Mat2033 - Discrete Mathematics

Relations and their Properties Continue…

- Let R be a relation from set A to set B
- Let S be a relation from set B to set C

- The composite of R and S is a relation from set A to set C
  - and is a set of ordered pairs (a,c) such that
  - there exists an (a,b) in R and an (b,c) in S

- The composite of R and S is denoted as SoR

The composite of R with S $S \circ R$

$$R = \{(1,1), (1,4), (2,3), (3,1), (3,4)\}$$

$$S = \{(1,0), (2,0), (3,1), (3,2), (4,1)\}$$

$$S \circ R = \{(1,0), (1,1), (2,1), (2,2), (3,0), (3,1)\}$$

Assume we have a relation R of people to motorcycles they own.
A person could have more that one motorcycle.
R is a set of ordered pairs {(Patrick,RGV),(Denis,Buell),
(Stan,ElectraGlide),(Denis,Hayabusa),(Gordon,Bandit),
(Gordon,R6)}

We could have a relation S of motorcycles to top speed
{(RGV,130),(Buell,126),(ElectraGlide,110),(Hayabusa,182),
(Bandit,140),(R6,155)}

So R is then the relation of people to possible top speeds
{(Patrick,130),(Denis,126),(Denis,182),(Stan,110),
(Gordon,140),(Gordon,155)}

Composite of a Relation with itself

Let R be a relation on the set A. The powers R^n are defined inductively as follows

$$R^1 = R$$

$$R^n = R^{n-1} \circ R$$

$$\therefore R^2 = R^1 \circ R = R \circ R$$

$$\therefore R^3 = R^2 \circ R$$

Composite of a Relation with itself

$$R^1 = R$$

$$R^n = R^{n-1} \circ R$$

$$R = \{(1,1),(2,1),(3,2),(4,3)\}$$

$$R^2 = \{(1,1),(2,1),(3,1),(4,2)\}$$

$$R^3 = \{(1,1),(2,1),(3,1),(4,1)\}$$

$$R^4 = \{(1,1),(2,1),(3,1),(4,1)\}$$

$$\therefore R^n = R^3$$

A Transitive Relation

Theorem: A relation R on a set A is transitive iff R^n is a subset of R for n = 1,2,3,...

Assume : $R^2 \subseteq R$

$(a,b) \in R \wedge (b,c) \in R \rightarrow (a,c) \in R^2$ // by definition of composition

$\therefore (a,c) \in R^2$

$(a,c) \in R^2 \wedge R^2 \subseteq R \rightarrow (a,c) \in R$

$\therefore transitive(R)$

The inductive step

Assume : $R^n \subseteq R$ // The inductive hypothesis

Show : $R^{n+1} \subseteq R$

Assume : $(a,b) \in R^{n+1}$

$R^{n+1} = R^n \circ R$ // Definition of composition

$\exists x(x \in A \wedge (a,x) \in R \wedge (x,b) \in R^n)$

$\mathrm{R}^{\mathrm{n}} \subseteq R \rightarrow (x,b) \in R$ // Using the inductive hypothesis

Since $transitive(R) \wedge (a,x) \in R \wedge (x,b) \in R \rightarrow (a,b) \in R$

$\therefore R^{n+1} \subseteq R$

Q.E.D

Example:

A = {1,2,3,4}

R = {(a,b) | a divides b}

R = {(1,1),(1,2),(1,3),(1,4),(2,2),(2,4),(3,3),(4,4)}

RoR = {(1,1),(1,2),(1,3),(1,4),(2,2),(2,4),(3,3),(4,4)}

RoR is a subset of R, therefore transitive

Obvious! If a divides b abd b divides c then a divides c!

Exercise:

Let R be a relation on people such that (a,b) is “a is a parent of b”

Let S be a relation on people such that (a,b) is “a is a sibling of b”

- What is SoR, RoS, RoR?

  - SoR composes R with S
    - (a,c) is in SoR if there exists
      - (a,b) in R and a (b,c) in S
      - “a is parent of b” and “b is sibling of c”
    - SoR should be in R!
  - RoS composes S with R
    - “a is sibling of b” and “b is a parent of c”
    - therefore (a,c) is “a is an aunt/uncle of c”

Here are a few examples of relations with (or without) these properties, all on the set $A = \{1, 2, 3, 4, 5\}$. We denote the relation by R; thus R is a set of ordered pairs from A. The graph we call G, and the matrix M, with rows and columns are labeled as before.

