WEEK 2 : Row Echelon – Guass Elimination

Last session:
- System of linear eqⁿs with n-variable and n- equations.
- Coefficient matrix (Invertible)
- Solution to System of linear Eqⁿs :
  - Cramer's Rule
  - Inverse of a matrix.

$$\begin{aligned} 3x_1 + 4x_2 + 5x_3 &= 1 \\ 5x_2 + x_3 &= 2 \\ x_1 + x_2 + x_3 &= 3 \end{aligned}$$

Coefficient matrix

$$\underbrace{\begin{bmatrix} 3 & 4 & 5 \\ 0 & 5 & 1 \\ 1 & 1 & 1 \end{bmatrix}}_{n\times n} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$A \bar{x} = \bar{b}$$

* If $\det A \neq 0$, i.e. A is invertible,

(1) Cramer's Rule : $A_{x_1} = \begin{bmatrix} 1 & 4 & 5 \\ 2 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$

$A_{x_2} = \_\_\_ \quad , A_{x_3} = \_\_\_$

$$x_1 = \frac{\det A_{x_1}}{\det A}, \quad x_2 = \frac{\det A_{x_2}}{\det A}, \quad \ldots$$

(2) Because A^{-1} exists;

$$A\bar{x} = \bar{b}$$

$$\boxed{\bar{x} = A^{-1}\bar{b}} = \bar{c}$$

$$\underbrace{3\times3 \quad 3\times1}_{3\times1}$$

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$$

□

TODAY:

(1) System of linear Eqn with m-equations and n-variables.

(2) Solving method for coefficient matrix in
   (i) Row Echelon form (REF)
   (ii) Reduced row Echelon form. (RREF)

(3) Matrix $A_{m\times n} \longrightarrow$ REF
   $A_{m\times n} \longrightarrow$ RREF
   by row reductions.

(4) Calculating determinant using row reduction.

(5) Solving $A_{m\times n}\bar{x} = \bar{b}$ by Gaussian elimination.

System of m linear equations and n- variables

2x3:

$$\left.\begin{array}{l} 4x_1+5x_2+x_3=1 \\ x_1+4x_2 \quad\; =-1 \end{array}\right] \quad \begin{array}{l} 2 \text{ eq}^{ns} \\ 3 \text{ variables} \end{array}$$

$$\Downarrow$$

$$A= \begin{bmatrix} 4 & 5 & 1 \\ 1 & 4 & 0 \end{bmatrix}_{2\times 3} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}_{3\times 1} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}_{2\times 1}$$

mxn:

$$\begin{array}{c} a_{11}x_1+\cdots+a_{1n}x_n=b_1 \\ a_{21}x_1+\cdots+a_{2n}x_n=b_2 \\ \vdots \\ a_{m1}x_1+\cdots+a_{mn}x_n=b_m \end{array}$$

$$A= \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & & \vdots & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad \bar{x}= \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \bar{b}= \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}.$$

$$\Longleftrightarrow \quad \underset{m\times n}{A}\,\underset{n\times 1}{\bar{x}} = \underset{m\times 1}{\bar{b}}$$

Row Echelon form.

A matrix is std in row-echelon form if

- The first non-zero element in each row called the leading entry, is 1.

- Each leading entry is in a column to the right of the leading entry in the previous row.

- Rows with all zero elements, if any, are below row having a non-zero element.

Example Row Echelon form.

$$\begin{bmatrix} 1 & 5 & 4 \\ 1 & 4 & 0 \end{bmatrix}_{2\times 3} \qquad \begin{bmatrix} 1 & 5 & 4 \\ 0 & 1 & 5 \end{bmatrix} \checkmark$$

$$(a) = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 0 & 1 \end{bmatrix}, \quad (b) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix} \times$$

↳ Not satisfy (2),

$$(c) = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \times, \quad (d) = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \times$$

↳ Not satisfies (2).

$$(e) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad (f) = \begin{bmatrix} 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$(a') = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad (f') = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Reduced Row Echelon form.

A matrix is said in reduced row echelon form if

- The first non-zero element in each row called the leading entry, is 1.

- Each leading entry is in a column to the right of the leading entry in the previous row.

- Rows with all zero elements, if any, are below row having a non-zero element.

- For a non-zero row, the leading entry in the row is the only non-zero entry in its column.

$$\begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[\text{operation}]{\text{Row}} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Row echelon form (REF) — Reduced row echelon. (RREF)

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \times \qquad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 2 & 3 \end{bmatrix} \times$$

System whose coefficient matrix is in REF or RREF:

– Independent and dependent variables:

$$\left.\begin{aligned} x_1 + 3x_2 &= 1 \\ x_3 + x_4 &= 5 \end{aligned}\right] \quad 2\ \text{eq}^n,\ 4\ \text{var.}$$

$$\begin{pmatrix} 1 & 3 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

independent var.

