Operations on Relations.

Cartesian Product.

The Cartesian Product of two sets A and B denoted by $\underline{A \times B}$ is the set of ordered pairs such that the first element in the pair belongs to A and the second element belongs to B.

$$A \times B = \{(a, b) \mid a \in A,\ b \in B\}$$

Note: If $A \neq B$ and A and B are non-empty then $\boxed{A \times B \neq B \times A}$

Ques: Given

$A1 = \{a, b\}$

$A2 = \{1, 2\}$

$A3 = \{\alpha, \beta, \gamma\}$

(i) $A1 \times A2$, (ii) $(A1 \times A2) \times A3$

$A1 \times A2 =$

$$\{(a,1),\ (a,2),\ (b,1),\ (b,2)\}$$

$(A1 \times A2) \times A3$

$$= \{(a,1,\alpha),\ (a,2,\alpha),\ (b,1,\alpha),\ (b,2,\alpha),$$
$$(a,1,\beta),\ (a,2,\beta),\ (b,1,\beta),\ (b,2,\beta),$$
$$(a,1,\gamma),\ (a,2,\gamma),\ (b,1,\gamma),\ (b,2,\gamma)\}$$

Operation on Relations

Given two relations R and S defined on X × Y and represented by relation matrices, the following operations are supported by R and S

1) Union 2) Intersection 3) Complement

4) Composition

Union $\quad R \cup S$
$= \max(R(x,y), S(x,y))$

Intersection $\quad R \cap S$
$= \min(R(x,y), S(x,y))$

Complement $\quad \bar{R}(x,y) = 1 - R(x,y)$

Composition . $\underset{x,y}{R} \circ \underset{y,z}{S}$.

[4. Crisp R/n.
$\downarrow$
Classical set Theory].

Given
R to be relation on X, Y
S to be relation on Y, Z
$R \circ S$ to be relation on X, Z

defined as,

$$R \circ S = \{(x,z) \mid (x,z) \in X \times Z, \exists\, y \in Y \text{ such that } (x,y) \in R \text{ and } (y,z) \in S\}$$

max-min composition

$T = R \circ S$.

$$\boxed{T(x,z) = \max_{y \in Y}(\min(R(x,y), S(y,z)))}$$

Q:
Let R, S be defined on the sets

$$\underline{\{\boxed{1}\ 3, 5\}}_{X} \times \underline{\{\boxed{1}, 3, 5\}}_{Y}$$

$R = \{(x, y) \mid y = x+2\}$, $S = \{(x, y) \mid x < y\}$

$R = \{(1, 3), (3, 5)\}$

$S = \{(1, 3), (1, 5), (3, 5)\}$

Cartesian Product.

$\{(1, 1), (1, 3), (1, 5), (3, 1), (3, 3),$
$(3, 5), (5, 1), (5, 3), (5, 5)\}$

Relation Matrices. $R = X \times Y$

$$\begin{array}{c|c} & \begin{matrix} 1 & 3 & 5 \end{matrix} \\ \begin{matrix} 1 \\ 3 \\ 5 \end{matrix} & \begin{bmatrix} 0 & \boxed{1} & 0 \\ 0 & 0 & \boxed{1} \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

Max-Min Composition.

fuzzy cartesian product.

Let $\tilde{A}$ be a fuzzy set defined on the universe X

$\tilde{B}$ be a fuzzy set defined on the universe Y

The Cartesian product between the fuzzy sets $\tilde{A}$ and $\tilde{B}$ indicated as $\tilde{A} \times \tilde{B}$, resulting in the fuzzy relation $\tilde{R}$ is given by

$$\tilde{R} = \tilde{A} \times \tilde{B} \subset X \times Y$$

where R has its membership function

$$\mu_{\tilde{R}}(x, y) = \mu_{\tilde{A} \times \tilde{B}}(x, y)$$

$$= \min(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(y))$$

Que: Let $\tilde{A} = \{(x_1, 0.2), (x_2, 0.7), (x_3, 0.4)\}$ and $\tilde{B} = \{(y_1, 0.5), (y_2, 0.6)\}$ be two fuzzy sets defined on the discourse $X = \{x_1, x_2, x_3\}$ and $Y = \{y_1, y_2\}$ respectively. Then, the fuzzy relation $\tilde{R}$ is given by the cartesian product $\tilde{A} \times \tilde{B}$

$\tilde{R}(x_1, y_1)$

$\tilde{A} \times \tilde{B} = \{ (x_1, y_1), (x_1, y_2), (x_2, y_1), (x_2, y^2), (x_3, y_1), (x_3, y_2) \}$

$\tilde{A} \times \tilde{B} = \min(\mu_{\tilde{A}}(x_1), \mu_{\tilde{B}}(y_1))$

$\tilde{R}(x_1, y_1) = \min(0.2, 0.5)$

$= 0.2$

$$\tilde{R} = \begin{array}{c c} & \begin{array}{cc} y_1 & y_2 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} & \begin{bmatrix} 0.2 & 0.2 \\ 0.5 & 0.6 \\ 0.4 & 0.4 \end{bmatrix} \end{array}$$