<u>Example:</u> $R_1 = \{(1,1), (2,2), (3,3), (4,4), (5,5), (1,3), (3,4)\}$

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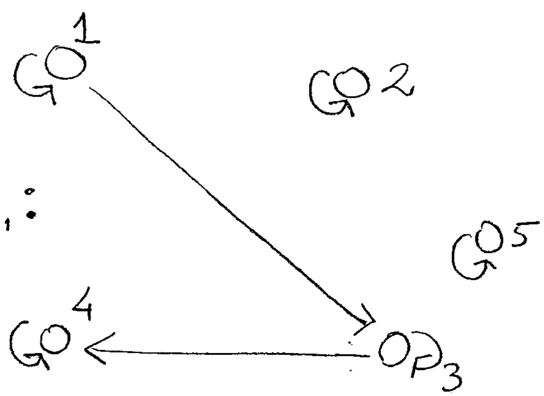
G_1:



M_1:

	1	2	3	4	5
1	1	0	1	0	0
2	0	1	0	0	0
3	0	0	1	1	0
4	0	0	0	1	0
5	0	0	0	0	1

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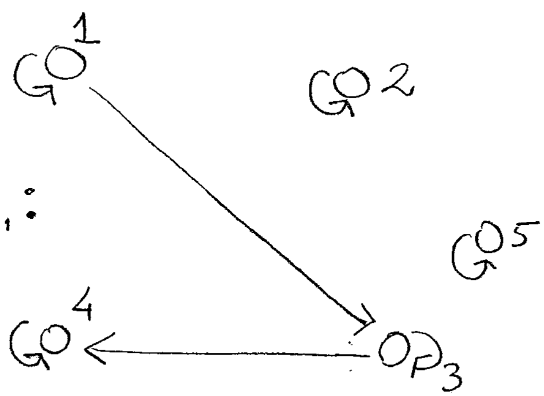
A reflexive but not symmetric and not transitive relation, $R_1 = \{(1,1), (2,2), (3,3), (4,4), (5,5), (1,3), (3,4)\}$

G_1:



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3	0	0	1	1	0
4	0	0	0	1	0
5	0	0	0	0	1

The presence of all the ordered pairs of the form (a,a) in R_1 is the requirement of the definition of reflexivity. We can also see that R_1 is reflexive by noting that there is a loop at every point of the graph G_1, or that the main diagonal of M_1 consists entirely of 1's. R_1 is not symmetric because $(1,3) \in R_1$ but $(3,1) \notin R_1$. In G_1 there is no backward arrow to correspond to the arrow from 1 to 3; the matrix M_1 is not symmetric about the main diagonal. Since $(1,3)$ and $(3,4)$ are in R_1 but $(1,4)$ is not in R_1, it is not transitive.

Example: Let $R_2 = \{(1,1), (1,2), (1,5), (2,1), (2,2), (2,5), (3,5), (4,4), (5,1), (5,2), (5,3), (5,5)\}$.

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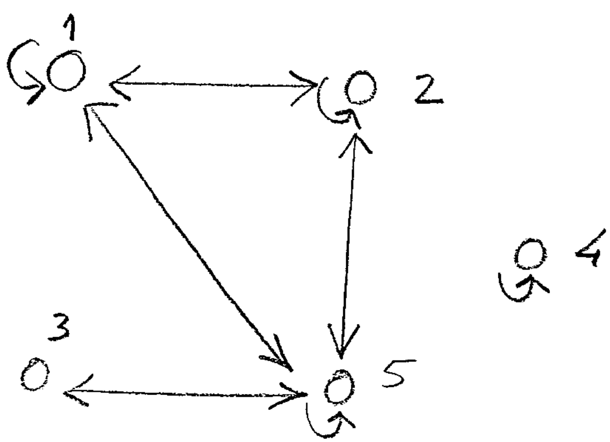
G_2:

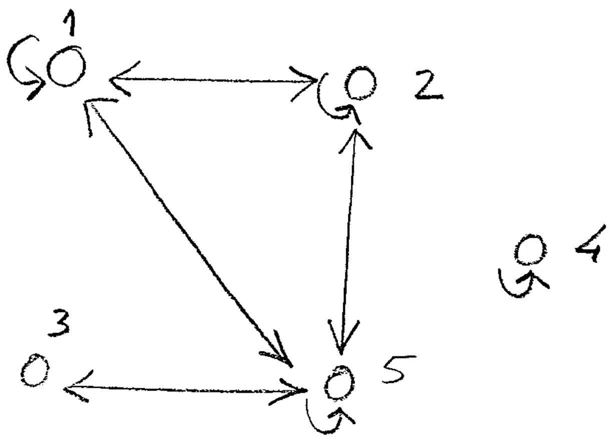


$$M_2: \begin{matrix} 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \end{matrix}$$

Example: Let $R_2 = \{(1,1), (1,2), (1,5), (2,1), (2,2), (2,5), (3,5), (4,4), (5,1), (5,2), (5,3), (5,5)\}$.

