Week 5 : Rank and Nullity of a matrix

Lecture 1 : The null space of a matrix finding nullity and a basis for the null space - Part 1

Content :

- What is the null space of a matrix? Equivalently, what is the solution space corresponding to a homogeneous system of linear equations?
- What is the nullity of a matrix?
- How do we find a basis for the null space?
- Examples
- The rank-nullity theorem
- Using determinants to check if a given set of vectors is a basis for a vector space.
- Examples

Recall : We can find the dimension and a basis for a vector space spanned by a set of vectors using Gaussian elimination.

Solution space of a homogeneous system of linear equations

Let A be an $m \times n$ matrix.

The subspace $W = \{x \in \mathbb{R}^n | Ax = 0\}$ of $\mathbb{R}^n$ is called the solution space of the homogeneous system of linear equation $Ax = 0$ or the null space of A.

Note that the null space is a subspace of $\mathbb{R}^n$. The dimension of the null space is called the nullity of A.

$x, y \in W \Rightarrow Ax = Ay = 0 \Rightarrow A(x+y) = Ax + Ay = 0 + 0 = 0.$

$\Rightarrow x + y \in W.$

$\lambda \in \mathbb{R},\ x \in W \Rightarrow A(\lambda x) = \lambda(Ax) = \lambda 0 = 0.$

Finding the nullity and a basis for the null space

We have seen how to find the dimension and a basis for the row space of A using row reduction.

We will use row reduction to also find the nullity and a basis for the null space of A.

Recall first how to find the solution space for a system $Ax = b$ i.e. Gaussian elimination.

- Form the augmented matrix $[A|b]$
- Apply the same row reduction operations on the augmented matrix that are used to row reduce A to obtain the augmented matrix $[R|c]$ where R is the matrix in reduced row echelon form obtained from A.
- If the i-th column has the leading entry of some row, we call x_i a dependent variable.
- If the i-th column does not have the leading entry of some row, we call x_i an independent variable.

$$\text{nullity}(A) = \text{number of independent variables}\ .$$

- Assign arbitrary value t_i to the i^{th} independent variable.
- Compute the value of each dependent variables in terms of t_is from the unique row it occurs in.
- Every solution is obtained by letting t_is vary in $\mathbb{R}$.

The vectors obtained by substituting $t_i = 1$ and $t_j = 0 \forall j \neq i$ as i varies constitutes a basis of the null space of A (i.e. the solution space of $Ax = 0$).

Example : 3 × 3 matrix

Consider the (matrix representation of the) homogeneous system of linear equations of the form $Ax = 0$, where $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix}$

The augmented matrix is $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 2 & 2 & 2 & 0 \\ 3 & 3 & 3 & 0 \end{array}\right]$.

Row reduction yields :

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 2 & 2 & 2 & 0 \\ 3 & 3 & 3 & 0 \end{array}\right] \underset{R_2-2R_1}{\overset{R_3-3R_1}{\rightsquigarrow}} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Independent variables : x_2, x_3, dependent variable : x_1.

Hence, $nullity(A) = 2$.

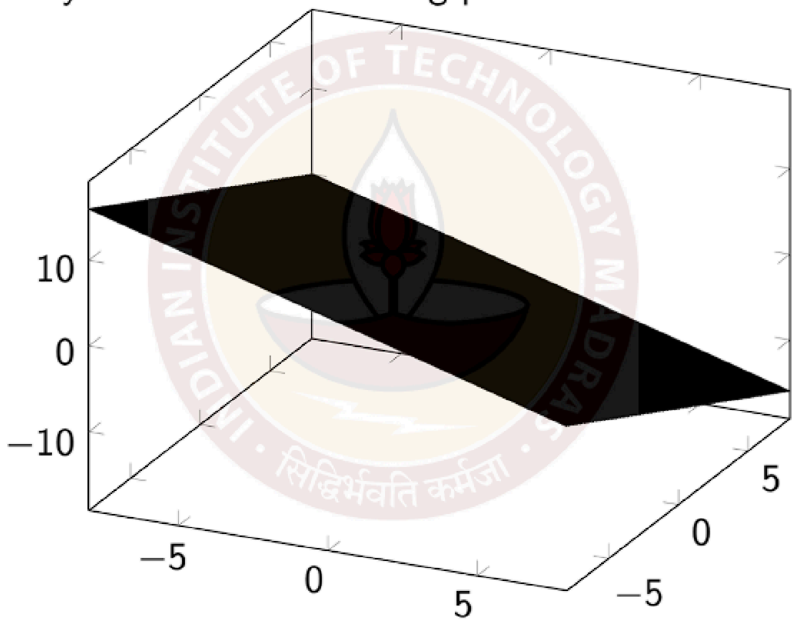
Put $x_2 = t_1$ and $x_3 = t_2$. Then the equation yields
$x_1 = -x_2 - x_3 = -t_1 - t_2$.

Hence, the null space of A (i.e. the solution space of $Ax = 0$) is $\{(-t_1 - t_2, t_1, t_2) | t_1, t_2 \in \mathbb{R}\}$.

$t_1 = 1, t_2 = 0$ yields the basis vector $(-1, 1, 0)$.
$t_1 = 0, t_2 = 1$ yields the basis vector $(-1, 0, 1)$.

Hence, a basis for the null space is $(-1, 1, 0), (-1, 0, 1)$.

Geometrically we have the following plane as the solution space :



Augmentation not required

Notice that in our computation, since the system is homogeneous, the augmented 0 vector remains unchanged during the row reduction process.

So we will drop the 0 column augmented to the matrix while performing the row reduction computations and use it only for solving for the dependent variables.

QN : 7

<u>Lecture 2 : The null space of a matrix finding nullity and a basis for the null space - Part 2</u>

Example : 3 × 4 matrix

Consider the (matrix representation of the) homogeneous system of linear equations of the form $Ax = 0$, where $A = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 2 & 3 & 0 & 3 \\ 1 & 1 & 1 & 2 \end{bmatrix}$.

