Basic Fuzzy Set Operations
($\underset{=}{\sim}$)

Given X to be the universe of discourse and, $\tilde{A}$ and $\tilde{B}$ to be the fuzzy sets with $\mu_{\tilde{A}}(x)$ and $\mu_{\tilde{B}}(x)$ as their respective membership functions, the basic fuzzy set operations that can be performed.

1) Union
2) Intersection
3) Complement
4) Product
5) Difference
6) Disjunctive Sum
7) Power of a fuzzy set
8) Equality

1) <u>Union</u> :→ The union of two fuzzy set $\tilde{A}$ and $\tilde{B}$ is a new fuzzy set $\tilde{A} \cup \tilde{B}$ also on X with a membership function defined as: $\mu_{\tilde{A} \cup \tilde{B}}(x) = \max\left(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\right)$

Q Given set

$A = \{(x_1, 0.5), (x_2, 0.7), (x_3, 0), (x_4, 0.6)\}$

$B = \{(x_1, 0.8), (x_2, 0.5), (x_3, 0.1), (x_4, 0.3)\}$

$\tilde{A} \cup \tilde{B}$

$\mu_{\tilde{A}\cup\tilde{B}}(x_1) = \max(\boxed{\mu_{\tilde{A}}(x_1)}, \mu_{\tilde{B}}(x_1))$
$= \max(0.5, 0.8) = 0.8$

$\mu_{\tilde{A}\cup\tilde{B}}(x_2) = \max(\mu_{\tilde{A}}(x_2), \mu_{\tilde{B}}(x_2))$
$= \max(0.7, 0.5) = 0.7$

$\mu_{\tilde{A}\cup\tilde{B}}(x_3) = \max(\mu_{\tilde{A}}(x_3), \mu_{\tilde{B}}(x_3)) = \max(0, 0.1) = 0.1$

$\mu_{\tilde{A}\cup\tilde{B}}(x_4) = \max(\mu_{\tilde{A}}(x_4), \mu_{\tilde{B}}(x_4)) = \max(0.6, 0.3) = 0.6$

$$\boxed{\tilde{A} \cup \tilde{B} = \{(x_1, 0.8), (x_2, 0.7), (x_3, 0.1), (x_4, 0.6)\}}$$

2) Intersection :- The intersection of fuzzy sets $\tilde{A}$ and $\tilde{B}$ is a new fuzzy set $\tilde{A} \cap \tilde{B}$ with a membership function defined as

$$\mu_{\tilde{A}\cap\tilde{B}}(x) = \min(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x))$$

$$\boxed{\tilde{A} \cap \tilde{B} = ?}$$

$$\mu_{\tilde{A}\wedge\tilde{B}}(x_1) = \min(\mu_{\tilde{A}}(x_1), \mu_{\tilde{B}}(x_1))$$
$$= \min(0.5, 0.8)$$
$$= 0.5$$

$$\mu_{\tilde{A}\wedge\tilde{B}}(x_2) = \min(\mu_{\tilde{A}}(x_2), \mu_{\tilde{B}}(x_2))$$
$$= \min(0.7, 0.5)$$
$$= 0.5$$

$$\mu_{\tilde{A}\wedge\tilde{B}}(x_3) = \min(\mu_{\tilde{A}}(x_3), \mu_{\tilde{B}}(x_3))$$
$$= \min(0, 0.1)$$
$$= 0.$$

$$\mu_{\tilde{A}\wedge\tilde{B}}(x_4) = \min(\mu_{\tilde{A}}(x_4), \mu_{\tilde{B}}(x_4))$$
$$= \min(0.6, 0.3)$$
$$= 0.3$$

| $A, B \rightarrow$ crisp set
| $\tilde{A}, \tilde{B} \rightarrow$ fuzzy set

$$\tilde{A} \cap \tilde{B} = \{(x_1, 0.5), (x_2, 0.5), (x_3, 0), (x_4, 0.3)\}.$$

3.) <u>Complement</u> :→ The complement of a fuzzy set $\tilde{A}$ is a new fuzzy set $\boxed{\tilde{A}^c}$ with a membership function

$$\boxed{\mu_{\tilde{A}^c}(x) = 1 - \mu_{\tilde{A}}(x)}$$

$\boxed{\tilde{A}^c = ?}$

$\tilde{A}^c = \{(x_1, 0.5), (x_2, 0.3), (x_3, 1), (x_4, 0.4)\}$

$\tilde{B}^c = \{(x_1, 0.2), (x_2, 0.5), (x_3, 0.9), (x_4, 0.7)\}$

4) **Product:** case 1: product of two fuzzy set
case 2: product of a fuzzy set with a crisp number.

