1. **Solution:** We will use the pigeonhole principle. Let the 103 numbers in S be $a_1, a_2, \ldots, a_{103}$. Note that since S is a set, the numbers are distinct, that is $a_i \neq a_j$ for $1 \leq i < j \leq 103$. For $h \in \{0,1,2,\ldots,6\}$ let $S_h = \{a_i \mid a_i \bmod 7 = h\}$ be the set of numbers from S whose remainder when divided by 7 is h. Every a_i is in exactly one S_h and therefore $S_0, S_1, \ldots, S_6$ is a partition of S. This implies that $|S| = \sum_{h=0}^{6} |S_h| = 103$. We claim that there is an index $r \in \{0,1,2,\ldots,6\}$ such that $|S_r| \geq 15$. This follows from the pigeon hole principle. A direct argument is the following. If $|S_h| \leq 14$ for each $h \in \{0,1,\ldots,6\}$ then $\sum_{h=0}^{6} |S_h| \leq 7 \times 14 \leq 98$ but $\sum_{h=0}^{6} |S_h| = 103$, a contradiction.

   We claim that $S_r \subseteq S$, with $|S_r| \geq 15$ is our desired set. For any two distinct numbers $a_i, a_j \in S_r$ we have the property that $a_i \equiv a_j \bmod 7$ ($a_i \bmod 7$ and $a_j \bmod 7$ are equal to r) which implies that $a_i - a_j$ is divisible by 7. ■

   Rubric: 10 points for a correct proof. Small issues with a proof (such as improperly using modular arithmetic, improperly stating the pigeonhole principle, not fully defining all variables, etc.) will lose anywhere from 1 to 4 points each. Large issues with a proof (such as not letting S be an arbitrary set, only proving that there is a pair of numbers in S' whose difference is a multiple of 7, not proving that the size of S' is at least 15, etc.) will lose anywhere from 5 to 8 points each.

2. **Solution:** We assume all logarithms are with respect to base 2. This is without loss of generality since only the choice of a, b will change by a constant factor for other base values, and the rest of the proof remains the same.

   We claim that $T(n) \leq n \log n + 1$ for all positive integers n (this is our inductive hypothesis). We prove this by induction on n. Let $g(n) = n \log n + 1$.

   For the base of induction, consider the cases $n = 1,2,3$. By the defn of T, $T(n) = 1$ for $n = 1,2,3$. For $n \geq 1$, $g(n) \geq 1$ since $g(n)$ is an increasing function of n and $g(1) = 1$. Therefore $T(n) \leq g(n)$ for $n = 1,2,3$.

   For the induction step, let $n \geq 4$ and suppose that the inductive hypotheis holds for all $k < n$. We will show that $T(n) \leq g(n)$ holds for n.

   By definition of T we have $T(n) = T(\lfloor n/2 \rfloor) + 2T(\lfloor n/4 \rfloor) + n$. Since $\lfloor n/2 \rfloor$ and $\lfloor n/4 \rfloor$ are smaller than n when $n \geq 4$, applying the inductive hypothesis, we have that

$$T(\lfloor n/2 \rfloor) \leq \lfloor n/2 \rfloor \log(\lfloor n/2 \rfloor) + 1$$

   and

$$T(\lfloor n/4 \rfloor) \leq \lfloor n/4 \rfloor \log(\lfloor n/4 \rfloor) + 1.$$

   Therefore,

$$\begin{aligned} T(n) &= T(\lfloor n/2 \rfloor) + 2T(\lfloor n/4 \rfloor) + n \\ &\leq (\lfloor n/2 \rfloor \log(\lfloor n/2 \rfloor) + 1) + 2(\lfloor n/4 \rfloor \log(\lfloor n/4 \rfloor) + 1) + n \end{aligned}$$

   Since the function $x \to \log(x)$ is an increasing function of x, and the $\lfloor x \rfloor \leq x$ for all x, we have that $\lfloor n/2 \rfloor \log(\lfloor n/2 \rfloor) \le \frac{n}{2}\log(\frac{n}{2})$ and $\lfloor n/4 \rfloor \log(\lfloor n/4 \rfloor) \le (n/4)\log(n/4)$. Thus,

$$\begin{aligned}
&(\lfloor n/2 \rfloor \log(\lfloor n/2 \rfloor) + 1) + 2(\lfloor n/4 \rfloor \log(\lfloor n/4 \rfloor) + 1) + n \\
&\le ((n/2)\log(n/2) + 1) + 2((n/4)\log(n/4) + 1) + n \\
&= (n/2)(\log(n/2) + \log(n/4)) + 3 + n \\
&= (n/2)(2\log(n) - 3) + 3 + n \\
&= n\log(n) + 3 - \frac{n}{2} \\
&\le n\log(n) + 1.
\end{aligned}$$

The last inequality is valid because $n \ge 4$. This completes the inductive proof. ■

Rubric (10): Standard induction rubric. Additional problem specific rubric: -1 for removing floor signs for T(.) instead of for anlogn+b (as T(.) is not known to be increasing). -1 each for not stating explicit, or stating incorrect a,b values. -1/2 for only one base case, instead of 3 (as we require n/4 to exist for the T(n/4) variable in the T(n) definition).

