MATHEMATICS FOR DATA SCIENCE II

WEEK 8

TOPICS TO BE COVERED IN WEEK 8

- Orthogonality and Linear Independence
- Orthonormal basis and Gram-Schmidt Process
- Projections using inner products
- Orthogonal transformations and rotations

Topics covered in previous session

Dot product in Euclidean space of dimension n: length and angle

$\mathbb{R}^n$

Some definitions on the vector space R^n , where $n \in N$

i) **Dot product**: Let $u = (u_1, u_2, \ldots, u_n)$, $v = (v_1, v_2, \ldots, v_n) \in R^n$, then the dot product of two vectors of R^n defined as

$$u\cdot v = u_1v_1+u_2v_2+\ldots+u_nv_n \quad \in \mathbb{R}$$

which is a scalar.

$$(3,4)\cdot(1,-2) = 3-8=-5$$

$\sqrt{u_1^2+u_2^2} = \sqrt{1+4} = \sqrt{5}$

ii) **Length of a vector**: Let $u = (u_1, u_2, \ldots, u_n) \in R^n$, then the length of the vector u is

$\|u\| = \sqrt{u_1^2+u_2^2+\ldots+u_n^2}$

norm of u

iii) **Relation between the dot product and length of a vector**:
$u = (u_1, u_2, \ldots, u_n) \in R^n$, then the length of the vector $\boxed{\|u\| = \sqrt{u\cdot u}} = \sqrt{u_1^2+u_2^2+\cdots+u_n^2}$

iv) **The angle between two vectors in R^n** :
Let $u = (u_1, u_2, \ldots, u_n)$, $v = (v_1, v_2, \ldots, v_n) \in R^n$, then the angle θ between two vectors of R^n is given by

$$\theta = \cos^{-1}\left(\frac{u\cdot v}{\sqrt{\underbrace{(u\cdot u)}_{\|u\|^2}\underbrace{(v\cdot v)}_{\|v\|^2}}}\right) = \cos^{-1}\left(\frac{u_1v_1+u_2v_2+\ldots+u_nv_n}{\sqrt{(u_1^2+u_2^2+\ldots+u_n^2)(v_1^2+v_2^2+\ldots+v_n^2)}}\right)$$

Inner product on a vector space

Inner product: An inner product on a vector space V is a function ⟨·, ·⟩ : V × V → R satisfying the following conditions: Let u, v, w ∈ V,

i) positive ⟨v, v⟩> 0 for all v ∈ V \ {0}; ⟨v, v⟩ = 0 if and only if v = 0. v is non-zero

ii) ⟨u + v, w⟩ = ⟨u, w⟩ + ⟨v, w⟩.

Symmetry iii) ⟨u, v⟩ = ⟨v, u⟩. linearity

iv) ⟨cu, v⟩ = c ⟨u, v⟩

Note: A vector space together with an inner product ⟨·, ·⟩ is called an inner product space.

Note: ⟨u , v+w⟩ = $\langle v+w, u\rangle \overset{\text{linearity(i)}}{=} \langle v,u\rangle + \langle w,u\rangle \overset{(iii)}{=} \langle u,v\rangle + \langle u,w\rangle$ Symmetry(iii)

⟨u, cv⟩ $\overset{(iii)}{=} \langle cv, u\rangle \overset{(iv)}{=} c\langle v,u\rangle \overset{(iii)}{=} c\langle u,v\rangle$

Norm on a vector space

A norm on vector space V is a function ||·||: V → R satisfying the following conditions:

Triangle inequality

- $||x+y|| \leq ||x|| + ||y||$, for all $x, y \in V$
- $||cx|| = |c|||x||$ for all $c \in \mathbb{R}$ and for all $x \in V$
- Non-negativity $||x|| \geq 0$ for all $x \in V$; $||x|| = 0$ if and only if $x = 0$

Norm induced by inner product

Let V be an inner product space with inner product ⟨. , .⟩. Then the function ||·||: V → R defined by ||v||= $\sqrt{\langle v, v\rangle}$ is a norm on V.

usual inner product

Dot product ---> Inner product

$V = \mathbb{R}^2$ $\quad v \in V$ $\quad ||v|| = \sqrt{v \cdot v} = \sqrt{v_1^2 + v_2^2}$

V.S. + Inner product

$(\mathbb{R}^2, \langle \cdot, \cdot \rangle) \to$ IPS

Eg: $\mathbb{R}^2$ $\quad u, v \in \mathbb{R}^2$ $\quad u = (u_1, u_2)$ $\quad v = (v_1, v_2)$

