Maths 2 D.S. Term 3-2025.

Today's Content.

- Vectors
- Visulization of a vector in $\mathbb{R}^2$
- Vectors in $\mathbb{R}^n$.
- Vectors addition
- Matrices
    - Square matrix
    - Diagonal matrix
    - Scalar matrix
    - Identity matrix.
- Operation on matrices
    - Addition
    - Scalar multiplication
    - multiplication
    - Some properties of matrices
- System of linear Eq^n's.
    - Augmented matrix

Vector:

A list of numbers / data points (real number).

* Column of a table to a vector:

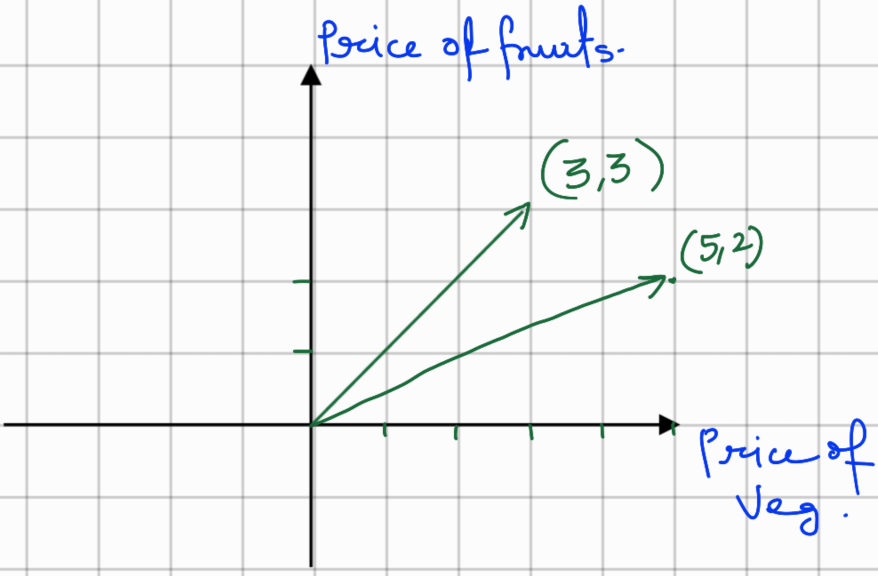
$$\begin{array}{ll} & \text{Tempature recorded at 3PM} \\ \text{Mon} & \\ \text{Tues} & \\ \text{Wed} & \end{array} \begin{bmatrix} 37°c \\ 40°C \\ 35°C \end{bmatrix} \rightsquigarrow (37, 40, 35)$$

* Row of a table to a vector:

	Veg.	Fruits.
A →	(100 ,	50)
B →	(80	40)
C →	(70	35)

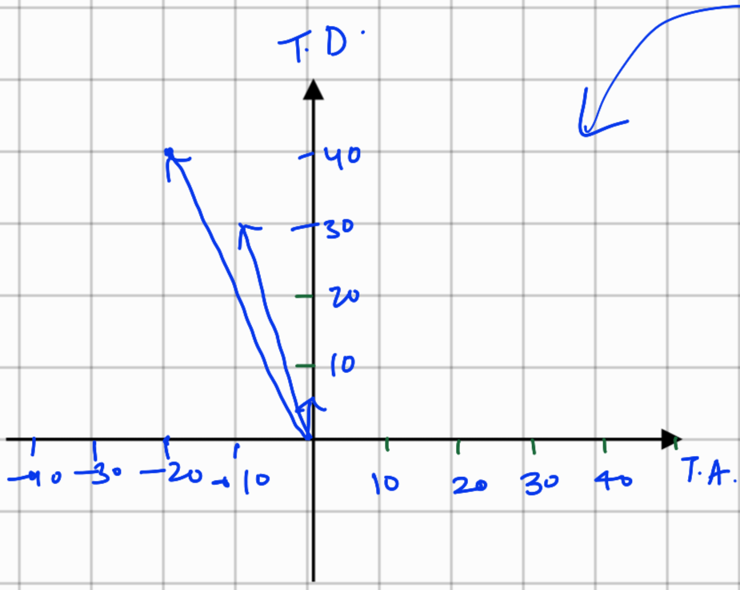
$$A + B = (100, 50) + (80, 40) = (180, 90)$$

