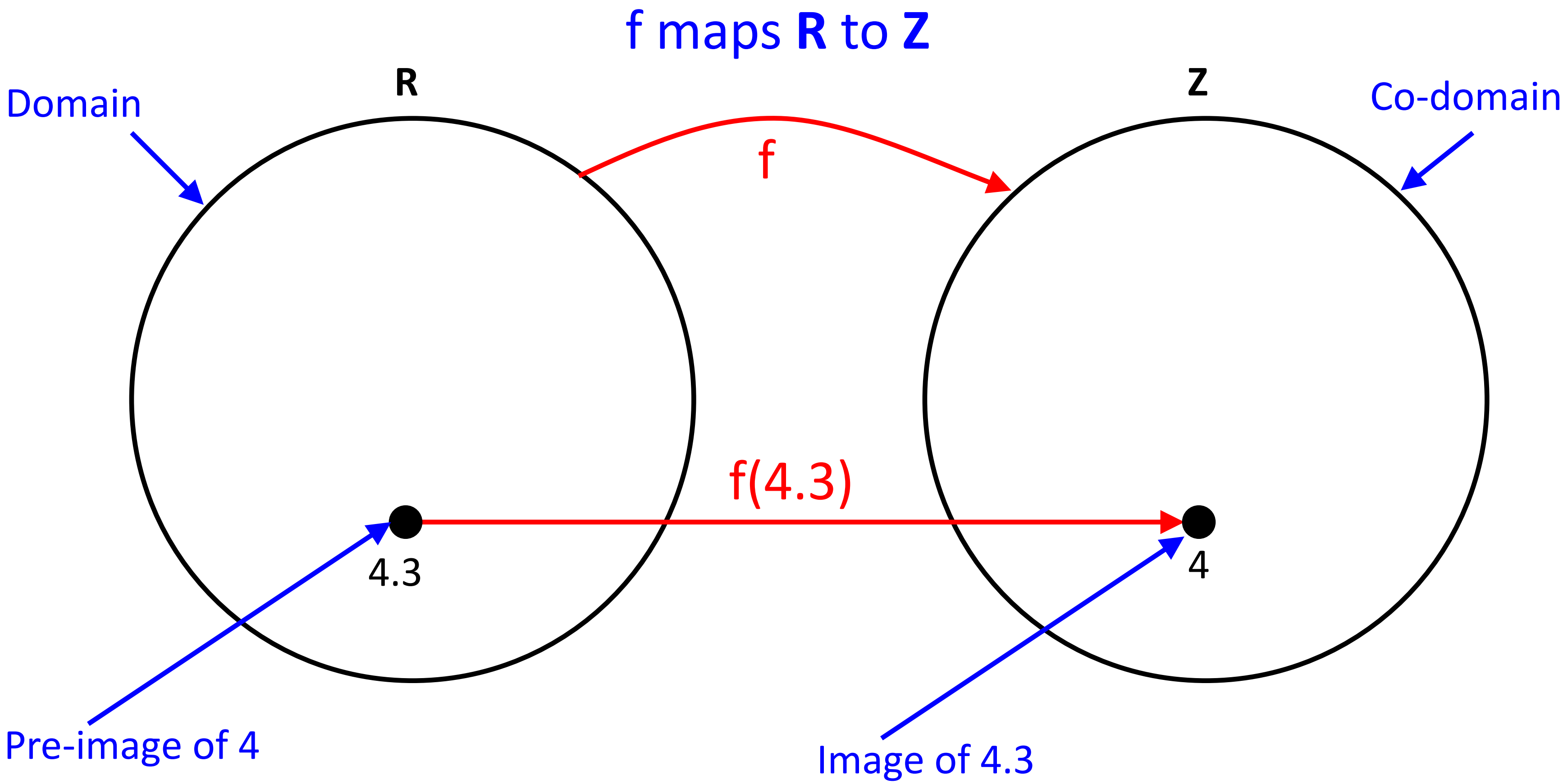
Mat2033 - Discrete Mathematics

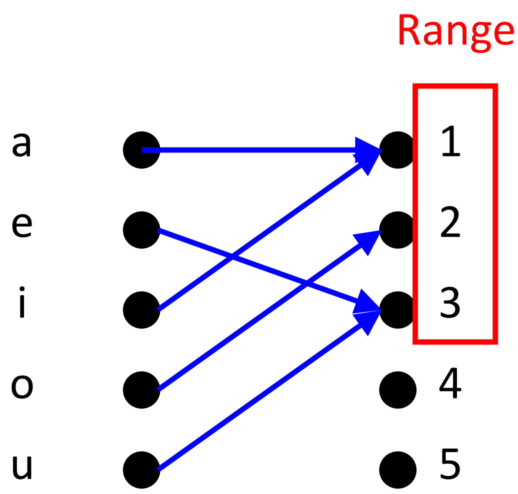
Functions

Definition of a function

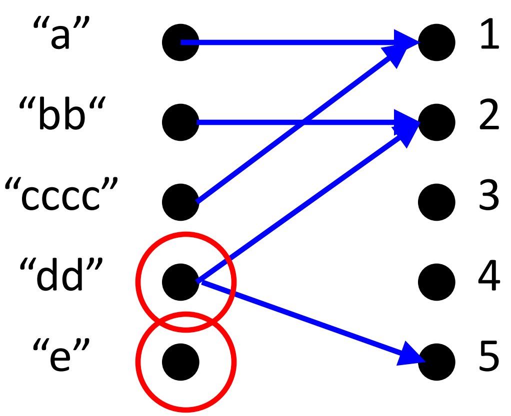
- A function takes an element from a set and maps it to a UNIQUE element in another set



Even more functions



Some function…



Not a valid function!
Also not a valid function!

Some Function Terminology

- If it is written that f:$A \rightarrow B$, and $f(a)=b$ (where $a \in A$ & $b \in B$), then we say:
  - A is the *domain* of f.
  - B is the *codomain* of f.
  - b is the *image* of a under f.
  - a is a *pre-image* of b under f.
    - In general, b may have more than 1 pre-image.
  - The *range* $R \subseteq B$ of f is $R=\{b \mid \exists a\, f(a)=b\}$.

