CS 229, Fall 2018

Problem Set #3 Solutions: Deep Learning & Unsupervised learning

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1.

(a)

$$
\begin{aligned}
\frac{\partial l}{\partial w_{1,2}^{[1]}} &= \frac{\partial l}{\partial o} \cdot \frac{\partial o}{\partial h_2} \cdot \frac{\partial h_2}{\partial w_{1,2}^{[1]}} \\
&= \frac{2}{m} \sum_{i=1}^{m} (o^{(i)} - y^{(i)}) \cdot o^{(i)} (1 - o^{(i)}) w_2^{[2]} \cdot h_2^{(i)} (1 - h_2^{(i)}) x_1^{(i)} \\
&= \frac{2}{m} w_2^{[2]} \sum_{i=1}^{m} (o^{(i)} - y^{(i)}) o^{(i)} (1 - o^{(i)}) h_2^{(i)} (1 - h_2^{(i)}) x_1^{(i)}
\end{aligned}
$$

$$
h_2^{(i)} = \sigma\left(w_{0,2}^{[1]} + x_1^{(i)} w_{1,2}^{[1]} + x_2^{(i)} w_{2,2}^{[1]}\right)
$$

$$
w_{1,2}^{[1]} := w_{1,2}^{[1]} - \alpha \cdot \frac{2}{m} w_2^{[2]} \sum_{i=1}^{m} (o^{(i)} - y^{(i)}) o^{(i)} (1 - o^{(i)}) h_2^{(i)} (1 - h_2^{(i)}) x_1^{(i)}
$$

(b)

It is possible. The three neurons in the hidden layer can be viewed as three independent linear classifiers, whose decision boundaries are the three sides of the triangle ($x_1 = 0.5$, $x_2 = 0.5$ and $x_1 + x_2 = 4$). The output is also a linear classifier, which differentiate the point is inside or outside the triangle.

(c)

It's not possible. When we adopt linear function for the hidden layer and step function for the output, the entire neuron network can be viewed as one linear classifier (not three). Because the dataset is not linearly separable, so it's impossible to achieve 100% accuracy.

2.

(a)

$$
\begin{aligned}
D_{\mathrm{KL}}(P\|Q) &= \sum_{x\in\mathcal{X}} P(x)\log\frac{P(x)}{Q(x)} \\
&= -\sum_{x\in\mathcal{X}} P(x)\log\frac{Q(x)}{P(x)} \\
&= E\Big[-\log\frac{Q(x)}{P(x)}\Big] \\
&\geq -\log E\Big[\frac{Q(x)}{P(x)}\Big] \\
&= -\log\Big(\sum_{x\in\mathcal{X}} P(x)\frac{Q(x)}{P(x)}\Big) \\
&= -\log\sum_{x\in\mathcal{X}} Q(x) \\
&= -\log 1 \\
&= 0
\end{aligned}
$$

If $P=Q$, then $D_{\mathrm{KL}}(P\|Q)=\sum_{x\in\mathcal{X}} P(x)\log 1 = 0$.

If $D_{\mathrm{KL}}(P\|Q)=0$, then $\frac{Q(x)}{P(x)} = E\Big[\frac{Q(x)}{P(x)}\Big] = 1$, namely $P=Q$.

So $D_{\mathrm{KL}}(P\|Q)=0$ iff $P=Q$.

(b)

$$
\begin{aligned}
D_{\mathrm{KL}}(P(X,Y)\|Q(X,Y)) &= \sum_x\sum_y P(x,y)\log\frac{P(x,y)}{Q(x,y)} \\
&= \sum_x\sum_y P(x)P(y|x)\log\frac{P(x)P(y|x)}{Q(x)Q(y|x)} \\
&= \sum_x\sum_y P(x)P(y|x)\Big(\log\frac{P(x)}{Q(x)} + \log\frac{P(y|x)}{Q(y|x)}\Big) \\
&= \sum_x\sum_y P(x)P(y|x)\log\frac{P(x)}{Q(x)} + \sum_x\sum_y P(x)P(y|x)\log\frac{P(y|x)}{Q(y|x)} \\
&= \sum_x P(x)\log\frac{P(x)}{Q(x)}\sum_y P(y|x) + \sum_x P(x)\sum_y P(y|x)\log\frac{P(y|x)}{Q(y|x)} \\
&= \sum_x P(x)\log\frac{P(x)}{Q(x)} + \sum_x P(x)\Big(\sum_y P(y|x)\log\frac{P(y|x)}{Q(y|x)}\Big) \\
&= D_{\mathrm{KL}}(P(X)\|Q(X)) + D_{\mathrm{KL}}(P(Y|X)\|Q(Y|X))
\end{aligned}
$$

(c)

$$
\begin{aligned}
D_{\mathrm{KL}}(\hat{P}\|P_\theta) &= \sum_{x\in\mathcal{X}} \hat{P}(x)\log\frac{\hat{P}(x)}{P_\theta(x)} \\
&= \sum_{x\in\mathcal{X}} \hat{P}(x)\log\hat{P}(x) - \sum_{x\in\mathcal{X}} \hat{P}(x)\log P_\theta(x)
\end{aligned}
$$

$$
\begin{aligned}
\arg\min_\theta D_{\mathrm{KL}}(\hat{P}\|P_\theta) &= \arg\min_\theta \sum_{x\in\mathcal{X}} \hat{P}(x)\log\hat{P}(x) - \sum_{x\in\mathcal{X}} \hat{P}(x)\log P_\theta(x) \\
&= \arg\max_\theta \sum_{x\in\mathcal{X}} \hat{P}(x)\log P_\theta(x) \\
&= \arg\max_\theta \sum_{x\in\mathcal{X}} \Big(\frac{1}{m}\sum_{i=1}^m 1\{x^{(i)}=x\}\Big)\log P_\theta(x) \\
&= \arg\max_\theta \sum_{i=1}^m \log P_\theta(x^{(i)})
\end{aligned}
$$

3.

