MATHEMATICS FOR DATA SCIENCE II

WEEK 6

Topics in week 6:

- Matrix representation of a linear transformation
- Range and Null space of a linear transformation
- Finding bases and dimensions of Range space and Nullity.

Matrix representation of linear transformation

dim n dim m $M(T) - m \times n$

Let T : V → W be a linear transformation. Let β = {$v_1, v_2, \dots, v_n$} be a basis of V and γ = {$w_1, w_2, \dots, w_m$} be a basis of W . So each $T(v_i)$ can be uniquely written as linear combination of w_j's, where i = 1, 2, . . . , n and j = 1, 2, . . . , m.

$T: \mathbb{R}^2 \to \mathbb{R}^3$

$\mathcal{B}_1 = \{\underset{v_1}{(1,0)}, \underset{v_2}{(0,1)}\}$

$\mathcal{B}_2 = \{\underset{w_1}{(1,0,0)}, \underset{w_2}{(0,1,0)}, \underset{w_3}{(0,0,1)}\}$

$T(x,y) = (2x-y, x+y)$

$T(1,0) = (2,0,1)$

$T(0,1) = (0,-1,1)$

$\underset{Tv_1}{T(1,0)} = \underline{2}\, w_1 + \underline{0}\, w_2 + \underline{1}\, w_3$

$\underset{Tv_2}{T(0,1)} = \underline{0}\, w_1 \underline{-1}\, w_2 + \underline{1}\, w_3$

$$T(v_1) = a_{11}w_1 + a_{21}w_2 + \dots + a_{m1}w_m$$
$$T(v_2) = a_{12}w_1 + a_{22}w_2 + \dots + a_{m2}w_m$$
$$\vdots$$
$$T(v_n) = a_{1n}w_1 + a_{2n}w_2 + \dots + a_{mn}w_m$$

The matrix corresponding to the linear transformation f with respect to the bases β and γ is given by,

$T: \mathbb{R}^2 \to \mathbb{R}^3$

$$A = \begin{array}{cc} Tv_1 & Tv_2 \\ \end{array}\begin{bmatrix} 2 & 0 \\ 0 & -1 \\ 1 & 1 \end{bmatrix}$$

$$T(x,y) = A\begin{bmatrix} x \\ y \end{bmatrix} = \underset{3\times 2}{\begin{bmatrix} 2 & 0 \\ 0 & -1 \\ 1 & 1 \end{bmatrix}}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x \\ -y \\ x+y \end{bmatrix}$$

$$\begin{array}{cccc} Tv_1 & Tv_2 & & Tv_n \end{array}$$
$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ & \vdots & \vdots & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

Exercise 1:

Consider the linear transformation

$T: \underset{\text{dim } 4}{M_{2\times 2}(\mathbb{R})} \to \underset{\text{dim } 4}{M_{2\times 2}(\mathbb{R})}$ such that $T(A) = PA$, where $P = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and a, b, c, d are in $\mathbb{R}$. $P = \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix}$

Let $\beta = \left\{ \underset{v_1}{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}, \underset{v_2}{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}, \underset{v_3}{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}}, \underset{v_4}{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}} \right\}$ be an ordered basis of $M_{2\times 2}(\mathbb{R})$. Which of the following matrices represents the matrix corresponding to the linear transformation T with respect to β for both domain and co-domain?

$Tv_1 = \underline{1}\, v_1 \underline{0}\, v_2 \underline{-1}\, v_3 \underline{0}\, v_4$

$Tv_2 = \underline{0}\, v_1 + \underline{1}\, v_2 + \underline{0}\, v_3 \underline{-1}\, v_4$

$Tv_3 = \underline{2}\, v_1 + \underline{0}\, v_2 + \underline{0}\, v_3 + \underline{0}\, v_4$

$Tv_4 = \underline{0}\, v_1 + \underline{2}\, v_2 + \underline{0}\, v_3 + \underline{0}\, v_4$

$$\begin{array}{cccc} Tv_1 & Tv_2 & Tv_3 & Tv_4 \end{array}$$
$$\underset{4\times 4}{\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}}$$

$TA = PA$

$TV_1 = \begin{bmatrix}1 & 2\\ -1 & 0\end{bmatrix}\overset{V_1}{\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}} = \begin{bmatrix}1 & 0\\ -1 & 0\end{bmatrix} = \underline{1}\,V_1 + \underline{0}\,V_2 - \underline{1}\,V_3 + \underline{0}\,V_4$

