ASP Project : Adaptive Least Mean Squares Estimation of Graph Signals

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About

The aim of this project is to propose a least mean squares (LMS) strategy for adaptive estimation of signals defined over graphs. Assuming the graph signal to be band-limited, over a known bandwidth, the method enables reconstruction, with guaranteed performance in terms of mean-square error, and tracking from a limited number of observations over a subset of vertices.

Furthermore, to cope with the case where the bandwidth is not known beforehand, we propose a method that performs a sparse online estimation of the signal support in the (graph) frequency domain, which enables online adaptation of the graph sampling strategy.

LMS Estimation of Graph Signals

- The signal is initially assumed to be perfectly band-limited, i.e. its spectral content is different from zero only on a limited set of frequencies F .
- Let us consider partial observations of signal x0 , i.e. observations over only a subset of nodes. Denoting with S the sampling set (observation subset), the observed signal at time n can be expressed as:

$$\boldsymbol{y}[n] = \mathbf{D}\left(\boldsymbol{x}_0 + \boldsymbol{v}[n]\right) = \mathbf{D}\mathbf{B}\boldsymbol{x}_0 + \mathbf{D}\boldsymbol{v}[n]$$

where D is the vertex-limiting operator, which takes nonzero values only in the set S, and v[n] is a zero-mean, additive noise with covariance matrix C v . The second equality comes from the bandlimited assumption, i.e. Bx0 = x0 , with B denoting the operator that projects onto the (known) frequency set F.

- Following an LMS approach, the optimal estimate for x0 can be found as the vector that solves the following optimization problem:

$$\begin{aligned} \min_{\boldsymbol{x}} \quad & \mathbf{E}\left\|\boldsymbol{y}[n] - \mathbf{D}\mathbf{B}\boldsymbol{x}\right\|^2 \\ \text{s.t.} \quad & \mathbf{B}\boldsymbol{x} = \boldsymbol{x}, \end{aligned}$$

Sampling Strategies

The properties of the LMS algorithm strongly depend on the choice of the sampling set S, i.e. on the vertex limiting operator D. The sampling strategy must be carefully designed in order to:

- enable reconstruction of the signal;
- guarantee stability of the algorithm; and
- impose a desired mean-square error at convergence.

Both the number of samples and their location is fundamental for the performance of the algorithm.

Three algorithms were used to find the samples to be taken for reconstruction algorithm:

- Minimum Mean square deviation
- Maximization of determinant
- Maximization of the minimum eigenvalue

Minimization of MSD

The method iteratively selects the samples from the graph that lead to the largest reduction in terms of steady state MSD.

$$\mathbf{G} = \mathbf{U}_{\mathcal{F}}^{H}\mathbf{D}\mathbf{C}_{v}\mathbf{D}\mathbf{U}_{\mathcal{F}}$$

$$\mathbf{Q} = (\mathbf{I} - \mu\,\mathbf{U}_{\mathcal{F}}^{H}\mathbf{D}\mathbf{U}_{\mathcal{F}}) \otimes (\mathbf{I} - \mu\,\mathbf{U}_{\mathcal{F}}^{H}\mathbf{D}\mathbf{U}_{\mathcal{F}}).$$

Sampling strategy 1: Minimization of MSD

Input Data : M, the number of samples.
Output Data : $\mathcal{S}$, the sampling set.
Function : initialize $\mathcal{S} \equiv \emptyset$
while $|\mathcal{S}| < M$
$s = \arg\min_{j}\ \mathrm{vec}(\mathbf{G}(\mathbf{D}_{\mathcal{S}\cup\{j\}}))^{T}(\mathbf{I} - \mathbf{Q}(\mathbf{D}_{\mathcal{S}\cup\{j\}}))^{\dagger}\mathrm{vec}(\mathbf{I});$
$\mathcal{S} \leftarrow \mathcal{S} \cup \{s\};$
end

Where D is the diagonal matrix with 1 when that vertex is present in the sampling set S, Cv is the covariance matrix of noise, Uf is a matrix with |F| eigenvectors of the Laplacian matrix and mu is the learning rate.

