NECESSARY LIBERAL PRECONDITIONS:
A PROOF SYSTEM

Mster's Thesis in Informatics

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Anran Wang: *Necessary Liberal Preconditions: A Proof System*

NECESSARY LIBERAL PRECONDITIONS: A PROOF SYSTEM
NOTWENDIGE LIBERALE VORBEDINGUNGEN: EIN BEWEISSYSTEM

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DECLARATION

Ich versichere, dass ich diese Masterarbeit selbstständig verfasst und nur die angegebenen Quellen und Hilfsmittel verwendet habe.

I confirm that this master's thesis is my own work and I have documented all sources and material used.

Munich, 15. September 2023

Anran Wang

ABSTRACT

This is where the abstract goes.

ZUSAMMENFASSUNG

Kurze Zusammenfassung des Inhaltes in deutscher Sprache…

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LISTINGS

ACRONYMS

Part I

HOARE TRIPLES, WEAKEST PRECONDITIONS, WEAKEST LIBERAL PRECONDITIONS

Some text about this part.

1 BACKGROUND

TODO: Make first letter big?

TODO: Decide on all the colors in the end.

TODO: Rewrite; add chapter contents.

2 PRELIMINARIES

2.1 LLOYD-HOARE LOGIC

TODO: A history lesson, rewrite to include Lloyd. See [4] P.27.

In 1969, C.A.R. Hoare wrote *An Axiomatic Basis for Computer Programming* [3] to explore the logic of computer programs using axioms and inference rules to prove the properties of programs. This system is known as Hoare Logic. He introduced sufficient preconditions that will guarantee correct results but does not rule out non-termination. A selection of the axioms and rules are shown in Table 1. [1] [2]

Axiom of Assignment	$F[x/e]\ \{x := e\}\ F$
Rules of Consequence	If $G\ \{C\}\ F$ and $F \Rightarrow P$ then $G\ \{C\}\ P$
	If $G\ \{C\}\ F$ and $P \Rightarrow G$ then $P\ \{C\}\ F$
Rule of Composition	If $G\ \{C_1\}\ F_1$ and $F_1\ \{C_2\}\ F$ then $G\ \{C_1; C_2\}\ F$
Rule of Iteration	If $F \wedge (B\ \{C\}\ F)$ then F {while B do C} $\neg B \wedge F$

Table 1: Valid Hoare Triples

$\{F[x/e]\}$ is obtained by substituting occurrences of x by e.

Semantically, a Hoare Triple $G\ \{C\}\ F$ is said to be valid for (partial) correctness, if the execution of the program C with an initial state satisfying the precondition G leads to a final state that satisfies the postcondition F, provided that the program terminates.

The definition indeed corresponds to this intended semantics. (Formal soundness proofs can be found in Krzysztof R. Apt's survey [1] in 1981.) As an example, consider the rule of composition: if the execution of program C_1 changes the state from G to F_1, and C_2 changes the state from F_1 to F, then executing them consecutively should bring the program state from G to F, with the intermediate state F_1.

The missing guarantee of termination can be seen in the rule of iteration: consider the example **TODO:** Add example here.

1 We omit the symbol $\vdash$ in front of a Hoare Triple, which denotes "valid/provable", for better readability.

2 Non-determinism was not considered in the original paper, so we treat the programs here as deterministic. With deterministic programs, one initial state corresponds to one final state, and by non-termination we assign a final state $\perp$.

TODO: Think about whether to add liberally deterministic (Hesselink 1992, Programs, Recursion and Unbounded Choice).

Figure 1 illustrates a valid Hoare Triple, Σ represents the set of all states, the section denoted with G includes the states that satisfy the predicate G. The arrow from left to write denotes the execution of the program C.

TODO: In case I change color for \ mathl, I should change the color for hoare triplc GCF.

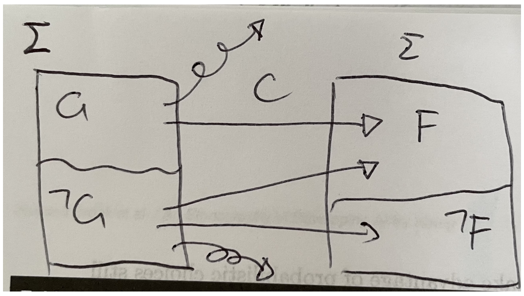

Figure 1: Valid Hoare Triple (Deterministic)

TODO: Digitize image.

Hoare Logic is sound, expressive, yet incomplete [1]. A sensible advancement would be to find the necessary and sufficient preconditions that grant us the post-properties, i.e. to eliminate the arrows from $\neg G$ to F in Figure 1, and to be able to prove termination, i.e. to eliminate the arrows from G to the abyss[3], which is what Edsger W Dijkstra accomplished with his weakest precondition transformer in 1975 [2], among other things.

