Week 4 : Basis and dimension

Lecture 1 : What is a basis for a vector space?

Linear dependence and independence (recall)

Let $v_1, v_2, \ldots, v_n$ be a set of vectors in the vector space V.

The set $v_1, v_2, \ldots, v_n$ is said to be linearly dependent, if there exist scalars $a_1, a_2, \ldots, a_n$, not all zero, such that

$$a_1 v_1 + a_2 v_2 + \ldots + a_n v_n = 0$$

The set $v_1, v_2, \ldots, v_n$ is said to be linearly independent, if the only choice of scalars $a_1, a_2, \ldots, a_n$ such that
$a_1 v_1 + a_2 v_2 + \ldots + a_n v_n = 0$ is with $a_i = 0$ for all i.

Span of a set of vectors

The span of a set S (of vectors) is defined as the set of all finite linear combinations of elements(vectors) of S, and denoted by $Span(S)$.

i.e. $Span(S) = \{\sum_{i=1}^{n} a_i v_i \in V | a_1, a_2, \ldots, a_n \in \mathbb{R}\}$

Example

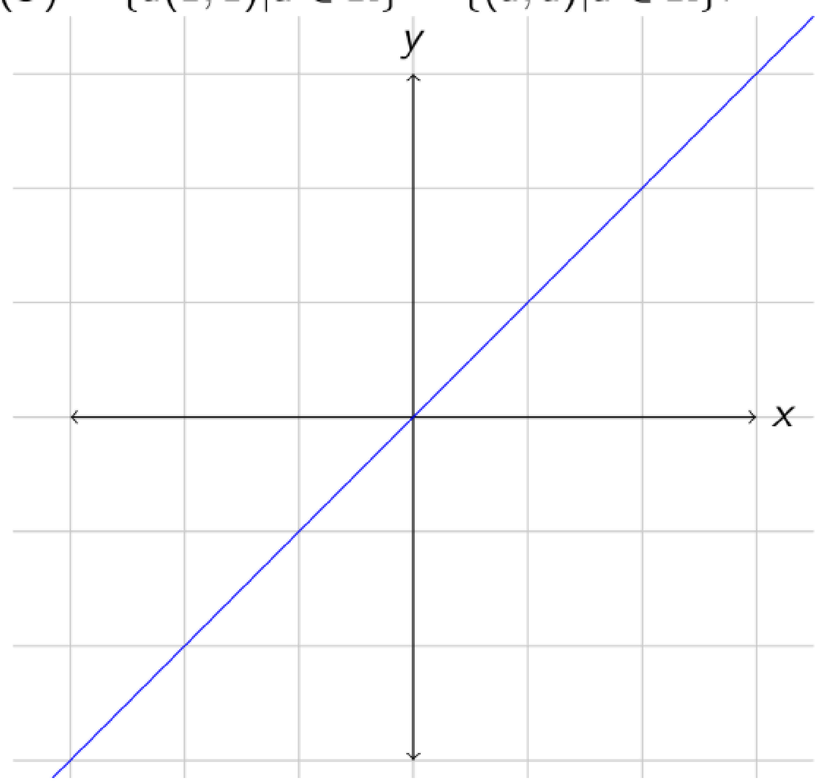
Let $S = \{(1,0)\} \subset \mathbb{R}^2$. Then

$$Span(S) = \{a(1,0) | a \in \mathbb{R}\} = \{(a,0) | a \in \mathbb{R}.\}$$

Thus, $Span(S)$ is the X-axis in $\mathbb{R}^2$.

More examples : in R^2

Let $S = \{(1,1)\} \subset \mathbb{R}^2$.
Then $Span(S) = \{a(1,1) | a \in \mathbb{R}\} = \{(a,a) | a \in \mathbb{R}\}$.



More examples : in R^3

Let $S = \{(1,0,0),(0,1,0)\} \subset \mathbb{R}^3$. Then
$Span(S) = \{a(1,0,0) + b(0,1,0) | a, b \in \mathbb{R}\} = \{(a,b,0) | a, b \in \mathbb{R}\}$.

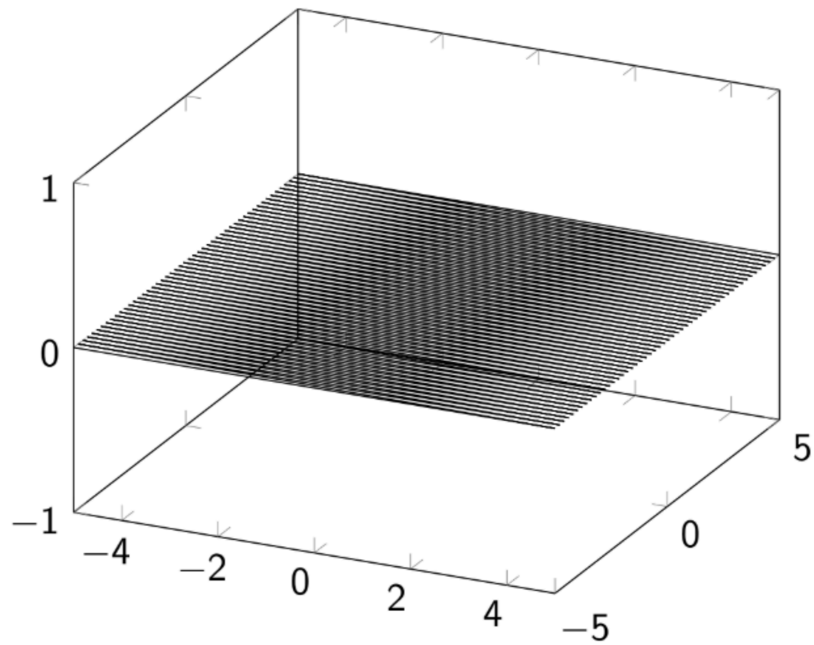


Figure: $Span(S)$ is the XY-plane

Spanning set for a vector space

Let V be a vector space. A set $S \subseteq V$ is a spanning set for V if $Span(S) = V$.

