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Theory: Componentwise Relative Condition Number

10 points

Part 1. Given a matrix $\boldsymbol{A}$, assume each entry is known to the same relative precision (e.g. same round-off error), then we know $\hat{\boldsymbol{A}} = \boldsymbol{A} + \boldsymbol{\delta A}$ such that $\max_{i,j} |\delta a_{ij}/a_{ij}| \le \epsilon$.

Define $|\boldsymbol{A}|_{ij} = |\boldsymbol{A}_{ij}|$, (i.e. the absolute value of a matrix is that matrix with all of entries made positive). Show that the relative backward error with respect to $\boldsymbol{Ax} = \boldsymbol{b}$ associated with solving $\hat{\boldsymbol{A}}\hat{\boldsymbol{x}} = \boldsymbol{b}$ satisfies

$$\frac{\|\boldsymbol{x} - \hat{\boldsymbol{x}}\|_1}{\|\boldsymbol{x}\|_1} \le \epsilon \| \, |\boldsymbol{A}^{-1}| \cdot |\boldsymbol{A}| \, \|_1.$$

You may use without proof that

- $\| |\boldsymbol{z}| \| = \|\boldsymbol{z}\|$ (which holds for any p-norm),
- $\boldsymbol{\delta A \delta x} \approx \boldsymbol{0}$, where $\boldsymbol{\delta x} = \boldsymbol{x} - \hat{\boldsymbol{x}}$, and that
- $|\boldsymbol{A} \cdot \boldsymbol{B}| \le |\boldsymbol{A}| \cdot |\boldsymbol{B}|$.

NOTE: Please show all work.

Hint: Find expressions or estimates for $\boldsymbol{\delta x}$, $|\boldsymbol{\delta x}|$, then $\|\boldsymbol{\delta x}\|_1$.

Part 2. Call $\kappa_{CR}(\boldsymbol{A}) = \| \, |\boldsymbol{A}^{-1}| \cdot |\boldsymbol{A}| \, \|_1$ the componentwise relative condition number of $\boldsymbol{A}$. Prove that

$$\kappa_{CR}(\boldsymbol{DA}) = \kappa_{CR}(\boldsymbol{A}),$$

where $\boldsymbol{D}$ is an arbitrary diagonal matrix.

NOTE: Please show all work.

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Your answer is correct.

Part 1:

$$\hat{\boldsymbol{A}}\hat{\boldsymbol{x}} = \boldsymbol{b},$$
$$(\boldsymbol{A} + \boldsymbol{\delta A})(\boldsymbol{x} + \boldsymbol{\delta x}) = \boldsymbol{b}$$

since $\boldsymbol{\delta A \delta x}$ will be neglibile in magnitude, the above imples

$$
\begin{aligned}
\boldsymbol{A}\boldsymbol{\delta x} &= -\boldsymbol{\delta A x}, \\
\boldsymbol{\delta x} &= -\boldsymbol{A}^{-1}\boldsymbol{\delta A x}, \\
|\boldsymbol{\delta x}| &= |\boldsymbol{A}^{-1}\boldsymbol{\delta A x}| \\
&\leq |\boldsymbol{A}^{-1}| \cdot |\boldsymbol{\delta A}| \cdot |\boldsymbol{x}| \\
\|\boldsymbol{\delta x}\| &\leq \||\boldsymbol{A}^{-1}| \cdot |\boldsymbol{\delta A}| \cdot |\boldsymbol{x}|\|, \\
\frac{\|\boldsymbol{\delta x}\|}{\|\boldsymbol{x}\|} &\leq \||\boldsymbol{A}^{-1}| \cdot |\boldsymbol{\delta A}|\| \\
&\leq \epsilon \||\boldsymbol{A}^{-1}| \cdot |\boldsymbol{A}|\|
\end{aligned}
$$

Part 2: It suffices to observe that $|\boldsymbol{DA}|_{ij} = |d_{ii}a_{ij}| = |d_{ii}| \cdot |a_{ij}| = (|\boldsymbol{D}| \cdot |\boldsymbol{A}|)_{ij}$, then

$$
\begin{aligned}
\kappa_{CR}(\boldsymbol{DA}) &= \||\boldsymbol{A}^{-1}\boldsymbol{D}^{-1}| \cdot |\boldsymbol{DA}|\| \\
&= \||\boldsymbol{A}^{-1}| \cdot |\boldsymbol{D}^{-1}| \cdot |\boldsymbol{D}| \cdot |\boldsymbol{A}|\| \\
&= \||\boldsymbol{A}^{-1}| \cdot |\boldsymbol{D}|^{-1} \cdot |\boldsymbol{D}| \cdot |\boldsymbol{A}|\| \\
&= \||\boldsymbol{A}^{-1}| \cdot |\boldsymbol{A}|\| \\
&= \kappa_{CR}(\boldsymbol{A})
\end{aligned}
$$