Week 5.

- Null space of a matrix, Nullity of a matrix.

Consider the following system of linear Eq^n's :

$$3x + 4y + 3z = 0$$
$$x + y + z = 0$$

- Homogenous system of linear Eq^n's.
- Coefficient matrix

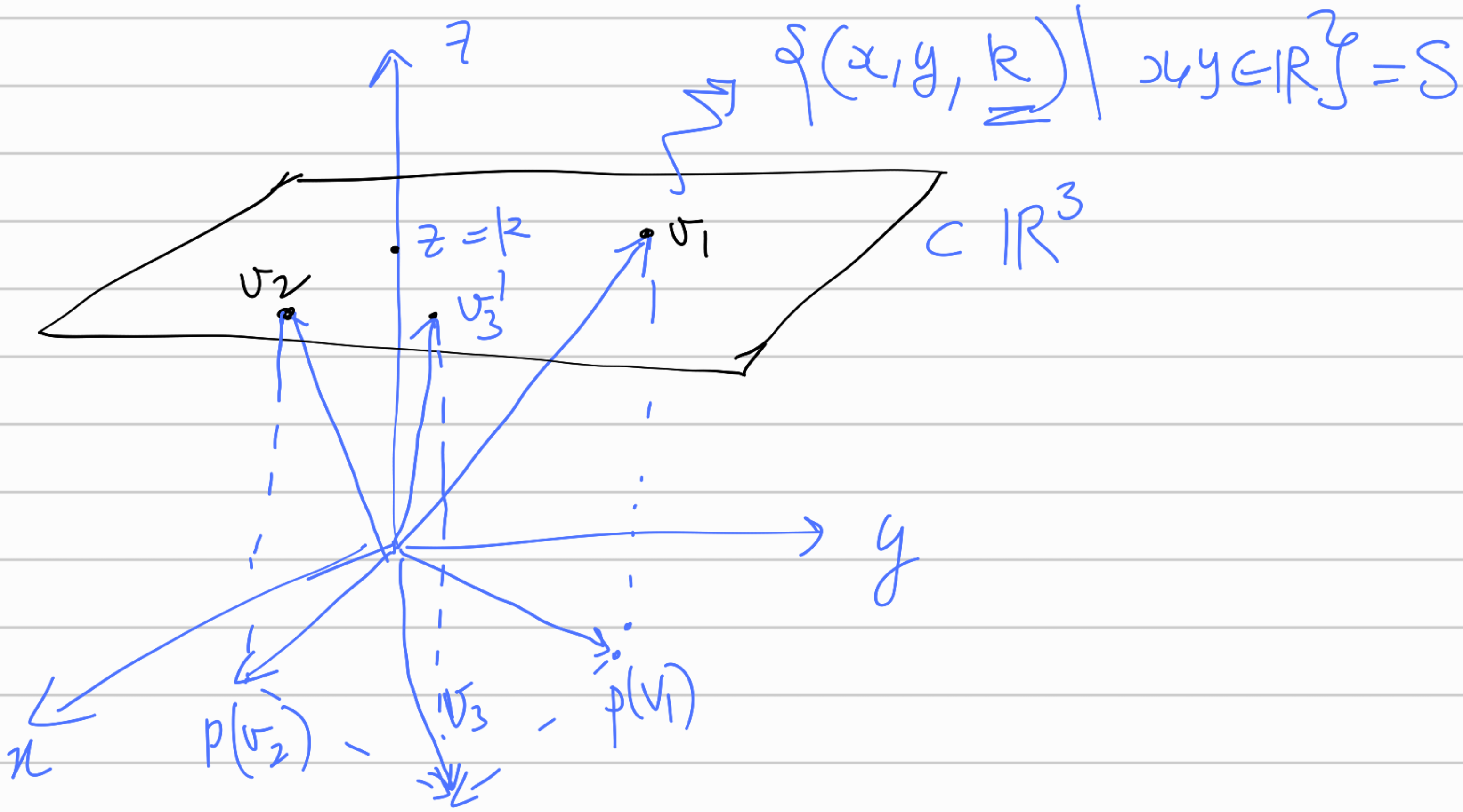
$$A = \begin{bmatrix} 3 & 4 & 3 \\ 1 & 1 & 1 \end{bmatrix}$$

- In terms of coefficient matrix

$$A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \bar{b}$$

How to find sol^n of above : ?
- Guassian elimination.

$$\underset{m \times (n+1)}{[A \mid b]} \xrightarrow{\text{Row operations}} [\text{RREF}(A) \mid b]$$



Lecture 3.2.

S = Affine flats := parallel copy of xy plane in $\mathbb{R}^3$.

