

NECESSARY LIBERAL PRECONDITIONS: A PROOF SYSTEM

MASTER'S THESIS IN INFORMATICS

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SCHOOL OF COMPUTATION, INFORMATION AND TECHNOLOGY - INFORMATICS
TECHNICAL UNIVERSITY OF MUNICH



Anran Wang: *Necessary Liberal Preconditions:
A Proof System*

**NECESSARY LIBERAL PRECONDITIONS:
A PROOF SYSTEM
NOTWENDIGE LIBERALE VORBEDINGUNGEN:
EIN BEWEISSYSTEM**

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Anran Wang: *Necessary Liberal Preconditions:
A Proof System*

DECLARATION

Ich versichere, dass ich diese Masterarbeit selbstständig verfasst und nur die angegebenen Quellen und Hilfsmittel verwendet habe.

I confirm that this master's thesis is my own work and I have documented all sources and material used.

Munich,

Anran Wang

For my parents Shizhu Wang and Derun Yuan, who love me patiently.

For Christian Schuler, who loves me funnily.

For my friends, who like me.

For me, who?

ABSTRACT

This thesis investigates the necessary liberal precondition, an overapproximation of Dijkstra's weakest liberal precondition (wlp) transformer. A discussion of the semantics of wlp and its relatives hints at a scenario where the necessary liberal precondition is useful: When the entirety of undesirable postconditions is unknown, by further relaxing wlp, the programmer ends up with a precondition that is likely to lead the program to known or unknown undesired final states upon termination.

ZUSAMMENFASSUNG

Diese Arbeit untersucht die notwendige liberale Vorbedingung, eine Überannäherung an Dijkstras schwächsten liberalen Vorbedingungstransformator (wlp). Eine Diskussion der Semantik von wlp und seinen Verwandten deutet auf ein Szenario hin, in dem die notwendige liberale Vorbedingung nützlich ist: Wenn die Gesamtheit der unerwünschten Nachbedingungen unbekannt ist, erhält der Programmierer durch weitere Lockerung von wlp eine Vorbedingung, die das Programm bei Beendigung wahrscheinlich in bekannte oder unbekannte unerwünschte Endzustände führt.

摘要

本论文研究了必要自由前提条件，即 Dijkstra 的最弱自由前提条件 (wlp) 的涵盖近似。首先讨论 wlp 及其近似方法的语义，这暗示了一个必要自由前提条件有用的场景：当我们不知道所有不好的后置条件时，程序员可以通过进一步放松 wlp 来找到一个前提条件，这个条件很可能会导致程序在终止时进入不好的最终状态，不论这些状态是已知或未知的。

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Part I

HOARE TRIPLES AND DIJKSTRA'S PREDICATE TRANSFORMERS

In this part, I explain the definition for the formalism used in this thesis, discuss some of the definitions, and demonstrate the semantics of the predicate transformers using graphs and operational semantics.

BACKGROUND

MOTIVATION In 1739, the Scottish philosopher David Hume questioned why we know that the sun will rise tomorrow, *“tho’ ’tis plain we have no further assurance of these facts, than what experience affords us”* [13]. Hume’s question about causality is daunting, yet most of us are not in crisis because we doubt if the sun rises tomorrow. The reason is probably that we believe in physics, astrology, and the rules and formulas that assure us the universe works in a certain way, hence the sun rises tomorrow. It is exactly the rules and formulas this thesis attempts to investigate, in the realm of computer programs, with which we are certain that the equivalent version of the sun in a program will rise tomorrow.

Computer programs are ubiquitous in almost every aspect of human life. We want them to solve our problem efficiently, and correctly. Fortunately, brilliant scientists and engineers have taken the matter into their hands. A recent example is seL4 [16], the first formally proven operating system kernel against overflows, memory leaks, non-termination, etc. using the interactive theorem prover Isabelle/HOL [27]. Its verification brings great potential to safety-critical fields like aircrafts and autonomous cars, the latter of which is to be in mass production in 2024.¹

Imagine soon being driven by an autonomous car carrying seL4. It is desirable that it delivers us to the correct destination, and never get stuck driving around the same block without making progress. Delivering the correct result and stopping eventually is called **total correctness**.

To know “for sure”, we could verify programs using formal methods. One famous method is **Hoare triples** [11]. A Hoare triple contains three parts: a precondition, a program, and a postcondition. They are written as such: $G \{C\} F$. It states that if the system starts in a state that satisfies the precondition, then the state after the execution of the program will satisfy the postcondition, provided that the program terminates. Hoare triples are elegant in that once we have appropriate preconditions, we can follow their reference rules on sequential programs with ease. But with Hoare triples in their original form, we know the program is correct, but we are not sure of its termination. This is called **partial correctness**.

To prove a program totally correct, Dijkstra presented the **weakest precondition transformer** [5] (wp): starting with a postcondition, it works backwards and calculates what the weakest precondition is that guarantees both correctness and termination. A predicate P is considered to be **weaker** than Q , if $Q \implies P$ is valid. Conversely, Q is **stronger** than P . In Hoare triples, the precondition is a **sufficient** condition for the program to be correct in that the final state will satisfy the desired postcondition, while with wp we obtain a **necessary and sufficient**

¹ <https://sel4.systems/news/2023#nio-skyos>, accessed 10.03.2024.

precondition. Relaxing the commitment for termination, Dijkstra also proposed the [weakest liberal precondition transformer](#) [7] (wlp) which delivers preconditions so that the program either terminates correctly or never terminates, proving partial correctness. The name contains the word [liberal](#), in the sense that the wlp transformer takes a more “liberal” view on non-termination, that divergence or execution forever is considered as acceptable in this case.

Since then, a plethora of research projects blossomed and yielded fruitful results. Not only did Hoare Logic receive numerous copious attention during the first decade of its proposal [1], it has become the basis for program verification now [9]. Hoare triple also finds applications in low-level programming languages [23], quantum programs [33], distributed real-time systems [12], and so on.

Likewise, Dijkstra’s wp transformer also spawned various scientific work on probabilistic programs [15], quantum programs [2], continuous programs [19], and so much more. Especially inspiring for this thesis is the work by O’Hearn named [incorrectness logic](#) [28]. It studies an underapproximation (subset) of the [strongest postcondition transformer](#) [7] (sp) by Dijkstra, focusing on reachability of final states to prove the existence of bugs. He proposed the incorrectness triple starting from the implication

$$F \implies \text{sp.C.G}$$

Here, F is an underapproximation of the strongest postcondition of precondition G w.r.t. program C , whereas in this thesis, we attempt to study the implication

$$\text{wlp.C.F} \implies G$$

where G is an overapproximation of the strongest liberal precondition of postcondition F w.r.t. program C . Despite focusing on different transformers and relations, incorrectness logic gives great intuition on how to approach the overapproximation over wlp.

CONTRIBUTION This thesis yields results both in pre- and postcondition transformers and the implication $\text{wlp.C.F} \implies G$. The results are listed below in chronological order:

1. We prove that the definition of the weakest precondition transformer of while-programs by Dijkstra [5] and the one using the least fixed point coincide, and gave intuition to explain the necessity of using the least fixed point instead of using any other fixed point.
2. Following the previous result, we deduce that the greatest fixed point is necessary to define the weakest liberal precondition.
3. We relate the pre- and postcondition transformers under angelic or demonic considerations over non-determinism using relations like termination, reachability, conjugation, implication, and equivalence. We then present the results as a graph in [Figure 3.6](#).

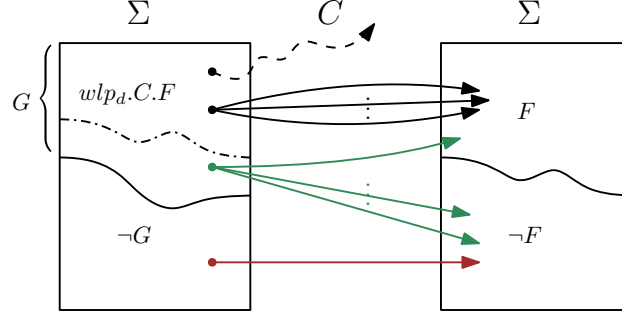


Figure 1.1: Necessarily Liberal Precondition G That Additionally Has Green Dots

4. We discuss all possible scenarios of the overapproximation $wlp.C.F \implies G$, what we call the [necessary liberal preconditions](#), concluding that without further restrictions, the overapproximate G can contain any initial states. However, we show that the only certainty is that initial states satisfying $\neg G$ will lead to termination in states satisfying $\neg F$. Following this, we demonstrate the usefulness of this overapproximation with an example.
5. We also provide a proof system that captures this overapproximation triple, and prove its soundness.
6. We also notice a special group of initial states, starting from which the execution is possible to terminate satisfying both the desired postcondition and its opposite, like the green parts in [Figure 1.1](#). We capture these preconditions by approaching from top and bottom, namely finding scenarios in which the necessary liberal precondition G underapproximates these special preconditions, and finding scenarios where G overapproximates them. In this way, we can establish an equivalence between G and wlp added with the special preconditions, given constraints. This G is exactly wlp with angelic non-determinism.

OUTLINE After explaining our motivation in this chapter, we proceed to build the formal foundations of the formalism used in this thesis in [Chapter 2](#). We first present a table of notations we adopt, then give a soft introduction of Hoare logic in [Section 2.2](#), followed by Dijkstra’s [Guarded Command Language](#), which we use throughout the thesis. Then we display the weakest precondition transformer in [Section 2.4](#), comparing it against Hoare logic, then give a detailed discussion about the use of fixed points while defining while loops. Finally, we add the notion of non-determinism to the wp transformer, and discuss the difference between angelic and demonic non-determinism and explain our design choice.

Following this, we show the weakest liberal precondition transformer in [Section 2.5](#) and the strongest postcondition transformer in [Section 2.6](#). Subsequently, we introduce a [big-step semantics](#) in [Section 2.7](#) to help us define the semantics of the aforementioned transformers in [Section 2.9](#) and list some properties in [Section 2.10](#).

In [Chapter 4](#), we first relate wp , wlp , sp , slp together using various relations, then illustrate them in a graph in [Chapter 3](#). This helps us distinguish the necessary liberal precondition with other implications that are researched, so that we can analyze the scenarios of our overapproximation in [Section 4.2](#). Then we provide an example to convey the usefulness of this overapproximation, and show its proof system as well as prove its soundness. We then discover a special scenario that is of interest, and find restraints that qualifies this special scenario, then prove their correctness in [Section 4.3](#). Finally, we summarize our conclusions in [Chapter 5](#) and propose possible future work.

PRELIMINARIES

2.1 NOTATIONS

Before proceeding, we clarify the notations used in this thesis, which are not uncommon in materials of computer science and mathematics. Readers are encouraged to skip this section and refer back to it if needed. The notations and their meaning are listed in [Table 2.1](#).

Notation	Meaning
\mathcal{X}	set of program variables
\mathcal{V}	set of values
$\sigma : \mathcal{X} \rightarrow \mathcal{V}$	program state
Σ	set of program states
\mathcal{C}	set of programs
\mathcal{P}	set of predicates
$F : \Sigma \rightarrow \{\text{true}, \text{false}\}$	predicate
$F := \{\sigma \in \Sigma \mid F(\sigma)\} (*)$	the set described by a predicate $F \in \mathcal{P}$
$F(\sigma) (**)$	
$F(\sigma) = \text{true} (**)$	state s satisfies predicate F ;
$\sigma \models F$	F is true when system is in state σ
$\sigma \in F$	
$\sigma \xrightarrow{c} \tau$	from initial state σ , an execution of program c terminates at final state τ
$\exists x. P : F$	syntactic sugar for
$\forall x. P : F$	
	$\exists x. (P \wedge F)$
	$\forall x. (P \wedge F)$

Table 2.1: Symbols and Notations

It is worth noting that we regard program states as total functions - we assume that we can assign some default values to variables in case they are undefined. We also simplify matters by assuming that there is only one interpretation as a total function from predicates to truth values. As a result, we can regard predicates as (total) functions from program states to truth values. We also overload the symbols for predicates and use them to identify the sets they describe as shown in [Line \(*\)](#).

By default, we take $F(\sigma)$ to mean the same as $F(\sigma) = \text{true}$ for convenience's sake as shown in [Lines \(**\)](#). We use the equation symbol $=$ to denote equivalences, and the symbols $:=$ for assignments and definitions.

The operators in descending binding power: $\neg, \in, \wedge, \vee, \implies, Q_ _ : _$ where Q is a quantifier: $Q \in \{\exists, \forall\}$. Implication binds to the left: $A \implies B \implies C$ is equivalent to $(A \implies B) \implies C$. Now we can proceed to discuss proof rules and systems that are relevant for this thesis.

2.2 HOARE LOGIC

Since the beginning of the 1960s, scholars have been researching the establishment of mathematics in computation [\[8, 22\]](#) to have a formal understanding and reasoning of programs. One of the most known methods is [Hoare logic](#).

In 1969, C.A.R. Hoare wrote *An Axiomatic Basis for Computer Programming* [\[11\]](#) to explore the logic of computer programs using axioms and inference rules to prove the properties of programs. He introduced [sufficient](#) preconditions that guarantee correct results but do not rule out non-termination. A selection of the axioms and rules are shown in [Table 2.2](#).¹

$\{F[x/e]\}$ is obtained by substituting occurrences of x by e .

Axiom of Assignment	$F[x/e] \{x := e\} F$
Rules of Consequence	$\text{If } G \{C\} F \text{ and } F \Rightarrow P \text{ then } G \{C\} P$ $\text{If } G \{C\} F \text{ and } P \Rightarrow G \text{ then } P \{C\} F$
Rule of Composition	$\text{If } G \{C_1\} F_1 \text{ and } F_1 \{C_2\} F \text{ then } G \{C_1; C_2\} F$
Rule of Iteration	$\text{If } (F \wedge B) \{C\} F \text{ then } F \{\text{while } B \text{ do } C\} \neg B \wedge F$

Table 2.2: Inference Rules for Valid Hoare Triple ²

Semantically, a Hoare triple $G \{C\} F$ is said to be valid for (partial) correctness, if the execution of the program C with an initial state satisfying the precondition G leads to a final state that satisfies the postcondition F , provided that the program terminates. The definitions in [Table 2.2](#) indeed correspond to this intended semantics. Formal soundness proofs can be found in Krzysztof R. Apt's survey [\[1\]](#) in 1981. As an example, consider the rule of composition: if the execution of program C_1 changes the state from G to F_1 , and C_2 changes the state from F_1 to F , then executing them consecutively should bring the program state from G to F , with the intermediate state F_1 .

The missing guarantee of termination can be seen in the rule of iteration: consider the triple $x \leq 2 \{\text{while } x \leq 1 \text{ do } x := x * 2\} 1 < x \leq 2$, it is provable in Hoare

¹ Non-determinism was not considered in the original paper, so we treat the programs here as deterministic. With deterministic programs, one initial state corresponds to one final state. In case of non-termination, there is simply no final state.

² We omit the symbol \vdash in front of a Hoare triple, which denotes "valid/provable", for better readability.

logic with the following proof tree. However, this while-loop will not terminate in case $x \leq 0$ in the initial state.

$$\frac{\frac{}{x \leq 1 \{x := x * 2\} x \leq 2} \text{Axiom of Assignment}}{x \leq 2 \{\text{while } x \leq 1 \text{ do } x := x * 2\} 1 < x \leq 2} \text{Rule of Iteration}$$

Using style taken from Kaminski's dissertation [14], Figure 2.1 illustrates a valid Hoare triple: Σ represents the set of all states, the section denoted with G includes the states that satisfy the predicate G . The arrows from left to right denote the executions of program C . The dashed arrows denote non-terminating executions.

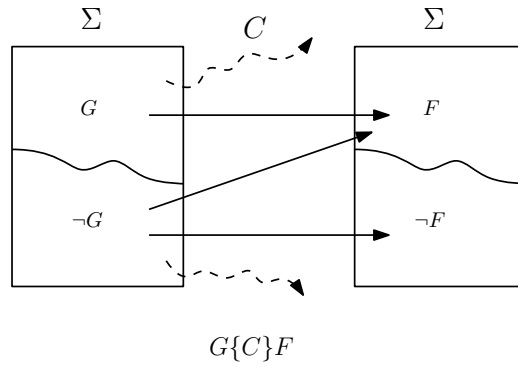


Figure 2.1: Valid Hoare Triple (Deterministic)

A sensible advancement of Hoare logic would be to also prove termination, i.e. to eliminate the arrows from G to the abyss. Supplementing Hoare logic with a termination proof is done by Zohar Manna and Amir Pnueli in 1974 [21], where they introduced what is called a **loop variant**, a value that decreases with each iteration. The name is in contrast to **loop invariant**, concretely the F in **Rule of Iteration** in Table 2.2, which is constant before and after the loop.

Another advancement would be to find the **necessary and sufficient** preconditions that grant us the post-properties, i.e. to eliminate the arrows from $\neg G$ to F in Figure 2.1, which is what Edsger W. Dijkstra accomplished with his **weakest precondition** transformer in 1975 [5], among other things.

2.3 GUARDED COMMAND LANGUAGE

From now on we use Dijkstra's (non-deterministic) **guarded command language (GCL)** [5] to represent programs and to include non-determinism (starting from Section 2.4.3). For better readability, we use an equivalent³ form of GCL that is similar to modern pseudo-code as shown in Table 2.3.

The **assignment**, **sequential composition**, **conditional choice**, **while-loop** commands conform to their usual meaning. The **non-deterministic choice** $\{C_1\} \square \{C_2\}$

³ Specifically, $\text{if } (\varphi) \{C_1\} \text{ else } \{C_2\}$ is equivalent to $\text{if } \varphi \rightarrow C_1 \square \neg\varphi \rightarrow C_2 \text{ fi}$ in Dijkstra's original style [5]; $\{C_1\} \square \{C_2\}$ is equivalent to $\text{if true} \rightarrow C_1 \square \text{true} \rightarrow C_2 \text{ fi}$.

$C ::= x := e$	$ C; C$	$ \{C\} \square \{C\}$
assignment	sequential composition	non-deterministic choice
$ \text{if } (\varphi) \{C\} \text{ else } \{C\}$	$ \text{while } (\varphi) \{C\}$	$ \text{skip} \quad \text{diverge}$
conditional choice	while-loop	

Table 2.3: Guarded Command Language

chooses from two programs non-deterministically. It is however not **probabilistic**, meaning we do not know the probabilistic distribution of the outcome of the choice.

When **skip** is executed, the program state does not change and the consecutive part is executed. When **diverge** is executed, the execution never stops and the program can not reach a final state.

