NECESSARY LIBERAL PRECONDITIONS:
A PROOF SYSTEM

Mster's Thesis in Informatics

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Anran Wang: *Necessary Liberal Preconditions: A Proof System*

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NOTWENDIGE LIBERALE VORBEDINGUNGEN: EIN BEWEISSYSTEM

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Submission date:	15. September 2023

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DECLARATION

Ich versichere, dass ich diese Masterarbeit selbstständig verfasst und nur die angegebenen Quellen und Hilfsmittel verwendet habe.

I confirm that this master's thesis is my own work and I have documented all sources and material used.

Munich, 15. September 2023

Anran Wang

ABSTRACT

This is where the abstract goes.

ZUSAMMENFASSUNG

Kurze Zusammenfassung des Inhaltes in deutscher Sprache. . .

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LISTINGS

ACRONYMS

Part I

HOARE TRIPLES, WEAKEST PRECONDITIONS, WEAKEST LIBERAL PRECONDITIONS

Some text about this part.

1 BACKGROUND

TODO: Make first letter big?

TODO: Decide on all the colors in the end.

TODO: Rewrite; add chapter contents.

2 PRELIMINARIES

2.1 HOARE LOGIC

Since the beginning of the 1960s, scholars have been researching the establishment of mathematics in computation [3, 7] to have a formal understanding and reasoning of programs. One of the most known methods is Hoare logic.

In 1969, C.A.R. Hoare wrote *An Axiomatic Basis for Computer Programming* [4] to explore the logic of computer programs using axioms and inference rules to prove the properties of programs. He introduced sufficient preconditions that guarantee correct results but do not rule out non-termination. A selection of the axioms and rules are shown in Table 1. [1][2] $\{F[x/e]\}$ is obtained by substituting occurrences of x by e.

Axiom of Assignment	$F[x/e]\ \{x := e\}\ F$
Rules of Consequence	If $G\ \{C\}\ F$ and $F \Rightarrow P$ then $G\ \{C\}\ P$
	If $G\ \{C\}\ F$ and $P \Rightarrow G$ then $P\ \{C\}\ F$
Rule of Composition	If $G\ \{C_1\}\ F_1$ and $F_1\ \{C_2\}\ F$ then $G\ \{C_1; C_2\}\ F$
Rule of Iteration	If $(F \wedge B)\ \{C\}\ F$ then $F\ \{\text{while } B \text{ do } C\}\ \neg B \wedge F$

Table 1: Valid Hoare Triple

Semantically, a Hoare triple $G\ \{C\}\ F$ is said to be valid for (partial) correctness, if the execution of the program C with an initial state satisfying the precondition G leads to a final state that satisfies the postcondition F, provided that the program terminates. The definitions in Table 1 indeed correspond to this intended semantics. Formal soundness proofs can be found in Krzysztof R. Apt's survey [1] in 1981. As an example, consider the rule of composition: if the execution of program C_1 changes the state from G to F_1, and C_2 changes the state from F_1 to F, then executing them consecutively should bring the program state from G to F, with the intermediate state F_1.

The missing guarantee of termination can be seen in the rule of iteration: consider the triple $x \leqslant 2\ \{\text{while } x \leqslant 1 \text{ do } x := x * 2\}\ 1 < x \leqslant 2$, it is provable in Hoare logic with the following proof tree. However, this while-loop will not terminate in case $x \leqslant 0$ in the initial state.

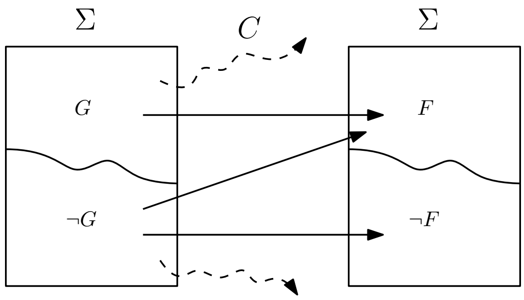
1 We omit the symbol $\vdash$ in front of a Hoare triple, which denotes "valid/provable", for better readability.