$x_1 = 1 - 3x_2$

$x_3 = 5 - x_4$

dependent variable.

$\underset{m\times n}{A}\,\bar{x} = \bar{b}$

Guassian elimination

$$\left[\begin{array}{c|c} A & b \end{array}\right] \longrightarrow \text{Augumented} \quad m\times(n+1)$$

↓ Row op.

$A \longrightarrow \text{RREF} \qquad b' \longrightarrow \left[\begin{array}{c|c} R & b' \end{array}\right]$

$R\bar{x} = \bar{b}$

Converting a matrix
$A_{m\times n} \longrightarrow$ REF / RREF :

Row reduction :

1. Interchange two rows.

$$\underset{A}{\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \xrightarrow{R_1 \leftrightarrow R_2} \underset{B}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$\det A = -\det B$

2. Scalar multiplication of a row by $t \in \mathbb{R}$.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \xrightarrow{R_3 \to \frac{1}{3}R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_i \longrightarrow t R_i$

$\det A = t \det B$

3. Adding multiples of a row to another row

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 3 \end{bmatrix}_{2\times 3} \xrightarrow[R_1 + (-1)R_2]{R_1 \to R_1 - R_2} \underbrace{\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 3 \end{bmatrix}}_{\text{RREF.}}$$

$R_i \longrightarrow R_i + cR_j$

$\det A = \det B$

Calculating determinant using row reduction :

Row operation		Change in 'det' (backward tracing)
$R_i \Leftrightarrow R_j$	$\longleftrightarrow$	$\det A = -\det B$.
$R_i \to cR_i$; $c \neq 0$	$\Longleftrightarrow$	$\det A = \frac{1}{c} \det B$
$R_i \to R_i + cR_j$, $i \neq j$	$\Longleftrightarrow$	$\det A = \det B$.

Example :

* Using Row reduction to calculate determinant.

Determinant

$$\begin{bmatrix} 2 & 4 & 1 \\ 3 & 8 & 7 \\ 5 & 6 & 9 \end{bmatrix} \qquad 70 = 35 \times 2$$

$$\xrightarrow{R_1/2} \begin{bmatrix} 1 & 2 & 1/2 \\ 3 & 8 & 7 \\ 5 & 6 & 9 \end{bmatrix} \quad 35 \qquad 35$$

$$\xrightarrow[R_3 - 5R_1]{R_2 - 3R_1} \begin{bmatrix} 1 & 2 & 1/2 \\ 0 & 2 & 11/2 \\ 0 & -4 & 13/2 \end{bmatrix} \quad 35 \qquad \frac{35 \times 2}{2}$$

$$\xrightarrow{R_2/2} \begin{bmatrix} 1 & 2 & 1/2 \\ 0 & 1 & 11/4 \\ 0 & -4 & 13/2 \end{bmatrix} \qquad 35/2$$

$$\xrightarrow{R_3+4R_2} \begin{bmatrix} 1 & 2 & 1/2 \\ 0 & 1 & 11/4 \\ 0 & 0 & 35/2 \end{bmatrix} \qquad 1 \times \frac{35}{2}$$

$$\xrightarrow{R_3 \times \frac{2}{35}} \begin{bmatrix} 1 & 2 & 1/2 \\ 0 & 1 & 11/4 \\ 0 & 0 & 1 \end{bmatrix} \qquad 1$$

$\downarrow$

upper triangular

Note:

$R_i \to R_i + cR_j$ does not change the

$R_i \to R_i + dR_i + cR_j$ $\quad j \neq i$

Solving an $(m\times n)$- system :
Method of Gauss Elimination

$$\begin{aligned} 3x + y + z &= 1 \\ 3y + 2z &= 2 \end{aligned}$$

$A_{m\times n}$ (2×3)- system. $A = \begin{bmatrix} 3 & 1 & 1 \\ 0 & 3 & 2 \end{bmatrix}_{2\times 3}$

$\downarrow$ Augumented matrix :

$[A|b]_{m\times(n+1)}$ $[A|b]_{m\times(n+1)} = \left[\begin{array}{ccc|c} 3 & 1 & 1 & 1 \\ 0 & 3 & 2 & 2 \end{array}\right]_{2\times 4}$

Apply row reductions to it until
$\underline{A \longrightarrow RREF}$.

$$[A|b] \xrightarrow[\text{reduction}]{\text{Row}} \underline{[R|c]} \longleftrightarrow R\bar{x} = c$$

Solution set of $R\bar{x} = \bar{c}$ is the same as
$A\bar{x} = b$.

