Angle Chasing and Triangle Centers

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1 Warm-up problems

1.1 (MATHCOUNTS 2011) A square $ABGH$ is located in the interior of a regular hexagon $ABCDEF$. What is the degree measure of $\angle DCG$?

1.2 ABC is an isosceles triangle such that $AB = BC$. E is a point on side AB and the foot of perpendicular from E to BC is D. It is given that $AE = DE$. Find $\angle DAC$.

2 Theorems and useful facts

2.1 (Miquel's Theorem) In triangle ABC, D, E, F are points on sides BC, CA, AB respectively. Prove that the circumcircles of triangles BDF, CDE, AEF pass through a common point.

2.2 (Reim's Theorem) Let ω_1, ω_2 be two circles intersecting at X, Y. A line through X intersects ω_1, ω_2 at A, C respectively. A line through Y intersects ω_1, ω_2 at B, D respectively. Prove that $AB||CD$

2.3 In $\triangle ABC$, D, E, F are the feet of perpendiculars from

A, B, C to the opposite sides respectively. Let H be the orthocenter of $\triangle ABC$. Prove that H is the incenter of $\triangle DEF$.

2.4 Use the notations of 2.2. Now let M be the midpoint of BC. Let H', A' be the reflections of H with respect to BC and M respectively. Prove that H', A' lies on the circle ABC and AA' is a diameter of this circle.

2.5 (Simson Line) Let ABC be a triangle and P be any point on circle ABC. Let X, Y, Z be the feet of the perpendiculars from P onto lines BC, CA, and AB. Prove that points X, Y, Z are collinear.

2.6 (Nine Point Circle) Consider a triangle ABC with orthocenter H. Let D, E, and F be the feet of the altitudes, let L, M, N be the midpoints of the sides, and let P, Q, R be the midpoints of AH, BH, and CH respectively. Prove that $D, E, F, L, M, N, P, Q, R$ are concyclic.

2.7 Use the same notations as 2.6. If O is the circumcenter of ABC, then prove that $2OL = AH$.

2.8 Prove that the center of the nine point circle is the midpoint of OH.

2.9 (Euler's Line) Let G be the centroid od $\triangle ABC$. Prove that O, G, H are collinear.

2.10 (Alternate Segment Theorem) Suppose PA is tangent to circle ABC. Suppose P and C are on opposite sides of line AB.

Prove that $\angle PAB = \angle ACB$.

2.11 (The Incenter/Excenter Lemma). Let ABC be a triangle with incenter I. Ray AI meets (ABC) again at L. Let I_A be the reflection of I over L. Then, (a) The points I, B, C, and I_A lie on a circle with diameter II_A and center L. (b) I_A is the $A-$excenter of $\triangle ABC$.

3 Problems

3.1 (Russian Olympiad 1996) Points E and F are on side BC of convex quadrilateral $ABCD$ (with E closer than F to B). It is known that $\angle BAE = \angle CDF$ and $\angle EAF = \angle FDE$. Prove that $\angle FAC = \angle EDB$.

3.2 (Canada 1997/4). The point O is situated inside the parallelogram $ABCD$ such that $\angle AOB + \angle COD = 180°$. Prove that $\angle OBC = \angle ODC$

3.3 (Iranian Geometry Olympiad 2015) Let w_1 and w_2 two circles such that $w_1 \cap w_2 = \{A, B\}$. let X a point on w_2 and Y on w_1 such that $BY \perp BX$. Suppose that O is the center of w_1 and $X' = w_2 \cap OX$. Now if $K = w_2 \cap X'Y$ prove X is the midpoint of arc AK.

3.4 (CGMO 2012/5). Let ABC be a triangle. The incircle of ABC is tangent to AB and AC at D and E respectively. Let O denote the circumcenter of BCI. Prove that $\angle ODB = \angle OEC$.