Case 1: Product of two fuzzy set

The product of two fuzzy sets $\tilde{A}$ and $\tilde{B}$ is a new fuzzy set $(\tilde{A} \cdot \tilde{B})$ whose membership function is defined as

$$\mu_{\tilde{A}\cdot\tilde{B}}(x) = \mu_{\tilde{A}}(x) \cdot \mu_{\tilde{B}}(x)$$

$$\mu_{\tilde{A}\cdot\tilde{B}}(x_1) = \mu_{\tilde{A}}(x_1) \cdot \mu_{\tilde{B}}(x_1) = 0.5 \times 0.8 = 0.40$$

$$\mu_{\tilde{A}\cdot\tilde{B}}(x_2) = \mu_{\tilde{A}}(x_2) \cdot \mu_{\tilde{B}}(x_2) = 0.7 \times 0.5 = 0.35$$

$$\mu_{\tilde{A}\cdot\tilde{B}}(x_3) = \mu_{\tilde{A}}(x_3) \cdot \mu_{\tilde{B}}(x_3) = 0 \times 0.1 = 0$$

$$\mu_{\tilde{A}\cdot\tilde{B}}(x_4) = \mu_{\tilde{A}}(x_4) \cdot \mu_{\tilde{B}}(x_4) = 0.6 \times 0.3 = 0.18$$

$\tilde{A} \cdot \tilde{B} = \{(x_1, 0.40), (x_2, 0.35), (x_3, 0), (x_4, 0.18)\}$

case 2: product of a fuzzy set with a crisp no.

Multiplying a fuzzy set $\tilde{A}$ by a crisp number 'a' gives a new fuzzy set $\boxed{a \cdot \tilde{A}}$ with the membership function

$$\mu_{a \cdot \tilde{A}} = a \cdot \mu_{\tilde{A}}(x)$$

Given

$a = 0.3$,

$a \cdot \tilde{A} = \{(x_1, 0.15), (x_2, 0.21), (x_3, 0), (x_4, 0.27)\}$

$a \cdot \tilde{B} = \{(x_1, 0.24), (x_2, 0.15), (x_3, 0.3), (x_4, 0.21)\}$

5) Equality

Two fuzzy sets $\tilde{A}$ and $\tilde{B}$ are said to be equal $(\tilde{A} = \tilde{B})$ when $\boxed{\mu_{\tilde{A}}(x) = \mu_{\tilde{B}}(x)}$

example:

$$\tilde{A} = \{(x_1, 0.4), (x_2, 0.8), (x_3, 0.6)\}$$

$$\tilde{B} = \{(x_1, 0.4), (x_2, 0.8), (x_3, 0.59)\}$$

$$\tilde{C} = \{(x_1, 0.4), (x_2, 0.8), (x_3, 0.6)\}$$

$\tilde{A} = \tilde{B}$

$$\left.\begin{array}{l}\mu_{\tilde{A}}(x_1) = 0.4 \\ \mu_{\tilde{B}}(x_1) = 0.4\end{array}\right\} \checkmark$$

$$\mu_{\tilde{A}}(x_2) = 0.8 = \mu_{\tilde{B}}(x_2) \checkmark$$

$$\left.\begin{array}{l}\mu_{\tilde{A}}(x_3) = 0.6 \\ \mu_{\tilde{B}}(x_3) = 0.59\end{array}\right\} \neq$$

$$\boxed{\tilde{A} \neq \tilde{B}}$$

$\tilde{A} = \tilde{C}$

$$\mu_{\tilde{A}}(x_1) = 0.4 = \mu_{\tilde{C}}(x_1)$$

$$\mu_{\tilde{A}}(x_2) = 0.8 = \mu_{\tilde{C}}(x_2)$$

$$\mu_{\tilde{A}}(x_3) = 0.6 = \mu_{\tilde{C}}(x_3)$$

$$\boxed{\tilde{A} = \tilde{C}}$$

6.) Power of a fuzzy set

The α power of a fuzzy set $\tilde{A}$ is a new fuzzy set $\tilde{A}^{\alpha}$ whose membership function is given by

$$\boxed{\mu_{\tilde{A}^{\alpha}}(x) = \left(\mu_{\tilde{A}}(x)\right)^{\alpha}}$$

Concentration of fuzzy Set (CON) = Raising a fuzzy set to its second power.

