WEEK 6.

RECALL:

LINEAR TRANFORMATION

$f: V \longrightarrow W$ is std. a L.T. if

$\begin{cases} \text{(i)} & f(v_1 + v_2) = f(v_1) + f(v_2) \text{ for all } v_1, v_2 \in V \\ \text{(ii)} & f(\alpha v) = \alpha f(v) \text{ for } \alpha \in \mathbb{R} \text{ and } v \in V \end{cases}$

linearity

$$\boxed{\begin{aligned} & f: V \to W \text{ is a L.T.} \\ \Leftrightarrow \;& f(v_1 + \alpha v_2) = f(v_1) + \alpha f(v_2) \text{ for all } \alpha \in \mathbb{R} \\ & \text{and } v_1, v_2 \in V. \end{aligned}}$$

Exercise: Suppose $f: V \to W$ satisfying $f(v_1 + \alpha v_2) = f(v_1) + \alpha f(v_2)$ then f satisfies (i) & (ii).

$f: V \longrightarrow W$

(*) (a) Injective of L.T. $\Leftrightarrow$ $f(v) = 0$ then $v = 0$

(b) Surjective of a L.T. $\Leftrightarrow$ $\mathrm{Im}(f) = W$

(c) An isom is a L.T. which injective and surjective.

(*) $f: V \longrightarrow W$ is a L.T. s.t

$\dim(V) = \underline{\underline{n}} = \dim(W)$

Then

isom. $\Leftrightarrow$ inj. $\Leftrightarrow$ surj.

$f: \mathbb{R}^2 \to \mathbb{R}^2,\ f(x,y) = (5x, x+y)$

Matrix w.r.t. std bases of domain & codomain.

$$A = \begin{bmatrix} 5 & 0 \\ 1 & 1 \end{bmatrix} \qquad A\begin{bmatrix} x \\ y \end{bmatrix} = (5x, x+y) = f(x,y)$$

$$\left.\begin{array}{l} \text{domain} = \mathbb{R}^2 \\ \text{Codomain} = \mathbb{R}^2 \end{array}\right\} \text{std. bases} \to \{(1,0), (0,1)\}$$

first column $\rightsquigarrow f(1,0) = (5,1) \in \text{Codomain} = 5(1,0) + 1(0,1) \rightsquigarrow \begin{pmatrix} 5 \\ 1 \end{pmatrix}$

second column $\rightsquigarrow \underbrace{f(0,1) = (0,1)}_{\text{step 1}} \in \text{Codomain} = 0(1,0) + 1.(0,1) \rightsquigarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Matrix w.r.t. $\{(0,1), (1,0)\}$ of domain and std. bases of codomain

$$A = \begin{bmatrix} 0 & 5 \\ 1 & 1 \end{bmatrix}$$

Matrix w.r.t. $\{(1,1), (0,1)\}$ of domain and $\{(1,0), (0,1)\}$ of codomain.

$$f(1,1) = (5,2) = 5(1,0) + 2(0,1)$$
$$f(0,1) = (0,1) = 0(1,0) + 1(0,1)$$
$$A = \begin{bmatrix} 5 & 0 \\ 2 & 1 \end{bmatrix}$$

(*) Basis determine the matrix of the linear Trans

(*) The order of the basis is also important to fix.

Suppose $\dim V = n$ and $\dim W = m$.

Defⁿ:

For a $f: V \longrightarrow W$ L.T., Let

$\beta = \{v_1, \ldots, v_n\}$ be an ordered bases of V

and $\gamma = \{w_1, \ldots, w_m\}$ be an ordered bases of W.

Then

first column $\rightsquigarrow f(v_1) \in W,\ f(v_1) = \underline{a_{11}}w_1 + \underline{a_{21}}w_2 + \cdots + \underline{a_{m1}}w_m \rightsquigarrow \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix}$

$f(v_2)$

$\vdots$

nth column $\rightsquigarrow f(v_n) = a_{1n}w_1 + a_{2n}w_2 + \cdots + a_{mn}w_m \rightsquigarrow \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}$

$$[A]_{\gamma\beta} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m\times n}$$

$$A = \begin{bmatrix} | & | & & | \\ f(v_1) & f(v_2) & \cdots & f(v_n) \\ | & | & & | \end{bmatrix}$$

(*) Let $\beta = v_1, \ldots, v_n$ be ordered bases of V

and $\gamma = w_1, \ldots, w_m$ be ordered bases of W.