$\langle u, v\rangle = u_1v_1 - (u_1v_2 + u_2v_1) + 2u_2v_2$

$\langle (1,-1), (1,0)\rangle = 1 - (0 - 1) + 0 = 1 + 1 = 2$

$||u|| = \sqrt{\langle u,u\rangle} = \sqrt{u_1^2 - (u_1u_2 + u_2u_1) + 2u_2^2}$

$= \sqrt{u_1^2 - 2u_1u_2 + 2u_2^2}$

Exercise 1: Consider a function $f : V \times V \to R$ where $V \subseteq R^2$ defined by

$f(v, w) = 2v_1w_1 + 5v_2w_2$, where $v = (v_1, v_2)$, $w = (w_1, w_2)$.

Choose the set of correct options.

✓ • Option 1: f satisfies the symmetry condition of the inner product.

linearity in both first and 2nd argument

✓ • Option 2: f satisfies the bilinearity condition of the inner product.

✓ • Option 3: f satisfies the positivity condition of the inner product.

✓ • Option 4: f is an inner product.

✗ • Option 5: f is not an inner product.

Op1 : verify $f(v,w) = f(w,v)$

$f(w,v) = 2w_1v_1 + 5w_2v_2 = f(v,w)$ $\qquad u+v = (u_1+v_1, u_2+v_2)$

Op 2: verify $f(u+v,w) = f(u,w) + f(v,w)$, $\left(f(cv,w) \overset{HW}{=} c f(v,w)\right)$

$f(u+v,w) = 2(u_1+v_1)w_1 + 5(u_2+v_2)w_2 = 2u_1w_1 + 2v_1w_1 + 5u_2w_2 + 5v_2w_2 = f(u,w) + f(v,w)$

Op3: verify: $f(v,v) > 0 \quad v \neq 0$

$f(v,v) = 2v_1^2 + 5v_2^2 > 0$

HW **Exercise 2:** Consider a function $f : V \times V \to R$ where $V \subseteq R^2$ defined by

$f(v, w) = v_1w_1 - v_1w_2 - v_2w_1 + 4v_2w_2$, where $v = (v_1, v_2)$, $w = (w_1, w_2)$.

Choose the set of correct options.

• Option 1: f satisfies the symmetry condition of the inner product.

• Option 2: f satisfies the bilinearity condition of the inner product.

• Option 3: f satisfies the positivity condition of the inner product.

• Option 4: f is an inner product.

• Option 5: f is not an inner product.

Exercise 3: Consider two vectors a = (0.4, 1.3, −2.2), b = (2, 3, −5) in R^3. Choose the set of correct options.

- Option 1: The two vectors satisfy the triangle inequality given by $\| a + b \| \leq \|a\| + \|b\|$.
- Option 2: The two vectors do not satisfy the triangle inequality.
- Option 3: The two vectors satisfy the Cauchy-Schwarz inequality given by $|\langle a, b\rangle| \leq \|a\| \, \|b\|$.
- Option 4: The two vectors do not satisfy the Cauchy-Schwarz inequality.

Find: $\|a\|$, $\|b\|$, $\|a+b\|$, $|\langle a,b\rangle|$

$\|a\| = 2.58$ $\|b\| = 6.16$ $\|a+b\| = 8.7$ $\|a\| + \|b\| = 8.74$

$|\langle a,b\rangle| = 15.7 \leq \|a\|\|b\| = 15.89$

Exercise 4: Consider a function f : V × V → R where $V \subseteq R^2$ defined by

Hw

$f(v, w) = v_1^2 w_1^2 + v_1 w_2^2 + v_2^2 w_1$, where $v = (v_1, v_2)$, $w = (w_1, w_2)$.

Choose the set of correct options.

