# HW 3: Extra problems

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The following problems are not for submission or grading. No solutions for them will be provided but you can discuss them on Piazza (however, some of them already contain a solution).

**1** Suppose $N_1 = (Q_1, \Sigma, \delta_1, s_1, A_1)$ and $N_2 = (Q_2, \Sigma, \delta_2, s_2, A_2)$ are NFAs. Formally describe a DFA that accepts the language $L(N_1) \setminus L(N_2)$. This combines subset construction and product construction to give you practice with formalism.

**2** Suppose $M = (Q, \Sigma, \delta, s, A)$ is a DFA. For states $p, q \in Q$ ($p$ can be same as $q$) argue that $L_{p,q} = \{w \mid \delta^*(p, w) = q\}$ is regular. Recall that $\text{PREFIX}(L) = \{w \mid wx \in L, x \in \Sigma^*\}$ is the set of all prefixes of strings in $L$. Express $\text{PREFIX}(L(M))$ as $\cup_{q \in Z} L_{s,q}$ for a suitable set of states $Z \subseteq Q$. Why does this prove that $\text{PREFIX}(L(M))$ is regular whenever $L$ is regular?

**3** For a language $L$ let $\text{MID}(L) = \{w \mid xwy \in L, x, y \in \Sigma^*\}$. Prove that $\text{MID}(L)$ is regular if $L$ is regular.

**4**

1. Draw an NFA that accepts the language $\{w \mid$ there is exactly one block of 0s of even length$\}$. (A "block of 0s" is a maximal substring of 0s.)
2. (a) Draw an NFA for the regular expression $(010)^* + (01)^* + 0^*$.
   (b) Now using the powerset construction (also called the subset construction), design a DFA for the same language. Label the states of your DFA with names that are sets of states of your NFA.

**5** This problem is to illustrate proofs of (the many) closure properties of regular languages.

1. For a language $L$ let $\text{FUNKY}(L) = \{w \mid w \in L$ but no proper prefix of $w$ is in $L\}$. Prove that if $L$ is regular then $\text{FUNKY}(L)$ is also regular using the following technique. Let $M = (Q, \Sigma, \delta, s, A)$ be a DFA accepting $L$. Describe a NFA $N$ in terms of $M$ that accepts $\text{FUNKY}(L)$. Explain the construction of your NFA.
2. In Lab 3 we saw that $\mathit{insert1}(L)$ is regular whenever $L$ is regular. Here we consider a different proof technique. Let $r$ be a regular expression. We would like to show that there is another regular expression $r'$ such that $L(r') = \mathit{insert1}(L(r))$.
   (a) For each of the base cases of regular expressions $\emptyset, \epsilon$ and $\{a\}, a \in \Sigma$ describe a regular expression for $\mathit{insert1}(L(r))$.
   (b) Suppose $r_1$ and $r_2$ are regular expressions, and $r'_1$ and $r'_2$ are regular expressions for the languages $\mathit{insert1}(L(r_1))$ and $\mathit{insert1}(L(r_2))$ respectively. Describe a regular expression for the language $\mathit{insert1}(L(r_1 + r_2))$ using $r_1, r_2, r'_1, r'_2$.
   (c) Same as the previous part but now consider $L(r_1 r_2)$.
   (d) Same as the previous part but now consider $L((r_1)^*)$.

**6** Recall that for any language $L$, $\overline{L} = \Sigma^* - L$ is the complement of $L$. In particular, for any NFA $N$, $\overline{L(N)}$ is the complement of $L(N)$.

Let $N = (Q, \Sigma, \delta, s, A)$ be an NFA, and define the NFA $N_{\text{comp}} = (Q, \Sigma, \delta, s, Q \setminus A)$. In other words we simply complemented the accepting states of $N$ to obtain $N_{\text{comp}}$. Note that if $M$ is DFA then $M_{\text{comp}}$ accepts $\Sigma^* - L(M)$. However things are trickier with NFAs.

1. Describe a concrete example of a machine $N$ to show that $L(N_{\text{comp}}) \neq \overline{L(N)}$. You need to explain for your machine $N$ what $\overline{L(N)}$ and $L(N_{\text{comp}})$ are.
2. Define an NFA that accepts $\overline{L(N)} - L(N_{\text{comp}})$, and explain how it works.
3. Define an NFA that accepts $L(N_{\text{comp}}) - \overline{L(N)}$, and explain how it works.

