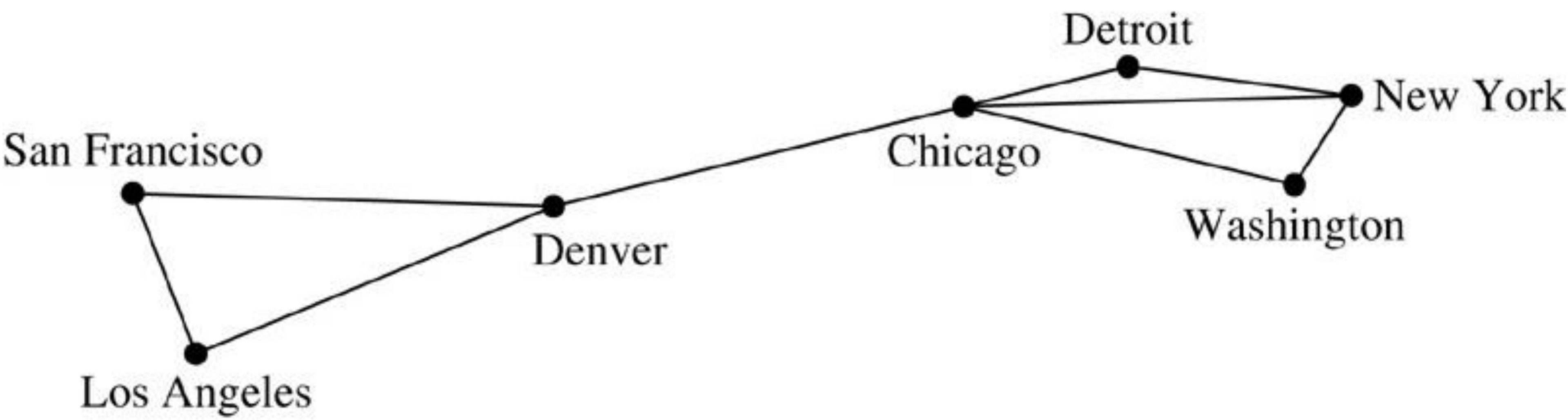
Mat2033 - Discrete Mathematics

Graph Theory and Its Applications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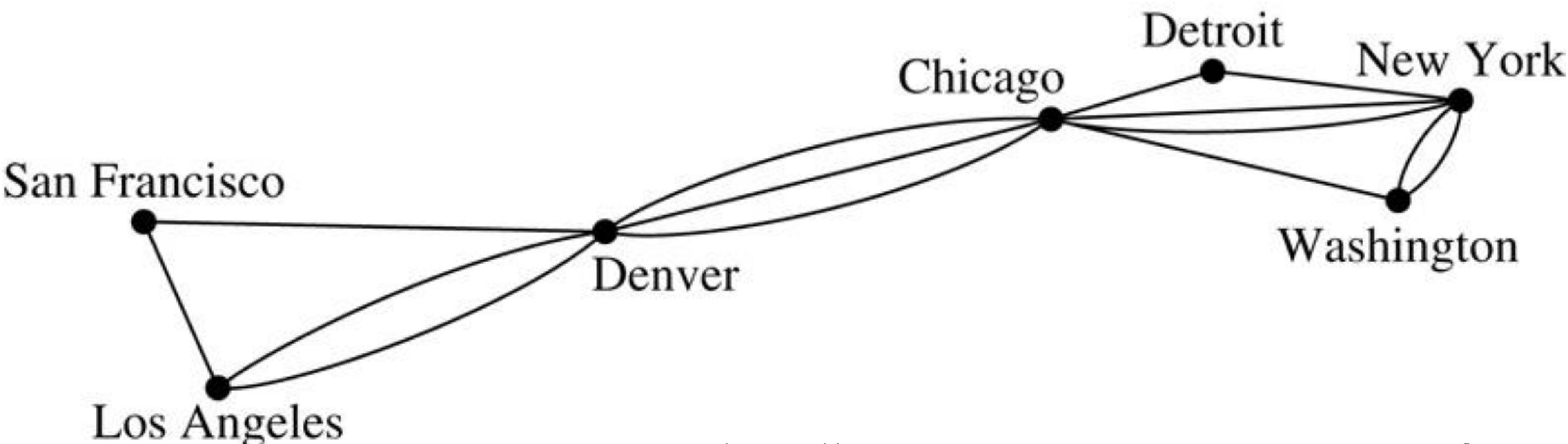
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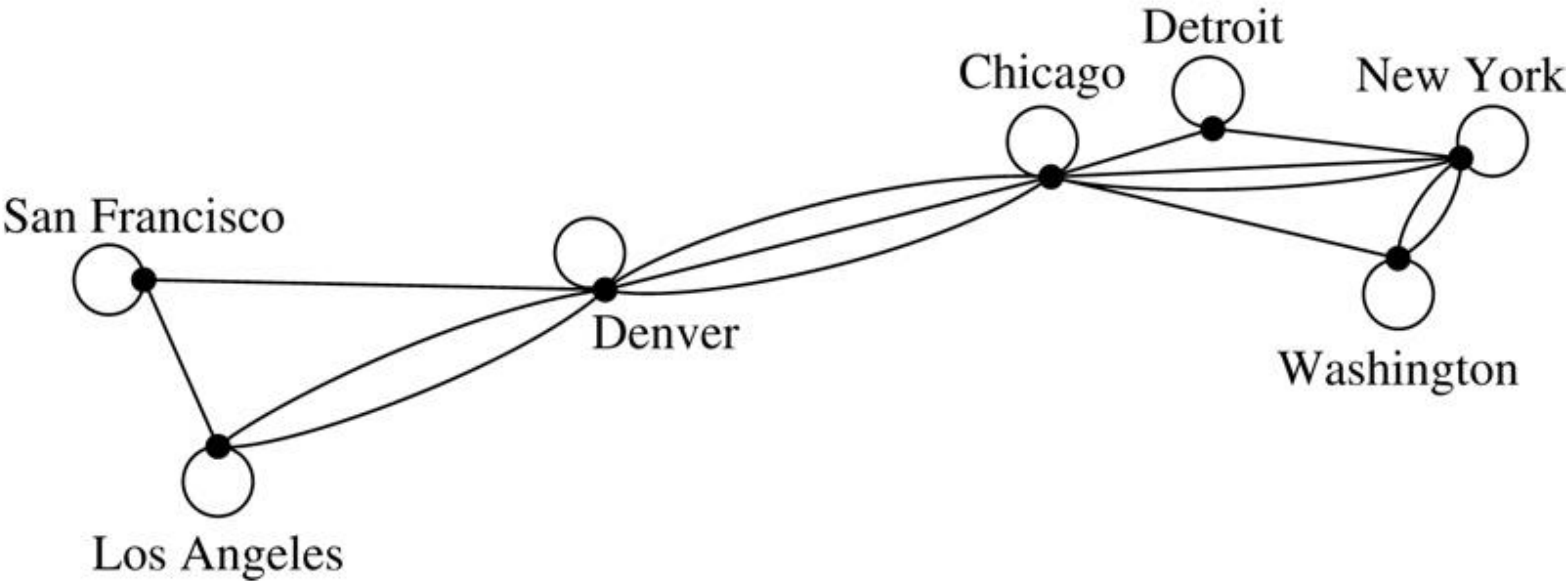
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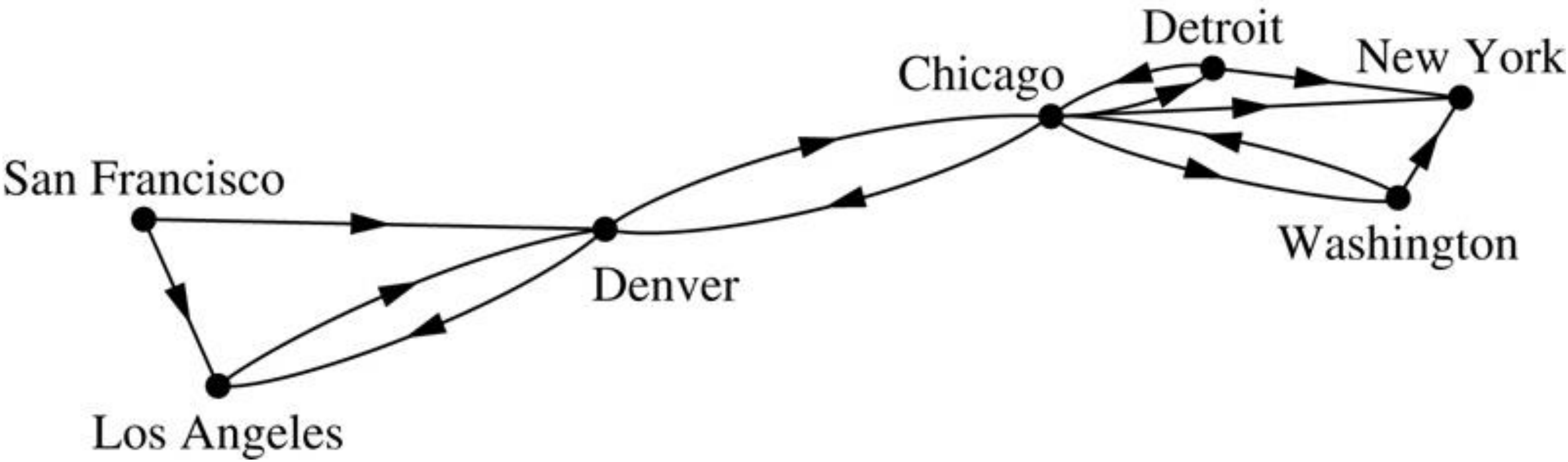
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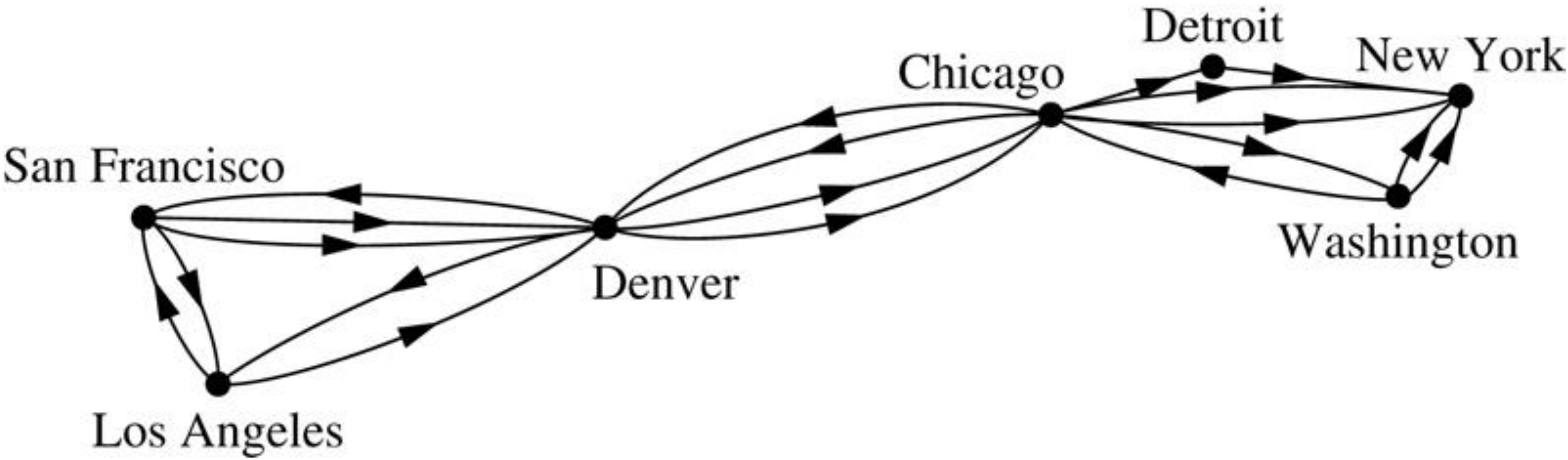
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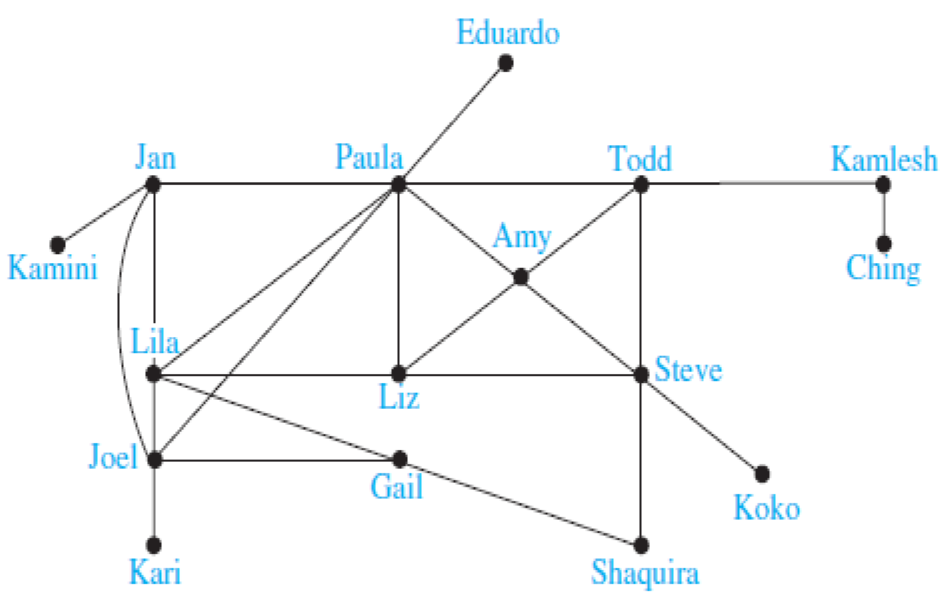
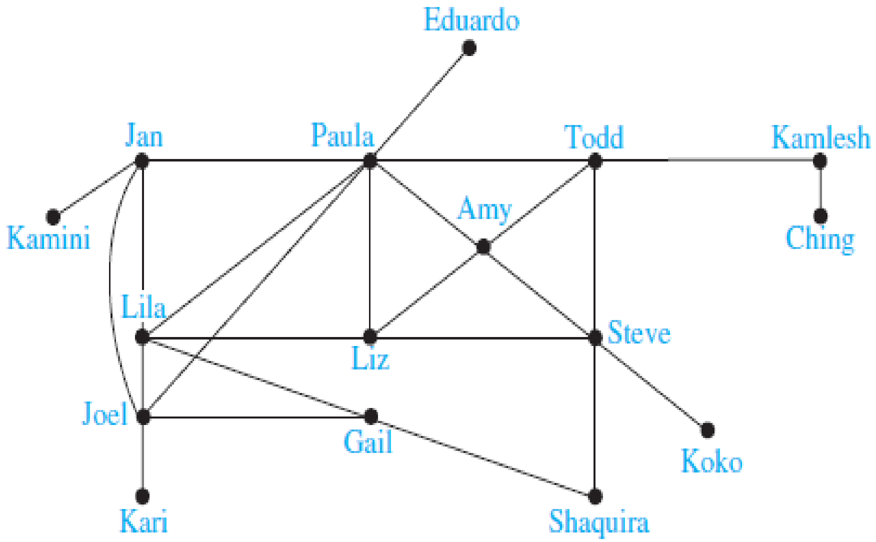
Graph Models

SOCIAL NETWORKS Graphs are extensively used to model social structures based on different kinds of relationships between people or groups of people. These social structures, and the graphs that represent them, are known as **social networks**. In these graph models, individuals or organizations are represented by vertices; relationships between individuals or organizations are represented by edges. The study of social networks is an extremely active multidisciplinary area, and many different types of relationships between people have been studied using them.



An Acquaintanceship Graph.

