Gaussian Quadrature

15 points

A Gaussian quadrature rule uses n unique points in an interval (a, b) ($a < b$ of course) to generate an approximation to an integral of an function over that interval; it is exact for polynomials of degree less than or equal to $2n - 1$. Complete the following derivation of these n Gauss points.

Let $p(x)$ be a real polynomial of degree n such that:

$$\int_a^b p(x)\, x^k\, dx = 0, \quad k = 0, \ldots, n-1.$$

- **Part A:** Show that the n zeros of $p(x)$ are real, simple (of multiplicity one), and lie in the open interval (a, b), by completing the following outline:

Outline of Proof for Part A (click to view)

- **Part B:** Show that the n-point interpolatory quadrature rule on $[a, b]$ whose nodes are the zeros of $p(x)$ has degree $2n - 1$.

  (*Note:* Recall that an interpolatory quadrature rule is one where the weights satisfy $w_i = \int_a^b \ell_i(x) dx$ where $\ell_i(x)$ is a Lagrange basis function)

Outline of Proof for Part B (click to view)

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Your answer is correct.

Part A

a) If f is continuous and $f(x) < 0$ on (a, b), then $-f(x)$ is continous and strictly positive on (a, b). Hence by applying the given theorem, we have $\int_a^b -f(x)dx > 0$. Since $\int_a^b -f(x)dx = -\int_a^b f(x)dx$, it follows that $\int_a^b f(x)dx < 0$.

b) If $p(x)$ had no root in (a,b), then either $p(x)>0$ or $p(x)<0$ for all x in (a,b). Hence, $\int_a^b p(x)dx$ would be either positive or negative. But $\int_a^b p(x)x^k dx = 0$ for $0 \leq k \leq n-1$. Taking $k=0$, we see that $\int_a^b p(x)dx = 0$, so p must have at least one root in this interval.

c) $Q(x)$ can be written $Q(x) = Q_{n-1}x^{n-1} + Q_{n-2}x^{n-2} + \dots Q_1x + Q_0$. Where each Q_i is a real number. So we have

$$\int_a^b p(x)Q(x)dx = Q_{n-1}\int_a^b p(x)x^{n-1}dx + Q_{n-2}\int_a^b p(x)x^{n-2}dx + \dots$$

$$+Q_1\int_a^b p(x)xdx + Q_0\int_a^b p(x)dx = 0+0+\dots+0+0 = 0$$

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d) Suppose $p(x)$ has a multiple root at x_i, i.e. $p(x) = Q(x)(x-x_i)^2$, where $Q(x)$ is a polynomial of degree $n-2$. We first suppose $Q(x)$ has no roots in (a,b), except possibly at x_i. Consider

$$\int_a^b p(x)Q(x)dx = \int_a^b Q^2(x)(x-x_i)^2dx =$$

$$\int_a^{xi} Q^2(x)(x-x_i)^2dx + \int_{xi}^b Q^2(x)(x-x_i)^2dx$$

Since $Q^2(x)(x-x_i)^2$ is strictly positive on both sub-intervals, each integral must be positive, so that the total integral is positive. However, this contradicts what was proven in the previous step. If $Q(x)$ has other roots in the interval (a,b) we can simply split the integral up into more subintervals, with endpoints corresponding to these roots and x_i; we will reach the same conclusion. In either case, we see that $p(x)$ cannot be written as $Q(x)(x-x_i)^2$, so it cannot have a root of multiplicity higher than one.

e) Suppose $p(x)$ has k roots in (a,b), $1 \leq k \leq n-1$, denoted by $x_1,\dots,x_2$. Since $p(x)$ can have no double roots, we can write $p(x) = Q(x)q_k(x)$, where $q_k(x) = (x-x_1)(x-x_2)\dots(x-x_k)$ and each x_i is not a root of $Q(x)$, a polynomial of degree $n-k$. Suppose $p(x)$ has no other roots in (a,b). Then $Q(x)$ has no roots in (a,b) and is either strictly positive or negative. If we integrate $p(x)$ against $q_k(x)$ we have:

$$\int_a^b p(x)q_k(x)dx = \int_a^b Q(x)q_k^2(x)dx =$$

$$\int_a^{x1} Q(x)q_k^2(x)dx + \int_{x1}^{x2} Q(x)q_k^2(x)dx + \dots$$

$$\dots + \int_{xk-1}^{xk} Q(x)q_k^2(x)dx + \int_{xk}^b Q(x)q_k^2(x)dx$$

Depending on the sign of $Q(x)$ each integrand is either strictly positive or negative on each subinterval. Hence all integrals must be positive, or all integrals must be negative; hence, the total integral is either positive or negative. This contradicts what was proven in part 3. Hence $Q(x)$ must have a root in (a,b), so that $p(x)$ has another root in (a,b).

Part B

a)

$$\int_a^b f(x)dx = \int_a^b (q(x)p(x) + r(x))dx =$$

$$\int_a^b p(x)q(x)dx + \int_a^b r(x)dx$$

By definition of $p(x)$, the first integral is zero, by what was proven in Part A, question 3, since the degree of $q(x)$ is less than or equal to $n-1$.

b) Since the degree of $r(x)$ is less than or equal to $n-1$, we can write it exactly as a sum of the n Lagrange basis functions based at n roots of $p(x)$. I.e.,

$$r(x) = \sum_{i=1}^{n} r(x_i)\ell_i(x)$$

Hence the integral can be written as:

$$\int_a^b r(x)dx = \int_a^b \sum_{i=1}^{n} r(x_i)\ell_i(x)dx = \sum_{i=1}^{n} r(x_i)\int_a^b \ell_i(x)dx$$

c) Since each x_i is a root of $p(x)$, we have:

$$f(x_i) = q(x_i)p(x_i) + r(x_i) = 0 + r(x_i) = r(x_i)$$

Putting this all together, we have:

$$\int_a^b f(x)dx = \int_a^b r(x)dx = \sum_{i=1}^{n} r(x_i)\int_a^b \ell_i(x)dx =$$

$$\sum_{i=1}^{n} f(x_i)\int_a^b \ell_i(x)dx$$

The final expression is exactly the expression for interpolatory quadrature. Hence, Gaussian quadrature coincides exactly with the true value of the integral.

Finally, consider the polynomial $p(x)p(x)$, i.e. take the polynomial that has these nodes as its roots, and square it. This is a polynomial of degree $2n$. We know that $p(x)p(x) > 0$ inbetween the nodes (the roots of p). So by splitting the total integrals into a sum of smaller integrals, we can show that

$$\int_a^b p^2(x)dx > 0$$

But the quadrature rule gives an approximation:

$$\sum_{i=1}^{n} w_i p(x_i)p(x_i) = \sum_{i=1}^{n} w_i * 0 = 0$$

which is clearly not exact. Hence the rule has degree $2n-1$.