*Hint:* For all three parts it is useful to classify strings in $\Sigma^*$ based on whether $N$ takes them to accepting and non-accepting states from $s$.

**7** Let $L$ be an arbitrary regular language. Prove that the language $\mathit{half}(L) := \{w \mid ww \in L\}$ is also regular.

## Solution:

Let $M = (\Sigma, Q, s, A, \delta)$ be an arbitrary DFA that accepts $L$. We define a new NFA $M' = (\Sigma, Q', s', A', \delta')$ with $\varepsilon$-transitions that accepts $\mathit{half}(L)$, as follows:

$$\begin{aligned} Q' &= (Q \times Q \times Q) \cup \{s'\} \\ s' &\text{ is an explicit state in } Q' \\ A' &= \{(h, h, q) \mid h \in Q \text{ and } q \in A\} \\ \delta'(s', \varepsilon) &= \{(s, h, h) \mid h \in Q\} \\ \delta'((p, h, q), a) &= \{(\delta(p, a), h, \delta(q, a))\} \end{aligned}$$

$M'$ reads its input string $w$ and simulates $M$ reading the input string $ww$. Specifically, $M'$ simultaneously simulates two copies of $M$, one reading the left half of $ww$ starting at the usual start state $s$, and the other reading the right half of $ww$ starting at some intermediate state $h$.

- The new start state $s'$ non-deterministically guesses the "halfway" state $h = \delta^*(s, w)$ without reading any input; this is the only non-determinism in $M'$.
- State $(p, h, q)$ means the following:
  - The left copy of $M$ (which started at state $s$) is now in state $p$.
  - The initial guess for the halfway state is $h$.
  - The right copy of $M$ (which started at state $h$) is now in state $q$.
- $M'$ accepts if and only if the left copy of $M$ ends at state $h$ (so the initial non-deterministic guess $h = \delta^*(s, w)$ was correct) and the right copy of $M$ ends in an accepting state.

*Rubric:* 5 points =

+ 1 for a formal, complete, and unambiguous description of a DFA or NFA
  - No points for the rest of the problem if this is missing.
+ 3 for a correct NFA
  - −1 for a single mistake in the description (for example a typo)
+ 1 for a *brief* English justification. We explicitly do *not* want a formal proof of correctness, but we do want one or two sentences explaining how the NFA works.

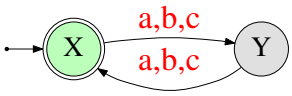
## 8 (100 PTS.) Codes.

Let $\Sigma$ be finite alphabet. A ***code*** is a mapping $f : \Sigma \to \{0,1\}^+$. For example, if $\Sigma = \{a, b, c\}$, a code $f$ might be $f(a) = 00010$, $f(b) = 000$, and $f(c) = 1$. (To simplify things, we assume that $f(a) \neq \varepsilon$, for any character $a \in \Sigma$.)

For a string $w_1 w_2 \cdots w_m \in \Sigma^*$, we define $f(w) = f(w_1)f(w_2)\cdots f(w_m)$. In the above code, we have

$$f(abcba) = 00010 \bullet 000 \bullet 1 \bullet 000 \bullet 00010. = 000100000100000010.$$

**8.A.** (10 PTS.) Let $L$ be the language of the following DFA $M$. What is $L$?




**8.B.** (20 PTS.) Working directly on the DFA $M$ from (A) construct an NFA for the language $f(L)$. Here $f(L) = \{f(w) \mid w \in L\}$ is the *code language*. Where $f$ is code from the above example.

**8.C.** (30 PTS.) Let $L \subseteq \Sigma^*$ be am arbitrary regular language. Prove that the encoded language $f(L) = \{f(w) \mid w \in L\}$ is regular.

Specifically, given a DFA $M = (Q, \Sigma, \delta, s, A)$ for $L$, describe how to build an NFA $M'$ for $f(L)$. Give an upper bound on the number of states of $M'$.

(I.e., You need to prove the correctness of your construction – that the language of the constructed NFA is indeed the desired language $f(L)$.)

(Rubric: Half the credit is for a correct construction, and the other half is for a correct proof of correctness.)

**8.D.** (40 PTS.) Let $L \subseteq \{0,1\}^*$ be a regular language. Consider the decoded language $L_f = \{w \in \Sigma^* \mid f(w) \in L\}$.

Prove that $L_f$ is a regular language. As above, given a DFA $M$ for $L$, describe how to construct an NFA for $L_f$.

(Rubric: Half the credit is for a correct construction, and the other half is for a correct proof of correctness.)