We also say the *signature* of f is $A \rightarrow B$.

Range versus Codomain

- The range of a function might *not* be its whole codomain.
- The codomain is the set that the function is *declared* to map all domain values into.
- The range is the *particular* set of values in the codomain that the function *actually* maps elements of the domain to.

Range vs. Codomain - Example

- Suppose I declare to you that: "f is a function mapping students in this class to the set of grades {A,B,C,D,E}."
- At this point, you know f's codomain is: _{A,B,C,D,E}_, and its range is _unknown!_.
- Suppose the grades turn out all As and Bs.
- Then the range of f is _{A,B}______, but its codomain is ___still {A,B,C,D,E}!____.

Function arithmetic

- Let $f_1(x) = 2x$
- Let $f_2(x) = x^2$

- $f_1+f_2 = (f_1+f_2)(x) = f_1(x)+f_2(x) = 2x+x^2$
- $f_1f_2 = (f_1f_2)(x) = f_1(x)f_2(x) = 2x.x^2 = 2x^3$

<u>Example:</u> $f_1, f_2 : \mathbb{R} \to \mathbb{R}$, $f_1(x) = x^2$, $f_2(x) = x - x^2$.

Find $f_1 + f_2$, $f_1 f_2$.

<u>solution:</u> $(f_1 + f_2)(x) = f_1(x) + f_2(x) = x^2 + (x - x^2) = x$

$(f_1 f_2)(x) = x^2(x - x^2) = x^3 - x^4$

Definition: Let f be a function from the set A to the set B and let S be a subset of A. The image of S is the subset of B that consists of the images of the elements of S. We denote the image of S by $f(S)$, so that

$$f(S) = \{ f(s) \mid s \in S \}$$

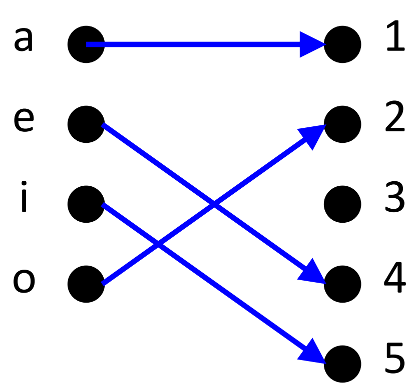
Example: Let $A = \{a, b, c, d, e\}$ and $B = \{1, 2, 3, 4\}$ with $f(a) = 2$, $f(b) = 1$, $f(c) = 4$, $f(d) = 1$, $f(e) = 1$. The image of the subset $S = \{b, c, d\}$ is the set $f(S) = \{1, 4\}$.

One-to-one functions

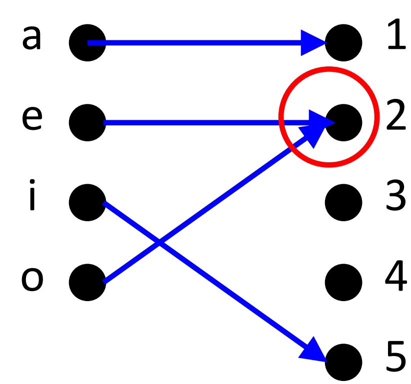
- A function is *one-to-one (1-1)*, or *injective*, or *an injective*, iff every element of its range has *only* 1 pre-image.
  - Formally: given $f{:}A\rightarrow B$,

$$\text{“}x \text{ is injective”} :\equiv (\neg\exists x,y{:}\ x\neq y \wedge f(x)=f(y)).$$

- Only <u>one</u> element of the domain is mapped <u>to</u> any given <u>one</u> element of the range.
- Formal definition: A function *f* is one-to-one if *f(x)* = *f(y)* implies *x* = *y*.



A one-to-one function



A function that is not one-to-one

Example: $f: \mathbb{Z} \to \mathbb{Z}$, $f(x)=x^2$ is NOT a one-to-one function. $f(1)=f(-1)$, but $1 \neq -1$.

Example: $f(x)=x+1$ is a one-to-one function.

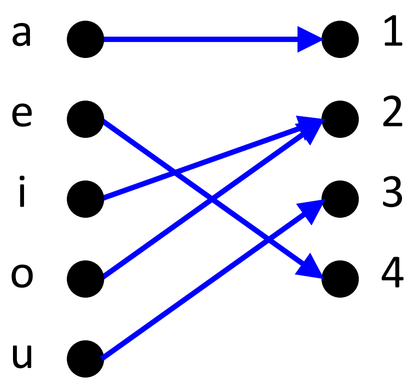
$$f(x)=f(y) \left(\Rightarrow \quad x+1=y+1 \Rightarrow \right) \quad x=y.$$

Definition: A function f whose domain and codomain are subsets of the set of real numbers is called <u>strictly increasing</u> if $f(x)<f(y)$ whenever $x<y$ and x and y are in the domain of f. Similarly, f is called <u>strictly decreasing</u> if $f(x)>f(y)$ whenever $x<y$ and x and y are in the domain of f.

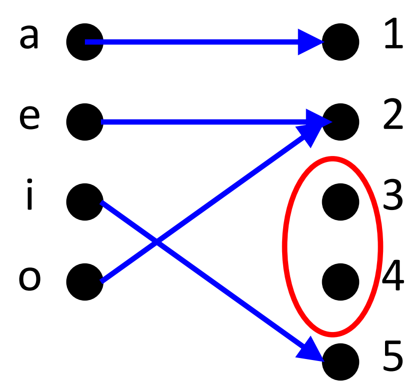
Onto functions

A function f:$A \rightarrow B$ is *onto* or *surjective* or *a surjection* iff its range is equal to its codomain ($\forall b \in B, \exists a \in A$: $f(a)=b$).

- Formal definition: A function f is onto if for all $y \in C$, there exists $x \in D$ such that $f(x)=y$.