Then

$$\{F: V \longrightarrow W\} \longleftrightarrow \{A_f \in M_{m\times n}\}$$

Q. Let $W = \{(x,y,z) \mid x+y+z=0\}$. Define

$$f : W \longrightarrow \mathbb{R}^2 \text{ as } f(x,y,z) = (y,z).$$

Then find out the matrix representation

(A) w.r.t. to the ordered basis $\{(-1,1,0),(-1,0,1)\}$ of W and std. bases of $\mathbb{R}^2$,

(B) ordered basis $\{(-1,1,0),(-1,0,1)\}$ of W and $\{(0,1),(1,1)\}$ of $\mathbb{R}^2$.

(A) $f(-1,1,0) = (1,0) = 1(1,0) + 0(0,1)$
$f(-1,0,1) = (0,1) = 0(1,0) + 1(0,1)$ $\rightsquigarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = A$

(B) $f(-1,1,0) = (1,0) = (-1)(0,1) + 1(1,1) \rightsquigarrow \begin{pmatrix} -1 \\ 1 \end{pmatrix}$
$f(-1,0,1) = (0,1) = 1(0,1) + 0(1,1) \rightsquigarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$A = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}$$

Let function $f: \mathbb{R}^2 \to \mathbb{R}$ defined as follows:

$$f(x,y) = x.$$

Consider a subset

$$K = \{(x,y) \in \mathbb{R}^2 \mid f(x,y) = 0\} \subset \mathbb{R}^2$$

Find K for f ?

$$K = \{(0,y) \mid y \in \mathbb{R}\} \subset \mathbb{R}^2$$

Moreover, K is a subspace of $\mathbb{R}^2$.

K is called '<u>kernel of f</u>'

In general, we define: Suppose V and W are vector spaces.

Let $f: V \longrightarrow W$ is a linear map, Then

$$\text{Kernel}(f) = \{v \in V \mid f(v) = 0\} \subseteq V.$$

* Kernel (f) is a subspace of V. For $v_1, v_2 \in$ Kernel (f) and $c \in \mathbb{R}$, to prove $v_1 + c v_2 \in$ Kernel (f).

$$f(v_1 + cv_2) = f(v_1) + f(cv_2)$$
$$= 0 + c f(v_2)$$
$$= 0 + 0 = 0$$

$\Rightarrow f(v_1 + cv_2) \in$ Kernel(f) $\Rightarrow$ kernel (f) is a subspace of V.

Example:

Find kernel of the following maps:

(i) $0 : V \longrightarrow W$, $0(v) = 0$ for $v \in V$.

Kernel $(0) = \{v \in V \mid 0(v) = 0\} = V$.

Map 0 is called the "zero map".

(ii) $id : V \longrightarrow V$, $id(v) = v$.

Kernel $(id) = \{v \in V \mid id(v) = 0\} = \{0\}$

map id is called the identity map.

(iii) $R : V^n \longrightarrow \mathbb{R}^n$ and $\{v_1, \ldots, v_n\}$ is an ordered bases of V.

For any $v \in V$, $v = \sum a_i v_i$ then we define

$$v \rightsquigarrow (a_1, \ldots, a_n)$$

$$R(v) = (a_1, a_2, \ldots, a_{n-1}, 0).$$

Kernel $(R) = \{v \in V \mid R(v) = 0 \in \mathbb{R}^n\}$

$$R(v) = (a_1, a_2, \ldots, a_{n-1}, 0) = (0, \ldots, 0)$$

$\Rightarrow a_1 = 0, a_2 = 0, \ldots, a_{n-1} = 0, a_n \in \mathbb{R}$

$\Rightarrow$ Kernel $(R) = \{a_n v_n \mid a_n \in \mathbb{R}\} = \text{span}\{v_n\} \subset V$.

(iv) $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$

$$A = \begin{bmatrix} 3 & 1 \\ 4 & 2 \\ 7 & 3 \end{bmatrix}, \quad T(v) = Av.$$

Finding Kernel of T : (using Nullspace).

(*) Let $F : V \longrightarrow W$ L.T. and $\beta = v_1, .., v_n$ be a basis of V and $\gamma = w_1, .., w_m$ be a basis of W. Then

$$\text{Kernel}(F) = \underline{\text{Null space}(A_F)}$$

$$\{v \in V \mid Fv = 0\} \xrightarrow{A_F} \{v \in V \mid [A_F]_{\gamma\beta} [v]_\beta = 0\}$$

$$v = \sum_{i=1}^{n} a_i v_i \rightsquigarrow [v]_\beta = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$$

Some important facts about Kernels:

if $F(v)=0$ then $v=0$

$\Downarrow$

(*) F is an Injective L.T. $\Leftrightarrow$ $\underline{\mathrm{Ker}(F)} = \{\underline{0_V}\}$

(*) $\mathrm{Ker}(F) = \mathrm{Null\ sp.}(A_F)$

F is injective L.T. $\Leftrightarrow$ $\mathrm{Null\ sp.}(A_F) = \langle 0_V \rangle$.

