Ellipsoid Method

Optimization Methods

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Overview

Introduction

Introduction

- The Ellipsoid Method is an iterative method for minimizing convex functions.
- The Ellipsoid Method is the first polynomial-time algorithm discovered for linear programming.

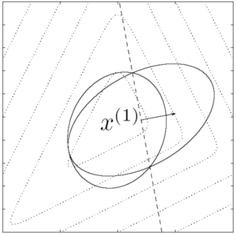

Figure: Volume of ellipsoid decreases

Problem Statement

The Ellipsoid Method is designed to solve decision problems rather than optimization problems.Therefore, we first consider the decision problem of finding a feasible point to a system of linear inequalities

$$C^T x \leq d \tag{1}$$

where C is a $n \times m$ matrix and d is an n-dimensional vector. We assume all data to be integral and n to be greater or equal than 2. The goal is to find a vector $x \in R^n$ satisfying (1) or to prove that no such x exists.

Overview

The problem starts with a big ellipsoid E that is guaranteed to contain P where

$$P = \{x \in R^n : Cx \leq d\}. \tag{2}$$

If the center of the ellipsoid is in P, then the algorithm is stopped Otherwise, we find an inequality $c_i^T x \leq d_i$ (c_i is the i^{th} row of matrix C) which is satisfied by all points in P but not satisfied by the center.

Theoretical Analysis

Let us first define an ellipsoid as

Lemma 1

Given a center a, and a positive definite matrix A, the ellipsoid $E(a, A)$ is defined as $\{x \in R^n : (x - a)^T A^{-1}(x - a) \leq 1\}$.

The ellipsoid algorithm has the important property that the ellipsoids constructed shrink in volume as the algorithm proceeds.

Lemma 2

$\frac{Vol(E_{k+1})}{Vol(E_k)} < e^{-\frac{1}{2(n+1)}}$

Theoretical Analysis

The final Ellipsoid Method can be written as:

- $k = 0$, $E_0 = E(a_0, A_0) \supseteq P$, $P = \{x : Cx \leq d\}$
- while $a_k \notin P$ do:
    1. Let $c_i^T x \leq d$ be an inequality that is valid for all $x \in P$ but $c_i^T a_k > d$
    2. $A_{k+1} = \frac{n^2}{n^2-1}(A_k - \frac{2}{n+1} bb^T)$
    3. $a_{k+1} = a_k - \frac{1}{n+1} b$
    4. $k = k + 1$

where $b = \frac{Ac_i}{\sqrt{c_i^T Ac_i}}$

Optimization

In combinatorial optimization a set $S \subseteq \{0,1\}^n$ is given and the task is to optimize over $P = conv(S)$.
Given a polytope (a geometric object with "flat" sides), the aim is to find a feasible point in it using some optimization problem. Let $c_i^T x$ with $c \in R^n$ be the objective function over P. Assuming that $c \in Z^n$, we can check the non-emptiness of

$$P' = P \cap \{x : c_i^T x \leq d + \frac{1}{2}\} \tag{3}$$

where $d \in Z$ and d has to be minimum.

Initialization

As starting ellipsoid, we can use the ball centered at the vector $(\frac{1}{2}, \frac{1}{2}, ..., \frac{1}{2})$ and of radius $\frac{1}{2}\sqrt{n}$, which goes through all the incident vectors. The ball has volume of

$$Vol(E_0) = \frac{1}{2^n}(\sqrt{n})^n Vol(B_n), \tag{4}$$

where B_n is the unit ball which has a value

$$Vol(B_n) = \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}. \tag{5}$$

On taking logarithm on both sides in equation (4), the following complexity is obtained $log(Vol(E_0)) = O(nlogn)$.

Termination Criteria

Let P' be non-empty and say $v_0 \in P' \cap \{0,1\}^n$, and $v_1, v_2, ..., v_n \in P \cap \{0,1\}^n = S$ be full dimensional. The v_i's are not necessarily in P', hence define w_i such that

$$w_i = \begin{cases} v_i & if \quad c_i^T x \leq d + \frac{1}{2} \\ v_0 + \alpha(v_i - v_0) & otherwise \end{cases}$$

where $\alpha = \frac{1}{2nc_{max}}$. This implies that $w_i \in P'$.

Now, P' can be a written as the convex set of $(v_0, w_1, w_2, ..., w_n)$. When a parallelepiped is formed by subtracting v_0 from w_i as $w_i - v_0 = \beta_i(v_i - v_0)$.

