AMC 10 2018

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– A

– February 7th, 2018

1 What is the value of

$$\left(\left((2+1)^{-1}+1\right)^{-1}+1\right)^{-1}+1?$$

(A) $\frac{5}{8}$ **(B)** $\frac{11}{7}$ **(C)** $\frac{8}{5}$ **(D)** $\frac{18}{11}$ **(E)** $\frac{15}{8}$

2 Liliane has 50% more soda than Jacqueline, and Alice has 25% more soda than Jacqueline. What is the relationship between the amounts of soda that Liliane and Alica have?

(A) Liliane has 20% more soda than Alice. **(B)** Liliane has 25% more soda than Alice. **(C)** Liliane has 45%
(D) Liliane has 75% more soda than Alice. **(E)** Liliane has 100% more soda than Alice.

3 A unit of blood expires after $10! = 10 \cdot 9 \cdot 8 \cdots 1$ seconds. Yasin donates a unit of blood at noon of January 1. On what day does his unit of blood expire?

(A) January 2 **(B)** January 12 **(C)** January 22 **(D)** February 11 **(E)** February 12

4 How many ways can a student schedule 3 mathematics courses – algebra, geometry, and number theory – in a 6-period day if no two mathematics courses can be taken in consecutive periods? (What courses the student takes during the other 3 periods is of no concern here.)

(A) 3 **(B)** 6 **(C)** 12 **(D)** 18 **(E)** 24

5 Alice, Bob, and Charlie were on a hike and were wondering how far away the nearest town was. When Alice said, "We are at least 6 miles away," Bob replied, "We are at most 5 miles away." Charlie then remarked, "Actually the nearest town is at most 4 miles away." It turned out that none of the three statements were true. Let d be the distance in miles to the nearest town. Which of the following intervals is the set of all possible values of d?

(A) $(0,4)$ **(B)** $(4,5)$ **(C)** $(4,6)$ **(D)** $(5,6)$ **(E)** $(5,\infty)$

AoPS Community 2018 AMC 10

6 Sangho uploaded a video to a website where viewers can vote that they like or dislike a video. Each video begins with a score of 0, and the score increases by 1 for each like vote and decreases by 1 for each dislike vote. At one point Sangho saw that his video had a score of 90, and that 65% of the votes cast on his video were like votes. How many votes had been cast on Sangho's video at that point?

(A) 200 **(B)** 300 **(C)** 400 **(D)** 500 **(E)** 600

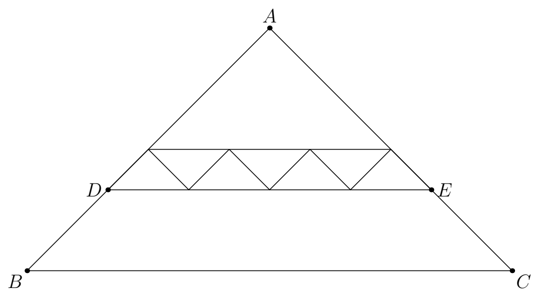
7 For how many (not necessarily positive) integer values of n is the value of $4000 \cdot \left(\frac{2}{5}\right)^n$ an integer?

(A) 3 **(B)** 4 **(C)** 6 **(D)** 8 **(E)** 9

8 Joe has a collection of 23 coins, consisting of 5-cent coins, 10-cent coins, and 25-cent coins. He has 3 more 10-cent coins than 5-cent coins, and the total value of his collection is 320 cents. How many more 25-cent coins does Joe have than 5-cent coins?

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

9 All of the triangles in the diagram below are similar to iscoceles triangle ABC, in which $AB = AC$. Each of the 7 smallest triangles has area 1, and $\triangle ABC$ has area 40. What is the area of trapezoid $DBCE$?



(A) 16 **(B)** 18 **(C)** 20 **(D)** 22 **(E)** 24

10 Suppose that real number x satisfies

$$\sqrt{49-x^2}-\sqrt{25-x^2}=3.$$

What is the value of $\sqrt{49-x^2}+\sqrt{25-x^2}$?