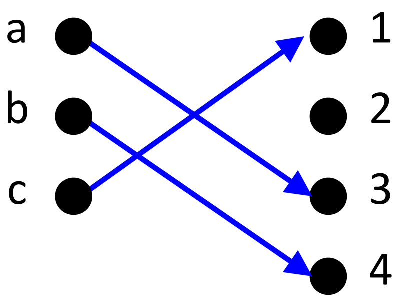
An onto function



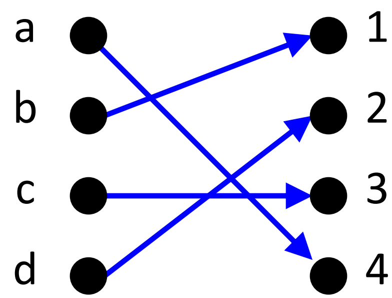
A function that is not onto

Onto vs. one-to-one

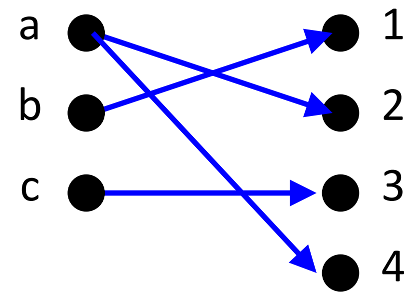
- Are the following functions onto, one-to-one, both, or neither?



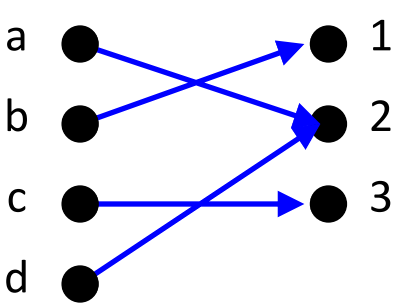
1-1, not onto



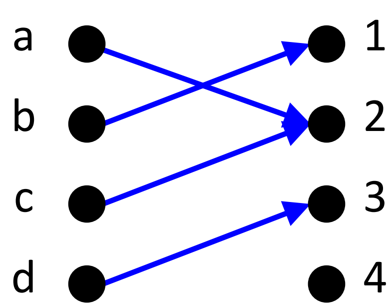
Both 1-1 and onto



Not a valid function



Onto, not 1-1



Neither 1-1 nor onto

Definition: A function from A to B is called **onto**, or **Surjective**, if and only if for every element $b \in B$ there is an element $a \in A$ with $f(a)=b$. A function f is called a **surjection** if it is onto.

$$\forall b \in B \ \exists a \in A \text{ such that } f(a)=b.$$

Example: The function $f: \mathbb{Z} \to \mathbb{Z}$, $f(x)=x^2$ is NOT onto. for $x=-1$ there is no element in $\mathbb{Z}$ such that $f(x)=x^2=-1$

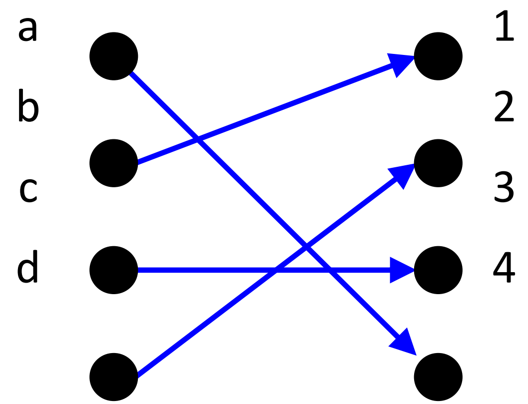
Example: $f: \mathbb{Z} \rightarrow \mathbb{Z}$, $f(x) = x + 1$. Is f onto?

<u>Example:</u> $f: \mathbb{Z} \to \mathbb{Z}$, $f(x) = x+1$. Is f onto?

<u>solution:</u> For every integer y there is an integer x such that $f(x)=y$. $f(x)=y$ if and only if $x+1=y$, which holds if and only if $x=y-1$.

Bijections

- Consider a function that is both one-to-one and onto:



- Such a function is a one-to-one correspondence, or a bijection

<u>Definition:</u> Let A be a set. The <u>identity function</u> on A is the function

$$I_A : A \to A \quad , \quad I_A(x) = x .$$

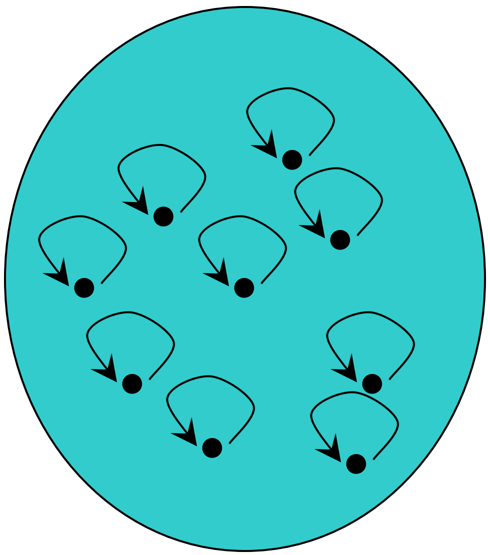
<u>Definition:</u> Let f be a one-to-one correspondence from the set A to the set B. The <u>inverse</u> function of f is the function that assigns to an element b belonging to B the unique element a in A such that $f(a)=b$. The inverse function of f is denoted by f^{-1}. Hence, $f^{-1}(b)=c$ when $f(a)=b$.

A function f is invertible iff it is one-to-one and onto.

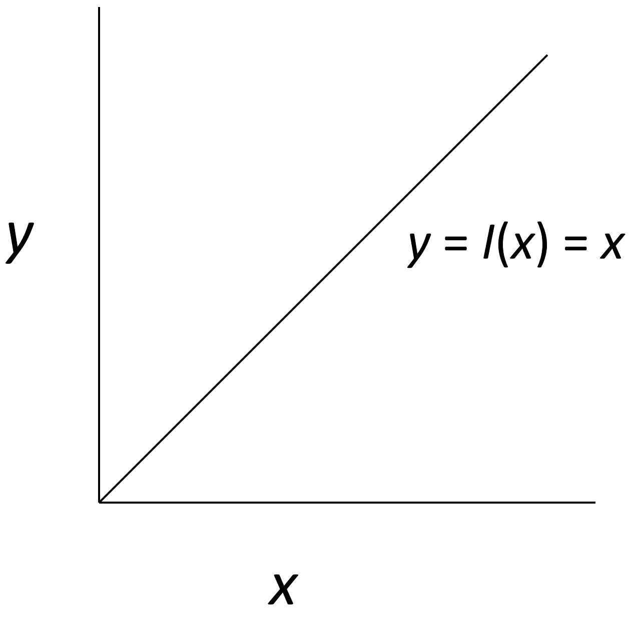
Example: $f: \mathbb{Z} \to \mathbb{Z}$, $f(x) = x^2$ is NOT invertible because $f(-1) = f(1) = 1$. So f is not one-to-one. Hence, f is not invertible.