The augmented matrix is $\left[\begin{array}{cccc|c} 1 & 2 & 0 & 3 & 0 \\ 2 & 3 & 0 & 3 & 0 \\ 1 & 1 & 1 & 2 & 0 \end{array}\right]$.

Row reduction yields :

$$\begin{bmatrix} 1 & 2 & 0 & 3 \\ 2 & 3 & 0 & 3 \\ 1 & 1 & 1 & 2 \end{bmatrix} \xrightarrow[R_2-2R_1]{R_3-\cancel{3}R_1} \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & -1 & 0 & -3 \\ 0 & -1 & 1 & -1 \end{bmatrix} \xrightarrow{-R_2} \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 1 & 0 & 3 \\ 0 & -1 & 1 & -1 \end{bmatrix}.$$

and continuing the process yields :

$$\begin{bmatrix}1 & 2 & 0 & 3\\0 & 1 & 0 & 3\\0 & -1 & 1 & -1\end{bmatrix} \overset{R_3+R_2}{\rightsquigarrow} \begin{bmatrix}1 & 2 & 0 & 3\\0 & 1 & 0 & 3\\0 & 0 & 1 & 2\end{bmatrix} \overset{R_1-2R_2}{\rightsquigarrow} \begin{bmatrix}1 & 0 & 0 & -3\\0 & 1 & 0 & 3\\0 & 0 & 1 & 2\end{bmatrix}.$$

$[R \mid 0]$

Independent variable : x_4, dependent variables : x_1, x_2, x_3.

Hence, $nullity(A) = 1$. Put $x_4 = t$.

The equations from the row reduced echelon form are :

$$x_1 - 3x_4 = 0 \qquad x_2 + 3x_4 = 0 \qquad x_3 + 2x_4 = 0,$$

and hence we obtain that $x_1 = 3t, x_2 = -3t, x_3 = -2t$.

Hence, the null space of A (i.e. the solution space of $Ax = 0$) is $\{(3t, -3t, 2t, t) | t \in \mathbb{R}\}$.

$t = 1$ yields the basis vector $(3, -3, 2, 1)$.

Hence, a basis for the null space of $A = \begin{bmatrix}1 & 2 & 0 & 3\\2 & 3 & 0 & 3\\1 & 1 & 1 & 2\end{bmatrix}$ is

$(3, -3, 2, 1)$.

The rank-nullity theorem

Let A be an $m \times n$ matrix.

Recall that the row rank of A is the dimension of the row space of A and the column rank of A is the dimension of the column space of A. These are equal and are denoted by $rank(A)$.

$rank(A)$ is calculated as the number of non-zero rows of the matrix R in reduced row echelon form obtained by row reduction.

Note that for a matrix R in the row echelon form, the number of non-zero rows $=$ number of dependent variables for the corresponding homogeneous system $Rx = 0$.

Hence, $rank(A) =$ number of non-zero rows of $R =$ number of dependent variables of $Rx = 0$.

$nullity(A) =$ number of independent variables of $Rx = 0$.

Therefore, we have the rank-nullity theorem :

Theorem

For an $m \times n$ matrix A, $rank(A) + nullity(A) = n$.

How to check if a set of n vectors is a basis for R

Short answer : Use determinants.

Suppose we are given n vectors of $\mathbb{R}^n$.

We write them as columns of a matrix, thus obtaining an $n \times n$ (square) matrix.

If the determinant of the matrix is 0, then the given set of vectors does not form a basis, otherwise it forms a basis.

Examples :
The standard basis $(1, 0), (0, 1)$ yields the identity matrix I with determinant 1.
The vectors $(1, -2), (5, -10)$ yields the matrix $\begin{bmatrix} 1 & 5 \\ -2 & -10 \end{bmatrix}$ with determinant 0. This is not a basis for $\mathbb{R}^2$.

Example in R^3

Is the set $\{\overset{x_1}{(1,2,3)}, \overset{x_2}{(0,1,2)}, \overset{x_3}{(1,3,0)}\}$ a basis for $\mathbb{R}^3$?

Form a matrix A with columns given by these vectors.

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 3 \\ 3 & 2 & 0 \end{bmatrix}$$

$det(A) = 1 \times (-6) - 0 \times (-9) + 1 \times (4-3) = -6+0+1 = -5 \neq 0.$

Hence the given set of vectors forms a basis of $\mathbb{R}^3$.

Let $b \in \mathbb{R}^3$. Need: $a_1, a_2, a_3 \in \mathbb{R}$ s.t. $\sum a_i x_i = b$.

$A \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = b$. Unique soln. is $\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = A^{-1} b$.

Example in R^4

Is $\{(1,2,3,0),(0,1,2,1),(1,3,0,2),(2,6,5,3)\}$ a basis for $\mathbb{R}^4$?

Form a matrix with columns given by these vectors.

$$A = \begin{bmatrix} 1 & 0 & 1 & 2 \\ 2 & 1 & 3 & 6 \\ 3 & 2 & 0 & 5 \\ 0 & 1 & 2 & 3 \end{bmatrix}$$

$A\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = 0$
& solve.

$$det(A) = 1 \times det \begin{bmatrix} 1 & 3 & 6 \\ 2 & 0 & 5 \\ 1 & 2 & 3 \end{bmatrix} - 0 + 1 \times det \begin{bmatrix} 2 & 1 & 6 \\ 3 & 2 & 5 \\ 0 & 1 & 3 \end{bmatrix} - 2 \times det \begin{bmatrix} 2 & 1 & 3 \\ 3 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix}$$

$$= 11 + 11 - 2 \times 11 = 0.$$

Hence the given set of vectors does not form a basis of $\mathbb{R}^4$.

