1 (100 PTS.) Greedy coloring

Given an undirected graph G with n vertices, the greedy coloring algorithm order the vertices of G in an arbitrary order $v_1, \ldots, v_n$. Initially all the vertices are not colored. In the ith iteration, the algorithm assigns v_i the smallest color (i.e., positive integer) k such that none of its neighbors that are already colored have color k. Let $f(v_i)$ denote the assigned color to v_i.

1.A. (30 PTS.) Prove that the above algorithm computes a valid coloring of the graph (i.e., there are is no edge uv in G such that $f(u) = f(v)$).

Solution:

Proof: Assume for the sake of contradiction that the claim is false, and there is a bad edge $uv \in \mathsf{E}(\mathsf{G})$, such that $f(u) = f(v)$. Assume that the algorithm first colored u, and then colored v. But then, when the algorithm colored v, u was already colored, and the algorithm would assign v a different color than u. A contradiction. ∎

1.B. (30 PTS.) Prove that if a vertex v is colored by color k, then there is a simple path in the graph $u_1, u_2, \ldots, u_k = v$, such that for $i = 1, \ldots, k$, we have $f(u_i) = i$ (and $u_i u_{i+1} \in \mathsf{E}(\mathsf{G})$ for all i).

Solution:

Proof: The proof is by induction.

Base of induction: For a vertex colored by color 1, the claim holds, since the desired path is just the vertex itself.

Inductive assumption: Assume, that the claim holds for any vertex of color j, for $j \geq 1$.

Inductive step: We need to prove the claim for a vertex colored with color $j+1$, for $j \geq 1$. So, consider a vertex u such that $f(u) = j + 1$. By the way the algorithm works, it must be that u has a neighbor that is already colored of color j, as otherwise, u would have been assigned either the color j, or a smaller color. Let v be this neighbor. By induction, there is a path $\pi = u_1, u_2, \ldots, u_j = v$, such that $u_i u_{i+1} \in \mathsf{E}(\mathsf{G})$, $f(u_i) = i$, for $i = 1, \ldots, j-1$, and $f(u_j) = j$. Setting $u_{j+1} = u$, and we add to this path π the edge $u_j u_{j+1}$. Since $f(u_{j+1}) = j + 1$, the resulting path has the desired property. ∎

1.C. (40 PTS.) Prove that G either have a simple path of length $\lfloor \sqrt{n} \rfloor$, or alternatively, G contains an independent set of size $\lfloor \sqrt{n} \rfloor$. A set of vertices $X \subseteq \mathsf{V}(\mathsf{G})$ is ***independent*** if no two vertices $x, y \in X$ form an edge in G.

Solution:

Proof: Let c be the number of colors used by the above greedy color algorithm. If $c \geq \lfloor \sqrt{n} \rfloor$, then there is a vertex with color $\lfloor \sqrt{n} \rfloor$, and by part (B), there is a path in the graph of length $\lfloor \sqrt{n} \rfloor$.

Otherwise, if $c < \lfloor \sqrt{n} \rfloor$, then there must be a color that is used by at least $n / \lfloor \sqrt{n} \rfloor \geq \lfloor \sqrt{n} \rfloor$ vertices. All the vertices using this color are not connected by an edge, since this is an independent set in the graph, which implies the claim. ∎

2 (100 PTS.) Prefix it.

Let $L \subseteq \{0, 1\}^*$ be a language defined as follows:

(i) $\varepsilon \in L$.

(ii) For all $w \in L$ we have $0w1 \in L$.

(iii) For all $x, y \in L$ we have $xy \in L$.

And these are all the strings that are in L. Prove, by induction, that for any $w \in L$, and any prefix u of w, we have that $\#_0(u) \geq \#_1(u)$. Here $\#_0(u)$ is the number of 0 appearing in u ($\#_1(u)$ is defined similarly). You can use without proof that $\#_0(xy) = \#_0(x) + \#_0(y)$, for any strings x, y.

