Mat2033 - Discrete Mathematics

Graph Theory and Its Applications

Definition of a graph

- A graph G is a finite nonempty set V(G) of vertices (also called nodes) and a (possibly empty) set E(G) of 2-element subsets of V(G) called edges (or lines).

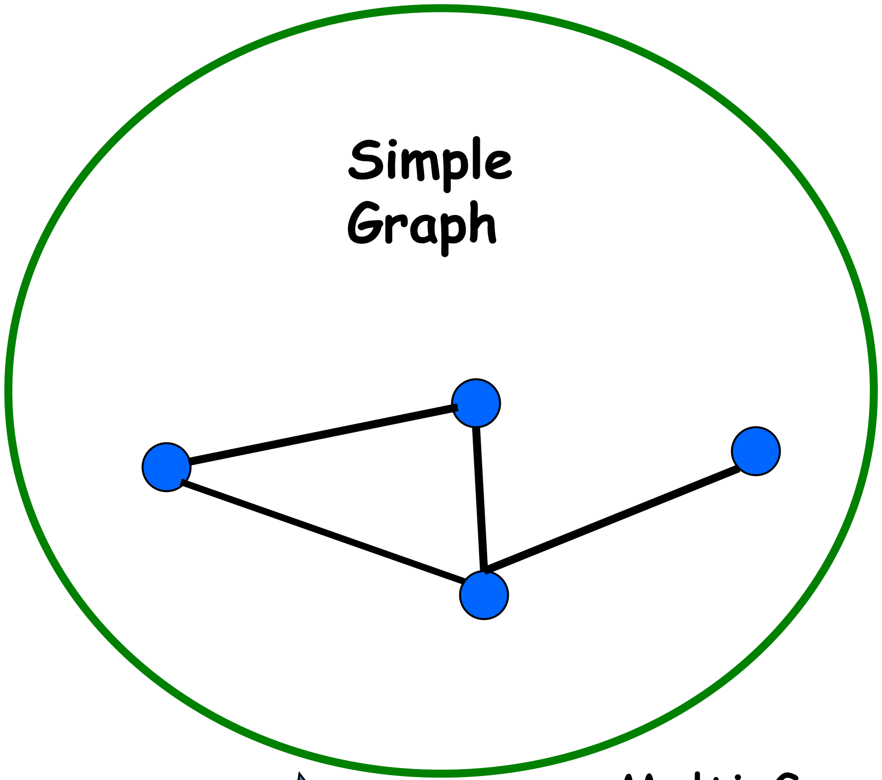
V(G) : vertex set of G

E(G) : edge set of G

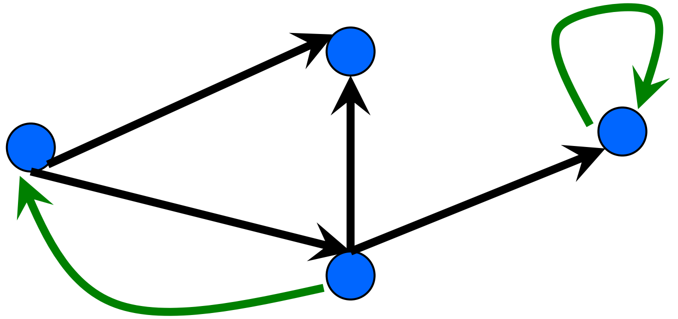
edge : {u, v} = {v, u} = uv (or vu)

G directed graph (digraph) edge: (u,v)

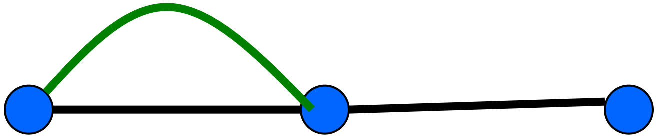
Types of Graphs



Directed Graph



Multi-Graph



Most of the problems in this course.

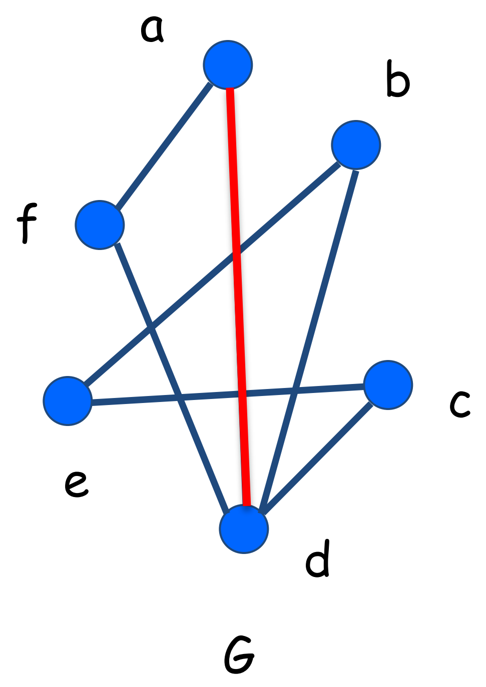
Simple Graphs

A graph G=(V,E) consists of:

A set of vertices, *V*

A set of *undirected* edges, *E*

- V(G) = {a,b,c,d,e,f}
- E(G) = {ad,af,bd,be,cd,ce,df}



Two vertices a,d are ***adjacent*** (***neighbours***) if the edge ad is present.

Example

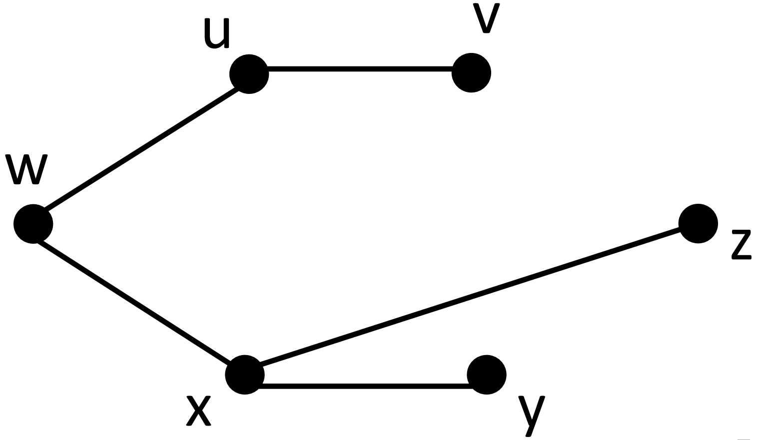
- A graph G=(V,E), where

V={u, v, w, x, y, z}

E={{u,v}, {u,w}, {w,x}, {x,y}, {x,z}}

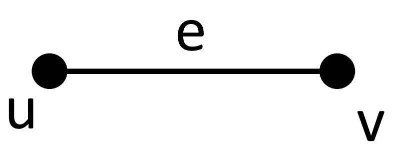
E={uv, uw, wx, xy, xz}

- G diagram :



Adjacent and Incident

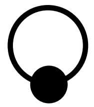
- u, v : vertices of a graph G

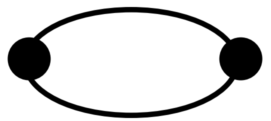


- u and v are adjacent in G if $uv \in E(G)$
  (u is adjacent to v, v is adjacent to u)
- e=uv (e joins u and v) (e is incident with u, e is incident with v)

Graphs types

loop multiedges, parallel edges





- undirected graph:
  - (simple) graph: loop (✗), multiedge (✗)
  - multigraph: loop (✗), multiedge (✓)
  - Pseudograph: loop (✓), multiedge (✓)

order and size

- The number of vertices in a graph G is called its order (denoted by |V(G)|).
- The number of edges is its size (denoted by |E(G)|).
- Proposition 1:
  If |V(G)| = p and |E(G)| = q, then $q \le \binom{p}{2}$
- A graph of order p and size q is called a (p, q) graph.

Application of graphs

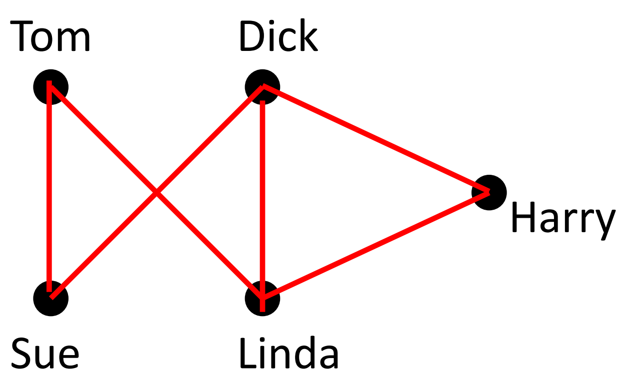
Example:

Tom, Dick know Sue, Linda.

Harry knows Dick and Linda.