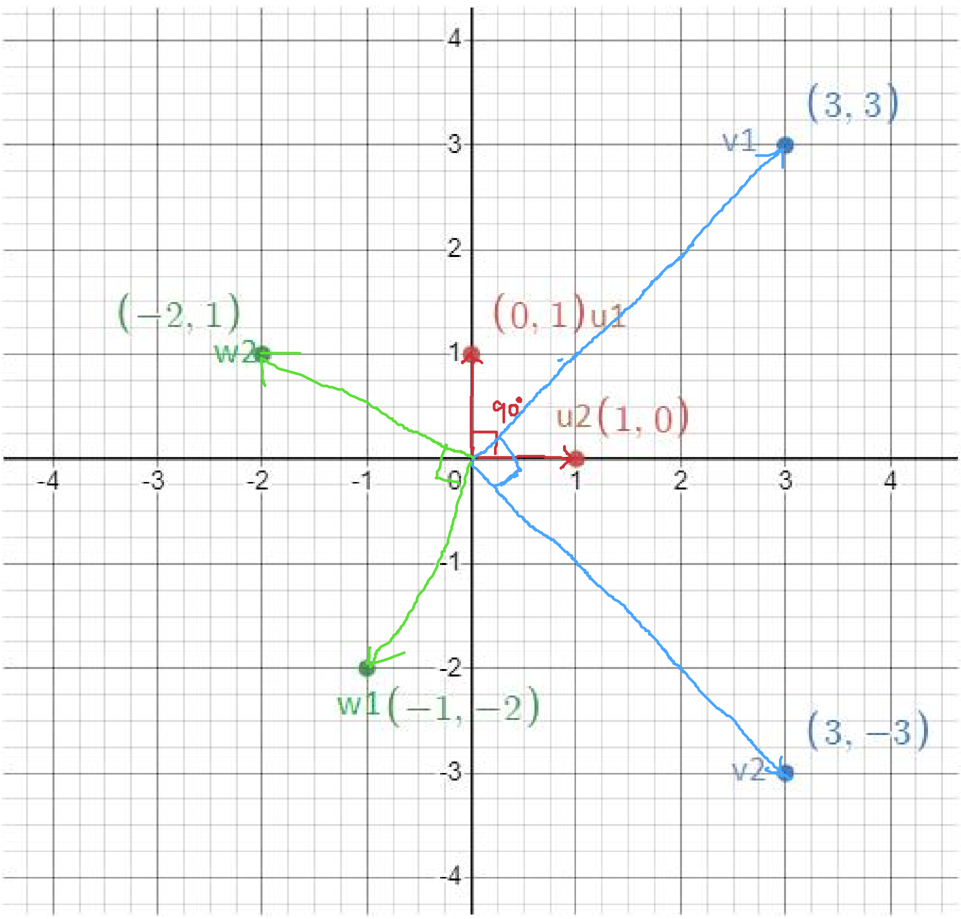
- Option 1: f satisfies the symmetry condition of the inner product.
- Option 2: f satisfies the positivity condition of the inner product.
- Option 3: f is an inner product.
- Option 4: f is not an inner product.

When do we call two lines to be perpendicular? Angle b/w them should be 90°

perpendicular

Orthogonality and linear Independence

orthogonal ⟹ linearly independence



$u_1 \bullet u_2 = (1,0)\cdot(0,1) = 0$

$v_1 \bullet v_2 = (3,3)\cdot(3,-3) = 0$

$w_1 \bullet w_2 = (-1,-2)\cdot(-2,1) = 0$

Orthogonal vectors:

Two vectors u and v of an inner product space V are said to be orthogonal if

$\langle u, v\rangle = 0$.

Example: Consider inner product space R^2 with inner product

$\langle u, v\rangle = u_1v_1 - (u_1v_2 + u_2v_1) + 2u_2v_2$, where $v = (v_1, v_2)$, $w = (w_1, w_2)$.

Let's check vectors (1, 1) and (1,0) are orthogonal to each.

$\langle(1, 1), (1, 0)\rangle =$ $1-(0+1)+0 = 1-1 = 0$

Orthogonal set of vectors:

An orthogonal set of vectors of an inner product space V is a set of vectors whose elements are mutually orthogonal. $S \subseteq \mathbb{R}^n$

Explicitly if $S = \{v_1, v_2, . . . , v_k\} \subseteq V$, then S is an orthogonal set of vectors if $\langle v_i , v_j\rangle = 0$ for $i, j \in \{1, 2, . . . , k\}$ and $i \neq j$.

$\langle v_i, v_i\rangle = \|v\|^2$

Example: Consider the set of vectors $S = \{\overset{v_1}{(4, 3, -2)}, \overset{v_2}{(-3, 2, -3)}, \overset{v_3}{(-5, 18, 17)}\}$. Set S is an orthogonal set of vectors in the inner product space R^3 with respect to dot product.

