MATHEMATICS FOR DATA SCIENCE II

WEEK 7

W 1-6: Matrices; System of lin eqns; V.S; Linear transf

TOPICS TO BE COVERED IN WEEK 7

- Equivalent and Similar matrices
- Affine spaces and transformations
- Lengths and angles
- Inner products and norms on an inner product space

Equivalence of Matrices

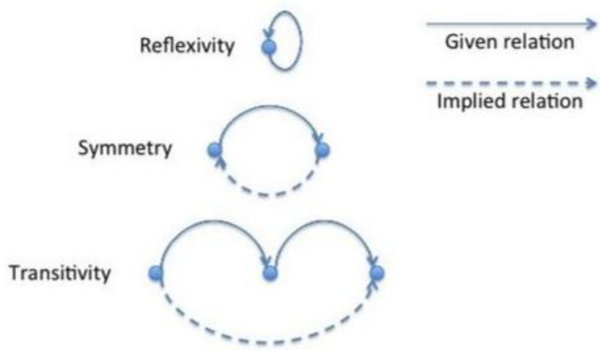
same order

Let A and B be two matrices of order m × n. We say that A is equivalent to B if there is an invertible matrix P of order n × n and an invertible matrix Q of order m × m such that:

$$\boxed{\underset{m\times n}{B} = \underset{m\times m}{Q}\,\underset{m\times n}{A}\,\underset{n\times n}{P}.}$$

Equivalence Relation: A binary relation ~ on a set A is said to be an equivalence relation, if and only if it is reflexive, symmetric and transitive.

Reflexive: a ~ a
Symmetric: if a ~ b, then b ~ a
Transitive: if a ~ b and b ~ c, then a ~ c



Let $M_{m\times n}(R)$ denote the set of all m×n matrices with entries from R. Equivalence of matrices on $M_{m\times n}(R)$ is an equivalence relation, that is, if A,B and C ∈ $M_{m\times n}(R)$, then

Reflexive

• A is equivalent to itself. Take $Q = I_{m\times m}$ to be the identity matrix of order m and $P = I_{n\times n}$ to be the identity matrix of order n. Then we can write $A = I_{m\times m} A I_{n\times n}$. That is, A is equivalent to itself.

$A \sim A$

$A = QAP$ $\quad Q = I_{m\times m}$

$P = I_{n\times n}$

Symmetric: $A \sim B \Rightarrow B \sim A$

• If A is equivalent to B then B is equivalent to A. If A is equivalent to B, then we know that there are two invertible matrices P and Q of order n and m, respectively, such that B = QAP. We can rewrite the above equality as:

$B = QAP$ $\quad Q' = Q^{-1}$

$Q^{-1}BP^{-1} = (Q^{-1}Q)A(PP^{-1}) = I_m A I_n = A$ $\quad A = Q^{-1}BP^{-1}. \quad \Rightarrow A = Q'BP' \quad P' = P^{-1}$

Since P and Q are invertible, P^{-1} and Q^{-1} are also invertible. Therefore B is equivalent to A.

Transitive: $A \sim B, B \sim C \Rightarrow A \sim C$

If A is equivalent to B and B is equivalent to C then A is equivalent to C.

If A is equivalent to B and B is equivalent to C, then we can write

$$B = QAP \quad \text{and} \quad C = Q'BP',$$

$C = \tilde{Q}A\tilde{P}$

$\tilde{Q}, \tilde{P}$ – invertible

where Q, Q' are invertible matrices of order m and P, P' are invertible matrices of order n. Using the above relation, we can write C as

$C = Q'BP'$

$= Q'QAPP'$

$$C = (Q'Q)A(PP').$$

$\tilde{Q} = Q'Q \quad \tilde{P} = PP'$

Note that both $Q'Q$ and PP' are invertible matrices. Therefore A is equivalent to C.

