CS/ECE 374 Fall 2018 Xinrui Ying (xinruiy2@illinois.edu)
Homework 0 Problem 2

(a) Consider the following recurrence.

$$T(n) = T(\lfloor n/2 \rfloor) + 2T(\lfloor n/4 \rfloor) + n \quad n \geq 4, \text{ and } T(n) = 1 \quad 1 \leq n < 4.$$

- Prove by induction that $T(n) = O(n \log n)$. More precisely show that $T(n) \leq an \log n + b$ for $n \geq 1$ where $a, b \geq 0$ are some fixed but suitably chosen constants (you get to choose and fix them).

Solution:

(a) Choose $a = 1$, $b = 1$.

- Base case: When $n = 1$, $T(1)$= 1, $log 1 = 0$, so that $an \log n + b = b = 1 \geq T(1)$. When $n = 2$, $T(2) = 1$, $log 2 = 1$, $an \log n + b = b + 2a = 1 + 2a = 3 \geq T(2)$. When $n = 3$, $T(3) = 1$, $an \log n + b = b + 3a log 3 = 1 + 3 log 3 \geq T(3)$. So it holds for $n = 1, 2, 3$.
- Induction step: Suppose it is true for $n = 4, 5, 6 \ldots k$, we want to show that $n = k + 1$ also holds. There are two cases:
  - k is an even number:
    Let $k = 2j$, $j \in \mathbb{N}$. $n = k + 1 = 2j + 1$. Then we have $T(k + 1) = T(\lfloor j + 1/2 \rfloor) + 2T(\lfloor 1/2j + 1/4 \rfloor) + k + 1$.

    Since j is a natural number, then $T(k + 1) = T(\lfloor j \rfloor) + 2T(\lfloor 1/2j \rfloor) + k + 1$. As $T(k) = T(\lfloor j \rfloor) + 2T(\lfloor 1/2j \rfloor) + k$, we have $T(k + 1) = T(k) + 1$.

    Since $n = k$ is already true that $T(k) \leq anlogn + b$, $T(k + 1) = T(k) + 1 \leq anlogn + b + 1 = anlogn + b + (1/logn) * logn = (an + (1/logn)) * logn + b$.

    Since $logn > 1$, $1/logn < 1$, and $log(n + 1) > log(n)$, then $(an + (1/logn)) * logn + b < (an + a) * (log(n + 1)) + b = a(n + 1) * (log(n + 1)) + b$.

    Hence, we conclude that $T(k + 1) \leq (an + (1/logn)) * logn + b < a(n + 1) * (log(n + 1)) + b$. The statement holds for $n = k + 1$ when k is an even number.
  - k is an odd number:
    Let $k = 2j + 1$, $j \in \mathbb{N}$. $n = k + 1 = 2(j + 1)$. We have $T(k + 1) = T(\lfloor (k + 1)/2 \rfloor) + 2T(\lfloor (k + 1)/4 \rfloor) + k + 1$. Since it's assume holds true for 1, 2 ... k, and k >= 4, then (k+1)/2 and (k+1)/4 are less than k that $T(\lfloor (k + 1)/2 \rfloor) \leq a(k + 1)/2(log(k + 1)/2) + b$, $T(\lfloor (k + 1)/4 \rfloor) \leq a(k + 1)/4(log(k + 1)/4) + b$.

    $b = 1$, Combine both inequality, $T(k + 1) = T(\lfloor (k + 1)/2 \rfloor) + 2T(\lfloor (k + 1)/4 \rfloor) + k + 1 \leq a(k + 1)/2(log(k + 1)/2) + b + 2 * (a(k + 1)/4(log(k + 1)/4) + b) + k + 1$.

Work on the right hand side, $a(k+1)/2(log(k+1)/2)+b+2*(a(k+1)/4(log(k+1)/4)+b)+k+1 = a(k+1)/2(log(k+1)/2)+(a(k+1)/2(log(k+1)/4)+3b+k+1 = a(k+1)/2((log(k+1)/2)+(log(k+1)/4))+3+k+1$.

Since $log(a/b) = loga-logb$ and $log(ab) = loga+logb$, $a(k+1)/2((log(k+1)/2)+(log(k+1)/4)) = a(k+1)/2(2*(log(k+1))-log2-log4)$.

According to piazza, log can be base on 2, $a(k+1)/2(2*(log(k+1))-log2-log4) = a(k+1)log(k+1)-a(k+1)/2*(log2+log4) = a(k+1)log(k+1)-a(k+1)/2*3$.

Now we can see that $a(k+1)/2((log(k+1)/2)+(log(k+1)/4))+3+k+1 = a(k+1)log(k+1)-a(k+1)/2*3+3+(k+1) = (k+1)log(k+1)-(k+1)*(3/2)+(k+1)+3 = (k+1)log(k+1)-(k+1)/2+3$.

$k \geq 4$, $(k+1)/2 > 2$, $3-(k+1)/2 < 1$. Hence, $(k+1)log(k+1)-(k+1)/2+3 < (k+1)log(k+1)+1 = a(k+1)log(k+1)+1$, then we can conclude that $T(k+1) < a(k+1)log(k+1)+b$.

Combine both cases, we proved that $T(n) \leq anlogn+b$ for $n \geq 1$ when choosing a = 1, b = 1.

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