WEEK 2 :

Recall,

Vectors : $(a_1, a_2, a_3, \ldots, a_n) \in \mathbb{R}^n$.

$(1,2) \in \mathbb{R}^2$, $(1,2,3) \in \mathbb{R}^3$.

Matrix: $\begin{bmatrix} 3 & 4 \\ 5 & -1 \end{bmatrix}_{2\times 2}$ $\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}_{n\times 1}$ $\begin{bmatrix} b_1 & \ldots & b_n \end{bmatrix}_{1\times n}$

Square matrix : # of rows = # of columns.

Diagonal matrix: $a_{ij} = 0$ if $i \neq j$

Scalar matrix: $a_{ii} = k \in \mathbb{R}$ and $a_{ij} = 0$ if $i \neq j$

Identity matrix: Scalar matrix with $a_{ii} = 1$.

* Determinant of a matrix

Minor, - Cofactor

$$A = \begin{pmatrix} 3 & 1 & 2 \\ 4 & -2 & 1/2 \\ 0 & -2 & -1 \end{pmatrix}.$$

$M_{1,1} := \det \begin{pmatrix} -2 & 1/2 \\ -2 & -1 \end{pmatrix} = 3.$ | $C_{1,1} = (-1)^2 . 3 = 3.$

$M_{1,2} = \det \begin{pmatrix} 4 & 1/2 \\ 0 & -1 \end{pmatrix} = -4$ | $C_{1,2} = (-1)^3 \cdot (-4) = 4$

$M_{1,3} = -8$ | $C_{1,3} = (-1)^4 (-8) = -8$

$$C_{i,j} = (-1)^{i+j} M_{i,j}$$

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & \ddots & \\ a_{n1} & & & a_{nn} \end{pmatrix}_{n \times n}$$

M_{ij} = determinant of submatrix obtained from A after removing ith row and jth column.

$$C_{ij} = (-1)^{i+j} M_{ij}$$

$$\det A = \sum_{j=1}^{n} a_{ij} C_{ij} = \sum_{j=1}^{n} a_{ij} (-1)^{1+j} M_{1j}$$

determinant of A, (expansion along Row 1).

Expansion along Row 'i' :

$$\det A = \sum_{j=1}^{n} a_{ij} C_{ij} = \sum_{j=1}^{n} a_{ij} (-1)^{i+j} M_{ij}$$

Expansion along Column 'j' :

$$\det A = \sum_{i=1}^{n} a_{ij} C_{ij} = \sum_{i=1}^{n} a_{ij} (-1)^{i+j} M_{ij}$$

Note: det A is independent of the choice of row or column used to calculate it.

For example:

$$B = \begin{pmatrix} 1 & 2 & -3 & 1 \\ 2 & 1/2 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

* finding det B by expanding along Row 4:

$$\det B = a_{41}\underline{C_{41}} + a_{42}\underline{C_{42}} + a_{43}\underline{C_{43}} + a_{44}C_{44}$$

$$= a_{44}\,C_{44} = (-1)\,(-1)^{4+4}\cdot M_{44} = 7$$

Where

$$M_{44} = \det\begin{pmatrix} 1 & 2 & -3 \\ 2 & 1/2 & 1 \\ 0 & 0 & 2 \end{pmatrix} = 2\det\begin{pmatrix} 1 & 2 \\ 2 & 1/2 \end{pmatrix}$$

$$= 2\left(1/2 - 4\right) = -7.$$

* finding det B by expanding along Row 3:

$$\det B = a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33} + a_{34}C_{34}$$

$$= 2\,C_{33} + 1\,C_{34}$$

$$C_{33} = (-1)^{3+3}\det\begin{pmatrix} 1 & 2 & 1 \\ 2 & 1/2 & 0 \\ 0 & 0 & -1 \end{pmatrix} = 7/2$$

$$C_{34} = (-1)^{3+4}\det\begin{pmatrix} 1 & 2 & -3 \\ 2 & 1/2 & 1 \\ 0 & 0 & 0 \end{pmatrix} = 0$$

$$\det B = 2(7/2) + 0 = 7.$$

Properties of determinants.

