CS 498 algorithm Spring 2019 Ray Ying (xinruiy2@illinois.edu)
Homework 1 Problem 4 Aditya Pillai (apillai4@illinois.edu)

Solution: X_i be the value of the counter after i events, and $Y_i = (1+a)^{X_i}$. For $n = 0, 1$ $Y_i = 1, 1+a$ deterministically.
$\mathbb{E}(Y_n) = an + 1$ Proof by induction on n

Proof:

$$\begin{aligned}
\mathbb{E}(Y_n) &= \mathbb{E}((1+a)^{X_n}) \\
&= \sum_{j=0}^{\infty}(1+a)^j \Pr(X_n = j) \\
&= \sum_{j=0}^{\infty}(1+a)^j(\Pr(X_{n-1} = j) \cdot (1 - \frac{1}{(1+a)^j}) + \Pr(X_{n-1} = j-1) \cdot \frac{1}{(1+a)^{j-1}}) \\
&= \mathbb{E}(Y_{n-1}) + \sum_{j=0}^{\infty}(1+a)\Pr(X_{n-1} = j-1) - \Pr(X_{n-1} = j) \\
&= \mathbb{E}(Y_{n-1}) + a \\
&= a(n-1) + 1 + a && \text{(By induction)} \\
&= an + 1
\end{aligned}$$

□

So the estimate for n the algorithm outputs is $\frac{(1+a)^X - 1}{a}$
$\mathbb{E}(Y_n^2) = an(a+2)(a(n-1)+2)/2 + 1$. $Y_n^2 = 1, (1+a)^2$ determinstically for $n = 0, 1$.

Proof:

$$\begin{aligned}
\mathbb{E}(Y_n^2) &= \mathbb{E}((1+a)^{2X_n}) \\
&= \sum_{j \geq 0}(1+a)^{2j} \Pr(X_n = j) \\
&= \sum_{j \geq 0}(1+a)^{2j}(\Pr(X_{n-1} = j) \cdot (1 - \frac{1}{(1+a)^j}) + \Pr(X_{n-1} = j-1) \cdot \frac{1}{(1+a)^{j-1}}) \\
&= \mathbb{E}(Y_{n-1}^2) + \sum_{j \geq 0}(1+a)^{j+1} \Pr(X_{n-1} = j-1) - (1+a)^j \Pr(X_{n-1} = j) \\
&= \mathbb{E}(Y_{n-1}^2) + (a^2 + 2a)\mathbb{E}(Y_{n-1}) \\
&= \mathbb{E}(Y_{n-1}^2) + (a^2 + 2a)(an - a + 1) \\
&= an(a+2)(a(n-1)+2)/2 + 1 && \text{(By induction)}
\end{aligned}$$

□

$\mathrm{Var}(Y_n) = \frac{a^3 n}{2}(n-1)$ and $\mathrm{Var}(\tilde{n}) = \frac{an}{2}(n-1)$ By applying Chebyshev we get

$$\begin{aligned}
\Pr(|\tilde{n} - n| \geq \epsilon n) &\leq \frac{an}{2n^2\epsilon^2}(n-1) \\
&\leq \frac{a}{2\epsilon^2}(1-1/n) \\
&\leq \frac{a}{2\epsilon^2} \\
&\leq 1/10 && \text{(this is true when } a \leq \epsilon^2/5)
\end{aligned}$$

This implies that for $0 < a \leq \epsilon^2/5$, $\Pr(|\tilde{n} - n| \leq \epsilon n|) \geq 9/10$
The number of bits the algorithm uses is $O(\log X)$, where X is the value of the counter after n increments. The previous part shows that $\tilde{n} \leq n(1+\epsilon)$ with probability at least $9/10$.

$$\begin{aligned}
\frac{(1+a)^X - 1}{a} &\leq (1+\epsilon)n \\
X\log(a+1) &\leq \log(an(1+\epsilon)+1) \\
X &\leq \frac{\log(an(1+\epsilon)+1)}{\log(a+1)} \\
&= \log_{a+1}(an(1+\epsilon)+1)
\end{aligned}$$

Therefore, $S(n) = \log(\log_{a+1}(an(1+\epsilon)+1))$ with probability at least $9/10$, if $\epsilon \leq 1$ then $O(\log(\log n))$ bits are used.