2,0) + c(0,2,1) = (0,0,0)$$

Hence we have the following system of linear equations:

$$a + b = 0 \qquad a + 2b + 2c = 0 \qquad 2a + c = 0.$$

Substituting $b = -a$ and $c = -2a$ in the middle equation yields that $a = 0$, $b = 0$, $c = 0$ is the unique solution of this system. Hence the vectors $(1,1,2)$, $(1,2,0)$ and $(0,2,1)$ are linearly independent.

**QN : 1,2,4,7,8,9,10**

# Lecture 5 : Linear independence - Part 2

## How to check linear independence in $R^m$

How do we check if $v_1, v_2, \dots, v_n \in \mathbb{R}^m$ are linearly independent?

In terms of coordinates, let $v_j = (v_{1j}, v_{2j}, \dots, v_{mj})$ ; $j = 1, 2, \dots, n$.

Let us write the linear combination of these vectors with *arbitrary* coefficients $a_1, a_2, \dots, a_n$ and equate it to 0 :

$$a_1 v_1 + a_2 v_2 + \ldots + a_n v_n = 0.$$

Considering each coordinate, we have the following identities :

$$\begin{array}{ccccccccc}
v_{11}a_1 & + & v_{12}a_2 & + & \ldots & + & v_{1n}a_n & = & 0 \\
v_{21}a_1 & + & v_{22}a_2 & + & \ldots & + & v_{2n}a_n & = & 0 \\
\vdots & & \vdots & & \ddots & & \vdots & & \vdots \\
v_{m1}a_1 & + & v_{m2}a_2 & + & \ldots & + & v_{mn}a_n & = & 0
\end{array}$$

Since the $a_i$ are arbitrary (unknown), we can treat this like a homogeneous system of linear equations with coefficients $v_{ij}$ and unknowns $a_i$.

For linear independence, we have to check if the only choice of $a_i$'s satisfying the above identities is $a_i = 0$.

Equivalently, in terms of the homogeneous system of linear equations, we have to check that its only solution is the 0 solution.

**Conclusion : To check $v_1, v_2, \ldots, v_n \in \mathbb{R}^m$ are linearly independent, we have to check that the homogeneous system of linear equations $Vx = 0$ has only the trivial solution, where the $j^{th}$ column of $V$ is $v_j$.**

## Example : 2 × 2

Consider the two vectors $(5, 2)$ and $(1, 3)$ in $\mathbb{R}^2$. Write the linear combination of these two vectors with *unknown* coefficients $x_1$ and $x_2$ and equate it to 0 : $x_1(5, 2) + x_2(1, 3) = (0, 0)$.

Hence we have the system of linear equations:

$$\begin{aligned} 5x_1 + x_2 &= 0 \\ 2x_1 + 3x_2 &= 0 \end{aligned}$$

Since the corresponding matrix $\begin{bmatrix} 5 & 1 \\ 2 & 3 \end{bmatrix}$ is invertible, the system of linear equations has a unique solution $x_1 = x_2 = 0$.

**Hence the vectors** $(5, 2)$ **and** $(1, 3)$ **are linearly independent**.

## Example : 3 × 2

Consider the two vectors $(1, 2, 0)$ and $(3, 3, 5)$ in $\mathbb{R}^3$. Write the linear combination of these two vectors with *unknown* coefficients $x_1$ and $x_2$ and equate it to $0$ : $x_1(1, 2, 0) + x_2(3, 3, 5) = (0, 0, 0)$.

Hence we have the system of linear equations:

$$\begin{aligned} x_1 + 3x_2 &= 0 \\ 2x_1 + 3x_2 &= 0 \\ 0x_1 + 5x_2 &= 0 \end{aligned}$$

It is easy to check that the system of linear equations has a unique solution $x_1 = x_2 = 0$ (or we can use Gaussian elimination and check that the only solution is the trivial one).

