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Newton's Method and Multiple Roots

10 points

Newton's method can be viewed as a way of transforming a root-finding problem $f(x) = 0$ into a fixed-point problem $x = g(x)$, where $g(x) = x - f(x)/f'(x)$. Recall that for simple roots, Newton's method has a quadratic convergence rate since $g'(x^*) = 0$, where $g'(x) = f(x)f''(x)/(f'(x))^2$. When there is a root with multiplicity, $f'(x^*) = 0$ which leads to a division by 0 in $g'(x^*)$. We will study how this affects the convergence rate of Newton's Method.

Consider the function $f(x) = (x - x^*)^m h(x)$, where m is an integer greater than 1 and h is arbitrary function such that $h(x^*) \neq 0$. Answer the following questions:

1. What is the fixed point problem obtained by applying Newton's Method? Give the formula for $g(x)$.
2. Evaluate $g'(x^*)$. Based on that value, what can you conclude about the convergence rate of Newton's Method for a root of multiplicity m?
3. In order to recover the quadratic convergence, we modify the Newton's method as

$$x_{n+1} = x_n - k\frac{f(x_n)}{f'(x_n)}$$

such that k is constant. For what value of k does the new fixed point iteration scheme has a quadratic convergence? Prove that for the found k, the iteration scheme achieves quadratic convergence.

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Your answer is correct.

- Part 1:

$$f(x) = (x - x^*)^m h(x)$$

$$f'(x) = m(x - x^*)^{m-1} h(x) + (x - x^*)^m h'(x)$$

$$g(x) = x - \frac{(x - x^*)^m h(x)}{m(x - x^*)^{m-1} h(x) + (x - x^*)^m h'(x)}$$

$$g(x) = x - \frac{(x - x^*) h(x)}{m h(x) + (x - x^*) h'(x)}$$

- Part 2:

$$g'(x) = 1 - \frac{[h + (x - x^*)h'][mh + (x - x^*)h'] - (x - x^*)h[mh' + h' + (x - x^*)h'']}{[mh + (x - x^*)h']^2}$$

$$g'(x^*) = 1 - \frac{mh^2 - 0}{(mh)^2}$$

$$g'(x^*) = 1 - \frac{1}{m}$$

Linear convergence since $0 < 1 - \frac{1}{m} < 1$ for any m greater than 1.

- Part 3: We modify the Newton's method by choosing $k = m$:

$$x_{k+1} = x_k - m\frac{f(x_k)}{f'(x_k)}$$

Note that the multiplier m will be carried over to the second term in $g'(x)$. Hence, we have:

$$g'(x^*) = 1 - m\frac{1}{m} = 0$$

This means we have recovered quadratic convergence.