Math 2 Week 1 : Vector and matrices

Lecture 1 : Vectors

Contents

- Vectors and data
- Why vectors?
- Examples
- Vectors and visualization
- Vectors in the physical context

Vectors and data

Often we encounter data in a table. For example :

Financial Year	Gross Domestic Product (in Rs. Cr) at 2004-05 Prices	Agriculture & Allied Services (in Rs. Cr.) at 2004-05 Prices	Agriculture (in Rs. Cr.) at 2004-05 Prices	Industry (in Rs. Cr.) at 2004-05 Prices	Mining and Quarrying (in Rs. Cr.) at 2004-05 Prices	Manufacturing (in Rs. Cr.) at 2004-05 Prices	Services (in Rs. Cr.) at 2004-05 Prices
2000-01	2342774	522755	439432	640043	69472	363163	1179976
2001-02	2472052	554157	467815	656737	70766	371408	1261158
2002-03	2570690	517559	429752	704095	76721	396912	1349035
2003-04	2777813	564391	476324	755625	78792	422062	1457797
2004-05	2971464	565426	476634	829783	85028	453225	1576255
2005-06	3253073	594487	502996	910413	86141	499020	1748173
2006-07	3564364	619190	523745	1021204	92578	570458	1923970
2007-08	3896636	655080	556956	1119995	95997	629073	2121561
2008-09	4158676	655689	555442	1169736	98055	656302	2333251
2009-10	4516071	660987	557715	1276919	103830	730435	2578165
2010-11	4937006	713477	606848	1393879	108938	801476	2829650
2011-12	5243582	739495	630540	1442498	108249	823023	3061589
2012-13	5503476	752746		1487533	108713	838541	3263196

India's GDP data from 2000-01 to 2012-3 with sector wise break-ups

Another example :

vs Teams	V.Kohli	M.S.Dhoni	R.Sharma	K.L.Rahul	S.Dhawan
Australia	54.57	44.86	61.33	45.75	45.80
England	45.30	46.84	50.44	6.60	32.45
New Zealand	59.91	49.47	33.47	68.33	32.72
South Africa	64.35	31.92	33.30	26.00	49.87
Sri Lanka	60.00	64.40	46.25	34.75	70.21
Pakistan	48.72	53.52	51.42	57.00	54.28

Team-wise batting averages

Vectors and data (Contd.)

A vector can be thought of as a list. In the context of the above examples, vectors could be columns or rows.

2010-11	4937006	713477	606848	1393879	108938	801476	2829650

South Africa	64.35	31.92	33.30	26.00	49.87

South Africa	64.35	31.92	33.30	26.00	49.87

(64.35, 31.92, 33.3, 26, 49.87) ← row vector OR row matrix

Gross Domestic Product (in Rs. Cr) at 2004-05 Prices
2342774
2472052
2570690
2777813
2971464
3253073
3564364
3896636
4158676
4516071
4937006
5243582
5503476

column vector OR column matrix →

$$\begin{bmatrix} 54.57 \\ 45.3 \\ 59.91 \\ 64.35 \\ 60 \\ 48.72 \end{bmatrix}$$

V.Kohli
54.57
45.30
59.91
64.35
60.00
48.72

Why Vectors

Vectors can be used to perform arithmetic operations on lists such as the table columns or rows e.g. suppose we want the average sectoral GDP across the years 200-01 to 2009-10.

sectoral GDP across the years 200-01 to 2009-10.

Financial Year	Gross Domestic Product (in Rs. Cr) at 2004-05 Prices	Agriculture & Allied Services (in Rs. Cr.) at 2004-05 Prices	Agriculture (in Rs. Cr.) at 2004-05 Prices	Industry (in Rs. Cr.) at 2004-05 Prices	Mining and Quarrying (in Rs. Cr.) at 2004-05 Prices	Manufacturing (in Rs. Cr.) at 2004-05 Prices	Services (in Rs. Cr.) at 2004-05 Prices
2000-01	2342774	522755	439432	640043	69472	363163	1179976
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2010-11	4937006	713477	606848	1393879	108938	801476	2829650

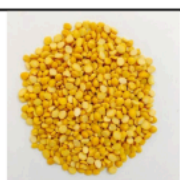
$\frac{1}{10} \times$ Total $\frac{1}{10} \times$ Total $\frac{1}{10} \times$ Total

$+ (\quad) + (\quad) + \ldots + (\quad) = (\quad)$

$\frac{1}{10} \times (\quad)$

Example 1

Arun has to buy 3 kg rice and 2 kg dal and Neela has to buy 5 kg rice and 6 kg dal.

Items	Arun	Neela	Total
 Rice in kg	3	5	8
 Dal in kg	2	6	8

$\begin{pmatrix} 3 \\ 2 \end{pmatrix} + \begin{pmatrix} 5 \\ 6 \end{pmatrix} = \begin{pmatrix} 8 \\ 8 \end{pmatrix}$

Then the vectors $(3, 2)$ for Arun and $(5, 6)$ for Neela represent their demands. We can add these vectors to get $(3, 2) + (5, 6) = (8, 8)$. This says that together they have to buy 8 kg rice and 8 kg dal.

Example 2

Stock taking in a grocery shop :

Items	In stock	Buyer A	Buyer B	Buyer C	New stock
 Rice in kg	150	8	12	3	100
 Dal in kg	50	8	5	2	75
 Oil in Litres	35	4	7	5	30
 Biscuits in packets	70	10	10	5	80
 Soap Bars	25	4	2	1	30

Example 2 contd. : addition of vectors

Taking stock of the items in the grocery shop can be done easily using vector representation :

$$(150, 50, 35, 70, 25) + (-8, -8, -4, -10 - 4) +$$
$$(-12, -5, -7, -10, -2) + (-3, -2, -5, -5, -1) +$$
$$(100, 75, 30, 80, 30) = (227, \ldots\ldots, 48)$$

Note that we add corresponding entries of the vectors. This is an example of addition of vectors.

Example 2 contd. : scalar multiplication

If Buyer A comes the next day and buys the same items in the same quantities then we can add the vector two times or multiply each co-ordinate of the vector by 2.

$$(8,8,4,10,4) + (8,8,4,10,4) = (16,16,8,20,8) = 2(8,8,4,10,4)$$

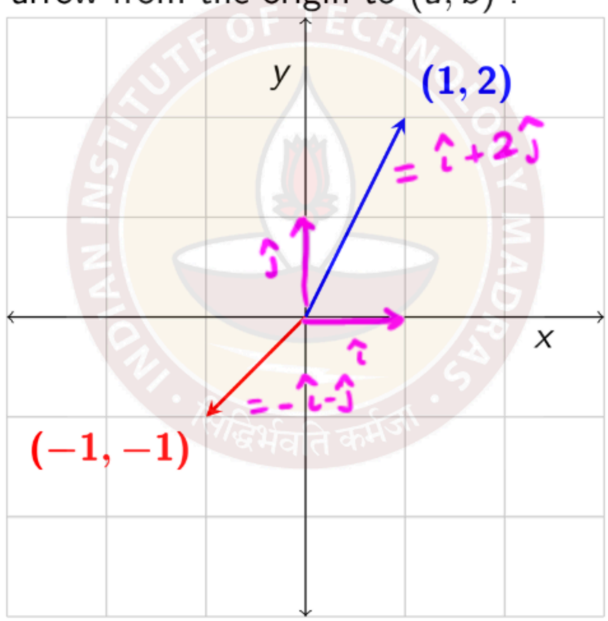
Multiplying a vector by a scalar (i.e. all its entries if it is a list) is called scalar multiplication.

$$c\,(v_1, v_2, \dots, v_n) = (cv_1, cv_2, cv_3, \dots, cv_n).$$

Visualization of vectors in R^2

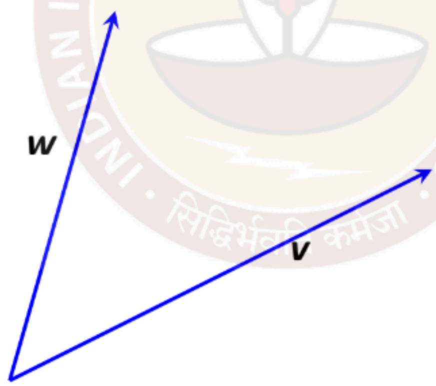
Point $(a, b) \equiv$ Vector $(a, b) \equiv a\hat{\mathbf{i}} + b\hat{\mathbf{j}}$.

Visualization : arrow from the origin to (a, b) .



Visualization of Vector Addition in R^2

We can add two vectors by joining them head-to-tail or by parallelogram law.



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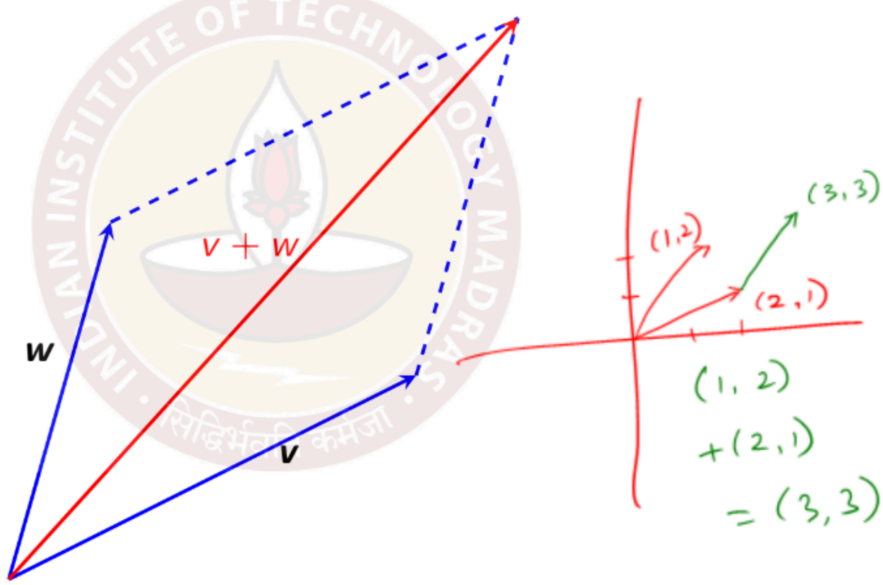


Vectors in R^n

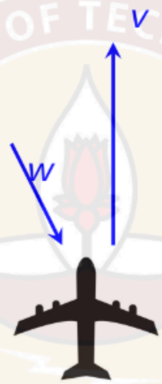
Vectors in $\mathbb{R}^n$ are lists (or rows or columns) with n real entries.

Vectors with n entries $\equiv$ Vectors in $\mathbb{R}^n$ $\equiv$ Points in $\mathbb{R}^n$.

Vectors in the physical context

A vector has magnitude (size) and direction. The length of the line shows its magnitude and the arrowhead points in the direction.

Example: A plane is flying towards the north and wind is blowing from the North-West.



$v+w$ is the direction in which the plane moves.

$v=$ velocity of the flight and $w=$ velocity of the wind

Some examples of vectors which appear in physics :

- Velocity
- Acceleration
- Force

FOR THIS COURSE REMEMBER THAT VECTORS MEAN ROWS OR COLUMNS OF NUMBERS.

Lecture 2 : Matices

Contents

- What is a matrix?
- Related terms.
- Diagonal and scalar matrices.
- The identity matrix.
- Linear equations and matrices.
- Addition of matrices.
- Scalar multiplication.
- Multiplication of matrices.
- Properties of matrix addition and multiplication.

What is a matrix?

Definition

A matrix is a rectangular array of numbers, arranged in rows and columns. (plural : matrices)

$$\text{Example: } \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix}_{2\times 3}$$

This is a 2×3 matrix (2 rows and 3 columns).

- An $m \times n$ matrix has m rows and n columns.

Example: $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix}_{2\times 3}$

This is a 2×3 matrix (2 rows and 3 columns).

(1, 2)-th entry of this matrix is 2.
(2, 3)-th entry of " " is 4.

- An $m \times n$ matrix has m rows and n columns.
- (i, j)-th entry of a matrix is the entry occuring in the i-th row and j-th column.

Square matrices

- A square matrix is a matrix in which the number of rows is the same as the number of columns.

  Example: $\begin{bmatrix} 0.3 & 5 & -7 \\ 2.8 & 0 & 1 \\ 0 & -2.5 & -1 \end{bmatrix}_{3\times 3}$

  This is a 3×3 matrix (3 rows and 3 columns).
- The i-th diagonal entry of a square matrix is the (i, i)-th entry.
- The diagonal of a square matrix is the set of diagonal entries.

Diagonal Matrices

Definition

A square matrix in which all entries except the diagonal are 0 is called a diagonal matrix.

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 4.2 \end{bmatrix}$$

Scalar Matrices

Definition

A diagonal matrix in which all the entries in the diagonal are equal is called a scalar matrix.

$$S = \begin{bmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{bmatrix} \quad 3 \times 3$$

The identity matrix

Definition

The scalar matrix with all diagonal entries 1 is called the identity matrix and is denoted by I.

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad 3 \times 3$$

$$I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Linear equations and matrices

A set of linear equations can be represented in terms of matrices.

Example

$$3x + 4y = 5$$
$$4x + 6y = 10$$

can be represented by the matrix

$$\left[\begin{array}{cc|c} \boxed{\begin{matrix} 3 & 4 \\ 4 & 6 \end{matrix}} & & \begin{matrix} 5 \\ 10 \end{matrix} \end{array}\right] \quad 2 \times 3$$

$$\begin{bmatrix} 3 & 4 & 5 \\ 4 & 6 & 10 \end{bmatrix}$$

Addition of matrices

$$\text{Example: } \begin{bmatrix} 1 & 9 \\ 0.6 & 7 \\ 4 & 1.5 \end{bmatrix} + \begin{bmatrix} 0 & 7 \\ 0.6 & -7 \\ 2.5 & 0.6 \end{bmatrix} = \begin{bmatrix} 1 & 16 \\ 1.2 & 0 \\ 6.5 & 2.1 \end{bmatrix}$$

Definition

The sum of two $m \times n$ matrix A and B is calculated entrywise : the (i,j)-th entry of the matrix $A+B$ is the sum of (i,j)-th entry of A and (i,j)-th entry of B

$$(A+B)_{ij} = A_{ij} + B_{ij}$$

$$\text{Example: } \underbrace{\begin{bmatrix} 1/2 & -3/4 & 3 \end{bmatrix}}_{1\times 3} + \underbrace{\begin{bmatrix} 2 & -3 & -1 \end{bmatrix}}_{1\times 3} = \underbrace{\begin{bmatrix} 5/2 & -15/4 & 2 \end{bmatrix}}_{1\times 3}$$

Scalar multiplication (Multiplying a matrix by a number)

Example: $3\begin{bmatrix}1 & 2 & 3\\ 4 & 5 & 6\end{bmatrix} = \begin{bmatrix}3 & 6 & 9\\ 12 & 15 & 18\end{bmatrix}$

Definition

The product of a matrix A with a number c is denoted by cA and the (i,j)-th entry of cA is product of (i,j)-th entry of A with the number c.

$$(cA)_{ij} = c(A_{ij})$$

Matrix multiplication (multiplying two matrices)

Definition

$A_{m\times n}B_{n\times p} = (AB)_{m\times p}$

The (i,j)-th entry of AB is defined as follows,

$$\boxed{(AB)_{ij} = \sum_{k=1}^{n} A_{ik}B_{kj}}$$

Remark

Multiplication of matrices A and B is defined only when the number of columns of A is the same as the number of rows of B.

