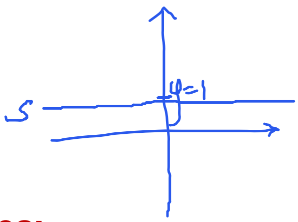
# MATHEMATICS FOR DATA SCIENCE II

## WEEK 3

### TOPICS IN WEEK 3

- Motivation for Vector Spaces
- Vector Spaces
- Properties of Vector Spaces
- Subspaces
- Linear combination
- Linear dependence
- Linear Independence




$S = \{(x,1) : x \in \mathbb{R}\}$

## Motivation for vector spaces:

Consider a system of linear equations Ax = b where A is an mxn matrix. We have seen a few ways of identifying solutions of this system. Let S = {x ϵ $R^n$ | Ax = 0}. That is, S is the set of all solutions of the system Ax = 0.

$A(0) = 0$
$0 \in S$

What do we know about S?

- S is always __non-empty__ and it contains the __zero__ vector.
- S is either a __singleton__ set or it contains __infinitely many__ elements. $S = \{0\}$ (trivial case)

Let us assume the case when S is an infinite set. Observe that

Additive closure

(1) if $v_1$, $v_2$ ϵ S, then $v_1 + v_2$ ϵ S. (Closed under addition)

$Av_1 = 0 \quad Av_2 = 0 \qquad A(v_1+v_2) = Av_1 + Av_2 = 0 + 0 = 0$

$\Rightarrow v_1 + v_2 \in S$

(2) if αϵR and $v_1$ ϵ S, then $\alpha v_1$ ϵ S. (Closed under scalar multiplication)

$Av_1 = 0 \Rightarrow A(\alpha v_1) = \alpha A(v_1) = \alpha . 0 = 0 \Rightarrow \alpha v_1 \in S$

**Exercises:** Check closure under vector addition and scalar multiplication

a) S = { (x, 0) : x ϵ R} $\subseteq \mathbb{R}^2$

Closed under vector addition: Yes / No

$v_1, v_2 \in S \quad v_1 = (x_1, 0) \quad v_2 = (x_2, 0)$

$v_1 + v_2 = (x_1 + x_2, 0) \in S$

Closed under scalar multiplication: Yes ✓ / No $\alpha \in \mathbb{R}$ $v_1 \in S$ $v_1 = (x_1, 0)$

$\alpha v_1 = \alpha(x_1, 0) = (\alpha x_1, 0) \in S$

b) S = {(x , y, 1) : x, y є R}

Closed under vector addition: Yes / No ✓ $v_1, v_2 \in S$ $v_1 = (x_1, y_1, 1)$ $v_2 = (x_2, y_2, 1)$

Closed under scalar multiplication: Yes / No ✓ $v_1 + v_2 = (x_1 + x_2, y_1 + y_2, \underbrace{2}_{\text{not equal to } 1}) \notin S$

$\alpha \in \mathbb{R}$ $v_1 \in S$

$\alpha v_1 = \alpha(x_1, y_1, 1) = (\alpha x_1, \alpha y_1, \underbrace{\alpha}_{\alpha \text{ need not be } 1}) \notin S$

c) S = {(x,y) є $R^2$: $\underline{x+y = 1}$} $= \{(x, 1-x) : x \in \mathbb{R}\}$ $y = 1 - x$

Closed under vector addition: Yes / No ✓ $v_1, v_2 \in S$ $v_1 = (x_1, 1-x_1)$ $v_2 = (x_2, 1-x_2)$

Closed under scalar multiplication: Yes /No ✓ $v_1 + v_2 = (\underbrace{x_1 + x_2}_{t}, \underbrace{2 - x_1 - x_2}_{\text{not } 1-t}) \notin S$

$\alpha(x_1, 1-x) = (\alpha x, \alpha - \alpha x) \notin S$

$\alpha x + \alpha - \alpha x = \alpha \to$ need not be 1

d) S= {(x,y,z) є $R^3$ : x+y+z = 0} (HW)