2.2 GUARDED COMMAND LANGUAGE

From now on we will use Dijkstra's (non-deterministic) guarded command language (GCL) [2] to represent programs and to include non-determinism (starting from Section 2.3.3). For better understanding, we use an equivalent [4] form of GCL that is similar to modern pseudo-code:

$$
\begin{aligned}
C \;::=\;\; & x := e \;\mid\; C;C \;\mid\; \{C\}\square\{C\} \;\mid\; \mathit{if}\ (\varphi)\ \{C\}\ \mathit{else}\ \{C\} \;\mid\; \mathit{while}\ (\varphi)\ \{C\} \\
& \mid \mathit{skip} \mid \mathit{diverge}
\end{aligned}
$$

The non-deterministic choice $\{C_1\}\square\{C_2\}$ chooses from two programs randomly. It is however not probabilistic, where we know the probabilistic distribution of the outcome of the choice. With the non-deterministic choice, we have no such knowledge.

3 Adding termination proof is also done by Zohar Manna and Amir Pnueli in 1974 [5], where they introduced what we call a loop variant, a value that decreases with each iteration. The name is in contrast to loop invariant, concretely the F in Rule of Iteration, which is constant before and after the loop.

4 Specifically, $\mathit{if}\ (\varphi)\ \{C_1\}\ \mathit{else}\ \{C_2\}$ is equivalent to $\mathit{if}\ \varphi \to C_1\ [\!]\ \neg\varphi \to C_2\ \mathit{fi}$ in Dijkstra's original style[2]; $\{C_1\}\square\{C_2\}$ is equivalent to $\mathit{if}\ \mathit{true} \to C_1\ [\!]\ \mathit{true} \to C_2\ \mathit{fi}$.

When skip is executed, the program state does not change and the consecutive part is executed. When diverge is executed, the program goes to state $\perp$ to denote non-termination, and the execution stops.

2.3 WEAKEST PRECONDITION

2.3.1 *The Deterministic Case*

To better relate Hoare Triples and Dijkstra's weakest precondition transformer, we first ignore non-determinism.

Again, the goal is to find the necessary and sufficient precondition such that the program is guaranteed to terminate in a state that satisfies the postcondition. Figure 2 shows it graphically.

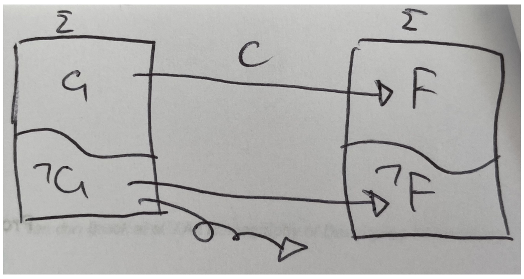

Figure 2: Weakest Precondition (Deterministic)

TODO: Digitize image.

We define the weakest precondition transformer structurally in lambda-calculus style[5] as in Table 2:

C	**wp.C.F**
skip	F
diverge	false
$x := e$	$F[x/e]$
$C_1; C_2$	$wp.C_1.(wp.C_2.F)$
if (φ) $\{C_1\}$ else $\{C_2\}$	$(\varphi \wedge wp.C_1.F) \vee (\neg\varphi \wedge wp.C_2.F)$
while (φ) $\{C'\}$	$\text{lfp } X.(\neg\varphi \wedge F) \vee (\varphi \wedge wp.C'.X)$

Table 2: The Weakest Precondition Transformer (Deterministic Programs) [4]

$F[x/e]$ is F where every occurrence of x is syntactically replaced by e.
$\text{lfp } X.f$ is the least fixed point of function f with variable X.

5 For example, $wp.C.F$ can be seen as $wp(C, F)$ in "typical" style, where wp is treated as a function that has two parameters. The advantage of lambda-calculus style is scalability, we can simply extend the aforementioned function like $wp.C.F.\sigma$ where σ means the initial state. Here wp is treated as a function that has three parameters, if we were to write it in the "typical" style. It is then questionable whether we changed the type of wp.

Let

$$\Phi(X) := (\neg\varphi \wedge F) \vee (\varphi \wedge wp.C'.X)$$

be the characteristic function, then wp for while-loop can be defined as:

$$wp.(while(\varphi)\{C'\}).F = lfp\ X.\Phi(X)$$

.

Most of the definitions in Table 2 are intuitive and correspond to their counterparts in Hoare Logic. To take special notice are the definitions for diverge and while. Since wp aims for total correctness, the precondition wp.diverge.F should terminate with postcondition F. Because diverge does not terminate, there is no such precondition and wp for diverge should be false.

The definition for the while-loop[4] is trickier, but we can verify its correctness by recalling Dijkstra's original definition.

TODO: Find out if there's earlier definition that used lfp.

2.3.2 *Defining Loops*

In Dijkstra's original paper[2], he defined wp for while-loops based on its (intended) semantics, i.e. the precondition such that, when satisfied, guarantees that the loop terminates with the required postcondition within a certain number of iterations.

