IIT Madras
BSc Degree

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Maths II
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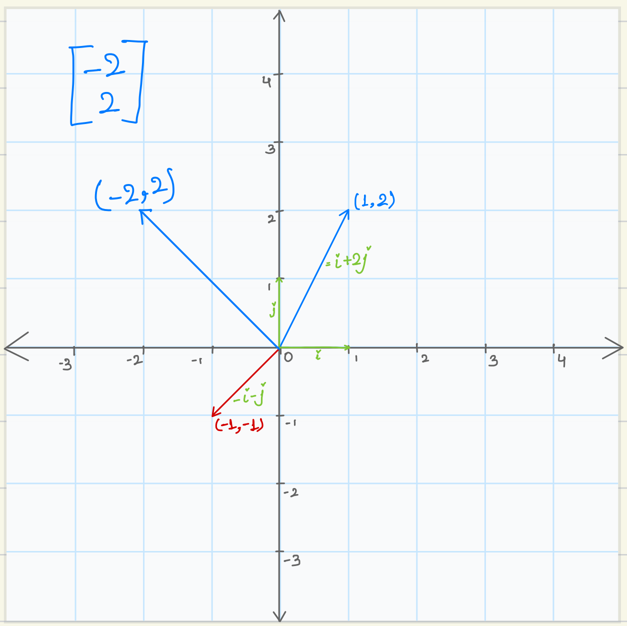
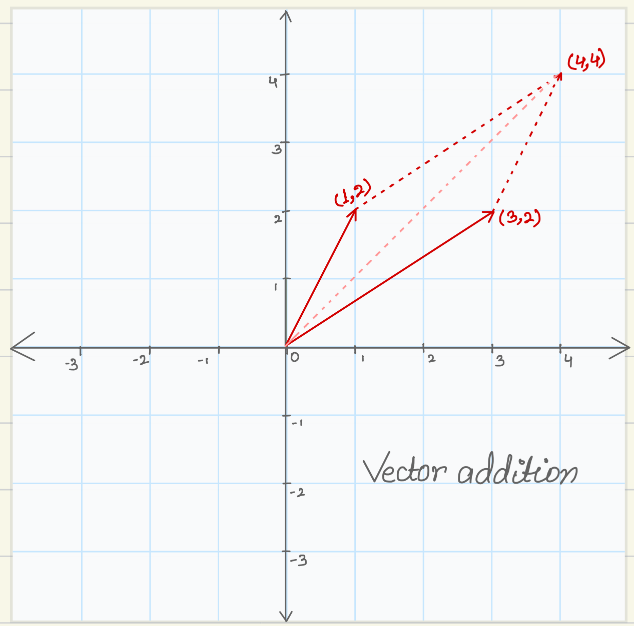
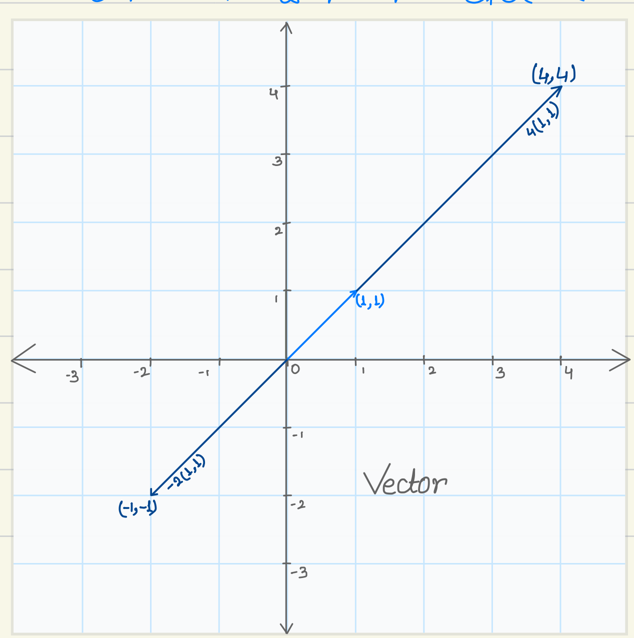
○ **Vectors :-** Vectors can be thought of list. It could be of rows or columns. It can be donate point in n-dimension and combining more than 1 vector form line. It can be used to perform arthemetic operation such as (sum/minus) and multiplication.





○ **Vector addition :** To add 2 vector graphically place the graphical line on top point of other line in same direction or parallel to line which is been added.

○ **Vector multiplication :** It is multiplying scaler (a number) with vector which streches vector in same direction in case of +ve scaler & in opposite direction in -ve scaler.



- **Matrix:** It is a rectangular/array of number, arranged in rows and column.

Eg → $\begin{bmatrix} 1, & 2, & 3 \\ 3, & 9 & 2 \end{bmatrix}$ This is 2X3 matrix. (2 row, 3 columns)

- An m x n metrics has m rows and n columns.
- (i, j)-th entry of metrics is entry in i^{th} row & j^{th} columns

- **Square metrics** is metrics with same no. of row & columns.

- **Diagonal metrix** has all no. 0 except diagonal.

- **Scaler metrics** is diagonal metric with all same value.

- **Identitity metrics** is scalar metrix with all value 1.

- Linear equation can be written in metrices:

Eg: $\begin{matrix} 3x + 4y = 5 \\ 4x + 6y = 10 \end{matrix} = \left[\begin{array}{cc|c} 3 & 4 & 5 \\ 4 & 6 & 10 \end{array}\right] = \begin{bmatrix} 3 & 4 & 5 \\ 4 & 6 & 10 \end{bmatrix}$

- Metrics addition: Here we add corresponding entries of matrics. (Both metrics must have same size)

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} + \begin{bmatrix} u & v & w \\ x & y & z \end{bmatrix} = \begin{bmatrix} a+u & b+v & c+w \\ d+x & e+y & f+z \end{bmatrix}$$

$$(A+B)_{ij} = A_{ij} + B_{ij}$$

- Scalar multiplication: Here we multiply scaler with each entry of metrics.

Eg: $3\begin{bmatrix}4 & 1\\ 2 & 3\end{bmatrix} = \begin{bmatrix}12 & 3\\ 6 & 9\end{bmatrix}$

- $(cA)_{ij} = c(A)_{ij}$

- Metrix multiplication: It is multipling row of metrix 1 & column of metrix 2 in consicutive place. Eg: $\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}_{2\times2}\begin{bmatrix}1 & 2 & 3\\ 3 & 4 & 5\end{bmatrix}_{2\times3} = \begin{bmatrix}7 & 10 & 13\\ 15 & 22 & 29\end{bmatrix}_{2\times3}$

- Here, no. of row in 1st metrix should be equal to no. of columns in 2nd metrix

$$A_{m\times n}B_{n\times b} = (AB)_{m\times p}$$

- The $(i,j)^{th}$ entry of AB is defined as:

$$(AB)_{ij} = \sum_{k=1}^{n} A_{ik}B_{kj}$$

- Multiplication by scaler metrices.

$$\begin{bmatrix}c & 0 & 0\\ 0 & c & 0\\ 0 & 0 & c\end{bmatrix}\begin{bmatrix}1 & 5\\ 2 & 3\\ 3 & 2\end{bmatrix} = \begin{bmatrix}c & 5c\\ 2c & 3c\\ 3c & 2c\end{bmatrix} = c\begin{bmatrix}1 & 5\\ 2 & 3\\ 3 & 2\end{bmatrix}$$

- If any metrics multiply by identity matrix. It gives the same matrix. It act as 1.

- Properties of matrix addition & multiplication:

$$(A+B)+C = A+(B+C)$$

$$(AB)C = A(BC)$$

$$A+B = B+A$$

$$AB \neq BA$$

$$\lambda(A+B) = \lambda A + \lambda B, \quad \lambda(AB) = (\lambda A)B = (\lambda B)A$$

$$A(B+C) = AB+BC, \quad (A+B)C = AC+BC$$

- Linear equation in form of matrix:

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$
$$\dots \qquad \dots$$
$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m$$

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix} \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \dots \\ b_m \end{bmatrix}$$

So, linear equation can be represented in form of $Ax = b$

Example:

$$3x + 2y + z = 6$$
$$x + \frac{1}{2}y + \frac{2}{3}z = \frac{7}{6}$$
$$4x + 6y - 10z = 0$$

$$A = \begin{bmatrix} 3 & 2 & 1 \\ 1 & -\frac{1}{2} & \frac{2}{3} \\ 4 & 6 & -10 \end{bmatrix} \quad x = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad b = \begin{bmatrix} 6 \\ 7/6 \\ 0 \end{bmatrix}$$

There are 3 possiblity for solution of linear system of eq:

- The system has infinitely many solution.
- The system has single unique solution.
- The system has no solution.

- **Determinant:** Every square matrix has a number associated with it, which is called determinant & can be denoted by det(A) or |A|. It can used in:
  - Solving system of linear equation.
  - Finding inverse of matrix & calulus, more ...

- Determinant of 1×1 matrix: $A = [a]$, $a = 1\times1$ than $\det(A) = a$
- Determinant of 2×2 matrix: $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\det(A) = ad - bc$

Eg: $A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} = 4 - 6 = -2$

- Determinant of 3×3 matrix: $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\det(A) = a_{11} \times \det\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12} \times \det\begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13} \times \det\begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

$$\det(A) = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

Eg, $A = \begin{bmatrix} 2 & 4 & 1 \\ 3 & 8 & 7 \\ 5 & 6 & 9 \end{bmatrix}$

$\det(A) = 2(72-42) - 4(27-35) - 1(18-40) = 70$

Determinant of product of matrices:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$$

$$AB = \begin{bmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{bmatrix}$$

$$\det(AB) = (ae+bg)(cf+dh) - (af+bh)(ce+dg)$$
$$= \cancel{aecf} + bgcf + aedh + \cancel{bgdh} - \cancel{afce} - afdg - bhce - \cancel{bhdg}$$
$$= bgcf + aedh - bhce - afdg$$
$$= bcfg + adeh - bceh - adfg$$
$$= adeh - adfg - bceh + bcfg$$
$$= ad(eh-fg) - bc(eh-fg)$$
$$= (ad-bc)(eh-fg)$$

So, $\det(AB) = \det(A)\det(B)$

Determinant of inverse of a matrix:

$$AA^{-1} = I = A^{-1}A$$

$$\Rightarrow \det(AA^{-1}) = \det(I)$$

we saw, $\Rightarrow \det(A)\det(A^{-1}) = 1$

$$\Rightarrow \det(A^{-1}) = \frac{1}{\det(A)}$$

- Properties of determinant:

— Switching 2 rows/columns of matrix 'A' gives -ve of determinants of matrix 'A'.

Eg: $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $\tilde{A} = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$

$$\det(\tilde{A}) = cb - ad = -(ad - bc) = -\det(A)$$

— Adding multiple of a row to another row in matrix 'A' gives same determinent as det(A)

Eg: $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $\tilde{A} = \begin{bmatrix} a+tc & b+td \\ c & d \end{bmatrix}$

$$\det(\tilde{A}) = (a+tc)d - (b+td)c = ad - \cancel{tcd} - bc - \cancel{tdc}$$
$$= ad - bc = \det(A)$$

— Scalar multiplication of row/column with constant 't' of a matrix 'A' gives $t \cdot \det(A)$.

Eg: $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $\tilde{A} = \begin{bmatrix} a & tb \\ c & td \end{bmatrix}$

$$\det(\tilde{A}) = tad - tbc = t(ad - bc)$$
$$= t \cdot \det(A)$$

- Upper triangular matrix: These are matrix which has zeros below the diagonal.
  - For such matrix the determinant is product of diagonal. Eg:

$$A = \begin{bmatrix} 2 & 4 & 3 \\ 0 & 8 & 7 \\ 0 & 0 & 9 \end{bmatrix} \quad \det(A) = 2(72-0) - 4(0-0) + 3(0-0)$$

$$= 2\times72 - 4\times0 + 3\times0 = 144 = 2\times8\times9$$

- The transpose of a matrix:
  The transpose of matrix '$A_{m\times n}$' is '$n\times m$' matrix with $(i,j)^{th}$ entry is A_{ji}.

Notation = A^T Definition $\Rightarrow (A^T)_{ij} = A_{ji}$

$$\text{let, } A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad \text{then, } A^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$$

$$\det(A) = a_{11} \times \det\begin{bmatrix} a_{22} & a_{32} \\ a_{23} & a_{33} \end{bmatrix} - a_{21} \times \det\begin{bmatrix} a_{12} & a_{32} \\ a_{13} & a_{33} \end{bmatrix} + a_{31} \times \det\begin{bmatrix} a_{12} & a_{22} \\ a_{13} & a_{23} \end{bmatrix}$$

$$= a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{21}(a_{12}a_{33} - a_{32}a_{13}) - a_{31}(a_{12}a_{23} - a_{22}a_{13})$$

$$= a_{11}a_{22}a_{33} - a_{11}a_{32}a_{23} - a_{21}a_{12}a_{33} - a_{21}a_{32}a_{13} - a_{31}a_{12}a_{23} - a_{31}a_{22}a_{13}$$

$$= \det(A)$$

$\because$ The determinant of transpose of a matrix is the determinant of matrix original matrix.

- Minors and cofactors:
  Minor as we seen in calculating determinant of 3x3 matrix.

So, as we have $n \times n$ square matrix. $[n \leq 4]$
Than the minor of entry of i^{th} row & j^{th} column is determinant of submatrix formed by deleting i^{th} row & j^{th} column.

$(i,j)^{th}$ minor

Eg, $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ $\quad M_{11} = \det \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}$

The (i,j)th cofactor $C_{ij} = (-1)^{i+j} M_{ij}$

For above example: $C_{11} = (-1)^{1+1} M_{11} = M_{11}$
& , $C_{23} = (-1)^{2+3} M_{11} = -M_{23}$

- Determinants in term of minors & cofactors:

For above example $(A_{3\times 3})$ determinant will be:

- $\det(A) = a_{11} \times \det \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12} \times \det \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} - a_{13} \times \det \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$

$= a_{11} \times M_{11} - a_{12} \times M_{12} + a_{13} \times M_{13}$

↳ Here we will replace with minor.

$= a_{11} \times C_{11} + a_{12} \times C_{12} + a_{13} \times C_{13}$

↳ To make all sign +ve we replace M_{ij} with cofactor.

for $A_{2\times2}$ also, $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$

$$\det(A) = A_{11}A_{22} - A_{12}A_{21}$$
$$= A_{11}M_{11} - A_{12}M_{12} = A_{11}C_{11} + A_{12}C_{12}$$

It is generalised for $A_{4\times4}$ also:

$$\det(A) = \sum_{j=1}^{4} (-1)^{1+j} a_{1j} M_{1j} = \sum_{j=1}^{4} a_{1j} C_{1j}$$

- Inductive definition of determinant:

So, for $A_{n\times n}$ metrix. The determinant will be similar to above like $A_{4\times4}$.

By directly replacing cofactor and minor. we get:

$$\det(A) = \sum_{j=1}^{n} (-1)^{1+j} a_{1j} M_{1j} = \sum_{j=1}^{n} a_{1j} C_{1j}$$

- Expension along any row or column:

It means determinant can not only calculated by 1 st row only. It can be calculated by selecting any fixed row or a column.

$$\mathrm{Det}(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} M_{ij} \quad \text{for a fixed } i$$
$$= \sum_{i=1}^{n} (-1)^{i+j} a_{ij} M_{ij} \quad \text{for a fixed } j$$

Eg, $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ $\quad \begin{bmatrix} + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ + & - & + & - & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \leftarrow (-1)^{i+j}$

Determinant by 2nd row of A:

$$= -a_{21} \times \det\begin{bmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{bmatrix} + a_{22} \times \det\begin{bmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{bmatrix} - a_{23} \det\begin{bmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{bmatrix}$$

Determinant by 2nd column of A:

$$= -a_{12} \times \det\begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{22} \times \det\begin{bmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{bmatrix} - a_{32} \det\begin{bmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{bmatrix}$$

- [ ] Important properties and identities:

Property 1: Determinant of a product is product of a determinant.

$$\det(AB) = \det(A)\det(B)$$
$$\det(A^n) = \det(A)^n$$
$$\det(A^{-1}) = \frac{1}{\det(A)} = \det(A)^{-1}$$

$$\det(P^{-1}AP) = \det(A)$$
$$\det(AB) = \det(BA)$$
$$\det(A^TA) = \det(A)^2$$
$$\det(A^T) = \det(A)$$

Property 2: Switching 2 rows or columns changes sign:

$$\det(\tilde{A}) = -\det(A)$$

Property 3: Adding multiples of a row to another row leaves determinant unchanged and viceversa for columns.

$$\det(\tilde{A}) = \det(A)$$

Property 4: Scaler multiplication of a row by constant 't' multiply determinant by 2 and vice-versa for column.

$$\det(tA) = t\det(A)$$

(for 1 row or colum)

For all row it will be:

$$\det(tA_{n \times n}) = t^n \det(A)$$

Useful tips:

- The determinant of a matrix with a row or column of 0 is 0.
- The determinant of a matrix with a row or column is linear combination of other column is zero.
- Scaler multiplication of a row/column by 't' multiplies determinant by 't'.
- While calculating determinant, you can choose to compute it using expansion along any suitable row/column.

Cramer's Rule:

Consider a solution of linear equation $Ax=b$ where A is a $n \times n$ invertable matrix and b is a column vector with 'n' entries.

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$
$$\dots \qquad \dots$$
$$a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = b_n$$

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix} \qquad b = \begin{bmatrix} b_1 \\ b_2 \\ \dots \\ b_n \end{bmatrix}$$

Than A_{x_i} to be matrix obtained by replacing the ith column of A by the column vector b. Cramer's rule states the unique solution is:

$$x_i = \frac{\det(A_{x_i})}{\det(A)}$$

Example:

Consider eq:

$$4x_1 - 3x_2 = 11$$
$$6x_1 + 5x_2 = 7$$

Sol, $x_1 = 2$, $x_2 = -1$

Matrix representation $Ax=b$:

$$A = \begin{bmatrix} 4 & -3 \\ 6 & 5 \end{bmatrix}, \quad b = \begin{bmatrix} 11 \\ 7 \end{bmatrix}$$

$\det(A) = 38$

$\boxed{A_{x_1} = \begin{bmatrix} 11 & -3 \\ 7 & 5 \end{bmatrix}, \det(A_{x_1}) = 76}$ $\boxed{A_{x_2} = \begin{bmatrix} 4 & 11 \\ 6 & 7 \end{bmatrix}, \det(A_{x_2}) = -38}$

Replace 1st row of A with b than 2nd row and on.

Calculate $\frac{\det(A_{x_1})}{\det(A)} = \frac{76}{38} = 2$ $\frac{\det(A_{x_2})}{\det(A)} = \frac{-38}{38} = -1$

Similarly we can solve any invertible matrix on 'n' dimension by constantly replacing ith column with with 'b' and finding determinant and dividing by determinant of 'A'.

- **Converse of matrix:**

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \det(A) = ab - bc \neq 0$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Here,
$$\left.\begin{aligned} ae + bg &= 1 \\ ce + dg &= 0 \\ af + bh &= 0 \\ cf + dh &= 1 \end{aligned}\right\} \qquad \begin{aligned} ade + bdg &= d \\ bce + bdg &= 0 \\ e(ad-bc) &= d \\ e &= \frac{d}{ad-bc} \end{aligned}$$

- **Adjugate of square matrix:-**

The adjugate matrix is transpose of

cofactor matrics which is obtain by minor.

$$adj(A) = C^T$$

Eg: $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 8 \\ 5 & 6 & 0 \end{bmatrix}$

$$\det(A) = 1(2\times0-8\times6)-2(0\times0-8\times5)+3(0\times6-2\times5)$$
$$= -48+80-30 = 2$$

Minors $\Rightarrow$
Remove ij row & col. & find det of rest
$\begin{bmatrix} -48 & -40 & -10 \\ -18 & -15 & -4 \\ 10 & 8 & 2 \end{bmatrix}$

$(\text{minor})(-1)^{i+j}$
Cofactor $= \begin{bmatrix} -48 & 40 & -10 \\ 18 & -15 & 4 \\ 10 & -8 & 2 \end{bmatrix}$

$$Adj(A) = C^T = \begin{bmatrix} -48 & 18 & 10 \\ 40 & -15 & -8 \\ -10 & 4 & 2 \end{bmatrix}$$

Compute $A \frac{1}{\det(A)} adj(A)$ & $\frac{1}{\det(A)} adj(A) A$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 8 \\ 5 & 6 & 0 \end{bmatrix} \begin{bmatrix} -24 & 9 & 5 \\ 20 & -\frac{15}{2} & -4 \\ -5 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} -24 & 9 & 5 \\ 20 & \frac{15}{2} & -4 \\ -5 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 8 \\ 5 & 6 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Hence, $A^{-1} = \frac{1}{\det(A)} adj(A)$

- Adjugate and inverse:

If A is a $n \times n$ metrix and $\det(A) \neq 0$, than A^{-1}.
Than A^{-1} exists and equal:

$$\text{Hence, } A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A)$$

- Solution of system of linear eq. with ivertable coff. matrix:

Consider a system of linear equation $Ax = b$ where coff. metrix A is invertable.

