Week 7

* Affine subspace and mapping:

Consider a line ℓ in $\mathbb{R}^2$ defined as:

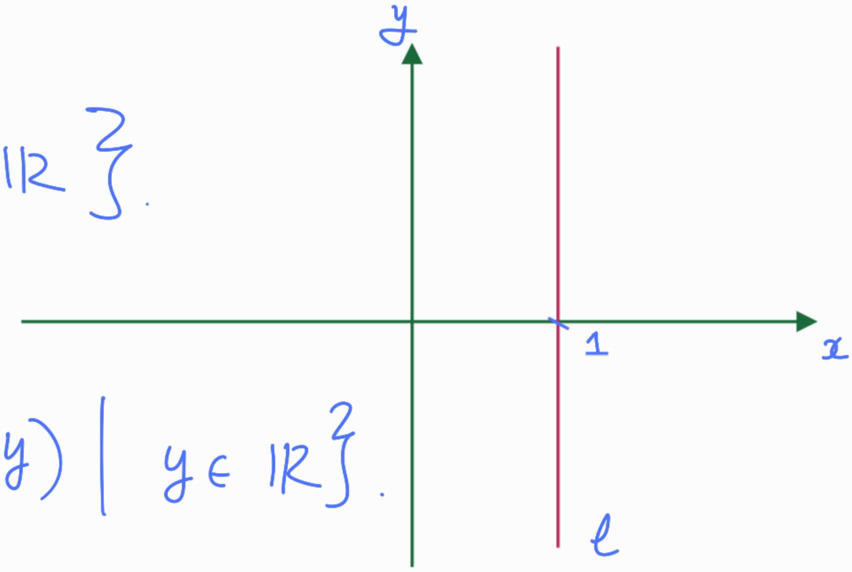
$$\ell := \{ (\underline{\underline{1}}, y) \mid y \in \mathbb{R} \}.$$

Is $\ell \subset \mathbb{R}^2$ a subspace?

No.

Then what is ℓ?

$$\ell := \{ (1, y) \mid y \in \mathbb{R} \}.$$



$$\ell := (1,0) + \{ (0,y) \mid y \in \mathbb{R} \}.$$

$\ell = v + \ell'$, for some $v \in \mathbb{R}^2$ and subsp $\ell' \subset \mathbb{R}^2$.

$v = (1,0)$ is <u>NOT</u> unique!!

$\ell := (1,1) + \ell'$

$\ell := (1,2) + \ell'$ and for any $v \in \ell$,

$\ell := v + \ell'$.

But subsp. ℓ is unique!!

In general,

Affine Subspaces

Let V be a vector space. An affine subspace of V is a subset L such that there exists $v \in V$ and a vector subspace $U \subseteq V$ such that

$$L = v + U := \{v + u | u \in U\}.$$

In general,
for any $v \in L$, we've $L = v + U$.
and subsp. $U \leq V$ is unique.

Some Examples:

(1) Every subspace of a Vector space is an Affine subsp.
$U \leq V$. Take $0 \in U$ and $U = 0 + U$
take $v \in U$, $U = v + U$.

(2) $V = \mathbb{R}^2$: . point, $\{(x,y)\} := (x,y) + \{(0,0)\}$
. any line $\ell = \{(x, mx+c)\} = (0,c) + \{(x, mx)\}$
. $\mathbb{R}^2$

(3) $V = \mathbb{R}^3$: point $\{(x,y,z)\} = (x,y,z) + \{(0,0,0)\}$.
any line
any plane
$\mathbb{R}^3$.

(4) Solution space of linear Equations:

$$\underset{m\times n}{A}\,\bar{x} = \bar{b}.$$

soluton space is shifted version of null space

e.g. $\begin{aligned} 3x+2y &= 1 \\ x+y &= 0 \end{aligned} \rightsquigarrow \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$x=1,\ y=-1$ is a solution (Particular Sol^n).

