Error Estimate and Varying Step Size for IVPs

10 points

In this problem, we are going to investigate how to automatically ("adaptively") choose the size of the time step when solving IVPs. Adaptivity means that based on the (usually) local error estimate, we find the largest step size such that the local error is within some tolerance. Aside from accuracy, stabilily also restricts the time step, but we will consider that beyond the scope of this problem.

Consider the following ODE to be of the (scalar) form

$$y'(t) = f(y(t)), \qquad y(t_0) = y_0.$$

Generalizations to systems of ODEs are straightforward, so we will not consider them here.

First, define the following notation:

- $y(t_k)$: the exact solution of the ODE at t_k
- y_k: the numerical solution of the ODE at t_k
- $u_k(t)$: a solution of the ODE passing through the point (t_k, y_k) (generally different from the numerical solution)
- h_k: time step size, $h_k = t_{k+1} - t_k$
- l_k: the local error, defined as $l_k = y_k - u_{k-1}(t_k)$. It is the ("local") error just for the current step.
- e_k: an estimate of l_k, so that $e_k = l_k + (\text{higher order terms})$
- tol: a scalar that's used as a tolerance

You will derive an estimation of local error that can be used to restrict the time step size. The *accuracy* of a numerical method is said to be "of order p" if $l_k = O(h_k^{p+1})$ (Note the off-by-one--that is intentional!) Suppose there are two available methods: method 1 with order p and method 2 with order p^*, where $p < p^*$. You may assume that p^* is an integer. In what follows, we will use a single numerical solution with values (y_k) and apply both methods for each step starting at (t_k, y_k). Answer the following questions:

1. Argue that $y_{k+1} - y^*_{k+1}$ can be used as e_{k+1}. Justify this argument. Denote the order in the higher order terms by $(p+i)$, where i is an integer you need to identify.
2. If $|e_k|$ exceeds tol, we reduce the step size from h_k to αh_k. What is the maximum value of α such that $|e_k| \le \text{tol}$? Give a closed-form expression for α depending on tol, the estimate $|e_k|$, and p.

Please submit your response to this written problem as a PDF file below. You may do either of the following:

- write your response out by hand, scan it, and upload it as a PDF.

  We will not accept unprocessed pictures taken with your phone.

  If you decide to use your phone for scanning, make sure to use an app such as CamScanner (https://www.camscanner.com/) to get a readable PDF. Alternatively, there's a fast and convenient scanner in the Engineering IT office in 2302 Siebel that can just email you a PDF. (It's the Fax-machine-looking thing--not the scanner that's attached to one of the computers.)

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  If you're looking for an easy-ish way to type math, check out TeXmacs (http://texmacs.org/) or LyX (http://www.lyx.org/). Both are installed in the virtual machine. (Under "Applications / Accessories / GNU TeXmacs editor" and "Applications / Office / LyX document processor" respectively.)

Submit your response to each problems in this homework as a separate PDF. If you have multiple PDFs that you need to merge into one, try PDF Split and Merge (http://www.pdfsam.org/download/).

NOTE: Please make sure your solutions are legible and easy to follow. If they are not, we may deduct up to five points *per problem*.

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- **Part A:**
  1. Begin by adding and subtracting $u_k(t_{k+1})$. Using the definition of l_{k+1}, l^*_{k+1}, we have

     $$\begin{aligned} y_{k+1} - y^*_{k+1} &= [u_k(t_{k+1}) - y^*_{k+1}] - [u_k(t_{k+1}) - y_{k+1}] \\ &= l_{k+1} + l^*_{k+1} \\ &= l_{k+1} + O(h^{p^*+1}) \end{aligned}$$

     Since p and p^* are all integers, we can safely write $y_{k+1} - y^*_{k+1} = l_{k+1} + O(h^{p+2})$. We observe that the unknown term $u_k(t_{k+1})$ cancels out and the result is just l_{k+1} plus higer order terms—that's how we estimate the local error.
  2. If the method is of order p then $l_k = O(h_k^{p+1})$, (One can write $l_k = h^{p+1}\phi(t) + O(h^{p+2})$.) then by using αh_k, the local error becomes $l_k = \alpha^{p+1} O(h_k^{p+1})$. Asymptotically, e_k behaves the same as l_k(by ignoring the higher order terms). We thus have $\alpha^{p+1} e_k = tol$, then $\alpha = (\frac{tol}{|e_k|})^{\frac{1}{p+1}}$