* Linear transformations.

$$T: \mathbb{R}^n \to \mathbb{R}^m$$

A function from $\mathbb{R}^n$ to $\mathbb{R}^m$ for which the output depends linearly on the input.

Eg. Cost of an object

$$P: \mathbb{R} \to \mathbb{R}.$$

$$q \mapsto P(q).$$

$$1 \, kg \mapsto 100 ₹$$

$$2 \, kg \mapsto 200 ₹$$

$$\underline{3.47 \, kg} \mapsto (3.47) \, 100$$
$$347.$$

$$\boxed{P(q_0) = 100 q_0}$$

$$P: \mathbb{R}^2 \to \mathbb{R}.$$

1 kg of sugar $\mapsto$ 50

1 kg of rice $\mapsto$ 60

$$P(x,y) = 50x + 60y.$$

$$\boxed{\begin{array}{l} i) \; P((x_1, y_1) + (x_2, y_2)) = \\ \quad P(x_1, y_1) + P(x_2, y_2) \\ ii) \; P(c(x,y)) = c P(x,y) \end{array}}$$

What does (2kg of $\underline{sugar}$ and 3kg of rice) cost? (280)

$$2(50) + 3(60).$$

In general, x of sugar and y of rice, the cost is

$$x(50) + y(60).$$

Transaction 1: Cost of 2kg sugar, 1kg rice is P_1

Transaction 2: Cost of 1kg sugar, 2kg rice is P_2.

Cost of 3kg sugar, 3kg rice: $P_1 + P_2$.

* Using P_1 and P_2 the cost of x kg sugar and y kg rice can be calculated (for any x, y).

Eg. 3kg sugar, 2kg rice.

Sugar	Rice
50	60

Eg. Same setup as above but delivery fee — 20.

$$\widetilde{P}(x, y) = 50x + 60y + 20.$$

$\widetilde{P}(1, 1) = 130$ $\widetilde{P}(1, 2) = 190$

$\widetilde{P}(2, 2) = 240$ $\widetilde{P}(1, 1) + \widetilde{P}(0, 1) = 210$.

$\widetilde{P}$ is not a linear map. (This is an affine map)

(If $P(c(x, y)) \neq c\, P(x, y)$ for some $x, y \in \mathbb{R}^2$, $c \in \mathbb{R}$, then P is not a linear)

Eg. $A: \mathbb{R} \to \mathbb{R}$.

$A(x)$ is the area of the square with side $|x|$.

$A(1) = 1$ $A(2) = 4$.

$A(2) \neq 2 A(1)$. $\therefore$ A is not a linear map.

$A(x) = x^2$.

Eg/Ex. Compound interest

* A map $T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is said to be a linear map if the following two conditions hold:

a) $T\left((x_1, \ldots, x_n) + (y_1, \ldots, y_n)\right) = T(x_1, \ldots, x_n) + T(y_1, \ldots, y_n).$

b) $T\left(c(x_1, \ldots, x_n)\right) = c\, T(x_1, \ldots, x_n).$

$\forall\ (x_1, \ldots, x_n),$
$(y_1, \ldots, y_n) \in \mathbb{R}^n$
$\&\ c \in \mathbb{R}.$

Eg. $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$(x, y) \longmapsto (x+y, x-y).$

T is a linear transformation.

a) $T\left((x_1, y_1) + (x_2, y_2)\right) = \underline{T(x_1+x_2,\ y_1+y_2)}$

$= (x_1 + x_2 + y_1 + y_2,\ x_1 + x_2 - y_1 - y_2)$
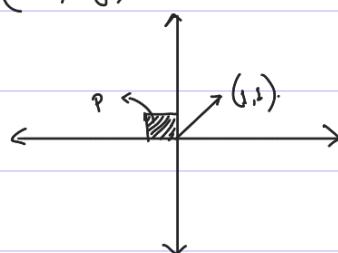
$\underline{T(x_1, y_1) + T(x_2, y_2)} = (x_1 + y_1,\ x_1 - y_1) + (x_2 + y_2,\ x_2 - y_2)$