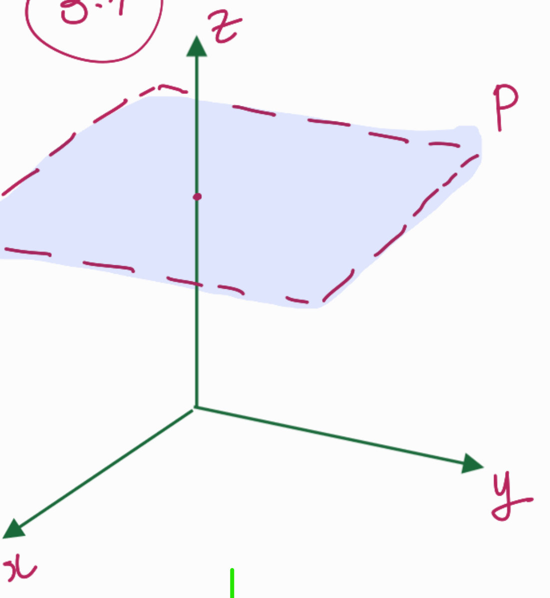
Taking $b=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$, we get null space $N(A) \leq \mathbb{R}^2$.

Thus, sol^n space $:= \underline{(1,-1)+N(A)}$.

$(1,-1)+(0,0)$

"An affine subspace"

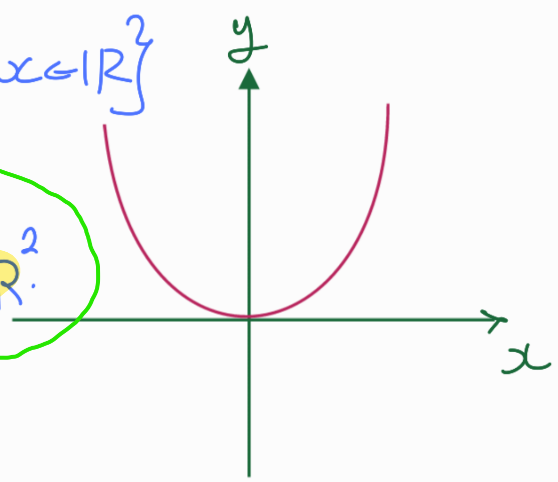
(3.1)



$$P := \{(x,y,2) \mid x,y\in\mathbb{R}\}$$
$$= \underline{(0,0,2)+\{(x,y,0) \mid x,y\in\mathbb{R}\}}$$

What is NOT an affine subsp?

* $L := \{(x,x^2) \mid x\in\mathbb{R}\}$



$L \neq v+U$

for any $v\in\mathbb{R}^2$ and $U\leq\mathbb{R}^2$.

Consider $A\bar{x}=\bar{b}$, suppose $x_0=(a_1,\dots,a_n)$ is a particular solution to the system. Then

<u>Claim:</u> for $v\in N(A)$, $v+x_0$ is also a solution to $A\bar{x}=\bar{b}$.

$$A(v+x_0) = Av + Ax_0 = 0 + Ax_0 = 0+b=b$$

$$(\because\ v\in N(A) \Rightarrow Av=0)$$

$$\Rightarrow\ v+x_0 \in \mathrm{sol}(A\bar{x}=\bar{b})$$

Thus, $x_0+N(A) \subseteq \mathrm{sol}(A\bar{x}=\bar{b})$. $\longrightarrow (1)$

Let $w\in \mathrm{sol}(A\bar{x}=\bar{b})$, $w-x_0\in N(A)$.

$$A(w-x_0) = Aw - Ax_0 = b-b = 0$$

$\Rightarrow (w-x_0)\in N(A)$. Thus, $w = x_0+N(A)$

$$\Rightarrow\ \mathrm{sol}(A\bar{x}=\bar{b}) \subseteq x_0+N(A) \longrightarrow (2)$$

By (1) & (2): $\mathrm{Sol}(A\bar{x}=\bar{b}) = x_0+N(A)$. $\square$

Affine mapping:

Consider a map

$$T : \mathbb{R}^3 \longrightarrow \mathbb{R}^2 \quad \text{defined as:}$$

$$T(x,y,z) := (x+y+1,\ z+3).$$

* T is Not a linear map.

* $$T(x,y,z) = (1,3) + (x+y,\ z)$$
$$= (1,3) + T'(x+y,\ z) \longrightarrow (1)$$

$T' : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ is a linear map.

$$T(0,0,0) = (1,3) \longrightarrow (2)$$

$$\mathbb{R}^3 := (0,0,0) + \mathbb{R}^3$$

$$(1) \ \& \ (2) \Rightarrow \underbrace{T(x,y,z)}_{T((0,0,0)+(x,y,z))} = \underbrace{T(0,0,0)}_{\text{vector}} + \underbrace{T'(x,y,z)}_{\text{linear map}}.$$

$$T(x,y,z) - T(0,0,0) = T'(x,y,z)$$

Such maps are called Affine mapping between two affine spaces.

