Error Bound for Approximate Eigenvalues Using Lanczos Iteration

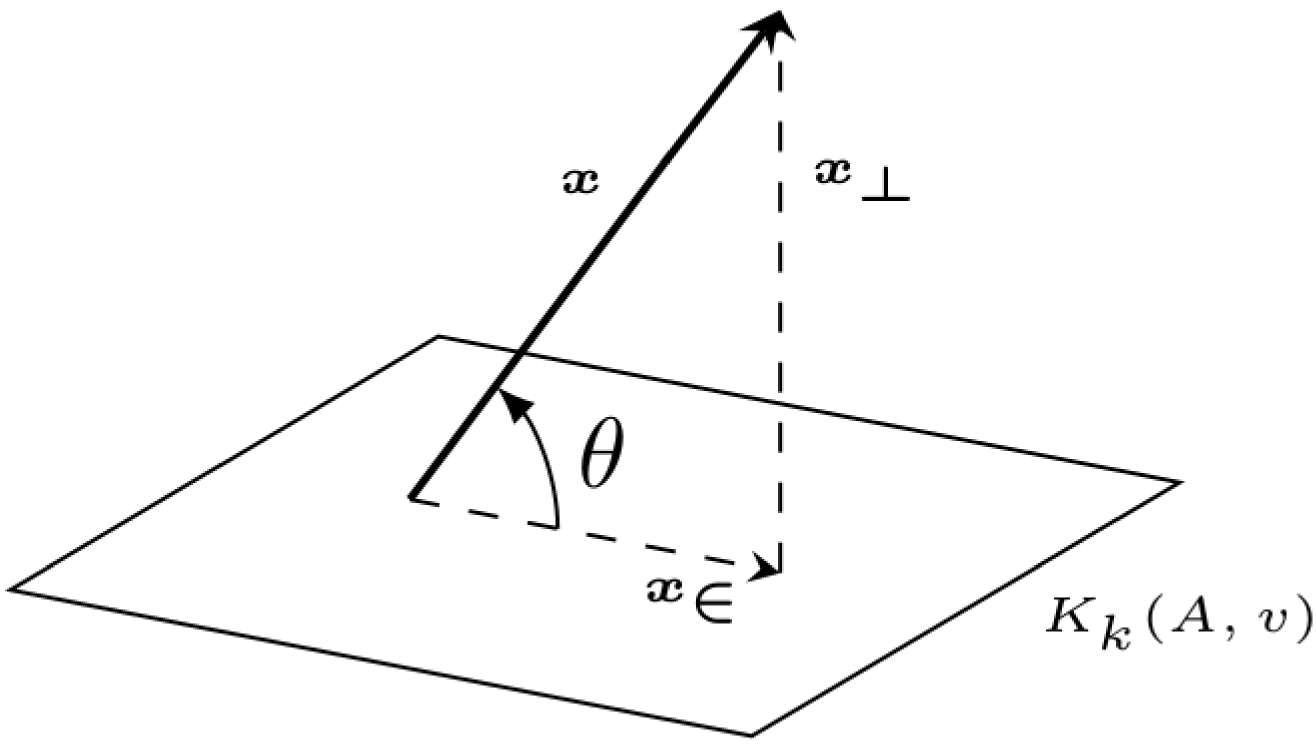
10 points

In this problem, we will examine, for a symmetric matrix A, an error bound on an eigenvalue obtained through Lanczos iteration, compared with a true eigenvalue of A. In particular, this error bound relates the angle of the projection of the true eigenvector associated with A to the closest eigenvector found in the Krylov subspace $(A, \boldsymbol{v})$, where $\boldsymbol{v}$ is some arbitrary vector. Let K_k represent the Krylov subspace formed by the symmetric matrix $A \in \mathbb{R}^{n\times n}$ with some vector, $\boldsymbol{v}$, such that $K_k = \operatorname{span}(\boldsymbol{v}, A\boldsymbol{v}, A^2\boldsymbol{v}, \ldots, A^{k-1}\boldsymbol{v})$. Furthermore, let the matrix $Q_k \in \mathbb{R}^{n\times k}$ be an orthogonal matrix that spans this same Krylov subspace that is the collection of vectors $(\boldsymbol{q}_1, \boldsymbol{q}_2, \ldots, \boldsymbol{q}_k)$.

Consider finding the eigenvalues of the matrix $T_k = Q_k^T A Q_k$. Let $A\boldsymbol{x} = \lambda\boldsymbol{x}$ and $\|\boldsymbol{x}\|_2 = 1$. In this problem, we will show that T_k has an eigenvalue μ such that

$$|\lambda - \mu| \leq \|A\|_2 \tan\theta,$$

where θ represents the angle between the eigenvector and the space spanned by the Krylov space.



1. Let $\boldsymbol{x}_\in = Q_k Q_k^T \boldsymbol{x}$ and $\boldsymbol{x}_\perp = (I - Q_k Q_k^T)\boldsymbol{x}$. Show that $A\boldsymbol{x} = A\boldsymbol{x}_\in + A\boldsymbol{x}_\perp$.
2. From Part 1, we can note that $A\boldsymbol{x}_\in + A\boldsymbol{x}_\perp = \lambda\boldsymbol{x}$. Using this, show that

$$T_k Q_k^T \boldsymbol{x} - \lambda Q_k^T \boldsymbol{x} = -Q_k^T A \boldsymbol{x}_\perp.$$

3. Let H be symmetric and $H\boldsymbol{z} - \mu\boldsymbol{z} = \boldsymbol{r}$ and $\boldsymbol{z} \neq 0$. Show that

$$\min_{\lambda \in eig(H)} |\lambda - \mu| \le \frac{\|\boldsymbol{r}\|_2}{\|\boldsymbol{z}\|_2}.$$

Hint: Consider the eigendecomposition of $H = U\Lambda U^T$, where Λ is a diagonal matrix of eigenvalues and U is a matrix of eigenvectors of H.