Multiplying both side by A^{-1}:

$$Ax = b$$
$$A^{-1}Ax = A^{-1}b$$
$$Ix = A^{-1}b$$
$$x = A^{-1}b$$

→ Complete example till now:

Calculate inverse of linear equation:

$$8x_1 + 8x_2 + 4x_3 = 1960$$
$$12x_1 + 5x_2 + 7x_3 = 2215$$
$$3x_1 + 2x_2 + 5x_3 = 1135$$

$$A = \begin{bmatrix} 8 & 8 & 4 \\ 12 & 5 & 7 \\ 3 & 2 & 5 \end{bmatrix} \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad b = \begin{bmatrix} 1960 \\ 2215 \\ 1135 \end{bmatrix}$$

$$\det(A) = 8(25-14) - 8(60-21) + 4(24-15)$$
$$= 88 - 312 + 36 = -188$$

$$\text{Minors} = \begin{bmatrix} 11 & 39 & 9 \\ 32 & 28 & -8 \\ 36 & 8 & -56 \end{bmatrix} \qquad \text{Coff. matrix} = \begin{bmatrix} 11 & -39 & 9 \\ -32 & 28 & 8 \\ 36 & -8 & -56 \end{bmatrix}$$

$$\text{Adjucate} = \begin{bmatrix} 11 & -32 & 36 \\ -39 & 28 & 8 \\ 9 & 8 & -56 \end{bmatrix}$$

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A) = \frac{1}{-188} \begin{bmatrix} 11 & -32 & 36 \\ -32 & 28 & -8 \\ 36 & 8 & -56 \end{bmatrix}$$

$$x = A^{-1} b = \frac{1}{-188} \begin{bmatrix} 11 & -32 & 36 \\ -32 & 28 & -8 \\ 36 & 8 & -56 \end{bmatrix} \begin{bmatrix} 1960 \\ 2215 \\ 1135 \end{bmatrix} = \frac{1}{-188} \begin{bmatrix} 1960 \\ 2215 \\ 1135 \end{bmatrix} = \begin{bmatrix} 45 \\ 125 \\ 150 \end{bmatrix}$$

means, $x_1 = 45$, $x_2 = 125$, $x_3 = 150$

○ Homogenous system of linear equation :

- It is in form $Ax = 0$. This solution is called 'trivial solution'.

- For this system there only 2 possibility :
  - 0 is unique solution
  - There is infinitely many solution other than 0.

- In a homogenous system of equation, if there are more variable than eq. than it guaranteed have non-trivial solution.

○ Echelon form:

□ Row echelon form:

The metrix is in row echelon form if:

- The 1st entry of each row (leading entry) is 1.
- Each leading entry in column to right of the leading entry of previous row.
- Row with all '0' element are below non-zero row.
- For row with all 0 leading entry will be 0.

Eg: $A_{ref} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ Row echelon form

□ Reduced Row echelon form:

In this the metrix is in row echelon form and the column having leading cofficient 1 should have all other element 0.

Eg: $A_{rref} = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ Reduced Row echelon form

Solving this, $A_{rref}\, x = b$

$x_1 + 2x_2 = b_1$ $\quad x_1 = b_1 - 2x_2$

$x_3 = b_2$ $\quad x_2 = c$ (constant)

$x_4 = b_3$ $\quad x_1 = b_1 - 2c$

$$x = \begin{bmatrix} b_1 - 2c \\ c \\ b_2 \\ b_3 \end{bmatrix}$$

- let $Ax=b$ a system of linear equation and suppose A is in reduced row escelon form.
- Assume for every zero row of A, the corrosponding entry of b is 0. (if the i^{th} row of $A=0$ so be b_i)
  - If i^{th} column has leading entry of some row, we call x_i a dependent var.
  - The i^{th} column don't have leading entry we call x_i an independent variable.

Example:

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix}$$

$x_1, x_2 \to$ dep. var

$x_3 \to$ Ind. var

$$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$x_1, x_3, x_4 \to$ dep. var.

$x_2 \to$ Ind. var.

Reduced row echelon form gives all sol of $Ax=b$

Elementry row operations:

Type 1: $\det(A) = -\det(B)$

Interchanging 2 rows

$$\begin{bmatrix} 3 & 2 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 7 & 1 & 1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 3 & 2 & 1 & 1 \\ 0 & 7 & 1 & 1 \end{bmatrix}$$

Type 2: $\det(A) = c\det(B)$

Scaler mult. of a row by constant

$$\begin{bmatrix} 3 & 2 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 7 & 1 & 1 \end{bmatrix} \xrightarrow{R_1/3} \begin{bmatrix} 1 & 2/3 & 1/3 & 1/3 \\ 1 & 1 & 0 & 0 \\ 0 & 7 & 1 & 1 \end{bmatrix}$$

Type 3: $\det(A) = \det(B)$

Adding multiple of a row to another row

$$\begin{bmatrix} 3 & 2 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 7 & 1 & 1 \end{bmatrix} \xrightarrow{R_1 - 3R_2} \begin{bmatrix} 0 & -1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 7 & 1 & 1 \end{bmatrix}$$

◦ Row reduction (Row echolon form) :

He we have to do row wise operations to convert a matrix to reduced row echelon form.

Action	Example with notation
1) Find leftmost non-zero column.	$\begin{bmatrix} 3 & 2 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 7 & 1 & 1 \end{bmatrix}$
2) Use row operation to get 1 in top of column. If we have 0 on top we can replace it with other row.	$\begin{bmatrix} 3 & 2 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 7 & 1 & 1 \end{bmatrix} \xrightarrow{R_1/3} \begin{bmatrix} 1 & 2/3 & 1/3 & 1/3 \\ 1 & 1 & 0 & 0 \\ 0 & 7 & 1 & 1 \end{bmatrix}$
3) Once we got 1 on top than we try to make '0' all element below it by sub. multiple of 1st row.	$\begin{bmatrix} 1 & 2/3 & 1/3 & 1/3 \\ 1 & 1 & 0 & 0 \\ 0 & 7 & 1 & 1 \end{bmatrix} \xrightarrow{R_2 - R_1} \begin{bmatrix} 1 & 2/3 & 1/3 & 1/3 \\ 0 & 1/3 & -1/3 & -1/3 \\ 0 & 7 & 1 & 1 \end{bmatrix}$
4) If all element that row after 1 become 0. we come to next row.	$\begin{bmatrix} 1 & 2/3 & 1/3 & 1/3 \\ 0 & 1/3 & -1/3 & -1/3 \\ 0 & 7 & 1 & 1 \end{bmatrix}$
5) Here we find next non-zero column by ignoring 1st row. By repeating same operation from 1-5.	$\begin{bmatrix} 1 & 2/3 & 1/3 & 1/3 \\ 0 & 1/3 & -1/3 & -1/3 \\ 0 & 7 & 1 & 1 \end{bmatrix} \xrightarrow{R_2 \times 3} \begin{bmatrix} 1 & 2/3 & 1/3 & 1/3 \\ 0 & 1 & -1 & -1 \\ 0 & 7 & 1 & 1 \end{bmatrix}$
- Removing 7 to 0	$\begin{bmatrix} 1 & 2/3 & 1/3 & 1/3 \\ 0 & 1 & -1 & -1 \\ 0 & 7 & 1 & 1 \end{bmatrix} \xrightarrow{R_3 - 7R_2} \begin{bmatrix} 1 & 2/3 & 1/3 & 1/3 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 8 & 8 \end{bmatrix}$

– Repeating (1–5)

$$\begin{bmatrix} 1 & 2/3 & 1/3 & 1/3 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 8 & 8 \end{bmatrix} \xrightarrow{R_3/8} \begin{bmatrix} 1 & 2/3 & 1/3 & 1/3 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Now we get matrix in Row echelon form:

$$\begin{bmatrix} 1 & 2/3 & 1/3 & 1/3 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

○ Reduced row echelon form:

We can reduce row echelon form matrix to reduced row echelon form.

Action	Example with notation
Take a column contaning 1 in extreme right.	$\begin{bmatrix} 1 & 2/3 & 1/3 & 1/3 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$
Do row operation to make elements '0' of that columns.	$\begin{bmatrix} 1 & 2/3 & 1/3 & 1/3 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \xrightarrow{R_2+R_3,\ R_1-\frac{R_3}{3}} \begin{bmatrix} 1 & 2/3 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$
Remove zero from previous column and keep doing untill all coll. left with 1 (1).	$\begin{bmatrix} 1 & 2/3 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \xrightarrow{R_1-\frac{2R_2}{3}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

The resulting matrix is in Reduced row echelon form:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

- Calculating determinant by row reduction :

As we saw row echelon form of matrix produces an upper triangular matrix with diagonal entry all 1 if it is invertable (square) or some 1 and 0.

- Row reduced A into row echelon form.
- If diagonal of reduced matrix has 0, than its determinant is 0.
- We can find determinant by doing reverse mapping of operation by finding effect on determinant. (bcz in row echelon form of squre matrix determinant is 1.

Type 1 : $\det(A) = -\det(B)$ Interchanging 2 rows.

Type 2 : $\det(A) = c\det(B)$ Scaler mult. of a row by constant.

Type 3 : $\det(A) = \det(B)$ Adding multiple of a row to another row.

- Augmanted matrix :

Here we have a system of linear equation wher A is 'mxn' matrix and b is 'mx1' column vector.

An augmanted matrix is a system defined as matrix of size 'mx(n+1)' whose first 'n' columns are column of A and last column is columnf b.

We denote it as $[A|b]$

Augmanted matrix : $[A|b] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & b_n \end{array}\right]$

Example :

$$3x_1 + 2x_2 + x_3 + x_4 = 6$$
$$x_1 + x_2 = 2$$
$$7x_2 + x_3 + x_4 = 8$$

$A = \begin{bmatrix} 3 & 2 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 7 & 1 & 1 \end{bmatrix}$ $b = \begin{bmatrix} 6 \\ 2 \\ 8 \end{bmatrix}$

$$[A|b] = \left[\begin{array}{cccc|c} 3 & 2 & 1 & 1 & 6 \\ 1 & 1 & 0 & 0 & 2 \\ 0 & 7 & 1 & 1 & 8 \end{array}\right]$$

○ Gaussian elimination method :

Consider system of linear equation $Ax = b$

- Form augmented matrix of system $[A|b]$
- Perform reduced row echelon form on $[A|b]$
- After step 1 & 2, R & C be the new matrix obtained $[R|C]$

Hence, solution of $Ax = b$ is precisely solution of $Rx = c$.

- From corresponding system of eq $Ax=b$ find all sol. of $Rx=c$ & hence $Ax=b$.

Example :

$$\left[\begin{array}{cccc|c} 3 & 2 & 1 & 1 & 6 \\ 1 & 1 & 0 & 0 & 2 \\ 0 & 7 & 1 & 1 & 8 \end{array}\right] \xrightarrow{R_1/3} \left[\begin{array}{cccc|c} 1 & 2/3 & 1/3 & 1/3 & 2 \\ 1 & 1 & 0 & 0 & 2 \\ 0 & 7 & 1 & 1 & 8 \end{array}\right] \xrightarrow{R_2 - R_1} \left[\begin{array}{cccc|c} 1 & 2/3 & 1/3 & 1/3 & 2 \\ 0 & 1/3 & -1/3 & -1/3 & 0 \\ 0 & 7 & 1 & 1 & 8 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & 2/3 & 1/3 & 1/3 & 2 \\ 0 & 1/3 & -1/3 & -1/3 & 0 \\ 0 & 7 & 1 & 1 & 8 \end{array}\right] \xrightarrow{R_2 \times 3} \left[\begin{array}{cccc|c} 1 & 2/3 & 1/3 & 1/3 & 2 \\ 0 & 1 & -1 & -1 & 0 \\ 0 & 7 & 1 & 1 & 8 \end{array}\right] \xrightarrow{R_3 - 7R_2} \left[\begin{array}{cccc|c} 1 & 2/3 & 1/3 & 1/3 & 2 \\ 0 & 1 & -1 & -1 & 0 \\ 0 & 0 & 8 & 8 & 8 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & 2/3 & 1/3 & 1/3 & 2 \\ 0 & 1 & -1 & -1 & 0 \\ 0 & 0 & 8 & 8 & 8 \end{array}\right] \xrightarrow{R_3/8} \left[\begin{array}{cccc|c} 1 & 2/3 & 1/3 & 1/3 & 2 \\ 0 & 1 & -1 & -1 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{array}\right]$$ Row echelon form

$$\left[\begin{array}{cccc|c} 1 & 2/3 & 1/3 & 1/3 & 2 \\ 0 & 1 & -1 & -1 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{array}\right] \xrightarrow[R_1 - \frac{R_3}{3}]{R_2 + R_3} \left[\begin{array}{cccc|c} 1 & 2/3 & 0 & 0 & 5/3 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{array}\right] \xrightarrow{R_1 - \frac{2R_2}{3}} \underbrace{\left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right.}_{R}\underbrace{\left|\begin{array}{c} 1 \\ 1 \\ 1 \end{array}\right]}_{c}$$

Reduced row echelon form

$Rx = c$

x_1, x_2, x_3 = dependent variable.

x_4 = independent variable

$x_4 = c$

Here, $x_1 = 1$, $x_2 = 1$,

$x_3 + x_4 = 1$, $x_3 = 1 - c$

Set of sol of $Rx = c$ & $Ax = b$ is $\{x_1 = 1, x_2 = 1, x_3 = 1 - c, x_4 = c \mid c \in R\}$

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1-c \\ c \end{bmatrix} \middle| \, c \in R \right\}$$

- If we found any 0 row than it doesn't have sol.

Vector space:

A vector space is a set with two operations (called addition & scaler multiplication with below (i) – (viii) properties.

▷ Properties of addition and scaler multiplication:

Let v, w, v' be vector in $\mathbb{R}^n$ & $a, b \in \mathbb{R}$

1) $v + w = w + v$
2) $(v+w) + v' = v + (w+v)$
3) $v + 0 = 0 + v = 0$
4) $v + (-v) = 0$
5) $1v = v$
6) $(ab)v = a(bv)$
7) $a(v+w) = av + aw$
8) $(a+b)v = av + av$

Any set satisfy these conditions is a vector space.

A vector space V over $\mathbb{R}$ is a set along 2 fn:

$$+ : V \times V \to V \quad \text{and} \quad \cdot : \mathbb{R} \times V \to V$$

(i.e. for each pair of element v_1 and v_2 in V, there is unique element $v_1 + v_2$ in V, and for each $c \in \mathbb{R}$ and $v \in V$ there is a unique element $c \cdot v$ in V)

Non-Example: Let's define scaler mult. in $\mathbb{R}^2$ as follows:

$\Rightarrow (x_1, x_2) + (y_1 + y_2) = (x_1 + y_1, x_2 - y_2)$

$\Rightarrow c(x_1, x_2) = (cx_1, cx_2)$

That's why,

$\circ\ (x_1, x_2) + (y_1 + y_2) \neq (y_1 + y_2) + (x_1, x_2)$

Cancellation law of vector addition:

If $v_1, v_2, v_3 \in V$ such that, $v_1 + v_3 = v_2 + v_3$, than $v_1 = v_2$.

Eg: $(x_1, x_2, x_3) + (y_1, y_2, y_3) = (z_1, z_2, z_3) + (y_1, y_2, y_3)$

$\Rightarrow x_i + y_i = z_i + y_i \quad , i = 1, 2, 3$

Substracting y_i from both side we get $x_i = z_i$.

- $0v = 0$ for each $v \in V$
- $(-c)v = -(cv) = c(-v)$ for each $c \in \mathbb{R}$ & each $v \in V$.
- $c0 = 0$ for each $c \in \mathbb{R}$

Linear Combination of vectors:-

Let V be vector space & $v_1, v_2, \ldots, v_n = V$.

The linear comb. of $v_1, v_2, \ldots, v_n$ with cofficient $a_1, a_2 \ldots, a_n \in \mathbb{R}$

A vector $v \in V$ is linear comb. of $v_1, v_2, \ldots v_n$ if there exist $a_1, a_2, \ldots, a_n \in \mathbb{R}$.

So that, $v = \sum_{i=1}^{n} a_i v_i$

Example in $\mathbb{R}^2$

: $2(1,2) + (2,1) = (4,5)$

Each vector is expression of linear comb. of other 2 vectors:

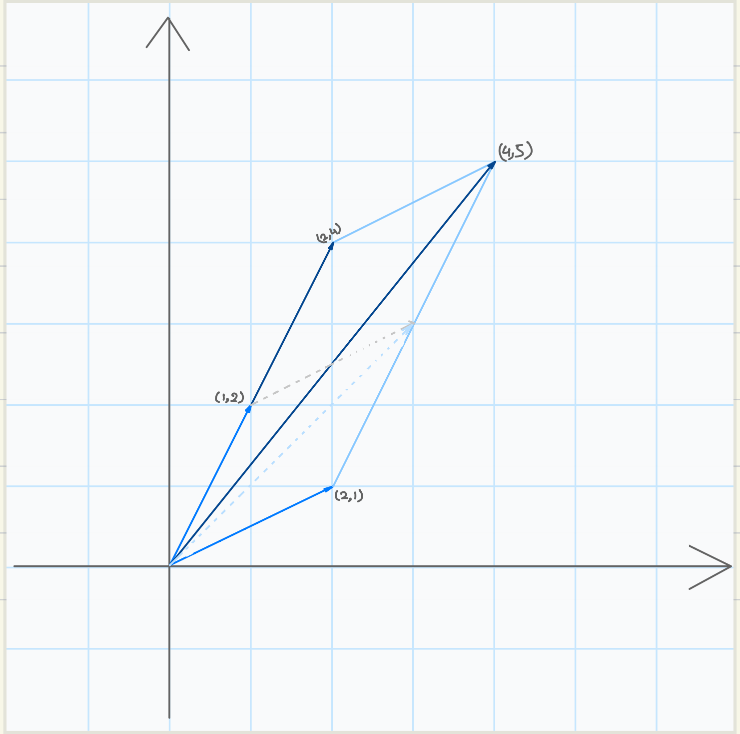
$\Rightarrow 1/2(4,5) - 1/2(2,1) = (1,2)$

$\Rightarrow (4,5) - 2(1,2) = (2,1)$

We can write expression as:

$2(1,2) + (2,1) - (4,5) = (0,0)$

Observe, zero vector is linear comb. of $(1,2), (2,1), (4,5)$ with non zero coff. $(2,1,-1)$.



Example in $\mathbb{R}^3$:

$2(0,2,1) + 3/2(2,2,0)$

$(0,4,2) + (3,3,0) = (3,7,2)$

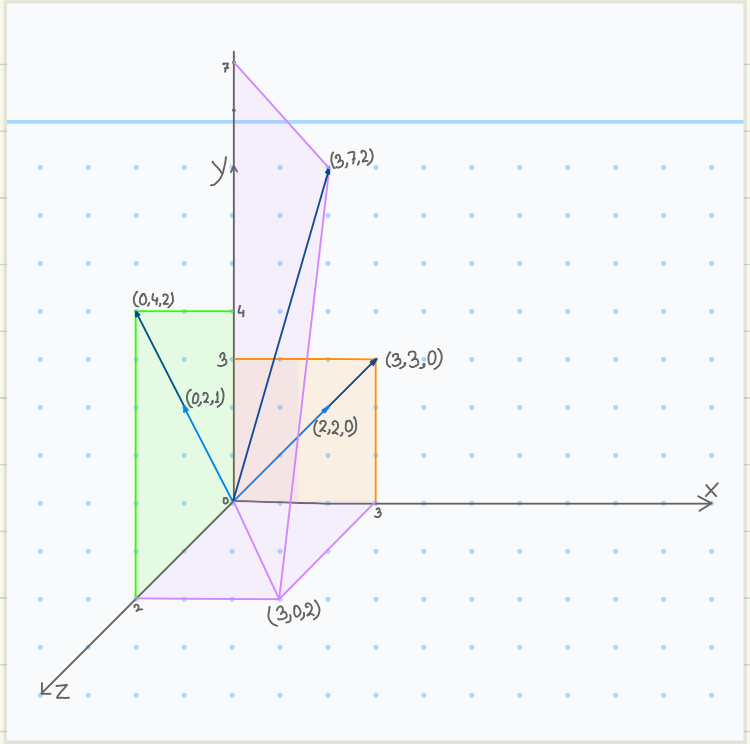
Moreover, each of vectors in expression is linier comb. of other 2 vectors:

- $1/2(3,7,2) - 3/4(2,2,0) = (0,2,1)$
- $2/3(3,7,2) - 4/3(0,2,1) = (2,2,0)$

Further we can rewrite expression:

$2(0,2,1) + 3/2(2,2,0) - (3,7,2) = (0,0,0)$

with non-zero cofficient.



The plain of 2 vector $(0,2,1)$ & $(2,2,0)$ can expressed by eq: $2x - 2y + 4z = 0$

let's choose vector $(1,2,0)$ which is not on plane. Than we can claim that, $(1,2,0)$ can be liner comb of $(0,2,1)$ & $(2,2,0)$.

The only way 0 vector is linear comb. if coff. = 0.

Linear dependence:

A set of vector $v_1, v_2, \ldots, v_n$ from vector space V is linierly dependent if there exist scaler $a_1, a_2, \ldots, a_n$, not all 0. such that:

$$a_1 v_1 + a_2 v_2 + \ldots + a_n v_n = 0$$

So, 0 vector is linear comb. of $v_1, v_2, \ldots, v_n$ with non-zero coff.

Eg: $\mathbb{R}^3$: $(2,3,7)$ & $(5/3, 5/2, 35/6)$ can easily be,

$5(2,3,7) - 6(5/3, 5/2, 35/6) = (0,0,0)$ Hence, it is linearly dependent.

▷ In R^2 if 2 vector are linierly dependent, they must be on same line.

▷ In R^3 if 3 vector are linierly dependent, they must be on same plane.

Linear independence :

A set of vector $v_1, v_2, \dots, v_n$ from a vector space V is said to be linearly independent if $v_1, v_2, \dots, v_n$ are not linearly dependent. 😂

if eq: $a_1v_1 + a_2v_2 + \dots + a_nv_n = 0$
can only be satisfied when $a_i = 0$ for all $a = 1, 2, \dots, n$.

A set of vector $v_1, v_2, \dots, v_n$ from vector space V is linierly independent if the only linear comb. of $v_1, v_2, \dots, v_n$ which equals 0 is linear combination with all cofficients 0.

Ex in R^2, 2 vector $(-1, 3)$ and $(2, 0)$

Consider eq: $a(-1, 3) + b(2, 0) = (0, 0)$

Hence eq : $-a + 2b = 0$ and $3a = 0$

$\therefore$ So, $a = 0$ and $b = 0$

Zero vector :

Let $v_1, v_2, \dots, v_n$ set of vector containing the 0 vector.
Suppose, $v_i = 0$. Than we can choose $a_i = 1$ & $a_j = 0$ for $j \neq i$ than linear comb. $a_1v_1 + a_2v_2 + \dots + a_nv_n = 0$ but not all cofficient is 0.