In our representation of GCL, non-determinism is explicitly constructed via the infix operator \square , whereas in its original definition, non-determinism occurs when the guards within the **if** and **while** commands are not mutually exclusive [7]. Additionally, the **if** statement in Dijkstra’s GCL is equivalent to divergence in case non of its guards are true, but in our version this can no longer happen because of the Law of Excluded Middle: the predicate φ must be either true or false, so either the “then” branch or the “else” branch is activated. Consequently, non-termination can only originate from either the **diverge** or the **while** command.

2.4 WEAKEST PRECONDITIONS

2.4.1 The Deterministic Case

To better relate Hoare triples and Dijkstra’s weakest precondition transformer, we first focus on deterministic programs. The goal is to find the **necessary and sufficient** precondition such that the program is guaranteed to **terminate** in a state that satisfies the postcondition. Figure 2.2 shows it graphically alongside the figure for valid Hoare triples. We can see that in Figure 2.2.2, the arrows from G to non-termination and from $\neg G$ to F are absent.



Figure 2.2: Valid Hoare Triple vs. Weakest Precondition (Deterministic)

The definition of the [weakest precondition](#) transformer is inductively over the program structure in lambda-calculus style⁴ as in [Table 2.4](#):

C	wp.C.F
skip	F
diverge	false
$x := e$	$F[x/e]$
$C_1; C_2$	$wp.C_1.(wp.C_2.F)$
if (φ) { C_1 } else { C_2 }	$(\varphi \wedge wp.C_1.F) \vee (\neg\varphi \wedge wp.C_2.F)$
while (φ) { C' }	$\text{lfp } X.(\neg\varphi \wedge F) \vee (\varphi \wedge wp.C'.X)$

Table 2.4: The Weakest Precondition Transformer for Deterministic Programs [14]

A predicate P is [weaker](#) than Q , if and only if $Q \implies P$ is valid, and Q is then [stronger](#) than P . In this case we also say that P is an [overapproximation](#) of Q , and Q is an [underapproximation](#) of P .

$F[x/e]$ is F where every occurrence of x is syntactically replaced by e . $\text{lfp } X.f$ is the least fixed point of function f with variable X .

Let

$$\Phi(X) := (\neg\varphi \wedge F) \vee (\varphi \wedge wp.C'.X)$$

be the characteristic function, then wp for while-loop can be defined as:

$$wp.(\text{while}(\varphi)\{C'\}).F = \text{lfp } X.\Phi(X)$$

Most of the definitions in [Table 2.4](#) are intuitive and correspond to their counterparts in Hoare logic, while those for [diverge](#) and [while](#) deserve special attention. Since wp aims for total correctness, a program starting in an initial state

⁴ For example, $wp.C.F$ can be seen as $wp(C, F)$ in “typical” style, where wp is treated as a function that has two parameters. The advantage of lambda-calculus style is scalability, we can simply extend the aforementioned function to $wp.C.F.\sigma$ where σ means the initial state. Here wp is treated as a function that has three parameters, if we were to write it in the “typical” style. It is then questionable whether we changed the type of wp .

satisfying the precondition $\text{wp}.\text{diverge}.F$ should terminate in a final state satisfying the postcondition F . Because diverge does not terminate, there is no such precondition and wp for diverge should be *false*.

The definition for the while-loop [14] is trickier, but we can verify its correctness by recalling Dijkstra's original definition in the following section.

2.4.2 Loops and Fixed Points

DEFINING LOOPS In Dijkstra's original paper [5], he defined wp for while-loops based on its (intended) semantics, i.e. the precondition that guarantees loop termination with the required postcondition within a certain number of iterations.

Let

$$\text{WHILE} = \text{while}(\varphi)\{C'\} \quad \text{and} \quad \text{IF} = \text{if } (\varphi)\{C'\} \text{ else } \{\text{diverge}\}.$$

Note that IF is originally defined as $\text{if } B_1 \rightarrow SL_1 \ \square \ \dots \ \square \ B_n \rightarrow SL_n \ \text{fi}$, where B_i are guards, and SL_i are sub-programs that are executed once their corresponding guards are evaluated to true, thus the name *guarded* command language. If multiple guard are true, then any of the corresponding sub-programs can be executed, which is how non-determinism is realized in Dijkstra's GCL. If none of the guards is true, then the program does not terminate in a normal state, rather *diverges*.

In our flavor of GCL, non-determinism is denoted specifically with the operator \square , and $\text{if } (\varphi) \{C_1\} \text{ else } \{C_2\}$ does not lead to divergence in case its guard φ evaluates to true, rather the else-branch. In essence, the if in our flavor of GCL is equivalent to $\text{if } \varphi \rightarrow C_1 \ \square \ \neg\varphi \rightarrow C_2 \ \text{fi}$. As a result, to replicate $\text{IF} = \text{if } B_1 \rightarrow SL_1 \ \square \ \dots \ \square \ B_n \rightarrow SL_n \ \text{fi}$ where divergence is possible in case none of the guards is true, we need diverge instead of skip in the else-branch.

Rewriting Dijkstra's definition in a form conforming to our style, he defines

$$H_0(F) = (\neg\varphi \wedge F) \quad \text{and} \quad H_k(F) = \text{wp}.\text{IF}.H_{k-1}(F) \vee H_0(F).$$

IF is defined in such way that $\text{wp}.\text{IF}.X$ is the weakest precondition that makes sure the guard of C' discharges once and C' is executed once, leaving the program in a state satisfying X . As a result, $H_k(F)$ corresponds to the weakest precondition such that the program terminates in a final state satisfying F after *at most* k iterations.

Then by definition:

$$\text{wp}.\text{WHILE}.F = (\exists k \geq 0 : H_k(F)) = \bigvee_{k \geq 0} H_k(F) \quad (2.1)$$

LEAST FIXED POINTS The definition in Table 2.4, however, uses the least fixed point of the characteristic function $\Phi(X)$. We can understand the use of fixed point in two ways.

Firstly, a precondition G being a fixed point of the characteristic function implies that

$$G = \Phi(G) = (\neg\varphi \wedge F) \vee (\varphi \wedge \text{wp}.C'.G) \quad (2.2)$$

This means if G is satisfied before the execution of **WHILE**, then termination is possible (left side of the disjunction) and repeated execution of C' is possible (right side of the disjunction): For **WHILE** to enter the loop to execute C' , ϕ must evaluate to true. Plugging it in [Equation 2.2](#) results in an equation $G = \text{wp}.C'.G$, which means that G is invariant before and after the execution of C' . In other words, after one execution of C' , G stays satisfied, which makes a further execution of C' possible. In short, *all fixed points of function Φ can potentially lead to non-termination*.

Secondly, we know that the semantics of **WHILE** is equivalent to the semantics of $\text{if}(\phi)\{C'; \text{WHILE}\} \text{ else } \{\text{skip}\}$ [\[20\]](#). Note that here the else-branch is defined with **skip** instead of **diverge** like the previous paragraph, because in case $\neg\phi$ is true before the execution of **WHILE**, the program simply skips it and executes the next component. This corresponds to a **skip** in the else-branch of **IF**. We can then derive the need for fixed points:

$$\begin{aligned} \text{wp.WHILE.F} &\stackrel{!}{=} \text{wp.}(\text{if } (\phi)\{C'; \text{WHILE}\} \text{ else } \{\text{skip}\}).F \\ &\stackrel{!}{=} \phi \wedge \text{wp.}(C'; \text{WHILE}).F \vee \neg\phi \wedge \text{wp.}\text{skip}.F \\ &\stackrel{!}{=} \phi \wedge \text{wp}.C'.(\text{wp.WHILE.F}) \vee \neg\phi \wedge F \\ &\stackrel{!}{=} \Phi(\text{wp.WHILE.F}) \end{aligned}$$

The result $\text{wp.WHILE.F} = \Phi(\text{wp.WHILE.F})$ means that wp.WHILE.F should be a fix point of function Φ by definition.

The question then arises: can we define wp with any fixed point? The answer is no. In fact, Dijkstra and Scholten [\[7\]](#) later also gave definitions for wp and wlp in an equivalent form of least and greatest fixed points. They called it "strongest" and "weakest solution". They also proved that it is necessary to use the extreme solutions. Here, we show the need for **least** fixed points by verifying that the definition in [Table 2.4](#) coincides with Dijkstra's definition in [Equation 2.1](#). Thanks to domain theory, we have a heuristic to calculate the least fixed point of Φ .

Theorem 2.1 [*lfp heuristics*] [\[14\]](#)

$$\text{lfp } \Phi = \sup_{n \in \mathbb{N}} \Phi^n(\text{false})$$

Coincidentally, $H_k(F)$ is the $(k+1)$ -th iteration of the characteristic function Φ from the bottom element, denoted by $\Phi^{k+1}(\text{false})$. For all predicates F and all programs C' :

Lemma 2.2 [*Correspondance of H and Φ*]

$$\forall k \geq 0 : H_k(F) = \Phi^{k+1}(\text{false})$$

Proof. Proof by induction.

BASE CASE:

$$\begin{aligned}
 \Phi(\text{false}) &= (\neg\varphi \wedge F) \vee (\varphi \wedge \text{wp}.C'.\text{false}) \\
 &= (\neg\varphi \wedge F) \vee (\varphi \wedge \text{false}) && | (***) \\
 &= \neg\varphi \wedge F && | \text{predicate calculus} \\
 &= H_0(F)
 \end{aligned}$$

Line (**) is supported by the Law of Excluded Miracle [6, p.18]: for all programs C , $\text{wp}.C.\text{false} = \text{false}$. It states that it is impossible for a program to terminate in a state satisfying no postcondition.

STEP CASE:

$$\begin{aligned}
 H_{k+1}(F) &= \text{wp}.IF.H_k(F) \vee H_0(F) \\
 &= \text{wp}.\{\text{if } (\varphi)\{C'\} \text{ else } \{\text{diverge}\}\}.H_k(F) \vee H_0(F) \\
 &&& | \text{unfold IF} \\
 &= (\varphi \wedge \text{wp}.C'.H_k(F)) \vee (\neg\varphi \wedge \text{wp}.\text{diverge}.H_k(F)) \vee H_0(F) \\
 &&& | \text{definition of wp} \\
 &= (\varphi \wedge \text{wp}.C'.H_k(F)) \vee (\neg\varphi \wedge \text{false}) \vee H_0(F) \\
 &&& | \text{definition of wp} \\
 &= (\varphi \wedge \text{wp}.C'.\Phi^{k+1}(\text{false})) \vee H_0(F) \\
 &&& | \text{induction hypothesis} \\
 &= (\varphi \wedge \text{wp}.C'.\Phi^{k+1}(\text{false})) \vee (\neg\varphi \wedge F) \\
 &= \Phi^{k+2}(\text{false})
 \end{aligned}$$

□

Combining Theorem 2.1 and Equation 2.1, as well as the fact that $\Phi^0(\text{false}) = \text{false}$, we can arrive at the conclusion that

$$\text{lfp } \Phi = \text{wp}.WHILE.F$$

i.e. the definitions with the least fixed point and Dijkstra's definition are equivalent. And since least fixed points are unique if they exist, it is necessary to use the **least** fixed point to define wp.

Intuitively, all the fixed points of function Φ can potentially lead to non-termination, but can **not guarantee** termination. Only the least fixed point can guarantee termination. Remember from Theorem 2.1 that $\text{lfp } \Phi$ is the disjunction of $\Phi^k(\text{false})$ for all non-negative k , whereas Lemma 2.2 tells us that $H_k(F) = \Phi^{k+1}(\text{false})$.

Recall from Equation 2.1 that $H_k(F)$ is the weakest precondition that the program terminates satisfying F after **at most** k iterations, so $\text{lfp } \Phi$ is

WHILE terminates in at most **0** iterations
 or WHILE terminates in at most **1** iterations

or WHILE terminates in at most 2 iterations

...

for all $k \in \mathbb{N}$. Now assume that some fixed point p of Φ guarantees termination in k steps, since $\text{lfp } \Phi$ is the disjunction of all preconditions that guarantee the termination of WHILE in certain steps, we know that $\text{lfp } \Phi \Leftrightarrow \dots \vee p \vee$, hence $p \Rightarrow \text{lfp } \Phi$. Since $\text{lfp } \Phi$ is the **least** fixed point, there can be no other fixed points that are weaker than the least one. In conclusion, p must be the least fixed point of Φ .

This tells us that all fixed points that are not the least one can potentially lead to non-termination, but only the least fixed point guarantees termination. Since wp is concerned with total correctness, the need for lfp arises.

The advantage of using the least fixed point to define wp is that there are heuristics to find it, whereas Equation 2.1 excels at giving intuitions for the preconditions that guarantee loop termination.

GREATEST FIXED POINT Following from the statement that *all fixed points that are not the least one can potentially lead to non-termination*, it comes naturally that we need the **greatest** fixed point when defining the weakest **liberal** precondition, a precondition that includes **all** initial states that can potentially lead to non-termination (see Figure 2.3). Since all the fixed points can lead to non-termination, we require that $p \Rightarrow \text{wlp.WHILE.F}$, where p is a fixed point of Φ . Following the same reasoning as Equation 2.2, we can conclude that wlp.C.F is a fixed point of Φ . Hence, by definition, $\text{wlp.WHILE.F} = \text{gfp } \Phi$.

2.4.3 The Non-deterministic Case: Angelic vs. Demonic

Now we bring the non-deterministic choice back into the picture and add its wp to Table 2.5. Here we assume a setting with **angelic non-determinism**, where we assume that whenever non-determinism occurs, it will be resolved in our favor. This means we are optimistic that the non-determinism is likely to resolve to our favor, meaning that we consider an initial state **can** lead the program execution to terminate in a final state satisfying F , we consider it a valid initial state to satisfy the weakest (liberal) precondition. Conversely, if we consider the non-deterministic choice to be demonic, then we require that all executions from an initial state **must** satisfy F for this state to be included in the set of states described by the weakest (liberal) precondition.

This results in the weakest precondition for our non-deterministic choice being a disjunction of the wp for its subprograms. We are hopeful that a precondition satisfying the wp of one of the subprograms can also lead to termination in our desired postcondition. This is a design choice that is different from Dijkstra's [5], where the wp for non-deterministic choice is a conjunction, hinting at a demonic setting. Both choices are justifiable, we choose to follow Zhang and Kaminski's work, favoring the resulting Galois connection between the weakest (liberal) precondition transformers and the strongest (liberal) postcondition transformers [31].

C	$wp.C.F$	$wlp.C.F$
skip	F	F
diverge	false	true
$x := e$	$F[x/e]$	$F[x/e]$
$C_1; C_2$	$wp.C_1.(wp.C_2.F)$	$wlp.C_1.(wlp.C_2.F)$
$\{C_1\} \square \{C_2\}$	$wp.C_1.F \vee wp.C_2.F$	$wlp.C_1.F \wedge wlp.C_2.F$
if (φ) $\{C_1\}$ else $\{C_2\}$	$(\varphi \wedge wp.C_1.F) \vee (\neg\varphi \wedge wp.C_2.F)$	$(\varphi \wedge wlp.C_1.F) \wedge (\neg\varphi \wedge wlp.C_2.F)$
while (φ) $\{C'\}$	$lfp X. (\neg\varphi \wedge F) \vee (\varphi \wedge wp.C'.X)$	$gfp X. (\neg\varphi \wedge F) \wedge (\varphi \wedge wlp.C'.X)$

Table 2.5: The Weakest (Liberal) Precondition Transformer for Non-deterministic Programs [14]

Figure 2.3.1 shows wp with non-deterministic programs. Each arrow from left to right shows a **possible** execution of program C . The effects of demonic and angelic non-determinism is highlighted in green. A condition under whose control the required postcondition is **reachable but not guaranteed** is considered as a valid precondition in an angelic setting (Figure 2.3.1), but not in a demonic setting (Figure 2.3.2).

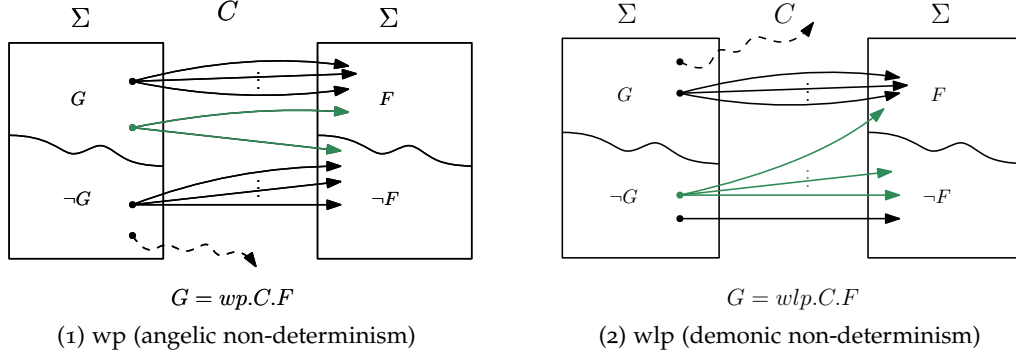


Figure 2.3: Weakest Precondition (Angelic Non-determinism) and Weakest Liberal Precondition (Demonic Non-determinism)

2.5 WEAKEST LIBERAL PRECONDITIONS

While the wp -transformer excludes non-termination, the wlp -transformer takes a more liberal approach. The weakest precondition delivers a precondition so that the program terminates and a state satisfying the postcondition is **reachable**. The weakest liberal precondition, however, delivers a precondition so that the program either terminates satisfying the postcondition, or diverges. The postcondition in the wlp setting is **guaranteed** upon termination, because we regard

the non-deterministic choice as demonic, again favoring to establish a Galois connection [31].

The definition of the weakest liberal precondition transformer is in Table 2.5. A graphical representation can be found on Figure 2.3.2.

As preluded earlier, greatest fixed points are used to define wlp for while-loops. It is an easy choice, since wlp is semantically the **weakest** liberal precondition, and wlp.WHILE.F should be a fixed point of its characteristic function, similar to Section 2.4.2.

Theorem 2.3 [14][gfp heuristics] $\text{gfp } \Phi = \sup_{n \in \mathbb{N}} \Phi^n(\text{true})$

2.6 STRONGEST POSTCONDITIONS

Following the style to define wp and wlp, Zhang and Kaminski [31] (re-)defined **strongest postconditions** that capture the characteristics of all reachable states after the execution. In essence, sp.C.G is a postcondition that is satisfied by **all** states that are **reachable** from G . The definition of the predicate transformer sp is shown in Table 2.6.