$\det \tilde{Q} = \det Q' \det Q \neq 0$

$\det \tilde{P} = \det P' \det P \neq 0$

Show that $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 7 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 \\ 1 & 0 \\ 2 & 7 \end{bmatrix}$ are equivalent.

Hint: Take $Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, both are invertible matrices.

Check: $B = QAP$

$= I_3 A \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 7 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Note: If it is given that A and B are equivalent then finding P and Q can be very challenging in most of the cases.

Suppose you know that

$$A = \underbrace{\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix}}_{2\times 3} \text{ and } B = \underbrace{\begin{bmatrix} 1 & -1 & 0 \\ 1 & 2 & 1 \end{bmatrix}}_{2\times 3} \qquad \underset{2\times 3}{B} = \underset{2\times 2}{Q}\,\underset{2\times 3}{A}\,\underset{3\times 3}{P}$$

are equivalent matrices. If it is given that Q = $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, then find the number possible choices for P.

Let $P = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$

$$\overset{B}{\begin{bmatrix} 1 & -1 & 0 \\ 1 & 2 & 1 \end{bmatrix}} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 0 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} d+g & e+h & f+i \\ a-d & b-e & c-f \end{bmatrix}$$

$$\left.\begin{matrix} d+g=1 & e+h=-1 & f+i=0 \\ a-d=1 & b-e=2 & c-f=1 \end{matrix}\right\} \Rightarrow \underbrace{\begin{matrix} g=1-d, & h=-1-e, & i=-f \\ d=a-1, & e=b-2, & f=c-1 \end{matrix}}$$

$$\Downarrow$$

$$g = 1-a+1 = 2-a$$
$$h = -1-b+2 = 1-b$$
$$i = -c+1$$

$$P = \begin{pmatrix} a & b & c \\ a-1 & b-2 & c-1 \\ 2-a & 1-b & 1-c \end{pmatrix}$$

where $-a-b+3c \neq 0$

$a+b \neq 3c$

P – invertible

$\det(P) = a\left[(b-2)(1-c) - (1-b)(c-1)\right]$
$-b \ldots$ (complete)

$= -a-b+3c$

We can choose a, b, c in inf. many ways and hence the no. of choices for P is also infinite.

Ex: $T(x,y) = (y,x)$ T w.r.t std basis $B_1 = \{\underset{v_1}{(1,0)}, \underset{v_2}{(0,1)}\}$

$$A_1 = \begin{array}{cc} Tv_1 & Tv_2 \\ \hline \end{array}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$Tv_1 = 0v_1 + 1v_2$

$(0,1) = 0(1,0) + 1(0,1)$

$Tv_2 = 1v_1 + 0v_2$

$(1,0) = 1(1,0) + 0(0,1)$

domain: $\overset{B_2}{\{\underset{w_1}{(1,1)}, \underset{w_2}{(1,-1)}\}}$ co-domain: std. basis

$T\overset{w_1}{(1,1)} = (1,1) = 1v_1 + 1v_2$

$T\overset{w_2}{(1,-1)} = (-1,1) = -1v_1 + 1v_2$

$$A_2 = \begin{array}{cc} Tw_1 & Tw_2 \end{array}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

Find matrix of T

Domain – B_1 B_2

$T(1,0) = (0,1) = \alpha_{11}(1,1) + \alpha_{21}(1,-1)$

$\alpha_{11} + \alpha_{21} = 0$

$\alpha_{11} - \alpha_{21} = 1$

Finding the matrices P and Q in a particular case:

Consider a linear transformation $T : V \to W$. Let $\beta_1 := \{v_1, v_2, \cdots, v_n\}$ and $\beta_2 := \{u_1, u_2, \cdots, u_n\}$ be two ordered bases of V, and $\gamma_1 = \{w_1, w_2, \cdots, w_m\}$ and $\gamma_2 = \{x_1, x_2, \cdots, x_m\}$ be the two ordered bases of W.

