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UNIT-08.

Inner product :

If u and v are $n \times 1$ matrix, then $u^T v$ be 1×1 matrix, when u^T be transpose of u, which is called inner product of u and v.

If $u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$ and $v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$, then the inner product of u and v is

$$u^T v = \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

$$= u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$

$$= u \cdot v.$$

Scalar or dot product :

let $u = (u_1, u_2, \ldots, u_n)$ and $v = (v_1, v_2, \ldots, v_n)$, then the scalar or dot product is denoted by $u \cdot v$ and defined as

$$u \cdot v = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n.$$

Ex. compute $u \cdot v$, $v \cdot u$ where $u = (2, 4, 5)$, $v = (-1, 3, -1)$

Soln: Here

$$u \cdot v = (2, 4, 5) \cdot (-1, 3, -1) = -2 + 12, -5$$
$$= 12 - 7$$
$$= 5$$

$$v \cdot u = (-1, 3, -1) \cdot (2, 4, 5) = -2 + 12 - 5$$
$$= 12 - 7$$
$$= 5 \quad \text{Ans}$$

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Length of vector :-
The length of norm of a vector v is a non-negative scalar denoted by $||v||$ and defined as
$||v|| = \sqrt{v \cdot v} = \sqrt{v_1^2+v_2^2+\dots+v_n^2}$

where $v = (v_1, v_2, \dots v_n)$

$\therefore ||v||^2 = v \cdot v$

Ex. find the length of vector $(4,5,6)$
Sol: here,
let $v = (4,5,6)$
length of $v = ||v|| = \sqrt{16+25+36} = \sqrt{4^2+5^2+6^2}$
$= \sqrt{77}$

Unit vector :-
A vector having length 1, is called unit vector. If v is a non-zero vector, then the unit vector along the direction of v is $\frac{v}{||v||}$ in the direction of v

Ex. Find the unit vector along the vector $v = (-2,1,0)$ and verify it.
Sol:
$v = (-2,1,0)$
$||v|| = \sqrt{(-2)^2+(1)^2+0^2} = \sqrt{4+1} = \sqrt{5}$
unit vector in the direction of v
$u = \frac{v}{||v||} = \frac{(-2,1,0)}{\sqrt{5}} = \left(\frac{-2}{\sqrt{5}}, \frac{1}{\sqrt{5}}, 0\right)$

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verification.
$||u|| = \sqrt{\left(\frac{-2}{\sqrt{5}}\right)^2+\left(\frac{1}{\sqrt{5}}\right)^2+0^2} = \sqrt{\frac{4}{5}+\frac{1}{5}} = \sqrt{\frac{5}{5}} = 1$
Verify.

Normalization of a vector :-
let v be a vector in R^n. Set $u = \frac{v}{||v||}$ then
the process creating u is called normalization v.

* **Distances between two vectors :-**
let u and v are in R^n, then the distances between u and v is the length between them and denoted by $dis(u,v)$ and defined as
$\therefore dis(u,v) = ||u-v||$

Ex. If $u = (2,3)$, $v = (3,-1)$, then find the distance between u and v.
Sol: Here
$u-v = (2,3)-(3,-1)$
$= (-1,4)$
$\therefore dis(u,v) = ||u-v|| = \sqrt{(-1)^2+(4)^2} = \sqrt{1+16} = \sqrt{17}$

* **Orthogonal vector**
Two vectors u and v in R^n are orthogonal to eachother if $u \cdot v = 0$.
Ex.

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Ex. Show that $u = (2, -3, 3)$, $v = (12, 3, -5)$ are Orthogonal.

Soln.

$u \cdot v = (2, -3, 3), (12, 3, -5)$

$= 24 - 9 - 15$

$= 24 - 24$

$= 0$

$\therefore$ u and v are orthogonal.

Imp The Pythagorean Theorem:

Two vectors u and v are orthogonal if and only if $\|u+v\|^2 = \|u\|^2 + \|v\|^2$.

proof: First suppose that u and v are orthogonal i.e. $u \cdot v = 0$ — (1)

and $\|u\|^2 = u \cdot u$, $\|v\|^2 = v \cdot v$

Now

$\|u+v\|^2 = (u+v) \cdot (u+v)$

$= u \cdot u + u \cdot v + v \cdot u + v \cdot v$

$= \|u\|^2 + 2u \cdot v + \|v\|^2$

$= \|u\|^2 + 0 + \|v\|^2$

$\|u+v\|^2 = \|u\|^2 + \|v\|^2$

Conversely

$\|u+v\|^2 = \|u\|^2 + \|v\|^2$

$\Rightarrow (u+v) \cdot (u+v) = \|u\|^2 + \|v\|^2$

$u \cdot u + u \cdot v + v \cdot u + v^2 = \|u\|^2 + \|v\|^2$

$\|u\|^2 + 2u \cdot v + \|v\|^2 = \|u\|^2 + \|v\|^2$

$2u \cdot v = 0$

$u \cdot v = 0$

$\therefore$ u and v are orthogonal Hence proved

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Exercise: 8.1.

1. Using these vector, compute the quantities where,

$u = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$, $v = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$, $w = \begin{bmatrix} 3 \\ -1 \\ -5 \end{bmatrix}$, $x = \begin{bmatrix} 6 \\ -2 \\ 3 \end{bmatrix}$

(i) $u \cdot u$

Here

$u = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

$u \cdot u = \|u\|^2$

~~$\|u\| = \sqrt{1^2 + 2^2} = \sqrt{1+4} = \sqrt{5}$~~

$u \cdot u = \|u\|^2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

$= 1 + 4$

$= 5$

(ii) $v \cdot u = [4, 6] \cdot [-1, 2]$

$= -4 + 12$

$= 8$

$\frac{v \cdot u}{u \cdot u} = 8/5$ Ans

$w \cdot w = \begin{bmatrix} 3 \\ -1 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -1 \\ 5 \end{bmatrix} = 9 + 1 + 25$