QN : 2,5,7,9,10,14

Lecture 3 : What is a linear mapping - Part 1

Grocery shop example

The prices of rice, dal and oil in Shop A in the town of Malgudi are as follows :

	Rice (per kg)	Dal (per kg)	Oil (per litre)
Shop A	45	125	150

The cost of 1 kg.of rice, 2 kg. of dal and 1 kg. of oil is $1 \times 45 + 2 \times 125 + 1 \times 150 = 445$.

The cost of 2 kg.of rice, 1 kg. of dal and 2 kg. of oil is $2 \times 45 + 1 \times 125 + 2 \times 150 = 515$.

The cost of x_1 kg.of rice, x_2 kg. of dal and x_3 kg. of oil is :

$$x_1 \times 45 + x_2 \times 125 + x_3 \times 150 = 45x_1 + 125x_2 + 150x_3.$$

Expressions and linear combinations

The term $45x_1 + 125x_2 + 150x_3$ is an expression.

We can equivalently think of it as a function c_A from $\mathbb{R}^3$ to $\mathbb{R}$ since for every value of x_1, x_2, x_3, we obtain a real number.

Since c_A is a linear combination of x_1, x_2, x_3 (with coefficients $45, 125, 150$), it is an example of a linear function.

Recall that linear combinations can also be expressed in terms of matrix multiplication e.g.

$$c_A(x_1, x_2, x_3) = 45x_1 + 125x_2 + 150x_3 = \begin{bmatrix} 45 & 125 & 150 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

Example : Cost linearity

A caterer gets an order from Office 1 in Malgudi on Monday for 30 tiffins prepared in some prescribed way and buys 20 kg. of rice, 10 kg. of dal and 4 litres of oil from Shop A for the purpose.

On Tuesday, they get an order from Office 2 in Malgudi for 40 tiffins prepared in some other prescribed way and buy 30 kg. of rice, 12 kg. of dal and 2 litres of oil from Shop A for the purpose.

On Wednesday, they get orders from both offices for 15 and 50 tiffins respectively, each to be prepared in the respective prescribed ways. How much does the caterer spend at Shop A?

Note that the required quantities of rice, dal and oil for preparing the 15 tiffins for office 1 on Wednesday will be half of the amounts on Monday i.e. 10 kg. of rice, 5 kg. of dal and 2 litres of oil.

Similarly, the required quantities of rice, dal and oil for preparing the 50 tiffins for office 2 on Wednesday will be $\frac{5}{4}$ times the amounts on Tuesday i.e. 37.5 kg. of rice, 15 kg. of dal and 2.5 litres of oil.

Hence, the total required quantities of rice, dal and oil on Wednesday will be i.e. 47.5 kg. of rice, 20 kg. of dal and 4.5 litres of oil. So the cost to the caterer at shop A is

$$45 \times 47.5 + 125 \times 15 + 150 \times 2.5 = \overset{5312.5}{\cancel{4387.5}} \text{ rupees.}$$

We could calculate this differently as follows :

Cost to the caterer for office 1 on Monday : 2750 rupees
Cost to the caterer for office 2 on Tuesday : 3150 rupees

Cost to the caterer on Wednesday :
$1/2 \times 2750 + 5/4 \times 3150 = 5312.5$ rupees.

The first method can be summarized as calculating the vector for Wednesday as adding 1/2 the vector for Monday and 5/4 times the vector for Tuesday and the cost is computed by applying the linear combination on it.

The second method as adding 1/2 the cost for Monday and 5/4 times the cost for Tuesday.

The following table summarizes this data : adding 1/2 the first row and 5/4 times the second row yields the third row.

	Kgs. of rice	Kgs. of dal	Litres of oil	Cost c_A
Monday	20	10	4	2750
Tuesday	30	12	2	3150
Wednesday	47.5	20	4.5	5312.5

Why does the second method work?

Why does the second method work?

Total Cost on Wed.

$$= 45 \times \text{rice reqd. on W.} + 125 \times \text{dal reqd. on W} + 150 \times \text{oil reqd. on W}$$

$$= 45 \times \left(\frac{1}{2} \times \text{rice on M} + \frac{5}{4} \times \text{rice on T}\right) + 125 \times \left(\frac{1}{2} \times \text{dal reg. on M} + \frac{5}{4} \times \text{dal on T}\right)$$
$$+ 150 \times \left(\frac{1}{2} \times \text{oil on M} + \frac{5}{4} \times \text{oil on T}\right)$$

$$= \frac{1}{2} \times \left\{\left(45 \times \text{rice on M}\right) + \left(125 \times \text{dal on M}\right) + \left(150 \times \text{oil on M}\right)\right\}$$
$$+ 5/4 \times \left\{\left(45 \times \text{rice on T}\right) + \left(125 \times \text{dal on T}\right) + \left(150 \times \text{oil on T}\right)\right\}$$

$$= \frac{1}{2} \times \text{cost on Monday} + 5/4 \times \text{cost on Tuesday.}$$

Lecture 4 : What is a linear mapping - Part 2

Grocery shops example

Suppose there are 3 shops in a locality, the original shop A and two other shops B and C. The prices of rice, dal and oil in each of these shops are as given in the table below :

	Rice (per kg)	Dal (per kg)	Oil (per litre)
Shop A	45	125	150
Shop B	40	120	170
Shop C	50	130	160

Based on these prices, how will we decide from which shop to buy your groceries?

Write the expression for the total cost of buying x_1 kg.of rice, x_2 kg. of dal and x_3 kg. of oil for each of the three shops and try to compare them.

Expressions

We have already seen the expression for shop A as a linear combination and remarked that it can be thought of as a function c_A and viewed as matrix multiplication.

$$c_A(x_1, x_2, x_3) = 45x_1 + 125x_2 + 150x_3 = \begin{bmatrix} 45 & 125 & 150 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

Similarly for shops B and C, we get functions c_B and c_C whose expressions and matrix forms are :

$$\begin{aligned} c_B(x_1, x_2, x_3) &= 40x_1 + 120x_2 + 170x_3 &= \begin{bmatrix} 40 & 120 & 170 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \\ c_C(x_1, x_2, x_3) &= 50x_1 + 130x_2 + 160x_3 &= \begin{bmatrix} 50 & 130 & 160 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \end{aligned}$$

Linear mappings

Comparing these expressions, it is clear that for any quantities x_1, x_2, x_3 that one would buy (i.e. when x_1, x_2, x_3 are positive), the third expression always yields larger values than the first one.

