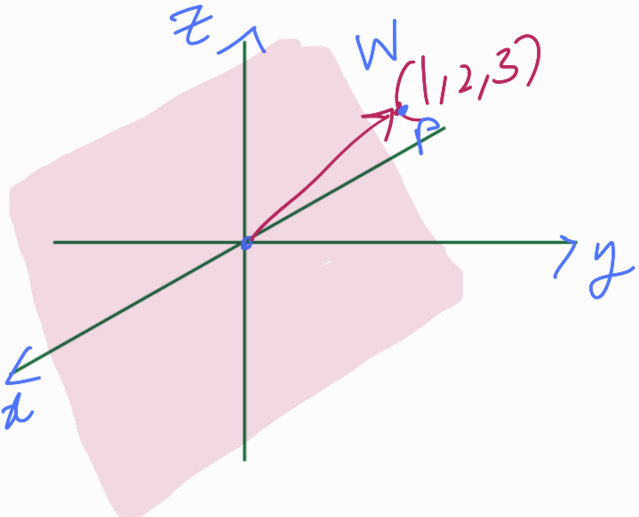
# Week 8

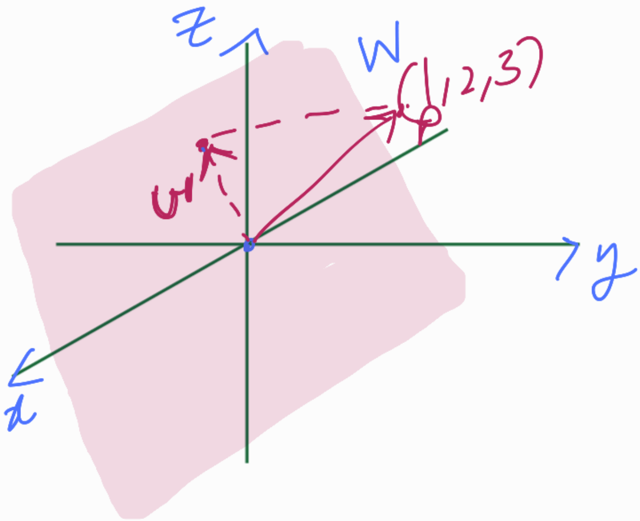
Consider $(\mathbb{R}^3, \text{dot product})$.

Let $W = \operatorname{span}\{\overset{u_1}{(1,0,0)}, \overset{u_2}{(0,-1,1)}\}$ and

$v = (1,2,3) \in \mathbb{R}^3$. Then

Q. What is the closest vector $v'$ in $W$ to $v$?




(1.) Find orthonormal bases of $W$.

$\|u_1\| = 1$, $\|u_2\| = \sqrt{2}$, $u_2' = \dfrac{u_2}{\|u_2\|} = \left(0, \frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

$u_1' = u_1 = (1,0,0)$

O.N.B.

$$\{u_1', u_2'\} = \left\{(1,0,0), \left(0, \frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)\right\}$$

## (2.) Projection of $v$ on $W$.:

$$\text{Proj}_{(1,0,0)} (1,2,3) = \big((1,0,0)\cdot(1,2,3)\big)(1,0,0)$$

$$= (1,0,0)$$

$$\text{Proj}_{(0,-1,1)} (1,2,3) = (-2+3)(0,-1,1)$$

$$= (0,-1,1)$$

$$\text{Proj}_W v = \text{Proj}_{u_1'} v + \text{Proj}_{u_2'} v = (1,0,0)+(0,-1,1)$$

$$= (1,-1,1)$$

In general,

let $(V, \langle \cdot, \cdot \rangle)$ be an inner product space. and $W$ be a subspace of $V$.

Consider $B = \{w_1, w_2, \ldots, w_k\}$ an orthonormal basis of $W$.