Example: $\begin{bmatrix}1 & 2 & 3\end{bmatrix}_{1\times 3}\begin{bmatrix}2 & 0.8\\ 5 & 0.7\\ 1/2 & -2\end{bmatrix}_{3\times 2} = \begin{bmatrix}13.5 & -3.8\end{bmatrix}_{1\times 2}$

Ex1:

Example: $\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}_{2\times 2}\begin{bmatrix}1 & 2 & 3\\ 3 & 4 & 5\end{bmatrix}_{2\times 3} = \begin{bmatrix}7 & 10 & 13\\ 15 & 22 & 29\end{bmatrix}_{2\times 3}$

$1\times 1 + 2\times 3 = 1+6 = 7$ (1,1)-th

$3\times 2 + 4\times 4 = 6+16 = 22$ (2,2)-th

$1\times 3 + 2\times 5 = 3+10 = 13$ (1,3)-th

Ex2 :

No. of columns in the first matrix = No. of rows in the second matrix

Example: $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}_{2\times 2} \begin{bmatrix} 5 \\ 6 \end{bmatrix}_{2\times 1} = \begin{bmatrix} 17 \\ 39 \end{bmatrix}_{2\times 1}$

$1\times 5 + 2\times 6 = 5+12 = 17$

$3\times 5 + 4\times 6 = 15+24 = 39.$

Multiplication by special matrices

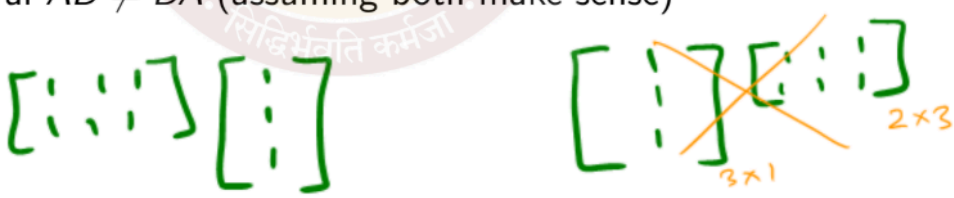
$$\begin{bmatrix} c & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & c \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} = \begin{bmatrix} c & 2c \\ 3c & 4c \\ 5c & 6c \end{bmatrix} = c \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$$

Scalar multiplication by c = multiplication by scalar matrix cI.

$$\boxed{\begin{gathered} IA_{3\times 3} = A_{3\times 3} = A_{3\times 3}I \\ IA_{3\times n} = A_{3\times n} \\ A_{m\times 3}I = A_{m\times 3} \end{gathered}}$$

Properties of matrix addition and multiplication

- $(A + B) + C = A + (B + C)$ (Associativity of addition)
- $(AB)C = A(BC)$ (Associativity of multiplication)
- $A + B = B + A$ (Commutativity of addition)
- In general $AB \neq BA$ (assuming both make sense)



- $\lambda(A + B) = \lambda A + \lambda B$
- $\lambda(AB) = (\lambda A)B = A(\lambda B)$
- $A(B + C) = AB + AC$
- $(A + B)C = AC + BC$

$$2\left(\begin{bmatrix}1\\1\end{bmatrix} + \begin{bmatrix}3\\4\end{bmatrix}\right) = 2\begin{bmatrix}1\\1\end{bmatrix} + 2\begin{bmatrix}3\\4\end{bmatrix}$$

QN : 6,7,8

Lecture 3 : Systems of Linear Equations

Contents

- What is a system of linear equations?
- What is its relation with matrices?
- How many solutions can it have?

Example :

Items	Buyer A	Buyer B	Buyer C
 Rice in Kg	8	12	3
 Dal in Kg	8	5	2
 Oil in Liter	4	7	5

Example Contd.

Suppose A paid $Rs.1960$, B paid $Rs.2215$ and C paid $Rs.1135$. We want to find the price of each item using this data. Suppose price of Rice is $Rs.x$ per kg., price of dal is $Rs.y$ per kg., price of oil is $Rs.z$ per liter. Hence we have the following system of linear equations:

$$8x + 8y + 4z = 1960$$
$$12x + 5y + 7z = 2215$$
$$3x + 2y + 5z = 1135$$

$12x + 12y + 6z = 2940.$

$4x + 4y + 2z = 980$

$12x + 8y + 20z = 4 \times 1135$

$7y - z = 725.$

$3y + 13z = 2325$

$3y + 13(7y - 725) = 2325.$

$\Rightarrow z = 7y - 725$

$94y = 11750 \Rightarrow y = 125 \Rightarrow z = 150$

$\Rightarrow x = 45$

Linear Equations

A linear equation is an equation of the form

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n = b$$

where $x_1, x_2, \ldots, x_n$ are the variables (or unknowns) and $a_1, a_2, \ldots, a_n$ are the coefficients, which are real numbers.

Example

2x+3y+5z=-9, where x, y, z are variables and $2, 3, 5$ are the coefficients.

System of Linear Equations

A system of linear equations is a collection of one or more linear equations involving the same set of variables For example,

$$\begin{aligned} 3x + 2y + z &= 6 \\ x - \frac{1}{2}y + \frac{2}{3}z &= \frac{7}{6} \\ 4x + 6y - 10z &= 0 \end{aligned}$$

$3 \times 1 + 2 \times 1 + 1 = 6.$

$1 - \frac{1}{2} \times 1 + \frac{2}{3} \times 1 = \frac{1}{2} + \frac{2}{3} = 7/6.$

$4 \times 1 + 6 \times 1 - 10 \times 1 = 0.$

is a system of three equations in the three variables x, y, z. A solution to a linear system is an assignment of values to the variables such that all the equations are simultaneously satisfied. A solution to the system above is given by

$$x = 1, y = 1, z = 1$$

General Form of System of Linear Equations

A general system of m linear equations with n unknowns can be written as

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\
&\ldots \\
&\ldots \\
a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m
\end{aligned}
$$

i^{th} eqn. $a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = b_n.$

Matrix Representation

The system of linear equations is equivalent to a matrix equation of the form

$$\boxed{Ax = b}$$

where A is an $m \times n$ matrix, x is a column vector with n entries and b is a column vector with m entries.

$$
A = \underbrace{\begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ldots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}}_{m\times n} \qquad x = \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}_{n\times 1} \qquad b = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}}_{m\times 1}
$$

The example we mentioned above

$$\begin{aligned} 3x + 2y + z &= 6 \\ x - \frac{1}{2}y + \frac{2}{3}z &= \frac{7}{6} \\ 4x + 6y - 10z &= 0 \end{aligned}$$

can be represented as $Ax = b$, where

$$A = \begin{bmatrix} 3 & 2 & 1 \\ 1 & -\frac{1}{2} & \frac{2}{3} \\ 4 & 6 & -10 \end{bmatrix} \qquad x = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad b = \begin{bmatrix} 6 \\ \frac{7}{6} \\ 0 \end{bmatrix}$$

$Ax = b$

eg.

The first example :

$$\begin{aligned} 8x + 8y + 4z &= 1960 \\ 12x + 5y + 7z &= 2215 \\ 3x + 2y + 5z &= 1135 \end{aligned}$$

$$A = \begin{bmatrix} 8 & 8 & 4 \\ 12 & 5 & 7 \\ 3 & 2 & 5 \end{bmatrix} \qquad x = \begin{bmatrix} x \\ y \\ z \end{bmatrix}_{3\times 1} \qquad b = \begin{bmatrix} 1960 \\ 2215 \\ 1135 \end{bmatrix}$$

$Ax = b$

Solutions to a linear system

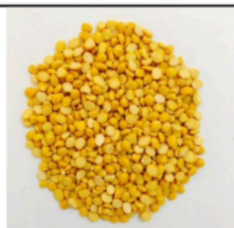
There are 3 possibilities for the solutions to a linear system of equations :

1) The system has infinitely many solutions. ∞

2) The system has a single unique solution. 1

3) The system has no solution. 0

Example of infinitely many Solutions

Items	Buyer A	Buyer B
 Rice in Kg	2	4
 Dal in Kg	1	2

Suppose A paid $Rs.215$, B paid $Rs.430$. We want to find the price of each items using this data. Suppose price of Rice is $Rs.x$ per kg., price of dal is $Rs.y$ per kg. Hence we have the following system of linear equations:

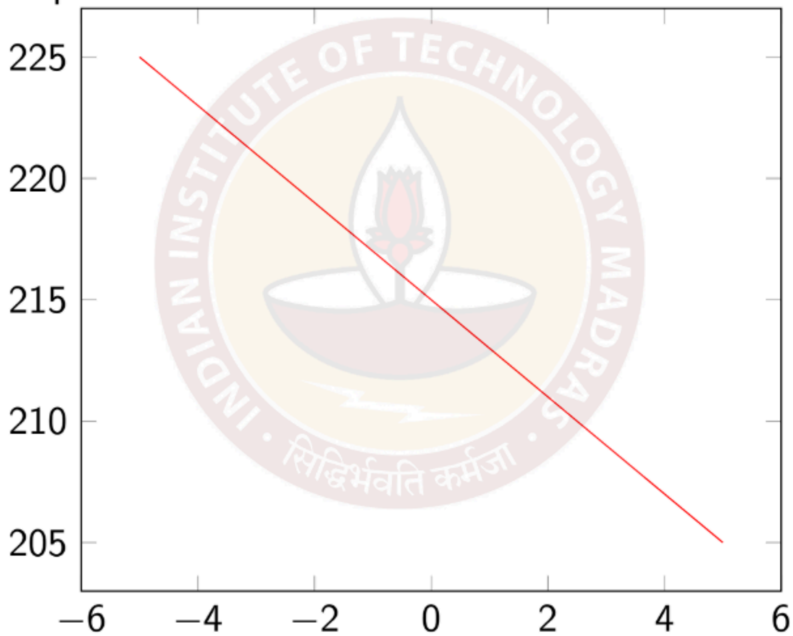
$$2x + y = 215$$
$$4x + 2y = 430$$

There are infinitely many x and y satisfying both the equations.

$x = 0, \; y = 215$

$x = \frac{215}{2} = 107.5 \; , \; y = 0$

Both the equations represents the same straight line in the two dimensional plane $\mathbb{R}^2$.



Every point on this line is a solution.

Example of a system of equations with no solution

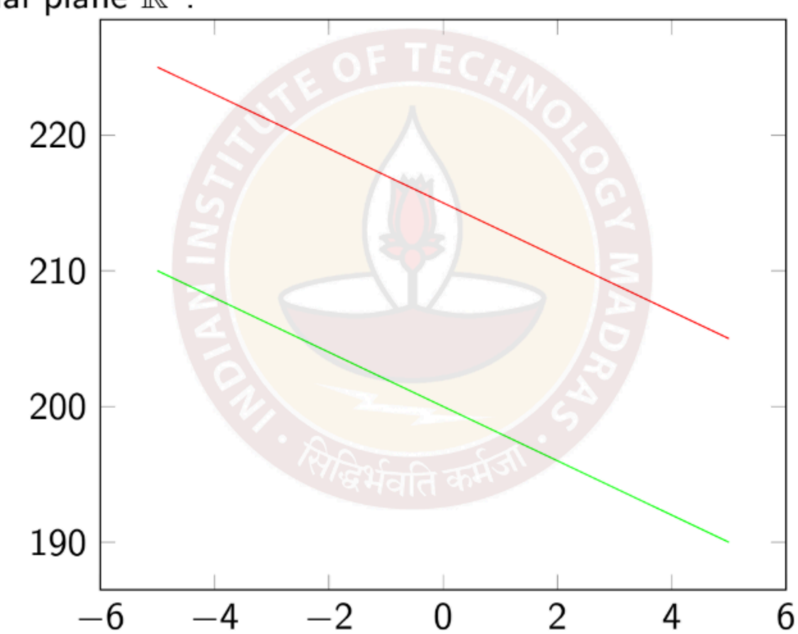
Suppose A and B bought the same amount of items as in the previous example. But for some reason the seller gave a discount to B. Suppose A paid $Rs.215$ and B paid $Rs.400$. Now after returning home they decided to find out the price of each item by solving the linear system of equations as before. Suppose price of rice is $Rs.x$ per kg., price of dal is $Rs.y$ per kg. Hence we have the following system of linear equations:

$$2x + y = 215$$
$$4x + 2y = 400$$

$4x + 2y = 430$ $\Rightarrow$ $400 = 430$
$4x + 2y = 400$

But in this case there are no solution of this system of equations.

The equations represents two parallel straight lines in the two dimensional plane $\mathbb{R}^2$.



Example of a system with a unique solution

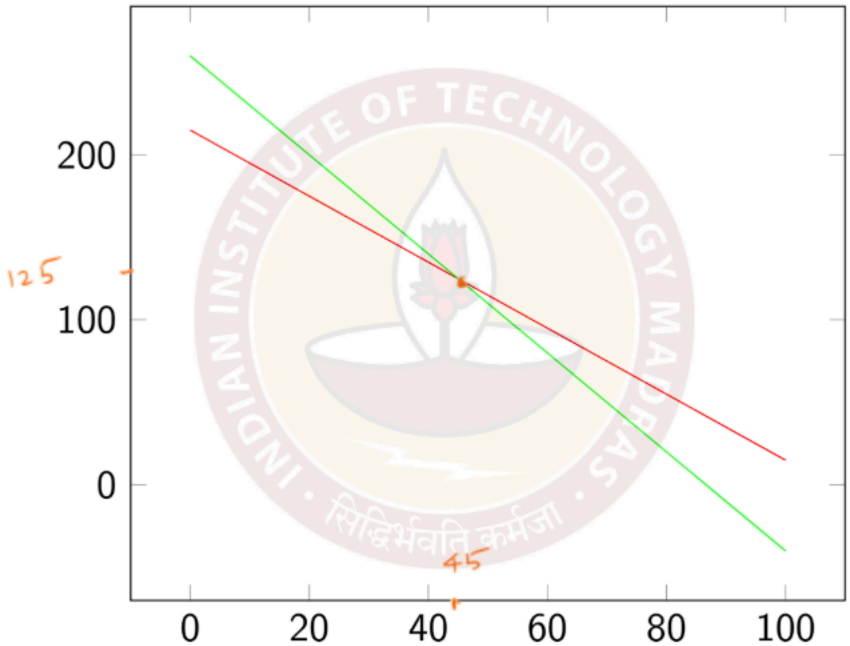
Items	Buyer A	Buyer B
 Rice in Kg	2	3
 Dal in Kg	1	1

Suppose A paid $Rs.215$, B paid $Rs.260$. We want to find the price of each items using this data. Suppose price of Rice is $Rs.x$ per kg., price of dal is $Rs.y$ per kg. Hence we have the following system of linear equations:

$$
\begin{aligned}
2x + y &= 215 \\
3x + y &= 260
\end{aligned}
$$

$\Rightarrow x = 45$

$\Rightarrow y = 215 - 90 = 125$



QN : 6,7,9,10

Lecture 4 : Determinants (Part 1)

Every square matrix A has an associated number, called its determinant and denoted by $det(A)$ or $|A|$. It is used in :

- solving a system of linear equations
- finding the inverse of a matrix
- calculus and more.