$TV_2 = \begin{bmatrix}1 & 2\\ -1 & 0\end{bmatrix}\overset{V_2}{\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}} = \begin{bmatrix}0 & 1\\ 0 & -1\end{bmatrix} =$

$TV_3 = \begin{bmatrix}1 & 2\\ -1 & 0\end{bmatrix}\overset{V_3}{\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}} = \begin{bmatrix}2 & 0\\ 0 & 0\end{bmatrix} =$

$TV_4 = \begin{bmatrix}1 & 2\\ -1 & 0\end{bmatrix}\overset{V_4}{\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}} = \begin{bmatrix}0 & 2\\ 0 & 0\end{bmatrix}$

$A = \begin{array}{cc} TV_1 & TV_2 \end{array}$
$A = \begin{bmatrix}1 & 0\\ -1 & 2\end{bmatrix}$

Exercise 2:

Let $\beta = \{\overset{v_1}{(1,0)}, \overset{v_2}{(0,1)}\}$ (standard basis) and $\gamma = \{\overset{\omega_1}{(1,1)}, \overset{\omega_2}{(1,-1)}\}$ be two bases of $\mathbb{R}^2$. If $\cancel{\begin{bmatrix}a & b\\ c & d\end{bmatrix}}$ is the matrix representation of a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ with respect to β for both the domain and the co-domain. Answer questions 7 and 8 using the information given above.

$TV_1 = 1V_1 - 1V_2 = (1,0) - (0,1) = (1,-1)$

$TV_2 = 0V_1 + 2V_2 = (0,0) + (0,2) = (0,2)$

$TV_1 = aV_1 + cV_2$

$TV_2 = bV_1 + dV_2$

. What is the matrix representation of T with respect to γ for both the domain and the co-domain?

$(x, y) = xV_1 + yV_2, \quad x, y \in \mathbb{R}$

$T(x,y) = xTV_1 + yTV_2 = x(1,-1) + y(0,2) = (x, -x+2y)$

$\boxed{T(x,y) = (x, -x+2y)}$

$T(x,y) = \begin{bmatrix}1 & 0\\ -1 & 2\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} = (x, -x+2y)$

$Tx = Ax$ only when A is repr. with st. basis for both dom & co-dom

$T\omega_1 = \underline{\alpha_{11}}\,\omega_1 + \underline{\alpha_{21}}\,\omega_2$

$T\omega_2 = \underline{\alpha_{12}}\,\omega_1 + \underline{\alpha_{22}}\,\omega_2$

$\begin{array}{cc} T\omega_1 & T\omega_2 \end{array}$
$\begin{bmatrix}\alpha_{11} & \alpha_{12}\\ \alpha_{21} & \alpha_{22}\end{bmatrix}$

$T\omega_1 = T(1,1) = (1,1)$

$T\omega_2 = T(1,-1) = (1,-3)$

$(1,1) = \alpha_{11}\overset{\omega_1}{(1,1)} + \alpha_{21}\overset{\omega_2}{(1,-1)} \Rightarrow \alpha_{11} = 1, \alpha_{21} = 0$

$(1,-3) = \alpha_{12}(1,1) + \alpha_{22}(1,-1) \Rightarrow \alpha_{12} = -1, \alpha_{22} = 2$

$[T]_\gamma = \begin{bmatrix}1 & -1\\ 0 & 2\end{bmatrix}$

What is the matrix representation of T with respect to γ $\{\omega_1, \omega_2\}$ for the domain and β $\{V_1, V_2\}$ for the co-domain?

Domain γ

$T\omega_1 = \underline{1}\,V_1 + \underline{1}\,V_2$

$T\omega_2 = \underline{1}\,V_1 - \underline{3}\,V_2$

co-dom β

$[T]_\gamma^\beta = \begin{array}{cc} T\omega_1 & T\omega_2 \end{array}$
$[T]_\gamma^\beta = \begin{bmatrix}1 & 1\\ 1 & -3\end{bmatrix}$

$T\omega_2 = (1,-3)$

HW: Find matrix of T w.r.t β for domain and γ for co-domain.

