Homework 3

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1. **Problem 1**

   In laying water mains, utilities must be concerned with the possibility of freezing. Although soil and weather conditions are complicated, reasonable approximations can be made on the basis of the assumption that soil is uniform in all directions. In that case the temperature in degrees Celsius $T(x,t)$ at a distance x (in meters) below the surface, t seconds after the beginning of a cold snap, approximately satisfies

   $$\frac{T(x,t) - T_s}{(T_i - T_s)} = erf(\frac{x}{2\sqrt{\alpha t}})$$

   where T_s is the constant temperature during a cold period, T_i is the initial soil temperature before the cold snap, $alpha$ is the thermal conductivity (in $"meters"^2$ per second), and

   $$erf(t) = \frac{2}{\sqrt{\pi}} \int_0^t \exp(-s^2)\, ds$$

   Assume that $T_i = 20$ (degrees C), $T_s = -15$, $\alpha = 0.138 \cdot 10^{-6}$ (m^2 per second). Use scipy for erf. For (b) and (c), use $\epsilon = 10^{-13}$

   (a) We want to determine how deep a water main should be buried so that it will only freeze after 60 days exposure at this constant surface temperature. Formulate the problem as a root finding problem $f(x) = 0$. What is f and f'? Plot f on $[0, \tilde{x}]$ where $\tilde{x}$ is chosen so that $f(\tilde{x}) > 0$.

   $$f(x) = T(x, 60 \text{ days}) = \text{erf}(\frac{x}{2\sqrt{\alpha 5184000}})(T_i - T_s) + T_s$$

   $$f'(x) = \frac{2}{2\sqrt{\alpha 5,184,000}\sqrt{\pi}} e^{-x^2}(T_i - T_s)$$

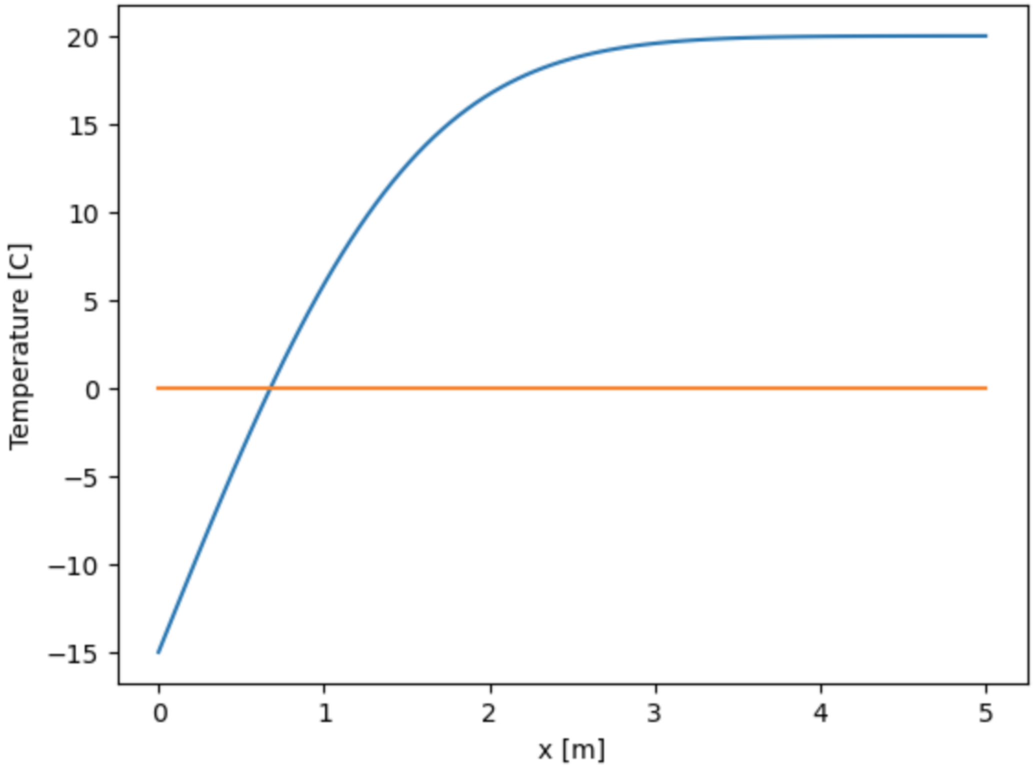


Figure 1: Plot of Temperature versus Depth

(b) Compute an approximate depth using Bisection with starting $a_0 = 0$ and $b_0 = \tilde{x}$.
x is approximately 0.676961854507681 m, using 34 iterations. (For this part and the next part, $\tilde{x} = 1$, but the graph was left plotting until $x = 5$m just to see the longer term behavior.)