Determinant of a 1×1 matrix :
If $A = [a]$, a 1×1 matrix then $det(A) = a$

Determinant of a 2 × 2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad det(A) = ad - bc$$

Example

$$A = \begin{bmatrix} 2 & 3 \\ 6 & 10 \end{bmatrix} \qquad det(A) = 20 - 18 = 2$$

Example

$$A = \begin{bmatrix} 5 & 2/3 \\ 6 & 3/7 \end{bmatrix} \qquad det(A) = 15/7 - 4 = -13/7$$

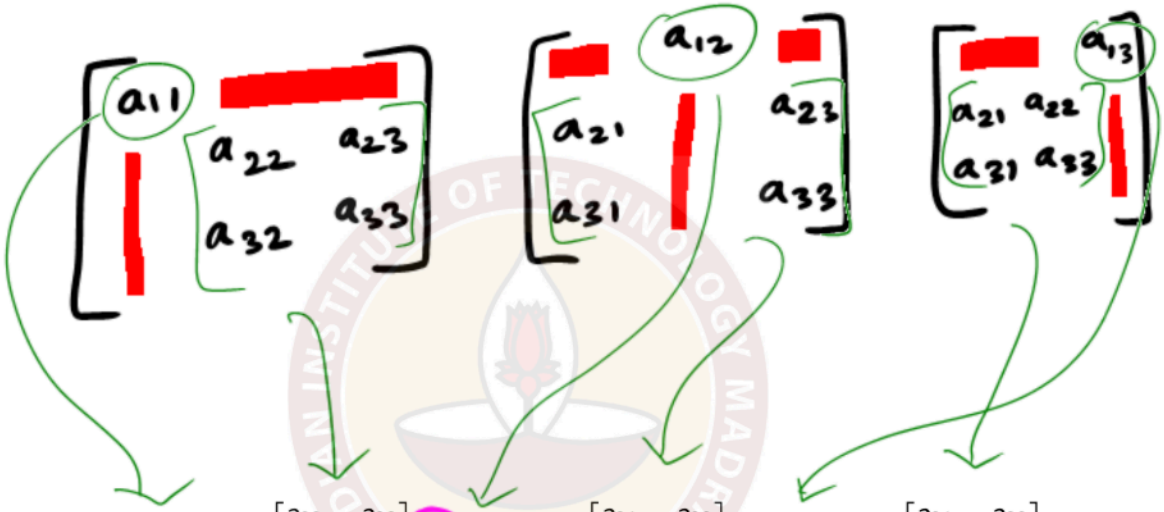
Determinant of a 3 × 3 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

We will obtain the determinant by expanding with respect to the $1st$ row :

$$\begin{bmatrix} \boxed{a_{11}} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad \begin{bmatrix} a_{11} & \boxed{a_{12}} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad \begin{bmatrix} a_{11} & a_{12} & \boxed{a_{13}} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{aligned} det(A) &= a_{11} \times det \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12} \times det \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13} \times det \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \\ &= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31}) \end{aligned}$$



$$
\begin{aligned}
det(A) &= a_{11} \times det \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12} \times det \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13} \times det \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \\
&= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31}) \\
&= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}
\end{aligned}
$$

Examples

1.

$$
A = \begin{bmatrix} 2 & 4 & 1 \\ 3 & 8 & 7 \\ 5 & 6 & 9 \end{bmatrix}
$$

$$
\begin{aligned}
det(A) &= 2 \times det \begin{bmatrix} 8 & 7 \\ 6 & 9 \end{bmatrix} - 4 \times det \begin{bmatrix} 3 & 7 \\ 5 & 9 \end{bmatrix} + 1 \times det \begin{bmatrix} 3 & 8 \\ 5 & 6 \end{bmatrix} \\
&= 2(72 - 42) - 4(27 - 35) + 1(18 - 40) \\
&= 2(30) - 4(-8) + 1(-22) \\
&= 60 + 32 - 22 \\
&= 70
\end{aligned}
$$

Determinant of the Identity matrix

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad det(I_2) = 1 - 0 = 1$$

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$det(I_3) = 1 \times det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - 0 \times det \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + 0 \times det \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$= 1 \times det(I_2)$$

$$= 1 \times 1 = 1.$$

Determinant of a product of matrices

$$\text{Let } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ and } B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$$

$$\text{Then } AB = \begin{bmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{bmatrix}$$

$$\begin{aligned} det(AB) &= (ae+bg)(cf+dh) - (af+bh)(ce+dg) \\ &= aecf + bgcf + aedh + bgdh - afce - bhce - afdg - bhdg \\ &= bgcf + aedh - bhce - afdg \\ &= bcfg + adeh - bceh - adfg \\ &= (ad-bc)(eh-fg) \\ &= det(A)det(B). \end{aligned}$$

It can be checked that for 3×3 matrices this equality holds.

Determinant of the inverse of a matrix

$$A A^{-1} = I = A^{-1} A$$

$$det(A A^{-1}) = det(I)$$

$$det(A)\, det(A^{-1}) = 1$$

$$\Rightarrow det(A^{-1}) = \frac{1}{det(A)} = det(A)^{-1}.$$

Properties : Switching two rows.

$$\text{Let } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}. \qquad \text{Define } \tilde{A} = \begin{bmatrix} c & d \\ a & b \end{bmatrix}.$$

$$det(\tilde{A}) = cb - da = -(ad - bc) = -det(A).$$

Switching two columns

$$\tilde{\tilde{A}} = \begin{bmatrix} b & a \\ d & c \end{bmatrix} \quad det(\tilde{\tilde{A}}) = bc - ad$$
$$= -(ad - bc) = -det(A)$$

This is also true for 3×3 matrices.

Properties : Adding multiples of a row to another row.

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Define $\tilde{A} = \begin{bmatrix} a+tc & b+td \\ c & d \end{bmatrix}$.

$$det(\tilde{A}) = (a+tc)d - (b+td)c = ad + \cancel{tcd} - bc - \cancel{tdc}$$
$$= ad - bc = det(A).$$

$$\tilde{A} = \begin{bmatrix} a+tb & b \\ c+td & d \end{bmatrix} \qquad det(\tilde{A}) = (a+tb)d - b(c+td)$$
$$= ad + \cancel{tb}d - bc - b\cancel{td}$$
$$= ad - bc = det(A)$$

Check this for 3×3 matrices.

Properties : Scalar multiplication of a row by a constant t.

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Define $\tilde{A} = \begin{bmatrix} a & tb \\ c & td \end{bmatrix}$.

$$det(\tilde{A}) = atd - tbc = t(ad - bc) = tdet(A).$$

$$\tilde{\tilde{A}} = \begin{bmatrix} ta & tb \\ c & d \end{bmatrix} \quad det(\tilde{\tilde{A}}) = tad - tbc$$
$$= t(ad - bc) = t\ det(A).$$

Same thing for 3×3 matrices.

QN : 7,10

Lecture 5 : Determinants (Part 2)

Recall from part 1 :

- $A = [a] \qquad det(A) = a$
- $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad det(A) = ad - bc$
- $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

  Expanding with respect to the $1st$ row :

$$det(A) = a_{11} \times det \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12} \times det \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13} \times det \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$
$$= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}$$

An example

$$A = \begin{bmatrix} 2 & 4 & 3 \\ 0 & 8 & 7 \\ 0 & 0 & 9 \end{bmatrix}$$

$$\begin{aligned} det(A) &= 2 \times det \begin{bmatrix} 8 & 7 \\ 0 & 9 \end{bmatrix} - 4 \times det \begin{bmatrix} 0 & 7 \\ 0 & 9 \end{bmatrix} + 3 \times det \begin{bmatrix} 0 & 8 \\ 0 & 0 \end{bmatrix} \\ &= 2(72 - 0) - 4(0 - 0) + 3(0 - 0) \\ &= 2(72) - 4(0) + 3(0) \\ &= 144 \end{aligned}$$

Lower

$A_{ij} = 0$ if $i < j$

$$A = \begin{bmatrix} 2 & 4 & 3 \\ 0 & 8 & 7 \\ 0 & 0 & 9 \end{bmatrix}$$

$$\begin{aligned} det(A) &= 2 \times det \begin{bmatrix} 8 & 7 \\ 0 & 9 \end{bmatrix} - 4 \times det \begin{bmatrix} 0 & 7 \\ 0 & 9 \end{bmatrix} + 3 \times det \begin{bmatrix} 0 & 8 \\ 0 & 0 \end{bmatrix} \\ &= 2(72 - 0) - 4(0 - 0) + 3(0 - 0) \\ &= 2(72) - 4(0) + 3(0) \\ &= 144 \\ &= 2 \times 8 \times 9 \end{aligned}$$

$A_{n \times n} =$

$A_{ij} = 0$ if $i > j$

U.T.

This is an upper triangular matrix. For such matrices, the determinant is the product of the diagonal elements.

The transpose of a matrix and its determinant

The transpose of $A_{m \times n}$ is the $n \times m$ matrix with (i, j)-th entry A_{ji}.

Notation : A^T Definition : $(A^T)_{ij} = A_{ji}$

$$\text{Let } A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \text{ then } A^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$$

$$\begin{aligned} det(A^T) &= a_{11} \times det \begin{bmatrix} a_{22} & a_{32} \\ a_{23} & a_{33} \end{bmatrix} - a_{21} \times det \begin{bmatrix} a_{12} & a_{32} \\ a_{13} & a_{33} \end{bmatrix} + a_{31} \times det \begin{bmatrix} a_{12} & a_{22} \\ a_{13} & a_{23} \end{bmatrix} \\ &= a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{21}(a_{12}a_{33} - a_{32}a_{13}) + a_{31}(a_{12}a_{23} - a_{22}a_{13}) \\ &= a_{11}a_{22}a_{33} - a_{11}a_{32}a_{23} - a_{21}a_{12}a_{33} + a_{21}a_{32}a_{13} + a_{31}a_{12}a_{23} - a_{31}a_{22}a_{13} \\ &= det(A) \end{aligned}$$

Minors and Cofactors

If A is an $n \times n$ square matrix with $n \leq 4$. Then the minor of the entry in the i-th row and j-th column is the determinant of the submatrix formed by deleting the i-th row and j-th column.

Name : the (i, j)-th minor Notation : M_{ij}

Example : $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ $M_{11} = det \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}$.

The (i, j)-th cofactor $C_{ij} = (-1)^{i+j} M_{ij}$.

Above example : $C_{11} = (-1)^{1+1} M_{11} = M_{11}$

$C_{23} = (-1)^{2+3} M_{23} = -M_{23}$

Determinant in terms of minors and cofactors

Observe that : For $A_{3\times3}$

$$\begin{aligned} det(A) &= a_{11} \times det \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12} \times det \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13} \times det \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \\ &= a_{11} \times M_{11} - a_{12} \times M_{12} + a_{13} \times M_{13} \\ &= a_{11} \times C_{11} + a_{12} \times C_{12} + a_{13} \times C_{13} \end{aligned}$$

This formula holds for $A_{2\times2}$.

We use it to generalize the determinant beyond n = 3. Generalization to A4×4 :

Definition

$det(A) = \Sigma_{j=1}^{4}(-1)^{1+j}a_{1j}M_{1j} = \Sigma_{i=1}^{4}a_{1j}C_{1j}$

$$= a_{11} \times det\begin{bmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{bmatrix} - a_{12} \times det\begin{bmatrix} a_{21} & a_{23} & a_{24} \\ a_{31} & a_{33} & a_{34} \\ a_{41} & a_{43} & a_{44} \end{bmatrix} + a_{13} \times det[\quad] - a_{14} \times det[\quad]$$

Inductive definition of the determinant

Suppose $A_{n\times n}$ is given and we know how to define determinants for $n-1 \times n-1$ matrices. Define minors and cofactors as before. :

The (i,j)-th minor is the determinant of the submatrix formed by deleting the i-th row and j-th column.

The (i,j)-th cofactor $C_{ij} = (-1)^{i+j}M_{ij}$.

$$det(I_n) = det(I_{n-1}) = det(I_{n-2}) = \cdots = det(I_3) = det(I_2) = 1.$$

Definition

$det(A) = \Sigma_{j=1}^{n}(-1)^{1+j}a_{1j}M_{1j} = \Sigma_{i=1}^{n}a_{1j}C_{1j}$

$$det(A) = a_{11} \times det\begin{bmatrix} a_{22} & a_{23} & \cdots & a_{2n} \\ a_{32} & a_{33} & \cdots & a_{3n} \\ & & \cdots & a_{nn} \end{bmatrix} - a_{12} \times det\begin{bmatrix} a_{21} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & & \vdots \\ & & & a_{nn} \end{bmatrix} + a_{13} \times det[\quad] - a_{14} \times det[\quad]$$

$$det(I_{n\times n}) = det\left(\begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}\right)$$

$$= 1 \times det\left(\begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}\right) - 0 \times det(\quad) + 0 \times det(\quad) \cdots$$

$$= 1 \times det(I_{n-1})$$

Expansion along any row or column

$$det(A) = \Sigma_{j=1}^{n}(-1)^{i+j}a_{ij}M_{ij} \quad \text{for a fixed } i$$ expansion along the i^{th} row

$$= \Sigma_{i=1}^{n}(-1)^{i+j}a_{ij}M_{ij} \quad \text{for a fixed } j$$ expansion along the j^{th} column

$$det(A_{3\times3}) = (-1)^{2+1}a_{21}\times M_{21} + (-1)^{2+2}a_{22}\times M_{22} + (-1)^{2+3}a_{23}\times M_{23}$$

$$= -a_{21}\times det\begin{bmatrix}a_{12} & a_{13}\\ a_{32} & a_{33}\end{bmatrix} + a_{22}\times det\begin{bmatrix}a_{11} & a_{13}\\ a_{31} & a_{33}\end{bmatrix} - a_{23}\, det\begin{bmatrix}a_{11} & a_{12}\\ a_{31} & a_{32}\end{bmatrix}$$

$$= (-1)^{1+2}a_{12}\times det\begin{bmatrix}a_{21} & a_{23}\\ a_{31} & a_{33}\end{bmatrix} + (-1)^{2+2}a_{22}\times det\begin{bmatrix} & \\ & \end{bmatrix} + (-1)^{2+3}a_{32}\times det\begin{bmatrix} & \\ & \end{bmatrix}$$

Important properties and identities

Property 1 : Determinant of a product is product of the determinants. Related identity : $det(AB) = det(A)det(B)$

$$det(A^n) = det(A)^n.$$

$$det(A^{-1}) = \frac{1}{det(A)} = det(A)^{-1}.$$

$$det(P^{-1}AP) = det(A).$$

$$det(AB) = det(BA)$$

$$det(A^TA) = det(A)^2.$$

$$det(A^T) \;\Big\Vert\; det(A)$$

expand along 1st column + induction

Property 2 : Switching two rows or columns changes the sign.

$$A = \begin{array}{r} i^{th} \rightarrow \\ j^{th} \rightarrow \end{array} \begin{bmatrix} a_{i1} & a_{i2} & \cdots & a_{in} \\ a_{j1} & a_{j2} & \cdots & a_{jn} \end{bmatrix} \qquad \tilde{A} = \begin{bmatrix} a_{j1} & a_{j2} & \cdots & a_{jn} \\ a_{i1} & a_{i2} & \cdots & a_{in} \end{bmatrix} \begin{array}{l} \leftarrow i^{th} \text{ row} \\ \leftarrow j^{th} \text{ row} \end{array}$$

$$\det(\tilde{A}) = -\det(A)$$

Expand along i^{th} row & use induction

Property 3 : Adding multiples of a row to another row leaves the determinant unchanged.

A

$$\tilde{A} = \begin{bmatrix} a_{i1}+ta_{j1} & a_{i2}+ta_{j2} & \cdots & a_{in}+ta_{jn} \\ a_{j1} & a_{j2} & \cdots & a_{jn} \end{bmatrix} \leftarrow i^{th} \text{ row}$$

$$\det(\tilde{A}) = \det(A)$$

$$\det(\tilde{\tilde{A}}) = \det(A)$$

$$\begin{array}{c} k^{th} \text{ column} \\ \downarrow \end{array}$$

$$\tilde{\tilde{A}} = \begin{bmatrix} a_{1k}+ta_{1l} \\ a_{2k}+ta_{2l} \\ \vdots \\ a_{nk}+ta_{nl} \end{bmatrix}$$

Property 3' : Adding multiples of a column to another column leaves the determinant unchanged.