Closed under vector addition: Yes / No

Closed under scalar multiplication: Yes /No

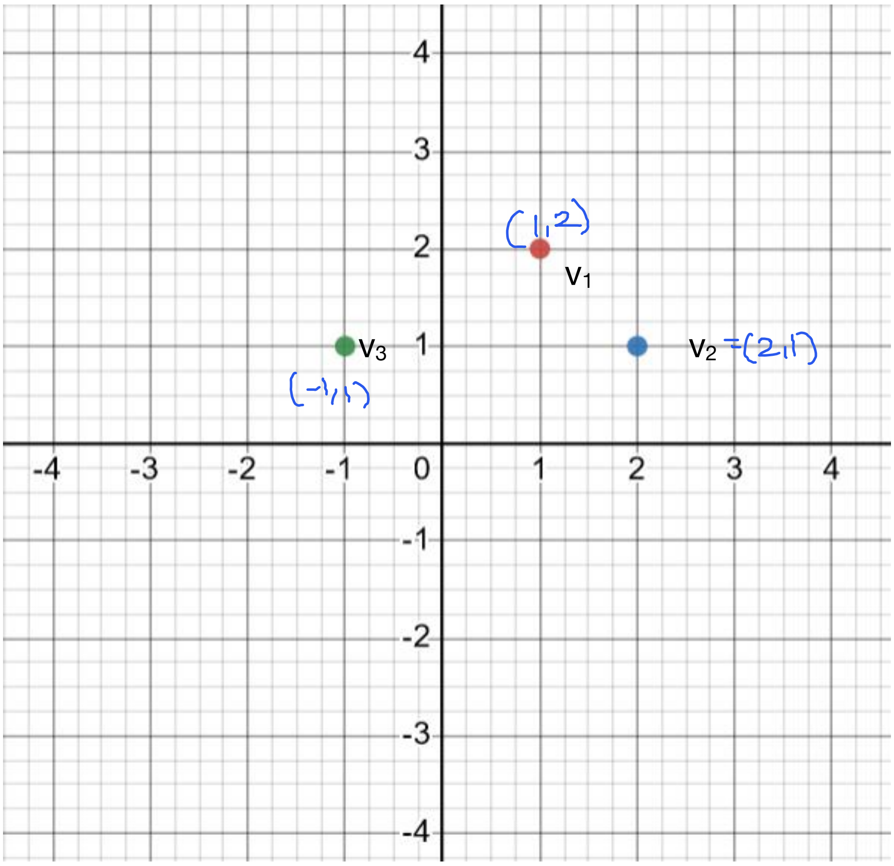
1) $v_1 + v_2 \in S$
2) $\alpha v_1 \in S$
$\Big\} \alpha_1 v_1 + \alpha_2 v_2 \in S$

Equivalently, if $v_1$ and $v_2$ are in S then $\alpha_1 v_1 + \alpha_2 v_2$ is also in S, where $\alpha_1$ and $\alpha_2$ are real numbers. We call $\alpha_1 v_1 + \alpha_2 v_2$ a linear combination of $v_1$ and $v_2$. In more general, if $v_1, v_2, \ldots, v_n \in S$ and $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{R}$, then $\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n$ ( linear combination of $v_1, v_2, \ldots, v_n$) is also in S.

Roughly, we can think of a vector space V as a collection of objects that behave similar to the vectors in the set S. We can perform two operations on V ;
→ closed under addition & sc. multiplication

- We can add the elements of V .
- We can multiply scalars ($\alpha \in R$) with the elements of V . These operations should satisfy a few conditions which are the axioms for a vector space.

**Properties of vectors in $R^2$**




$v_1$= (1,2) $v_2$= (2,1) $v_3$= (-1,1)

1) Addition is commutative: $v_1+v_2$ = (3,3) $v_2+v_1$= (3,3)

$v_2+v_3 = v_3+v_2$ $u+v = v+u$

2) Addition is associative: $(v_1+v_2)+v_3$ = (3,3) + (-1,1) $v_1+(v_2+v_3)$ = (1,2)+(1,2)

$(v_1+v_2)+v_3 = v_1+(v_2+v_3)$ = (2,4) = (2,4)

3) Additive identity: Let v be any vector in $R^2$.

Find a vector such that when it is added to v, the sum is still v.

v + <u>0</u> = v

(x,y) +(0,0)= (x,y)

"zero" vector → additive identity

4) Additive inverse: Find a vector such that when it is added to v, the sum is the zero vector.

$V = (x, y)$

v + $\underline{-v}$ = 0 (what is 0 here?) $-V = (-x, -y) \rightarrow$ add. inv. of $V$

$(0, 0)$

5) Identity of scalar multiplication: Find a scalar such that when it is multiplied to v, the new vector is still v.

$\underline{1}$(v) = $v$

6) Associativity for scalar multiplication:

2(3v$_1$) = $2(3(1,2))$ $= 2(3,6)$ $= (6,12)$ (2.3)v$_1$ = $6(1,2)$ $= (6,12)$

7) Distributivity of Scalar Multiplication with Respect to Vector Addition

2(v$_1$ + v$_2$) = $2((1,2)+(2,1))$ $= 2(3,3)$ $= (6,6)$ 2.v$_1$+ 2.v$_2$= $2(1,2) + 2(2,1)$ $= (2,4) + (4,2)$ $= (6,6)$

8) Distributivity of Scalar Multiplication with Respect to Scalar Addition

(2+3).v$_1$ = $5(1,2)$ $= (5,10)$ 2.v$_1$+ 3.v$_1$ = $2(1,2) + 3(1,2)$ $= (2,4) + (3,6)$ $= (5,10)$

## Vector Spaces

A vector space $V$ over $\mathbb{R}$ is a set along with two functions

$$\left\{ \begin{array}{l} + : V \times V \to V \\ (v_1, v_2) \to v_1 + v_2 \end{array} \right. \quad \text{and} \quad \left. \begin{array}{l} \cdot : \mathbb{R} \times V \to V \\ \alpha \; V \to \alpha V \end{array} \right\}$$

(1) $V$ is closed under addition & sc. multiplication (2)

(i.e. for each pair of elements $v_1$ and $v_2$ in $V$, there is a unique element $v_1 + v_2$ in $V$, and for each $c \in \mathbb{R}$ and $v \in V$ there is a unique element $c \cdot v$ in $V$)

that satisfies the following conditions:

Note: It is also represented as (V ; +; ·; R) $\rightarrow$ real numbers

Set addition (+), Sc. mul (·)

i) $v_1 + v_2 = v_2 + v_1$ for all $v_1, v_2 \in V$

ii) $(v_1 + v_2) + v_3 = v_1 + (v_2 + v_3)$ for all $v_1, v_2, v_3 \in V$

iii) There exists an element in $V$ denoted by 0 such that $v + 0 = v$ for all $v \in V$

iv) For each element $v \in V$ there exists an element $v' \in V$ such that $v + v' = 0$

v) For each element $v \in V$, $1v = v$

vi) For each pair of elements $a, b \in \mathbb{R}$ and each element $v \in V$, $(ab)v = a(bv)$

vii) For each element $a \in \mathbb{R}$ and each pair of elements $v_1$ and $v_2$, $a(v_1 + v_2) = av_1 + av_2$

viii) For each pair of elements $a, b \in \mathbb{R}$ and each element $v \in V$, $(a + b)v = av + bv$

**Note:** $1, 2 + 8 \rightarrow 10$ conditions for V.S

• To prove a set is a vector space, we need to verify additive and multiplicative closure and all the other axioms given above.

• If just one of the vector space axiom fails to hold, then V is not a vector space.

Additive identity

• Zero element $0 \in V$ of a vector space V is always unique.

$V = \mathbb{R}^2 \quad 0 = (0,0)$

• The real number $0 \in R$ and the zero vector $0 \in V$ of a vector space V are commonly denoted by the symbol 0. One can always tell from the context whether 0 means the zero scalar ($0 \in R$) or the zero vector ($0 \in V$).

• It is standard to suppress · and only write au instead of a · u.

A vector is an element of a vector space.

**Examples:** Following are some examples of a vector space

1) $R^n = \{(x_1, x_2, \ldots, x_n) : x_i \in \mathbb{R}\}$

$n \rightarrow$ natural number

$n=1 \quad \mathbb{R} \rightarrow$ real nos

$n=2 \quad \mathbb{R}^2 \rightarrow$ plane

$n=3 \quad \mathbb{R}^3 \rightarrow$ 3-d space

2) The set of all m×n matrices with real entries, $M_{m\times n}$

$$M_{2\times 2} = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{R} \right\}$$

$$\text{Add. identity} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\text{Add inv} = \begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix}$$

3) Solutions of a Homogeneous system Ax=0 where A is an m×n matrix. (i.e. a system with m equations and n variables,

Note: This space is a subset of $R^n$ and hence is a subspace of $R^n$.

$S = \{(x, y) : x, y > 0\} \rightarrow$ closed under addition

$\alpha \in \mathbb{R} \quad \alpha = -1$

$\alpha(x, y) \in S$ ? No

Not closed under sc. mul

## Exercises:

1) Let $V = R^2 = \{(x_1, x_2) \mid x_1, x_2 \in R\}$, and the addition and scalar multiplication on V are defined as follows: defining the "addition" in this way

$$(x_1, x_2) + (y_1, y_2) := (x_1 + y_1, x_2 - y_2)$$
$$c \cdot (x_1, x_2) = (cx_1, cx_2).$$

Show that addition is not commutative.

Additive commutative. $v_1, v_2 \in V$ $\quad v_1 + v_2 = v_2 + v_1$

$v_1 = (x_1, x_2) \quad v_2 = (y_1, y_2)$

$v_1 + v_2 = (x_1 + y_1, x_2 - y_2)$ $\quad v_1 + v_2 \neq v_2 + v_1$

$v_2 + v_1 = (y_1 + x_1, y_2 - x_2)$

$V$ – not a VS.

2) Consider the set $V = \{(x, 1) \mid x \in R\}$. The addition and scalar multiplication on V is defined as follows:

$$\underset{v_1}{(x, 1)} + \underset{v_2}{(y, 1)} := \underset{v_1+v_2}{(x + y, 1)} \in V$$
$$\underset{c}{c} \cdot \underset{v_1}{(x, 1)} := (cx, 1) \in V$$

Check whether V is a vector space or not with respect to the given operations.

1) closure under addition : ✓
2) " " sc. mul : ✓
3) $v_1 + v_2 = (x+y, 1) \quad v_2 + v_1 = (y+x, 1) \quad v_1 + v_2 = v_2 + v_1$ ✓
4) $(v_1 + v_2) + v_3 = (x+y, 1) + (z, 1) = (x+y+z, 1)$

$v_3 = (z, 1)$

$v_1 + (v_2 + v_3) = (x, 1) + [(y, 1) + (z, 1)] = (x, 1) + (y+z, 1)$
$= (x+y+z, 1)$

L.H.S = R.H.S

5) $v \in V \quad v = (x, 1)$

$v + \underset{\in V}{(\alpha, 1)} = v$ $\qquad$ Additive identity : $(0, 1)$

verify

$\boxed{7, 8, 9, 10}$

$(x, 1) + (\alpha, 1) = (x, 1)$

$(x + \alpha, 1) = (x, 1)$

6) $(x, 1) + (-x, 1) = (0, 1)$

Add. inverse

Consider a set $V = \{(x, y) \mid x, y \in \mathbb{R}\} \subseteq \mathbb{R}^2$ with the usual addition as in $\mathbb{R}^2$ and scalar multiplication is defined as

$$c(x, y) = \begin{cases} (0, 0) & c = 0 \\ (\frac{cx}{2}, \frac{y}{c}) & c \neq 0 \end{cases} \quad (x, y) \in V, \ c \in \mathbb{R}$$

Consider the statements given below.