$\langle v_1, v_2\rangle = 0$

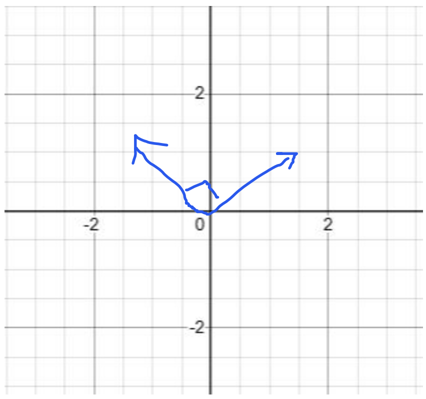
$\langle v_1, v_3\rangle = 0$

$\langle v_2, v_3\rangle = 0$

Question: Can there be an orthogonal set of 3 vectors in R^2? No

Orthognality $\Rightarrow$ lin.ind

max no. of lin. ind vectors in $\mathbb{R}^2 = 2$



Orthogonality implies independence: Let $\{v_1, v_2, . . . ,v_k\}$ be an orthogonal set of vectors in the inner product space V . Then $\{v_1, v_2, . . . ,v_k\}$ is a linearly independent set of vectors.

Given : $\langle v_i, v_j \rangle = 0 \quad i \neq j$

To show : $\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_k v_k = 0 \implies \alpha_i = 0 \quad \forall i$

Proof : Let $\sum_{i=1}^{k} \alpha_i v_i = 0$

bilinearity $\quad \langle v_i, v_j \rangle = 0 \quad i \neq j$

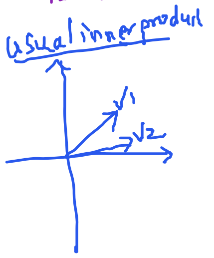
$$0 = \langle \sum_{i=1}^{k} \alpha_i v_i , v_1 \rangle = \sum_{i=1}^{k} \alpha_i \langle v_i , v_1 \rangle = \alpha_1 \langle v_1, v_1 \rangle \implies \alpha_1 = 0$$

$v_1 \neq 0 \implies \langle v_1, v_1 \rangle > 0$, $\neq 0$

Similarly, $\alpha_i = 0 \quad \forall i$

In dim (V) = n, then any orthogonal set in V can have maximum n vectors

Is the converse true? Does lin. ind imply orthogonality? No



v_1 & $v_2 \to$ lin. ind

v_1 & $v_2 \to$ not orthogonal

Orthogonal basis: Let V be an inner product space. A basis consisting of mutually orthogonal vectors is called an orthogonal basis.

B - Basis
B - orthogonal set $\Big\} \implies$ B - orthogonal Basis

Recall:

1) Basis is a maximal linearly independent set

2) Orthogonal set of vectors is already linearly independent.

1) and 2) implies that

3) If dim(V) = n, then Orthogonal basis is nothing but the orthogonal set of n vectors.

Question: If V is a vector space of dimension n, then what is the maximum number of vectors in an orthogonal set of V? n

Examples: Orthogonal basis

i) The standard basis of inner product space R^n with respect to dot product.

ii) We verified the set S = {(4, 3, –2), (–3, 2, –3), (–5, 18, 17)} is an orthogonal subset of R^3 with respect to dot product and $\dim(R^3) = 3$.

$v_1 \quad v_2 \quad v_3$

$\|v_i\| \neq 1$

but not orthonormal

So the set S is an orthogonal basis of inner product R^3.

iii) We proved that these vectors (1, 1) and (1, 0) are orthogonal vectors in the inner product space R^2 with inner product

$\langle u, v\rangle = u_1v_1 - (u_1v_2 + u_2v_1) + 2u_2v_2$, where $v = (v_1, v_2)$, $w = (w_1, w_2)$.

So the set {(1, 1), (1, 0)} is an orthogonal basis of the inner product space R^2.

Question: Is the above set an orthogonal basis of R^2 with respect to the usual inner product? No

$(1,1)\cdot(1,0) = 1 \neq 0$

not orthogonal

Orthonormal set: An orthonormal set of vectors of an inner product space V is an orthogonal set of vectors such that the norm of each vector of the set is 1.

That is, a set $\{v_1, v_2, \ldots, v_k\}$ is orthonormal if $\langle v_i, v_j\rangle = \begin{cases} 0, & i \neq j \quad \text{orthogonal} \\ 1, & i = j \quad \text{unit 1 (normalized)} \end{cases}$ for all $1 \leq i, j \leq k$.

$\|v_i\| = 1$

Orthonormal basis: An orthonormal basis is an orthonormal set of vectors which form a basis.

Note:

1) Each vector has norm 1 in an orthogonal basis.
2) An orthonormal basis is a maximal orthonormal set in the inner product space.

For example, the standard basis of R^n with respect to dot product forms an orthonormal basis.