Property 1:

* Determinant of a product is a product of determinants.:

$$A_{n\times n} \text{ and } B_{n\times n} : (AB)_{n\times n}$$

$$\det(AB) = (\det A)(\det B)$$

Property 2:

* Switching two rows or columns changes the sign:

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \xrightarrow{R_1 \leftrightarrow R_2} B = \begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix}$$

$$\det A = 4-6 = -2 \qquad \det B = 6-4 = 2$$

$$\det A = -\det B.$$

$$A \xrightarrow{C_2 \Leftrightarrow C_1} B = \begin{pmatrix} 2 & 1 \\ 4 & 3 \end{pmatrix} \qquad \det B = 2.$$

Property 3 :

- Adding multiples of a row to another row leaves the determinant unchanged
- Adding multiples of a column to another column leaves the determinant unchanged

$$A = \begin{pmatrix} -1 & 0 \\ 3 & 1 \end{pmatrix} \xrightarrow[R_2 + 3R_1]{R_2 \longrightarrow} B = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$\det A = -1$ $\qquad\qquad$ $\det B = -1$

$$A \xrightarrow{C_1 \longrightarrow C_1 - 3C_2} B = \begin{pmatrix} -1 & 0 \\ \underbrace{3-3(1)}_{=0} & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$\det B = -1$

Property 4:

Suppose $A_{n\times n}$ and B is obtained by multiplying i^{th} row of A by a scalar $k \in \mathbb{R}$ (or j^{th} column by a scalar $k \in \mathbb{R}$). Then, $\det B = k(\det A)$

$\det A = 3$.

$$A = \begin{pmatrix} 3 & 3 \\ 1 & 2 \end{pmatrix} \xrightarrow{R_1 \to R_1/3} B = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, \quad \det B = ?$$

$\det B = 1$

$= \frac{1}{3}(\det A)$.

$$\Big\downarrow C_1 \to C_1/3$$

$$C = \begin{pmatrix} 1 & 3 \\ 1/3 & 2 \end{pmatrix}$$

$$\det C = 1 = \frac{1}{3}\det(A)$$

Property 5:

* If $A_{n\times n}$ has a zero row or a zero column. (i.e. all entries in that row or column are zero) then $\det A = 0$.

e.g.:

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 3 & 4 & 5 \\ 100 & 105 & 103 & 104 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \det A = ?$$

$$\det A = \underline{a_{41}} C_{41} + \underline{a_{42}} C_{42} + \underline{a_{43}} C_{43} + \underline{a_{44}} C_{44}$$

$$\boxed{\det A = 0}$$

Property 6:

If a row / column of a matrix is a linear combination of some rows or columns, then the det of the matrix is zero.

e.g.:

$$A = \begin{pmatrix} 1 & -1 & 0 \\ 2 & -3 & 1 \\ 0 & -1 & 1 \end{pmatrix} \qquad \det A = ?$$

$$(2, -3, 1) = 2(1, -1, 0) + (0, -1, 1)$$

$$\boxed{\text{Row } 2 = 2 \cdot \text{Row } 1 + \text{Row } 3.}$$

Row2 is a linear combination of Row1 and Row3

$$\Rightarrow \det A = 0.$$

(Verify): $\det A = (-1)^{3+2}(-1) \det\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} + 1 \det\begin{pmatrix} 1 & -1 \\ 2 & -3 \end{pmatrix}$

$$= 1 + 1(-3+2) = 0$$

Square matrix, System of linear Eq^n:

Recall that, • system of linear eq^{n}s :

$$3x + 4y = 12$$
$$x + y = 5$$

• Matrix representation of above system:

$$A = \begin{pmatrix} 3 & 4 \\ 1 & 1 \end{pmatrix} \rightsquigarrow \text{Coefficient matrix}$$

$$\begin{pmatrix} 3 & 4 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 12 \\ 5 \end{pmatrix}$$

If # of eq^n = # of unknown/variables, then the coefficient matrix is a <u>square matrix</u>.

• What do we mean by solution to above system?

To find out the all possible values of the unknowns/variables, i.e. $x, y, z \ldots$

Method to find solution: <u>Cramer's Rule</u>.

(Cramer's Rule on the previous example).

$$\begin{aligned} 3x + 4y &= 12 \\ x + y &= 5 \end{aligned} \rightsquigarrow \begin{pmatrix} 3 & 4 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 12 \\ 5 \end{pmatrix}$$

$$A = \begin{pmatrix} 3 & 4 \\ 1 & 1 \end{pmatrix} ; \quad \det A = 3 - 4 = -1.$$

$$A_x = \begin{pmatrix} 12 & 4 \\ 5 & 1 \end{pmatrix} ; \quad \det A_x = 12 - 20 = -8$$

$$A_y = \begin{pmatrix} 3 & 12 \\ 1 & 5 \end{pmatrix} ; \quad \det A_y = 15 - 12 = 3$$

If $\det A \neq 0$;

$$x = \frac{\det A_x}{\det A}, \quad y = \frac{\det A_y}{\det A}.$$

$$x = \frac{-8}{-1} = 8 \quad ; \quad y = \frac{3}{-1} = -3.$$

Verify:

$$\begin{aligned} 3x + 4y &= 12 \rightsquigarrow 3(8) + 4(-3) = 12 \\ x + y &= 5 \rightsquigarrow 8 - 3 = 5. \end{aligned}$$

Notice that:

(1) Coefficient matrix is square matrix

i.e. # of eq^n = # of variable.