- ✓ **P:** $V$ is closed under addition. – True
- ✓ **Q:** $V$ has zero $(0,0)$ element with respect to addition. i.e., there exists some element 0 such that $v + 0 = v$, for all $v \in V$. – True
- ✗ **R:** $1.v = v$ where $1 \in \mathbb{R}$ and $v \in V$.
- ✓ **S:** $a(v_1 + v_2) = av_1 + av_2$ where $v_1, v_2 \in V$ and $a \in \mathbb{R}$.
- ✗ **T:** $(a + b)v = av + bv$ where $a, b \in \mathbb{R}$ and $v \in V$.

R: $v = (x,y)$

$$1 \cdot v = \left(\frac{1.x}{2}, \frac{y}{1}\right) = \left(\frac{x}{2}, y\right) \neq v$$

S: $v_1 = (x_1, y_1)$ $v_2 = (x_2, y_2)$, $a \neq 0$

$$a(x_1 + x_2, y_1 + y_2) = \left(\frac{a(x_1+x_2)}{2}, \frac{y_1+y_2}{a}\right)$$

$$= \left(\frac{ax_1}{2} + \frac{ax_2}{2}, \frac{y_1}{a} + \frac{y_2}{a}\right)$$

$$= \left(\frac{ax_1}{2}, \frac{y_1}{a}\right) + \left(\frac{ax_2}{2}, \frac{y_2}{a}\right)$$

$$= a(x_1, y_1) + a(x_2, y_2)$$

$$a(v_1 + v_2) = av_1 + av_2$$

T:

$$(a+b)(x,y) = \left(\frac{(a+b)x}{2}, \frac{y}{a+b}\right)$$

$$\neq$$

$$a(x,y) + b(x,y) = \left(\frac{ax}{2}, \frac{y}{a}\right) + \left(\frac{bx}{2}, \frac{y}{b}\right)$$

$$= \left(\frac{(a+b)x}{2}, \frac{y}{a} + \frac{y}{b}\right)$$

$$\frac{y}{a+b} \neq \frac{y}{a} + \frac{y}{b}$$

## Properties

$x + z = y + z$
$\Rightarrow x = y$

### Cancellation law of vector addition

If $v_1, v_2, v_3 \in V$ such that $v_1 + v_3 = v_2 + v_3$, then $v_1 = v_2$.

**Corollaries:** add. identity

- The vector 0 described in (iii) is unique.

$V + \underline{0} = V$ ↳ This is the only vector satisfying this condition

- The vector $v'$ described in (iv) is unique and it is standard to refer to it as $-v$.

$V + (-V) = 0$
↓
unique

## Subspaces

A non-empty subset W of a vector space V is called a subspace of V if W is a vector space under the operations addition and scalar multiplication defined in V.

To show that a non-empty set W is a vector subspace, one doesn't need to check all the vector space axioms.

<u>Conditions for a subspace:</u>

If W is a non-empty subset of a vector space V , then W is a subspace of V if and only if the following conditions hold (1) If $w_1$ and $w_2$ are in W , then $w_1 + w_2 \in W$ .
(2) For all $c \in R$ and for all $w_1 \in W$ , $c \cdot w_1 \in W$ .
A subspace W of a vector space V is called a proper subspace if $W \subsetneq V$.

**Note:**

V

Every vector space V over R has two trivial subspaces:

• V itself is a subspace of V .

• The subset consisting of the zero vector $\{0_V\}$ of V is also a subspace of V.

**Examples:**

$y = -x$

1) Check whether W = {(x, y) | x + y = 0} ⊂ $R^2$ is a vector subspace of V = $R^2$ or not.

$= \{(x, -x) ; x \in \mathbb{R}\}$

1) $w_1, w_2 \in W$

$w_1 = (x_1, -x_1)$ $w_2 = (x_2, -x_2)$

$w_1 + w_2 = (x_1 + x_2, -x_1 - x_2) = (x_1 + x_2, -(x_1 + x_2)) \in W$

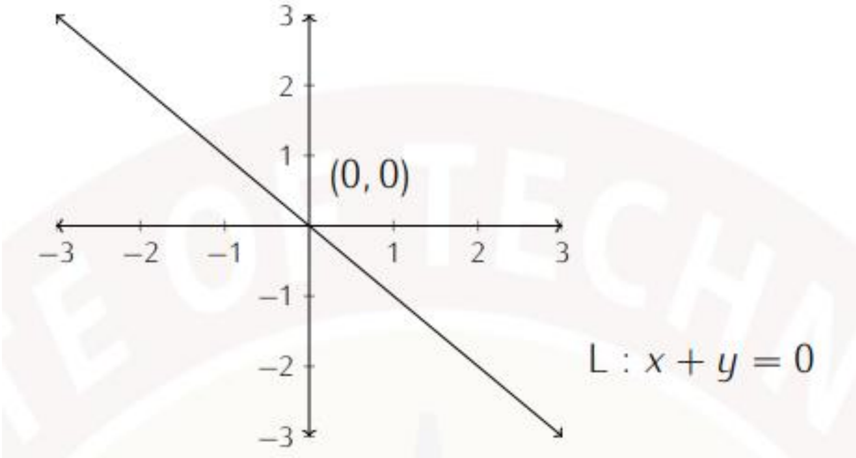
Cl. under add.

