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Homework 2 Problem 2

Zhe Zhang (zzhan157@illinois.edu)
Ray Ying (xinruiy2@illinois.edu)
Anqi Yao (anqiyao2@illinois.edu)

3.In Lab 3 we considered the language $delete\mathbf{1}(L) = \{xy \mid x\mathbf{1}y \in L\}$. Intuitively, $delete\mathbf{1}(L)$ is the set of all strings that can be obtained from strings in L by deleting exactly one $\mathbf{1}$. For example, if $L = \{\mathbf{101101}, \mathbf{00}, \varepsilon\}$, then $delete\mathbf{1}(L) = \{\mathbf{01101}, \mathbf{10101}, \mathbf{10110}\}$. We argued that if L is regular then $delete\mathbf{1}(L)$ is also regular and the proof strategy was as follows: given a DFA M such that $L = L(M)$, construct an NFA N such that $L(N) = delete\mathbf{1}(L)$. Here we consider a different proof technique. Let r be a regular expression. We will develop an algorithm that given r constructs a regular expression r' such that $L(r') = delete\mathbf{1}(L(r))$. Assume $\Sigma = \{\mathbf{0}, \mathbf{1}\}$.

1. For each of the base cases of regular expressions $\emptyset, \epsilon$ and $\{a\}, a \in \Sigma$ describe a regular expression for $delete\mathbf{1}(L(r))$.
2. Suppose r_1 and r_2 are regular expressions, and r_1' and r_2' are regular expressions for the languages $delete\mathbf{1}(L(r_1))$ and $delete\mathbf{1}(L(r_2))$ respectively. Describe a regular expression for the language $delete\mathbf{1}(L(r_1 + r_2))$ using r_1, r_2, r_1', r_2'.
   Briefly justify the correctness of your construction. The argument should take the form of proving $L_1 = L_2$ by showing that $L_1 \subseteq L_2$ and $L_2 \subseteq L_1$.
3. Same as the previous part but now consider $L(r_1 r_2)$. This is a bit more tricky than the previous part.
4. Same as the previous part but now consider $L((r_1)^*)$.
5. Apply your construction to the regular expression $r = 0^* + (01)^* + 011^*0$ to obtain a regular expression for the language $delete\mathbf{1}(L(r))$.

Solution: $\vee$

1. (a) Suppose r is empty set $\emptyset$ so that r represents the empty language. Deleting exactly one 1 from the empty set results in an empty set. Thus $delete\mathbf{1}(L(r))$ is represented with $\emptyset$.

   (b) Suppose $r = \epsilon$ then deleting exactly one 1 from the empty string results in an empty set. Thus $delete\mathbf{1}(L(r))$ is represented with $\emptyset$.

   (c) Suppose $r = a$ then there are two possibilities: $a = 0$ and $a = 1$. If $a = 0$ then deleting exactly one 1 from 0 results in an empty set. If $a = 1$ then deleting exactly one 1 results in an empty string. Thus $delete\mathbf{1}(L(r))$ is represented with $\emptyset$ when $a = 0$ and ϵ when $a = 1$.

2. Since r_1 and r_2 are regular expressions, $r = r_1 + r_2$ is also a regular expression and denotes $L(r) = L(r_1) \cup L(r_2)$. Since r_1' and r_2' are regular expressions for the languages $delete\mathbf{1}(L(r_1))$ and $delete\mathbf{1}(L(r_2))$, we have $r' = r_1' + r_2'$. For every string $w \in L(r)$, then $w \in L(r_1)$ or $w \in L(r_2)$. If we delete exactly one 1 from the string w, the new string is in either $L(r_1')$ or $L(r_2')$. Hence $L(r_1' + r_2') \subseteq delete\mathbf{1}(L(r_1 + r_2))$ (A)

Now let's think from the other side. Let the string $w' \in delete\mathbf{1}(L(r))$, then there exists the string $w \in L(r)$ such that w' is the result of deleting exactly one 1 from the string w. Since $r = r_1 + r_2$, ($w \in L(r_1)$ so that $w' \in L(r_1')$) or ($w \in L(r_2)$ so that $w' \in L(r_2')$). In both cases, $w' \in L(r_1' + r_2')$ so that $delete\mathbf{1}(L(r_1 + r_2)) \subseteq L(r_1' + r_2')$ (B)

Based on the equations (A) and (B), we conclude that $delete\mathbf{1}(L(r)) = L(r_1' + r_2')$. Thus $delete\mathbf{1}(L(r_1 + r_2))$ is represented with the regular expression $r_1' + r_2'$.