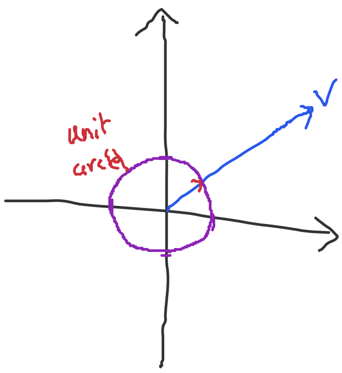
Example: Consider R^3 with the usual inner product (i.e. dot product) and the set

$S = \{ \overset{v_1}{\tfrac{1}{3}(1, 2, 2)}, \overset{v_2}{\tfrac{1}{3}(-2, -1, 2)}, \overset{v_3}{\tfrac{1}{3}(2, -2, 1)} \}$.

Verify S forms an orthonormal basis of R^3.

To show: $\langle v_1, v_2 \rangle = 0$, $\langle v_1, v_3 \rangle = \langle v_2, v_3 \rangle = 0$

$\|v_1\| = \|v_2\| = \|v_3\| = 1$



$v = (1, 1)$ $\|v\| = \sqrt{2}$

$\downarrow$

v' $\|v'\| = 1$

$v' = \alpha v$

$1 = \|v'\| = \|\alpha v\| = |\alpha| \, \|v\|$

$\Rightarrow \alpha = \frac{1}{\|v\|}$

Obtaining orthonormal set from orthogonal set

If $\gamma = \{v_1, v_2, \ldots, v_n\}$ is an orthogonal set of vectors then we can obtain an orthonormal set of vectors β from γ by dividing each vector by its norm

i.e., $\beta = \{\frac{v_1}{\|v_1\|}, \frac{v_2}{\|v_2\|}, \ldots, \frac{v_n}{\|v_n\|}\}$.

Example: Consider R^2 with usual inner product and the orthogonal basis

$S = \{(1, 3), (-3, 1)\}$. $\quad \langle v_1, v_2 \rangle = 0$

Convert this ONB → orthonormal basis

$\|v_1\| = \sqrt{10} \quad \|v_2\| = \sqrt{10}$

ONB: $\left\{ \frac{1}{\sqrt{10}}(1,3), \frac{1}{\sqrt{10}}(-3,1) \right\}$

Basis not orthogonal: $(1,1,0,0), (1,0,1,0), (0,0,1,1), (0,1,0,1)$

$(5,3,-7,9) = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 + \alpha_4 v_4$

Find α_i's.

Solve some sys of linear eqn

Importance of orthonormal basis

Suppose $S = \{v_1, v_2, \ldots, v_n\}$ is an orthonormal basis of an inner product space V and let $v \in V$. Then v can be written as

$\langle v_i, v_j \rangle = 0 \ \forall i \neq j$

$\langle v_i, v_i \rangle = 1$

$v = c_1v_1 + c_2v_2 + \ldots + c_nv_n$

$\langle v, v_i \rangle = \langle c_1v_1 + c_2v_2 + \cdots + c_nv_n, v_i \rangle = c_i \langle v_i, v_i \rangle = c_i$

How do we find $c_1, c_2, \ldots, c_n$?

For any basis, this means writing a system of linear equation and solving it.

But since S is an orthonormal, we can use inner product and can compute $c_i = \langle v, v_i \rangle$, $i \in \{1, 2, 3, \ldots, n\}$,

$\langle v, v_i \rangle = c_i$

$S = \left\{ \frac{1}{\sqrt{10}}(1,3), \frac{1}{\sqrt{10}}(-3,1) \right\}$

$(5,-7) = c_1v_1 + c_2v_2 \qquad c_1 = \langle v, v_1 \rangle = \frac{1}{\sqrt{10}}(5,-7)\cdot(1,3) = \frac{1}{\sqrt{10}}(5-21) = \frac{-16}{\sqrt{10}}$

$c_2 = -\frac{22}{\sqrt{10}}$

$\frac{-16}{\sqrt{10}} \cdot \frac{1}{\sqrt{10}} \cdot (1,3) \ \frac{-22}{\sqrt{10}\sqrt{10}}(-3,1) = \frac{1}{10}(-16,-48) + \frac{1}{10}(66,-22)$

$= \frac{1}{10}(50,-70) = (5,-7)$

Exercise 1: Consider two vectors a = (2, 0, 3, 0, 8), b = (3, 2, –2, 4, 0) in R^5. Choose the set of correct options.