Hence, a set of vector $v_1, v_2, \dots, v_n$ containing 0 vectors is always a linearly dependent set.

- Two non-zero vectors are linearly independent when they are not multiples of each other.

- If 3 vectors are linearly independent than none of these vectors is linear combination of other two.

Linear independance in R^n:

Vectors are said to be linearly independent if we equate these vectors to 0 it is only possible if all the cofficients are zero.

And if By creating matrix from eq. & vectors find determinant. If determinant of that matrix is non-zero. Means that metrix is invertible. Than those vectors are linearly independent.

$$\therefore \det(A) \neq 0 \quad \text{(Linarly independent)}$$

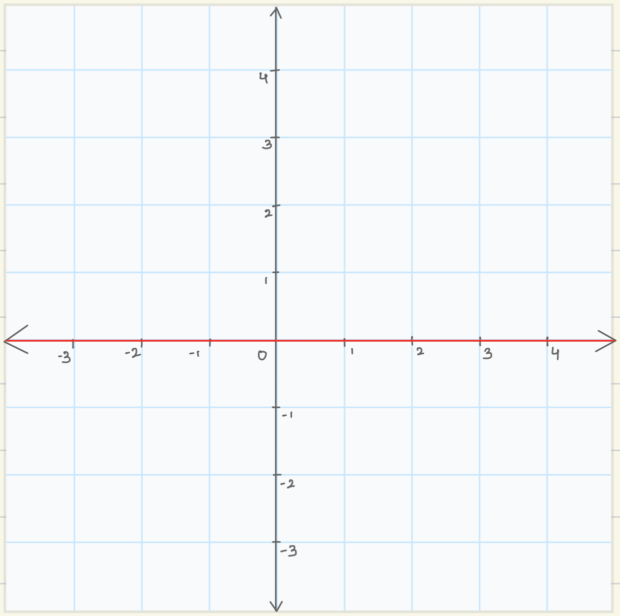
Span of set of vectors:

It is set of all finite linear combination of (vector) elements. denoted by Span(S)

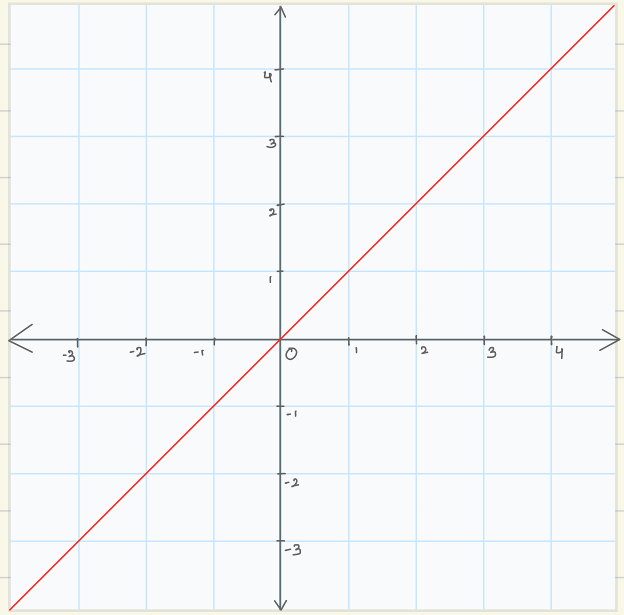
i.e,

$$\text{Span}(S) = \{\textstyle\sum_{i=1}^{n} a_i v_i \in V \mid a_1, a_2, \dots, a_n \in \mathbb{R}\}$$

Eg: Span of $\{(1,0)\} \subset R^2$
$= \{a(1,0) \mid a \in \mathbb{R}\} = (a,0)$

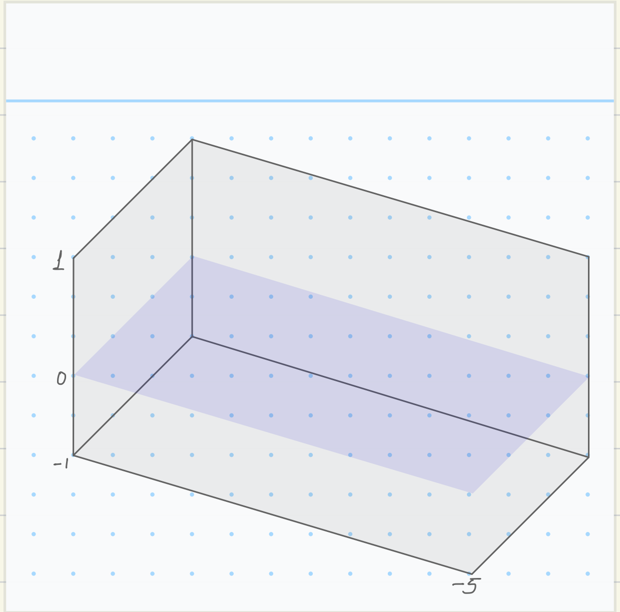


Span of $\{(1,1)\} \subset R^2$
$= \{a(1,1) \mid a \in \mathbb{R}\} = (a,a)$



Span of $\{(1,0,0),(0,1,0)\} \subset R^3$

than, $\{a(1,0,0)+b(0,1,0) \mid a,b \in \mathbb{R}\} = \{(a,b,0) \mid a,b \in \mathbb{R}\}$



○ Spanning set of a vector space :

It is a set $S \subseteq V$ is spanning set V if $\text{span}(S) = V$

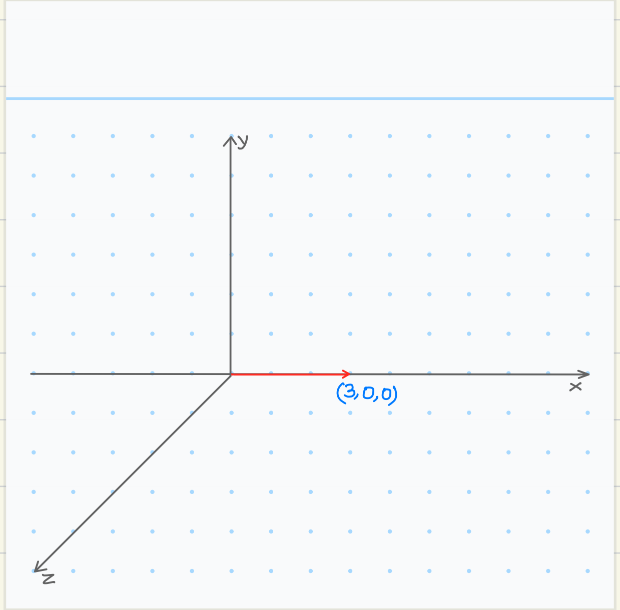
It is combination of those vector whose comb.

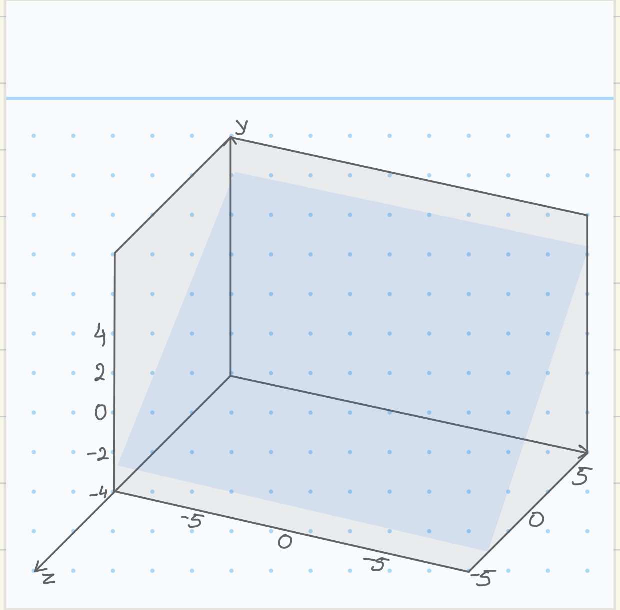
can produce any vector in V of vector space V.

Eg :- $S = \{(1,0),(0,1)\}$ than span $(S) = R^2$
$S = \{(1,0),(0,1),(1,3)\}$ than span $(S) = R^2$
$S = \{(1,0,0),(0,1,0),(0,0,1)\}$ than span $(S) = R^3$

For creating spanning set start with empty set and keep adding new vector which is not in the span of vactor we have so far and in this way we create spanning set of n dimension (R^n).

Eg:- Start with empty set ϕ as before.
- Thus, $S_0 = \phi$ and hence Span (S_0) = Span $(\phi) = \{(0,0,0)\}$
- Append any vector not in Span(S_0) e.g. $(3,0,0)$ & call new S_1.





- Choose vector outside span S_1, eg $(2,2,1)$ and append it to S_1 & call the new set S_2.
Then $S_2 = S_1 \cup \{(2,2,1)\}$ & Span(S_2) in the plain above.

- Again choose a vector outside Span(R_2), eg (1,3,3) append it to S_2 & call new set S_3.

Then $S_3 = S_2 \cup \{(1,3,3)\}$

An arbitrary vector $(x,y,z) \in R^3$ can be written as:

$$(x,y,z) = \frac{3x-5y+4z}{9}(3,0,0) + (y-z)(2,2,1) + \frac{2z-y}{3}(1,3,3)$$

Hence, $\text{Span}(S_3) = R^3$

- **Basis:** Basis B of vector space V is linearly independent subset of V that span V.

The basic of of a vector space is a set of linearly independent vector that span the full space.

- **Equavalant condition for B to be basic:**

The following conditions are equivalant to to subset $B \subseteq V$ being a basic:

- B is linearly independent & span(B) = V
- B is maximal linearly independent set. (adding vectos make linearly dependent)
- B is minimal spanning set. (Not spanning by removing 1 vector)

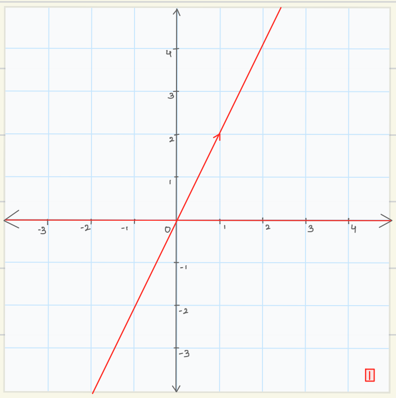
- **How to find basic vectors:-**

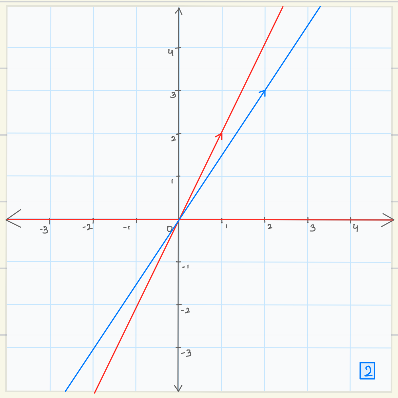
For creating spanning set • start with empty set and keep adding new vector which is not in the span of set obtained. Until we obtain a spanning set.

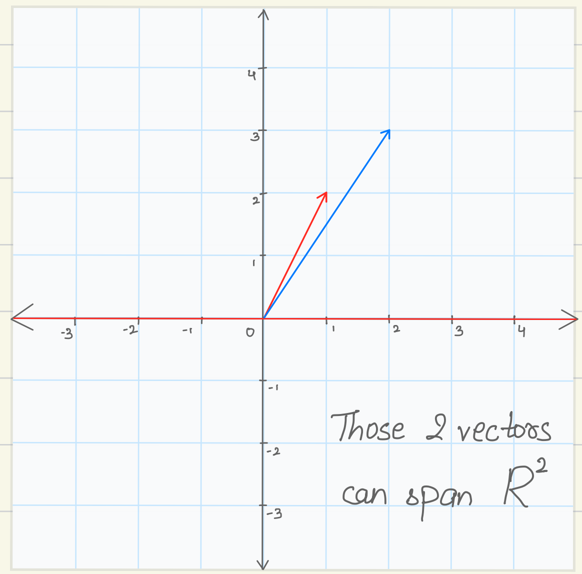
• Take a spanning set and keep deleting vectors which are linear combination of other vector until you remain with with vectors that satisfy that they are not linear combination of other one.

Eg. in R^2: (Methord 1)

- Let start with empty set $\{\phi\}$ and append non-zero vector $(1,2)$ 1







- Now choose other vector not span of prev. vector eg. $(2,3)$
$\text{span}(\{(1,2),(2,3)\}) = R^2$

Eg in R^3 (Methord 2):

Start with set:

$S = \{(1,0,0),(1,2,0),(1,0,3),(0,2,3),(0,4,2)\}$

- Check span of R^3

Now, observe, $(0,4,2) = 2(1,2,0,0) + \frac{2}{3}(1,0,3) - \frac{8}{3}(1,0,0)$

So, removing $(0,4,2)$ $\{(1,0,0),(1,2,0),(1,0,3),(0,2,3)\}$

- ~~&~~ & $(0,2,3) = (1,2,0) + (1,0,3) - 2(1,0,0)$

by deleting $(0,2,3)$ we left with $\{(1,0,0),(1,1,0),(1,0,1)\}$

Final vector is not combination of other vector.

Hence S_2 forms a basis of R^3.

Rank/dimensions of vector space:

It is size or cardianality of basic of vector space.
Eg: If B is basis of V, than the rank is number of element in B.
Dimension of $R^n = n$.

Subspace W of R^3 spanned by $\{(1,0,0),(0,1,0),(3,5,0)\}$
(Dependent) Observe that, $3(1,0,0)+5(0,1,0)=(3,5,0)$
By removing $\{(1,0,0),(0,1,0)\}$ from subspace W.
Dimension of W is 2.

In term of matrix: Eg: Write a vector which span W as row of matrix:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 5 & 0 \end{bmatrix} \text{ Applying row-reduction } \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 5 & 0 \end{bmatrix} \xrightarrow{R_3-3R_1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 5 & 0 \end{bmatrix} \xrightarrow{R_3-5R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

So, no' of non-zero row is dimension of span$(W) = 2$

Rank of matrix:

Let A be $m \times n$ matrix.
- Column space of A is subspace of R^m spanned by col. vector A.
- Row space of A is subspace of R^m spanned by row vector A.
- Dimension of col. space is def. by col. rank of A.
- Dimension of row space is def. by row rank of A.

Fact: Column rank = row rank which is also called rank of A.

Eg: $A = \begin{bmatrix} 1 & 0 & 1 \\ -2 & -3 & 1 \\ 3 & 3 & 0 \end{bmatrix}$ In reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Non-zero rank & Rank(A) = 2.

□ Rank & dimension using Gaussian elimination:

– Finding dimension and basic with given spanning set:

Considered a vector span W spanned by set S.

eg: let consider a vector space W spanned by set X:
$\{(1,0,1), (-2,-3,1), (3,3,0)\}$

→ Forming metrix of vector in the spanning set as rows and reduce to row echelon form:

$$\begin{bmatrix} 1 & 0 & 1 \\ -2 & -3 & 1 \\ 3 & 3 & 0 \end{bmatrix} \xrightarrow[R_2-(-2R_1)]{R_3-3R_1} \begin{bmatrix} 1 & 0 & 1 \\ 0 & -3 & 3 \\ 0 & 3 & -3 \end{bmatrix} \xrightarrow[R_3-(3R_2)]{R_2/-3} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

– The dimension of vector space W = 2 (non-zero row)

– The basic of vector space W = 2 (non-zero row)

– Column method:

let W be subspace of R^3 spanned by set S.
$S = \{(1,0,1), (-2,-3,1), (3,3,0)\}$
Form metrix with vector in S as columns:

$\begin{bmatrix} 1 & -2 & 3 \\ 0 & -3 & 3 \\ 1 & 1 & 0 \end{bmatrix}$ Row redrced form $\begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ Here first two vectors form basic of W.

○ Solution space of homogeneous system of linear equation or Null space:

→ Let A be an $m \times n$ matrix.
The subspace $W = \{x \in R^n \mid Ax = 0\}$ of R^n is called solution space of homogenous system of linear equation $Ax = 0$ or null space of A.
Nullspace is subspace in R^n. The dimension of null space is called nullity of A.
Null space is the collection of vector x which is multiplied by the given matrix gives zero vector.

□ Finding nullity and basis for nulspace:

We can find nullity by row reduction and gaussian elimination. For homogenous system of linear eq. $Ax = 0$

Eg:- $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix}$

Row reduction - Gausian elimination

The augment metrix $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 2 & 2 & 2 & 0 \\ 3 & 3 & 3 & 0 \end{array}\right] \xrightarrow[R_2 - 2R_1]{R_3 - 3R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

Independent variable $= X_2, X_3$ (nullity $= 2$)
dependent variable $= X_1$
Putt $x_2 = t_1$ and $x_3 = t_2$ than equation will be:

$x_1 = -x_2 - x_3$ [From above metrix]
$x_1 = -t_1 - t_2$

Hence, nullspace of A (the solution space of $Ax=0$) is
$\{(-t_1, -t_2, t_1, t_2) \mid t_1, t_2 \in R\}$
Example of solution, $(-5,2,3)$, $(-20,10,10)$

We get basic vector by putting value of t_1 & t_2 0&1 interchangebly.

$t_1 = 1$, $t_2 = 0$ yields the basis vector $(-1,1,0)$
$t_1 = 0$, $t_2 = 0$ yields the basis vector $(-1,0,1)$

Hence, basic of nullspace is $(-1,1,0), (-1,0,1)$

Example 3X4:
Homogenous system matrix $Ax=0$

$$A = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 2 & 3 & 0 & 3 \\ 1 & 1 & 1 & 1 \end{bmatrix} \quad \text{Aug. mat:-} \left[\begin{array}{cccc|c} 1 & 2 & 0 & 3 & 0 \\ 2 & 3 & 0 & 3 & 0 \\ 1 & 1 & 1 & 1 & 0 \end{array}\right] \quad \text{Row Reduced} \left[\begin{array}{cccc|c} 1 & 0 & 0 & -3 & 0 \\ 0 & 1 & 0 & 3 & 0 \\ 0 & 0 & 1 & 2 & 0 \end{array}\right]$$

ind. variable $= x_4$, dep. var $= x_1, x_2, x_3$. Nullity $= 1$.
We put $x_4 = t$
eq. will be:-

$x_1 - 3x_4 = 0 \qquad x_1 = 3t$
$x_2 + 3x_4 = 0 \qquad x_2 = -3t$
$x_3 + 2x_4 = 0 \qquad x_3 = -2t$

Hence nullspace of A for sol. space of $Ax=0$ is
$\{(3t, -3t, -2t, t) \mid t \in R\}$
Putt any thing in place of t and get basic vector.

○ Rank-nullity theorem:
Lets A be a $M \times N$ metrix.

Rank(A) calculated as no. of non-zero row (dependent var.) of a matrix in row-echelon form.
Hence,

rank(A) = (no. of row $\neq 0$ of R) = no. of dep. var. in $Rx=0$
nullity(A) = no. of independent variable of $Rx=0$

So, we have rank-nullity theorem:

For an $m \times n$ matrix A, rank(A) + nullity(A) = n.

○ How to check if set of n vector is basic of R^n:

- We are given n vector of R^n
- Than we write them as columns of matrix & get $n \times n$ matrix.
- If determinant of that matrix is 0, than given set of vector doesn't form a basis, otherwise it form a basis.

Eg: vectors for standard basic are (1,0), (0,1) give matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ identity matrix with determinant = 1.
so it form basis.

2.) We take vector (1,−2), (5,−10) yields to $\begin{bmatrix} 1 & 5 \\ -2 & -10 \end{bmatrix}$ with 0 determinant. doesn't form basis of R^2.
Because they are linear multiple of each other.

3.) In a set {(1,2,3), (0,1,2), (1,3,0)} basic of R^3?

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 3 \\ 3 & 2 & 0 \end{bmatrix}$$

det(A) = −5
det $\neq 0$ so it is basis of R^3.

○ Expression linear combination :

The term $45x_1 + 125x_2$ is an expression.
We can think of it as fn C_A from $\mathbb{R}^3$ to $\mathbb{R}$. since every value of x_1, x_2 & x_3, we obtain a real no.
Combination can be expressed as:

Eg: $C_A(x_1, x_2, x_3) = 45x_1, 125x_2, 150x_3 = \begin{bmatrix} 45 & 125 & 150 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

let x_1 = rice, x_2 = daal, x_3 = oil.

day1 office1 order. 30 tiffin with
20kg rice, 10 kg dal and 4 ltre oil.

day2 office2 order 40 tiffin with
30 kg rice, 12 kg dal and 2 ltre oil.

day 3 office 1 & office 2 with 15 & 50 tiffin.
from above 2 we can get for 15 tiffin office 1: $\frac{1}{2}$
10 kg rice, 5 kg daal, 2 ltre oil.
office 2 with 50 tiffin $5/4$
37.5 kg rice, 15 kg daal and 2.5 ltre oil.

total req = 47.5 kg rice, 20 kg daal & 4.5 Litre oil.
From above coff:

$45 \times 47.5 + 125 \times 20 + 150 \times 4.5 = 5312.5$ rupees.

We can also calculate it by:
Cost of caterer office 1 day 1 = 2750 rs
Cost of caterer office 2 day 2 = 3150 rs
day 3 off. 1 & 2 = $\frac{1}{2} \times 2750 + \frac{5}{4} \times 3150 = 5312.5$ rs.