$\alpha w_1 = (\alpha x_1, -\alpha x_1) \in W$

cl. under sc.mul

W-subspace of $\mathbb{R}^2$

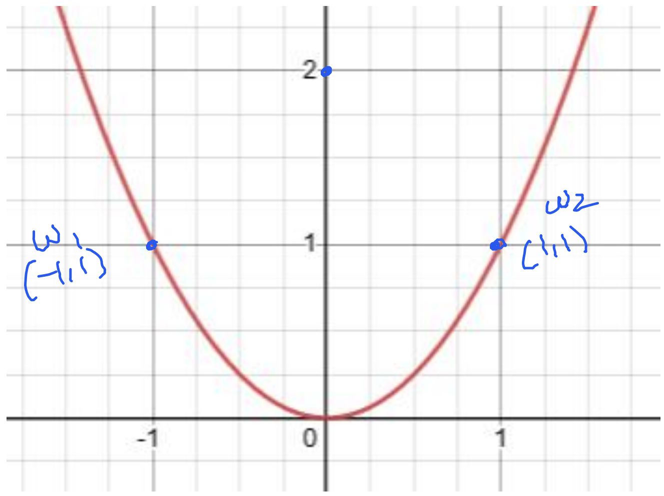
Geometrically, the set W represents a straight line in $R^2$. The geometrical representation of W is given below.



From the above graph, it is clear that the line L passes through the origin. In general, a line in $R^2$ is a subspace of $R^2$ if and only if it passes through the origin.

under usual addition & sc. mult

2)Consider the parabola $W = \{(x, y) \mid y = x^2\} \subset V = R^2$. We want to check whether W is a vector subspace of V or not.




To disprove a property, we can pick specific vectors

$w_1 + w_2 = (0,2) \notin W$

$W \rightarrow$ not closed under add.

$W \rightarrow$ not a subspace

Note: The only non-trivial subspaces of $\mathbb{R}^2$ are st. lines passing thru origin

3) Check whether the set $W = \{A \in M_{2\times2}(R) \mid A = A^T\}$ of all 2 × 2 real symmetric matrices is a subspace of $M_{2\times2}(R)$ or not with standard addition and scalar multiplication.

1) $A_1, A_2 \in W \quad (A_1 + A_2)^T = A_1^T + A_2^T = A_1 + A_2$

$A_1 + A_2 \in W$ ✓

2) $(\alpha A)^T = \alpha A^T = \alpha A \in W$ ✓

$W$ - subspace of $M_{2\times2}(\mathbb{R})$

4) Check whether the set W of 2 × 2 invertible matrices with real entries with standard addition and scalar multiplication is a subspace of $M_{2\times2}(R)$ or not.

$A_1, A_2 \in W$

$A_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\quad A_2 = \begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix}$

$\det(A_1) = ad - bc \neq 0$ $\quad \det(A_2) = ad - bc \neq 0$

$A_1 + A_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \rightarrow$ not invble

W – not a subsp.

5) Show that W = {(0, y, z) : y, z ∈ R} is a subspace of real vector space $R^3$, where $R^3$ is a vector space with respect to the usual addition and scalar multiplication.

$w_1 = (0, y_1, z_1)$ $\quad$ HW

$w_2 = (0, y_2, z_2)$

Note: w - plane in $\mathbb{R}^2$, passing thru origin.

6) Show that the plane W = {(x, y, z) | x + y + z = 1} ⊂ $R^3$ is not a vector subspace of $R^3$, where $R^3$ is a vector space with respect to the usual addition and scalar multiplication.

plane, not passing thru origin.

$w_1 = (x_1, y_1, z_1)$ $\quad w_1 + w_2 = (x_1 + x_2, y_1 + y_2, z_1 + z_2) \notin W$

$w_2 = (x_2, y_2, z_2)$ $\quad x_1 + x_2 + y_1 + y_2 + z_1 + z_2 = 2 \neq 1$

$x_1 + y_1 + z_1 = 1$ $\quad$ W – not a subsp

$x_2 + y_2 + z_2 = 1$

7) Let W = {(x, y, z) | x ≥ z} be a subset of the vector space $R^3$ (with respect to the usual addition and scalar multiplication). Then show that W is not a vector subspace of $R^3$.