Let

$$WHILE = while(\varphi)\{C'\} \qquad IF = if\ (\varphi)\{C'; WHILE\}\ else\ \{skip\}$$

Rewriting Dijkstra's definition in a form conforming to our style, he defines

$$H_0(F) = (F \wedge \neg\varphi) \qquad H_k(F) = (wp.IF.(H_{k-1}(F)) \vee H_0(F))$$

Intuitively, when $H_0(F)$ is satisfied before the execution of WHILE, the loop is exited with 0 iteration in a state that satisfies $F \wedge \neg\varphi$ hence F. Then we can understand $H_k(F)$ as the weakest precondition such that the program terminates in a final state satisfying F after at most k iterations.

Then by definition:

$$wp.WHILE.F = (\exists k \geqslant 0 : H_k(F)) \tag{1}$$

We state that our definition coincides with this definition. Without going too deep into domain theory, we only use one of its theorem that yields a computation for least fix points, when they exist.

Theorem 1. ***TODO:*** *Insert theorem, then explain least point iteration from bottom.*

Coincidentally, $H_k(F)$ is the $k-$th iteration from bottom $\perp$ to calculate the least fixed point of the characteristic function: $\Phi^k(\perp)$. Thus by finding the least fixed point, we've found a k that satisfies (1).

2.3.3 *The Non-deterministic Case*

Now we bring the non-deterministic choice back into the picture and add its definition as shown in Table 3.

C	**wp.C.F**
skip	F
diverge	false
$x := e$	$F[x/e]$
$C_1; C_2$	$wp.C_1.(wp.C_2.F)$
if (φ) $\{C_1\}$ else $\{C_2\}$	$(\varphi \wedge wp.C_1.F) \vee (\neg\varphi \wedge wp.C_2.F)$
$\{C_1\}\Box\{C_2\}$	$wp.C_1.F \vee wlp.C_2.F$
while (φ) $\{C'\}$	$\mathrm{lfp}\ X.(\neg\varphi \wedge F) \vee (\varphi \wedge wp.C'.X)$

Table 3: The Weakest Precondition Transformer (Non-deterministic Programs) [4]

To justify this definition for the non-deterministic choice, we must first clarify the intended semantics/meaning of the wp-transformer.

Let $[\![C]\!]$ denote the execution of program C, $[\![C]\!].\sigma$ denote the set of final states that can occur after the execution of C.

(A state is a function that maps a program variable to a value. The set of states is denoted by $\Sigma = \{\sigma \mid \sigma : \mathrm{Vars} \to \mathrm{Vals}\}$.)

If C is deterministic, then $[\![C]\!].\sigma$ is a set of a single state, either a final state σ' or $\perp$, if the execution does not terminate. If C is non-deterministic, $[\![C]\!].\sigma$ can be a set with multiple elements, since multiple final states can be possible.

The weakest precondition $wp.C.F$ is then

Note for readers: Up to here is readable.

Figure 3 shows wp with non-deterministic programs. Each arrow from left to right shows a possible execution of program C.

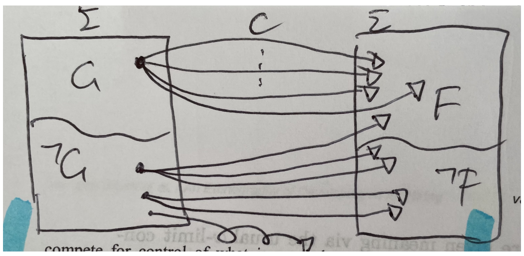

Figure 3: Weakest Precondition (Non-deterministic)

TODO: Digitize image.

2.4 WEAKEST LIBERAL PRECONDITION

We define the weakest liberal precondition transformer in Table 4.

C	**wlp.C.F**
$skip$	F
$diverge$	$true$
$x := e$	$F[x/e]$
$C_1; C_2$	$wp.C_1.(wp.C_2.F)$
$if\ (\varphi)\ \{C_1\}\ else\ \{C_2\}$	$(\varphi \wedge wp.C_1.F) \vee (\neg\varphi \wedge wp.C_2.F)$
$\{C_1\}\square\{C_2\}$	$wlp.C_1.F \wedge wlp.C_2.F$
$while\ (\varphi)\ \{C'\}$	$gfp\ X.(\neg\varphi \wedge F) \vee (\varphi \wedge wp.C'.X)$

Table 4: The Weakest Liberal Precondition Transformer

Part II

NECESSARY LIBERAL PRECONDITIONS

Some text about this part.

3 A PROOF SYSTEM

3.1 A PROOF SYSTEM

In this section we study the necessary liberal precondition:

$$wlp.\mathsf{C}.\mathsf{F} \implies \mathsf{G}$$

4 CONCLUSIONS

4.1 CONCLUSIONS

4.2 FUTURE WORK

Part III

APPENDIX

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COLOPHON

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