Null space and Range space of a linear transformation

Null sp (A) = {x | Ax = 0}

Kernel or nullspace of a linear transformation:

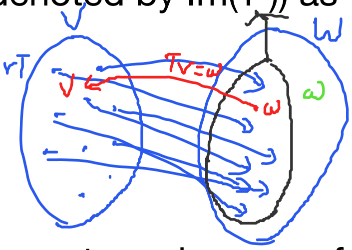
Let T : V → W be a linear transformation. We define kernel of T (denoted by ker(T)) to be the set of all vectors v in V such that T (v) = 0.

$$Ker(T) = \{v \in V \mid T(v) = 0\} \subseteq V$$

Image or range space of a linear transformation:

Let T : V → W be a linear transformation. We define image of T (denoted by Im(T)) as follows:

w is the image of v under T



$$Im(T) = \{w \in W \mid \exists v \in V \text{ for which } T(v) = w\}$$

$\subseteq W$ $= \{Tv : v \in V\}$ v is the pre-image of w under T

Theorem:

- Kernel or nullspace of a linear transformation T : V → W is a vector subspace of V .

$S \subseteq V$ To prove S is a subsp of V : $s_1, s_2 \in S$ $\Rightarrow \alpha s_1 + \beta s_2 \in S$

- Image or range space of a linear transformation T : V → W is a vector subspace of W . $w_1, w_2 \in Im(T) \Rightarrow$ There exists $v_1, v_2 \in V$ such that $Tv_1 = w_1$ and $Tv_2 = w_2$

$\alpha w_1 + \beta w_2 = \alpha T v_1 + \beta T v_2 = T(\alpha v_1 + \beta v_2)$

$v_1, v_2 \in V \Rightarrow \alpha v_1 + \beta v_2 \in V$ → preimage of ↓ So, Im(T) is a subspace of W.

Dimension of kernel or nullspace of linear transformation T is defined as **Nullity(T)** and dimension of image or range space of linear transformation T is defined as **Rank(T).** Rank(T) = Rank(A)

Exercise 3: Choose the set of correct options.

~~○~~ Option 1: There exists a linear transformation T : $R^2 \to R^2$ whose range is contained in the circle $x^2 + y^2 = 1$. $(0,0)$ $0^2 + 0^2 \neq 1$

✓○ Option 2: There exists a linear transformation T : $R^2 \to R^2$ whose range is equal the line x = y. $T(x,y) = (x,x)$ $Im(T) = \{(x,x) : x \in \mathbb{R}\} \to$ line $x = y$

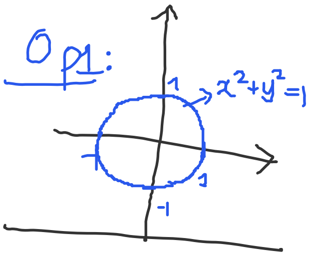
~~○~~ Option 3: There exists a linear transformation T : $R^2 \to R^2$ whose range is contained in the set S = {(x, y) | x > 0}.

✓○ Option 4: There exists a linear transformation T : $R^2 \to R^2$ whose range is equal the line y = πx.

Range (T) - Subsp of $\mathbb{R}^2$ — possible subspaces: a) {(0,0)} b) st. line thru (0,0) c) $\mathbb{R}^2$

check: 1) $(0,0) \in$ Range(T) 2) a or b or c

Op1:



$T: V \to W$

Op3:

$(0,0) \notin S$

Op4:

$T(x,y) := (x, \pi x)$

nullity = 0

one-one $\quad null(T) = \{0\} \iff Tx = Ty \Rightarrow x = y$

onto $\quad Range(T) = W \quad Rank(T) = \dim W$

iso : one-one & onto $\quad \dim V = \dim W$

Rank-nullity: Rank(T) + Nullity(T) = dim(V) $\quad \dim W + 0 = \dim V$

Exercise 4: Which of the following linear transformations is an isomorphism between the vector spaces V = {(x, y, z) | x = y − z, and x, y, z ∈ R} ⊂ R^3 and W = {(x, y, z) | x = y, and x, y, z ∈ R} ⊂ R^3 ?

$Ker T = \{v \in V : Tv = 0\}$

~~Option 1: T : V → W, such that T(x, y, z) = (y, y, 0).~~

$T(1, 0, -1) = (0, 0, 0)$ (y = 0; $Tv = 0$, $v \neq 0$) $\quad Ker T \neq \{0\}$ $\quad T \to$ not 1-1 $\to$ not iso

✓ Option 2: T : V → W, such that T(x, y, z) = (y, y, z).