Property 4 : Scalar multiplication of a row by a constant t multiplies the determinant by t.

A

$$\tilde{A} = \begin{bmatrix} & & & \\ ta_{i1} & ta_{i2} & \cdots & ta_{in} \\ & & & \end{bmatrix}$$

$$\det(\tilde{A}) = \sum_{j=1}^{n} (-1)^{i+j}\, t a_{ij} \tilde{M}_{ij} = t\left(\sum_{j=1}^{n} (-1)^{i+j} a_{ij} \tilde{M}_{ij}\right) = t\left(\sum_{j=1}^{n} (-1)^{i+j} a_{ij} M_{ij}\right) = t \det(A).$$

Warning: $\det(tA_{n\times n}) = t^n \det(A)$

Property 4' : Scalar multiplication of a column by a constant t multiplies the determinant by t.

Useful computational tips

1) The determinant of a matrix with a row or column of zeros is 0.
2) The determinant of a matrix in which one row (or column) is a linear combination of other rows (resp. columns) is 0.
3) Scalar multiplication of a row by a constant t multiplies the determinant by t.
4) While computing the determinant, you can choose to compute it using expansion along a suitable row or column.

QN : 7,9

Area of $\triangle$, vertices given as (x_1,y_1) (x_2,y_2), (x_3,y_3)

$$= \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Determinant of Diagonal Matrix

$$\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} = abc$$ (same for upper/lower $\triangle$ Matrix

Imp Matrix and their det() :-

1. $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (a-b)(b-c)(c-a) = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ bc & ac & ab \end{vmatrix}$

2. $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c)$

Diagonal Matrix or Inverse

$$\begin{bmatrix} 1/a & 0 & 0 \\ 0 & 1/b & 0 \\ 0 & 0 & 1/c \end{bmatrix}$$

3. $\begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(ab+bc+ca)$

Diagonal Matrix or Adj

$$\begin{bmatrix} bc & 0 & 0 \\ 0 & ac & 0 \\ 0 & 0 & ab \end{bmatrix}$$

4. $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = -(a^3+b^3+c^3-3abc)$

5. $\begin{vmatrix} x! & (x+1)! & (x+2)! \\ (x+1)! & (x+2)! & (x+3)! \\ (x+2)! & (x+3)! & (x+4)! \end{vmatrix} = 2(x!\,(x+1)!\,(x+2)!)$

6. $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^4 & b^4 & c^4 \end{vmatrix} = (a-b)(b-c)(c-a)(a^2+b^2+c^2-ab-bc-ca)$

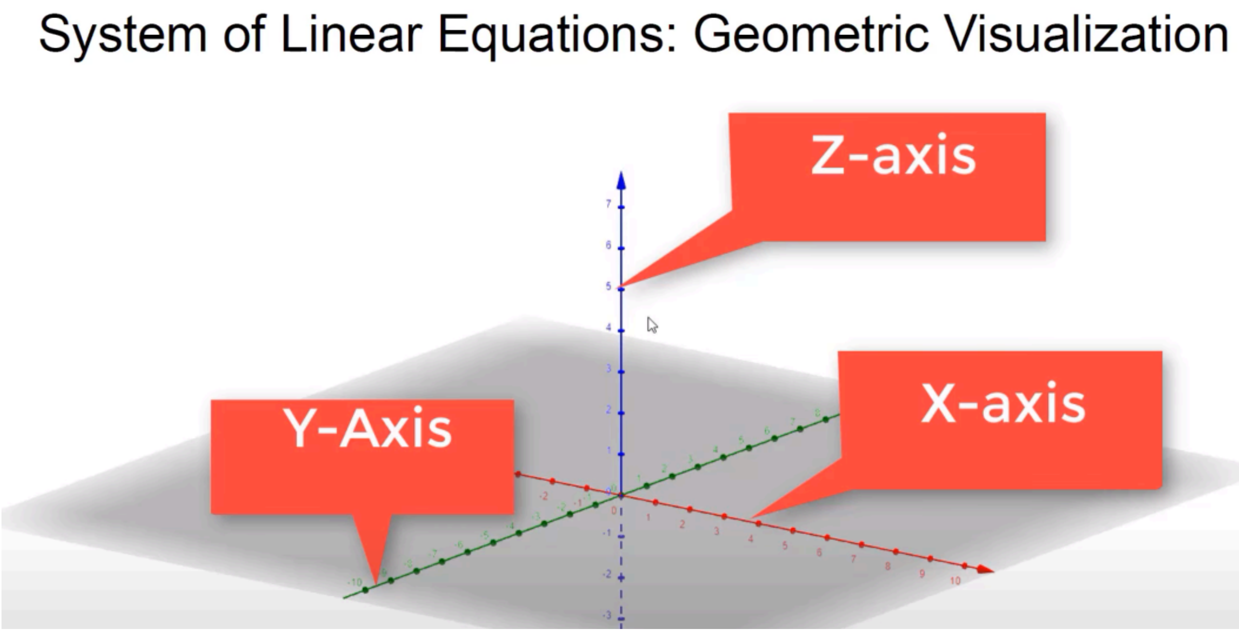
$\boxed{A^2 - (\text{trace of } A)A + |A|I = 0}$

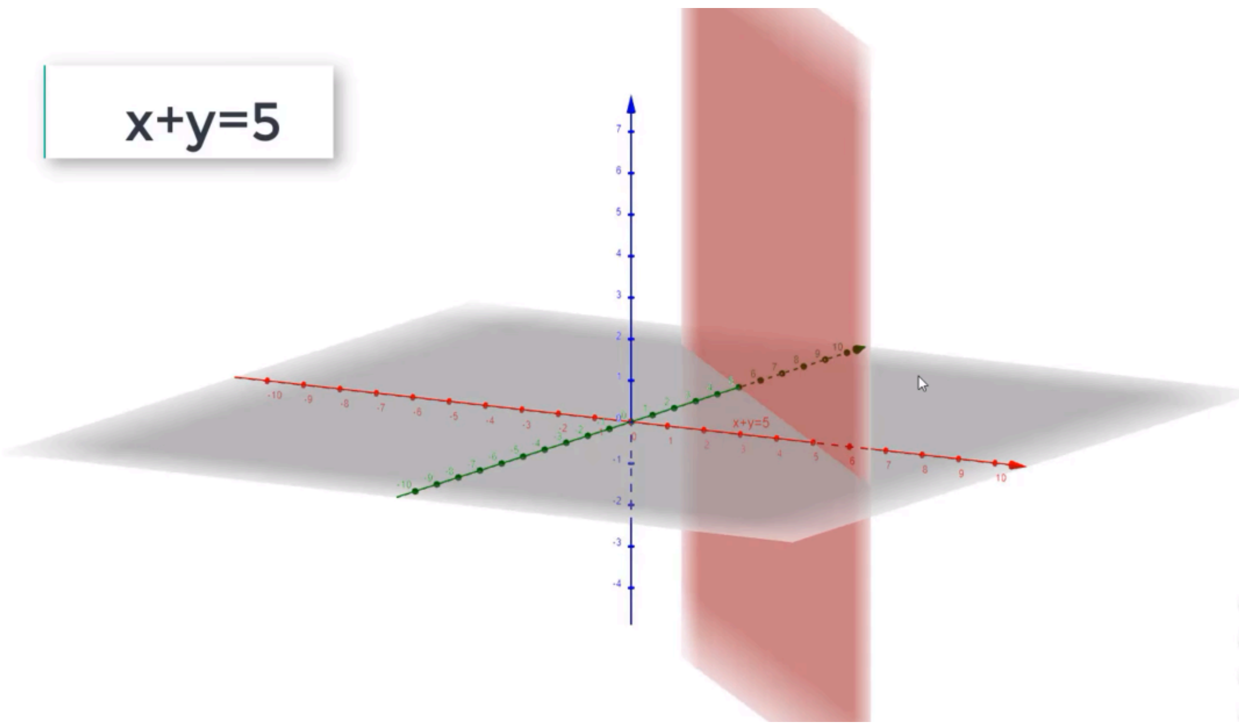
Upper $\triangle$ Matrix, $a_{ij}=0$, $i>j$

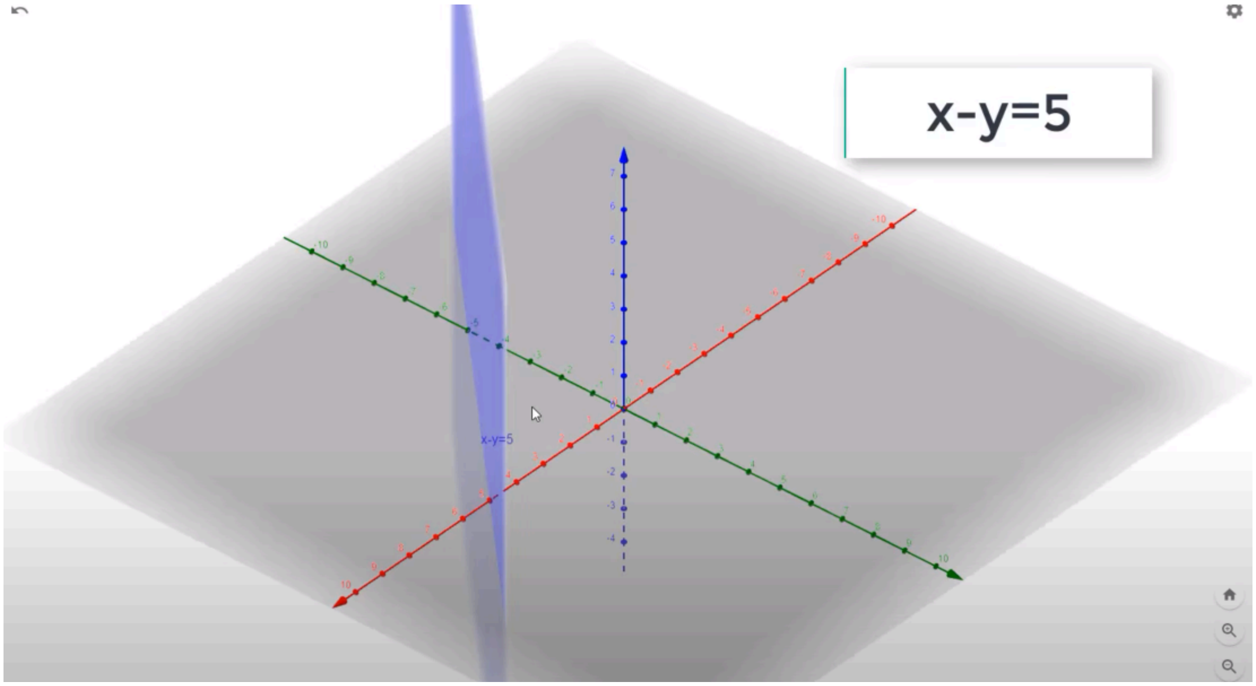
Lower $\triangle$ Matrix, $a_{ij}=0$, $i<j$

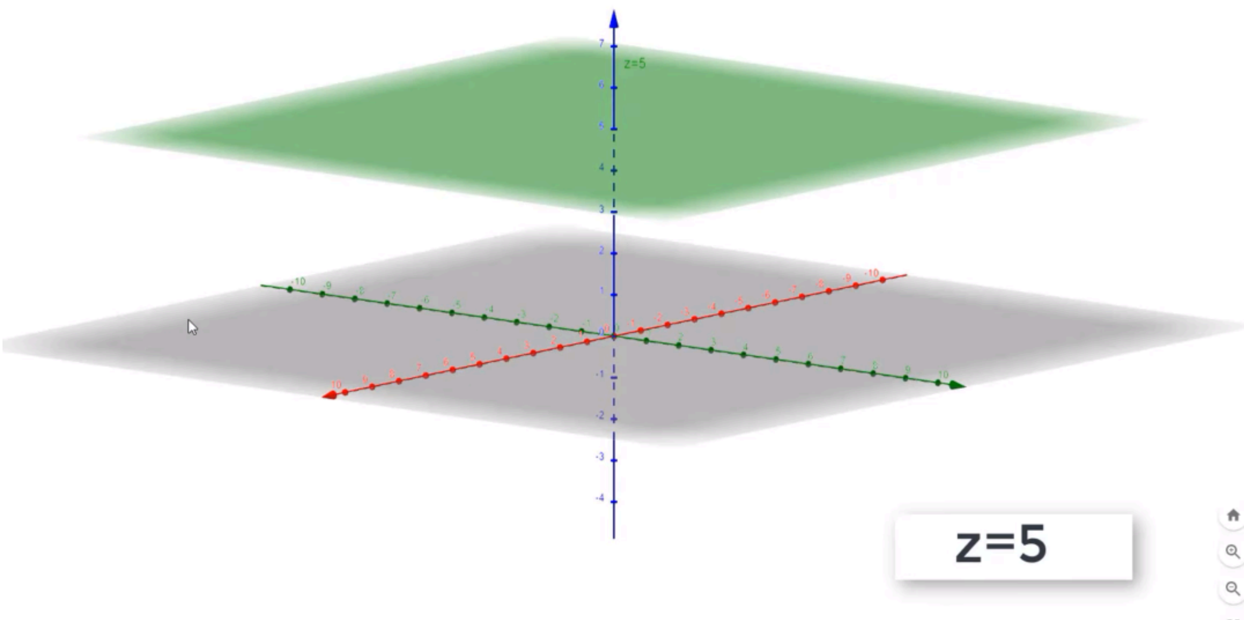
Week 01 - Tutorial 01

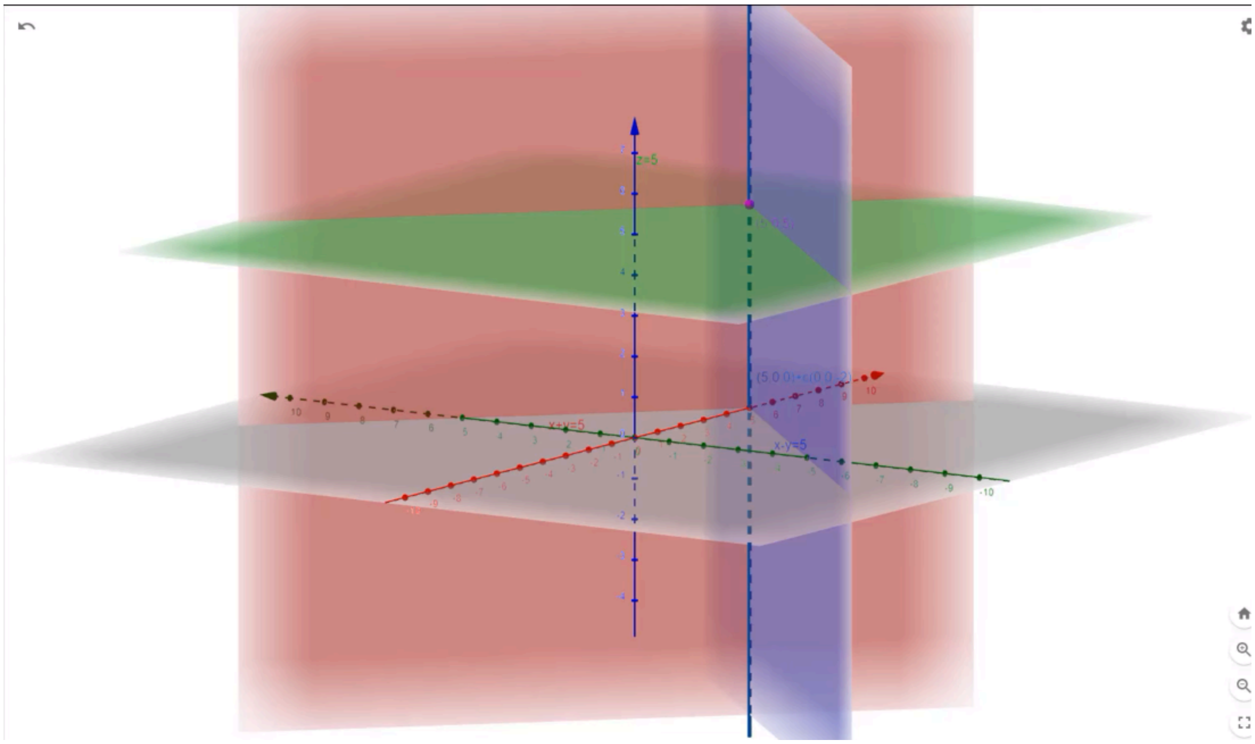
Ex 1 :