$\Rightarrow$

acquaintance graph:

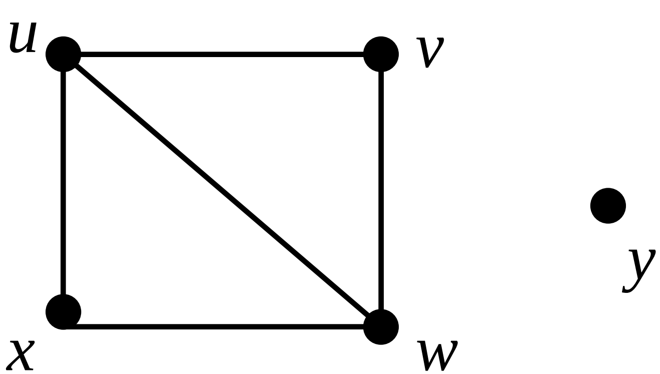


The degree of a vertex

Definition.

For a vertex v of G, its neighborhood $N(v) = \{ u \in V(G) \mid vu \in E(G) \}$.

The degree of vertex v is $\deg(v) = | N(v) |$.

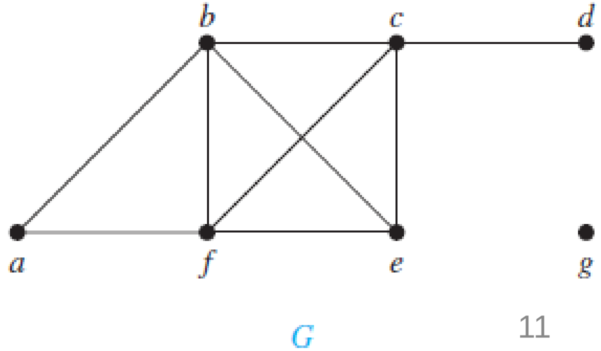


$N(u) = \{x, w, v\}$, $N(y)=\{ \}$

$\deg(u) = 3$, $\deg(y) = 0$

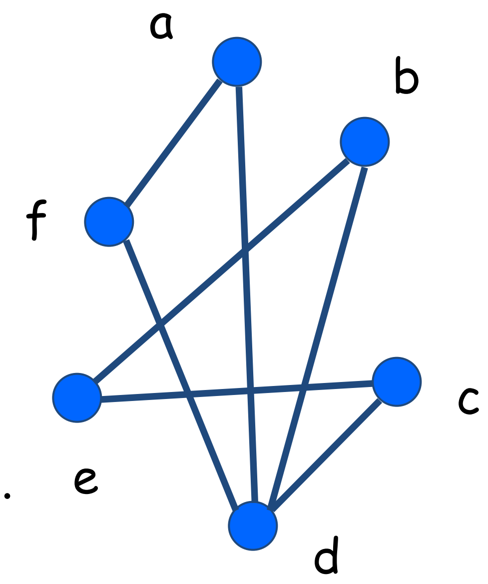
Notes

- If $|V(G)| = p$, then
  $0 \leq \deg(v) \leq p-1, \quad \forall\, v \in V(G)$.
- If $\deg(v) = 0$, then v is called an isolated vertex
- If $\deg(v) = 1$, then v is called an pendant vertex
- v is an odd vertex if $\deg(v)$ is odd.
  v is an even vertex if $\deg(v)$ is even.



Vertex Degrees

An edge uv is *incident* on the vertex u and the vertex v.



The ***neighbour set*** N(v) of a vertex v is the set of vertices adjacent to it.

e.g. N(a) = {d,f}, N(d) = {a,b,c,f}, N(e) = {b,c}.

degree of a vertex = # of ***incident*** edges

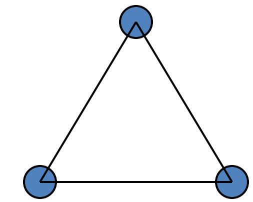
e.g. deg(d) = 4, deg(a)=deg(b)=deg(c)=deg(e)=deg(f)=2.

the degree of a vertex v = the number of neighbours of v?

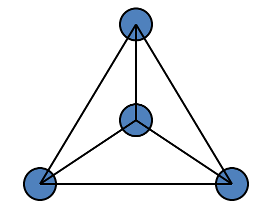
For multigraphs, NO. For simple graphs, YES.

Degree Sequence

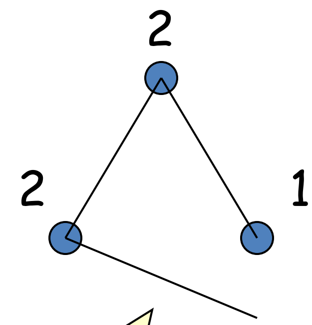
Is there a graph with degree sequence (2,2,2)? YES.



Is there a graph with degree sequence (3,3,3,3)? YES.



Is there a graph with degree sequence (2,2,1)? NO.



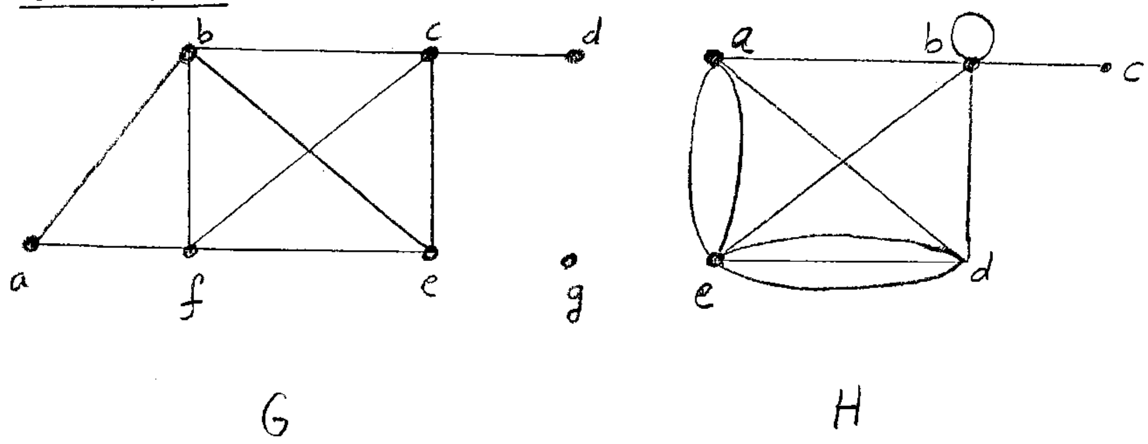
Is there a graph with degree sequence (2,2,2,2,1)?

Where to go?

NO.

What's wrong with these sequences?

Example:



What are the degrees of the vertices in the graphs G and H ?

The *degree of a vertex in an undirected graph* is the number of edges incident with it, except that a loop at a vertex contributes twice to the degree of that vertex. The degree of the vertex *v* is denoted by deg(*v*).

Solution: In G, deg (a) = 2, deg (b) = deg(c) = deg (f) = 4, deg (d) = 1, deg (e) = 3, deg (g) = 0.

In H, deg (a) = 4, deg (b) = deg (e) = 6, deg (c) = 1, and deg (d) = 5.

Handshaking Lemma

For any graph, sum of degrees = twice # edges

Lemma.

$$2|E| = \sum_{v \in V} \deg(v)$$

Corollary.

1. Sum of degree is an even number.
2. Number of odd degree vertices is even.

Examples. 2+2+1 = odd, so impossible.

2+2+2+2+1 = odd, so impossible.

Handshaking Lemma

Lemma. $$\boxed{2|E| = \sum_{v \in V} \deg(v)}$$

Proof. Each edge contributes 2 to the sum on the right.

Question. Given a degree sequence, if the sum of degree is even, is it true that there is a graph with such a degree sequence?

For simple graphs, NO, consider the degree sequence (3,3,3,1).

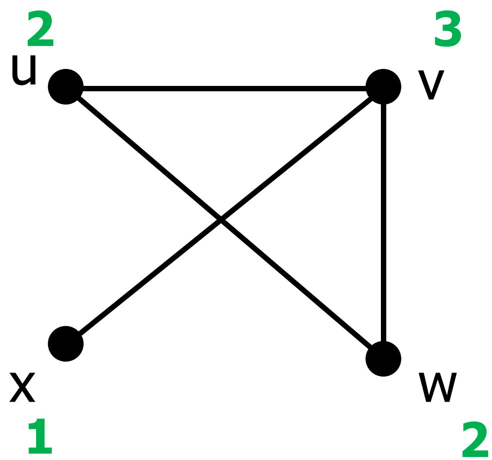
For multigraphs (with self loops), YES! (easy by induction)

Handshaking theorem

- **Theorem 1.1** (Handshaking theorem)
  Let G be a graph, then

$$\sum_{v \in V(G)} \deg(v) = |E(G)| \times 2$$

Example:



$$\sum_{v \in V(G)} \deg(v) = 8$$

$$|E(G)| = 4$$

Handshaking theorem

Corollary 1.1

Every graph contains an even number of odd vertices.