2 Non-determinism was not considered in the original paper, so we treat the programs here as deterministic. With deterministic programs, one initial state corresponds to one final state, and in case of non-termination we assign a final state $\perp$.

TODO: Think about whether to add liberally deterministic (Hesselink 1992, Programs, Recursion and Unbounded Choice).

$$\dfrac{\dfrac{}{x \leqslant 1 \; \{x := x * 2\} \; x \leqslant 2} \; \textbf{Axiom of Assignment}}{x \leqslant 2 \; \{\text{while } x \leqslant 1 \text{ do } x := x * 2\} \; 1 < x \leqslant 2} \; \textbf{Rule of Iteration}$$

Using style taken from Benjamin L. Kaminski's dissertation [5], Figure 1 illustrates a valid Hoare triple: Σ represents the set of all states, the section denoted with G includes the states that satisfy the predicate G. The arrow from left to right denotes the execution of the program C.



$G\{C\}F$

Figure 1: Valid Hoare Triple (Deterministic)

A sensible advancement of Hoare logic would be to also prove termination, i.e. to eliminate the arrows from G to the abyss. Supplementing Hoare logic with a termination proof is done by Zohar Manna and Amir Pnueli in 1974 [6], where they introduced what we call a loop variant, a value that decreases with each iteration. The name is in contrast to loop invariant, concretely the F in **Rule of Iteration** in Table 1, which is constant before and after the loop.

Another advancement would be to find the necessary and sufficient preconditions that grant us the post-properties, i.e. to eliminate the arrows from $\neg G$ to F in Figure 1, which is what Edsger W. Dijkstra accomplished with his weakest precondition transformer in 1975 [2], among other things.

2.2 GUARDED COMMAND LANGUAGE

From now on we will use Dijkstra's (non-deterministic) guarded command language (GCL) [2] to represent programs and to include non-determinism (starting from Section 2.3.3). For better readability, we use an equivalent[3] form of GCL that is similar to modern pseudo-code as shown in Table 2.

The assignment, sequential composition, conditional choice, while-loop commands conform to their usual meaning. The non-deterministic choice $\{C_1\}\Box\{C_2\}$ chooses from two programs randomly. It is however not probabilistic, meaning we do not know the probabilistic distribution of the outcome of the choice.

When skip is executed, the program state does not change and the consecutive part is executed. When diverge is executed, the program goes to a state $\perp$ symbolizing non-termination, and the execution stops.

3 Specifically, $\text{if } (\varphi)\ \{C_1\} \text{ else } \{C_2\}$ is equivalent to $\text{if } \varphi \to C_1 \;[]\; \neg\varphi \to C_2 \text{ fi}$ in Dijkstra's original style [2]; $\{C_1\}\Box\{C_2\}$ is equivalent to $\text{if true} \to C_1 \;[]\; \text{true} \to C_2 \text{ fi}$.

C ::=	$x := e$	$\mid C; C$	$\mid \{C\}\Box\{C\}$	
	assignment	sequential composition	non-deterministic choice	
	$\mid$ if (φ) $\{C\}$ else $\{C\}$	$\mid$ while (φ) $\{C\}$	$\mid$ skip	$\mid$ diverge
	conditional choice	while-loop		

Table 2: Guarded Command Language

2.3 WEAKEST PRECONDITION

2.3.1 *The Deterministic Case*

To better relate Hoare triples and Dijkstra's weakest precondition transformer, we first focus on deterministic programs. Again, the goal is to find the necessary and sufficient precondition such that the program is guaranteed to terminate in a state that satisfies the postcondition. Figure 2 shows it graphically alongside the figure for valid Hoare triples.