$\tilde{R}(x_1, y_2) = \min(\mu_{\tilde{A}}(x_1), \mu_{\tilde{B}}(y_2))$

$= \min(0.2, 0.6)$

$= 0.2$

$\tilde{R}(x_2, y_1) = \min(\mu_{\tilde{A}}(x_2), \mu_{\tilde{B}}(y_1))$

$= \min(0.7, 0.5) = 0.5$

$\tilde{R}(x_2, y_2) = \min(\mu_{\tilde{A}}(x_2), \mu_{\tilde{B}}(y_2))$

$= \min(0.7, 0.6)$

$= 0.6$

$\tilde{R}(x_3, y_1) = \min(\mu_{\tilde{A}}(x_3), \mu_{\tilde{B}}(y_1))$

$= \min(0.4, 0.5) = 0.4$

$\tilde{R}(x_3, y_2) = \min(\mu_{\tilde{A}}(x_3), \mu_{\tilde{B}}(y_2))$

$= \min(0.4, 0.6) = 0.4$

$\underline{X} = \{x_1, x_2, x_3\}$

$\underline{Y} = \{y_1, y_2\}$

$Z = \{z_1, z_2, z_3\}$

Let $\tilde{R}$ be a fuzzy relation.

$$\begin{array}{c|cc} & y_1 & y_2 \\ \hline x_1 & \boxed{0.5} & 0.1 \\ x_2 & 0.2 & 0.9 \\ x_3 & 0.8 & 0.6 \end{array}$$

y_3

Let $\tilde{S}$ be a fuzzy relation

$$\begin{array}{c|ccc} & z_1 & z_2 & z_3 \\ \hline y_1 & \boxed{0.6} & 0.4 & 0.7 \\ y_2 & \boxed{0.5} & 0.8 & 0.9 \end{array}$$

y_3

$$\underset{(x,y)}{\tilde{R}} \circ \underset{(y,z)}{\tilde{S}} (x_1, z_1) = \max\left(\min(0.5, 0.6), \min(0.1, 0.5)\right)$$

$$= \max(0.5, 0.1)$$

$$= 0.5$$

(x, z)

$\boxed{(x_1, z_1)} = (x_1, y_1), (y_1, z_1) \rightarrow (x_1, z_1)$

$(x_1, y_2), (y_2, z_1) = (x_1, z_1)$

Let $\tilde{R}$ & $\tilde{S}$ be fuzzy relations on $X \times Y$

Union:

$$\mu_{\tilde{R} \cup \tilde{S}} = \max \left(\mu_{\tilde{R}}(x,y), \mu_{\tilde{S}}(x,y) \right)$$

Intersection

$$\mu_{\tilde{R} \cap \tilde{S}} = \min \left(\mu_{\tilde{R}}(x,y), \mu_{\tilde{S}}(x,y) \right)$$

Complement:

$$\mu_{\tilde{R}^c} = 1 - \mu_{\tilde{R}}(x,y)$$

Composition of relation:

$$\mu_{\tilde{R} \circ \tilde{S}}(x,z) = \max_{y \in Y} \left(\min \mu_{\tilde{R}}(x,y), \mu_{\tilde{S}}(y,z) \right)$$

$\tilde{R}\circ\tilde{S}\,(x_1, z_1) = \max(\min(0.3, [illegible]), \min([illegible]))$

$= \max([illegible])$

$= 0.4$

$\tilde{R}\circ\tilde{S}\,(x_1, z_2) = \max(\min([illegible]), \min([illegible]))$

$= \max([illegible])$

$\tilde{R}\circ\tilde{S}\,(x_2, z_1) = \max(\min([illegible]), \min([illegible]))$

$= \max(0 [illegible])$

$\tilde{R}\circ\tilde{S}\,(x_2, z_2) = \max(\min(0.2, [illegible]), \min([illegible]))$

$= \max(0.2, 0.8) = 0.8$

$\tilde{R}\circ\tilde{S}\,(x_1, z_3) = \max(\min(0.2, [illegible]), \min([illegible]))$

$= \max(0.2, 0.9) = 0.9$

$\tilde{R}\circ\tilde{S}\,(x_3, z_1) = \max(\min(0.8, 0.6), \min([illegible]))$

$= \max(0.6, 0.5) = 0.6$

$\tilde{R}\circ\tilde{S}\,(x_3, z_2) = \max(\min(0.8, 0.4), \min(0.6, 0.3))$

$= \max(0.4, 0.6) = 0.6$

$\tilde{R}\circ\tilde{S}\,(x_3, z_3) = \max(\min(0.8, 0.1), \min([illegible]))$

$= \max(0.1, 0.6) = 0.1$

$$\tilde{R} \circ \tilde{S} = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \\ \begin{bmatrix} 0.5 & 0.4 & 0.5 \\ 0.5 & 0.3 & 0.4 \\ 0.6 & 0.6 & 0.7 \end{bmatrix} \end{array}$$