$= 35$ Ans

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$X \cdot W = \begin{bmatrix} 6 \\ -2 \\ 3 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \\ -5 \end{bmatrix}$

$= 18+2-15$

$= 5$

then,

$\frac{X \cdot W}{W \cdot W} = \frac{5}{35} = 1/7$ Ans

(iii) $W \cdot W = \begin{bmatrix} 3 \\ -1 \\ -5 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \\ -5 \end{bmatrix} = 9+1+25 = 35$

$\frac{1}{W \cdot W} \cdot W = \frac{1}{35} \begin{bmatrix} 3 \\ -1 \\ -5 \end{bmatrix} = \begin{bmatrix} 3/35 \\ -1/35 \\ -5/35 \end{bmatrix} = \begin{bmatrix} 3/35 \\ -1/35 \\ -1/7 \end{bmatrix}$

Ans

(iv) $\frac{1}{u \cdot u} u$

$u \cdot u = 5$

$\frac{1}{u \cdot u} \cdot u = \frac{1}{5} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1/5 \\ 2/5 \end{bmatrix}$

(v) $\left(\frac{u \cdot v}{v \cdot v}\right) \cdot v =$

$u \cdot v = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 6 \end{bmatrix} = -4+12 = 8$

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$v \cdot v = \begin{bmatrix} 4 \\ 6 \end{bmatrix} \begin{bmatrix} 4 \\ 6 \end{bmatrix} = 16+36 = 52$

$\therefore \left(\frac{u \cdot v}{v \cdot v}\right) \cdot v = \frac{8}{52} \begin{bmatrix} 4 \\ 6 \end{bmatrix} = 2/13 \begin{bmatrix} 4 \\ 6 \end{bmatrix} = \begin{bmatrix} 8/13 \\ 12/13 \end{bmatrix}$

(vi) $\left(\frac{X \cdot W}{X \cdot X}\right) X$

$X \cdot W = 5$

$X \cdot X = \begin{bmatrix} 6 \\ -2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 6 \\ -2 \\ 3 \end{bmatrix} = 36+4+9 = 49$

$\left(\frac{X \cdot W}{X \cdot X}\right) \cdot X = \frac{5}{49} \begin{bmatrix} 6 \\ -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 30/49 \\ -10/49 \\ 15/49 \end{bmatrix} = \begin{bmatrix} 30/49 \\ -10/49 \\ 15/49 \end{bmatrix}$

Ans

(vii) ~~$\|W\|^2$~~

~~$W \cdot W = \begin{bmatrix} 3 \\ -1 \\ -5 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \\ -5 \end{bmatrix} = 9+1+25 = 35$ Ans~~

(vii) $\|W\|$

$\|W\| = \sqrt{9+1+25} = \sqrt{35}$

(viii) $\|X\| = \sqrt{6^2+(-2)^2+3^2} = \sqrt{36+4+9} = \sqrt{49}$

$\|X\| = 7$ Ans

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2 Find a unit vector in the direction of the given vector.

(i) $\begin{bmatrix} -30 \\ 40 \end{bmatrix}$

Sol^n: Here

let $V = \begin{bmatrix} -30 \\ 40 \end{bmatrix}$

$||V|| = \sqrt{(-30)^2 + (40)^2} = \sqrt{900+1600} = \sqrt{2500} = 50$

We know that

Unit vector in the direction of v is u

$u = \frac{V}{||V||} = \frac{(-30, 40)}{50} = (-3/5, 4/5)$

$u = \begin{bmatrix} -3/5 \\ 4/5 \end{bmatrix}$ Ans

(ii) $\begin{bmatrix} -6 \\ 4 \\ -3 \end{bmatrix}$

Sol^n: Here,

let $v = \begin{bmatrix} -6 \\ -4 \\ -3 \end{bmatrix}$

The unit vector in the direction of v is u

then,

$||V|| = \sqrt{(-6)^2 + (4)^2 + (-3)^2} = \sqrt{36+16+9} = \sqrt{61}$

$u = \frac{V}{||V||} = \frac{(-6, 4, -3)}{\sqrt{61}} = \left(\frac{-6}{\sqrt{61}}, \frac{4}{\sqrt{61}}, \frac{-3}{\sqrt{61}}\right)$

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$\therefore u = \begin{bmatrix} -6/\sqrt{61} \\ 4/\sqrt{61} \\ -3/\sqrt{61} \end{bmatrix}$ Ans

(iii) $\begin{bmatrix} 7/4 \\ 1/2 \\ 1 \end{bmatrix}$

Sol^n: Here,

let $V = \begin{bmatrix} 7/4 \\ 1/2 \\ 1 \end{bmatrix}$

We know that

the unit vector along the direction of v is u.

$u = \frac{V}{||V||}$

$||V|| = \sqrt{\frac{49}{16} + \frac{1}{4} + 1} = \sqrt{\frac{69}{16}} = \frac{\sqrt{69}}{4}$

$u = \frac{(7/4, 1/2, 1)}{\frac{\sqrt{69}}{4}} = \left(\frac{7}{\sqrt{69}}, \frac{2}{\sqrt{69}}, \frac{4}{\sqrt{69}}\right)$

$\therefore u = \begin{bmatrix} 7/\sqrt{69} \\ 2/\sqrt{69} \\ 4/\sqrt{69} \end{bmatrix}$ Ans

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(iv) $\begin{bmatrix} 8/3 \\ 2 \end{bmatrix}$

Solⁿ: Here

let $v = \begin{bmatrix} 8/3 \\ 2 \end{bmatrix}$

we know that, the unit vector along the direction of v is u

$u = \frac{v}{\|v\|}$

$\|v\| = \sqrt{64/9 + 4} = \sqrt{64+36/9} = \sqrt{100/9} = \frac{10}{3}$

$u = \frac{v}{\|v\|} = \frac{\begin{bmatrix} 8/3 & 2 \end{bmatrix}}{10/3} = \begin{bmatrix} 8/10 , & 6/10 \end{bmatrix}$