(c) Compute an approximate depth using Newton's Method with starting value $x_0 = 0.01$. What happens if you start with $x_0 = \tilde{x}$?
Using $x_0 = 0.01$, the necessary depth is 0.676961854462041 m, in 21 iterations.
Using $x_0 = \tilde{x}$, the necessary depth is 0.6769618545013728 m, in 20 iterations.

2. **Problem 2** Let $f(x)$ denote a function with root α with multiplicity m.

   (a) Write a formal definition of what it means for α to be a root of multiplicity m for $f(x)$.

   $$f(x) = (x - \alpha)^m q(x)$$

   Assuming $q(x)$ has no root at $x = \alpha$, the root at α has multiplicity m. If $q(x)$ has a root at $x = \alpha$, then the equation changes to $f(x) = (x - \alpha)^{m-n} q(x)$, where n is the multiplicity of the root in $q(x)$

   (b) Show that Newton's method applied to $f(x)$ only converges linearly to the root α.

   $$f(x) = (x - \alpha)^m q(x)$$

   Turning this into a fixed point iteration problem,

   $$g_{NR}(x) = x - \frac{(x-\alpha)^m q(x)}{m(x-\alpha)^{m-1} q(x) + (x-\alpha)^m q'(x)}$$

   $$g_{NR}(x) = x - \frac{(x-\alpha)^m q(x)}{(x-\alpha)^{m-1}[mq(x) + (x-\alpha) q'(x)]}$$

   $$g_{NR}(x) = x - \frac{(x-\alpha) q(x)}{mq(x) + (x-\alpha) q'(x)}$$

   To find the derivative of $g_{NR}(x)$, ignore any terms that have an $x - \alpha$ since they go to zero since α is a root of the function.

   $$g'_{NR}(x) = 1 - \frac{(q(x) + ...)(mq(x) + ...) - (...)(...)}{[mq(x) + ...]^2}$$

   $$g'_{NR}(x) = 1 - \frac{mq(x)^2}{m^2 q(x)^2}$$

   $$g'_{NR}(x) = 1 - \frac{1}{m}$$

   If $m > 1$, this is linear convergence because $g'_{NR}(\alpha) > 0$. So Newton's method $f(x)$, corresponding to $g_{NR}(x)$ is linearly convergent.

   (c) Show that the fixed point iteration applied to $g(x) = x - m\frac{f(x)}{f'(x)}$ is second order convergent.
   Assuming $f(x)$ is twice differentiable, we can find $g'_{NR}(x)$

   $$g'_{NR}(x) = 1 - m\frac{f'(x)f'(x) - f(x)f''(x)}{f'(x)^2}$$

   $$g'_{NR}(x) = 1 - m + m\frac{f''(x)f(x)}{f'(x)^2}$$

   Plugging in $x = \alpha$, where $f(\alpha) = 0, f'(\alpha) \neq 0$ and $m = 1$ for multiplicity of one

   $$g'_{NR}(\alpha) = 0$$

   Since $g'_{NR}(x) = 0$, Newton's method has second order, quadratic convergence.

   (d) What does (c) provide for Newton's method in the case of roots with multiplicity greater than 1?
   Roots with multiplicity greater than one will converge quadratically since $g'_{NR}(\alpha) \neq 0$ (from the notes).

3. **Problem 3** Beginning with the definition of order of convergence of a sequence $x_k|_{k=1}^{\infty}$ that converges to α, derive a relationship between the $\log(|x_{k+1} - \alpha|)$ and $\log(|x_k - \alpha|)$. What is the order p in this relationship?

$$\lim_{k\to\infty} \tfrac{|x_{k+1}-\alpha|}{|x_k-\alpha|^p} = \lambda$$

$$\log \frac{|x_{k+1}-\alpha|}{|x_k-\alpha|^p} = \log\lambda$$

$$\log|x_{k+1}-\alpha| - p\log|x_k-\alpha| = \log\lambda$$

$$p = \frac{\log\lambda - \log|x_{k+1}-\alpha|}{-\log|x_k-\alpha|}$$

For large k, $\lambda \approx 1$ so $\log\lambda \approx 0$

$$\boxed{p = \frac{\log|x_{k+1}-\alpha|}{\log|x_k-\alpha|}}$$

4. **Problem 4** There are two ways of improving the convergence of Newton's method when a root has multiplicity greater than 1: Problem 2c and apply Newton's method to $g(x) = \frac{f(x)}{f'(x)}$. In this problem consider finding the root of the function $f(x) = e^{3x} - 27x^6 + 27x^4e^x - 9x^2e^{2x}$ in the interval $(3, 5)$. Explore the order of convergence when applying (i) Newton's method, (ii) the modified Newton's method from class, (iii) the modified Newton's method in Problem 2. Which method do you prefer and why?

   (a) Newton's Method
   $\alpha = 0.9865020493952698$, $\lambda = 0.6263531754842516$
   For this method, since the root has a multiplicity greater than 1, the method is linearly convergent with rate $\lambda = 0.6263531754842516$. This required 23 iterations to get a value of $r = 3.733054928$.