Then R_2 is symmetric but is neither reflexive nor transitive.

G_2:



$$M_2: \begin{matrix} 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \end{matrix}$$

In this example the symmetry is obvious from all three points of view. R_2 is not reflexive because (3,3) is not in R_2; there is a '0' on the main diagonal. The nontransitivity is most apparent from the graph; there is a path from 3 to 5 to 1 but there is no shortcut from 3 to 1. Similarly for 3 and 2, and for the paths in the reverse directions.

Example: Take $R_3 = \{(1,2), (2,3), (1,3), (3,3), (4,4), (5,5), (2,2), (3,2)\}$

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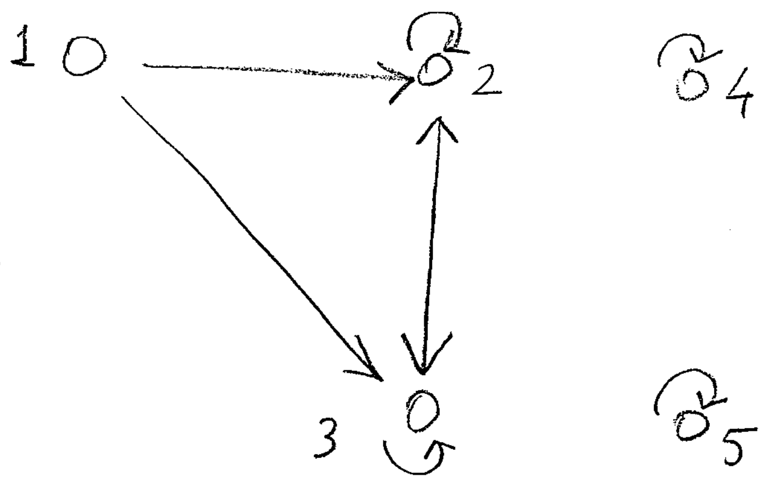
G_3:

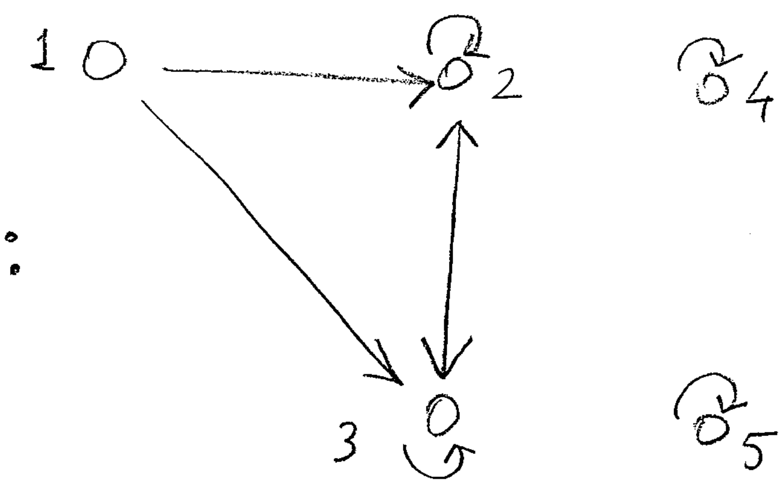


M_3: $\left[\begin{array}{ccc|cc} 0 & 1 & 1 & & \\ 0 & 1 & 1 & \multicolumn{2}{c}{0} \\ 0 & 1 & 1 & & \\ \hline & & & 1 & 0 \\ & & & 0 & 1 \end{array}\right]$

Example: Take $R_3 = \{(1,2), (2,3), (1,3), (3,3), (4,4), (5,5), (2,2), (3,2)\}$

R_3 is transitive but neither reflexive nor symmetric.

G_3:



$$M_3: \left[\begin{array}{ccc|cc} 0 & 1 & 1 & & \\ 0 & 1 & 1 & \multicolumn{2}{c}{0} \\ 0 & 1 & 1 & & \\ \hline & & & 1 & 0 \\ & & & 0 & 1 \end{array}\right]$$

A blank submatrix box, or one with a lone 0, is understood to have all its entries 0. Thus in M_3 the upper right and lower left corners are all 0's. Reflexivity fails because (1,1) is not in R_3; the loop at 1 does not exist in G_3; there is a 0 on the main diagonal of M_3. Symmetry fails because the back arrow is absent from 2 to 1 — for every arrow there must be a back arrow, or else the graph (relation) is not symmetric. Or you can simply notice that the matrix is not symmetric about the main diagonal.