$\therefore u = \begin{bmatrix} 4/5 & 3/5 \end{bmatrix}$

(3) Find the distance between $x = \begin{bmatrix} 10 \\ -3 \end{bmatrix}$ and $y = \begin{bmatrix} -1 \\ -5 \end{bmatrix}$

Solⁿ: Here,

$x - y = \begin{bmatrix} 10 & -3 \end{bmatrix} - \begin{bmatrix} -1 & -5 \end{bmatrix}$

$= \begin{bmatrix} 11 & 2 \end{bmatrix}$

dis $\|x-y\| = \sqrt{11^2 + 2^2} = \sqrt{121 + 4} = \sqrt{125}$

dis $\|x-y\| = 5\sqrt{5}$ Ans

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(4) Determine which pairs of vectors are orthogona

(i) $a = \begin{bmatrix} 8 \\ -5 \end{bmatrix}, b = \begin{bmatrix} -2 \\ -3 \end{bmatrix}$

Solⁿ: Here

$a = \begin{bmatrix} 8 \\ -5 \end{bmatrix}, b = \begin{bmatrix} -2 \\ -3 \end{bmatrix}$

For orthogonal, $a \cdot b = 0$

$a \cdot b = \begin{bmatrix} 8 \\ -5 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ -3 \end{bmatrix}$

$= -16 + 15$

$= -1 \neq 0$

$\therefore$ the given pair of vector are not orthogonal

(ii) $u = \begin{bmatrix} 12 \\ 3 \\ -5 \end{bmatrix}, v = \begin{bmatrix} 2 \\ -3 \\ 3 \end{bmatrix}$

Solⁿ: Here

$u = \begin{bmatrix} 12 \\ 3 \\ -5 \end{bmatrix}, v = \begin{bmatrix} 2 \\ -3 \\ 3 \end{bmatrix}$

For Orthogonal, $u \cdot v = 0$

$u \cdot v = \begin{bmatrix} 12 \\ 3 \\ -5 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -3 \\ 3 \end{bmatrix} = 24 - 9 - 15$

$= 24 - 24$

$= 0$

$\therefore$ the given pair of vector are orthogonal

Ans

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(iii) $u = \begin{bmatrix} 3 \\ 2 \\ -5 \\ 0 \end{bmatrix}, \quad v = \begin{bmatrix} -4 \\ 1 \\ -2 \\ 6 \end{bmatrix}$

Sol^n: Here,

$u = \begin{bmatrix} 3 \\ 2 \\ -5 \\ 0 \end{bmatrix}, \quad v = \begin{bmatrix} -4 \\ 1 \\ -2 \\ 6 \end{bmatrix}$

For Orthogonal, $u \cdot v = 0$

$u \cdot v = \begin{bmatrix} 3 \\ 2 \\ -5 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} -4 \\ 1 \\ -2 \\ 6 \end{bmatrix}$

$= -12 + 2 + 10 + 0$

$= -10 + 10$

$= 0$

$\therefore$ the given pair of vectors are orthogonal

(iv) $y = \begin{bmatrix} -3 \\ 7 \\ 4 \\ 0 \end{bmatrix}, \quad z = \begin{bmatrix} 1 \\ -8 \\ 15 \\ -7 \end{bmatrix}$

Sol^n: Here,

$y = \begin{bmatrix} -3 \\ 7 \\ 4 \\ 0 \end{bmatrix}, \quad z = \begin{bmatrix} 1 \\ -8 \\ 15 \\ -7 \end{bmatrix}$

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For orthogonal, $y \cdot z = 0$,

$y \cdot z = \begin{bmatrix} -3 \\ 7 \\ 4 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -8 \\ 15 \\ -7 \end{bmatrix}$

$= -3 - 56 + 60 + 0$

$= -59 + 60$

$= 1 \neq 0$

$\therefore$ the given pair of vectors are not Orthogonal.

(5) Find the angle between given vectors.

(i) $u = (1, -3)$ and $v = (2, 4)$

Sol^n: Here

$u = (1, -3)$

$v = (2, 4)$

we know that

$u \cdot v = \|u\| \cdot \|v\| \cos\theta$

$\cos\theta = \dfrac{u \cdot v}{\|u\| \, \|v\|}$

$u \cdot v = (1, -3) \cdot (2, 4) = 2 - 12 = -10$

$\|u\| = \sqrt{1 + 9} = \sqrt{10}$

$\|v\| = \sqrt{4 + 16} = \sqrt{20}$

$\therefore \cos\theta = \dfrac{u \cdot v}{\|u\| \cdot \|v\|} = \dfrac{-10}{\sqrt{10} \cdot \sqrt{20}} = \dfrac{-10}{\sqrt{200}} = \dfrac{-\sqrt{2}}{2}$

$\theta = 135^\circ$ or $\frac{3\pi}{4}$ Ans.

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(ii) $u = \begin{bmatrix} 4 \\ 1 \\ 8 \end{bmatrix}$ and $v = \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}$

Sol^n: Here,

$u = \begin{bmatrix} 4 \\ 1 \\ 8 \end{bmatrix}, \quad v = \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}$

We know that

$\cos\theta = \dfrac{u \cdot v}{\|u\| \cdot \|v\|}$

$u \cdot v = \begin{bmatrix} 4 \\ 1 \\ 8 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} = 4 + 0 - 24 = -20$

$\|u\| = \sqrt{16+1+64} = \sqrt{81} = 9$

$\|v\| = \sqrt{1+0+9} = \sqrt{10}$

$\therefore \cos\theta = \dfrac{-20}{9\sqrt{10}}$

$\theta = \cos^{-1}\left(\dfrac{-20}{9\sqrt{10}}\right)$ Ans.