$T \subseteq S$

$Span(T) \subseteq Span(S)$

$T \subseteq Span(S)$

$Span(T) \subseteq Span(S)$

Example

- If $S = \{(1,0),(0,1)\}$ then $Span(S) = \mathbb{R}^2$
- If $S = \{(1,0),(0,1),\boxed{(1,2)}\}$ then $Span(S) = \mathbb{R}^2$
- If $S = \{(1,1),(0,1)\}$ then $Span(S) = \mathbb{R}^2$
- If $S = \{(1,0,0),(0,1,0),(0,0,1)\}$ then $Span(S) = \mathbb{R}^3$

$(x,y) \in \mathbb{R}^2 \quad (x,y) = x(1,0) + y(0,1).$

$(x,y,z) \in \mathbb{R}^3 \quad (x,y,z) = x(1,0,0) + y(0,1,0) + z(0,0,1).$

$(1,0) = (1,1) - (0,1) \quad \therefore (1,0) \in Span(\{(1,1),(0,1)\}) \Rightarrow$

$\therefore \{(1,0),(0,1)\} \subseteq Span$ ''

Example : Adding vectors to obtain a spanning set for R^3

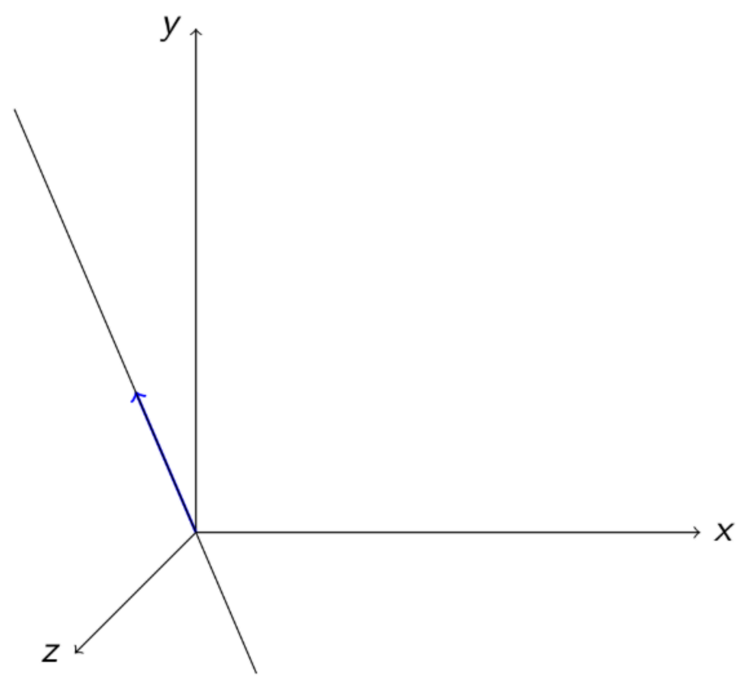
We will try to "build" a spanning set for the vector space $\mathbb{R}^3$.

Start with S_0 to be the empty set $\emptyset$. Then $Span(S_0) = Span(\emptyset) = \{(0,0,0)\}$.

Since this is not the full vector space, append a vector outside $Span(S_0)$ in $\mathbb{R}^3$ e.g. $(0,2,1)$ to S_0 and call the new set S_1.

So $S_1 = S_0 \cup \{(0,2,1)\}$.

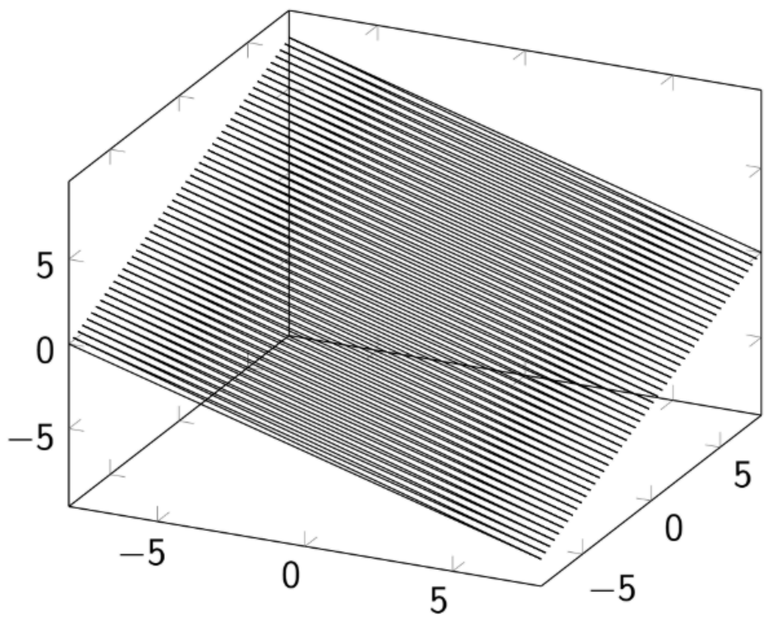
$Span(S_1)$ is the line shown in the picture below.



Choose a vector outside $Span(S_1)$ e.g. $(2, 2, 0)$, append it to S_1 and call the new set S_2.

So $S_2 = S_1 \cup \{(2, 2, 0)\}$.

$Span(S_2)$ is the plane shown in the picture.



Again choose a vector outside $Span(S_2)$, e.g. $(0, 0, 5)$, append it to S_2 and call the new set S_3.

So $S_3 = S_2 \cup \{(0, 0, 5)\}$.