The projection of $v \in V$ on $W$ is defined as:

$$\text{Proj}_W v = \sum_{i=1}^{k} \text{Proj}_{w_i} v.$$

In case of orthogonal basis of W:

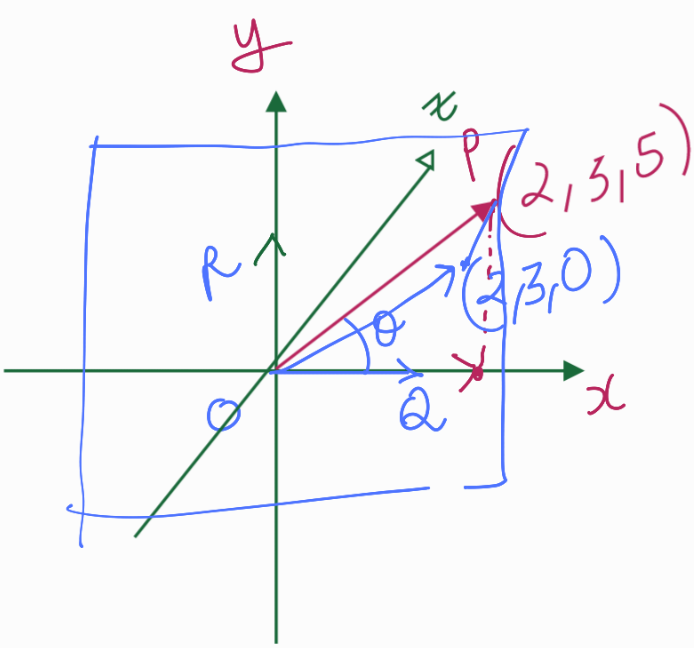
Let $\{v_1, \ldots, v_k\}$ be an orthogonal basis of W. Then we convert them to an orthonormal basis $\left\{\frac{v_1}{\|v_1\|}, \ldots, \frac{v_k}{\|v_k\|}\right\}$.

Define $\text{proj}_W v := \sum_{i=1}^{k} \text{Proj}_{\frac{v_i}{\|v_i\|}} v$

$$= \sum_{i=1}^{k} \left\langle \frac{v_i}{\|v_i\|}, v \right\rangle \frac{v_i}{\|v_i\|}$$

$$= \sum \frac{1}{\|v_i\|} \langle v_i, v \rangle \frac{v_i}{\|v_i\|}$$

$$= \sum_{i=1}^{k} \frac{\langle v_i, v \rangle}{\|v_i\|^2} v_i$$




$(1,0,0)$ and $(0,1,0)$ for xy-plane

$2 \cdot (1,0,0) + 3(0,1,0)$

$(2,3,0)$

## Projection as linear map:

In the first example: $(\mathbb{R}^3$, dot product$)$

$W = \text{span}\{(1,0,0), (0,-1,1)\}$.

We define a function

$$P_W : \mathbb{R}^3 \longrightarrow \mathbb{R}^3.$$
$$v \longmapsto \text{Proj}_W(v)$$

Claim: $P_W$ is a linear map.

For any $v_1, v_2 \in \mathbb{R}^3$ and $a \in \mathbb{R}$,

$$P_W(v_1 + av_2) = P_W(v_1) + a P_W(v_2)$$

$$\text{L.H.S.} = \text{Proj}_{u_1'}(v_1 + av_2) + \text{Proj}_{u_2'}(v_1 + av_2)$$

$$= (u_1' \cdot (v_1 + av_2))\, u_1' + (u_2' \cdot (v_1 + av_2))\, u_2'$$

$$= (u_1' \cdot v_1 + a\, u_1' \cdot v_2)\, u_1' + (u_2' \cdot v_1 + a(u_2' \cdot v_2))\, u_2'$$

$$= (u_1' \cdot v_1)\, u_1' + (u_2' \cdot v_1)\, u_2' + a\{\underline{(u_1' \cdot v_2)\, u_1' + (u_2' \cdot v_2)\, u_2'}\}$$

