CS/ECE 374 FALL 2018 Homework 10 Problem 1

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Consider the language

$$L_{374\mathrm{H}} = \{\langle M \rangle \mid M \text{ halts on at least 374 distinct input strings}\}.$$

Note that for $\langle M \rangle \in L_{374\mathrm{H}}$, it is not necessary for M to *accept* any string; it is sufficient for it to *halt* on (and possibly reject) 374 different strings. Prove that $L_{374\mathrm{H}}$ is undecidable.

Solution: For the sake of argument, suppose there is an algorithm DECIDE-L_{374H} that correctly decides the language L_{374H}. Then we can solve the halting problem as follows:

```
DECIDEHALT(⟨M,w⟩):
    Encode the following Turing machine M′:

        M′(x):
            run M on output w
            if string = "1","2",...,or "374"
                return True
            else
                return False

    if DECIDE-L374H(⟨M′⟩):
        return True
    else
        return False
```

We prove this reduction correct as follows:

$\Longrightarrow$

Suppose M halts on input w.
Then M' accepts every input string x.
In particular, M' halts on at least 374 distinct input strings.
So DECIDE-L_{374H} accepts the encoding $\langle M' \rangle$.
So DECIDEHALT correctly accepts the encoding ⟨M,w⟩.

$\Longleftarrow$

Suppose M does not halt on input w.
Then M' diverges on every input string x.
In particular, M' does not halt on at least 374 distinct input strings.
So DECIDE-L_{374H} rejects the encoding $\langle M' \rangle$.
So DECIDEHALT correctly rejects the encoding ⟨M,w⟩.

In both cases, DECIDEHALT is correct. But that's impossible, because HALT is undecidable. We conclude that the algorithm DECIDE-L_{374H} does not exist. ■