Finite Differences for the Wave Equation

Advanced Computational Methods I

Anouar MOUHOUB
Oussama DERAOUI
Oumaima CHQAF

Supervised by :
Pr. Imad ELMAHI

January 29, 2023

.

Acknowledgments :

We would like to extend our most heartfelt and sincere gratitude to dear professor Imad for his support and guidance throughout this entire semester .
It has been an honour to be under your wing for the past months dear professor and we hope this humble effort lives up to your expectations and meets your standards of excellence.

.

Contents

1 1-D wave equation

We consider the following 1D wave equation with initial and boundary conditions:

$$\begin{cases} \frac{\partial^2 U}{\partial t^2} - c^2 \frac{\partial^2 U}{\partial x^2} = 0 & for\ t > 0, 0 < x < L \\ u(x, t = 0) = f(x) & for\ -L \leq x \leq L \\ \frac{\partial U}{\partial t}(x, 0) = 0 & for\ -L \leq x \leq L \\ u(-L, t) = u(L, t) = 0 & for\ t < 0 \end{cases} \tag{1}$$

u(x, t) represents the planar wave and c is the speed of the wave (here we take c $\geq$ 0).

1.1 Question 1

Show that the solution of equation (E1) can be written as the sum of two progressive waves F and G propagating respectively with speeds c and -c:

$$u(x, t) = F(x - ct) + G(x + ct)$$

We propose to approximate the solution of (E1) on [L, L] $\times$ [0, T] We subdivide the interval [L, L] into a regular mesh formed of N nodes x_j. We denote by u_j^n the approximate solution at node $x_j = j\delta x$ and at time $t^n = n\delta t$, and we set $c\frac{\Delta t}{\Delta x} = \lambda$. A finite difference scheme centered in space and time can be written as follows:

$$\frac{u_j^{n-1} + u_j^{n+1} - 2u_j^n}{\Delta t^2} - c^2 \frac{u_{j-1}^n + u_{j+1}^n - 2u_j^n}{\Delta x^2} = 0$$

First method of solving the equation : D'Alembert Method

Let's consider the following homogeneous wave equation.

$$\frac{\partial^2 U}{\partial t^2} - c^2 \frac{\partial^2 U}{\partial x^2} = 0$$

This explicit solution, is based on a method that consists of the decomposition of the operator of waves (called D'Alembert).
We noticed that this equation is written as the following remarkable identity $A^2 - B^2 = (A-B)(A+B)$.
So,

$$\frac{\partial^2 U}{\partial t^2} - c^2 \frac{\partial^2 U}{\partial x^2} = (\frac{\partial U}{\partial t} - c\frac{\partial U}{\partial x})(\frac{\partial U}{\partial t} + c\frac{\partial U}{\partial x})$$

Because u(x,t) is a continuous function in x and t, we have :

$$\frac{\partial \rho}{\partial t} = \frac{\partial \psi}{\partial x} = \frac{\partial^2 U}{\partial x \partial t} = \frac{\partial^2 U}{\partial t \partial x}$$

In order to solve the wave equation we need to solve the following system:

$$\begin{cases} \frac{\partial v}{\partial t} - c\frac{\partial v}{\partial x} = 0 \\ \frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = v \end{cases}$$

Thus, we know that the first equation is a simple transport equation. It's solution depends only on $(x + ct)$. To prove this we must verify :

$$\frac{\partial}{\partial t} v(y - ct, t) = \frac{\partial v}{\partial t} - c\frac{\partial v}{\partial x} = 0$$

So,

$$v(y-ct,t) = v(y,0) \quad \textit{for all} \quad y$$

If we reverse the notations we will obtain :

$$v(x,t) = v(x+ct,0) \quad \textit{for all} \quad x$$

Let's consider $h(y) = v(y,0)$. We must now solve the transport equation below :

$$\frac{\partial u(x,t)}{\partial t} + c\frac{\partial u(x,t)}{u\partial x} = h(x+ct)$$

Then we conclude that the general solution of the homogenous is :

$$u(x,t) = F(x-ct)\textit{(with the same argumentation above)}$$

Now lets denote G a primitive of $\frac{h}{2c}$, we have an evident particular solution

$$u(x,t) = G(x+ct)$$

Thus the solution of equation (1) is the sum of two waves F and G propagating respectively with speed c but we are in 1D, one wave will follow x the other $-x$:

$$\boxed{U(x,t) = F(x-ct) + G(x+ct)}$$

Second method of solving the equation : Cauchy Method

Let's proceed to a change of variables.
we have

$$\frac{\partial}{\partial t}(\frac{\partial U}{\partial t}) - c^2 \frac{\partial}{\partial x}(\frac{\partial U}{\partial x}) = 0$$

we put

$$\rho = \frac{\partial U}{\partial t} \quad \psi = \frac{\partial U}{\partial x}$$

we get

$$\frac{\partial \rho}{\partial t} - c^2 \frac{\partial \psi}{\partial x} = 0$$

Since u(x,t) is a continuous function in x and t, we have :

$$\frac{\partial \rho}{\partial t} = \frac{\partial \psi}{\partial x} = \frac{\partial^2 U}{\partial x \partial t} = \frac{\partial^2 U}{\partial t \partial x}$$

