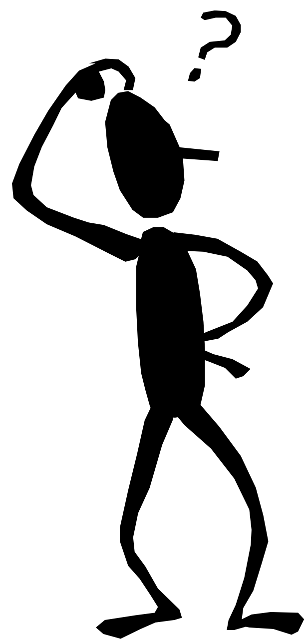
Mat2033 - Discrete Mathematics

The Foundations: Logic and Proof, Sets and Functions

Section 1.3 - 1.4

Predicates & Quantifiers



Predicates and Quantifiers

Statements involving variables, such as

"$x > 3$", "$x = y + 3$", and "$x + y = z$"

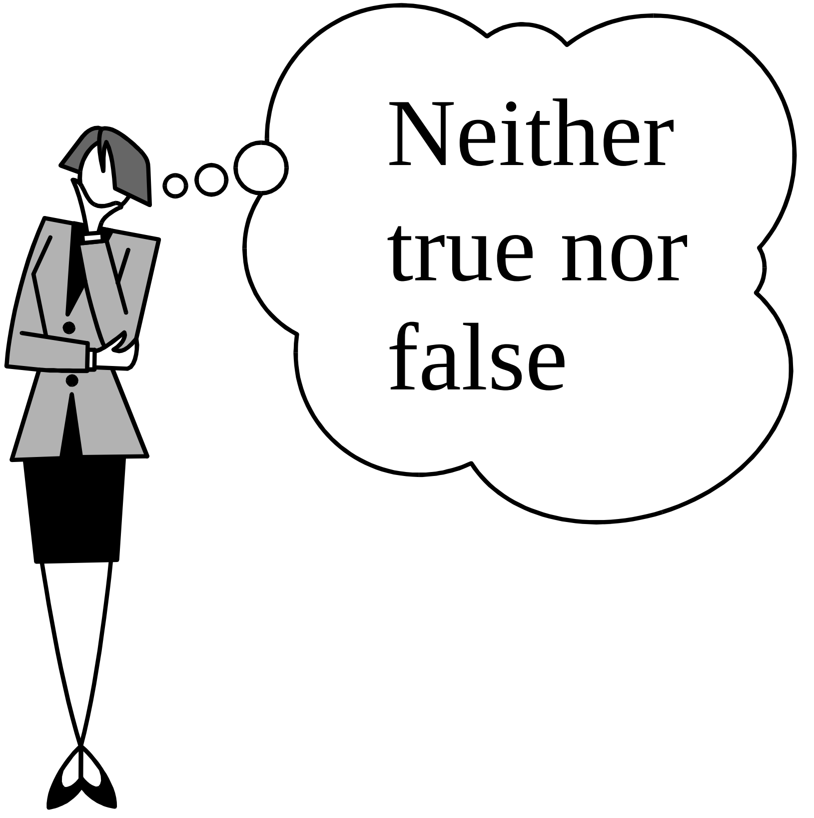
are often found in mathematical assertions and in computer programs. These statements are neither true nor false when the values of the variables are not specified. The statement "x is greater than 3" has two parts. The first part, the variable x, is the subject of the statement. The second part – the predicate, "is greater than 3" — refers to a property that the subject of the statement can have. We can denote the statement "x is greater than 3" by $P(x)$, where P denotes the predicate "is greater than 3" and x is the variable. The statement $P(x)$ is also said to be the value of the propositional

Open Statement

$x > 8$

$p < q - 5$

$x = y + 6$



Propositional Functions

$x > 8$

x is greater than 8.

x is greater than 8.

subject predicate

Propositional Functions

x is greater than 8.

subject predicate

P(x)

propositional function P at x

Propositional Functions

There are two ways a propositional function

`P(x)` “`x > 8`”

can become true or false (a proposition)

`P(x)    “x > 8”`

1. The variable may be given a value

`P(5)  “5 > 8”` **FALSE**

`P(12) “12 > 8”` **TRUE**

Propositional Functions (two variables)

Let `Q(x,y)` be the statement

`“x is the capital of y.”`

What are the truth values of:

`Q(Paris, France)`

`Q(Ankara, Greece)`

Example: Let $P(x)$ denote the statement "$x > 3$". What are the truth values of $P(4)$ and $P(2)$?

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Solution:

$Q(1,2)$ means "$1 = 2 + 3$" which is False.

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Solution:

$Q(1,2)$ means "$1 = 2 + 3$" which is False.

$Q(3,0)$ means "$3 = 0 + 3$" which is True.

Example: Let $R(x,y,z)$ denote the statement "$x+y=z$". What are the truth values of the propositions $R(1,2,3)$ and $R(0,0,1)$?

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<u>Solution</u>:

$R(1,2,3)$ means "$1+2=3$" which is True.

$R(0,0,1)$ means "$0+0=1$" which is False. ∎

In general, a statement involving the n variables $x_1, x_2, \ldots x_n$ can be denoted by

$$P(x_1, x_2, \ldots, x_n).$$

A statement of the form $P(x_1, x_2, \ldots, x_n)$ is the value of the <u>propositional function</u> P at the n-tuple $(x_1, x_2, \ldots, x_n)$, and P is also called a predicate.

Quantifiers

2. A propositional function

`P(x)    “x > 8”`

can become true or false (a proposition) by using *quantifiers*.

Quantifiers

When all the variables in a propositional function are assigned values, the resulting statement has a truth value. However, there is another important way, called <u>quantification</u> to create a proposition from a propositional function. Two types of quantification will be discussed here, namely, universal quantification and existential quantification.

Many mathematical statements assert that a property is true for all values of a variable in a particular domain, called the <u>universe of discourse.</u> Such a statement is expressed using a universal quantification. The universal quantification of a propositional function is the propositi: that asserts that $P(x)$ is true for all values of x in the universe of discourse.

Universal Quantification

A Universal Quantifier states that **`P(x)`** is true for all values of x in the universe of discourse.

$$\forall x P(x)$$

states the universal quantification of **`P(x)`**

Here $\forall$ is called the <u>universal quantifier</u>. The proposition $\forall x\, P(x)$ is also expressed as

"for all x $P(x)$" or "for every x $P(x)$"

Example: Express the statement

"Every student in this class has studied calculus"

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Then the statement "Every student in this class has studied calculus" can be written as $\forall x\, P(x)$, where the universe of discourse consists of the students in this class.

__Example:__ Let $P(x)$ be the statement "$x+1 > x$". What is the truth value of the quantification $\forall x\, P(x)$, where the universe of discourse is the set of real numbers?