Proof: If the number of vertices with odd degree is odd, then the degree sum must be odd. $\rightarrow\leftarrow$

Example: How many edges are there in a graph with 10 vertices each of degree 6?

Example: How many edges are there in a graph with 10 vertices each of degree 6?

Solution:

Sum of degrees of vertices $= 6 \cdot 10 = 60$

$2e = 60 \Rightarrow e = \frac{60}{2} = 30$ edges.

Example: A certain graph G has order 14 and size 27. The degree of each vertex of G is 3, 4 or 5. There are six vertices of degree 4. How many vertices of G have degree 3 and how many have degree 5 ?

Solution: Let x be the number of vertices of G having degree 3.

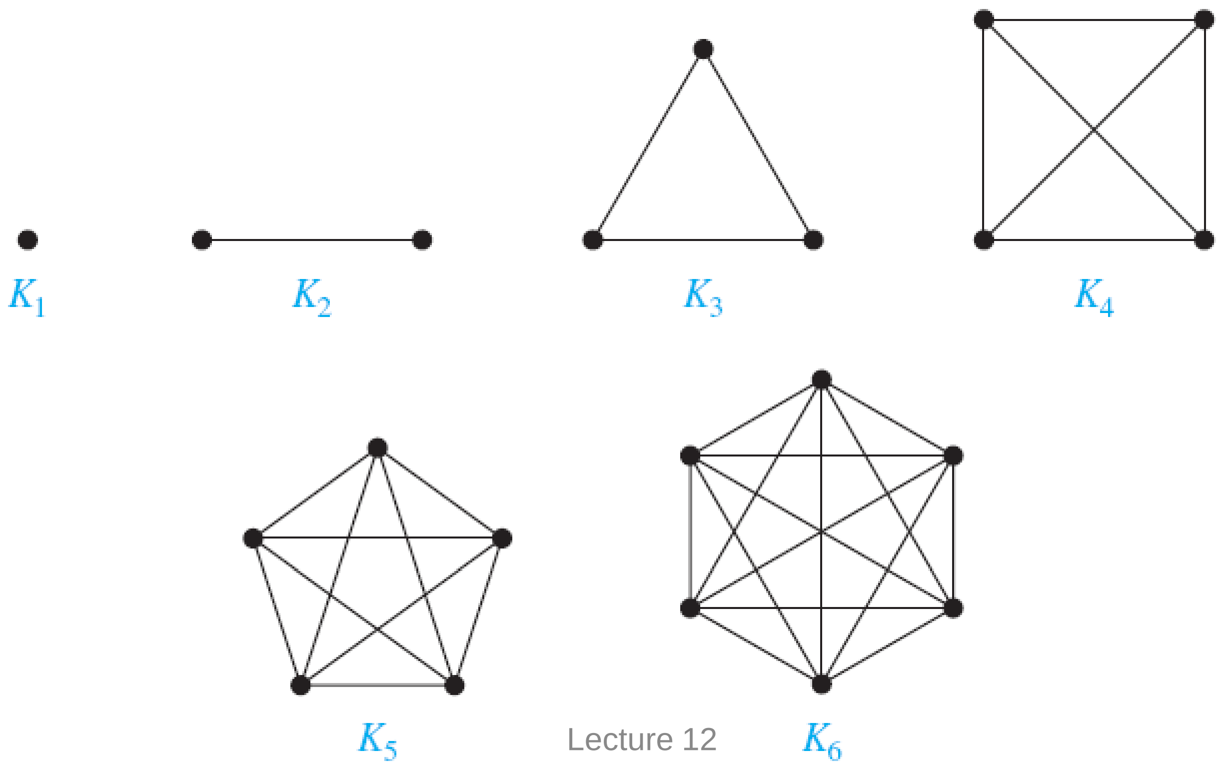
$14-6=8$ vertices have degree 3 or 5. So there are $8-x$ vertices of degree 5.

Then we have $3.x+4.6+5.(8-x)=2.27$

Hence $x=5$, $8-x=3$

Some Special Simple Graphs

Complete Graphs A **complete graph on n vertices**, denoted by K_n, is a simple graph that contains exactly one edge between each pair of distinct vertices. The graphs K_n, for $n = 1, 2, 3, 4, 5, 6$, are displayed in Figure 3. A simple graph for which there is at least one pair of distinct vertex not connected by an edge is called **noncomplete**.



Cycles A **cycle** C_n, $n \geq 3$, consists of n vertices $v_1, v_2, \ldots, v_n$ and edges $\{v_1, v_2\}$, $\{v_2, v_3\}, \ldots, \{v_{n-1}, v_n\}$, and $\{v_n, v_1\}$. The cycles C_3, C_4, C_5, and C_6 are displayed in Figure 4. ◀

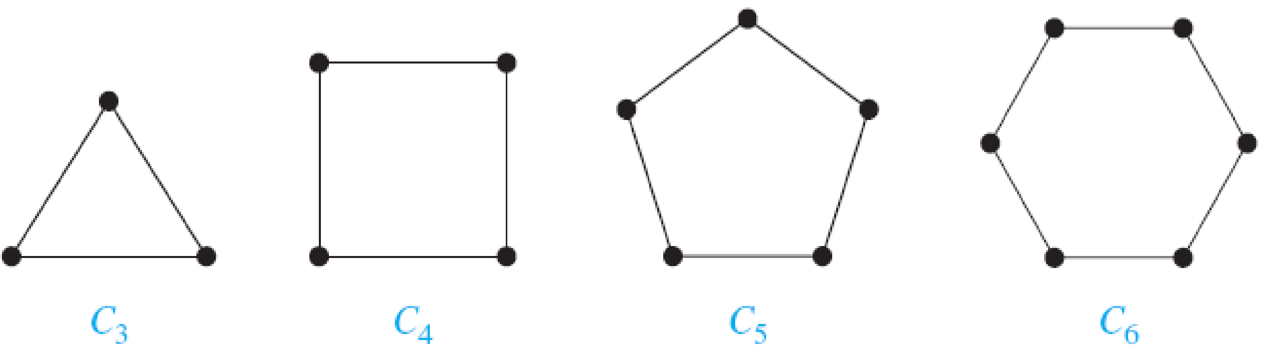


FIGURE 4 **The Cycles C_3, C_4, C_5, and C_6.**

Wheels We obtain a **wheel** $\boldsymbol{W_n}$ when we add an additional vertex to a cycle C_n, for $n \geq 3$, and connect this new vertex to each of the n vertices in C_n, by new edges. The wheels W_3, W_4, W_5, and W_6 are displayed in Figure 5.

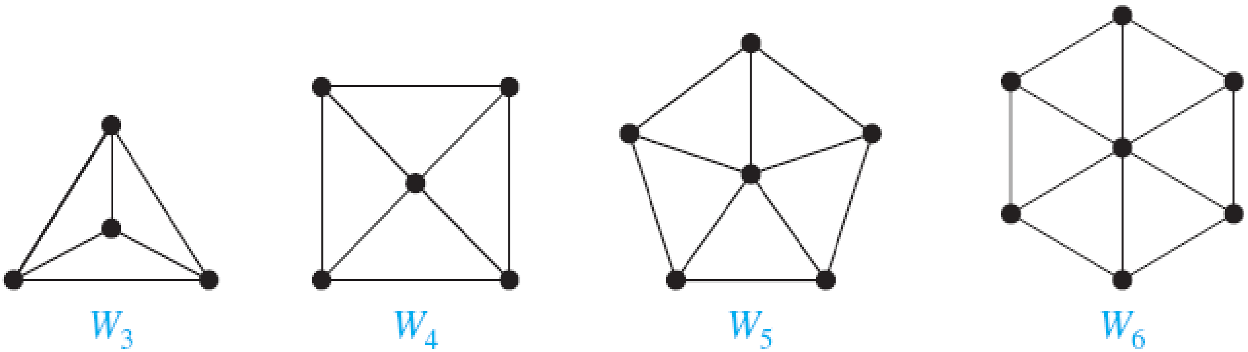


FIGURE 5 **The Wheels W_3, W_4, W_5, and W_6.**

***n*-Cubes** An ***n*-dimensional hypercube**, or ***n*-cube**, denoted by Q_n, is a graph that has vertices representing the 2^n bit strings of length n. Two vertices are adjacent if and only if the bit strings that they represent differ in exactly one bit position. We display Q_1, Q_2, and Q_3 in Figure 6.