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(iii) $x = (1, 0, 1, 0)$ and $y = (-3, -3, -3, -3)$

Sol^n: Here

$x = (1, 0, 1, 0)$

$y = (-3, -3, -3, -3)$

We know that

$\cos\theta = \dfrac{x \cdot y}{\|x\| \cdot \|y\|}$

$x \cdot y = (1, 0, 1, 0) \cdot (-3, -3, -3, -3)$

$= -3 - 3 = -6$

$\|x\| = \sqrt{1+1} = \sqrt{2}$

$\|y\| = \sqrt{9+9+9+9} = \sqrt{36}$

$\therefore \cos\theta = \dfrac{x \cdot y}{\|x\| \cdot \|y\|} = \dfrac{-6}{\sqrt{2} \cdot \sqrt{36}} = \dfrac{-6}{\sqrt{2} \cdot 6} = \dfrac{-1}{\sqrt{2}}$

$\theta = \cos^{-1}\left(\dfrac{-1}{\sqrt{2}}\right)$

$\theta = 135^\circ$ or $\dfrac{3\pi}{4}$ Ans.

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(6) If $u, v \in R^n$, prove that $[dis(u,-v)]^2 = [dis(u$
iff $u.v = 0$

Sol^n: Here,
$[dis(u,-v)]^2 = [dis(u, v)]^2$
where $u.v = 0$
~~We have,~~
~~$[dis(u,-v)]^2 = (u+v)$~~
taking L.H.S,
$dis(u,-v) = \|u+v\|$
$[dis(u,-v)]^2 = \|(u+v)\|^2$
$= u^2 + 2.u.v + v^2$
$= \|u\|^2 + 2.(u.v) + \|v\|^2$
$= \|u\|^2 + 2.0 + \|v\|^2$
$= \|u\|^2 + \|v\|^2$

taking R.H.S.
$dis(u,v) = \|u-v\|$
$[dis(u,v)]^2 = (u-v)^2$
$= u^2 - 2.u.v + v^2$
$= \|u\|^2 - 2.(u.v) + \|v\|^2$
$= \|u\|^2 - 2.0 + \|v\|^2$
$= \|u\|^2 + \|v\|^2$

$\therefore$ L.H.S = R.H.S
proved

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Exercise:- 8.2

Orthogonal set
- A set of vectors $\{v_1, v_2, \ldots v_n\}$ in R^n is said to be an orthogonal if $u_i . u_j = 0$, for $i \neq j$, for $i \neq j$, for $i, j = 1, 2, \ldots n$

Ex. Find a set of vectors $\{u_1, u_2, u_3\}$ is an orthogonal set where
$u_1 = (2, -7, -1)$, $u_2 = (-6, -3, 9)$, $u_3 = (3, 1, -1)$

Sol^n: Here
$u_1 . u_2 = (2, -7, -1) . (-6, -3, 9)$
$= -12 + 21 - 9$
$= 0$

$u_2 . u_3 = (-6, -3, 9) . (3, 1, -1)$
$= -18 - 3 - 9$
$= -30 \neq 0$

$u_3 . u_1 = (3, 1, -1) . (2, -7, -1)$
$= 6 - 7 + 1$
$= 0$

So, the set of vectors (u_1, u_2, u_3) are not orthogonal.
but $\{u_1, u_2\}$ and $\{u_3 . u_1\}$ are orthogonal

Theorem:- If $S = \{u_1, u_2, \ldots u_n\}$ is an orthogonal set of non-zero vectors in R^n, then S is linearly independent and hence is a basis for the subspace spanned by S.

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Theorem 1:

let $\{u_1, u_2, \cdots u_p\}$ be an orthogonal basis for subspace W of R^n. For any y in W the weight in the linear combination

$y = c_1u_1 + c_2u_2 + \cdots c_pu_p$ are given by

$$c_j = \frac{y \cdot u_j}{u_j \cdot u_j}, \text{ for } j = 1, 2, \cdots p.$$

Ex. The set $\{u_1, u_2, u_3\}$ where

$u_1 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$, $u_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$, $u_3 = \begin{bmatrix} -1/2 \\ -2 \\ 7/2 \end{bmatrix}$

is an orthogonal basis in R^3. Express the vector $y = \begin{bmatrix} 6 \\ 1 \\ -8 \end{bmatrix}$ as a linear combination of vector.

Sol^n: Here,

$y \cdot u_1 = (6, 1, -8) \cdot (3, 1, 1)$
$= 18 + 1 - 8 = 11$

$u_1 \cdot u_1 = (3, 1, 1) \cdot (3, 1, 1) = 9 + 1 + 1 = 11$

$c_1 = \frac{y \cdot u_1}{u_1 \cdot u_1} = \frac{11}{11} = 1.$

$y \cdot u_2 = (6, 1, -8) \cdot (-1, 2, 1)$
$= -6 + 2 - 8$
$= -12$

$u_2 \cdot u_2 = (-1, 2, 1) \cdot (-1, 2, 1)$
$= 1 + 4 + 1 = 6$

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$c_2 = \frac{y \cdot u_2}{u_2 \cdot u_2} = \frac{-12}{6} = -2$

For,
c_3

$y \cdot u_3 = (6, 1, -8) \cdot (-1/2, -2, 7/2)$
$= -3 - 2 - 28$
$= -33$

$u_3 \cdot u_3 = (-1/2, -2, 7/2) \cdot (-1/2, -2, 7/2)$

$= 1/4 + 4 + 49/4$

$= \frac{1 + 16 + 49}{4} = \frac{66}{4} = \frac{33}{2}$

$c_3 = \frac{y \cdot u_3}{u_3 \cdot u_3} = \frac{-33}{33} \times 2 = -2$

Now,

$c_1u_1 + c_2u_2 + c_3u_3 = 1(3, 1, 1) + -2(-1, 2, 1) + -2(-1/2, -2, 7/2)$

$= (3, 1, 1) + (2, -4, -2) + (1, 4, -7)$
$= (3+2+1, 1-4+4, 1-2-7)$
$= (6, 1, -8)$

$y = (6, 1, -8)$

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1. Determine which sets of vectors are orthogo...