Maximization of Determinant $\mathbf{U}_{\mathcal{F}}^{H}\mathbf{D}\mathbf{U}_{\mathcal{F}}$

The rationale underlying this strategy is to design a well suited basis for the graph signal that we want to estimate. This criterion coincides with the maximization of the the pseudo-determinant of the matrix $\mathbf{U}_{\mathcal{F}}^{H}\mathbf{D}\mathbf{U}_{\mathcal{F}}$ i.e. the product of all non zeros eigenvalues.

Sampling strategy 2: Maximization of $\left|\mathbf{U}_{\mathcal{F}}^{H}\mathbf{D}\mathbf{U}_{\mathcal{F}}\right|_{+}$

Input Data : M, the number of samples.
Output Data : $\mathcal{S}$, the sampling set.
Function : initialize $\mathcal{S} \equiv \emptyset$
while $|\mathcal{S}| < M$
$s = \arg\max_{j} \left|\mathbf{U}_{\mathcal{F}}^{H}\mathbf{D}_{\mathcal{S}\cup\{j\}}\mathbf{U}_{\mathcal{F}}\right|_{+};$
$\mathcal{S} \leftarrow \mathcal{S} \cup \{s\};$
end

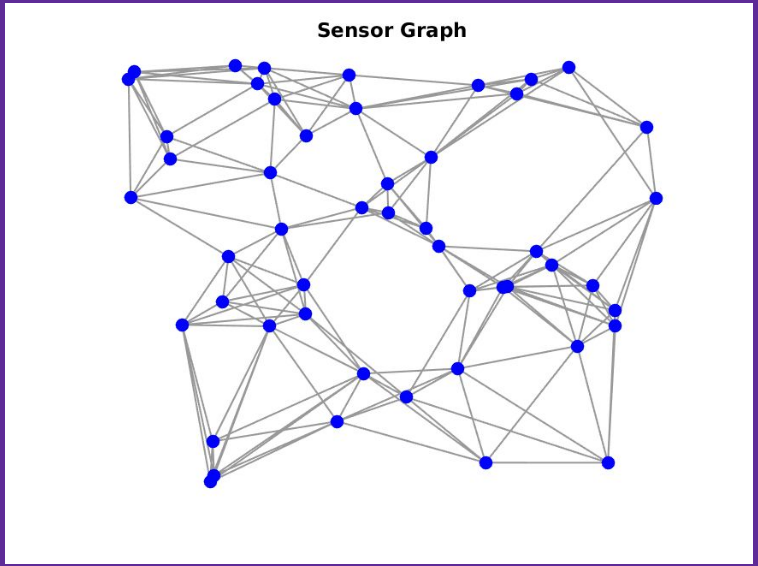
Maximization of $\lambda^{+}_{\min}\left(\mathbf{U}_{\mathcal{F}}^{H}\mathbf{D}\mathbf{U}_{\mathcal{F}}\right)$

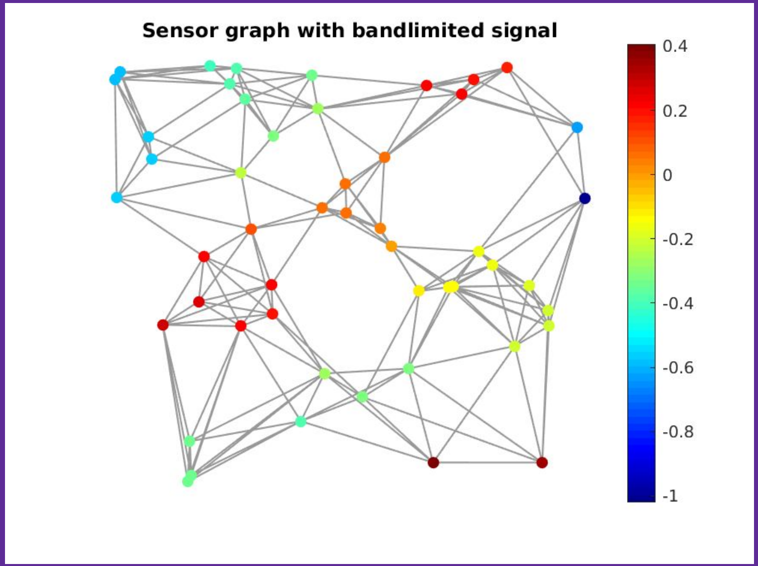
This algorithm can be formulated as the maximization of the minimum nonzero eigenvalue of the matrix $\mathbf{U}_{\mathcal{F}}^{H}\mathbf{D}\mathbf{U}_{\mathcal{F}}$.