- ✓ Option 1: a and b are orthogonal. $\quad a \cdot b = 6 + 0 - 6 + 0 + 0 = 0$
- ✗ Option 2: a and b are not orthogonal.
- ✗ Option 3: $(a - b) \cdot a = 0$. $\quad a - b = (-1, -2, 5, -4, 8)$
- ✓ Option 4: $(a - b) \cdot a = 77$. $\quad (a-b)\cdot a = -2 + 0 + 15 + 0 + 64 = 77$

Exercise 2: Choose the set of correct statements.

- Option 1: In an orthogonal set, the norms of all the vectors are equal.
- Option 2: In an orthogonal set, the vectors are linearly independent.
- Option 3: In an orthogonal set, the vectors are linearly dependent.
- Option 4: If the columns of an $n \times n$ coefficient matrix A comprises the individual vectors of an orthogonal set in R^n, then there must be a unique solution to the system $AX = b$, where X, b are $n \times 1$ vectors.

$A \begin{bmatrix} v_1 & v_2 \cdots & v_n \end{bmatrix} \rightarrow$ orthogonal vector

$\Rightarrow$ lin. ind

$\Rightarrow \det(A) \neq 0 \Rightarrow Ax = b$ unique soln

- Option 5: If the columns of an $n \times n$ coefficient matrix A comprises the individual vectors of an orthogonal set in R^n, then there are no solutions to the system $AX = b$, where X, b are $n \times 1$ vectors.
- Option 6: The determinant of a square matrix formed by a set of orthogonal vectors in R^n is zero.
- Option 7: A set of n vectors can never form an orthogonal basis in R^{n-1}.

Exercise 3: Which of the following is an orthogonal basis of the given vector spaces with respect to the standard inner product (dot product)?

- Option 1: $\{(1, 0), (0, 1)\}$ is an orthogonal basis of R^2.
- Option 2: $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ is an orthogonal basis of R^3.
- Option 3: $\{(3, 4), (4, -3), (2, -3)\}$ is an orthogonal basis of R^2.
- Option 4: $\{\underbrace{(2, 1, -1)}_{v_1}, \underbrace{(-1, 1, -1)}_{v_2}, \underbrace{(3, -3, 3)}_{v_3}\}$ is an orthogonal basis of R^3.

$v_2, v_3 \rightarrow$ lin. dep

$v_3 = -3v_2$

u

Exercise 4: Find a vector in R^4 that is orthogonal to the subspace spanned by (1, 1, 0, 0) and (0, 1, 1, 0) with respect to the dot product as the inner product.

v

- Option 1: (1, –1, 1, 0)
- Option 2: (2, 3, 4, 5)
- Option 3: (1, –1, 1, 1)
- Option 4: (1, 1, 1, 0)

$w = (w_1, w_2, w_3, w_4)$

$(1,1,0,0)\cdot(w_1,w_2,w_3,w_4) = 0$

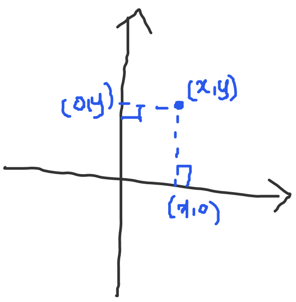
$u\cdot w = 0 \qquad v\cdot w = 0$

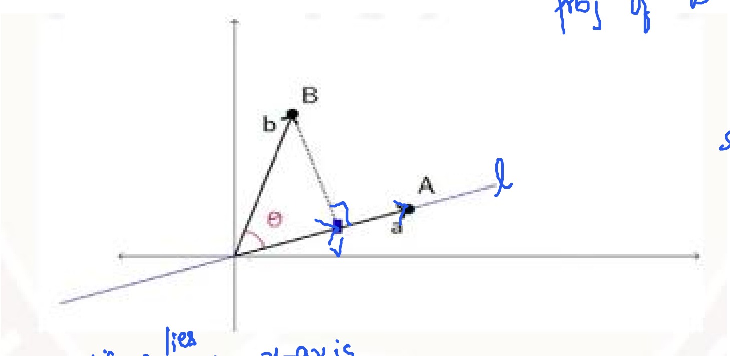
$w_1 + w_2 = 0 \qquad w_2 + w_3 = 0$

$w_2 = -w_1 \qquad w_3 = -w_2 = w_1$

Projections using inner products

Projection of a vector along another vector: Let A and B be two points in R^2. Suppose we want to find the point nearest to B on the line l passing through A and the origin.





Proj of b on a : lies on the line passing thru (0,0) & a

↓

so it should be of the form αa.