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Solution: Since $P(x)$ is true for all real numbers x, the quantification

$$\forall x\, P(x)$$

is true.

Example: Let $Q(x)$ be the statement "$x < 2$". What is the truth value of the quantification $\forall x\, Q(x)$, where the universe of discourse is the set of real numbers?

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Solution: $Q(x)$ is not true for all real numbers x, since, for instance, $Q(3)$ is false. Thus

$$\forall x\, Q(x)$$

is false.

Note: When all the elements in the universe of discourse can be listed – say, $x_1, x_2, \ldots, x_n$ – it follows that the universal quantification $\forall x\, P(x)$ is the same as the conjunction

$$P(x_1) \wedge P(x_2) \wedge \cdots \wedge P(x_n)$$

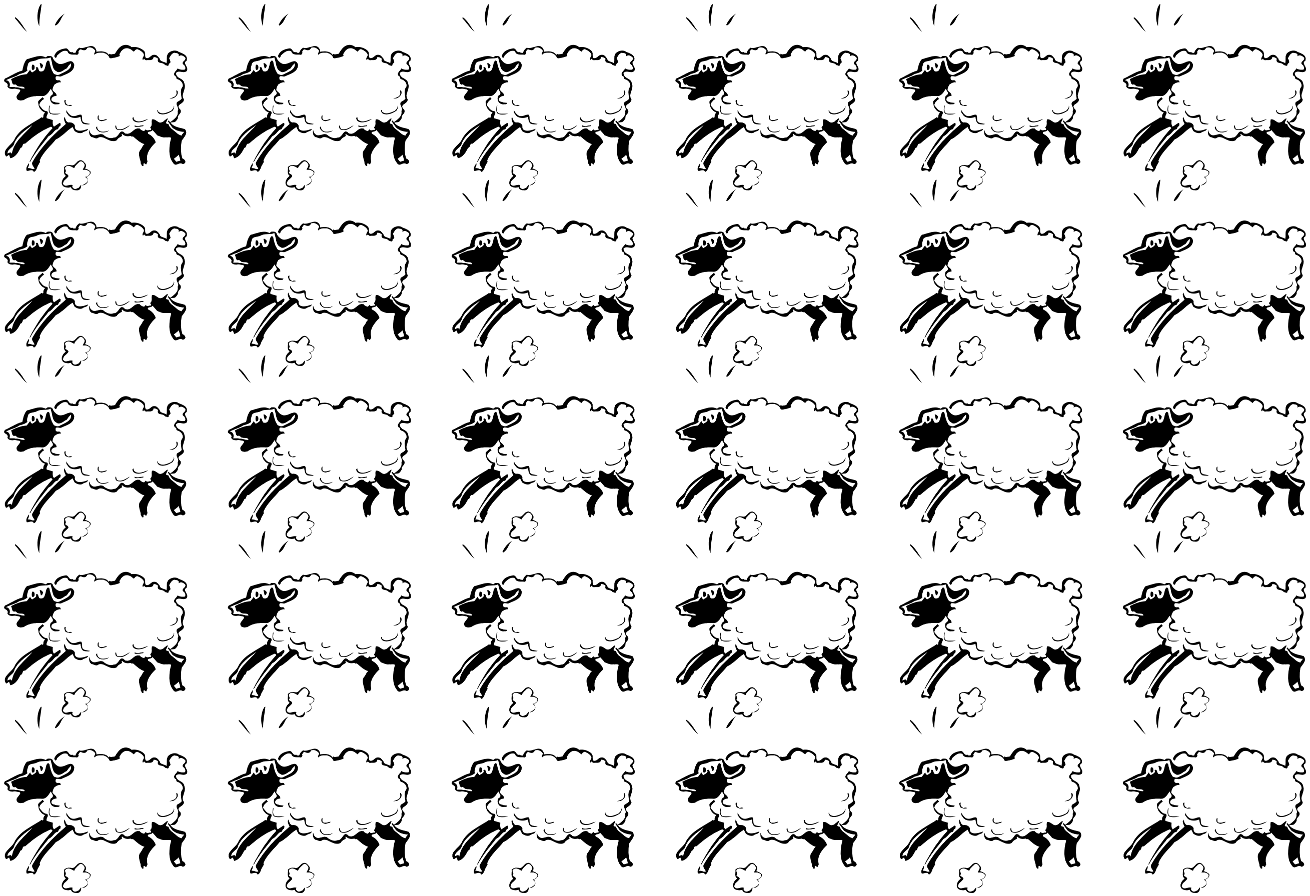
Since this conjunction is true if and only if $P(x_1), P(x_2), \ldots, P(x_n)$ are all true.

Example: $P(x)$ is the statement "$x^2 < 10$", where the universe of discourse is the positive integers not exceeding 4. What is the truth value of of $\forall x\, P(x)$?