In this example, affine spaces are $\mathbb{R}^3$ (domain) and $\mathbb{R}^2$ (codomain).

Affine mappings of affine subspaces

Let L and L' be affine subspaces of V and W respectively. Let $f : L \to L'$ be a function. Consider any vector $v \in L$ and the unique subspace $U \subseteq V$ such that $L = v + U$. Note that $f(v) \in L'$ and hence $L' = f(v) + U'$ where U' is the unique subspace of W corresponding to L'. Then f is an affine mapping from L to L' if the function $g : U \to U'$ defined by $g(u) = f(u + v) - f(v)$ is a linear transformation.

$f : L \longrightarrow L'$

$T : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$

$T(v) = T(0,0,0) = (1,3) \in L' = \mathbb{R}^2$

$v = (0,0,0) + \underbrace{\mathbb{R}^3}_{U}$

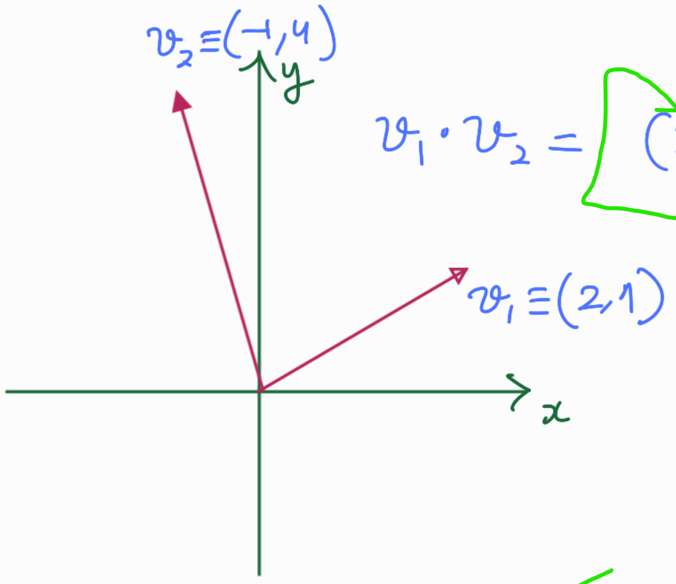
$L' = \mathbb{R}^2 = T(0,0,0) + \mathbb{R}^2$

$T(x,y,z) = (x+y+1, z+3)$

$T'(x,y,z) = T((x,y,z)+(0,0,0)) - T(0,0,0)$

$= T(x,y,z) - (1,3) = (x+y, z)$

Dot product : Length and Angle (in $\mathbb{R}^2$)



Dot product :

$$v_1 \cdot v_2 = \boxed{(x_1, y_1) \cdot (x_2, y_2) = x_1 x_2 + y_1 y_2}$$

$$= 2.$$

Length :

$$\left.\begin{array}{l} \|v_1\| = \sqrt{v_1 \cdot v_1} \\ \|(x_1, y_1)\| = \sqrt{(x_1, y_1) \cdot (x_1, y_1)} \end{array}\right\} = \sqrt{x_1^2 + y_1^2}$$

$$\|v_1\| = \sqrt{5}, \quad \|v_2\| = \sqrt{17}$$

Angle :

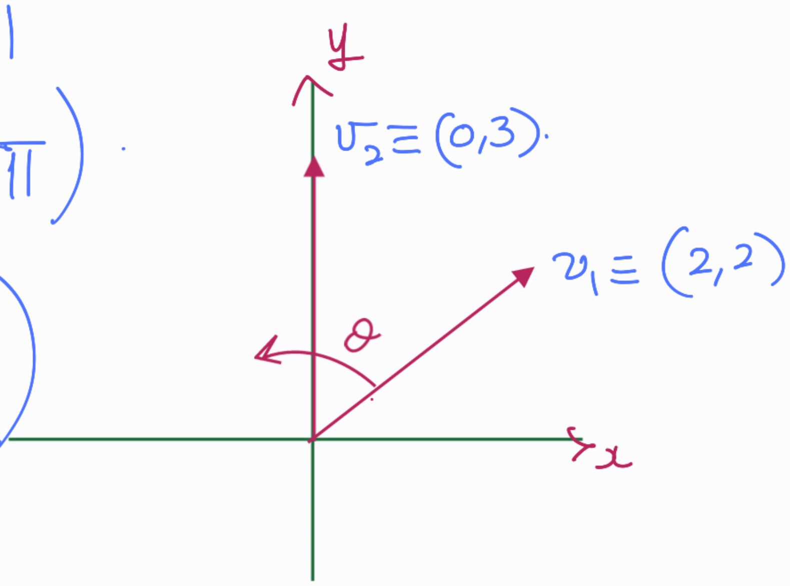
$$\cos\theta = \frac{v_1 \cdot v_2}{\|v_1\| \, \|v_2\|}$$