Thus we can conclude the following system :

$$\begin{cases} \frac{\partial \rho}{\partial t} - c^2 \frac{\partial \psi}{\partial x} = 0 \\ \frac{\partial \psi}{\partial t} - \frac{\partial \rho}{\partial x} = 0 \end{cases} \tag{2}$$

From (2) we can obtain :

$$\frac{\partial}{\partial t}\begin{pmatrix} \rho \\ \psi \end{pmatrix} + \begin{pmatrix} 0 & -c^2 \\ -1 & 0 \end{pmatrix} \frac{\partial}{\partial t}\begin{pmatrix} \rho \\ \psi \end{pmatrix}$$

Let's set a new vector $\omega = \begin{pmatrix} \rho \\ \psi \end{pmatrix}$ and a matrix $A = \begin{pmatrix} 0 & -c^2 \\ -1 & 0 \end{pmatrix}$.
Our equation becomes :

$$\frac{\partial \omega}{\partial t} + A\frac{\partial \omega}{\partial x} = 0 \tag{3}$$

Now in order to solve (2) we will find the eigenvalues of the matrix A.

$$\begin{aligned} det(A - \lambda I_2) &= \begin{pmatrix} -\lambda & -c^2 \\ -1 & -\lambda \end{pmatrix} \\ &= \lambda^2 - c^2 \\ det(A - \lambda I_2) &= (\lambda + c)(\lambda - c) \end{aligned}$$

Thus $\lambda_1 = -c$ and $\lambda_2 = c$, and since those eigenvalues are real and distinct, then A is diagonalisable.

$$A = PDP^{-1}$$

And we know that $\lambda_1 = -c, \lambda_2 = c$, so:

$$D = \begin{pmatrix} -c & 0 \\ 0 & c \end{pmatrix}$$

Let's compute the eigenvalues now.

Lets consider V= $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$

First case : $\lambda_1 = -c$

$$
\begin{aligned}
AV_1 &= -cV_1 \\
&\Leftrightarrow \begin{pmatrix} 0 & -c^2 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} -cv_1 \\ -cv_2 \end{pmatrix} \\
&\Leftrightarrow \begin{cases} -c^2 v_2 = -cv_1 \\ -v_1 = -cv_2 \end{cases} \\
&\Leftrightarrow v_1 = cv_2 \\
&\Leftrightarrow V_1 = \begin{pmatrix} c \\ 1 \end{pmatrix}
\end{aligned}
$$

Second case : $\lambda_2 = c$

$$
\begin{aligned}
AV_2 &= cV_2 \\
&\Leftrightarrow \begin{pmatrix} 0 & -c^2 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} v_3 \\ v_4 \end{pmatrix} = \begin{pmatrix} cv_3 \\ cv_4 \end{pmatrix} \\
&\Leftrightarrow \begin{cases} -c^2 v_4 = cv_3 \\ -v_3 = cv_4 \end{cases} \\
&\Leftrightarrow v_3 = -cv_4 \\
&\Leftrightarrow V_2 = \begin{pmatrix} -c \\ 1 \end{pmatrix}
\end{aligned}
$$

From V_1 and V_2 we conclude that:

$$
P = \begin{pmatrix} c & -c \\ 1 & 1 \end{pmatrix}
$$

Let's compute P^{-1} because we will need it later to make the jump back from the variable change.

$$
\begin{aligned}
P^{-1} &= \frac{1}{det(P)} com(P)^T \\
&= \frac{1}{2c} \begin{pmatrix} 1 & -1 \\ c & c \end{pmatrix}^T \\
&= \frac{1}{2c} \begin{pmatrix} 1 & c \\ -1 & c \end{pmatrix} \\
P^{-1} &= \begin{pmatrix} \frac{1}{2c} & \frac{1}{2} \\ \frac{-1}{2c} & \frac{1}{2} \end{pmatrix}
\end{aligned} \tag{4}
$$

Now let's head back to (3):

$$
\begin{aligned}
&\frac{\partial \omega}{\partial t} + A \frac{\partial \omega}{\partial x} = 0 \\
\Leftrightarrow &\frac{\partial \omega}{\partial t} + PDP^{-1} \frac{\partial \omega}{\partial x} = 0 \qquad \oplus \\
\Leftrightarrow &P^{-1} \frac{\partial \omega}{\partial t} + DP^{-1} \frac{\partial \omega}{\partial x} = 0 \\
\Leftrightarrow &\frac{\partial (P^{-1}\omega)}{\partial t} + D \frac{\partial (P^{-1}\omega)}{\partial x} = 0
\end{aligned}
$$

Again we consider :

$$
\frac{\partial R}{\partial t} + D \frac{\partial R}{\partial x} = 0 \ with \ R = P^{-1}\omega = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix}
$$

Now we reached the condition where we can use the method of characteristics.