(a)

$$\nabla_\theta \log p(y;\theta) = \frac{\nabla_\theta p(y;\theta)}{p(y;\theta)}$$

$$\begin{aligned}
\mathbb{E}_{y\sim p(y;\theta)}[\nabla_{\theta'} \log p(y;\theta')|_{\theta'=\theta}] &= \mathbb{E}_{y\sim p(y;\theta)}\Big[\frac{\nabla_\theta p(y;\theta)}{p(y;\theta)}\Big] \\
&= \int_{-\infty}^{\infty} p(y;\theta)\frac{\nabla_\theta p(y;\theta)}{p(y;\theta)}dy \\
&= \int_{-\infty}^{\infty} \nabla_\theta p(y;\theta)dy \\
&= \nabla_\theta \int_{-\infty}^{\infty} p(y;\theta)dy \\
&= 0
\end{aligned}$$

(b)

$$\begin{aligned}
\mathrm{Cov}[X] &= E[(X-E[X])(X-E[X])^T] \\
&= E[XX^T] \quad \text{when } E[X]=0
\end{aligned}$$

$$\begin{aligned}
\mathcal{I}(\theta) &= \mathrm{Cov}_{y\sim p(y;\theta)}[\nabla_{\theta'} \log p(y;\theta')|_{\theta'=\theta}] \\
&= \mathbb{E}_{y\sim p(y;\theta)}[\nabla_{\theta'} \log p(y;\theta')\nabla_{\theta'} \log p(y;\theta')^T|_{\theta'=\theta}]
\end{aligned}$$

(c)

$$\frac{\partial \log p(y;\theta)}{\partial\theta_i} = \frac{1}{p(y;\theta)}\frac{\partial p(y;\theta)}{\partial\theta_i}$$

$$\begin{aligned}
\mathcal{I}(\theta)_{ij} &= \mathbb{E}_{y\sim p(y;\theta)}[\nabla_{\theta'} \log p(y;\theta')\nabla_{\theta'} \log p(y;\theta')^T|_{\theta'=\theta}]_{ij} \\
&= \mathbb{E}_{y\sim p(y;\theta)}\Big[\frac{\partial \log p(y;\theta)}{\partial\theta_i}\frac{\partial \log p(y;\theta)}{\partial\theta_j}\Big] \\
&= \mathbb{E}_{y\sim p(y;\theta)}\Big[\frac{1}{(p(y;\theta))^2}\frac{\partial^2 p(y;\theta)}{\partial\theta_i\partial\theta_j}\Big]
\end{aligned}$$

$$\frac{\partial^2 \log p(y;\theta)}{\partial\theta_i\partial\theta_j} = -\frac{1}{(p(y;\theta))^2}\frac{\partial^2 p(y;\theta)}{\partial\theta_i\partial\theta_j} + \frac{1}{p(y;\theta)}\frac{\partial^2 p(y;\theta)}{\partial\theta_i\partial\theta_j}$$

$$\begin{aligned}
\mathbb{E}_{y\sim p(y;\theta)}[-\nabla^2_{\theta'} \log p(y;\theta')|_{\theta'=\theta}]_{ij} &= \mathbb{E}_{y\sim p(y;\theta)}\Big[\frac{1}{(p(y;\theta))^2}\frac{\partial^2 p(y;\theta)}{\partial\theta_i\partial\theta_j} - \frac{1}{p(y;\theta)}\frac{\partial^2 p(y;\theta)}{\partial\theta_i\partial\theta_j}\Big] \\
&= \mathbb{E}_{y\sim p(y;\theta)}\Big[\frac{1}{(p(y;\theta))^2}\frac{\partial^2 p(y;\theta)}{\partial\theta_i\partial\theta_j}\Big] - \mathbb{E}_{y\sim p(y;\theta)}\Big[\frac{1}{p(y;\theta)}\frac{\partial^2 p(y;\theta)}{\partial\theta_i\partial\theta_j}\Big] \\
&= \mathbb{E}_{y\sim p(y;\theta)}\Big[\frac{1}{(p(y;\theta))^2}\frac{\partial^2 p(y;\theta)}{\partial\theta_i\partial\theta_j}\Big] - \int_{-\infty}^{\infty} p(y;\theta)\frac{1}{p(y;\theta)}\frac{\partial^2 p(y;\theta)}{\partial\theta_i\partial\theta_j}dy \\
&= \mathbb{E}_{y\sim p(y;\theta)}\Big[\frac{1}{(p(y;\theta))^2}\frac{\partial^2 p(y;\theta)}{\partial\theta_i\partial\theta_j}\Big] - \frac{\partial^2}{\partial\theta_i\partial\theta_j}\int_{-\infty}^{\infty} p(y;\theta)dy \\
&= \mathbb{E}_{y\sim p(y;\theta)}\Big[\frac{1}{(p(y;\theta))^2}\frac{\partial^2 p(y;\theta)}{\partial\theta_i\partial\theta_j}\Big] \\
&= \mathcal{I}(\theta)_{ij}
\end{aligned}$$

$$\mathbb{E}_{y\sim p(y;\theta)}[-\nabla^2_{\theta'} \log p(y;\theta')|_{\theta'=\theta}] = \mathcal{I}(\theta)$$

