CS/ECE 374 FALL 2018 Zhe Zhang (zzhan157@illinois.edu)
Homework 10 Problem 2 Ray Ying (xinruiy2@illinois.edu)
Anqi Yao (anqiyao2@illinois.edu)


We call an undirected graph an *eight-graph* if it has an odd number of nodes, say $2n-1$, and consists of two cycles $C_1$ and $C_2$ on $n$ nodes each and $C_1$ and $C_2$ share exactly one node. See figure below for an eight-graph on 7 nodes.

Given an undirected graph $G$ and an integer $k$, the EIGHT problem asks whether or not there exists a subgraph which is an eight-graph on $2k-1$ nodes. Prove that EIGHT is NP-Complete.

---

**Solution:**

To prove EIGHT is NP-Complete, we need to prove first EIGHT is NP, and second an NP-hard (in this case, NP-Complete) problem (Hamiltonian cycle) is reducible to EIGHT.

First, show that EIGHT is an NP by finding an efficient certifier to the EIGHT problem.

Problem: Given an undirected graph $G = (V, E)$ and an integer $k$, check whether there is an eight-graph subgraph with $2k-1$ nodes.
Certificate: Set $V_8 \in V$, $|V_8| = 2k-1$.
Certifier: Check whether there is an eight-graph $H = (V_8, E_8)$, where $E_8$ is a subset of $E$ containing edges with vertices from $V_8$.
We can see that the certifier is efficient because testing the certificate takes polynomial time, and thus the EIGHT problem is an NP.

Second, we will show that the Hamiltonian cycle problem can be reduced to the EIGHT problem.

Let $G = (V, E)$ be an undirected graph where $V = v_1, v_2, ..., v_k$.
Create another undirected graph $G' = (V', E')$, $V' = v'_1, v'_2, ..., v'_k$, $G'$ has an undirected Hamiltonian cycle, and $G'$ shares only one vertex with $G$.
Then let $G'' = (V'' = V \cup V', E'' = E \cup E')$, and notice that $G''$ has $2k-1$ vertices due to the vertex shared between $G$ and $G'$.
Also denote the shared vertex as $v_i$.

Claim:
$G$ has an undirected Hamiltonian cycle $\iff$ $G''$ has a eight-graph subgraph with $2k-1$ vertices.
Proof:
" $\Longrightarrow$ " : Suppose $G$ has an undirected Hamiltonian cycle. Since $G'$ is created with a Hamiltonian cycle and $G$ and $G'$ shares only 1 vertex. We have 2 cycles in $G''$ and there is exactly one node shared between them. Thus they together form the eight-graph $G''$ with $2k-1$ nodes.
" $\Longleftarrow$ " : Suppose that a graph $G$ has a eight-graph with $2k-1$ nodes. We can have $C_1$ and $C_2$ be the two cycles in the eight-graph such that $|(C_1 \cap C_2)| = 1$, $|C_1| = |C_2| = k$. By the construction, we are certain that $G'$ has a Hamiltonian cycle and $|V(G')| = k$. As the statement of eight-graph, we can construct the graph $G''$ by deleting $G'/v$ from $G$ where $v$ is the intersection vertex of the two cycles in the eight-graph. Since the remaining graph is a cycle of length $k$ and there is only

$k$ vertices left, we are guarantee to have a Hamiltonian cycle $G''$. So we proved that $G''$ is a Hamiltonian cycle by showing $G$ is a eight-graph.
Thus the claim is proved.

Now we have shown that the EIGHT problem is an NP and it could be reduced from an NP-complete problem. Hence, the EIGHT problem is NP-complete. ■