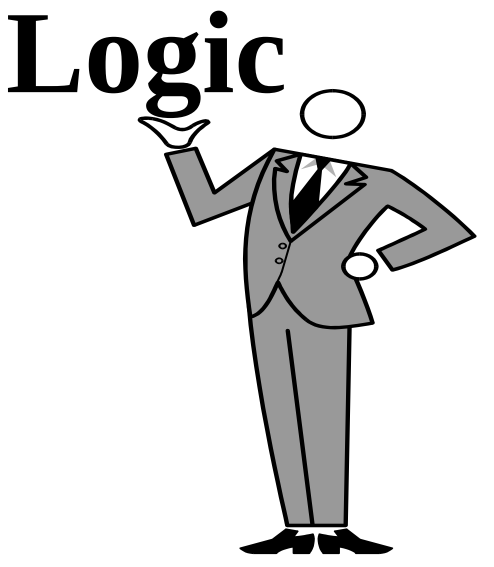
Mat2033 - Discrete Mathematics

The Foundations: Logic and Proof, Sets and Functions

Sections 1.1, 1.2



Propositional Equivalences

Propositions

A proposition is a statement that is either true or false, but not both.

Today is Tuesday.

Six is a prime number.

~~Are you Bob?~~

$7<5$

~~Consider this statement.~~

<u>Example:</u> All the following statements are propositions

1. Washington D.C., is the capital of the United States of America. (True).

2. Toronto is the capital of Canada. (False)

3. $1+1=2$ (True)

4. $2+2=3$ (False)

Example: Consider the following sentences.

1. What time is it? (Not Proposition)
2. Read this carefully. (Not Proposition)
3. $x+1=2$ (Not Proposition)
4. $x+y=z$ (Not Proposition)

Compound Propositions

Compound propositions are formed from existing propositions using logical operators

Today is Wednesday and it is snowing outside.

12 is not a prime number.

Negation of a Proposition

P	$\neg P$
T	F
F	T

NOT

3

Example: Find the negation of the proposition

"Today is Friday"

and express this in simple English.

Solution: The negation is

"It is not the case that today is Friday."

This negation can be more simply expressed by

"Today is not Friday" or "It is not Friday Today".

ntroduce the logical operators that are used to form
ew propositions from two or more existing propositions.
These logical operators are also called connectives.

Definition 3: Let p and q be propositions. The proposition "p or q", denoted by $p \vee q$, is the proposition that is false when p and q are both false and true otherwise. The proposition $p \vee q$ is called the disjunction of p and q.

Disjunction of Two Propositions

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

OR

- The binary *disjunction operator* "$\vee$" (*OR*) combines two propositions to form their logical *disjunction*.
- p = "My car has a bad engine."
- q = "My car has a bad carburetor."
- $p \vee q$ = "Either my car has a bad engine, or my car has a bad carburetor."

<u>Definition 2</u>: Let p and q be propositions. The proposition "p and q", denoted by $p \wedge q$, is the proposition that is true when both p and q are true and is false otherwise. The proposition $p \wedge q$ is called the <u>conjunction</u> of p and q.

Conjunction of Two Propositions

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

AND

Example: Find the conjunction of the propositions p and q where p is the proposition "Today is Friday" and q is the proposition "It is raining today".

solution: The conjunction of these propositions, $p \wedge q$, is the proposition "Today is Friday and it is raining today". This proposition is true on rainy Fridays and is false on any day that is not a Friday and on Fridays when it does not rain.

Definition 4: Let p and q be propositions. The exclusive or of p and q, denoted by $p \oplus q$, is the proposition that is true when exactly one of p and q is true and is false otherwise.

Exclusive OR of Two Propositions

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

Exactly one of them is true.

For example, the inclusive or is beign used in the statement

"Students who have taken calculus or computer science can take this class."

Here we mean that students who have taken both calculus and computer science can take the class, as well as the students who have taken only one of the two subjects. On the other hand, we are using the exclusive or when we say

"Students who have taken calculus or computer science, but not both, can enroll in this class."

Here we mean that the students who have taken both calculus and a computer science course cannot take the class. Only those who have taken exactly one of the two courses can take the class.

Similarly, when a menu at a restaurant states, "Soup or salad comes with an entrée," the restaurant almost always means that customers can have either soup or salad, but not both. Hence, it is an exclusive, rather than an inclusive or.

Definition 5: Let p and q be propositions. The implication $p \to q$ is the proposition that is false when p is true and q is false and true otherwise. In this implication p is called the hypothesis (or antecedent or premise) and q is called the conclusion (or consequence).

Implication

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

p is called the *hypothesis* and q is the *conclusion*

Implication

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

- “if p, then q”
- “p implies q”
- “if p,q”
- “p only if q”
- “p is sufficient for q”
- “q if p”
- “q whenever p”
- “q is necessary for p”

q whenever p

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Suppose that the proposition is true. Then, q is true whenever p is true.

p is sufficient for q

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Suppose that the proposition is true. Then, to guarantee that q is true it is sufficient to say that p is true.