(A) 8 **(B)** $\sqrt{33}+8$ **(C)** 9 **(D)** $2\sqrt{10}+4$ **(E)** 12

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11 When 7 fair standard 6-sided dice are thrown, the probability that the sum of the numbers on the top faces is 10 can be written as
$$\frac{n}{6^7},$$
where n is a positive integer. What is n?

(A) 42 **(B)** 49 **(C)** 56 **(D)** 63 **(E)** 84

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12 How many ordered pairs of real numbers (x, y) satisfy the following system of equations?

$$x + 3y = 3$$
$$\big||x| - |y|\big| = 1$$

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 8

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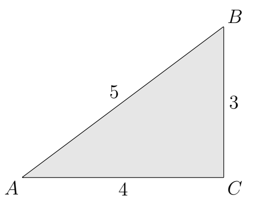
13 A paper triangle with sides of lengths 3, 4, and 5 inches, as shown, is folded so that point A falls on point B. What is the length in inches of the crease?



(A) $1+\frac{1}{2}\sqrt{2}$ **(B)** $\sqrt{3}$ **(C)** $\frac{7}{4}$ **(D)** $\frac{15}{8}$ **(E)** 2

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14 What is the greatest integer less than or equal to
$$\frac{3^{100}+2^{100}}{3^{96}+2^{96}}?$$

(A) 80 **(B)** 81 **(C)** 96 **(D)** 97 **(E)** 625

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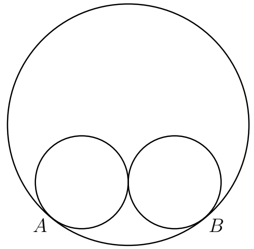
15 Two circles of radius 5 are externally tangent to each other and are internally tangent to a circle of radius 13 at points A and B, as shown in the diagram. The distance AB can be written in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. What is $m+n$?



(A) 21 **(B)** 29 **(C)** 58 **(D)** 69 **(E)** 93

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16 Right triangle ABC has leg lengths $AB = 20$ and $BC = 21$. Including $\overline{AB}$ and $\overline{BC}$, how many line segments with integer length can be drawn from vertex B to a point on hypotenuse $\overline{AC}$?

(A) 5 **(B)** 8 **(C)** 12 **(D)** 13 **(E)** 15

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17 Let S be a set of 6 integers taken from $\{1, 2, \dots, 12\}$ with the property that if a and b are elements of S with $a < b$, then b is not a multiple of a. What is the least possible value of an element in S?

(A) 2 **(B)** 3 **(C)** 4 **(D)** 5 **(E)** 7

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18 How many nonnegative integers can be written in the form
$$a_7 \cdot 3^7 + a_6 \cdot 3^6 + a_5 \cdot 3^5 + a_4 \cdot 3^4 + a_3 \cdot 3^3 + a_2 \cdot 3^2 + a_1 \cdot 3^1 + a_0 \cdot 3^0,$$
where $a_i \in \{-1, 0, 1\}$ for $0 \le i \le 7$?

(A) 512 **(B)** 729 **(C)** 1094 **(D)** 3281 **(E)** $59,048$

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19 A number m is randomly selected from the set $\{11, 13, 15, 17, 19\}$, and a number n is randomly selected from $\{1999, 2000, 2001, \dots, 2018\}$. What is the probability that m^n has a units digit of 1?

(A) $\frac{1}{5}$ **(B)** $\frac{1}{4}$ **(C)** $\frac{3}{10}$ **(D)** $\frac{7}{20}$ **(E)** $\frac{2}{5}$

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20 A scanning code consists of a 7×7 grid of squares, with some of its squares colored black and the rest colored white. There must be at least one square of each color in this grid of 49 squares. A scanning code is called *symmetric* if its look does not change when the entire square is rotated by a multiple of 90° counterclockwise around its center, nor when it is reflected across a line joining opposite corners or a line joining midpoints of opposite sides. What is the total number of possible symmetric scanning codes?

(A) 510 **(B)** 1022 **(C)** 8190 **(D)** 8192 **(E)** 65,534

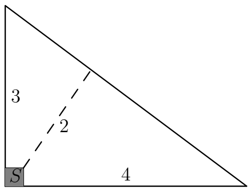
21 Which of the following describes the set of values of a for which the curves $x^2+y^2=a^2$ and $y=x^2-a$ in the real xy-plane intersect at exactly 3 points?