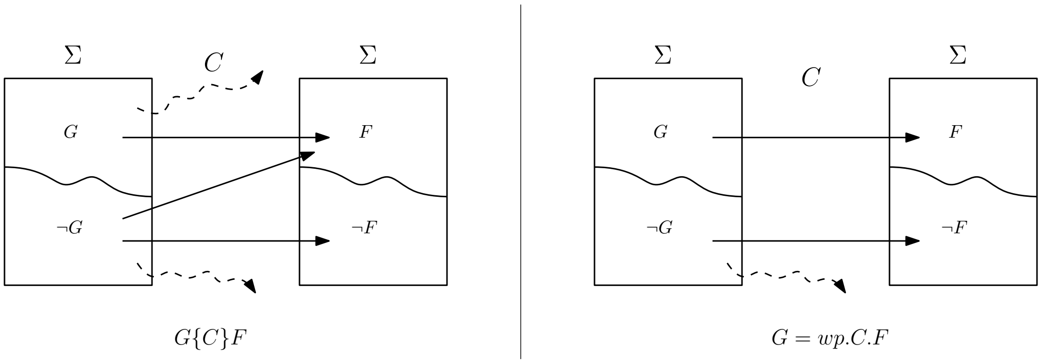


Figure 2: Valid Hoare Triple vs. Weakest Precondition (Deterministic)

We define the weakest precondition transformer inductively over the program structure in lambda-calculus style[4] as in Table 3:
$F[x/e]$ is F where every occurrence of x is syntactically replaced by e.
$\text{lfp } X.f$ is the least fixed point of function f with variable X.
Let

$$\Phi(X) := (\neg\varphi \wedge F) \vee (\varphi \wedge wp.C'.X)$$

be the characteristic function, then wp for while-loop can be defined as:

$$wp.(\text{while}(\varphi)\{C'\}).F = \text{lfp } X.\Phi(X)$$

Most of the definitions in Table 3 are intuitive and correspond to their counterparts in Hoare logic, while those for diverge and while deserve special attention.

4 For example, $wp.C.F$ can be seen as $wp(C, F)$ in "typical" style, where wp is treated as a function that has two parameters. The advantage of lambda-calculus style is scalability, we can simply extend the aforementioned function to $wp.C.F.\sigma$ where σ means the initial state. Here wp is treated as a function that has three parameters, if we were to write it in the "typical" style. It is then questionable whether we changed the type of wp.

C	wp.C.F
skip	F
diverge	false
$x := e$	$F[x/e]$
$C_1; C_2$	$wp.C_1.(wp.C_2.F)$
if (φ) $\{C_1\}$ else $\{C_2\}$	$(\varphi \wedge wp.C_1.F) \vee (\neg\varphi \wedge wp.C_2.F)$
while (φ) $\{C'\}$	lfp $X.(\neg\varphi \wedge F) \vee (\varphi \wedge wp.C'.X)$

Table 3: The Weakest Precondition Transformer for Deterministic Programs [5]

Since wp aims for total correctness, a program starting in an initial state satisfying the precondition wp.diverge.F should terminate in a final state satisfying the postcondition F. Because diverge does not terminate, there is no such precondition and wp for diverge should be false.

The definition for the while-loop [5] is trickier, but we can verify its correctness by recalling Dijkstra's original definition in the following section.

TODO: Find out if there's earlier definition that used lfp.

2.3.2 *Defining Loops*

In Dijkstra's original paper [2], he defined wp for while-loops based on its (intended) semantics, i.e. the precondition that guarantees loop termination with the required postcondition within a certain number of iterations.

Let

$$WHILE = while(\varphi)\{C'\} \quad \text{and} \quad IF = if\ (\varphi)\{C'; WHILE\}\ else\ \{skip\}.$$

Rewriting Dijkstra's definition in a form conforming to our style, he defines

$$H_0(F) = (F \wedge \neg\varphi) \quad \text{and} \quad H_k(F) = (wp.IF.(H_{k-1}(F)) \vee H_0(F)).$$

Intuitively, when $H_0(F)$ is satisfied before the execution of $WHILE$, the loop is exited with 0 iteration in a state that satisfies $F \wedge \neg\varphi$ hence F. Then, we can understand $H_k(F)$ as the weakest precondition such that the program terminates in a final state satisfying F after at most k iterations. **TODO:** Explain a bit more about how this relate to the

Then by definition:

$$wp.WHILE.F = (\exists k \geqslant 0 : H_k(F)) \tag{1}$$

We state that our definition in Table 3 coincides with this definition. Without going too deep into domain theory, we only use one of its theorem that yields a computation for least fixed points, when they exist.