$$= \text{Proj}_W(v_1) + a\, \text{Proj}_W(v_2) = P_W(v_1) + a P_W(v_2)$$

$$= \text{R.H.S.}$$

(2) Kernel $(P_W) =$

$$P_W(v) = 0 \Rightarrow \underline{Proj_{u_1'}(v)} + \underline{Proj_{u_2'}(v)} = 0$$

$$\Rightarrow Proj_{u_1'}(v) = 0 = Proj_{u_2'}(v)$$

$$(u_1' \cdot v)u_1' = 0 = (u_2' \cdot v)u_2'$$

suppose $v = (a, b, c)$.

$$u_1' \cdot v = (1,0,0) \cdot (a,b,c) = 0 \Rightarrow a = 0$$

$$u_2' \cdot v = \left(0, \frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \cdot (a,b,c) = 0 \Rightarrow b = c$$

$$v = (a, b, c) = (0, c, c) = \underline{c(0,1,1)}$$

$$Ker(P_W) = span\{(0,1,1)\}.$$

(3.) Range $(P_W) = W$.

(4) Take a different orthonormal basis $B' = \{(4,0,0), (0,1,-1)\}$ of $W$.

Is $(P_W)_B(v) = (P_W)_{B'}(v)$. ?

$$B' = \left\{(1,0,0), \left(0, \frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)\right\}$$

In general,

Let $(V, \langle \cdot, \cdot \rangle)$ be inner product space and $W$ be a subspace of $V$. Moreover, assume that $\{w_1, \ldots, w_k\}$ be an orthonormal basis of $W$. Define

$$P_W : V \longrightarrow V \text{ as}$$

$$P_W(u) = \text{Proj}_W(u).$$

Properties:

(a) $\text{Kernel}(P_W) = W^\perp = \{v \in V \mid P_W(v) = 0\}$

$$P_W(v) = \text{Proj}_W(v) = 0 = \sum_{i=1}^{k} \text{Proj}_{w_i}(v) = 0$$

$\Leftrightarrow$ Each $1 \le i \le k$, $\text{Proj}_{w_i}(v) = 0$

$$\Leftrightarrow \quad \underline{\langle w_i, v \rangle} w_i = 0$$

$$\Leftrightarrow \quad \langle w_i, v \rangle = 0 \quad \text{for all } 1 \le i \le k. \quad \text{--- (1)}$$

Take any $w \in W$, $\langle w, v \rangle = 0$ where $v \in \text{Ker} P_W$

$$\Rightarrow \quad w = \sum_{i=1}^{k} a_i w_i, \quad \langle w, v \rangle = \left\langle \sum a_i w_i, v \right\rangle$$

$$= \sum_{i=0}^{k} a_i \langle w_i, v \rangle = 0$$

$v \in \text{Ker}(P_W) \Leftrightarrow \underline{\langle w, v \rangle = 0 \text{ for all } w \in W.}$

$$W^\perp = \{v \in V \mid \langle w, v \rangle = 0 \text{ for all } w \in W\}$$

(b) $\text{Range}(P_W) = W$

(c) Independent of choice of basis of W.

(d) $\|P_W(v)\| \le \|v\|$.

(e) $P_W^2 = P_W$.

(d) $\|P_W(u)\| \le \|v\|$

(e) $P_W^2 = P_W \circ P_W$

$$P_W^2(u) = (P_W \circ P_W)(u) = \underline{P_W}(P_W(u))$$

$\because P_W(u) \in W$, let $P_W(u) = w \in W$,

$$P_W(w) = w$$

# Finding an orthonormal basis :

Consider a subsp. $W = \{(x,y,z) \in \mathbb{R}^3 \mid x+y+z=0\}$.

Q. Find orthonormal basis of W. ?

Step A. Find a basis B of W. (We know)

Step B. Convert basis B to an orthogonal basis. (GRAM-SCHMIDT PROCESS).