Now we discuss equivalence relations, those that are reflexive and symmetric and transitive.

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<u>Example:</u> $R_4 = A \times A$. For R_4 we have taken the set of all possible ordered pairs from the set A. Obviously R_4 is an equivalence relation. We show the graph G_4 and the matrix M_4 in the case that $A = \{1, 2, 3, 4, 5\}$

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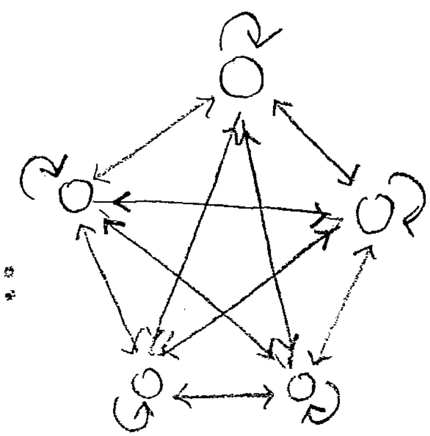
G_4:



$$M_4: \quad \begin{matrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{matrix}$$

Example: Let $R_5 = \{(1,1), (2,2), (3,3), (4,4), (5,5)\}$. The graph and matrix are

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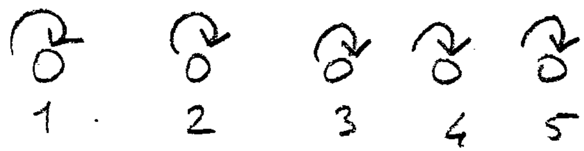
G_5: 

$$M_5: \begin{vmatrix} 1 & & & & \\ & 1 & & 0 & \\ & & 1 & & \\ & 0 & & 1 & \\ & & & & 1 \end{vmatrix}$$

This equivalence relation is the other extreme from R_4. It is the smallest subset of $A \times A$ that is reflexive, symmetric and transitive.

Example: $R_6 = R_5 \cup \{(1,2), (2,1), (1,3), (3,1), (2,3), (3,2), (4,5), (5,4)\}$

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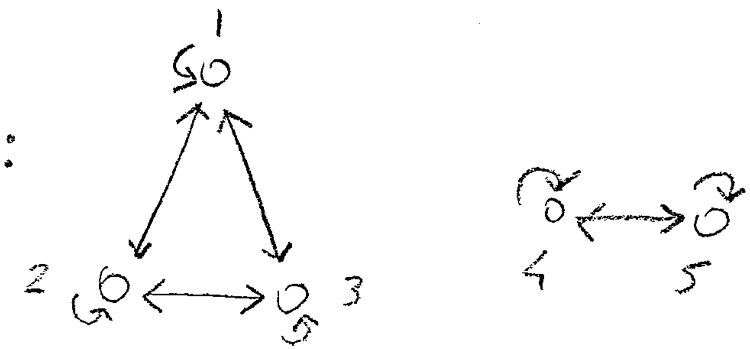
G_6:



$$M_6: \left[\begin{array}{ccc|cc} 1 & 1 & 1 & & \\ 1 & 1 & 1 & & \\ 1 & 1 & 1 & & \\ \hline & & & 1 & 1 \\ & & & 1 & 1 \end{array}\right]$$

Blank box in the upper right means there are no arrows from any point of $\{1,2,3\}$ to any point of $\{4,5\}$. Similarly, the blank box in the lower left means that there are no arrows fro $4,5$ to any of $1,2,3$.

Exercise:

Consider the following relations on $\{1, 2, 3, 4\}$:

$$R_1 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)\},$$
$$R_2 = \{(1, 1), (1, 2), (2, 1)\},$$
$$R_3 = \{(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (3, 3), (4, 1), (4, 4)\},$$
$$R_4 = \{(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)\},$$
$$R_5 = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)\},$$

$X = \{1, 2, 3, 4, 5, 6\}$. The relation R on X given by

$$R = \{(1,1), (1,3), (1,5), (3,1), (3,3), (3,5), (5,1), (5,3), (5,5), (2,2), (2,6), (6,2), (6,6), (4,4)\} \blacksquare$$

Let R be an equivalence relation on a set X. For each $a \in X$, let

$$[a] = \{x \in X \mid xRa\}$$

Then

<u>Definition:</u> Let R be an equivalence relation on a set X. The sets $[a]$ defined in the above are called the equivalence classes of X given by the relation R.