Vector in $\mathbb{R}^2$



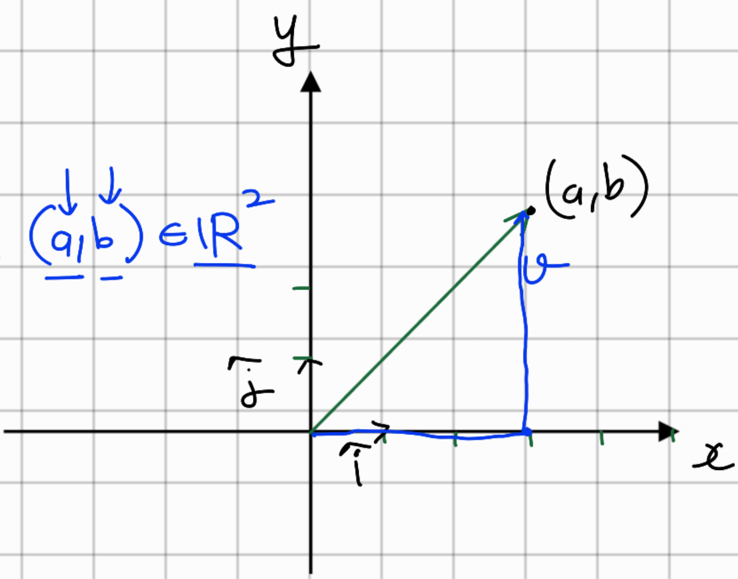
$(5,2)$

Geometric



	Tempature at Antartica	tempature at Delhi
May.	-20°	40°
Dec.	0°	5°
Mar	-10°	30°

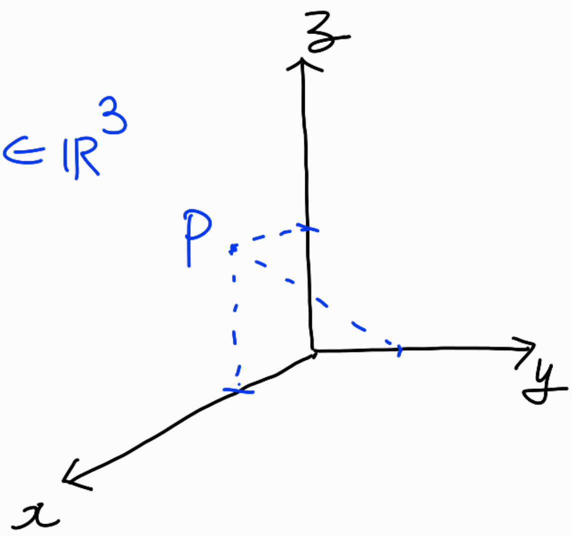
$$v = \underbrace{\boxed{a}\,\hat{i} + b\,\hat{j}} \equiv (a,b) \in \mathbb{R}^2$$



Vector in $\mathbb{R}^n$:

$n=2$: $\quad v = (a, b)$

$n=3$: $\quad v = (a, b, c) \in \mathbb{R}^3$



$v = (a, b, c, d) \in \mathbb{R}^4$

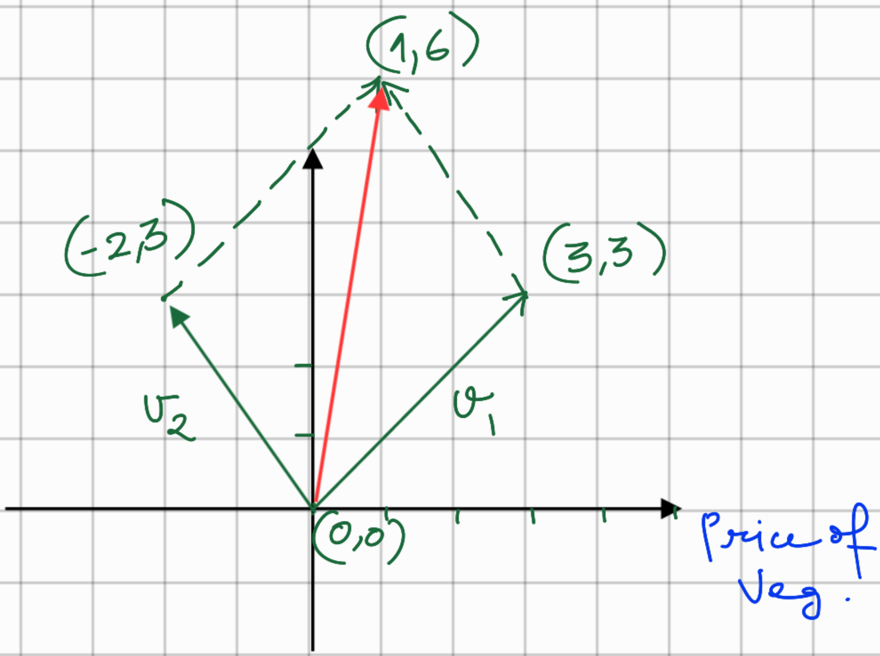
$\vdots$

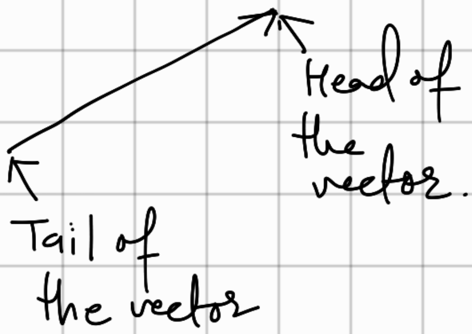
$v = (a_1, a_2, \ldots, a_n) \in \mathbb{R}^n$

$n=2 \qquad (a_1, a_2)$

$n=3 \qquad (a_1, a_2, a_3)$

Addition of two vectors in $\mathbb{R}^2$:





$$v_1 + v_2 = (3,3) + (-2,3) = (3-2, 3+3) = (1,6)$$

$$\left.\begin{aligned} v_1 &= (a_1, a_2, \dots, a_n) \in \mathbb{R}^n \\ v_2 &= (b_1, b_2, \dots, b_n) \in \mathbb{R}^n \end{aligned}\right\}.$$

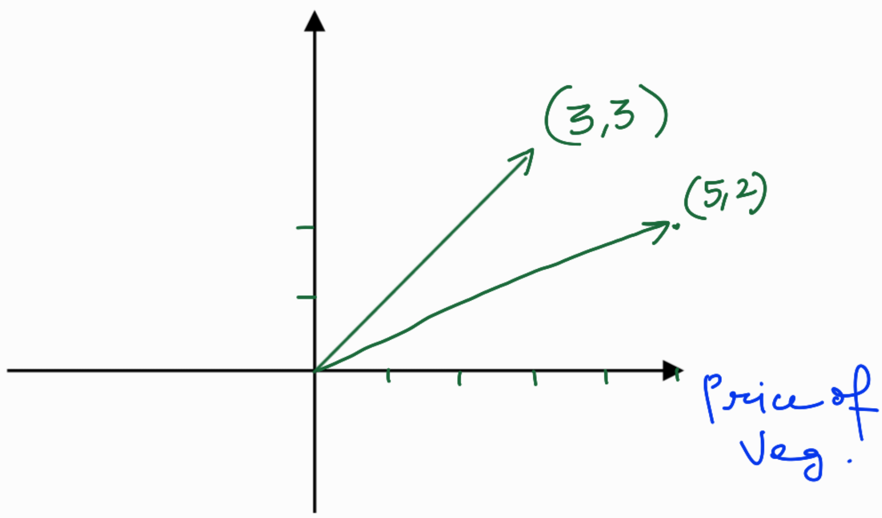
$$\begin{aligned} v_1 + v_2 &= (a_1, \dots, a_n) + (b_1, \dots, b_n) \\ &= (a_1 + b_1, a_2 + b_2, \dots, a_n + b_n). \end{aligned}$$

Some more examples of vectors:

Vectors in physics

- Velocity,
- Acceleration,
- Force,

<u>Magnitude</u> and <u>direction</u>.