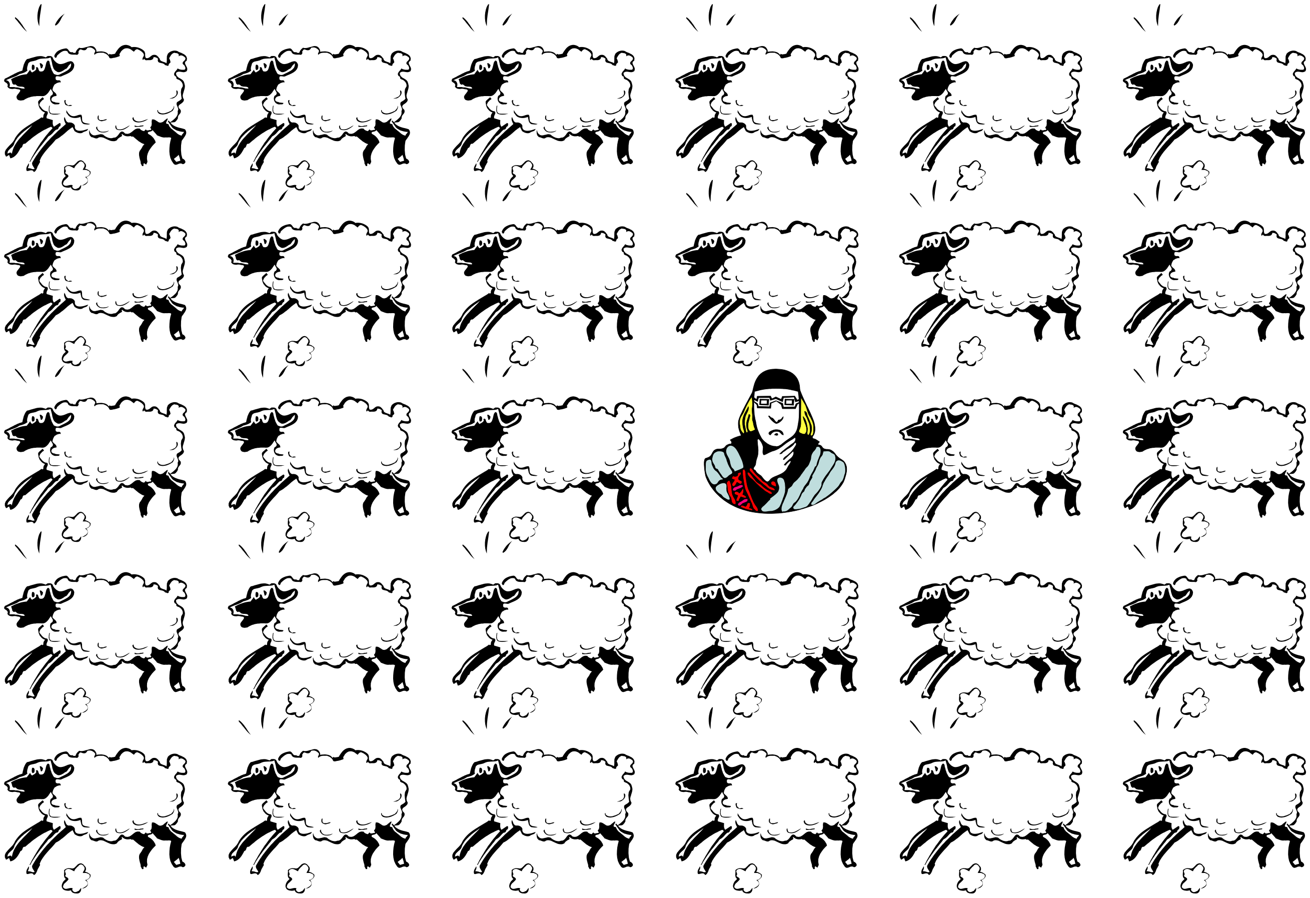
Example: $P(x)$ is the statement "$x^2 < 10$", where the universe of discourse is the positive integers not exceeding 4. What is the truth value of of $\forall x\, P(x)$?

solution: $\forall x\, P(x)$ is true if $P(1) \wedge P(2) \wedge P(3) \wedge P(4)$ is true. $P(1)$, $P(2)$ and $P(3)$ are true but $P(4)$ is not true. So $\forall x\, P(x)$ is False.

$$\forall x P(x)$$ True



$\forall x P(x)$ False



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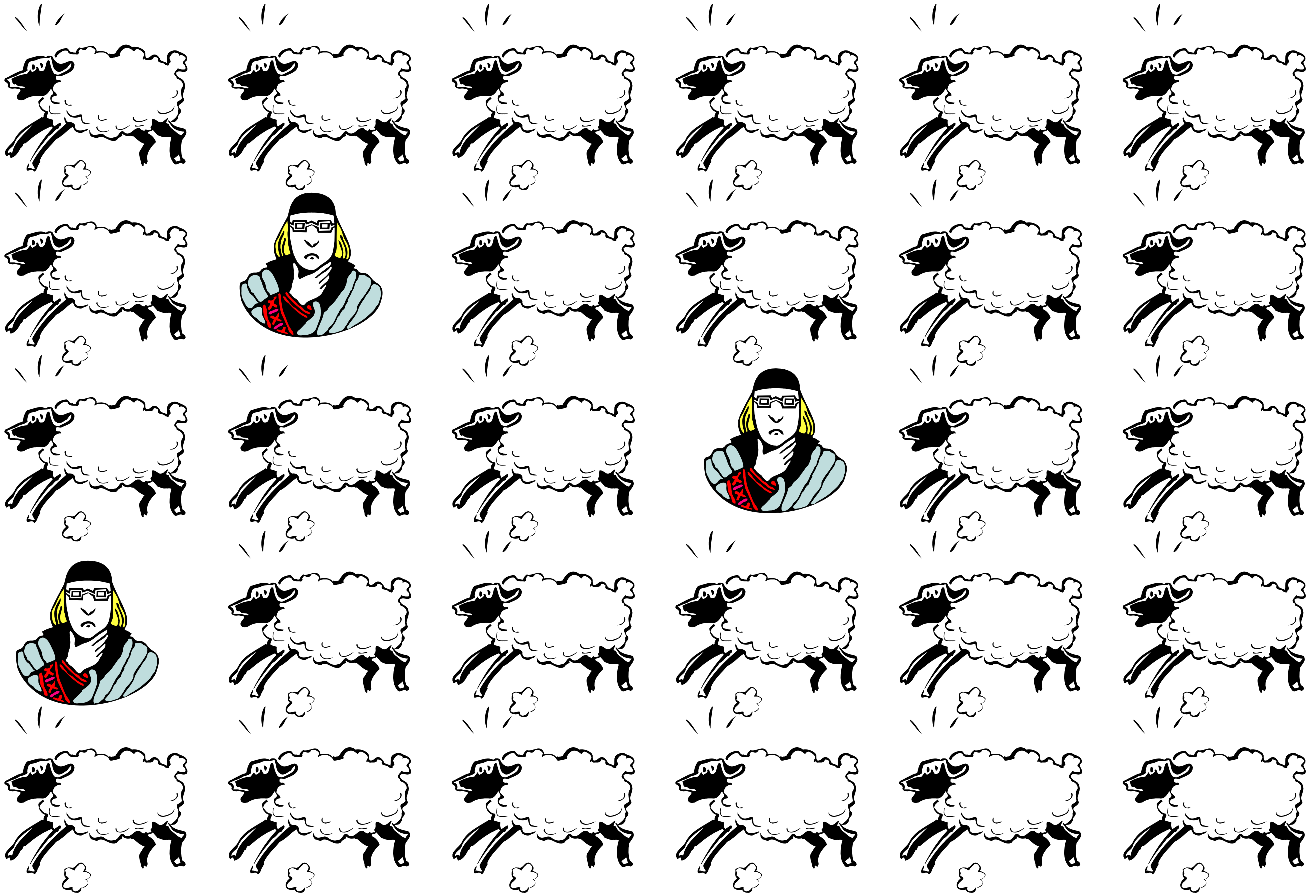

$$\forall x P(x)$$

False

Universal Quantification

Example: Every person in this class has completed MATH 2033.

Let $P(x)$ represent the statement "x has completed MATH 2033" where the universe of discourse is this class.

$$\forall x P(x)$$

then represents the above statement.

If $S(x)$ represented the statement "x is in this class" and

$P(x)$ denotes the statement "x has had MATH 2033"

or its equivalent

$$\forall x(S(x) \rightarrow P(x))$$

Also represents the statement.

Universe of Discourse

It is important to note that the universe of discourse must be defined before the logical value of a propositional function may be discussed.