• Let A be the matrix representation of T with respect the bases β_1 of V and γ_1 of W.

$$Tv_1 = \alpha_{11}w_1 + \alpha_{21}w_2 + \dots + \alpha_{n1}w_n$$
$$Tv_2 = \alpha_{12}w_1 + \alpha_{22}w_2 + \dots + \alpha_{n2}w_n$$
$$\vdots$$
$$Tv_n = \alpha_{1n}w_1 + \alpha_{2n}w_2 + \dots + \alpha_{nn}w_n$$

$$A = \begin{array}{cccc} Tv_1 & Tv_2 & \cdots & Tv_n \\ \end{array}$$
$$A = \begin{bmatrix} \alpha_{11} & \alpha_{21} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & & \alpha_{2n} \\ \vdots & \vdots & & \vdots \\ \alpha_{n1} & \alpha_{m2} & & \alpha_{nn} \end{bmatrix}$$

$\{u_1, u_2, \dots, u_n\}$ $\{x_1, x_2, \dots, x_n\}$

• Let B be the matrix representation of T with respect the bases β_2 of V and γ_2 of W. $B = (\beta_{ij})$

$$Tu_1 = \beta_{11}x_1 + \beta_{21}x_2 + \dots + \beta_{n1}x_n$$
$$Tu_2 = \beta_{12}x_1 + \beta_{22}x_2 + \dots + \beta_{n2}x_n$$
$$\vdots$$
$$Tu_n = \beta_{1n}x_1 + \beta_{2n}x_2 + \dots + \beta_{nn}x_n$$

$$\begin{array}{ccc} Tu_1 & Tu_2 & Tu_n \end{array}$$
$$B = \begin{bmatrix} \beta_{11} & \beta_{12} & \cdot & \beta_{1n} \\ \beta_{21} & \beta_{22} & & \beta_{2n} \\ \vdots & \vdots & & \vdots \\ \beta_{n1} & \beta_{n2} & & \beta_{nn} \end{bmatrix}$$

$T : V \to W$ $A = [T]_{\beta_1}^{\gamma_1}$ $B = [T]_{\beta_2}^{\gamma_2}$ $B = QAP$ $[T]_{\beta_1}^{\gamma_1}$

$Q \to \gamma_1$ interms of γ_2 $P \to \beta_2$ interms of β_1

Then A and B are equivalent, and satisfy the equality B = QAP, where P and Q are defined as follows:

• For P, express the elements of the ordered basis β_2 in terms of the ordered basis β_1, that is,

$$u_1 = a_{11}v_1 + a_{21}v_2 + \cdots + a_{n1}v_n$$
$$u_2 = a_{12}v_1 + a_{22}v_2 + \cdots + a_{n2}v_n$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$u_n = a_{1n}v_1 + a_{2n}v_2 + \cdots + a_{nn}v_n.$$

The matrix P is given by

$$P = \begin{bmatrix} a_{11} & a_{12} \dots a_{1n} \\ a_{21} & a_{22} \dots a_{2n} \\ \vdots & \vdots \dots \vdots \\ a_{n1} & a_{n2} \dots a_{nn} \end{bmatrix}.$$

For Q, express the elements of the ordered basis γ_1 in terms of the ordered basis γ_2, that is

$$
\begin{aligned}
w_1 &= b_{11}x_1 + b_{21}x_2 + \cdots + b_{m1}x_m \\
w_2 &= b_{12}x_1 + b_{22}x_2 + \cdots + a_{m2}x_m \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
w_m &= b_{1m}x_1 + b_{2m}x_2 + \cdots + b_{mm}x_m.
\end{aligned}
$$

The matrix Q is given by

$$
Q = \begin{bmatrix} b_{11} & b_{12} \dots b_{1m} \\ b_{21} & b_{22} \dots b_{2m} \\ \vdots & \vdots \dots \vdots \\ b_{m1} & b_{m2} \dots b_{mm} \end{bmatrix}.
$$