From $\bigoplus$ we get :

$$\begin{cases} \frac{\partial r_1}{\partial t} - c\frac{\partial r_1}{\partial x} = 0 \\ \frac{\partial r_2}{\partial t} + c\frac{\partial r_2}{\partial x} = 0 \end{cases} \tag{5}$$

To each of the two above equations, we will use the method of characteristics:

Case of r_1

$$\frac{dr_1}{ds} = \frac{\partial r_1}{\partial t}.\frac{dt}{ds} + \frac{\partial r_1}{\partial x}.\frac{dx}{ds}$$

We have $(\frac{\partial r_1}{\partial t} = c\frac{\partial r_1}{\partial x})$

$$\begin{aligned} \frac{dr_1}{ds} &= c\frac{\partial r_1}{\partial x}.\frac{dt}{ds} + \frac{\partial r_1}{\partial x}.\frac{dx}{ds} \\ \frac{dr_1}{ds} &= \frac{\partial r_1}{\partial x}(c\frac{dt}{ds} + \frac{dx}{ds}) \end{aligned}$$

for $\frac{dr_1}{ds} = 0$

$$\begin{aligned} c\frac{dt}{ds} + \frac{dx}{ds} &= 0 \\ c.dt &= -dx \\ x &= -ct + \xi_1 \\ \xi_1 &= x + ct \end{aligned}$$

Case of r_2

$$\frac{dr_2}{ds} = \frac{\partial r_2}{\partial t}.\frac{dt}{ds} + \frac{\partial r_2}{\partial x}.\frac{dx}{ds}$$

We have $(\frac{\partial r_2}{\partial t} = -c\frac{\partial r_2}{\partial x})$

$$\begin{aligned} \frac{dr_2}{ds} &= -c\frac{\partial r_2}{\partial t}.\frac{dt}{ds} + \frac{\partial r_2}{\partial x}.\frac{dx}{ds} \\ \frac{dr_2}{ds} &= \frac{\partial r_2}{\partial x}(\frac{dx}{ds} - c\frac{dt}{ds}) \end{aligned}$$

for $\frac{dr_2}{ds} = 0$

$$\begin{aligned} \frac{dx}{ds} - c\frac{dt}{ds} &= 0 \\ dx &= cdt \\ x &= ct + \xi_2 \\ \xi_2 &= x - ct \end{aligned}$$

Now we have :

$$\begin{aligned} \xi_1 &= x + ct \\ r_1(\xi_1) &= r_1(x + ct) \end{aligned}$$

and

$$\xi_2 = x - ct$$
$$r_2(\xi_2) = r_2(x - ct)$$

Solutions

$$\begin{cases} r_1(x,t) & = r_1(x+ct, 0) \\ r_2(x,t) & = r_2(x-ct, 0) \end{cases}$$

we knew that $R = P^{-1}\omega$ so $\omega = PR$

$$\begin{pmatrix} \rho \\ \psi \end{pmatrix} = \begin{pmatrix} c & -c \\ 1 & 1 \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \end{pmatrix}$$

we know that :

$$\begin{cases} u(x, t=0) = f(x) \\ \frac{\partial U}{\partial t}(x, 0) = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} \rho & = c(r_1' - r_2') \\ \psi & = r_1 + r_2 = f(x) \end{cases}$$

We will know derive the second formula.

$$\Leftrightarrow \begin{cases} \rho & = c(r_1' - r_2') \\ \psi' & = r_1' + r_2' = f'(x) \end{cases}$$

$$\Leftrightarrow \begin{cases} r_1(x) & = \frac{1}{2}\psi(x)' + \frac{1}{2c}\int_0^x \rho(y)dy + cste \\ r_2(x) & = \frac{1}{2}\psi(x)' - \frac{1}{2c}\int_0^x \rho(y)dy + cste \end{cases}$$

Finally we get back to the non derived $\psi = r_1 + r_2 = f(x)$ and we deduce that the solution of equation (1) is the sum of two waves F and G propagating respectively with speed c but we are in 1D, one wave will follow the other :

$$\boxed{U(x,t) = F(x - ct) + G(x + ct)}$$

1.2 Question 2

Study the truncation error, the order and the consistency of the numerical scheme (1)

In order to study the truncation error, we will proceed to Taylor approximation in time and space.
In space :

$$\begin{aligned} U_{j+1}^n &= U(j + \Delta x, t) \\ &= U_j^n + \Delta x \left.\frac{\partial U}{\partial x}\right|_j^n + \frac{\Delta x^2}{2!} \left.\frac{\partial^2 U}{\partial x^2}\right|_j^n + O(\Delta x^3) \end{aligned} \tag{6}$$

$$\begin{aligned} U_{j-1}^n &= U(j + \Delta x, t) \\ &= U_j^n - \Delta x \left.\frac{\partial U}{\partial x}\right|_j^n + \frac{\Delta x^2}{2!} \left.\frac{\partial^2 U}{\partial x^2}\right|_j^n + O(\Delta x^3) \end{aligned} \tag{7}$$

(5) + (6) We derive till order 2:

$$U^n_{j+1} + U^n_{j-1} = 2U^n_j + \frac{\Delta x^2}{2!}\left.\frac{\partial^2 U}{\partial x^2}\right|^n_j + O(\Delta x^3)$$
$$\frac{U^n_{j+1} - 2U^n_{j-1} + U^n_j}{\Delta x^2} + O(\Delta x) = \frac{\partial^2 u}{\partial x^2}$$

In time :

$$\begin{aligned} U^{n+1}_j &= U(j, t+\Delta t) \\ &= U^n_j + \Delta t\frac{\partial U}{\partial t} + \frac{\Delta t^2}{2!}\frac{\partial^2 U}{\partial t^2} + O(\Delta t^3) \end{aligned} \tag{8}$$

$$\begin{aligned} U^{n-1}_j &= U(j, t-\Delta t) \\ &= U^n_j - \Delta t\frac{\partial U}{\partial t} + \frac{\Delta t^2}{2!}\frac{\partial^2 U}{\partial t^2} + O(\Delta t^3) \end{aligned} \tag{9}$$