$(2,0,1) \in W$

$2 \geq 1$

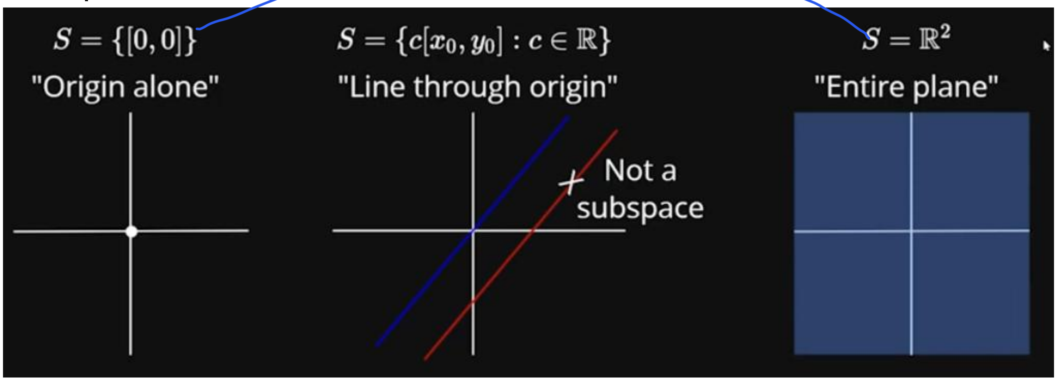
$-1(2,0,1) = (-2,0,-1) \notin W$

$-2 < -1$

W - not a subsp

trivial

Subspaces of $R^2$




- The zero subspace {0}
- Lines through the origin
- The entire space $R^2$

Subspaces of $R^3$

- The zero subspace {0}
- Lines through the origin
- Planes through the origin
- The entire space $R^3$

7) Consider the following sets:

$V_1 = \{A \mid A \in M_{2\times 2}(\mathbb{R})$ and $A$ is a symmetric matrix, i.e., $A = A^T\}$ – Subsp
$V_2 = \{A \mid A \in M_{2\times 2}(\mathbb{R})$ and $A$ is a scalar matrix$\}$ – subsp
$V_3 = \{A \mid A \in M_{2\times 2}(\mathbb{R})$ and $A$ is a diagonal matrix$\}$ – subsp
$V_4 = \{A \mid A \in M_{2\times 2}(\mathbb{R})$ and $A$ is a upper triangular matrix$\}$ – subs
$V_5 = \{A \mid A \in M_{2\times 2}(\mathbb{R})$ and $A$ is a lower triangular matrix$\}$ – subsp

Choose the set of correct options.

- [x] Only $V_1$ is a subspace of $M_{2\times 2}(\mathbb{R})$.
- [x] Only $V_4$ is a subspace of $M_{2\times 2}(\mathbb{R})$.
- [x] Both $V_2$ and $V_3$ are subspaces of $M_{2\times 2}(\mathbb{R})$
- [x] All are subspaces of $M_{2\times 2}(\mathbb{R})$

$V_2 = \left\{ \begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix} : \alpha \in \mathbb{R} \right\}$ ✓ $A_1, A_2 \in V_2$ $A_1 = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_1 \end{pmatrix}$ $A_2 = \begin{pmatrix} \alpha_2 & 0 \\ 0 & \alpha_2 \end{pmatrix}$

$V_3 = \left\{ \begin{bmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{bmatrix} : \alpha_1, \alpha_2 \in \mathbb{R} \right\}$

$V_4 = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} : a, b, c \in \mathbb{R} \right\}$

$V_5 = \left\{ \begin{bmatrix} a & 0 \\ b & c \end{bmatrix} : a, b, c \in \mathbb{R} \right\}$