They satisfy the relation

$$
B = \begin{bmatrix} b_{11} & b_{12} \dots b_{1m} \\ b_{21} & b_{22} \dots b_{2m} \\ \vdots & \vdots \dots \vdots \\ b_{m1} & b_{m2} \dots b_{mm} \end{bmatrix} A \begin{bmatrix} a_{11} & a_{12} \dots a_{1n} \\ a_{21} & a_{22} \dots a_{2n} \\ \vdots & \vdots \dots \vdots \\ a_{n1} & a_{n2} \dots a_{nn} \end{bmatrix}.
$$

Let $T : R^3 \to R^2$ be the linear transformation defined by $T(x_1, x_2, x_3) = (x_1 + x_2, 2x_3 - x_1)$.

- Let A be the matrix representation of the linear transformation T with respect to the ordered bases $\beta_1 = \{\overset{v_1}{(1, 0, 0)}, \overset{v_2}{(0, 1, 0)}, \overset{v_3}{(0, 0, 1)}\}$ for the domain and $\gamma_1 = \{\overset{w_1}{(1, 0)}, \overset{w_2}{(0, 1)}\}$ for the co-domain.

- Let B be the matrix representation of the linear transformation T with respect to the ordered bases $\beta_2 = \{\underset{u_1}{(1, 0, -1)}, \underset{u_2}{(1, 1, 1)}, \underset{u_3}{(1, 0, 0)}\}$ for the domain and $\gamma_2 = \{\underset{x_1}{(0, 1)}, \underset{x_2}{(1, 0)}\}$ for the co-domain.

Let Q and P be matrices such that B = QAP. Then find all matrices A, B, P and Q.

$T(x_1, x_2, x_3) = (x_1 + x_2, 2x_3 - x_1)$

$Tv_1 = T(1, 0, 0) = (1, -1) = 1w_1 - 1w_2$

$Tv_2 = T(0, 1, 0) = (1, 0) = 1w_1 + 0w_2$

$Tv_3 = T(0, 0, 1) = (0, 2) = 0w_1 + 2w_2$

$Tu_1 = (1, -3) = \underline{-3}\,\overset{x_1}{(0, 1)} + \underline{1}\,\overset{x_2}{(1, 0)}$

$Tu_2 = (2, 1) = \underline{1}\,(0, 1) + \underline{2}\,(1, 0)$

$Tu_3 = (1, -1) = \underline{-1}\,(0, 1) + \underline{1}\,(1, 0)$

$$
A = \begin{array}{c} \begin{matrix} Tv_1 & Tv_2 & Tv_3 \end{matrix} \\ \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \end{bmatrix} \end{array}
$$

$$
B = \begin{array}{c} \begin{matrix} Tu_1 & Tu_2 & Tu_3 \end{matrix} \\ \begin{bmatrix} -3 & 1 & -1 \\ 1 & 2 & 1 \end{bmatrix} \end{array}
$$

P — β_2 interms of β_1

Q — γ_1 interms of γ_2

	domain	co-domain
$B \rightarrow$	β_2	γ_2
$Q \rightarrow$	γ_1	γ_1
$A \rightarrow$	β_1	β_1
$P \rightarrow$	β_2	

$$P = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ -1 & 1 & 0 \end{bmatrix}$$

$$\beta_2 = \{ \overset{u_1}{(1,0,-1)}, \overset{u_2}{(1,1,1)}, \overset{u_3}{(1,0,0)} \}$$

$$u_1 = \underline{1}\, V_1 + \underline{0}\, V_2 + \underline{-1}\, V_3$$

$$u_2 = \underline{1}\, V_1 + \underline{1}\, V_2 + \underline{1}\, V_3$$

$$u_3 = \underline{1}\, V_1 + \underline{0}\, V_2 + \underline{0}\, V_3$$

$$\gamma_1 = \{ \underset{\omega_1}{(1,0)}, \underset{\omega_2}{(0,1)} \} \qquad \gamma_2 = \{ \underset{x_1}{(0,1)}, \underset{x_2}{(1,0)} \}$$

$$(1,0)\ \omega_1 = \underline{0}\, x_1 + \underline{1}\, x_2$$

$$(0,1)\ \omega_2 = \underline{1}\, x_1 + \underline{0}\, x_2$$

$$Q = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

HW

Check: $B = \underline{QAP}$

A, B → same order

Other Characterization of Equivalent matrices:

(1) Two matrices A and B are equivalent if A can be transformed in to B by a combination of elementary row and column operations.

or

(2) Two matrices A and B are equivalent if Rank(A)=Rank(B).