Any arbitrary vector $(x, y, z) \in \mathbb{R}^3$ can be written as follows:

$$(x, y, z) = \frac{y-x}{2}(0, 2, 1) + \frac{x}{2}(2, 2, 0) + \frac{x-y+2z}{10}(0, 0, 5)$$

Hence

$$Span(S_3) = \mathbb{R}^3$$

Another example

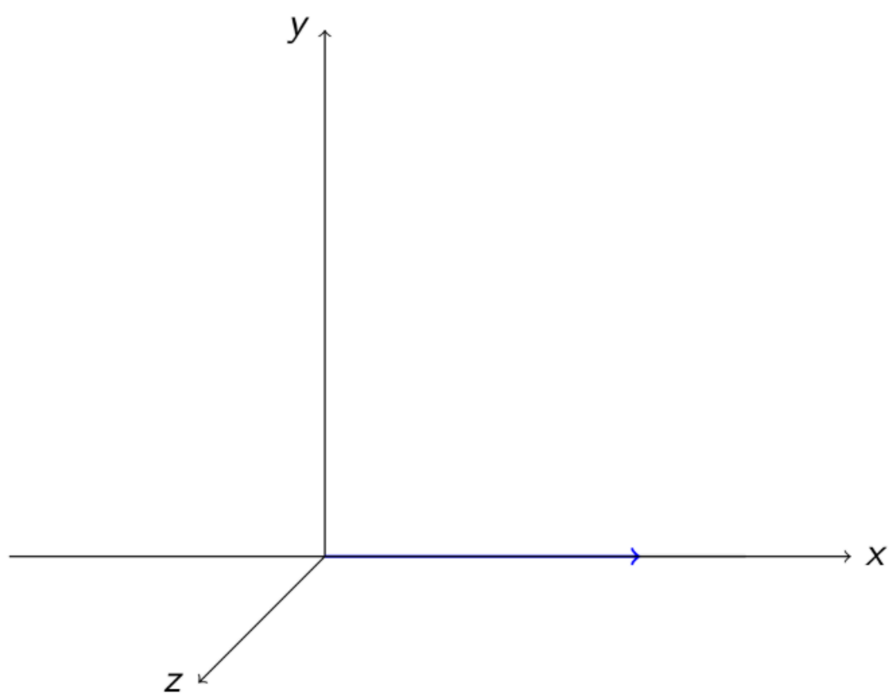
Start with S_0 to be the empty set $\emptyset$ as before.

Thus $S_0 = \emptyset$ and hence $Span(S_0) = Span(\emptyset) = \{(0,0,0)\}$.

Append any vector not in $Span(S_0)$ e.g. $(3,0,0)$ to S_0 and call the new set S_1.

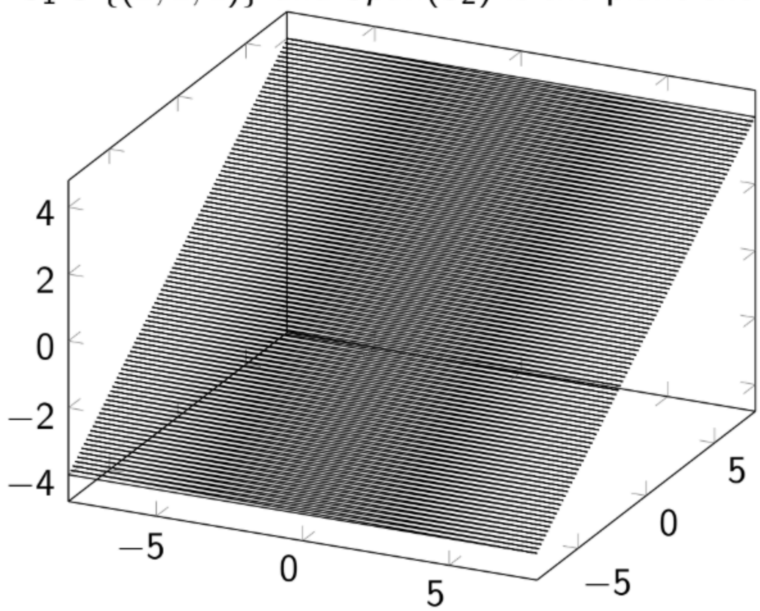
Hence $S_1 = S_0 \cup \{(3,0,0)\}$.

$Span(S_1)$ is the X-axis, as shown below.



Choose a vector outside $Span(S_1)$ e.g. $(2, 2, 1)$ and append it to S_1 and call the new set S_2.

Then $S_2 = S_1 \cup \{(2, 2, 1)\}$ and $Span(S_2)$ is the plane shown below.



Again choose a vector outside $Span(S_2)$ e.g. $(1, 3, 3)$, append it to S_2 and call the new set S_3.

Then $S_3 = S_2 \cup \{(1, 3, 3)\}$.

Any arbitrary vector $(x, y, z) \in \mathbb{R}^3$ can be written as follows:

$$(x, y, z) = \frac{3x-5y+4z}{9}(3, 0, 0) + (y - z)(2, 2, 1) + \frac{2z-y}{3}(1, 3, 3)$$

Hence

$$Span(S_3) = \mathbb{R}^3$$

What is a basis?

A basis B of a vector space V is a linearly independent subset of V that spans V.

Example

Let $e_i \in \mathbb{R}^n$ be the vector with i^{th} coordinate 1 and all other coordinates 0 e.g. $e_1 = (1, 0, 0, \ldots, 0)$.

The set $\varepsilon = \{e_1, e_2, \ldots, e_n\} \subseteq \mathbb{R}^n$ is a basis for $\mathbb{R}^n$. ~~consisting of~~

$(x_1, x_2, \cdots, x_n)$

$= x_1 e_1 + x_2 e_2 + \cdots + x_n e_n$

$\therefore \text{Span}(\mathcal{E}) = \mathbb{R}^n$.

$\sum_{i=1}^{n} a_i e_i = 0 \Rightarrow j^{th}$ coordinate of LHS is a_j

$\Rightarrow a_j = 0 \quad \forall j$.

$\therefore \mathcal{E}$ is lin. indept.

QN : 2,5,6,7,9,10

Lecture 2 : Finding bases for vector spaces

Let V be a vector space and S be a subset of V.

$Span(S) = \{\sum_{i=1}^{n} a_i v_i \in V | a_1, a_2, \ldots, a_n \in \mathbb{R}\}$

S is a spanning set for V if $Span(S) = V$.

S is a basis for V if it is a linearly independent set which spans V.