**Hence the vectors** $(1, 2, 0)$ **and** $(3, 3, 5)$ **are linearly independent**.

## Example : 2 × 3

Consider the three vectors $(1,2)$, $(1,3)$ and $(3,4)$ in $\mathbb{R}^2$. Equate the linear combination of these three vectors with *unknown* coefficients $x_1, x_2$ and $x_3$ to 0 : $x_1(1,2) + x_2(1,3) + x_3(3,4) = (0,0)$.

Hence we have the system of linear equations:

$$\begin{aligned} 1x_1 + 1x_2 + 3x_3 &= 0 \\ 2x_1 + 3x_2 + 4x_3 &= 0 \end{aligned}$$

The augmented matrix for this system is $\left[\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 2 & 3 & 4 & 0 \end{array}\right]$.

Gaussian elimination yields infinitely many solutions for this system of the form $x_1 = -5c, x_2 = 2c, x_3 = c$ where $c \in \mathbb{R}$. **Hence the vectors** $(1,2)$**,** $(1,3)$ **and** $(3,4)$ **are linearly dependent**.

## Example : 3 × 3

Consider the three vectors $(1,2,0)$, $(0,2,4)$ and $(3,0,0)$ in $\mathbb{R}^3$. Equate the linear combination of these three vectors with *unknown* coefficients $x_1, x_2$ and $x_3$ to 0 :

$$x_1(1,2,0) + x_2(0,2,4) + x_3(3,0,0) = (0,0,0).$$

Hence we have the system of linear equations:

$$\begin{aligned} x_1 + 0x_2 + 3x_3 &= 0 \\ 2x_1 + 2x_2 + 0x_3 &= 0 \\ 0x_1 + 4x_2 + 0x_3 &= 0 \end{aligned}$$

Since the matrix $\begin{bmatrix} 1 & 0 & 3 \\ 2 & 2 & 0 \\ 0 & 4 & 0 \end{bmatrix}$ is invertible, the system of linear equations has a unique solution $x_1 = x_2 = x_3 = 0$. **Hence the vectors** $(1,2,0)$**,** $(0,2,4)$ **and** $(3,0,0)$ **are linearly independent**.

## More than 2 vectors in $R^2$

Suppose we have $n$ vectors in $\mathbb{R}^2$ where $n \geq 3$. To check linear indepence, we have to check whether the corresponding homogeneous linear system $Vx = 0$ has the unique solution $x = 0$.

Since $n \geq 3 > 2$, this is a homogeneous system with more unknowns ($n$) than equations (2).

We have seen in the previous week that Gaussian elimination will yield infinitely many solutions.

**Hence, any set of** $n$ **vectors in** $\mathbb{R}^2$ **with** $n \geq 3$ **are linearly dependent**.

## More than n vectors in $R^n$

The same argument as for $\mathbb{R}^2$ in the previous slide yields :

**Hence, any set of $r$ vectors in $\mathbb{R}^n$ with $r > n$ are linearly dependent**.

## Example in $R^3$

Consider the four vectors $(1,2,0)$, $(0,2,4)$, $(3,0,0)$ and $(1,2,3)$ in $\mathbb{R}^3$. To check linear independence, we write the corresponding system of linear equations :

$$\begin{aligned} x_1 + 0x_2 + 3x_3 + x_4 &= 0 \\ 2x_1 + 2x_2 + 0x_3 + 2x_4 &= 0 \\ 0x_1 + 4x_2 + 0x_3 + 3x_4 &= 0 \end{aligned}$$

To solve this system, we consider the augmented matrix $\left[\begin{array}{cccc|c} 1 & 0 & 3 & 1 & 0 \\ 2 & 2 & 0 & 2 & 0 \\ 0 & 4 & 0 & 3 & 0 \end{array}\right]$ and apply Gaussian elimination.

Row reduction results in the augmented matrix $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1/4 & 0 \\ 0 & 1 & 0 & 3/4 & 0 \\ 0 & 0 & 1 & 1/4 & 0 \end{array}\right]$

Thus we obtain solutions : $x_1 = -\frac{c}{4}, x_2 = -\frac{3c}{4}, x_3 = -\frac{c}{4}, x_4 = c$ where $c \in \mathbb{R}$.