3. (a) Part a:

Solution: We use Jeff's style of inductive proof. Let w be an *arbitrary* string in L.

Assume that $\#(\mathtt{1}, x)$ is odd for every string $x \in L$ such that $|x| < |w|$.

There are four cases to consider (mirroring the four cases in the definition):

- If $w = 1$, then $\#(\mathtt{1}, w) = 1$ which is odd, so w is odd.
- If $w = \mathtt{0}x$ for some string $x \in L$, then
$$\#(\mathtt{1}, w) = \#(\mathtt{1}, \mathtt{0}) + \#(\mathtt{1}, x) = \#(\mathtt{1}, x)$$
$\#(\mathtt{1}, x)$ is odd by the inductive hypothesis (since $|x| < |w|$), hence $\#(\mathtt{1}, w)$ is odd.
- If $w = x\mathtt{0}$ for some string $x \in L$, then
$$\#(\mathtt{1}, w) = \#(\mathtt{1}, x) + \#(\mathtt{1}, \mathtt{0}) = \#(\mathtt{1}, x)$$
$\#(\mathtt{1}, x)$ is odd by the inductive hypothesis (since $|x| < |w|$), hence $\#(\mathtt{1}, w)$ is odd.
- Otherwise, $w = x\mathtt{1}y$ for some strings $x, y \in L$. Then
$$\begin{aligned}
\#(\mathtt{1}, w) &= \#(\mathtt{1}, x) + \#(\mathtt{1}, \mathtt{1}) + \#(\mathtt{1}, y) \\
&= \#(\mathtt{1}, x) + 1 + \#(\mathtt{1}, y)
\end{aligned}$$
Both $\#(\mathtt{1}, x), \#(\mathtt{1}, y)$ are odd by the inductive hypothesis (since $|x| < |w|$ and $|y| < |w|$) and the sum of three odd numbers is always odd, so $\#(\mathtt{1}, w)$ is also odd.

In all four cases, we conclude that $\#(\mathtt{1}, w)$ is odd. ■

Rubric (10): Standard induction rubric.

(b) Part b:

Solution: We will prove by induction on the length of the string w that if $\#(\mathtt{1}, w)$ is odd, then $w \in L$. Let w be an *arbitrary* string such that $\#(\mathtt{1}, w)$ is odd. Assume that every string x with $|x| < |w|$ and $\#(\mathtt{1}, x)$ odd belongs to L. We consider four cases below, and every string w with $\#(\mathtt{1}, w)$ odd falls into one of these cases.

Case 1: w starts with **0**. That is $w = \mathbf{0}x$ for some string x. Then we have $\#(\mathbf{1}, x) = \#(\mathbf{1}, w)$, therefore $\#(\mathbf{1}, x)$ is odd. Since $|x| < |w|$, by induction hypothesis, we have that $x \in L$. By the second construction rule we have that $w = \mathbf{0}x$ is also in L.

Case 2: w ends with **0**. That is $w = x\mathbf{0}$ for some string x. Then we have $\#(\mathbf{1}, x) = \#(\mathbf{1}, w)$, therefore $\#(\mathbf{1}, x)$ is odd. Since $|x| < |w|$, by induction hypothesis, we have that $x \in L$. By the third construction rule we have that $w = \mathbf{0}x$ is also in L.

Case 3: w starts and ends with **1** and $|w| = 1$. Then $w = \mathbf{1}$. By the first construction rule, $w \in L$.

Case 4: w starts and ends with **1** and $|w| > 1$. Then $\#(\mathbf{1}, w) \geq 2$, but since $\#(\mathbf{1}, w)$ is odd, we have that $\#(\mathbf{1}, w) \geq 3$. Consider the second **1** in w. Let x be the prefix of w till the second **1** in w (not including it), let y be the suffix of of w after the second **1**. Then $w = x\mathbf{1}y$ where $\#(\mathbf{1}, x) = 1$. Since $\#(\mathbf{1}, w) \geq 3$ and odd, and $\#(\mathbf{1}, x) = 1$, we have that $\#(\mathbf{1}, y)$ is odd as well. Since $|x| < |w|$ and $|y| < |w|$, by induction hypothesis, we have that $x, y \in L$. By the fourth construction rule we have that $w = x\mathbf{1}y \in L$.

■

Rubric (10): Five points for each induction. Given according to the standard induction rubric.