Theorem 1. ***TODO:*** *Insert theorem, then explain least point iteration from bottom.*

Coincidentally, $H_k(F)$ is the k−th iteration of the characteristic function Φ from the bottom element, denoted by $\Phi^k(\bot)$. Thus by identifying the least fixed point, we've found a k that satisfies (1).

2.3.3 *The Non-deterministic Case*

Now we bring the non-deterministic choice back into the picture and add its wp to Table 4. To justify this definition for the non-deterministic choice, we must first clarify the intended semantics of the wp-transformer.

C	**wp.C.F**
$\mathtt{skip}$	F
$\mathtt{diverge}$	$\mathtt{false}$
$x := e$	$F[x/e]$
$C_1; C_2$	$wp.C_1.(wp.C_2.F)$
$\mathtt{if}\ (\varphi)\ \{C_1\}\ \mathtt{else}\ \{C_2\}$	$(\varphi \wedge wp.C_1.F) \vee (\neg\varphi \wedge wp.C_2.F)$
$\{C_1\}\square\{C_2\}$	$wp.C_1.F \vee wp.C_2.F$
$\mathtt{while}\ (\varphi)\ \{C'\}$	$\mathrm{lfp}\ X.(\neg\varphi \wedge F) \vee (\varphi \wedge wp.C'.X)$

Table 4: The Weakest Precondition Transformer for Non-deterministic Programs [5]

Let $[\![C]\!]$ denote the execution of program C and $[\![C]\!].\sigma$ denote the set of final states that can occur after the execution of C. A state is a function that maps a program variable to a value. The set of states is denoted by $\Sigma = \{\sigma \mid \sigma : \mathsf{Vars} \to \mathsf{Vals}\}$.

If C is deterministic, then $[\![C]\!].\sigma$ is a set of a single state, either a final state σ' or $\bot$, if the execution does not terminate. If C is non-deterministic, $[\![C]\!].\sigma$ can be a set with multiple elements, since multiple final states can be possible.

The weakest precondition $wp.C.F$ is then

Note for readers: Up to here is readable.

Figure 3 shows wp with non-deterministic programs. Each arrow from left to right shows a possible execution of program C.

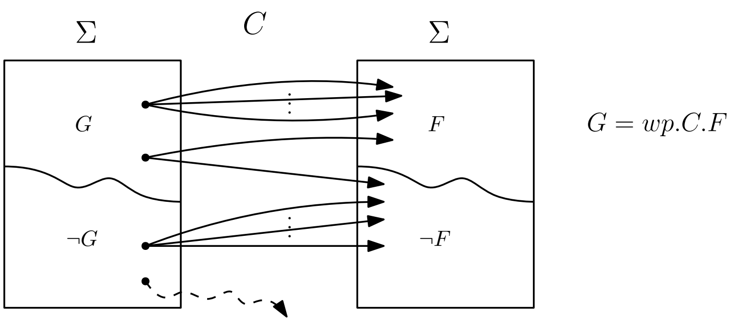


Figure 3: Weakest Precondition (Non-deterministic)

2.4 WEAKEST LIBERAL PRECONDITION

We define the weakest liberal precondition transformer in Table 5.