3. Since r_1 and r_2 are regular expressions, $r = r_1 r_2$ is also a regular expression. Since r_1' and r_2' are regular expressions for the languages $delete\mathbf{1}(L(r_1))$ and $delete\mathbf{1}(L(r_2))$, we have $r' = r_1' r_2 + r_1 r_2'$. For every string $w = xy \in L(r)$, we assume that x corresponds to r_1 and y corresponds to r_2 then $x \in L(r_1)$ and $y \in L(r_2)$. If we delete exactly one 1 from the string w, it will delete from either the string x or string y. (1) If it delete from the string x, then $w' = x'y$ where $x' \in L(r_1')$ so that $w' \in L(r_1' r_2)$. (2) If it delete from the string y, then $w' = xy'$ where $y' \in L(r_2')$ so that $w' \in L(r_1 r_2')$. According to these two conditions, it is safe to say that $w' \in L(r_1' r_2 + r_1 r_2')$. Hence $delete\mathbf{1}(L(r_1 r_2)) \subseteq L(r_1' r_2 + r_1 r_2')$ (A)

   Now let's think from the other side. Let the string $w' \in L(r_1' r_2 + r_1 r_2')$, then either $w' \in L(r_1' r_2)$ or $w' \in L(r_1 r_2')$. (1) If $w' \in L(r_1' r_2)$, then $w' = x'y$ where $x' \in L(r_1')$ and $y \in L(r_2)$. w' is the result of deleting exactly one 1 from the string x, which is also the result of deleting exactly one 1 from the string w. (2) If $w' \in L(r_1 r_2')$, then $w' = xy'$ where $x \in L(r_1)$ and $y' \in L(r_2')$. w' is the result of deleting exactly one 1 from the string y, which is also the result of deleting exactly one 1 from the string w. Hence $L(r_1' r_2 + r_1 r_2') \subseteq delete\mathbf{1}(L(r_1 r_2))$ (B)

   Based on the equations (A) and (B), we conclude that $delete\mathbf{1}(L(r)) = L(r_1' r_2 + r_1 r_2')$. Thus $delete\mathbf{1}(L(r_1 r_2))$ is represented with the regular expression $r_1' r_2 + r_1 r_2'$.

4. Since r_1 is regular expressions, $r = r_1^*$ is also a regular expression. Since r_1' is regular expressions for the languages $delete\mathbf{1}(L(r_1))$, we have $r' = r_1^* r_1' r_1^*$. Let the string $w \in L(r_1^*)$, then there are strings $x_1, x_2, ..., x_n \in L(r_1)$ such that $w = x_1 x_2 ... x_n$. $w' = x_1 x_2 ... x_{i-1} x_i' x_{i+1} ... x_n$ is the result of deleting exactly one 1 from the string w. Since $x_1 x_2 ... x_{i-1} \in L(r_1^*)$, x_i' $L(r_1')$ and $x_{i+1} ... x_n \in L(r_1^*)$, $w' = x_1 x_2 ... x_{i-1} x_i' x_{i+1} ... x_n \in L(r_1^* r_1' r_1^*)$. Hence $delete\mathbf{1}(L(r_1^*)) \subseteq L(r_1^* r_1' r_1^*)$ (A)

   Now let's think from the other side. If $w' \in L(r_1^* r_1' r_1^*)$, then $w' = xy'z$ where $x \in L(r_1^*)$, $y \in L(r_1')$ and $z \in L(r_1^*)$. The string y' is the result of deleting exactly one 1 from the string y where $y \in L(r_1^*)$ so that $w' = xy'z$ is the result of deleting an exactly one 1 from w where $w = xyz$. Hence $L(r_1^* r_1' r_1^*) \subseteq delete\mathbf{1}(L(r_1^*))$ (B)

   Based on the equations (A) and (B), we conclude that $delete\mathbf{1}(L(r)) = L(r_1^* r_1' r_1^*)$. Thus $delete\mathbf{1}(L(r_1^*))$ is represented with the regular expression $r_1^* r_1' r_1^*$.

5.

$$\begin{aligned}
delete\mathbf{1}(L(r)) &= delete\mathbf{1}(L(0^* + (01)^* + 011^*0)) && (1)\\
&= delete\mathbf{1}(L(0^*)) + delete\mathbf{1}(L((01)^*)) + delete\mathbf{1}(L(011^*0)) && (2)\\
&= delete\mathbf{1}(L(0^*)) + delete\mathbf{1}(L((01)^*)) + delete\mathbf{1}(L(01 \times 1^*0)) && (3)\\
&= \emptyset + (01)^*0(01)^* + 01^*0 + 01(1^*1^*0 + \emptyset) && (4)\\
&= (01)^*0(01)^* + 01^*0 + 011^*0 && (5)
\end{aligned}$$

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