~~Option 3: T : V → W, such that T(x, y, z) = (x, y, z).~~

$T(1, 0, -1) = (1, 0, -1)$ $\quad (1, 0, -1) \notin W$

~~Option 4: There does not exist any isomorphism between V and W.~~

$V: (y-z, y, z) = y(1, 1, 0) + z(-1, 0, 1)$ $\quad \dim V = 2$

$W: (x, x, z) = x(1, 1, 0) + z(0, 0, 1)$ $\quad \dim W = 2$

Op 2: $\quad Im(T) = \{(y, y, z) : y, z \in R\}$

$= y(1, 1, 0) + z(0, 0, 1)$

Basis of Im(T) = Basis of W

$Im(T) = W \Rightarrow T$ is onto

$T: V \to W$

If $\dim V = \dim W$, then T is one-one $\iff$ T is onto

dim(V) = dim(W) and T is onto $\Rightarrow$ T is 1-1

$\Rightarrow$ T is iso

Exercise 5: Which of the following linear transformations is an isomorphism
HW between the vector spaces $V = \{(x, y, z) \mid x = y - z = 0, \text{ and } x, y, z \in R\} \subset R^3$ and $W = \{(x, y, z) \mid x = y = 0, \text{ and } x, y, z \in R\} \subset R^3$?

○ Option 1: $T : V \to W$, such that $T(x, y, z) = (x, 0, y)$.

○ Option 2: $T : V \to W$, such that $T(x, y, z) = (0, y, z)$.

○ Option 3: $T : V \to W$, such that $T(x, y, z) = (0, 0, x)$.

○ Option 4: There does not exist any isomorphism between V and W.

Exercise 6: Consider the linear transformation $T : R^3 \to R^3$ such that $T(x, y, z) = (\underset{\alpha}{x - y + z}, \underset{\beta}{2x + y - z}, \underset{\gamma}{-x - 2y + 2z})$. Which of the following options are true about T?

✓ ○ Option 1: Range of T is $\{(x, y, z) \in R^3 \mid x = y + z\}$. $\{(y+z, y, z) : y, z\}$

✗ ○ Option 2: Range of T is $\{(x, y, z) \in R^3 \mid x = 0, y + z = 0\}$.

✓ ○ Option 3: Rank of T is 2.

✗ ○ Option 4: Rank of T is 1.

Op 1: check: $\alpha = \beta + \gamma$! Verified

$\dim(V)=n$ $\quad I: V \to V \quad I(v)=v \to$ Isomorphism

Exercise 7: Choose the set of correct options.

Rank$(I)=n$ $\quad$ Nullity$(I)=0$

$Im(I)=V$ $\quad$ Rank$(I)=\dim(Im(I))=\dim(V)=n$

- Option 1: Nullity and rank of the identity transformation on a vector space of dimension n are 0 and n respectively.

- Option 2: Nullity and rank of the identity transformation on a vector space of dimension n are 1 and n – 1 respectively.

- Option 3: Nullity and rank of the identity transformation on a vector space of dimension n are n and 0 respectively.

$T: \overset{n}{V} \to \overset{n}{W}$

- Option 4: Nullity and rank of an isomorphism between two vector spaces V and W (both of dimension n) are n and 0 respectively.

- Option 5: Nullity and rank of an isomorphism between two vector spaces V and W (both of dimension n) are 0 and n respectively.

- Option 6: There cannot exist an isomorphism between two vector spaces whose dimensions are not the same.

Inverse of a linear transformation

Consider a linear transformation T : $R^n \to R^n$. When and how can we define the inverse of a linear transformation?

A - $n \times n$ $\quad A: A^{-1}$ exists if $\det(A) \neq 0$. $\quad AA^{-1} = A^{-1}A = I_n$

$\det(A) \neq 0 \Rightarrow$ Rank$(A) = n$, nullity$(A) = 0$

$\Downarrow$

Given $T: \mathbb{R}^n \to \mathbb{R}^n$, how to find T^{-1}?

$\det(A) \neq 0$

Rank$(T) = n$, nullity$(T)=0$

$\Downarrow$

$A \to$ matrix of T w.r.t std bases.