So we can write

$$-\tfrac{c}{4}(1,2,0) - \tfrac{3c}{4}(0,2,4) - \tfrac{c}{4}(3,0,0) + c(1,2,3) = 0 \text{ for } c \in \mathbb{R}.$$

In particular with $c = 4$

$$-1(1,2,0) - 3(0,2,4) - 1(3,0,0) + 4(1,2,3) = 0$$

**Hence the vectors** $(1,2,0)$**,** $(0,2,4)$**,** $(3,0,0)$ **and** $(1,2,3)$ **are linearly dependent**.

## Relationship with determinant

To check whether a set of $n$ vectors in $\mathbb{R}^n$ are linearly independent, we have to find the solutions of the homogeneous system $Vx = 0$ where $V$ is an $n \times n$ matrix obtained by arranging the vectors in columns.

Since $V$ is a square matrix, it has unique solution $x = 0$ if and only if $\cancel{A}$ $V$ is invertible if and only if $det(\cancel{A}\, V) \neq 0$.

- If $A$ is invertible then there exists $A^{-1}$ such that $AA^{-1} = I = A^{-1}A$. Hence $det(A).det(A^{-1}) = 1$ which implies $det(A) \neq 0$.
- Now if $det(A) \neq 0$ then $A^{-1} = \frac{1}{det(A)} adj(A)$ exists.

## Example

Let us consider the vectors $(1, 4, 2)$,$(0, 4, 3)$ and $(1, 1, 0)$ in $\mathbb{R}^3$.

We obtain the matrix

$$V = \begin{bmatrix} 1 & 0 & 1 \\ 4 & 4 & 1 \\ 2 & 3 & 0 \end{bmatrix}.$$

Since $det(\cancel{A}\, V) = 1 \neq 0$, the matrix $\cancel{A}$ $V$ is invertible and hence **the vectors** $(1, 4, 2)$**,**$(0, 4, 3)$ **and** $(1, 1, 0)$ **are linearly independent**.

**QN : 2,3,4,5,8,9,10**

# Week 06 Tutorial 01

## Subspace

Definition: A subset $W$ of a vector space $V$ over $\mathbb{R}$ is called subspace of $V$ if $W$ is also a vector space over $\mathbb{R}$ with the same operations, defined over $V$.

$V$ $\quad +, \cdot$

$x, y \in V \quad x+y \in V$

$c \in \mathbb{R}, x \in V \quad c \cdot x \in V$

$W \subseteq V$

$+, \cdot$

$x \in W, y \in W \quad x+y \in W$

$c \in \mathbb{R}, x \in W \quad c \cdot x \in W$

Suppose $W$ is a given subset of $V$

Test for become a subspace

(i) $0 \in W$

(ii) $x \in W, y \in W \Rightarrow x+y \in W$ (closed under addition)

(iii) $c \in \mathbb{R}, x \in W \Rightarrow c \cdot x \in W$ (closed under sclar multiplication)

then we will say $W$ is a subspace of $V$.

Some example

let $V = \mathbb{R}^2$

$$W = \{(x, y) \mid x+y=0, \; x, y \in \mathbb{R}\} \subseteq \mathbb{R}^2$$

(i) $0 \in W$ $\qquad x=0, y=0 \qquad 0+0=0$

$(0,0) \in W$

(ii) $w_1 = (x_1, y_1)$, $w_2 (x_2, y_2)$ $\longrightarrow$ $w_1$ $x_1+y_1=0$ $\quad w_2$, $x_2+y_2=0$

$w_1 + w_2 = (x_1, y_1) + (x_2, y_2) = (x_1+x_2, y_1+y_2)$

$\in W$

$(x_1+x_2)+(y_1+y_2) = (x_1+y_1) + (x_2+y_2)$

$= 0 + 0 = 0$

(III) $c \in \mathbb{R}$, $w_1 = (x_1, y_1) \in W$

$x_1 + y_1 = 0$

$c \cdot w_1 = (cx_1, cy_1)$

$cx_1 + cy_1 = c(x_1 + y_1) = 0$

$W$ is a subspace of $\mathbb{R}^2$

$W_1 = \{(x,y) \mid x+y=0, \ x,y \in \mathbb{Z}\} \subseteq \mathbb{R}^2$

$0 \in \mathbb{Z}$

$(0,0) \rightarrow 0+0=0$

✓ (I) $0,0 \in W$

✓ (II) $(x_1,y_1), (x_2,y_2) \in W_1 \Rightarrow (x_1+x_2, \ y_1+y_2) \in W_1$

× (III) $\frac{1}{2}$ $(1,-1) \in W_1$ $\frac{1}{2}(1,-1) = (\frac{1}{2}, -\frac{1}{2})$