(7) + (8) We derive till order 2:

$$U^{n+1}_j + U^{n-1}_j = 2U^n_j + \frac{\Delta t^2}{2!}\frac{\partial^2 U}{\partial t^2} + O(\Delta t^3)$$
$$\frac{U^{n+1}_j - 2U^n_j + U^{n-1}_j}{\Delta t^2} + O(\Delta t) = \frac{\partial^2 u}{\partial t^2}$$

$$\begin{aligned} ET &= (\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2}) - (\frac{u^{n-1}_j + u^{n+1}_j - 2u^n_j}{\Delta t^2} - c^2\frac{u^n_{j-1} + u^n_{j+1} - 2u^n_j}{\Delta x^2}) \\ &= O(\Delta t) + O(\Delta x) \end{aligned}$$

We know that:

$$|O(\Delta t)| \leq \lambda_1 |\Delta t| \longrightarrow 0 \ when \ \Delta t \longrightarrow 0$$

Then:

$$\lim_{\Delta t \longrightarrow 0} O(\Delta t) = 0$$

and

$$|O(\Delta x)| \leq \lambda_1 |\Delta x| \longrightarrow 0 \ when \ \Delta x \longrightarrow 0$$

Then

$$\lim_{\Delta x \longrightarrow 0} O(\Delta x) = 0$$

So

$$\lim_{\Delta x, \Delta t \longrightarrow 0} ET = 0$$

The numerical scheme is consistent and is in order 1 in space and order 1 in time.

1.3 Question 3

Using Fourier Von-Neumann analysis, show that the scheme (1) is stable under the condition.

$$c\frac{\Delta t}{\Delta x} \leq 1$$

Our schema can be written this way :

$$u_j^{n+1} = 2u_j^n - u_j^{n-1} + \lambda^2(u_{j+1}^n - 2u_j^n + u_{j-1}^n)$$

With

$$\lambda = c\frac{\Delta t}{\Delta x} \longrightarrow \lambda^2 = c^2\frac{\Delta t^2}{\Delta x^2}$$

We consider $u_j^n = C^n e^{i\xi j\Delta x}$, We inject it in our schema :

$$\begin{aligned} C^{n+1}e^{i\xi j\Delta x} &= 2 \times C^n e^{ij\xi\Delta x} - C^{n-1}e^{ij\xi\Delta x} + \lambda^2(C^n e^{i\xi\Delta x(j+1)} - 2 \times C^n e^{ij\xi\Delta x} + C^n e^{i(j-1)\xi\Delta x}) \\ &= 2 \times C^n - C^{n-1} + \lambda^2(C^n e^{i\xi\Delta x} - 2 \times C^n + C^n e^{-i\xi\Delta x}) \\ &= 2 \times C^n - C^{n-1} + \lambda^2(C^n(e^{-i\xi\Delta x} + e^{i\xi\Delta x}) - 2C^n) \end{aligned}$$

We know following the Euler formula :

$$cos(x) = \frac{e^{ix} + e^{-ix}}{2}$$

$$\begin{aligned} C^{n+1} &= 2 \times C^n - C^{n-1} + \lambda^2(2 \times C^n cos(\xi\Delta x) - 2C^n) \\ &= C^{n+1} + 2(\lambda^2 - \lambda^2 cos(\xi\Delta x) - 1)C^n + C^{n-1} \end{aligned}$$

we know that

$$\begin{cases} cos^2(x) + sin^2(x) = 1 \\ and \\ cos(2x) = cos^2(x) - sin^2(x) \end{cases}$$

Thus

$$2sin^2(\frac{x}{2}) = 1 - cos(x)$$

We use this formula

$$r^2 + 2(2\lambda^2 sin^2(\frac{\xi\Delta x}{2}) - 1)r + 1$$

We get

$$C^{n+1} - 2 \times C^n + C^{n-1} + 2C^n(\lambda^2 - 2\lambda^2 sin(\frac{\xi\Delta x}{2})^2) = 0$$

$$C^{n+1} + C^{n-1} - 2 \times C^n(1 - 2\lambda^2 sin(\frac{\xi\Delta x}{2})^2) = 0$$

The characteristic equation can be written :

$$r^2 + 2 \times (2\lambda^2 sin(\frac{\xi\Delta x}{2})^2 - 1) \times r + 1 = 0 \tag{10}$$

The general term of this Fourrier sequel $(C^n)_{n\in N}$ is written :

$$C^n = \alpha \times r_1 + \beta \times r_2$$

where r_1, r_2 are the roots of the equation

$$\begin{cases} r_1 = 1 - \lambda^2 sin^2(J) - \sqrt{\Delta} \\ r_2 = 1 - \lambda^2 sin^2(J) + \sqrt{\Delta} \\ where\ J = \frac{\xi \Delta x}{2} \\ and\ \Delta = 16\lambda^2 sin^2(J)(\lambda^2 sin^2(J) - 1) \end{cases} \tag{11}$$

If this perturbation does not grow in time the scheme is stable.Thus, we need $r_1 \leq 1$ and $r_2 \leq 1$. Following the system (10) the product of the two roots r_1 and r_2 is equal to 1. If the roots are real which is the case, then one of the roots has a module superior to one.Thus , the scheme is unstable. Means:

$$\Leftrightarrow (1 - 2\lambda^2 sin^2(J))^2 < 1$$
$$\Leftrightarrow 2\lambda^2 sin^2(J) < 2$$
$$\Leftrightarrow \lambda^2 sin^2(J) < 1$$
$$\Leftrightarrow \lambda < 1$$

And from the previous statement we conclude that :

$$\lambda = c\frac{\Delta t}{\Delta x} \leq 1$$

1.4 Question 4

Implement the numerical algorithm and perform the simulations on the following two test cases, using a mesh with $N = 100$ nodes .