Theorem 1 :

Let R be an equivalence relation on a set A. These statements for elements a and b of A are equivalent:

$(i)\ aRb \qquad (ii)\ [a] = [b] \qquad (iii)\ [a] \cap [b] \neq \emptyset$

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Proof: We first show that (i) implies (ii). Assume that aRb. We will prove that $[a] = [b]$ by showing $[a] \subseteq [b]$ and $[b] \subseteq [a]$. Suppose $c \in [a]$. Then aRc. Because aRb and R is symmetric, we know that bRa. Furthermore, because R is transitive and bRa and aRc, it follows that bRc. Hence, $c \in [b]$. This shows that $[a] \subseteq [b]$. The proof that $[b] \subseteq [a]$ is similar; it is left as an exercise for the reader.

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Next, we will show that (iii) implies (i). Suppose that $[a] \cap [b] \neq \emptyset$. Then there is an element c with $c \in [a]$ and $c \in [b]$. In other words, aRc and bRc. By the symmetric property, cRb. Then by transitivity, because aRc and cRb, we have aRb.

Because (i) implies (ii), (ii) implies (iii), and (iii) implies (i), the three statements, (i), (ii), and (iii), are equivalent. $\triangleleft$

We are now in a position to show how an equivalence relation *partitions* a set. Let R be an equivalence relation on a set A. The union of the equivalence classes of R is all of A, because an element a of A is in its own equivalence class, namely, $[a]_R$. In other words,

$$\bigcup_{a \in A} [a]_R = A.$$

In addition, from Theorem 1, it follows that these equivalence classes are either equal or disjoint, so

$$[a]_R \cap [b]_R = \emptyset,$$

when $[a]_R \neq [b]_R$.

These two observations show that the equivalence classes form a partition of A, because they split A into disjoint subsets.

More precisely, a **partition** of a set S is a collection of disjoint nonempty subsets of S that have S as their union. In other words, the collection of subsets A_i, $i \in I$ (where I is an index set) forms a partition of S if and only if

$$A_i \neq \emptyset \text{ for } i \in I,$$

$$A_i \cap A_j = \emptyset \text{ when } i \neq j,$$

and

$$\bigcup_{i \in I} A_i = S.$$

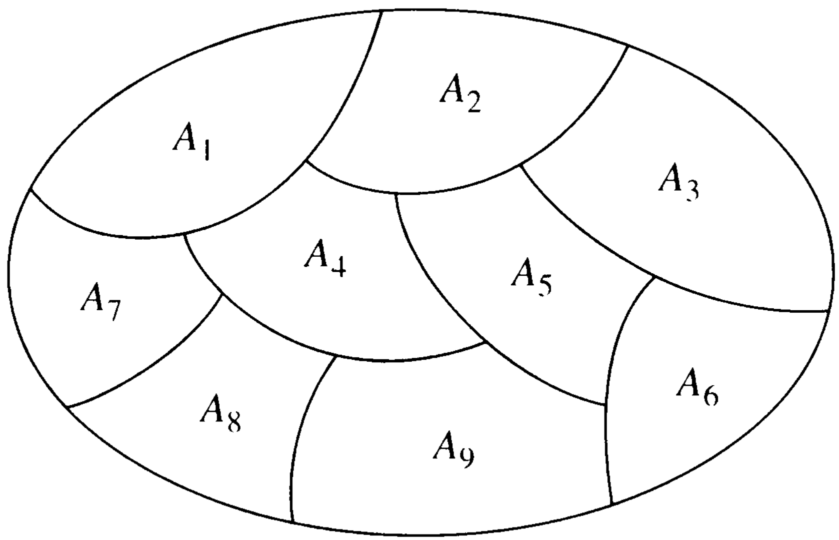
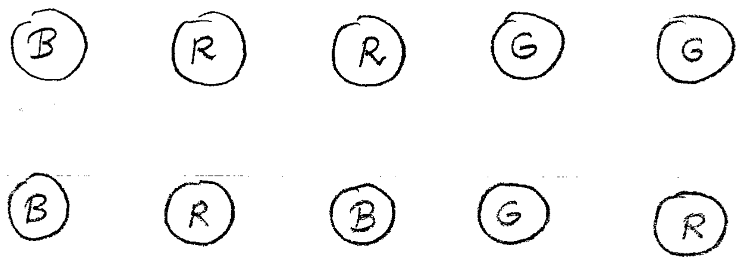


FIGURE 1 A Partition of a Set.