However, the comparison between the second expression and the others depends on the quantities of the items bought, i.e. on x_1, x_2, x_3.

A natural way to make this comparison would be to create a vector of costs i.e. $\big(c_A(x_1, x_2, x_3), c_B(x_1, x_2, x_3), c_C(x_1, x_2, x_3)\big)$.

We can then think of the cost vector as a function c from $\mathbb{R}^3$ to $\mathbb{R}^3$ by setting these expressions as the coordinates in $\mathbb{R}^3$ i.e.

$$c(x_1, x_2, x_3) = (c_A(x_1, x_2, x_3), c_B(x_1, x_2, x_3), c_C(x_1, x_2, x_3)) =$$
$$(45x_1 + 125x_2 + 150x_3, 40x_1 + 120x_2 + 170x_3, 50x_1 + 130x_2 + 160x_3).$$

Example

We can use matrix multiplication to express the cost function c in a compact form and extract its properties :

$$c(x_1, x_2, x_3) = \begin{bmatrix} 45 & 125 & 150 \\ 40 & 120 & 170 \\ 50 & 130 & 160 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

e.g. the costs of buying 2 kg Rice, 1 kg Dal, and 2 litres oil are given by the cost vector $\begin{bmatrix} 45 & 125 & 150 \\ 40 & 120 & 170 \\ 50 & 130 & 160 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 515 \\ 540 \\ 550 \end{bmatrix}.$

Linearity

As in the case of the function c_A, the property of "linearity" of costs can now be extracted from the matrix form for the cost function c.

$$c\left(\tfrac{1}{2}\times\text{rice on }M + 5/4\times\text{rice on }T,\ \tfrac{1}{2}\times\text{dal on }M + 5/4\times\text{dal on }T,\ \tfrac{1}{2}\times\text{oil on }M + 5/4\times\text{oil on }T\right)$$

$$= \left(c_A(\quad),\ c_B(\quad),\ c_C(\quad)\right)$$

$$= \left(\tfrac{1}{2}\times c_A(M) + 5/4\times c_A(T),\ \tfrac{1}{2}\times c_B(M) + 5/4\times c_B(T),\ \tfrac{1}{2}\times c_C(M) + 5/4\times c_C(T)\right)$$

$$= \tfrac{1}{2}\times c(M) + 5/4\times c(T)$$

$$M = (\text{rice on }M,\ \text{dal on }M,\ \text{oil on }M)$$

$$T = (\text{rice on }T,\ \text{dal on }T,\ \text{oil on }T)$$

$$c(\alpha(x_1,x_2,x_3) + (y_1,y_2,y_3)) = \alpha\, c(x_1,x_2,x_3) + c(y_1,y_2,y_3).$$

$$\left[\quad\right]\left\{\alpha\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} + \begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}\right\} = \alpha\left[\quad\right]\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} + \left[\quad\right]\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}$$

What is a linear mapping

A linear mapping f from $\mathbb{R}^n$ to $\mathbb{R}^m$ can be defined as follows :

$$f(x_1, x_2, \ldots, x_n) \quad = \left(\sum_{j=1}^{n} a_{1j}x_j, \sum_{j=1}^{n} a_{2j}x_j, \ldots, \sum_{j=1}^{n} a_{mj}x_j \right).$$

where the coefficients a_{ij}s are real numbers (scalars). A linear mapping can be thought of as a collection of linear combinations.

We can write the expressions on the RHS in matrix form as Ax

where $A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ldots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}$ and $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$.

Linearity of linear mappings

It follows that a linear mapping satisfies linearity, i.e. for any $c \in \mathbb{R}$ (scalar)

$$f(x_1 + cy_1, x_2 + cy_2, \ldots, x_n + cy_n) = f(x_1, x_2, \ldots, x_n) + cf(y_1, y_2, \ldots, y_n).$$

$$f(x_1 + cy_1, x_2 + cy_2, \ldots, x_n + cy_n) = A \begin{bmatrix} x_1 + cy_1 \\ x_2 + cy_2 \\ \vdots \\ x_n + cy_n \end{bmatrix}$$

$$= A \left\{ \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + c \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \right\} = A \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + cA \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$$

$$= f(x_1, \ldots, x_n) + c\, f(y_1, \ldots, y_n).$$

QN : 5,6,7

Lecture 5 : What is a Linear Transformation

Recall : linear mappings

A linear mapping f from $\mathbb{R}^n$ to $\mathbb{R}^m$ is :

$$f(x_1, x_2, \ldots, x_n) \quad = \quad (\sum_{j=1}^{n} a_{1j}x_j, \sum_{j=1}^{n} a_{2j}x_j, \ldots, \sum_{j=1}^{n} a_{mj}x_j).$$

where the coefficients a_{ij}s are real numbers (scalars).

We can write the expressions on the RHS in matrix form as Ax.

Linear mappings satisfy linearity, i.e. for any $c \in \mathbb{R}$ (scalar) :

$$f(x_1 + cy_1, x_2 + cy_2, \ldots, x_n + cy_n) = \\ f(x_1, x_2, \ldots, x_n) + cf(y_1, y_2, \ldots, y_n).$$

Formal definition

A function $f : V \to W$ between two vector spaces V and W is said to be a linear transformation if for any two vectors v_1 and v_2 in the vector space V and for any $c \in \mathbb{R}$ (scalar) the following conditions hold :

- $f(v_1 + v_2) = f(v_1) + f(v_2)$ ✓
- $f(cv_1) = cf(v_1)$ ✓

Equivalent to linearity: $f(v_1 + cv_2) = f(v_1) + cf(v_2)$.