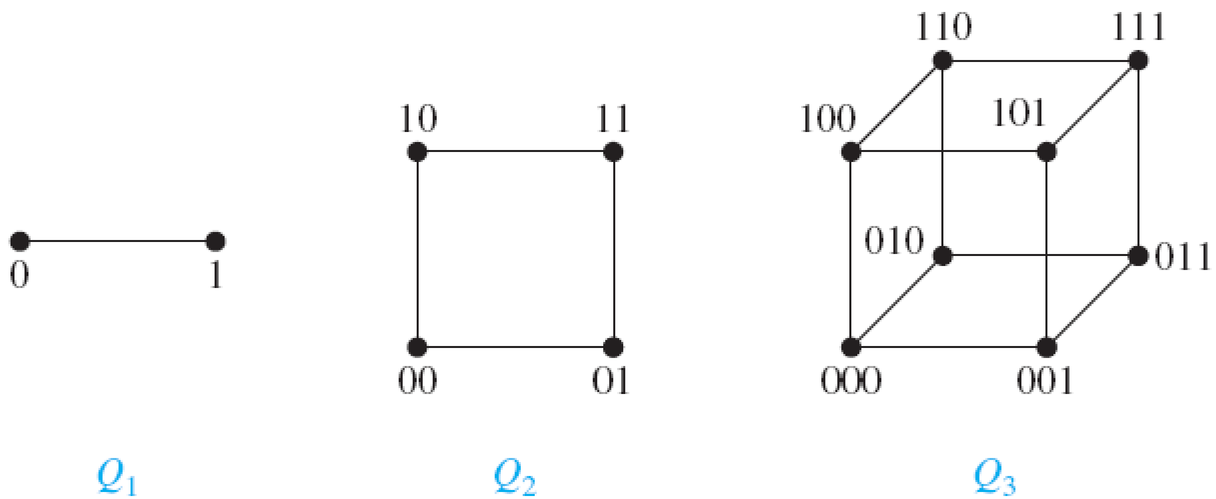


FIGURE 6 **The *n*-cube Q_n, $n = 1, 2, 3$.**

Note that you can construct the $(n+1)$-cube Q_{n+1} from the n-cube Q_n by making two copies of Q_n, prefacing the labels on the vertices with a 0 in one copy of Q_n and with a 1 in the other copy of Q_n, and adding edges connecting two vertices that have labels differing only in the first bit. In Figure 6, Q_3 is constructed from Q_2 by drawing two copies of Q_2 as the top and bottom faces of Q_3, adding 0 at the beginning of the label of each vertex in the bottom face and 1 at the beginning of the label of each vertex in the top face. (Here, by *face* we mean a face of a cube in three-dimensional space. Think of drawing the graph Q_3 in three-dimensional space with copies of Q_2 as the top and bottom faces of a cube and then drawing the projection of the resulting depiction in the plane.)

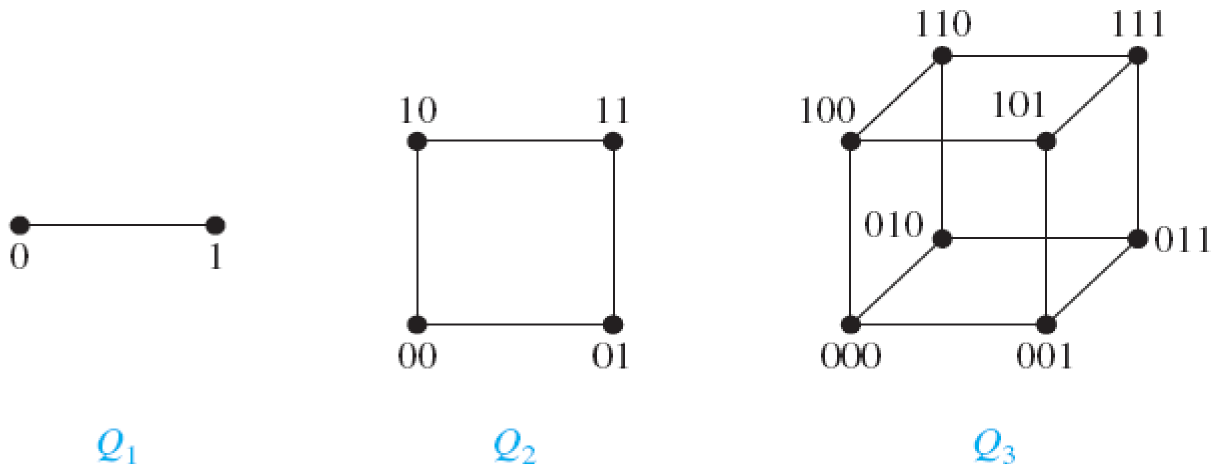


FIGURE 6 **The n-cube Q_n, $n = 1, 2, 3$.**

Bipartite Graphs

Sometimes a graph has the property that its vertex set can be divided into two disjoint subsets such that each edge connects a vertex in one of these subsets to a vertex in the other subset. For example, consider the graph representing marriages between men and women in a village, where each person is represented by a vertex and a marriage is represented by an edge. In this graph, each edge connects a vertex in the subset of vertices representing males and a vertex in the subset of vertices representing females.

DEFINITION

A simple graph G is called *bipartite* if its vertex set V can be partitioned into two disjoint sets V_1 and V_2 such that every edge in the graph connects a vertex in V_1 and a vertex in V_2 (so that no edge in G connects either two vertices in V_1 or two vertices in V_2). When this condition holds, we call the pair (V_1, V_2) a *bipartition* of the vertex set V of G.

EXAMPLE

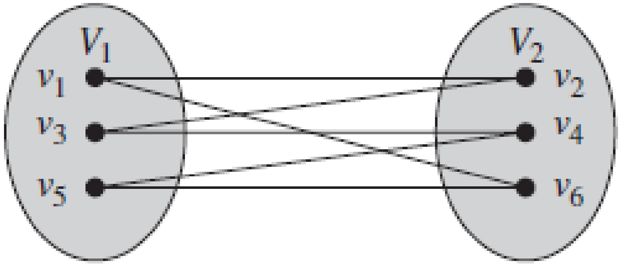


FIGURE 7 Showing That C_6 Is Bipartite.

C_6 is bipartite, as shown in Figure 7, because its vertex set can be partitioned into the two sets $V_1 = \{v_1, v_3, v_5\}$ and $V_2 = \{v_2, v_4, v_6\}$, and every edge of C_6 connects a vertex in V_1 and a vertex in V_2.

K_3 is not bipartite. To verify this, note that if we divide the vertex set of K_3 into two disjoint sets, one of the two sets must contain two vertices. If the graph were bipartite, these two vertices could not be connected by an edge, but in K_3 each vertex is connected to every other vertex by an edge.

EXAMPLE 11 Are the graphs G and H displayed in Figure 8 bipartite?

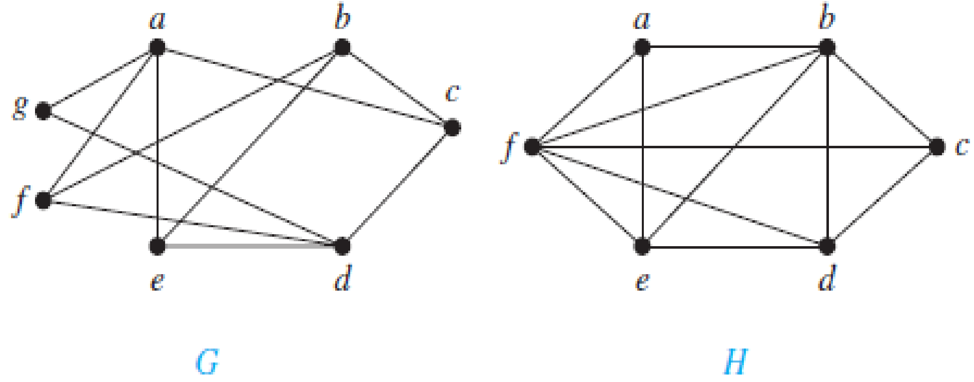


FIGURE 8 **The Undirected Graphs G and H.**

Solution: Graph G is bipartite because its vertex set is the union of two disjoint sets, $\{a, b, d\}$ and $\{c, e, f, g\}$, and each edge connects a vertex in one of these subsets to a vertex in the other subset. (Note that for G to be bipartite it is not necessary that every vertex in $\{a, b, d\}$ be adjacent to every vertex in $\{c, e, f, g\}$. For instance, b and g are not adjacent.)

Graph H is not bipartite because its vertex set cannot be partitioned into two subsets so that edges do not connect two vertices from the same subset. (The reader should verify this by considering the vertices a, b, and f.)

Theorem 4 provides a useful criterion for determining whether a graph is bipartite.

