Week 4 -

Today's Session:

* Basis of a vector space.

* How to find a basis for a subspace of $\mathbb{R}^n$.

* Dimension of a subspace in $\mathbb{R}^n$.

* Rank of a matrix.

* Span of vectors $\{v_1, \ldots, v_n\} \subset V$

$a_1 v_1 + a_2 v_2 + \cdots + a_n v_n \in V$

$\operatorname{Span}\{v_1, \ldots, v_n\} = \{ \sum_{i=1}^{n} a_i v_i \mid 1 \le i \le n \}$

* "Span of vectors" forms a subspace of vector space.

$\operatorname{span}\{\underline{(1,2)}, \underline{(2,4)}\} = \{a(1,2) + b(2,4) \mid a,b \in \mathbb{R}\}$

$= \{a(1,2) + 2b(1,2) \mid a,b \in \mathbb{R}\}$

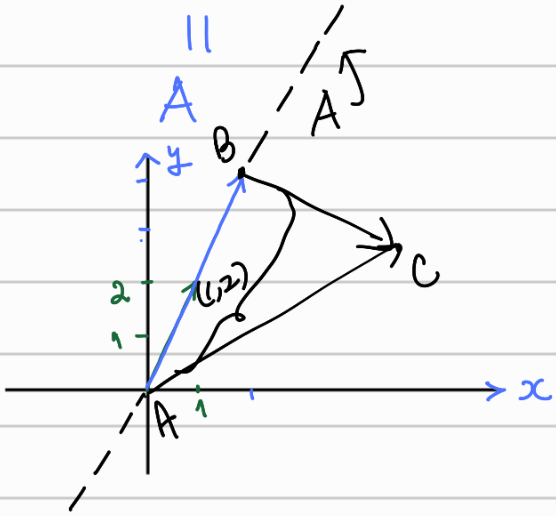
$= \{(a+2b)(1,2) \mid a,b \in \mathbb{R}\}$

$= \{t(1,2) \mid t \in \mathbb{R}\}$

$= \underbrace{\operatorname{span}(1,2)} \subset \mathbb{R}^2$

$\parallel$

A



Basis :

A set of vectors $\{v_1, .., v_n\} \subset V$ such that

(i) $\text{span}\{v_1, .., v_n\} = V$

(ii) $v_1, .., v_n$ are linearly independent.

$\hookrightarrow a_1v_1 + \cdots + a_nv_n = 0 \Leftrightarrow a_i = 0$.

Example :

(1) $\{(1,0,0), (0,1,0), (0,0,1)\}$ in $\mathbb{R}^3 \rightarrow$ Basis

(2) $\{(3,4,5), (-3,0,0), (0,4,5)\}$ in $\mathbb{R}^3 \rightarrow$ Not a basis

$$-(3,4,5) + (-3,0,0) - 1(0,4,5) = 0$$

$$(3,4,6) = a(3,4,5) + b(-3,0,0) + c(0,4,5)$$

$$\left.\begin{array}{lll} 3a - 3b = 3 & \rightsquigarrow & a - b = 1 \\ 4a + 4c = 4 & \rightsquigarrow & a + c = 1 \\ 5a + 5c = 6 & \rightsquigarrow & a + c = 6/5 \end{array}\right\}$$

$\rightarrow$ No solution.

$$S = \text{span}\{(3,4,5), (-3,0,0), (0,4,5)\}$$
$$\parallel$$
$$\text{span}\{(-3,0,0), (0,4,5)\}$$

$\dim S = 2$. $\square$

Dimension of a vector space:

Cardinality of the basis set is called the dimension of the vector space.

Equivalent def^n of basis:

(1) Set of vectors $\{v_1, \ldots, v_n\}$ that span$\{v_1, \ldots, v_n\} = V$ and are linearly independent.

(2) A minimal spanning set of vectors of V is basis.

(3) A maximal linearly independent set of vectors form a basis.