(i) $\begin{bmatrix} -1 \\ 4 \\ -3 \end{bmatrix}, \begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -4 \\ -7 \end{bmatrix}$

Sol^n: Here,

$u_1 = \begin{bmatrix} -1 \\ 4 \\ -3 \end{bmatrix}, u_2 = \begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix}, u_3 = \begin{bmatrix} 3 \\ -4 \\ -7 \end{bmatrix}$

$u_1 \cdot u_2 = \begin{bmatrix} -1 \\ 4 \\ -3 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix} = -5+8-3 = 0$

$u_2 \cdot u_3 = (5,2,1)(3,-4,-7) = 15-8-7 = 0$

$u_3 \cdot u_1 = (3,-4,-7)(-1,4,-3)$

$= -3-16+21$

$= -19+21 = 2 \neq 0$

Hence, the given sets of vectors are not orthogonal.

$\therefore$ The set $\{u_1, u_2\}$, and $\{u_2, u_3\}$ are orthogonal set.

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(ii) $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -5 \\ -2 \\ 1 \end{bmatrix}$

Sol^n: Here,

$u_1 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, u_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, u_3 = \begin{bmatrix} -5 \\ -2 \\ 1 \end{bmatrix}$

$u_1 \cdot u_2 = (1,-2,1) \cdot (0,1,2)$

$= (0-2+2) = 0$

$u_2 \cdot u_3 = (0,1,2) \cdot (-5,-2,1)$

$= (0-2+2) = 0$

$u_3 \cdot u_1 = (-5,-2,1) \cdot (1,-2,1)$

$= -5+4+1$

$= -5+5 = 0$

$\therefore$ Hence, the given set of vectors are orthogonal.

(iii) $\begin{bmatrix} 2 \\ -7 \\ -1 \end{bmatrix}, \begin{bmatrix} -6 \\ -3 \\ 9 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}$

Sol^n: Here,

$u_1 = \begin{bmatrix} 2 \\ -7 \\ -1 \end{bmatrix}, u_2 = \begin{bmatrix} -6 \\ -3 \\ 9 \end{bmatrix}, u_3 = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}$

$u_1 \cdot u_2 = (2,-7,-1) \cdot (-6,-3,9)$

$= -12+21-9$

$= -21+21 = 0$

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$u_2 \cdot u_3 = (-6,-3,9) \cdot (3,1,-1)$

$= -18-3-9$

$= -18-12$

$= -30 \neq 0$

$u_3 \cdot u_1 = (3,1,-1) \cdot (2,-7,-1)$

$= 6-7+1$

$= 7-7 = 0$

$\therefore$ The given set of vectors $\{u_1, u_2, u_3\}$ ~~are~~ not orthogonal.

(iv) $\begin{bmatrix} 2 \\ -5 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ -2 \\ 6 \end{bmatrix}$

Soln: Here,

$u_1 = \begin{bmatrix} 2 \\ -5 \\ -3 \end{bmatrix}, u_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, u_3 = \begin{bmatrix} 4 \\ -2 \\ 6 \end{bmatrix}$

$u_1 \cdot u_2 = (2,-5,-3) \cdot (0,0,0)$

$= (0+0+0) = 0$

$u_2 \cdot u_3 = (0,0,0)\,(4,-2,6) = 0+0+0 = 0$

$u_3 \cdot u_1 = (4,-2,6) \cdot (2,-5,-3)$

$= 8+10-18$

$= 18-18$

0

$\therefore$ Hence the given set of vectors are orthogonal.

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(v) $\begin{bmatrix} 3 \\ -2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ -3 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 8 \\ 7 \\ 0 \end{bmatrix}$

Soln: Here,

$u_1 = \begin{bmatrix} 3 \\ -2 \\ 1 \\ 3 \end{bmatrix}, u_2 = \begin{bmatrix} -1 \\ 3 \\ -3 \\ 4 \end{bmatrix}, u_3 = \begin{bmatrix} 3 \\ 8 \\ 7 \\ 0 \end{bmatrix}$

$u_1 \cdot u_2 = (3,-2,1,3) \cdot (-1,3,-3,4)$

$= -3-6-3+12$

$= -12+12 = 0$

$u_2 \cdot u_3 = (3,8,7,0) \cdot (-1,3,-3,4)$

$= -3+24-21+0$

$= -24+24 = 0$

$u_3 \cdot u_1 = (3,8,7,0) \cdot (3,-2,1,3)$

$= 9-16+7$

$= 16-16 = 0$

$\therefore$ Hence, the given set of vectors $\{u_1, u_2,$ ~~are~~ orthogonal.

(vi) $\begin{bmatrix}5\\-4\\0\\3\end{bmatrix}, \begin{bmatrix}-4\\1\\-3\\8\end{bmatrix}, \begin{bmatrix}3\\3\\5\\-1\end{bmatrix}$

Sol^n Here,

$u_1 = \begin{bmatrix}5\\-4\\0\\3\end{bmatrix}, \quad u_2 = \begin{bmatrix}-4\\1\\-3\\8\end{bmatrix}, \quad u_3 = \begin{bmatrix}3\\3\\5\\-1\end{bmatrix}$

$u_1 . u_2 = (5,-4,0,3).(-4,1,-3,8)$
$= -20-4+0+24$
$= -24+24 = 0$

$u_2 . u_3 = (-4,1,-3,8).(3,3,5,-1)$
$= -12+3-15-8$
$= 3-35$
$= -32 \neq 0$

$u_3 . u_1 = (3,3,5,-1).(5,-4,0,3)$
$= 15-12+0-3$
$= 15-15$
$= 0$

$\therefore$ Hence the given set of vectors $\{u_1, u_2, u_3\}$ ~~are~~ not orthogonal.

2. Show that $\{u_1, u_2\}$ or $\{u_1, u_2, u_3\}$ is an orthogonal basis for R^2 or R^3, respectively. Then express x as a linear combination of the u's.