Equivalent conditions for B to be a basis

The following conditions are equivalent to a subset $B \subseteq V$ being a basis :

i) B is linearly independent and $Span(B) = V$.

ii) B is a maximal linearly independent set.

iii) B is a minimal spanning set.

Suppose $\mathcal{B}$ is a basis.

$\therefore$ $\mathcal{B}$ is lin. indept.

Suppose $\mathcal{B}' = \mathcal{B} \cup \{v\}$.

$\therefore$ $v = \sum_{i=1}^{n} a_i v_i$ where $v_1, \ldots, v_n \in \mathcal{B}$.

$\therefore$ $\mathcal{B}'$ is a lin. dep. set.

maximal lin. indept. means

① it is lin. indept.

② appending any vector makes it lin. dep.

minimal spanning means

① it is spanning

② it is no longer spanning if we delete any vector

How do we find a basis?

We can find a basis by any one of the methods described below :

i) Start with the $\emptyset$ and keep appending vectors which are not in the span of the set thus far obtained, until we obtain a spanning set.

Examples 1 & 2

ii) Take a spanning set and keep deleting vectors which are linear combinations of the other vectors, until the remaining vectors satisfy that they are not a linear combination of the other remaining ones.

Example : Method 1 : V = R^2

Let us start with the empty set and append a non-zero vector e.g. $(1,2)$.

Now choose another vector which is not in the span of the earlier vector e.g. $(2,3)$.

$Span(\{(1,2),(2,3)\}) = \mathbb{R}^2$.

Hence this set forms a basis for $\mathbb{R}^2$.

Example : Method 2 : V = R^3

Let us start with the set

$$S = \{(1,0,0), (1,2,0), (1,0,3), (0,2,3), (0,4,2)\}$$

Check that $Span(S) = \mathbb{R}^3$.

Now observe that, $(0,4,2) = 2(1,2,0) + \frac{2}{3}(1,0,3) - \frac{8}{3}(1,0,0)$.

So delete $(0,4,2)$.

Hence our new set of vectors is

$$S_1 = \{(1,0,0), (1,2,0), (1,0,3), (0,2,3)\}$$

Observe that $(0,2,3) = (1,2,0) + (1,0,3) - 2(1,0,0)$.

Hence delete $(0,2,3)$.

Hence our new set of vectors is

$$S_2 = \{(1,0,0), (1,1,0), (1,0,1)\}$$

None of these vectors is a linear combination of the other two vectors.

Hence S_2 forms a basis of R^3.

QN : 2,4,5,6,7,8,9,10

Lecture 3 : What is the rank/dimension for a vector space

Basis for a vector space (recall)

A basis B of a vector space V is a linearly independent subset of V that spans V.

Equivalently : A basis B of a vector space V is a subset $B \subseteq V$ such that every element of V can be uniquely written as a linear combination of elements of B.

i.e. if $B = \{v_1, v_2, \ldots, v_n\}$ then for every $v \in V$, there is a unique set of scalars $a_1, a_2, \ldots, a_n \in \mathbb{R}$ such that $v = \sum_{i=1}^{n} a_i v_i$.

What is the rank/dimension of a vector space

The dimension (or rank) of a vector space is the **size (or cardinality) of a basis of the vector space**.

for this course : if B is a basis of V, then the rank is the number of elements in B.

For every vector space there exists a basis,and all bases of a vector space have the same number of elements (or cardinality) ; hence, the dimension (or rank) of a vector space (say V) is uniquely defined and denoted by $dim(V)$ (or $rank(V)$) respectively.

Dimension of R^n

Recall the i^{th} **standard basis vector** in $\mathbb{R}^n$.

$$e_i = (0, 0, \ldots, 0, 1, 0, \ldots, 0)$$

i.e. the i-th co-ordinate is 1 and 0 elsewhere.

Recall that the set $\{e_1, e_2, \ldots, e_n\}$ is a basis of $\mathbb{R}^n$ called the standard basis.

Hence the dimension of $\mathbb{R}^n$ is n.

Example

Let use calculate the dimension of the subspace W of $\mathbb{R}^3$ spanned by $\{(1, 0, 0), (0, 1, 0), (3, 5, 0)\}$.

Observe that, $3(1, 0, 0) + 5(0, 1, 0) = (3, 5, 0)$.

Hence the set is not linearly independent.

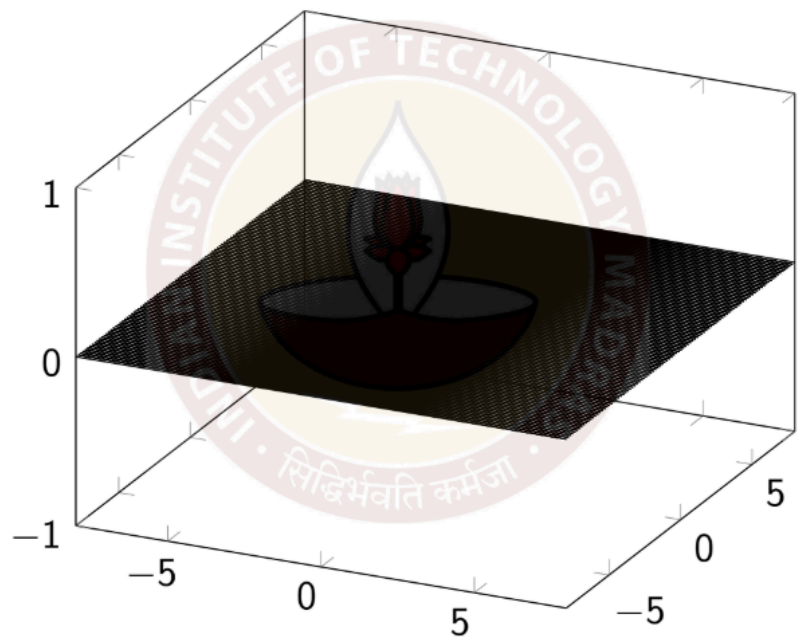
Hence we delete the vector $(3, 5, 0)$ from this set.

The remaining two vectors form a linearly independent set.