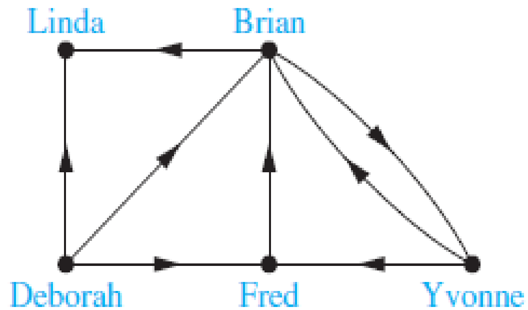
Graph Models



An Acquaintanceship Graph.

Acquaintanceship and Friendship Graphs We can use a simple graph to represent whether two people know each other, that is, whether they are acquainted, or whether they are friends (either in the real world in the virtual world via a social networking site such as Facebook). Each person in a particular group of people is represented by a vertex. An undirected edge is used to connect two people when these people know each other, when we are concerned only with acquaintanceship, or whether they are friends. No multiple edges and usually no loops are used. (If we want to include the notion of self-knowledge, we would include loops.) A small acquaintanceship graph is shown in Figure . The acquaintanceship graph of all people in the world has more than six billion vertices and probably more than one trillion edges!

Graph Models



An Influence Graph.

Influence Graphs In studies of group behavior it is observed that certain people can influence the thinking of others. A directed graph called an **influence graph** can be used to model this behavior. Each person of the group is represented by a vertex. There is a directed edge from vertex *a* to vertex *b* when the person represented by vertex *a* can influence the person represented by vertex *b*. This graph does not contain loops and it does not contain multiple directed edges. An example of an influence graph for members of a group is shown in Figure . In the group modeled by this influence graph, Deborah cannot be influenced, but she can influence Brian, Fred, and Linda. Also, Yvonne and Brian can influence each other.

Graph Models

COMMUNICATION NETWORKS We can model different communications networks using vertices to represent devices and edges to represent the particular type of communications links of interest. We have already modeled a data network in the first part of this section.

Call Graphs Graphs can be used to model telephone calls made in a network, such as a long-distance telephone network. In particular, a directed multigraph can be used to model calls where each telephone number is represented by a vertex and each telephone call is represented by a directed edge. The edge representing a call starts at the telephone number from which the call was made and ends at the telephone number to which the call was made. We need directed edges because the direction in which the call is made matters. We need multiple directed edges because we want to represent each call made from a particular telephone number to a second number.

A small telephone call graph is displayed in Figure 8(a), representing seven telephone numbers. This graph shows, for instance, that three calls have been made from 732-555-1234 to 732-555-9876 and two in the other direction, but no calls have been made from 732-555-4444 to any of the other six numbers except 732-555-0011. When we care only whether there has been a call connecting two telephone numbers, we use an undirected graph with an edge connecting telephone numbers when there has been a call between these numbers. This version of the call graph is displayed in Figure 8(b).

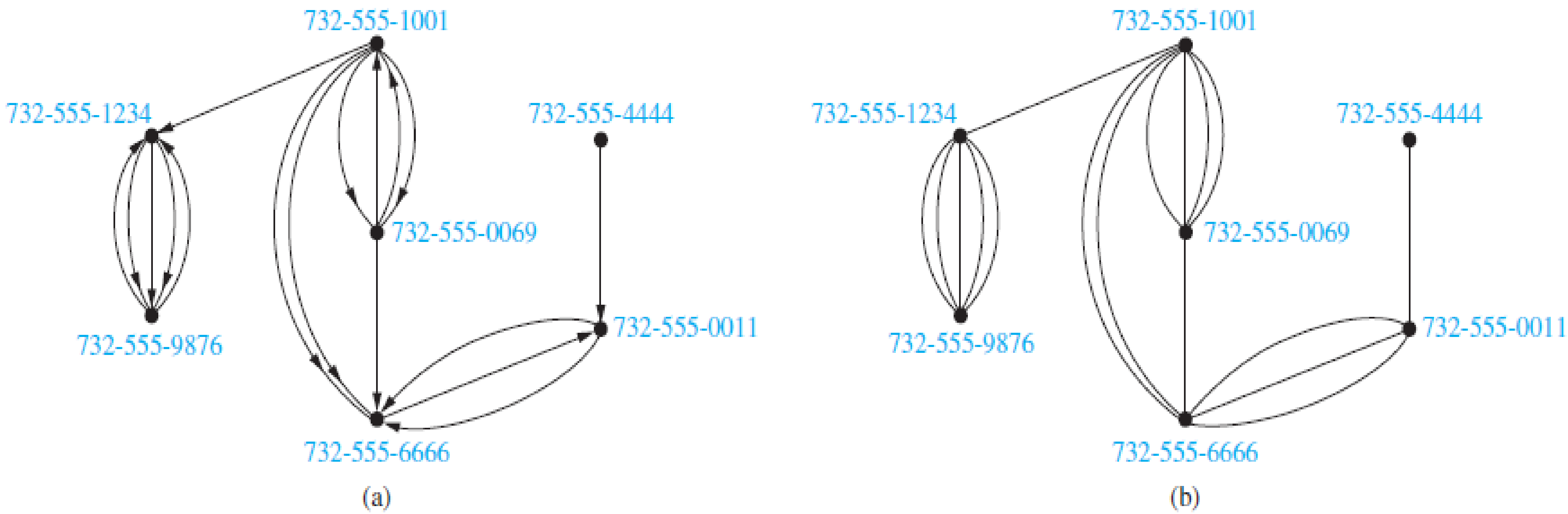


FIGURE 8 **A Call Graph.**

Graph Models

INFORMATION NETWORKS Graphs can be used to model various networks that link particular types of information. Here, we will describe how to model the World Wide Web using a graph. We will also describe how to use a graph to model the citations in different types of documents.

The Web Graph The World Wide Web can be modeled as a directed graph where each Web page is represented by a vertex and where an edge starts at the Web page a and ends at the Web page b if there is a link on a pointing to b. Because new Web pages are created and others removed somewhere on the Web almost every second, the Web graph changes on an almost continual basis. Many people are studying the properties of the Web graph to better understand the nature of the Web.

Citation Graphs Graphs can be used to represent citations in different types of documents, including academic papers, patents, and legal opinions. In such graphs, each document is represented by a vertex, and there is an edge from one document to a second document if the first document cites the second in its citation list. (In an academic paper, the citation list is the bibliography, or list of references; in a patent it is the list of previous patents that are cited; and in a legal opinion it is the list of previous opinions cited.) A citation graph is a directed graph without loops or multiple edges.

Graph Models

TRANSPORTATION NETWORKS We can use graphs to model many different types of transportation networks, including road, air, and rail networks, as well shipping networks.

Airline Routes We can model airline networks by representing each airport by a vertex. In particular, we can model all the flights by a particular airline each day using a directed edge to represent each flight, going from the vertex representing the departure airport to the vertex representing the destination airport. The resulting graph will generally be a directed multigraph, as there may be multiple flights from one airport to some other airport during the same day. ◀

Road Networks Graphs can be used to model road networks. In such models, vertices represent intersections and edges represent roads. When all roads are two-way and there is at most one road connecting two intersections, we can use a simple undirected graph to model the road network. However, we will often want to model road networks when some roads are one-way and when there may be more than one road between two intersections. To build such models, we use undirected edges to represent two-way roads and we use directed edges to represent one-way roads. Multiple undirected edges represent multiple two-way roads connecting the same two intersections. Multiple directed edges represent multiple one-way roads that start at one intersection and end at a second intersection. Loops represent loop roads. Mixed graphs are needed to model road networks that include both one-way and two-way roads. ◀

BIOLOGICAL NETWORKS Many aspects of the biological sciences can be modeled using graphs.