Proj of (x, y) on x-axis : lies on x-axis

" " " " y-axis : lies on y-axis

The nearest point will be the foot of the perpendicular drawn from the point B to the line l.

To solve this problem in the perspective of vectors, we will use the concept of inner products.

Aim: To find the length of the vector v which is the foot of the perpendicular from B to the line l?

Once we know the length, ||v||, we can determine the vector v. This is because,

v = αa (since v lies along the vector a).

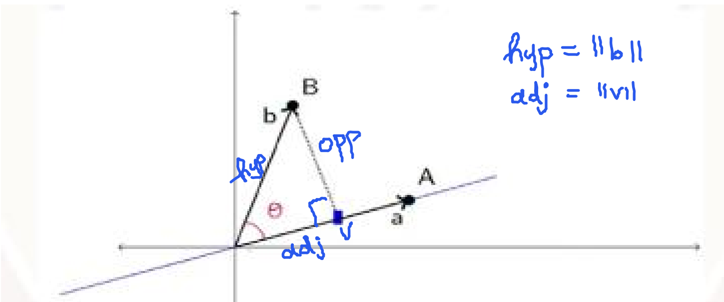
Hence ||v||= α||a||. From this we get the value of

$\alpha = \frac{||v||}{||a||}$. Thus $v = \frac{||v||}{||a||}a$. Note this can also be writtenas $v = ||v|| \frac{a}{||a||}$.

which is the unit vector in the direction of a and multiply it with the length of the vector v.

||v||= ||b||cos θ (θ is the angle between the two vectors a and b.

$\cos\theta = \frac{adj}{hyp} = \frac{||v||}{||b||}$



$\cos \theta = \frac{\langle a, b\rangle}{||a||\ ||b||}$.

$v = ||v|| \frac{a}{||a||}$

$||v|| = ||b|| \cos\theta$

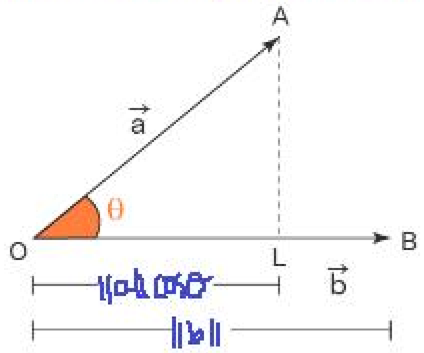
$$v = ||b|| \frac{\langle a, b\rangle}{||a||\ ||b||} \frac{a}{||a||} = \frac{\langle a, b\rangle}{||a||^2} a = \frac{\langle a, b\rangle}{\langle a, a\rangle} a.$$

To find the length of the vector v which is the foot of the perpendicular from B to the line.

projection of the vector b along the vector a, $\boxed{v = \frac{\langle a,b\rangle}{\langle a,a\rangle} a}$

So, if we know the vectors a and b, we can find the point along the vector a that is nearest to the vector b. This vector v, is called the projection of the vector b along the vector a.

Geometrical representation of dot product



$\cos\theta = \frac{adj}{hyp} = \frac{OL}{\|a\|}$

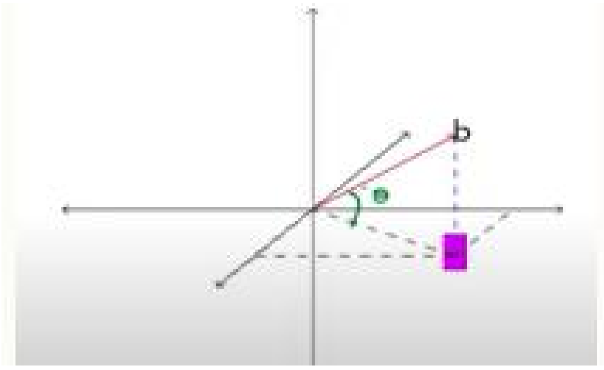
$\Rightarrow OL = \|a\| \cos\theta$

$a \cdot b = \|a\| \, \|b\| \cos\theta$

Projection of $\vec{a}$ on $\vec{b}$ $= OL = \|a\| \cos\theta$

$= \frac{a \cdot b}{\|b\|}$.

Similarly, we can talk about shortest distances in R^3. Suppose b is a point whose shortest distance from the plane generated by (1, 0, 0) and (0, 1, 0) needs to be calculated.



Projection of a vector onto a subspace

Let V be an inner product space, v ∈ V and W ⊆ V be a subspace.