(A) $a=\frac{1}{4}$ **(B)** $\frac{1}{4}<a<\frac{1}{2}$ **(C)** $a>\frac{1}{4}$ **(D)** $a=\frac{1}{2}$ **(E)** $a>\frac{1}{2}$

22 Let $a,b,c,$ and d be positive integers such that $\gcd(a,b)=24$, $\gcd(b,c)=36$, $\gcd(c,d)=54$, and $70<\gcd(d,a)<100$. Which of the following must be a divisor of a?

(A) 5 **(B)** 7 **(C)** 11 **(D)** 13 **(E)** 17

23 Farmer Pythagoras has a field in the shape of a right triangle. The right triangle's legs have lengths 3 and 4 units. In the corner where those sides meet at a right angle, he leaves a small unplanted square S so that from the air it looks like the right angle symbol. The rest of the field is planted. The shortest distance from S to the hypotenuse is 2 units. What fraction of the field is planted?



(A) $\frac{25}{27}$ **(B)** $\frac{26}{27}$ **(C)** $\frac{73}{75}$ **(D)** $\frac{145}{147}$ **(E)** $\frac{74}{75}$

24 Triangle ABC with $AB=50$ and $AC=10$ has area 120. Let D be the midpoint of $\overline{AB}$, and let E be the midpoint of $\overline{AC}$. The angle bisector of $\angle BAC$ intersects $\overline{DE}$ and $\overline{BC}$ at F and G, respectively. What is the area of quadrilateral $FDBG$?

(A) 60 **(B)** 65 **(C)** 70 **(D)** 75 **(E)** 80

25 For a positive integer n and nonzero digits a, b, and c, let A_n be the n-digit integer each of whose digits is equal to a; let B_n be the n-digit integer each of whose digits is equal to b, and let C_n be the $2n$-digit (not n-digit) integer each of whose digits is equal to c. What is the greatest possible value of $a+b+c$ for which there are at least two values of n such that $C_n-B_n=A_n^2$?

(A) 12 **(B)** 14 **(C)** 16 **(D)** 18 **(E)** 20

– B

– February 15th, 2018

1 Kate bakes a 20-inch by 18-inch pan of cornbread. The cornbread is cut into pieces that measure 2 inches by 2 inches. How many pieces of cornbread does the pan contain?

(A) 90 **(B)** 100 **(C)** 180 **(D)** 200 **(E)** 360

2 Sam drove 96 miles in 90 minutes. His average speed during the first 30 minutes was 60 mph (miles per hour), and his average speed during the second 30 minutes was 65 mph. What was his average speed, in mph, during the last 30 minutes?

(A) 64 **(B)** 65 **(C)** 66 **(D)** 67 **(E)** 68

3 In the expression $(\_\_\_\times\_\_\_)+(\_\_\_\times\_\_\_)$ each blank is to be filled in with one of the digits $1,2,3,$ or $4,$ with each digit being used once. How many different values can be obtained?

(A) 2 **(B)** 3 **(C)** 4 **(D)** 6 **(E)** 24

4 4. A three-dimensional rectangular box with dimensions X, Y, and Z has faces whose surface areas are $24, 24, 48, 48, 72,$ and 72 square units. What is $X+Y+Z$?

(A) 18 **(B)** 22 **(C)** 24 **(D)** 30 **(E)** 36

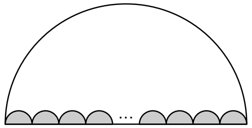
5 How many subsets of $\{2,3,4,5,6,7,8,9\}$ contain at least one prime number?

(A) 128 **(B)** 192 **(C)** 224 **(D)** 240 **(E)** 256

6 A box contains 5 chips, numbered $1,2,3,4,$ and 5. Chips are drawn randomly one at a time without replacement until the sum of the values drawn exceeds 4. What is the probability that 3 draws are required?