(2.) Determinant of the coefficient matrix is non-zero.

We are studying system of linear eq^n's with coefficient matrix with non-zero determinant.

Note: Cramer's Rule is applicable to an invertible matrix!

Stepwise Cramer's Rule:

for 2x2:
$$a_{11}x_1 + a_{12}x_2 = b_1$$
$$a_{21}x_1 + a_{22}x_2 = b_2$$

$$\underbrace{\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

If $\det A \neq 0$, $\quad A_{x_1} = \begin{pmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{pmatrix}$ and $A_{x_2} = \begin{pmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{pmatrix}$

$$x_1 = \frac{\det A_{x_1}}{\det A}, \quad x_2 = \frac{\det A_{x_2}}{\det A}$$

for 3x3:

$$\left.\begin{aligned} x + y + z &= 1 \\ y + z &= 4 \\ 4x + 3y + 2z &= 0 \end{aligned}\right\} \rightarrow A\bar{x} = \bar{b}$$

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 4 & 3 & 2 \end{pmatrix}; \quad \bar{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad \bar{b} = \begin{pmatrix} 1 \\ 4 \\ 0 \end{pmatrix}$$

$$\det A = (-1) + 0 + 4(0) \neq 0$$

$$A_x = \begin{pmatrix} 1 & 1 & 1 \\ 4 & 1 & 1 \\ 0 & 3 & 2 \end{pmatrix} \quad A_y = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 4 & 1 \\ 4 & 0 & 2 \end{pmatrix} \quad A_z = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 4 \\ 4 & 3 & 0 \end{pmatrix}$$

$$x = \frac{\det A_x}{\det A} \qquad y = \frac{\det A_y}{\det A}, \qquad z = \frac{\det A_z}{\det A}$$

$n \times n$:

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$
$$\vdots$$
$$a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n$$

$$\underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}}_{\bar{x}} = \underbrace{\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}}_{\bar{b}}$$

$$A\bar{x} = \bar{b}$$

If $\det A \neq 0$, for $1 \leq i \leq n$

A_{x_i} = matrix obtained from A after replacing i^{th} column by $\begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$.

Then $x_i := \dfrac{\det A_{x_i}}{\det A}$.

<u>Solution to a system of linear Eq^n with coefficient matrix with nonzero determinant.</u>

$$A_{n\times n}\,\bar{x}_{n\times 1} = \bar{b}_{n\times 1}$$

$$\underline{(A^{-1}A)}\,\bar{x} = (A^{-1})\,\bar{b}$$

$$I_n\,\bar{x} = (A^{-1})\,\bar{b} \Rightarrow \bar{x} = (A^{-1})\,\bar{b}.$$

INVERSE OF A MATRIX

* Inverse of a $(n\times n)$-matrix A: The inverse of matrix A is an square $(n\times n)$-matrix A^{-1} s.t.

$$\underline{(A^{-1}).A = I_n} = A.A^{-1}$$

* If A is invertible $\Rightarrow$ $\det A \neq 0$.

$$\left.\begin{aligned} \det(A^{-1}A) &= \det(A^{-1})\det(A) \\ \det(I_n) &= \underline{1} \end{aligned}\right\}$$

$$\det A \neq 0 \Leftarrow \det(A^{-1})\det(A) = 1 \Rightarrow \boxed{\det(A^{-1}) = \frac{1}{\det(A)}}$$

Converse ?

$\det A \neq 0 \Rightarrow$ Do we have inverse of A ?

— Yes.