Step A

$$x+y+z=0$$

$$x = -y-z$$

$$(x,y,z) = (-y-z, y, z)$$

$$B = \{(-1,1,0), \overset{u_2}{(-1,0,1)}\}$$

$$u_1 = u_1' = (-1,1,0),$$

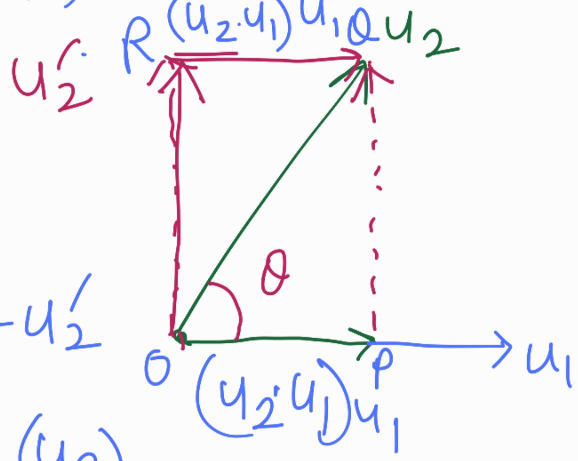
$$u_2 \longrightarrow u_2' \text{ s.t. } \langle u_2', u_1' \rangle = 0$$

$$u_2 \cdot u_1' = 1$$

$$OQ = OP + OR$$

$$u_2 = (u_2 \cdot u_1)u_1 + u_2'$$

$$u_2' = u_2 - \mathrm{Proj}_{u_1}(u_2)$$



Step B: Using projection.

Let $u_1 = (-1,1,0)$ and $u_2 = (-1,0,1)$.

$$u_1 = (-1,1,0)$$

$$u_1' = \frac{u_1}{\|u_1\|} = \left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$$

$$u_2' = u_2 - \operatorname{Proj}_{u_1'}(u_2)$$

$$= (-1,0,1) - (u_2 \cdot u_1') \cdot u_1'$$

$$= (-1,0,1) - \frac{1}{\sqrt{2}}\left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$$

$$= (-1,0,1) - \left(\frac{-1}{2}, \frac{1}{2}, 0\right) = \left(\frac{-1}{2}, \frac{-1}{2}, 1\right)$$

$$u_2'' = \frac{u_2'}{\|u_2'\|} = \frac{\left(-1/2, -1/2, 1\right)\sqrt{2}}{\sqrt{3}}$$

$$\text{O.N.B.} = \left\{\left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right), \left(\frac{-\sqrt{2}}{2\sqrt{3}}, \frac{-\sqrt{2}}{2\sqrt{3}}, \sqrt{2}\right)\right\}.$$

In general,

G-S Procedure:

Suppose $\{v_1, \dots, v_k\}$ is a linearly independent set vectors vectors in $V$. Let $W = \text{span}\{v_1, \dots, v_k\}$.

Let $e_1 = v_1$ and $w_1 = e_1 / \|e_1\|$.

and for $2 \le j \le k$, $e_j = v_j - \text{Proj}_{\text{Span}\{w_1, \dots, w_{k-1}\}}(v_j)$

$$e_j = \left( v_j - \text{Proj}_{w_1}(v_j) - \text{Proj}_{w_2} v_j - \cdots - \text{Proj}_{w_{k-1}} v_j \right)$$

$$w_j = e_j / \|e_j\|$$

Then $\{e_1, \dots, e_k\}$ is O.G.S. and $\{w_1, \dots, w_k\}$ is an O.N.S.

$\{v_1, v_2, \dots, v_k\}$

$v_1 = e_1$, $w_1 = v_1 / \|v_1\|$

$\{w_1, v_2\}$, $e_2 = v_2 - \text{Proj}_{w_1}(v_2)$, $w_2 = e_2 / \|e_2\|$.