Partition of a set:

Suppose that we have a set X of 10 balls, each of which is either red, blue, or green



If we divide the balls into sets R, B, and G according to color, the family $\{R, B, G\}$ is a partition of X

A partition can be used to define a relation. If $\mathcal{S}$ is a partition of X, we may define xRy to mean that for some set $S \in \mathcal{S}$, both x and y belong to S. For example the relation obtained could be described as "is the same color as". The next theorem shows that such a relatio is always reflexive, symmetric, and transitive.

Theorem: Let $\mathcal{S}$ be a partition of a set X. Define xRy to mean that for some set S in $\mathcal{S}$, both x and y belong to S. Then R is reflexive, symmetric, and transitive.

Proof: Let $x \in X$. By the definition of partition, x belongs to some member of $S \in \mathcal{S}$. Thus xRx and R is reflexive.

Suppose that xRy. Then both x and y belong to some set $S \in \mathcal{S}$. Since both y and x belong to S, yRx and R is symmetric.

Finally, suppose that xRy and yRz. Then both x and y belong to some set $S \in \mathcal{S}$ and both y and z belong to some set $T \in \mathcal{S}$. Since y belongs to exactly one member of $\mathcal{S}$, we must have $S = T$. Therefore, both x and z belong to S and xRz. We have shown that R is transitive.

Example: Consider the partition

$$\mathcal{S} = \{\{1,3,5\}, \{2,6\}, \{4\}\}$$

of $X = \{1,2,3,4,5,6\}$. The relation R on X given by the above thm. contains the ordered pairs $(1,1)$, $(1,3)$, and $(1,5)$ because $\{1,3,5\}$ is in $\mathcal{S}$. The complete relation is

Example: Consider the partition

$$\mathcal{S} = \{\{1,3,5\}, \{2,6\}, \{4\}\}$$

of $X = \{1,2,3,4,5,6\}$. The relation $\mathcal{R}$ on X given by the above thm. contains the ordered pairs $(1,1)$, $(1,3)$, and $(1,5)$ because $\{1,3,5\}$ is in $\mathcal{S}$. The complete relation is

$$\mathcal{R} = \{(1,1), (1,3), (1,5), (3,1), (3,3), (3,5), (5,1), (5,3), (5,5),$$
$$(2,2), (2,6), (6,2), (6,6), (4,4)\} \blacksquare$$

Theorem: Let R be an equivalence relation on a set X. For each $a \in X$, let

$$[a] = \{x \in X \mid xRa\}$$

Then

$$S = \{[a] \mid a \in X\}$$

is a partition of X.

Proof: We must show that every element in X belongs to exactly one member of S.

Let $a \in X$. Since aRa, $a \in [a]$. Thus every element in X belongs to at least one member of S. It remains to show that every element in X belongs to exactly one member of S; that is,

$$\text{if } x \in X \text{ and } x \in [a] \cap [b], \text{ then } [a] = [b] \quad (*)$$

We first show that if aRb, then $[a] = [b]$. Suppose that aRb. Let $x \in [a]$. Then xRa. Since aRb and R is transitive, xRb. Therefore, $x \in [b]$ and $[a] \subseteq [b]$. The argument that $[b] \subseteq [a]$ is the same as that just given, but with the roles of a and b interchanged. Thus $[a] = [b]$.

We now prove $(*)$. Assume that $x \in X$ and $x \in [a] \cap [b]$. Then xRa and xRb. Our preceding result shows that $[x] = [a]$ and $[x] = [b]$. Thus $[a] = [b]$.

Example: Consider the partition

$$\mathcal{S} = \{\{1,3,5\},\{2,6\},\{4\}\}$$

of $X = \{1,2,3,4,5,6\}$. The equivalence class of $[1]$ containing 1 consists of all x such that $(x,1) \in R$. Therefore,

$$[1] = \{1,3,5\}.$$

The remaining equivalence classes are found similarly:

$$[3] = [5] = \{1,3,5\},$$

$$[2] = [6] = \{2,6\}$$

$$[4] = \{4\}.$$

Example: Let $X = \{1, 2, ..., 10\}$. Define xRy to mean that 3 divides $x-y$. We can readily verify that the relation R is reflexive, symmetric, and transitive. Thus R is an equivalence relation on X.

Let us determine the members of the equivalence classes. The equivalence class $[1]$ consists of all x with $xR1$. Thus

$$[1] = \{x \in X \mid 3 \text{ divides } x-1\} = \{1, 4, 7, 10\}$$

Similarly,

$$[2] = \{2, 5, 8\}$$

$$[3] = \{3, 6, 9\}$$

These three sets partition X. Note that

$$[1] = [4] = [7] = [10]$$

$$[2] = [5] = [8]$$

$$[3] = [6] = [9].$$

Theorem: Let R be an equivalence relation on a finite set X. If each equivalence class has r elements, there are $|X|/r$ equivalence classes.