<u>vector</u>:

$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}, \quad \begin{pmatrix} a_1 & a_2 & \cdots & a_n \end{pmatrix} \rightarrow \underline{(a_1, a_2, \ldots, a_n)} \in \mathbb{R}^n$$

Column of number

Row of number

Matrix:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 7 \end{bmatrix}_{2\times 3}, \quad \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}_{3\times 1},$$

↳ Column matrix

$$\begin{bmatrix} 10 & 15 & 3 \end{bmatrix}_{1\times 3} \longrightarrow \text{Row matrix}$$

Order of a matrix: # of rows × # of column

$$A = \begin{bmatrix} 3 & 4 \\ 5 & 6 \\ 7 & 8 \end{bmatrix}_{3\times 2}; \quad A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times 3}; \quad A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}_{2\times 2}$$

↓ square matrix

$(ij)^{th}$ element of a matrix:

ij^{th} entry of a matrix is entry at i^{th} row and j^{th} column.

For a matrix A

$$a_{ij} := (A)_{ij}$$

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2\times 2}, \quad (B)_{12} = 0.$$

* An $m \times n$ - matrix has <u>m rows</u> and <u>n columns</u>.

* A $m \times n$- matrix is called square matrix if $m=n$. (i.e., # of rows = # of columns).

* $$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$; e.g. 3^{rd} diagonal entry $= 9$.

↳ Diagonal entries.

* $(i,i)^{th}$ entry of a matrix is called i^{th}-diagonal entry.

Diagonal matrix:

A square matrix is called a diagonal matrix if all <u>non-diagonal</u> entries are zero.

For example.

1. $\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}_{3\times 3}$, 2 $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}_{2\times 2}$

3. $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ → diagonal matrix.

↳ zero matrix.

Non-diagonal matrix:

$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 & 3 \\ 0 & 3 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

Scalar matrix:

A diagonal matrix in which all the entries in the diagonal are equal is called a <u>scalar matrix</u>.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} -3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & -3 \end{bmatrix}, \quad \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$$

Scalar   Not a Scalar but Diagonal   Scalar

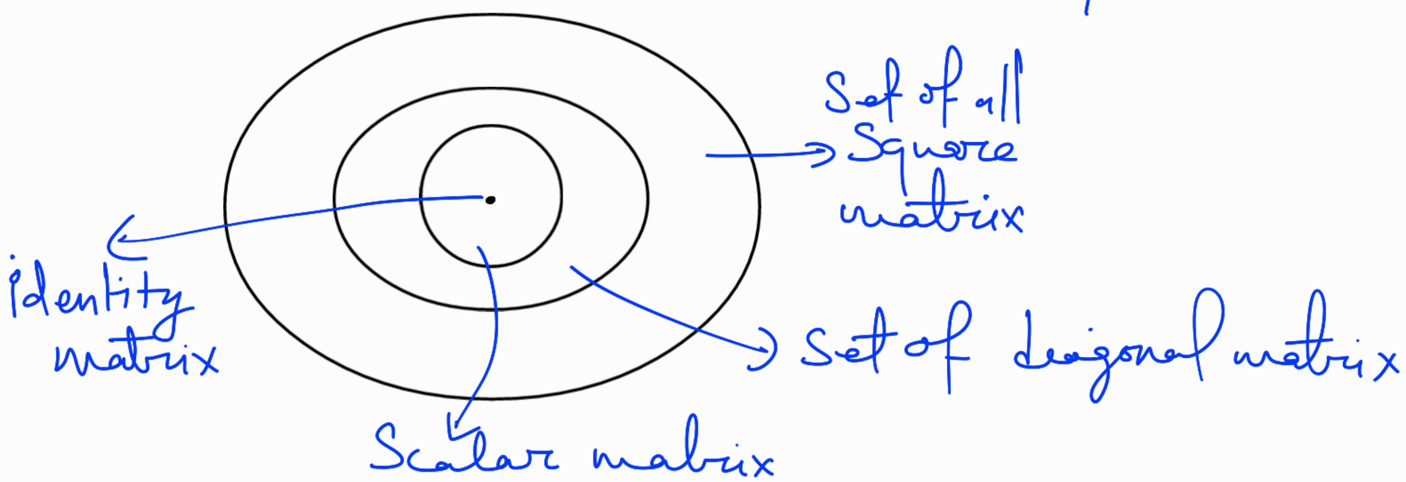
Identity matrix:

A scalar matrix whose diagonal entries are 1.