Test Case 1

$$L = 10\ \ m\ \ ;\ \ c = 1\ \ m/s\ \ ;\ \ f(x) = e^{-x^2}$$

Represent the results at physical times

$$t_0 = 0\ \ s\ \ ,\ \ t_1 = 1\ \ s\ \ ,\ \ t_2 = 4\ \ s\ \ ,\ \ t_3 = 7\ \ s$$

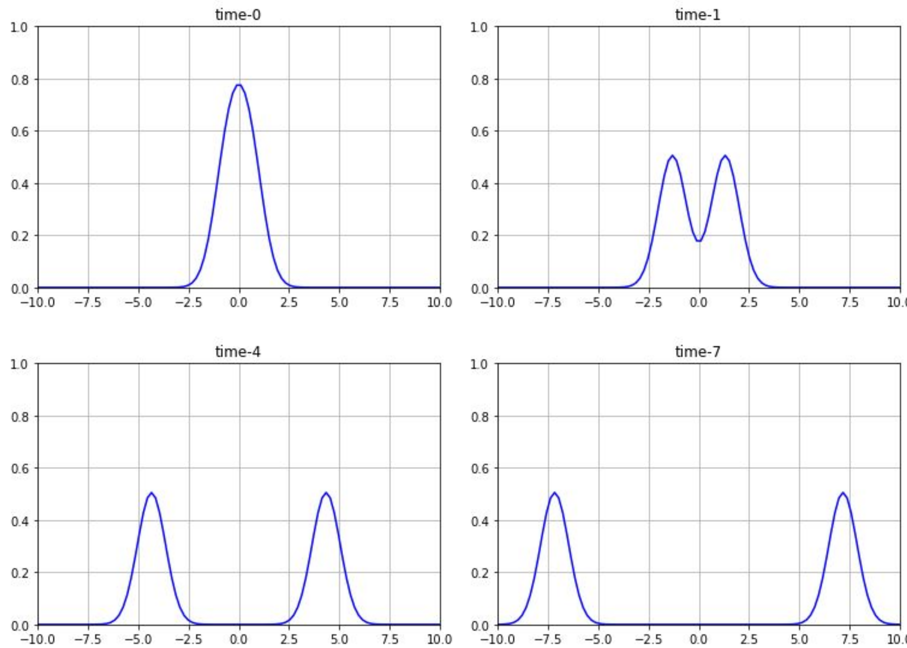
Test Case 2

$$L = 1\ \ \ m\ \ ;\ \ c = 1\ \ m/s\ \ ;\ \ f(x) = (\frac{cos((2k+1)\pi}{2}\ \frac{x}{L}\ \ \ ,\ \ k = 3$$