Niche Overlap Graphs in Ecology Graphs are used in many models involving the interaction of different species of animals. For instance, the competition between species in an ecosystem can be modeled using a **niche overlap graph**. Each species is represented by a vertex. An undirected edge connects two vertices if the two species represented by these vertices compete (that is, some of the food resources they use are the same). A niche overlap graph is a simple graph because no loops or multiple edges are needed in this model. The graph in Figure 11 models the ecosystem of a forest. We see from this graph that squirrels and raccoons compete but that crows and shrews do not.

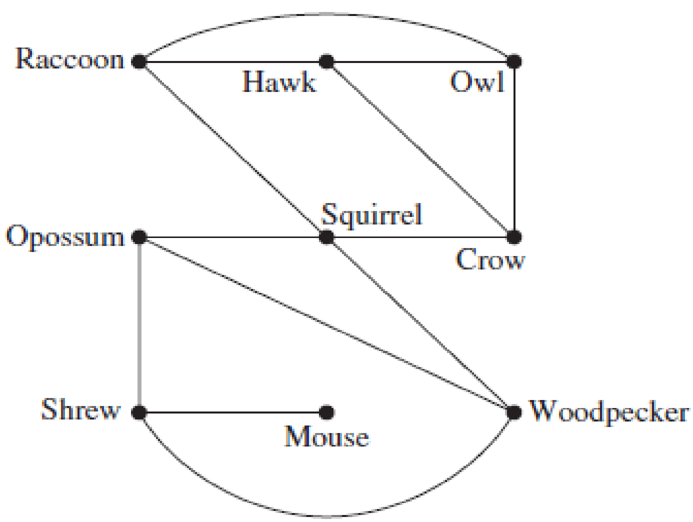


FIGURE 11 A Niche Overlap Graph.

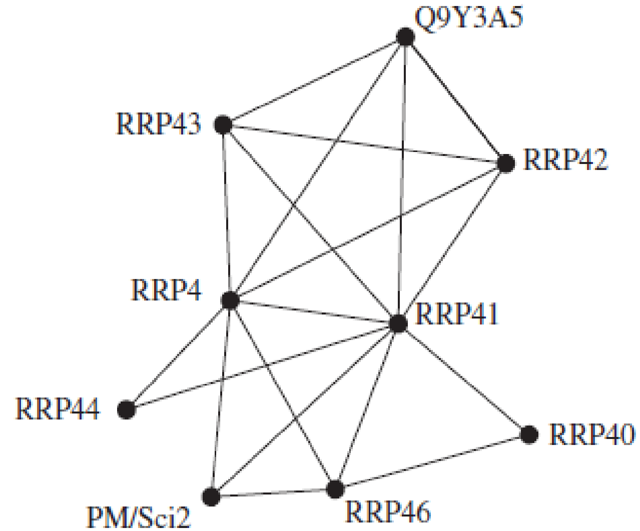


FIGURE 12 A Module of a Protein Interaction Graph.

Protein Interaction Graphs A protein interaction in a living cell occurs when two or more proteins in that cell bind to perform a biological function. Because protein interactions are crucial for most biological functions, many scientists work on discovering new proteins and understanding interactions between proteins. Protein interactions within a cell can be modeled using a **protein interaction graph** (also called a **protein–protein interaction network**), an undirected graph in which each protein is represented by a vertex, with an edge connecting the vertices representing each pair of proteins that interact. It is a challenging problem to determine genuine protein interactions in a cell, as experiments often produce false positives, which conclude that two proteins interact when they really do not. Protein interaction graphs can be used to deduce important biological information, such as by identifying the most important proteins for various functions and the functionality of newly discovered proteins.

Because there are thousands of different proteins in a typical cell, the protein interaction graph of a cell is extremely large and complex. For example, yeast cells have more than 6,000 proteins, and more than 80,000 interactions between them are known, and human cells have more than 100,000 proteins, with perhaps as many as 1,000,000 interactions between them. Additional vertices and edges are added to a protein interaction graph when new proteins and interactions between proteins are discovered. Because of the complexity of protein interaction graphs, they are often split into smaller graphs called modules that represent groups of proteins that are involved in a particular function of a cell. Figure 12 illustrates a module of the protein interaction graph described in [Bo04], comprising the complex of proteins that degrade RNA in human cells. To learn more about protein interaction graphs, see [Bo04], [Ne10], and [Hu07].

Graph Models

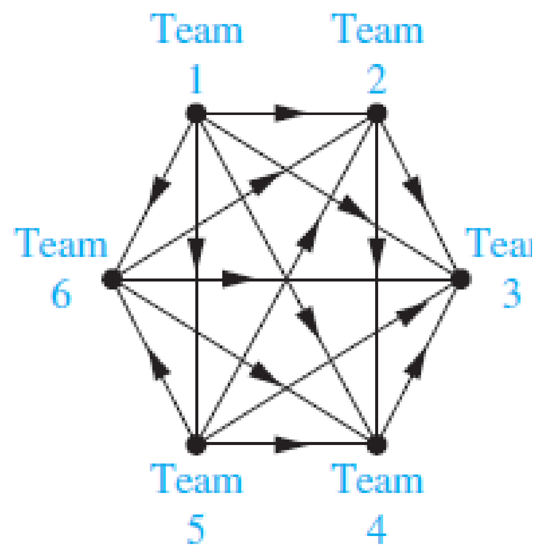


FIGURE 13 **A Graph Model of a Round-Robin Tournament.**

TOURNAMENTS We now give some examples that show how graphs can also be used to model different kinds of tournaments.

Round-Robin Tournaments A tournament where each team plays every other team exactly once and no ties are allowed is called a **round-robin tournament**. Such tournaments can be modeled using directed graphs where each team is represented by a vertex. Note that (a, b) is an edge if team a beats team b. This graph is a simple directed graph, containing no loops or multiple directed edges (because no two teams play each other more than once). Such a directed graph model is presented in Figure 13. We see that Team 1 is undefeated in this tournament, and Team 3 is winless.

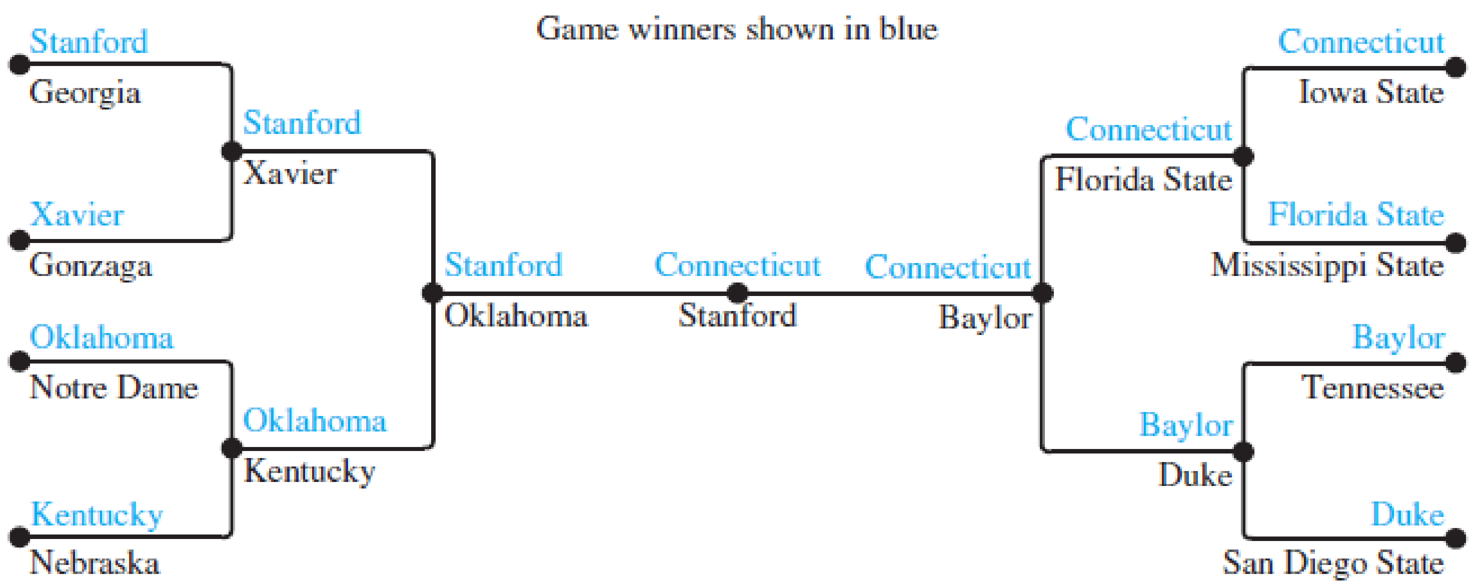


FIGURE 14 **A Single-Elimination Tournament.**

Single-Elimination Tournaments A tournament where each contestant is eliminated after one loss is called a **single-elimination tournament**. Single-elimination tournaments are often used in sports, including tennis championships and the yearly NCAA basketball championship. We can model such a tournament using a vertex to represent each game and a directed edge to connect a game to the next game the winner of this game played in. The graph in Figure 14 represents the games played by the final 16 teams in the 2010 NCAA women's basketball tournament. ◀

Definition of a graph

- A graph G is a finite nonempty set V(G) of vertices (also called nodes) and a (possibly empty) set E(G) of 2-element subsets of V(G) called edges (or lines).