Show that A= $\begin{bmatrix} 2 & 6 & -3 \\ 1 & 3 & 2 \end{bmatrix}$ and B= $\begin{bmatrix} 1 & 3 & 2 \\ 0 & 0 & -7 \end{bmatrix}$ are equivalent.

Rank(A) = 2 Rank(B) = 2

Rank(A) = Rank(B)

$\Rightarrow$ A and B are equivalent

Check whether A and B are equivalent $A = \begin{bmatrix} 2 & 0 & 0 \\ 7 & -1 & 0 \\ 6 & 8 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ -2 & 0 \end{bmatrix}$

not same order

No

Check whether A and B are equivalent $A = \begin{bmatrix} 2 & 0 & 0 \\ 7 & -1 & 0 \\ 6 & 8 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 & 2 \\ -2 & 0 & -1 \\ 1 & 3 & 5 \end{bmatrix}$.

$\det(A) = -6 \neq 0$

$\Downarrow$

$\text{rank}(A) = 3$

$\det(B) = 1(+3) - 1(-10+1) + 2(-6)$

$= 3 + 9 - 12 = 0$

$\text{rank}(B) < 3$

$rk(A) \neq rk(B)$

$\Rightarrow$ A and B are not equivalent.

If A and B are equivalent matrices of order m×n, investigate whether the following are true?

$B = QAP \qquad Q, P \to$ Invble

(1) A^T and B^T are equivalent. True

(2) A^2 and B^2 are equivalent. False

(3) AB and BA are equivalent. False

(1) $B^T = (QAP)^T = \underbrace{P^T}_{Q'} A^T \underbrace{Q^T}_{P'} \Rightarrow B^T = Q' A^T P'$

(2) $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \quad rk(A) = 1$

A & B equivalent

$B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad rk(B) = 1$

$A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad rk(A^2) = 0$

$B^2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad rk(B^2) = 1$

A^2 & B^2 — not equivalent

3) $AB = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ $BA = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$rk(AB) = 1$ $rk(BA) = 0$

AB & BA $\rightarrow$ not equivalent

Equivalent : $A \sim B$

$B = QAP$ Q, P inv'ble

$T : V \rightarrow W$

$\beta_1 - \gamma_1 \rightarrow A$

$\beta_2 - \gamma_2 \rightarrow B$

$B = \underset{\substack{\downarrow \\ \gamma_1 \\ \text{in terms of } \gamma_2}}{Q} A \underset{\substack{\downarrow \\ \beta_2 \\ \text{in terms of } \beta_1}}{P}$

$\text{rank}(A) = \text{rank}(B)$

Similar Matrices

Two matrices A and B of order n×n (square) are said to be similar if there exists an invertible matrix P such that B = $P^{-1}AP$.

Note: We check similarity only for square matrices of same order

Verify that similarity defines an equivalence relation between square matrices of same order.