ALL Solution of $R\bar{x} = \bar{c}$.

$$\left[\begin{array}{ccc|c} 3 & 1 & 1 & 1 \\ 0 & 3 & 2 & 2 \end{array}\right]$$

$$\Big\downarrow R_1 \to R_1/3$$

$$\left[\begin{array}{ccc|c} 1 & 1/3 & 1/3 & 1/3 \\ 0 & 3 & 2 & 2 \end{array}\right]$$

$$\Big\downarrow R_2 \to R_2/3$$

x

$$\left[\begin{array}{ccc|c} 1 & 1/3 & 1/3 & 1/3 \\ 0 & 1 & 2/3 & 2/3 \end{array}\right]$$ → Augmented matrix

dependent var. — y

REF

z – independent var.

$$R = \begin{pmatrix} 1 & 1/3 & 1/3 \\ 0 & 1 & 2/3 \end{pmatrix}, \quad b' = \begin{pmatrix} 1/3 \\ 2/3 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1/3 & 1/3 \\ 0 & 1 & 2/3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1/3 \\ 2/3 \end{pmatrix}$$

$$\left.\begin{aligned} x + \tfrac{1}{3}y + \tfrac{1}{3}z &= \tfrac{1}{3} \\ y + \tfrac{2}{3}z &= \tfrac{2}{3} \end{aligned}\right\}$$

$$\boxed{y = \tfrac{2}{3}(1-z)}$$

$$x = \tfrac{1}{3} - \tfrac{1}{3}z - \tfrac{1}{3}y$$

$$= 1/3 - 1/3 z - 1/3\left(\frac{2}{3}(1-z)\right)$$

$$= \frac{1}{3}\left[1 - z - \frac{2}{3} + \frac{2z}{3}\right]$$

$$= \frac{1}{3}\left(\frac{3-3z-2+2z}{3}\right) = \frac{1}{9}(1-z)$$

$$\left\{\left(\frac{1}{9}(1-z),\ \frac{2}{3}(1-z),\ z\right) \,\middle|\, z \in \mathbb{R}\right\}$$

$$\left(\boxed{\begin{matrix} 1 & 1/3 & 1/3 \\ 0 & 1 & 2/3 \end{matrix}} \;\middle|\; \begin{matrix} 1/3 \\ 2/3 \end{matrix}\right)$$

$$\Big\downarrow R_1 - \frac{1}{3}R_2$$

$$\left(\begin{matrix} 1 & 0 & 1/3 - \frac{2}{9} \\ 0 & 1 & 2/3 \end{matrix} \;\middle|\; \begin{matrix} 1/3 - \frac{2}{9} \\ 2/3 \end{matrix}\right)$$

$$\left(\begin{matrix} 1 & 0 & 1/9 \\ 0 & 1 & 2/3 \end{matrix} \;\middle|\; \begin{matrix} 1/9 \\ 2/3 \end{matrix}\right)$$

$$R = \begin{pmatrix} 1 & 0 & 1/9 \\ 0 & 1 & 2/3 \end{pmatrix} \qquad b'' = \begin{pmatrix} 1/9 \\ 2/3 \end{pmatrix}$$

x dependent y

z - independent

$$R\,\bar{x} = b''$$

$$\begin{pmatrix} 1 & 0 & 1/9 \\ 0 & 1 & 2/3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1/9 \\ 2/3 \end{pmatrix}$$

RREF

$$\left.\begin{array}{l} x + z/9 = 1/9 \\ y + \frac{2}{3}z = 2/3 \end{array}\right] \rightarrow \begin{array}{l} x = 1/9 - z/9 \\ y = 2/3 - \frac{2}{3}z \end{array}$$

$$\begin{array}{ccc} \left(\boxed{\begin{matrix}1\\0\end{matrix}}\right. & \boxed{\begin{matrix}2\\0\end{matrix}} & \left.\boxed{\begin{matrix}0\\1\end{matrix}}\right) \\ \downarrow & \downarrow & \downarrow \\ x_1 & x_2 & x_3 \end{array} \qquad \begin{array}{cc} \left(\begin{matrix}1\\0\\0\end{matrix}\right. & \left.\begin{matrix}0\\1\\0\end{matrix}\right) \\ x & y \end{array}$$

Example:

$$x_1 + x_2 + x_3 = 2$$
$$x_2 - 3x_3 = 1$$
$$2x_1 + x_2 + 5x_3 = 0$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & -3 & 1 \\ 2 & 1 & 5 & 0 \end{array}\right]$$

$$\Big\downarrow R_3 - 2R_1$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & -3 & 1 \\ 0 & -1 & 3 & -4 \end{array}\right]$$

$$\Big\downarrow R_3 + R_2$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & -3 & 1 \\ 0 & 0 & 0 & -3 \end{array}\right] \quad \text{Inconsistent}$$

$$\Big\downarrow$$

$$R = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{pmatrix}, \quad b' = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}$$

$$\left.\begin{aligned} x + y + z &= 2 \\ y - 3z &= 1 \\ 0 &= -3 \end{aligned}\right]$$