Hence the set $\{(1, 0, 0), (0, 1, 0)\}$ forms a basis of the subspace W spanned by $\{(1, 0, 0), (0, 1, 0), (3, 5, 0)\}$.

Hence dimension of the subspace W is 2.

Geometrically the subspace W is the XY-plane.



Example : in terms of matrices

Write the vectors which span (or generate) W as rows of a matrix :

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 5 & 0 \end{bmatrix}.$$

Apply row reduction to this matrix.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 5 & 0 \end{bmatrix} \overset{R_3-3R_1}{\rightsquigarrow} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 5 & 0 \end{bmatrix} \overset{R_3-5R_2}{\rightsquigarrow} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The final matrix is the row echelon form of the original matrix and its rows form a basis of the subspace W.

In particular, the number of non-zero rows is $2 = dim(W)$.

Rank of a matrix

Let A be an $m \times n$ matrix.

- The **column space** of A is the subspace of $\mathbb{R}^m$ spanned by the column vectors of A.
- The **row space** of A is the subspace of $\mathbb{R}^n$ spanned by the row vectors of A.
- The dimension of the column space of A is defined as the **column rank** of A.
- The dimension of row space of A is defined as the **row rank** of A.

Fact : **Column rank= Row rank** and this number is called the **rank** of A.

Example

Let us find the rank of the matrix $A = \begin{bmatrix} 1 & 0 & 1 \\ -2 & -3 & 1 \\ 3 & 3 & 0 \end{bmatrix}$.

Reduce it to row echelon form:

$$\begin{bmatrix} 1 & 0 & 1 \\ -2 & -3 & 1 \\ 3 & 3 & 0 \end{bmatrix} \xrightarrow{R_2+2R_1} \begin{bmatrix} 1 & 0 & 1 \\ 0 & -3 & 3 \\ 3 & 3 & 0 \end{bmatrix} \xrightarrow{R_3-3R_1} \begin{bmatrix} 1 & 0 & 1 \\ 0 & -3 & 3 \\ 0 & 3 & -3 \end{bmatrix}$$

$$\xrightarrow{-R_2/3} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 3 & -3 \end{bmatrix} \xrightarrow{R_3-3R_2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

There are two non-zero rows. Hence $rank(A) = 2$.

QN : 2,7

Lecture 4 : Rank and dimension using Gaussian elimination

Finding dimension and basis with a given spanning set

Consider a vector space W spanned by a set S.

e.g. let us consider the vector space W spanned by the set $S = \{(1,0,1),(-2,-3,1),(3,3,0)\}$.

We will use the following steps to find the dimension and a basis for W and carry out the steps for our example.

- Form a matrix with the vectors in the spanning set as the rows.

$$\begin{bmatrix} 1 & 0 & 1 \\ -2 & -3 & 1 \\ 3 & 3 & 0 \end{bmatrix}$$

- Reduce to a matrix in the row echelon form.

$$\begin{bmatrix} 1 & 0 & 1 \\ -2 & -3 & 1 \\ 3 & 3 & 0 \end{bmatrix} \xrightarrow{R_2+2R_1} \begin{bmatrix} 1 & 0 & 1 \\ 0 & -3 & 3 \\ 3 & 3 & 0 \end{bmatrix} \xrightarrow{R_3-3R_1} \begin{bmatrix} 1 & 0 & 1 \\ 0 & -3 & 3 \\ 0 & 3 & -3 \end{bmatrix}$$

$$\xrightarrow{-R_2/3} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 3 & -3 \end{bmatrix} \xrightarrow{R_3-3R_2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

- The number of non-zero rows is the dimension of the vector space W.
- The vectors corresponding to the non-zero rows form the basis of the vector space W.

$$\text{In the example, the final matrix is } \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Hence, dimension of the vector space spanned by $\{(1,0,1),(-2,-3,1),(3,3,0)\}$ is 2 and a basis is given by $(1,0,1),(0,1,-1)$.

Example in R^4

Apply the steps above to find the rank and a basis of the vector space spanned by the vectors $\{(1,-2,0,4),(3,1,1,0),(-1,-5,-1,8),(3,8,2,-12)\}$.

We construct the matrix with rows corresponding to the vectors in the spanning set :

$$\begin{bmatrix} 1 & -2 & 0 & 4 \\ 3 & 1 & 1 & 0 \\ -1 & -5 & -1 & 8 \\ 3 & 8 & 2 & -12 \end{bmatrix}$$

Apply row reduction :

$$\begin{bmatrix} 1 & -2 & 0 & 4 \\ 3 & 1 & 1 & 0 \\ -1 & -5 & -1 & 8 \\ 3 & 8 & 2 & -12 \end{bmatrix} \xrightarrow{R_2-3R_1} \begin{bmatrix} 1 & -2 & 0 & 4 \\ 0 & 7 & 1 & 12 \\ \cancel{0} & \cancel{-7} & \cancel{-1} & \cancel{12} \\ \cancel{0} & \cancel{14} & \cancel{2} & \cancel{-24} \end{bmatrix} \xrightarrow{R_3+R_1} \begin{bmatrix} 1 & -2 & 0 & 4 \\ 0 & 7 & 1 & 12 \\ 0 & -7 & -1 & 12 \\ 3 & 8 & 2 & -12 \end{bmatrix}$$

$$\xrightarrow{R_4-3R_1} \begin{bmatrix} 1 & -2 & 0 & 4 \\ 0 & 7 & 1 & 12 \\ 0 & -7 & -1 & 12 \\ 0 & 14 & 2 & -24 \end{bmatrix} \xrightarrow{R_2/7} \begin{bmatrix} 1 & -2 & 0 & 4 \\ 0 & 1 & 1/7 & 12/7 \\ 0 & -7 & -1 & 12 \\ 0 & 14 & 2 & -24 \end{bmatrix} \xrightarrow[R_4-14R_3]{R_3+7R_2} \begin{bmatrix} 1 & -2 & 0 & 4 \\ 0 & 1 & 1/7 & 12/7 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Hence the dimension of this vector space is 2 and $\{(1, -2, 0, 4), (0, 1, 1/7, 12/7)\}$ is a basis.