$f(v_1+cv_2) = f(v_1) + \underline{\underline{f(cv_2)}} = f(v_1) + cf(v_2)$

Linear mappings are linear transformation.

Examples

1. $f : \mathbb{R}^2 \to \mathbb{R}^2$ $\quad f(x, y) = (2x, y)$
2. $f : \mathbb{R}^2 \to \mathbb{R}^2$ $\quad f(x, y) = (2x, 0)$
3. $f : \mathbb{R}^3 \to \mathbb{R}^3$ $\quad f(x, y, z) = (\frac{x}{2}, 3y, 5z)$
4. $f : \mathbb{R}^3 \to \mathbb{R}^4$ $\quad f(x, y, z) = (4y - z, 3y + \frac{11}{19}z, 5x - 2z, 23y)$
5. $f : \mathbb{R} \to \mathbb{R}^3$ $\quad f(t) = (t, 3t, \frac{23}{89}t)$
6. $f : \mathbb{R}^2 \to \mathbb{R}$ $\quad f(x, y) = x.$

1-1 and onto functions

Recall that a function $f : V \to W$ is 1-1 (or injective) if $f(v_1) = f(v_2)$ implies $v_1 = v_2$.

Recall that a function $f : V \to W$ is onto (or surjective) if for every $w \in W$ there exists $v \in V$ such that $f(v) = w$.

For a linear transformation, being 1-1 is equivalent to $f(v) = 0$ implies $v = 0$.

$f : V \to W$ is a lin. trans.

Assume f is 1-1. Then $f(v_1) = f(v_2) \Rightarrow v_1 = v_2$.

$f(0) = 0$. If $f(v) = 0 \Rightarrow f(v) = f(0)$
$\Rightarrow v = 0$.

$f(0+0) = f(0) + f(0)$
$\Rightarrow f(0) = f(0) + f(0)$
$\Rightarrow f(0) = 0$.

Conversely, assume $f(v) = 0 \Rightarrow v = 0$.

$f(v_1) = f(v_2) \Rightarrow f(v_1 - v_2) = 0$.
$\Rightarrow v_1 - v_2 = 0$
$\Rightarrow v_1 = v_2$.

$f(v_1) - f(v_2) = 0$.
$f(v_1) + f(-v_2) = 0$
$f(v_1 - v_2) = 0$.

$f(v) + f(-v) = f(v + (-v)) = f(0) = 0$
$\Rightarrow f(-v) = -f(v)$

What is an isomorphism

Recall that a function $f : V \to W$ is bijective (or a bijection) if it is 1-1 and onto.

Note that being a bijection is equivalent to : for any $w \in W$ there exists a unique $v \in V$ such that $f(v) = w$.

A linear transformation $f : V \to W$ between two vector spaces V and W is said to be an isomorphism if it is a bijection.

Example 1 seen earlier is an isomorphism :

$$f : \mathbb{R}^2 \to \mathbb{R}^2 \qquad f(x,y) = (2x, y).$$

$f(x,y) = (0,0) \Rightarrow (2x,y) = (0,0) \Rightarrow 2x = 0, y = 0$
$\Rightarrow x = 0, y = 0$
$\Rightarrow (x,y) = (0,0)$ } f is 1-1

For $(u,v) \in \mathbb{R}^2$ consider $x = u/2$, $y = v$.
$\therefore f(x,y) = (2x,y) = (2 \cdot u/2, v) = (u,v)$ } onto

Isomorphisms : Non-examples

Example 2 seen earlier is not an isomorphism :

$$f : \mathbb{R}^2 \to \mathbb{R}^2 \qquad f(x, y) = (2x, 0).$$

There is no pre-image for the vector (u, v), where v is non-zero. e.g. $(0, 1)$ has no pre-image. So f is not surjective. Also $f(x, y) = (0, 0)$ implies $(2x, 0) = (0, 0)$, hence $x = 0$. But there is no restriction on y, e.g. $f(0, 1) = (0, 0)$. Hence f is not 1-1 either.

Similarly, the fifth example $f : \mathbb{R} \to \mathbb{R}^3 \quad ; \quad f(t) = (t, 3t, \frac{23}{89}t)$ is 1-1 but not onto.

Also the sixth example $f : \mathbb{R}^2 \to \mathbb{R} \quad ; \quad f(x, y) = x$ is onto but not 1-1.

Bases determine linear transformations

Let V be a vector space with basis $\{v_1, v_2, \ldots, v_n\}$.
Let $f : V \to W$ be a linear transformation. Then the ordered vectors $f(v_1), f(v_2), \ldots f(v_n)$ uniquely determine f.

Let $v \in V$. $v = \sum_{i=1}^{n} c_i v_i$.

$$\boxed{f(v) = f\left(\sum_{i=1}^{n} c_i v_i\right) = \sum_{i=1}^{n} c_i f(v_i)}$$

is determined by $c_1, \ldots, c_n$ & $f(v_1), f(v_2), \ldots, f(v_n)$.

Suppose $w_1, \ldots, w_n$ is a specified set of vectors in W

There is a unique lin. trans. f s.t. $f(v_i) = w_i$.

Define $f(v) = \sum_{i=1}^{n} c_i w_i$ where $v = \sum_{i=1}^{n} c_i v_i$.

$f(v_k) = w_k$.