The number of iterations of the Ellipsoid Method before convergence is at most

$$log(Vol(E_0)) - log(Vol(P')) = O(nlogn + nlogc_{max}). \tag{6}$$

Separation Oracle

To decide when $x \in R^n$ is in P', we need to find a Separation Oracle for P.

Lemma 3

A polynomial time Separation Oracle for a convex set P is a procedure which given x, either tells that $x \in P$ or returns a hyperplane separating x from P. The procedure should run in polynomial time.

Optimum Solution

Theorem 1

Let $S = \{0,1\}^n$ and $P = conv(S)$. Assume that P is full-dimensional and we are given a separation oracle for P. Then, given $c \in Z^n$, one can find $min\{c^T x : x \in S\}$ by the Ellipsoid Method by using a polynomial number of operations and calls to the separation oracle.

The number of iterations of the Ellipsoid Method for combinatorial optimization is $O(n^2 log^2 n + n^2 log^2 c_{max})$, each iteration requiring a call to the Separation Oracle and a polynomial number of operations.

Minimum Cost Arborescence Problem

The minimum cost arborescence problem relies on solving a primal-dual method

$$
\begin{aligned}
& min \quad \sum_{a \in A} c_a x_a \\
& s.t. \\
& \qquad \sum_{a \in \delta^-(S)} x_a \geq 1 \quad \forall S \subseteq V \backslash \{r\} \\
& \qquad x_a \geq 0 \quad a \in A
\end{aligned}
\tag{7}
$$

where for a given arbitrary subset of nodes $S \subseteq V$, $\delta^-(S) = \{(i,j) \in A | i \in V \backslash S, j \in S\}$ and $c : A \rightarrow R_{\geq 0}$ is the sharing cost function, such that c_a is the cost of taking $a \in A$ as part of the shared network (e.g., the installation cost).

Linear Programming

From duality theory, the linear optimization problem is equivalent to finding a feasible point of the following system of linear inequalities

$$\begin{aligned} C^T x &\le d \\ -x &\le 0 \\ -C^T y &\le q \\ -y &\le 0 \\ -q^T x + d &\le 0. \end{aligned} \tag{8}$$

The third and fourth inequality come from the dual problem of linear optimization problem, and the last inequality results from the Strong Duality Theorem, which means zero slackness.

Linear Programming

In the Bisection Method, the bounds are found from the solutions of the primal and dual problem to get the lower bound and upper bound unless it is infeasible.
Once bounds are obtained, the problem can be solved with Ellipsoid Method iteratively using the additional constraint

$$-q^T x \geq -\frac{u + l}{2}, \tag{9}$$

where l and u are lower and upper bounds respectively.
If the new problem is infeasible, the upper bound is updated to $\frac{u+l}{2}$, and this happens iteratively until the solution is found.

Separation and Optimization

Lemma 4

The separation problem and the optimization problem over the same family of polytopes are polynomially equivalent.

An example is the maximum stable set problem, which can be modeled as integer programming problem

$$\begin{aligned} max \quad & c^T x \\ s.t. \quad & x_i + x_j \leq 1 \quad \forall (i,j) \in E \\ & x \in \{0,1\} \end{aligned} \tag{10}$$

where $x_i = 1$ iff node i is in a maximum stable set, otherwise it is zero.

Separation and Optimization

Relaxing the binary constraints for x to create a LP problem

$$0 \leq x \leq 1, \tag{11}$$

and considering the odd-cycle inequalities

$$\sum_{i \in C} x_i \leq \frac{|C| - 1}{2} \tag{12}$$

for each odd cycle C in G, we can write the polytope which satisfies all these odd cycle inequalities as

$$P := \{x \in R^{|V|} | x \; satisfies (10), (11) \; and \; .(12)\} \tag{13}$$

This is called the cycle-constraint stable set polytope.

Summary

- In the term paper, I have explored the Ellipsoid Method and the theoretical analysis behind the method.
- It is the first polynomial time algorithm discovered for linear programming.
- However, it has a few of disadvantages, the rate of convergence of the Ellipsoid Method is rather slow, especially when compared to practical experience with the simplex method.
- The Ellipsoid Method does not provide optimal dual variables.
- However, it is a powerful theoretical tool in the analysis of the computational complexity of combinatorial optimization problems.

Thank You