$= (x_1 + y_1 + x_2 + y_2,\ x_1 + x_2 - y_1 - y_2).$

b) $\underline{T(c(x_1, y_1))} = T(cx_1, cy_1) = (cx_1 + cy_1,\ cx_1 - cy_1)$

$\underleftarrow{c\, T(x_1, y_1)} = c(x_1 + y_1,\ x_1 - y_1) = (cx_1 + cy_1,\ cx_1 - cy_1).$
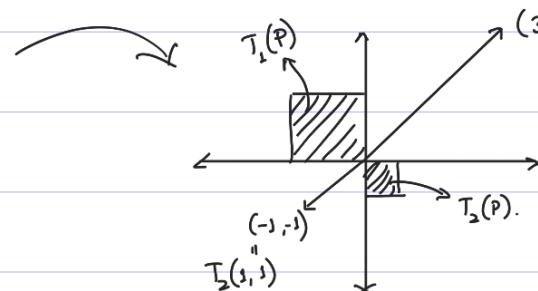
Eg. $T_1: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$
Ex. T_1 is a linear transformation

$(x, y) \longmapsto (3x, 3y).$

Domain

Co-domain
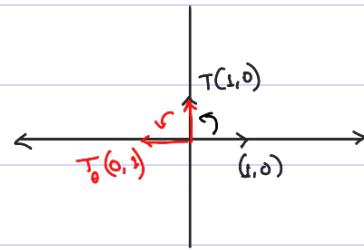
$T_2(x, y) = (-x, -y).$

Eg. $T_\theta: \mathbb{R}^2 \to \mathbb{R}^2$.

  $(x,y) \mapsto (x\cos\theta - y\sin\theta, \quad x\sin\theta + y\cos\theta)$ ✓

  $(1,0) \mapsto (\cos\theta, \sin\theta)$

  $(0,1) \mapsto (-\sin\theta, \cos\theta)$.

  $\theta = \pi/2 \qquad (1,0) \mapsto (0,1)$

T_θ is a linear map which rotates every vector in the plane counterclockwise by θ.

Ex. $T: \mathbb{R}^2 \to \mathbb{R}^2 \quad (x,y) \mapsto (y,x)$. Show that T is a l.m and describe it geometrically.

* Given vector spaces V and W, a linear transformation from V to W is a map $T: V \to W$ such that

a) $T(v_1 + v_2) = T(v_1) + T(v_2)$

b) $T(cv) = cT(v)$

$\forall \ v_1, v_2, v \in V, \ c \in \mathbb{R}$.

Remark: If $T: V \to W$ is a l.t. , then $T(0_v) = 0_w$.

a) $T(0_v + 0_v) = T(0_v) + T(0_v)$

  $0_w + T(0_v) = T(0_v) + T(0_v)$

  $T(0_v) = 0_w$.

$$T : V \to W$$

* **Injective (one-to-one) l.t.:** A l.t. which is injective, i.e.,

$$T(v) = T(v') \Rightarrow v = v'.$$

(No vector in W has more than one pre-image)

Result: A l.t. $T : V \to W$ is injective if and only if

$$T(v) = 0 \Rightarrow v = 0. \quad (*)$$

$$T(v_1) = T(v_2) \Rightarrow T(v_1) - T(v_2) = 0$$
$$\Rightarrow T(v_1) + T(-v_2) = 0$$
$$\Rightarrow T(v_1 + (-v_2)) = 0.$$
$$\overset{(*)}{\Rightarrow} v_1 + (-v_2) = 0 \Rightarrow v_1 = v_2$$

Eg. $T_1 : \mathbb{R}^2 \to \mathbb{R}^2$

$$T_1(x,y) = (x+y, x-y)$$

Solve: $T_1(x,y) = (0,0)$

$$(x+y, x-y) = (0,0)$$

$$\left. \begin{array}{r} x+y = 0 \\ x-y = 0 \end{array} \right\} \begin{array}{l} (x,y) = (0,0) \\ \text{is the} \\ \text{unique solution} \end{array}$$

$$\Rightarrow (x,y) = (0,0)$$

$$\Rightarrow T_1 \text{ is injective}$$

$T_2 : \mathbb{R}^2 \to \mathbb{R}^2$

$$(x,y) \mapsto (x+y, 2x+2y).$$

Find (x,y) ($\in$ Domain) s.t

$$T_2(x,y) = (0,0).$$

$$(x+y, 2x+2y) = (0,0)$$

$$\left. \begin{array}{r} x+y = 0 \\ 2x+2y = 0. \end{array} \right\} \text{Infinitely many solutions.}$$

In particular, $T_2(\underline{1,-1}) = (0,0)$

$$\Rightarrow T_2 \text{ is not injective.}$$

* Surjective l.t: A l.t. $T: V \to W$ s.t $\forall \omega \in W \; \exists \, v \in V$ s.t $T(v) = \omega$.

