Mat2033 - Discrete Mathematics

Relations and their Properties

Binary Relations

A binary relation from A to B is a subset of A x B

- The Cartesian product of two sets, say A and B
- We might represent this as a set of ordered pairs
- In a pair, first is from A, second is from B

The relation is a set of pairs where first element is from A and second is from B

$$a \in A \wedge b \in B$$

We say ***"a is related to b by R"*** where R is a relation

$$a\,R\,b \Leftrightarrow (a,b) \in R$$

- Students:
  - Alex, Bea, Cath, Don, Eddie, Fiona
- Subjects:
  - IP1, FP1, AF2
- Let R be the relation of students who passed subjects

R = {(Alex,IP1),(Alex,AF2),(Bea,AF2),(Cath,AF2),(Cath,IP1),
(Don,AF2),(Don,IP1),(Fiona,IP1),(Eddie,IP1),(Fiona,FP2)}

Order between pairs is insignificant (look at Fiona)

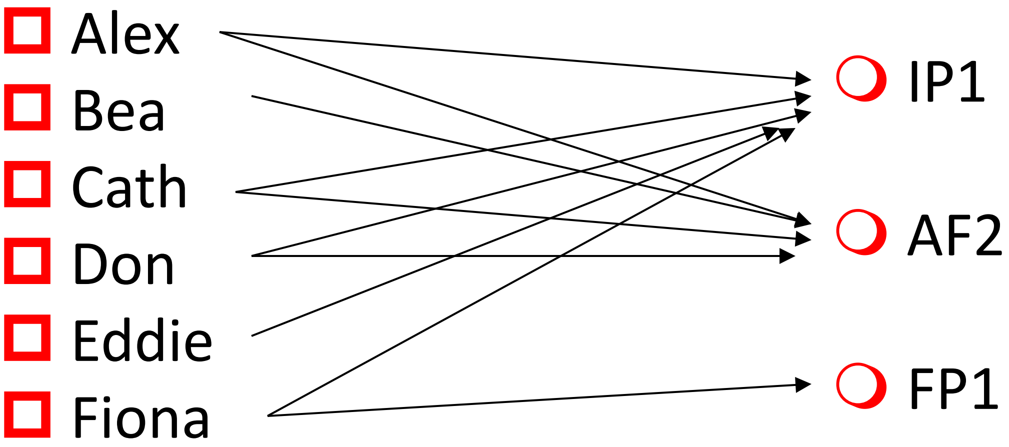
R is a set. Right?

Order within pairs ***is*** significant (a pair (FP2,Fiona)?)

A set of ordered pairs

Pictorially

R = {(Alex,IP1),(Alex,AF2),(Bea,AF2),(Cath,AF2),(Cath,IP1),
(Don,AF2),(Don,IP1),(Fiona,IP1),(Eddie,IP1),(Fiona,FP1)}



But you could have functional relations

It isn’t a function!

Example: How many relations are there on a set of n elements?

- Each relation R_i is a subset of $\{(a,b) \mid a \in A \wedge b \in B\}$
  R_i is a subset of the Cartesian product of A and B
- When we have a relation on a single set, this is just A = B
  R_i is then a subset of $\{(x,y) \mid x \in A \wedge y \in A\}$
- The cardinality of A x A is $|A \times A| = n^2$
- There are 2^n subsets of a set of size n
- If the set of tuples to choose from is of size n^2
  then there are 2^{n^2} possible subsets

✓ There are $2^{(n^2)}$ possible relations

Equivalence Relations and Partitions

Now we take up an idea fundamental for computer science and mathematics. Indeed, you have seen equivalence relations from the beginning of your study of mathematics and computer science.