Dilation of fuzzy Set (DIL) = Taking the square root of a fuzzy set.

eg: $\tilde{A} = \{(x_1, 0.4), (x_2, 0.2), (x_3, 0.7)\}$

$\boxed{\alpha = 3}$

$$\mu_{\tilde{A}^3}(x) = \left[\mu_{\tilde{A}}(x)\right]^3$$

$$\mu_{\tilde{A}^3}(x_1) = (0.4)^3 = 0.064$$

$$\mu_{\tilde{A}^3}(x_2) = (0.2)^3 = 0.008$$

$$\mu_{\tilde{A}^3}(x_3) = (0.7)^3 = 0.343$$

$$\boxed{\tilde{A}^3 = \{(x_1, 0.064), (x_2, 0.008), (x_3, 0.343)\}}$$

7) Difference

The difference of two fuzzy sets $\tilde{A}$ and $\tilde{B}$ is a new fuzzy set $\tilde{A}-\tilde{B}$, is defined as

$$\boxed{\tilde{A}-\tilde{B} = (\tilde{A}\cap\tilde{B}^c)}$$

Que: $\tilde{A} = \{(x_1, \underline{0.2}), (x_2, \underline{0.5}), (x_3, \underline{0.6})\}$

$\tilde{B} = \{(x_1, 0.1), (x_2, 0.4), (x_3, 0.5)\}$

$$\boxed{\tilde{B}^c = 1-\mu_{\tilde{B}}(x)}$$

$\tilde{B}^c = \{(x_1, \underline{0.9}), (x_2, \underline{0.6}), (x_3, \underline{0.5})\}$

$\tilde{A}\cap\tilde{B}^c = \min(\mu_{\tilde{A}}(x), \mu_{\tilde{B}^c}(x))$

$$\boxed{= \{(x_1, 0.2), (x_2, 0.5), (x_3, 0.5)\}}$$

8) Disjunctive Sum

The disjunctive sum of two fuzzy sets $\tilde{A}$ and $\tilde{B}$ is a new fuzzy set $\tilde{A}\oplus\tilde{B}$ defined as

$$\boxed{\tilde{A}\oplus\tilde{B} = \underbrace{\underbrace{(\tilde{A}^c\cap\tilde{B})}_{1} \cup \underbrace{(\tilde{A}\cap\tilde{B}^c)}_{2}}_{3}}$$

$$\tilde{A}^c = \{(x_1, 0.8), (x_2, 0.5), (x_3, 0.4)\}$$

$$\tilde{A}^c \cap \tilde{B} = \min(\mu_{\tilde{A}^c}(x), \mu_{\tilde{B}}(x))$$

$$\boxed{\tilde{A}^c \cap \tilde{B} = \{(x_1, 0.1), (x_2, 0.4), (x_3, 0.4)\}}$$

$$\tilde{A} \oplus \tilde{B} = \max\left(\mu_{\tilde{A}^c \cap B}(x), \mu_{\tilde{A} \cap B^c}(x)\right)$$

$$\tilde{A} \oplus \tilde{B} = \{(x_1, 0.2), (x_2, 0.5), (x_3, 0.5)\}$$

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α- cut:

The α-cut of a fuzzy set $\tilde{A}$ is denoted by A_α is a set consisting of those elements of X whose membership values exceed the threshold value α.

$$A_\alpha = \{x \mid \mu_{\tilde{A}}(x) \geq \alpha\}$$

eg:- $\tilde{A} = \{(g_1, \cancel{0.4}), (g_2, 0.5), (g_3 = \cancel{0.2}), (g_4 = 0.9), (g_5 = 0.6)\}$

$\boxed{\alpha = 0.5}$ $A_{0.5} = \{g_2, g_4, g_5\}$.

$\boxed{\alpha = 0.2}$ $= \{g_1, g_2, g_3, g_4, g_5\}$