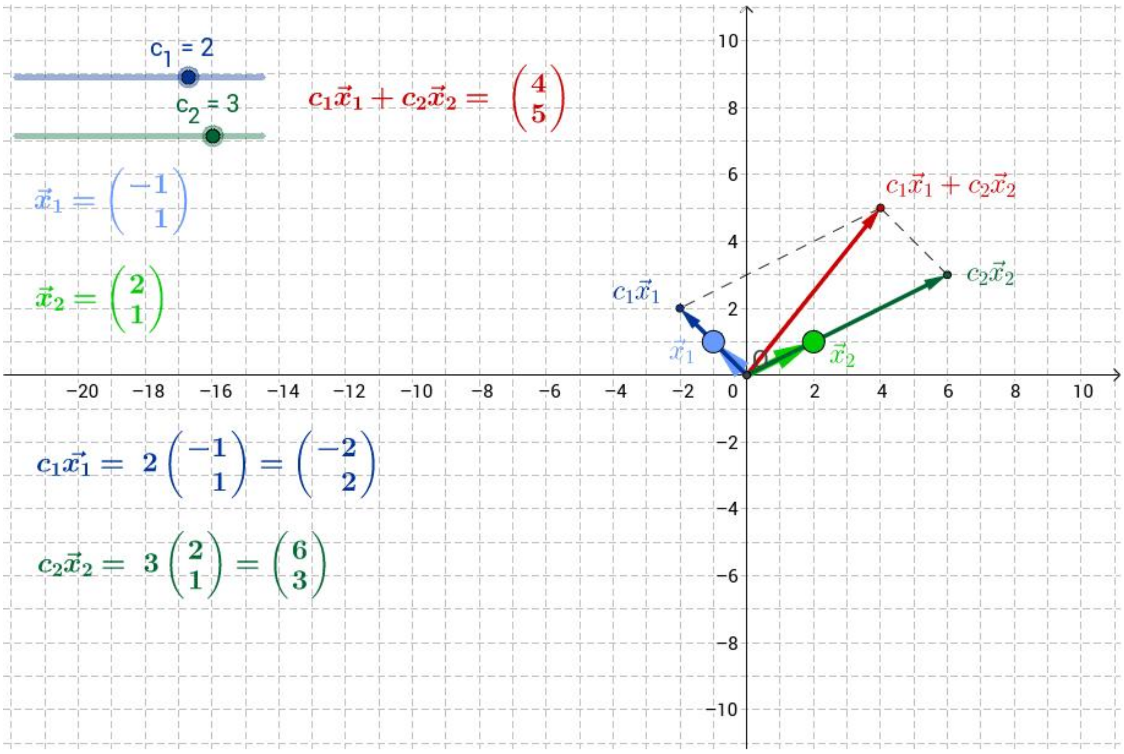
---

HW: $V = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a + d = 0 \right\}$ Is this a subspace?

## Linear Combination

Let $V$ be a vector space and $v_1, v_2, \ldots, v_n \in V$. The linear combination of $v_1, v_2, \ldots, v_n$ with coefficients $a_1, a_2, \ldots, a_n \in \mathbb{R}$ is the vector $\sum_{i=1}^{n} a_i v_i \in V$. $a_1v_1 + a_2v_2 + \ldots + a_nv_n$

A vector $v \in V$ is a linear combination of $v_1, v_2, \ldots, v_n$ if there exist some $a_1, a_2, \ldots, a_n \in \mathbb{R}$ so that $v = \sum_{i=1}^{n} a_i v_i$.




**Examples:** $v_1 \neq \alpha v_2$ for any $\alpha$

1) Let $v_1=(1,0)$ and $v_2=(0,1)$. Note: $v_1$ and $v_2$ are lin. independent

Can you write $(x,y)$ as a lin. comb. of $v_1$ and $v_2$? Yes

$(x,y) = \alpha_1 v_1 + \alpha_2 v_2$

$(x,y) \in \mathbb{R}^2$ $\quad = \alpha_1(1,0) + \alpha_2(0,1)$ $\quad \boxed{\begin{matrix}\alpha_1 = x \\ \alpha_2 = y\end{matrix}}$ $\quad \text{Span}\{v_1, v_2\} = \mathbb{R}^2$

$(x,y) \in \text{span}\{v_1,v_2\} = (\alpha_1, 0) + (0, \alpha_2)$

$\mathbb{R}^2 \subseteq \text{span}\{v_1,v_2\} = (\alpha_1, \alpha_2)$ $\qquad (-2,3) = -2v_1 + 3v_2$

$\Rightarrow x = \alpha_1,\ y = \alpha_2$

2) Let $v_1=(1, 2)$ and $v_2=(1,1)$. Let $v = (5, 6)$ Note: $v_1$ & $v_2$ are lin. ind.

Can you write $v$ as a lin. comb of $v_1$ and $v_2$? Yes

$(5,6) = \alpha_1(1,2) + \alpha_2(1,1)$

$(5,6) = (\alpha_1 + \alpha_2,\ 2\alpha_1 + \alpha_2)$

$\alpha_1 + \alpha_2 = 5$ $\quad \boxed{\alpha_1 = 1 \quad \alpha_2 = 4}$

$2\alpha_1 + \alpha_2 = 6$

3) Let $v_1=(1, -1)$ and $v_2=(-2,1)$. Let $v = (3, 2)$

write $v$ as a lin. comb of $v_1$ and $v_2$ (HW)

$v_1=(2, 3)$ $v_2=(4, 6)$ $\quad v_2 = 2v_1$

$v_1$ & $v_2 \rightarrow$ lin. dep.

$a_1 \neq 0 \quad a_2 \neq 0$

Note that if $v$, $v_1$ and $v_2$ are vectors such that $v = a_1v_1 + a_2v_2$, then each of the vector is a linear combination of the other two vectors.

Further 0 (vector) can be written as a linear combination of $v$, $v_1$ and $v_2$.

Case (i) zero coefficients

$0 = \underline{0}v + \underline{0}v_1 + \underline{0}v_2$

Case (ii) non-zero coefficients $\quad v = a_1v_1 + a_2v_2$

$0 = \underline{-1}\,v + \underline{a_1}v_1 + \underline{a_2}v_2$

$\{v, v_1, v_2\} \rightarrow$ lin. dep

Note: any three vectors in $\mathbb{R}^2$ are always lin. dep.