Identity Function Illustrations

- The identity function:

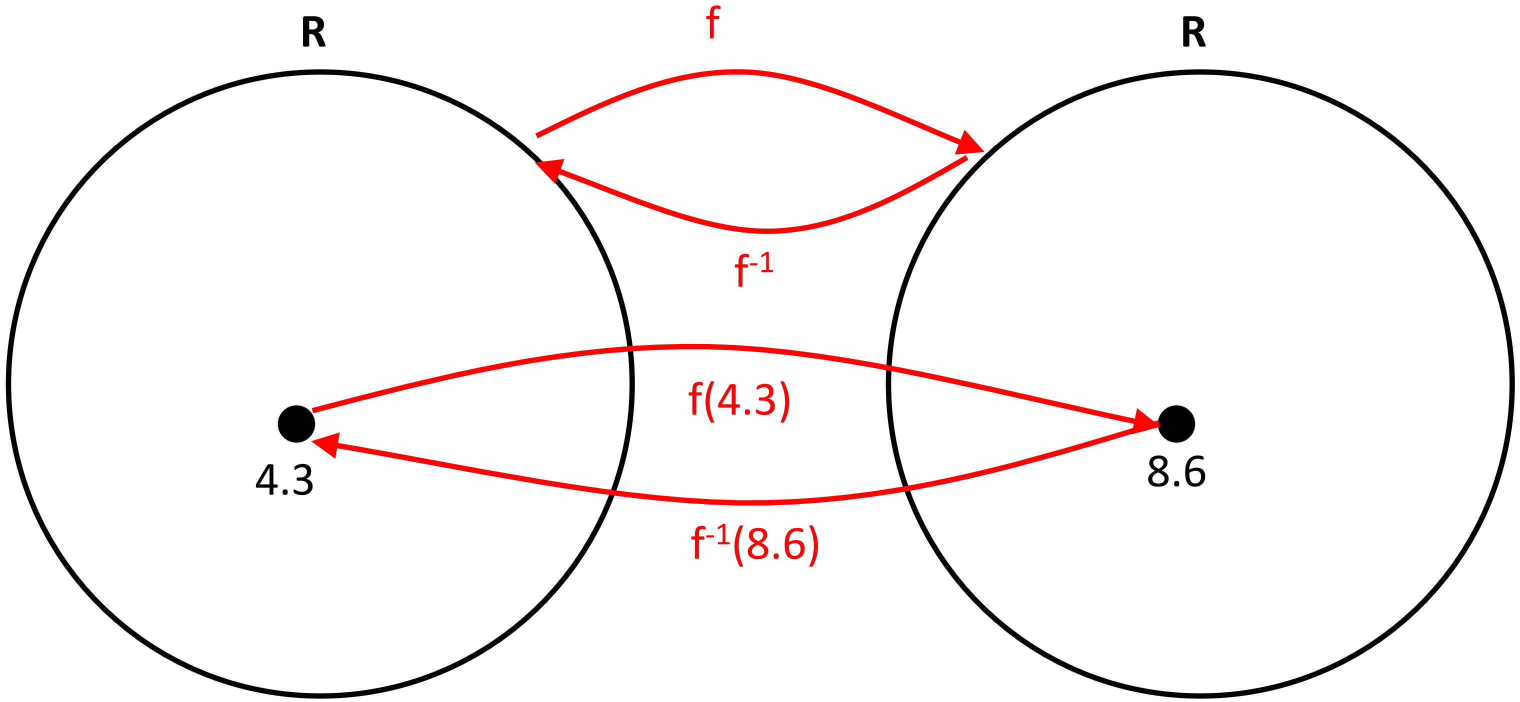


Domain and range



Inverse functions

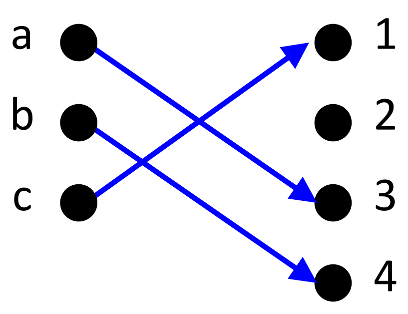
Let f(x) = 2.x



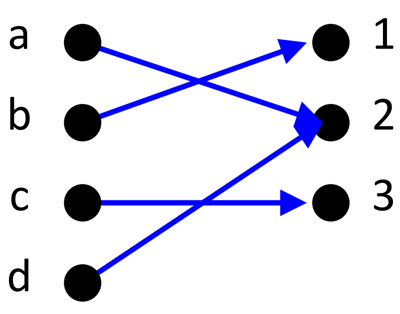
Then $f^{-1}(x) = x/2$

More on inverse functions

- Can we define the inverse of the following functions?



What is $f^{-1}(2)$?
Not onto!