(d)

$$\tilde{\theta} = \theta + d$$

$$\begin{aligned}
\log p(y;\tilde{\theta}) &\approx \log p(y;\theta) + (\tilde{\theta}-\theta)^T \nabla_{\theta'} \log p(y;\theta')|_{\theta'=\theta} + \frac{1}{2}(\tilde{\theta}-\theta)^T \left(\nabla^2_{\theta'} \log p(y;\theta')|_{\theta'=\theta}\right)(\tilde{\theta}-\theta) \\
&= \log p(y;\theta) + d^T \nabla_{\theta'} \log p(y;\theta')|_{\theta'=\theta} + \frac{1}{2}d^T \left(\nabla^2_{\theta'} \log p(y;\theta')|_{\theta'=\theta}\right) d
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}_{y\sim p(y;\theta)}[\log p(y;\tilde{\theta})] &= \mathbb{E}_{y\sim p(y;\theta)}[\log p(y;\theta)] + \frac{1}{2}d^T \mathbb{E}_{y\sim p(y;\theta)}[\nabla^2_{\theta'} \log p(y;\theta')|_{\theta'=\theta}]d \\
&= \mathbb{E}_{y\sim p(y;\theta)}[\log p(y;\theta)] + \frac{1}{2}d^T \mathcal{I}(\theta) d
\end{aligned}$$

$$\begin{aligned}
D_{\mathrm{KL}}(p_\theta \| p_{\theta+d}) &= D_{\mathrm{KL}}(p_\theta \| p_{\tilde{\theta}}) \\
&= \mathbb{E}_{y\sim p(y;\theta)}[\log p(y;\theta)] - \mathbb{E}_{y\sim p(y;\theta)}[\log p(y;\tilde{\theta})] \\
&\approx \frac{1}{2}d^T \mathcal{I}(\theta) d
\end{aligned}$$

(e)

$$d^* = \arg\max_d \ell(\theta+d) \quad \text{subject to} \quad D_{\mathrm{KL}}(p_\theta \| p_{\theta+d}) = c$$

$$\begin{aligned}
\ell(\theta+d) &\approx \ell(\theta) + d^T \nabla_{\theta'} \ell(\theta')|_{\theta'=\theta} \\
&= \log p(y;\theta) + d^T \nabla_{\theta'} \log p(y;\theta')|_{\theta'=\theta} \\
&= \log p(y;\theta) + d^T \frac{\nabla_{\theta'} p(y;\theta')|_{\theta'=\theta}}{p(y;\theta)}
\end{aligned}$$

$$D_{\mathrm{KL}}(p_\theta \| p_{\theta+d}) \approx \frac{1}{2}d^T \mathcal{I}(\theta) d$$

$$\begin{aligned}
\mathcal{L}(d,\lambda) &= \ell(\theta+d) - \lambda\Big[D_{\mathrm{KL}}(p_\theta \| p_{\theta+d}) - c\Big] \\
&\approx \log p(y;\theta) + d^T \frac{\nabla_{\theta'} p(y;\theta')|_{\theta'=\theta}}{p(y;\theta)} - \lambda\Big[\frac{1}{2}d^T \mathcal{I}(\theta) d - c\Big]
\end{aligned}$$

$$\nabla_d \mathcal{L}(d,\lambda) \approx \frac{\nabla_{\theta'} p(y;\theta')|_{\theta'=\theta}}{p(y;\theta)} - \lambda \mathcal{I}(\theta) d = 0$$

$$\tilde{d} = \frac{1}{\lambda}\mathcal{I}(\theta)^{-1} \frac{\nabla_{\theta'} p(y;\theta')|_{\theta'=\theta}}{p(y;\theta)}$$

$$\begin{aligned}
\nabla_\lambda \mathcal{L}(d,\lambda) &\approx c - \frac{1}{2}d^T \mathcal{I}(\theta) d \\
&= c - \frac{1}{2}\cdot\frac{1}{\lambda}\frac{\nabla_{\theta'} p(y;\theta')|_{\theta'=\theta}{}^T}{p(y;\theta)}\mathcal{I}(\theta)^{-1}\cdot\mathcal{I}(\theta)\cdot\frac{1}{\lambda}\mathcal{I}(\theta)^{-1}\frac{\nabla_{\theta'} p(y;\theta')|_{\theta'=\theta}}{p(y;\theta)} \\
&= c - \frac{1}{2\lambda^2(p(y;\theta))^2}\nabla_{\theta'} p(y;\theta')|_{\theta'=\theta}{}^T \mathcal{I}(\theta)^{-1}\nabla_{\theta'} p(y;\theta')|_{\theta'=\theta} \\
&= 0
\end{aligned}$$

$$\lambda = \sqrt{\frac{1}{2c(p(y;\theta))^2}\nabla_{\theta'} p(y;\theta')|_{\theta'=\theta}{}^T \mathcal{I}(\theta)^{-1}\nabla_{\theta'} p(y;\theta')|_{\theta'=\theta}}$$

$$\begin{aligned}
d^* &= \sqrt{\frac{2c(p(y;\theta))^2}{\nabla_{\theta'} p(y;\theta')|_{\theta'=\theta}{}^T \mathcal{I}(\theta)^{-1}\nabla_{\theta'} p(y;\theta')|_{\theta'=\theta}}}\mathcal{I}(\theta)^{-1}\frac{\nabla_{\theta'} p(y;\theta')|_{\theta'=\theta}}{p(y;\theta)} \\
&= \sqrt{\frac{2c}{\nabla_{\theta'} p(y;\theta')|_{\theta'=\theta}{}^T \mathcal{I}(\theta)^{-1}\nabla_{\theta'} p(y;\theta')|_{\theta'=\theta}}}\mathcal{I}(\theta)^{-1}\nabla_{\theta'} p(y;\theta')|_{\theta'=\theta}
\end{aligned}$$