T is an isomorphism

$T^{-1}x = A^{-1}x$

T^{-1} exists if T is an isomorphism

$TT^{-1} = T^{-1}T = I$

A^{-1} $\quad T^{-1}$

Exercise 8: Let $T : R^4 \to R^4$ be a linear transformation. Suppose $v_1 \neq v_2$ are two distinct
non-zero solutions of $T(x) = 0$. What is the maximum value that rank(T) can take? 3

$0 \neq v_1, v_2 \in \ker(T) \Rightarrow \text{nullity}(T) \geq 1$

$v_1 = \alpha v_2$

$\text{nullity}(T) + \text{Rank}(T) = 4$

$1 + 3$

$v_1 = (1,0,0,0)$ $v_2 = (2,0,0,0)$ $v_3 = (3,0,0,0)$ $v_1, v_2, v_3 \in \ker(T)$

$\Rightarrow \{v_1, v_2, v_3\} \to$ L.D.

Finding basis for null space and range space by row reduced echelon form

Let $T : V \to W$ be a linear.

Follow the steps below to obtain a basis for nullspace of T:

Step 1: Find the matrix A corresponding to T with respect to some standard ordered bases $\beta = \{v_1, v_2, \ldots, v_n\}$ and $\gamma = \{w_1, w_2, \ldots, w_m\}$ for V and W respectively.

Step 2: Use row reduction on A to obtain the matrix R which is in reduced row echelon form.

Step 3: The basis of the solution space of $Rx = 0$ is the basis of null space of matrix A and can be obtained by finding the pivot and non-pivot columns (dependent and independent variables) as seen earlier.

Follow the steps below to obtain a basis for range space of T:

Follow step 1 and step 2 as above:

Step 3: Recall that if $i_1, i_2, \ldots, i_r$ are the columns of R containing the pivot elements, then the same columns of A form a basis for the column space of A.

Exercise 9: Consider the following linear transformation:

$T : R^4 \to R^3$ $T(x_1, x_2, x_3, x_4) = (2x_1 + 4x_2 + 6x_3 + 8x_4, x_1 + 3x_2 + 5x_4, x_1 + x_2 + 6x_3 + 3x_4)$

Find a basis for the nullspace and range space of T.

$v_1, v_2, v_3, v_4 \to$ std basis of $\mathbb{R}^4$

Find Tv_1, Tv_2, Tv_3, Tv_4

$$A = \begin{array}{cccc} Tv_1 & Tv_2 & Tv_3 & Tv_4 \\ \end{array}$$
$$A = \begin{bmatrix} 2 & 4 & 6 & 8 \\ 1 & 3 & 0 & 5 \\ 1 & 1 & 6 & 3 \end{bmatrix} \underset{\text{Hw: verify}}{\xrightarrow{RREF}} \begin{array}{cccc} \text{1st} & \text{2nd} & & \\ \end{array}$$
$$R = \begin{bmatrix} \boxed{1} & 0 & 9 & 2 \\ 0 & \boxed{1} & -3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Rank $(T) = 2$

Nullity $(T) = 2$

$Rx = 0$

$$x_1 + 9x_3 + 2x_4 = 0$$
$$x_2 - 3x_3 + x_4 = 0$$
$$x_1 = -9x_3 - 2x_4 \qquad x_3 = t_1 \quad x_4 = t_2$$
$$x_2 = 3x_3 - x_4 \qquad \text{free varibles}$$

$$\text{Nullsp}(T) = \{(-9t_1 - 2t_2,\ 3t_1 - t_2,\ t_1, t_2) \mid t_1, t_2 \in \mathbb{R}\}$$
$$= \{t_1(-9, 3, 1, 0) + t_2(-2, -1, 0, 1) \mid t_1, t_2 \in \mathbb{R}\}$$

Basis of Nullsp $(T) = \{(-9, 3, 1, 0), (-2, -1, 0, 1)\}$

Basis of Range$(T) = \{$columns of A containing pivot$\}$
$$= \{(2, 1, 1), (4, 3, 1)\}$$

Rank-nullity theorem

Let $T : V \to W$ be a linear transformation. Then Rank(T) + Nullity(T) = dim(V).

Comparison with matrix case: Let A be an $m \times n$ matrix. Then rank (A)+nullity (A) = n.

1-1 $\Rightarrow$ Ker(T) = {0} $\Rightarrow$ nullity (T) = 0 $\Rightarrow$ Rank (T) + null (T) = dim(V)

$\Rightarrow$ Rank (T) = dim(V)

Let $T : V \to W$ be a linear transformation.

1-1

- If T is injective, then Rank(T) = dim(V).
- If T is an isomorphism, then dim(W) = dim(V).