Example

Consider the standard basis $\{(1,0),(0,1)\}$ of $\mathbb{R}^2$. What linear transformation $f : \mathbb{R}^2 \to \mathbb{R}^2$ do we obtain by extending :

$$f((1,0)) = (2,0) = 2(1,0) \quad w_1$$
$$f((0,1)) = (0,1) \quad w_2$$

$$(x,y) = x\,(1,0) + y\,(0,1).$$

$$f(x,y) = x(2,0) + y(0,1) = (2x,0) + (0,y).$$
$$= (2x,y).$$

Example : changing the basis

Note that if we choose a different basis for $\mathbb{R}^2$, then we may get a different linear transformation. In the previous example, consider the basis $\{(1,0),(1,1)\}$ instead of the standard basis for $\mathbb{R}^2$. Let us calculate the linear transformation f that we obtain by extending :

$$f((1,0)) = (2,0) = 2(1,0) \quad w_1$$
$$f((1,1)) = (0,1) \quad w_2$$

Note that every element (x,y) is uniquely represented in terms of this basis as $(x,y) = (x-y)(1,0) + y(1,1)$.

asis :
$(1,0), (1,1)$
$_1 = (2,0)$
$_2 = (1,1)$

$$f(x,y) = (x-y)\,f(1,0) + y\,f(1,1)$$
$$= \boxed{(x-y)}\,(2,0) + \boxed{y}\,(0,1)$$
$$= (2(x-y), 0) + (0,y) = (2x-2y, y).$$

$$(x,y) = (x-y)(2,0) + y(1,1)$$
$$= (2x-2y+y, y)$$
$$= (2x-y, y).$$

QN : 5,6,9,10,11,12

Week 5 : Tutorial 1

Maths 2 Week 7 Tutorials

Rank and nullity of a matrix, and finding nullspace using Gauss elimination :

$$A = \begin{pmatrix} 2 & 4 & 0 & 2 \\ 1 & 0 & 3 & 1 \\ 3 & 2 & 1 & 0 \end{pmatrix}_{3\times 4}$$

$$Ax = 0$$

$$\begin{pmatrix} 2 & 4 & 0 & 2 \\ 1 & 0 & 3 & 1 \\ 3 & 2 & 1 & 0 \end{pmatrix}_{3\times 4} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}_{4\times 1} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}_{3\times 1}$$

Row echelon form / Reduced row echelon form

$$\Rightarrow \left.\begin{aligned} 2x_1 + 4x_2 + 2x_4 &= 0 \\ x_1 + 3x_3 + x_4 &= 0 \\ 3x_1 + 2x_2 + x_3 &= 0 \end{aligned}\right\} \text{System of linear equations.}$$

$$\begin{pmatrix} 2 & 4 & 0 & 2 \\ 1 & 0 & 3 & 1 \\ 3 & 2 & 1 & 0 \end{pmatrix} \xrightarrow{\frac{1}{2}R_1} \begin{pmatrix} 1 & 2 & 0 & 1 \\ 1 & 0 & 3 & 1 \\ 3 & 2 & 1 & 0 \end{pmatrix} \xrightarrow{R_2 - R_1} \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & -2 & 3 & 0 \\ 3 & 2 & 1 & 0 \end{pmatrix} \xrightarrow{R_3 - 3R_1} \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & -2 & 3 & 0 \\ 0 & -4 & 1 & -3 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & -2 & 3 & 0 \\ 0 & -4 & 1 & -3 \end{pmatrix} \xrightarrow{\left(-\frac{1}{2}\right)R_2} \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & -\frac{3}{2} & 0 \\ 0 & -4 & 1 & -3 \end{pmatrix} \xrightarrow{R_1 - 2R_2} \begin{pmatrix} 1 & 0 & 3 & 1 \\ 0 & 1 & -\frac{3}{2} & 0 \\ 0 & -4 & 1 & -3 \end{pmatrix}$$

$$\xrightarrow{R_3 + 4R_2} \begin{pmatrix} 1 & 0 & 3 & 1 \\ 0 & 1 & -\frac{3}{2} & 0 \\ 0 & 0 & -5 & -3 \end{pmatrix} \xrightarrow{\left(-\frac{1}{5}\right)R_3} \begin{pmatrix} 1 & 0 & 3 & 1 \\ 0 & 1 & -\frac{3}{2} & 0 \\ 0 & 0 & 1 & \frac{3}{5} \end{pmatrix} \xrightarrow{R_1 - 3R_3} \begin{pmatrix} 1 & 0 & 0 & -\frac{4}{5} \\ 0 & 1 & -\frac{3}{2} & 0 \\ 0 & 0 & 1 & \frac{3}{5} \end{pmatrix}$$

Row-echelon form.

$$\xrightarrow{R_2 + \frac{3}{2}R_3} \begin{pmatrix} 1 & 0 & 0 & -\frac{4}{5} \\ 0 & 1 & 0 & \frac{9}{10} \\ 0 & 0 & 1 & \frac{3}{5} \end{pmatrix}$$

Reduced row echelon form

$$R = \begin{pmatrix} 1 & 0 & 0 & -4/5 \\ 0 & 1 & 0 & 9/10 \\ 0 & 0 & 1 & 3/5 \end{pmatrix}_{3\times 4}$$

A is $m \times n$

Rank (A) + nullity (A) = n

n = 4 in this case.

$Rx = 0$

$$\left.\begin{aligned} x_1 - \tfrac{4}{5} x_4 &= 0 \\ x_2 + \tfrac{9}{10} x_4 &= 0 \\ x_3 + \tfrac{3}{5} x_4 &= 0 \end{aligned}\right\}$$

Rank (R) = 3 = Rank (A)

$x_1 \rightarrow$ dependent variable.

$x_2 \rightarrow$ dependent variable.

$x_3 \rightarrow$ dependent variable.

$x_4 \rightarrow$ independent variable/free variable.

$x_1 = \frac{4}{5} x_4$

$x_2 = -\frac{9}{10} x_4$

$x_3 = -\frac{3}{5} x_4$

$\left(\frac{4}{5} x_4, -\frac{9}{10} x_4, -\frac{3}{5} x_4, x_4\right)$

$\boxed{Ax = 0}$

$x_4 = t$

$\left(\frac{4}{5} t, -\frac{9}{10} t, -\frac{3}{5} t, t\right)$

$t \in \mathbb{R}$.