x-y=5


z=5



Week 01 - Tutorial 02

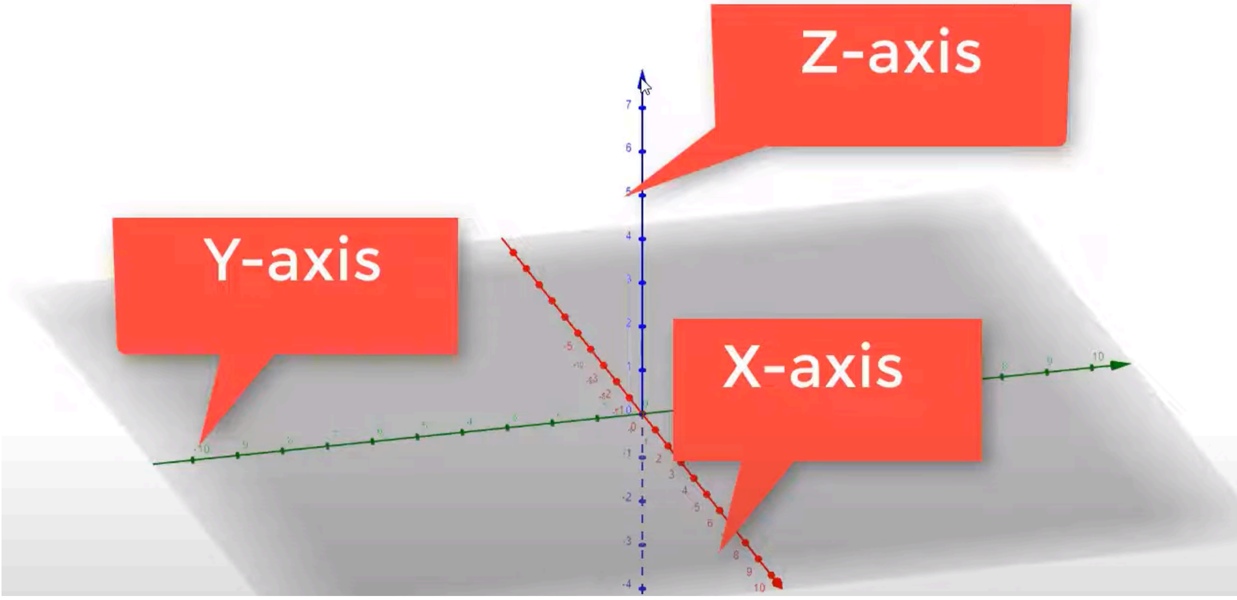
Ex 2 :