$$\theta = \cos^{-1}\left(\frac{v_1 \cdot v_2}{\|v_1\| \, \|v_2\|}\right).$$

$$\theta = \cos^{-1}\left(\frac{\cancel{6}}{\cancel{2}\sqrt{2} \cdot \cancel{3}}\right)$$

$$\theta = 45^\circ$$

$\square$.



Inner product on a Vector space V.

Inner product on a vector space

An inner product on a vector space V is a function $\langle\, .\, ,\, .\, \rangle : V \times V \to \mathbb{R}$ satisfying the following :

- $\langle v, v\rangle > 0$ for all $v \in V \setminus \{0\}$; $\langle v, v\rangle = 0$ if and only if $v = 0$.
- $\langle v_1 + v_2, v_3\rangle = \langle v_1, v_3\rangle + \langle v_2, v_3\rangle$ $\longrightarrow$ $\langle v_1, v_2+v_3\rangle = \langle v_1, v_2\rangle + \langle v_1, v_3\rangle$
- $\langle v_1, v_2\rangle = \langle v_2, v_1\rangle$
- $\langle cv_1, v_2\rangle = c\langle v_1, v_2\rangle = \langle v_1, cv_2\rangle$. where $c \in \mathbb{R}$.

A vector space V with an inner product $\langle\cdot,\cdot\rangle : V\times V \to \mathbb{R}$ is called an inner product space. We denote it by $(V, \langle\cdot,\cdot\rangle)$.

Example:

(1) Dot product is an inner product on $\mathbb{R}^2$:

$\langle v, w\rangle = v\cdot w$; $\quad v\cdot v = 0 \Rightarrow v = 0$

$\langle v_1+v_2, v_3\rangle = (v_1+v_2)\cdot v_3 = v_1\cdot v_3 + v_2\cdot v_3$

$\langle v_1, v_2\rangle = v_1\cdot v_2 = v_2\cdot v_1$

$(\mathbb{R}^2, \text{dot product})$.

(2) Dot product on a $\mathbb{R}^n$:

$(x_1, x_2, \cdots, x_n)\cdot(y_1, y_2, \cdots, y_n) := \sum_{i=1}^{n} x_i y_i$

* Exercise: dot product on $\mathbb{R}^n$ is an inner product on $\mathbb{R}^n$.

(3) $\langle \cdot , \cdot \rangle : \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}$.

$$\langle (x_1, y_1), (x_2, y_2) \rangle := x_1 y_1 - (x_1 y_2 + x_2 y_1) + 2 x_2 y_2 .$$

$$\langle (x, y), (x, y) \rangle = xy - (xy + xy) + 2xy$$

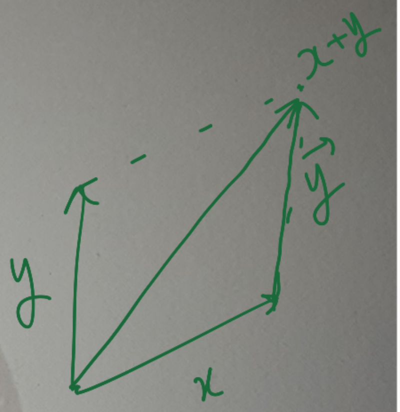
$$= xy = 0 \Leftrightarrow x = 0 \text{ or } y = 0 .$$

$v = (1, 0)$, $\langle (1,0), (1,0) \rangle = 0$ but $(1,0) \neq (0,0)$

So, $\langle \cdot , \cdot \rangle$ is NOT an inner product. $\square$.

Norm on a vector space

A norm on a vector space V is a function



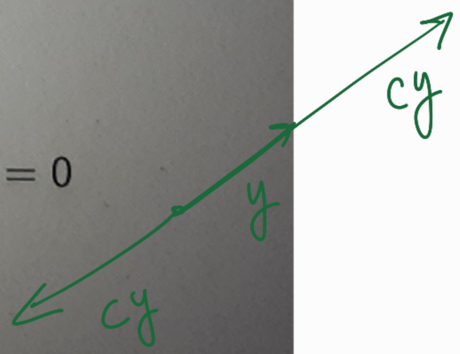
$$||.|| : V \to \mathbb{R}$$

$$x \mapsto ||x||$$

satisfying the following conditions:

$||x+y|| \leq ||x|| + ||y||$.