Nullspace (A) = $\{x \mid Ax = 0\}$ $= \left\{\left(\frac{4}{5} t, -\frac{9}{10} t, -\frac{3}{5} t, t\right) \mid t \in \mathbb{R}\right\}$

$t = 1$

$\left(\frac{4}{5}, -\frac{9}{10}, -\frac{3}{5}, 1\right) \in$ Nullspace (A)

Basis of Nullspace (A) = $\left\{\left(\frac{4}{5}, -\frac{9}{10}, -\frac{3}{5}, 1\right)\right\}$

Nullity of (A) = 1.

3 / 3

Week 5 : Tutorial 2

Maths 2 Week 7 Tutorials

A is an $m \times n$ matrix then rank (A) + nullity (A) = n

Finding the nullspace of a matrix using Gauss elimination:

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 4 \\ 1 & 2 & 5 \end{pmatrix}_{3\times 3}$$

$Ax = 0$

$$\left.\begin{aligned} x_1 + x_3 &= 0 \\ 2x_2 + 4x_3 &= 0 \\ x_1 + 2x_2 + 5x_3 &= 0 \end{aligned}\right\}$$

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 4 \\ 1 & 2 & 5 \end{pmatrix} \xrightarrow{R_3 - R_1} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 4 \\ 0 & 2 & 4 \end{pmatrix} \xrightarrow{\frac{1}{2} R_2} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \end{pmatrix} \xrightarrow{R_3 - 2R_2} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} = R$$

Reduced row echelon form

x_1 = dependent variable

x_2 = dependent variable.

x_3 = independent variable/free variable.

$Rx = 0$

$$\Rightarrow \left.\begin{aligned} x_1 + x_3 &= 0 \\ x_2 + 2x_3 &= 0 \end{aligned}\right\}$$

$$\left.\begin{aligned} x_1 &= -x_3 \\ x_2 &= -2x_3 \end{aligned}\right\}$$

$x_3 = t$

$x_1 = -t$

$x_2 = -2t$

Nullspace (A) = $\{(-t, -2t, t) \mid t \in \mathbb{R}\}$

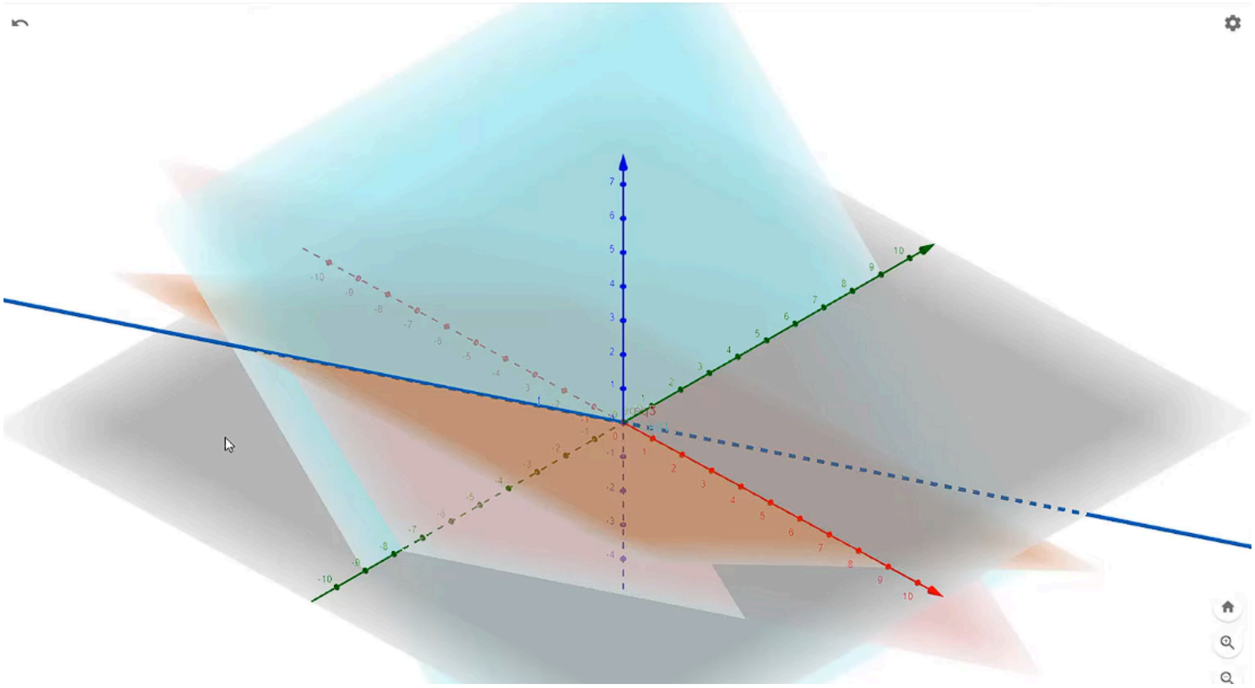
$= \{t(-1, -2, 1) \mid t \in \mathbb{R}\}$

$\{(-1, -2, 1)\}$ = Basis of Nullspace (A)

Nullity (A) = 1 , Rank (A) = 2

Rank (A) + Nullity (A) = 3 ✓

4 / 4



Week 5 : Tutorial 3

Mathematics for Data Science - 2 - Week 7.

Consider $S = \{(1,2,0), (0,3,1), (3,3,-1), (3,0,-2)\} \subseteq \mathbb{R}^3$.

Let $V = \text{Span}\{(1,2,0), (0,3,1)\} = \{\alpha(1,2,0) + \beta(0,3,1) : \alpha, \beta \in \mathbb{R}\}$

What is the dimension of V? $= \{(\alpha, 2\alpha+3\beta, \beta) : \alpha, \beta \in \mathbb{R}\}$.

$\dim V \leq 2$.