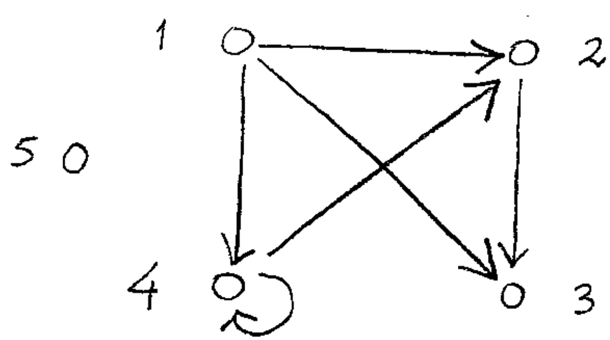
Relations, Graphs, and Matrices

Definition: A binary relation on a set A is a subset of the Cartesian product $A \times A$.

As you can see, the concept of relation is very general. Let us look at different ways to define a relation. We defined it as a set of ordered pairs. It can also be viewed as a directed graph or as a relation matrix.

The *directed graph* of the relation is the set of points of A together with the ordered pairs of the given relation; these two sets are often represented in a sketch as a set of dots, one dot for each point of A, and a set of arrows joining the dots, one arrow for each ordered pair (a,b) in the given relation, and drawn as an arrow from a to b:

Example: The relation $\{(1,2),(1,3),(1,4),(2,3),(4,4),(4,2)\}$ on the set $A=\{1,2,3,4,5\}$ has the graph



The arrow from "1" to "2" stands for the ordered pair (1,2)

The <u>relation-matrix</u> is a square array of 0's and 1's with rows and columns labeled by the elements of A, one row and one column for each such element. For each a, b in A, the entry in row a and column b of the matrix is 1 (a,b) is one of the ordered pairs of the relation, and 0 if it is not.

	1	2	3	4	5
1	0	1	1	1	0
2	0	0	1	0	0
3	0	0	0	0	0
4	0	1	0	1	0
5	0	0	0	0	0

The relation-matrix of the above example.

<u>Example:</u> The relation $\{(1,2),(1,3),(1,4),(2,3),(4,4),(4,2)\}$ on the set $A=\{1,2,3,4,5\}$,

Example: An example of a relation R defined by the set-builder notation

$$R = \{(a,b) \mid a, b \in \mathbb{N},\ b = a+1\}$$

The graph of R is

$$\overset{1}{\circ} \longrightarrow \overset{2}{\circ} \longrightarrow \overset{3}{\circ} \longrightarrow \overset{4}{\circ} \longrightarrow \cdots$$

Its relation-matrix is an infinite matrix that in its upper left corner is

$$\begin{array}{c|ccccc}
 & 1 & 2 & 3 & 4 & \cdots \\
\hline
1 & 0 & 1 & 0 & 0 & \\
2 & 0 & 0 & 1 & 0 & \cdots \\
3 & 0 & 0 & 0 & 1 & \\
\vdots & & \vdots & & &
\end{array}$$

Note: To convince you how general the idea of relation is, we now observe that our five-point set A of above example has a total of more than 32,000,000 different relations on it. Precisely, there are 2^{25} of them, because every different subset of the 25-point set $A \times A$ is a different relation. So we shall narrow our inquiry to a much smaller class of relations: the equivalence relations.

Equivalence Relations

Definition: A relation on a set A is called an equivalence relation if and only if it is reflexive, symmetric, and transitive.

Definition: A relation R is reflexive if and only if for all $a \in A$, (a,a) is in the relation R.

Thus in graphical terms, there must be a loop at every point of A. In the relation-matrix, the main diagonal must be all 1's.

Example: The prior examples are not reflexive. The complete relation $R = A \times A$ on A is reflexive, of course.

Practice: Prove $R' = \{(x, y) \mid x, y \in \mathbb{N},\ x \leq y\}$ is reflexive.

Note: We can easily count the number of reflexive relations on our five-point set A. There are 2^{20}, about 1,000,000, of them. This is so because a reflexive relation R may be any subset of $A \times A$ that includes $D = \{(x,x) \mid x \in A\}$, a set of five points: $D \subseteq R \subseteq A \times A$. Our answer is the total number of subsets of $(A \times A) - D$. This set has $25 - 5 = 20$ points and, therefore, 2^{20} subsets.