(f)

Newton's method

$$\theta := \theta - H^{-1}\nabla_\theta \ell(\theta)$$

Natural gradient

$$\begin{aligned}\mathcal{I}(\theta) &= \mathbb{E}_{y\sim p(y;\theta)}[-\nabla^2_{\theta'}\log p(y;\theta')|_{\theta'=\theta}] \\ &= \mathbb{E}_{y\sim p(y;\theta)}[-\nabla^2_{\theta}\ell(\theta)] \\ &= -\mathbb{E}_{y\sim p(y;\theta)}[H]\end{aligned}$$

$$\begin{aligned}\theta: &= \theta + \tilde{d} \\ &= \theta + \frac{1}{\lambda}\mathcal{I}(\theta)^{-1}\nabla_\theta \ell(\theta) \\ &= \theta - \frac{1}{\lambda}\mathbb{E}_{y\sim p(y;\theta)}[H]^{-1}\nabla_\theta \ell(\theta)\end{aligned}$$

4.

(a)

$$\begin{aligned}\ell_{\text{semi-sup}}(\theta^{(t+1)}) &= \ell_{\text{unsup}}(\theta^{(t+1)}) + \alpha \ell_{\text{sup}}(\theta^{(t+1)}) \\ &\geq \sum_{i=1}^{m}\left(\sum_{z^{(i)}} Q_i^{(t)}(z^{(i)})\log\frac{p(x^{(i)},z^{(i)};\theta^{(t+1)})}{Q_i^{(t)}(z^{(i)})}\right) + \alpha\left(\sum_{i=1}^{\tilde{m}}\log p(\tilde{x}^{(i)},\tilde{z}^{(i)};\theta^{(t+1)})\right) \\ &\geq \sum_{i=1}^{m}\left(\sum_{z^{(i)}} Q_i^{(t)}(z^{(i)})\log\frac{p(x^{(i)},z^{(i)};\theta^{(t)})}{Q_i^{(t)}(z^{(i)})}\right) + \alpha\left(\sum_{i=1}^{\tilde{m}}\log p(\tilde{x}^{(i)},\tilde{z}^{(i)};\theta^{(t)})\right) \\ &= \ell_{\text{unsup}}(\theta^{(t)}) + \alpha \ell_{\text{sup}}(\theta^{(t)}) \\ &= \ell_{\text{semi-sup}}(\theta^{(t)})\end{aligned}$$

(b)

Latent variables: $z^{(i)}$

$$\begin{aligned}w_j^{(i)} &= p(z^{(i)}=j|x^{(i)};\phi,\mu,\Sigma) \\ &= \frac{p(x^{(i)}|z^{(i)}=j;\mu,\Sigma)p(z^{(i)}=j;\phi)}{\sum_{l=1}^{k}p(x^{(i)}|z^{(i)}=l;\mu,\Sigma)p(z^{(i)}=l;\phi)} \\ &= \frac{\frac{1}{(2\pi)^{n/2}|\Sigma_j|^{1/2}}\exp\Big(-\frac{1}{2}(x^{(i)}-\mu_j)^T\Sigma_j^{-1}(x^{(i)}-\mu_j)\Big)\phi_j}{\sum_{l=1}^{k}\frac{1}{(2\pi)^{n/2}|\Sigma_l|^{1/2}}\exp\Big(-\frac{1}{2}(x^{(i)}-\mu_l)^T\Sigma_l^{-1}(x^{(i)}-\mu_l)\Big)\phi_l}\end{aligned}$$

(c)

Parameters: ϕ, μ, Σ

$$\mathcal{L}(\phi) = \sum_{i=1}^{m}\sum_{l=1}^{k} w_l^{(i)}\log\phi_l + \sum_{i=1}^{\tilde{m}}\sum_{l=1}^{k}1\{\tilde{z}^{(i)}=l\}\log\phi_l + \beta(\sum_{l=1}^{k}\phi_l - 1)$$

$$\nabla_{\phi_j}\mathcal{L}(\phi) = \sum_{i=1}^{m}\frac{w_j^{(i)}}{\phi_j} + \sum_{i=1}^{\tilde{m}}\frac{1\{\tilde{z}^{(i)}=j\}}{\phi_j} + \beta = 0$$

$$\phi_j = \frac{\sum_{i=1}^{m}w_j^{(i)} + \alpha\sum_{i=1}^{\tilde{m}}1\{\tilde{z}^{(i)}=j\}}{-\beta}$$

$$\sum_{l=1}^{k}\phi_l = \frac{\sum_{i=1}^{m}\sum_{l=1}^{k} w_l^{(i)} + \alpha\sum_{i=1}^{\tilde{m}}\sum_{l=1}^{k} 1\{\tilde{z}^{(i)}=l\}}{-\beta}$$
$$= \frac{m+\alpha\tilde{m}}{-\beta}$$
$$= 1$$

$$-\beta = m+\alpha\tilde{m}$$

$$\phi_j = \frac{\sum_{i=1}^{m} w_j^{(i)} + \alpha\sum_{i=1}^{\tilde{m}} 1\{\tilde{z}^{(i)}=j\}}{m+\alpha\tilde{m}}$$

$$\begin{aligned}\nabla_{\mu_j}\ell_{\text{unsup}} &= \sum_{i=1}^{m} w_j^{(i)}\Sigma_j^{-1}(x^{(i)}-\mu_j)\\ &= \Sigma_j^{-1}\Big(\sum_{i=1}^{m} w_j^{(i)}x^{(i)} - \mu_j\sum_{i=1}^{m} w_j^{(i)}\Big)\end{aligned}$$

