Mat2033 - Discrete Mathematics

Partial Orderings

Definition 7.5. Relation R on the set S is called a **partial ordering** if it is reflexive, anti-symmetric and transitive. A set S together with a partial ordering R is called a **partially ordered set** (poset), denoted by (S, R).

Example 7.8. Consider a relation on the set $S = \mathbb{R}$ defined as following:

$$R = \left\{(a, b) \in \mathbb{R} \times \mathbb{R} \;\middle|\; a \leq b\right\}$$

Since for any real number a we have $a \leq a$, R is reflexive. For any real numbers a and b, if $a \leq b$ and $b \leq a$ then $a = b$. Therefore, R is anti-symmetric. For any real numbers a, b and c, if $a \leq b$ and $b \leq c$ then $a \leq c$. Therefore, R is transitive. So R is reflexive, anti-symmetric and transitive. Therefore, R is a partial ordering.

Example 7.9. Consider a relation on the set $S = \mathbb{N}$ defined as following:

$$R = \left\{ (a,b) \in \mathbb{N} \times \mathbb{N} \;\middle|\; a|b \right\},$$

where $a|b$ means that a divides b.

Since for any natural number a we have $a|a$, R is reflexive. For any natural numbers a and b, if $a|b$ and $b|a$ then $a = b$. Therefore, R is anti-symmetric. For any natural numbers a, b and c, if $a|b$ and $b|c$ then $a|c$. Therefore, R is transitive. So R is reflexive, anti-symmetric and transitive. Therefore, R is a partial ordering.

Notation 7.1. Let (S, R) be any poset. For elements a and b of set S if $(a, b) \in R$ then we will say that 'a is less than or equal to b with respect to partial ordering R' and will write $a \preceq b$. If $(a, b) \in R$ but $a \neq b$ then we will say that 'a is less than b with respect to partial ordering R' and will write $a \prec b$.

Definition 7.6. Let (S, R) be a poset. The elements a and b of set S are called **comparable** with respect to partial ordering R if either $a \preceq b$ or $b \preceq a$. The elements a and b of set S are called **incomparable** with respect to partial ordering R if neither $a \preceq b$ nor $b \preceq a$.

Example 7.10. Consider again a relation

$$R = \left\{ (a, b) \in \mathbb{N} \times \mathbb{N} \;\middle|\; a|b \right\}$$

which is a partial ordering on the set $S = \mathbb{N}$ (see Example 11.9). Natural numbers 7 and 28 are comparable with respect to partial ordering R and $7 \preceq 28$. Natural numbers 7 and 27 are incomparable with respect to R because neither $(7, 27) \in R$ nor $(27, 7) \in R$.

Definition 7.7. Let (S, R) be a poset. If every two elements a and b of set S are comparable with respect to partial ordering R then R is called a **total ordering** and S is called a **totally ordered set** with respect to R.

Example 7.11. Consider again a relation

$$R = \left\{(a, b) \in \mathbb{R} \times \mathbb{R} \;\middle|\; a \leq b\right\}$$

which is a partial ordering on the set $S = \mathbb{R}$ (see Example 11.8). Since for any two real numbers a and b, we have either $a \leq b$ or $b \leq a$, this relation R is a total ordering.

Example 7.12. Obviously, a partial ordering

$$R = \left\{(a, b) \in \mathbb{N} \times \mathbb{N} \;\middle|\; a|b\right\}$$

is not a total ordering. (Why?)

Definition 7.8. Let R be a total ordering on the set S. Set S is called **well-ordered set** with respect to R if every nonempty subset of S has a least element.

Example 7.13. Consider a relation on the set $S = \mathbb{N}$ defined as following:

$$R = \left\{ (a, b) \in \mathbb{N} \times \mathbb{N} \;\middle|\; a \leq b \right\}$$

which is a total ordering. Since any subset of $\mathbb{N}$ has a least element, set $\mathbb{N}$ is a well-ordered set with respect to this relation R.

Example 7.14. Now, consider a relation on the set $S = \mathbb{Z}$ defined as following:

$$R = \left\{ (a, b) \in \mathbb{Z} \times \mathbb{Z} \;\middle|\; a \leq b \right\}$$

which is a total ordering. Subset $\{\ldots, -5, -3, -1, 1, 3, 5, \ldots\}$ of set $\mathbb{Z}$ does not have a least element and therefore $\mathbb{Z}$ is not a well-ordered set with respect to this relation R.