(i) $u_1 = \begin{bmatrix}2\\-3\end{bmatrix}, \quad u_2 = \begin{bmatrix}6\\4\end{bmatrix}$, and $x = \begin{bmatrix}9\\-7\end{bmatrix}$

Sol^n Here,

$c_1 = \dfrac{y\,u_1}{u_1.u_1} = \dfrac{x.u_1}{u_1.u_1}$

$y.u_1 = x.u_1 = (9,-7).(2,-3) = 18+21 = 39$

$u_1.u_1 = (2,-3).(2,-3) = 4+9 = 13$

$c_1 = \dfrac{x.u_1}{u_1.u_1} = \dfrac{39}{13} = 3$

$c_2 = \dfrac{x\,u_2}{u_2.u_2}$

$x.u_2 = (9,-7).(6,4) = 54-28 = 26$

$u_2.u_2 = (6,4).(6,4) = 36+16 = 52 \quad \therefore c_2 = \dfrac{26}{52} = \dfrac{1}{2}$

$\therefore x = c_1u_1 + c_2u_2$
$= 3(2,-3) + 26(6,4)$
$= (6,-9) + (156,104) \quad \therefore x = 3u_1 + \frac{1}{2}u_2$
$= (162,95)$

$\therefore x = \begin{bmatrix}162\\95\end{bmatrix}$ Ans

Ans

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(ii) $u_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, u_2 = \begin{bmatrix} -2 \\ 6 \end{bmatrix}$, and $x = \begin{bmatrix} -6 \\ 3 \end{bmatrix}$

Soln: Here,

$u_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, u_2 = \begin{bmatrix} -2 \\ 6 \end{bmatrix}$ and $x = \begin{bmatrix} -6 \\ 3 \end{bmatrix}$

$c_1 = \dfrac{u_1 \cdot x}{u_1 \cdot u_1}$

$u_1 \cdot x = (3,1) \cdot (-6,3)$

$= -18 + 3 = -15$

$u_1 \cdot u_1 = (3,1) \cdot (3,1) = 9 + 1 = 10$

$c_1 = \dfrac{-15}{10} = \dfrac{-3}{2}$

$c_2 = \dfrac{u_2 \cdot x}{u_2 \cdot u_2}$

$u_2 \cdot x = (-2,6) \cdot (-6,3)$

$= 12 + 18 = 30$

$u_2 \cdot u_2 = (-2,6) \cdot (-2,6)$

$= 4 + 36 = 40$

$c_2 = \dfrac{u_2 \cdot x}{u_2 \cdot u_2} = \dfrac{30}{40} = 3/4$

$\therefore x = c_1 u_1 + c_2 u_2$

$= -3/2\, u_1 + 3/4\, u_2$ Ans

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(iii) $u_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, u_2 = \begin{bmatrix} \cancel{2}-1 \\ 4 \\ \cancel{-2}1 \end{bmatrix}, u_3 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$, and $x = \begin{bmatrix} 8 \\ -4 \\ -3 \end{bmatrix}$

Soln: Here,

$c_1 = \dfrac{u_1 \cdot x}{u_1 \cdot u_1}$

$u_1 \cdot x = (1,0,1) \cdot (8,-4,-3) = 8 + 0 - 3 = 5$

$u_1 \cdot u_1 = (1,0,1) \cdot (1,0,1) = 1 + 0 + 1 = 2$

$c_1 = 5/2$

$c_2 = \dfrac{u_2 \cdot x}{u_2 \cdot u_2}$

$u_2 \cdot x = (2,1,-2) \cdot (8,-4,-3) = 16 - 4 + 6 = 18$

$u_2 \cdot u_2 = (2,1,-2) \cdot (2,1,-2) = 4 + 1 + 4 = 9$

$c_2 = 18/9 = 2$

$c_3 = \dfrac{u_3 \cdot x}{u_3 \cdot u_3}$

$u_3 \cdot x = (2,1,-2) \cdot (8,-4,-3) = 16 - 4 + 6 = 18$

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(iii) $u_1 = \begin{bmatrix}1\\0\\1\end{bmatrix}$, $u_2 = \begin{bmatrix}-1\\4\\1\end{bmatrix}$, $u_3 = \begin{bmatrix}2\\1\\-2\end{bmatrix}$, $x = \begin{bmatrix}8\\-4\\-3\end{bmatrix}$

Sol^n: Here,

$u_1 = \begin{bmatrix}1\\0\\1\end{bmatrix}$, $u_2 = \begin{bmatrix}-1\\4\\1\end{bmatrix}$, $u_3 = \begin{bmatrix}2\\1\\-2\end{bmatrix}$, $x = \begin{bmatrix}8\\-4\\-3\end{bmatrix}$

$c_1 = \dfrac{u_1 \cdot x}{u_1 \cdot u_1}$

$u_1 \cdot x = (1,0,1)\cdot(8,-4,-3) = 8+0-3 = 5$

$u_1 \cdot u_1 = (1,0,1)\cdot(1,0,1) = 1+0+1 = 2$

$\therefore c_1 = 5/2$

$c_2 = \dfrac{u_2 \cdot x}{u_2 \cdot u_2}$

$u_2 \cdot x = (-1,4,1)\cdot(8,-4,-3)$

$= -8-16-3$

$= -27$

$u_2 \cdot u_2 = (-1,4,1)\ (-1,4,1)$

$= (1+16+1) = 18$

$\therefore c_2 = \dfrac{-27}{18} = -3/2$

$c_3 = \dfrac{u_3 \cdot x}{u_3 \cdot u_3}$

$u_3 \cdot x = (2,1,-2)\cdot(8,-4,-3) = 16-4+6 = 18$

$u_3 \cdot u_3 = (2,1,-2)\ (2,1,-2) = 4+1+4 = 9$

$\therefore c_3 = 18/9 = 2$

$x = u_1c_1 + u_2c_2 + u_3 \cdot c_3 = 5/2\, u_1 - 3/2\, u_2 + 2u_3$

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(iv) $u_1 = \begin{bmatrix}3\\-3\\0\end{bmatrix}$, $u_2 = \begin{bmatrix}2\\2\\-1\end{bmatrix}$, $u_3 = \begin{bmatrix}1\\1\\4\end{bmatrix}$ and $x = \begin{bmatrix}5\\-3\\1\end{bmatrix}$