$n=2$, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$

$n=3$, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$

$$I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$



Matrix Addition:

* $[3]_{1\times1} = 3$

* $$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}_{3\times2} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix} = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ \vdots & \vdots \end{bmatrix}$$

Suppose A and B are two $m\times n$- matrices.
Then

$$(A+B)_{ij} = (A)_{ij} + (B)_{ij}$$

$$\begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix} + \begin{bmatrix} 3 & 7 \\ 8 & 9 \end{bmatrix} = \begin{bmatrix} 4 & 9 \\ 12 & 14 \end{bmatrix}$$

$$\underline{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}_{3\times1}} + \underline{\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}_{1\times3}} = \text{Not possible}$$

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}_{3\times1} + \begin{bmatrix} 1 \\ 2 \end{bmatrix}_{2\times1} = \text{Not possible}$$

$$\begin{bmatrix} 1 \\ 3 \end{bmatrix} + \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$$

Properties of matrix addition:

Suppose A and B are two $(m \times n)$-matrices.

1. Associative:

$$\boxed{(A+B)+C = A+(B+C)}$$

For example:

$$\left(\overset{A}{\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}} + \overset{B}{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}\right) + \overset{C}{\begin{bmatrix} 3 & 4 \\ 1 & -1 \end{bmatrix}} = \begin{bmatrix} 2 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 3 & 4 \\ 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 6 \\ 4 & 3 \end{bmatrix}$$

$$\underset{A}{\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}} + \left(\underset{B}{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}} + \underset{C}{\begin{bmatrix} 3 & 4 \\ 1 & -1 \end{bmatrix}}\right) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 4 & 4 \\ 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 6 \\ 4 & 3 \end{bmatrix}$$

2. Commutative:

$$\boxed{A+B = B+A}$$

Scalar multiplication:

Suppose A is $(m\times n)$-matrix and $k\in\mathbb{R}$.

Then

$$\boxed{(kA)_{ij} = k\cdot(A)_{ij}}$$

For example:

$$2\cdot\begin{bmatrix} 1 & 2 & -3 \\ -1 & 4 & 5 \end{bmatrix} = \begin{bmatrix} 2 & 4 & -6 \\ -2 & 8 & 10 \end{bmatrix}$$

Distributive property with addition:

For two $(m \times n)$-matrices A and B, and $k \in \mathbb{R}$,

$$\boxed{k(A+B) = kA + kB.}$$

Eg:

$$2\left(\overset{A}{\begin{bmatrix} 1 & 2 & 3 \\ -1 & 4 & 5 \end{bmatrix}} + \overset{B}{\begin{bmatrix} 1 & 4 & 5 \\ -1 & -1 & -1 \end{bmatrix}}\right)$$

$$= 2\begin{pmatrix} 2 & 6 & 8 \\ -2 & 3 & 4 \end{pmatrix} = \begin{pmatrix} 4 & 12 & 16 \\ -4 & 6 & 8 \end{pmatrix}$$

$$2A = \begin{pmatrix} 2 & 4 & 6 \\ -2 & 8 & 10 \end{pmatrix}, \quad 2B = \begin{pmatrix} 2 & 8 & 10 \\ -2 & -2 & -2 \end{pmatrix}$$

$$2A + 2B = \begin{pmatrix} 4 & 12 & 16 \\ -4 & 6 & 8 \end{pmatrix}$$

Proof :-

$$[k(A+B)]_{ij} = k(A+B)_{ij}$$

$$= k\{(A)_{ij} + (B)_{ij}\}$$

$$= k(A)_{ij} + k(B)_{ij}$$

$$\Rightarrow k(A+B) = k(A+B)$$

$\square$

Matrix Multiplication:

Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} -1 & -4 \\ 0 & 0 \end{bmatrix}$

$$(A.B)_{ij} = \sum_{k=1}^{2} A_{ik} \cdot B_{ki}$$

$$A.B = \begin{bmatrix} -1 & -4 \\ -3 & -12 \end{bmatrix}$$

Let A be $(m \times p)$-matrix and B be $(p \times n)$ matrix. Then

(i) $A_{m\times p} B_{p \times n}$ is well defined

(ii) $(AB)_{ij} = \sum_{k=1}^{p} (A)_{ik} (B)_{kj}$.

