# Condition Numbers and the Distance to Singularity

10 points

Let $A \in \mathbb{R}^{n\times n}$ be nonsingular.

If $B \in \mathbb{R}^{n\times n}$ and

$$\frac{\|A-B\|_2}{\|A\|_2} < \frac{1}{\kappa_2(A)},$$

then $B$ is nonsingular.

In this problem, you will prove the above statement, which tells us that the reciprocal of the condition number can be understood as a measure of relative distance to the nearest singular matrix. An alternate view of the statement is that nonsingular matrices form an open set. Therefore, if we have a matrix $B$ that is "slightly" different from $A$ (as measured by the matrix norm), then $B$ will also be nonsingular.

## Part 1:

First show that

$$\frac{1}{\|A^{-1}\|_2} \cdot \|\boldsymbol{x}\|_2 \leq \|A\boldsymbol{x}\|_2.$$

## Part 2:

Now, show that

$$\|A\boldsymbol{x}\|_2 \leq \|A-B\|_2 \cdot \|\boldsymbol{x}\|_2 + \|B\boldsymbol{x}\|_2.$$

## Part 3:

Using parts 1 and 2, show that the original statement holds.

Be sure to show all your work and provide justifications for every step to receive full credit.

Please submit your response to this written problem as a PDF file below. You may do either of the following:

- write your response out by hand, scan it, and upload it as a PDF.

  We will not accept unprocessed pictures taken with your phone.

If you decide to use your phone for scanning, make sure to use an app such as CamScanner (https://www.camscanner.com/) to get a readable PDF. Alternatively, there's a fast and convenient scanner in the Engineering IT office in 2302 Siebel that can just email you a PDF. (It's the Fax-machine-looking thing--not the scanner that's attached to one of the computers.)

- create the PDF using software.

  If you're looking for an easy-ish way to type math, check out TeXmacs (http://texmacs.org/) or LyX (http://www.lyx.org/). Both are installed in the virtual machine. (Under "Applications / Accessories / GNU TeXmacs editor" and "Applications / Office / LyX document processor" respectively.)

Submit your response to each problems in this homework as a separate PDF. If you have multiple PDFs that you need to merge into one, try PDF Split and Merge (http://www.pdfsam.org/download/).

**NOTE:** Please make sure your solutions are legible and easy to follow. If they are not, we may deduct up to five points *per problem*.

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Your answer is correct.

Since $\kappa_2(A) = \|A\|_2 \cdot \|A^{-1}\|_2$, then

$$\begin{aligned}\frac{\|A-B\|_2}{\|A\|_2} &< \frac{1}{\kappa_2(A)} \\ \Rightarrow \|A-B\|_2 &< \frac{1}{\|A^{-1}\|_2}.\end{aligned}$$

Let $\boldsymbol{x} \in \mathbb{R}^n$ be an arbitrary nonzero vector.

Then we can multiply both sides by $\|\boldsymbol{x}\|_2$, resulting in

$$(*) \qquad \|A-B\|_2 \cdot \|\boldsymbol{x}\|_2 < \frac{1}{\|A^{-1}\|_2} \cdot \|\boldsymbol{x}\|_2.$$

Part 1.

Let's manipulate the right-hand side by muliplying $\boldsymbol{x}$ by the identity matrix, since this will not change the Euclidean norm.

$$\begin{aligned}&\frac{1}{\|A^{-1}\|_2} \cdot \|\boldsymbol{x}\|_2 \\ =&\frac{1}{\|A^{-1}\|_2} \cdot \left\|A^{-1}A\boldsymbol{x}\right\|_2\end{aligned}$$

Using the definition of a matrix norm,

$$\begin{aligned}&\frac{1}{\|A^{-1}\|_2}\cdot\|A^{-1}A\boldsymbol{x}\|_2\\ \le&\frac{\|A^{-1}\|_2}{\|A^{-1}\|_2}\cdot\|A\boldsymbol{x}\|_2\\ =&\|A\boldsymbol{x}\|_2\end{aligned}$$

Part 2.

$$\begin{aligned}&\|A\boldsymbol{x}\|_2\\ =&\|A\boldsymbol{x}-B\boldsymbol{x}+B\boldsymbol{x}\|_2\\ \le&\|A\boldsymbol{x}-B\boldsymbol{x}\|_2+\|B\boldsymbol{x}\|_2\\ \le&\|A-B\|_2\cdot\|\boldsymbol{x}\|_2+\|B\boldsymbol{x}\|_2.\end{aligned}$$

Part 3.

Combining $(*)$ with the inequalities from Parts 1 and 2 results in

$$\|A-B\|_2\cdot\|\boldsymbol{x}\|_2<\|A-B\|_2\cdot\|\boldsymbol{x}\|_2+\|B\boldsymbol{x}\|_2.$$

We can subtract $\|A-B\|_2\cdot\|\boldsymbol{x}\|_2$ from both sides to obtain:

$$0<\|B\boldsymbol{x}\|_2$$

Therefore, $\boldsymbol{x}$ is not in the null space of $B$. Since $\boldsymbol{x}$ was an arbitrary nonzero vector, $B$ is nonsingular.