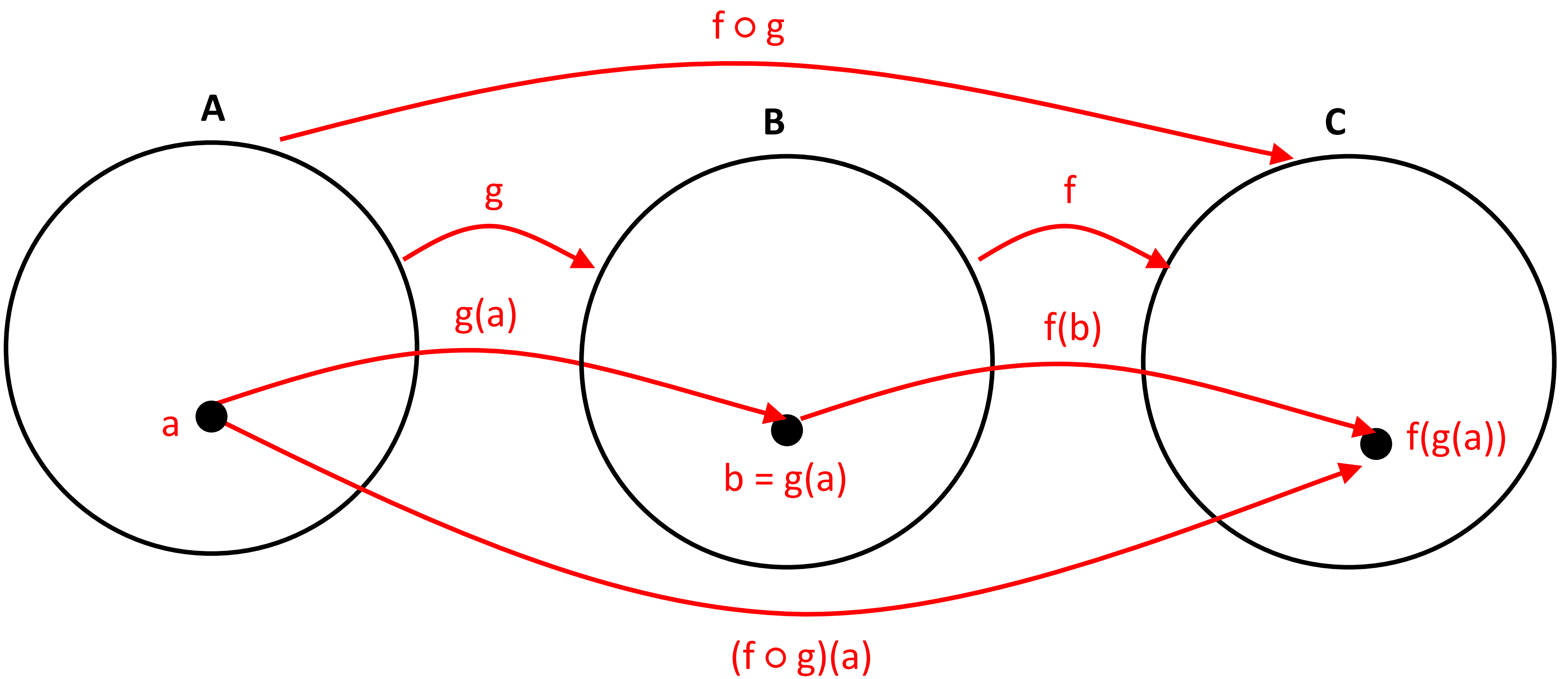


What is $f^{-1}(2)$?
Not 1-to-1!

- An inverse function can ONLY be done defined on a bijection

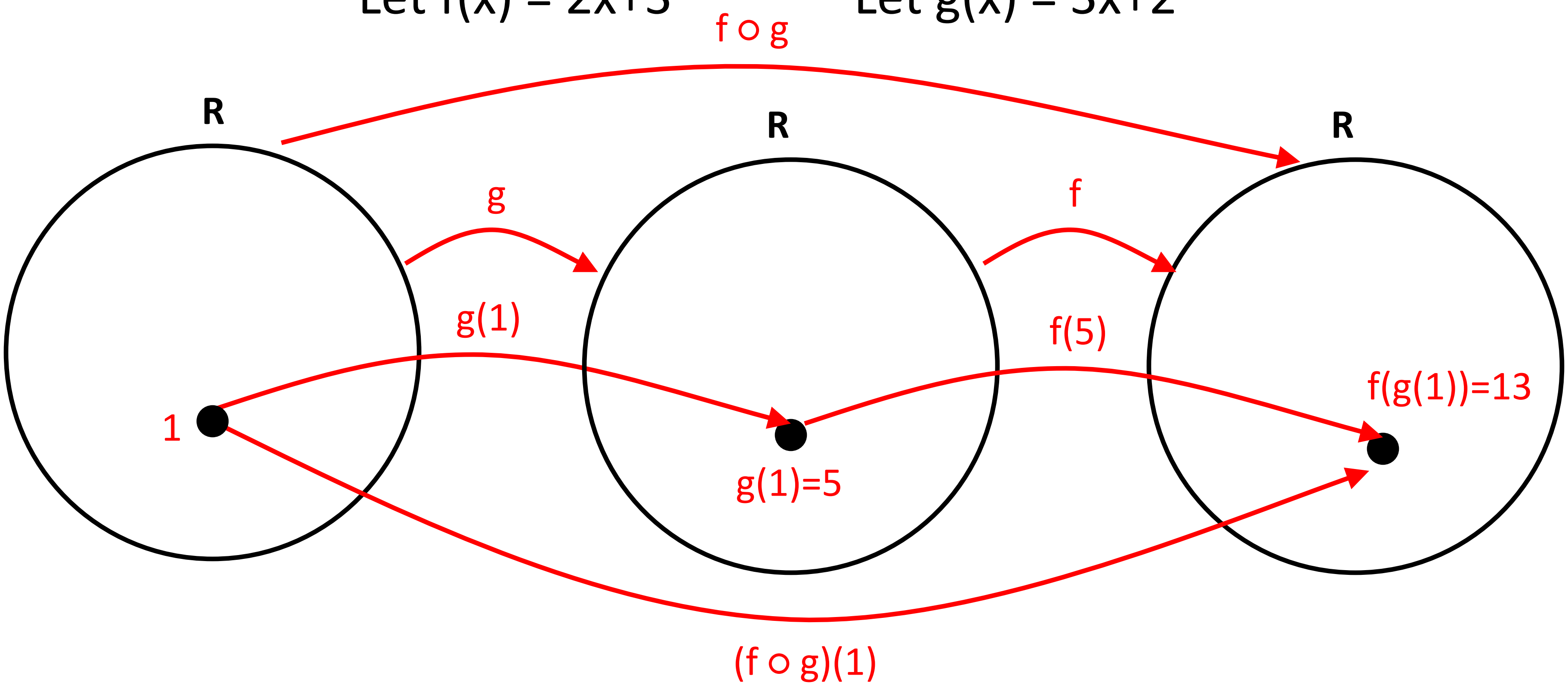
Compositions of functions

$$(f \circ g)(x) = f(g(x))$$



Compositions of functions

Let $f(x) = 2x+3$ Let $g(x) = 3x+2$



$f(g(x)) = 2(3x+2)+3 = 6x+7$

Compositions of functions

Does $f(g(x)) = g(f(x))$?