* Find out in which of the following case matrix multiplication is defined:

			order
(i)	$A_{2\times 3}$ and $B_{3\times 2}$,	BA = Yes	3×3
(ii)	$A_{4\times 3}$ and $B_{3\times 5}$,	BA = No	$\times$
(iii)	$A_{3\times 3}$ and $B_{3\times 1}$,	AB = Yes	3×1

Properties : Suppose $A_{m\times n}, B_{n\times p}, C_{p\times q}$ matrix. Then

(i) Associative :

$$\boxed{(AB)C = A(BC)}$$

$\longrightarrow (AB)_{m\times p}\, C_{p\times q} = (ABC)_{m\times q}$

$\longrightarrow A_{m\times n}\, (BC)_{n\times q} = (ABC)_{m\times q}$

Ex: Check associativity in the following case:

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}_{2\times 2}, \quad B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}_{2\times 1}, \quad C = (1)_{1\times 1}$$

* Multiplication with a (1×1) matrix is same as the scalar multiplication.

(ii) Not always Commutative:

i.e., $AB \neq BA$ (need not be).

$$\left.\begin{aligned} A_{2\times 3}\, B_{3\times 1} &= (AB)_{2\times 1}. \\ B_{3\times 1}\, A_{2\times 3} &= \text{Not defined} \end{aligned}\right\}$$

$$(A_{2\times 3}\, B_{3\times 2}) = (AB)_{2\times 2}$$

$$\neq$$

$$(B_{3\times 2}\, A_{2\times 3}) = (BA)_{3\times 3}$$

(iii) Distributive with addition:

Suppose $A_{m\times n}$, $B_{n\times p}$ and $C_{n\times p}$ matrices.

$$\boxed{A(B+C) = AB + AC.}$$

Exercise: For $k \in \mathbb{R}$, and matrices

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad B = \begin{pmatrix} e & f \\ g & h \end{pmatrix}.$$

Check the following equality:

$$k(AB) = (kA)B = A(kB).$$

Linear Equation:

with 2 variables:

$$\left.\begin{aligned} 2x + 3y &= 12 \\ 4x + 5y &= 36 \end{aligned}\right\} \begin{aligned} &\longrightarrow A \\ &\longrightarrow B \end{aligned}$$

2 variables & 2 eq^{n}s.

$$\underset{2\times 2}{\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}} \underset{2\times 1}{\begin{bmatrix} x \\ y \end{bmatrix}} = \underset{2\times 1}{\begin{bmatrix} 12 \\ 36 \end{bmatrix}}$$

$$\begin{bmatrix} 2x + 3y \\ 4x + 5y \end{bmatrix} = \begin{bmatrix} 12 \\ 36 \end{bmatrix}$$

w/ 3 variables:

$$\begin{aligned} 4x + 5y + z &= 35 \longrightarrow A \\ 3x + y + z &= 10 \longrightarrow B \end{aligned}$$

3 variables and 2 eq^{n}s.

$$\underset{2\times 3}{\begin{bmatrix} 4 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}} \underset{3\times 1}{\begin{bmatrix} x \\ y \\ z \end{bmatrix}} = \underset{2\times 1}{\begin{bmatrix} 35 \\ 10 \end{bmatrix}}$$

w/ n-variables : Suppose m-equations and n-variables. Then

$$
\left.
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
\vdots \qquad \vdots \qquad\quad \vdots \quad &\quad \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{aligned}
\right]
$$

system of linear eq^n's. of n-variables.

$$
\underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}}_{m \times n}
\underbrace{\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}}_{n \times 1}
=
\underbrace{\begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}}_{m \times 1}
$$