$\{w_1, w_2, v_3\}$, let $X_2 = \text{span}\{w_1, w_2\}$.

$$e_3 = v_3 - \text{Proj}_{X_2}(v_3), \quad w_3 = e_3 / \|e_3\|$$

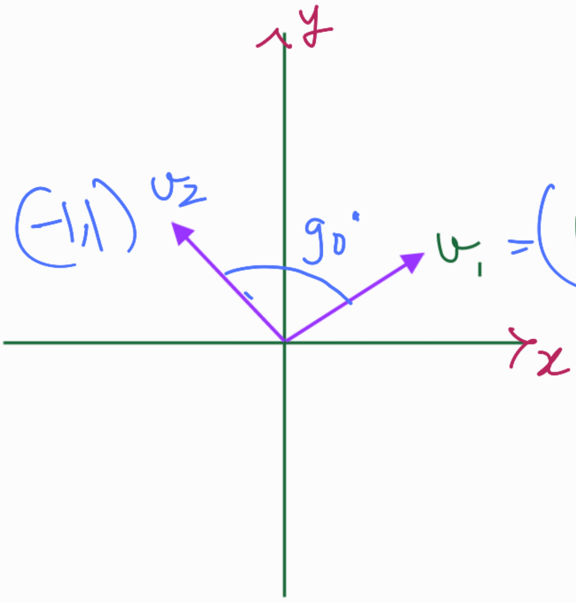
Theorem: Any inner product sp. $(V, \langle \cdot, \cdot \rangle)$ has a orthonormal basis.

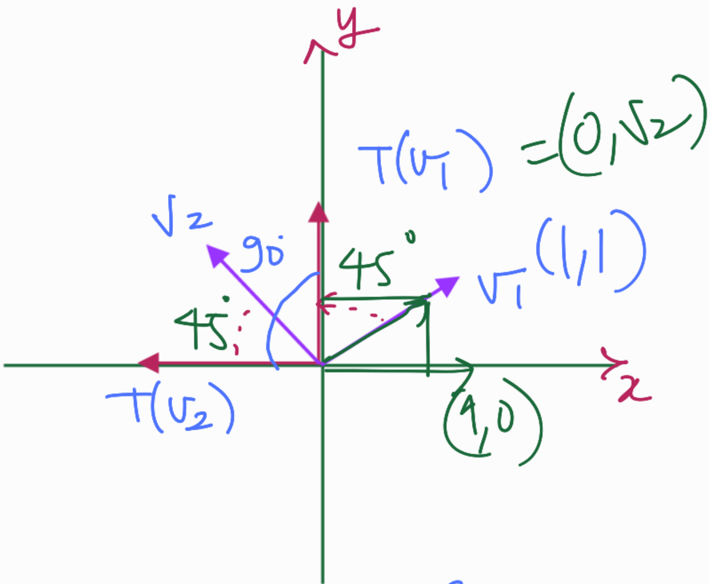
# Orthogonal transformation :

* Linear map that preserve "lengths" and "angle" are called orthogonal transformations.

**Defn:** A linear map $T: V \longrightarrow W$ is std. orthogonal transformn: $\langle v_1, v_2 \rangle_V = \langle Tv_1, Tv_2 \rangle_W$ for all $v_1, v_2 \in V$.

## Rotation in $\mathbb{R}^2$




$$T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

$$T(1,1) = (0, \sqrt{2})$$

$$T(-1,1) = (-\sqrt{2}, 0)$$

$$T(0,1) = \left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

$$T(1,0) = (1, 1)\frac{1}{\sqrt{2}}$$

$$T(0,1) = \frac{1}{2}\left[T(1,1) + T(-1,1)\right]$$

$$T(1,0) = \frac{1}{2}\left[T(1,1) - T(-1,1)\right]$$

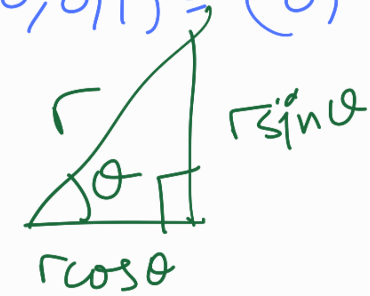
$$\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} \cos 45° & -\sin 45° \\ \sin 45° & \cos 45° \end{bmatrix}$$