Extended example:

	Rice/Kg	Daal/kg	Oil/l
Shop A	45	125	150
Shop B	40	120	170
Shop C	50	130	160

$$C_A(x_1, x_2, x_3) = 45x_1, 125x_2, 150x_3 = \begin{bmatrix} 45 & 125 & 150 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$C_B(x_1, x_2, x_3) = 40x_1, 120x_2, 170x_3 = \begin{bmatrix} 40 & 120 & 170 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$C_C(x_1, x_2, x_3) = 50x_1, 130x_2, 160x_3 = \begin{bmatrix} 50 & 130 & 160 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

We can compare by creating vectors:

$$C(x_1, x_2, x_3) = (C_A(x_1, x_2, x_3)\ C_B(x_1, x_2, x_3)\ C_C(x_1, x_2, x_3))$$
$$= (45x_1 + 125x_2 + 150x_3,\ 40x_1 + 120x_2 + 170x_3,\ 50x_1 + 130x_2 + 160x_3)$$

Now when we are given x_1, x_2 and x_3 we put in this cost fn & see result to find lowest cost.
Apart from this we can also use metrix:

$$C(x_1, x_2, x_3) = \begin{bmatrix} 45 & 125 & 150 \\ 40 & 120 & 170 \\ 50 & 130 & 160 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

If we have to buy 2 kg rice, 1 kg Dal, 2 ltr oil.

$$\begin{bmatrix} 45 & 125 & 150 \\ 40 & 120 & 170 \\ 50 & 130 & 160 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 515 \\ 540 \\ 550 \end{bmatrix}$$

This will be the cost to compare.

It has linearity:

means: $C(\alpha(x_1, x_2, x_3) + (y_1, y_2, y_3)) = \alpha C(x_1, x_2, x_3) + C(y_1, y_2, y_3)$

matrix $\begin{bmatrix} 45 & 125 & 150 \\ 40 & 120 & 170 \\ 50 & 130 & 160 \end{bmatrix} \left\{ \alpha \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \right\} = \alpha \begin{bmatrix} 45 & 125 & 150 \\ 40 & 120 & 170 \\ 50 & 130 & 160 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

□ Linear mapping:

A linear mapping f from R^n to R^m can defined as:

$$f(x_1, x_2, \cdots, x_n) = \left(\sum_{j=1}^{n} a_{1j} x_j, \sum_{j=1}^{n} a_{2j} x_j, \sum_{j=1}^{n} a_{2j} x_j \right)$$

Where cofficients a_{ij}s are real no (scalar). A linear mapping can be thought of collection of linear comb.

We can write it in matrix form.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix}$$

Linearity of linear mapping:

Linear mapping satisfy linearity i.e for $c \in R$

$$f(x_1 + cy_1, x_2 + cy_2, \ldots, x_n + cy_n) = f(x_1, x_2, \cdots, x_n) + cf(y_1, y_2, \cdots, y_n)$$

$$f(x_1 + cy_1, x_2 + cy_2, \ldots, x_n + cy_n) = A \begin{bmatrix} x_1 + cy_1 \\ x_2 + cy_2 \\ \cdots \\ x_n + cy_n \end{bmatrix}$$

$$A = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix} + c \begin{bmatrix} y_1 \\ y_2 \\ \cdots \\ y_n \end{bmatrix} \right\} = A \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix} + cA \begin{bmatrix} y_1 \\ y_2 \\ \cdots \\ y_n \end{bmatrix} = f(x_1, x_2, \ldots, x_n) + cf(y_1, y_2, \cdots y_n)$$

○ Linear transformation:

A fn $f: V \rightarrow W$ b/w two vector space V & W is said to be linear transformation if for any 2 vector V_1 and V_2 in vector space V & for any scaler.

Finaly, A linear transformation transformation from $T: R^n \longrightarrow R^m$ which satisfy:

- $f(V_1+V_2) = f(V_1) + f(V_2)$
- $f(cV_1) = cf(V_1)$

Linear mapping are basically linear transformation.

→ For linear transformation, being 1-1 (one to one) is equivalent to $f(V)=0$ implies $v=0$.

→ A linear transformation $f: V \rightarrow W$ b/w 2 vector space V and W said to be an isomorphism if it is bijective (onto)

○ Basic determine linear transformation:

let V be vector space with basic $\{V_1, V_2, \dots, V_n\}$

let, $f: V \rightarrow W$ be linear transformation. Than offered vector $f(V_1), f(V_2), \dots, f(V_n)$ uniquely determine f.

let $v \in V$, $v = \sum_{i=1}^{n} c_i v_i$

$$f(v) = f\left(\sum_{i=1}^{n} c_i v_i\right) = \sum_{i=1}^{n} c_i f(v_i)$$

Eg: linear transformation of matrix:
Consider linear transformation:
$f: \mathbb{R}^2 \to \mathbb{R}^2$; $f(x,y) = (2x, y)$

We can represent it in $f(x,y) = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

By considering standard basic $(1,0)$ and $(0,1)$ for R^2
$f(1,0) = (2,0) = 2(1,0) + 0(0,1)$
$f(0,1) = (0,1) = 0(1,0) + 1(0,1)$

- Recovering linear transformation:
let $\beta = v_1, v_2 \dots v_n$ & $\gamma = w_1, w_2, \dots, w_n$ ordered basis of V, W.
Suppose A is a $m \times n$ matrix. What is corresponding lin. tran?
Let $v \in V$. Express $v = \sum_{j=1}^{n} c_j v_j$. define

$$f(v) = \sum_{j=1}^{n} c_j \sum_{i=1}^{m} A_{ij} w_i$$

Check that f is linear transformation.
Letting $c_k = 1$ & $c_j = 0$ for $j \neq k$, we get
$f(v_k) = A_{1k} w_1 + A_{2k} w_2 + \dots + A_{mk} w_m$
Hence the matrix corresponding to f is indeed A.

- Kernal: Kernal is the set of fn which gives output '0' after transformation.
Kernal is denoted as $Ker(f)$.

$$Ker(f) = \{ v \in V \mid f(v) = 0 \}$$

- Image : Image is function is output after transformation. It is range of of function. Image of function denoted as Im(f)

$$Im(f) = \{w \in W \mid \exists v \in V \text{ for which } f(v) = w\}$$

Eg: Consider $f: R^2 \to R^2$ $f(x,y) = (2x, y)$
Then $Ker(f) = (0,0)$, $Im(f) = \mathbb{R}^2$

$f(x,y) = (2x, 0)$
$Ker(f) = (0, y)$
$Im(f) = (x, 0) = \mathbb{R}$

Example: Consider $T: R^4 \to R^3$, difined by:
$T(x_1, x_2, x_3, x_4) = (2x_1 + 4x_2 + 6x_3 + 8x_4,\ x_1 + 3x_2 + 5x_4,\ x_1 + x_2 + 6x_3 + 3x_4)$
Choose β & γ to standard (ordered) basis for R^4 & R^3

The corresponding metrix is: $\begin{bmatrix} 2 & 4 & 6 & 8 \\ 1 & 3 & 0 & 5 \\ 1 & 1 & 6 & 3 \end{bmatrix}$ Row reduced form $\begin{bmatrix} 1 & 0 & 9 & 2 \\ 0 & 1 & -3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Non-pivot or independent variable are x_3 & x_4
so putting $x_3 = s$ & $s_4 = t$ we get, $x_1 = -9s - 2t$ & $x_2 = 3s - t$

By, substituting $s = 1, t = 0$ and $s = 0, t = 1$ gives basic vector $(-9, 3, 1, 0), (-2, -1, 0, 1)$ for null space.

Since we have choosen β to be standard basis, the basis for Ker (T) is also same,

$$-9e_1 + 3e_2 + 1e_3 + 0e_4 = (-9, 3, 1, 0)$$
$$-2e_1 - 1e_2 + 0e_3 + 1e_4 = (-2, -1, 0, 1)$$

Moreover, the pivot columns are 1st & 2nd columns. Hence column space is first 2 column of the matrix A, i.e. $(2,1,1), (4,3,1)$

Since we choose γ to standard basic, the basis for im(T) is also same: ie,

$2e_1 + 1e_2 + 1e_3 = (2,1,1)$, and

$4e_1 + 3e_2 + 1e_3 = (4,3,1)$

Ex-2 (Where basis we use are not standard basis)

Let $V = \mathbb{R}^2$, $W = \{(x,y,z) \mid x+y+z=0\}$

let respective ordered basic be $\beta = (1,1), (1,-1)$

$\gamma = (-1,1,0), (-1,0,1)$

$T(x,y) = (0, x+2y, -x-2y)$ linear transform from V to W.

$(1,1) = 0, 1+2, -1-2 = (0,3,-3)$

$(1,-1) = 0, 1-2, -1+2 = (0,-1,1)$

So, $(0,3,-3)$ is 3 times $\underbrace{(-1,1,0)}_{(-3,3,0)}$ & -3 times $\underbrace{(-1,0,1)}_{(3,0,-3)}$

Similarly $(0,-1,1)$ is -1 times $\underbrace{(-1,1,0)}_{(1,-1,0)}$ & 1 times $\underbrace{(-1,0,1)}_{(-1,0,1)}$

Corrosponding matrix $\begin{bmatrix} 3 & -1 \\ -3 & 1 \end{bmatrix}$ Reduced row echelon form $\begin{bmatrix} 1 & -1/3 \\ 0 & 0 \end{bmatrix}$

Pivot column = x_1 (dependent)

non pivot col = x_2 (independent), So, $x_2 = t$

than $x_1 = t/3$

The basic for null space is singleton set consist by substuting $t = 1$ i.e $\{(1/3, 1)\}$

Therefor $Ker(t) = \frac{1}{3}(1,1) + 1(1,1) = (4/3, -2/3)$

Since pivot col is 1st col. of orig. matrix $= (3,-3)$
Therefore Ker(t) in singleton set consist of
$3(-1,1,0) + (-3)(-1,0,1) = (0,3,-3)$ (Basis of img of lin trans. T)

Rank nullity theorem of linear transformation:

Let $T: V \to W$ be a Lin. Trans.
The rank of T (denoted rank(T)) is dim. of Im(T)
The nullity of T (denotes nullity(T)) is dim. of Ker(T)

Reinterpreting rank-nullity theorem of matrices, we obtain: $rank(t) + nullity(t) = dim(V)$

- Equivalance of matrices: (denoted by $\sim$)
  Let A & B are $m \times n$ matrices. We say A is equivalant to B if $B = QAP$ for invertable metrices ($P_{n \times n}$ & $Q_{m \times m}$).
- Characters: → A can be transformed into B by combining elementry row and column operations.
  → $rank(A) = rank(B)$

Equavalance of matrics is an equavalance relation i.e:
▷ A is equivalant to itself. ($A = I_{m \times m} A I_{n \times n}$) $A \sim A$

▷ A is equiv. to B means B is equiv. to A.
($B = QAP$ and $A = Q^{-1}BP^{-1}$) | $A \sim B \Rightarrow B \sim A$

▷ A is equiv. to B & B is equiv. to C means C is equiv. to A.
($B = QAP$, $C = Q'^{-1}BP'^{-1}$, $C = Q'Q\,A\,PP'^{-1}$) | $A \sim B, B \sim C \Rightarrow A \sim C$

- Linear transformation and equivalance of matrices:

Consider linear trans. $T: V \longrightarrow W$, two orderd basis β_1 & β_2 for V, and two ordered basis γ_1 & γ_2 for W.
Let A be a matrix corrosponding to T with respect to basis β_1 & γ_1 and B be matrix corrosponding to T with respect to β_2 and γ_2

$P \rightarrow$ express the ordared basis B_2 in term of B_1.
$Q \rightarrow$ Express ordered basic γ_1 in term of γ_2.

Then $B = QAP$

Eg :- Consider linear transformation $f: R^3 \rightarrow R^2$, defined as:

$$f(x,y,z) = (x+y, y+z)$$

Considered 2 ordered basis for R^3
$\beta_1 = (1,0,0), (0,1,0), (0,0,1)$, & $\beta_2 = (1,1,0), (0,1,1), (0,0,1)$

Similarly consider 2 orderd bases for R^2
$\gamma_1 = (1,0), (0,1)$ & $\gamma_2 = (1,0), (1,1)$

From Transformation $f(x,y,z) = (x+y, y+z)$ we get:

$f(1,0,0) = (1,0)$
$f(0,1,0) = (1,1) = 1(1,0) + 1(0,1)$
$f(0,0,1) = (0,1)$

Hence matrix corrosponds to f with respect to basis β_1 and γ_1 is $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$

Similarly 2nd basic: in $(x+y, y+z)$

$$f(1,1,0) = (2,1) = 1(1,0) + 1(1,1)$$
$$f(0,1,1) = (1,2) = -1(1,0) + 2(1,1)$$
$$f(0,0,1) = (0,1) = 1(1,0) + 1(1,1)$$

Hence matrix corrosponds to f with respect to basis β_2 and γ_2 is $B = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 2 & 1 \end{bmatrix}$

Choose $P = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ and $Q = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$. Then

(Produced out of air randomly)

$$QAP = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 2 & 1 \end{bmatrix} = B$$

Hence A & B are equivalent to each other.

- Similar matrices:

An $n \times n$ matrix A is similar to an $n \times n$ matrix B if there exist an $n \times n$ matrix P such that, $B = P^{-1}AP$

Similarity of matrics is an equavalance relation i.e.:

▷ A is similar to itself. $(P = I : A = I^{-1}AI)$

▷ A is similar to B means B is similar to A.
$(B = P^{-1}AP \Rightarrow PBP^{-1} = A \Rightarrow A = (P^{-1})^{-1}BP^{-1})$

▷ A is similar to B & B is similar to C means C is similar to A.
$(B = P^{-1}AP\,,\; C = Q^{-1}BQ\,,\; C = Q^{-1}(P^{-1}AP)Q = Q^{-1}P^{-1}APQ = (PQ)^{-1}A(PQ))$

Important properties (Similar matrices):-

→ A and B are equivalent.

→ A and B have same rank.

→ $\det(B) = \det(A)$ (determinants are same)

$$\det(B) = \det(P^{-1}AP) = \det(P^{-1})\det(A)\det(P)$$
$$= \frac{1}{\det(P)}\det(A)\det(P) = \det(A)$$

→ Several other invariants of A and B are the same such as characteristic polynomial, minimal polynomial & eigen values with multiplicity.

Eg: Consider a linear transf. $f : R^3 \to R^3$ where
$$f(x,y,z) = (-x+y+z,\, x-y+z,\, x+y-z)$$
Let $\beta = \gamma$ both be the standard ordered basis of R^3.

We get:-
$$f(1,0,0) = (-1,1,1) = -1(1,0,0) + 1(0,1,0) + 1(0,0,1)$$
$$f(0,1,0) = (1,-1,1) = 1(1,0,0) - 1(0,1,0) + 1(0,0,1)$$
$$f(0,0,1) = (1,1,-1) = 1(1,0,0) + 1(0,1,0) - 1(0,0,1)$$

Hence the matrix of f corrosponding standard basic is : $\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$

Consider other basis $B' =$

Than we have:

$f(1,1,1) = (1,1,1) = 1(1,1,1)+0(-1,1,0)+0(-1,0,1)$

$f(-1,1,0) = (2,-2,0) = 0(1,1,1)-2(-1,1,0)+0(-1,0,1)$

$f(-1,0,1) = (2,0,-2) = 0(1,1,1)+0(-1,1,0)-2(-1,0,1)$

Hence the matrix of f corrosponding ordered basic B'

is : $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$

Let $P = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ than $P^{-1} = \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ -1/3 & 2/3 & -1/3 \\ -1/3 & -1/3 & 2/3 \end{bmatrix}$

Then $P^{-1}AP = \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ -1/3 & 2/3 & -1/3 \\ -1/3 & -1/3 & 2/3 \end{bmatrix} \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

$$\begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 2/3 & -4/3 & 2/3 \\ 2/3 & 2/3 & -4/3 \end{bmatrix} \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

Hence A and B are similar matrix.

- Linear transformation and similarity of matrices:

Consider linear trans. $T: V \longrightarrow V$, two ordered basis β & γ for V.

Let A be a matrix corrosponding to T with respect to basis β and B be matrix corrosponding to T with respect to γ.

P → express the ordared basis γ in term of B.
Q → express ordered basis β in term of γ.
Then, $B = P^{-1}AP$.

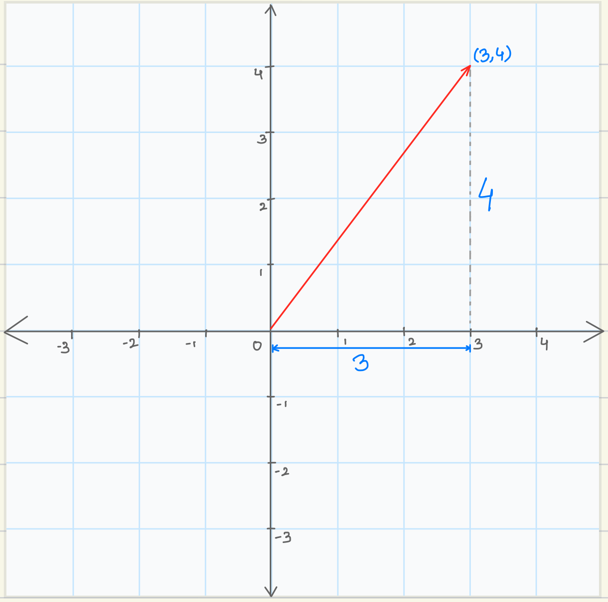
— Affine subspace: It is moving subspace from base to top of vector.

- Dot product of 2 vectors in R^2:
  By generalizing in (x_1, y_1) and (x_2, y_2) in R^2. Dot product we get $(x_1, y_1) \cdot (x_2, y_2) = x_1 x_2 + y_1 y_2$

  Consider 2 vector (3,4) and (2,7) in R^2. Dot product of these two vectors gives us scaler as follows:
  $(3,4) \cdot (2,7) = (3 \times 2) + (4 \times 7) = 6 + 28 = 34$

- The length of vector in R^2:
  For previous example length of vector (3,4) in R^2 is:

Using pythagorans' theorem, the length of vector (3,4) is
$\sqrt{3^2 + 4^2} = 5$



To generalise it length of vector (x, y)

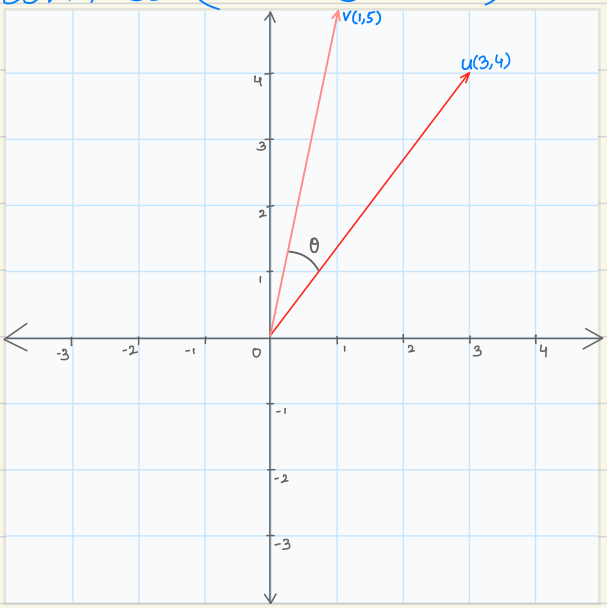
$= \sqrt{x^2 + y^2}$

- The relation between length and dot product in R^2:
  Observe that dot product of $(3,4)\cdot(3,4)$ is 3^2+4^2.
  Than the length of $(3,4)$ is square root of the dot product of the vector with itself.

  Length of vector $(3,4) = \sqrt{(3,4)\cdot(3,4)} = \sqrt{3^2+4^2} = 5$

  More generally, the length of vector $(x,y) \in R^2$ is $\sqrt{x^2+y^2} = \sqrt{(x,y)\cdot(x,y)}$

- The angle b/w 2 vectors in R^2:
  The angle between vector 'u' and 'v' and measure how far the direction of v from u (or viceversa).



Eg. θ be angle b/w $u(3,4)$ & $v(1,5)$.

- It can be measured in degree (0-360) or in radian b/w $(0-2\pi)$
- Angle is often described by computing trignometric fn eg. sin, cos, tan

- The dot product of 2 vectors in R^3:
  Similar to R^2 here also after multiplying at same level we add it.

Eg: $(1,2,3)\cdot(2,0,1) = (1\times2)+(2\times0)+(3\times1) = 2+0+3 = 5$

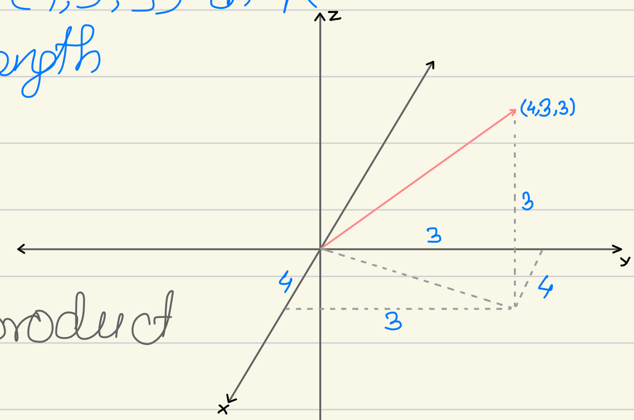
Generalizing: $(x_1,y_1,z_1)\cdot(x_2,y_2,z_2) = x_1x_2+y_1y_2+z_1z_2$

- Length of vector in R^3:

lets find length of vector $(4,3,3)$ in R^3.

Here also by pythagoras we find length of $(4,3,3)$ vector is:

$$\sqrt{4^2+3^2+3^2} = \sqrt{34} \text{ unit}$$



- Similarly the length and dot product in R^3

Observe that dot product of $(4,3,3)\cdot(4,3,3) = 4^2+3^2+3^2$.
Than the length of $(4,3,3)$ is square root of the dot product of the vector with itself.

$$\text{Length of vector } (4,3,3) = \sqrt{(4,3,3)\cdot(4,3,3)}$$
$$\sqrt{4^2+3^2+3^2} = \sqrt{34}$$

More generally, the length of vector $(x,y,z) \in R^3$ is
$$\sqrt{x^2+y^2+z^2} = \sqrt{(x,y,z)\cdot(x,y,z)}$$

- The angle b/w 2 vectors in R^3 and the dot product:

The angle between vector 'u' and 'v' in R^3 is angle b/w them computed by passing a plane through them.