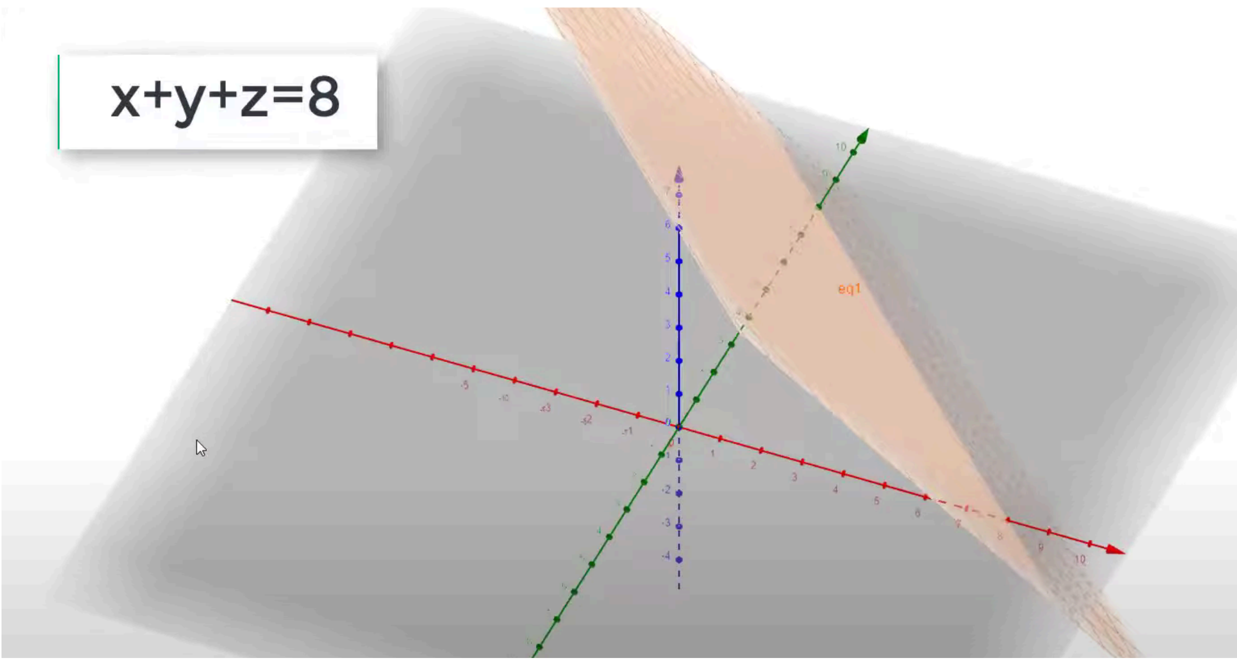
System 2

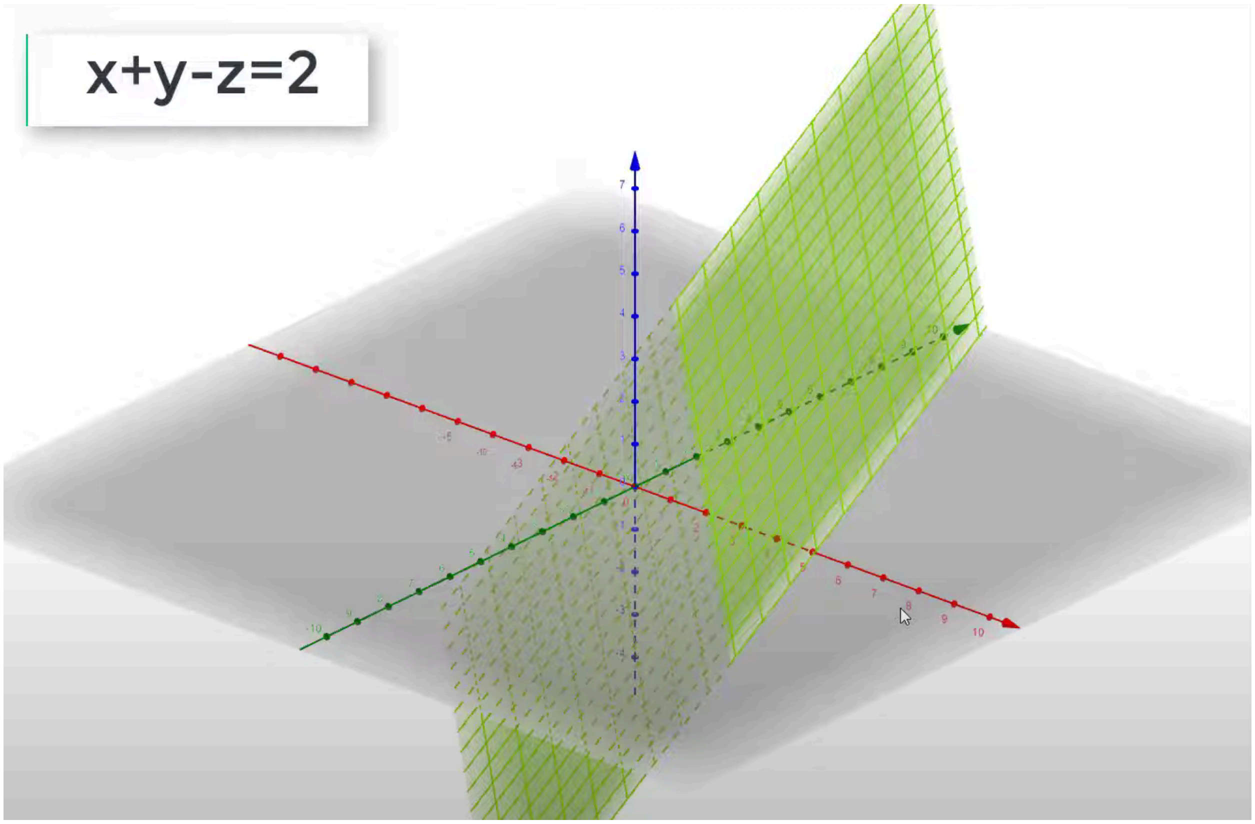
x+y+z=8

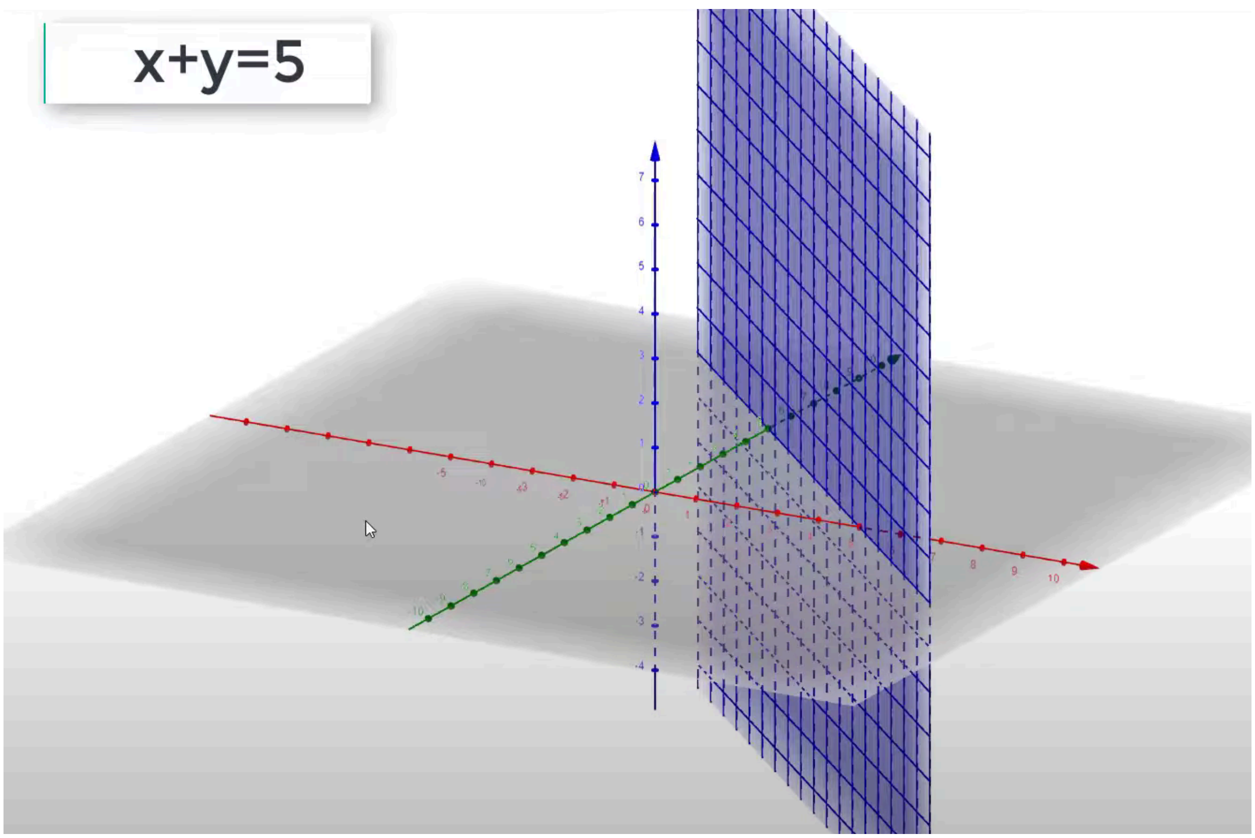
x+y-z=2

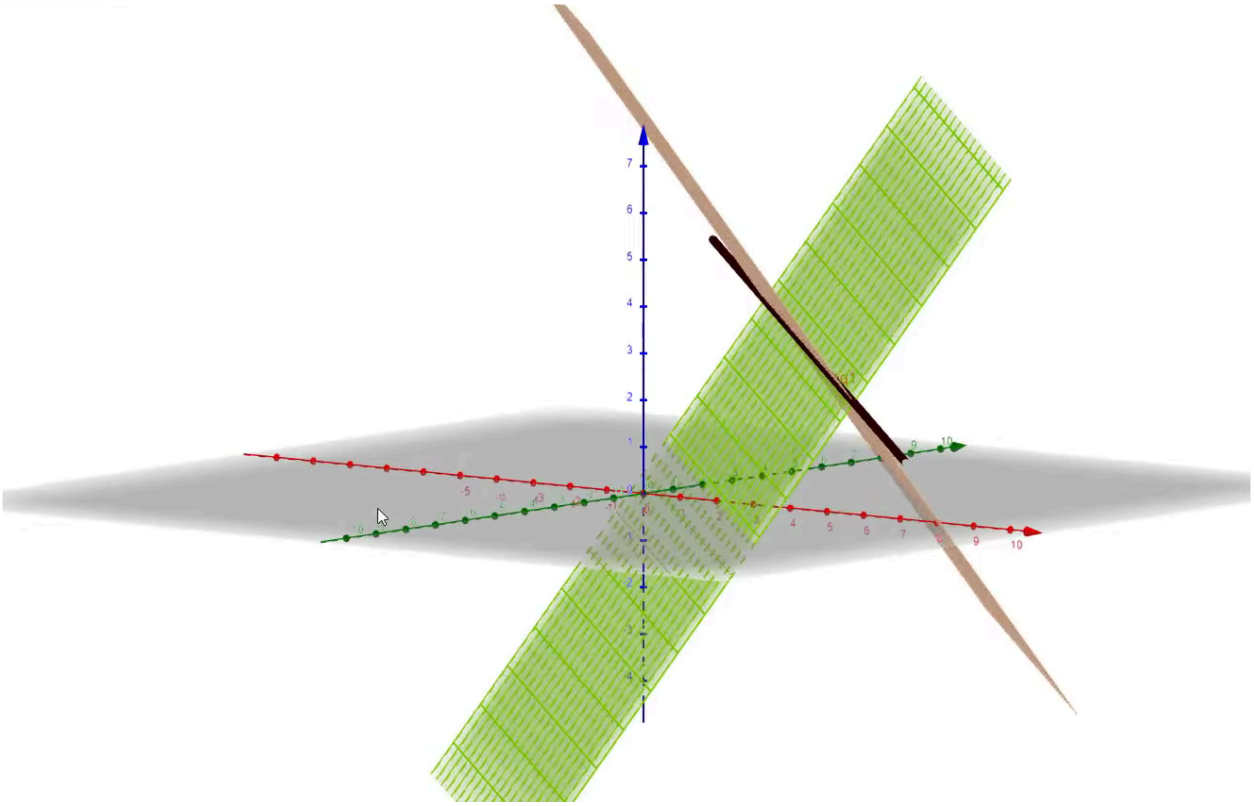
x+y=5

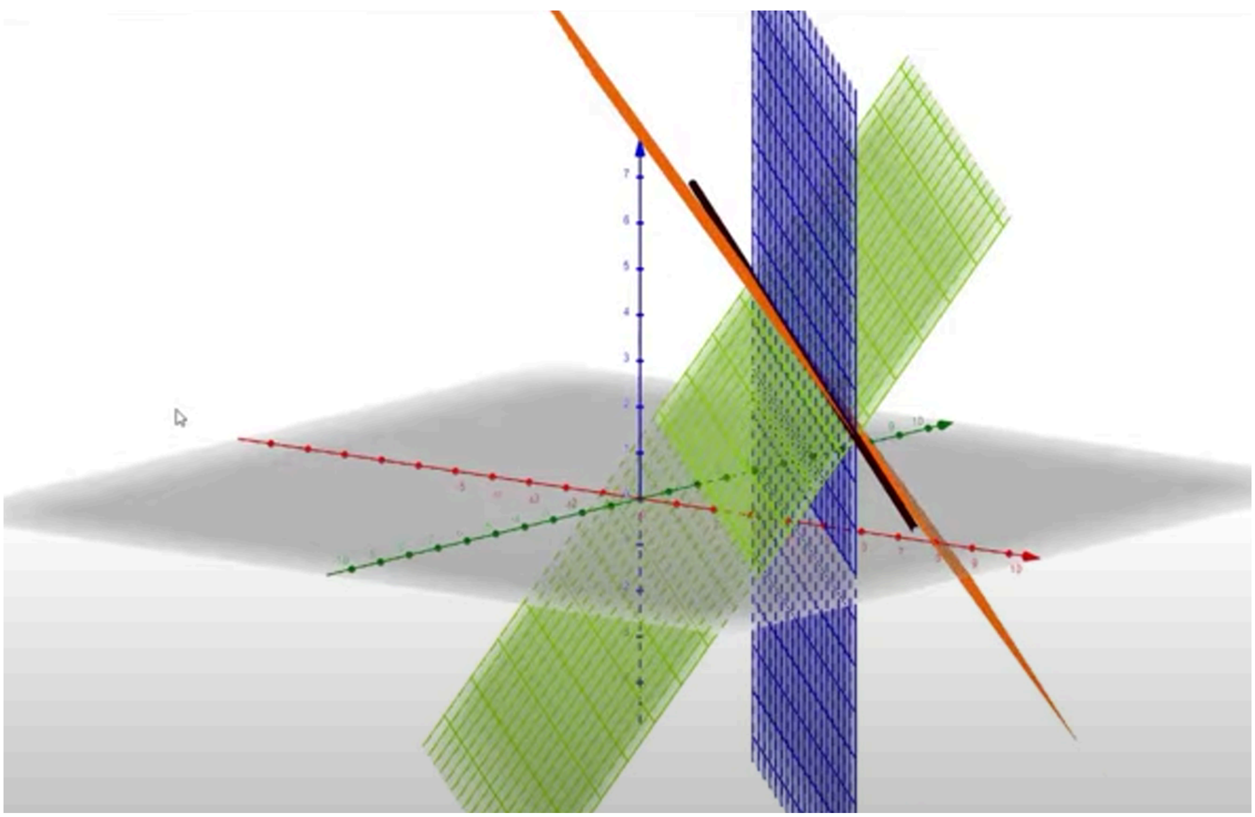

Z-axis
Y-axis
X-axis


x+y+z=8
eq1


x+y-z=2


x+y=5





Infinitely Many Solution

Week 01 - Tutorial 03

System of linear equations - 3

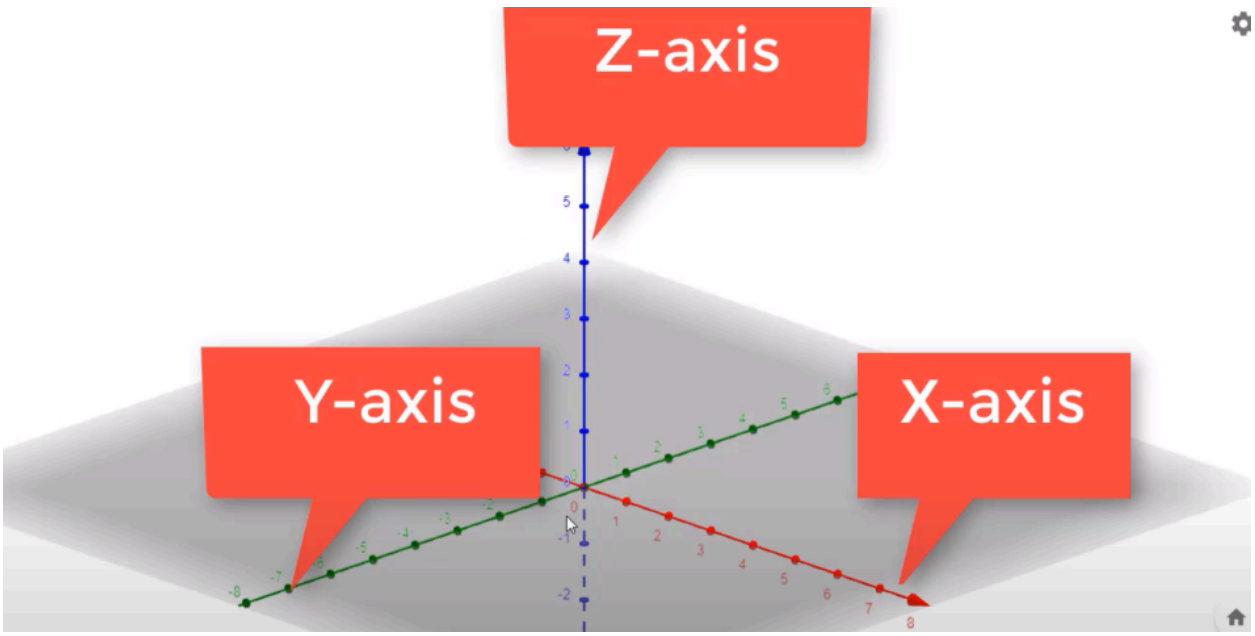
$$\left.\begin{aligned} x+y+z&=0 \\ x+y+z&=3 \\ x+y-z&=1 \end{aligned}\right\}$$

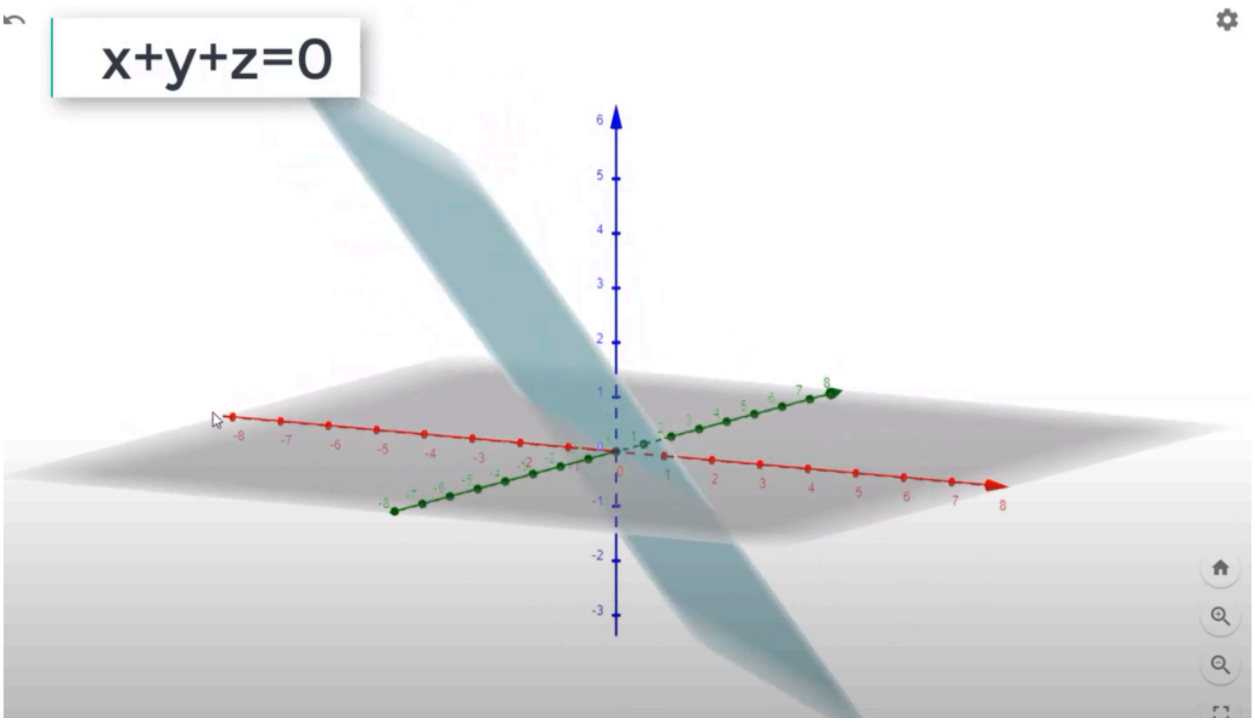
$\underline{(a, b, c)}$

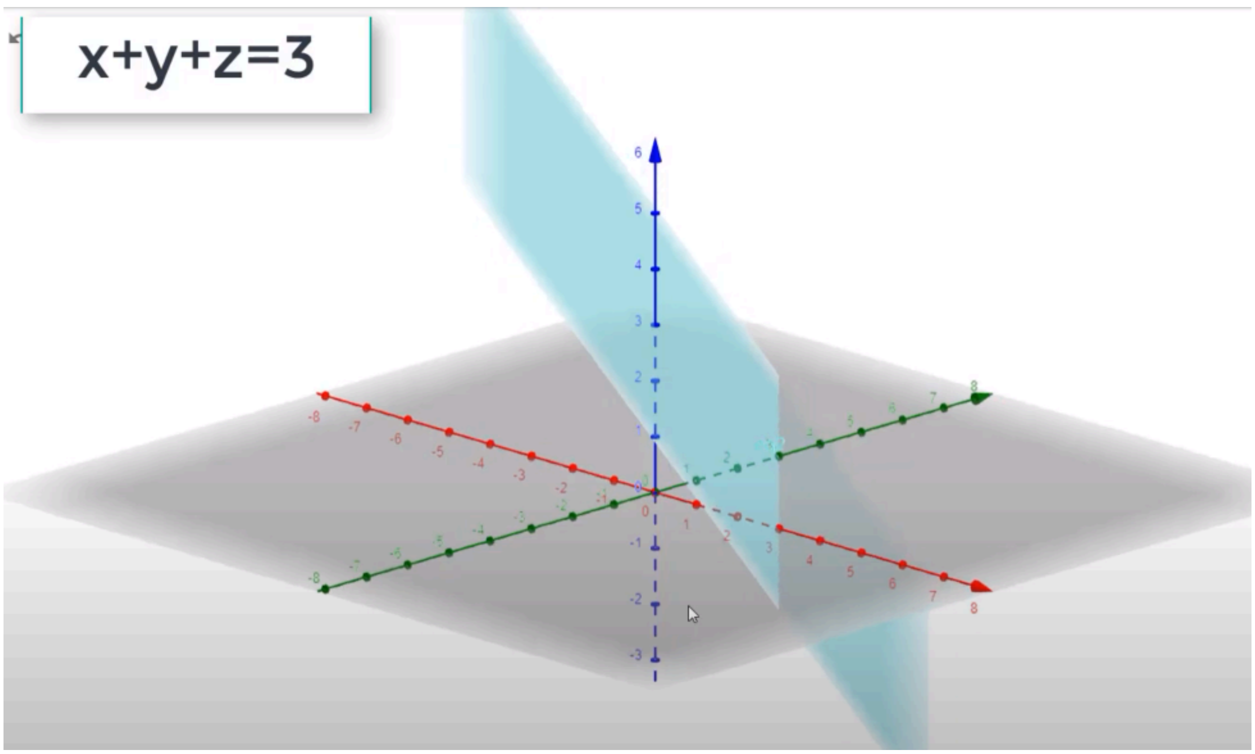
$a+b+c=0$

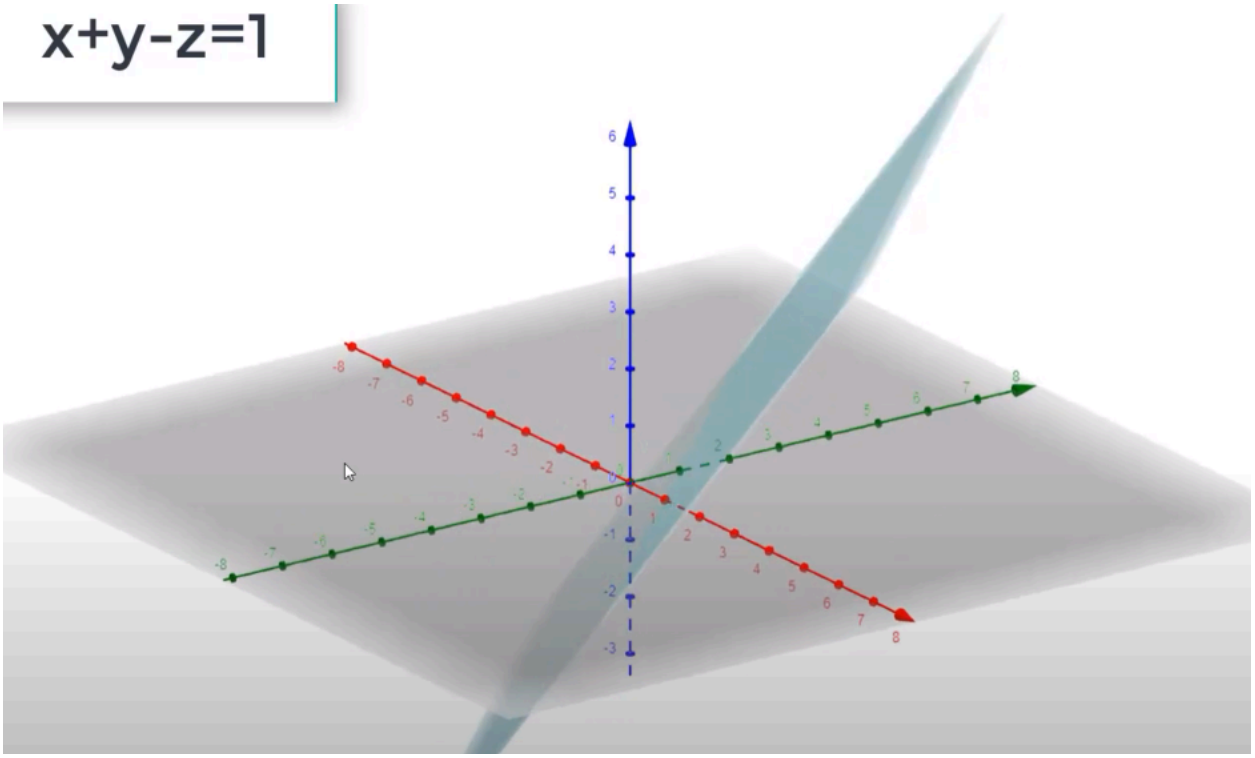
$a+b+c=3$

$\Rightarrow \underline{0=3}$ (absurd)