$$\begin{aligned}\nabla_{\mu_j}\ell_{\text{sup}} &= \sum_{i=1}^{\tilde{m}} 1\{\tilde{z}^{(i)}=j\}\Sigma_j^{-1}(\tilde{x}^{(i)}-\mu_j)\\ &= \Sigma_j^{-1}\Big(\sum_{i=1}^{\tilde{m}} 1\{\tilde{z}^{(i)}=j\}\tilde{x}^{(i)} - \mu_j\sum_{i=1}^{\tilde{m}} 1\{\tilde{z}^{(i)}=j\}\Big)\end{aligned}$$

$$\begin{aligned}\nabla_{\mu_j}\ell_{\text{semi-sup}} &= \nabla_{\mu_j}\ell_{\text{unsup}} + \alpha\nabla_{\mu_j}\ell_{\text{sup}}\\ &= \Sigma_j^{-1}\Big[\Big(\sum_{i=1}^{m} w_j^{(i)}x^{(i)} - \mu_j\sum_{i=1}^{m} w_j^{(i)}\Big) + \alpha\Big(\sum_{i=1}^{\tilde{m}} 1\{\tilde{z}^{(i)}=j\}\tilde{x}^{(i)} - \mu_j\sum_{i=1}^{\tilde{m}} 1\{\tilde{z}^{(i)}=j\}\Big)\Big]\\ &= \Sigma_j^{-1}\Big[\Big(\sum_{i=1}^{m} w_j^{(i)}x^{(i)} + \alpha\sum_{i=1}^{\tilde{m}} 1\{\tilde{z}^{(i)}=j\}\tilde{x}^{(i)}\Big) - \mu_j\Big(\sum_{i=1}^{m} w_j^{(i)} + \alpha\sum_{i=1}^{\tilde{m}} 1\{\tilde{z}^{(i)}=j\}\Big)\Big]\\ &= 0\end{aligned}$$

$$\mu_j = \frac{\sum_{i=1}^{m} w_j^{(i)}x^{(i)} + \alpha\sum_{i=1}^{\tilde{m}} 1\{\tilde{z}^{(i)}=j\}\tilde{x}^{(i)}}{\sum_{i=1}^{m} w_j^{(i)} + \alpha\sum_{i=1}^{\tilde{m}} 1\{\tilde{z}^{(i)}=j\}}$$

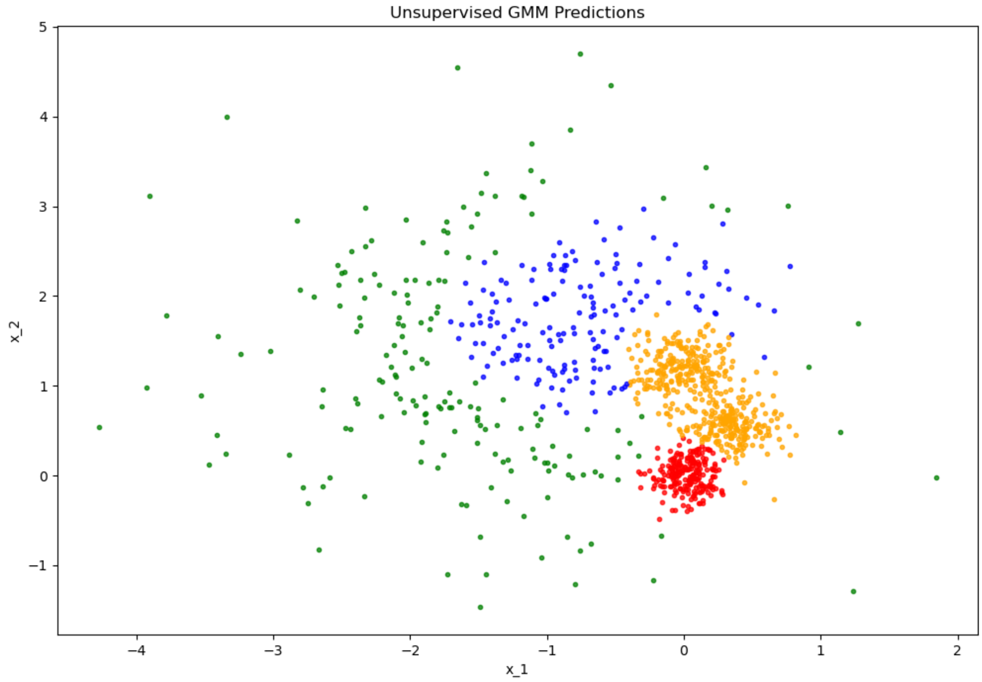
$$\nabla_{\Sigma_j}\ell_{\text{unsup}} = -\frac{1}{2}\sum_{i=1}^{m} w_j^{(i)}\Sigma_j^{-1} + \frac{1}{2}\Sigma_j^{-1}\Big(\sum_{i=1}^{m} w_j^{(i)}(x^{(i)}-\mu_j)(x^{(i)}-\mu_j)^T\Big)\Sigma_j^{-1}$$