It measures how far the direction of v from u (Viceversa)

Let u & v be 2 vector in R^3. Than we can compute angle θ b/w the vector u & v using dot product as:

$$\cos\theta = \frac{u\cdot v}{\sqrt{(v\cdot v)\cdot(u\cdot u)}} \quad \text{i.e. } \theta = \cos^{-1}\left(\frac{u\cdot v}{\sqrt{(v\cdot v)\cdot(u\cdot u)}}\right)$$

Example:

Compute angle in R^3

lets compute angle θ b/w $(1,0,0)$ & $(1,0,1)$

$$(1,0,0)\cdot(1,0,1) = 1 \quad (1,0,1)\cdot(1,0,1) = 2 \quad (1,0,0)\cdot(1,0,0) = 1$$

$$\text{Hence, } \theta = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4} \text{ radians or } 45^\circ$$

- Dot product in R^n : length and angle :

- let $u = (u_1, u_2, \dots, u_n)$ & $v = (v_1, v_2, \dots v_n)$ be vectors in R^n
- The dot products of 2 vectors u and v is defined as $u \cdot v = u_1 v_1 + u_2 v_2 + \dots + u_n v_n$.
- The length of vector u is denoted by $\|u\|$ and defined by $\|u\| = \sqrt{u \cdot u}$
- The angle θ b/w 2 vector u and v is measured on the 2-dimensional plane spanned by u and v and can be computed as:

$$\cos\theta = \frac{u \cdot v}{\|u\| \times \|v\|} \quad \text{i.e. } \theta = \cos^{-1}\left(\frac{u \cdot v}{\sqrt{(v \cdot v)\cdot(u \cdot u)}}\right)$$

- Inner product of vector space :

An inner product of a vector space V in a f^n

$\langle \cdot , \cdot \rangle : V \times V \longrightarrow R$ satisfies following :

$\rightarrow \langle v, v \rangle > 0$ for all $v \in V \setminus \{0\}$; $\langle v, v \rangle = 0$ only if $v = 0$.

$\rightarrow \langle V_1 + V_2, V_3 \rangle = \langle V_1, V_3 \rangle + \langle V_2 + V_3 \rangle$

$\rightarrow \langle V_1, V_2 \rangle = \langle V_2, V_1 \rangle$

$\rightarrow \langle CV_1, V_2 \rangle = C\langle V_1, V_2 \rangle = \langle V_1, CV_2 \rangle$

- A vector V together with an inner product $\langle \cdot, \cdot \rangle$ is called an inner product space.

- The dot product is an example of an inner product.

Example of inner product in R^2:

$$\langle \cdot, \cdot \rangle : R^2 \times R^2 \longrightarrow R$$

$$\langle u, v \rangle = x_1 y_1 - (x_1 y_2 + x_2 y_1) + 2 x_2 y_2$$

where $u = (x_1, x_2)$ and $v = (y_1, y_2)$ be in R^2.

Can be written as: $\begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$

Eg

$$u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}, \quad V = \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}$$

$$u^T V = \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = u_1 V_1 + u_2 V_2 + u_3 V_3$$

If $u^T V = 0$ than u and v are orthogonal $= \perp$.

- Norm on vector space:

A norm of a vector space V is a function

$$\| \cdot \| : V \longrightarrow R$$

$$x \longmapsto \| X \|$$

Satisfying foll. cond:

→ $||x+y|| \leq ||x|| + ||y||$, for all $x, y \in V$
→ $||cx|| = |c|||x||$ for all $c \in R$ and for all $x \in V$
→ $||x|| \geq 0$ for all $x \in V$; $||x||=0$ if and only if $x=0$

length as an example of norm:

Recall length of a vector $u=(x_1, x_2, \dots, x_n) \in R^n$ is
$$||u|| = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2}$$
The length fn $R^n \to R$ is norm on R^n

Example of norm on R^n:

Define $||u_1|| = |x_1| + |x_2| + \dots + |x_3|$ for $u=(x_1, x_2, \dots, x_n) \in R^n$

○ The inner product includes a norm:
Let V is an inner product space with inner product $\langle \cdot, \cdot \rangle$
Than the fn $||\cdot|| : V \to R$ defined by $||V|| = \sqrt{\langle V, V \rangle}$ is a norm of V.

○ Geometric intution of orthogonal vector:
If angle θ b/w 2 vectors u and v in R^n is right angle $90°$ then:
$$\cos(\theta) = 0 = \frac{u \cdot v}{||u|| \, ||v||}$$
than $u \cdot v = 0$
eg → $(1,2,3)$ and $(2,2,-2) = 1\times2 + 2\times2 - 6 = 2+4-6 = 0$.

Orthogonal vector:

Two vector u and v of an inner product space V are said to be orthogonal if $\langle u,v\rangle = 0$

eg: Consider R^2 with an innerproduct

$$\langle u,v\rangle = x_1y_1 - (x_1y_2 + x_2y_1) + 2x_2y_2$$

where $u = (x_1, x_2)$ and $v = (y_1, y_2)$ be in R^2.

then vector $(1,1)$ and $(1,0)$ are orthogonal by:

$$\langle (1,1),(1,0)\rangle = 1\times 1 - (1\times 0 + 1\times 1) + 2\times 1\times 0$$
$$= 1 - 1 + 0 = 0$$

Orthogonal set of vector:

It is set of vectors whose elements are mutually orthogonal.

Explicitly, if $S = \{V_1, V_2, \ldots, V_K\} \subseteq V$, than S is an orthogonal set of vector if:

$$\langle V_i, V_j\rangle = 0 \text{ if } i,j \in \{1,2,\ldots,K\} \text{ are } i \neq j.$$

Eg: In R^3 with usual dot product. The set $S = \{(4,3,-2),(-3,2,-3),(-5,18,17)\}$ is an orthogonal set of vector.

like:

$$(4,3,-2)\cdot(-3,2,-3) = -12+6+6 = 0$$
$$(4,3,-2)\cdot(-5,18,17) = -20+54-34 = 0$$
$$(-3,2,-3)\cdot(-5,18,17) = 15+36-51 = 0$$

- Orthogonality and linear independence:

  let $\{V_1, V_2 \ldots V_K\}$ be an orthogonal set of non-zero vectors in the inner product space V.

  Than $\{V_1, V_2, \ldots, V_K\}$ is linearly independent set of vector.

- Orthogonal basis:

Let V be an inner product space. A basis consist of mutually orthogonal vectors is called orthogonal basis.

Since an orthogonal set of vectors is already linearly independent, an orthogonal set is a basis when it is maximal orthogonal set. (there is no orthogonal set strictly containing this one).

If $\dim(V) = n$, then

orthogonal basis $\equiv$ orthogonal set of n vectors.

Example:

1. Standard basis.
2. $\{(4,3,-2),(-3,2,-3),(-5,18,17)\} \subseteq R^3$
3. Consider R^2 with inner product

- Orthonormal set:

It is an orthogonal set of vector such that the norm (length) of each vector of set is 1.

Simply it is all the vectors in B which have length 1 and all vectors are orthagonal 90° to each other.

Explicitly: $S = \{v_1, v_2, \dots, v_k\} \subseteq V$, than S is is orthonormal set of vectors if

orthogonal $\langle v_i, v_j \rangle = 0$ for $i,j \in \{1,2,\dots,k\}$ and $i \neq k$

and, length $\|v_i\| = 1$ $\forall i \in \{1,2,\dots,k\}$

It is same as orthogonal set with norm (length) of each vector 1.

Orthonormal basic:

A orthonormal set which form a basic is called orthonormal basic.

Equivalantly: An orthonormal basis is an orthogonal basis where the norm of each vector is 1.

– An orthormal basis is a maximal orthonormal set.

Eg: The standard basis with respect to usual inner product forms an orthonormal basis.

$$\langle e_i, e_j \rangle = (0,0,\dots,0,1,0,\dots 0) \cdot (0,0,\dots,1,0,0,\dots 0)$$
$$= 0\times 0 + 0\times 0 + \dots + 0\times 1 + 1\times 0 + 0\times 0 + \dots + 0\times 0 = 1$$
$$\|e_i\| = \sqrt{\langle e_i, e_i \rangle} = \sqrt{0\times 0 + \dots + 1\times 1 + 0\times 0 + \dots + 0\times 0} = \sqrt{1} = 1$$

Eg: Consider R^3 with usual inner product and the set $\beta = \{\frac{1}{3}(1,2,2), \frac{1}{3}(-2,-1,2), \frac{1}{3}(2,-2,1)\}$. Then β form an orthogonal basis of R^3.

In this, $\|v_1\|^2 = \|v_2\|^2 = \|v_3\|^2 = 1$ (norm / length)

and, $\langle v_1, v_2 \rangle = \langle v_2, v_3 \rangle = \langle v_1, v_3 \rangle = 0$

Cardianility $|\beta| = 3$ & β is linearly independent.

& $Dim(R^3) = 3$ so, β is an orthonormal basis.

Obtaining orthonormal sets from orthogonal sets:

Let V be an inner product space. if $\Gamma = \{v_1, v_2, \dots, v_k\}$ is an orthogonal set of vectors, than we can obtain orthonormal set of vectors β from Γ by:

$$\beta = \left\{ \frac{v_1}{\|v_1\|}, \frac{v_2}{\|v_2\|}, \dots, \frac{v_k}{\|v_k\|} \right\}$$

$$\langle v_i, v_j \rangle = 0$$
$$\left\langle \frac{v_i}{\|v_i\|}, \frac{v_j}{\|v_j\|} \right\rangle = 0$$

$$\left\| \frac{v_i}{\|v_i\|} \right\| = \frac{1}{\|v_i\|} \|v_i\| = 1$$

Consider example in R^2 with usual innerproduct and the orthogonal basis $\Gamma = \{(1,3), (-3,1)\}$

norm, $\|(1,3)\| = 10$, $\|-3,1\| = 10$

than $\beta = \left\{ \frac{1}{\sqrt{10}}(1,3), \frac{1}{\sqrt{10}}(-3,1) \right\}$ is orthonormal basis of R^2.

- Importance of orthonormal basis:

Suppose $\Gamma = \{v_1, v_2, \dots, v_n\}$ is an orthonormal basis of an inner product space V and let $v \in V$.

Than V can be written as:

$$v = c_1 v_1 + c_2 v_2 + \dots + c_n v_n.$$

Since Γ is orthonormal, we can use inner product and compute $c_i = \langle v, v_i \rangle$

$$\langle v, v_i \rangle = \langle c_1 v_1 + c_2 v_2 + \dots + c_n v_n, v_i \rangle$$
$$= c_1 \langle v_1, v_i \rangle + c_2 \langle v_2, v_i \rangle + \dots + c_i \langle v_i, v_i \rangle + \dots + c_n \langle v_n, v_i \rangle$$
$$= c_i \langle v_i, v_i \rangle \text{ all other become } 0.$$
$$= c_i \|v_i\|^2 = c_i$$

Eg: $\left\{ \frac{1}{\sqrt{10}}(1,3), \frac{1}{\sqrt{10}}(-3,1) \right\}$ is orthonormal basis of R^2. Write $(2,5)$ as lin. comb. in term of basis vector.

$$(2,5) = c_1 \frac{1}{\sqrt{10}}(1,3), c_2 \frac{1}{\sqrt{10}}(-3,1)$$

$$c_1 = \left\langle (2,5), \frac{1}{\sqrt{10}}(1,3) \right\rangle = \frac{1}{\sqrt{10}}(2 \times 1 + 5 \times 3) = \frac{1}{\sqrt{10}} \times 17 = \frac{17}{\sqrt{10}}$$

$$c_2 = \left\langle (2,5), \frac{1}{\sqrt{10}}(-3,1) \right\rangle = \frac{1}{\sqrt{10}}(2 \times (-3) + 5 \times 1) = \frac{-1}{10}$$

We got the cofficient

$(2,5) = \frac{17}{\sqrt{10}} v_1 + \frac{-1}{\sqrt{10}} v_2 = \frac{17}{\sqrt{10}} \times \frac{1}{\sqrt{10}} (1,3) - \frac{1}{\sqrt{10}} \times \frac{1}{\sqrt{10}} (-3,1)$

$= \frac{17}{10} (1,3) - \frac{1}{10} (-3,1)$

- Gram-Schmidt process overview:

In an inner product space:

$$\boxed{\begin{array}{c}\text{Any basis}\\ x_1, x_2, \ldots\ldots, x_n\end{array}} \xrightarrow[\text{process}]{\text{Gram-Schmidt}} \boxed{\begin{array}{c}\text{Orthonormal basis}\\ v_1, v_2, \ldots\ldots, v_n\end{array}}$$

- Example and intution:

Consider the basis $\beta = \{(1,2,2), (-1,0,2), (0,0,1)\}$ for $\mathbb{R}^3$.

Orthonomal basis for $\mathbb{R}^3$ will be?

Let $v_1 = (1,2,2)$. Vector orthogonal to v_1, i.e. a vector in $\langle v_1 \rangle$, so we use projection P_{v_1} to v_1

Define $v_2 = (-1,0,2) - P_{v_1}((-1,0,2))$

$= (-1,0,2) - \frac{\langle(-1,0,2),(1,2,2)\rangle}{\langle(1,2,2),(1,2,2)\rangle}(1,2,2)$

$= \left(-\frac{4}{3}, -\frac{2}{3}, \frac{4}{3}\right)$

We want a vector which is orthogonal to both v_1 & v_2, i.e. a vector in $\operatorname{span}(\{v_1,v_2\})^{\perp}$, so we use Projection $P_{\operatorname{span}(\{v_1,v_2\})}$ to $\operatorname{span}(\{v_1,v_2\})$.

Define $v_3 = (0,0,1) - P_{v_1}((0,0,1)) - P_{v_2}((0,0,1))$

$= (-1,0,2) - \frac{\langle(-1,0,2),(1,2,2)\rangle}{\langle(1,2,2),(1,2,2)\rangle}(1,2,2) - \frac{\langle(0,0,1),(-\frac{4}{3},-\frac{2}{3},\frac{4}{3})\rangle}{(\frac{4}{3},-\frac{2}{3},\frac{4}{3})(\frac{4}{3},-\frac{2}{3},\frac{4}{3})}\left(-\frac{4}{3},-\frac{2}{3},\frac{4}{3}\right)$

$= \left(\frac{2}{9}, -\frac{2}{9}, \frac{1}{9}\right)$

Thus v_1, v_2, v_3 is an orthogonal basis and dividing each vactor by its norm yields an orthonormal basis:

$$= \left\{\left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right), \left(-\frac{2}{3}, -\frac{1}{3}, \frac{2}{3}\right), \left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right)\right\}$$

You can chech by $\langle v_1, v_2\rangle = \langle v_2, v_3\rangle = \langle v_1, v_3\rangle = 0$

- The Gram-Schmidt process:

Let V be inner product space with a basis $\{x_1, x_2, \dots x_n\}$
Define the orthogonal basis $\{v_1, v_2, \dots, v_n\}$ and the cornosponding orthonormal basis $\{w_1, w_2, \dots, w_n\}$ as follows:

$$v_1 = x_1 \; ; \quad w_1 = \frac{v_1}{\|v_1\|}$$

$$v_2 = x_2 - \langle x_2, w_1\rangle w_1 \; ; \quad w_2 = \frac{v_2}{\|v_2\|}$$

$$\dots \qquad \dots$$

$$v_i = x_i - \langle x_i, w_i\rangle w_i - \langle x_i, w_2\rangle w_2 - \dots - \langle x_i, w_{i-1}\rangle w_{i-1} \; ; \quad w_i = \frac{v_i}{\|v_i\|}$$

$$\dots \qquad \dots$$

$$v_n = x_n - \langle x_n, w_n\rangle w_n - \langle x_n, w_2\rangle w_2 - \dots - \langle x_n, w_{n-1}\rangle w_{n-1} \; ; \quad w_n = \frac{v_n}{\|v_n\|}$$

- Any finite-dimensional vector space with an inner product has an orthogonal basis.
- Any basis can be changed to an orthonormal basis using Gram-Schmidt Process.

- Orthogonal transformation :-

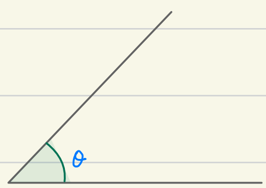
Let V be an inner product space and T be a linear transformation from V to V. T is said to be orthogonal transformation if

$$\langle Tv, Tw \rangle = \langle v, w \rangle \ \forall v, w \in V.$$

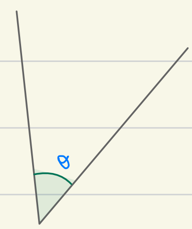
When $V = \mathbb{R}^n$ with usual innerproduct, a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is orthogonal if it preserve angle and length.

If length is preserved than angle automaticaly got preserved.

Example:



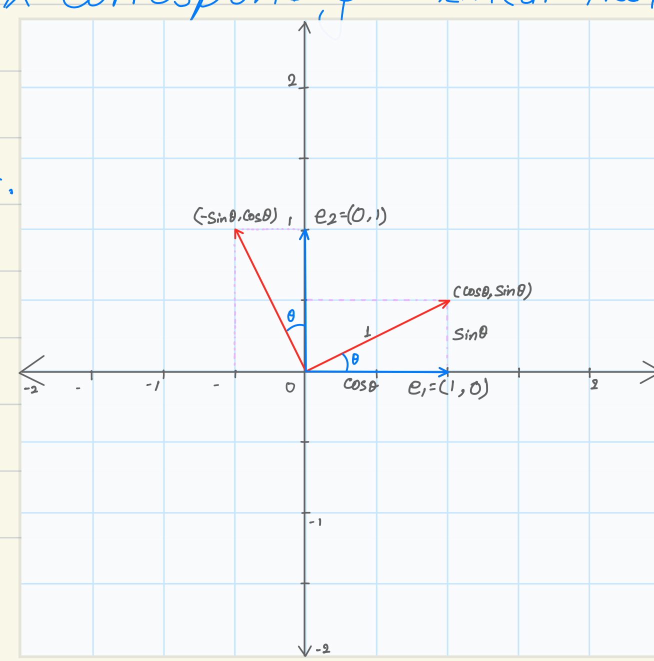
After Transformation



- Finding rotation matrix in $\mathbb{R}^2$:

Consider the standard basis $\{(1,0),(0,1)\}$ of $\mathbb{R}^2$. Rotate the plane by angle θ. The vector obtained after totation tells us matrix corrosponding to linear transformat

Blue one is basic vectors and reds are transformed vector.



Let T_θ be corrosponding linear transformation. Then:

$$T_\theta(1,0) = (\cos(\theta), \sin(\theta))$$

and, $T_\theta(0,1) = (-\sin(\theta), \cos(\theta))$.

Thus matrix corresponding to linear transformation is: $R_\theta = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$

Note: $R_\theta^T = R_{-\theta}$ & $R_\theta^T R_\theta = R_\theta R_\theta^T = I$

So, length and angle are preserved & std. basis are orthonormal than rotated vector are also orthonormal & yields an orthonormal basis of R^2.

- Orthogonal matrix:

As $\{v_1, v_2, v_3\}$ is orthonormal set, the linear transformation T is an orthogonal transformation.

Observe that:- $AA^T = A^TA = I_3$

A square matrix A is called an orthogonal matrix.

if: $AA^T = A^TA = I$

☐ Multivariable functions : Visualization.

- Scalar-valued multivariable function :

  A scalar-valued multivariable function is a function $f : D \to \mathbb{R}$ where D is domain in $\mathbb{R}^n$: $(N > 1)$

  Examples:

  - Linear transformation.
  - Polynomial transformation.
  - (Arthemetic) Combinations or compositions.

- Vector-valued multivariable function :

  A vector-valued multivariable function is a function $f : D \to \mathbb{R}^m$ where D is domain in $\mathbb{R}^n$: $(m, n > 1)$

  It can be thought of as a vector of scalar-valued multivariable function.

  Example of linear transformation :

$$f(x, y, z) = (x^2 + y^2, y^2 + z^2, z^2 + x^2)$$
$$\mathbb{R}^3 \longrightarrow \mathbb{R}^3$$

- Multivariable function (Function of several variable):

  It is either a scalar-valued multivariable function or a vector-valued multivariable function.

  When considering multivariable fn we write $f : D \to \mathbb{R}^m$ where D is domain in $\mathbb{R}^n$ where $n > 1$ and with no restriction on m (i.e m can be 1).

  To refer element in D without bothering about coordinate we use $\underset{\sim}{x} \in D$.

Examples:

$$f(x,y) = 2.5x - 3.9y$$
$$f(x,y) = 3x^3 - 3y^2 + 4.8x^2y - 9.9xy + \pi$$
$$f(x,y) = \sin(x^2+y^2)$$
$$f(x,y) = \frac{1}{2\pi} e^{-\frac{x^2+y^2}{2}}$$
$$f(x,y) = 10e^{-2x-5y}$$
$$f(x,y) = \frac{xy}{x^2+y^2}$$
$$f(x,y) = x^2+y^2+z^2$$
$$f(x,y) = (2x+2y+2z)$$
$$f(x,y) = (\sin(x)\cos(y), \tan(y+z), \ln(x^2+y^2+z^2), e^{xyz}) \mid R^3 \to R^4$$

- Arthematic operations on multivariable function:

let $D \subset R^n$ and $f: D \to R^m$, $g: D \to R^m$ be multivariable function on D.

- The sum function $f+g$ is defined on D by:
$$(f+g)(\underset{\sim}{x}) \longrightarrow f(\underset{\sim}{x}) + g(\underset{\sim}{x}), \quad x \in D$$

- Let $c \in R$. The function cf defined on D by:
$$(cf)(\underset{\sim}{x}) \triangleq c \times f(\underset{\sim}{x}), \quad \underset{\sim}{x} \in D.$$

- If $m=1$, the product function fg is defined on D by:
$$fg(\underset{\sim}{x}) = f(\underset{\sim}{x}) \times g(\underset{\sim}{x}), \quad \underset{\sim}{x} \in D$$

- If $m=1$, & $g(\underset{\sim}{x}) \neq 0$, $\underset{\sim}{x} \in D$, the quotient f/g is defined on D by: $(f/g)(\underset{\sim}{x}) = f(\underset{\sim}{x})/g(\underset{\sim}{x})$, $\underset{\sim}{x} \in D$.