Let $f(x) = 2x+3$

Let $g(x) = 3x+2$

$f(g(x)) = 2(3x+2)+3 = 6x+7$

$g(f(x)) = 3(2x+3)+2 = 6x+11$

Not equal!

Function composition is not commutative!

Note: i) $f \circ g \neq g \circ f$. The commutative law does not hold for the composition of functions.

ii) If f is 1-1 and onto then f^{-1} exists. f composition f^{-1} gives the identity function

$$f \circ f^{-1} = f^{-1} \circ f = I.$$

And

$$(f^{-1})^{-1} = f.$$

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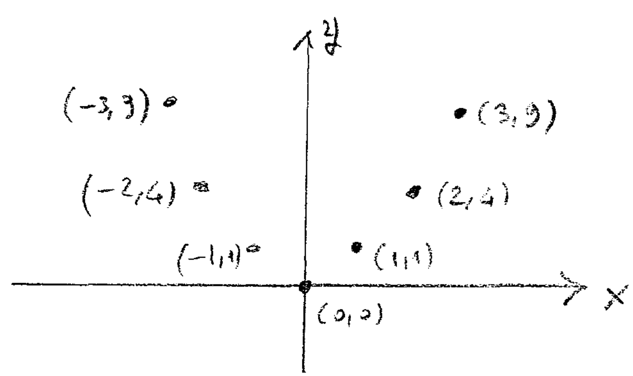
The Graphs of Functions

We can associate a set of pairs in $A \times B$ to each function from A to B. This set of pairs is called the <u>graph</u> of the function and is often displayed pictorially to aid in understanding the behavior of the function.

Definition: Let f be a function from the set A to the set B. The graph of the function f is the set of ordered pairs $\{(a,b) \mid a \in A \text{ and } f(a)=b\}$.

$$\text{graph } f \subseteq A \times B$$

Example: $f: \mathbb{Z} \to \mathbb{Z}$, $f(x)=x^2$. Graph of f is



Some Important Functions

Definition: The floor function assigns to the real number x the largest integer that is less than or equal to x. The value of the floor function at x is denoted by $\lfloor x \rfloor$. The ceiling function assigns to the real number x the smallest integer that is greater than or equal to x. The value of the ceiling function at x is denoted by $\lceil x \rceil$

Example: $\lfloor \frac{1}{2} \rfloor = 0$, $\lceil \frac{1}{2} \rceil = 1$, $\lfloor -\frac{1}{2} \rfloor = -1$, $\lceil -\frac{1}{2} \rceil = 0$

$\lfloor 3.1 \rfloor = 3$ $\lceil 3.1 \rceil = 4$ $\lfloor 7 \rfloor = 7$, $\lceil 7 \rceil = 7$ ▫

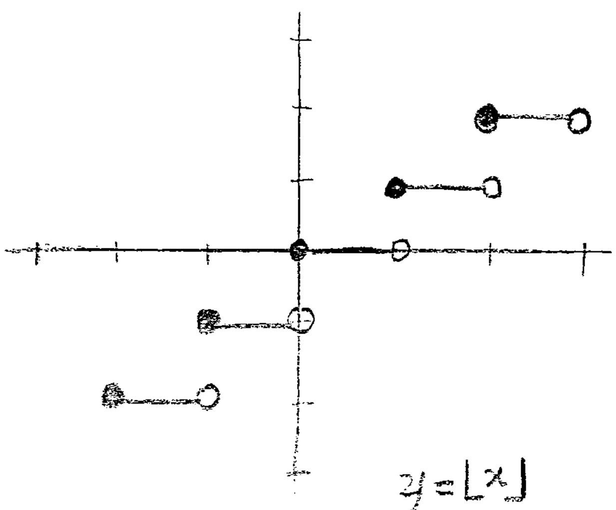
Useful functions

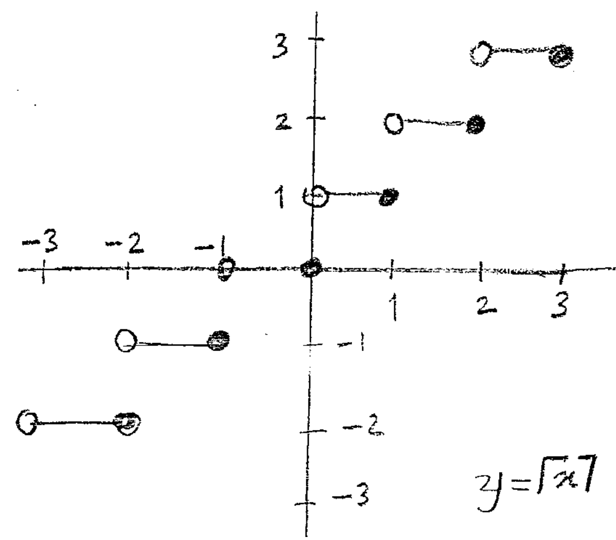
- Floor: $\lfloor x \rfloor$ means take the greatest integer less than or equal to the number

- Ceiling: $\lceil x \rceil$ means take the lowest integer greater than or equal to the number

Note: The floor and ceiling functions are useful in a wide variety of applications, including those involving data storage and data transmission.