# Week 06 Tutorial 02

Second example of a subspace

$$V = \mathbb{R}^3$$

$$W = \{ (x, y, z) \mid x - 4y + 2z = 0 , \; x, y, z \in \mathbb{R} \}$$

$0 \in \mathbb{R}$, $(0,0,0) \in W$

$$(x, y, z) + (0,0,0) = (x, y, z)$$

(I)

(II) $w_1 = (x_1, y_1, z_1)$ $\quad w_2 = (x_2, y_2, z_2)$ $\quad w_1 + w_2 = (x_1, y_1, z_1) + (x_2, y_2, z_2)$

$= (x_1 + x_2, y_1 + y_2, z_1 + z_2) \in W$

$x_1 - 4y_1 + 2z_1 = 0$ $\quad x_2 - 4y_2 + 2z_2 = 0$ $\quad (x_1 + x_2) - 4(y_1 + y_2) + 2(z_1 + z_2)$

$= (x_1 - 4y_1 + 2z_1) + (x_2 - 4y_2 + 2z_2)$

(III) $c \in \mathbb{R}$, $w_1 = (x_1, y_1, z_1) \longrightarrow (x_1, y_1, z)$

$x_1 - 4y_1 + 2z_1 = 0$

$$c \cdot w_1 = c(x_1, y_1, z_1)$$

$$= (cx_1, cy_1, cz_1)$$

$$cx_1 - 4cy_1 + 2cz_1 = c(x_1 - 4y_1 + 2z_1)$$

$$= 0$$

$$c \cdot w_1 \in W$$

# Week 06 Tutorial 03

third example

$V = M_{3\times3}(\mathbb{R})$

$$W = \left\{ \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \middle| \; a_{11}+a_{12}+a_{13}=0,\; a_{21}+a_{22}+a_{23}=0 \;\&\; a_{31}+a_{32}+a_{33}=0 \right\}$$

(I)

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$

(II)

$$w_1 = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{matrix} \to 0 \\ \to 0 \\ \to 0 \end{matrix}, \quad w_2 \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} \begin{matrix} 0 \\ 0 \\ 0 \end{matrix}$$

$$w_1 + w_2 = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} \to 0 \\ & & \\ & & \end{bmatrix}$$

(III) $c \in \mathbb{R}$

$$c \cdot w_1 = \begin{bmatrix} ca_{11} & ca_{12} & ca_{13} \\ - & - & - \\ - & - & - \end{bmatrix} \to \quad c(a_{11}+a_{12}+a_{13}) = 0$$

## Week 06 Tutorial 04

Dependent & Independent vectors

Three vectors $\{(1,1,-3), (-1,1,-1), (-1,-1,3)\} \in \mathbb{R}^3$

$x \; y \; z$

$$A = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & -1 \\ -3 & -1 & 3 \end{bmatrix}$$

$$\det(A) = 1(3-1) + 1(3-3) - 1(-1+3)$$
$$= 2 + 0 - 2$$
$$= 0$$

$x + 2y + z = 1 + 2 - 3 = 0$

$x + 2y + z = 0$

$-1 + 2 - 1 = 0$

$-1 - 2 + 3 = 0$

## Week 06 Tutorial 05

Geometric representation of linearly independent vectors

$(-1,1,0) \quad (1,0,1) \quad (0,1,-1)$

$$A = \begin{bmatrix} -1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix} \qquad \det(A) = -1(-1) - 1(-1) = 2 \neq 0$$