$a(1,2,0) + b(0,3,1) = (0,0,0) \Rightarrow a = 0 = b$

$(a, 2a+3b, b) = (0,0,0)$ $\quad \{\underline{(1,2,0), (0,3,1)}\}$ are lin. ind.

$a = 0, \quad 2a+3b = 0, \quad b = 0$. $\quad \dim V = 2$.

If vectors in S are written as columns of a matrix A, what will the rank and nullity of A be?

$$A = \begin{bmatrix} 1 & 0 & 3 & 3 \\ 2 & 3 & 3 & 0 \\ 0 & 1 & -1 & -2 \end{bmatrix}_{3\times 4} \xrightarrow{R_2 - 2R_1} \begin{bmatrix} 1 & 0 & 3 & 3 \\ 0 & 3 & -3 & -6 \\ 0 & 1 & -1 & -2 \end{bmatrix} \xrightarrow{\frac{1}{3}R_2} \begin{bmatrix} 1 & 0 & 3 & 3 \\ 0 & 1 & -1 & -2 \\ 0 & 1 & -1 & -2 \end{bmatrix} \xrightarrow{R_3 - R_2} \begin{bmatrix} 1 & 0 & 3 & 3 \\ 0 & 1 & -1 & -2 \\ \boxed{0 \quad 0 \quad 0 \quad 0} \end{bmatrix}$$

$\text{rank}(A) = 2$

$\text{rank}(A) + \text{nullity}(A) = 4$

$\Rightarrow \text{nullity}(A) = 2$.

What will dimension of vector space spanned by S?

$W = \text{Span}(S)$

$\dim W = 2$.

Week 5 : Tutorial 4

Mathematics for Data Science -2 - Week 7.

Note Title

Rank of $\begin{bmatrix} 2 & -3 & 4 \\ 0 & 1 & -2 \\ 1 & -3 & a \end{bmatrix}$ is 2. Find the value of a.

$$\begin{bmatrix} 2 & -3 & 4 \\ 0 & 1 & -2 \\ 1 & -3 & a \end{bmatrix} \xrightarrow{R_1 \to \frac{1}{2}R_1} \begin{bmatrix} 1 & -3/2 & 2 \\ 0 & 1 & -2 \\ 1 & -3 & a \end{bmatrix} \xrightarrow{R_3 \to R_3 - R_1} \begin{bmatrix} 1 & -3/2 & 2 \\ 0 & 1 & -2 \\ 0 & -3/2 & a-2 \end{bmatrix}$$

$$\xrightarrow{R_3 \to -\frac{2}{3}R_3} \begin{bmatrix} 1 & -3/2 & 2 \\ 0 & 1 & -2 \\ 0 & 1 & -\frac{2}{3}(a-2) \end{bmatrix} \xrightarrow{R_3 \to R_3 - R_2} \begin{bmatrix} 1 & -3/2 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 2-\frac{2}{3}(a-2) \end{bmatrix}$$

$2 - \frac{2}{3}(a-2) = 0$

$2 = \frac{2}{3}(a-2)$

$3 = a-2 \Rightarrow \boxed{a=5}$

Week 5 : Tutorial 5

Mathematics for Data Science -2 - Week 7.

Note Title

If $T : \mathbb{R}^2 \to \mathbb{R}^3$ is a linear transformation defined by

$T(x, y) = (2x + 3y,\ 5x - y,\ x + 6y)$, then is T one-one, onto?

$T : \mathbb{R}^n \to \mathbb{R}^m$ rank(T) + nullity$(T) = n$.

rank$(T) = m$, then T is onto. rank$(T) \leq m$

nullity$(T) = 0$, then T is one-one. nullity$(T) \leq n$

$\boxed{n > m}$ $\Rightarrow$ if T is one-one, then rank$(T) = n$ # T cannot be 1-1.

$m > n$ $\Rightarrow$ if T is onto, then nullity$(T) = n - m < 0$ # T cannot be onto.

$n = m$ $\Rightarrow$ if T is onto, then nullity$(T) = 0 \Rightarrow T$ is 1-1.

$T : \mathbb{R}^2 \to \mathbb{R}^3$ $T(x, y) = (2x + 3y,\ 5x - y,\ x + 6y)$

$T(x, y) = 0 \Rightarrow (x, y) = 0$.

$$\left.\begin{array}{l} 2x + 3y = 0 \\ 5x - y = 0 \\ x + 6y = 0 \end{array}\right\} \quad \left.\begin{array}{l} 5x = y \\ x = -6y \end{array}\right\} \quad \begin{array}{l} 5(-6y) = y \\ \Rightarrow -30y = y \Rightarrow y = 0. \\ \Rightarrow x = 0. \end{array}$$

T is one-one.

Solve with US 5

1. Key points:

Rank of a matrix:

- Maximum number of linearly independent column vectors (or row vectors) in a matrix.

NOTE: Rank of a matrix is always less than or equal to the number of columns (or rows).

How to find rank of a matrix A:

- Reduce the matrix to row echelon form.
- Find the number of non zero rows in the reduced matrix which will be the rank of the matrix A.

2 Key points:

Null space of a matrix A:

- Let A be an $m \times n$ matrix. The subspace W = $\{x \mid x \in \mathbb{R}^n, Ax = 0\}$ is called the null space of A.
- The dimension of the null space W is called the nullity of A.

Rank-Nullity theorem:

- Let A be an $m \times n$ matrix, then rank(A) + nullity(A) = n.

3 Key points:

Linear transformation:

A function f from one vector space V to another vector space W $(f : V \longrightarrow W)$ is said to be a linear transformation if for any two vectors v_1 and v_2 in the vector space V and for any $c \in \mathbb{R}$ (scalar) the following conditions hold:

(i) $f(v_1 + v_2) = f(v_1) + f(v_2)$

(ii) $f(cv_1) = cf(v_1)$

RWU 5

PA : 2,3,4,5,7,10

GA : 2,6,8,9,13