Sol^n: Here,

$u_1 = \begin{bmatrix}3\\-3\\0\end{bmatrix}$, $u_2 = \begin{bmatrix}2\\2\\-1\end{bmatrix}$, $u_3 = \begin{bmatrix}1\\1\\4\end{bmatrix}$ and $x = \begin{bmatrix}5\\-3\\1\end{bmatrix}$

$c_1 = \dfrac{u_1 \cdot x}{u_1 \cdot u_1}$

$u_1 \cdot x = (3,-3,0)\cdot(5,-3,1) = 15+9 = 24$

$u_1 \cdot u_1 = (3,-3,0)\ (3,-3,0) = 9+9 = 18$

$\therefore c_1 = \dfrac{24}{18} = \dfrac{4}{3}$

$c_2 = \dfrac{u_2 \cdot x}{u_2 \cdot u_2}$

$u_2 \cdot x = (2,2,-1)\cdot(5,-3,1) = 10-6-1 = 3$

$u_2 \cdot u_2 = (2,2,-1)\cdot(2,2,-1) = 4+4+1 = 9$

$\therefore c_2 = 3/9 = 1/3$

$c_3 = \dfrac{u_3 \cdot x}{u_3 \cdot u_3}$

$u_3 \cdot x = (1,1,4)\cdot(5,-3,1) = 5-3+4 = 6$

$u_3 \cdot u_3 = (1,1,4)\cdot(1,1,4) = 1+1+16 = 18$

$c_3 = 6/18 = 1/3$

$\therefore x = c_1u_1 + c_2u_2 + c_3u_3$

$= 4/3\, u_1 + 1/3\, u_2 + 1/3\, u_3$

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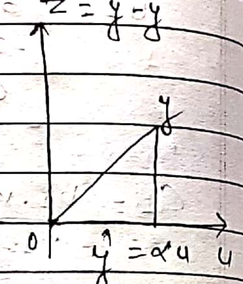
Orthogonal projection.

let u be a non-zero vector in R^n and y be the decomposition vector in R^n such that

$y = \hat{y} + z$



where $\hat{y}$ is the multiple of u and $z = y - \hat{y}$ is the Orthogonal vector of u then

$y - \hat{y}$ is the orthogonal to u if and only if

$(y - \hat{y}) \cdot u = 0$

$(y - \alpha u) \cdot u = 0$

$y \cdot u - \alpha \cdot u \cdot u = 0$

$\alpha u \cdot u = y \cdot u$

$\Rightarrow \alpha = \dfrac{y \cdot u}{u \cdot u}$

Hence, the vector $\hat{y}$ is the Orthogonal projection of y on to u.

$$\hat{y} = \left(\frac{y \cdot u}{u \cdot u}\right) \cdot u$$

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3. compute the Orthogonal projection of $\begin{bmatrix} 1 \\ 7 \end{bmatrix}$ on to the line through $\begin{bmatrix} -4 \\ 2 \end{bmatrix}$ and the origin

Sol: Here,

let $y = \begin{bmatrix} 1 \\ 7 \end{bmatrix}$ and $u = \begin{bmatrix} -4 \\ 2 \end{bmatrix}$

we Know that,

The orthogonal projection of y on to u is,

$$\hat{y} = \left(\frac{y \cdot u}{u \cdot u}\right) \cdot u$$

then,

$\therefore y \cdot u = (1, 7) \cdot (-4, 2)$

$= -4 + 14 = 10$

$u \cdot u = (-4, 2) \cdot (-4, 2) = 16 + 4 = 20$

$$\hat{y} = \left(\frac{y \cdot u}{u \cdot u}\right) \cdot u = \left(\frac{10}{20}\right) \cdot u$$

$$= \frac{1}{2} \cdot \begin{bmatrix} -4 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$ Ans.

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4(i) Let $y = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $u = \begin{bmatrix} 4 \\ -7 \end{bmatrix}$. Write y as the sum of two orthogonal vectors, one is span{u} and one orthogonal to u.

Sol^n: Here,

$y = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $u = \begin{bmatrix} 4 \\ -7 \end{bmatrix}$

$y \cdot u = (2,3)\cdot(4,-7) = 8-21 = -13$

$u \cdot u = (4,-7)(4,-7) = 16+49 = 65$

Now,

the orthogonal projection $\hat{y}$ of y on to u is

$\hat{y} = \left(\frac{u \cdot y}{u \cdot u}\right) \cdot u = \left(\frac{-13}{65}\right) u = -\frac{1}{5}\begin{pmatrix} 4 \\ -7 \end{pmatrix}$

Here

$y = \hat{y} + (y-\hat{y}) \Rightarrow y = \begin{bmatrix} -4/5 \\ 7/5 \end{bmatrix} + \left[(2,3) - (-4/5, 7/5)\right]$

$y = (-4/5, 7/5) + (2+4/5, 3-7/5)$

$= (-4/5, 7/5) + (14/5, 8/5)$

$y = \begin{bmatrix} -4/5 \\ 7/5 \end{bmatrix} + \begin{bmatrix} 14/5 \\ 8/5 \end{bmatrix}$

This shows that y can be written as sum $\hat{y}$ and $(y-\hat{y})$, since $\alpha \cdot u = \hat{y}$ for $\alpha = \left(\frac{y \cdot u}{u \cdot u}\right)$

So span $\{u\} = \hat{y}$ Here,

$\hat{y} \cdot (y-\hat{y}) = \begin{pmatrix} -4/5 \\ 7/5 \end{pmatrix} \cdot \begin{pmatrix} 14/5 \\ 8/5 \end{pmatrix} = -56/5 + 56/5 = 0$

This implies $\hat{y}$ is orthogonal to $(y-\hat{y})$, therefore $(y-\hat{y})$ is orthogonal to u,

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4(ii) let $y = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$ and $u = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$. write y as the sum of a vector in span{u} and a vector orthogonal to u

Sol^n: Here,

$y = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$ and $u = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$