Example:





Example: Data stored on a computer disk or transmitted over a data network are usually represented as a string of bytes. Each byte is made up of 8 bits. How many bytes are required to encode 100 bits of data?

solution:

$$\left\lceil \frac{100}{8} \right\rceil = \lceil 12{,}5 \rceil = 13 \text{ bytes is required.}$$

Ceiling and floor properties

Let n be an integer

(1a) $\lfloor x \rfloor = n$ if and only if $n \le x < n+1$

(1b) $\lceil x \rceil = n$ if and only if $n-1 < x \le n$

(1c) $\lfloor x \rfloor = n$ if and only if $x-1 < n \le x$

(1d) $\lceil x \rceil = n$ if and only if $x \le n < x+1$

(2) $x-1 < \lfloor x \rfloor \le x \le \lceil x \rceil < x+1$

(3a) $\lfloor -x \rfloor = -\lceil x \rceil$

(3b) $\lceil -x \rceil = -\lfloor x \rfloor$

(4a) $\lfloor x+n \rfloor = \lfloor x \rfloor + n$

(4b) $\lceil x+n \rceil = \lceil x \rceil + n$

Ceiling property proof

- Prove rule 4a: $\lfloor x+n \rfloor = \lfloor x \rfloor + n$
  - Where n is an integer
  - Will use rule 1a: $\lfloor x \rfloor = n$ if and only if $n \leq x < n+1$
- Direct proof!
  - Let $m = \lfloor x \rfloor$
  - Thus, $m \leq x < m+1$ (by rule 1a)
  - Add n to both sides: $m+n \leq x+n < m+n+1$
  - By rule 1a, $m+n = \lfloor x+n \rfloor$
  - Since $m = \lfloor x \rfloor$, $m+n$ also equals $\lfloor x \rfloor + n$
  - Thus, $\lfloor x \rfloor + n = m+n = \lfloor x+n \rfloor$

Factorial

- Factorial is denoted by n!
- n! = n . (n-1) .(n-2) … 2 .1
- Thus, 6! = 6 . 5 . 4 . 3 . 2 . 1 = 720
- Note that 0! is defined to equal 1

Proving Function problems

- Let f be an invertible function from Y to Z
- Let g be an invertible function from X to Y
- Show that the inverse of $f \circ g$ is:
  - $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$

(Pf) Thus, we want to show, for all $x \in X$

$((f \circ g) \circ (g^{-1} \circ f^{-1}))(x) = x$ and $((g^{-1} \circ f^{-1}) \circ (f \circ g))(x) = x$

$$
\begin{aligned}
((f \circ g) \circ (g^{-1} \circ f^{-1}))(x) &= (f \circ g)((g^{-1} \circ f^{-1}))(x)) \\
&= (f \circ g)(g^{-1}(f^{-1}(x))) \\
&= (f(g(g^{-1}(f^{-1}(x))))) \\
&= (f(f^{-1}(x)) \\
&= x
\end{aligned}
$$

The second equality is similar

In the questions below determine whether the rule describes a function with the given domain and codomain.

$f : \mathbf{N} \rightarrow \mathbf{N}$ where $f(n) = \sqrt{n}$.

$f: \mathbf{N} \rightarrow \mathbf{N}$ where $f(n) = \sqrt{n}$.

Ans: Not a function; $f(2)$ is not an integer.

$f : \mathbf{N} \to \mathbf{N}$ where $f(n) = \sqrt{n}$.

Ans: Not a function; $f(2)$ is not an integer.

$h : \mathbf{R}^{+} \to \mathbf{R}$ where $h(x) = \sqrt{x}$.

$f : \mathbf{N} \to \mathbf{N}$ where $f(n) = \sqrt{n}$.

Ans: Not a function; $f(2)$ is not an integer.

$h : \mathbf{R}^{+} \to \mathbf{R}$ where $h(x) = \sqrt{x}$.

Ans: Function.

$$F : \mathbf{R} \to \mathbf{R} \text{ where } F(x) = \frac{1}{x-5} \quad .$$

$F : \mathbf{R} \to \mathbf{R}$ where $F(x) = \dfrac{1}{x-5}$.

Ans: Not a function; $F(5)$ not defined.

$$F : \mathbf{R} \rightarrow \mathbf{R} \text{ where } F(x) = \frac{1}{x-5} \quad .$$

Ans: Not a function; $F(5)$ not defined.

$$F : \mathbf{Z} \rightarrow \mathbf{R} \text{ where } F(x) = \frac{1}{x^2-5} \quad .$$

$$F : \mathbf{R} \to \mathbf{R} \text{ where } F(x) = \frac{1}{x-5} \quad .$$

Ans: Not a function; $F(5)$ not defined.

$$F : \mathbf{Z} \to \mathbf{R} \text{ where } F(x) = \frac{1}{x^2-5} \quad .$$

Ans: Function.

$F : \mathbf{R} \to \mathbf{R}$ where $F(x) = \dfrac{1}{x-5}$.

Ans: Not a function; $F(5)$ not defined.

$F : \mathbf{Z} \to \mathbf{R}$ where $F(x) = \dfrac{1}{x^2-5}$.

Ans: Function.

$F : \mathbf{Z} \to \mathbf{Z}$ where $F(x) = \dfrac{1}{x^2-5}$.

$$F : \mathbf{R} \to \mathbf{R} \text{ where } F(x) = \frac{1}{x-5} \quad .$$

Ans: Not a function; $F(5)$ not defined.

$$F : \mathbf{Z} \to \mathbf{R} \text{ where } F(x) = \frac{1}{x^2-5} \quad .$$

Ans: Function.

$$F : \mathbf{Z} \to \mathbf{Z} \text{ where } F(x) = \frac{1}{x^2-5} \quad .$$

Ans: Not a function; $F(1)$ not an integer.

$$\therefore \; G : \mathbf{R} \rightarrow \mathbf{R} \text{ where } G(x) = \begin{cases} x+2 & \text{if } x \geq 0 \\ x-1 & \text{if } x \leq 4 \end{cases}$$

- $G : \mathbf{R} \to \mathbf{R}$ where $G(x) = \begin{cases} x+2 & \text{if } x \geq 0 \\ x-1 & \text{if } x \leq 4 \end{cases}$

  Ans: Not a function; the cases overlap. For example, $G(1)$

. $f : \mathbf{R} \to \mathbf{R}$ where $f(x) = \begin{cases} x^2 & \text{if } x \le 2 \\ x-1 & \text{if } x \ge 4 \end{cases}$

. $f: \mathbf{R} \to \mathbf{R}$ where $f(x) = \begin{cases} x^2 & \text{if } x \le 2 \\ x-1 & \text{if } x \ge 4 \end{cases}$

Ans: Not a function; $f(3)$ not defined.

i. $G : \mathbf{Q} \to \mathbf{Q}$ where $G(p/q) = q$.

i. $G : \mathbf{Q} \rightarrow \mathbf{Q}$ where $G(p/q) = q$.