- $||x+y|| \leq ||x|| + ||y||$, for all $x, y \in V$
  $v+w \qquad v \qquad w$
- $||cx|| = |c|||x||$ for all $c \in \mathbb{R}$ and for all $x \in V$
  $cv \qquad |c|\,||v||$
- $||x|| \geq 0$ for all $x \in V$; $||x|| = 0$ if and only if $x = 0$
  $||v|| \qquad v \qquad ||v|| = 0$ iff $v = 0$.



Example:

(1) Length is a norm on $\mathbb{R}^2$.

$$||v|| = ||(x,y)|| = \sqrt{x^2+y^2}.$$

$$\underbrace{||(x_1,y_1)+(x_2,y_2)||}_{||} \leq ||(x_1,y_1)|| + ||(x_2,y_2)||$$

(2) Length of a vector in $\mathbb{R}^n$:

$(x_1, \ldots, x_n) \in \mathbb{R}^n$.

$$\|(x_1, \ldots, x_n)\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$$

$\bar{x}, \bar{y} \in \mathbb{R}^n$

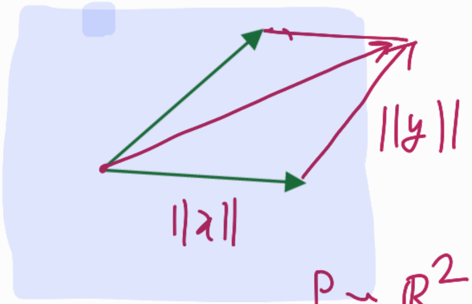
(i) $\bar{y} = c\,\bar{x}$ for some $c \in \mathbb{R}$.

$$\|\bar{x}+\bar{y}\| = \|(c+1)\bar{x}\| = |c+1|\,\|\bar{x}\| = |c|\,\|x\| + \|x\|$$
$$= \|y\| + \|x\|$$

(ii) $\bar{y} \neq c\,\bar{x}$ for all $c \in \mathbb{R}$.

$\text{span}\{\bar{y}, \bar{x}\} = \text{plane in } \mathbb{R}^n$.

$$\|\bar{x}+\bar{y}\| < \|\bar{x}\| + \|\bar{y}\|$$



$P \cong R^2$

$\square$

(3) Defining Norm on an inner product space $(V, \langle\cdot,\cdot\rangle)$.

Let $v \in V$,

$$\|v\| := \sqrt{\langle v, v\rangle}$$

$$\|v+w\|^2 = \langle v+w, v+w\rangle = \langle v, v+w\rangle + \langle w, v+w\rangle$$
$$= \langle v, v\rangle + \langle v, w\rangle + \langle w, v\rangle + \langle w, w\rangle$$
$$= \|w\|^2 + \|v\|^2 + 2\langle v, w\rangle$$

Check: other conditions for norm.

(4) On $\mathbb{R}^3$, define

$$\|\cdot\|_\infty : \mathbb{R}^3 \longrightarrow \mathbb{R} \quad \text{as follow:}$$

$$\|(x,y,z)\|_\infty := \max\{|x|, |y|, |z|\}.$$

$$\|(x_1,y_1,z_1)+(x_2,y_2,z_2)\|_\infty = \max\{|x_1+x_2|, |y_1+y_2|, |z_1+z_2|\}$$

without loss of generality, assume that

$$= |x_1+x_2| \le |x_1| + |x_2|$$

$$|x_1| \le \|(x_1,y_1,z_1)\|_\infty, \quad |x_2| \le \|(x_2,y_2,z_2)\|_\infty$$

$$\|(x_1,y_1,z_1)\|_\infty + \|(x_2,y_2,z_2)\|_\infty$$

$$\|(x_1,y_1,z_1)\|_\infty = \max\{|x_1|, |y_1|, |z_1|\}$$

$$\|v+w\|_\infty \le \|v\|_\infty + \|w\|_\infty. \qquad \square$$

Check: the other two conditions. (Exercise)

Exercises:

(1) Consider a function,

$$\langle \cdot , \cdot \rangle_c : \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}.$$

$$\langle (x_1, y_1), (x_2, y_2) \rangle_c = (c-5)(x_1 y_1 + x_2 y_2)$$

For what value of c, $\langle \cdot , \cdot \rangle_c$ does not define an inner product?