* $A \equiv A$ $P = I$ $P^{-1} = I^{-1} = I$ $A = P^{-1}AP$

* $A \equiv B \Rightarrow B \equiv A$ $B = P^{-1}AP \Rightarrow PBP^{-1} = (PP^{-1})A(PP^{-1}) = IAI = A$

$B \equiv A$

* $A \equiv B,\ B \equiv C \Rightarrow A \equiv C$

$B = P_1^{-1}AP_1,\ C = P_2^{-1}BP_2 \Rightarrow C = \underbrace{P_2^{-1}P_1^{-1}}_{P^{-1}} A \underbrace{P_1P_2}_{P}$ $(P_1P_2)^{-1} = P_2^{-1}P_1^{-1}$

$C = P^{-1}AP \Rightarrow A \equiv C$

If two matrices A and B are similar then they are also equivalent.

$$B = \underset{\substack{\downarrow \\ Q}}{P^{-1}} A P$$

Note that the converse of the above theorem is not true, that is, if two square matrices A and B are equivalent that does not imply that A and B are similar. We will see this with an example. Consider the matrices

Important example: $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

$rk(A) = 2 \qquad rk(B) = 2$

$\Rightarrow$ A & B are equivalent.

Q: Can A & B be similar?

Suppose A & B are similar

$B = P^{-1} A P$

$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = P^{-1} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{I} P = P^{-1} I P = P^{-1} P = I$

not possible

$\Rightarrow$ This assumption is wrong.

So, A & B are not similar

equivalence $\not\Rightarrow$ similar

even for sq. matrices

Check whether $A = \begin{bmatrix} 5 & 3 \\ 3 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 9 \\ 1 & -9 \end{bmatrix}$ are similar or not.

$rk(A) = 2 \quad rk(B) = 1$

Not equivalent

Not equivalent $\Rightarrow$ Not similar

(1) If A or B is invertible, then AB is similar to BA.

(2) If A and B are similar, then A^n is similar to B^n.

(3) If A and B are similar, then A^T is similar to B^T.

(1) A - invble : choose $P = A$

$P^{-1}(AB)P = A^{-1}ABA = IBA = BA$

$\Rightarrow AB$ & BA are similar

HW: B - invble choose $P =$ ___

(2) $A \simeq B \Rightarrow B = P^{-1}AP \Rightarrow B^2 = (P^{-1}AP)(P^{-1}AP)$

$= P^{-1}APP^{-1}AP$

$= P^{-1}AIAP$

$B^2 = P^{-1}A^2P$

Similarly, $B^n = P^{-1}A^nP$

(3) $A \equiv B \Rightarrow B = P^{-1}AP \Rightarrow B^T = (P^{-1}AP)^T = P^T A^T (P^{-1})^T$

$= P^T A^T (P^T)^{-1}$

$(P^{-1})^T = (P^T)^{-1}$

Finding the matrix P for a particular case: Consider a linear transformation $T : V \to V$. Let $\beta := \{v_1, v_2, \cdots, v_n\}$ and $\gamma = \{u_1, u_2, \cdots, u_n\}$ be two ordered bases of V.

• Let A be the matrix representation of T with respect the basis β for both domain and co-domain. $Tv_1 = a_{11}v_1 + a_{21}v_2 + \cdots + a_{n1}v_n$.

• Let B be the matrix representation of T with respect the basis γ for both domain and co-domain.

Then A and B are similar, and satisfy the equality $B = P^{-1}AP$, where P and P^{-1} are defined as follows:

• For P, express the elements of the ordered basis γ in terms of the ordered basis β, that is,

$$\begin{aligned} u_1 &= a_{11}v_1 + a_{21}v_2 + \cdots + a_{n1}v_n \\ u_2 &= a_{12}v_1 + a_{22}v_2 + \cdots + a_{n2}v_n \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ u_n &= a_{1n}v_1 + a_{2n}v_2 + \cdots + a_{nn}v_n. \end{aligned}$$

The matrix P is given by $P = \begin{bmatrix} a_{11} & a_{12} \dots a_{1n} \\ a_{21} & a_{22} \dots a_{2n} \\ \vdots & \vdots \dots \vdots \\ a_{n1} & a_{n2} \dots a_{nn} \end{bmatrix}$.