Solution:

Proof: The proof is by induction on the length of w.

Base of induction: If $|w| = 0$ then $w = \varepsilon$, and then $\#_0(w) = 0 \geq \#_1(u) = 0$. Since the only prefix of the empty string is itself, the claim readily follows.

Inductive assumption: Assume that the claim holds for all strings of length $\leq n$.

Inductive step: We need to prove the claim for a string w of length $n + 1$. There are two possibilities:

- $w = 0z1$, for some string $z \in L$.
  Let u be any prefix of w. If $u = \varepsilon$ or $u = 0$ then the claim clearly holds for u.
  If $u = w$, then we have that

  $$\#_0(u) = \#_0(w) = 1 + \#_0(z) + 0 \geq 1 + \#_1(z) = \#_1(w) = \#_1(u),$$

  which implies the claim (we used the inductive claim on z, as $z \in L$, and $|z| \leq |w| - 2 \leq n - 1 < n$).
  So the interesting case is where $u = 0z'$, where z' is a prefix of z. We then have that

  $$\#_0(u) = \#_0(0z') = 1 + \#_0(z') \geq 1 + \#_1(z') = 1 + \#_1(u) > \#_1(u),$$

  Again, we used the fact that $z \in L$, z' is a prefix of Z, and using induction on z which is strictly shorter than w. This implies the claim.
- $w = x, y$, for some strings $x, y \in L$, such that $|x|, |y| > 0$.
  Let u be a prefix of w. If u is a prefix of x, then the claim holds readily by induction. As such, we assume that $u = xz$, for some z which is prefix of y. But then, we have that

  $$\#_0(u) = \#_0(xz) = \#_0(x) + \#_0(z) \geq \#_1(x) + \#_1(z) = \#_1(u),$$

  by using the inductive claim on x (which is a prefix of itself), and on z (which is a prefix of y. Here, we used that both x and y are strictly shorter than w, and the inductive assumption holds for them.

(The proof here is an example of structural induction.) ■

3 (100 PTS.) A recurrence.

Consider the recurrence

$$T(n) = \begin{cases} T(\lfloor n/3 \rfloor) + T(\lfloor n/4 \rfloor) + T(\lfloor n/5 \rfloor) + T(\lfloor n/6 \rfloor) + n & n \geq 6 \\ 1 & n < 6. \end{cases}$$

Prove by induction that $T(n) = O(n)$.

(An easier proof follows from using the techniques described in section 3 of these notes on recurrences.)

Solution:

Claim 1.1. *For $c \geq 20$, and for all integers $n > 1$, we have that $T(n) \leq cn$.*

Proof: **Base of induction.** For $n < 6$ the claim holds for any $c \geq 1$ by definition.

Inductive assumption. Assume the claim hold for $n \leq k$, That is, for any $n \leq k$, we have $T(n) \leq cn$.

Inductive step. We need to prove the claim for $n = k + 1$, for $k \geq 6$. We have

$$\begin{aligned} T(n) &\leq T(\lfloor n/3 \rfloor) + T(\lfloor n/4 \rfloor) + T(\lfloor n/5 \rfloor) + T(\lfloor n/6 \rfloor) + n \\ &\leq c\lfloor n/3 \rfloor + c\lfloor n/4 \rfloor) + c\lfloor n/5 \rfloor) + c\lfloor n/6 \rfloor) + n && \text{// By the inductive assumption} \\ &\leq cn/3 + cn/4 + cn/5 + cn/6 + n \\ &\leq \left(\frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6}\right)cn + n = \left(\frac{3}{4} + \frac{1}{5}\right)cn + n = \left(\frac{19}{20}c + 1\right)n \leq cn, \end{aligned}$$

the last step holds if

$$\frac{19}{20}c + 1 \leq c \iff 1 \leq \frac{1}{20}c \iff c \geq 20.$$

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