An alternative to the row-based method

The row-based method we discussed produces a basis from a spanning set, but may not contain the vectors in the spanning set.

Can we get a basis consisting of vectors in the spanning set?

Can we make the process discussed earlier of deleting vectors in the spanning set which are linear combinations of other vectors in the spanning set algorithmic rather than ad hoc?

Indeed we can, again by using the row echelon form, by using the following fact :

If R is the matrix obtained by row reducing A, then the columns of A corresponding to the columns of R containing the pivots (i.e. the leading 1s or equivalently the columns corresponding to the dependent variables) form a basis for the column space of A.

Example : Columnn method

Let W be the subspace of $\mathbb{R}^3$ spanned by the set $S = \{(1,0,1),(-2,-3,1),(3,3,0)\}$.

We will use the fact in the previous slide to find a basis for W which is a subset of S.

Form the matrix with the vectors in S as the columns.

$$\begin{bmatrix} 1 & -2 & 3 \\ 0 & -3 & 3 \\ 1 & 1 & 0 \end{bmatrix}$$

Row reduce this matrix :

$$\begin{bmatrix} 1 & -2 & 3 \\ 0 & -3 & 3 \\ 1 & 1 & 0 \end{bmatrix} \xrightarrow{R_3 - R_1} \begin{bmatrix} 1 & -2 & 3 \\ 0 & -3 & 3 \\ 0 & 3 & -3 \end{bmatrix} \xrightarrow{-R_2/3} \begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & -1 \\ 0 & 3 & -3 \end{bmatrix}$$

Example : Columnn method (contd.)

The final step in row reduction is :

$$\begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & -1 \\ 0 & 3 & -3 \end{bmatrix} \xrightarrow{R_3\text{-}3R_2} \begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

This matrix is in row echelon form and the columns with the pivot entries (leading 1s) are the first and second columns.

Therefore $(1,0,1),(-2,-3,1)$, which are the first and second vectors in S respectively, form a basis for W.

Second example : columnn method

Find a basis of the vector space spanned by the vectors $\{(1,-2,0,4),(3,1,1,0),(-1,-5,-1,8),(3,8,2,-12)\}$.

Construct the matrix with columns corresponding to the vectors in the spanning set :

$$\begin{bmatrix} 1 & 3 & -1 & 3 \\ -2 & 1 & -5 & 8 \\ 0 & 1 & -1 & 2 \\ 4 & 0 & 8 & -12 \end{bmatrix}$$

Row reduce this matrix :

$$\begin{bmatrix} 1 & 3 & -1 & 3 \\ -2 & 1 & -5 & 8 \\ 0 & 1 & -1 & 2 \\ 4 & 0 & 8 & -12 \end{bmatrix} \underset{R_4-4R_1}{\overset{R_2+2R_1}{\rightsquigarrow}} \begin{bmatrix} 1 & 3 & -1 & 3 \\ 0 & 7 & -7 & 14 \\ 0 & 1 & -1 & 2 \\ 0 & -12 & 12 & -24 \end{bmatrix} \overset{R_2/7}{\rightsquigarrow} \begin{bmatrix} 1 & 3 & -1 & 3 \\ 0 & 1 & -1 & 2 \\ 0 & 1 & -1 & 2 \\ 0 & -12 & 12 & -24 \end{bmatrix}.$$

The final step in row reduction yields :

$$\begin{bmatrix} 1 & 3 & -1 & 3 \\ 0 & 1 & -1 & 2 \\ 0 & 1 & -1 & 2 \\ 0 & -12 & 12 & -24 \end{bmatrix} \underset{R_4+12R_2}{\overset{R_3-R_2}{\rightsquigarrow}} \begin{bmatrix} 1 & 3 & -1 & 3 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

This matrix is in row echelon form and the columns with the pivot entries (leading 1s) are the first and second columns.

Therefore $(1,-2,0,4),(3,1,1,0)$, which are the first and second vectors in S respectively, form a basis for W.

QN : 9,10

✅ Rank Inequality

A known result in linear algebra:

$$\text{rank}(AB) \geq \text{rank}(A) + \text{rank}(B) - n$$

Week 4 Tutorial 1 :

<u>Basis</u>

$V = M_{2\times 2}(\mathbb{R})$

$V = M_{2\times 2}(\mathbb{R})$

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{11}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + a_{12}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + a_{21}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + a_{22}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\alpha\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \beta\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \gamma\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + \delta\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = 0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow \alpha = 0,\ \beta = 0,\ \gamma = 0,\ \delta = 0$$

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

Week 4 Tutorial 2 :

Another example

$$W = \left\{ \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \;\middle|\; \begin{array}{l} a_{11}+a_{12}+a_{13}=0 \\ a_{21}+a_{22}+a_{23}=0 \\ a_{31}+a_{32}+a_{33}=0 \end{array} \right\} \subseteq M_{3\times 3}(\mathbb{R})$$

$$a_{13} = -a_{11}-a_{12}, \quad a_{23} = -a_{21}-a_{22}, \quad a_{33} = -a_{31}-a_{32}$$

Let
$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = a_{11}\begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + a_{12}\begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + a_{21}\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} + a_{22}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$
$$+ a_{31}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{bmatrix} + a_{32}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix}$$

In the way
$$\left\{ \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix} \right\}$$

$$x_1\begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + x_2\begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + x_3\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} + x_4\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} + x_5\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{bmatrix}$$
$$+ x_6\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix} = 0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1 & x_2 & -x_1-x_2 \\ x_3 & x_4 & -x_3-x_4 \\ x_5 & x_6 & -x_5-x_6 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow x_1 = x_2 = \cdots = x_6 = 0$$

Week 4 Tutorial 3 :

Finding a basis of the given vector space:

$\{v_1, v_2\}$ is a basis of V.

