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Homework 0 Problem 3

3. Consider the set of strings $L \subseteq \{\mathtt{0}, \mathtt{1}\}^*$ defined recursively as follows:
   - The string $\mathtt{1}$ is in L.
   - For any string x in L, the string $\mathtt{0}x$ is also in L.
   - For any string x in L, the string $x\mathtt{0}$ is also in L.
   - For any strings x and y in L, the string $x\mathtt{1}y$ is also in L.
   - These are the only strings in L.

   (a) Prove by induction that every string $w \in L$ contains an odd number of $\mathtt{1}$s.

   (b) Is every string w that contains an odd number of $\mathtt{1}$s in L? In either case prove your answer.

   Let $\#(a, w)$ denote the number of times symbol a appears in string w; for example,

$$\#(\mathtt{0}, \mathtt{101110101101011}) = 5 \quad \text{and} \quad \#(\mathtt{1}, \mathtt{101110101101011}) = 10.$$

   You may assume without proof that $\#(a, uv) = \#(a, u) + \#(a, v)$ for any symbol a and any strings u and v, or any other result proved in class, in lab, or in the lecture notes. Otherwise, your proofs must be formal and self-contained.

Solution:

(a) Proof: By induction on the length of w

- Base case: When the length of w is one. 1 is the only element of L and thus L has odd number of 1s. The base case holds.
- Induction: Suppose the statement is true for ws with length from 1 to k. we need to show it is true for w with length of $k+1$. There are two cases to have w with length of $k+1$:
  - We already have a $w \in L$ with length of k. Then by the hypothesis, w now has odd number of 1s. Then we may have $w0 \in L$ or $0w \in L$ with length of $k+1$. For both $w0$ and $0w$, they have odd number of 1s from w, and thus satisfy the statement.
  - We have $w_1 \in L$ and $w_2 \in L$ with length of k_1 and k_2, where $k_1 + k_2 = k$. We create a new w with a length of $k+1$, we can have $w_{new} = w_1 1 w_2$. Then the number of 1s in w_{new} is $odd + 1 + odd$ and thus odd. Hence, the statement holds for this case.

  Since the induction holds for both cases, the statement is true.

Therefore, every string $w \in L$ contains an odd number of $\mathtt{1}$s.

(b) Proof:

- Base case: w contains one 1. If w is a single, then by the definition, $w \in L$. Since we can recursively concatenating 0s to the front or tail of the strings concatenated from $w =$ "1" to get a $w_{new} \in L$, according with the definition pf L. $w \in L$ for all ws formed in this way. Thus the base case holds.
- Induction: Suppose it holds for w with $k = 1, 3, 5, ..., 2i - 1$ numbers of 1s, $i \in Z$. We need to show it is true for $k = 2i + 1$. Let m represent the string with $2i - 1$ 1s. By the hypothesis, $m \in L$. Then let n represent the string with a single "1". By the definition, $n \in L$. Thus from the definition we can have $m1n \in L$ and $n1m \in L$ and these are the only combinations to have from w with $k = 2i - 1$. Then in both $m1n$ and $n1m$, we have $2i - 1 + 1 + 1 = 2i + 1$ "1"s and thus the induction holds.

Therefore, every string w that contains an odd number of **1**s in L.

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