* A bijective l.t. is called a linear isomorphism.

* A linear transformation $T: V \to W$ is completely determined by the images of a basis of V.

Suppose $\{v_1, \ldots v_n\}$ is a basis of V and we know $T(v_1), \ldots, T(v_n)$. Then, $T(v)$ can be calculated for any $v \in V$.

Let $v \in V$.

$$v = a_1 v_1 + a_2 v_2 + \cdots + a_n v_n. \quad \text{for some } a_i \in \mathbb{R}.$$

$$T(v) = T(a_1 v_1 + \cdots + a_n v_n)$$

$$\overset{(a)}{=} T(a_1 v_1) + \cdots + T(a_n v_n).$$

$$\overset{(b)}{=} a_1 T(v_1) + \cdots + a_n T(v_n).$$

Eg. $P(2,1) = P_1 \;,\; P(1,2) = P_2.$

$P(3,3) = P_1 + P_2$

Calculate the cost of 2kg of sugar and 3kg of rice.

Note that $\{(2,1), (1,2)\}$ is a basis of $\mathbb{R}^2$

$$\underline{\underline{(2,3)}} = a(2,1) + b(1,2)$$

$2a + b = 2 \quad (4a + 2b = 4).$

$a + 2b = 3$

$a = \frac{1}{3} \quad b = \frac{4}{3}$

$$(2,3) = \frac{1}{3}(2,1) + \frac{4}{3}(1,2)$$

$$P(2,3) = \frac{1}{3}P(2,1) + \frac{4}{3}P(1,2).$$

$$= \frac{P_1 + 4P_2}{3}.$$

Eg. $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$$T(1,1) = (3,1) \quad , \quad T(1,0) = (4,2), \quad T(1,-1) = (x,y).$$

Find x & y.

Basis: $\{(1,1), (1,0)\}$.

$$\left(x=3, \; y=3\right)$$

$$\left(x=5, \; y=2\right)$$

$$(1,-1) = a(1,1) + b(1,0)$$

$$a+b = 1$$

$$a = -1.$$

$$a = -1, \; b = 2.$$

$$T(1,-1) = (-1)(3,1) + 2(4,2)$$

$$= (5,3).$$

$$T(1,-1) = T\left(a(1,1) + b(1,0)\right)$$

$$= T\left(a(1,1)\right) + T\left(b(1,0)\right)$$

$$= a\,T(1,1) + b\,T(1,0).$$

Ex. Prove that $T(x,y) = (4x-y, \; 2x-y)$.

Nullity of a matrix:

$\{\underset{\sim}{x} \in \mathbb{R}^n \mid A\underset{\sim}{x} = \underset{\sim}{0}\}$ is a vector subspace, called nullspace of A, denoted $N(A)$.

dim $(N(A))$ is called the nullity of A.

$$\begin{pmatrix} 1 & 3 & 4 & 1 \\ 1 & 2 & 3 & 2 \\ 1 & 1 & 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 3 & 4 & 1 \\ 0 & -1 & -1 & 1 \\ 0 & -2 & -4 & 0 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & 3 & 4 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 1 & 2 & 0 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} \boxed{1} & 3 & 4 & 1 \\ 0 & \boxed{1} & 1 & -1 \\ 0 & 0 & \boxed{1} & 1 \end{pmatrix}$$

4 variables

3 dep.   (1 ind)

$$\begin{pmatrix} 1 & 3 & 4 & 1 & \bigm| & 0 \\ 0 & 1 & 1 & -1 & \bigm| & 0 \\ 0 & 0 & 1 & 1 & \bigm| & 0 \end{pmatrix}$$

$\boxed{x_4 = t}$

$x_3 + x_4 = 0 \Rightarrow \boxed{x_3 = -t}$

$x_2 + x_3 - x_4 = 0 \Rightarrow \boxed{x_2 = 2t}$

$x_1 + 3x_2 + 4x_3 + x_4 = 0 \Rightarrow \boxed{x_1 = -3t}$

$N(A) = \{(-3t, 2t, -t, t) \mid t \in \mathbb{R}\}$

$= \text{Span}\{(-3, 2, -1, 1)\}$.

Nullity = 1.

Ex: $\begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ What is the nullity and nullspace of this matrix?

Quiz 1:   (5×7) $\max(\text{rank}) - \min(\text{rank})$.

(2×3)

$$\begin{pmatrix} \circ & \circ & \circ \\ \circ & \circ & \circ \end{pmatrix}$$