Example: Consider these relations on the set of integers:

$$R_1 = \{(a, b) \mid a \le b\},$$
$$R_2 = \{(a, b) \mid a > b\},$$
$$R_3 = \{(a, b) \mid a = b \text{ or } a = -b\},$$
$$R_4 = \{(a, b) \mid a = b\},$$
$$R_5 = \{(a, b) \mid a = b + 1\},$$
$$R_6 = \{(a, b) \mid a + b \le 3\}.$$

Which of these relations are reflexive?

Solution: The reflexive relations from Example 5 are R_1 (because $a \le a$ for every integer a), R_3, and R_4. For each of the other relations in this example it is easy to find a pair of the form (a, a) that is not in the relation. (This is left as an exercise for the reader.) ◀

Definition: A relation R is <u>symmetric</u> if and only if, for every ordered pair (a,b) in R, the reversed ordered pair (b,a) is also in R.

* Thus whenever we have one arrow in the graph, there must be another arrow "going backward" (except for loops, of course). The relation-matrix must be symmetric about the main diagonal.

Examples: Consider $R = A \times A$ again, the complete relation. It is symmetric, of course. Its relation-matrix consists entirely of 1's.

Define now R' on $\mathbb{N}$ as

$$R' = \{(a,b) \mid a,b \in \mathbb{N},\ a+b \text{ is odd}\}$$

Definition: A relation R is **symmetric** if and only if, for every ordered pair (a,b) in R, the reversed ordered pair (b,a) is also in R.

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Examples: Consider $R = A \times A$ again, the complete relation. It is symmetric, of course. Its relation-matrix consists entirely of 1's.

Define now R' on $\mathbb{N}$ as

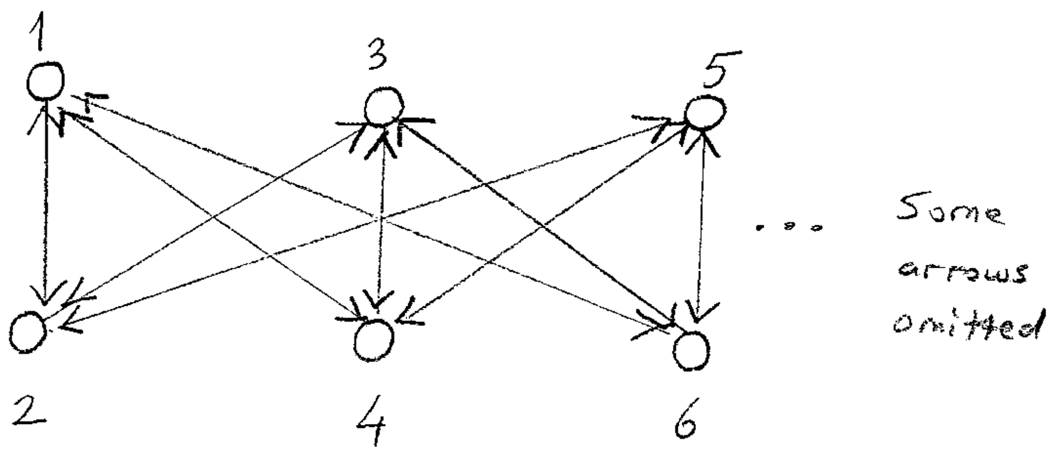
$$R' = \{(a,b) \mid a,b \in \mathbb{N},\ a+b \text{ is odd}\}$$

Thus two integers are related by R' iff one is even and the other is odd. R' is symmetric. The upper left corner of the relation-matrix of R' is

$$\begin{array}{c|cccccc} & 1 & 2 & 3 & 4 & \cdots \\ \hline 1 & 0 & 1 & 0 & 1 & \\ 2 & 1 & 0 & 1 & 0 & \cdots \\ 3 & 0 & 1 & 0 & 1 & \\ \vdots & & \vdots & & & \end{array}$$

You can see in various ways that R' is not reflexive.