Sampling strategy 2: Maximization of $\left|\mathbf{U}_{\mathcal{F}}^{H}\mathbf{D}\mathbf{U}_{\mathcal{F}}\right|_{+}$

Input Data : M, the number of samples.
Output Data : $\mathcal{S}$, the sampling set.
Function : initialize $\mathcal{S} \equiv \emptyset$
while $|\mathcal{S}| < M$
$s = \arg\max_{j} \left|\mathbf{U}_{\mathcal{F}}^{H}\mathbf{D}_{\mathcal{S}\cup\{j\}}\mathbf{U}_{\mathcal{F}}\right|_{+}$;
$\mathcal{S} \leftarrow \mathcal{S} \cup \{s\}$;
end

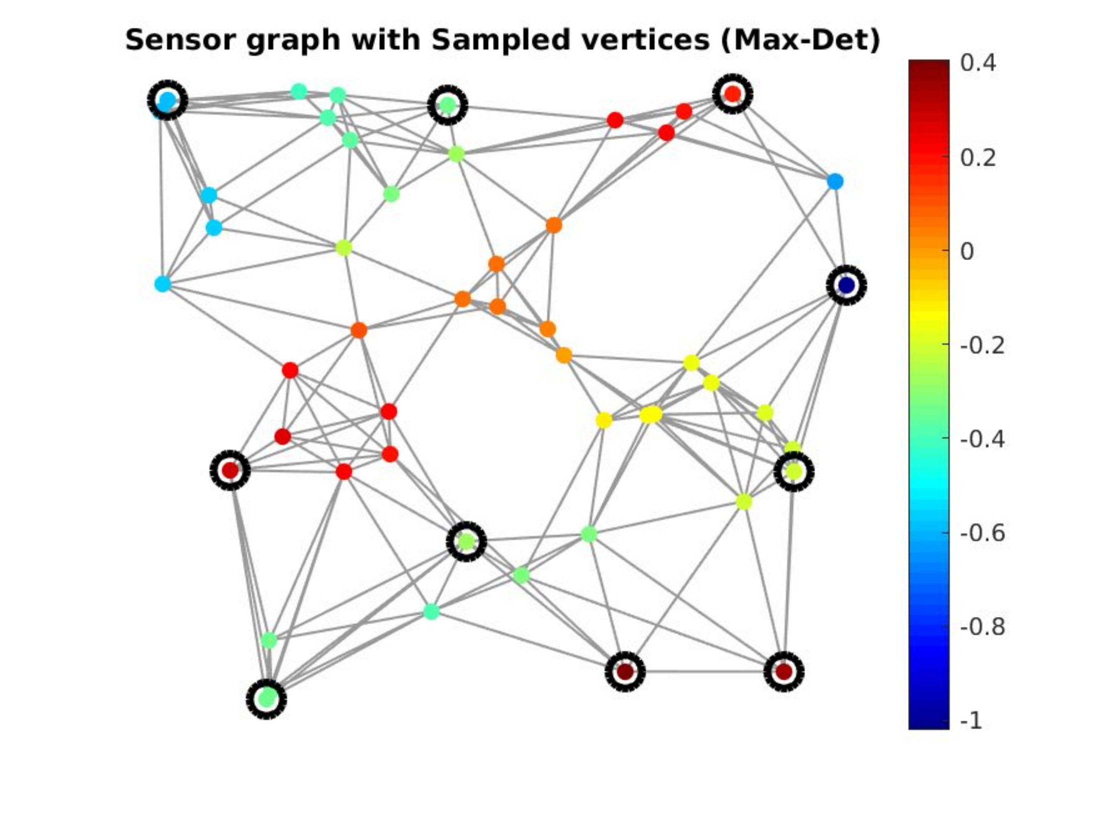
Graph and Graph Signal





Bandwidth is 10, Nodes=50, Learning rate=0.5

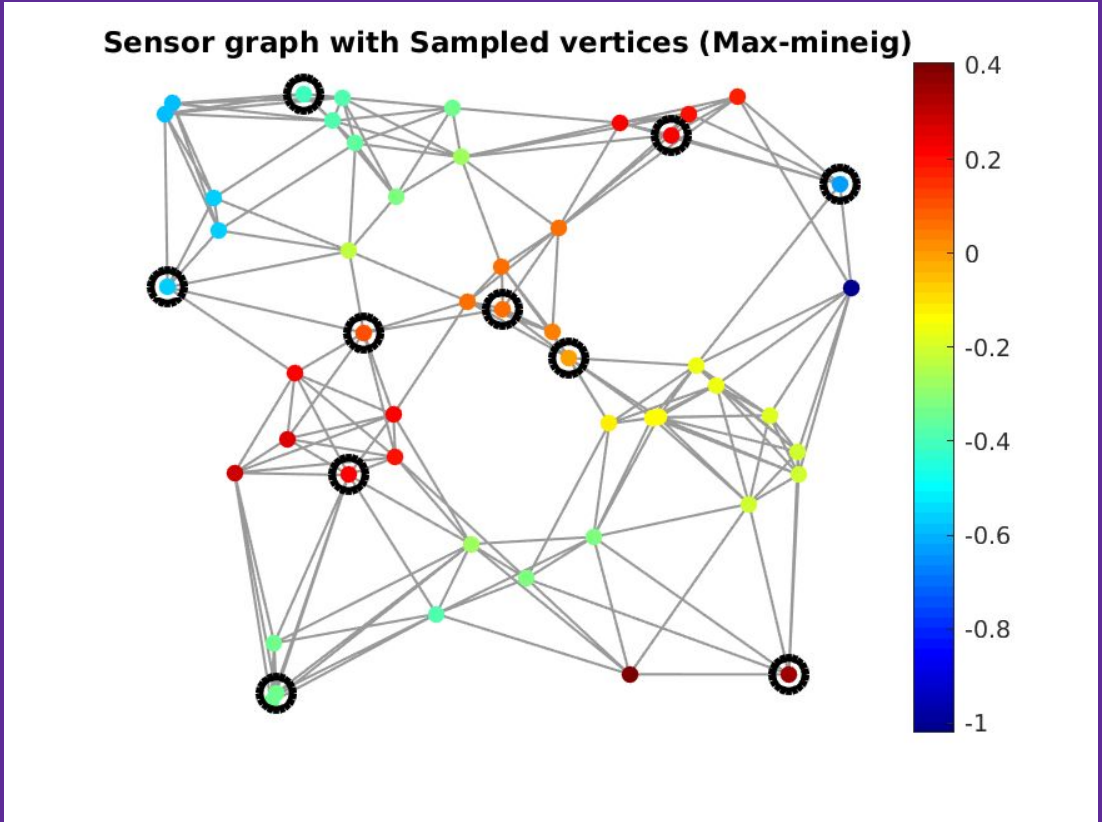
Results of sampling



Sampled set of vertices for Max-Det Algorithm

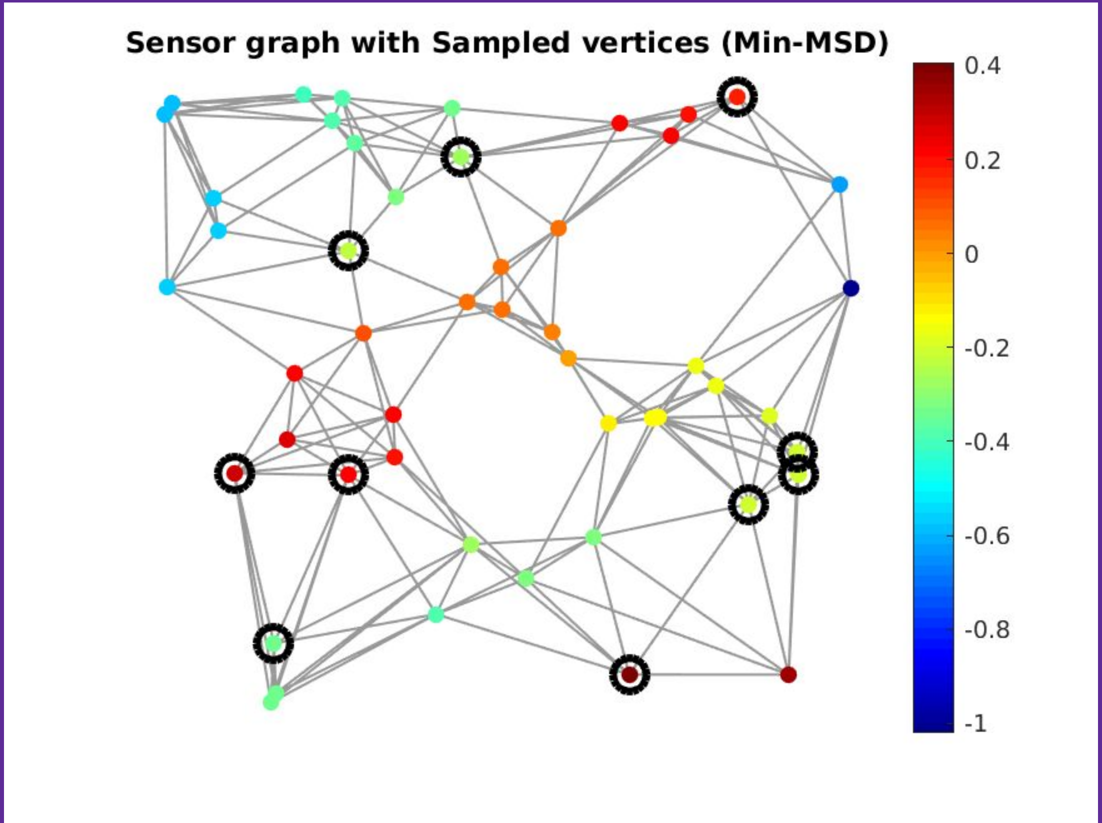
46
36
23
50
26
3
6
43
7
48

Results of sampling




```
Sampled set of vertices for Max-mineig Algorithm
    28
     9
    49
    12
    39
     2
    32
