Objectives:

* Determinants

AQ.1.2. Q.6.

q $A \to$ square matrix of order 2. If $A^2 = I$, then $A = I$ or $A = -I$ (False)

$A = I$ or $A = -I$ whenever $A^2 = I$

Eg. $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ ✓

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$a^2+bc = 1$
$b(a+d) = 0$
$c(a+d) = 0$
$bc + d^2 = 1$

Recap: Matrices, Vectors, System of linear equations.

Eg. $$\left.\begin{aligned} 2x + 3y &= -1 \\ x - y &= 2 \end{aligned}\right\}$$ System of 2 equations in 2 variables.

A solution to the above system is (x_0, y_0) such that the equations are satisfied.

$x = 1,\ y = -1$ is a solution to the above system.

$$\boxed{A\underset{\sim}{x} = \underset{\sim}{b}}$$

$A = \begin{pmatrix} 2 & 3 \\ 1 & -1 \end{pmatrix}$ co-efficent matrix

$\underset{\sim}{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ unknown vector

$\underset{\sim}{b} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$ constant vector

$$\begin{pmatrix} 2 & 3 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

$$2x = 3 \Rightarrow 2^{-1}.(2x) = 2^{-1}.3 \Rightarrow x = \tfrac{1}{2}.3.$$

$$A\underset{\sim}{x} = \underset{\sim}{b} \Rightarrow \underset{\sim}{x} = \frac{\underset{\sim}{b}}{A} \quad \times$$

Determinant of a matrix: A number associated to a square matrix. . . . .

* Determinant of 1×1 matrix:

$A = (a) \qquad \det(A) = a.$

* **Determinant of 2x2 matrix:**

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \det(A) = ad - bc.$$

Eg. $A = \begin{pmatrix} 1 & k \\ 2 & 4 \end{pmatrix}$ Find k if $\det(A) = 8$.

$$\det(A) = 1 \cdot 4 - 2 \cdot k$$

$$8 = 4 - 2k \Rightarrow k = -2.$$

* **Determinant of a 3x3 matrix:**

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = a \det\begin{pmatrix} e & f \\ h & i \end{pmatrix} - b \det\begin{pmatrix} d & f \\ g & i \end{pmatrix} + c \det\begin{pmatrix} d & e \\ g & h \end{pmatrix}$$

Eg.

$$\begin{array}{c} \begin{matrix} + & - & + \end{matrix} \\ \begin{pmatrix} 1 & 2 & 3 \\ 1 & -2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \end{array} = 1 \det\begin{pmatrix} -2 & 3 \\ 1 & 2 \end{pmatrix} - 2 \det\begin{pmatrix} 1 & 3 \\ 3 & 2 \end{pmatrix}$$

$$+ 3 \det\begin{pmatrix} 1 & -2 \\ 3 & 1 \end{pmatrix}$$

$$= 1. (-7) - 2(-7) + 3(7)$$

$$= 28.$$

Properties of determinants:

i) Determinant of identity matrix (I_2 or I_3) is 1.

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

ii) $\det(AB) = \det(A)\det(B)$.

Does there exist B s.t $\underline{\underline{BA = I}}$?

Suppose such a B exists.

$$\det(BA) = \det(I)$$

$$\det(B).\ \det(A) = 1. \qquad \left(\text{If } \det(A) = 0,\ \text{then } 0 = 1\right)$$

$\therefore$ It is necessary that $\det(A) \neq 0$ for existence of inverse.

$$\det(B) = \frac{1}{\det(A)}$$

iii) Switching rows /columns changes the sign of the determinant

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \tilde{A} = \begin{pmatrix} b & a \\ d & c \end{pmatrix}$$

$$\det(A) = ad - bc \qquad \det(\tilde{A}) = bc - ad$$

$$= -\det(A).$$

Works for 3×3 as well

$$\det\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = 1\ \det\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = -1.$$

$$\|$$

$$-\det\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = -1.$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{C_2 \leftrightarrow C_3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \xrightarrow{C_2 \leftrightarrow C_1} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$(-1)^n \det(A).$$

iv) Determinant of transpose of a matrix is the same as the determinant of the matrix.