(A)$\frac{1}{15}$ **(B)**$\frac{1}{10}$ **(C)**$\frac{1}{6}$ **(D)**$\frac{1}{5}$ **(E)**$\frac{1}{4}$

7 In the figure below, N congruent semicircles lie on the diameter of a large semicircle, with their diameters covering the diameter of the large semicircle with no overlap. Let A be the combined area of the small semicircles and B be the area of the region inside the large semicircle but outside the semicircles. The ratio $A:B$ is $1:18$. What is N?



(A) 16 **(B)** 17 **(C)** 18 **(D)** 19 **(E)** 36

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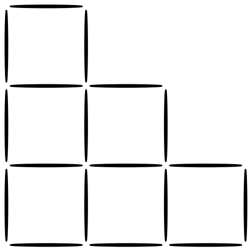
8 Sara makes a staircase out of toothpicks as shown:



This is a 3-step staircase and uses 18 toothpicks. How many steps would be in a staircase that used 180 toothpicks?

(A) 10 **(B)** 11 **(C)** 12 **(D)** 24 **(E)** 30

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9 The faces of each of 7 standard dice are labeled with the integers from 1 to 6. Let p be the probability that when all 7 dice are rolled, the sum of the numbers on the top faces is 10. What other sum occurs with the same probability as p?

(A) 13 **(B)** 26 **(C)** 32 **(D)** 39 **(E)** 42

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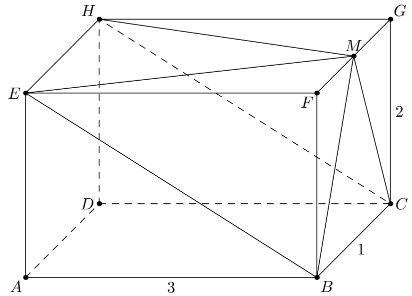
10 In the rectangular parallelepiped shown, $AB = 3, BC = 1,$ and $CG = 2$. Point M is the midpoint of $\overline{FG}$. What is the volume of the rectangular pyramid with base $BCHE$ and apex M?



(A) 1 **(B)** $\frac{4}{3}$ **(C)** $\frac{3}{2}$ **(D)** $\frac{5}{3}$ **(E)** 2

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11 Which of the following expressions is never a prime number when p is a prime number?

(A) p^2+16 **(B)** p^2+24 **(C)** p^2+26 **(D)** p^2+46 **(E)** p^2+96

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12 Line segment $\overline{AB}$ is a diameter of a circle with $AB = 24$. Point C, not equal to A or B, lies on the circle. As point C moves around the circle, the centroid (center of mass) of $\triangle ABC$ traces out a closed curve missing two points. To the nearest positive integer, what is the area of the region bounded by this curve?

(A) 25 **(B)** 38 **(C)** 50 **(D)** 63 **(E)** 75

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13 How many of the first 2018 numbers in the sequence $101, 1001, 10001, 100001, \ldots$ are divisible by 101?

(A) 253 **(B)** 504 **(C)** 505 **(D)** 506 **(E)** 1009

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14 A list of 2018 positive integers has a unique mode, which occurs exactly 10 times. What is the least number of distinct values that can occur in the list?

(A) 202 **(B)** 223 **(C)** 224 **(D)** 225 **(E)** 234

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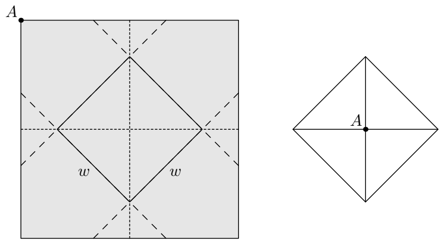
15 A closed box with a square base is to be wrapped with a square sheet of wrapping paper. The box is centered on the wrapping paper with the vertices of the base lying on the midlines of the square sheet of paper, as shown in the figure on the left. The four corners of the wrapping paper are to be folded up over the sides and brought together to meet at the center of the top of the box, point A in the figure on the right. The box has base length w and height h. What is the area of the sheet of wrapping paper?



