Homework 1

Algorithms for Big Data
CS498ABD Spring 2019
Due: 10am, Wednesday, Feb 6th

Instructions:

- Each home work can be done in a group of size at most two. Only one home work needs to be submitted per group. However, we recommend that each of you think about the problems on your own first.
- Homework needs to be submitted in pdf format on Gradescope. See `https://courses.engr.illinois.edu/cs374/fa2018/hw-policies.html` for more detailed instructions on Gradescope submissions.
- Follow academic integrity policies as laid out in student code. You can consult sources but cite all of them including discussions with other class mates. Write in your own words. See the site mentioned in the preceding item for more detailed policies.

Exercise 1: Sampling, Chebyshev vs Chernoff. Suppose you want to estimate the average of n numbers via sampling (for example the heights of students in the class). The average can be very skewed by outliers. However, we can obtain an accurate estimate if we assume that the numbers are within some limited range. Assume the input numbers $z_1, z_2, \ldots, z_n$ are from $[a, b]$ where $a, b \in \mathbb{R}$ with $a \leq b$. Suppose you sample k input numbers (with replacement) and output their average as the estimate for the true average $\alpha = (\sum_i z_i)/n$. Let X be the random variable denoting the output value.

(a) Using Chebyshev's inequality, show that for $k \geq \frac{(b-a)^2}{\delta\epsilon^2}$, we have

$$\mathrm{P}[|X - \alpha| \geq \epsilon] \leq \delta.$$

(b) Using the Chernoff inequality, show that there exists a constant $c > 0$ such that for $k \geq \frac{c(b-a)^2 \log(2/\delta)}{\epsilon^2}$, we have

$$\mathrm{P}[|X - \alpha| \geq \epsilon] \leq \delta.$$

Exercise 2: Quick Sort Given an array A of n numbers (which we assume are distinct for simplicity), the algorithm picks a pivot x uniformly at random from A and computes the rank of x. If the rank of x is between $n/4$ and $3n/4$ (call such a pivot a good pivot), it behaves like the normal QuickSort in partitioning the array A and recursing on both sides. If the rank of x does not satisfy the desired property (the pivot picked is not good), the algorithm simply repeats the process of picking a pivot until it finds a good one. Note that in principle the algorithm may never terminate!

(a) Write a formal description of the algorithm.

(b) Prove that the expected run time of this algorithm is $O(n \log n)$ on an array on n numbers.

(c) Prove that the algorithm terminates in $O(n \log n)$ time with high probability.

Exercise 3: Pairwise Independence. Suppose we want to generate N pairwise independent random variables in the range $\{0, 1, 2, \ldots, M-1\}$. We will assume that N and M are powers of 2 and let $N = \{0,1\}^n$ and $M = \{0,1\}^m$ (hence $n = \log N$ and $m = \log M$). We saw a scheme in the lecture using mn bits. Here we will revisit that scheme in a different way and then see how it can be made more randomness-efficient.

Pick a uniformly random matrix $A \in \{0,1\}^{m \times n}$ and a random vector $b \in \{0,1\}^m$. Then for a vector $v \in \{0,1\}^n$, set $X_v = Av + b \mod 2$ (by this we mean component wise mod 2).

(a) Show that for all $u, v \in \{0,1\}^n$ where $u \neq v$ and $\alpha, \beta \in \{0,1\}^m$, there exist $w \in \{0,1\}^n$, $w \neq \mathbb{0}$, and $\gamma \in \{0,1\}^m$ such that

$$\mathrm{P}_{A,b}[Au + b = \alpha \bmod 2 \wedge Av + b = \beta \bmod 2] = \frac{1}{2^m}\mathrm{P}_A[Aw = \gamma \bmod 2]$$

for any distribution over A if b is independently chosen uniformly at random.

(b) Suppose we pick A and b uniformly at random. Show that under this scheme, for all $w \in \{0,1\}^n$ where $w \neq \mathbb{0}$ and for all $\gamma \in \{0,1\}^m$,

$$\mathrm{P}_A[Aw = \gamma \bmod 2] = \frac{1}{2^m}.$$

Why does this guarantee that X_u and X_v are independent for $u \neq v$ and $u \neq \mathbb{0}, v \neq \mathbb{0}$?