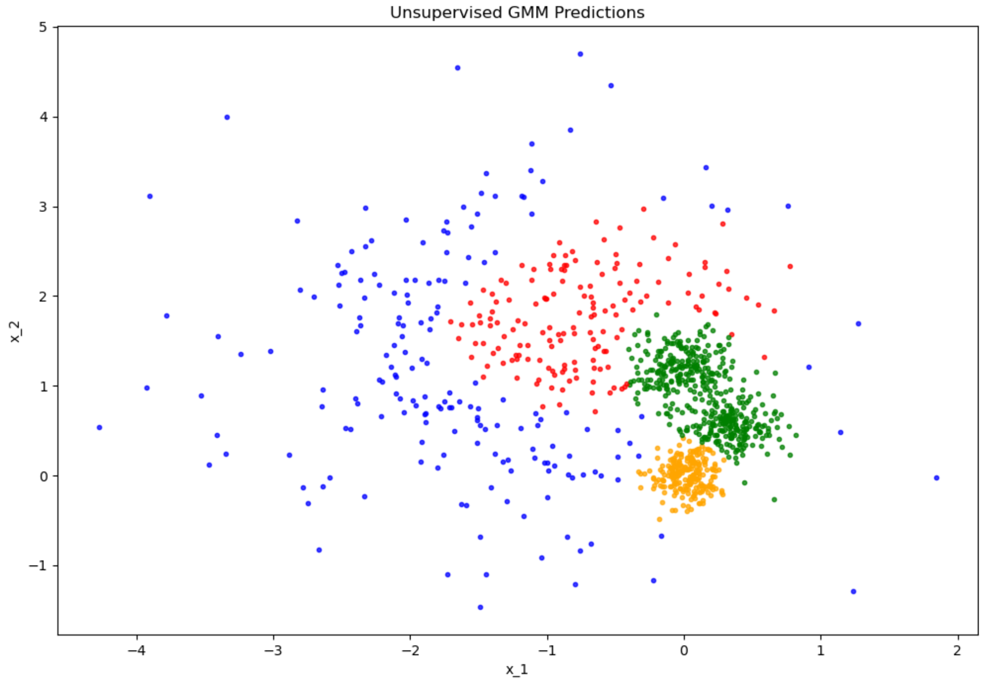
$$\nabla_{\Sigma_j}\ell_{\text{sup}} = -\frac{1}{2}\sum_{i=1}^{\tilde{m}} 1\{\tilde{z}^{(i)}=j\}\Sigma_j^{-1} + \frac{1}{2}\Sigma_j^{-1}\Big(\sum_{i=1}^{\tilde{m}} 1\{\tilde{z}^{(i)}=j\}(\tilde{x}^{(i)}-\mu_j)(\tilde{x}^{(i)}-\mu_j)^T\Big)\Sigma_j^{-1}$$

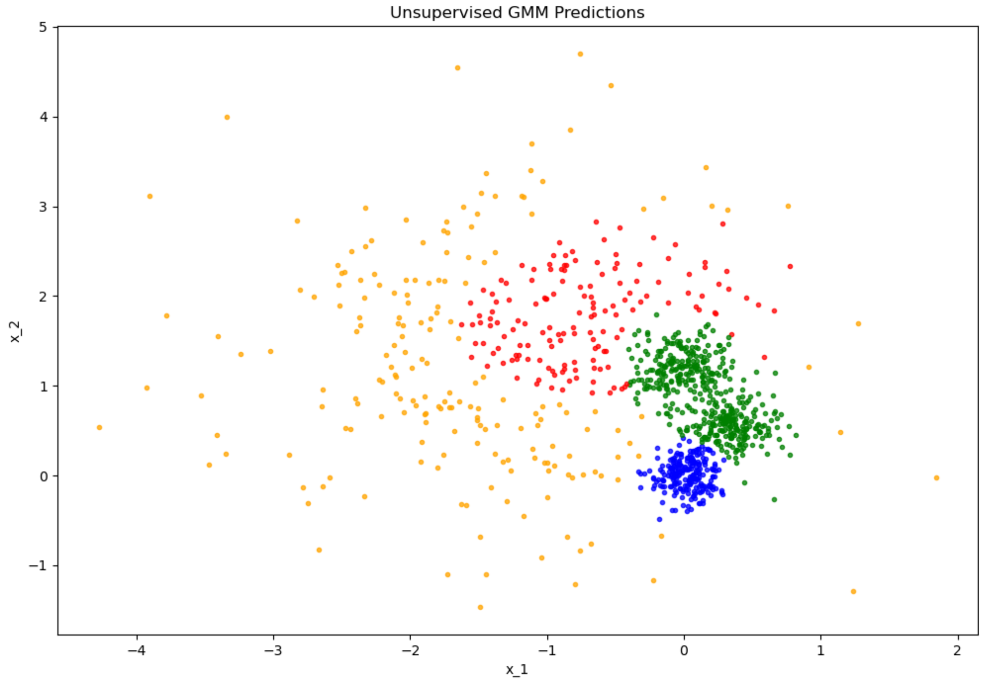
$$\begin{aligned}\nabla_{\Sigma_j}\ell_{\text{semi-sup}} &= \nabla_{\Sigma_j}\ell_{\text{unsup}} + \alpha\nabla_{\Sigma_j}\ell_{\text{sup}}\\ &= -\frac{1}{2}\sum_{i=1}^{m} w_j^{(i)}\Sigma_j^{-1} + \frac{1}{2}\Sigma_j^{-1}\Big(\sum_{i=1}^{m} w_j^{(i)}(x^{(i)}-\mu_j)(x^{(i)}-\mu_j)^T\Big)\Sigma_j^{-1}\\ &\quad -\frac{1}{2}\alpha\sum_{i=1}^{\tilde{m}} 1\{\tilde{z}^{(i)}=j\}\Sigma_j^{-1} + \frac{1}{2}\alpha\Sigma_j^{-1}\Big(\sum_{i=1}^{\tilde{m}} 1\{\tilde{z}^{(i)}=j\}(\tilde{x}^{(i)}-\mu_j)(\tilde{x}^{(i)}-\mu_j)^T\Big)\Sigma_j^{-1}\\ &= -\frac{1}{2}\Sigma_j^{-1}\Big(\sum_{i=1}^{m} w_j^{(i)} + \alpha\sum_{i=1}^{\tilde{m}} 1\{\tilde{z}^{(i)}=j\}\Big)\\ &\quad +\frac{1}{2}\Sigma_j^{-1}\Big(\sum_{i=1}^{m} w_j^{(i)}(x^{(i)}-\mu_j)(x^{(i)}-\mu_j)^T + \alpha\sum_{i=1}^{\tilde{m}} 1\{\tilde{z}^{(i)}=j\}(\tilde{x}^{(i)}-\mu_j)(\tilde{x}^{(i)}-\mu_j)^T\Big)\Sigma_j^{-1}\\ &= 0\end{aligned}$$