C	sp.C.G
skip	G
diverge	false
$x := e$	$\exists a. x = e[x/a] \wedge G[x/a]$
$C_1; C_2$	$\text{sp.C}_2.(\text{sp.C}_1.G)$
$\{C_1\} \Box \{C_2\}$	$\text{sp.C}_1.G \vee \text{sp.C}_2.G$
if (φ) $\{C_1\}$ else $\{C_2\}$	$\text{sp.C}_1.(\varphi \wedge G) \vee \text{sp.C}_2.(\neg \varphi \wedge G)$
while (φ) $\{C'\}$	$\neg \varphi \wedge \text{lfp } X.G \vee \text{sp.C.}(\varphi \wedge X)$

Table 2.6: The Strongest Postcondition Transformer [31]

We can also illustrate the behavior of a program controlled by sp in Figure 2.4. Instead of discussing termination starting from a precondition, sp focuses on reachability of states satisfying postconditions. The dotted arrow points to postconditions describing unreachable final states after the execution of C . For example, no state would satisfy $x = 2$ after the execution of $x := 1$.

2.7 BIG STEP SEMANTICS

To express the meaning of GCL programs, we first attempt to use **big-step semantics** to describe executions of a program. This later proved infeasible for our purpose, because of the absence of inference rules for non-termination, which we still show with an example. Still, big-step semantics generates simple proofs, so we still show it for GCL programs and hope that it can prove useful for future

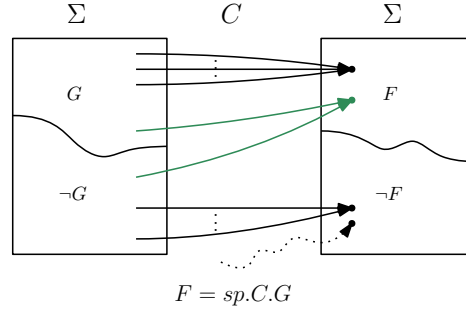


Figure 2.4: Strongest Postcondition (Angelic Non-determinism)

$\frac{}{\sigma \xrightarrow{\text{skip}} \sigma} \text{skip}$	$\frac{}{\sigma \xrightarrow{x:=e} \sigma(x := \sigma.e)} \text{assign}$
$\frac{\sigma \xrightarrow{C_1} \mu, \mu \xrightarrow{C_2} \tau}{\sigma \xrightarrow{C_1; C_2} \tau} \text{seq}$	$\frac{\sigma \xrightarrow{C_i} \tau, i \in \{1, 2\}}{\sigma \xrightarrow{C_1 \sqcup C_2} \tau} \text{choice}_i$
$\frac{\sigma \in \varphi, \sigma \xrightarrow{C_1} \tau}{\sigma \xrightarrow{\text{IF}} \tau} \text{if}_1$	$\frac{\sigma \notin \varphi, \sigma \xrightarrow{C_2} \tau}{\sigma \xrightarrow{\text{IF}} \tau} \text{if}_0$
$\frac{\sigma \notin \varphi}{\sigma \xrightarrow{\text{WHILE}} \sigma} \text{while}_0$	$\frac{\sigma \in \varphi, \sigma \xrightarrow{C} \mu, \mu \xrightarrow{\text{WHILE}} \tau}{\sigma \xrightarrow{\text{WHILE}} \tau} \text{while}_n$
$\frac{\perp \notin \Sigma}{\sigma \xrightarrow{\text{diverge}} \perp} \text{diverge}$	

where IF=if (φ) { C_1 } else { C_2 }, WHILE=while (φ) do { C }, and $\sigma, \tau, \mu \in \Sigma$.

Table 2.7: Big Step Semantics

endeavors. Taking inspiration from Nipkow and Klein's book [26] we define the big-step semantics in Table 2.7.

Note that in rule assign, we have a formula $\sigma(x := \sigma.e)$. Here we overload the symbol for function application "." so that it applies to expressions as well, but without specifying the set of expressions hence restricting our programming language. An expression e would be evaluated in a usual way, e.g. $x + y$ at state σ would evaluate to $\sigma.x + \sigma.y$.

We also use symbols $\sigma(x := v)$ to denote a state where the value of variable x is v , and all other variables have the same values as in σ . In our big-step semantics, divergence is defined to lead to state \perp that is not a part of Σ .

Let $\sigma \in \Sigma$ be a non-trivial state, where non-trivial means that $\sigma \models \text{true}$ is valid. To denote non-termination of a program C with initial state σ , a simple approach is to consider it as equivalent to the non-existence of a state $\tau \in \Sigma$ that $\sigma \xrightarrow{C} \tau$. This covers the case of divergence, since the only inference rule for diverge goes to a state that is not in Σ . However, we can not express the non-termination of programs such as

$$D := (x := y \sqcup \text{while}(\text{true}) \text{ do } \{x := z\})$$

First, there is no rule to prove the while-loop on the right hand of the choice. Since $\sigma \notin \text{true}$ is the same as with our notation, $\sigma \notin \{\sigma \in \Sigma \mid \sigma \models \text{true}\}$ can not be valid, since we are looking at a non-trivial σ . Consequently, we can not use the while_0 rule, hence the last premise of while_n rule will never be fulfilled for this while-loop. As a result, we can only prove that

$$\frac{\frac{\sigma \xrightarrow{x:=y} \sigma(x := \sigma.y)}{\sigma \xrightarrow{x:=y \square \text{while } (\text{true}) \text{ do } \{x:=z\}} \sigma(x := \sigma.y)} \text{choice}_1}{\sigma \xrightarrow{x:=y} \sigma(x := \sigma.y)} \text{assign}$$

As a result, there is $\tau \in \Sigma$ such that $\sigma \xrightarrow{D} \tau$, and we can not express the non-termination of D with the non-existence of τ such that $\sigma \xrightarrow{D} \tau$, as we proposed. In other words, we can not distinguish D from $x := y$ in this big-step semantics without reasoning about the structure or length of the proof tree.

In conclusion, this version of big-step semantics proves inadequate to reason with some non-terminating executions. Granted, one can bypass this problem [17, 24] or turn to small-step semantics [24] and use paths to distinguish between executions, but these methods are not simplistic for our purpose, and we choose another type of semantics that is more suitable, namely the collecting semantics.

2.8 COLLECTING SEMANTICS

Zhang and Kaminski assign meaning to GCL programs with a collecting semantics [32]. Informally, we “collect” reachable states via program execution starting from a set of initial states. The definition is shown in Table 2.8.

Let $\text{Conf} = \mathcal{P}(\Sigma)$, the set of **configurations** in form of the power set of the set of states. An element of the Conf set is then a set of program states. For a configuration S , a **filtering** $\llbracket \varphi \rrbracket S := \{\sigma \mid \sigma \in S \wedge \sigma \models \varphi\}$ is the set of program states in S that satisfy the predicate φ . The circle \circ is composition as usual: $(\llbracket C_2 \rrbracket \circ \llbracket C_1 \rrbracket)S = \llbracket C_2 \rrbracket(\llbracket C_1 \rrbracket S)$.

C	$\llbracket C \rrbracket S$
skip	S
diverge	\emptyset
$x := e$	$\{\sigma[x/\sigma(e)] \mid \sigma \in S\}$
$C_1; C_2$	$(\llbracket C_2 \rrbracket \circ \llbracket C_1 \rrbracket)S$
$\{C_1\} \square \{C_2\}$	$\llbracket C_1 \rrbracket S \cup \llbracket C_2 \rrbracket S$
if $(\varphi) \{C_1\}$ else $\{C_2\}$	$(\llbracket C_1 \rrbracket \circ \llbracket \varphi \rrbracket)S \cup (\llbracket C_2 \rrbracket \circ \llbracket \neg \varphi \rrbracket)S$
while $(\varphi) \{C'\}$	$\llbracket \neg \varphi \rrbracket(\text{lfp } X. S \cup (\llbracket C' \rrbracket \circ \llbracket \varphi \rrbracket)X)$

Table 2.8: Collecting Semantics for GCL [32]

Originally, the collecting semantics for diverge and skip was not explicitly given, but since $\llbracket \cdot \rrbracket$ “collects” reachable final states, $\llbracket \text{skip} \rrbracket S$ should return S , and $\llbracket \text{diverge} \rrbracket S$ should be the empty set for any S .

We see the difference between this collecting semantics and the big-step semantics for GCL [Section 2.7](#) by looking at the same program D:

Remember from [Section 2.1](#) that a predicate is a function that maps program states to truth values, and it can be written in a set form: $F = \{\sigma \in \Sigma \mid F(\sigma)\}$. This gives us two interesting insights:

First, a configuration corresponds to a class of equivalent predicates. Note that we can have predicates in different form that identify the same mapping from program states to truth values, for example

$$P \implies Q \text{ and } \neg P \vee Q$$

Then the set form of multiple predicates can be identical to the set of program states described by the configuration. Nevertheless, we can see that a configuration is not much different from a predicate. Second, a filtering is a set intersection. Rewriting the filtering, we get $\llbracket \varphi \rrbracket S = \{\sigma \mid \sigma \in S \wedge \sigma \in \varphi\} = \varphi \cap S$.

With collecting semantics, we can define the soundness of the predicate transformers.

2.9 SOUNDNESS

Remember that a predicate is a function from program states to truth values (see [Section 2.1](#)). Then we can apply a predicate to a program state to get its truth value: $\text{wlp}.C.F.\sigma$, showing the advantage of using lambda-calculus style notation for function application.

Theorem 2.4 [*Soundness of wp*] [\[31\]](#)

$$\text{wp}.C.F.\sigma = \bigvee_{\tau \in \llbracket C \rrbracket \{\sigma\}} F.\tau$$

$\text{wp}.C.F$ is satisfied exactly by the initial states starting from which the execution of C terminates, and a final state satisfying F is always reachable. We can rewrite [Theorem 2.4](#) as⁵:

$$\text{wp}.C.F.\sigma = \exists \tau \in \llbracket C \rrbracket \{\sigma\} : F.\tau$$

Recall from [Section 2.1](#) that predicate F also defines a set $\{\sigma \in \Sigma \mid F.\sigma\}$, then we can rewrite the equation above as:

Corollary 2.5 [*Soundness of wp*]

$$\text{wp}.C.F = \{\sigma \in \Sigma \mid \exists \tau \in \llbracket C \rrbracket \{\sigma\} : F.\tau\}$$

Theorem 2.6 [*Soundness of wlp*] [\[7, 31\]](#)

$$\text{wlp}.C.F.\sigma = \bigwedge_{\tau \in \llbracket C \rrbracket \{\sigma\}} F.\tau$$

⁵ $\exists \tau \in \Sigma : P$ is short for $\exists \tau. \tau \in \Sigma : P$

$\text{wlp}.C.F$ is satisfied exactly by initial states under whose control the execution of C either never terminates or terminates satisfying F

Similar as before, we can rewrite [Theorem 2.6](#) as:

$$\text{wlp}.C.F.\sigma = \forall \tau \in \llbracket C \rrbracket \{\sigma\} : F.\tau$$

And consequently:

Corollary 2.7 [Soundness of wlp]

$$\text{wlp}.C.F = \{\sigma \in \Sigma \mid \forall \tau \in \llbracket C \rrbracket \{\sigma\} : F.\tau\}$$

From [Theorem 2.4](#) and [Theorem 2.6](#), we see again that the resolving the non-determinism of wp as angelic and that of wlp is demonic is a good choice. It yields a simple and elegant way to define the soundness properties of wp and wlp using the collecting semantics, which the demonic resolution for wp and angelic resolution for wlp can not achieve.

To support the reasoning in [Section 4.3](#), we also need the soundness property of sp:

Theorem 2.8 [Soundness of sp] [[29](#), [31](#)]

$$\text{sp}.C.G.\tau = \bigvee_{\sigma \text{ where } \tau \in \llbracket C \rrbracket \{\sigma\}} F.\sigma$$

$\text{sp}.C.G$ is satisfied exactly by final states that are definitely reachable from some initial state satisfying G . Similarly, we can rewrite the soundness property of sp as:

$$\text{sp}.C.G.\tau = \exists \sigma \in \Sigma : \tau \in \llbracket C \rrbracket \{\sigma\} \wedge F.\tau$$

And consequently:

Corollary 2.9 [Soundness of sp]

$$\text{sp}.C.G = \{\tau \in \Sigma \mid \exists \sigma \in \Sigma : \tau \in \llbracket C \rrbracket \{\sigma\} \wedge F.\tau\}$$

2.10 PROPERTIES OF WP AND WLP

We also include some properties of predicate transformers in this section that we will need later in [Chapter 4](#) for our proofs.

Theorem 2.10 [Conjugates] [[31](#)] *wp and wlp are each other's conjugates:*

$$\forall C \in \mathcal{C} : \forall F \in \mathcal{P} : \text{wp}.C.F = \neg \text{wlp}.C.\neg F$$

This property can be rewritten as $\text{wp}.C.F \vee \neg \text{wlp}.C.\neg F = \text{true}$. It tells us that wp and wlp complement each other, like in [Figure 2.5](#).

Theorem 2.11 [Monotonicity] [[7](#)] *For any $C \in \mathcal{C}$, wlp.C is monotonic:*

$$\forall X, Y \in \mathcal{P} : X \implies Y \implies (\text{wlp}.C.X \implies Y)$$

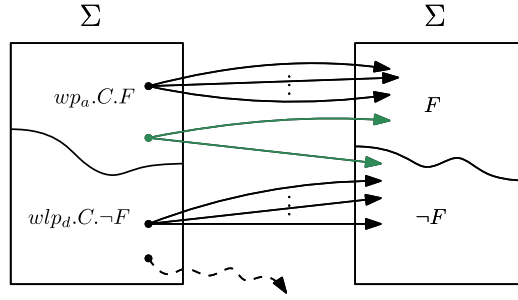


Figure 2.5: wp and wlp Are Conjugates

It states that if we relax the postcondition into a weaker, more general condition, then we should be able to find more initial states that can lead to final states that satisfy this weaker postcondition upon termination.

Theorem 2.12 [7] $\forall C \in \mathcal{C} : \text{wlp}.C.\text{true} = \text{true}$

The left side of the equation describes the set of initial states that “either never terminates, or terminates satisfying true”, which describes all initial states, hence it should equal to the set of all states, i.e. a predicate that maps all states to true.

With these theorems, we finish this chapter and the first part of this thesis. In the first part, we explain our motivation to study the predicate transformer, then proceed to discuss background information by first list the notations used throughout this thesis. Following this, we showcased the relevant formalism for this thesis, namely Hoare triples, the weakest (liberal) precondition transformer, and the strongest postcondition transformer.

During this process, we give intuition behind the definitions we use, and compare them to their original form. We also discuss the different perspectives to resolve non-determinism: angelic versus demonic, and give an excursion about the use of least and greatest fixed points in our definition. We also show our attempts to grasp the soundness properties of the predicate transformers using the big-step semantics, and conclude its ineptness for our purpose. In conclusion, we choose the collecting semantics to describe the soundness properties of our predicate transformers, and mention some other properties as well.

Concluding the first part, we proceed in the second part to further study the focus of this thesis. We call it the **necessary liberal precondition**. **Necessary**, because its violation will lead to violation of the given postcondition, the definition of a necessary condition in logical terms. **Liberal**, because we take a liberal view on non-termination. We accept initial states that can lead to non-termination as satisfying our precondition. **Precondition**, well, obviously.

Part II

NECESSARY LIBERAL PRECONDITIONS

In this part, I discuss the possible scenarios of the inspected implication, identify a distinctive case to study before proceeding with the general setting.

PRECONDITIONS, POSTCONDITIONS, AND THEIR APPROXIMATIONS

In this chapter, we give an overview of precondition and postcondition transformers presented in the previous chapter, establishing relations between all of them using termination, conjugation, reachability, and Galois connections. This reveals that the necessary liberal precondition triple that we investigate in this thesis is still an open question in the sense that it is not yet well-researched to our knowledge, and that it is not automatically related to another triple by the aforementioned relations. Furthermore, the methods mentioned in this chapter used to study other triples prove helpful while investigating necessary liberal preconditions.

3.1 PRECONDITION TRANSFORMERS

Figure 3.1 shows the impact of the interpretation of non-determinism on the semantics of the wp and wlp transformers. The subscripts _a and _d correspond to angelic and demonic resolution of non-determinism mentioned in Section 2.4.3, respectively. The notion of angelic and demonic is reflected by the placement of green dots and arrows in Figure 3.1.

In general, the rectangles denote the set of all states Σ . C is the program that we are investigating, all the arrows are executions of program C . When a rectangle is divided by a line into multiple parts, it means that the states are categorized into different types. For example, the section labelled with $wlp.C.F$ denotes all the states that satisfy $wlp.C.F$. The dashed arrow starting from inside a rectangle, ending outside any rectangle denotes executions that do not terminate. The arrows that share the same starting point denote different executions starting from the same initial state, and those that share an ending point denote executions that end in the same final state.

With angelic non-determinism, we consider the non-deterministic choice to resolve in our favor, which means that if one of the possible paths from this non-deterministic choice can lead to the desired postcondition, we regard the precondition as a valid precondition. Graphically, the green arrows in Figure 3.1.1 and Figure 3.1.3 are included in the weakest (liberal) precondition when we regard the non-determinism as angelic. Formally, the decision to look at the non-deterministic choice as angelic is reflected by the use of disjoints in its definition, for example,

$$wp.C.(\{C_1\} \sqcup \{C_2\}) = wp.C_1.F \vee wp.C_2.F$$

for a postcondition F in Table 2.5.