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad A^T = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

v) Determinant of a triangular matrix is the product of diagonal entries.

$$\det \begin{pmatrix} 1 & -1 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix} = 1 \det \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix} - (-1) \det \begin{pmatrix} 0 & 1 \\ 0 & 3 \end{pmatrix} + 3 \det \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$$

$$= 1 \cdot 2 \cdot 3 = 6.$$

vi) Adding scalar multiples of one row (column) to another row (column) does not change the determinant

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \begin{pmatrix} 1 & 2 & 3 \\ 1 & -2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \xrightarrow{R_1 \mapsto R_1 - R_2} \begin{pmatrix} 1-1 & 2-(-2) & 3-3 \\ 1 & -2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

$R_1 + R_2$

$$\begin{pmatrix} 2 & 0 & 6 \\ 1 & -2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \qquad \begin{pmatrix} 0 & 4 & 0 \\ 1 & -2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

$$-4 \det \begin{pmatrix} 1 & 3 \\ 3 & 2 \end{pmatrix} = -4(-7) = 28.$$

Eg.

$$\begin{pmatrix} 3 & 4 \\ 2 & 2 \end{pmatrix} \xrightarrow{R_1 - 2R_2} \begin{pmatrix} -1 & 0 \\ 2 & 2 \end{pmatrix} \qquad \det = -2.$$

Eg.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longrightarrow \begin{pmatrix} a+tc & b+td \\ c & d \end{pmatrix}$$

Eg. $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow[R_2 - R_1]{R_1 - R_2} \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$1 \qquad\qquad\qquad\qquad 0$

DON'T DO THIS!

vii) Multiplying a row (column) by a real number also multiplies the det by the same real number

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \tilde{A} = \begin{pmatrix} ta & tb \\ c & d \end{pmatrix}$$

$$\det(\tilde{A}) = t \det(A).$$

Eg. $\det(-I) = \underline{\underline{(-1)^n}}$

$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

$\det(-I_3) = -1$

$\det(-I_2) = 1$

$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \to \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

$\to \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$

$\det(tA) = t^n \det(A).$

A is an $(n \times n)$-matrix.

Ex. det. of a matrix is zero if

i) there is a row of zeroes

ii) there are two identical rows.

$$* \quad A = \begin{pmatrix} \overset{+}{a_{11}} & \overset{-}{a_{12}} & \overset{+}{a_{13}} \\ \overset{-}{a_{21}} & \overset{+}{a_{22}} & \overset{-}{a_{23}} \\ \overset{+}{a_{31}} & \overset{-}{a_{32}} & \overset{+}{a_{33}} \end{pmatrix} = a_{11} \underline{\underline{\det \begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix}}} - \cdots$$

M_{11} = determinant of sub-matrix obtained by removing 1^{st} row and 1^{st} column.
(called $(1,1)^{th}$-minor)

M_{ij} = det. of ...... removing i^{th} row and j^{th} column.

$$\det(A) = a_{11} M_{11} - a_{12} M_{12} + a_{13} M_{13}$$

$$C_{ij} \longrightarrow (i,j)^{th}\text{-cofactor} = \underline{\underline{(-1)}}^{i+j} M_{ij}$$

$$\det(A) = a_{11} C_{11} + a_{12} C_{12} + a_{13} C_{13}$$
$$= \sum_{j=1}^{3} a_{1j} C_{1j}$$

* Determinant of a 4×4 matrix

$$\det(A) = \sum_{j=1}^{4} a_{1j} \underline{\underline{C_{1j}}}$$

$$\begin{pmatrix} \overset{+}{a_{11}} & \overset{-}{a_{12}} & \overset{+}{a_{13}} & \overset{-}{a_{14}} \\ - & - & - & - \\ - & - & - & - \\ - & - & - & - \end{pmatrix}$$

