1.

(a)

$$\nabla f(x) = \nabla(\frac{1}{2}x^T Ax + b^T x) = Ax + b$$

(b)

$$\frac{\partial g(h(x))}{\partial x_i} = \frac{\partial g(h(x))}{\partial h(x)} \frac{\partial h(x)}{\partial x_i} = g'(h(x))\frac{\partial h(x)}{\partial x_i}$$

$$\nabla f(x) = \nabla g(h(x)) = g'(h(x))\nabla h(x)$$

(c)

$$\begin{aligned}
\nabla^2 f(x) &= \begin{bmatrix} \frac{\partial \nabla f(x)}{\partial x_1} & \frac{\partial \nabla f(x)}{\partial x_2} & \dots & \frac{\partial \nabla f(x)}{\partial x_n} \end{bmatrix} \\
&= \begin{bmatrix} \frac{\partial \nabla (Ax+b)}{\partial x_1} & \frac{\partial \nabla (Ax+b)}{\partial x_2} & \dots & \frac{\partial \nabla (Ax+b)}{\partial x_n} \end{bmatrix} \\
&= \begin{bmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ A_{21} & A_{22} & \dots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \dots & A_{nn} \end{bmatrix} = A
\end{aligned}$$

(d)

$$\nabla f(x) = \nabla g(a^T x) = g'(a^T x)\nabla(a^T x) = g'(a^T x)a$$

$$\frac{\partial^2 g(h(x))}{\partial x_i \partial x_j} = \frac{\partial^2 g(h(x))}{\partial (h(x))^2} \frac{\partial h(x)}{\partial x_i} \frac{\partial h(x)}{\partial x_j} = g''(h(x))\frac{\partial h(x)}{\partial x_i} \frac{\partial h(x)}{\partial x_j}$$

$$\frac{\partial^2 g(a^T x)}{\partial x_i \partial x_j} = g''(a^T x)\frac{\partial (a^T x)}{\partial x_i} \frac{\partial (a^T x)}{\partial x_j} = g''(a^T x)a_i a_j$$

$$\nabla^2 f(x) = \nabla^2 g(a^T x) = g''(a^T x) \begin{bmatrix} a_1 a_1 & a_1 a_2 & \dots & a_1 a_n \\ a_2 a_1 & a_2 a_2 & \dots & a_2 a_n \\ \vdots & \vdots & \ddots & \vdots \\ a_n a_1 & a_n a_2 & \dots & a_n a_n \end{bmatrix} = g''(a^T x)aa^T$$

2.

(a)

$$A^T = (zz^T)^T = zz^T = A$$

$$x^T Ax = x^T zz^T x = x^T z(x^T z)^T = (x^T z)^2 \geq 0$$

(b)

$$Ax = zz^T x = z(z^T x) = 0$$

$$N(A) = \{x \in \mathbb{R}^n : z^T x = 0\}$$

$$R(A) = R(zz^T) \leq \min\{R(z), R(z^T)\} = 1$$

$$R(A) = 1$$

(c)

$$(BAB^T)^T = BA^T B^T = BAB^T$$

$$x^T BAB^T x = (x^T B)A(x^T B)^T \geq 0$$

3.

(a)

$$A = T\Lambda T^{-1}$$

$$AT = T\Lambda$$

$$A \begin{bmatrix} t^{(1)} & t^{(2)} & \dots & t^{(n)} \end{bmatrix} = \begin{bmatrix} t^{(1)} & t^{(2)} & \dots & t^{(n)} \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix}$$

$$\begin{bmatrix} At^{(1)} & At^{(2)} & \dots & At^{(n)} \end{bmatrix} = \begin{bmatrix} \lambda_1 t^{(1)} & \lambda_2 t^{(2)} & \dots & \lambda_n t^{(n)} \end{bmatrix}$$

$$At^{(i)} = \lambda_i t^{(i)}$$

(b)

$$A = U\Lambda U^T$$

$$AU = U\Lambda U^T U = U\Lambda$$

$$A \begin{bmatrix} u^{(1)} & u^{(2)} & \dots & u^{(n)} \end{bmatrix} = \begin{bmatrix} u^{(1)} & u^{(2)} & \dots & u^{(n)} \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix}$$

$$\begin{bmatrix} Au^{(1)} & Au^{(2)} & \dots & Au^{(n)} \end{bmatrix} = \begin{bmatrix} \lambda_1 u^{(1)} & \lambda_2 u^{(2)} & \dots & \lambda_n u^{(n)} \end{bmatrix}$$

$$Au^{(i)} = \lambda_i u^{(i)}$$

(c)

$$At^{(i)} = \lambda_i t^{(i)}$$

$$(t^{(i)})^T At^{(i)} = \lambda_i \|t^{(i)}\|_2 = \lambda_i \geq 0$$