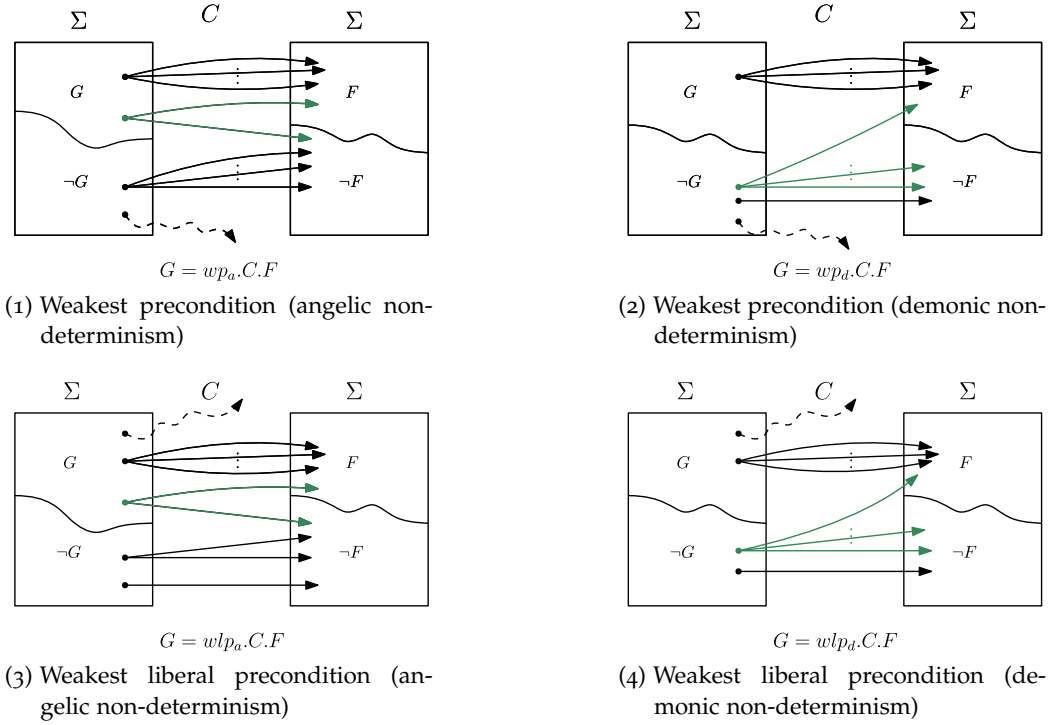


Figure 3.1: wp, wlp with Angelic and Demonic Non-determinism

On the other hand, if the non-deterministic choice is resolved as demonic, we are pessimistic over the result of the non-deterministic choice: we assume it will not resolve to our favor, hence we are only confident to include a condition into the valid precondition set if all possible paths end in a final state satisfying our desired postcondition upon termination. Graphically, the green arrows are not included in wp_d and wlp_D in Figure 3.1.2 and Figure 3.1.4. The definition is then using conjuncts instead of disjuncts, like

$$wlp.C.(\{C_1\} \sqcap \{C_2\}) = wlp.C_1.F \wedge wlp.C_2.F$$

for a postcondition F in Table 2.5.

The choice is somewhat artificial, depending on what properties the author wishes for their transformers. For example, Dijkstra and Scholten choose demonic non-determinism for wp and wlp transformers, acquiring a property by definition [7]:

$$wp_d.C.F = wlp_d.C.F \wedge wp_d.C.true$$

This means that by finding the weakest liberal precondition and proving termination, one conveniently finds the weakest precondition of the same program and postcondition. Graphically, one needs only to re-place the dashed arrows indicating non-termination to get from Figure 3.1.1 to Figure 3.1.3 and vice versa, or from Figure 3.1.2 to Figure 3.1.4 and vice versa.

This is especially useful when trying to underapproximate the weakest precondition, or in other words, finding valid Hoare Triple, considering that $G \{C\} F$ is a valid Hoare Triple (with regard to total correctness, meaning that we supple-

ment the original axioms [11] with rules to prove termination [21]) if and only if $G \Rightarrow wp.C.F$.

Conversely, if one were to choose different views on non-determinism with wp and wlp transformers, for example angelic resolution for wp and demonic resolution for wlp, we end up with a different property that wp and wlp are each other's *conjugates* [6, 31]:

$$wp_a.C.F = \neg wlp_d.C.\neg F$$

This indicates a duality between wp_a and wlp_d in the sense that by defining one transformer, we can arrive at the other by negation. Graphically, we can “flip” Figure 3.1.2 to arrive at Figure 3.1.3 and vice versa, or “flip” Figure 3.1.1 to arrive at Figure 3.1.4 and vice versa, like in Figure 3.2.

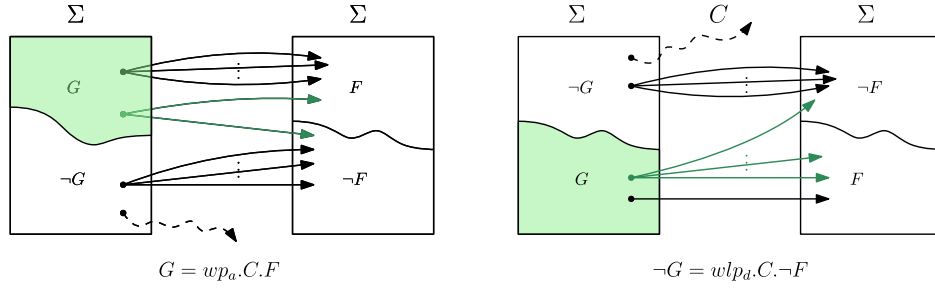


Figure 3.2: wp_a and wlp_d Are Conjugates

These relations can be summed up as in Figure 3.3. The choice whether to resolve the non-determinism angelically or demonically for wp and wlp depends then on which of the relations the author prefers.

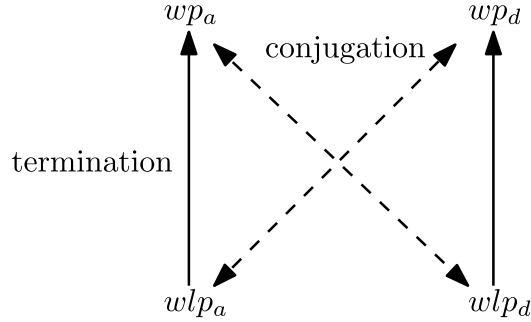


Figure 3.3: Relations in wp-Family

In this thesis, we prefer wp with angelic non-determinism and wlp with demonic non-determinism, because this choice gifts us Galois connection between the precondition transformers and the postcondition transformers [31], which we will illustrate in Section 3.3:

Theorem 3.13 [Galois connection between wp and slp]

$$wp_a.C.F \Rightarrow G \text{ if and only if } F \Rightarrow slp_d.C.G$$

Theorem 3.14 [Galois connection between wlp and sp]

$$G \implies \text{wlp}_d.C.F \text{ if and only if } \text{sp}_a.C.G \implies F$$

3.2 POSTCONDITION TRANSFORMERS

The strongest postcondition was originally discussed as a side-product of the wp and wlp transformers by Dijkstra and Scholten [7]. $\text{sp}.S.Y$ is meant to describe “those final states for which there exists a computation under control of S that belongs to the class *initially* Y ”, but it has found more important utilities. For example, de Vries and Koutavas [29] use sp to reason with randomized algorithms, although they called it **weakest postconditions**. O’Hearn [28] uses sp to define **incorrectness logic** by underapproximating it, building the theoretical foundation to a new method of identifying errors in programs.

Instead of questioning about termination and final states upon termination, like the precondition transformers, the postcondition transformers question about reachability and initial states that can reach the desired postconditions. The choice of angelic and demonic non-determinism also plays a rule while determining the semantics of the postcondition transformers, as shown in Figure 3.4.

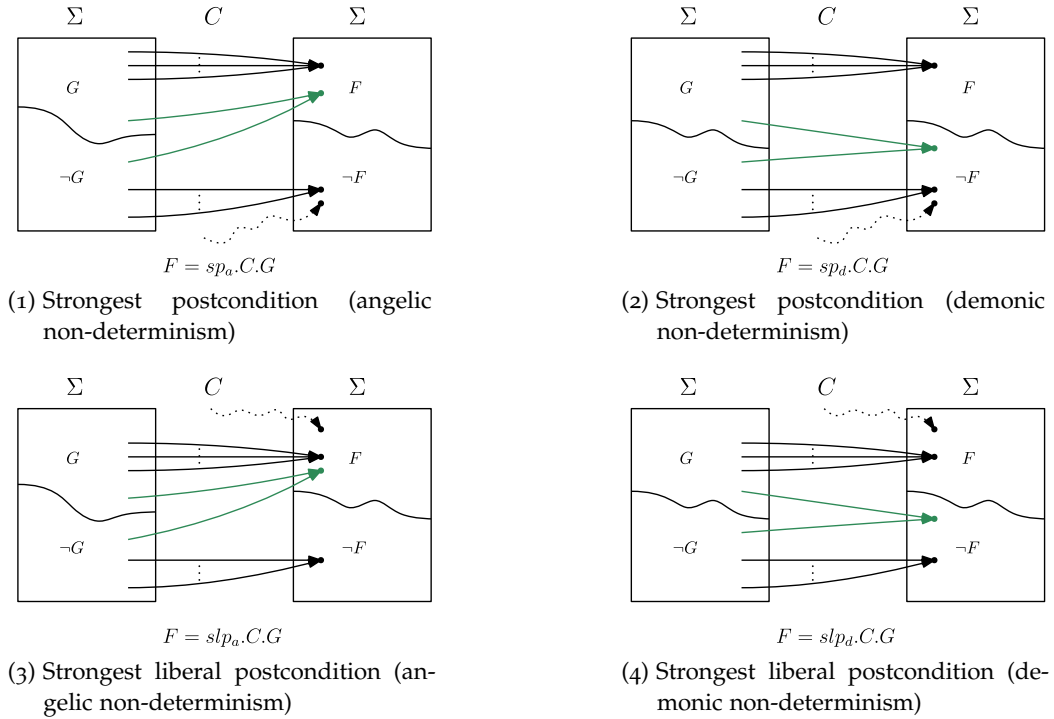


Figure 3.4: sp , slp with Angelic and Demonic Non-determinism

Note that while sp is well-researched, slp is less investigated to the best of our knowledge. For example, Wulandari and Plump [30] defined and proved soundness for slp , but only for loop-free programs. While Li et al. [18] proposed the definition of slp including loops, but did not include proof for soundness

and completeness. Even so, the semantics of slp is rather undisputed: slp should be sp joint with all unreachable postconditions.

Following the same principles as the ones for precondition transformers, we can deduce a similar relation between the postcondition transformers with different resolution for non-determinism, as seen in Figure 3.5. The main difference is that the arrows pointing up are not labelled with termination, but reachability.

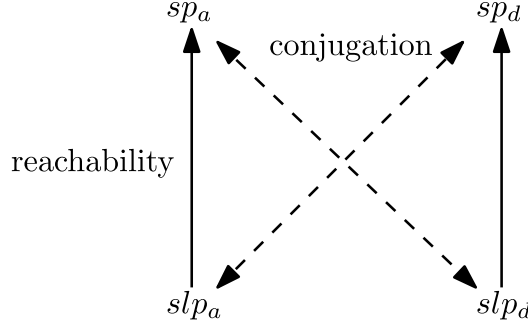


Figure 3.5: Relations in sp -Family

3.3 APPROXIMATION TRIPLES IN LITERATURE

Supplementing the Galois connection specified in Theorem 3.13 and Theorem 3.14, we arrive at Figure 3.6. It is now obvious that we chose to resolve non-determinism exactly like the colored ones in Figure 3.6, so that we can conveniently connect the pre- and postcondition transformers.

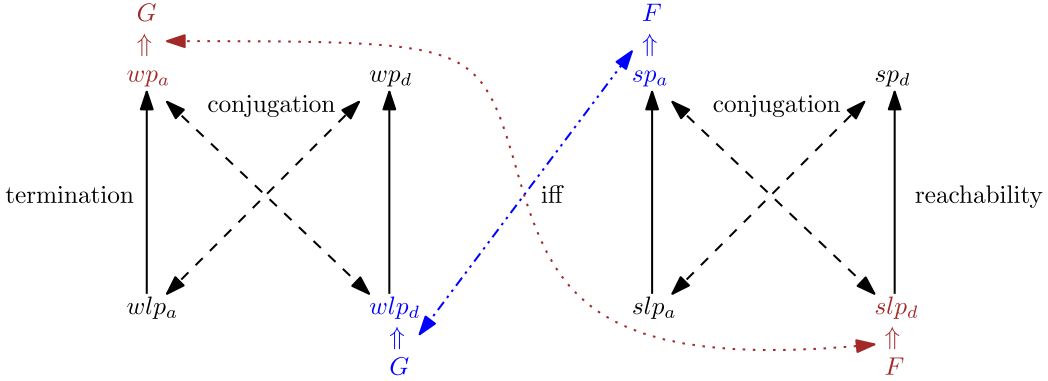


Figure 3.6: Connecting wp -Family and sp -Family

The advantage of the design choice of wp and sp with angelic non-determinism as well as wlp and slp with demonic non-determinism shows itself now. By choosing the formalism that are colored in Figure 3.6, all the under- and over-approximations over the predicate transformers are related well together, as we depict in Figure 3.7.

Some of the implications in Figure 3.6 are well-studied by now, as we will see in the upcoming paragraphs. Since the view on angelic or demonic non-

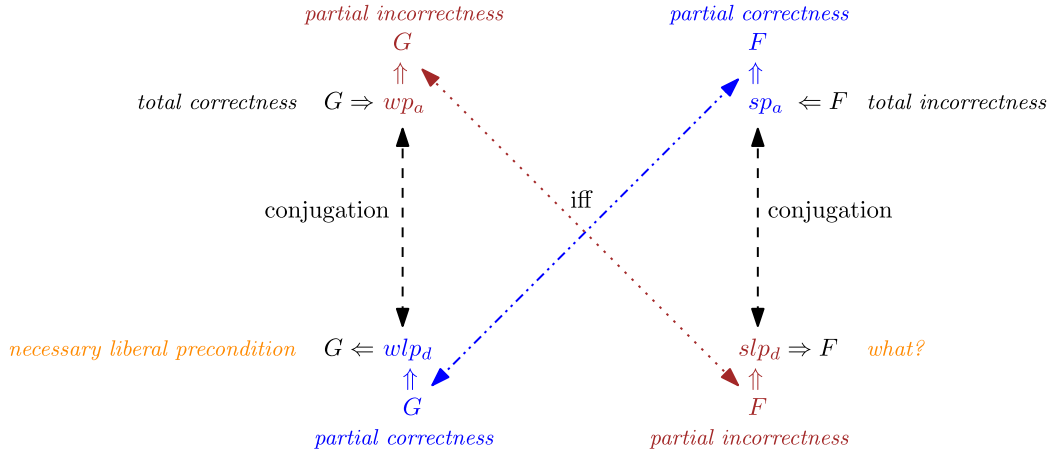


Figure 3.7: Connecting Overapproximations and Underapproximations

determinism varies depending on the authors' choice, and do not make fundamental differences on their results, we will mention it in the description, but drop the subscripts $_a$ and $_d$ in the remainder of the section.

PARTIAL CORRECTNESS For example, the blue implication

$$G \implies \text{wlp}.C.F$$

corresponds to a Hoare triple [11] with respect to partial correctness: $G \{C\} F$ is valid if and only if $G \implies \text{wlp}.C.F$ [9, 27]. This means that if the program C starts in a state satisfying G , then the execution must end in a final state satisfying F , if it terminates. In terms of approximation, we can say that the precondition in the valid Hoare triple is an underapproximation of the weakest precondition.

Hoare triple does not originally contain rules for non-deterministic choice, but it can be added without much difficulty, for example with demonic non-determinism [25]:

$$\frac{G \{C_1\} F, G \{C_2\} F}{G \{C_1 \sqcup C_2\} F}$$

As we can see in Figure 3.6, the blue implications satisfy the if-and-only-if relation. This means that an overapproximation of the strongest postcondition

$$\text{sp}.C.G \implies F$$

also guarantees partial correctness, aka $G \{C\} F$ is a valid Hoare triple.

TOTAL CORRECTNESS Manna and Pnueli [21] supplemented Hoare triple with rules to prove termination, making the extended proof system valid for total correctness:

$$\frac{P \wedge \phi \wedge v \in \mathbb{N} \wedge v = n \quad \{C\} \quad P \wedge \phi \wedge v \in \mathbb{N} \wedge v < n}{P \wedge v \in \mathbb{N} \quad \{\text{while } (\phi) \text{ do } C\} \quad \neg \phi \wedge P \wedge t \in \mathbb{N}}$$

P is the so-called **invariant**, a condition that stays unchanged before and after the program execution. v in this case is the **variant**, it is modified by the execution of the program. Specifically, v reduces with each execution of the sub-program C inside the loop body. Since v is a natural number, it is never negative, i.e. bound from the bottom. Naturally, execution of the loop has to stop at some point, the loop body C can be executed maximum v times. This version of Hoare triple proves total correctness (termination and correctness), and it corresponds to the implication labelled with “total correctness” in Figure 3.6:

$$G \implies wp.C.F$$

TOTAL INCORRECTNESS In the correctness realm, we are concerned with termination and guaranteeing that only good things can happen in the end. Changing the perspective to focus on reachability of erroneous final states, we end up in the incorrectness realm. The implication marked with “total incorrectness” in Figure 3.6 was studied by O’Hearn [28] under the name “incorrectness logic”, and by de Vries and Koutavas [29] with the name “reverse Hoare logic”:

$$F \implies sp.C.G$$

With this triple, we can prove the reachability of states described by F starting from the initial condition satisfying G . We can view it as an indication that the program C was not constructed correctly, and bugs described by F are reachable [28]; or use it to prove that the final states in F are reachable via randomized programs [29].

They all take an angelic view on the non-determinism, as well as Dijkstra and Scholten [7]. For example, O’Hearn defines the rule for non-deterministic choice as follows (ϵ is either *ok* or *er*, an indicator whether the final condition is reached normally or erroneously):

$$\frac{[P]C_i[\epsilon : Q]}{[P]C_1 + C_2[\epsilon : Q]} \text{ choice (where } i = 1 \text{ or } 2)$$

Zhang and Kaminski [31] supplemented the word “total” to the name of this triple, reason being that the strongest postcondition transformer only include final states that are reachable, dual to the total correctness triple that only allows states that lead to termination.

PARTIAL INCORRECTNESS As the name indicates, when there is “total incorrectness”, there should be “partial incorrectness”. The red implications correspond to the triples for partial incorrectness. As mentioned in the previous section, the *slp* transformer lacks attention, starting with a sound definition with loops [18, 30]. As far as our knowledge, the underapproximation triple of *slp*

$$F \implies slp.C.G$$

still needs proper investigation. However, the Galois connection relates it to the overapproximation triple of *wp*:

$$wp.C.F \implies G$$

Cousot et al. [4] studied an implication similar to this triple to help lighten the burden of providing specifications in design by contract programming schemes. They call it the “necessary preconditions”, in the sense that these preconditions can not guarantee correctness, but once violated, definitely lead to erroneous final states. They achieve this by finding preconditions that will definitely lead to erroneous final states, then take its negation. Because they separated the issue of termination from their necessary precondition reasoning by assuming that all programs terminate, their necessary precondition is not exactly an overapproximation of wp. Nevertheless, it gives intuition on one of the ways to approach the necessary liberal precondition.

NECESSARY LIBERAL PRECONDITION The overapproximation of wlp is the main focus of this thesis, denoted with orange label in Figure 3.6: $\text{wlp.C.F} \Rightarrow G$. It reflexes an extremely relaxed view: in this triple, G can be satisfied by all possible initial states, under whose control the program can diverge, successfully terminate, erroneously terminate ... The only certainty is that G is the necessary condition, in the sense that once violated, the program can definitely reach erroneous final states. We will study this triple further in the upcoming sections.