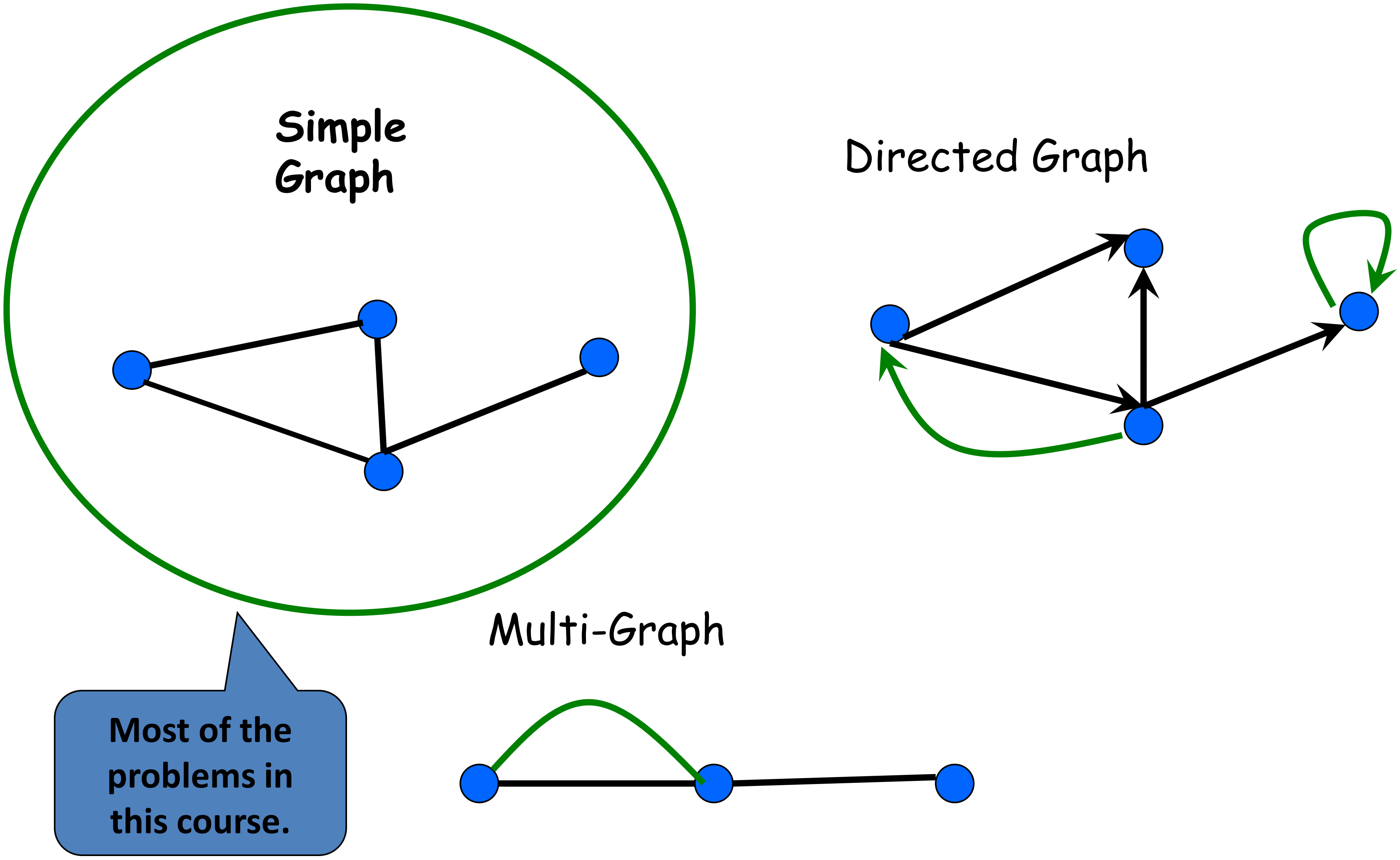
V(G) : vertex set of G

E(G) : edge set of G

edge : {u, v} = {v, u} = uv (or vu)

G directed graph (digraph) edge: (u,v)

Types of Graphs

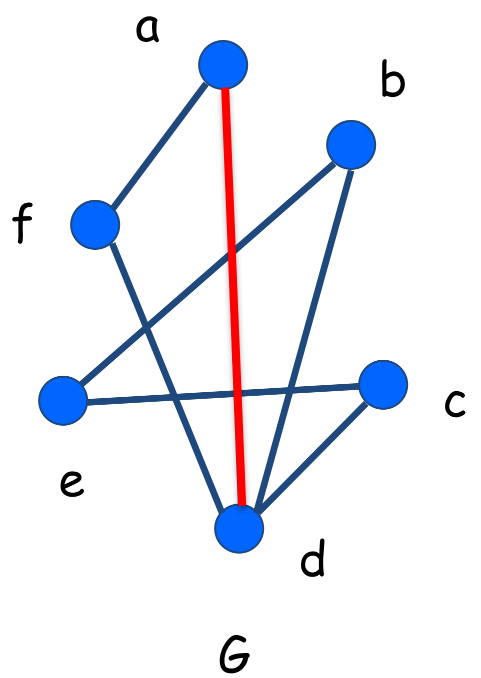


Simple Graphs

A graph G=(V,E) consists of:

A set of vertices, *V*

A set of *undirected* edges, *E*



- V(G) = {a,b,c,d,e,f}
- E(G) = {ad,af,bd,be,cd,ce,df}

Two vertices a,d are ***adjacent*** (***neighbours***) if the edge ad is present.

Example

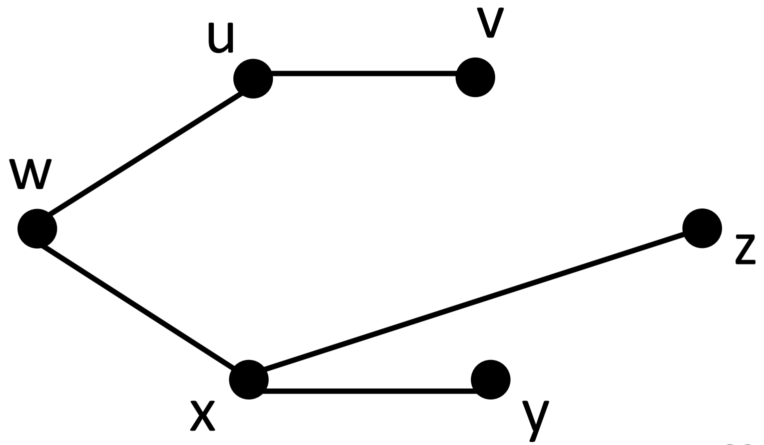
- A graph G=(V,E), where

V={u, v, w, x, y, z}

E={{u,v}, {u,w}, {w,x}, {x,y}, {x,z}}

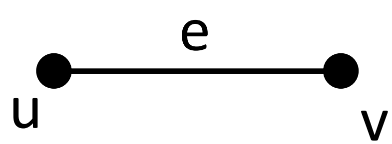
E={uv, uw, wx, xy, xz}

- G diagram :



Adjacent and Incident

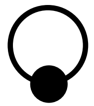
- u, v : vertices of a graph G



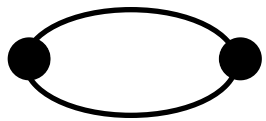
- u and v are adjacent in G if uv $\in$ E(G)
(u is adjacent to v, v is adjacent to u)
- e=uv (e joins u and v) (e is incident with u, e is incident with v)

Graphs types

loop



multiedges, parallel edges



- undirected graph:
  - (simple) graph: loop (✗), multiedge (✗)
  - multigraph: loop (✗), multiedge (✓)
  - Pseudograph: loop (✓), multiedge (✓)

order and size

- The number of vertices in a graph G is called its order (denoted by |V(G)|).
- The number of edges is its size (denoted by |E(G)|).
- Proposition 1:
  If |V(G)| = p and |E(G)| = q, then $q \leq \binom{p}{2}$
- A graph of order p and size q is called a (p, q) graph.