Note that $p \rightarrow q$ is false only in the case that p is true but q is false, so that it is true when both p and q are true, and when p is false (no matter what truth value q has).

For instance, the implication

"<u>If it is sunny today, then we will go to beach.</u>"

is an implication used in normal language, since there is a relationship between the hypothesis and the conclusion. Further, this implication is considered valid unless it is indeed sunny today, but we do not go to beach. On the other hand, the implication

"If today is Friday, then $2+3=5$."

is true from the definition of implication, since its conclusion is true. (The truth value of the hypothesis does not matter then.) The implication

"If today is Friday, then $2+3=6$".

is true everyday except Friday, even though $2+3=6$ is false.

Examples of Implications

- "If this lecture ends, then the sun will rise tomorrow." *True* or *False*?
- "If Tuesday is a day of the week, then I am a penguin." *True* or *False*?
- "If 1+1=6, then Bush is president." *True* or *False*?
- "If the moon is made of green cheese, then I am richer than Bill Gates." *True* or *False*?

We can build up compound propositions using the negation operator and different connectives defined so far.

For instance

(*) $(p \vee q) \wedge (\neg r)$ is the conjunction of $p \vee q$ and $\neg r$

(**) $\neg p \wedge q$ is the conjunction of $\neg p$ and q. (Not the negation of the conjunction of p and q, namely $\neg (p \wedge q)$.

There are some related implications that can be formed from $p \to q$. The proposition $q \to p$ is called the <u>converse</u> of $p \to q$. The <u>contrapositive</u> of $p \to q$ is the proposition $\neg q \to \neg p$.

<u>Example:</u> Find the converse and the contrapositive of the implication

"If today is Thursday, then I have a test today."

<u>solution:</u> The converse is

"If I have a test today, then today is Thursday."

And the contrapositive of this implication is

"If I have not a test today, then today is not Thursday"

Converse of an Implication

p	q	$p \rightarrow q$	$p \leftarrow q$
T	T	T	T
T	F	F	T
F	T	T	F
F	F	T	T

Example of Converse

If it stays warm for a week, the apple trees will bloom.

If the apple trees bloom, it will be warm for a week.

If x is even then x^2 is even.

If x^2 is even then x is even.

Contrapositive of an Implication

p	q	$p \to q$	$\neg p$	$\neg q$	$\neg q \to \neg p$
T	T	T	F	F	T
T	F	F	F	T	F
F	T	T	T	F	T
F	F	T	T	T	T

Examples of Contrapositive

If it snows tonight, then I will stay at home.

If I do not stay at home, then it will not snow tonight.

If x is odd then x^2 is odd.

If x^2 is not odd then x is not odd.

If x^2 is even then x is even.

Definition 6: Let p and q be propositions. The biconditional $p \leftrightarrow q$ is the proposition that is true when p and q have the same truth values and is false otherwise.

TABLE 6: The truth table for the biconditional $p \leftrightarrow q$

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Note that the biconditional $p \leftrightarrow q$ is true precisely when both the implications $p \rightarrow q$ and $q \rightarrow p$ are true. Because of this, the terminology

"p if and only if q"

is used for this biconditional. Other common ways of expressing the proposition $p \leftrightarrow q$ are:

"p is necessary and sufficient for q" and

"If p then q, and conversely."

Biconditional

$$p \leftrightarrow q$$

p	q	$p \rightarrow q$	$p \leftarrow q$	$(p \rightarrow q) \wedge (p \leftarrow q)$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

Biconditional

$$p \leftrightarrow q \qquad (p \to q) \land (p \leftarrow q)$$

p if and only if **q** **p** iff **q**

Tlanslating English Sentences:

English is often ambigous. Translating sentences into logical expressions removes the ambiguity.

Example: " You can access the Internet from Campus only if you are a computer science major or you are not a freshman."

Translate this sentence into a logical expression.

Example: "You can access the Internet from Campus only if you are a computer science major or you are not a freshman."

Translate this sentence into a logical expression.

Solution:

a = You can access the Internet from Campus

c = You are a computer science major

f = You are a freshman

only if = $\rightarrow$

or = $\vee$

$$a \rightarrow (c \vee \neg f).$$

<u>Example:</u> "You cannot ride the roller coaster if you are under 4 feet tall unless you are older than 16 years old."

Translate this sentence into a logical expression.

Example: "You cannot ride the roller coaster if you are under 4 feet tall unless you are older than 16 years old."

Translate this sentence into a logical expression.

Solution:

q = You can ride the roller coaster.

r = You are under 4 feet tall

s = You are older than 16 years old.

$$(r \wedge \neg s) \rightarrow \neg q.$$