REMAINING TRIPLE The remaining triple denoted with label “what?” still awaits investigation as far as we know:

$$\text{slp.C.G} \Rightarrow F$$

We think that one approach is to first establish a sound definition for slp, then find connection between slp and sp, either by conjugation or reachability depending on the choice how to resolve non-determinism. From here, one can start investigating this triple following similar methods for other triples.

With this overview of triples studied in literature, we know that the triple $\text{wlp.C.F} \Rightarrow G$ is still an open question, and not automatically solved by a Galois connection to another implication. As a result, we investigate the overapproximation we call the necessary liberal precondition following methods used with other triples by first discussing the possible scenarios of this overapproximation, then look into examples to shed light on the meaning of this implication.

NECESSARY LIBERAL PRECONDITIONS

We are interested in studying the [necessary liberal precondition](#), a weakening of the weakest liberal precondition:

$$\text{wlp}.C.F \implies G$$

We show our interest, by relating all the pre- and postcondition transformers, so we can distinguish the existing triples that are already well-researched graphically.

Luckily, we can find scenarios where the overapproximation can be useful, and demonstrate it with an example, before giving a proof system. We conclude that this overapproximation can help us find more preconditions that can lead to failure, either by including states that lead to unidentified errors, or by including states that can lead to errors via non-terministic choice. The latter method then leads to finding conditions that are able to capture these special states. The side effect of this result is that we can now find the wlp transformer with angelic non-determinism using wlp with demonic non-determinism and sp.

This chapter includes our most important results:

1. The predicate transformers are all related together nicely by relations like termination, reachability, conjunction, over- and under-approximation, and implication in both directions.
2. Without any additional constraints, the necessary liberal preconditions can be anything. The only thing we are sure of is that its negation guarantees termination in the negation of the desired postcondition.
3. Necessary liberal preconditions find their usefulness in avoiding bugs.
4. With necessary liberal precondition, we also can overapproximate and underapproximate wlp with angelic non-determinism (wlp_a). In the end, we can qualify wlp_a using necessary liberal precondition and strongest postcondition transformers, without having to define wlp_a .

In the upcoming sections, we first relate together wp, wlp, sp, and slp with both angelic and demonic non-determinism, so we can distinguish the necessary liberal precondition from the existing triples that are already well-studied. In this chapter, we first discuss the general semantics of the necessary liberal precondition, then focus on a characteristic scenario and show properties that captures it.

4.1 RELATING OVERAPPROXIMATION OF WLP WITH SP

Remember in [Section 3.3](#), we mentioned that there is an equivalence relation between the underapproximation of wlp and overapproximation of sp via Galois connection:

$$G \implies \text{wlp}.C.F \text{ iff } \text{sp}.C.G \implies F \quad (*)$$

One might wonder, can we also establish some similar relation between the overapproximation of wlp and underapproximation of sp:

$$\text{wlp}.C.F \implies G \quad \leftarrow? \quad F \implies \text{sp}.C.G \quad (**)$$

This would imply that we merely need to study the underapproximation of sp to result in the overapproximation of wlp, the former of which is studied as the total incorrectness triple [\[28\]](#).

The answer is no. Without additional constraints, we can not directly establish any implication relations between $\text{wlp}.C.F \implies G$ and $F \implies \text{sp}.C.G$.

Remember from [Chapter 3](#) that $\text{wlp}.C.F$ describes all initial states that can potentially lead to non-termination, and $\text{sp}.C.G$ is satisfied only by final states that are reachable. Termination and reachability are two different aspects of program execution. If we denote explicitly the states that can lead to non-termination with blue squares, and states that are unreachable with orange circles, then we can see on the left half of [Figure 4.1](#) that F can also be satisfied with unreachable states, and from the right half that G can also be satisfied by states leading to non-termination. In general, without further constraints, F in [Figure 4.1](#) is decorated with **some** of the orange circles, and G is decorated with **some** of the blue squares.

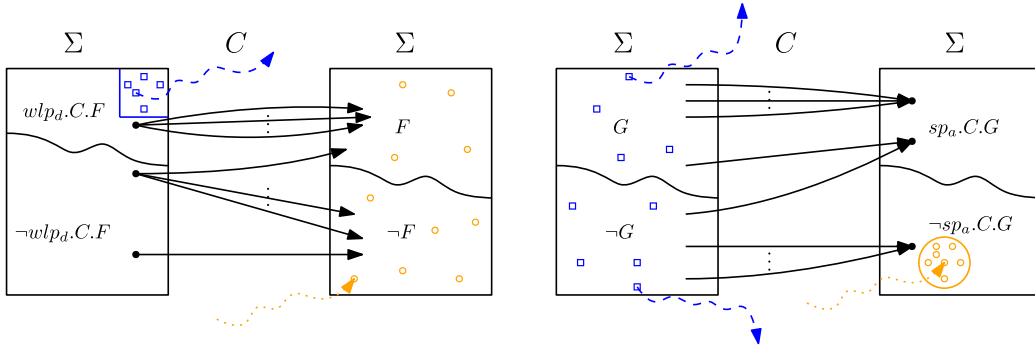


Figure 4.1: Non-terminating and unreachable States

As an example, consider the program

$$C = \text{if } (x = 2) \text{ then } \{x := x + 2\} \text{ else } \{x := x + 1\}$$

and a postcondition $F = \{x > 2\}$. We can deduce the weakest liberal precondition:

$$\begin{aligned} \text{wlp}.C.F &= \{x = 2\} \wedge (\text{wlp}.x := x + 2).F \vee \{x \neq 2\} \wedge \text{wlp}.(x := x + 1).F \\ &\Leftrightarrow \{x = 2\} \wedge \{x > 0\} \vee \{x \neq 2\} \wedge \{x > 1\} \end{aligned}$$

$$\begin{aligned} &\Leftrightarrow \{x = 2\} \vee \{x > 2\} \\ &\Leftrightarrow \{x \geq 2\} \end{aligned}$$

As a result, as long as the program C starts in an initial state satisfying $x \geq 2$, it will terminate in a state satisfying $F = \{x > 2\}$. However, a state where x evaluates to 3 would satisfy F , but it is never reachable: assume the else-branch was discharged, then x must evaluate to 2 initially, so else-branch would not be discharged; otherwise assume the if-branch was discharged, then x must initially evaluate to 1, but then the if-branch would not be discharged. Hence, any state where x evaluates to 3 is unreachable, but still satisfies the desired postcondition F . Such states are denoted as orange circles in Figure 4.1. Similarly, $sp.C.F$ can also be satisfied by initial states that can lead to non-termination.

Attempting to establish a relation for line (**), we first start from the left side. Assume precondition G satisfies $wlp.C.F \Rightarrow G$, we can end up in a situation like Figure 4.2.

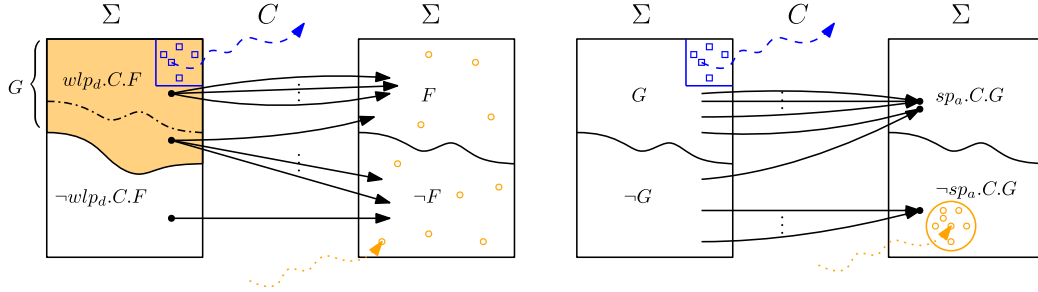
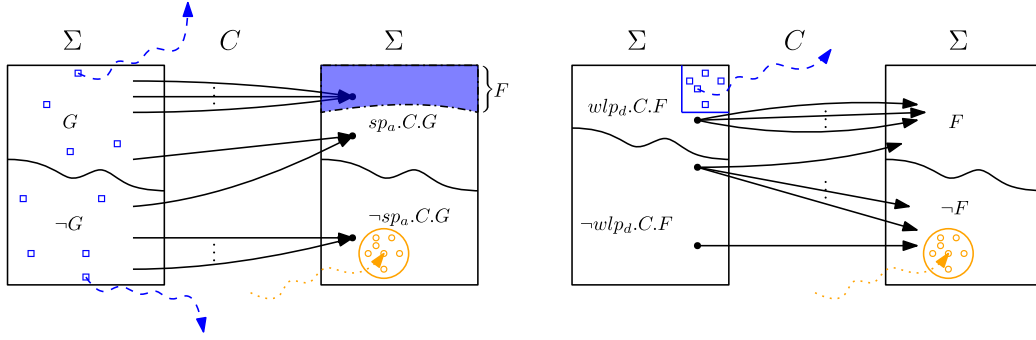


Figure 4.2: $sp.C.G$ where G Is Overapproximation of $wlp.C.F$

We see on the left side of Figure 4.1 that G “includes” all the blue squares, and F only some of the orange circles. Now what do we know about $sp.C.G$? We see on the right side of Figure 4.1, $sp.C.G$ is only satisfied by reachable states, i.e. none of the orange circles. This means that in general, without further constraints, we at least have that $F \not\Rightarrow sp.C.G$, since F may be satisfied by unreachable states, but $sp.C.G$ is only satisfied by reachable states.

Can we then have a relation like $sp.C.G \Rightarrow F$? The answer is no. Assume we have both $wlp.C.F \Rightarrow G$, and $sp.C.G \Rightarrow F$, then with the Galois connection mentioned in line (*), we can conclude that $G \Rightarrow wlp.C.F$, which means that $G \Leftrightarrow wlp.C.F$, G must be **exactly** the weakest liberal precondition of F . Since we can only assume that G overapproximates wlp , we can not draw the conclusion that $sp.C.G \Rightarrow F$. As a result, from $wlp.C.F \Rightarrow G$ we can neither conclude $sp.C.G \Rightarrow F$ nor $F \Rightarrow sp.C.G$.

Analogously, assume $F \Rightarrow sp.C.G$, what can we state about $wlp.C.F$? From the left side of Figure 4.3, we can read that F does not “include” orange circles, i.e. F is only satisfied by reachable states. What do we know then about the relation between $wlp.C.F$ and G ? We see from Figure 4.3, $wlp.C.F$ “includes” all the blue squares, but G only some of them. Consequently, we can not conclude that $wlp.C.F \Rightarrow G$. Also, we can not conclude that $G \Rightarrow wlp.C.F$, because this would require $sp.C.G \Rightarrow F$. Together with the assumption $F \Rightarrow sp.C.G$,

Figure 4.3: $wlp.C.F$ where F Is Underapproximation of $sp.C.G$

this means that F must be **exactly** the strongest postcondition of program C w.r.t. precondition G , also a luxury we can not assume.

Summarizing the two conclusions above together, we can say that in general, without further constraints:

- From $wlp.C.F \implies G$, we can neither conclude $F \implies sp.C.G$ nor $sp.C.G \implies F$;
- Analogously, from $F \implies sp.C.G$ we can neither conclude $wlp.C.F \implies G$ nor $G \implies wlp.C.F$.

This tells us that we can **not** overapproximate wlp by underapproximating sp , at least not directly.

4.2 THE GENERAL CASE

As hinted before, in our triple $wlp.C.F \implies G$, G overapproximates wlp . It can take different forms: on the one hand, G can be so general that it is satisfied by any program state; on the other hand, a G that is barely weaker than $wlp.C.F$ is also not much different from the latter. Alternatively, G can also be all kinds of preconditions that starting from it, any postcondition is reachable. One thing we are certain about, though, is that a program with an original state satisfying $\neg G$ will terminate, and the final state can satisfy $\neg F$:

$$\begin{aligned} wlp.C.F \implies G &\Leftrightarrow \neg G \implies \neg wlp.C.F \\ &\Leftrightarrow \neg G \implies wp.C.\neg F \end{aligned} \quad | \text{ Theorem 2.10}$$

These equivalences indicate that $\neg G \{C\} \neg F$ is a valid Hoare triple with respect to total correctness, i.e. if C starts in an initial condition satisfying $\neg G$, then its execution **must** terminate, and it must be able to terminate satisfying $\neg F$.

4.2.1 Overapproximation of wlp

In [Section 2.5](#) we define the weakest liberal precondition and state that it characterizes all the preconditions starting from which the program either **diverges** or **will** terminate in a state satisfying F . We are certain to use “will” instead of

“can” as the non-deterministic choice is viewed as demonic, so the behavior of wlp can be depicted by [Figure 3.1.4](#).

The rectangle on the left (right) denote the set of all initial (final) states Σ . C is the program, and its executions are depicted by arrows. The rectangle is divided into halves, the part with label wlp.C.F denotes all the states that satisfy wlp.C.F, and so on. The dashed arrow starting from inside a rectangle, ending outside neither rectangle denotes executions that do not terminate. The arrows that share the same point from the left rectangle denote different executions starting from the same initial state.

Starting from one initial state, the execution of the program can take various forms: it can be strictly deterministic, or it can have multiple non-deterministic choices; it can terminate in a final state, or it can execute forever. We categorize the executions of the program in four ways:

1. the dashed arrow means non-terminating executions;
2. the black arrows are executions starting from an initial state satisfying wlp.C.F and only terminating in final states satisfying F ;
3. the green arrows are the executions starting from an initial state satisfying \neg wlp.C.F but can terminate in states either satisfying F or satisfying $\neg F$;
4. the red arrow represents executions starting from an initial state satisfying \neg wlp.C.F and only terminating in final states satisfying $\neg F$.

If we were to weaken the precondition, it can happen in various ways as shown in [Figure 3.1.4-9](#). For example, [Figure 3.1.4](#) shows the situation where G is exactly wlp.C.F. But in [Figure 4.4.2](#), G is strictly weaker than wlp.C.F. Here, G is satisfied by all initial states that satisfy wlp.C.F, plus **some** of the initial states, starting from which the execution can end satisfying F or $\neg F$, aka some of the green arrows. Further, the G in [Figure 4.4.3](#) is satisfied by **all** the above initial states, i.e. all the green arrows start from G . The same principle applies to the red arrows. [Figure 4.4.4](#) shows a G that is satisfied additionally by **some** of the initial states that are starting points of the red arrows, i.e. executions that start in \neg wlp.C.F, and end in $\neg F$. In [Figure 4.4.5](#), G contains all such initial states.

The situation can be trickier, for example in [Figure 4.4.6](#), G is also the starting point of both green and red arrows. In other words, if C starts from some initial state satisfying this G , then maybe its execution diverges, maybe it terminates satisfying F , maybe not. The same can be said about the G in [Figure 4.4.7](#) and [Figure 4.4.8](#). Finally, G is weakened so much that it is simply true, hence satisfied by all initial states as seen in [Figure 4.4.9](#).

In general, when $\text{wlp.C.F} \implies G$, G can be satisfied by all possible initial states, which can be the starting points of black, green, red, or dashed arrows, representing executions terminating in states satisfying F , F or $\neg F$, $\neg F$, or non-terminating, respectively. As a result, we can not make many statements without adding extra restrictions to G .

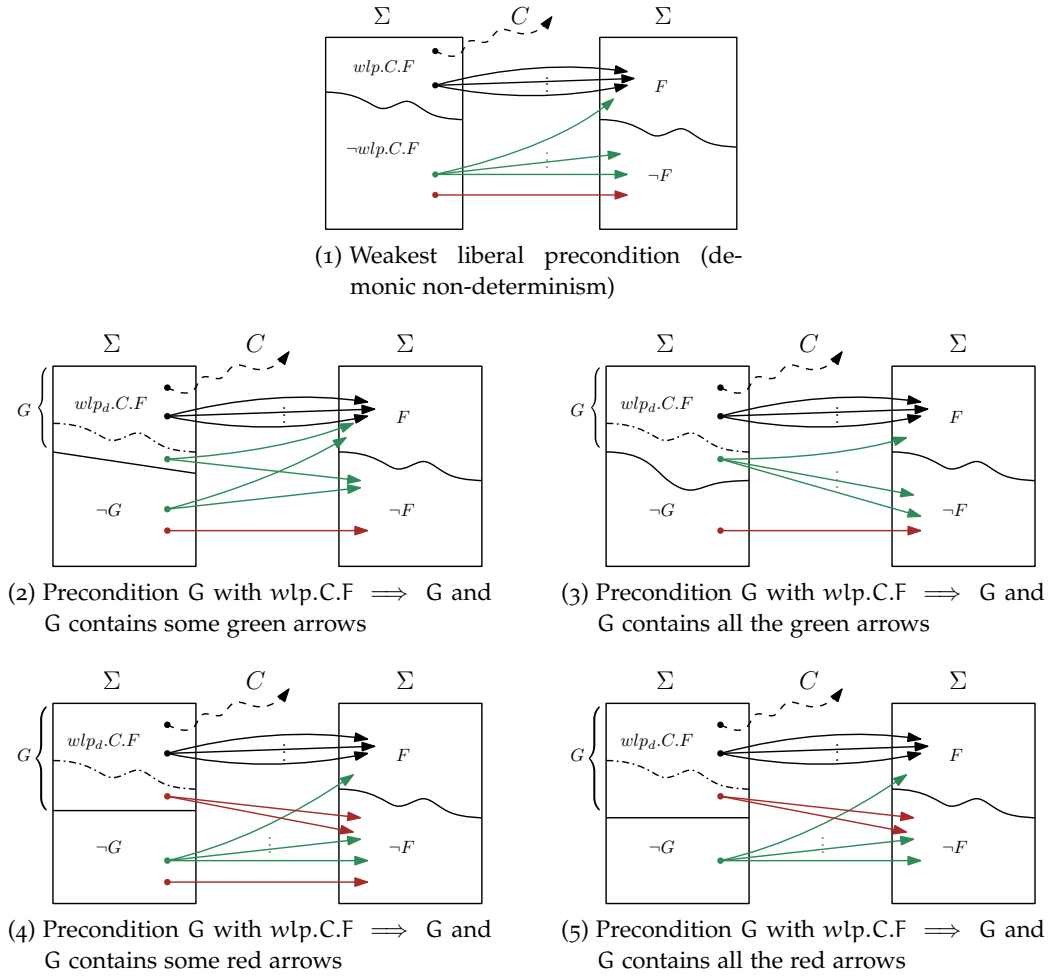


Figure 4.4: Case Distinction of Preconditions Weaker Than wlp

However, we can see from Figure 3.1.4-9 that when $\neg G$ is not empty, there is always some arrow ending in $\neg F$. So there always exists some execution that starts from an initial state satisfying $\neg G$, and ends in a final state satisfying $\neg F$.