C	**wlp.C.F**
skip	F
$\mathit{diverge}$	true
$x := e$	$F[x/e]$
$C_1; C_2$	$wlp.C_1.(wlp.C_2.F)$
$\mathit{if}\ (\varphi)\ \{C_1\}\ \mathit{else}\ \{C_2\}$	$(\varphi \wedge wlp.C_1.F) \vee (\neg\varphi \wedge wlp.C_2.F)$
$\{C_1\}\Box\{C_2\}$	$wlp.C_1.F \wedge wlp.C_2.F$
$\mathit{while}\ (\varphi)\ \{C'\}$	$\mathit{gfp}\ X.(\neg\varphi \wedge F) \vee (\varphi \wedge wlp.C'.X)$

Table 5: The Weakest Liberal Precondition Transformer

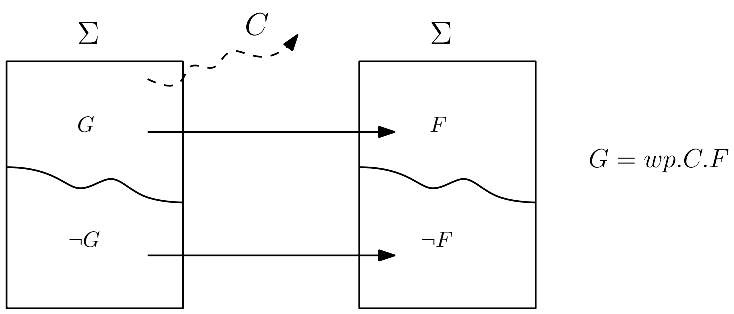


Figure 4: Weakest Liberal Precondition (Deterministic)

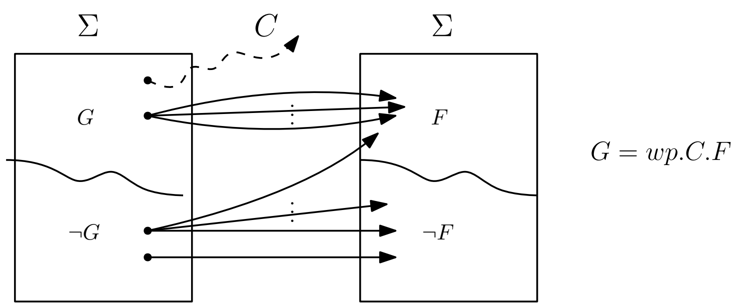


Figure 5: Weakest Liberal Precondition (Non-deterministic)

Part II

NECESSARY LIBERAL PRECONDITIONS

Some text about this part.

3 A PROOF SYSTEM

3.1 A PROOF SYSTEM

In this section we study the necessary liberal precondition:

$$wlp.C.F \implies G$$

4 CONCLUSIONS

4.1 CONCLUSIONS

4.2 FUTURE WORK

Part III

APPENDIX

[1] Krzysztof R. Apt. “Ten Years of Hoare’s Logic: A Survey—Part I.” In: *ACM Trans. Program. Lang. Syst.* 3.4 (1981), 431–483. ISSN: 0164-0925. DOI: 10.1145/357146.357150. URL: https://doi.org/10.1145/357146.357150.

[2] Edsger W. Dijkstra. “Guarded commands, nondeterminacy and formal derivation of programs.” In: *Communications of the ACM* 18.8 (1975), pp. 453–457.

[3] Robert W Floyd. “Assigning meanings to programs.” In: *Program Verification: Fundamental Issues in Computer Science* (1993), pp. 65–81.

[4] Charles Antony Richard Hoare. “An axiomatic basis for computer programming.” In: *Communications of the ACM* 12.10 (1969), pp. 576–580.

[5] Benjamin Lucien Kaminski. “Advanced weakest precondition calculi for probabilistic programs.” PhD thesis. RWTH Aachen University, 2019.

[6] Zohar Manna and Amir Pnueli. “Axiomatic approach to total correctness of programs.” In: *Acta Informatica* 3 (1974), pp. 243–263.

[7] John McCarthy. “Towards a mathematical science of computation.” In: *Program Verification: Fundamental Issues in Computer Science* (1993), pp. 35–56.

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