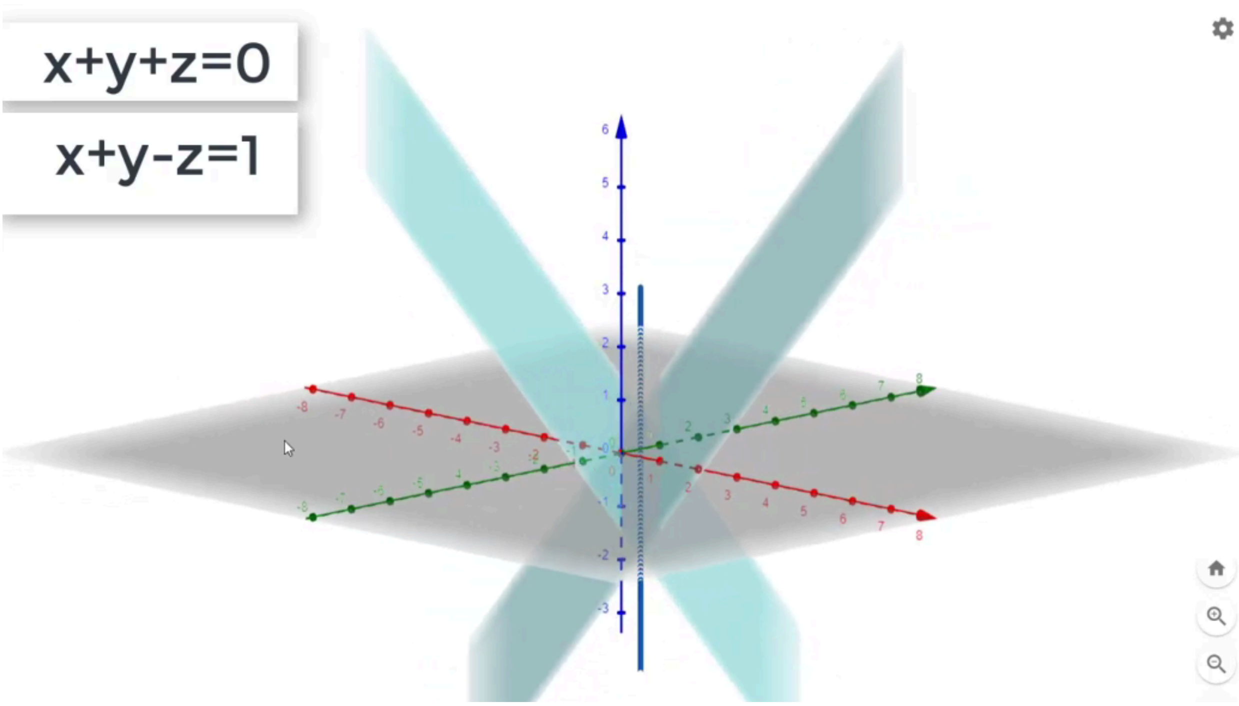



x+y+z=0

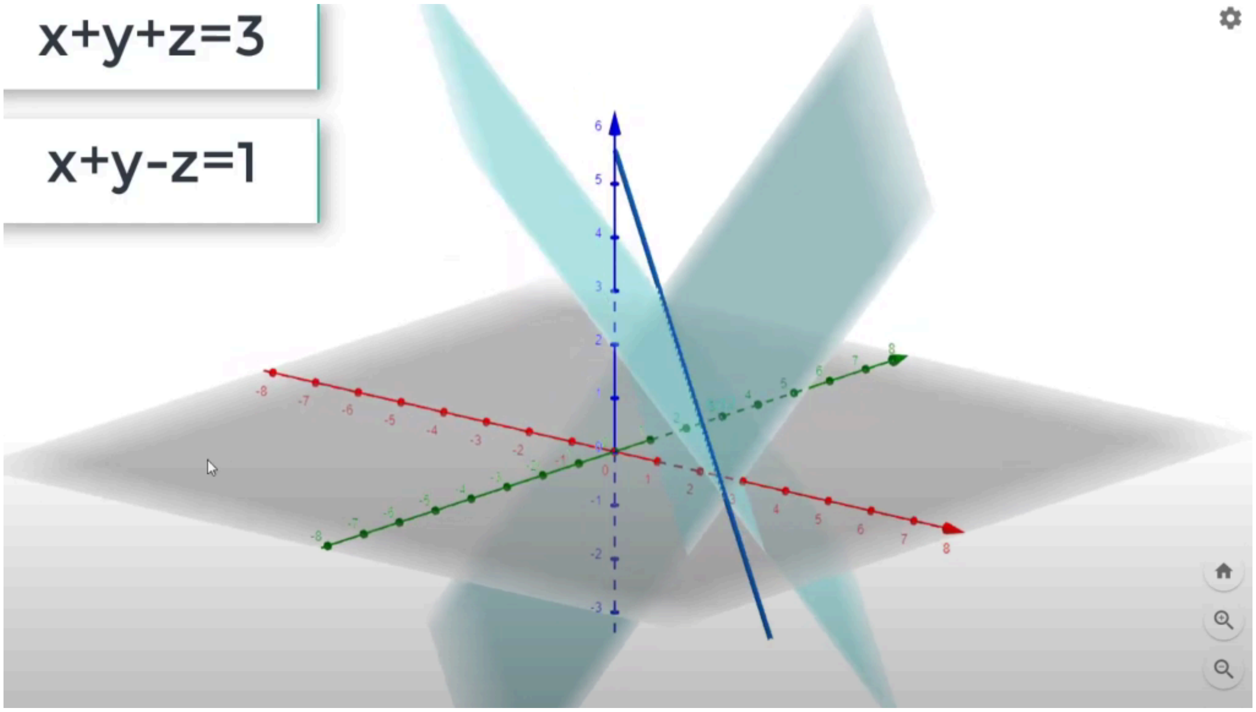

x+y+z=3



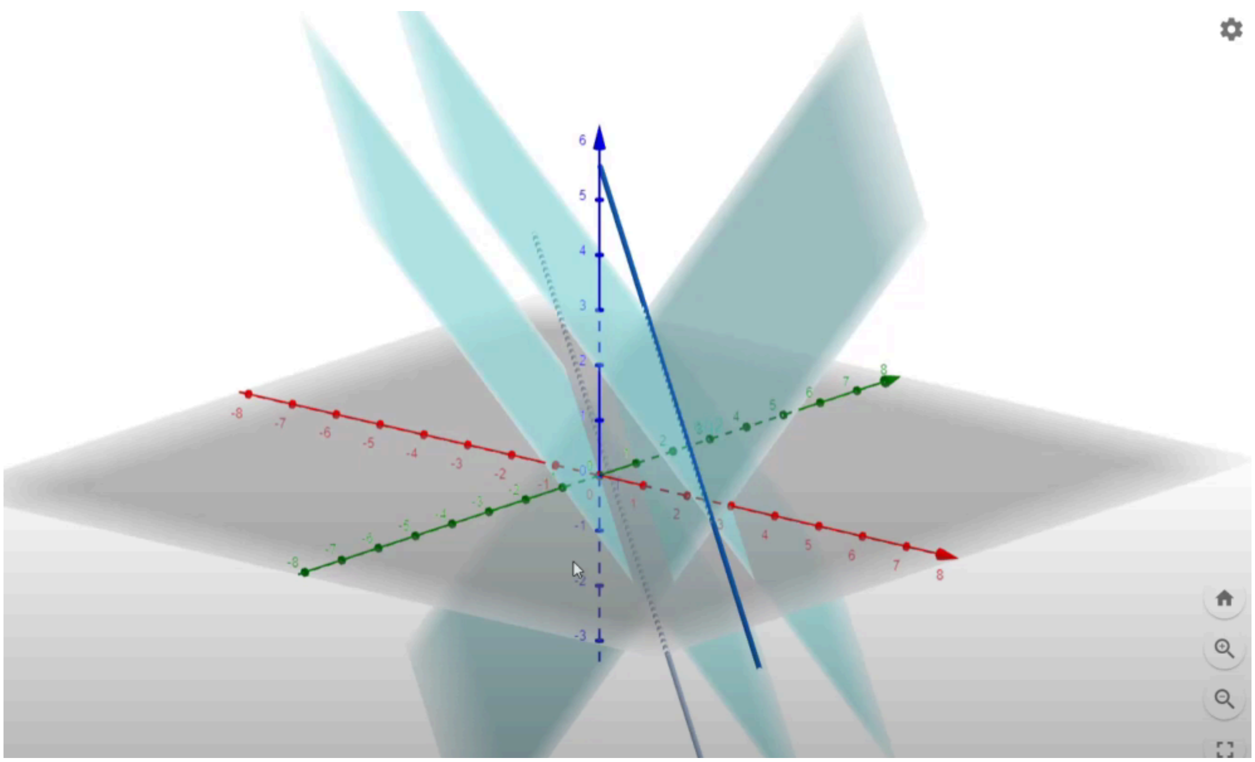
1st and 2nd



2nd and 3rd



For all



No solution

Week 01 - Tutorial 04

System of linear equations - 4

$$\left.\begin{aligned} z &= 5 \\ z &= 3 \\ x &= 4 \end{aligned}\right\}$$

x, y, z

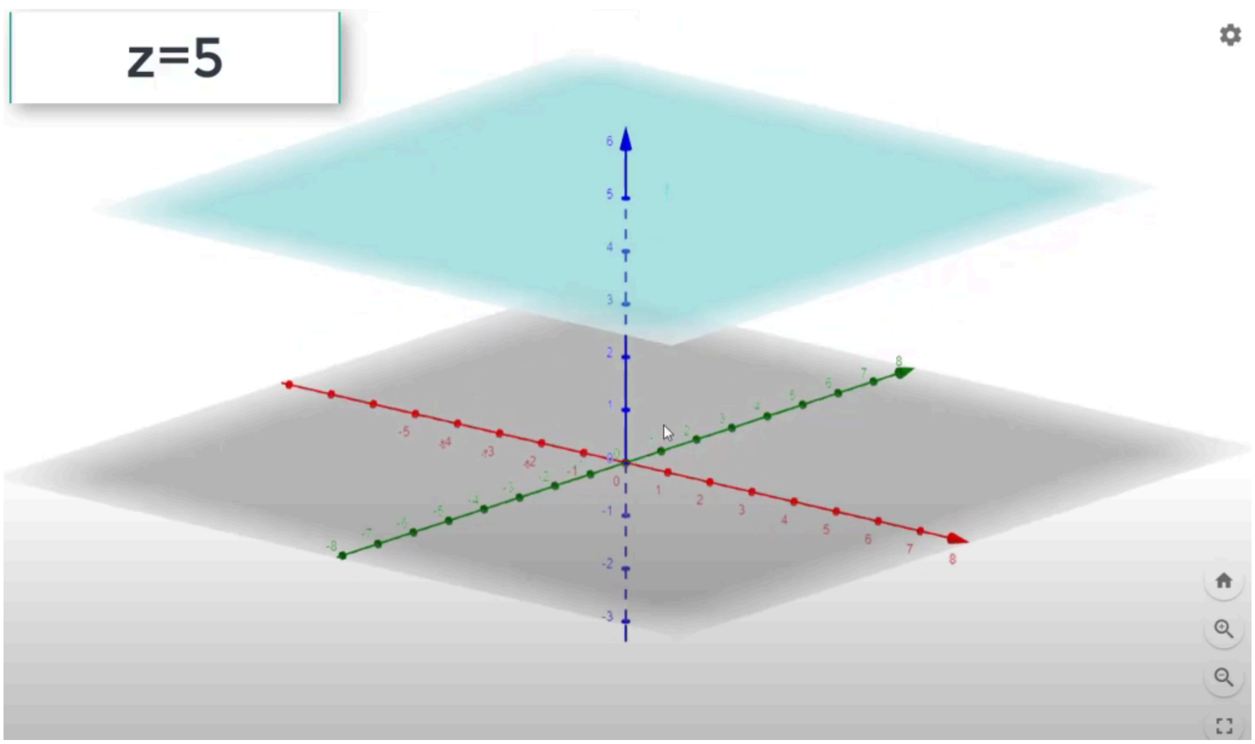
$$\left.\begin{aligned} 0\cdot x + 0\cdot y + 1\cdot z &= 5 \\ 0\cdot x + 0\cdot y + 1\cdot z &= 3 \\ 1\cdot x + 0\cdot y + 0\cdot z &= 4 \end{aligned}\right\}$$

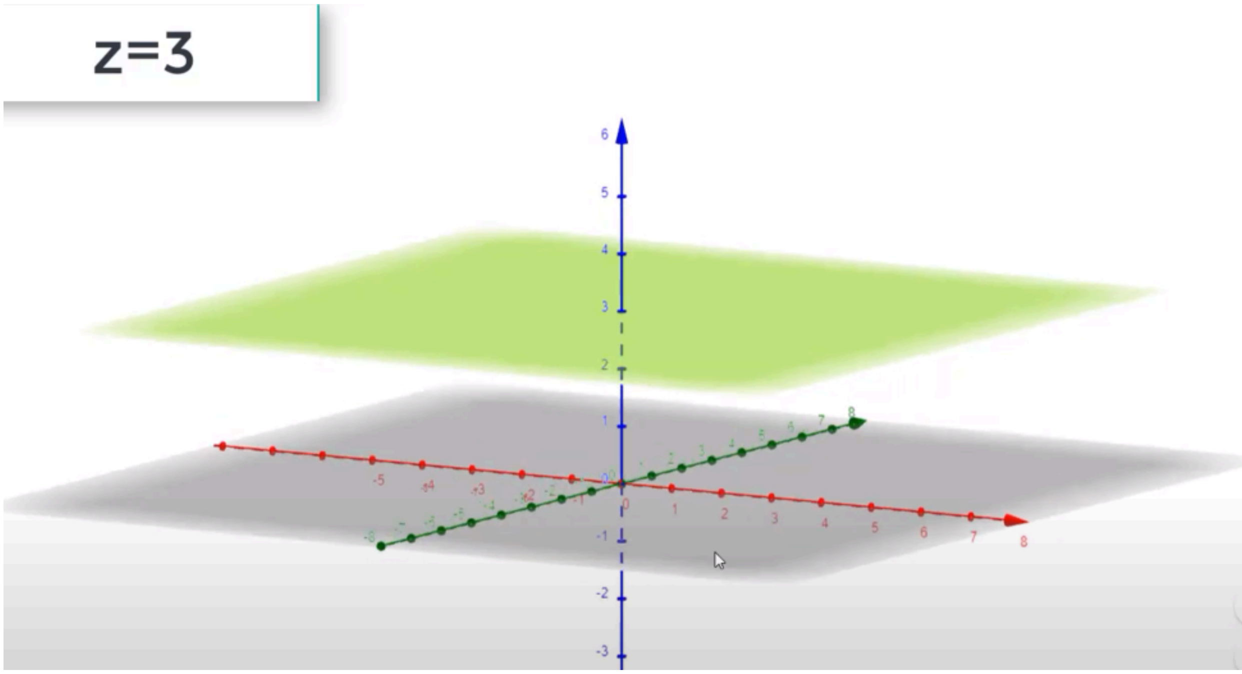
(a, b, c)