* Determinant of a $(n \times n)$ matrix : $\det(A) = \sum_{j=1}^{n} a_{1j} C_{1j}$

* System of equations have the following possibilities for the solution set

a) Unique solution b) No solution c) Infinitely solution

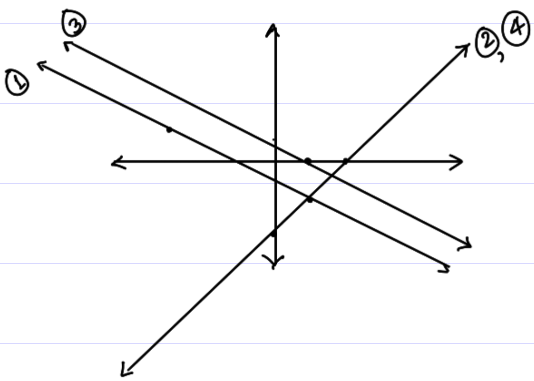
2 variables:

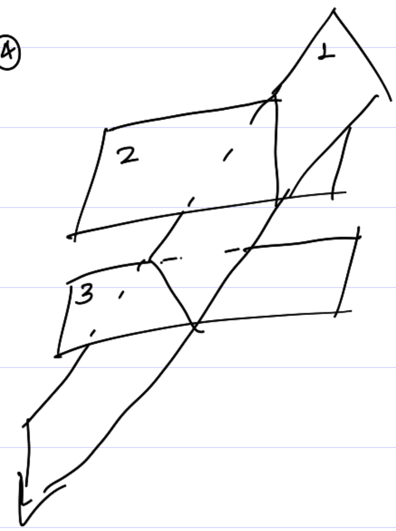
① $2x+3y=-1$

② $x-y=2$

③ $2x+3y=2$

④ $2x-2y=4$





$\{①, ②\}$ has a unique solution.

$\{①, ③\}$ has no solution.

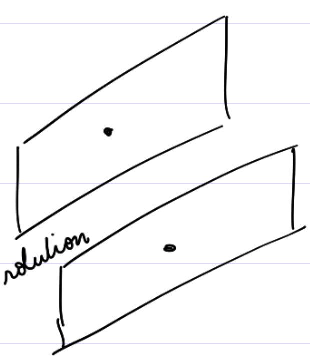
$\{②, ④\}$ has infinitely many solutions.

3 variables:

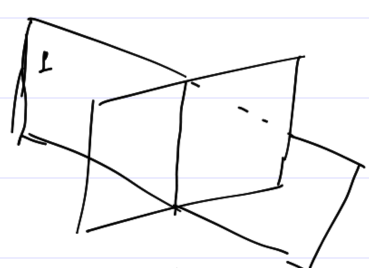
$ax+by+cz=d$.

$2x+3y+z=1$

$2x+3y+z=-1$

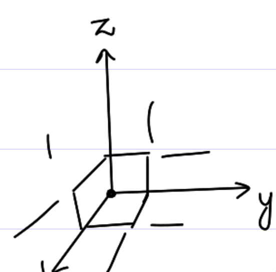


No solution



Inf. solutions

Eg.



Unique sol.

$A\underset{\sim}{x} = \underset{\sim}{b}$

* Homogeneous system of equations always have a solution ($\underset{\sim}{x} = \underset{\sim}{0}$)

$$A\underset{\sim}{x} = \underset{\sim}{0} \qquad \begin{aligned} ax + by + cz &= 0 \\ a'x + b'y + c'z &= 0 \\ a''x + b''y + c''z &= 0. \end{aligned}$$

AQ 1.3 6.

$k = -3$

(1) $\underline{2x_1 + 3x_2 = 6}$

$-2x_1 + kx_2 = d \longrightarrow -2x_1 - 3x_2 = d \rightarrow \underline{2x_1 + 3x_2 = -d.}$ (2)

$\underline{4x_1 + 6x_2 = 12}$

(1) $-$ (2) $\rightarrow 0 = 6 + d.$