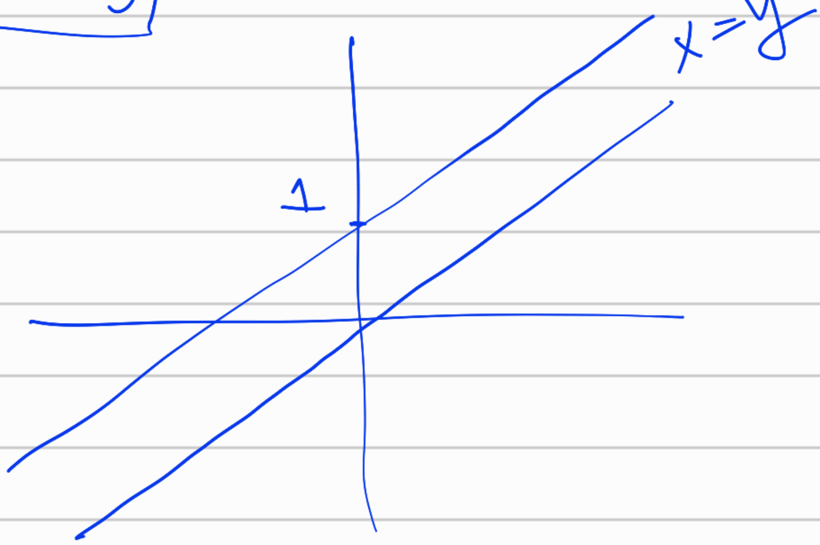
RREF

$\left.\begin{array}{l} \text{— Solution to a system linear Eq}^n. \\ \text{— type of solution} \\ \text{— helps to express solution set} \end{array}\right\}$

$$\begin{array}{cccc} x_1 & x_2 & x_3 & \\ \multicolumn{3}{c}{\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}} & \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \end{array}$$

$$x_1 + x_2 + x_3 = 0$$

$$\boxed{x_1 = -x_2 - x_3}$$



$$y - 1 = x$$

$$\left.\begin{aligned} x - y &= 0 \\ x - y &= -1 \end{aligned}\right]$$

$$\left(\begin{array}{cc|c} 1 & -1 & 0 \\ 1 & -1 & -1 \end{array}\right) \xrightarrow{R_2 - R_1} \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & -1 \end{array}\right)$$

Computing the inverse of A

$\det A \neq 0$

$$A\,(A^{-1}) = I$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} \bar{x} & \bar{y} & \bar{z} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$A\,\bar{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = b_1$$

$$A\,\bar{y} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = b_2$$

$$A\,\bar{z} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = b_3$$

$$\begin{array}{cccc} & b_1 & b_2 & b_3 \\ \left(\begin{array}{c} \\ A \\ \\ \end{array}\right. & \underbrace{\left|\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right.}_{} & & \left.\right) \end{array} \longrightarrow \left(\begin{array}{c|c} \underset{\substack{\text{RREF} \\ \downarrow \\ I}}{R} & \underbrace{\qquad}_{A^{-1}} \end{array}\right)$$

* Complete the following Calculation of A^{-1}?

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 2 & 4 & 8 & 0 & 1 & 0 \\ 3 & 9 & 27 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[R_3-3R_1]{R_2-2R_1} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 6 & -2 & 1 & 0 \\ 0 & 6 & 24 & -3 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2/2} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & -1 & 1/2 & 0 \\ 0 & 6 & 24 & -3 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[R_3/6]{} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & -1 & 1/2 & 0 \\ 0 & 1 & 4 & -1/2 & 0 & 1/6 \end{array}\right]$$

$$\xrightarrow{R_3-R_2} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & -1 & 1/2 & 0 \\ 0 & 0 & 1 & 1/2 & -1/2 & 1/6 \end{array}\right]$$

$$\xrightarrow{R_2-3R_3} \left(\begin{array}{ccc|c} 1 & 1 & 1 & \\ 0 & 1 & 0 & \\ 0 & 0 & 1 & \end{array}\right)$$

$$\xrightarrow{R_1-R_2} \left(\begin{array}{ccc|c} 1 & 0 & 1 & \\ 0 & 1 & 0 & \\ 0 & 0 & 1 & \end{array}\right)$$

$$\xrightarrow{R_1 - R_3} \left(\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & 1 & 0 & \\ 0 & 0 & 1 & A^{-1} \end{array} \right)$$