Example:

Let ***P(x)*** be the statement " x spends more than five hours every weekday in class," where the universe of discourse for x is the set of students. Express $\forall x \neg P(x)$ in English.

No student spends more than five hours every weekday in class

Existential Quantification

States that $P(x)$ is true for *some* value of x in the universe of discourse.

$$\exists x P(x)$$

states the existential quantification of $P(x)$

<u>Definition 2</u>: The existential quantification of $P(x)$ is the proposition

"There exists an element x in the universe of discourse such that $P(x)$ is true."

We use the notation

$$\exists x \; P(x)$$

for the existential quantification of $P(x)$. Here $\exists$ is called the <u>existential quantifier</u>. The existential quantification $\exists x \; P(x)$ is also expressed as

"There is an x such that $P(x)$"

"There is at least one x such that $P(x)$"

or

"For some x $P(x)$".

Example: Let $P(x)$ denote the statement "$x > 3$". What is the truth value of the quantification $\exists x\, P(x)$, where the universe of discourse is the set of real numbers?

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Solution: Since "$x > 3$" is true – for instance, when $x = 4$ – the existential quantification of $P(x)$, which is $\exists x\, P(x)$, is true.

Existential Quantification

There is at least one person in this class who has completed MATH 2033.

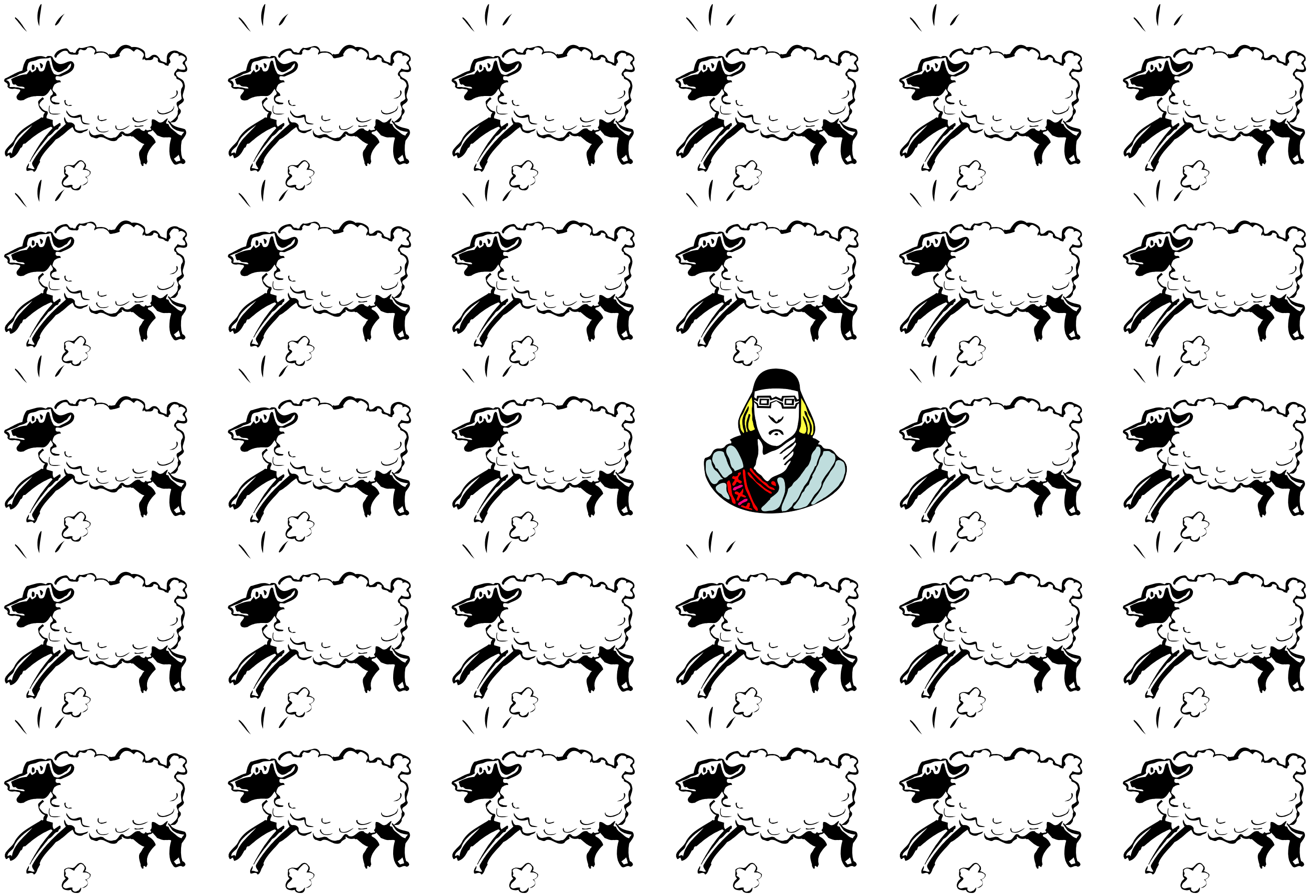
Let $P(x)$ represent the statement "x has completed MATH 2033" where the universe of discourse is this class.

$$\exists x P(x)$$

then represents the above statement.