Then the projection of v onto W is the vector in W , denoted by $proj_W(v)$, computed as follows:

- Find an orthonormal basis $\{v_1, v_2, \ldots, v_n\}$ for W .
- Define $proj_W(v) = \sum_{i=1}^{n} \langle v, v_i \rangle \, v_i$

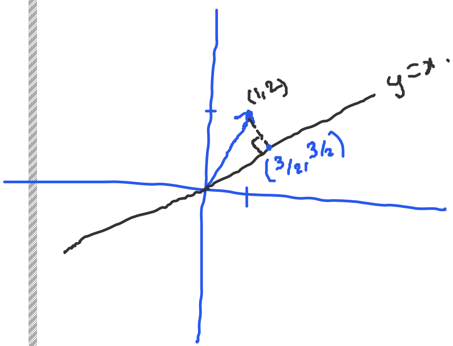
(Note that $\|v_i\| = 1$ ∀ i and hence we are not dividing by its norm.)

Observe that $\{v_1, v_2, \ldots, v_n\}$ is a randomly chosen orthonormal basis for W and hence $proj_W(v)$ is independent of the chosen orthonormal basis. The projection of a vector onto a subspace does not change with the choice of a basis (as expected!).

The projection of a vector v onto a subspace W is the nearest vector $v' \in W$, that is v' satisfies $\|v - v'\| \leq \|v - w\|$ for all $w \in W$.

Question: What is $proj_W(v)$ if $v \in W$?

Example: Let $V = R^2$ and W be the line y = x. The projection of (1, 2) onto W is



find the proj of $\overset{b}{(1,2)}$ onto $\overset{a \to \text{ any pt on } W}{(1,1)}$

$$v = \frac{\langle a, b\rangle}{\langle a, a\rangle} a.$$

$$= \frac{1+2}{2} \cdot (1,1)$$

$$= \left(\frac{3}{2}, \frac{3}{2}\right)$$

Exercise 1: Let $V = R^3$ and $W = \{(x, y, z) : x+y = 0\}$.

a) What is the projection of (2, –2, 0)?

b) What is the projection of (2, 1, 3)?

Basis for W

.

Exercise 2: Let $v_1 = (1, 0, 1, 1)$ and $v_2 = (0, 1, 1, 1)$ be the vectors from the inner product space R^4 with respect to the dot product. If $v_3 = v_2 + av_1$ where $a \in R$ and v_1, v_3 are orthogonal, then

○ Option 1: $a = -2/3$

○ Option 2: $a = 2/3$

○ Option 3: $a = 1/3$

○ Option 4: $a = -1/3$

Projection as a linear transformation

Let V be an inner product space and W be a subspace.

Define $T : V \to V$ defined by $T(v) = \text{proj}_W(v)$ for all $v \in V$.

- T is a linear transformation.

This linear transformation is called the **projection map** from V to W and is denoted by P_W.

- What is the range of T ?

- What is the rank of T ?

- What is the kernel of T ?

Note:

1) $P_W^2 = P_W$

2) $||P_W(v)|| \leq ||v||$

The projection of a vector cannot be longer than the vector itself.

Exercise 1: Consider the inner product

$\langle a, b \rangle = a_1b_1 - a_1b_2 - a_2b_1 + 4a_2b_2$

where $a = (a_1, a_2)$, $b = (b_1, b_2)$ are vectors in R^2.

a. Let x = (1, 2). Find the projection of x in the direction of (3, 4) using the inner product defined above.

b. Let x = (1, 2), find the projection of x in a direction perpendicular to (3, 4).

Exercise 2: Consider an orthogonal basis {(1, 2, 1), (−2, 0, 2)} of a subspace W , of the inner product space R^3 with respect to the dot product. If y = (1, 2, 3) ∈ R^3, then find $Proj_W(y)$.

Exercise 3: Suppose W_1 and W_2 are subspaces of a vector space V . Let P_{W1} and P_{W2} denote the projection from V to W_1 and V to W_2 respectively and consider the following statements:

- Statement P: If $P_{W1} + P_{W2}$ is a projection from V to $W_1 + W_2$, then $P_{W1} \circ P_{W2} + P_{W2} \circ P_{W1} = 0$.
- Statement Q: $P^2_{W1} := P_{W1} \circ P_{W1}$ is not a projection from V to W_1.
- Statement R: If A is the matrix representation of P_{W2}, then A cannot be a symmetric matrix.
- Statement S: $P_{W1} - P_{W2}$ is a projection from V to $W_1 + W_2$.

Find the number of correct statements