$v_1, v_2 \in S$

$\mathrm{proj}_{xy}(v_1)$, $\mathrm{proj}_{xy}(v_2)$,

$$\mathrm{proj}_{xy}(v_1+v_2) := \mathrm{proj}_{xy}(v_1) + \mathrm{proj}_{xy}(v_2)$$

$$(x_1, y_1, k) \oplus (x_2, y_2, k) := (x_1+x_2, y_1+y_2, k)$$

$$(x_1, y_1, 0) + (x_2, y_2, 0) = (x_1+x_2, y_1+y_2, 0)$$

$$\ell \cdot (x, y, k) := (\ell x, \ell y, k)$$

$$\ell \cdot (x, y, 0) = (\ell x, \ell y, 0)$$

$$[A|b] = \left[\begin{array}{ccc|c} 3 & 4 & 3 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 3 & 4 & 3 & 0 \end{array}\right]$$

$$\downarrow R_2 - 3R_1$$

$$\underbrace{\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right]}_{\text{RREF}} \xleftarrow{R_1 - R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right]$$

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightsquigarrow \begin{array}{l} x + z = 0 \\ \quad y = 0 \end{array}$$

$$\text{Sol}^n \text{ set} = \{(-z, 0, z) \mid z \in \mathbb{R}\}$$

In particular,
the sol^n of homogeneous system of linear eq^n $= \underbrace{\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; A\bar{x} = \bar{0} \right\}}_{\text{Null space of } A.}$

In general:

the subspace $W = \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n \;\middle|\; A \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \right\} \subset \mathbb{R}^n$

is called the
(i) sol^n space of the homogenous system $A\bar{x} = 0$

or (ii) the null space of A.

* Null space of $A_{m\times n}$ is a subspace of $\mathbb{R}^n$?

$$\underbrace{A_{m\times n}\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

$v_1, v_2 \in$ Null space of $A_{m\times n}$

$(v_1 + v_2) \in$ Null sp.

Addition: $A(v_1+v_2) = 0$

$A(v_1+v_2) = Av_1 + Av_2 = 0 + 0 = 0$

$\Rightarrow A(v_1+v_2) = 0$

$\Rightarrow v_1 + v_2 \in \text{Nullsp}(A)$

Scalar: $k \in \mathbb{R}$, $v \in \text{Nullsp}(A)$.

$kv \in \text{Nullsp}(A)$.

$A(kv) = k(Av) = k\,0. = 0 \Rightarrow kv \in \text{Nullsp}(A)$

* $\text{Nullsp}(A_{m\times n})$ is a subsp of $\mathbb{R}^n$.

Finding null space of A:

Guassian elimination:

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 2 & 2 & 2 & 0 \\ -4 & -4 & -4 & 4 \end{bmatrix} \xrightarrow{R_2 - 2R_1} \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -4 & -4 & 4 & 4 \end{bmatrix}$$

$$A \begin{pmatrix} x_1 \\ \vdots \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\xrightarrow{R_3 + 4R_1} \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} \xrightarrow{R_3/4} \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R_2 \Leftrightarrow R_3} \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\mathrm{RREF}(A) \Rightarrow \begin{aligned} x_1 + x_2 + x_3 &= 0 \\ x_4 &= 0 \end{aligned}$$

$$x_1 = -(x_2 + x_3)$$

$$\text{Sol}^n \text{ set} = \{(-x_2 - x_3, x_2, x_3, 0) \mid x_2, x_3 \in \mathbb{R}\}$$

OR, $\text{Null sp}(A_{3\times 4})$ //

$$\text{Null sp.}(A_{3\times4}) = \{(-x_2 - x_3, x_2, x_3, 0) \mid x_2, x_3 \in \mathbb{R}\}$$

method 1:

$$(-x_2 - x_3, x_2, x_3, 0) = (-x_2, x_2, 0, 0) + (-x_3, 0, x_3, 0)$$

$$x_2(-1, 1, 0, 0) + x_3(-1, 0, 1, 0)$$

$$= \text{span}\{(-1,1,0,0), (-1,0,1,0)\}$$

method 2:

$(-x_2 - x_3, x_2, x_3, 0)$, $(-x_2 - x_3, x_2, x_3, 0)$

$x_2 = 1, x_3 = 0$ $\qquad$ $x_2 = 0, x_3 = 1$

$(-1, 1, 0, 0)$ $\qquad$ $(-1, 0, 1, 0)$

$$\text{Null sp}(A_{3\times5}) = \{(-x_1, -x_3 - x_4, x_3, x_4, 0) \mid x_1, x_3, x_4 \in \mathbb{R}\}$$

$(-x_1, -x_3 - x_4, x_3, x_4, 0)$, $(-x_1, -x_3 - x_4, x_3, x_4, 0)$

$x_1 = 1, \quad x_3 = x_4 = 0$ $\qquad$ $x_3 = 1, \quad x_1 = x_4 = 0$

$(-1, 0, 0, 0, 0)$ $\qquad$ $(0, -1, 1, 0, 0)$

$x_4 = 1, \quad x_1 = x_3 = 0$

$(0, -1, 0, 1, 0)$

Stepwise:

$$A_{m\times n} \rightsquigarrow \underset{m\times n}{A}\underset{n\times 1}{\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}} = \underset{m\times 1}{\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}}$$

1) Write the augumented matrix
$[A|0]$

2) Apply row reduction to obtain
$[RREF(A)|0]$

3) Determine independent variables & dependent variables.

$$\boxed{\text{Def}^{n} \ \text{Nullity}(A) = \#\text{ of independent variable}}$$

4) Express dependent variables in terms of independent from unique rows they occurs in RREF(A).