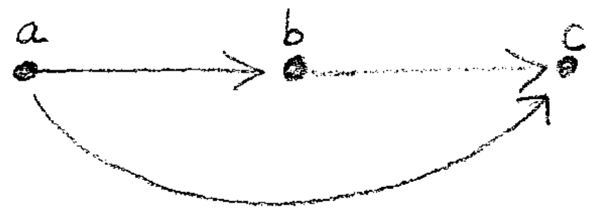
The graph of R' is, in part



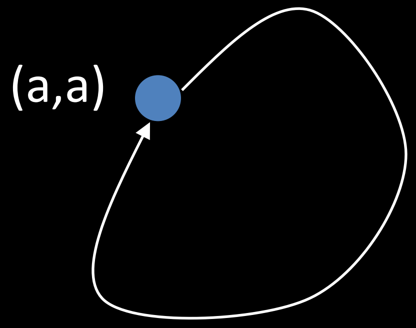
There are too many arrows to draw here. They exist in both directions between each odd and each even integer.

Definition: A relation R is transitive if and only if, for all a, b, c in A $[(a,b) \in R$ and $(b,c) \in R]$ implies $(a,c) \in R$

In a graph of R this property tells of certain shortcuts: whenever we can go from a to c in two steps, we can go there in one step.



- Reflexive:
  - if a is in A then (a,a) is in R
- Symmetric:
  - if (a,b) is in R and $a \neq b$ then (b,a) is in R
- Antisymmetric:
  - if (a,b) is in R and $a \neq b$ then (b,a) is not in R
- Transitive:
  - if (a,b) is in R and (b,c) is in R then (a,c) is in R



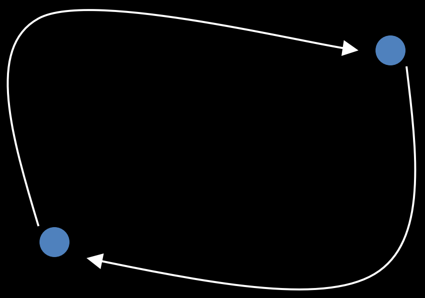
- Reflexive
  - if a is in A then (a,a) is in R

- Example: a divides b i.e. a|b
  - R = {(a,b)| a $\in$ A, b $\in$ B, a|b}

$$\boxed{a \in A \rightarrow (a,a) \in R}$$

- Symmetric
  - if (a,b) is in R and a $\neq$ b

    then (b,a) is in R

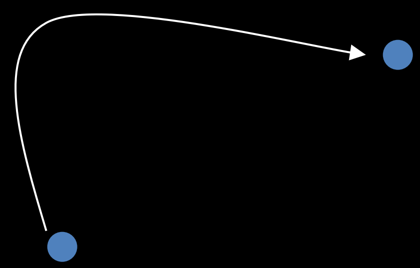


Example: a is married to b

$$\boxed{(a,b) \in R \wedge a \neq b \rightarrow (b,a) \in R}$$

Properties of Relations

Antisymmetric



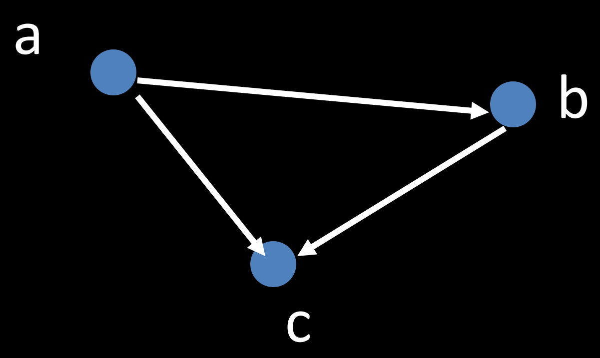
- Antisymmetric
    - if (a,b) is in R and a $\neq$ b

      then (b,a) is not in R

Example: a divides b i.e. a|b and a < b

R = {(a,b)| a in A, b in B, a|b}

$$\boxed{(a,b) \in R \wedge a \neq b \rightarrow (b,a) \notin R}$$



- Transitive
  - if (a,b) is in R and (b,c) is in R
    then (a,c) is in R

Example: a is less than b i.e. a < b

- a and b are positive integers
- R = {(a,b) | a < b}

$$\boxed{(a,b) \in R \wedge (b,c) \in R \rightarrow (a,c) \in R}$$

Exercise:

Consider the following relations on $\{1, 2, 3, 4\}$:

$$R_1 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)\},$$
$$R_2 = \{(1, 1), (1, 2), (2, 1)\},$$
$$R_3 = \{(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (3, 3), (4, 1), (4, 4)\},$$
$$R_4 = \{(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)\},$$
$$R_5 = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)\},$$
$$R_6 = \{(3, 4)\}.$$

Exercise: Consider these relations on the set of integers:

$$R_1 = \{(a, b) \mid a \leq b\},$$
$$R_2 = \{(a, b) \mid a > b\},$$
$$R_3 = \{(a, b) \mid a = b \text{ or } a = -b\},$$
$$R_4 = \{(a, b) \mid a = b\},$$
$$R_5 = \{(a, b) \mid a = b + 1\},$$
$$R_6 = \{(a, b) \mid a + b \leq 3\}.$$

$$A = \{1,2,3\} \qquad B = \{1,2,3,4\}$$

$$R_1 = \{(1,1),(2,2),(3,3)\}$$

$$R_2 = \{(1,1),(1,2),(1,3),(1,4)\}$$

$$R_1 \cup R_2 = \{(1,1),(1,2),(2,2),(1,3),(1,4),(3,3)\}$$

$$R_1 \cap R_2 = \{(1,1)\}$$

$$R_1 - R_2 = \{(2,2),(3,3)\}$$

$$R_2 - R_1 = \{(1,2),(1,3),(1,4)\}$$

Let R_1 be the "less than" relation on the set of real numbers and let R_2 be the "greater than" relation on the set of real numbers, that is, $R_1 = \{(x, y) \mid x < y\}$ and $R_2 = \{(x, y) \mid x > y\}$. What are $R_1 \cup R_2$, $R_1 \cap R_2$, $R_1 - R_2$, $R_2 - R_1$, and $R_1 \oplus R_2$?

Let R_1 be the "less than" relation on the set of real numbers and let R_2 be the "greater than" relation on the set of real numbers, that is, $R_1 = \{(x, y) \mid x < y\}$ and $R_2 = \{(x, y) \mid x > y\}$. What are $R_1 \cup R_2$, $R_1 \cap R_2$, $R_1 - R_2$, $R_2 - R_1$, and $R_1 \oplus R_2$?

Solution: We note that $(x, y) \in R_1 \cup R_2$ if and only if $(x, y) \in R_1$ or $(x, y) \in R_2$. Hence, $(x, y) \in R_1 \cup R_2$ if and only if $x < y$ or $x > y$. Because the condition $x < y$ or $x > y$ is the same as the condition $x \neq y$, it follows that $R_1 \cup R_2 = \{(x, y) \mid x \neq y\}$. In other words, the union of the "less than" relation and the "greater than" relation is the "not equals" relation.

Next, note that it is impossible for a pair (x, y) to belong to both R_1 and R_2 because it is impossible for $x < y$ and $x > y$. It follows that $R_1 \cap R_2 = \emptyset$. We also see that $R_1 - R_2 = R_1$, $R_2 - R_1 = R_2$, and $R_1 \oplus R_2 = R_1 \cup R_2 - R_1 \cap R_2 = \{(x, y) \mid x \neq y\}$. ◀

- Let R be a relation from set A to set B
- Let S be a relation from set B to set C

- The composite of R and S is a relation from set A to set C
  - and is a set of ordered pairs (a,c) such that
  - there exists an (a,b) in R and an (b,c) in S

- The composite of R and S is denoted as SoR

The composite of R with S $S \circ R$

$$R = \{(1,1),(1,4),(2,3),(3,1),(3,4)\}$$

$$S = \{(1,0),(2,0),(3,1),(3,2),(4,1)\}$$

$$S \circ R = \{(1,0),(1,1),(2,1),(2,2),(3,0),(3,1)\}$$