Exercise 10: Find the rank and nullity of the linear transformation

$T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ $(x, y) \to (0, x)$ $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ Range(T) = {(0, x) : x ∈ ℝ}

T(1,0) = (0,1) T(0,1) = (0,0)

and verify the rank nullity theorem.

= Span{(0,1)}

rank (T) = 1 Nullity (T) = 1

$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

1 + 1 = 2 ✓

$\Rightarrow x_1 = 0$

Null(T) = {(0, x_2) : x_2 ∈ ℝ}

Exercise 11: Can there be a linear transformation such that Ker(T) = Im(T)?

Yes

Exercise 12: Consider a linear transformation $T : V \to W$ such that $\dim(V) = 5$, $\dim(W) = 7$ and T is one-to-one. Find Rank(T) and Nullity(T).

5 0

Exercise 13: Consider a linear transformation $T : V \to W$ such that $\dim(V) = 5$, $\dim(W) = 3$ and $\dim(\text{Ker}(T))$ is 2. Then show that T is onto. ✓

nullity

$\text{Rank}(T) = \underbrace{5 - 2}_{\dim V - \text{nullity}} = 3 = \dim(W)$

dim
4×3

Exercise 14: Can we find a linear transformation $T : R^4 \to R^3$ with kernel $\{0\}$?

Suppose $\text{Ker}(T) = \{0\} \Rightarrow \text{nullity}(T) = 1$, $\text{Rank}(T) = 3 \Rightarrow T$ is onto

$\Rightarrow T$ is iso

not possible

Exercise 15: Can we find a linear transformation $T : R^3 \to R^4$ with Range(T) = Co-domain(T)? $\to$ T is onto

$\text{Rank}(T) = 4$ not possible

$T : R^m \to R^n$ $\boxed{\text{Rank}(T) \le \min\{m, n\}}$

Observation:

Let $T: V \to W$ be a linear transformation.

- If $\dim(W) \le \dim(V)$, then T cannot be 1-1
- If $\dim(V) \le \dim(W)$, then T cannot be onto

Exercise 16: Choose the set of correct options.

○ Option 1: Any injective linear transformation between any two vector spaces which have the same dimensions, must be an isomorphism.

○ Option 2: Any surjective linear transformation between any two vector spaces which have the same dimensions, must be an isomorphism.

○ Option 3: There does not exist any surjective linear transformation from R^2 to R^3.

○ Option 4: There does not exist any injective linear transformation from R^2 to R^3.

Exercise 17: Choose the set of correct options.

- Option 1: There exists a linear transformation $T : R^2 \to R^2$ such that Image(T)= Kernel(T).
- Option 2: There exists a linear transformation $T : R^3 \to R^3$ such that Image(T)= Kernel(T).
- Option 3: If $T : V \to V$ is a linear transformation and $v_1, v_2 \in V$ are linearly independent then $T(v_1)$, $T(v_2)$ are also linearly independent.
- Option 4: If $T : V \to V$ is a linear transformation and $T(v_1)$, $T(v_2)$ are linearly independent then $v_1, v_2 \in V$ are also linearly independent.

$T: \mathbb{R}^n \to \mathbb{R}^m$ $\quad$ $\underset{m\times n}{A} \to$ matrix of T w.r.t std basis for both domain & codomain

$x \in \mathbb{R}^n$

$Tx = Ax$

$T: \mathbb{R}^3 \to \mathbb{R}^2$

$$Ax = \begin{array}{c} \begin{array}{ccc} c_1 & c_2 & c_3 \end{array} \\ \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \end{array} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \end{bmatrix}$$

$$= x_1 \overset{c_1}{\begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix}} + x_2 \overset{c_2}{\begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix}} + x_3 \overset{c_3}{\begin{bmatrix} a_{13} \\ a_{23} \end{bmatrix}}$$

$Ax =$

$Ax \to$ lin. comb of columns of A

$Im(T) = \{Tx : x \in \mathbb{R}^n\}$ $\quad$ $Tx = Ax$

= "all possible lin. comb" of columns of A

= span(columns of A) = column space of A

$T: V \to W$ $\quad$ $\dim(V) = \dim W = k$

$\beta_V = \{v_1, v_2, \ldots, v_k\}$ $\quad$ $T(v_i) = w_i, \ 1 \le i \le k$

$\beta_W = \{w_1, w_2, \ldots, w_k\}$ $\quad$ $T \to$ isomorphism