4) Example in $R^3$

$v_1 = (1, 0, 0)$, $v_2 = (0, 1, 0)$ and $v_3 = (0, 0, 1)$

$(x, y, z) = xv_1 + yv_2 + zv_3$

$v_1 \& v_2 \rightarrow$ lin. ind

$v_1 = (0, 2, 1)$, $v_2 = (2, 2, 0)$ and $v = (3, 7, 2)$ $\in \text{span}\{v_1, v_2\}$ ? LS

$(3, 7, 2) = a_1 v_1 + a_2 v_2$

$\text{Span}\{v_1, v_2\}$ = all possible lin. combinations of $v_1 \& v_2$

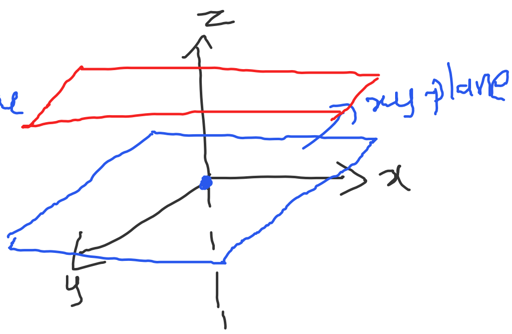
$= \{a_1 v_1 + a_2 v_2 : a_1, a_2 \in \mathbb{R}\}$

The plane that contains these $\underbrace{\text{two points}}_{v_1 \& v_2}$ is given by $\underbrace{2x-2y+4z = 0}_{\text{span}\{v_1, v_2\}}$.

Question 1: Let v be any linear combination of the vectors $v_1$ and $v_2$. Does v lie the plane containing $v_1$ and $v_2$? Yes

plane thru origin: $v = a_1 v_1 + a_2 v_2 \in$ plane

$0 \in \underbrace{\text{span}\{v_1, v_2\}}_{\text{plane}}$ ? yes




Question 2: Let $w = (1, 2, 0)$. Does this vector lie on the plane $2x-2y+4z = 0$?

Can we write w as a linear combination of $v_1$ and $v_2$? $\boxed{\text{No}}$

$2(1) - 2(2) + 4(0)$

$= 2 - 4 + 0 = -2 \neq 0$

w does not lie on the plane

$v_1 = (0, 2, 1)$ $v_2 = (2, 2, 0)$

$2(0) - 2(2) + 4(1) = 0$

$v_1, v_2 \Rightarrow$ lie on the plane



$2(2) - 2(2) + 4(0) = 0$

$(1,2,0) = \alpha_1 v_1 + \alpha_2 v_2$
$= \alpha_1(0,2,1) + \alpha_2(2,2,0)$
$= (0, 2\alpha_1, \alpha_1) + (2\alpha_2, 2\alpha_2, 0)$
$= (2\alpha_2, 2\alpha_1 + 2\alpha_2, \alpha_1)$

$2\alpha_2 = 1$ $\quad 2\alpha_1 + 2\alpha_2 = 2$ $\quad \boxed{\alpha_1 = 0}$ $\quad$ no solution
$\alpha_2 = 1/2$ $\quad 2(0) + 2(1/2) = 0 + 1 \neq 2$

Question 3: Can we write 0 as a linear combination of the vectors $v_1$, $v_2$ and w?

$\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 w = 0$ $\qquad w \neq \alpha_1 v_1 + \alpha_2 v_2$ for any $\alpha_1, \alpha_2$
$\Rightarrow \alpha_1 = \alpha_2 = \alpha_3 = 0$

Let $V_1 = (1,1)$, $V_2 = (1,0)$, and $V_3 = (0,1)$ be three vectors. Find out the correct set of options.

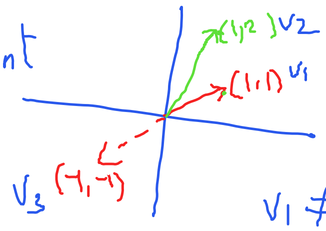
(a) $(2,3) = 2V_1 + 0V_2 + V_3$
(b) $(2,3) = 0V_1 + 2V_2 + 3V_3$
(c) $(2,3) = 2V_1 + V_2 + 0V_3$ $\quad (2,2) + (1,0) + (0,0) = (3,2)$
(d) $(2,3) = 0V_1 + 3V_2 + 2V_3$
$(3,0) + (0,2) = (3,2)$

In $\mathbb{R}^2$
1) $v_1 \& v_2$ — same line thru origin (or) scalar multiples of each other } lin. dep.

$v_1, v_3 \to$ lin. dependent

$\alpha_1 v_1 + \alpha_2 v_3 = 0$

$\alpha_1 = 1$ $\alpha_2 = 1$ $\to$ non-zero coefficients




$v_1$ & $v_2$ do not lie on the same line thru origin.

$v_1 \neq \alpha v_2$ for any $\alpha$.

$\alpha_1 v_1 + \alpha_2 v_2 = 0 \Rightarrow \alpha_1 = \alpha_2 = 0$

**Linearly dependent vectors**

A set of vectors $v_1, v_2, \ldots, v_n$ from a vector space $V$ is said to be $\boxed{\text{linearly dependent}}$, if there exist scalars $a_1, a_2, \ldots, a_n$, not all zero, such that

$v_1$ & $v_2$ lin. independent

$$a_1 v_1 + a_2 v_2 + \ldots + a_n v_n = 0$$

Example:

Consider the following two vectors in $\mathbb{R}^3$,

$$(2, 3, 7) \text{ and } \left(\tfrac{5}{3}, \tfrac{5}{2}, \tfrac{35}{6}\right).$$

Are these vectors linearly dependent?

Example:

Consider the following three vectors in $\mathbb{R}^3$,

$$(2, 1, 2), (3, 0, 1) \text{ and } (10, -4, -2)$$

Are these vectors linearly dependent?