Does $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ form a basis of $M_{2\times2}(\mathbb{R})$?

$$M_{2\times2}(\mathbb{R}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \,\middle|\, a, b, c, d \in \mathbb{R} \right\}$$

$\hookrightarrow$ Vector space : $\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} = \begin{bmatrix} a+a_1 & b+b_1 \\ c+c_1 & d+d_1 \end{bmatrix}$

$$k \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix} \quad k \in \mathbb{R}.$$

$$a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$\square$

find out dimension of

$$V = \{ M_{3\times 2} \mid \text{sum of each row is } 0 \}.$$

$$V = \left\{ \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}_{3\times 2} \;\middle|\; \begin{matrix} a+b=0 \Rightarrow b=-a \\ c+d=0 \Rightarrow c=-d \\ e+f=0 \Rightarrow f=-e \end{matrix} \right\}$$

$$\downarrow$$

$$V = \left\{ \begin{bmatrix} a & -a \\ c & -c \\ e & -e \end{bmatrix} \;\middle|\; a,c,e \in \mathbb{R} \right\}$$

$$V = \left\{ a \begin{bmatrix} 1 & -1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 1 & -1 \\ 0 & 0 \end{bmatrix} + e \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & -1 \end{bmatrix} \;\middle|\; a,c,e \in \mathbb{R} \right\}$$

dimension is 3.

How to find a basis vector subspace of $\mathbb{R}^n$? ?

Consider, for example, span of

$$\{(3,4,5),\ (1,1,2),\ (0,0,1),\ (4,0,1)\} \subset \mathbb{R}^3.$$

$$S = \text{span}\{(\quad),(\quad),(\quad),(\quad)\} \subset \mathbb{R}^3$$

$$A = \begin{bmatrix} 3 & 4 & 5 \\ 1 & 1 & 2 \\ 0 & 0 & 1 \\ 4 & 0 & 1 \end{bmatrix} \xrightarrow[R_3 \leftrightarrow R_4]{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 1 & 2 \\ 3 & 4 & 5 \\ 4 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow[R_3 - 4R_1]{R_2 - 3R_1} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -1 \\ 0 & -4 & -7 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_3/4} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -1 \\ 0 & 1 & 7/4 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R_3 - R_2} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 11/4 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{4R_3/11} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Big\downarrow R_4 - R_3$$

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\text{span}\{(1,1,2),(0,1,-1),(0,0,1)\} = S$$

— Independent

$\dim S = 3.$

Method 1: (Writing as rows) $\{v_1, .., v_n\} \subset \mathbb{R}^n$. Then $S = \text{span}\{v_1, .. , v_n\}$.

(1) Form a matrix A with each vector as a row.

(2.) Perform row reduction on A to obtain its REF or RREF

(3.) The non zero rows of REF(A) or RREF(A) form a basis of span of vectors.

Method 2:

$$\{(3,4,5,6), (1,2,3,4), (0,0,0,1), (1,2,-1,0), (2,2,2,2)\} \subset \mathbb{R}^4$$

Find out the basis of span of above vectors.

(1) Write vectors as a column of a matrix A

(2.) Apply row reductions on A. and obtain a REF/RREF.

(3.) Columns of A corresponding to columns of REF/RREF having leading entry/pivot form the basis of the span.

$$A = \begin{bmatrix} 3 & 1 & 0 & 1 & 2 \\ 4 & 2 & 0 & 2 & 2 \\ 5 & 3 & 0 & -1 & 2 \\ 6 & 4 & 1 & 0 & 2 \end{bmatrix} \xrightarrow[R_2/2]{R_1/3} \begin{bmatrix} 1 & 1/3 & 0 & 1/3 & 2/3 \\ 2 & 1 & 0 & 1 & 1 \\ 5 & 3 & 0 & -1 & 2 \\ 6 & 4 & 1 & 0 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1/3 & 0 & 1/3 & 2/3 \\ & & & & \\ & & & & \\ & & & & \end{bmatrix}$$