Ans: Not a function; $f(1/2) = 2$ and $f(2/4) = 4$.

Give an example of a function $f : \mathbf{Z} \to \mathbf{Z}$ that is 1-1 and not onto $\mathbf{Z}$.

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Ans: $f(n) = 2n$.

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Ans: $f(n) = 2n$.

Give an example of a function $f : \mathbf{Z} \to \mathbf{Z}$ that is onto $\mathbf{Z}$ but not 1-1.

Give an example of a function $f: \mathbf{Z} \to \mathbf{Z}$ that is 1-1 and not onto $\mathbf{Z}$.
Ans: $f(n) = 2n$.

Give an example of a function $f: \mathbf{Z} \to \mathbf{Z}$ that is onto $\mathbf{Z}$ but not 1-1.
Ans: $f(n) = \left\lfloor \frac{n}{2} \right\rfloor$.

Give an example of a function $f : \mathbf{Z} \rightarrow \mathbf{Z}$ that is 1-1 and not onto $\mathbf{Z}$.

Ans: $f(n) = 2n$.

Give an example of a function $f : \mathbf{Z} \rightarrow \mathbf{Z}$ that is onto $\mathbf{Z}$ but not 1-1.

Ans: $f(n) = \left\lfloor \frac{n}{2} \right\rfloor$.

Give an example of a function $f : \mathbf{Z} \rightarrow \mathbf{N}$ that is both 1-1 and onto $\mathbf{N}$.

Give an example of a function $f : \mathbf{Z} \to \mathbf{Z}$ that is 1-1 and not onto $\mathbf{Z}$.
Ans: $f(n) = 2n$.

Give an example of a function $f : \mathbf{Z} \to \mathbf{Z}$ that is onto $\mathbf{Z}$ but not 1-1.
Ans: $f(n) = \left\lfloor \frac{n}{2} \right\rfloor$.

Give an example of a function $f : \mathbf{Z} \to \mathbf{N}$ that is both 1-1 and onto $\mathbf{N}$.
Ans: $f(n) = \begin{cases} -2n & \text{if } n \le 0 \\ 2n - 1 & \text{if } x > 0 \end{cases}$

Suppose $f : \mathbf{N} \to \mathbf{N}$ has the rule $f(n) = 4n + 1$. Determine whether f is 1-1.
Ans:

Suppose $f : \mathbf{N} \to \mathbf{N}$ has the rule $f(n) = 4n + 1$. Determine whether f is onto $\mathbf{N}$.
Ans:

Suppose $f : \mathbf{Z} \to \mathbf{Z}$ has the rule $f(n) = 3n^2 - 1$. Determine whether f is 1-1.
Ans:

Suppose $f : \mathbf{Z} \to \mathbf{Z}$ has the rule $f(n) = 3n - 1$. Determine whether f is onto $\mathbf{Z}$.
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Suppose $f : \mathbf{N} \to \mathbf{N}$ has the rule $f(n) = 4n + 1$. Determine whether f is 1-1.
Ans: Yes.

Suppose $f : \mathbf{N} \to \mathbf{N}$ has the rule $f(n) = 4n + 1$. Determine whether f is onto $\mathbf{N}$.
Ans: No.

Suppose $f : \mathbf{Z} \to \mathbf{Z}$ has the rule $f(n) = 3n^2 - 1$. Determine whether f is 1-1.
Ans: No.

Suppose $f : \mathbf{Z} \to \mathbf{Z}$ has the rule $f(n) = 3n - 1$. Determine whether f is onto $\mathbf{Z}$.
Ans: No.

Suppose $f : \mathbf{N} \to \mathbf{N}$ has the rule $f(n) = 3n^2 - 1$. Determine whether f is 1-1.
Ans: Yes.

Suppose $f : \mathbf{N} \to \mathbf{N}$ has the rule $f(n) = 4n^2 + 1$. Determine whether f is onto $\mathbf{N}$
Ans: No.

$$f(n) = \begin{cases} \dfrac{-n}{2}, & n \text{ even} \\ \dfrac{n^2+1}{2}, & n \text{ odd} \end{cases}$$

is $f: \mathbf{N} \to \mathbf{Z}$ 1-1 and onto $\mathbf{Z}$?

$$f(n) = \begin{cases} -2n, & n \leq 0 \\ 2n+1, & n > 0 \end{cases}$$

an example of a function $f : \mathbf{Z} \rightarrow \mathbf{N}$ that is 1-1 and not onto $\mathbf{N}$.

$$f(n) = \begin{cases} \dfrac{-n}{2}, & n \text{ even.} \\ \dfrac{n\text{-}1}{2}, & n \text{ odd} \end{cases}$$

an example of a function $f: \mathbf{N} \to \mathbf{Z}$ that is onto $\mathbf{Z}$ and not 1-1.

Let $f(x) = \lfloor x^3/3 \rfloor$. Find $f(S)$ if S is:

(a) $\{-2,-1,0,1,2,3\}$.
(b) $\{0,1,2,3,4,5\}$.
(c) $\{1,5,7,11\}$.
(d) $\{2,6,10,14\}$.

Ans: (a) $\{-3,-1,0,2,9\}$.
(b) $\{0,2,9,21,41\}$.
(c) $\{0,41,114,443\}$.
(d) $\{2,72,333,914\}$.