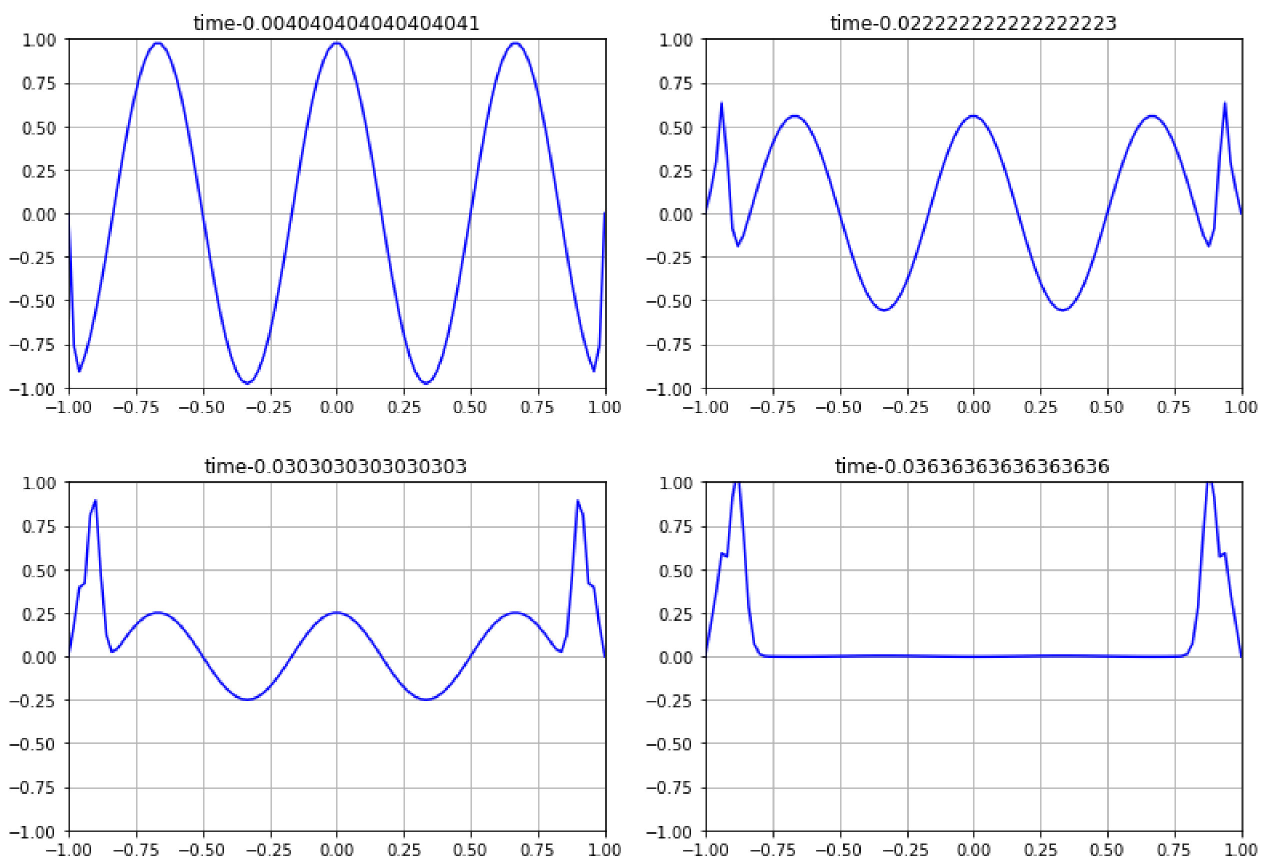
Represent the results at times

$$t_0 = 0\ \ s\ \ ,\ \ t_1 = \frac{T}{5}\ \ ,\ \ t_2 = \frac{2T}{5}\ \ ,\ \ T = \frac{4L}{(2k+1)c}$$

Test case 1 :



Test case 2 :



2 2-D wave equation

The 2D wave equation can be written as follows:

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - c^2 \Delta u = 0 & for\ t > 0, (x,y) \in \Omega \\ u(x,y,t=0) = f(x,y) & for\ (x,y) \in \Omega \\ \frac{\partial u}{\partial t}(x,y,0) = 0 & for\ (x,y) \in \Omega \\ u(x,y,t) = 1 & for\ t \geq 0, (x,y) \in \Omega \end{cases} \tag{12}$$

$\Omega = [-L_x, L_x]x[-L_y, L_y]$ being the computational domain, which is assumed to be rectangular, and $\partial\Omega$ is the boundary Ω.

The computational domain is discretized by a rectangular mesh with N_x nodes in x direction.and N_y nodes in y direction. The space steps are $\Delta x = \frac{L_x}{N_x - 1}$ and $\Delta y = \frac{L_y}{N_y - 1}$.

We denote by X_i, j and Y_i, j the coordinates of a node of index (i, j) and by u_i^n, j the approximate solution at the node (i, j) and at the time $t^n = n * \Delta t$.

2.1 Question 5

Based on the scheme (1) for the 1D wave equation (E1), propose a finite difference scheme for the 2D wave equation (E2)

Using Taylor approximation, we propose the following schema :
In space :

$$u_{i+1,j}^n = u(t^n, x_i + \Delta x, y_j)$$
$$u_{i+1,j}^n = u_{i,j}^n + \Delta x \frac{\partial u}{\partial x}|_{i,j}^n + \frac{\Delta x^2}{2}\frac{\partial^2 u}{\partial x^2}|_{i,j}^n + \frac{\Delta x^3}{3!}\frac{\partial^3 u}{\partial x^3}|_{i,j}^n + O(\Delta x^4)$$

$$u_{i-1,j}^n = u(t^n, x_i - \Delta x, y_j)$$
$$u_{i-1,j}^n = u_{i,j}^n - \Delta x \frac{\partial u}{\partial x}|_{i,j}^n + \frac{\Delta x^2}{2}\frac{\partial^2 u}{\partial x^2}|_{i,j}^n - \frac{\Delta x^3}{3!}\frac{\partial^3 u}{\partial x^3}|_{i,j}^n + O(\Delta x^4)$$

Then $\boxed{\frac{\partial^2 u}{\partial^2 x} = \frac{u_{i-1,j}^n + u_{i+1,j}^n - 2u_{i,j}^n}{\Delta x^2} + O(\Delta x^2)}$

$$u_{i,j+1}^n = u(t^n, x_i, y_j + \Delta y)$$
$$u_{i,j+1}^n = u_{i,j}^n + \Delta y \frac{\partial u}{\partial y}|_{i,j}^n + \frac{\Delta y^2}{2}\frac{\partial^2 u}{\partial y^2}|_{i,j}^n + \frac{\Delta y^3}{3!}\frac{\partial^3 u}{\partial y^3}|_{i,j}^n + O(\Delta y^4)$$

$$u_{i,j-1}^n = u(t^n, x_i, y_j - \Delta y)$$
$$u_{i,j-1}^n = u_{i,j}^n - \Delta y \frac{\partial u}{\partial y}|_{i,j}^n + \frac{\Delta y^2}{2}\frac{\partial^2 u}{\partial y^2}|_{i,j}^n - \frac{\Delta y^3}{3!}\frac{\partial^3 u}{\partial y^3}|_{i,j}^n + O(\Delta y^4)$$