Function obtained by composition:

Let $D \in R^n$ & $f: D \to R^m$ be a multivariable function. Let $g: E \to R^p$ be a function on E where $\text{Range}(f(\underset{\sim}{x})) \subseteq E \subseteq R^m$. Then for each $\underset{\sim}{x} \in D$, $f(\underset{\sim}{x}) \in E$ & therefore $g(f(\underset{\sim}{x}))$ yields a well-defined element in R^p.

We can obtain multivariable fn: $g \circ f: D \to R^p$ called composition of f and g defined as:

$$g \circ f(\underset{\sim}{x}) = g(f(\underset{\sim}{x})), \ \underset{\sim}{x} \in D.$$

Example: $f(x,y) = x^2+y^2$ is fn from R^2 to R.
$g(x) = \sqrt{x}$ is fn from $E = \{x \in R \mid x \geq 0\}$ to R
Then $g \circ f(x) = \sqrt{x^2+y^2}$

Curves in R^m:

A curve in R^m refers to the range of a function $f: D \to R^m$ where D is a domain in R.

Example:
- Line in R^m
- $\Gamma(f)$ where f is function of one variable.
- Conicks in R^2
- Helix in R^3: $s(t) = (\cos(t), \sin(t), t)$
- The subset $\{(x,y) \mid y^2 = x^3\}$ of R^2.

Things to remember for further course:

- From now on function means scalar-valued multivariable function
- If a is point in R^n, then open ball of radius r around $\underset{\sim}{a}$ is defined:

$$\{x \in R^n \mid \|x - a\| < r\}$$

- $e_1, e_2, \ldots, e_n$ is the standard ordered basis of R^n.

- Rate of change w.r.t a particular variable at a point:
  let $f(x_1, x_2, \dots, x_n)$ be a fn defined on D in R^n containing a point $\underset{\sim}{a}$ and an open ball arount it (radius).

Then rate of change of f at $\underset{\sim}{a}$ w.r.t the variable x_i is

$$\lim_{h \to 0} \frac{f(\underset{\sim}{a}+he_i)-f(\underset{\sim}{a})}{h}$$

here, $\underset{\sim}{a} = (a_1, a_2, \dots, a_n)$ | $e_i = (0,0,0,\dots,0,\underset{\text{ith place}}{1},0,0,\dots,0,0)$

$$\Rightarrow \lim_{h\to 0} \frac{f((a_1,a_2,\dots,a_n)+h(0,0,\dots 1,0,0,\dots,0)) - f(a_1,a_2,\dots,a_n)}{h}$$

$$\Rightarrow \lim_{h\to 0} \frac{f(a_1,a_2,\dots a_{i-1}, a_i+h, a_{i+1},\dots,a_n) - f(a_1,a_2,\dots,a_n)}{h}$$

Example:-

- Rate of change $f(x,y) = x+y$ at $(0,0)$ w.r.t. x

$$\rightarrow \lim_{x\to 0} \frac{f((0,0)+h(1,0))-f(0,0)}{h} = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h}$$

$$\Rightarrow \lim_{x\to 0} \frac{h-0}{h} \Rightarrow \lim_{x\to 0} \frac{h}{h} = 1$$

- Rate of change of $f(x,y,z) = xy+yz+zx$ at $(1,2,3)$ w.r.t. y

$$\Rightarrow \lim_{x\to 0} \frac{f((1,2,3)+h(0,1,0))-f(1,2,3)}{h} = \lim_{x\to 0} \frac{f(1,2+h,+3)-f(1,2,3)}{h}$$

$$\lim_{x\to 0} \frac{1(\cancel{2}+h)+(\cancel{2}+h)3+\cancel{1\times 3}-\cancel{1\times 2}-\cancel{2\times 3}-\cancel{1\times 3}}{h} = \lim_{x\to 0} \frac{h+3h}{h} = 4$$

- Rate of change of $f(x,y) = \sin(x,y)$ at $(1,0)$ w.r.t x.

$$\Rightarrow \lim_{x \to 0} \frac{f((1,0)+h(1,0)) - f(1,0)}{h} = \lim_{x \to 0} \frac{f(1+h,0) - f(1,0)}{h} = \lim_{x \to 0} \frac{0-0}{h} = 0.$$

w.r.t (with respect to) y:

$$\Rightarrow \lim_{x \to 0} \frac{f((1,0)+h(0,1)) - f(1,0)}{h} = \lim_{x \to 0} \frac{\sin(h) - 0}{h} = \lim_{h \to 0} \frac{\sin(h)}{h} = 1.$$

- Partial derivatives:

Let $f(x_1, x_2, \ldots, x_n)$ be function defined on domain D in R^n. The partial derivative of f w.r.t x_i is the fn denoted by:

$$f_{x_i} = (\underset{\sim}{x}) \text{ or } \frac{\partial f}{\partial x_i}(\underset{\sim}{x}) \text{ as defined as:}$$

∂ = dell

$$\frac{\partial f}{\partial x_i}(\underset{\sim}{x}) = \lim_{x \to 0} \frac{f(\underset{\sim}{x} + he_i) - f(\underset{\sim}{x})}{h}$$

Its domain consists of those points of D at which the limits exists.

Example:

$$f(x,y) = x+y$$

Bcz in respect to x.

$$\frac{\partial f}{\partial x}(x,y) = \lim_{h \to 0} \frac{f((x,y)+h(1,0)) - f(x,y)}{h} = \lim_{h \to 0} \frac{f(x+h+y) - f(x,y)}{h}$$

$$= \lim_{h \to 0} \frac{x+h+y-(x+y)}{h} = 1$$

Similary it can be calculated with respect to y.

- Calculating partial derivatives:

To calculate partial derivative w.r.t x_i, think of f only as a function of x_i while treating all other variables as constants. Then calculate it as derivative of one variable.

Example:

$\rightarrow$ $f(x,y,z) = xy + yz + zx$

$\frac{\partial f}{\partial x}(x,y,z) = y + 0 + z$ (treated 'y' as constant $x^3 = x^2; 3x = 3$)

$\frac{\partial f}{\partial x}(x,y,z) = x + z$ $\quad$ $\frac{\partial f}{\partial x}(x,y,z) = x + y$

$\rightarrow$ $f(x,y,z) = \sin(xy)$

$\frac{\partial f}{\partial x}(x,y,z) = \cos(xy) \cdot y = y\cos(xy)$

$\frac{\partial f}{\partial x}(x,y) = x\cos(xy)$

$$\rightarrow f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

$$\frac{\partial f}{\partial x}(x,y) = \frac{(x^2+y^2)y - xy(2x)}{(x^2+y^2)^2} = \frac{y^3 + x^2 y}{(x^2+y^2)^2}$$

$$\frac{\partial f}{\partial x} = \frac{x^3 - y^2 x}{(x^2+y^2)^2}$$

$(x,y) = (0,0)$

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h \to 0} \frac{0-0}{h} = 0$$

$$\frac{\partial f}{\partial x}(0,0) = 0$$

- Rate of change in a particular point:

let $f(x_1, x_2, \ldots, x_n)$ be a fn defined on a domain D in R^n contains a pt. $\underset{\sim}{a}$ and an open ball around it.

Insted of direction of axes, we are intrested in rate of change of fn f at $\underset{\sim}{a}$ in some other direction.

We can use same idea as for partial derivatives & choose unit vector $u \in R^n$ in direction we want to compute.

$$\lim_{h \to 0} \frac{f(\underset{\sim}{a} + hu) - f(\underset{\sim}{a})}{h}$$

if, $\underset{\sim}{a} = (a_1, a_2, \ldots, a_n)$, $u = (u_1, u_2, \ldots, u_n)$

$$\lim_{h \to 0} \frac{f(a_1 + hu_1, a_2 + hu_2, \ldots, a_n + hu_n) - f(a_1, \ldots, a_n)}{h}$$

This is fn of on variable:

we can think as $g(h) = f(a_1 + hu_1, \ldots, a_n + hu_n)$

it means; $\lim\limits_{x \to 0} \frac{g(h) - g(0)}{h}$

Example:

- Rate of change of $f(x,y) = x+y$ at $(0,0)$ in direction of the $y=x$ line.

$\Rightarrow$ $y = x$ means 45° line so $u = (1/\sqrt{2}, 1/\sqrt{2})$

$$\lim_{h \to 0} \frac{f(0 + h\frac{1}{\sqrt{2}}, 0 + h\frac{1}{\sqrt{2}}) - f(0,0)}{h}$$

$$= \lim_{h \to 0} \frac{(h/\sqrt{2} + h/\sqrt{2}) - (0+0)}{h} = \lim_{h \to 0} \frac{\sqrt{2}\, h}{h} = \sqrt{2}$$

- Rate of change of $f(x,y,z) = xy + yz + zx$ at $(1,2,3)$ in direction of vector $(4,3,0) = v$.

Here first we have to convert vectors in unit vectors by dividing by its norm.

$\|v\| = \sqrt{4^2+3^2+0^2} = \sqrt{16+9} = 5 \qquad u = \frac{1}{5}(4,3,0)$

$$\lim_{h\to 0} \frac{f(1+h^4/5,\ 2+h^3/5,\ 3+h^0/5) - f(1,2,3)}{h}$$

$$\lim_{h\to 0} \frac{h^2\,12/25 + 2h\,\frac{4}{5} + h\,3/5 + 3h\,3/5 + 3h\,4/5}{h} = \frac{8}{5}+\frac{3}{5}+\frac{9}{5}+\frac{12}{5} = \frac{32}{5}$$

- The rate of change of $f(x,y) = \sin(x,y)$ at $(1,0)$ is the direction 60° from x axis) $u = (\cos 60^\circ, \sin 60^\circ) = (1/2, \sqrt{3}/2)$

$$= \lim_{h\to 0} \frac{f(1+h/2,\ 0+h\sqrt{3}/2) - f(1,0)}{h} = \lim_{h\to 0} \frac{\sin((1+h/2)\frac{\sqrt{3}}{2}h) - \sin(1,0)}{h}$$

by defining $g(h) = \sin((1+h/2)\sqrt{3}/2\,h)$

$$\Rightarrow g'(h) = \cos(\ )\times\left\{\sqrt{3}/2\,h \times 1/2 + (1+h/2)\sqrt{3}/2\right\}$$

$$\Rightarrow g'(0) = \cos(0)\times\sqrt{3}/2 = \sqrt{3}/2$$

- Directional derivatives:

Let $f(x_1, x_2, \dots, x_n)$ be fn in domain D in R^n. The directional derivative of f in direction of unit vector u is the fn denoted by $f_u(\underset{\sim}{X})$ as defined as:

$$f_u(\underset{\sim}{x}) = \lim_{x\to 0} \frac{f(\underset{\sim}{x}+hu) - f(\underset{\sim}{x})}{h}$$

Its domain consists of those points of D at which limit exists.

Properties : Linearity and product

Linearity :- Let $c \in \mathbb{R}$. If the directional derivative at point $\underset{\sim}{a}$ in direction of unit vector u exists for both fn $f(\underset{\sim}{x})$ and $g(\underset{\sim}{x})$, then it also exists for $(cf+g)(\underset{\sim}{x})$ and

$$(cf+g)_u(\underset{\sim}{a}) = cf_u(\underset{\sim}{a}) + g_u(\underset{\sim}{a})$$

Product rule :- If dirctional derivatitive at p.t $\underset{\sim}{a}$ in the direction of the unit vector u exists for both fn $f(\underset{\sim}{x})$ and $g(\underset{\sim}{x})$, then it also exists for $(fg)(\underset{\sim}{x})$ and

$$(fg)_u(\underset{\sim}{a}) = f_u(\underset{\sim}{a})\, g(\underset{\sim}{a}) + f(\underset{\sim}{a})\, g_u(\underset{\sim}{a})$$

Quotent rule :- If dirctional derivatitive at p.t $\underset{\sim}{a}$ in the direction of the unit vector u exists for both fn $f(\underset{\sim}{x})$ and $g(\underset{\sim}{x})$, and $g(\underset{\sim}{a}) \neq 0$, than id also exist for $f/g(\underset{\sim}{x})$ and

$$(f/g)_u(\underset{\sim}{a}) = \frac{f_u(\underset{\sim}{a})\, g(\underset{\sim}{a}) - f(\underset{\sim}{a})\, g_u(\underset{\sim}{a})}{g(\underset{\sim}{a})^2}$$

Example:-

- $f(x,y) = x + y$, $u = (u_1, u_2)$

$$f_u(x,y) = \lim_{h\to 0} \frac{f(x+hu_1,\ y+hu_2) - f(x,y)}{h}$$

$$\lim_{h\to 0} \frac{x+hu_1+y+hu_2-(x+y)}{h} = \lim_{h\to 0} \frac{hu_1+hu_2}{h} = u_1+u_2 = u_1\times 1+u_2\times 1$$

□ $f(x,y,z) = xy + yz + zx$, $u = (u_1, u_2, u_3)$

$$f_u(x,y,z) = \lim_{h \to 0} \frac{(x+hu_1)(y+hu_2)+(y+hu_2)(z+hu_3)+(z+hu_3)(x+hu_1)-(xy+yz+zx)}{h}$$

$$\lim_{h\to 0} \frac{h^2(u_1u_2+u_2u_3+u_3u_1)+h(u_1y+xy+u_3y+zu_2+zu_1+xu_3)}{h}$$

$$= u_1y + u_2x + u_3y + u_2z + u_1z + u_3x$$

$$= u_1(y+z) + u_2(x+z) + u_3(x+y)$$

○ Limits of sequence in R^p:

Let $\{\underset{\sim}{a_n}\}$ be sequence in R^p. Denote the coordinates of $\underset{\sim}{a_n} = \{a_{n1}, a_{n2}, \ldots, a_{np}\}$

We say $\{\underset{\sim}{a_n}\}$ has limit $\underset{\sim}{a} = \{a_1, a_2, \ldots, a_p\} \in R^p$ if as n increases, the sequence in the i^{th} coordinate has limit a_i. i.e. $\{\underset{\sim}{a_{ni}}\} \to a_i$ for each i.

- A sequence $\{\underset{\sim}{a_n}\}$ in R^p is called conversant if it converges to some limit.
- A sequence $\{\underset{\sim}{a_n}\}$ in R^p is called divergent if it is not convergent.

$\lim 1/n \to 0$

▭ Example: $\{(1/n, n\sin(1/n))\}$ $= \lim n\sin(1/n) \to 1$ So $= (0,1)$

$\{((-1)^n, n\sin(1/n))\}$ it don't converge.

○ Limit of a scalar-valued multivariable fn at a point:

Let f be scalar-valued multivariable fn defined on a domain D in R^k and $\underset{\sim}{a}$ be a point that exists a sequence in D which converges to $\underset{\sim}{a}$.

If there exists a real no. L such that $f(a_n) \to L$ for all seq. $\underset{\sim}{a_n}$ such that $\underset{\sim}{a_n} \to \underset{\sim}{a}$, then we say limit of f at $\underset{\sim}{a}$ exists and equals L. We denote this by:

$$\lim_{\underset{\sim}{x} \to \underset{\sim}{a}} f(\underset{\sim}{x}) = L.$$

It is equavalant to: as $\underset{\sim}{x}$ comes closer and closer to $\underset{\sim}{a}$, $f(\underset{\sim}{x})$ comes closer & closer to L. If there is not L we say limit don't exists.

- Rules about scalar-valued multivariable functions:

- If $\lim_{\underset{\sim}{x} \to \underset{\sim}{a}} f(\underset{\sim}{x}) = F$, $\lim_{\underset{\sim}{x} \to \underset{\sim}{a}} g(x) = G$ and $c \in R$, then

$$\lim_{\underset{\sim}{x} \to \underset{\sim}{a}} (cf+g)(x) = cF + G$$

- If $\lim_{\underset{\sim}{x} \to \underset{\sim}{a}} f(\underset{\sim}{x}) = F$, $\lim_{\underset{\sim}{x} \to \underset{\sim}{a}} g(x) = G$ and $c \in R$, then

$$\lim_{\underset{\sim}{x} \to \underset{\sim}{a}} (fg)(\underset{\sim}{x}) = FG.$$

- If $\lim_{\underset{\sim}{x} \to \underset{\sim}{a}} f(\underset{\sim}{x}) = F$, $\lim_{\underset{\sim}{x} \to \underset{\sim}{a}} g(x) = G \neq 0$, than fn $\frac{f}{g}$ is defined in at least a small interval around a and

$$\lim_{\underset{\sim}{x} \to \underset{\sim}{a}} \frac{f}{g}(\underset{\sim}{x}) = \frac{F}{G}.$$

Example: $h(x,y,z) = x^2y^3 + y^3z^2 + xyz$

$$\lim_{(x,y,z)\to(1,2,3)} h(x,y,z) = \lim_{(x,y,z)\to(1,2,3)} x^2y^3 + \lim_{(x,y,z)\to(1,2,3)} y^3z^2 + \lim_{(x,y,z)\to(1,2,3)} xyz$$

$$= 1^2\times 2^3 + 2^3\times 3^2 + 1\times 2\times 3$$

$$= 8 + 72 + 6 = 86.$$

Composition: Suppose f is a scalar-valued multivariable fn and g is a fn of one variable such that the composition $g\circ f$ is well-defined. If

- If $\lim_{\underset{\sim}{x}\to\underset{\sim}{a}} f(\underset{\sim}{x}) = F$, $\lim_{\underset{\sim}{x}\to\underset{\sim}{a}} g(x) = L$, than $\lim_{\underset{\sim}{x}\to\underset{\sim}{a}} (g\circ f)(\underset{\sim}{x}) = L$.

Example:

$$h(x,y,z) = e^{xyz}$$

We can write it in two ways other is:

$$f(x,y,z) = xyz \quad , \quad g(x) = e^x$$

$$\Rightarrow (g\circ f)(x,y,z) = g(f(x,y,z)) = e^{xyz}$$

$$\lim_{(x,y,z)\to(1,2,3)} f = (1\times 2\times 3) = 6 \qquad \lim_{x\to 6} e^6$$

Sanwich principle :

- If $\lim_{\underset{\sim}{x}\to\underset{\sim}{a}} f(\underset{\sim}{x}) = L$, $\lim_{\underset{\sim}{x}\to\underset{\sim}{a}} g(x) = L$, and $h(\underset{\sim}{x})$ is fn such that $f(\underset{\sim}{x}) \le h(\underset{\sim}{x}) \le g(\underset{\sim}{x})$, then $\lim_{\underset{\sim}{x}\to\underset{\sim}{a}} f(x) = L$

- Finding limit by substitution:

Suppose we want to find the limit of fn $f(\underset{\sim}{x})$ at pt $\underset{\sim}{a}$ i.e. $\lim_{\underset{\sim}{x} \to \underset{\sim}{a}} f(\underset{\sim}{x})$. Often we can substitute value of $\underset{\sim}{a}$ in the expression for $f(\underset{\sim}{x})$ & obtain limit.

But in complicated fn it don't work or when $\underset{\sim}{a}$ don't belong to domain of $f(x)$

Example:

$$\lim_{x \to (0,0)} \frac{x^3 - y^2 x}{(x^2+y^2)^2}$$

$$\underset{\sim}{a_n} = (1/n, 0)$$

$$f(\underset{\sim}{a_n}) = \frac{(1/n)^3 - 0^2 \times 1/n}{((1/n)^2 + 0^2)^2} = \frac{1/n^3}{1/n^4} = n \quad \text{divergent.}$$

$$\underset{\sim}{b_n} = (0; 1/n)$$

$$f(\underset{\sim}{b_n}) = \frac{0^3 - 1/n^2 \times 0}{(0^2 + (1/n)^2)^2} = 0 \quad \text{convergent}$$

Both giving diff. result so limit don't exist.

- Limit of a vector-valued function at a point:

Let $f: D \to R^m$ be a vector-valued multivariable function defined on a domain D in R^k and $\underset{\sim}{a}$ be a point such that there exists a sequence in D which converges to $\underset{\sim}{a}$.

If f_i is ith component function f, then f_i is a scalar-valued function from D to R. suppose each i. the limit $\lim_{\underset{\sim}{x} \to \underset{\sim}{a}} f_i(\underset{\sim}{x})$ exists and equation L_i

Define: $\underset{\sim}{L} = (L_1, L_2, \dots, L_m)$. Then $\lim_{\underset{\sim}{x} \to \underset{\sim}{a}} f(\underset{\sim}{x}) = \underset{\sim}{L}$

- This is equivalant to as $\underset{\sim}{x}$ comes closer & closer to $\underset{\sim}{a}$, $f(\underset{\sim}{x})$ eventually comes closer & closer to $\underset{\sim}{L}$.
- If for some i, the limit f_i at $\underset{\sim}{a}$ don't exist, then the limit of f at $\underset{\sim}{a}$ don't exist.

Example: $\lim_{\underset{\sim}{x} \to (1,2)} \left(x^2 y + y^3, e^{xy}, \frac{x^2 - 1}{y^3 - 2} \right)$

If limit for each vector exist than only lim exist.

$\lim_{\underset{\sim}{x} \to (1,2)} x^2 y + y^3 = 1^2 \times 2 + 2^3 = 10$, $\lim_{\underset{\sim}{x} \to (1,2)} e^{xy} = e^{1 \times 2} = e^2$, $\lim_{\underset{\sim}{x} \to (1,2)} \frac{x^2 - 1}{y^3 - 2} = \frac{1^2 - 1}{2^3 - 2} = 0$

each limit exist so it will be $(10, e^2, 0)$

- $\lim_{\underset{\sim}{x} \to (0,0)} \left(\frac{\sin(xy)}{xy}, \frac{x^3 - y^2 x}{(x^2 + y^2)^2} \right)$

$\lim_{\underset{\sim}{x} \to (0,0)} \frac{\sin(xy)}{xy} = \frac{\sin(0)}{0} = 1$. $\lim_{\underset{\sim}{x} \to (0,0)} \frac{x^3 - y^2 x}{(x^2 + y^2)^2} =$ Does not exist

Limit don't exist for this vector value fn don't have limit.

- limit of a fn at a point along a curve:

Let f be a scaler valued multivariable fn in domain D in $\mathbb{R}^k$ and $\underset{\sim}{a}$ pt that sequince a exist in D which converges to $\underset{\sim}{a}$. Let C is a curve passing through pt $\underset{\sim}{a}$ belong to domain D.