Proof: Let $X_1, X_2, \ldots, X_k$ denote the distinct equivalence classes. Since these sets partition X,

$$|X| = |X_1| + |X_2| + \cdots + |X_k| = r + r + \cdots + r = kr$$

and the conclusion follows.

#1. Let R be the following relation defined on the set $\{a, b, c, d\}$:

$$R = \{(a, a), (a, c), (a, d), (b, a), (b, b), (b, c), (b, d), (c, b), (c, c), (d, b), (d, d)\}.$$

Determine whether R is:

(a) reflexive.

(b) symmetric.

(c) antisymmetric.

Solution:

(a) R is reflexive because R contains (a, a), (b, b), (c, c), and (d, d).

(b) R is not symmetric because $(a, c) \in R$, but $(c, a) \notin R$.

(c) R is not antisymmetric because both $(b, c) \in R$ and $(c, b) \in R$, but $b \neq c$.

Determine whether R is transitive.

Solution:

The relation R is not transitive because, for example, $(a, c) \in R$ and $(c, b) \in R$, but $(a, b) \notin R$.

- Floor Function: $\lfloor x \rfloor$ means take the greatest integer less than or equal to the number

Let n be an integer

(1a) $\lfloor x \rfloor = n$ if and only if $n \le x < n+1$

Example:

$\lfloor \frac{1}{2} \rfloor = 0$, $\lfloor -\frac{1}{2} \rfloor = -1$

$\lfloor 3.1 \rfloor = 3$ $\lfloor 7 \rfloor = 7$

#2. Let R be the following relation on the set of real numbers:

$$aRb \leftrightarrow \lfloor a \rfloor = \lfloor b \rfloor, \text{ where } \lfloor x \rfloor \text{ is the floor of } x.$$

Determine whether R is:

(a) reflexive. (b) symmetric. (c) antisymmetric.

Solution:

(a) R is reflexive: $\lfloor a \rfloor = \lfloor a \rfloor$ is true for all real numbers.

(b) R is symmetric: suppose $\lfloor a \rfloor = \lfloor b \rfloor$; then $\lfloor b \rfloor = \lfloor a \rfloor$.

(c) R is not antisymmetric: we can have aRb and bRa for distinct a and b. For example, $\lfloor 1.1 \rfloor = \lfloor 1.2 \rfloor$.

Determine whether R is transitive.

Solution:

R is transitive: suppose $\lfloor a \rfloor = \lfloor b \rfloor$ and $\lfloor b \rfloor = \lfloor c \rfloor$; from transitivity of equality of real numbers, it follows that $\lfloor a \rfloor = \lfloor c \rfloor$.

#4. Let $A = \{(x, y) \mid x, y \text{ integers}\}$. Define a relation R on A by the rule

$$(a, b)R(c, d) \leftrightarrow a \leq c \text{ and } b \leq d.$$

Determine whether R is:

(a) reflexive. (b) symmetric. (c) antisymmetric.

Solution:

(a) R is reflexive: $(a, b)R(a, b)$ for all elements (a, b) because $a \leq a$ and $b \leq b$ is always true.

(b) R is not symmetric: For example, $(1, 2)R(3, 7)$ (because $1 \leq 3$ and $2 \leq 7$), but $(3, 7)\not R(1, 2)$.

(c) R is antisymmetric: Suppose $(a, b)R(c, d)$ and $(c, d)R(a, b)$. Therefore $a \leq c$, $c \leq a$, $b \leq d$, $d \leq b$. Therefore $a = c$ and $b = d$, or $(a, b) = (c, d)$.

Determine whether R is transitive.

Solution:

R is transitive: Suppose $(a, b)R(c, d)$ and $(c, d)R(e, f)$. Therefore $a \leq c$ and $c \leq e$, and $b \leq d$ and $d \leq f$. Therefore, $a \leq e$ and $b \leq f$, or $(a, b)R(e, f)$.

#5. Let $A = \{(x, y) \mid x, y \text{ integers}\}$. Define a relation R on A by the rule

$$(a, b)R(c, d) \;\leftrightarrow\; a = c \text{ or } b = d.$$

Determine whether R is:

(a) reflexive. (b) symmetric. (c) antisymmetric.

Solution:

(a) R is reflexive: $(a, b)R(a, b)$ for all elements (a, b) because $a = a$ and $b = b$ are always true.

