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Homework 3 Problem 1 Ray Ying (xinruiy2@illinois.edu)
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1. Prove that the following languages are not regular by providing a fooling set. You need to provide an infinite set and also prove that it is a valid fooling set for the given language.
   (a) $L = \{0^i 1^j 2^k \mid i + j = k + 1\}$.
   (b) Recall that a block in a string is a maximal non-empty substring of indentical symbols. Let L be the set of all strings in $\{\mathtt{0}, \mathtt{1}\}^*$ that contain two non-empty blocks of `1`s of unequal length. For example, L contains the strings `01101111` and `01001011100010` but does not contain the strings `000110011011` and `00000000111`.
   (c) $L = \{0^{n^3} \mid n \geq 0\}$.
2. Suppose L is not regular. Prove that $L \setminus L'$ is not regular for any finite language L'. Give a simple example of a non-regular language L and a regular language L' such that $L \setminus L'$ is regular.

Solution: 1.(a) Let F be the language 0^*.
Let x and y be arbitrary strings in F.
Then $x = 0^i$ and $y = 0^j$ for some non-negative integers $i \neq j$.
Let $z = 1^{j-i+1}2^j$.
Then $xz = 0^i 1^{j-i+1} 2^j \in$ L.
And $yz = 0^j 1^{j-i+1} 2^j \notin$ L because $i \neq j$ so that $2j - i + 1 \neq j + 1$.
Thus, F is a fooling set for L.
Because F is infinite, L cannot be regular.

(b) Let F be the language $(11)^*$.
Let x and y be arbitrary strings in F.
Then $x = (11)^i$ and $y = (11)^j$ for some non-negative integers $i \neq j$.
Let $z = 0^+(11)^i 0^+$.
Then $xz = 0^+(11)^i 0^+(11)^i \notin$ L.
And $yz = 0^+(11)^i 0^+(11)^j \in$ L since $i \neq j$.
Thus, F is a fooling set for L.
Because F is infinite, L cannot be regular.

(c) Let F be the language 0^*.
Let x and y be arbitrary strings in F.
Then $x = 0^i$ and $y = 0^j$ for some non-negative integers $i \neq j$.
Let $z = 0^{i^3+3i^2+2i+1}$.
Then $xz = 0^{i^3+3i^2+3i+1} = 0^{(i+1)^3} \in$ L.
And $yz = 0^{i^3+3i^2+2i+j+1} \notin$ L because $i \neq j$ so that $i^3 + 3i^2 + 2i + j + 1 \neq (i+1)^3$.
Thus, F is a fooling set for L.
Because F is infinite, L cannot be regular.

2. Since L' is finite, $L' \setminus L$ is finite. Since all the finite languages are regular, both L' and $L' \setminus L$ are regular.
Suppose that $L \setminus L'$ is regular. Base on the Corollary 3.4, the difference between two regular languages is also regular, $L' \setminus (L' \setminus L)$ is regular.
Then $L = (L \setminus L') \cup (L' \setminus (L' \setminus L))$ is regular, which contradicts with the fact that L is not regular. Thus $L \setminus L'$ is not regular for any finite language L'.

For the example, consider $L = \{0^n 1^n \mid n \geq 0\}$ which is not regular and $L' = \{0,1\}^*$ which is regular. Then $L \setminus L' = \emptyset$ which is regular. ■