• For P^{-1}, express the elements of the ordered basis β in terms of the ordered basis γ (or one can compute P^{-1} directly after computing P), that is,

$$
\begin{aligned}
v_1 &= b_{11}u_1 + b_{21}u_2 + \cdots + b_{n1}u_n \\
v_2 &= b_{12}u_1 + b_{22}u_2 + \cdots + a_{n2}u_n \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
v_n &= b_{1n}u_1 + b_{2n}u_2 + \cdots + b_{nn}u_n.
\end{aligned}
$$

The matrix P^{-1} is given by $P^{-1} = \begin{bmatrix} b_{11} & b_{12} \dots b_{1n} \\ b_{21} & b_{22} \dots b_{2n} \\ \vdots & \vdots \dots \vdots \\ b_{n1} & b_{n2} \dots b_{nn} \end{bmatrix}$.

$$
B = P^{-1}AP = \begin{bmatrix} b_{11} & b_{12} \dots b_{1n} \\ b_{21} & b_{22} \dots b_{2n} \\ \vdots & \vdots \dots \vdots \\ b_{n1} & b_{n2} \dots b_{nn} \end{bmatrix} A \begin{bmatrix} a_{11} & a_{12} \dots a_{1n} \\ a_{21} & a_{22} \dots a_{2n} \\ \vdots & \vdots \dots \vdots \\ a_{n1} & a_{n2} \dots a_{nn} \end{bmatrix}.
$$

Let $T : R^2 \to R^2$ be the linear transformation defined by $T(x_1, x_2) = (-x_2, x_1)$.

• Let A be the matrix representation of the linear transformation T with respect to the ordered basis $\beta = \{(1, 0), (0, 1)\}$ for both domain co-domain.

• Let B be the matrix representation of the linear transformation T with respect to the ordered basis $\gamma = \{(1, 2), (1, -1)\}$ for both domain co-domain. Let P a matrix such that $B = P^{-1}AP$.

Then find all matrices A,B,P and P^{-1}.

$\boxed{\text{HW}}$

Properties of similar matrices

- If M is an invertible matrix of order n then

$$\text{Rank}(AM) = \text{Rank}(MA) = \text{Rank}(A),$$

for any arbitrary matrix of A order n.

- If A and B are two matrices of order n then

$$\text{Trace}(AB) = \text{Trace}(BA) \quad \text{and} \quad \text{Det}(AB) = \text{Det}(A)\text{Det}(B)$$

- If two matrices A and B are similar, then they have the same rank.

similar $\Rightarrow$ equivalence $\Rightarrow$ same rank

- If two matrices A and B are similar, then they have the same trace.

$tr(XY) = tr(YX)$ $\quad B = P^{-1}AP$

$tr(B) = tr(P^{-1}AP) = tr(\underbrace{AP}_{Y}\underbrace{P^{-1}}_{X}) = tr(AI) = tr(A)$

$X = P^{-1}$

$Y = AP$

- If two matrices A and B are similar, then they have the same determinant.

$det(B) = det(P^{-1}AP) = det(P^{-1})\, det(A)\, det(P)$

$= \frac{1}{det(P)}\, det(A)\, det(P) = det(A)$

Similarity → same rk, same trace, same det

⇍ converse not true

Note: If two matrices are similar then they have same rank, trace and determinant but the converse is not true. In particular, if two matrices A and B have same rank, trace and determinant that does not imply that they are similar.

Take $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

Compare the rank, trace and determinant and also check if they are similar.

$rk(A) = rk(B) = 2$

$tr(A) = tr(B) = 2$

$det(A) = det(B) = 1$

But, A and B are not similar

Check whether $A = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$ are similar or not. Not sim

$tr(A) = 5 \neq tr(B) = 4$

Check whether $A = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 11 \\ 0 & 4 \end{bmatrix}$ are similar or not.

$det(A) = 6 \neq det(B) = 4$ not similar.

Can a scalar matrix be similar to a non-scalar matrix? No.

$A = \alpha I$

Suppose

$B = P^{-1} A P$

$= P^{-1} (\alpha I) P$

$= \alpha P^{-1} I P$

$B = \alpha I \rightarrow$ not possible

But B is non-scalar