$$\Sigma_j = \frac{\sum_{i=1}^{m} w_j^{(i)}(x^{(i)}-\mu_j)(x^{(i)}-\mu_j)^T + \alpha\sum_{i=1}^{\tilde{m}} 1\{\tilde{z}^{(i)}=j\}(\tilde{x}^{(i)}-\mu_j)(\tilde{x}^{(i)}-\mu_j)^T}{\sum_{i=1}^{m} w_j^{(i)} + \alpha\sum_{i=1}^{\tilde{m}} 1\{\tilde{z}^{(i)}=j\}}$$

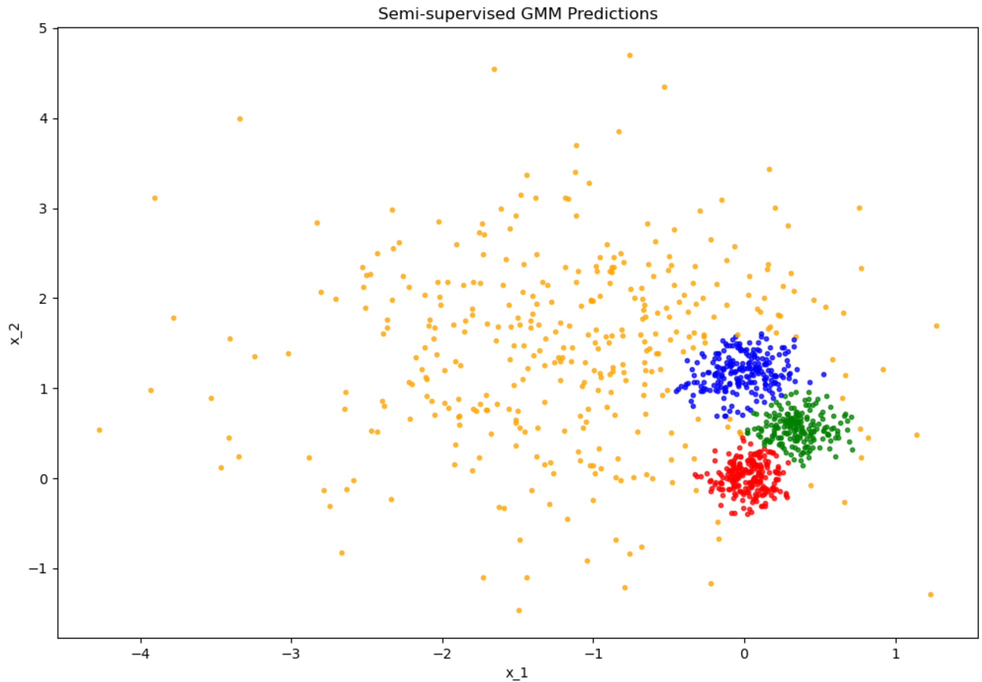
(d)

