Quantities with Errors

10 points

Let $x, y \in \mathbb{R}^n$ be vectors of real numbers. Consider the problem of computing the dot product

$$\mathbf{x}^T\mathbf{y} = \sum_{i=1}^{n} x_i y_i.$$

On an actual machine, the above computation will be subject to rounding errors that accumulate over the course of computation.

Part 1

Let us model the effect of rounding a number by saying that the rounded number $\tilde{x}$ should satisfy

$$\tilde{x} = x(1+\delta)$$

where δ is a small quantity. Assuming that δ does not depend on the number x, what (in exact arithmetic) does the expression $\sum_{i=1}^{n} \tilde{x}_i \tilde{y}_i$ evaluate to?

Part 2

Arithmetic operations are also subject to rounding on a computer. Let $\oplus, \otimes$ denote the operation of rounded addition and rounded multiplication as done by a computer. Let us model these operations by saying that

$$x \oplus y = (x+y)(1+\delta)$$

$$x \otimes y = xy(1+\delta)$$

where δ is another small quantity.

Suppose the dot product is evaluated by means of the expression:

$$(x_1 \otimes y_1) \oplus (x_2 \otimes y_2) \oplus \cdots \oplus (x_n \otimes y_n).$$

The expression is evaluated from left to right.

How many times do $\otimes$ and $\oplus$ appear in the expression? Assuming that δ does not depend on the inputs, what does this expression evaluate to? (You may ignore the effect of input rounding as described in Part 1.)

Part 3

Suppose you had a function $\mathrm{fma}(a, x, b)$ which satisfies

$$\mathrm{fma}(a, x, b) = (ax+b)(1+\delta)$$

where δ is a small quantity. (See Fused multiply-add (https://en.wikipedia.org/wiki/Multiply%E2%80%93accumulate_operation) for some context.)

Then the dot product can be evaluated as

$$\mathrm{fma}(x_n, y_n, \mathrm{fma}(x_{n-1}, y_{n-1}, \; \cdots \; \mathrm{fma}(x_2, y_2, \mathrm{fma}(x_1, y_1, 0)))).$$

How many times is fma applied in the expression? Assuming that δ does not depend on the inputs, what does this expression evaluate to? (You may once again ignore the effect of input rounding as described in Part 1.)

Please submit your response to this written problem as a PDF file below. You may do either of the following:

- write your response out by hand, scan it, and upload it as a PDF.

  We will not accept unprocessed pictures taken with your phone.

  If you decide to use your phone for scanning, make sure to use an app such as CamScanner (https://www.camscanner.com/) to get a readable PDF. Alternatively, there's a fast and convenient scanner in the Engineering IT office in 2302 Siebel that can just email you a PDF. (It's the Fax-machine-looking thing--not the scanner that's attached to one of the computers.)

- create the PDF using software.

  If you're looking for an easy-ish way to type math, check out TeXmacs (http://texmacs.org/) or LyX (http://www.lyx.org/). Both are installed in the virtual machine. (Under "Applications / Accessories / GNU TeXmacs editor" and "Applications / Office / LyX document processor" respectively.)

Submit your response to each problems in this homework as a separate PDF. If you have multiple PDFs that you need to merge into one, try PDF Split and Merge (http://www.pdfsam.org/download/).

NOTE: Please make sure your solutions are legible and easy to follow. If they are not, we may deduct up to five points *per problem*.

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Your answer is correct.

Part 1

The answer is

$$\sum_{i=1}^{n} x_i y_i (1+\delta)^2.$$

Part 2

The operator $\otimes$ appears n times, and the operator $\oplus$ appears $n-1$ times.

Consider the term $s_j = x_j \otimes y_j$ in the expression. Let $\Delta = 1+\delta$. By considering the nested sequence

$$((((s_1 + s_2)\Delta + s_3)\Delta + \cdots)\Delta + s_n)\Delta$$

one sees that s_j is multipled by Δ^{n-j+1}, when $j > 1$, and $\Delta^{n-1}$when $j = 1$. The operation $\otimes$ also contributes a factor of Δ.

So the expression evaluates to

$$x_1y_1(1+\delta)^n + \sum_{i=2}^{n} x_iy_i(1+\delta)^{n-i+2}.$$

Part 3

The function fma is applied n times in the expression.

Consider the term x_jy_j. To each term of this form the fma will be applied a total of $(n-j+1)$ times.

So the expression evaluates to

$$\sum_{i=1}^{n} x_iy_i(1+\delta)^{n-i+1}.$$