Hint 2: As an intermediate step, show that $\|\boldsymbol{r}\|_2 = \|(\Lambda - \mu I)U^T\boldsymbol{z}\|_2$.

4. Now, using the results from parts 2 and 3, prove that

$$|\lambda - \mu| \le \|A\|_2 \tan\theta.$$

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Your answer is correct.

1. We will approach this part by expanding out $A\boldsymbol{x}_\in + A\boldsymbol{x}_\perp$.

$$\begin{aligned} A\boldsymbol{x}_\in + A\boldsymbol{x}_\perp &= AQ_kQ_k^T\boldsymbol{x} + A(I - Q_kQ_k^T)\boldsymbol{x} \\ &= AQ_kQ_k^T\boldsymbol{x} + Ax - AQ_kQ_k^T\boldsymbol{x} \\ &= A\boldsymbol{x} \end{aligned}$$

Hence $A\boldsymbol{x} = A\boldsymbol{x}_\in + A\boldsymbol{x}_\perp$.

2. Starting from $A\boldsymbol{x}_\in + A\boldsymbol{x}_\perp = \lambda\boldsymbol{x}$, we can see that this is equivalent to $AQ_kQ_k^T\boldsymbol{x} + A\boldsymbol{x}_\perp = \lambda\boldsymbol{x}$. We can then multiply on the left Q_k^T to both sides to get $Q_k^TAQ_kQ_k^T\boldsymbol{x} + Q_k^TA\boldsymbol{x}_\perp = Q_k^T\lambda\boldsymbol{x}$. Then, using the fact that $T_k = Q_k^TAQ_k$, we now have

$$\begin{aligned} T_kQ_k^T\boldsymbol{x} + Q_k^TA\boldsymbol{x}_\perp &= Q_k^T\lambda\boldsymbol{x} \\ \Rightarrow T_kQ_k^T\boldsymbol{x} - \lambda Q_k^T\boldsymbol{x} &= -Q_k^TA\boldsymbol{x}_\perp. \end{aligned}$$

3. Since H is symmetric, it will have orthogonal eigevectors and we can write its eigendecomposition as $H = U\Lambda U^T$.

$$\begin{aligned} &(H - \mu I)\boldsymbol{z} = \boldsymbol{r} \\ \Rightarrow &(U\Lambda U^T - \mu I)\boldsymbol{z} = \boldsymbol{r} \\ \Rightarrow &U(\Lambda - \mu I)U^T\boldsymbol{z} = \boldsymbol{r} \\ \Rightarrow &(\Lambda - \mu I)U^T\boldsymbol{z} = U^T\boldsymbol{r} \end{aligned}$$

We can take the norm of both sides of this expression to obtain

$$\|\boldsymbol{r}\|_2 = \|(\Lambda - \mu I)U^T\boldsymbol{z}\|_2.$$

Because this equality holds for all eigenvalues λ of H, we can write that the right-hand side is bounded from below by the smallest difference between an eigenvalue λ and the approximate eigenvalue μ.

$$\Rightarrow \|\boldsymbol{r}\|_2 \geq \min_{\lambda \in \text{eig}(H)} |\lambda - \mu| \cdot \|\boldsymbol{z}\|_2$$

Therefore,

$$\min_{\lambda \in eig(H)} |\lambda - \mu| \leq \frac{\|\boldsymbol{r}\|_2}{\|\boldsymbol{z}\|_2}.$$

4. If we consider the result from part 2, we can let $\boldsymbol{z} = Q_k^T \boldsymbol{x}$. Since A is symmetric, $H = T_k$ will also be symmetric. We can now apply the statement in part 3 which gives us that

$$|\lambda - \mu| \leq \frac{\|Q_k^T A \boldsymbol{x}_\perp\|_2}{\|Q_k^T \boldsymbol{x}\|_2}.$$

From the diagram, we see that $\|\boldsymbol{x}_\in\|_2 = \cos\theta$ and $\|\boldsymbol{x}_\perp\|_2 = \sin\theta$. Furthermore, $\|\boldsymbol{x}_\in\|_2 = \|Q_k Q_k^T \boldsymbol{x}\|_2 = \|Q_k^T \boldsymbol{x}\|_2$, since Q_k is orthogonal. Therefore,

$$\begin{aligned} |\lambda - \mu| &\leq \frac{\|Q_k^T A \boldsymbol{x}_\perp\|_2}{\|Q_k^T \boldsymbol{x}\|_2} \\ &\leq \frac{\|A\|_2 \|Q_k^T\|_2 \|\boldsymbol{x}_\perp\|_2}{\|\boldsymbol{x}_\in\|_2} \\ &= \|A\|_2 \frac{\sin\theta}{\cos\theta} \\ &= \|A\|_2 \tan\theta \end{aligned}$$