(A) $2(w+h)^2$ **(B)** $\frac{(w+h)^2}{2}$ **(C)** $2w^2+4wh$ **(D)** $2w^2$ **(E)** w^2h

16 Let $a_1, a_2, \ldots, a_{2018}$ be a strictly increasing sequence of positive integers such that

$$a_1+a_2+\cdots+a_{2018}=2018^{2018}.$$

What is the remainder when $a_1^3+a_2^3+\cdots+a_{2018}^3$ is divided by 6?

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

17 In rectangle $PQRS$, $PQ=8$ and $QR=6$. Points A and B lie on $\overline{PQ}$, points C and D lie on $\overline{QR}$, points E and F lie on $\overline{RS}$, and points G and H lie on $\overline{SP}$ so that $AP=BQ<4$ and the convex octagon $ABCDEFGH$ is equilateral. The length of a side of this octagon can be expressed in the form $k+m\sqrt{n}$, where k, m, and n are integers and n is not divisible by the square of any prime. What is $k+m+n$?

(A) 1 **(B)** 7 **(C)** 21 **(D)** 92 **(E)** 106

18 Three young brother-sister pairs from different families need to take a trip in a van. These six children will occupy the second and third rows in the van, each of which has three seats. To avoid disruptions, siblings may not sit right next to each other in the same row, and no child may sit directly in front of his or her sibling. How many seating arrangements are possible for this trip?

(A) 60 **(B)** 72 **(C)** 92 **(D)** 96 **(E)** 120

19 Joey and Chloe and their daughter Zoe all have the same birthday. Joey is 1 year older than Chloe, and Zoe is exactly 1 year old today. Today is the first of the 9 birthdays on which Chloe's age will be an integral multiple of Zoe's age. What will be the sum of the two digits of Joey's age the next time his age is a multiple of Zoe's age?

(A) 7 **(B)** 8 **(C)** 9 **(D)** 10 **(E)** 11

20 A function f is defined recursively by $f(1)=f(2)=1$ and

$$f(n)=f(n-1)-f(n-2)+n$$

for all integers $n\geq 3$. What is $f(2018)$?

(A) 2016 **(B)** 2017 **(C)** 2018 **(D)** 2019 **(E)** 2020

21 Mary chose an even 4-digit number n. She wrote down all the divisors of n in increasing order from left to right: $1, 2, ..., \frac{n}{2}, n$. At some moment Mary wrote 323 as a divisor of n. What is the smallest possible value of the next divisor written to the right of 323?

(A) 324 **(B)** 330 **(C)** 340 **(D)** 361 **(E)** 646

22 Real numbers x and y are chosen independently and uniformly at random from the interval $[0,1]$. Which of the following numbers is closest to the probability that x, y, and 1 are the side lengths of an obtuse triangle?

(A) 0.21 **(B)** 0.25 **(C)** 0.29 **(D)** 0.50 **(E)** 0.79

23 How many ordered pairs (a,b) of positive integers satisfy the equation

$$a\cdot b+63=20\cdot \text{lcm}(a,b)+12\cdot \text{gcd}(a,b),$$

where gcd(a,b) denotes the greatest common divisor of a and b, and lcm(a,b) denotes their least common multiple?

(A) 0 **(B)** 2 **(C)** 4 **(D)** 6 **(E)** 8

24 Let $ABCDEF$ be a regular hexagon with side length 1. Denote by X, Y, and Z the midpoints of $\overline{AB}$, $\overline{CD}$, and $\overline{EF}$, respectively. What is the area of the convex hexagon whose interior is the intersection of the interiors of $\triangle ACE$ and $\triangle XYZ$?

(A) $\frac{3}{8}\sqrt{3}$ **(B)** $\frac{7}{16}\sqrt{3}$ **(C)** $\frac{15}{32}\sqrt{3}$ **(D)** $\frac{1}{2}\sqrt{3}$ **(E)** $\frac{9}{16}\sqrt{3}$

25 Let $\lfloor x\rfloor$ denote the greatest integer less than or equal to x. How many real numbers x satisfy the equation $x^2+10{,}000\lfloor x\rfloor=10{,}000x$?

(A) 197 **(B)** 198 **(C)** 199 **(D)** 200 **(E)** 201