Application of graphs

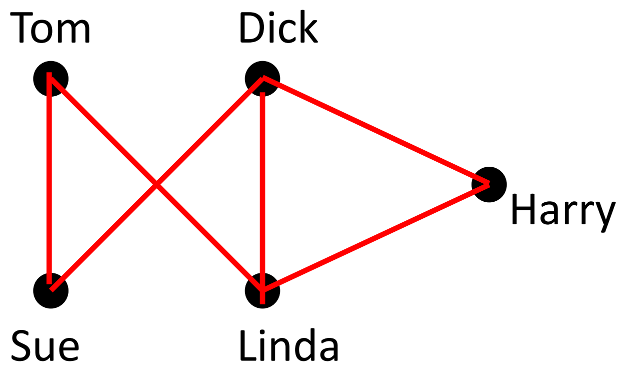
Example:

Tom, Dick know Sue, Linda.

Harry knows Dick and Linda.

$\Rightarrow$

acquaintance graph:



The degree of a vertex

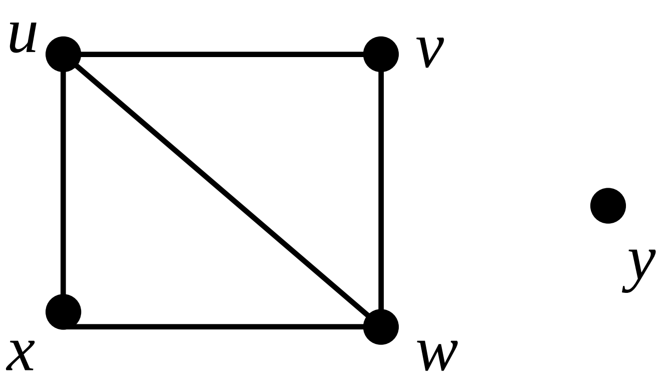
Definition.

For a vertex v of G, its neighborhood

$N(v) = \{ u \in V(G) \mid vu \in E(G) \}$.

The degree of vertex v is

$\deg(v) = | N(v) |$.

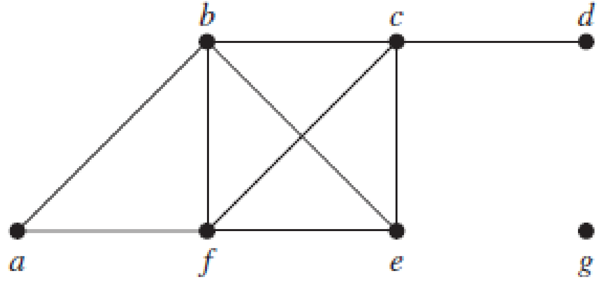


$N(u) = \{x, w, v\}$, $N(y)=\{ \}$

$\deg(u) = 3$, $\deg(y) =0$

Notes

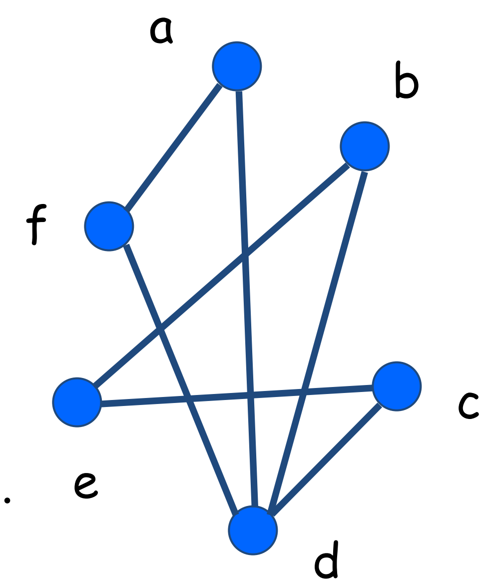
- If $|V(G)| = p$, then
  $0 \leq \deg(v) \leq p-1, \quad \forall\, v \in V(G)$.
- If $\deg(v) = 0$, then v is called an isolated vertex
- If $\deg(v) = 1$, then v is called an pendant vertex
- v is an odd vertex if $\deg(v)$ is odd.
  v is an even vertex if $\deg(v)$ is even.



G

Vertex Degrees

An edge uv is ***incident*** on
the vertex u and the vertex v.



The ***neighbour set*** N(v) of a vertex v
is the set of vertices adjacent to it.

e.g. N(a) = {d,f}, N(d) = {a,b,c,f}, N(e) = {b,c}.

degree of a vertex = # of ***incident*** edges

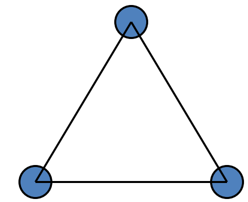
e.g. deg(d) = 4, deg(a)=deg(b)=deg(c)=deg(e)=deg(f)=2.

the degree of a vertex v = the number of neighbours of v?

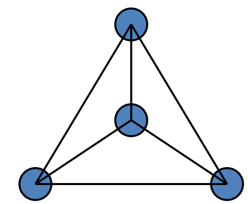
For multigraphs, NO. For simple graphs, YES.

Degree Sequence

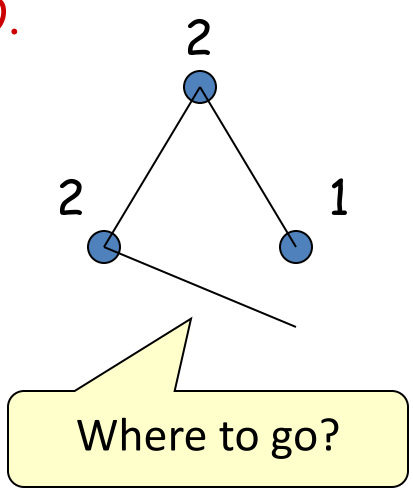
Is there a graph with degree sequence (2,2,2)? YES.



Is there a graph with degree sequence (3,3,3,3)? YES.



Is there a graph with degree sequence (2,2,1)? NO.

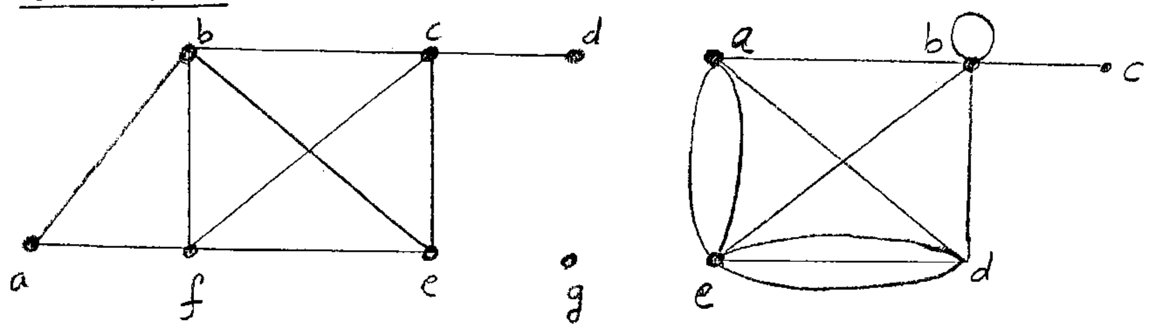


Is there a graph with degree sequence (2,2,2,2,1)?