(*) dimension of $\mathrm{Ker}(F) = \mathrm{Nullity}(A_F)$.

Finding basis for Kernel:

$F_2: \mathbb{R}^2 \longrightarrow \mathbb{R}^3$
$(x,y) \longmapsto (x, 2x, 3x)$

domain has std bases $\rightsquigarrow \{(1,0),(0,1)\}$
codomain has std bases $\rightsquigarrow \{(1,0,0),(0,1,0),(0,0,1)\}$

$\rightsquigarrow F(1,0) = (1,2,3) = 1(1,0,0) + 2(0,1,0) + 3(0,0,1)$

$\rightsquigarrow F(0,1) = (0,0,0) = 0.(1,0,0) + 0(0,1,0) + 0(0,0,1)$

$$A = \underbrace{\begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 3 & 0 \end{bmatrix}}_{3\times 2} \xrightarrow[R_3-3R_1]{R_2-2R_1} \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$(x,0,0) = (0,0,0) \Rightarrow x = 0$

$y \in \mathbb{R}$

$\mathrm{Ker}(F) = \{(0,y) \mid y \in \mathbb{R}\}$
$= \{y(0,1) \mid y \in \mathbb{R}\} = \mathrm{span}\{(0,1)\}$

Question:

$V = \mathbb{R}^3 \rightarrow$ std bases

$W = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + 2z = 0\} \subset \mathbb{R}^3$

$$x + y + 2z = 0$$
$$x = -y - 2z$$

$$(-y - 2z, y, z)$$

$y = 1,\ z = 0 \qquad (-1, 1, 0)$

$y = 0,\ z = 1 \qquad (-2, 0, 1)$

Basis of $W = \{(-1, 1, 0), (-2, 0, 1)\}$.

$$T : W \longrightarrow \mathbb{R}^3$$

$$T(x, y, z) = (x, y, z)$$

$T(-1, 1, 0) = (-1, 1, 0) =$

$T(-2, 0, 1) = (-2, 0, 1) =$

$$A = \begin{bmatrix} -1 & -2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\text{Null sp}(A) = \begin{bmatrix} -1 & -2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{R_1 + R_2} \begin{pmatrix} 0 & -2 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\downarrow R_1 + 2R_3$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \xleftarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix} \xleftarrow{R_1 \leftrightarrow R_3} \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\text{Null sp}(A) = \{0\} = \text{Ker}(F)$$

$$R: W \longrightarrow \mathbb{R}^3$$

$$R(x,y,z) = (x, 0, 0)$$

$$R(-1,1,0) = (-1,0,0) \rightsquigarrow \qquad A = \begin{bmatrix} -1 & -2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$R(-2,0,1) = (-2,0,0) \rightsquigarrow$$

$$\downarrow -R_1$$

$$\begin{pmatrix} 1 & 2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & 2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$(x+2y, 0, 0) = (0,0,0)$$

$$\Rightarrow \quad x + 2y = 0 \Rightarrow x = -2y$$

$$\text{Nullsp } A = \{(-2y, y) \mid y \in \mathbb{R}\} = \text{span}\,\underline{\underline{(-2,1)}}$$

$$\text{Kernel } F = \text{span}\{(-2)(-1,1,0) + 1(-2,0,1)\}$$

$$= \text{span}\{(2,-2,0) + (-2,0,1)\}$$

$$= \text{span}\,(0,-2,1)$$

$T: V \longrightarrow W.$

(1) Matrix representation A of T w.r.t. the orderad bases of domain & codomain.

(2) Find basis of null sp (A).

$\longrightarrow \{(a_{11}, \cdots, a_{1m}), (a_{21}, \ , a_{2m}), \cdots (a_{n1}, \cdots a_{nm})\}$

(3) Find basis for Kernel (T) $\rightarrow \{u_1, \cdots u_n\}$

$$u_j = \sum_{i=1}^{m} a_{ji} v_i \qquad 1 \leq j \leq n, \ 1 \leq i \leq m$$