CS 498 algorithm Spring 2019 Ray Ying (xinruiy2@illinois.edu)
Homework 1 Problem 1 Aditya Pillai (apillai4@illinois.edu)

Solution:

1. We know from the last two sentences that X is the random Variable donating the output value and α is the true average. By Chebyshev's Inequality,

$$Pr[|X-\alpha| \geq \epsilon] \leq \frac{Var(X)}{\epsilon^2}$$

However, we want to show that

$$Pr[|X-\alpha| \geq \epsilon] \leq \delta$$

Therefore, we only need to show that

$$\frac{Var(X)}{\epsilon^2} \leq \delta$$

δ cannot be 0 since δ can be denominator, we can transform the above inequality to

$$\frac{Var(X)}{\delta\epsilon^2} \leq 1$$

As we are given that

$$\frac{(b-a)^2}{\delta\epsilon^2} \leq k$$

We want to prove that $\frac{(b-a)^2}{k} \geq Var(X)$, as X is the mean of the all X_i where X_i are the height of people in the k sample, $X = \sum_i \frac{X_i}{k}$.

Based on the definition of $Var(X)$, since X_i and X_j where $i \neq j$, they are independent Variables, then $Var(X) = \sum Var(X_i/k)$.

We can prove that $Var(X_i) \leq (b-a)^2/4$. Since $X_i \leq b$, $X_i \leq b$ $\sum_i b \cdot X_i \geq \sum_i X_i^2$. So $Var(X_i) = E(X_i^2) - E(X_i)^2 \leq E(b \cdot X_i) - E(X_i)^2 = b \cdot E(X_i) - E(X_i)^2 = E(X_i)(b - E(X_i))$. Since $b-a \geq b - E(X_i)$, $Var(X_i) \leq E(X_i)(b-E(X_i)) \leq (b-a)^2/4$.

Therefore, $Var(X_i/k) \leq \frac{((b-a)/k)^2}{4}$ as $x_i/k \in [a/k, a/k]$, therefore, $Var(X) \leq k \cdot \frac{((b-a)/k)^2}{4}$
$\Longrightarrow$ Var(X) $\leq \frac{(b-a)^2}{4k} \Longrightarrow$
k $\cdot Var(X) = (b-a)^2/4$

Hence, we have $(b-a)^2 \geq k \cdot Var(X) \Longrightarrow \frac{Var(X)}{\delta\epsilon^2} \leq 1 \Longrightarrow Pr[|X-\alpha| \geq \epsilon] \leq \delta$.

2. For Chernoff's inequality, we have the general form:

$$Pr[|X-\alpha| \geq \epsilon] \leq 2 \cdot e^{(\frac{-\epsilon^2}{2k})}$$

However, for Chernoff's inequality, we need to normalize each X to the range $[-1,1]$. Also, X should be the total sum, therefore, $X = kX$, $\alpha = k\alpha$ and $\epsilon = k\epsilon$. We have to normalize it to satisfy the precondition, we want to assume that X and α is in the range of $[-1,1]$, then X would be some constant $z + 2/(b-a)$ and α would be $z + 2/(b-a)$, however, as we are taking the absolute value of the difference, the constant doesn't matter. So our Chernoff's inequality will be:

$$Pr[|k\frac{2}{(b-a)} \cdot X - k\frac{2}{(b-a)}\alpha| \geq \frac{2}{(b-a)}k\epsilon] \leq 2 \cdot e^{(\frac{-(\frac{2}{(b-a)}k\epsilon)^2}{2k})}$$

We will perform transformation on the left side, the whole inequation will become

$$Pr[|X-\alpha| \geq \epsilon] \leq 2 \cdot e^{(\frac{-(\frac{2}{(b-a)}k\epsilon)^2}{2k})}$$

Therefore, all we need to show is

$$\delta \geq 2 \cdot e^{(\frac{-2k\epsilon^2}{(b-a)^2})}$$

From the given condition of $k \geq \frac{c(b-a)^2 log(2/\delta)}{\epsilon^2}$, we can do some transformations on the inequality. $k \geq \frac{c(b-a)^2 log(2/\delta)}{\epsilon^2} \implies (\mathrm{k} \cdot \epsilon^2)/(c \cdot (b-a)^2) \geq log(2/\delta) \implies e^{(k\cdot\epsilon^2)/(c\cdot(b-a)^2)} \geq 2/\delta \implies \delta \geq 2/e^{(k\cdot\epsilon^2)/(c\cdot(b-a)^2)}$.

In order to prove that $\delta \geq 2 \cdot e^{(\frac{-2k\epsilon^2}{(b-a)^2})}$, we can prove that $2/e^{(k\cdot\epsilon^2)/(c\cdot(b-a)^2)} \geq 2 \cdot e^{(\frac{-2k\epsilon^2}{(b-a)^2})} \implies 1 \geq e^{(k\cdot\epsilon^2)/(c\cdot(b-a)^2)} \cdot e^{(\frac{-2k\epsilon^2}{(b-a)^2})} \implies e^{(k\cdot\epsilon^2)/(c\cdot(b-a)^2)+(\frac{-2k\epsilon^2}{(b-a)^2})} \leq 1$.

To prove that $e^{(k\cdot\epsilon^2)/(c\cdot(b-a)^2)+(\frac{-2k\epsilon^2}{(b-a)^2})} \leq 1$, we need to prove that $(k \cdot \epsilon^2)/(c \cdot (b-a)^2) - (\frac{2k\epsilon^2}{(b-a)^2}) \leq 0 \implies (\frac{2k\epsilon^2}{(b-a)^2}) \geq (k \cdot \epsilon^2)/(c \cdot (b-a)^2) \implies c \geq \frac{1}{2}$. Therefore, $c \geq \frac{1}{2}$, $\delta \geq 2/e^{(k\cdot\epsilon^2)/(c\cdot(b-a)^2)} \geq 2 \cdot e^{(\frac{-2k\epsilon^2}{(b-a)^2})}$.
Hence, we showed that there exist a constant $c = \frac{1}{2} > 0$ that

$$Pr[|X-\alpha| \geq \epsilon] \leq \delta$$

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