5.) Suppose $x_{j_1}, .., x_{j_k}$ are independent variable. Then define basis:

$$v_i = (f_1, .., f_n)$$

by taking $x_{j_i} = 1$ and other $x_{j_{i'}} = 0$.

Remark:

In augumented matrix $[A|0]$, last column is zero.

The last column matrix remain unchanged while performing row operations.

So, to compute null space & nullity, we may drop last column in computation.

Exercise: find the nullity of A and basis of nullspace of A.

$$A = \begin{bmatrix} 3 & 1 & 2 & -1 \\ 6 & 2 & 4 & -2 \\ -1 & -1/3 & -2/3 & 1/3 \end{bmatrix}$$

Rank = 1

Nullity = 3

$$\xrightarrow{R_2 - 2R_1} \begin{bmatrix} 3 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \\ -1 & -1/3 & -2/3 & 1/3 \end{bmatrix}$$

$$\xrightarrow{R_1/3} \begin{bmatrix} 1 & 1/3 & 2/3 & -1/3 \\ 0 & 0 & 0 & 0 \\ -1 & -1/3 & -2/3 & 1/3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1/3 & 2/3 & -1/3 \\ -1 & -1/3 & -2/3 & 1/3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_2 + R_1} \begin{bmatrix} 1 & 1/3 & 2/3 & -1/3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{Null sp}(A) = \left\{ \left(-\tfrac{1}{3}x_2 - \tfrac{2}{3}x_3 + \tfrac{1}{3}x_4, x_2, x_3, x_4\right) \in \mathbb{R}^4 \right\}$$

$$\begin{pmatrix} 1 & 1/3 & 2/3 & -1/3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$x_1 + \frac{x_2}{3} + \frac{2x_3}{3} - \frac{x_4}{3} = 0 \Rightarrow x_1 = -\frac{x_2}{3} - \frac{2x_3}{3} + \frac{x_4}{3}.$$

$$\{(-1/3, 1, 0, 0), (-2/3, 0, 1, 0), (1/3, 0, 0, 1)\}$$

Recall: (from last session):

$A_{m \times n}$

Rank(A) = # of non-zero rows in RREF(A)

= # of dependent variables in RREF(A).

Since,

$$\underbrace{\text{\# of dependent variable}} + \underbrace{\text{\# of independent variable}} = \underbrace{\text{Total \# of variable}}$$

$$\boxed{\text{Rank}(A) + \text{Nullity}(A) = n \; (= \text{\# of columns})}$$

↳ Rank-Nullity Theorem.

Checking if set of n-vectors form a basis of $\mathbb{R}^n$ or not?

Recall: basis:

$$A = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} \quad ; \quad \det A = ?$$

$$A^T = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix}$$

$\det A = 0 \iff$ there is a non-zero vector $\bar{x} \in \mathbb{R}^n$ s.t. $A\bar{x} = 0$.

$\iff \text{Nullity}(A) \geq 1$

$0 \leq \text{Rank}(A) \leq n \quad \Leftarrow \left[\ \text{Rank}(A) + \text{Nullity}(A) = n \right.$

$\Rightarrow \text{Rank}(A) \leq n-1$

$\det A \neq 0 \iff \text{Rank}(A) = n$

$\Updownarrow$

If $\text{Nullity}(A) = 0$ then $A\bar{x} = 0$ has unique sol^n.

If $\text{Nullity}(A) > 0$ then $A\bar{x} = 0$ has infinite sol^n.

Example 1:

$\{(1,0,3), (4,1,2), (3,3,2)\}$ in $\mathbb{R}^3$.

determinant.

$$\det \begin{bmatrix} 1 & 0 & 3 \\ 4 & 1 & 2 \\ 3 & 3 & 2 \end{bmatrix}_{3\times 3}$$

$$= 1(2-6) - 0(8-6) + 3(12-3)$$

$$= -4 + 27 = 23.$$

Example ② :

$$\{(1,1,1,1), (2,2,2,2), (3,0,3,0), (0,4,0,4)\}$$

in $\mathbb{R}^4$.

$$\det\begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 3 & 0 & 3 & 0 \\ 0 & 4 & 0 & 4 \end{pmatrix}$$