If we take ax+by+cz=d, then we know that a subspace, containing these three vectors, is a plane or a line (other than trivial subspaces) which passes through the origin, hence d=0

$$ax + by + cz = 0$$

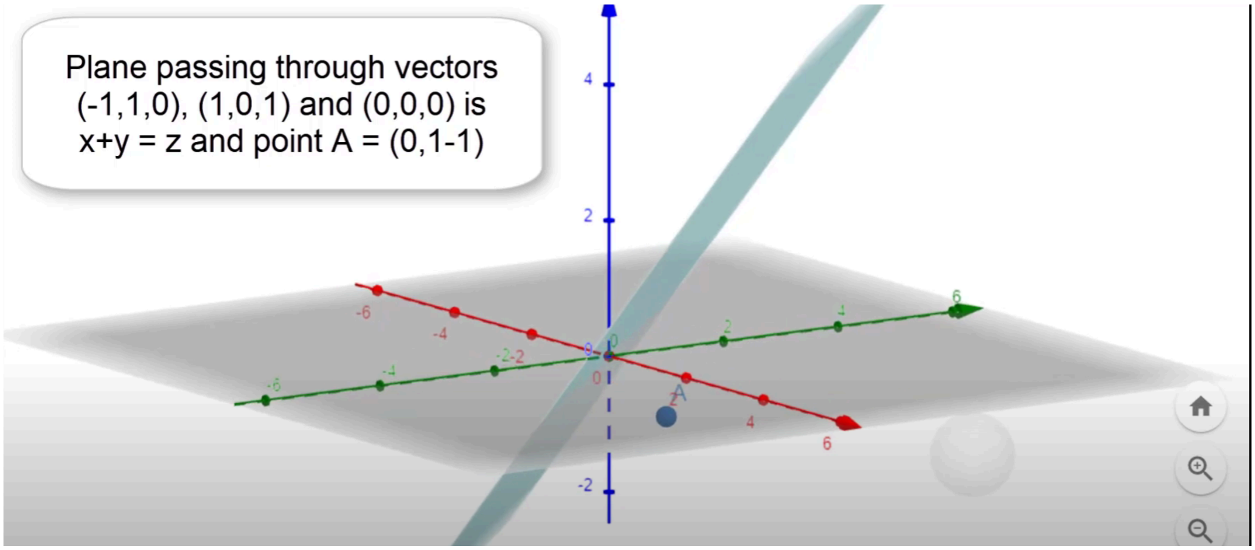
$(-1,1,0)$ $\quad -a + b = 0$

$(1,0,1)$ $\quad a + c = 0$

$(0,1,-1)$ $\quad b - c = 0$

$$\left.\begin{aligned} -a+b&=0 \\ a+c&=0 \\ b-c&=0 \end{aligned}\right\} \Rightarrow \begin{aligned} a&=0 \\ b&=0 \\ c&=0 \end{aligned}$$

$$0 = 0$$


Plane passing through vectors
(-1,1,0), (1,0,1) and (0,0,0) is
x+y = z and point A = (0,1-1)
4
2
-2
-6
-4
-2
2
4
6
A

# Week 06 Tutorial 06

If $W_1$ and $W_2$ are subspaces of $V$, then $W_1 \cap W_2$ must be a subspace of $V$. But $W_1 \cup W_2$ may not be a subspace of $V$.

Soln.
$$W_1 \cap W_2 \subseteq W_1 \subseteq V$$
$$\subseteq W_2 \subseteq V$$

i) $0 \in W_1$ , $0 \in W_2$ $\Rightarrow$ $0 \in W_1 \cap W_2$

ii) $a, b \in W_1 \cap W_2$ $\Rightarrow$ $\underline{a, b \in W_1}$ and $a, b \in W_2$

$\Rightarrow$ $a+b \in W_1$ , $a+b \in W_2$ $\Rightarrow$ $a+b \in W_1 \cap W_2$

iii) $a \in W_1 \cap W_2$ , $c \in \mathbb{R}$. $a \in W_1$ and $a \in W_2$


i) $0 \in W_1$ , $0 \in W_2$ $\Rightarrow$ $0 \in W_1 \cap W_2$

ii) $a, b \in W_1 \cap W_2$ $\Rightarrow$ $\underline{a, b \in W_1}$ and $a, b \in W_2$

$\Rightarrow$ $a+b \in W_1$ , $a+b \in W_2$ $\Rightarrow$ $a+b \in W_1 \cap W_2$

iii) $a \in W_1 \cap W_2$ , $c \in \mathbb{R}$. $a \in W_1$ and $a \in W_2$


$ca \in W_1$ and $ca \in W_2$

$\Rightarrow$ $ca \in W_1 \cap W_2$

$W_1 \cap W_2$ is a subspace of $V$.