$V = \{(x, y, z) \mid x = y + z,\ x, y, z \in \mathbb{R}\}$

$y = 1, z = 0, x = 1$ $\quad v_1 = (1, 1, 0) \in V$

$y = 0, z = 1, x = 1$ $\quad v_2 = (1, 0, 1) \in V$

$a(1, 1, 0) + b(1, 0, 1) = (0, 0, 0)$ $\quad \{v_1, v_2\}$ is linearly independent.

$\Rightarrow (a + b, a, b) = (0, 0, 0)$

So, $a = 0$, $b = 0$ $\quad // \quad v = (y + z, y, z) \in V$

$v = y v_1 + z v_2$ $\quad v \in \text{Span}\{v_1, v_2\}$

Week 4 Tutorial 4 :

If $\{v_1, v_2, v_3\}$ is a basis of $\mathbb{R}^3$, then $\underbrace{\{v_1, v_1+v_2, v_1-v_3\}}_{S}$ is a basis of $\mathbb{R}^3$.

Sol^n:

$$av_1 + b(v_1+v_2) + c(v_1-v_3) = 0$$

$$\Rightarrow av_1 + bv_1 + bv_2 + cv_1 - cv_3 = 0$$

$$\Rightarrow (a+b+c)v_1 + bv_2 - cv_3 = 0$$

S is lin. ind.

$\underline{a+b+c=0}$ $\quad b=0 \quad -c=0$

$\Rightarrow c=0$

$a=0$

$$\Rightarrow av_1 + bv_1 + bv_2 + cv_1 - cv_3 = 0$$

$$\Rightarrow (a+b+c)v_1 + bv_2 - cv_3 = 0$$

S is lin. ind.

$\underline{a+b+c=0}$ $\quad b=0 \quad -c=0$

$\Rightarrow c=0$

$a=0$

$\text{Span}\{v_1, v_2, v_3\} = \mathbb{R}^3$ to show $\text{Span}(S) = \mathbb{R}^3$ ✓

let $v \in \mathbb{R}^3$

$$v = av_1 + bv_2 + cv_3 \quad \text{for some } a, b, c \in \mathbb{R}.$$

$$v = (a-b+c)v_1 + b(v_1+v_2) - c(v_1-v_3)$$

Week 4 Tutorial 5 :

Basis: Recall

Basis of a vector space:

i) Maximal linear independent set.

ii) Minimal spanning set.

$V \neq 0$ $\quad 0 \neq v_1 \in V$ $\quad \{v_1\} \rightarrow$ linearly independent.

$\mathrm{Span}(v_1) = V$ $\quad \{v_1\} \rightarrow$ forms a basis.

$0 \neq v_2 \in V \setminus \mathrm{Span}(v_1)$

$\{v_1, v_2\} \rightarrow$ linearly independent.

$\mathrm{Span}\{v_1, v_2\} = V$ $\quad \{v_1, v_2\} \rightarrow$ basis.

If not, $\left(\mathrm{Span}\{v_1, v_2\} \neq V\right)$

$0 \neq v_3 \in V \setminus \mathrm{Span}\{v_1, v_2\}$

$\{v_1, v_2, v_3\} \longrightarrow$ linearly independent.

... so on.

$\boxed{\mathrm{Span}\{v_1, \dots, v_n\} = V}$

$\{v_1, \dots, v_n\}$ forms a basis of V.

$v' \in V$ $\quad \{v_1, \dots, v_n, v'\} \rightarrow$ spanning of V.

Basis of a vector space:

i) Maximal linear independent set.

ii) Minimal spanning set.

v_i $\{i \in 1, \dots, n\}$

$\{v_1, \dots, v_n\}$ minimal spanning set.

$V \neq 0$ $\quad 0 \neq v_1 \in V$

$\{v_1\} \rightarrow$ linearly independent.

$\mathrm{Span}(v_1) = V$ $\quad \{v_1\} \rightarrow$ forms a basis.

$0 \neq v_2 \in V \setminus \mathrm{Span}(v_1)$

$\{v_1, v_2\} \longrightarrow$ linearly independent.

$\mathrm{Span}\{v_1, v_2\} = V$ $\quad \{v_1, v_2\} \rightarrow$ basis.

If not, $\left(\mathrm{Span}\{v_1, v_2\} \neq V\right)$

$0 \neq v_3 \in V \setminus \mathrm{Span}\{v_1, v_2\}$

$\{v_1, v_2, v_3\} \longrightarrow$ linearly independent.

... so on.

$\boxed{\mathrm{Span}\{v_1, \dots, v_n\} = V}$

$\{v_1, \dots, v_n\}$ forms a basis of V.

$\{v_1, \dots, v_n, v'\}$ $\hookrightarrow$ spanning of V.

Basis of a vector space:

i) Maximal linear independent set.

ii) Minimal spanning set.

$V \neq 0$ $\quad 0 \neq v_1 \in V$

$\{v_1\} \rightarrow$ linearly independent.

$\mathrm{Span}(v_1) = V$ $\quad \{v_1\} \rightarrow$ forms a basis.

$0 \neq v_2 \in V \setminus \mathrm{Span}(v_1)$

$\{v_1, v_2\} \longrightarrow$ linearly independent.

$\mathrm{Span}\{v_1, v_2\} = V$ $\quad \{v_1, v_2\} \rightarrow$ basis.

If not, $\left(\mathrm{Span}\{v_1, v_2\} \neq V\right)$

$0 \neq v_3 \in V \setminus \mathrm{Span}\{v_1, v_2\}$

$\{v_1, v_2, v_3\} \longrightarrow$ linearly independent.

... so on.

$\boxed{\mathrm{Span}\{v_1, \dots, v_n\} = V}$

$\{v_1, \dots, v_n\}$ forms a basis of V.

$v' \in V$ $\quad \{v_1, \dots, v_n, v'\}$ $\hookrightarrow$ spanning of V.

$v' \in \mathrm{Span}\{v_1, \dots, v_n\}$

$v' = \alpha_1 v_1 + \dots + \alpha_n v_n$

$\Rightarrow \alpha_1 v_1 + \dots + \alpha_n v_n - v' = 0$

Week 4 Tutorial 6 :

MATHEMATICS FOR DATA SCIENCE - 2 - WEEK 7

Find the dimension of the following vector spaces.