(c) We can make the following improvement. A matrix A is *Toeplitz* if the entries along each diagonal are constant, i.e., $A_{i,j} = A_{i+1,j+1}$. Suppose we choose A uniformly at random from the set of Toeplitz matrices whose entries are in $\{0,1\}$. Show that under this scheme, it is also the case that for all $w \in \{0,1\}^n$ where $w \neq \mathbb{0}$ and for all $\gamma \in \{0,1\}^m$,

$$\mathrm{P}_A[Aw = \gamma \bmod 2] = \frac{1}{2^m}.$$

(d) How many random bits do we need to generate a random Toeplitz matrix $A \in \{0,1\}^{m \times n}$ and how much storage (in bits) do you need to store it implicitly? Express this as a function of N and M and compare with the number of random bits needed for the case when A is picked as a random $\{0,1\}$ matrix.

Exercise 4: Probabilistic counter. In lecture we analyzed probabilistic counting: initialize a counter X to 1, and for every increment instruction, increment X with probability $1/2^X$. By averaging many such estimators, we obtained a $(1+\epsilon)$-approximation to n with good probability and space usage was $O(\log\log n)$. In this problem you will investigate a minor modification. Imagine we still initialize X to 1, but we increment it with probability $1/(1+a)^X$ for some fixed $a > 0$. (Note that your estimator for n would have to change from $2^X - 1$ to something else.)

How small must a be so that our estimate $\widetilde{n}$ of n satisfies $|\widetilde{n} - n| \leq \epsilon n$ with at least $9/10$ probability when we return the output of a single estimator instead of averaging many estimators we did in the lecture? Also derive a bound $S = S(n)$ on the space (in bits) so that this algorithm uses at most S space with at least $9/10$ probability by the end of the n increments.

Additional exercises (not to be submitted)

Exercise 5. In class, we proved a powerful tail inequality called the "(multiplicative) Chernoff bound" that we will use time and time again. In this exercise, we rewrite the Chernoff in a convenient form that is a little more interpretable and easier to apply.

Recall the Chernoff inequality, as follows. Let $X_1, X_2, \ldots, X_n \in [0,1]$ be n independent, nonnegative, and uniformly bounded random variables. Let

$$\mu = \mathrm{E}\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} \mathrm{E}[X_i]$$

be the expected value of the sum. The Chernoff inequality states that for any $\delta > 0$, we have

$$\mathrm{P}\left[\sum_{i=1}^{n} X_i \geq (1+\delta)\mu\right] \leq \left(\frac{e^{\delta}}{(1+\delta)^{1+\delta}}\right)^{\mu} \text{ and } \mathrm{P}\left[\sum_{i=1}^{n} X_i \leq (1-\delta)\mu\right] \leq \left(\frac{e^{-\delta}}{(1-\delta)^{1-\delta}}\right).$$

Now let $X_1, \ldots, X_n$ and μ be as above.

(a) Show that for $x \geq 0$ sufficiently small, we have

$$x - (1+x)\ln(1+x) \leq -\frac{x^2}{3}.$$

Hint: Consider the Taylor expansion $\ln(1+x) = x - x^2 + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots$ *for* $x \in (-1,1]$.

(b) Show that for $\epsilon \in [0,1]$,

$$\mathrm{P}\left[\sum_{i=1}^{n} X_i \geq (1+\epsilon)\mu\right] \leq e^{-\epsilon^2\mu/3}.$$

(c) Show that for $x \in [0,1]$, we have

$$x + (1-x)\ln(1-x) \geq \frac{x^2}{2}.$$

(d) Show that for $\epsilon \in [0, 1]$,

$$\mathrm{P}\left[\sum_{i=1}^{n} X_i \leq (1-\epsilon)\mu\right] \leq e^{-\epsilon^2\mu/2}.$$

Exercise 6, 7. Exercises 2 and 3 from HW 4 of the 2016 algorithms course (`https://courses.engr.illinois.edu/cs473/fa2016/Homework/hw4.pdf`)

Exercise 8: Reservoir sampling. Show that `sample-without-replacement` (below) outputs uniformly random sample of k elements without replacement from a stream.

```
sample-without-replacement($k$)

1. $S[1..k] \leftarrow$ null
2. $m \leftarrow 0$
3. While (stream is not done)
    A. $m \leftarrow m + 1$
    B. $e_m$ is current item in stream
    C. If $(m \leq k)$ $S[m] \leftarrow x_m$
    D. else
        i. select integer $r$ uniformly at
           random from $\{1, 2, \ldots, m\}$
        ii. If $(r \leq k)$ $S[r] \leftarrow x_m$
4. Output $S$
```