The limit of f at $\underset{\sim}{a}$ along the curve C exists & equals L if for every equence $\underset{\sim}{a_n}$ contained in C which converges to $\underset{\sim}{a}$, the seq $f(\underset{\sim}{a_n})$ converges to L.

Example:

$$g(x,y) = \frac{x^3 - y^2 x}{(x^2+y^2)^2}$$

$$\lim_{(x,y)\to(0,0)} g(x,y) = \lim_{x\to 0} \frac{1}{x} \text{ (DNE)} \quad \left(g(x,0) = \frac{x^3}{x^4} = \frac{1}{x} \right)$$

along x axis (Along x axis means Y axis = 0 so it will be $g(x,0)$)

$$\lim_{(x,y)\to(0,0)} g(x,y) = \lim_{y\to 0} = 0 \quad \left(g(0,y) = \frac{0}{y^4} \right)$$

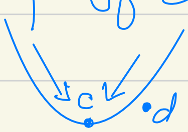
along y axis (Along axis means x axis = 0 so it will be $g(0,y)$)

- Limit of a fn along curve and limit of fn:

let f be a scaler-valued multivariable fn defined on a domain D in $\mathbb{R}^k$ and $\underset{\sim}{a}$ be a point such that there exists a sequence in D which converges to $\underset{\sim}{a}$.

Theorem:

The limit of f at $\underset{\sim}{a}$ exists and equal L precisly when for every curve C in the domain D passes through $\underset{\sim}{a}$ the limit of f at $\underset{\sim}{a}$ along C exist and equals L.



→ If curve from both side reach point c than limit exist but it it don't reach 'd' than it don't exist.

Example:

$$\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$$

Along x axis $(x,0)$ — ①

Along y axis $(0,y)$ — ⑪

① $\lim_{x\to 0} \frac{x^2}{x^2} = 1$ ⑪ $\lim_{y\to 0} \frac{-y^2}{y^2} = -1$

they don't match so limit don't exist.

Another example:

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$$

along x axis $\lim_{x\to 0} \frac{0}{x^2+0^2} = 0$

along y axis $\lim_{y\to 0} \frac{0}{0^2+y} = 0$

but it match and by substituting $x=y$ if dont match than it is not defined $\lim_{x\to 0} \frac{x^2}{x^2+x^2} = \frac{1}{2} \neq 0$ limit DNE.

3rd example

$$\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^2} =$$

the line $y = mx = \lim_{x\to 0} \frac{x(mx)^2}{x^2+(mx)^4} = \frac{mx^3}{x^2(1+m^2x^2)}$

along line $x=0$ $\lim_{y\to 0} \frac{0}{y^4} = 0$

- Continuity of function:

Suppose f. multivar. fn in domain D in R^k & $\underset{\sim}{a} \in D$ be a pt exist a seq. in D which converges to $\underset{\sim}{a}$.

f is continuous at $\underset{\sim}{a}$ if the limit of f at $\underset{\sim}{a}$ exists and $f(\underset{\sim}{x}) = f(\underset{\sim}{a})$

$f(\underset{\sim}{a_n}) \to f(\underset{\sim}{a})$ whenever $\underset{\sim}{a_n} \to \underset{\sim}{a}$.

The fn f is continous if it is continous at all pt in domain. For all pt $\underset{\sim}{a}$ for which $f(\underset{\sim}{a})$ is defined.

$$\lim_{x\to a} f(\underset{\sim}{x}) = f(\underset{\sim}{a})$$

- Computing dirational derivative vs partial derivative:

Directional derivative:

Eg: $f(x,y,z) = xy+yz+zx$ $\qquad u = (u_1, u_2, u_3)$

$$f_u(\underset{\sim}{x}) = \lim_{h\to 0} \frac{f(x+hu_1, y+hu_2, z+hu_3) - f(x,y,z)}{h}$$

$$= \lim_{h\to 0} \frac{(x+hu_1)(y+hu_2)+(y+hu_2)(z+hu_3)+(z+hu_3)(x+hu_1)-(xy+yz+zx)}{h}$$

$$= \lim_{h\to 0} \frac{h^2(u_1u_2+u_2u_3+u_3u_1)+h(xu_2+yu_1+yu_3+zu_2+zu_1+xu_3)}{h}$$

$$= \lim_{h\to 0} h(u_1u_2+u_2u_3+u_3u_1)+u_1(y+z)+u_2(x+z)+u_3(x+y)$$

It become 0 bcz of constant

$$= u_1(y+z)+u_2(x+z)+u_3(x+y)$$

It can be written as $\nabla f(\underset{\sim}{u})\cdot u$

Partial derivative

$f_x = y+z, \quad f_y = x+z, \quad f_z = y+x.$

○ Gradient vector / function:

Let $f(x_1, x_2, \ldots, x_n)$ be fn in domain D in R^n containing open ball around point $\underset{\sim}{a}$.

Suppose all partial derivative of f at $\underset{\sim}{a}$ exist. Than gradient vector of f at $\underset{\sim}{a}$ is vector $(f_{x_1}(\underset{\sim}{a}), f_{x_2}(\underset{\sim}{a}), \ldots, f_{x_n}(\underset{\sim}{a}))$ in R^n. It is denoted by $\nabla f(\underset{\sim}{a})$.

The gradiant fn of f is a fn taking value in R^n obtained by associating every pt $\underset{\sim}{a}$ to its gradiant vector $\nabla f(\underset{\sim}{a})$.

- The domain of $\nabla f(\underset{\sim}{a})$ is set of points in D where all partial derivative exist.

Example:

- $f(x,y) = \sin(xy)$ $\qquad \frac{\partial f}{\partial x} = y\cos(xy), \; \frac{\partial f}{\partial y} = x\cos(xy)$

  $\nabla f(x,y) = (y\cos(xy), x\cos(xy))$

  $\nabla f(0,0) = (0,0) \qquad \nabla f(\pi,1) = (1\cos(\pi), \pi\cos(\pi)) = (-1,-\pi)$

- $f(x,y,z) = x^2+y^2+z^2 \to$ Partial der: $\frac{\partial f}{\partial x} = 2x, \; \frac{\partial f}{\partial y} = 2y, \; \frac{\partial f}{\partial z} = 2z$

  $\nabla f(x,y,z) = (2x, 2y, 2z)$

  $\nabla f(1,2,3) = (2,4,6)$

Prev. calc.

Partial der:
$$\frac{\partial f}{\partial x} = \begin{cases} \frac{x^3 - y^2 x}{(x^2+y^2)^2} & \because (x,y) \neq (0,0) \\ 0 & \because (x,y) = 0\,0 \end{cases}$$

$$\frac{\partial f}{\partial x} \quad \frac{y^3 - x^2 y}{(x^2+y^2)^2} \quad \because (x,y) \neq (0,0)$$
$$0 \quad \because (x,y) = 0\,0$$

– $f(x,y) = \frac{xy}{x^2+y^2}$

$$\nabla f(x,y) = \begin{cases} \frac{1}{(x^2+y^2)^2}(x^3 - y^2 x, y^3 - x^2 y) & \because (x,y \neq 0,0) \\ (0,0) & \because (x,y = 0,0) \end{cases}$$

- Properties of gradients:

Linearity: Let $c \in R$
$$\nabla(cf+g)(\underset{\sim}{x}) = c\nabla f(\underset{\sim}{x}) + \nabla g(\underset{\sim}{x})$$

Product rule:
$$\nabla(fg)(\underset{\sim}{x}) = g(\underset{\sim}{x})\nabla f(\underset{\sim}{x}) + f(\underset{\sim}{x})\nabla g(\underset{\sim}{x})$$

Quotient rule:
$$\nabla(f/g)(\underset{\sim}{x}) = \frac{1}{g(\underset{\sim}{x})^2}(g(\underset{\sim}{x})\nabla f(\underset{\sim}{x}) - f(\underset{\sim}{x})\nabla g(\underset{\sim}{x}))$$

- Directional derivatives and gradients:

Let $f(x_1, \ldots x_n)$ be fn in D in R^n containing open ball around $\underset{\sim}{a}$. Suppose ∇f exists and is continous on some open ball around point $\underset{\sim}{a}$. Then for every unit vector u, the directional derivative $f_u(\underset{\sim}{a})$ exists and equals $\nabla f(\underset{\sim}{a}) \cdot u$

Example

– $f(x,y) = x+y$

$$\frac{\partial f}{\partial x} = 1, \quad \frac{\partial f}{\partial x} = 1, \quad \nabla f = (1,1)$$

$$f_u(x,y) = (1,1) \cdot u = u_1 \cdot u_2$$

$- f(x,y,z) = xy + yz + zx$

$\frac{\partial f}{\partial x} = y+z$, $\frac{\partial f}{\partial y} = x+z$, $\frac{\partial f}{\partial z} = y+x$

$\nabla f = (y+z, x+z, y+x)$

$f_u(x,y,z = \nabla f \cdot u$

$= (y+z, z+x, x+y) \cdot (u_1, u_2, u_3)$

$= u_1(y+z) + u_2(z+x) + u_3(x+y)$

$- f(x,y) = \sin(xy)$ $\nabla f = (y\cos(xy), x\cos(xy))$

$f_u(x,y) = \nabla f \cdot u$

$= (y\cos(xy), x\cos(xy)) \cdot u$

$= u_1 y\cos(xy) + u_2 x\cos(xy)$

- Water flow in 3D
- In what direction is directional derivative minimized?

Let $f(x_1, \ldots, x_n)$ be fn in D in R^n. With open ball around $\underset{\sim}{a}$.
Suppose ∇f exists and is continous on some open ball around point $\underset{\sim}{a}$.

$f_u = \nabla f(\underset{\sim}{a}) \cdot u$ $\because$ when θ is $\angle$ b/w

$= \|\nabla f(\underset{\sim}{a})\| \cdot \|u\| \cdot \cos(\theta)$ $\nabla f(\underset{\sim}{a})$ & u

u = unit vector = 1 $= \|\nabla f(\underset{\sim}{a})\| \cdot \cos(\theta)$

$\cos(\theta)$ is minimized when $\theta = \pi$, i.e u is pointing in direction opp to $\nabla f(\underset{\sim}{a})$

$\therefore$ The min. value of f_u is attained when $u = -\nabla f(\underset{\sim}{a}) / \|\nabla f(\underset{\sim}{a})\|$ & equal to $-\|\nabla f(\underset{\sim}{a})\|$

- Direction in which the directional derivative is maximized or remain unchanged:

Assume same hyphesis the gradian exist and is continous on openball.

$$f_u = \|\nabla f(\underset{\sim}{a})\| \cdot \|u\| \cos(\theta) = \|\nabla f(\underset{\sim}{a})\| \cos(\theta)$$

It is maximized when $\theta = 0$, i.e. 'u' in same direction $\nabla f(\underset{\sim}{a})$ i.e. $u = \dfrac{\nabla f(\underset{\sim}{a})}{\|\nabla f(\underset{\sim}{a})\|}$

It remain 0 when (rate of chang) / $f_u = 0$ i.e when cosin is 0 when $\theta = \pi/2$ i.e 'u' is orthogonal or perpendicular to $\nabla f(\underset{\sim}{a})$

- Derivative:
  - It is minimised when it is opp. direction of gradiant vector.
  - It is maximised when it is in same direction to gradiant vector.
  - It is constant when it is in perpendicular to gradiant vector.

Directions: steepest ascent, steepest descent, nochange.

Property	In term of directional derivative	Direction
Steepest ascent	f_u is +ve and maximum	$u = \nabla f / \|\nabla f\|$
Steepest descent	f_u is −ve and minimum	$u = -\nabla f / \|\nabla f\|$
No Change	$f_u = 0$	u is orthogonal to ∇f

Example:

1) $f(x,y) = \sin(xy)$ Compute gradient

$\nabla f(x,y) = (y\cos(xy), x\cos(xy))$

at $(\pi,1)$ what is direction of stepest decent.

$\nabla f(\pi,1)\ (\cos(\pi), \pi\cos(\pi)) = (-1,-\pi)$

$$u = \frac{-\nabla f(\pi,1)}{\|\nabla f(\pi,1)\|} = \frac{(1,\pi)}{\sqrt{1+\pi^2}}$$

2) $f(x,y,z) = x^2+y^2+z^2$

$\nabla f(x,y,z) = (2x, 2y, 2z)$

At $(1,1,1)$ what is direction in which fn increases fastest.

$\nabla f(1,1,1) = (2,2,2)$.

In direction of $u = \dfrac{(2,2,2)}{\sqrt{2^2+2^2+2^2}} = \dfrac{1}{\sqrt{3}}(1,1,1)$

In which direction fn remain constant?

eg, $(1,-1,0)\frac{1}{\sqrt{2}}$, $\frac{1}{\sqrt{2}}(1,0,-1)$

- Tangents for $f(x,y)$:

Let $f(x,y)$ be a fn in domain D of R^2 with open ball around point $\underset{\sim}{a}$.

Consider a line L in D passing through $\underset{\sim}{a}$ & restrict f to L.

If line L is in the direction of unit vector 'u', than tangent will be line with slope $f_u(\underset{\sim}{a})$ & passing through point $(\underset{\sim}{a}, f(\underset{\sim}{a}))$

Example:

— $f(x,y) = x+y$; tangent at $(1,1)$ in direction of $(1,0)$

- First we have to find directional derivative.

$f_u(1,1) = \frac{\partial f}{\partial x}(x,y) = 1$

$(a,b,f(ab))$ $\quad u_1, u_2, f_u(a,b)$

$(x(t), y(t), z(t)) = (1,1,2) + t(1,0,1)$

$x(t) = 1+t, \; y(t) = 1, \; z(t) = 2+t$

— $f(x,y) = xy$; tangent at $(1,1)$ in direction of $(3,4)$

$u = \left(\frac{3}{5}, \frac{4}{5}\right)$ $\quad f_u(1,1) = 1\times\frac{3}{5} + 1\times\frac{4}{5} = \frac{7}{5}$ $\quad \nabla f(x,y) = (y,x)$

$\nabla f(1,1) = (1,1)$

$(x(t), y(t), z(t)) = (1,1,1) + t(3/5, 4/5, 7/5)$

$= (1+3t/5, 1+4t/5, 1+7t/5)$

— $f(x,y) = \sin(xy)$; tangent at $(\pi,1)$ in direction of $(1,2)$

$u = \frac{1}{\sqrt{5}}(1,2)$ $\quad \nabla f(x,y) = (y\cos(xy), x\cos(xy))$ $\quad \nabla f(\pi,1)$

$f_u(\pi,1) = -1\times\frac{1}{\sqrt{5}} + (-\pi)\times\frac{2}{\sqrt{5}} = \frac{-2\pi-1}{\sqrt{5}}$ $= (-1,-\pi)$

$(x(t), y(t), z(t)) = (\pi,1,0) + t(1/\sqrt{5}, 2/\sqrt{5}, (-2\pi-1)/\sqrt{5})$

- Tangents for scalar-valued multivariable functions:

All same and if line L is in the direction of unit vector u, than the tangent will be the line with slope $f_u(\underset{\sim}{a})$ and passing through pt $(\underset{\sim}{a}, f(\underset{\sim}{a}))$

- Parametric equation with example:

Similar to 2 variable we can diduce equation of tangent line as:

$\underset{\sim}{x} = (x_1, x_2, \ldots, x_n)$, z is variable in which we are measuring the fn.

$\underset{\sim}{a} = (a_1, \ldots, a_n)$, $u = (u_1, \ldots, u_n)$

Line through $\underset{\sim}{a}$ in direction of u is

$$x_i(t) = a_i + tu_i \quad , \quad z = 0$$

$$(\underset{\sim}{x}(t), z(t)) = (\underset{\sim}{a}, 0) + t(u, 0)$$

$\therefore$ The tangent line to f at $\underset{\sim}{a}$ above L is:

$$\left.\begin{array}{l} (\underset{\sim}{x}(t), z(t)) = (\underset{\sim}{a}, f(\underset{\sim}{a})) + t(u, f_u(\underset{\sim}{a})) \\ x_i(t) = a_i + tu_i \ , \ z(t) = f(\underset{\sim}{a}) + t f_u(\underset{\sim}{a}) \end{array}\right] \text{Tangent line}$$

Example: $f(x,y) = xy + yz + zx$; tangent at $(1,1,1)$ in direction of $(-1,-2,2)$

$u = 1/3(-1,-2,2)$ $\quad \nabla f(x,y,z) = (y+z, z+x, x+y)$

$\nabla f(1,1,1) = (2,2,2)$

$f_u(1,1,1) = -2/3$

$$(x(t), y(t), z(t), u(t)) = (1,1,1,3) + t(-1/3, -2/3, 2/3, -2/3)$$

$$x(t) = 1 - t/3, \ y(t) = 1 - 2t/3, \ z(t) = 1 + 2t/3, \ u(t) = 3 - 2t/3$$

- When do all tangent exist:

This is equivalant to asking when all directional derivative exists.

Let $f(x_1, x_2, \ldots, x_n)$ be fn defined in domain D in $\mathbb{R}^n$ containing open ball around point $\underset{\sim}{a}$.

- Suppose ∇f exist and is continous on some open ball around $\underset{\sim}{a}$. Then for every unit vector u, the directional derivative $f_u(\underset{\sim}{a})$ exists and equals $\nabla f(\underset{\sim}{a}) \cdot u$.

○ The collection of all tangents:

let $f(x,y)$ a fn in domain D in R^2 with open ball around point (a,b).

Suppose ∇f exists and is continous on some open ball arount point (a,b). Thall all tangent arount point (a,b) exist and eq. of tangent in unit vector u direction is:

$$f_u(a,b) = \nabla f(a,b) \cdot u$$
$$= \frac{\partial f}{\partial x}(a,b) \cdot u_1 + \frac{\partial f}{\partial y}(a,b) u_2$$

$$x(t) = a + u_1 t, \quad y(t) = b + u_2 t, \quad z(t) = f(a,b) + f_u(a,b) t$$
$$= f(a+b) + \left(\frac{\partial f}{\partial x}(a,b) u_1 + \frac{\partial f}{\partial y}(a,b) u_2\right) t$$

○ The equation of tangent plane:

let $f(x,y)$ be a fn defined on domain D in R^2 with open ball around point (a,b)

Suppse gradiant ∇f exist and continous. Than eq. of tangent plane to f at (a,b) is given by:

$$z = f(a,b) + \frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-b)$$

$$\frac{\partial f}{\partial x}(a,b) x + \frac{\partial f}{\partial y}(a,b) y - z = \frac{\partial f}{\partial x}(a,b) a + \frac{\partial f}{\partial y}(a,b) b.$$

Example:

$f(x,y) = x + y$; tangent at $(1,1)$

$\Rightarrow \nabla f(1,1) = (1,1)$, $z = 2 + 1(x-1) + 1(y-1)$

$z = 2 + x - 1 + y - 1 = x + y$

$z = x + y$

$f(x,y) = xy$; tangent at $(1,1)$

$\Rightarrow \nabla f(1,1) = (1,1)$, $z = 1 + 1(x-1) + 1(y-1)$

$z = x + y - 1$

$f(x,y) = \sin(xy)$; tangent at $(1,0)$

$\rightarrow \nabla f(1,0) = (0,1)$, $z = 0 + 0(x-1) + 1(y-0)$

$z = y$

○ The tangent hyperplane:

let $f(\underset{\sim}{x})$ be a fn defined on domain D in $\mathbb{R}^2$ with open ball around point $\underset{\sim}{a}$.

Suppose ∇ exists and continous around pint $\underset{\sim}{a}$.

Then eq. of tangent hyperplane to f at (a,b) is given by:

$$z = f(\underset{\sim}{a}) + \sum \frac{\partial f}{\partial x_i}(\underset{\sim}{a})(x_i - a_i)$$

$$= f(\underset{\sim}{a}) + \nabla f(\underset{\sim}{a})$$

Example:

$f(x,y,z) = xy + yz + zx$; tangent at $(1,1,1)$

$\nabla f(x,y,z) = (x+y, y+z, z+x)$

$\nabla(1,1,1) = (2,2,2)$

Tangent hyperplane is:

$$z = f(1,1,1) + \nabla f(1,1,1) \cdot (x-1, y-1, z-1)$$

$$= 3 + (2,2,2) \cdot (x-1, y-1, z-1)$$

Eq is $u = 3 + 2(x-1) + 2(y-1) + 2(z-1)$

$f(x,y,z) = x^2 + y^2 + z^2$; tangent at $(2,3,-1)$

$\nabla f = (2x, 2y, 2z)$

$\nabla f(2,3,-1) = (4,6,-2)$

$u = f(2,3,-1) + \nabla f(2,3,-1) \cdot (x-2, y-3, 3+1)$

$= 14 + (4,6,-2) \cdot (x-2, y-3, z+1)$

Eq is $= u = 14 + 4(x-2) + 6(\ -3) - 2(z+1)$

- Linear approximation:

Let $f(\underset{\sim}{x})$ be a fn defined on domain D in $\mathbb{R}^2$ with open ball around point $\underset{\sim}{a}$.

Suppose ∇ exists and continous around pint $\underset{\sim}{a}$.

Then the fn $L_f(\underset{\sim}{x}) = f(\underset{\sim}{a}) + \nabla f(\underset{\sim}{a}) \cdot (\underset{\sim}{x} - \underset{\sim}{a})$ is best linear approximation for fn f close to $\underset{\sim}{a}$.

Example.

L.A to $f(x,y) = xy$ at $(1,1)$

$\nabla f(x,y) = (y,x)$

$\nabla f(1,1) = (1,1)$

$L_f(x,y) = f(1,1) + \nabla f(1,1) \cdot (x-1, y-1)$

$= 1 + (1,1) \cdot (x-1, y-1)$

$= 1 + x - 1 + y - 1 = x + y - 1$ is best linear approx. of $(1,1)$

LA of $f(x,y,z) = x^2 + y^2 + z^2$ at $(2,3,-1)$

$\nabla f(2,3,-1) = 4, 6, -2)$

$L_f(x,y,z) = 3 + (4,6,-2) \cdot (x+2, y-3, z+1)$

$= 3 + 4(x-2) + 6(y-3) - 2(z+1)$

$= 4x + 6y - 2z - 29$ is best linear Approx close to $(2,3,-1)$

○ Point of local extrema for multivariable function:

let $f(\underset{\sim}{x})$ be a fn on domain D in R^n & suppose $\underset{\sim}{a} \in D$.