    18
    15
    46
```

Results of sampling

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Sampled set of vertices for Min-MSD Algorithm
    16
    47
    15
    36
    43
    24
    44
     6
     8
    48
```

Remaining Tasks

Algorithm 1: LMS algorithm for graph signals

Start with $x[0] \in \mathcal{B}_{\mathcal{F}}$ chosen at random. Given a sufficiently small step-size $\mu > 0$, for each time $n > 0$, repeat:

$$x[n+1] = x[n] + \mu\,\mathbf{B}\mathbf{D}\,(y[n] - x[n]) \qquad (12)$$

- Implementing this algorithm after finding the desired D matrix.
- B is $\mathbf{U}\boldsymbol{\Sigma}_{\mathcal{F}}\mathbf{U}^H$, and $\boldsymbol{\Sigma}_{\mathcal{F}}$ is a diagonal matrix defined as $\boldsymbol{\Sigma}_{\mathcal{F}} = \mathrm{diag}\{\mathbf{1}_{\mathcal{F}}\}$. .
- For any bounded initial condition, the LMS strategy asymptotically converges in the mean-square error sense if the step-size μ is: $0 < \mu < \frac{2}{\lambda_{\max}\left(\mathbf{U}_{\mathcal{F}}^H \mathbf{D} \mathbf{U}_{\mathcal{F}}\right)}$,
- The mean square error is given by: $\mathbf{E}\|\widehat{s}[n+1]\|_{\varphi}^2 = \mathbf{E}\|\widehat{s}[n]\|_{\mathbf{Q}\varphi}^2 + \mu^2 \mathrm{vec}(\mathbf{G})^T\varphi$ where Q and G are defined before and $\|\widehat{s}[n]\|_{\boldsymbol{\Phi}}^2 = \widehat{s}[n]^H \boldsymbol{\Phi} \widehat{s}[n]$.

  $\boldsymbol{\Phi} \in \mathbb{C}^{|\mathcal{F}|\times|\mathcal{F}|}$ is any Hermitian nonnegative-definite matrix

  $\varphi = \mathrm{vec}(\boldsymbol{\Phi})$ where the notation vec(·) stacks the columns of Φ on top of each other.

- Further tasks include situations where we consider the graph is fixed, and the spectral content of the signal can vary over time in an unknown manner and then trying to estimate the signal.