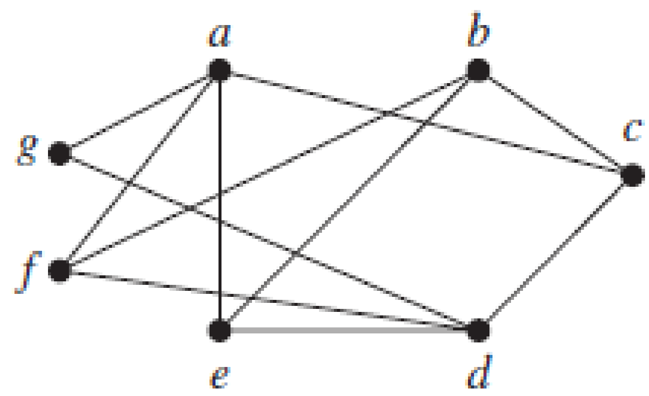
THEOREM 4

A simple graph is bipartite if and only if it is possible to assign one of two different colors to each vertex of the graph so that no two adjacent vertices are assigned the same color.

Proof: First, suppose that $G = (V, E)$ is a bipartite simple graph. Then $V = V_1 \cup V_2$, where V_1 and V_2 are disjoint sets and every edge in E connects a vertex in V_1 and a vertex in V_2. If we assign one color to each vertex in V_1 and a second color to each vertex in V_2, then no two adjacent vertices are assigned the same color.

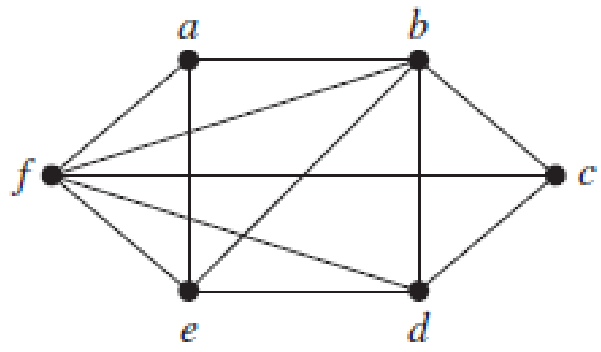
Now suppose that it is possible to assign colors to the vertices of the graph using just two colors so that no two adjacent vertices are assigned the same color. Let V_1 be the set of vertices assigned one color and V_2 be the set of vertices assigned the other color. Then, V_1 and V_2 are disjoint and $V = V_1 \cup V_2$. Furthermore, every edge connects a vertex in V_1 and a vertex in V_2 because no two adjacent vertices are either both in V_1 or both in V_2. Consequently, G is bipartite. $\triangleleft$

EXAMPLE 12 Use Theorem 4 to determine whether the graphs in Example 11 are bipartite.



G

Solution: We first consider the graph G. We will try to assign one of two colors, say red and blue, to each vertex in G so that no edge in G connects a red vertex and a blue vertex. Without loss of generality we begin by arbitrarily assigning red to a. Then, we must assign blue to c, e, f, and g, because each of these vertices is adjacent to a. To avoid having an edge with two blue endpoints, we must assign red to all the vertices adjacent to either c, e, f, or g. This means that we must assign red to both b and d (and means that a must be assigned red, which it already has been). We have now assigned colors to all vertices, with a, b, and d red and c, e, f, and g blue. Checking all edges, we see that every edge connects a red vertex and a blue vertex. Hence, by Theorem 4 the graph G is bipartite.



H

Next, we will try to assign either red or blue to each vertex in H so that no edge in H connects a red vertex and a blue vertex. Without loss of generality we arbitrarily assign red to a. Then, we must assign blue to b, e, and f, because each is adjacent to a. But this is not possible because e and f are adjacent, so both cannot be assigned blue. This argument shows that we cannot assign one of two colors to each of the vertices of H so that no adjacent vertices are assigned the same color. It follows by Theorem 4 that H is not bipartite.

Theorem 4 is an example of a result in the part of graph theory known as graph colorings. Graph colorings is an important part of graph theory with important applications. We will study

Complete Bipartite Graphs **A complete bipartite graph** $K_{m,n}$ is a graph that has its vertex set partitioned into two subsets of m and n vertices, respectively with an edge between two vertices if and only if one vertex is in the first subset and the other vertex is in the second subset. The complete bipartite graphs $K_{2,3}$, $K_{3,3}$, $K_{3,5}$, and $K_{2,6}$ are displayed in Figure 9. ◀

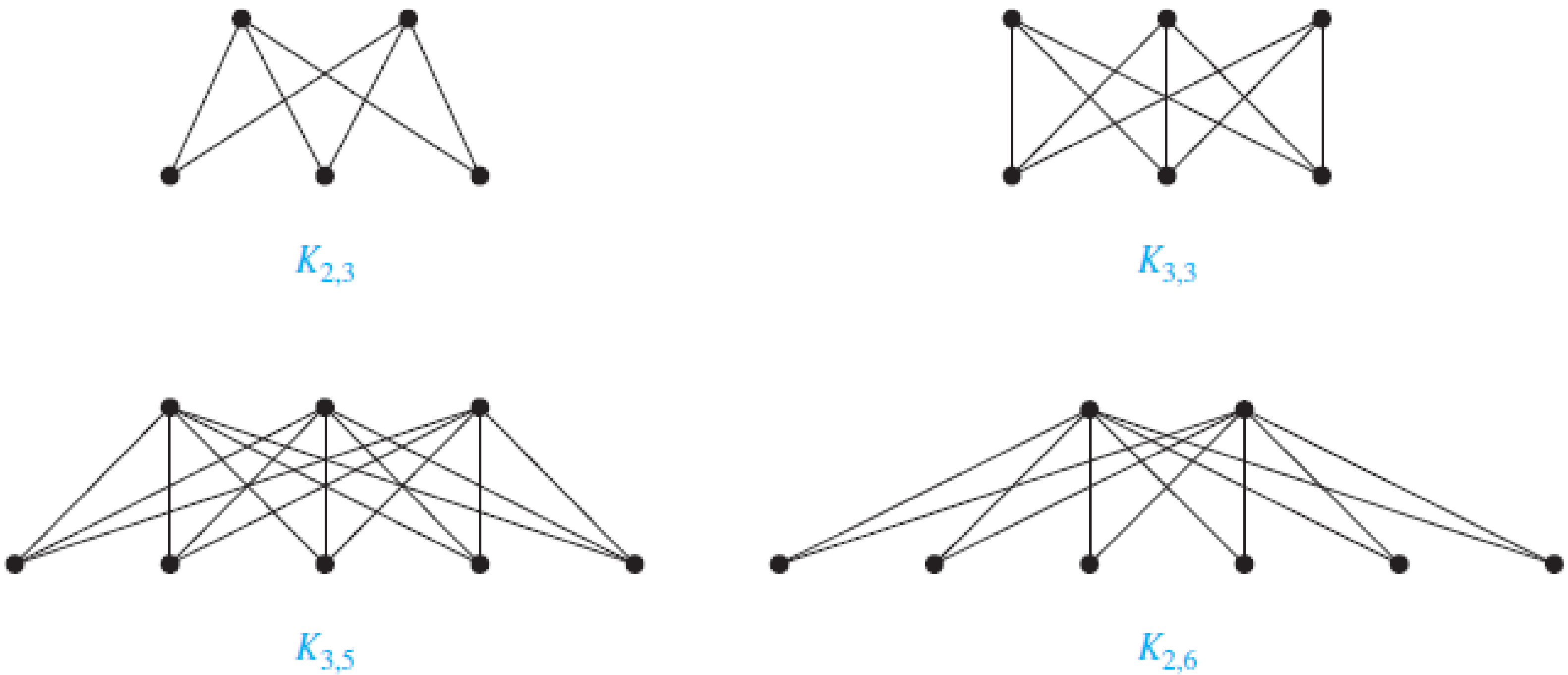
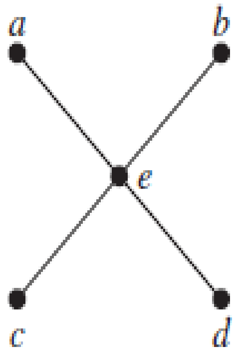


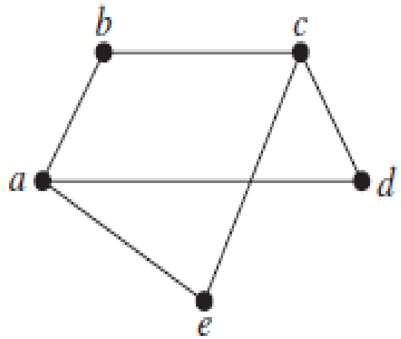
FIGURE 9 **Some Complete Bipartite Graphs.**

In Exercises 21–25 determine whether the graph is bipartite. You may find it useful to apply Theorem 4 and answer the question by determining whether it is possible to assign either red or blue to each vertex so that no two adjacent vertices are assigned the same color.