Then $\boxed{\frac{\partial^2 u}{\partial^2 y} = \frac{u_{i,j-1}^n + u_{i,j+1}^n - 2u_{i,j}^n}{\Delta y^2} + O(\Delta y^2)}$

In time :

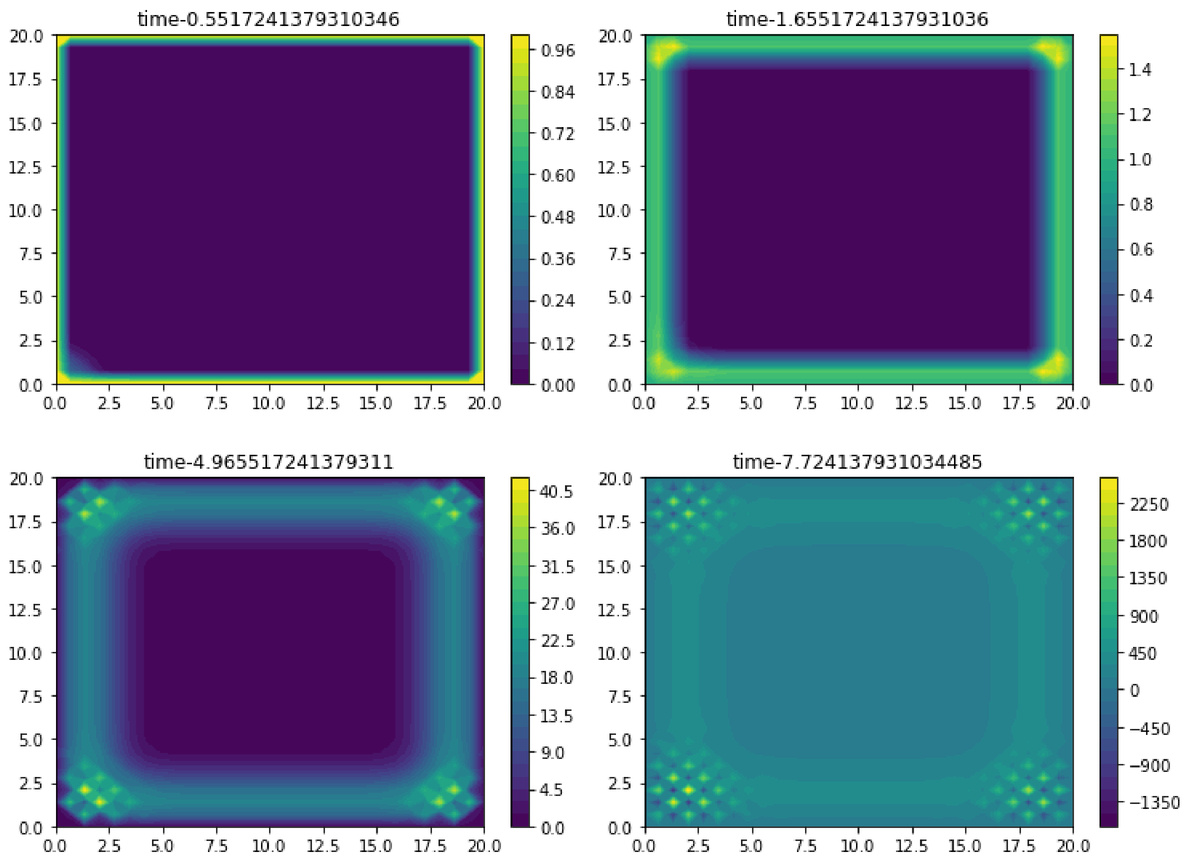
$$u_j^{n+1} = u(t^n + \Delta t, x_j)$$
$$u_j^{n+1} = u_j^n + \Delta t \frac{\partial u}{\partial t}|_j^n + \frac{\Delta t^2}{2}\frac{\partial^2 u}{\partial t^2}|_j^n + \frac{\Delta t^3}{3!}\frac{\partial^3 u}{\partial t^3}|_j^n + O(\Delta t^4)$$

$$u_j^{n-1} = u(t^n - \Delta t, x_j)$$
$$u_j^{n-1} = u_j^n - \Delta t \frac{\partial u}{\partial t}|_j^n + \frac{\Delta t^2}{2}\frac{\partial^2 u}{\partial t^2}|_j^n - \frac{\Delta t^3}{3!}\frac{\partial^3 u}{\partial t^3}|_j^n + O(\Delta t^2)$$

Our numerical schema is : $\boxed{\frac{u_j^{n-1}+u_j^{n+1}-2u_j^n}{\Delta t^2} - c^2(\frac{u_{i-1,j}^n+u_{i+1,j}^n-2u_{i,j}^n}{\Delta x^2} + \frac{u_{i,j-1}^n+u_{i,j+1}^n-2u_{i,j}^n}{\Delta y^2}) = 0}$

2.2 Question 6

Test case :
- $L_x = L_y = 10$ m ; c = 1 m/s ; $f(x,y) = e^{-(x^2+y^2)}$;$N_x = N_y = 30$
- Represent the contour plots of u at times $t_0 = 0s, t_1 = 1s, t_2 = 4s, t_3 = 7s$.



