CS/ECE 374 Fall 2018 Xinrui Ying (xinruiy2@illinois.edu)
Homework 0 Problem 3

Consider the set of strings $L \subseteq \{0, 1\}^*$ defined recursively as follows:

- The string 1 is in L.
- For any string x in L, the string $0x$ is also in L.
- For any string x in L, the string $x0$ is also in L.
- For any strings x and y in L, the string $x1y$ is also in L.
- These are the only strings in L.

(a) Prove by induction that every string $w \in L$ contains an odd number of 1s.

(b) Is every string w that contains an odd number of 1s in L? In either case prove your answer.

Let $\#(a, w)$ denote the number of times symbol a appears in string w; for example,

$$\#(0, 101110101101011) = 5 \quad \text{and} \quad \#(1, 101110101101011) = 10.$$

You may assume without proof that $\#(a, uv) = \#(a, u) + \#(a, v)$ for any symbol a and any strings u and v, or any other result proved in class, in lab, or in the lecture notes. Otherwise, your proofs must be formal and self-contained.

Solution: (a)

- Base case: When w length $= 1$. The only string for w is 1, which has odd number of 1s. So the base case holds.
- Induction step: Suppose w having odd number of $1s$ is true for length $1, 2, 3...k$, we want to length $= k + 1$ also holds. There are only the following three cases:
  - We already have w with length k having odd number of $1s$, since $w \in L$, we also have $w0 \in L$. The number of $1s$ in $w0$ is same as number of 1s in w, the length of $w0$ = length of w + 1 = k + 1. So w holds for length $k + 1$ in this case.
  - We already have w with length k having odd number of $1s$, since $w \in L$, we also have $0w \in L$. The number of $1s$ in $0w$ is same as number of 1s in w, the length of $0w$ = length of w + 1 = k + 1. So w holds for length $k + 1$ in this case.
  - Let $m \in L$ be any string with length $k1$ where $k1 < k$, and $n \in L$ be any string with length $k2$ where $k2 = k - k1$. Since m and n have length less then k, we can conclude that they both have odd number of $1s$. Since both m and n are in L, then $m1n$ is also in L. The number of $1s$ in $m1n$ = number of $1s$ in m + number of $1s$ in n + 1. Since odd + odd + 1 = odd, we have number of $1s$ in $m1n$ is odd. The length of $m1n = k2 + k1 + 1 = k + 1$. So w holds for length $k + 1$ in this case.

  The prove of every string $w \in L$ contains an odd number of $1s$ is complete.

(b) It's true that every string w contains an odd number of $1s$ is in L.

- Base case: w contains only one $1s$. Let w starts with just 1, then w is in L, you can concatenate 0 at the front or the end to still be in L. After you concatenate for the first time, you still have w in L and you can recursively doing this and you can have any number of $0s$ in the front or back of the 1 and you still have $w \in L$. So the base case holds.
- Induction step: Suppose it true for w with $1, 3, 5...k-2, k$ number of $1s$, we want to prove that number of $1s = k+2$ also holds. Let m be any string with k number of $1s$, since the statement is assume true for k number of $1s$, we have $m \in L$. Let n be any string with 1 number of $1s$, since the statement is proved true for 1 number of $1s$, we have $n \in L$. Now since both m and n are in L, then $m1n$ and $n1m$ are both in L. The number of 1s for $m1n$ or $n1m = 1 + 1 + k = k+2$. Since m and n can be any string with just required number of $1s$, $m1n$ and $n1m$ contain all the possible outcomes with $k+2$ of $1s$. $m1n$ and $n1m$ are both in L, hence the statement is true for $k+2$ number of $1s$.

The prove of every string w contains an odd number of $1s$ is in L is complete.

■