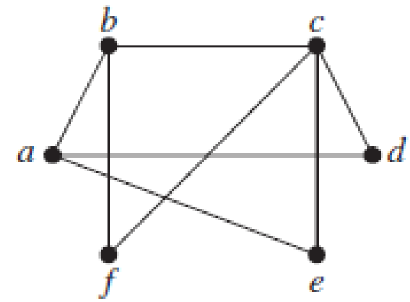
21.



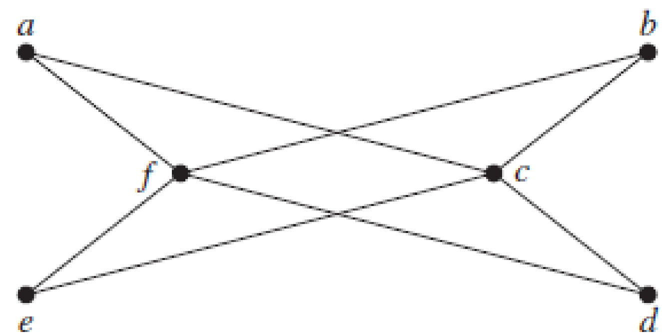
22.



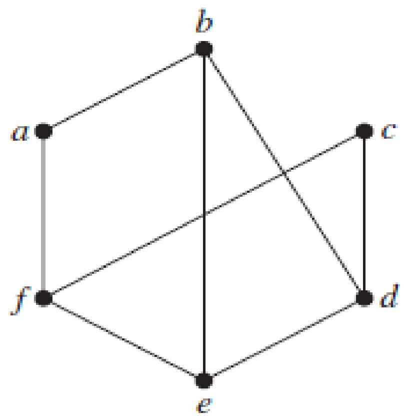
23.



24.



25.



Exercises For which values of n are these graphs bipartite?

a) K_n b) C_n c) W_n d) Q_n

a) By the definition given in the text, K_1 does not have enough vertices to be bipartite. Clearly K_2 is bipartite. There is a triangle in K_n for $n > 2$, so those complete graphs are not bipartite.

b) First we need $n \geq 3$ for C_n to be defined. If n is even, then C_n is bipartite, since we can take one part to be every other vertex. If n is odd, then C_n is not bipartite.

c) Every wheel contains triangles, so no W_n is bipartite.

d) Q_n is bipartite for all $n \geq 1$, since we can divide the vertices into these two classes: those bit strings with an odd number of 1's, and those bit strings with an even number of 1's.

Example How many vertices and how many edges do these graphs have?

a) K_n b) C_n c) W_n
d) $K_{m,n}$ e) Q_n

a) n vertices, $n(n-1)/2$ edges

b) n vertices, n edges

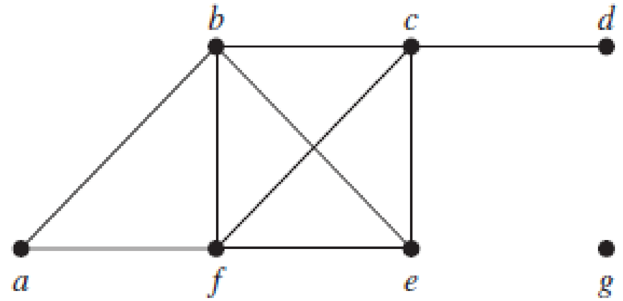
c) $n+1$ vertices, $2n$ edges

d) $m+n$ vertices, mn edges

e) 2^n vertices, $n2^{n-1}$ edges

DEFINITION

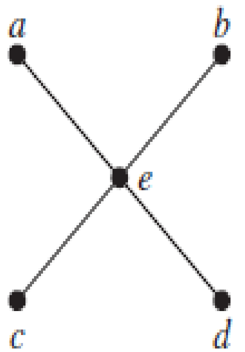
The **degree sequence** of a graph is the sequence of the degrees of the vertices of the graph in nonincreasing order. For example, the degree sequence of the graph G in Example 1 is 4, 4, 4, 3, 2, 1, 0.



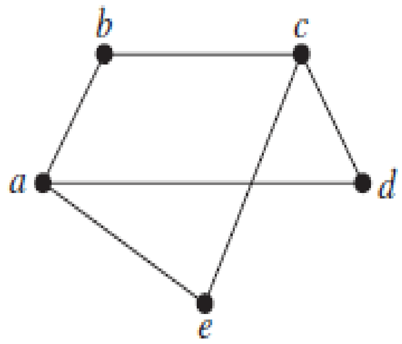
G

Find the degree sequences for each of the graphs in Exercises 21–25.

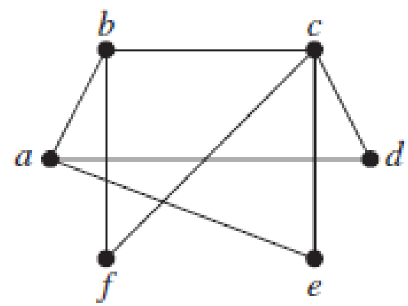
21.



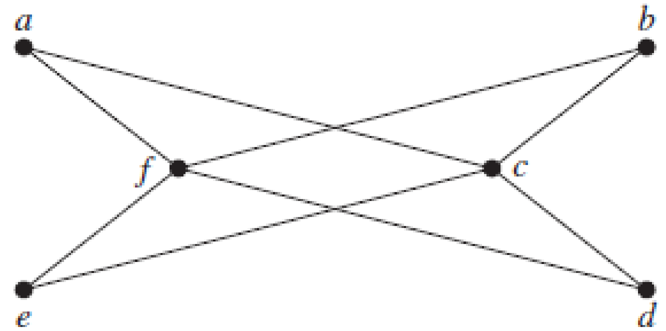
22.



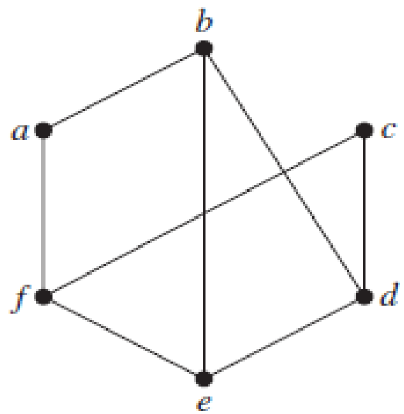
23.



24.



25.



Example

Find the degree sequence of each of the following graphs.

a) K_4 b) C_4 c) W_4
d) $K_{2,3}$ e) Q_3

a) 3, 3, 3, 3

Example

Find the degree sequence of each of the following graphs.

a) K_4 b) C_4 c) W_4
d) $K_{2,3}$ e) Q_3

a) 3, 3, 3, 3 b) 2, 2, 2, 2

Example

Find the degree sequence of each of the following graphs.

a) K_4 **b)** C_4 **c)** W_4

d) $K_{2,3}$ **e)** Q_3

a) 3, 3, 3, 3 b) 2, 2, 2, 2 c) 4, 3, 3, 3, 3

Example

Find the degree sequence of each of the following graphs.

a) K_4 b) C_4 c) W_4

d) $K_{2,3}$ e) Q_3

a) 3, 3, 3, 3 b) 2, 2, 2, 2 c) 4, 3, 3, 3, 3 d) 3, 3, 2, 2, 2

Example

Find the degree sequence of each of the following graphs.

a) K_4 **b)** C_4 **c)** W_4

d) $K_{2,3}$ **e)** Q_3

a) 3, 3, 3, 3 b) 2, 2, 2, 2 c) 4, 3, 3, 3, 3 d) 3, 3, 2, 2, 2 e) 3, 3, 3, 3, 3, 3, 3, 3

Example

Find the degree sequence of each of the following graphs.

a) K_4 b) C_4 c) W_4
d) $K_{2,3}$ e) Q_3

a) 3, 3, 3, 3 b) 2, 2, 2, 2 c) 4, 3, 3, 3, 3 d) 3, 3, 2, 2, 2 e) 3, 3, 3, 3, 3, 3, 3, 3

What is the degree sequence of the bipartite graph $K_{m,n}$ where m and n are positive integers? Explain your answer.

What is the degree sequence of K_n, where n is a positive integer? Explain your answer. $n-1, n-1, \ldots, n-1$ (n terms)

How many edges does a graph have if its degree sequence is 4, 3, 3, 2, 2? Draw such a graph.