3 Code Python

1D model test case 1 :

```python
import numpy as np
import math
import matplotlib.pyplot as plt
#Test case 1
def f(x,y):
    return np.exp(-(x**2 + y**2) )


c = 1
#longueur et largeur du canal
Lx = 10
Ly = 10
# nombre de noeud suivant x et y
Nx = 30
Ny = 30
#le pas de maillage suivant x et y
dx = 2*Lx/(Nx-1)
dy = 2*Ly/(Ny-1)
# maillage et condition initiale
x= np.zeros((Nx,Ny))
y= np.zeros((Nx,Ny))
u= np.zeros((Nx,Ny))
un_1 = np.zeros((Nx,Ny))

for i in range(Nx):
    for j in range(Ny):
        x[i,j] = i*dx
        y[i,j] = j*dy
        u[i,j] = f(x[i,j], y[i,j])
        un_1[i,j] = f(x[i,j], y[i,j])

CFL = 0.8
dt = min(CFL*dx/c, CFL*dy/c)

lamda1 = c*dt/dx
lamda2 = c*dt/dy


t=0
T = 30
unew = np.zeros((Nx,Ny))
while(t < T):
    for i in range(1,Nx-1):
        for j in range(1,Ny-1):
            unew[i,j] = 2*u[i,j] - un_1[i,j] + (lamda1**2)*(u[i-1,j]- 2*u[i,j]+ u[i+1,
    j]) \
                                                + (lamda2**2)*(u[i,j-1]- 2*u[i,j]+ u[i,j+1])
        # les conditions aux limites
    unew[0,:] = 1 #bord sud
    unew[:,Ny-1] = 1
    unew[:,0] =1
    unew[Nx-1,:] = 1


    t = t + dt
    u = unew.copy()
    un_1 = un_1.copy()
    plt.title('time-{}'.format(t))
    plt.contourf(x,y,u,30)
    plt.colorbar()
    plt.show()
```

1D model est case 2 :

```
    #Test case 2

c = 1
L = 1
k=3
def f(x):
    return np.cos(2*k*np.pi*x/2*L)


# maillage et condition initiale
N = 100
x= np.linspace(-L,L,N)
u= np.zeros(N)
un_1 = np.zeros(N)
for i in range(N):
    u[i] = f(x[i])
    un_1[i] = f(x[i])
#trac  de la condition initiale
plt.plot(x,u,'-b')
plt.grid()

#le pas de maillage
dx = 2*L/(N-1)
# temps final des simulations
T = 4*L/(2*k+1)*c
Tf = T/5
#initialisation de temps:
t= 0
# Nombre de CFL tel que (0<CFL<=1)
CFL = 0.1
#calcul du pas du temps pour avoir la stabilit
dt = CFL*dx/c
lamda = c*dt/dx
unew= np.zeros(N)
#boucle principale en temps
#On descritise notre interval en (N-1) sous interval
while (t < Tf):
    for i in range(1,N-1):
        unew[i] = 2*u[i]-un_1[i]+(lamda*2)*(u[i+1]- 2*u[i] + u[i-1])
        un_1[i]= u[i]
        #conditions aux limites
    unew[0] = 0
    unew[N-1] = 0
    if int(t) in [0,T/5,T/5,T*2/3]:
        print(t)
        plt.plot(x, u, '-b')
        plt.title('time-{}'.format(t))
        plt.axis([-1,1,-1,1])
        plt.grid()
        plt.pause(0.1)

    t = t + dt
    un_1 = un_1.copy()
    u = unew.copy()
```

2D Model test case :

```python
import numpy as np
import math
import matplotlib.pyplot as plt
#Test case 1
def f(x,y):
    return np.exp(-(x**2 + y**2) )


c = 1
#longueur et largeur du canal
Lx = 10
Ly = 10
# nombre de noeud suivant x et y
Nx = 30
Ny = 30
#le pas de maillage suivant x et y
dx = 2*Lx/(Nx-1)
dy = 2*Ly/(Ny-1)
# maillage et condition initiale
x= np.zeros((Nx,Ny))
y= np.zeros((Nx,Ny))
u= np.zeros((Nx,Ny))
un_1 = np.zeros((Nx,Ny))

for i in range(Nx):
    for j in range(Ny):
        x[i,j] = i*dx
        y[i,j] = j*dy
        u[i,j] = f(x[i,j], y[i,j])
        un_1[i,j] = f(x[i,j], y[i,j])

CFL = 0.8
dt = min(CFL*dx/c, CFL*dy/c)

lamda1 = c*dt/dx
lamda2 = c*dt/dy


t=0
T = 30
unew = np.zeros((Nx,Ny))
while(t < T):
    for i in range(1,Nx-1):
        for j in range(1,Ny-1):
            unew[i,j] = 2*u[i,j] - un_1[i,j] + (lamda1**2)*(u[i-1,j]- 2*u[i,j]+ u[i+1,
    j]) \
                                                        + (lamda2**2)*(u[i,j-1]- 2*u[i,j]+ u[i,j+1])
        # les conditions aux limites
    unew[0,:] = 1 #bord sud
    unew[:,Ny-1] = 1
    unew[:,0] =1
    unew[Nx-1,:] = 1


    t = t + dt
    u = unew.copy()
    un_1 = un_1.copy()
    plt.title('time-{}'.format(t))
    plt.contourf(x,y,u,30)
    plt.colorbar()
    plt.show()
```