$$\exists x P(x)$$

True



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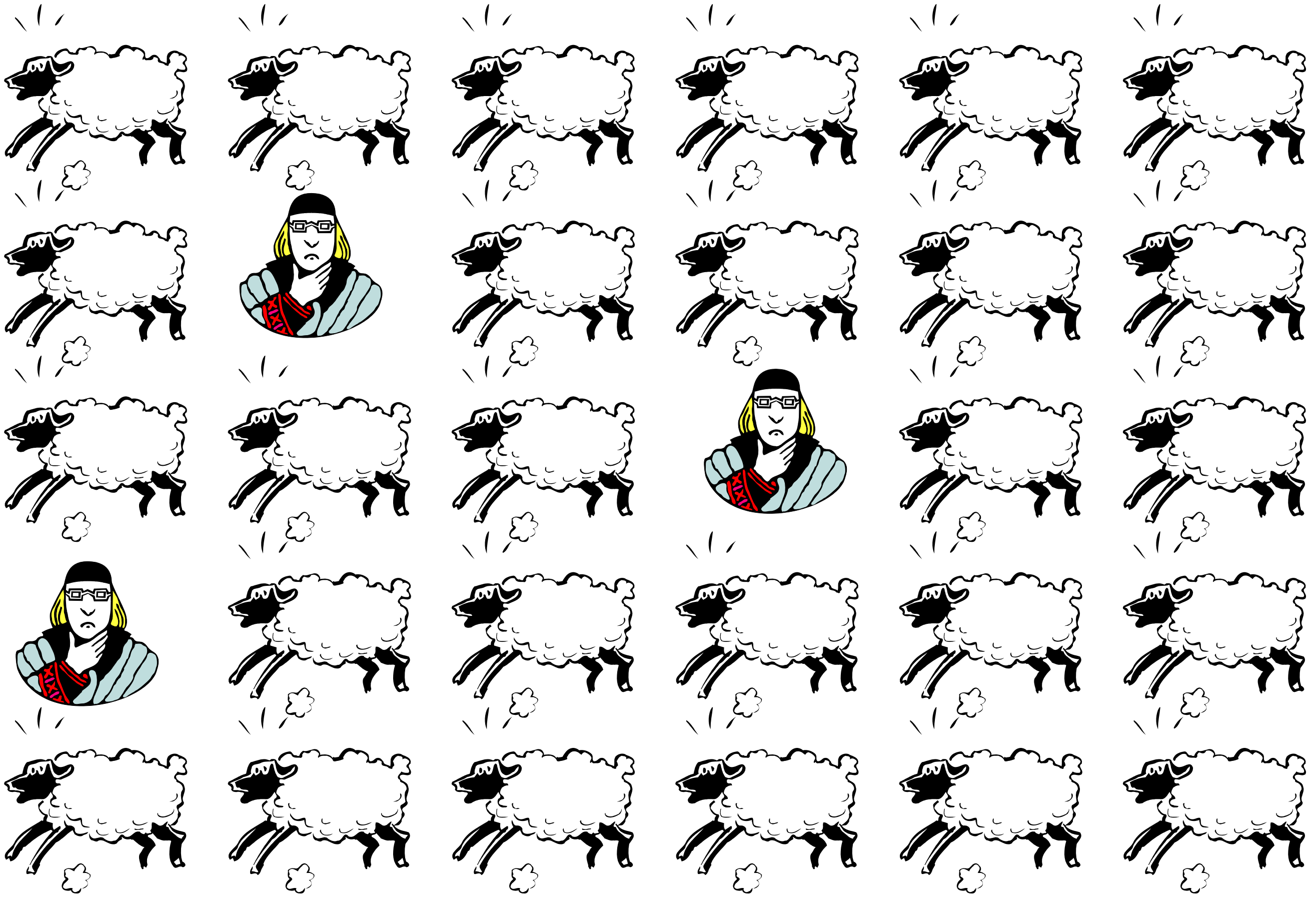
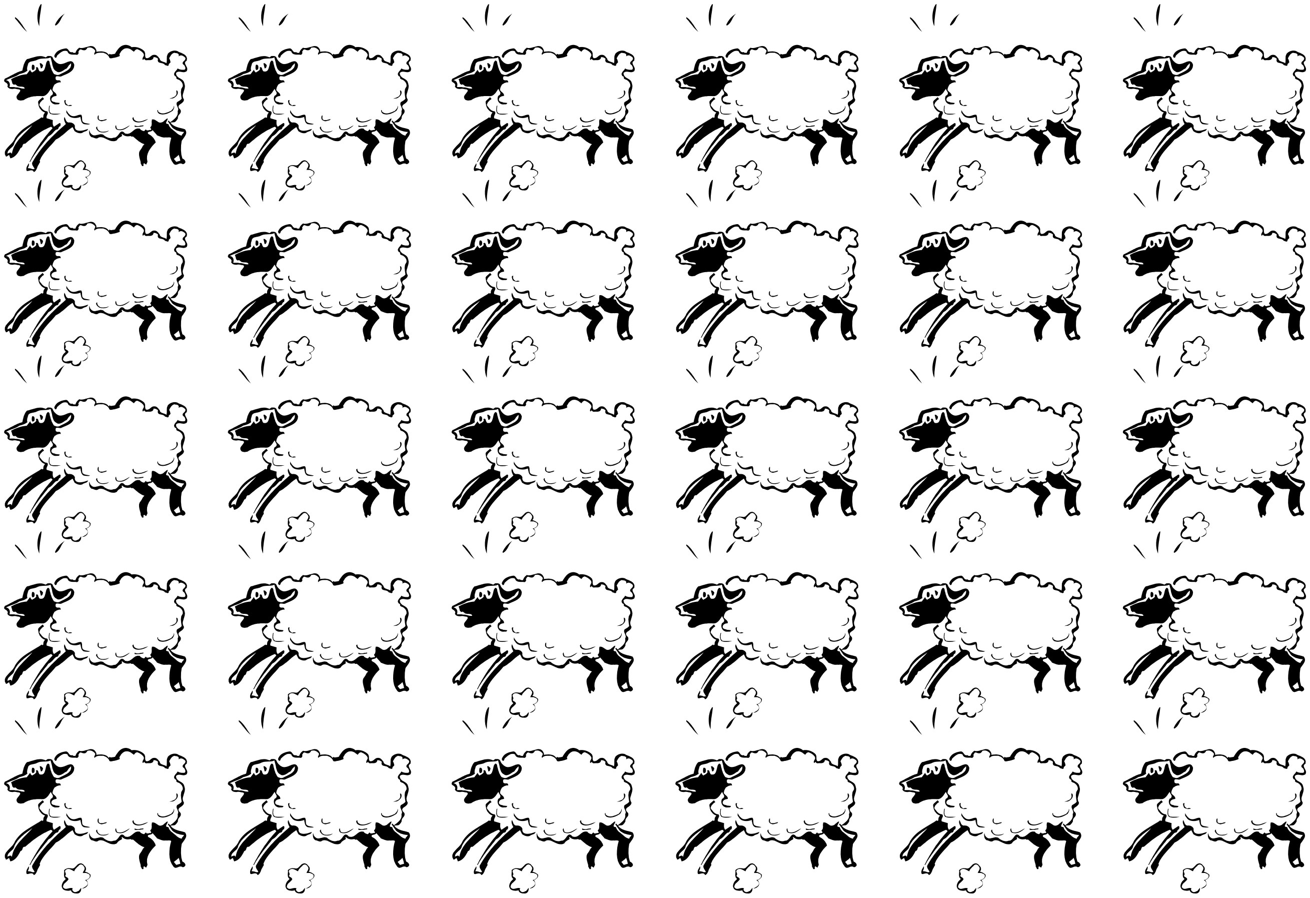
True



$$\exists x P(x)$$

False



$$\exists x P(x)$$

True

Example: Let $Q(x)$ denote the statement "$x = x+1$".
What is the truth value of the quantification $\exists x\, Q(x)$, where the universe of discourse is the set of real numbers?

Example: Let $Q(x)$ denote the statement "$x = x+1$".

What is the truth value of the quantification $\exists x\, Q(x)$, where the universe of discourse is the set of real numbers?

Solution: $Q(x)$ is false for every number x, the existial quantification of $Q(x)$, which is $\exists x\, Q(x)$, is false.