Alternatively, we can say that does not contain any **black** or **dashed** arrows in all cases. In other words, if program C starts in any initial state satisfying $\neg G$, then either G is empty, or

- its executions terminate, and
- there exists an execution that ends up in a final state that satisfies $\neg F$.

This again states that $\neg G \{C\} \neg F$ is a valid Hoare triple. The question then naturally arises: why do we concern ourselves with G , if we can just prove our specifications using wp or Hoare triples? To demonstrate the answer, we analyze the example written in Listing 4.1.

```

1  ... // starting device
2  n := n + 1 □ n := n - 1 □ diverge;
3  ... // retirement

```

Listing 4.1: Door with Sensors Counting Number of People Present

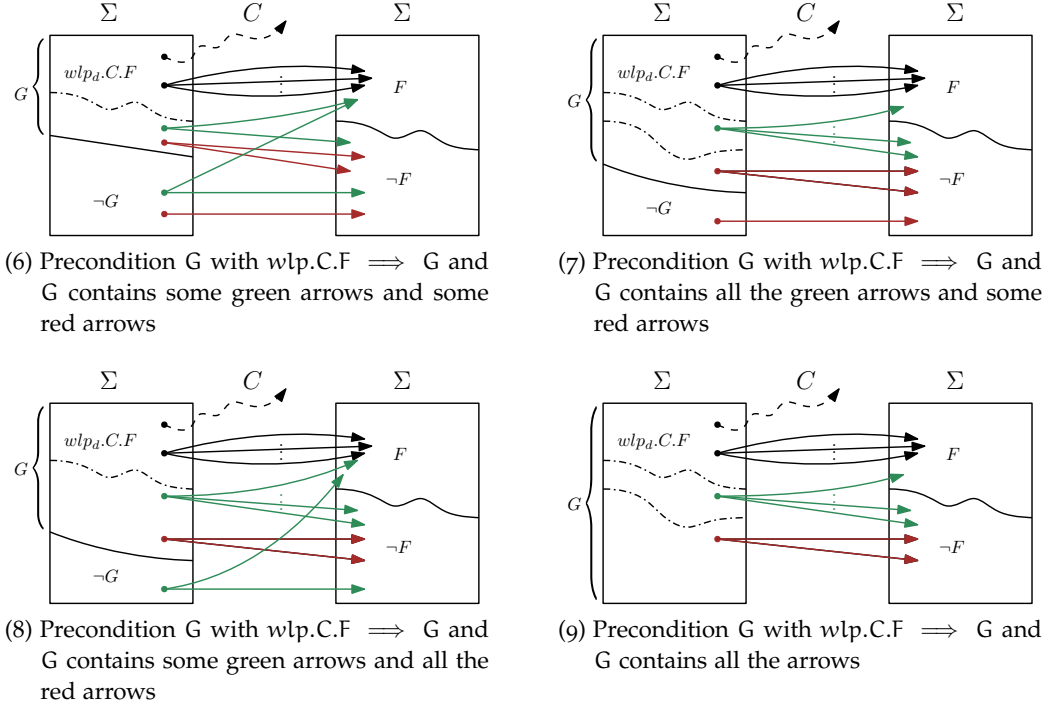


Figure 4.4: Case Distinction of Preconditions Weaker Than wlp (Cont.)

4.2.2 Example: Door with Sensors

In the author's student life, there was a door at university that she always finds interesting. The door is located at a lecture hall, equipped with two sensors on the left and right sides of the door frame. Each time a person walks in or out, the sensor registers and adds or decreases the number of people in the lecture hall, making a small tick sound. It seems that this door serves the purpose of helping security guards keep an eye on the lecture rooms after closing time without having to be physically there.

But can the door really distinguish from a person entering or leaving the room? And what if naughty students try to trick the door by using a backpack to pretend to exit the door multiple times? What if one of the sensors break? We can this simple scenario by putting the three parts in non-deterministic choice. It can happen that a student enters the room hence increasing the count of the number of people present: $n := n + 1$; or a person exits and decreasing the number: $n := n - 1$. Alternatively, the sensor can break because of old age and result in unforeseeable behavior, for example always detecting someone entering forever.

We know that there is definitely something wrong, in case the security sees on their device that the number of people in a lecture hall is negative. If we want to know what caused it, we can calculate the weakest liberal precondition of the program snippet with regard to $F = \{\sigma \in \Sigma \mid \sigma.n < 0\}$ in [Listing 4.1](#). We write $F = \{n < 0\}$ in short:

1	... // starting device
2	{ $n < -1$ }

```

3   {n < -1} ∧ {n < 1} ∧ true
4   n := n + 1 □ n := n - 1 □ diverge;
5   {n < 0}
6   ... // retirement

```

Listing 4.2: Weakest Liberal Precondition w.r.t Postcondition $F = \{\sigma \in \Sigma \mid \sigma.n < 0\}$

It seems like we should avoid starting the system in a state where n is less than -1 . But is enough? If we start the system in a state where $n = -1$, or even $n = 0$, we can still reach an erroneous state where the value of n is negative. [Figure 4.5](#) We can see that with wlp, it is not enough to identify all preconditions

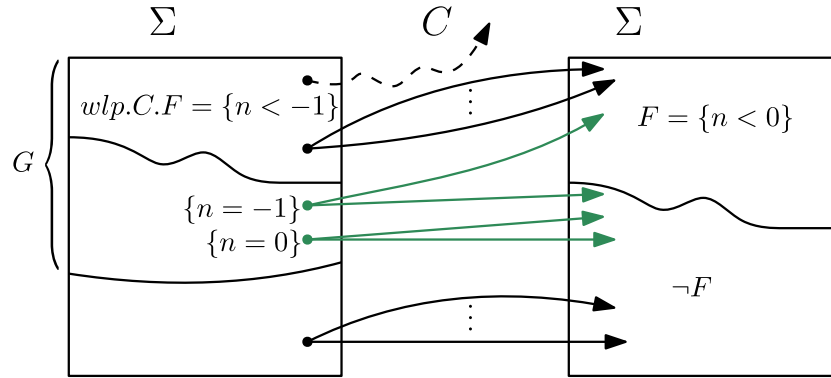


Figure 4.5: $wlp.C.F = \{n < -1\}$

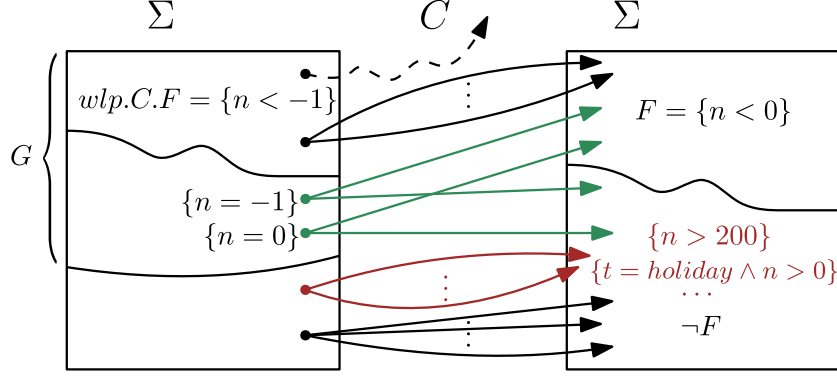
that can result in errors. In other words, the green arrows in [Figure 4.5](#) are not recognized. But by overapproximating wlp to include preconditions like $\{n = -1\}$ and $\{n = 0\}$, we can also include all the green arrows in G .

This is, however, not the only thing that is missing. Obviously, a lecture hall does not have unlimited capacity. Once the system is showing a number higher than the maximum capacity, the security would also instantly know that something is wrong. Also, there could be conditions like “during holiday time, the lecture halls are unoccupied” that an expert of the system (the security guards in this case) would know, but not in general. So even by including all the green arrows via overapproximation, we are still missing some of the red arrows like in [Figure 4.6](#), the hidden knowledge that seasoned users possess.

This demonstrates overapproximating wlp is useful to include the green and red arrows in [Figure 4.6](#), aka when:

- some preconditions **can but are not guaranteed** to lead to the erroneous final states non-deterministically;
- or the user only have insufficient knowledge of the system, but experts can deduce other erroneous final states that are unknown to the public.

To utilize this overapproximation, we either identify the preconditions that result in the green arrows, which we will do in the next section, or we first calculate the weakest liberal precondition of the erroneous postcondition that we know

Figure 4.6: $wlp.C.F \Rightarrow G$

of, then relax this precondition by “guessing” similar erroneous postconditions with the extra knowledge from experts over the system.

4.2.3 Example: To Exclude Bugs

Here comes an example similar as the ones described by Cousot to overapproximate wp.

4.2.4 Proof System

We call the overapproximation of wlp **necessary liberal precondition**, “necessary” because it is required to ensure avoidance of erroneous final states, and “liberal”, because it is satisfied by all initial states that can potentially lead to non-termination, just like wlp. We provide a proof system in Table 4.1. We use $\langle G \rangle C \langle F \rangle$ to denote that G is a valid necessarily liberal precondition for program C given postcondition F .

The rules are mostly unsurprising. Remember that we would like to construct rules to prove that a precondition is the necessary liberal precondition, it should overapproximate the weakest liberal precondition, i.e. a triple $\langle G \rangle C \langle F \rangle$ should satisfy:

$$wlp.C.F \Rightarrow G$$

Consequently, we have the diverge axiom: because $wlp.diverge.F = \text{true}$, and true is the only condition that overapproximates it, so the necessary liberal precondition is true .

Consequently, we have the skip axiom: because $wlp.skip.F = F$, then F overapproximates it. Alternatively, we can construct the rule in such way:

$$\frac{F \Rightarrow G}{\langle G \rangle \text{skip} \langle F \rangle} \text{skip}_{alt}$$

But this rule can be derived using the skip rule and the conseq in Table 4.1, so we choose the simpler skip rule. The same principle applies to the assign, seq, if, and the choice rule: we keep the rules simple by not discussing overapproximation directly, but let the conseq rule be concerned with relaxation.

The consequence rule says that we are allowed to weaken the precondition and strengthen the postcondition:

$$\frac{P \implies G, \langle P \rangle C \langle Q \rangle, F \implies Q}{\langle G \rangle C \langle F \rangle} \text{conseq}$$

Intuitively, if a precondition P already overapproximates the weakest liberal precondition, then by overapproximating it with G , the result is still an overapproximation of wlp:

$$\text{wlp}.C.F \implies P \implies G$$

Conversely, we can strengthen the postcondition from Q to F , because $\text{wlp}.C$ is monotonic (2.11), then given $F \implies Q$ we can derive:

$$\text{wlp}.C.F \implies \text{wlp}.C.Q \implies G$$

Hence, G is still an overapproximation of the stronger $\text{wlp}.C.F$.

Note that the conseq rule is the complete opposite direction from the “Rules of Consequence” in Hoare logic that we referred to in Table 2.2. Writing the consequence rule of Hoare logic in the same natural deduction style, we arrive at the following rule labeled with conseq_h :

$$\frac{G \implies P, P \{C\} Q, Q \implies F}{G \{C\} F} \text{conseq}_h$$

This corresponds to our expectation, because we know that the overapproximation triple is exactly a Hoare triple (with respect to partial correctness) with negated pre- and postconditions:

$$\begin{aligned} & \text{wlp}.C.F \implies G \\ \Leftrightarrow & \neg G \implies \neg \text{wlp}.C.F \\ \Leftrightarrow & \neg G \implies \text{wp}.C.\neg F & | \text{Theorem 2.10} \\ \Leftrightarrow & \neg G \{C\} \neg F & (*) \end{aligned}$$

Note that the last line is an implication in both directions, because we take the version of Hoare triple regarding total correctness, i.e. supplementing the rule to prove termination of while loops [21]:

$$\frac{P \wedge \phi \wedge v \in \mathbb{N} \wedge v = n \quad \{C\} \quad P \wedge \phi \wedge v \in \mathbb{N} \wedge v < n}{P \wedge v \in \mathbb{N} \quad \{\text{while } (\phi) \text{ do } C\} \quad \neg \phi \wedge P \wedge t \in \mathbb{N}} \text{while}_h$$

This is necessary to give intuition of the consequence rule in our proof system. Assuming that the triple $\langle \cdot \rangle \cdot \langle \cdot \rangle$ is sound (which we will prove later in this section), i.e. $\langle P \rangle C \langle Q \rangle \implies (\text{wlp}.C.F \implies G)$, we can derive the conseq rule from conseq_h :

$$\begin{aligned} & \frac{P \implies G, \langle P \rangle C \langle Q \rangle, F \implies Q}{P \implies G, \text{wlp}.C.Q \implies P, F \implies Q} \text{soundness} \\ & \frac{P \implies G, \text{wlp}.C.Q \implies P, F \implies Q}{P \implies G, \neg P \{C\} \neg Q, F \implies Q} (*) \\ & \frac{P \implies G, \neg P \{C\} \neg Q, F \implies Q}{\neg G \implies \neg P, \neg P \{C\} \neg Q, \neg Q \implies \neg F} \text{first-order logic} \\ & \frac{\neg G \implies \neg P, \neg P \{C\} \neg Q, \neg Q \implies \neg F}{\neg G \{C\} \neg F} \text{conseq}_h \\ & \frac{\neg G \{C\} \neg F}{\text{wlp}.C.F \implies G} (*) \\ & \frac{\text{wlp}.C.F \implies G}{\langle G \rangle C \langle F \rangle} \text{(soundness)} \end{aligned}$$

$\frac{}{\langle F \rangle \text{ skip } \langle F \rangle} \text{skip}$	$\frac{}{\langle \text{true} \rangle \text{ diverge } \langle F \rangle} \text{div}$
$\frac{}{\langle F[x/e] \rangle x := e \langle F \rangle} \text{assign}$	$\frac{\langle G \rangle C_1 \langle P \rangle, \langle P \rangle C_2 \langle F \rangle}{\langle G \rangle C_1; C_2 \langle F \rangle} \text{seq}$
$\frac{\langle \varphi \wedge G \rangle C_1 \langle F \rangle, \langle \neg \varphi \wedge G \rangle C_2 \langle F \rangle}{\langle G \rangle \text{ IF } \langle F \rangle} \text{if}$	
$\frac{\langle P \rangle C' \langle \varphi \vee P \rangle}{\langle P \rangle \text{ WHILE } \langle \varphi \vee P \rangle} \text{while}_1$	$\frac{\langle P \rangle C' \langle F \rangle}{\langle P \vee F \rangle \text{ WHILE } \langle F \rangle} \text{while}_2$
$\frac{\langle G \rangle C_1 \langle F \rangle, \langle G \rangle C_2 \langle F \rangle}{\langle G \rangle C_1 \square C_2 \langle F \rangle} \text{choice}$	$\frac{P \implies G, \langle P \rangle C \langle Q \rangle, F \implies Q}{\langle G \rangle C \langle F \rangle} \text{conseq}$
where IF = if (φ) { C_1 } else { C_2 }, WHILE = while (φ) do { C }.	

Table 4.1: The Proof System

It is however not trivial to construct the while rule. Assume we would like to find some G to overapproximate wlp.WHILE.F , unfolding according to the definition in Table 2.5:

$$\begin{aligned} \text{wlp.WHILE.F} &= \text{gfp } X. \Phi(X) \\ &= \text{gfp } X. (\neg \varphi \wedge F) \vee (\varphi \wedge \text{wlp.C'.X}) \end{aligned}$$

Since G overapproximates wlp.WHILE.F , the greatest fixed point of Φ , then G must also overapproximate any fixed point of Φ : For any fixed point P of Φ , it is valid that $P \implies \text{gfp } \Phi$, add the fact that $\text{gfp } \Phi \implies G$, we can conclude that $P \implies G$. In other words, the weakest liberal precondition G must overapproximate all fixed points of Φ :

$$\begin{aligned} \forall P. P = \Phi(P) : P &\stackrel{!}{\implies} G \\ \Leftrightarrow \forall P. P = (\neg \varphi \wedge F) \vee (\varphi \wedge \text{wlp.C'.X}) : P &\stackrel{!}{\implies} G \\ &= \text{gfp } X. (\neg \varphi \wedge F) \vee (\varphi \wedge \text{wlp.C'.X}) \end{aligned}$$

Lemma 4.15 [The proof system is sound]

$$\langle G \rangle C \langle F \rangle \implies (\text{wlp.C.F} \implies G)$$

Proof. We prove by induction and case distinction over the lowest level of the proof tree.

CASES skip, div, assign Straightforward.

CASE seq Assume we proved the triple $\langle G \rangle C_1; C_2 \langle F \rangle$ with the following proof tree:

$$\begin{array}{c}
\frac{\dots}{\dots\dots\dots} \\
\vdots \\
\frac{\langle G \rangle C_1 \langle P \rangle, \langle P \rangle C_2 \langle F \rangle}{\langle G \rangle C_1; C_2 \langle F \rangle} \text{seq}
\end{array}$$

Then we have proven $M_1 := \langle G \rangle C_1 \langle P \rangle$ and $M_2 := \langle P \rangle C_2 \langle F \rangle$ as the premises of the last deduction, as well as $M_3 := \langle G \rangle C_1; C_2 \langle F \rangle$ as the conclusion of the last deduction step. We also have the induction hypotheses

$$H_1 := \langle G \rangle C_1 \langle P \rangle \implies (\text{wlp}.C_1.P \implies G)$$

$$H_2 := \langle P \rangle C_2 \langle F \rangle \implies (\text{wlp}.C_2.F \implies P)$$

Our goal is to prove $\langle G \rangle C_1; C_2 \langle F \rangle \implies (\text{wlp}.(C_1; C_2).F \implies G)$. By discharging M_1 in H_1 , M_2 in H_2 , and M_3 in the goal, we acquire premises $\text{wlp}.C_1.P \implies G$ and $\text{wlp}.C_2.F \implies P$ to prove goal $\text{wlp}.(C_1; C_2).F \implies G$. This is done by discharging of the definition of wlp with composition and the monotonicity of wlp:

$$\begin{array}{ll}
\text{wlp}.(C_1; C_2).F = \text{wlp}.C_1.(\text{wlp}.C_2.F) & | \text{definition of wlp} \\
\implies \text{wlp}.C_1.P & | \text{monotonicity of wlp} \\
\implies G & | \text{premise}
\end{array}$$

CASE if Unrolling the definition of $\text{wlp}.IF.F$ we get that

$$\text{wlp}.IF.F = \varphi \wedge \text{wlp}.C_1.F \vee \neg\varphi \wedge \text{wlp}.C_2.F$$

Similar as before, by discharging induction hypotheses and premises, we can acquire the following conditions:

$$\text{wlp}.C_1.F \implies \varphi \wedge G \text{ and } \text{wlp}.C_2.F \implies \neg\varphi \wedge G$$

From the first condition we get:

$$\varphi \wedge \text{wlp}.C_1.F \implies \text{wlp}.C_1.F \implies \varphi \wedge G \implies G$$

and

$$\neg\varphi \wedge \text{wlp}.C_2.F \implies \text{wlp}.C_2.F \implies \neg\varphi \wedge G \implies G$$

Hence $\text{wlp}.IF.F \implies G$.

