CS/ECE 374 FALL 2018
Homework 3 Problem 2

Zhe Zhang (zzhan157@illinois.edu)
Ray Ying (xinruiy2@illinois.edu)
Anqi Yao (anqiyao2@illinois.edu)

Describe a context free grammar for the following languages. Clearly explain how they work and the role of each non-terminal. Unclear grammars will receive little to no credit.

1. $\{a^i b^j c^k \mid k = 3(i+j)\}$.
2. $\{a^i b^j c^k d^\ell \mid i, j, k, \ell \geq 0 \text{ and } i + \ell = j + k\}$.
3. $L = \{0,1\}^* \setminus \{0^n 1^{2n} \mid n \geq 0\}$. In other words the complement of the language $\{0^n 1^{2n} \mid n \geq 0\}$.

Solution:

1.
$$S_0 \rightarrow aS_0ccc|S_1 \qquad (1)$$
$$S_1 \rightarrow bS_1ccc|\epsilon \qquad (2)$$

- We have to make the number of c three times of number of a and number of b combined. Since a is outer than the location of b, we need a and c to wrap b and c.
  - $L(S_0) = \{a^i b^j c^k \mid k = 3(i+j)\}$
  - $L(S_1) = \{b^i c^k \mid k = 3i\}$

2. There are two cases to consider:

(1) If $i \geq j$, the number of b's is at most as the number of a's in the string. Then we could represent i with $j + n$ where n is a non-negative integer. Since $i + \ell = j + k$, we get $j + n + \ell = j + k$ so that $n + \ell = k$. Then we rewrite the original form of the string $a^i b^j c^k d^\ell$ by substituting j with $i - n$ and k with $n + l$, and we get $a^i b^{i-n} c^{n+\ell} d^\ell = a^i b^{i-n} c^n c^\ell d^\ell$. The grammar to generate the string $a^i b^{i-n} c^n$ are:

$$S_0 \rightarrow aS_0c \mid S_1$$

$$S_1 \rightarrow aS_1b \mid \epsilon$$

since the number of a's is equal to the number of b's plus the number of c's. Every time we get the a or b, we also get the c. The grammar to generate the string $c^\ell d^\ell$ is:

$$S_2 \rightarrow cS_2d \mid \epsilon$$

since the number of c's is equal to the number of d's. Every time we get the c, we also get the d. Hence the grammar to generate the string $a^i b^{i-n} c^n c^\ell d^\ell$ is:

$$A \rightarrow S_0S_2$$

(2) If $i \leq j$, the number of a's is at most as the number of b's in the string. Then we could represent j with $i+n$ where n is a non-negative integer. Since $i+\ell = j+k$, we get $i+\ell = i+n+k$ so that $\ell = n+k$. Then we rewrite the original form of the string $a^i b^j c^k d^\ell$ by substituting j with $i+n$ and l with $n+k$, and we get $a^i b^{i+n} c^k d^{n+k} = a^i b^i b^n c^k d^{n+k}$. The grammar to generate the string $a^i b^i$ is:

$$S_1 \to aS_1b \mid \epsilon$$

since the number of a's is equal to the number of b's. Every time we get the a, we also get the b. The grammar to generate the string $b^n c^k d^{n+k}$ is:

$$S_3 \to bS_3d \mid S_2$$

$$S_2 \to cS_2d \mid \epsilon$$

since the number of d's is equal to the number of b's plus the number of c's. Every time we get the b or c, we also get the d. Hence the grammar to generate the string $a^i b^i b^n c^k d^{n+k}$ is:

$$B \to S_1S_3$$

Therefore, the CFG for the language $\{a^i b^j c^k d^\ell \mid i,j,k,\ell \geq 0 \text{ and } i+\ell = j+k\}$ are:

$$S \to A \mid B \qquad \{a^i b^j c^k d^\ell \mid i,j,k,\ell \geq 0 \text{ and } i+\ell = j+k\}$$

$$A \to S_0S_2 \qquad \{a^i b^j c^k d^\ell \mid i,j,k,\ell \geq 0 \text{ and } i+\ell = j+k \; and \; i \geq j \;\; (k \geq \ell)\}$$

$$B \to S_1S_3 \qquad \{a^i b^j c^k d^\ell \mid i,j,k,\ell \geq 0 \text{ and } i+\ell = j+k \; and \; i \leq j \;\; (k \leq \ell)\}$$

$$S_0 \to aS_0c \mid S_1 \qquad \{a^i b^j c^k \mid i,j,k \geq 0 \text{ and } i+\ell = j+k \; and \; i \geq j \;\; (i \geq k)\}$$

$$S_1 \to aS_1b \mid \epsilon \qquad \{a^i b^j \mid i,j \geq 0 \text{ and } i+\ell = j+k \; and \; i = j\}$$

$$S_2 \to cS_2d \mid \epsilon \qquad \{c^k d^\ell \mid i,j,k,\ell \geq 0 \text{ and } i+\ell = j+k \; and \; k = \ell\}$$

$$S_3 \to bS_3d \mid S_2 \qquad \{b^j c^k d^\ell \mid j,k,\ell \geq 0 \text{ and } i+\ell = j+k \; and \; i \leq \ell \;\; (k \leq \ell)\}$$

3. We can write L as the union of two languages L_1 and L_2,
where $L_1 = 0^m1^n | n \neq 2m, m, n \geq 0$ and $L_2 = (0+1)^*10(0+1)^*$. Since L_2 is the complement of $0^*1^*1^*$, and L_1 is contained by L by definition of L, it is obvious that L is either in L_1 or L_2.
So we have the following constructions for the cfg:
S -> $S_0|S_1$ $\{0,1\}^* \setminus \{0^n1^{2n} \mid n \geq 0\}$
S_0 -> $0S_011|S_2|S_3\{0^m1^n | n \neq 2m, m, n \geq 0\}$
S_2 -> $0|0S_2$ $\quad 0^+$
S_3 -> $1|1S_3$ $\quad 1^+$
S_1 -> S_410S_4 $\quad (0+1)^*10(0+1)^*$
S_4 -> $\epsilon|0S_4|1S_4$ $\quad (0+1)^*$

■