— The point $\underset{\sim}{a}$ is local maximum of f for some open ball B containing $\underset{\sim}{a}$, $f(\underset{\sim}{x}) \leq f(\underset{\sim}{a})$ where $\underset{\sim}{x} \in B \cap D$.

— The point $\underset{\sim}{a}$ is local minimum of f for some open ball B containing $\underset{\sim}{a}$, $f(\underset{\sim}{a}) \leq f(\underset{\sim}{x})$ where $\underset{\sim}{x} \in B \cap D$.

— A local extremum of f is either a local maximum or a local minimum of f.

○ The gradient vector at point of local extrema:

Let $f(\underset{\sim}{x})$ be a fn defined on a domain D in R^n containing some open ball around point $\underset{\sim}{a}$ of local extremum.

Restrict f to a line L passing through $\underset{\sim}{a}$ and view it as a vector of one variable on L.

Than $\underset{\sim}{a}$ is local extremum for restricted fn on L & hence the directional directive of f in the direction of line L (if it exist) at $\underset{\sim}{a}$ is 0.

In particular, those partial derivative which exists at $\underset{\sim}{a}$ must be 0.

If $\nabla f(\underset{\sim}{a})$ exist for a local extremum $\underset{\sim}{a}$ than $\nabla(\underset{\sim}{a}) = 0$.

○ Critical point: A point $\underset{\sim}{a}$ is critical p t of a fn $f(\underset{\sim}{x})$ if either $\nabla f(\underset{\sim}{a})$ doesn't exist or $\nabla f(\underset{\sim}{a})$ exists & $\nabla f(\underset{\sim}{a}) = 0$.

Example: critical pt of $f(x,y) = x^2 + 6xy + 4y^2 + 2x - 4y$.

$$\frac{\partial f}{\partial x} = 2x + 6y + 2$$

$\nabla f(x,y) = (2x+6y+2, 6x+8y-4)$

set $\nabla f = 0$ i.e $(2x+6y+2, 6x+8y-4) = (0,0)$

$2x+6y+2=0$

$6x+8y-4=0$ $\quad x=2, y=-1$

so, critical point is $(2,-1)$

- Saddle point: Every local extremum is a critical point. Unfortunately not all critical point are local extrema.

Example: $f(x,y) = x^3$

A saddle pt is a critical pt $\underset{\sim}{a}$ such that $\nabla f(\underset{\sim}{a})$ exists & $\nabla f(\underset{\sim}{a}) = 0$ but $\underset{\sim}{a}$ is not a local extremum.

- Absolute (Or global) extrema:

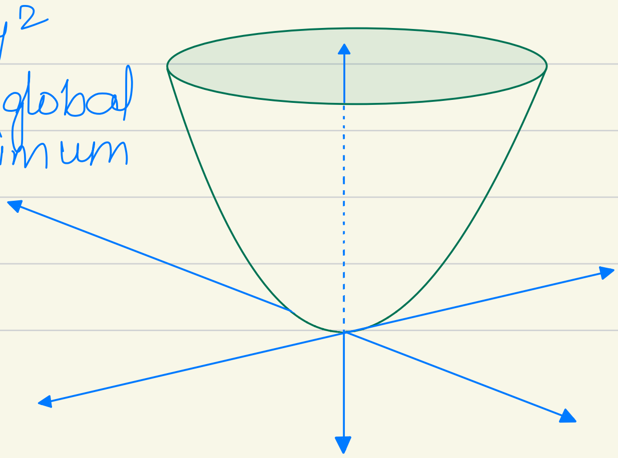
Let $f(\underset{\sim}{x})$ be a fn defined on a domain D in R^n & suppose $\underset{\sim}{a} \in D$.

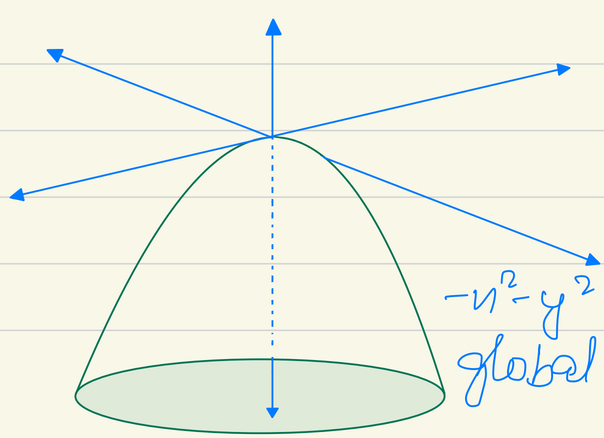
— The point $\hat{a}$ is an absolute maximum (global) of f if $f(\underset{\sim}{x}) \leq f(\underset{\sim}{a})$ for all $x \in D$.

— The point $\hat{a}$ is an absolute minimum (global) of f if $f(\underset{\sim}{a}) \leq f(\underset{\sim}{x})$ for all $x \in D$.

x^2+y^2

(0,0) is global minimum





$-x^2-y^2$ (0,0) is global maximum

- Existance of absolute maximum/minimum:

A domain D in R^n is called closed if it contains all its boundry point. A doman in R^n called bounded if it is contained inside a ball around 0 with finite radius.

If the domain D is closed and bounded & f is continous on D, than global maximum & minimum exist.

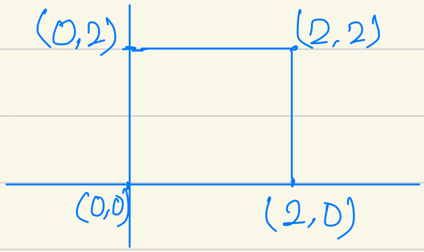
Global max & min are in particular local max & min unless they are boundry pt.

So, to find global max & min we find critical pt:
- Inside domain D.
- On boundry of D.
- On the boundry of boundry of D.

& check value of f in all critical point.

Example:

Find absolute max & min of the fn $f(x,y) = x^3+y^3-3x-3y^2+1$ over square with vertices $(0,0),(2,0),(2,2),(0,2)$



$\Rightarrow \nabla f = (3x^2-3, 3y^2-6y)$

set to 0. $3x^2-3=0$, $3y^2-6y=0$

$x^2=1$, $y(y-2)=0$

$x=\pm 1$, $y=0$ or 2

$(1,0),(1,2)$ (critical pt for squares $(0,0),(0,2),(2,2)(2,0)$

$f(x,0)=x^3-3x+1$: $0<x<2$ (boundry with end point)

$f'(x)=0 \Rightarrow 3x^2-3=0 \Rightarrow x=\pm 1 \Rightarrow (1,0)$

$f(x,2)=x^3+8-3x-12+1 = x^3-3x-3$

$f'(x)=0 \Rightarrow x$

$f(0,y) = y^3-6y=0$

$f'(y) = 0$

$3y^2 - 6y = 0$

$y(y-2) = 0 \qquad (0,0), (0,2)$

$f(2,y) = y^3 - 3y^2 + C$

$(1,0), (1,2), (0,0), (0,2), (2,2), (2,0)$

$-1 \quad -5 \quad 1 \quad -3 \quad -1 \quad 3$

so abs. max is $(2,0)$ with value 3

" " min " $(1,2)$ " " -5

- Second order partial derivative for $f(x,y)$:

Let $f(x,y)$ be fn defined on domain D in R^2. Then 2nd order partial derivative of f are partial derivatives of partial derivative.

Notation:

f_{xx} or $\dfrac{\partial^2 f}{\partial x^2}$

f_{yy} or $\dfrac{\partial^2 f}{\partial y^2}$

$$\left.\begin{array}{l} f_{xy} \text{ or } \dfrac{\partial^2 f}{\partial y \partial x} \\ f_{yx} \text{ or } \dfrac{\partial^2 f}{\partial x \partial y} \end{array}\right\} \text{Mixed partial derivative}$$

Example:

$f(x,y) = x + y \quad : \dfrac{\partial f}{\partial x} = 1, \quad \dfrac{\partial f}{\partial y} = 1$

$$\frac{\partial^2 f}{\partial x^2} = 0, \quad \frac{\partial^2 f}{\partial y^2} = 0, \quad \frac{\partial^2 f}{\partial x \partial y} = 0, \quad \frac{\partial^2 f}{\partial y \partial x} = 0$$

$f(x,y) = \sin(xy)$: $\dfrac{\partial f}{\partial x} = y\cos(xy)$, $\dfrac{\partial f}{\partial y} = x\cos(xy)$

$$\frac{\partial^2 f}{\partial x^2} = y(-\sin(xy)y) = -y^2\sin(xy)$$

$$\frac{\partial^2 f}{\partial y^2} = -x^2\sin(xy)$$

$$\frac{\partial^2 f}{\partial x\partial y} = 1(\cos(xy)) + x(-y\sin(xy)) = \cos(xy) - xy\sin(xy)$$

$$\frac{\partial^2 f}{\partial y\partial x} = 1(\cos(xy)) + y(-x\sin(xy)) = \cos(xy) - xy\sin(xy)$$

- Cliraut's theorem about mixed partial:

  Let $f(x,y)$ be a fn defined on a domain D in $\mathbb{R}^2$ containing a point $\underset{\sim}{a}$ and an open ball around it. If 2nd order mixed partial derivative f_{xy} & f_{yx} are continous on open ball around $\underset{\sim}{a}$ then, $f_{xy}(\underset{\sim}{a}) = f_{yx}(\underset{\sim}{a})$

- Second order partial derivative:

  Let $f(x_1, x_2, \ldots, x_n)$ be a fn defined on domain D in $\mathbb{R}^n$. Then 2nd order partial derivative of f be defined analogously as the partial derivatives of partial derivative.

$$f_{x_ix_i} \quad \text{or} \quad \frac{\partial^2 f}{\partial x_i^2}$$

$$f_{x_ix_j} = (f_{x_i})_{x_j} \quad \text{or} \quad \frac{\partial^2 f}{\partial x_j \partial x_i} = \frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_i}\right)$$

Example: $f(x,y,z) = xy + yz + zx$

$$\frac{\partial f}{\partial x} = y+z \;,\; \frac{\partial f}{\partial y} = x+z \;,\; \frac{\partial f}{\partial z} = x+y$$

$$\frac{\partial^2 f}{\partial x^2} = 0 \;,\; \frac{\partial^2 f}{\partial y \partial x} = 1 \;,\; \frac{\partial^2 f}{\partial z \partial x} = 1$$

$$\text{My}, \frac{\partial^2 f}{\partial y^2} = 0 \;,\; \frac{\partial^2 f}{\partial z \partial y} = 1 \;,\; \frac{\partial^2 f}{\partial z^2} = 0$$

from clairaut's theorem:

$$\frac{\partial^2 f}{\partial x \partial y} = 1 \;,\; \frac{\partial^2 f}{\partial x \partial z} = 1 \;,\; \frac{\partial^2 f}{\partial y \partial z} = 1$$

- Higher order partial derivatives:

Let $f(x_1, x_2, \ldots, x_n)$ be a fn in domain D in R^n. Then higher order partial derivative of f are defined analogously by taking successive partial derivatives.

$$f_{x_{i_1} x_{i_2} \cdots x_{i_n}} = ((f_{x_{i_1}})_{x_{i_2}} \ldots)_{x_{ik}}$$

or

$$\frac{\partial^k f}{\partial x_{i_k} \cdots \partial x_{i_2} \partial x_{i_1}} = \frac{\partial}{\partial x_{ik}} \left(\frac{\partial}{\partial x_{ik-1}} \left(\cdots \left(\frac{\partial f}{\partial x_{i_1}} \right) \right) \right)$$

Its domain is where derivative exists.

- Hassian Matrix:

Let $f(x_1, x_2, \ldots, x_n)$ be fn defined on domain D in R^n.

Then Hassian matrix defined as:

$$Hf = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_j} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_j} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ & & \ddots & & & \\ \cdots & \cdots & & \cdots & \cdots & \cdots \\ \frac{\partial^2 f}{\partial x_i \partial x_1} & \frac{\partial^2 f}{\partial x_i \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_i \partial x_j} & \cdots & \frac{\partial^2 f}{\partial x_i \partial x_n} \\ & & \ddots & & & \\ \cdots & \cdots & & \cdots & \cdots & \cdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_j} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix}$$

Example:

$$f(x,y) = x + y$$

$$Hf = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}_{2\times 2}$$

$$f(x,y) = \sin(xy)$$

$$Hf = \begin{bmatrix} -y^2 \sin(xy & \cos(xy) - xy\sin(xy) \\ \cos(xy) - xy\sin(xy) & -x^2\sin(xy) \end{bmatrix}$$

$$f(xyz) = xy + yz + zx$$

$$Hf \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} 3\times 3$$

☐ 2nd derivative test to chek neture of critical point:

1) If a is a critical pt $f''(a) > 0$, than it is local minimum.
2) If a is a critical pt $f''(a) < 0$, than it is local maximum.
3) If a is a critical pt $f''(a) = 0$, than test is inconclusive.

- Critical point in multivariable fn:

A point $\underset{\sim}{a}$ is critical pt of a fn $f(x)$ if either $\nabla f(\underset{\sim}{a})$ doesn't exist or $\nabla f(\underset{\sim}{a})$ exists & $\nabla f(\underset{\sim}{a}) = 0$.

A saddle pt is a critical pt $\underset{\sim}{a}$ such that $\nabla f(\underset{\sim}{a})$ exists and $\nabla f(\underset{\sim}{a}) = 0$ but $\underset{\sim}{a}$ not local extremum.

- The Hessian test: Classifying critical point of $f(x,y)$

Let $f(x,y)$ be a fn defined on domain D in R^2. Let $\underset{\sim}{a}$ is a critical pt of f such that the first & second order partial derivative are continous in open ball around $\underset{\sim}{a}$.

Hessian test can applied to check nature of critical pt $\underset{\sim}{a}$.

1) If $\det(Hf(\underset{\sim}{a})) > 0$ and $f_{xx}(\underset{\sim}{a}) > 0$ than $\underset{\sim}{a}$ is Local minimum.
2) If $\det(Hf(\underset{\sim}{a})) > 0$ and $f_{xx}(\underset{\sim}{a})$ 0 than $\underset{\sim}{a}$ is Local maximum
3) If $\det(Hf(\underset{\sim}{a})) < 0$ than $\underset{\sim}{a}$ is a saddle point.
4) If $\det(Hf(\underset{\sim}{a})) = 0$ than test is inconclusive

Example:

- $f(x,y) = x^2 + y^2$

$\nabla f = (2x, 2y)$. Critical pt $(0,0)$

$Hf = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ $\quad Hf(0,0) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

$\det(Hf(0,0)) = 4 > 0$

$f_{xx}(0,0) = 2 > 0$

$\therefore (0,0)$ is local minimum.

− $f(x,y) = -x^2 - y^2$ $\quad \nabla f = (-2x, -2y)$, Critical pt $(0,0)$

$$Hf = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} \qquad Hf(0,0) = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$$

$\det(Hf(0,0) = 4 > 0$

$f_{xx} = -2 < 0$

$\therefore$ $(0,0)$ is local maximum.

− $f(x,y) = x^2 - y^2$ $\quad \nabla f = (2x, -2y)$ $\quad$ c.p $= (0,0)$

$$Hf = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix} \qquad Hf(0,0) = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$$

$\det(Hf(0,0)) = -4 < 0$

$\therefore$ $(0,0)$ is a saddle point.

− $f(x,y) = x^4 + y^4$ $\quad \nabla f = (4x^3, 4y^3)$. C.P $= (0,0)$

$$Hf = \begin{bmatrix} 12x^2 & 0 \\ 0 & 12y^2 \end{bmatrix} \qquad Hf(0,0) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$\det(Hf(0,0)) = 0$

$\therefore$ The test is inconclusive.

Example: $f(x,y) = x^2 + 6xy + 4y^2 + 2x - 4y$

$\nabla f = (2x + 6y + 2,\ 6x + 8y - 4)$ equating to 0

C.P $= (2,-1)$

$$Hf = \begin{bmatrix} 2 & 6 \\ 6 & 8 \end{bmatrix} = Hf(2,-1)$$

$\det(Hf(2,-1) = 16 - 36 = -20 < 0$

$\therefore$ $(2,-1)$ is a saddle point.

- The Hessian test : Classifing critical point of $f(x,y,z)$ [3 variable]

Let $f(x,y,z)$ be a fn defined in D in $\mathbb{R}^3$. Let $\underset{\sim}{a}$ be a critical pt. of f such that 1st, 2nd order partial derivatives are continous in an open ball around $\underset{\sim}{a}$.

Hessain test can be applied to check nature of critical point $\underset{\sim}{a}$.

1.) If $f_{xx}(\underset{\sim}{a}) > 0$, $(f_{xx}f_{yy} - f_{xy}^2)(\underset{\sim}{a}) > 0$, $\det(Hf(\underset{\sim}{a})) > 0$, then $\underset{\sim}{a}$ is a local minimum.

2.) If $f_{xx}(\underset{\sim}{a}) < 0$, $(f_{xx}f_{yy} - f_{xy}^2)(\underset{\sim}{a}) > 0$, $\det(Hf(\underset{\sim}{a})) < 0$, then $\underset{\sim}{a}$ is a local macimum.

3.) If $\det(Hf(\underset{\sim}{a})) \neq 0$ & cas 1 or 2 occur, than $\underset{\sim}{a}$ is saddle point.

4) If $\det(Hf(\underset{\sim}{a})) = 0$, then the test is inconclusive.

- Understanding the term :

Terms are: $\boxed{f_{xx}}$, $\boxed{(f_{xx}f_{yy} - f_{xy}^2)(\underset{\sim}{a})}$ & $\boxed{\det(Hf(\underset{\sim}{a}))}$

$$
Hf(\underset{\sim}{a}) = \begin{array}{c} \begin{array}{ccc} & \text{det} & \text{det} \end{array} \\ \begin{bmatrix} f_{xx}(\underset{\sim}{a}) & f_{xy}(\underset{\sim}{a}) & f_{xz}(\underset{\sim}{a}) \\ f_{xy}(\underset{\sim}{a}) & f_{yy}(\underset{\sim}{a}) & f_{yz}(\underset{\sim}{a}) \\ f_{xz}(\underset{\sim}{a}) & f_{yz}(\underset{\sim}{a}) & f_{zz}(\underset{\sim}{a}) \end{bmatrix} \\ \begin{array}{ccc} 1\times1 & 2\times2 & 3\times3 \end{array} \end{array}
$$

Signs:	+	+	+	local minimum
	−	+	−	local maximum
approch	←	←	←	
	other non-zero cases			
	$\det(Hf(\underset{\sim}{a})) \neq 0$			Saddle point
	Degrate case			
	$\det(Hf(\underset{\sim}{a})) = 0$			Inconclusive

Example:

$$f(x,y,z) = x^2+y^2+z^2$$

$\nabla f = (2x, 2y, 2z)$, Critical pt : $(0,0,0)$

$$Hf = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = Hf(0,0,0)$$

$\det(Hf(0,0,0)) = 8 > 0$ $\therefore (0,0,0)$

$$\det\left(\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\right) = 4 > 0$$

$f_{xx}(0,0,0) = 2 > 0$ Local minima.

$$f(x,y,z) = -x^2-y^2-z^2$$

$\nabla f = (-2x, -2y, -2z)$, Critical pt : $(0,0,0)$

$$Hf = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix} = Hf(0,0,0)$$

$\det(Hf(0,0,0)) = -8 < 0$ $\therefore (0,0,0)$

$$\det\left(\begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}\right) = 4 > 0$$

$f_{xx}(0,0,0) = -2 < 0$ Local maxima.

$$f(x,y,z) = x^2-y^2+z^2$$

$$Hf = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = Hf(0,0,0)$$

$\det(Hf(0,0,0)) = -8 < 0$ $\therefore (0,0,0)$

$$\det\left(\begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}\right) = -4 < 0$$

$f_{xx}(0,0,0) = 2$ Saddle point.

- $f(x,y,z) = x^4 + y^4 + z^4$

$\nabla f = (4x^3, 4y^3, 4z^3)$, critical pt. $(0,0,0)$

$$Hf = \begin{bmatrix} 12x^2 & 0 & 0 \\ 0 & 12y^2 & 0 \\ 0 & 0 & 12z^2 \end{bmatrix} \qquad Hf(0,0,0) = 0_{3\times 3}$$

Inconclusive

- $f(x,y,z) = xy + yz + zx$

$\nabla f = (y+z, z+x, x+y)$ equating to 0, we get

$x = -y = z$, $x = -z$ $\Big\}$ $x = y = z = 0$

critical pt $= (0,0,0)$

$$Hf = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} = Hf(0,0,0)$$

$$\det(Hf(0,0,0)) = 0 \times \det\begin{bmatrix} & \\ & \end{bmatrix} - 1 \times \det\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + 1 \times \det\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$= 1 + 1 = 2 > 0$

$$\det\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = -1 < 0$$

$\therefore$ $(0,0,0)$ is a saddle point.

- Diffrentiablity, tangents and best linear approximation:

f is diffrentiable at $\underset{\sim}{a}$

hyper

Tangent plane to f at $\underset{\sim}{a}$ exists

Best linear approximation to f at $\underset{\sim}{a}$ exist

- Diffrentiablity for scalar-valued multivariable function:

Let f be a scalar-valued multivariable fn defined on a domain D in R^n containing an open ball around a point $\underset{\sim}{a}$.

Then f is diffrentiable at $\underset{\sim}{a}$ if:

$$\lim_{h \to 0} \frac{f(\underset{\sim}{a}+\underset{\sim}{h}) - f(\underset{\sim}{a}) - \underset{\sim}{h} \cdot \nabla f(\underset{\sim}{a})}{\|h\|} = 0.$$