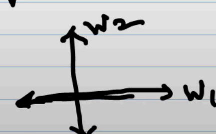
$\underline{W_1 \cup W_2}$ $V = \mathbb{R}^2$ $W_1 = \{(x, 0) \mid x \in \mathbb{R}\}$

$W_2 = \{(0, y) \mid y \in \mathbb{R}\}$

$(1, 0) \in W_1 \cup W_2$

$(0, 1)$

$(1, 1) \notin W_1 \cup W_2$



# Notes : SWU 1

## Key Points: 1

**Definition of a vector space V:**

A vector space $V$ over $\mathbb{R}$ is a set along with two functions:

$+ : V \times V \to V$ ( $V$ is closed under vector addition i.e., if $x, y \in V \implies x + y \in V, \ \forall\, x, y \in V$)
$. : \mathbb{R} \times V \to V$ ( $V$ is closed under scalar multiplication i.e., if $x \in V$ and $c \in \mathbb{R} \implies c.x \in V, \ \forall\, x \in V,$ and $\forall c \in \mathbb{R}$)
Axioms of vector additions of a vector space
$v_1 + v_2 = v_2 + v_1, \ \forall\, v_1, v_2 \in V.$
$v_1 + (v_2 + v_3) = (v_1 + v_2) + v_3, \ \forall\, v_1, v_2$ and $v_3 \in V.$
There exists an element 0 (called the zero vector of $V$) in $V$ such that $0 + v = v, \ \forall\, v \in V.$
There exists a vector $v'$ in $V$ such that $v' + v = v + v' = 0, \ \forall\, v \in V$ ($v'$ is called the inverse ( or negative ) of $v$ with respect to addition).

Axioms relating vector addition and scalar multiplication of a vector space (since this is usual multiplication in real numbers, we suppress the "." sign ).
For each vector $v \in V, 1v = v.$
For each vector $v \in V$ and for each pair $a, b \in \mathbb{R}, (a + b)v = av + bv.$
For each $a \in \mathbb{R}$ and for each pair $v_1, v_2 \in V, a(v_1 + v_2) = av_1 + av_2.$
For each vector $v \in V$ and for each pair $a, b \in \mathbb{R}, (ab)v = a(bv).$

Consider a set $V = \{(x, y, z) \mid x + y + z = 0, x, y, z \in \mathbb{R}\} \subseteq \mathbb{R}^3$ with the usual addition: $(x_1, y_1, z_1) + (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2),$
where $(x_1, y_1, z_1), (x_2, y_2, z_2) \in V$ and
scalar multiplication: $c(x, y, z) = (cx, cy, cy), \ \ c \in \mathbb{R}$ and $(x, y, z) \in V.$

## Key Points: 2

Definition of a subspace: A subset $W$ of a vector space $V$ is a subspace of $V$ if it is a vector space with respect to the same operations defined on $V$.
Criteria to check a subset $W$ of a vector space $V$ is a subspace:
Zero vector should be in $W$ i.e., $0 \in W$.
If two vectors $v_1$ and $v_2 \in W$, then $v_1 + v_2 \in W$.
Let $c \in \mathbb{R}$ and if $v \in W$, then $cv \in W$

## Key Points: 3

Linear combination: Let $V$ be a vector space and $v_1, v_2, ...v_k \in V$. The linear combination of $v_1, v_2, ...v_k$ with coefficients $a_1, a_2, ...a_k \in \mathbb{R}$ is the vector $v = \sum_{i=1}^{k} a_i v_i$.
A vector $v \in V$ is said to be a linear combination of $v_1, v_2, ...v_k \in V$ if there exist $a_1, a_2, ...a_k \in \mathbb{R}$ so that $v = \sum_{i=1}^{k} a_i v_i$