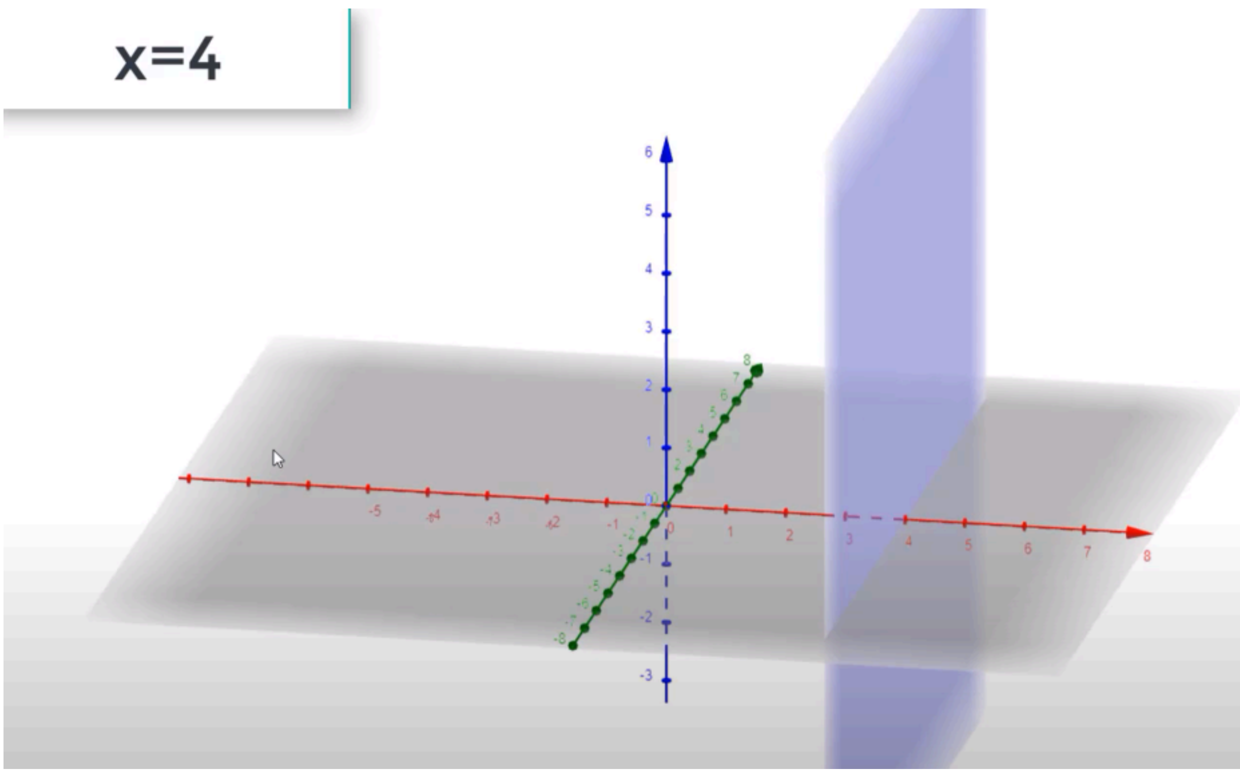
$c = 5$

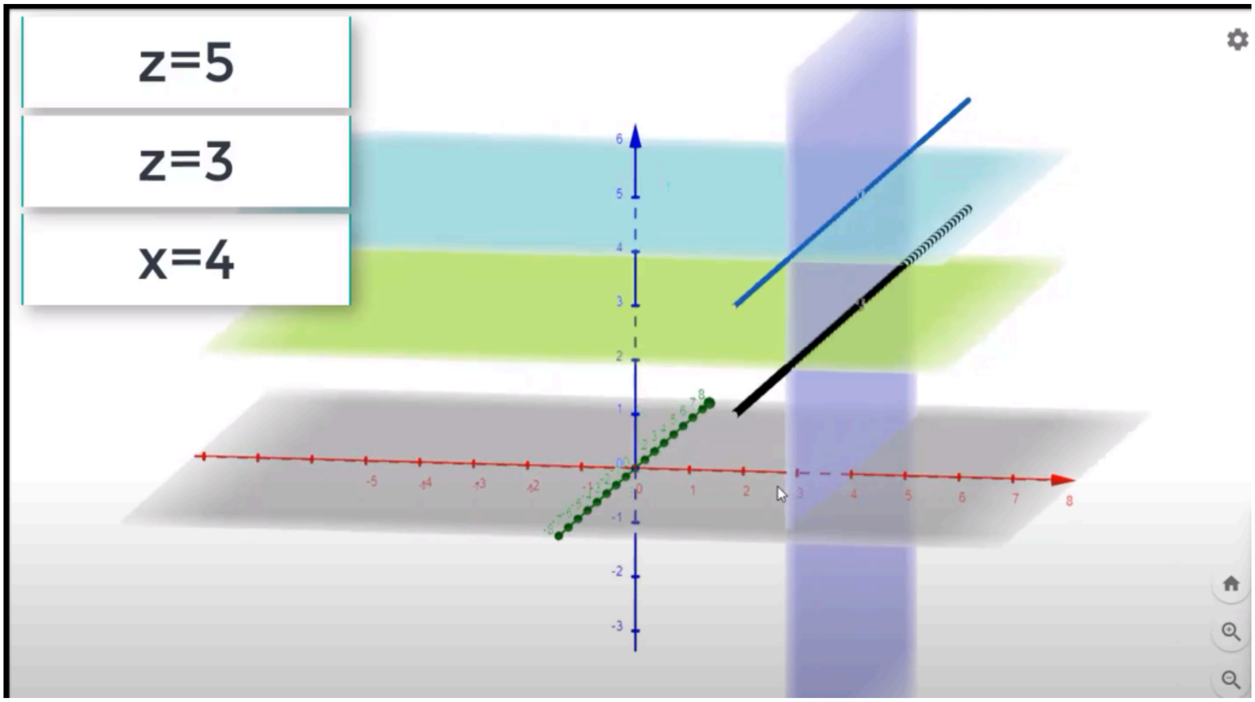
$c = 3$

$5 = 3$ (absurd)




z=3


x=4



No solution

Week 01 - Tutorial 05

Application of vector addition and scalar multiplication : working with some large data set

$c_3 = \frac{1}{2}(c_1 + c_2)$

$\frac{1}{2}(v_1 + w_1)$

$\frac{1}{2}(v_2 + w_2)$

$\vdots$

$c_1 = \begin{pmatrix}\cdot\\\cdot\\\cdot\\\vdots\end{pmatrix}$ $c_2 = \begin{pmatrix}\cdot\\\cdot\\\cdot\end{pmatrix}$

Airlines	Safety Scores		
	1985-1999 c_1	2000-2014 c_2	combined c_3
Southwest Airlines	0.99	0.82	0.9
Cathay Pacific	0.91	0.86	0.88
Lufthansa	0.8	0.96	0.88
Brithish Airways	0.9	0.85	0.88
Air canada	0.73	0.73	0.73
Qantas	0.77	0.65	0.71
United/Continental	0.37	0.98	0.67
KLM	0.46	0.76	0.61
Virgin Atlantic	0.57	0.62	0.6
Singapore Airlines	0.6	0.58	0.59
All Nippon Airways	0.57	0.57	0.57
TAP-Air Portugal	0.51	0.51	0.51
Finnair	0.42	0.47	0.45
Hawaiian Airlines	0.47	0.41	0.44
LAN Airlines	0.12	0.62	0.37
Austrian Airlines	0.35	0.35	0.35
Aer Lingus	0.26	0.4	0.33
American	0.4	0.26	0.33
Delta/ Northwest	-0.16	0.79	0.31
Iberia	0.03	0.46	0.24
Air New-Zealand	0.39	0.06	0.23
Air Condor	0	0.44	0.22
COPA	-0.05	0.49	0.22
Alaska Airlines	0.39	0	0.2

c_1	c_2
v_1	w_1
v_2	w_2
v_3	w_3
$\vdots$	$\vdots$

$c_1 + c_2$
$v_1 + w_1$
$v_2 + w_2$
$v_3 + w_3$
$\vdots$

Week 01 - Tutorial 06

Question : If $M = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}_{2\times 2}$, then find the value of λ such that $M^2 + \lambda M - 5I = 0$

Week 01 - Tutorial 07

System of linear equations

1. In a laptop showroom there are laptops with RAM: 4GB, 8GB, and 16GB of different companies: A, B and C. Last week, the showroom sold 2, 1 and 3 laptops with 4GB RAM of companies A, B, and C respectively; 1, 2 and 1 laptops with 8GB RAM of companies A, B, and C respectively; and 2, 3 and 3 laptops with 16GB RAM of companies A, B and C respectively. The price of laptops with a particular GB Ram is the same irrespective of the company (i.e., laptop of companies A, B and C with 4GB RAM have the same price; similarly, laptop of companies A, B and C with 8GB RAM have the same price; and laptop of companies A, B and C with 16GB RAM have the same price). The owner of showroom earned ₹14 , ₹17 and ₹18 (in ten thousand) in that week by selling laptop of companies A, B and C respectively.
Using the above information, form the system of linear equations and find the matrix representation of the system of linear equations to find the price of 1 laptop with 4GB, 1 laptop with 8GM and 1 laptop 16 GB.

1 laptop with 8GM and 1 laptop 16 GB.

	A	B	C
4GB	2	1	3
8GB	1	2	1
16GB	2	3	3
	14	17	18

(in thousand)

4 GB: p | 8GB: q | 16GB: r

$$\begin{cases} 2p + q + 2r = 14 \\ p + 2q + 3r = 17 \\ 3p + q + 16r = 18 \end{cases}$$

$$A = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 3 \end{bmatrix}$$

$$x = \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

$$b = \begin{bmatrix} 14 \\ 17 \\ 18 \end{bmatrix}$$

$$Ax = b \Rightarrow \begin{bmatrix} 2 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 3 \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} 14 \\ 17 \\ 18 \end{bmatrix}$$

1

Week 01 - Tutorial 08

Question: Consider the following matrices:

$$A = \begin{bmatrix} 3x & 0 \\ 4 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 2y & 7 \\ 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 7 & x \\ 2x & 1 \end{bmatrix}$$

and $D = \begin{bmatrix} 0 & 3y \\ y & 0 \end{bmatrix}$

(i) If $A + B = C + D$, then find the possible values of x and y.

(ii) If $\det(C) = -1$, then find the possible values of x.

Notes :

Key Points:

Vectors in $\mathbb{R}^2$ are represented by ordered pairs (a, b).

The first element in the ordered pair denotes the X-coordinate and the second one denotes the Y-coordinate in the Cartesian plane.

The addition of two vectors $v_1 = (x_1, y_1)$ and $v_2 = (x_2, y_2) \in \mathbb{R}^2$, is given by $v_1 + v_2 = (x_1 + x_2, y_1 + y_2)$.

The scalar multiplication of a vector $v = (x, y) \in \mathbb{R}^2$ with a scalar $c \in \mathbb{R}$, is given by $cv = (cx, cy)$.

The addition of two vectors $V_1 = (a_1, b_1, c_1)$ and $V_2 = (a_2, b_2, c_2)$ are done coordinate-wise, as follow :

$$V_1 + V_2 = (a_1, b_1, c_1) + (a_2 + b_2 + c_2) = (a_1 + a_2, b_1 + b_2, c_1 + c_2)$$

The scalar multiplication of a vector $V = (a, b, c)$ by a scalar α is given by:

$$\alpha(a, b, c) = (\alpha a, \alpha b, \alpha c)$$

3. Key Points:

Addition of matrices $A=\begin{bmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{bmatrix}$ and $B=\begin{bmatrix}b_{11}&b_{12}&b_{13}\\b_{21}&b_{22}&b_{23}\\b_{31}&b_{32}&b_{33}\end{bmatrix}$ is given by

$$A+B=\begin{bmatrix}a_{11}+b_{11}&a_{12}+b_{12}&a_{13}+b_{13}\\a_{21}+b_{21}&a_{22}+b_{22}&a_{23}+b_{23}\\a_{31}+b_{31}&a_{32}+b_{32}&a_{33}+b_{33}\end{bmatrix}$$

If $R=\begin{bmatrix}a&b&c\end{bmatrix}$ and $C=\begin{bmatrix}d\\e\\f\end{bmatrix}$, then the product RC is given by

$$\begin{bmatrix}ad+be+cf\end{bmatrix}$$

Suppose $A=\begin{bmatrix}R_1\\R_2\end{bmatrix}$ and $B=\begin{bmatrix}C_1&C_2&C_3\end{bmatrix}$, where R_i's denote the rows of matrix A and C_i's denote the columns of matrix B. Moreover, assume that the number of columns of A and the number of rows of B are the same. The product AB is given by,

$$AB=\begin{bmatrix}R_1C_1&R_1C_2&R_1C_3\\R_2C_1&R_2C_2&R_2C_3\end{bmatrix}$$

Addition of two matrices A and B is defined if both A and B have the same number of rows and the same number of columns. If both the matrices A and B have m rows and n columns, then the matrix $A+B$ also has m rows and n columns

If A is an $m\times n$ matrix and B is an $n\times p$ matrix, then AB is well-defined and is an $m\times p$ matrix.

[Hint: Multiplication of matrices $A=\begin{bmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{bmatrix}$ and $B=\begin{bmatrix}b_{11}&b_{12}&b_{13}\\b_{21}&b_{22}&b_{23}\\b_{31}&b_{32}&b_{33}\end{bmatrix}$ is given by $AB=$ $\begin{bmatrix}a_{11}b_{11}+a_{12}b_{21}+a_{13}b_{31}&a_{11}b_{12}+a_{12}b_{22}+a_{13}b_{32}&a_{11}b_{13}+a_{12}b_{23}+a_{13}b_{33}\\a_{21}b_{11}+a_{22}b_{21}+a_{23}b_{31}&a_{21}b_{12}+a_{22}b_{22}+a_{23}b_{32}&a_{21}b_{13}+a_{22}b_{23}+a_{23}b_{33}\\a_{31}b_{11}+a_{32}b_{21}+a_{33}b_{31}&a_{31}b_{12}+a_{32}b_{22}+a_{33}b_{32}&a_{31}b_{13}+a_{32}b_{23}+a_{33}b_{33}\end{bmatrix}$ Similarly, BA can be calculated.]

4. Key points:

Consider a system of linear equations as follows:

$$a_{11}x_1+a_{12}x_2+a_{13}x_3=b_1$$
$$a_{21}x_1+a_{22}x_2+a_{23}x_3=b_2$$
$$a_{31}x_1+a_{32}x_2+a_{33}x_3=b_3$$

Let the matrix representation of the above system be $Ax=b$, where $A=\begin{bmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{bmatrix}$,

$x=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$, and $b=\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}$.

A is called $Coefficient\ matrix$.

Suppose there are m number of equations and n number of variables in the system of linear equations, then the coefficient matrix will be an $m\times n$ matrix, x will be an $n\times 1$ and b will be an $m\times 1$ matrix.

5. Key points:

Let A be a 2×2 matrix as follows:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$det(A) = ad - bc$$

Let A be a 3×3 matrix as follows:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$det(A) = a_{11} \times det\left(\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}\right) - a_{12} \times det\left(\begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix}\right) + a_{13} \times det\left(\begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}\right)$$

i.e., $det(A) = a_{11} \times (a_{22}a_{33} - a_{32}a_{23}) - a_{12} \times (a_{21}a_{33} - a_{23}a_{31}) + a_{13} \times (a_{21}a_{32} - a_{22}a_{31})$

(Expanding with respect to first row)

6. Key points:

Row operations and relation with determinants:

- Type 1: Interchanging two rows of a matrix changes the sign of the determinant.
- Type 2: Multiplying a real number with a row and adding it to some other row does not change the determinant.
- Type 3: The determinant of a new square matrix obtained by multiplying a real number c with a row of square matrix of order n, is c times the determinant of the original matrix.

Same types of column operations, as discussed above for row operations, gives same type of relation with the determinants.

If two rows or two columns of a matrix are equal, then the determinant of the matrix is 0.

If all the elements of any row or any column of a matrix are 0, then the determinant of the matrix is 0.

$det(A) = det(A^T)$, where A^T denotes the transpose of A.

$det(AB) = det(BA) = det(A)det(B)$.

PA : 10

GA : 9,10