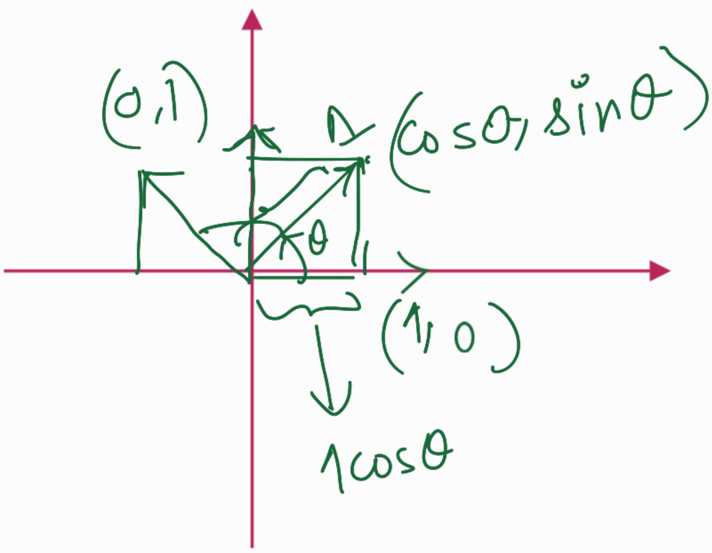
# Rotation in $\mathbb{R}^3$

$x$-axis -
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

$T(1,0,0) = (1,0,0)$, $T(0,1,0) = (0, \cos\theta, \sin\theta)$

$T(0,0,1) = (0, -\sin\theta, \cos\theta)$




$$T(0,1) = (\cos(90+\theta), \sin(90+\theta))$$
$$= (-\sin\theta, \cos\theta)$$

$$[T] = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

y-axis rotation in $\mathbb{R}^3$.

$$T(0,1,0) = (0,1,0)$$

$$T(1,0,0) = (\cos\theta, 0, \sin\theta)$$

$$T(0,0,1) = (\sin\theta, 0, \cos\theta)$$

$$\begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$$

Another example of orthogonal transformation:

$$T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$

$$T(x,y,z) = \left( \frac{x+y}{\sqrt{2}}, \frac{x-y}{\sqrt{2}}, z \right).$$

Matrix representation of T: (w.r.t canonical basis)

$$A = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

In general, let $T : V \longrightarrow V$. and $\{v_1, v_2, v_3\}$ orthonormal basis of $(V, \langle \cdot, \cdot \rangle)$.

Suppose A denote matrix representation of T w.r.t. $\{v_1, v_2, v_3\}$. Then

① $\langle Tv_1, Tv_2 \rangle = 0$ and $\langle Tv_2, Tv_3 \rangle = 0$

$\Rightarrow$ $c_1, c_2, c_3$ are orthogonal

$$\begin{array}{cc} & \begin{array}{ccc} c_1 & c_2 & c_3 \end{array} \\ A = & \begin{bmatrix} | & | & | \\ Tv_1 & Tv_2 & Tv_3 \\ | & | & | \end{bmatrix} \end{array}$$

② $\langle Tv_1, Tv_1 \rangle = 1$

$\Rightarrow$ $\|c_1\| = 1$.

$$A = \begin{bmatrix} c_1 & c_2 & c_3 \end{bmatrix}$$

Then $AA^T = Id$. (Exercise)

$$A^TA = Id$$

**Defn:** An $(n \times n)$ matrix $A$ is said to be orthogonal if

$$AA^T = Id = A^TA.$$

→ Linear transformation defined by an orthogonal matrix:

$$T : V \to V$$

$T(v) = Av$ is orthogonal transformation.

$\square$.