<u>Note:</u> When all of the elements in the universe of discourse can be listed — say, $x_1, x_2, \ldots, x_n$ — the existential quantification $\exists x\, P(x)$ is the same as the disjunction

$$P(x_1) \vee P(x_2) \vee \cdots \vee P(x_n)$$

since this disjunction is true if and only if at least one of $P(x_1), P(x_2), \ldots, P(x_n)$ is true.

Example: What is the truth value of $\exists x\, P(x)$ where $P(x)$ is the statement "$x^2 > 10$" and the universe of discourse consists of the positive integers not exceeding 4?

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solution: Since the universe of discourse is $\{1, 2, 3, 4\}$, the proposition $\exists x\, P(x)$ is the same as the disjunction

$$P(1) \vee P(2) \vee P(3) \vee P(4)$$

Since $P(4)$, which is the statement "$4^2 > 10$" is true, it follows that $\exists x\, P(x)$ is true.

Example:

Let ***P(x)*** be the statement " x spends more than five hours every weekday in class," where the universe of discourse for x is the set of students. Express $\exists x \neg P(x)$ in English.

There is some student (maybe more than one) who does not spend more than five hours every weekday in class.

Quantifiers

	TRUE	FALSE
$\forall x P(x)$	`P(x)` must be true for every `x`.	There is some `x` for which `P(x)` is false
$\exists x P(x)$	There is some `x` for which `P(x)` is true.	`P(x)` must be false for every `x`.

Example: Assume that the universe of discourse for the variables x and y is the set of all real numbers. The statement

$$\forall x \; \forall y \; (x + y = y + x)$$

says that $x + y = y + x$ for all real numbers x and y. This is the commutative law for addition of real numbers.

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$$* \quad \forall x \, \exists y \, (x+y=0)$$

Says that for every real number x there is a real number y such that $x+y=0$. This states that every rea number has an additive inverse.

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* $\forall x \, \exists y \, (x+y=0)$

says that for every real number x there is a real number y such that $x+y=0$. This states that every real number has an additive inverse.

* $\forall x \, \forall y \, \forall z \, (x+(y+z) = (x+y)+z)$

is the associative law for addition of real numbers.

Many mathematical statements involve multiple quantifications of propositional functions involving more than one variable. It is important to note that the order of the quantifiers is important, unless all the quantifiers are universal quantifiers or all are existential quantifiers.

Quantifiers with Multiple Variables

$$\forall x \forall y P(x, y)$$

$$\forall y \forall x P(x, y)$$

Multiple Variables

$$\exists x \exists y P(x, y)$$

$$\exists y \exists x P(x, y)$$

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$$\exists x \forall y P(x, y)$$

$$\forall y \exists x P(x, y)$$

$$\exists y \forall x P(x, y)$$

$$\forall x \exists y P(x, y)$$

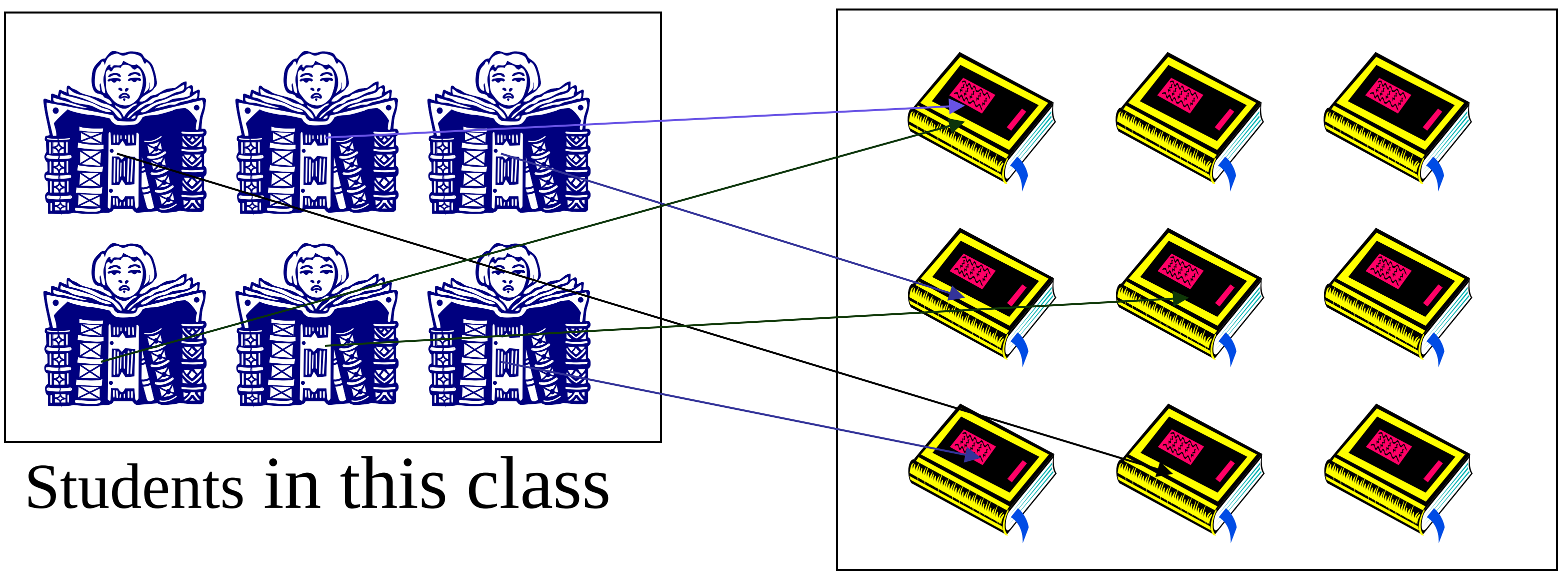
Take care in reading these.

Example:

Let $P(x, y)$ be the statement "x has taken class y," where the universe of discourse for x is the set of all students in this class and for y is the set of all MATH courses at the University.