$V_1 = \{A \in M_{3\times3}(\mathbb{R}) : A = A^T\}$.

$V_2 = \{A \in M_{3\times3}(\mathbb{R}) : A \text{ is a scalar matrix}\}$

$V_3 = \{A \in M_{3\times3}(\mathbb{R}) : A \text{ is a diagonal matrix}\}$

$V_4 = \{A \in M_{3\times3}(\mathbb{R}) : A \text{ is an upper triangular matrix}\}$

$V_5 = \{A \in M_{3\times3}(\mathbb{R}) : A \text{ is a lower triangular matrix}\}$

$V_1 = \{A \in M_{3\times3}(\mathbb{R}) : A = A^T\}$ $\qquad \dim V_1 = 6$.

2x2 : $\begin{bmatrix} a & b \\ b & d \end{bmatrix} \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \begin{bmatrix} a & \boxed{b} \\ \boxed{b} & d \end{bmatrix}$

3x3 : $\begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}$

$$\left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}.$$

$V_2 = \{A \in M_{3\times3}(\mathbb{R}) : A \text{ is a scalar matrix}\}$. $\dim V_2 = 1$.

2x2 : $\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$

3x3 : $\begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}$

$$\left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}.$$

$V_3 = \{A \in M_{3\times3}(\mathbb{R}) : A \text{ is a diagonal matrix}\}$. $\dim V_3 = 3$.

2×2: $\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$

3×3: $\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$

$$\left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}$$

$V_4 = \{A \in M_{3\times3}(\mathbb{R}) : A \text{ is an upper triangular matrix}\}$.
$\dim V_4 = 6$.

2×2: $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$

3×3: $\begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}$

$$\left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}$$

Solve with US 4

Key points: 1

The span of a set S (of vectors) is defined as the set of all finite linear combinations of elements (vectors) of S and denoted by $Span(S)$ i.e., $Span(S) = \{\sum_{i=1}^{n} a_i v_i \in V \mid a_1, a_2, ..., a_n \in \mathbb{R} \text{ and } v_1, v_2, ..., v_n \in S\}$.

Let V be a vector space. A set $S \subseteq V$ is spanning set for V if $Span(S) = V$.

Algorithm to obtain a spanning set for a non-zero vector space V:

i) Start with the empty set $S_0 = \phi$.

ii) Add any non-zero vector $v_1 \in V$ to S_0 and define it as S_1. So $S_1 = S_0 \cup \{v_1\}$.

iii) If $Span(S_1) = V$, then we are done. Otherwise choose $v_2 \in V \setminus span(S_1)$ and add it to the set and call it S_2. So $S_2 = S_1 \cup \{v_2\}$.

iv) If $Span(S_2) = V$, then we are done. Otherwise, choose $v_3 \in V \setminus Span(S_2)$, and construct $S_3 = S_2 \cup \{v_3\}$.

v) Repeat the process until $Span(S_n) = V$ for some n.

QN 1

Key points: 2

The following conditions are equivalent for a set to be a basis of a vector space:

- Maximal linearly independent set : A linearly independent set such that if any vector is appended to it, the new set is no longer linearly independent.
- Minimal spanning set : A spanning set such that if any vector is removed from the set, then the set will not be a spanning set.

QN 5,6

Key Points: 3

Basis: A basis of a vector space V is a linearly independent subset of V that spans V .

Basis of a vector space:

- Maximal linearly independent set.
- Minimal spanning set.

How to find a basis of a subspace of $\mathbb{R}^n$?

- Suppose a subspace V of $\mathbb{R}^n$ is defined by some constraints, as follows:

$$V = \{(x_1, x_2, \dots, x_n) \mid f_1(x_1, x_2, \dots, x_n) = 0, f_2(x_1, x_2, \dots, x_n) = 0, \dots, f_m(x_1, x_2, \dots, x_n) = 0\} \subseteq \mathbb{R}^n$$

Where f_i's are linear equations in general.

- **Step 1**: By solving f_i's we have to try to find out the dependent and independent variables. Suppose there are k independent variables and there are $n-K$ dependent variables.

- **Step 2**: Choose the first independent variable, let it be x_1 and assign the value 1 to that, and assign 0 to all the other independent variables.

- **Step 3**: Find out the values of all the dependent variables with respect to the assignment given in Step 2 and call that vector to be v_1.
- **Step 4**: Carry out this process for each independent variable, to get the vectors $v_2, \dots, v_k$.

The set of vectors $\{v_1, v_2, \dots, v_k\}$ is a linearly independent set and this is a maximal linearly independent set in V, hence forms a basis of V. Hence, dimension of V is k in this case.

QN 9,10

Key Points: 4

How to find a basis of a subspace V of $M_{m\times n}(\mathbb{R})$:

- **Step 1**: Consider each element of a $m \times n$ matrix as a variable. So there are mn number of variables.

- **Step 2**: Using the constraints given as the definition of the subspace separate out the independent and the dependent variables. Suppose there are k independent variables and $mn - k$ dependent variables.

- **Step 3**: Define a matrix M_1 by assigning 1 to one independent variable and 0 to all the other independent variables, together with finding out the values of all the dependent variables.

- **Step 4**: Repeat this process for all the independent variables to get the vectors $M_2, M_3, \dots, M_k$.

The set of matrices $\{M_1, M_2, \dots, M_k\}$ forms a basis of V. Hence the dimension of V is k in this case.

13

Key points: 5

Rank of a matrix:

- Maximum number of linearly independent column vectors (or row vectors) in a matrix.

NOTE: Rank of a matrix is always less than or equal to the number of columns (or rows).

How to find rank of a matrix A:

- Reduce the matrix to row echelon form.
- Find the number of non zero rows in the reduced matrix which will be the rank of the matrix A.

PA : 4,5,7,9

GA : 2,3,4,5,6,10

EMQ : 1,2,3