CS/ECE 374 Fall 2018

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Problem 3 HW1

a) $M = (Q, \Sigma, \delta, s, A)$, where

$Q = \{(q_1, q_2, q_3, q_4) \mid q_1 \in Q_1, q_2 \in Q_2, q_3 \in Q_3, q_4 \in Q_4\}$;

$\Sigma = \Sigma$;

$\delta = Q \times \Sigma \to Q$, where $\delta((q_1, q_2, q_3, q_4), a) = (\delta_1(q_1, a), \delta_2(q_2, a), \delta_3(q_3, a), \delta_4(q_4, a))$;

$A = \{(q_1, q_2, q_3, q_4) \mid q_1 \in A_1, q_2 \in Q_2 - A_2, (q_3 \in Q_3 - A_3 \text{ or } q_4 \in A_4)\}$;

$s = (s_1, s_2, s_3, s_4)$.

b) To prove M is correct, is the same to prove the statement:

"$\forall w, \delta_M^*((s_1, s_2, s_3, s_4), w) \in A$ iff $w \in L$"

Proof. By Induction on $|w|$.

Base case: When $|w| = 0$ or $w = \varepsilon$, if $\delta_M^*(s, w)$ is accepted, then $w \in L$ by definition. ①

If $w \in L$, then $w \in (L_1 - L_2) \cap (L_4 \cup \bar{L}_3) = (L_1 \cap \bar{L}_2) \cap (\bar{L}_3 \cup L_4)$.

Thus we have

$$\begin{cases} w \in L_1 \\ w \in \bar{L}_2 \\ w \in \bar{L}_3 \text{ or } w \in L_4 \end{cases} \xleftrightarrow[\text{of } M_1, M_2, M_3, M_4]{\text{By the definition}} \begin{cases} \delta_1^*(s_1, w) \in A_1 \\ \delta_2^*(s_2, w) \in \bar{A}_2 = Q_2 - A_2 \\ \delta_3^*(s_3, w) \in Q_3 - A_3 \text{ or } \delta_4^*(s_4, w) \in A_4. \end{cases}$$

Then when $w = \varepsilon$, we have
$$\begin{cases} \delta_1^*(s_1, \varepsilon) = s_1 \in A_1, \\ \delta_2^*(s_2, \varepsilon) = s_2 \in Q_2 - A_2 \\ \delta_3^*(s_3, w) = s_3 \in Q_3 - A_3 \text{ or } \delta_4^*(s_4, \varepsilon) = s_4 \in A_4. \end{cases}$$

Since $A = \{(q_1, q_2, q_3, q_4) \mid q_1 \in A, q_2 \in Q_2 - A_2, q_3 \in Q_3 - A_3 \text{ or } q_4 \in A_4\}$,

(s_1, s_2, s_3, s_4) here satisfies A, thus $(s_1, s_2, s_3, s_4) \in A$.

Then we proved $\delta_M^*(s, \varepsilon = w) = (s_1, s_2, s_3, s_4) \in A$ ②.

Combine ① and ② we have proved the base case.

Induction: Assume that $\delta_M^*(s, w) \in A$ iff $w \in L$ for all w such that $|w| < i$.

Consider w such that $|w| = i$ for $i > 0$.

Without loss of generality, we can assume that $w = ua$, where $a \in \Sigma$ and $u \in \Sigma^{i-1}$.

It is obvious if $\delta_M^*(s, w)$ is accepted, $w \in L$ by the definition of $L(M)$. ③

Then we need to show if $w \in L$, $\delta_M^*(s, w) \in A$.

