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Homework 0 Problem 2

2. Consider the following recurrence.

$$T(n) = T(\lfloor n/2 \rfloor) + 2T(\lfloor n/4 \rfloor) + n \quad n \geq 4, \text{ and } T(n) = 1 \quad 1 \leq n < 4.$$

- Prove by induction that $T(n) = O(n \log n)$. More precisely show that $T(n) \leq an \log n + b$ for $n \geq 1$ where $a, b \geq 0$ are some fixed but suitably chosen constants (you get to choose and fix them).

Solution:

2. Proof: By induction on n. Choose $a = 1$, $b = 1$.

- Base case:
  - When $n = 1$, $T(1)$= 1, $an \log n + b = \log 1 + 1 = 1$. $T(1) \leq an \log n$.
  - When $n = 2$, $T(2) = 1$, $an \log n + b = 2 \log 2 + 1 = 3$. $T(2) \leq an \log n$.
  - When $n = 3$, $T(3) = 1$, $an \log n + b = 3 \log 3 + 1 \leq an \log n$.

  So the recurrence holds for $n = 1, 2, 3$.

- Induction:
  Suppose the recurrence holds for n = 1, 2, 3, ..., $k-1$, $k \geq 4$.
  We need to show it holds for n = k.
  By the definition of floor, $\lfloor k/2 \rfloor \leq k/2$ and $\lfloor k/4 \rfloor \leq k/4$.
  Thus from the induction hypothesis, we get $T(\lfloor k/2 \rfloor) \leq T(k/2) \leq a * (k/2) \log(k/2) + b$ and $T(\lfloor k/4 \rfloor) \leq T(k/4) \leq a * (k/4) \log(k/4) + b$.
  Then by the definition of the recurrence,

$$\begin{aligned}
T(k) &= T(\lfloor k/2 \rfloor) + T(\lfloor k/4 \rfloor) + k \\
&\leq T(k/2) + T(k/4) + k \\
&\leq a(k/2)\log(k/2) + 2a(k/4)\log(k/4) + 3b + k \\
&= \frac{ak}{2}(\log(k) - \log(2) + \log(k) - \log(4)) + 2b + k \\
&= \frac{ak}{2}(2\log(k) - \log(k) - \log(4)) + 2b + k \\
&= ak\log(k) - k/2 + k + 3b \\
&= ak\log(k) - n/2 + 3b
\end{aligned}$$

  Substitute with $a = b = 1$. We get $T(k) = k \log(k) - k/2 + 3$.
  For $k \geq 4$, $k/2 \geq 2$, thus $T(k) \leq k \log(k) + 1$, which is the same as $T(k) \leq ak \log(k) + b$.
  Hence, the induction has been proved.

  Therefore, $T(n) = O(n \log n)$.

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