$$\forall x \exists y P(x, y)$$

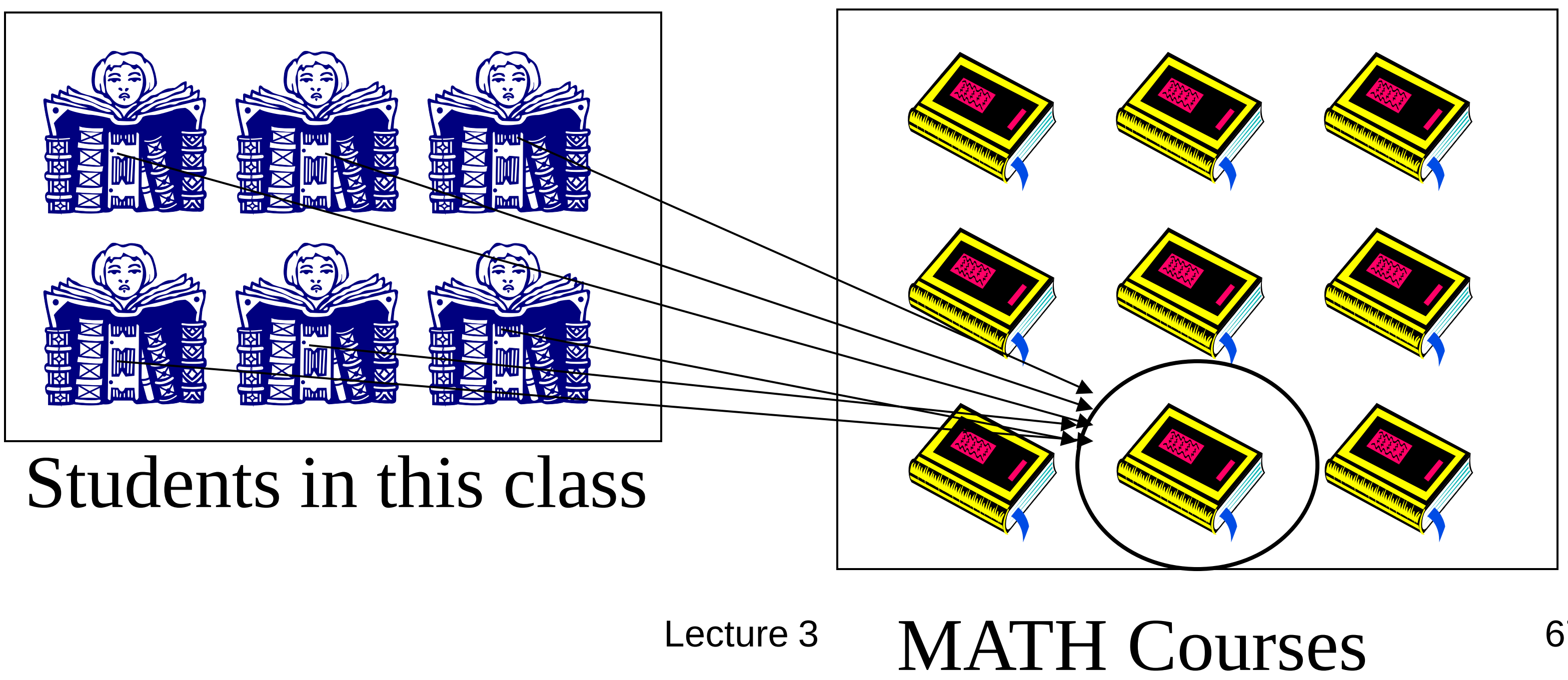
For every student in this class, there is a MATH course that student has taken.



MATH Courses

$$\exists y \forall x P(x, y)$$

There is a MATH course that every student in this class has taken.

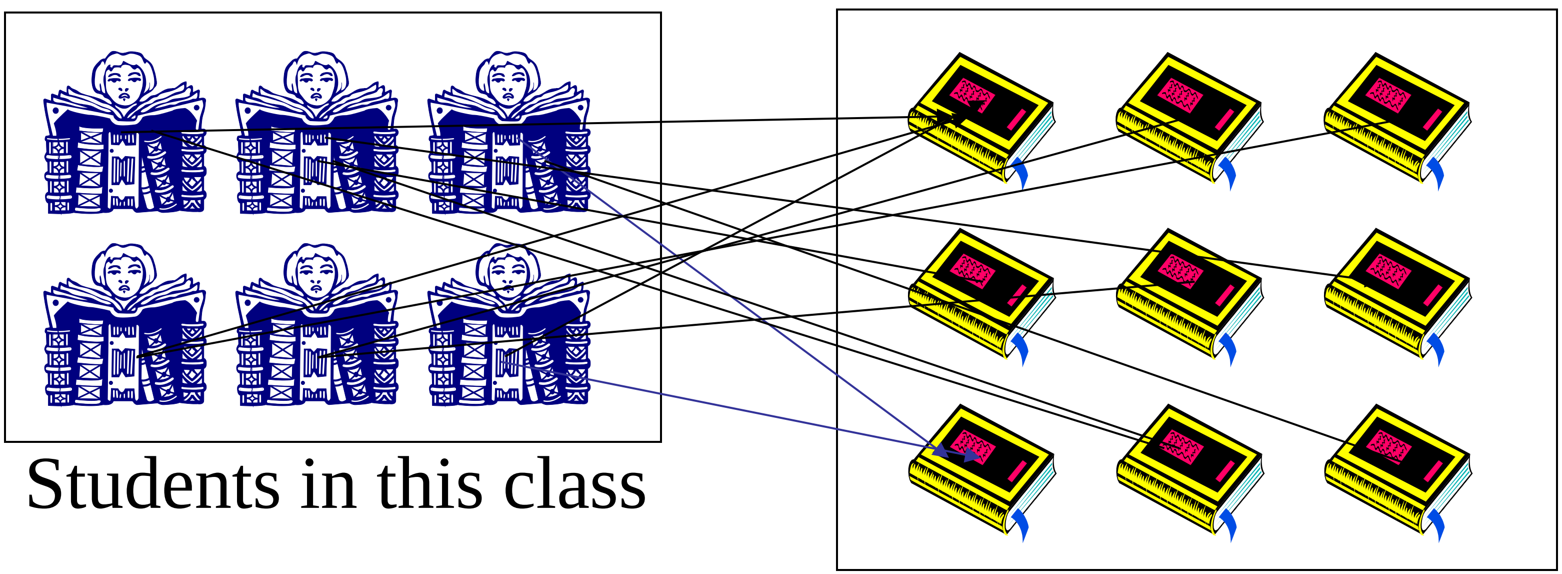


Example:

Let $P(x, y)$ be the statement "x has taken class y," where the universe of discourse for x is the set of all students in this class and for y is the set of all MATH courses at the University.

$$\forall y \exists x P(x, y)$$

For every (pick any one you want) MATH course, there is a student in this class who has taken the course.



Students in this class

MATH Courses

Example: Translate the statement

$$\forall x \{ C(x) \vee \exists y [C(y) \wedge F(x,y)] \}$$

into English, where $C(x)$ is "x has a computer" $F(x,y)$ is "x and y are friends," and the universe of discourse for both x and y is the set of all students in your school.

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solution:

* For every student x in your school x has a computer or there is a student y such that y has a computer and x and y are friends.

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Solution:

* For every student x in your school x has a computer or there is a student y such that y has a computer and x and y are friends.

or

* Every student in your school has a computer or has a friend who has a computer.

Example: Translate the statement

$$\exists x \forall y \forall z \{ [(F(x,y) \wedge F(x,z) \wedge (y \neq z)] \rightarrow \neg F(y,z)] \}$$

into English, where $F(a,b)$ means a and b are friends and the universe of discourse for x, y and z is the set of all students in your school.