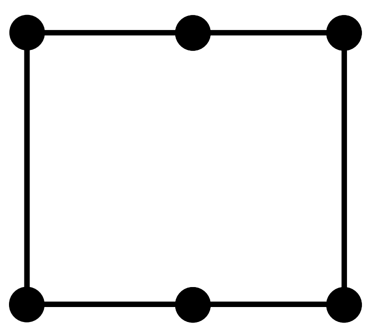
Regular graph

Definition:

A graph G is r-regular if every vertex of G has degree r.

A graph G is regular if it's r-regular for some r.

Example:



2-regular

Note.

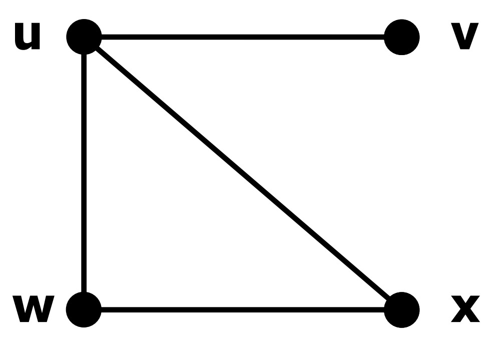
There is no 1-regular graph or 3-regular graph of order 5. (by Corollary 1.1)

Complement

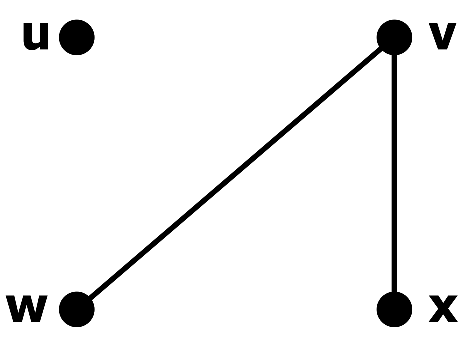
Definition.

The complement $\overline{G}$ of a graph G is a graph with $V(G) = V(\overline{G})$, and $uv \in E(G)$ iff $uv \notin E(\overline{G})$.

G



$\overline{G}$



Exercise .

Every vertex of a graph G of order 14 and size 25 has degree 3 or 5.

How many vertices of degree 3 does G have?

sol. Suppose there are x vertices of degree 3, then there are $14-x$ vertices of degree 5.

$|E(G)| = 25 \Rightarrow$ degree sum$=50$

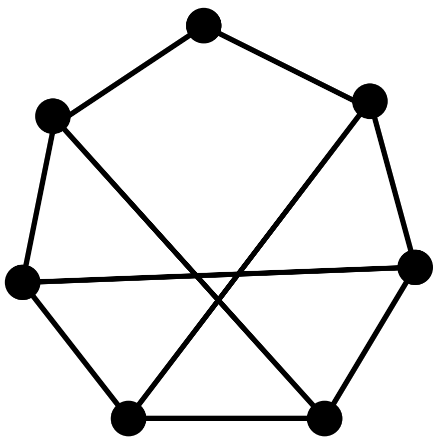
$3x + 5(14-x) = 50$

$\Rightarrow x = 10$

Exercise 10.

A graph G of order 7 and size 10 has six vertices of degree a and one of degree b. What is b?

Try to draw the graph



sol. $6a + b = 20$

$(a, b) = (0, 20)$ (×)

$(1, 14)$ (×)

$(2, 8)$ (×)

$(3, 2)$ (✓)

$\therefore\ a=3,\ b=2.$

The **complementary graph** $\overline{G}$ of a simple graph G has the same vertices as G. Two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in G. Describe each of these graphs.

a) $\overline{K_n}$ **b)** $\overline{K_{m,n}}$ **c)** $\overline{C_n}$ **d)** $\overline{Q_n}$

a) The graph with n vertices and no edges

b) The disjoint union of K_m and K_n

c) The graph with vertices $\{v_1, \ldots, v_n\}$ with an edge between v_i and v_j unless $i \equiv j \pm 1 \pmod{n}$

d) The graph whose vertices are represented by bit strings of length n with an edge between two vertices if the associated bit strings differ in more than one bit

Example: If G is a simple graph with 15 edges and $\overline{G}$ has 13 edges, how many vertices does G have?

Exercises

1. If the simple graph G has v vertices and e edges, how many edges does $\overline{G}$ have?

2. If the degree sequence of the simple graph G is 4, 3, 3, 2, 2, what is the degree sequence of $\overline{G}$?

3. If the degree sequence of the simple graph G is $d_1, d_2, \ldots, d_n$, what is the degree sequence of $\overline{G}$?

1. $v(v-1)/2 - e$

3. $n-1-d_n, n-1-d_{n-1}, \ldots, n-1-d_2, n-1-d_1$

A simple graph is called **regular** if every vertex of this graph has the same degree. A regular graph is called n-**regular** if every vertex in this graph has degree n.

Example:

- For which values of n are these graphs regular?

  a) K_n **b)** C_n **c)** W_n **d)** Q_n

- For which values of m and n is $K_{m,n}$ regular?
- How many vertices does a regular graph of degree four with 10 edges have?

a) For all $n \geq 1$ **b)** For all $n \geq 3$ **c)** For $n = 3$ **d)** For all $n \geq 0$

New Graphs from Old

Sometimes we need only part of a graph to solve a problem. For instance, we may care only about the part of a large computer network that involves the computer centers in New York, Denver, Detroit, and Atlanta. Then we can ignore the other computer centers and all telephone lines not linking two of these specific four computer centers. In the graph model for the large network, we can remove the vertices corresponding to the computer centers other than the four of interest, and we can remove all edges incident with a vertex that was removed. When edges and vertices are removed from a graph, without removing endpoints of any remaining edges, a smaller graph is obtained. Such a graph is called a **subgraph** of the original graph.

A *subgraph of a graph* $G = (V, E)$ is a graph $H = (W, F)$, where $W \subseteq V$ and $F \subseteq E$. A subgraph H of G is a *proper subgraph* of G if $H \neq G$.