NO.

What's wrong with these sequences?

Example:



G H

What are the degrees of the vertices in the graphs G and H ?

The *degree of a vertex in an undirected graph* is the number of edges incident with it, except that a loop at a vertex contributes twice to the degree of that vertex. The degree of the vertex v is denoted by $\deg(v)$.

Solution: In G, $\deg(a) = 2$, $\deg(b) = \deg(c) = \deg(f) = 4$, $\deg(d) = 1$, $\deg(e) = 3$, $\deg(g) = 0$.

In H, $\deg(a) = 4$, $\deg(b) = \deg(e) = 6$, $\deg(c) = 1$, and $\deg(d) = 5$.

Handshaking Lemma

For any graph, sum of degrees = twice # edges

Lemma.

$$\boxed{2|E| = \sum_{v \in V} \deg(v)}$$

Corollary.

1. Sum of degree is an even number.
2. Number of odd degree vertices is even.

Examples. 2+2+1 = odd, so impossible.

2+2+2+2+1 = odd, so impossible.

Handshaking Lemma

Lemma.
$$\boxed{2|E| = \sum_{v \in V} \deg(v)}$$

Proof. Each edge contributes 2 to the sum on the right.

Question. Given a degree sequence, if the sum of degree is even, is it true that there is a graph with such a degree sequence?

For simple graphs, NO, consider the degree sequence (3,3,3,1).

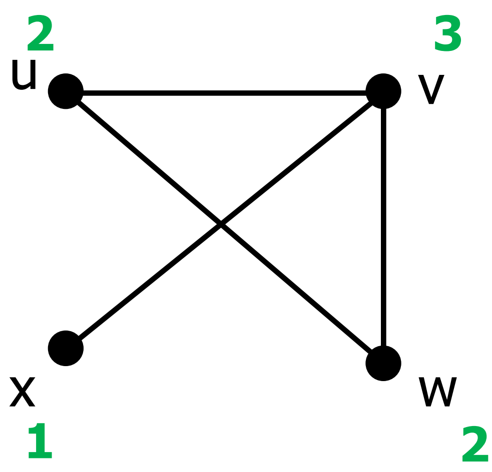
For multigraphs (with self loops), YES! (easy by induction)

Handshaking theorem

- **Theorem 1.1** (Handshaking theorem)
  Let G be a graph, then

$$\sum_{v \in V(G)} \deg(v) = |E(G)| \times 2$$

Example:



$$\sum_{v \in V(G)} \deg(v) = 8$$

$$|E(G)| = 4$$

Handshaking theorem

Corollary 1.1

Every graph contains an even number of odd vertices.

Proof: If the number of vertices with odd degree is odd, then the degree sum must be odd. $\rightarrow\leftarrow$

Example: How many edges are there in a graph with 10 vertices each of degree 6?

Example: How many edges are there in a graph with 10 vertices each of degree 6?

Solution:

Sum of degrees of vertices $= 6 \cdot 10 = 60$

$2e = 60 \Rightarrow e = \frac{60}{2} = 30$ edges.

Example: A certain graph G has order 14 and size 27. The degree of each vertex of G is 3, 4 or 5. There are six vertices of degree 4. How many vertices of G have degree 3 and how many have degree 5 ?

Solution: Let x be the number of vertices of G having degree 3.

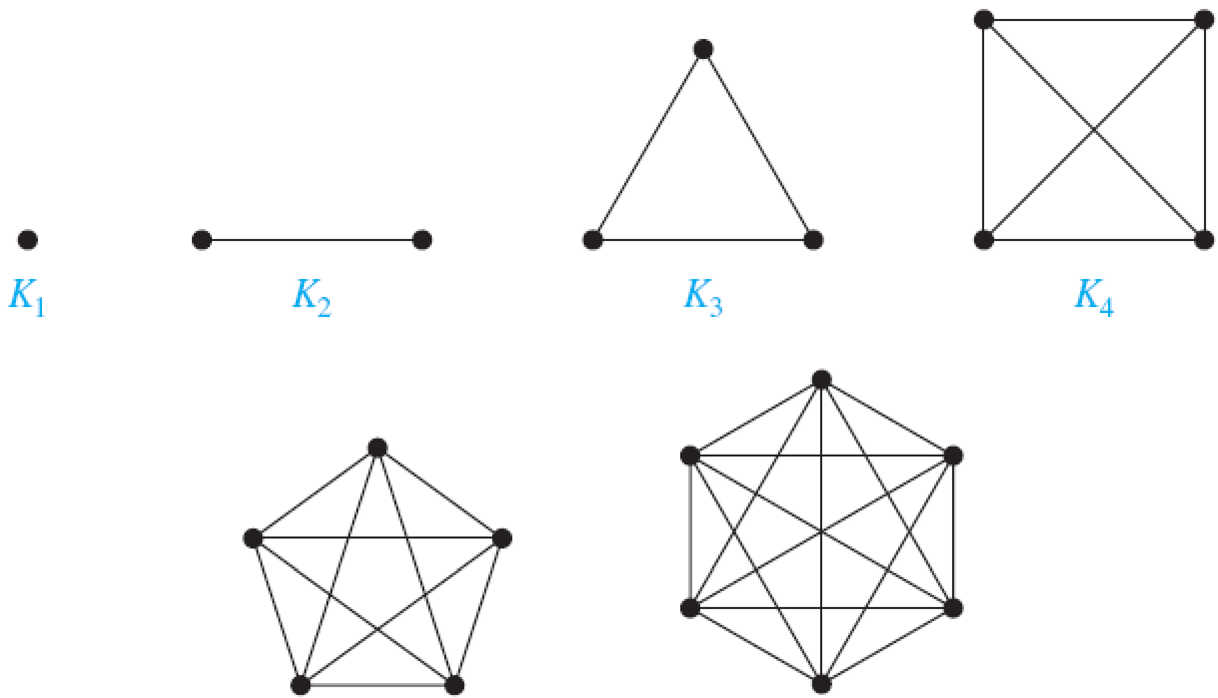
14-6=8 vertices have degree 3 or 5. So there are 8-x vertices of degree 5.

Then we have $3.x+4.6+5.(8-x)=2.27$

Hence $x=5$, $8-x=3$

Some Special Simple Graphs

Complete Graphs A **complete graph on n vertices**, denoted by K_n, is a simple graph that contains exactly one edge between each pair of distinct vertices. The graphs K_n, for $n = 1, 2, 3, 4, 5, 6$, are displayed in Figure 3. A simple graph for which there is at least one pair of distinct vertex not connected by an edge is called **noncomplete**.



K_1 K_2 K_3 K_4

K_5 K_6

Cycles A **cycle** C_n, $n \geq 3$, consists of n vertices $v_1, v_2, \ldots, v_n$ and edges $\{v_1, v_2\}$, $\{v_2, v_3\}, \ldots, \{v_{n-1}, v_n\}$, and $\{v_n, v_1\}$. The cycles C_3, C_4, C_5, and C_6 are displayed in Figure 4.