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* There is a student x such that for all students y and all students z other than y, if x and y are friends and x and z are friends, than y and z are not friends.

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into English, where $F(a,b)$ means a and b are friends and the universe of discourse for x, y and z is the set of all students in your school.

solution:

* There is a student x such that for all students y and all students z other than y, if x and y are friends and x and z are friends, than y and z are not friends.

* In other words, there is a student none of whose friends are also friends with each other.

Translating Sentences into Logical Expressions:

Example: Express the statements

i) "Some students in this class has visited Mexico"

ii) "Every student in this class has visited either Canada or Mexico"

using quantifiers.

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Solution: Let the universe of discourse for the variable x be the set of students in your class.

$M(x) \equiv$ x has visited mexico.

$C(x) \equiv$ x has visited canada.

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i) $\exists x \ M(x)$

ii) $\forall x \ (C(x) \vee M(x))$

Example: Express the statement "If somebody is female and is a parent, then this person is someone's mother" as a logical expression.

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Solution:

$F(x) \equiv$ x is female

$P(x) \equiv$ x is a parent

$M(x,y) \equiv$ x is the mother of y

$\forall x \Big(F(x) \wedge P(x)\Big) \rightarrow \exists y \; M(x,y)$

an equivalent expression: $\forall x \; \exists y \big((F(x) \wedge p(x)) \rightarrow M(x, y)\big)$

Example: Use quantifiers to express the statement.

"There is a woman who has taken a flight on every airline in the world"

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Solution:

$P(w, f) \equiv$ w has taken f

$Q(f, a) \equiv f$ is a flight on a

The universe of discourse for w, f and a consist of all the women in the world, all airplane flights, and all airlines.

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$$\exists w \, \forall a \, \exists f \, (P(w,f) \wedge Q(f,a))$$

Example: "All Lions are fierce"

"Some Lions do not drink coffee"

"Some fierce creatures do not drink coffee"

Express these statements as a logical expression.
Universe of discourse is all creatures over the world.

Solution:

$P(x) \equiv x$ is a lion

$Q(x) \equiv x$ is fierce

$R(x) \equiv x$ drinks coffee

$\forall x \; (P(x) \rightarrow Q(x))$

$\exists x \; (P(x) \wedge \neg R(x))$

$\exists x \; (Q(x) \wedge \neg R(x))$

Universe of discourse is all creatures.

Example: Let $P(x,y)$ be the statement "$x+y=y+x$"

What is the truth value of the quantification

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<u>Example:</u> Let $P(x,y)$ be the statement "$x+y=y+x$"
What is the truth value of the quantification

$$\forall x \, \forall y \, P(x,y)$$

<u>Solution:</u> The quantification

$$\forall x \, \forall y \, P(x,y)$$

denotes the proposition

"For all real numbers x and for all real numbers y, it is true that $x+y=y+x$".

So the proposition

$$\forall x \, \forall y \, P(x,y)$$

is true.

Example: Let $Q(x,y)$ denote "$x+y=0$." What are the truth values of the quantifications

$$\exists y \, \forall x \, Q(x,y)$$

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Solution:

$\exists y \, \forall x \, Q(x,y)$ denotes the proposition

"There is a real number y such that for every real number x, $Q(x,y)$ is true".

No matter what value of y is chosen, there is only one value of x for which $x+y=0$. Since there is no real number y such that $x+y=0$ for all real numbers x. So $\exists y \, \forall x \, Q(x,y)$ is FALSE.

Example: Let $Q(x,y)$ denote "$x+y=0$." What are the truth values of the quantifications

$$\exists y\ \forall x\ Q(x,y)$$
$$\forall x\ \exists y\ Q(x,y)$$

$\forall x\ \exists y\ Q(x,y)$ denotes the proposition

"For every real number x there is a real number y such that $Q(x,y)$ is true."

Given a real number x, there is a real number y such that $x+y=0$; namely, $y=-x$. So $\forall x\ \exists y\ Q(x,y)$ is TRUE.

Quantification of Two Variables

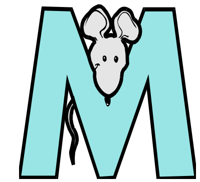
Statement	When True?	When False?
$\forall x \forall y\, P(x,y)$ $\forall y \forall x\, P(x,y)$	$P(x,y)$ is true for every pair x,y.	There is a pair x,y for which $P(x,y)$ is false.
$\forall x \exists y\, P(x,y)$	For every x there is a y for which $P(x,y)$ is true.	There is an x such that $P(x,y)$ is false for every y.
$\exists x \forall y\, P(x,y)$	There is an x for which $P(x,y)$ is true for every y.	For every x there is a y for which $P(x,y)$ is false.
$\exists x \exists y\, P(x,y)$ $\exists y \exists x\, P(x,y)$	There is a pair x,y for which $P(x,y)$ is true.	$P(x,y)$ is false for every pair x,y.

Negating Quantifiers

$$\neg \exists x P(x) \Leftrightarrow \forall x \neg P(x)$$

$$\neg \forall x P(x) \Leftrightarrow \exists x \neg P(x)$$

Example



Let $P(x)$ be the statement "the word x contains the letter m."

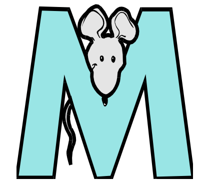
$\neg\exists x P(x)$

There is no word that contains the letter m.

$\forall x \neg P(x)$

For any word you pick, that word does not contain the letter m.

Example



Let $P(x)$ be the statement "the word x contains the letter m."

$\neg \forall x P(x)$

Not every word contains the letter m.

$\exists x \neg P(x)$

There is a word that does not contain the letter m.

We will often want to consider the negation of a quantified expression. For instance, consider the negation of the statement

"Every student in the class has taken a course in calculus"

This statement is a universal quantification, namely,

$\forall x \; P(x)$

where $P(x)$ is the statement "x has taken a course in calculus"

The negation of this statement is

" It is not the case that every student in the class has taken a course in calculus."

or this is equivalent to

" There is a student in the class who has not taken a course in calculus"

And this is simply the existential quantification of the negation of the original propositional function

$$\exists x \; \neg P(x).$$

This example illustrates the following equivalance:

$$\neg \forall x\, P(x) \Leftrightarrow \exists x\, \neg P(x).$$

Similarly the negation of $\exists x\, Q(x)$ is

$$\neg \exists x\, Q(x) \Leftrightarrow \forall x\, \neg Q(x)$$

Negation	Equivalent Statement	When true?	When False?
$\neg \exists x\, P(x)$	$\forall x\, \neg P(x)$	$P(x)$ is false for every x.	There is an x for which $P(x)$ is true.
$\neg \forall x\, P(x)$	$\exists x\, \neg P(x)$	There is an x for which $P(x)$ is false.	$P(x)$ is true for every x.