   (b) Modified Newton's Method
   $\alpha = 2.2684875772107413$
   This modified method is quadratically convergent. It required 4 iterations to get to a value of $r = 3.733072345$.

   (c) Method from 2c
   This method is more difficult to implement since the function is not simple to find the multiplicity of roots. Therefore, it is harder to implement this method.

   Of all these methods, I prefer using the normal Newton's method. For this problem, 23 iterations was fine since it didn't require too much computation time, and linear convergence was sufficient. I didn't prefer the other modified functions since it was much more annoying to calculate the second derivatives of the function. I also didn't know the root multiplicity so I didn't like the method from 2c. I may have preferred the 2nd method if it were more trivial to calculate the derivative but in this instance, I prefer the regular Newton's method.

5. **Problem 5** Use Newton and Secant method to approximate the largest root of

$$f(x) = x^6 - x - 1$$

Start Newton's method with $x_0 = 2$. Start Secant method with $x_0 = 2$ and $x_1 = 1$.

(a) Create a table of the error for each step in the iteration. Does the error decrease as you expect?

Newton's Method

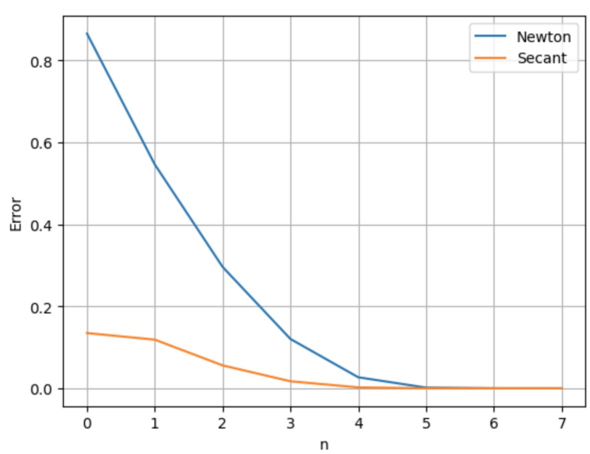
x_0	$\|x_0 - \alpha\|$	$\log \|x_0 - \alpha\|$	n
2.000000000	0.319371728	-0.495703532	1
1.680628272	0.249889284	-0.602252367	2
1.430738988	0.175768032	-0.755060110	3
1.254970956	0.093432523	-1.029501922	4
1.161538433	0.025185159	-1.598855310	5
1.136353274	0.001622746	-2.789749499	6
1.134730528	0.000006390	-5.194509785	7

Secant Method

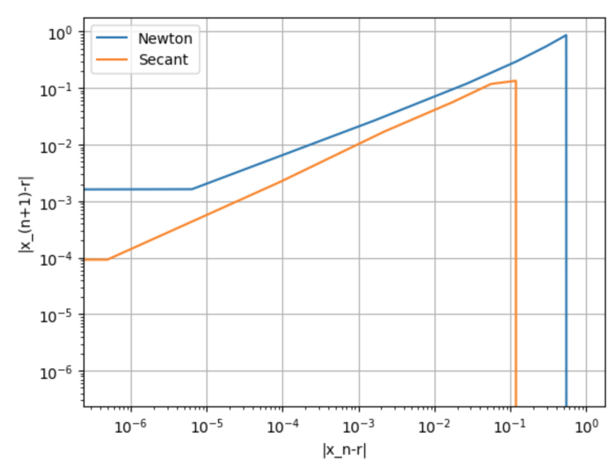
x_0	$\|x_0 - \alpha\|$	$\log \|x_0 - \alpha\|$	n
1.016129032	0.016129032	-1.792391689	0
1.190577769	0.174448736	-0.758332172	1
1.117655831	0.072921938	-1.137141799	2
1.132531550	0.014875719	-1.827522026	3
1.134816808	0.002285258	-2.641064802	4
1.134723646	0.000093162	-4.030760935	5
1.134724138	0.000000492	-6.307732664	6

The error does decrease as I expect. Both of these are quadratically convergent methods, so it makes sense that they would be able to converge in few iterations and at a good rate.

(b) Plot $|x_{k+1} - \alpha|$ vs $|x_k - \alpha|$ on log-log axes where α is the exact root for both methods. What are the slopes of the lines that result from this plot? How does this relate to the order?



Error vs. Iteration



Plot of $\log |x_k - \alpha|$ vs $\log |x_{k+1} - \alpha|$

The slope is not clear since it is in loglog scale. The slope calculates the same way as the formula in question 4 to find α, so I used this to determine the order. For Newton's method, $\alpha = 1.4719174596363513$, and for Secant Method, $\alpha = 1.574424365099342$. You can see that the slope of the secant graph is slightly greater, which matches the results.