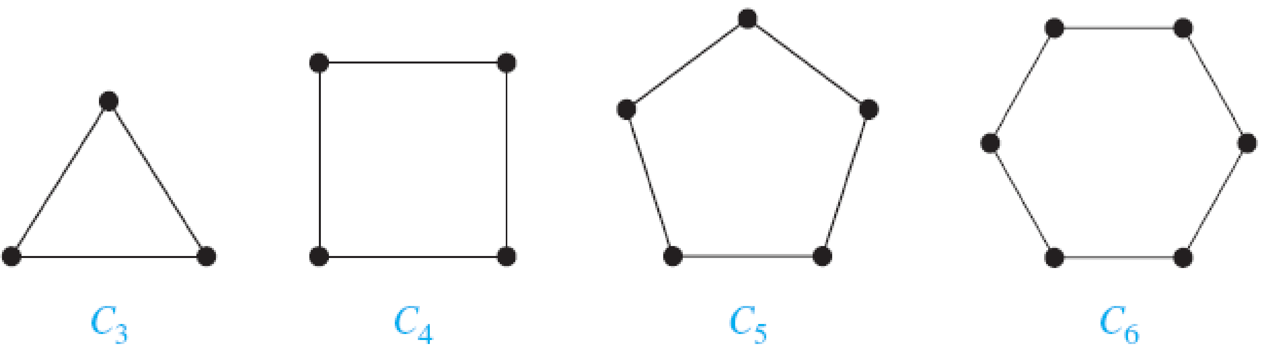


FIGURE 4 **The Cycles C_3, C_4, C_5, and C_6.**

Wheels We obtain a **wheel W_n** when we add an additional vertex to a cycle C_n, for $n \geq 3$, and connect this new vertex to each of the n vertices in C_n, by new edges. The wheels W_3, W_4, W_5, and W_6 are displayed in Figure 5.

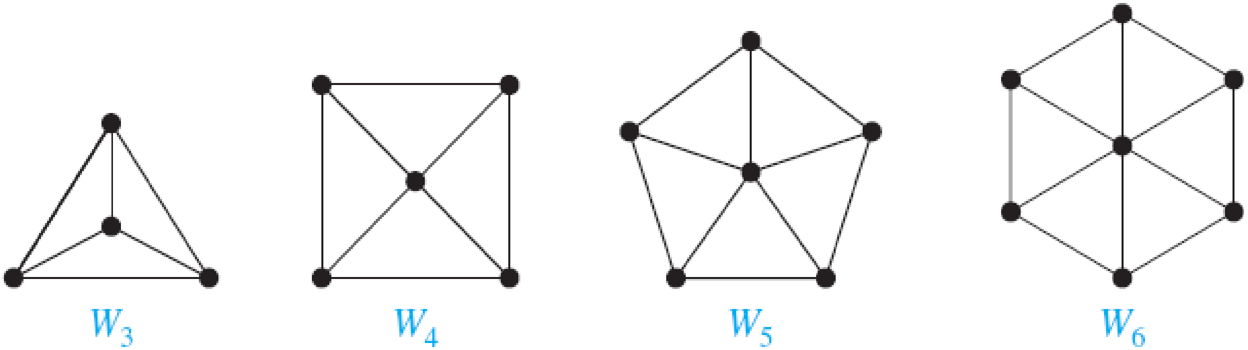


FIGURE 5 **The Wheels W_3, W_4, W_5, and W_6.**

***n*-Cubes** An ***n*-dimensional hypercube**, or ***n*-cube**, denoted by Q_n, is a graph that has vertices representing the 2^n bit strings of length n. Two vertices are adjacent if and only if the bit strings that they represent differ in exactly one bit position. We display Q_1, Q_2, and Q_3 in Figure 6.

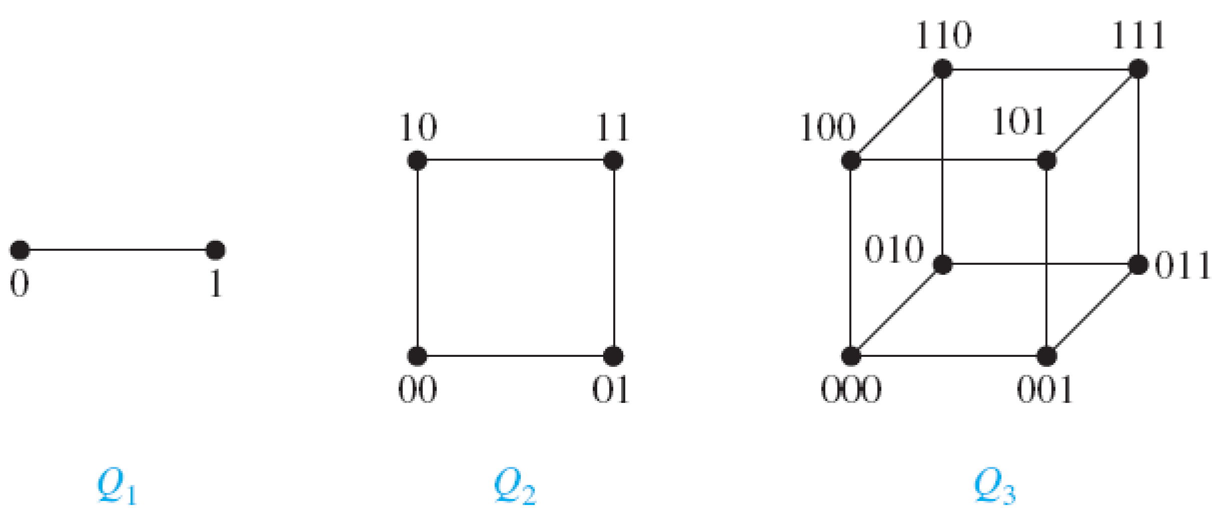


FIGURE 6 **The *n*-cube Q_n, $n = 1, 2, 3$.**

Note that you can construct the $(n+1)$-cube Q_{n+1} from the n-cube Q_n by making two copies of Q_n, prefacing the labels on the vertices with a 0 in one copy of Q_n and with a 1 in the other copy of Q_n, and adding edges connecting two vertices that have labels differing only in the first bit. In Figure 6, Q_3 is constructed from Q_2 by drawing two copies of Q_2 as the top and bottom faces of Q_3, adding 0 at the beginning of the label of each vertex in the bottom face and 1 at the beginning of the label of each vertex in the top face. (Here, by *face* we mean a face of a cube in three-dimensional space. Think of drawing the graph Q_3 in three-dimensional space with copies of Q_2 as the top and bottom faces of a cube and then drawing the projection of the resulting depiction in the plane.)

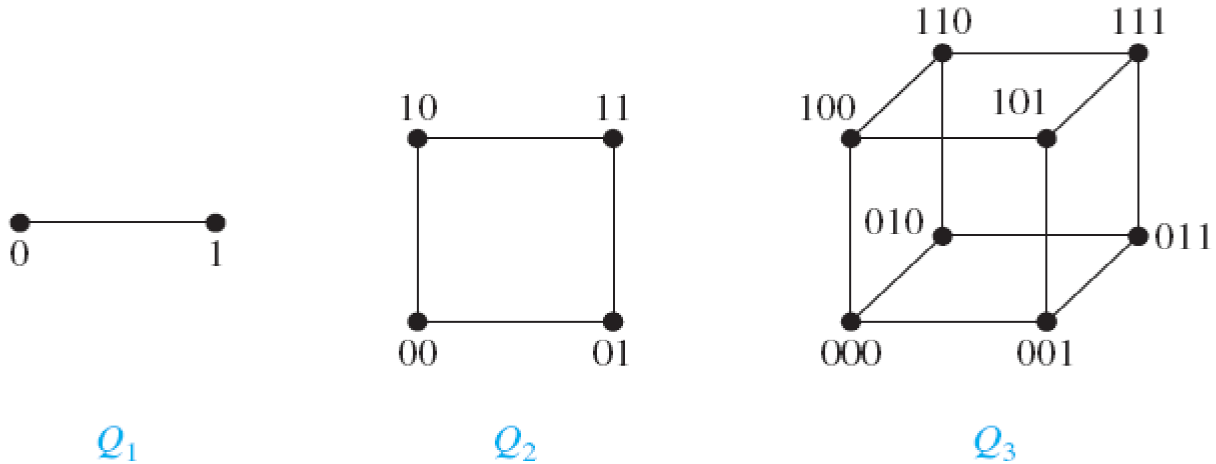


FIGURE 6 **The n-cube Q_n, $n = 1, 2, 3$.**