## QN : 1,2,5,7,8,9,11

# SWU-2

# Key points: 1

Definition of linear dependence: A set of vectors $v_1, v_2, ...v_n \in V$ is said to be linearly dependent if there exist scalars $a_1, a_2, ...a_n \in \mathbb{R}$, not all zero such that $a_1v_1 + a_2v_2 + ... + a_nv_n = 0$.
To check the vectors $v_1, v_2, ...v_n \in \mathbb{R}^m$ (with usual addition and scalar multiplication) are linearly dependent, we have to verify that the homogeneous system of linear equations $Ax = 0$ has infinitely many solutions, where the $j^{th}$ column of $A$ is $v_j$.
Definition of linear independence: A set of vectors $v_1, v_2, ...v_n \in V$ is said to be linearly independent if $a_1v_1 + a_2v_2 + ... + a_nv_n = 0$ implies that $a_i = 0$ for $i = 1, 2, ...n$.
To check the vectors $v_1, v_2, ...v_n \in \mathbb{R}^m$ (with usual addition and scalar multiplication) are linearly independent, we have to verify that the homogeneous system of linear equations $Ax = 0$ has only the trivial solution (i.e., $x = 0$), where the $j^{th}$ column of $A$ is $v_j$.
If $S$ is a set containing zero vector, then the set $S$ is a linearly dependent set.
Let $S$ be a set of vectors from a vector space $V$. If an element of $S$ is a scalar multiple of another element from the set $S$, then $S$ is a linearly dependent set.

# Key points: 2

Given two vectors $(x_1, y_1)$, and $(x_2, y_2) \in \mathbb{R}^2$ construct $A = \begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix}$. Then these vectors are linearly dependent if and only if $det(A) = 0$ and linearly independent if and only if $det(A) \neq 0$.

Given three vectors $(x_1, y_1, z_1), (x_2, y_2, z_2)$ and $(x_3, y_3, z_3) \in \mathbb{R}^3$ construct $A = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix}$. Then these vectors are linearly dependent if and only if $det(A) = 0$ and linearly independent if and only if $det(A) \neq 0$

# Key points: 3

Any set of $n$ vectors in $\mathbb{R}^2$ with $n \geq 3$ is linearly dependent.
Any set of $n$ vectors in $\mathbb{R}^3$ with $n \geq 4$ is linearly dependent.
Any set of $n + 1$ vectors in $\mathbb{R}^n$ is linearly dependent.
Any set containing a linearly dependent set is linearly dependent.
If a set is linearly independent, then every subset of it, is linearly independent.

# QN : 12 , 15

**Recall the definition of linearly dependent and independent:**

Statement 1) Definition of linear dependence: A set of vectors $\{v_1, v_2, \ldots v_n\}$ in $V$ is said to be linearly dependent if there exist scalars $a_1, a_2, \ldots a_n \in \mathbb{R}$, not all zero such that $a_1v_1 + a_2v_2 + \ldots + a_nv_n = 0$.
Statement 2) Definition of linear independence: A set of vectors $\{v_1, v_2, \ldots v_n\}$ in $V$ is said to be linearly independent if $a_1v_1 + a_2v_2 + \ldots + a_nv_n = 0$ implies that $a_i = 0$ for $i = 1, 2, \ldots n$.
Statement 3) If $S$ is a set containing zero vector, then the set $S$ is a linearly dependent set.
Statement 4) Let $S$ be a subset of a vector space $V$. If an element of $S$ is a scalar multiple of another element from the set $S$, then $S$ is a linearly dependent set.
Statement 5) If $S$ is linearly dependent, then there exists $v \in S$, such that $v$ can be written as a linear combination of other vectors in $S$ and vice versa.

# RWU :

# EMQ : 1,2,3

# PA : 1,2,6,7,8

# GA : 1,2,3,6,7,8,9