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}_{n\times n}$$

$$\underline{\underline{C_{ij}}} := \text{cofactor of } \underline{\underline{a_{ij}}} = (-1)^{i+j} M_{ij}$$

$\hookrightarrow$ cofactor matrix of $A = \begin{pmatrix} C_{11} & \cdots & C_{1n} \\ \vdots & \ddots & \vdots \\ C_{n1} & \cdots & C_{nn} \end{pmatrix} =: C$

$$A = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \qquad C = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 3 & 0 \\ 2 & -6 & 3 \end{pmatrix}$$

Def^n:

The adjugate matrix of A is defined as:

$$\operatorname{adj}(A) = C^T.$$

In the previous example:

$$\operatorname{adj}(A) = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 3 & 0 \\ 2 & -6 & 3 \end{pmatrix}^T = \begin{pmatrix} 1 & -1 & 2 \\ 0 & 3 & -6 \\ 0 & 0 & 3 \end{pmatrix}$$

<u>Def^n</u>:- $$A^{-1} = \frac{1}{\det A} \operatorname{adj}(A)$$

In the previous example:

$$\det A = \det \begin{pmatrix} 3 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} = 3$$

$$A^{-1} = \frac{1}{3} \begin{pmatrix} 1 & -1 & 2 \\ 0 & 3 & -6 \\ 0 & 0 & 3 \end{pmatrix} = \begin{pmatrix} 1/3 & -1/3 & 2/3 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$$

Verify:

$$A^{-1}A = I_n$$

$$\begin{pmatrix} 1/3 & -1/3 & 2/3 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 3 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I_3$$

$AA^{-1} = I_n$ (Check!)

* System of linear Eq^n with Coefficient matrix of non-zero determinant:

$$a_{11} x_1 + \cdots + a_{1n} x_n = b_1$$
$$a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n = b_2$$
$$\vdots \qquad \vdots$$
$$a_{n1} x_1 + a_{n2} x_2 + \cdots + a_{nn} x_n = b_n$$

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & \vdots & & \\ a_{n1} & a_{n2} & & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

$$A_{n\times n} \bar{x}_{n\times 1} = \bar{b}_{n\times 1}$$

If $\det A \neq 0$.

(i) Cramer's Rule to find out sol^n $\bar{x}$.

(ii) Inverse of A:

- $C = \begin{pmatrix} C_{11} & \cdots & C_{1n} \\ \vdots & & \\ C_{n1} & \cdots & C_{nn} \end{pmatrix}$ cofactor matrix of A.

- $\operatorname{adj}(A) = C^T$.

- $A^{-1} = \dfrac{1}{\det A} \operatorname{adj}(A)$

$$\boxed{\bar{x} = A^{-1}\bar{b}}$$

Example:

$$\left.\begin{aligned} x + y + z &= 1 \\ y + z &= 4 \\ 4x + 3y + 2z &= 0 \end{aligned}\right\} \rightarrow A\bar{x} = \bar{b}$$

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 4 & 3 & 2 \end{pmatrix}; \quad \bar{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad \bar{b} = \begin{pmatrix} 1 \\ 4 \\ 0 \end{pmatrix}$$

$$\det A = -1. \qquad C = \begin{pmatrix} -1 & 4 & -4 \\ 1 & -2 & -1 \\ 0 & -1 & 1 \end{pmatrix} \quad (\text{Check!})$$

$$\operatorname{adj}(A) = C^T = \begin{pmatrix} -1 & 1 & 0 \\ 4 & -2 & -1 \\ -4 & -1 & 1 \end{pmatrix}$$

$$A^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ -4 & 2 & 1 \\ 4 & 1 & -1 \end{pmatrix}$$

$$\bar{x} = A^{-1} b.$$

$$= \begin{pmatrix} 1 & -1 & 0 \\ -4 & 2 & 1 \\ 4 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 4 \\ 0 \end{pmatrix} = \begin{pmatrix} -3 \\ 4 \\ 8 \end{pmatrix}$$

$\square$

Homogenous system of linear Eq^n:

$$\left.\begin{array}{l} 3x_1 + 4x_2 = 0 \\ 7x_1 + 8x_2 = 0 \end{array}\right\}$$

$$\begin{array}{l} a_{11} x_1 + \cdots + a_{1n} x_n = 0 \\ a_{21} x_1 + \cdots + a_{2n} x_n = 0 \\ \vdots \\ a_{n1} x_1 + \cdots + a_{nn} x_n = 0 \end{array}$$

$\Downarrow$

If coefficient matrix A is invertible, then $A\bar{x} = \bar{b}$ has a unique sol^n.

which is $\bar{x} = A^{-1}\bar{b} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}_{n\times 1}$

* A homogenous system of linear eq^n with n eq^n and n variable:

→ has a unique sol^n if $\det A \neq 0$.

→ has infinitely many sol^n if $\det A = 0$.

□

Next time: We will discuss how to solve system of linear Eqⁿ in general.

$$\begin{aligned} 3x + 4y + 5z &= 1 \\ y + 3z &= 2 \\ 5z + 4w &= -3 \end{aligned}$$

- $\left.\begin{array}{l} \text{3 equations} \\ \text{4 variables} \end{array}\right\}.$