$$\{(3,4,5),(1,1,2),(0,0,1),(4,0,1)\}\subset\mathbb{R}^3$$

$$\begin{pmatrix}3&1&0&4\\4&1&0&0\\5&2&1&1\end{pmatrix}\xrightarrow{R_1/3}\begin{pmatrix}1&1/3&0&4/3\\4&1&0&0\\5&2&1&1\end{pmatrix}$$

$$\xrightarrow[R_3-5R_1]{R_2-4R_1}\begin{pmatrix}1&1/3&0&4/3\\0&-1/3&0&-16/3\\0&1/3&1&-17/3\end{pmatrix}$$

$$\xrightarrow{R_3+R_2}\begin{pmatrix}1&1/3&0&4/3\\0&-1/3&0&-16/3\\0&0&1&-11\end{pmatrix}\xrightarrow{-3R_2}\begin{pmatrix}1&1/3&0&4/3\\0&1&0&16\\0&0&1&-11\end{pmatrix}$$

$$=\{(3,4,5),(1,1,2),(0,0,1)\}$$

Rank of a matrix:

$$A = \begin{pmatrix} 3 & 2 & 1 \\ 4 & 5 & 6 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}_{4\times 3} \qquad A_{m\times n}$$

Column space := span of columns of A.

$$= \{ a_1(3,4,0,0) + a_2(2,5,0,1) + a_3(1,6,1,0) \mid a_i \in \mathbb{R} \}$$

- Column space is sub sp. of $\mathbb{R}^m$.

Row space = span of rows of A.

$$= \{ a_1(3,2,1) + a_2(4,5,6) + a_3(0,0,1) + a_4(0,1,0) \mid a_i \in \mathbb{R} \}.$$

- Row space is a subsp. of $\mathbb{R}^n$.

* Dimension of column sp / row sp of $A_{m\times n}$ makes sense.

* Column rank (A) = dimension of column sp. of A

Row rank (A) = dimension of row sp. of A.

$\boxed{\text{Theorem: Row rank (A) = Column Rank (A).}}$

Def: Rank of a matrix A = row rank (A) = column rank (A)

$$A=\begin{pmatrix}3&2&1\\4&5&6\\0&0&1\\0&1&0\end{pmatrix}\xrightarrow{R_1/3}\begin{pmatrix}1&2/3&1/3\\4&5&6\\0&0&1\\0&1&0\end{pmatrix}$$

$$\xrightarrow{R_2-4R_1}\begin{pmatrix}1&2/3&1/3\\0&7/3&14/3\\0&0&1\\0&1&0\end{pmatrix}\xrightarrow{3R_2/7}\begin{pmatrix}1&2/3&1/3\\0&1&2\\0&0&1\\0&1&0\end{pmatrix}$$

$$\xrightarrow{R_4\leftrightarrow R_2}\begin{pmatrix}1&2/3&1/3\\0&1&0\\0&0&1\\0&1&2\end{pmatrix}\xrightarrow{R_4-2R_3}\begin{pmatrix}\ \end{pmatrix}$$

$$\xrightarrow{R_4-R_2}\begin{pmatrix}1&2/3&1/3\\0&1&0\\0&0&1\\0&0&0\end{pmatrix}$$ REF

Rank of $A = 3$.

Q1. $A\in M_{n\times n}(\mathbb{R})$, $A^3=A$, rank A^2 and A ?

Q2. $$\begin{bmatrix}1&-1/3&-1/2&-1/2\\-1/2&1&-1/2&-1/2\\0&-1/3&1&0\\-1/2&-1/3&0&1\end{bmatrix}\xrightarrow[+1/2R_1]{R_2}\begin{matrix}1&-1/3&-1/2&-1/2\\0&1-1/6&-1/2-1/4&-1/2-1/4\end{matrix}$$