CASE while how to support taking these rules instead of others?

CASE choice Similar to the case with seq, we can prove that $\text{wlp}.(C_1 \square C_2).F \implies G$ from

$$\text{wlp}.(C_1 \square C_2).F = \text{wlp}.C_1.F \wedge \text{wlp}.C_2.F$$

and

$$\text{wlp}.C_1.F \implies G \wedge \text{wlp}.C_2.F \implies G$$

CASE conseq This case is also not different from before: from the induction hypotheses and premises we can derive $\text{wlp}.C.Q \implies P$. Together with $P \implies G, F \implies Q$ and the monotonicity of $\text{wlp}.C$ we conclude that

$$\text{wlp}.C.F \implies \text{wlp}.C.Q \implies P \implies G$$

□

Lemma 4.16 [The proof system is incomplete] here

Proof. here

□

4.3 A SPECIAL CASE

As promised before, we will look into a way to identify the green arrows in Figure 4.5, aka the executions that start from initial states but may end non-deterministically in F or $\neg F$. We can easily see that if we overapproximate wlp by adding all the green arrows, we end up with Figure 4.4.3, which is exactly wlp with angelic non-determinism, as shown in Figure 3.1 at the beginning of this chapter.

When G corresponds to Figure 4.4.3, we know that under its control, the program always **can** reach a final state satisfying F if it terminates, while with an initial state satisfying $\neg G$, the program is **will** terminate satisfying $\neg F$. This behavior describes exactly the behavior of wlp with angelic non-determinism, which is more obvious if we put together Figure 4.4.3 and Figure 3.1.3:

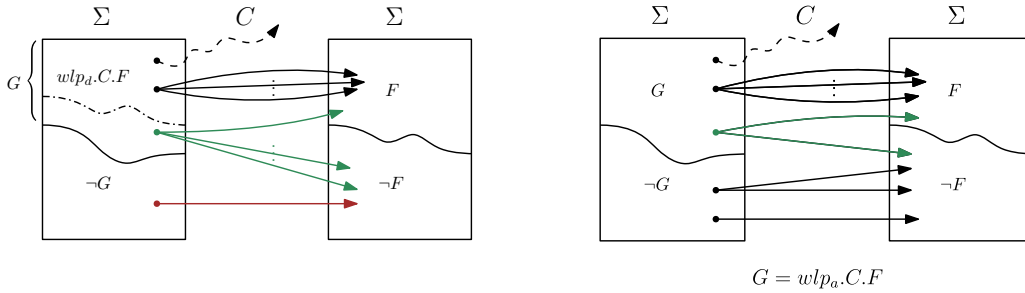


Figure 4.7: Comparing G With Angelic wlp

Describing the intended behavior of wlp with angelic non-determinism (denoted by wlp_a) as “if C starts in an initial state satisfying $\text{wlp}_a.C.F$, then its execution either never terminates, or is always possible to end in a state satisfying F ”, then we can deduce its semantics as follows, recalling the representation for non-termination mentioned in Section 2.7:

Statement 4.17 [Semantics of wlp_a]

$$\text{WRONG!!!} \text{wlp}_a.C.F = \{\sigma \in \Sigma \mid \neg(\exists \tau \in \Sigma : \sigma \xrightarrow{C} \tau) \vee (\exists \tau \in \Sigma : \sigma \xrightarrow{C} \tau \wedge \tau \models F)\}$$

While trying to distinguish what is and is not in wlp_a , we approach it in two directions: first, we find restraints that make the necessary liberal precondition G

where $\text{wlp}.C.F \implies G$ an overapproximation of angelic wlp ($\text{wlp}_a.C.F \implies G$); then we find restraints that makes G an underapproximation of angelic wlp. Finally, we combine both directions and conclude with statements that make G and $\text{wlp}_a.C.G$ equivalent. We also gain a side effect of expressing wlp_a without having to define it.

Lemma 4.18 [Angelic wlp implies G]

if $(\text{wlp}.C.F \implies G) \wedge (\text{sp}.C.\neg G \implies \neg F)$ then $\text{wlp}_a.C.F \implies G$

The second prerequisite $\text{sp}.C.\neg G \implies \neg F$ states that from $\neg G$ we are only allowed to reach $\neg F$, making sure that all green arrows as in [Figure 4.4](#) are included in G .

Proof. The assumption expresses that for any state $\sigma \in \Sigma$:

$$\begin{aligned}
 \text{wlp}.C.F \implies G &\Leftrightarrow \sigma \in \text{wlp}.C.F \implies \sigma \in G \\
 &\Leftrightarrow (\forall \tau \in \Sigma : \sigma \xrightarrow{C} \tau \implies \tau \in F) \implies \sigma \in G && | \text{Theorem 2.6} \\
 &\Leftrightarrow (\forall \tau \in \Sigma : \neg(\sigma \xrightarrow{C} \tau) \vee \tau \in F) \implies \sigma \in G \\
 &\Leftrightarrow \neg(\exists \tau \in \Sigma : (\sigma \xrightarrow{C} \tau) \wedge \neg(\tau \in F)) \implies \sigma \in G && (a)
 \end{aligned}$$

Also, for any state $\tau \in \Sigma$:

$$\begin{aligned}
 \text{sp}.C.\neg G \implies \neg F &\Leftrightarrow \tau \in \text{sp}.C.\neg G \implies \tau \in \neg F \\
 &\Leftrightarrow (\exists \mu \in \Sigma : \mu \xrightarrow{C} \tau \wedge \mu \in \neg G) \implies \tau \in \neg F && | \text{Theorem 2.8} \\
 &\Leftrightarrow \neg(\tau \in \neg F) \implies \neg(\exists \mu \in \Sigma : \mu \xrightarrow{C} \tau \wedge \mu \in \neg G) \\
 &\Leftrightarrow \tau \in F \implies \forall \mu \in \Sigma : \neg(\mu \xrightarrow{C} \tau \wedge \mu \in \neg G) \\
 &\Leftrightarrow \tau \in F \implies \forall \mu \in \Sigma : \neg(\mu \xrightarrow{C} \tau) \vee \neg(\mu \in \neg G) \\
 &\Leftrightarrow \tau \in F \implies \forall \mu \in \Sigma : \neg(\mu \xrightarrow{C} \tau) \vee \mu \in G && (b)
 \end{aligned}$$

Our goal is to prove that for any state $\sigma \in \Sigma$:

$$\begin{aligned}
 \text{wlp}_a.C.F \implies G &\Leftrightarrow \sigma \in \text{wlp}_a.C.F \implies \sigma \in G \\
 &\Leftrightarrow \neg(\exists \tau \in \Sigma : \sigma \xrightarrow{C} \tau) \vee (\exists \tau \in \Sigma : \sigma \xrightarrow{C} \tau \wedge \tau \in F) \\
 &\implies \sigma \in G && | \text{Statement 4.17} \\
 &\Leftrightarrow (\neg(\exists \tau \in \Sigma : \sigma \xrightarrow{C} \tau) \implies \sigma \in G) && (c) \\
 &\quad \wedge ((\exists \tau \in \Sigma : \sigma \xrightarrow{C} \tau \wedge \tau \in F) \implies \sigma \in G) && (d)
 \end{aligned}$$

We can prove [Lemma 4.18](#) by proving that [Line \(a\)](#) implies [Line \(c\)](#) and that [Line \(b\)](#) implies [Line \(d\)](#). For any state $\sigma \in \Sigma$, we first prove (a) \implies (c):

$$\begin{aligned} \text{true} &\Leftrightarrow (\exists \tau \in \Sigma : (\sigma \xrightarrow{C} \tau) \wedge \neg(\tau \in F)) \implies (\exists \tau \in \Sigma : \sigma \xrightarrow{C} \tau) \\ &\Leftrightarrow \neg(\exists \tau \in \Sigma : \sigma \xrightarrow{C} \tau) \implies \neg(\exists \tau \in \Sigma : (\sigma \xrightarrow{C} \tau) \wedge \neg(\tau \in F)) \\ &\implies \neg(\exists \tau \in \Sigma : \sigma \xrightarrow{C} \tau) \implies \sigma \in G \quad | \text{Line (a)} \end{aligned}$$

It is also valid that for any state $\sigma \in \Sigma$, (b) \implies (d). Assume there exists $\tau \in \Sigma$ such that for some state $\sigma \in \Sigma$,

$$\sigma \xrightarrow{C} \tau \wedge \tau \in F \text{ is valid.}$$

Then we conclude from [Line \(b\)](#) that

$$\forall \mu \in \Sigma : \neg(\mu \xrightarrow{C} \tau) \vee \mu \in G$$

Since $\sigma \in \Sigma$, it follows that $\neg(\sigma \xrightarrow{C} \tau) \vee \sigma \in G$. We already know that $\sigma \xrightarrow{C} \tau$, hence $\sigma \in G$ must be true, therefore proving [Line \(d\)](#). \square

Now that we can use G to overapproximate wlp_a , the logical next step is to underapproximate it, so that in the final step we can establish equivalences between G and angelic wlp :

Lemma 4.19 *[G implies angelic wlp]*

$$\text{if } (P \implies G) \implies \neg(\text{sp.C.P} \implies \neg F) \text{ then } G \implies \text{wlp}_a.\text{C.F}$$

Here, the prerequisite states that looking at any predicate P stronger than G (or any subset of G , if we view predicates as sets of all states that satisfy them), there are no executions reaching a final state satisfying $\neg F$. Remember that sp.C.P is exactly satisfied by states that can be reached starting from an initial state in P , recalling [Figure 3.4.1](#). The implication $\text{sp.C.P} \implies \neg F$ means that from P , we can **only** reach states satisfying $\neg F$. By negating this condition, we assert that starting from a state satisfying P , we should be able to reach final states other than those satisfying $\neg F$.

Since P can be singleton, i.e. a condition that is satisfied by only one state σ , then from σ we are not allowed to **only** have executions that end in a state satisfying $\neg F$. Consequently, we do not allow executions starting from G that **only** finish in $\neg F$, making sure that G does not include the red arrows as in [Figure 4.4](#).

Proof. The assumption expresses that for any state $\sigma \in \Sigma$:

$$\begin{aligned} P &\implies G \implies \neg(\text{sp.C.P} \implies \neg F) \\ &\Leftrightarrow P \implies G \implies \neg(\forall \tau \in \Sigma : \tau \in \text{sp.C.P} \implies \tau \in \neg F) \\ &\Leftrightarrow P \implies G \implies \exists \tau \in \Sigma : \neg(\tau \in \text{sp.C.P} \implies \tau \in \neg F) \\ &\Leftrightarrow P \implies G \implies \exists \tau \in \Sigma : \tau \in \text{sp.C.P} \wedge \neg(\tau \in \neg F) \end{aligned}$$

$$\begin{aligned}
&\Leftrightarrow P \Rightarrow G \Rightarrow \exists \tau \in \Sigma : \tau \in \text{sp.C.P} \wedge \tau \in F \\
&\Leftrightarrow P \Rightarrow G \Rightarrow \exists \tau \in \Sigma : (\exists \mu \in \Sigma : \mu \xrightarrow{C} \tau \wedge \mu \in P) \wedge \tau \in F \\
&\hspace{15em} | \text{Theorem 2.8} \\
&\Leftrightarrow \sigma \in P \Rightarrow \sigma \in G \Rightarrow \exists \tau \in \Sigma : (\exists \mu \in \Sigma : \mu \xrightarrow{C} \tau \wedge \mu \in P) \wedge \tau \in F \quad (e)
\end{aligned}$$

Our goal is to prove that for any state $\sigma \in \Sigma$:

$$\begin{aligned}
&G \Rightarrow \text{wlp}_a.C.F \\
&\Leftrightarrow \sigma \in G \Rightarrow \sigma \in \text{wlp}_a.C.F \\
&\Leftrightarrow \sigma \in G \Rightarrow \neg(\exists \tau \in \Sigma : \sigma \xrightarrow{C} \tau) \vee (\exists \tau \in \Sigma : \sigma \xrightarrow{C} \tau \wedge \tau \in F) \\
&\hspace{15em} | \text{Statement 4.17}
\end{aligned}$$

For some state $\sigma \in \Sigma$, assume $\sigma \in G$, then we can construct set $P = \{\sigma\}$ such that the prerequisites in [Line \(e\)](#) holds. Consequently, the postrequisite in holds. Now we can find witnesses μ and τ such that

$$\mu \xrightarrow{C} \tau \wedge \mu \in P \wedge \tau \in F$$

Since P is a singleton set, μ can only be σ . Then we have found a witness τ such that $\sigma \xrightarrow{C} \tau$ and $\tau \in F$, satisfying the postrequisite of our goal. \square

Corollary 4.20 *[G equivalent to angelic wlp]*

if $\text{wlp}.C.F \Rightarrow G \wedge \text{sp.C.}\neg G \Rightarrow \neg F$ and $P \Rightarrow G \Rightarrow \neg(\text{sp.C.P} \Rightarrow \neg F)$
then $G = \text{wlp}_a.C.F$

With this corollary, we can identify all initial states that lead to executions terminating satisfying our given postcondition F , or its opposite. By setting the necessary liberal precondition G to be equal to $\text{wlp}_a.C.F$, we make sure that G is satisfied by all such states. If we think F to be the specification for erroneous termination, then this means that G is concerned with **all** problematic initial states, and it is necessary to avoid G to steer clear of those errors.

4.4 FROM NECESSARY PRECONDITION TO NECESSARY LIBERAL PRECONDITION

To reason about correctness, it is sometimes helpful to first reason about partial correctness, then supplement with termination proof to reach total correctness. For example, when Hoare triple was first proposed [\[11\]](#), it is in form of the sufficient precondition that guarantees partial correctness:

$$H \{C\} F \Leftrightarrow H \Rightarrow \text{wlp}.C.F$$

Then Manna and Pnueli [\[21\]](#) provided a termination proof for while-loops in Hoare triple, so that the extended Hoare Triple is a sufficient precondition that guarantees total correctness:

$$H' \{C\} F \Leftrightarrow H' \Rightarrow \text{wp}.C.F$$

Explicitly, to arrive at $H \implies wp.C.F$, we can first investigate $H \implies wlp.C.F$, then supplement it with a termination proof.

It stands to reason that while attempting to reason about the necessary liberal precondition, we can take a similar approach but from the opposite direction: starting from necessary precondition $wp.C.F \implies G$, supplementing it with a termination or non-termination proof, we try to reach the necessary liberal precondition $wlp.C.F \implies G$.

First, we attempt to find a necessary precondition G where $wp.C.F \implies G$. For this reason, we propose an overapproximation of wp , denoted by wp_d^+ in Table 4.2.

C	$wp_d.C.F$	$wp_d^+.C.F$	$wlp_d.C.F$
skip		F	
diverge	false	false	true
$x := e$		$F[x/e]$	
$C_1; C_2$		$wp_d^+.C_1.(wp_d^+.C_2.F)$	
$\{C_1\} \Box \{C_2\}$		$wp_d^+.C_1.F \wedge wp_d^+.C_2.F$	
if (φ) $\{C_1\}$ else $\{C_2\}$		$(\varphi \wedge wp_d^+.C_1.F)$ $\vee (\neg\varphi \wedge wp_d^+.C_2.F)$	
while (φ) $\{C'\}$	$lfp X. (\neg\varphi \wedge F)$ $\vee (\varphi \wedge wp_d.C'.X)$	$fp X. (\neg\varphi \wedge F)$ $\vee (\varphi \wedge wp_d^+.C'.X)$	$gfp X. (\neg\varphi \wedge F)$ $\vee (\varphi \wedge wlp_d.C'.X)$

Table 4.2: wp_d^+ , An Overapproximation of wp

We can see from Table 4.2 that the only difference wp_d^+ is defined very close to wp_d and wlp_d , aside from the different resolution for non-determinism and treatment for non-termination, the most significant difference is the definition for while-loops: with wp_d^+ , we look for a fixed point of the characteristic function, instead of the least fixed point with wp and the greatest fixed point with wlp . The computation of **some** fixed point is simpler than the computation of the least or the greatest fixed points. There is a repertoire of fixed point combinators, with which we can calculate a fixed point of the given function [3]. The choice of which fixed point to take is left to the user. This also tells us that wp_d^+ represents rather a class of precondition transformers, depending on the choice of the fixed point, one chooses one of the precondition transformers in wp_d^+ . We can see from Table 4.2 that wp_d^+ overapproximates wp_d , and underapproximates wlp_d :

Lemma 4.21

$$\forall C \in \mathcal{C}. \forall F \in \Sigma : wp_d.C.F \implies wp_d^+.C.F$$

Proof. By structural induction. □

Lemma 4.22

$$\forall C \in \mathcal{C}. \forall F \in \Sigma : wp_d^+.C.F \implies wlp_d.C.F$$

Proof. By structural induction. □

The semantics of wp_d^+ is captured in Figure 4.8. The area marked in dark yellow is wp_d , the areas with dark and medium yellow mark wp_d^+ , and all three yellow areas together represent wlp_d . Remember from Theorem 2.10 that $wlp_d.C$ and $wp_a.C$ are each other's conjugates, i.e. $wp_a.C.F = wlp_d.C.\neg F$, hence we can label the rest of the left rectangle with $wp_a.C.\neg F$.