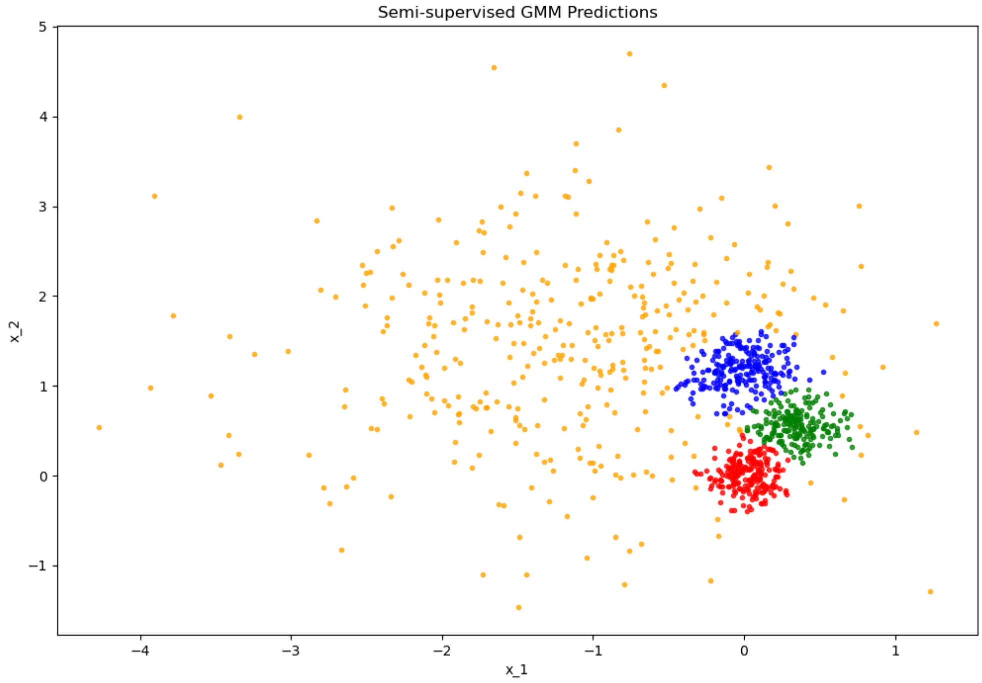

Unsupervised GMM Predictions
x_2
x_1
5
4
3
2
1
0
−1
−4
−3
−2
−1
0
1
2

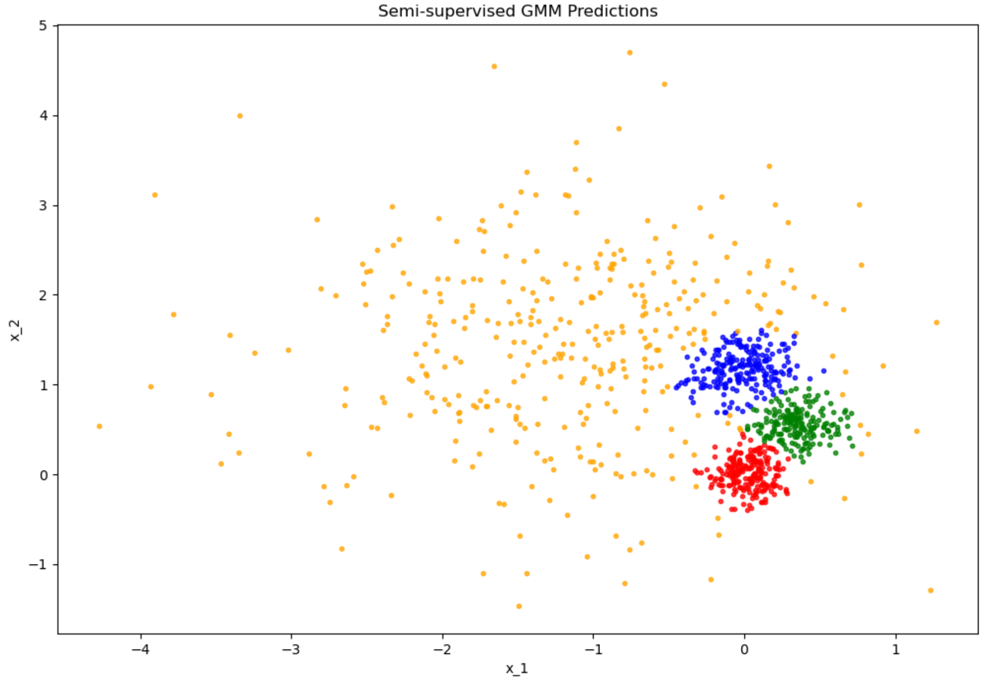

Unsupervised GMM Predictions
x_2
x_1
5
4
3
2
1
0
−1
−4
−3
−2
−1
0
1
2


Unsupervised GMM Predictions
x_1
x_2

(e)


Semi-supervised GMM Predictions
x_1
x_2





(f)

i.

Semi-supervised EM take much less iterations to converge than unsupervised EM.

Nearly 50 iterations for semi-supervised EM and more than 1000 for unsupervised EM.

ii.

Semi-supervised EM are more stable than unsupervised EM.

The assignments by unsupervised EM are random with different random initializations.

But the assignments by semi-supervised EM are the same.

iii.

The overall quality of assignments by semi-supervised EM are higher than unsupervised EM.

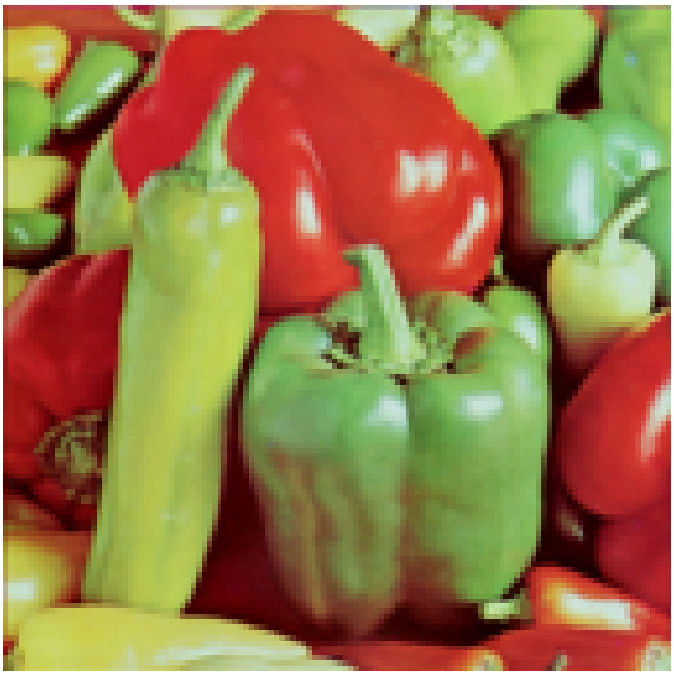
In the pictures of semi-supervised EM, there are three nearly the same low-variance Gaussian distributions, and a high-variance Gaussian distribution.

In the pictures of unsupervised EM, there are four Gaussian distributions which variances are different.

5.

(a)

Original small image



Original large image

Updated large image

(b)

In the original image, we need $3 \times 8 = 24$ bits to represent a pixel.

In the compressed image, we only need 4 bits (16 colors) to represent a pixel.

So the image are compressed by factor 6.