(b) R is symmetric: Suppose $(a, b)R(c, d)$. Therefore $a = c$ or $b = d$. Therefore $c = a$ or $d = b$. Therefore $(c, d)R(a.b)$.

(c) R is not antisymmetric: For example, $(1, 2)R(1, 3)$ and $(1, 3)R(1, 2)$ because $1 = 1$, but $(1, 2) \neq (1, 3)$.

Determine whether R is transitive.

Solution:

R is not transitive: For example, $(1, 2)R(1, 3)$ because $1 = 1$, and $(1, 3)R(4, 3)$ because $3 = 3$. But $(1, 2) \neq (4, 3)$ because $1 \neq 4$ and $2 \neq 3$.

#1. (a) Verify that the following is an equivalence relation on the set of real numbers:

$$aRb \leftrightarrow \lfloor a \rfloor = \lfloor b \rfloor, \text{ where } \lfloor x \rfloor \text{ is the floor of } x.$$

(b) Describe the equivalence classes arising from the equivalence relation in part (a).

Solution:

(a) R is reflexive: $\lfloor a \rfloor = \lfloor a \rfloor$ is true for all real numbers.

R is symmetric: suppose $\lfloor a \rfloor = \lfloor b \rfloor$; then $\lfloor b \rfloor = \lfloor a \rfloor$.

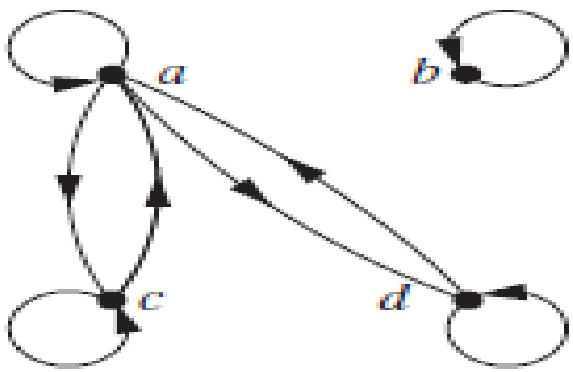
R is transitive: suppose $\lfloor a \rfloor = \lfloor b \rfloor$ and $\lfloor b \rfloor = \lfloor c \rfloor$; from transitivity of equality of real numbers, it follows that $\lfloor a \rfloor = \lfloor c \rfloor$.

(b) Two real numbers, a and b, are related if they have the same floor. This happens if and only if a and b lie in the same interval $[n, n+1)$ where n is an integer. That is, the equivalence classes are the intervals $\ldots, [-2,-1),\ [-1,0),\ [0,1),\ [1,2),\ [2,3), \ldots.$

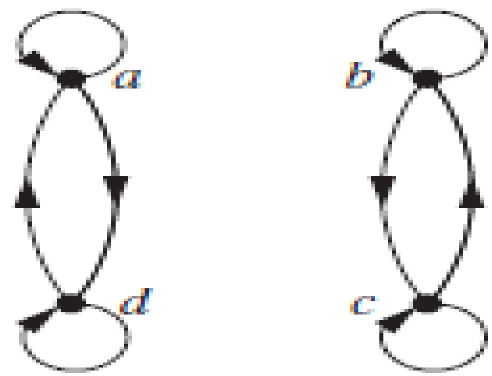
Exercise:

In Exercises 21–23 determine whether the relation with the directed graph shown is an equivalence relation.

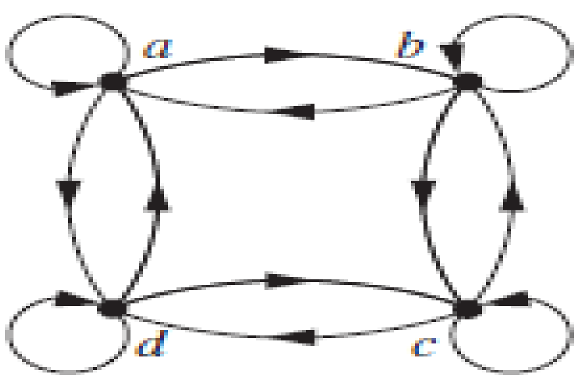
21.



22.



23.



Exercise:

24. Determine whether the relations represented by these zero–one matrices are equivalence relations.

a) $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

b) $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$

c) $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Exercise:

Let R be the relation on the set of ordered pairs of positive integers such that $((a, b), (c, d)) \in R$ if and only if $a + d = b + c$. Show that R is an equivalence relation.

Exercise:

Let R be the relation on the set of ordered pairs of positive integers such that $((a, b), (c, d)) \in R$ if and only if $ad = bc$. Show that R is an equivalence relation.