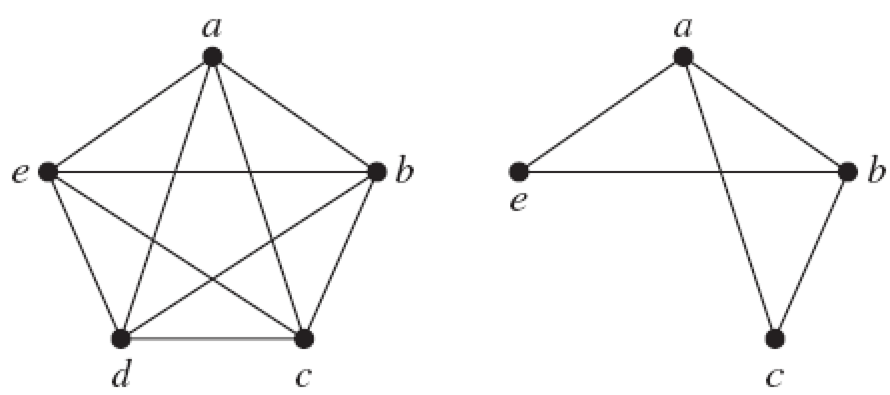
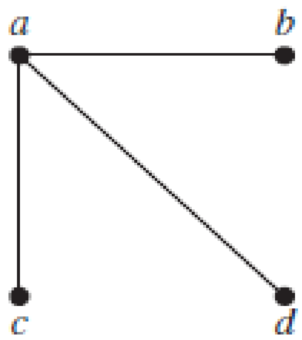
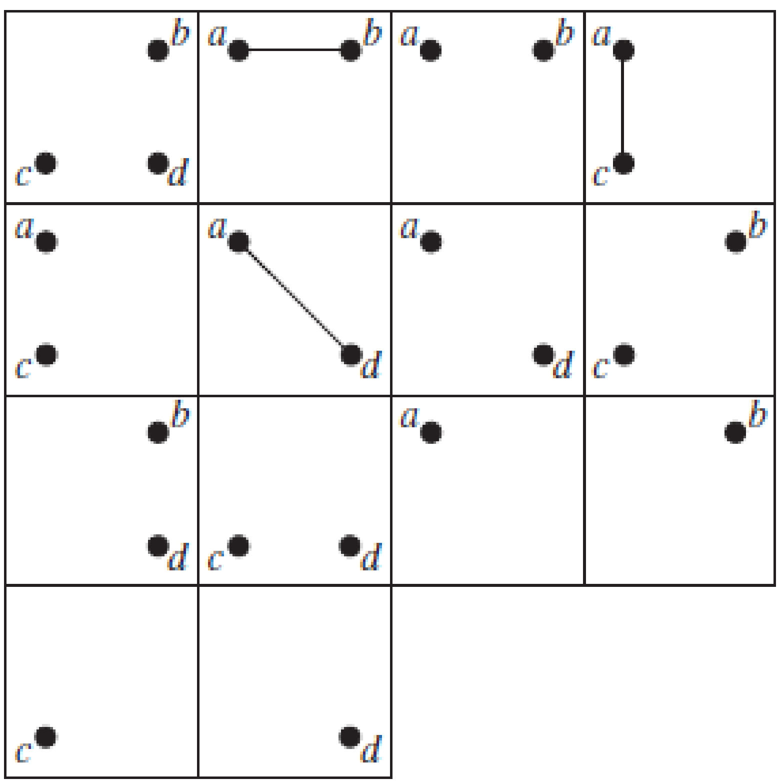
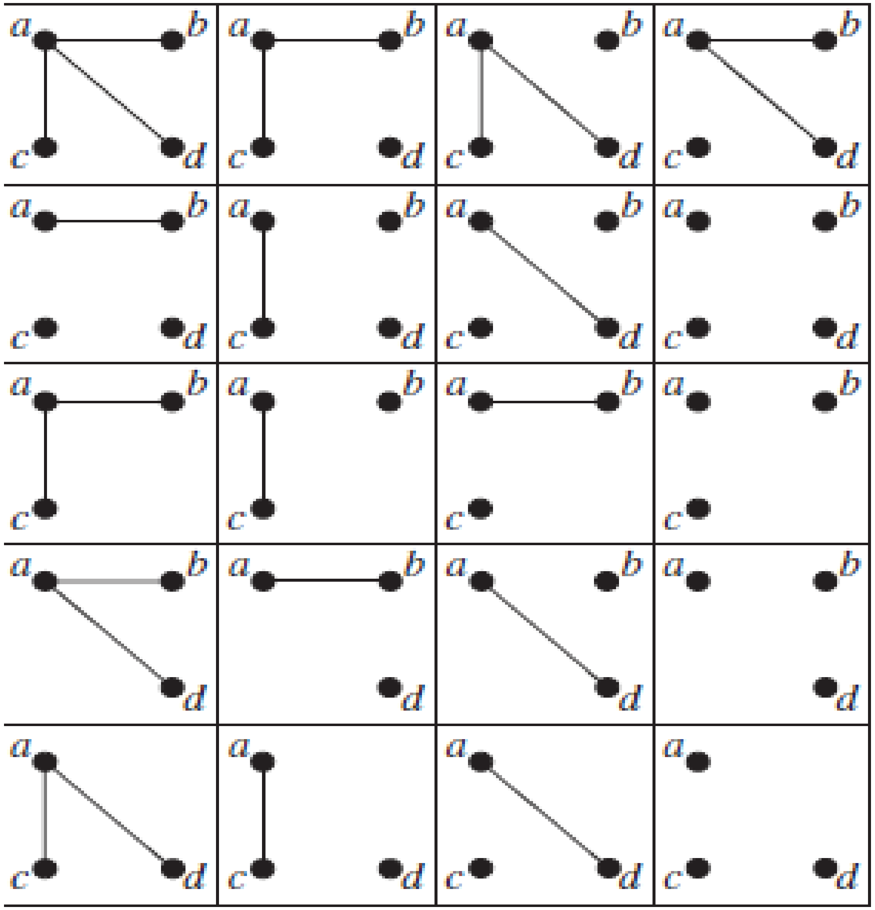


FIGURE 15 **A Subgraph of K_5.**

Example: Draw all subgraphs of this graph.







GRAPH UNIONS Two or more graphs can be combined in various ways. The new graph that contains all the vertices and edges of these graphs is called the **union** of the graphs. We will give a more formal definition for the union of two simple graphs.

The *union* of two simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the simple graph with vertex set $V_1 \cup V_2$ and edge set $E_1 \cup E_2$. The union of G_1 and G_2 is denoted by $G_1 \cup G_2$.

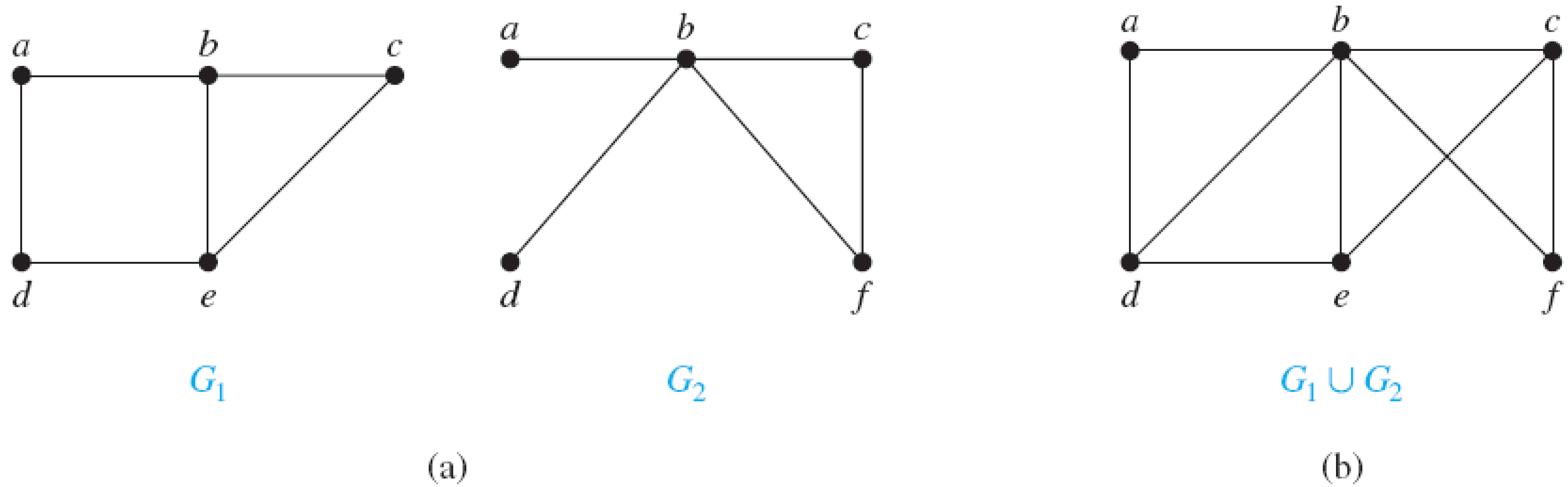
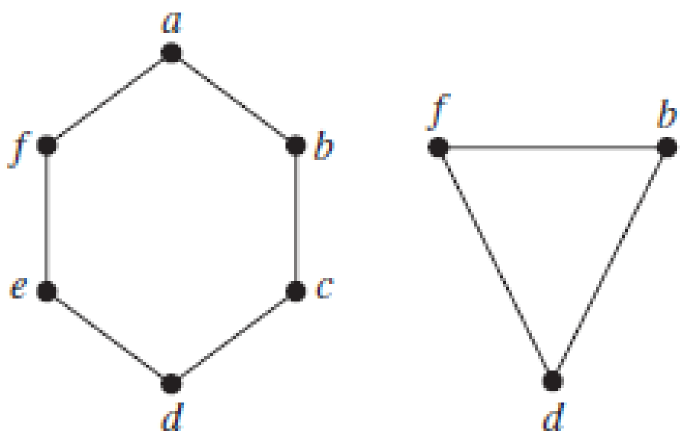


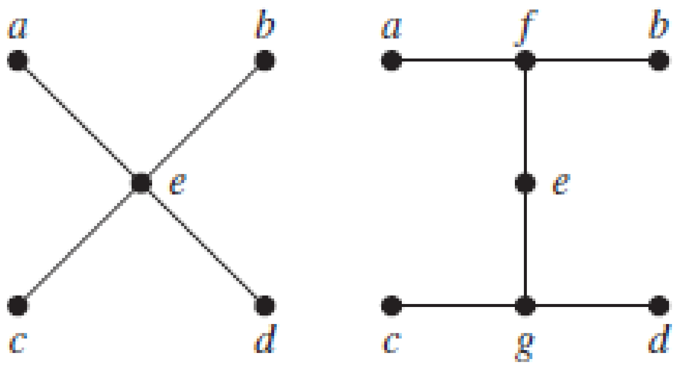
FIGURE 16 **(a) The Simple Graphs G_1 and G_2; (b) Their Union $G_1 \cup G_2$.**

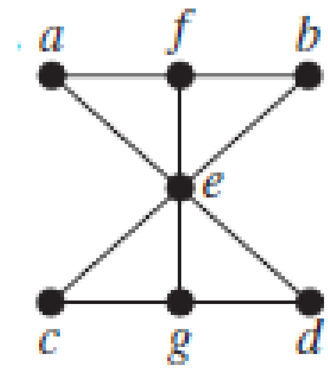
In Exercises 56–58 find the union of the given pair of simple graphs. (Assume edges with the same endpoints are the same.)

56.



57.





58.