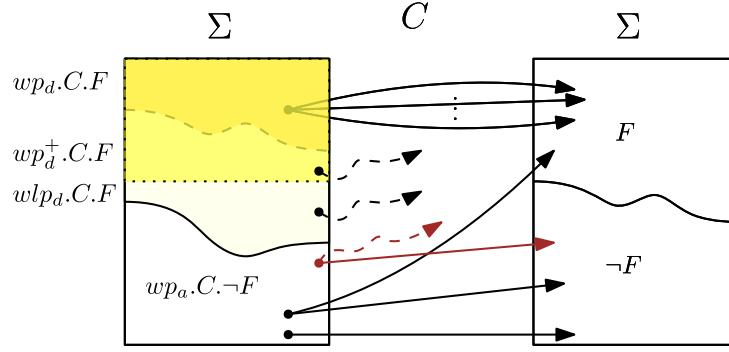


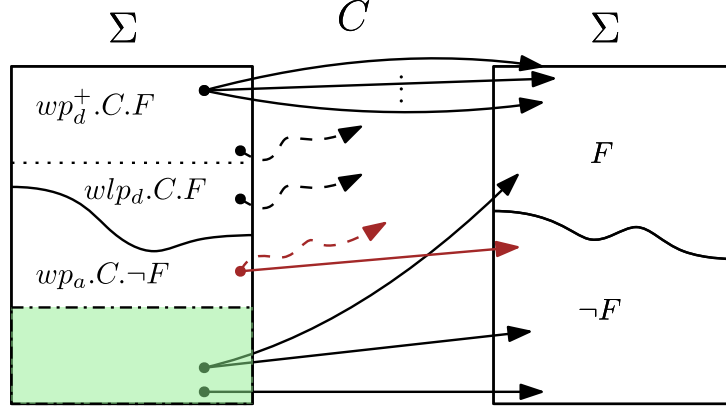
Figure 4.8: Graphical Illustration of wp_d^+

Remember that our goal is to overapproximate wlp_d , i.e. to find G such that $wlp.C.F \implies G$. Speaking in terms of Figure 4.8, it corresponds to finding an area larger than all three yellow areas, i.e. to find G where $wlp_d.C.F \implies G$, which is then equivalent to $\neg G \implies wp_a.C.\neg F$, as we see from Figure 4.8. This yields an alternative method to finding the necessary liberal precondition, given program C and postcondition F :

1. Calculate $wp_d^+.C.F$, where the discovery of any fixed point should be easier than the discovery of the least and the greatest one.
2. Negate the result: $P := \neg wp_d^+.C.F$.
3. Find $Q \implies P$ such that C **always** terminates starting from any initial states satisfying Q .
4. The negation of Q is then the necessary liberal precondition of program C with respect to postcondition F .

One might wonder, what is the difference between this method and directly underapproximating wp_a , i.e. proving the validity of a Hoare triple? The main difference is that we are not concerned with the postcondition anymore. With this method, wp_d^+ is responsible for reasoning about the final states, and we are only concerned with termination proofs. By proving that **all** executions terminate, we identify the green area in Figure 4.9, which is strictly an underapproximation of wp_a , provided that the orange arrows in Figure 4.9 do exist, and that the negation of the green area is an overapproximation of wlp_d .

Unfortunately, proving termination is not trivial. In fact, given a program C , and a condition P , finding a Q where $Q \implies P$, such that the execution of

Figure 4.9: Overapproximating wlp_d Using wp_d^+

C starting in an initial state satisfying Q is in its essence the [halting problem](#), which is undecidable. At this point, we can not provide an algorithm to always find such Q .

However, the use of wp_d^+ still helps us separate the reasoning about final states and termination: wp_d^+ lets us rule out executions not ending in a state satisfying F with less computational complexity, and we have freedom to choose a termination proof without having to be concerned with final states.

An interesting observation of wp_d^+ presents itself: remember from [Figure 3.7](#) that partial correctness is indicated by $G \implies wlp_d.C.F$, and partial incorrectness by $wp_a.C.F \implies G$. We know from [Lemma 4.22](#) that

$$wp_d^+.C.F \implies wlp_d.C.F$$

taking its contrapositive, we get $\neg wlp_d.C.F \implies \neg wp_d^+.C.F$. From [Theorem 2.10](#) we know that $wp_a.C.F = \neg wlp_d.C.\neg F$, hence

$$\neg wp_a.C.\neg F \implies \neg wp_d^+.C.F$$

This tells us that wp_d^+ is valid for partial correctness, and $\neg wp_d^+$ is valid for partial incorrectness. This is not surprising, since partial correctness and incorrectness are each other's contrapositives, it still gives us a notion to represent the class of predicate transformers that are the “cut” between partial correctness and incorrectness triples, as shown in [Figure 4.10](#).

To summarize this chapter, we started with the description of relations that connect the predicate transformers together, to introduce the necessary liberal precondition. Then we discovered that G can be an indicator of “bad” precondition that is necessary to avoid, to successfully terminate. We can do it in two ways:

1. If we are sure that we captured all undesired final states in F , then we use the necessary liberal precondition that underapproximates (overapproximates) wlp_a , so that we catch as much “bad” initial states as possible.

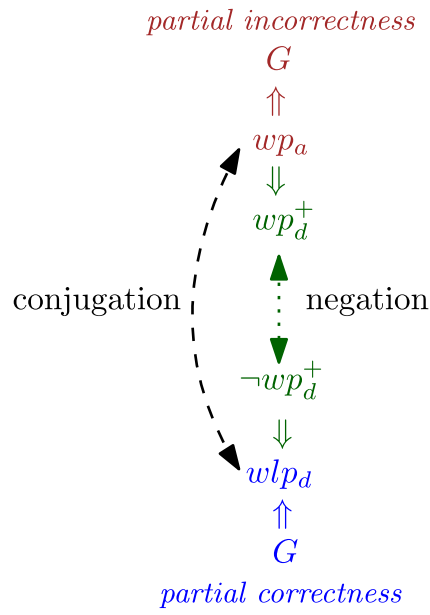


Figure 4.10: The Class of wp_d^+ Is the Cut Between Partial Correctness and Incorrectness

2. If we only have limited knowledge of the erroneous final states, then we appeal to experts of the system for other potential errors in order to reach necessary liberal precondition.

CONCLUSIONS

5.1 CONCLUSIONS

In this thesis, we study the weakest liberal precondition transformer and its over-approximation: G such that $\text{wlp}.C.F \Rightarrow G$. We first discuss the definitions of while-loops in its original form [5] and a variant using fixed points. We establish an equivalence between the two forms of definitions, validating the prudence of the second version. Subsequently, we investigate the G in question. Coincidentally, supplementing G with extra constraints using the strongest postcondition transformer, G coincides with the weakest liberal precondition transformer with angelic non-determinism:

$$(\text{sp}.C.\neg G \Rightarrow \neg F) \wedge (P \Rightarrow G \Rightarrow \neg(\text{sp}.C.P \Rightarrow \neg F)) \Rightarrow G = \text{wlp}_a.C.F$$

However, without extra constraints, G can be a precondition from which all executions are possible. The only certainty is that $\neg G \{C\} \neg F$ is a valid Hoare Triple. Regardless, G still finds its usefulness while trying to identify preconditions that lead to erroneous final states, when there are initial states that can both lead to errors and successes non-deterministically, or when we do not have sufficient knowledge of all the undesired final states. To do so, one first finds the weakest liberal precondition with respect to the known “bad” final states, then over-approximate the found precondition by “guessing” more possible unwanted final states.

The main contribution of this thesis can be summed up in the following list:

1. We prove that the definitions of while-loops with or without the use of fixed points are equivalent in [Section 2.4.2](#), and give intuition to the use of fixed points and the necessity of the use of the least fixed point and the greatest fixed point while defining wp and wlp transformers.
2. We give a graphic overview of the relations between the predicate transformers with angelic or demonic non-determinism in [Chapter 3](#), which helps demonstrate the triples spawned by underapproximations and overapproximations of the predicate transformers.
3. We conclude in [Section 4.2](#) that the necessary liberal precondition in general, without further conditions, is possible to be satisfied by any type of initial states. However, the negation $\neg G$ forms a valid Hoare triple with the negation of the postcondition: $\neg G \{C\} \neg F$.
4. We also find that the necessary liberal preconditions are useful when faced with non-deterministic choices that can lead to both error and success, or in situations where we do not have information about all the erroneous

final states. We propose a heuristic to use this triple, and provide a proof system that captures the overapproximation triple in [Section 4.2.4](#).

5. The examples make us notice a special type of initial states, under whose control the execution can terminate in both final states that satisfy the desired postconditions, or final states that oppose them.

We capture this type of initial states in [Section 4.3](#) by underapproximating and overapproximating them with the necessary liberal preconditions. Consequently, we find a way to approach wlp_a without having to define it previously.

5.2 FUTURE WORK

We think it is interesting and possible to find rules with denotational semantics to capture the initial states that can lead both to the desired F and its opposite $\neg F$, taking inspiration from incorrectness logic. Additionally, this thesis is only concerned with binary predicates, i.e. a predicate that evaluates to either true or false. Albeit classic, it might be more interesting to examine the above results in a quantitative setting, where predicates evaluate to more than true or false. In a quantitative setting, the notion of angelic or demonic non-determinism might be too extreme. Instead of regarding the non-determinism as completely in or against our favor, which are strong assumptions, what are the implications when the non-determinism resolves partially in or against our favor? As the poet Qu Yuan said:

路漫漫其修远兮，
Long, long had been my road and far, far was the journey;
 吾将上下而求索。
I would go up and down to seek my heart's desire. [10]

BIBLIOGRAPHY

- [1] Krzysztof R. Apt. “Ten Years of Hoare’s Logic: A Survey—Part I.” In: *ACM Trans. Program. Lang. Syst.* 3.4 (Oct. 1981), pp. 431–483. ISSN: 0164-0925. DOI: [10.1145/357146.357150](https://doi.org/10.1145/357146.357150). URL: <https://doi.org/10.1145/357146.357150>.
- [2] Michele Boreale. “Complete algorithms for algebraic strongest postconditions and weakest preconditions in polynomial odes.” In: *Science of Computer Programming* 193 (2020), p. 102441. ISSN: 0167-6423. DOI: <https://doi.org/10.1016/j.scico.2020.102441>. URL: <https://www.sciencedirect.com/science/article/pii/S0167642320300514>.
- [3] Felice Cardone and J Roger Hindley. “History of lambda-calculus and combinatory logic.” In: *Handbook of the History of Logic* 5 (2006), pp. 723–817.
- [4] Patrick Cousot, Radhia Cousot, Manuel Fähndrich, and Francesco Logozzo. “Automatic inference of necessary preconditions.” In: *International Workshop on Verification, Model Checking, and Abstract Interpretation*. Springer. 2013, pp. 128–148.
- [5] Edsger W. Dijkstra. “Guarded commands, nondeterminacy and formal derivation of programs.” In: *Communications of the ACM* 18.8 (1975), pp. 453–457.
- [6] Edsger W. Dijkstra. *A Discipline of Programming*. Prentice-Hall Series in Automatic Computation. Englewood Cliffs, N.J: Prentice-Hall, 1976. ISBN: 978-0-13-215871-8.
- [7] Edsger W. Dijkstra and Carel S. Scholten. *Predicate Calculus and Program Semantics*. New York, NY: Springer, 1990. ISBN: 978-1-4612-7924-2 978-1-4612-3228-5. DOI: [10.1007/978-1-4612-3228-5](https://doi.org/10.1007/978-1-4612-3228-5).
- [8] Robert W. Floyd. “Assigning meanings to programs.” In: *Program Verification: Fundamental Issues in Computer Science* (1993), pp. 65–81.
- [9] Mike Gordon and Hélène Collavizza. “Forward with Hoare.” In: *Reflections on the Work of C.A.R. Hoare*. Ed. by A.W. Roscoe, Cliff B. Jones, and Kenneth R. Wood. London: Springer London, 2010, pp. 101–121. ISBN: 978-1-84882-911-4 978-1-84882-912-1. DOI: [10.1007/978-1-84882-912-1_5](https://doi.org/10.1007/978-1-84882-912-1_5). (Visited on 03/10/2024).
- [10] David Hawkes, Qu Yuan, and Various. *The Songs of the South: An Anthology of Ancient Chinese Poems by Qu Yuan and Other Poets*. Reissue edition. Harmondsworth: Penguin Classics, Jan. 2012. ISBN: 978-0-14-044375-2.
- [11] Charles Antony Richard Hoare. “An axiomatic basis for computer programming.” In: *Communications of the ACM* 12.10 (1969), pp. 576–580.
- [12] Jozef Hooman. “Extending Hoare logic to real-time.” In: *Formal Aspects of Computing* 6 (1994), pp. 801–825.
- [13] David Hume. *A treatise of human nature*. Clarendon Press, 1896.

- [14] Benjamin Lucien Kaminski. “Advanced weakest precondition calculi for probabilistic programs.” PhD thesis. RWTH Aachen University, 2019.
- [15] Benjamin Lucien Kaminski, Joost-Pieter Katoen, Christoph Matheja, and Federico Olmedo. “Weakest precondition reasoning for expected run-times of probabilistic programs.” In: *Programming Languages and Systems: 25th European Symposium on Programming, ESOP 2016, Held as Part of the European Joint Conferences on Theory and Practice of Software, ETAPS 2016, Eindhoven, The Netherlands, April 2–8, 2016, Proceedings* 25. Springer. 2016, pp. 364–389.
- [16] Gerwin Klein et al. “seL4: formal verification of an OS kernel.” In: *Proceedings of the ACM SIGOPS 22nd Symposium on Operating Systems Principles. SOSP ’09*. Big Sky, Montana, USA: Association for Computing Machinery, 2009, pp. 207–220. ISBN: 9781605587523. DOI: [10.1145/1629575.1629596](https://doi.org/10.1145/1629575.1629596). URL: <https://doi.org/10.1145/1629575.1629596>.
- [17] Xavier Leroy and Hervé Grall. “Coinductive Big-Step Operational Semantics.” In: *Information and Computation* 207.2 (Feb. 2009), pp. 284–304. ISSN: 08905401. DOI: [10.1016/j.ic.2007.12.004](https://doi.org/10.1016/j.ic.2007.12.004). (Visited on 04/08/2024).
- [18] Shoumei Li, Xia Wang, Yoshiaki Okazaki, Jun Kawabe, Toshiaki Murofushi, Li Guan, and Janusz Kacprzyk, eds. *Nonlinear Mathematics for Uncertainty and Its Applications*. Vol. 100. Advances in Intelligent and Soft Computing. Berlin, Heidelberg: Springer Berlin Heidelberg, 2011. ISBN: 978-3-642-22832-2 978-3-642-22833-9. DOI: [10.1007/978-3-642-22833-9](https://doi.org/10.1007/978-3-642-22833-9). (Visited on 04/01/2024).
- [19] Junyi Liu, Li Zhou, Gilles Barthe, and Mingsheng Ying. “Quantum Weakest Preconditions for Reasoning about Expected Runtimes of Quantum Programs.” In: *Proceedings of the 37th Annual ACM/IEEE Symposium on Logic in Computer Science. LICS ’22*. Haifa, Israel: Association for Computing Machinery, 2022. ISBN: 9781450393515. DOI: [10.1145/3531130.3533327](https://doi.org/10.1145/3531130.3533327). URL: <https://doi.org/10.1145/3531130.3533327>.
- [20] Johan J. Lukkien. “Operational Semantics and Generalized Weakest Preconditions.” In: *Science of Computer Programming* 22.1-2 (Apr. 1994), pp. 137–155. ISSN: 01676423. DOI: [10.1016/0167-6423\(94\)90010-8](https://doi.org/10.1016/0167-6423(94)90010-8). (Visited on 04/08/2024).
- [21] Zohar Manna and Amir Pnueli. “Axiomatic approach to total correctness of programs.” In: *Acta Informatica* 3 (1974), pp. 243–263.
- [22] John McCarthy. “Towards a mathematical science of computation.” In: *Program Verification: Fundamental Issues in Computer Science* (1993), pp. 35–56.
- [23] Magnus O. Myreen and Michael J. C. Gordon. “Hoare Logic for Realistically Modelled Machine Code.” In: *Tools and Algorithms for the Construction and Analysis of Systems*. Ed. by Orna Grumberg and Michael Huth. Berlin, Heidelberg: Springer Berlin Heidelberg, 2007, pp. 568–582. ISBN: 978-3-540-71209-1.

- [24] Keiko Nakata and Tarmo Uustalu. “Trace-Based Coinductive Operational Semantics for While.” In: *Theorem Proving in Higher Order Logics*. Ed. by Stefan Berghofer, Tobias Nipkow, Christian Urban, and Makarius Wenzel. Vol. 5674. Berlin, Heidelberg: Springer Berlin Heidelberg, 2009, pp. 375–390. ISBN: 978-3-642-03358-2 978-3-642-03359-9. DOI: [10.1007/978-3-642-03359-9_26](https://doi.org/10.1007/978-3-642-03359-9_26). (Visited on 04/08/2024).
- [25] Tobias Nipkow. “Hoare Logics in Isabelle/HOL.” In: *Proof and System-Reliability*. Ed. by Helmut Schwichtenberg and Ralf Steinbrüggen. Dordrecht: Springer Netherlands, 2002, pp. 341–367. ISBN: 978-1-4020-0608-1 978-94-010-0413-8. DOI: [10.1007/978-94-010-0413-8_11](https://doi.org/10.1007/978-94-010-0413-8_11). (Visited on 04/02/2024).
- [26] Tobias Nipkow and Gerwin Klein. *Concrete semantics: with Isabelle/HOL*. Springer, 2014.
- [27] Tobias Nipkow, Markus Wenzel, and Lawrence C Paulson. *Isabelle/HOL: a proof assistant for higher-order logic*. Springer, 2002.
- [28] Peter W. O’Hearn. “Incorrectness Logic.” In: *Proceedings of the ACM on Programming Languages* 4.POPL (Jan. 2020), pp. 1–32. ISSN: 2475-1421. DOI: [10.1145/3371078](https://doi.org/10.1145/3371078). (Visited on 03/10/2024).
- [29] Edsko de Vries and Vasileios Koutavas. “Reverse hoare logic.” In: *International Conference on Software Engineering and Formal Methods*. Springer. 2011, pp. 155–171.
- [30] Gia Wulandari and Detlef Plump. *Verifying Graph Programs with First-Order Logic (Extended Version)*. Nov. 2020. arXiv: [2010.14549](https://arxiv.org/abs/2010.14549) [cs]. (Visited on 04/01/2024).
- [31] Linpeng Zhang and Benjamin Kaminski. “Quantitative Strongest Post.” In: *arXiv preprint arXiv:2202.06765* (2022).
- [32] Linpeng Zhang and Benjamin Lucien Kaminski. *Quantitative Strongest Post*. 2022. arXiv: [2202.06765](https://arxiv.org/abs/2202.06765) [cs.LG].
- [33] Li Zhou, Nengkun Yu, and Mingsheng Ying. “An applied quantum Hoare logic.” In: *Proceedings of the 40th ACM SIGPLAN Conference on Programming Language Design and Implementation*. PLDI 2019. Phoenix, AZ, USA: Association for Computing Machinery, 2019, pp. 1149–1162. ISBN: 9781450367127. DOI: [10.1145/3314221.3314584](https://doi.org/10.1145/3314221.3314584). URL: <https://doi.org/10.1145/3314221.3314584>.