If $w \in L \Leftrightarrow w \in (L_1 - L_2) \cap (L_4 \cup \overline{L_3}) \Leftrightarrow \begin{cases} w \in L_1 \\ w \notin L_2 \\ w \in L_4 \text{ or } w \notin L_3 . \end{cases}$

$$\Leftrightarrow \begin{cases} ua \in L_1 \\ ua \notin L_2 \\ ua \in L_4 \text{ or } ua \notin L_3 . \end{cases}$$

By the definition of M_1, M_2, M_3, M_4 : $\Leftrightarrow \begin{cases} \delta_1^*(s_1, ua) \in A_1 & ① \\ \delta_2^*(s_2, ua) \in \bar{A}_2 & ② \\ \delta_3^*(s_3, ua) \in \bar{A}_3 \text{ or } \delta_4^*(s_4, ua) \in A_4 . & ③ \end{cases}$

From the induction hypothesis, $\delta^*(s, u) \in A \Leftrightarrow u \in L$ since $|u| < i$.

Then by $u \in L$, we have $u \in (L_1 - L_2) \cap (L_4 \cup \overline{L_3})$, which is

$$\begin{cases} u \in L_1 \\ u \notin L_2 \\ u \notin L_3 \text{ or } u \in L_4 \end{cases} \Leftrightarrow \begin{cases} \delta_1^*(s_1, u) \in A_1 \subseteq Q_1 , \\ \delta_2^*(s_2, u) \in \bar{A}_2 \subseteq Q_2 \\ \delta_3^*(s_3, u) \in \bar{A}_3 \subseteq Q_3 \text{ or } \delta_4^*(s_4, u) \in A_4 \subseteq Q . \end{cases}$$

Then we can have from ① that $\delta_1^*(s_1, ua) = \delta_1^*(\delta_1^*(s_1, u), a) \in A_1$.

Since $\delta_1^*(s_1, u) \in Q_1$, and $\delta_1^*(\delta_1^*(s_1, u), a)$ is accepted, we can have $a \in L_1$.

From ② we have $\delta_2^*(s_2, ua) = \delta_2^*(\delta_2^*(s_2, u), a) \in \bar{A}_2$.

Since $\delta_2^*(s_2, u) \in Q_2$, by the definition of $L(M_2)$, $a \notin L_2$, $a \in \overline{L_2}$.

From ③, there are 3 cases given $\delta_3^*(s_3, ua) \in \bar{A}_3$ or $\delta_4^*(s_4, ua) \in A_4$.

Case 1 : $\delta_4^*(s_4, u) \in Q_4$ and $\delta_3^*(s_3, u) \in \bar{Q}_3$.

Then $a \in \overline{L_3}$ or $a \in L_4$, since $\delta_3^*(s_3, ua) = \delta_3^*(\delta_3^*(s_3, u), a)$ and same for δ_4^*.

Case 2 : $\delta_4^*(s_4, u) \notin Q_4$ and $\delta_3^*(s_3, u) \in \bar{Q}_3$. Then $a \in \overline{L_3}$ similarly.

Case 3 : $\delta_4^*(s_4, u) \in Q_4$ and $\delta^*(s_3, u) \notin \bar{Q}_3$. Then $a \in L_4$.

In either case, we have $a \in \overline{L_3}$ or L_4.

Thus we have $a \in L_1$ and $a \in \overline{L_2}$ and ($a \in \overline{L_3}$ or L_4), this is

$$(a \in L_1) \cap (a \in \overline{L_2}) \cap (a \in \overline{L_3} \cup L_4) = a \in (L_1 - L_2) \cap (L_4 \cup \overline{L_3}) = L .$$

Then $\delta^*(s, w) = \delta^*(s, ua) = \delta^*(\delta^*(s, u), a)$.

Since $\delta^*(s, u) \in A$ by the hypothesis and $A \subseteq Q$, $\delta^*(s, u) \subseteq Q$ and $a \in L$.

By the definition of $L(M)$, $\delta^*(s, w) = \delta^*(\delta^*(s, u), a) \in A$ ④

Combine ③, ④, we get $\delta^*(s, w) \in A$ iff $w \in L$ as needed.

Therefore, M is a correct construction. $\blacksquare$