$y \cdot u = (2,6)\cdot(7,1) = 14+6 = 20$

$u \cdot u = (7,1)(7,1) = 49+1 = 50$

The orthogonal projection ~~of~~ $\hat{y}$ of y on to u is

$\hat{y} = \left(\frac{y \cdot u}{u \cdot u}\right) \cdot u = \left(\frac{20}{50}\right) \cdot (7/5) = \begin{bmatrix} 14/5 \\ 2/5 \end{bmatrix}$

Here $y = \hat{y} + (y-\hat{y}) \Rightarrow y = \begin{bmatrix} 14/5 \\ 2/5 \end{bmatrix} + \left\{\begin{bmatrix} 2 \\ 6 \end{bmatrix} - \begin{bmatrix} 14/5 \\ 2/5 \end{bmatrix}\right\}$

$y = \begin{bmatrix} 14/5 \\ 2/5 \end{bmatrix} + \begin{bmatrix} -4/5 \\ 28/5 \end{bmatrix}$

This show that y can be written as sum of $\hat{y}$ and $(y-\hat{y})$. Since $\alpha u = \hat{y} \Rightarrow \alpha = \left(\frac{y \cdot u}{u \cdot u}\right)$ so

Span $\{u\} = \hat{y}$ Here,

$\hat{y} \cdot (y-\hat{y}) = \begin{pmatrix} 14/5 \\ 2/5 \end{pmatrix} \cdot \begin{pmatrix} -4/5 \\ 28/5 \end{pmatrix} = -56/5 + 56/5 = 0$

This implies $\hat{y}$ is orthogonal to $(y-\hat{y})$ therefore $(y-\hat{y})$ is orthogonal to u. and y as the sum of a vector in span{u} and a vector orthogonal to u.

Ans

5 (i) Let $y = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $u = \begin{bmatrix} 8 \\ 6 \end{bmatrix}$. compute the distance from y to the line through u and the Origin.

Sol^n: Here

$y = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $u = \begin{bmatrix} 8 \\ 6 \end{bmatrix}$

$y \cdot u = (3,1)\cdot(8,6) = 24+6 = 30$

$u \cdot u = (8,6)\cdot(8,6) = 64+36 = 100$

$\hat{y} = \left(\frac{y \cdot u}{u \cdot u}\right) u = \frac{30}{100}\cdot(8,6) = \frac{3}{10}(8,6)$

then

let $z = y - \hat{y} = (3,1) - (24/10, 18/10)$

$= (3 - 24/10, 1 - 18/10)$

$= (6/10, -8/10)$

$\|z\| = \sqrt{\frac{36}{100} + \frac{64}{100}} = \sqrt{\frac{100}{100}} = \sqrt{1} = 1.$

$\therefore$ the distance from y to the line through the u and the origin is 1.

(ii) let $y = \begin{bmatrix} -3 \\ 9 \end{bmatrix}$ and $u = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ compute the distance from y to the line through u and the Origin.

Sol^n: Here,

$y = \begin{bmatrix} -3 \\ 9 \end{bmatrix}$ and $u = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$y \cdot u = (-3,9)\cdot(1,2) = -3+18 = 15$

$u \cdot u = (1,2)\cdot(1,2) = 1+4 = 5$

$\hat{y} = \left(\frac{y \cdot u}{u \cdot u}\right) \cdot u = \frac{15}{5}(1,2)$

$= (3,6)$

Here

$z = y - \hat{y} = (-3,9) - (3,6)$

$z = (-6, 3)$

$\|z\| = \sqrt{36+9} = \sqrt{45} = 3\sqrt{5}$

$\therefore$ The distance from y to the line through u and the origin is $3\sqrt{5}$ - Ans

~~Orthogonal~~ normal set :- An orthogonal set of unit vector is called orthonormal set.
Orthonormal basis :- If every vector of an orthogonal basis of unit vector then the basis orthonormal basis

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(6) Determine which sets of vectors are orthogonal. If a set is only orthogonal, normalize the vector to produce an orthonormal set.

(i) $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$

Sol^n: Here,
let $V_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, V_2 = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$

$V_1 \cdot V_2 = (0,1,0) \cdot (0,-1,0) = 0-1+0 = -1 \neq 0$

since the given set of vector $\{V_1, V_2\}$ is not an orthogonal vector and hence it doesnot hold the orthornormal set

(ii) $\begin{bmatrix} -0.6 \\ 0.8 \end{bmatrix}, \begin{bmatrix} 0.8 \\ 0.6 \end{bmatrix}$

Sol^n: Here,
$V_1 = \begin{bmatrix} -0.6 \\ 0.8 \end{bmatrix}$ and $V_2 = \begin{bmatrix} 0.8 \\ 0.6 \end{bmatrix}$

$V_1 \cdot V_2 = \begin{bmatrix} -0.6 \\ 0.8 \end{bmatrix} \cdot \begin{bmatrix} 0.8 \\ 0.6 \end{bmatrix} = \overset{-0.48+0.48}{\cancel{-4.8+4.8}} = 0$

$||V_1|| = \sqrt{(-0.6)^2+(0.8)^2} = \sqrt{\cancel{\cdot 36+64}}\ \sqrt{0.36+0.64} = \sqrt{1} = 1$

$||V_2|| = \sqrt{(0.8)^2+(0.6)^2} = \sqrt{0.36+0.64} = \sqrt{1} = 1$

which show that $\{V_1, V_2\}$ are unit vector
Thus the set $\{V_1, V_2\}$ is an orthonormal set. since every orthogonal of non-zero vector form a basis so $\{V_1, V_2\}$ are an orthonormal basis.

(iii) $\begin{bmatrix} -2/3 \\ 1/3 \\ 2/3 \end{bmatrix}, \begin{bmatrix} 1/3 \\ 2/3 \\ 0 \end{bmatrix}$

Sol^n: Here,
let $V_1 = \begin{bmatrix} -2/3 \\ 1/3 \\ 2/3 \end{bmatrix}, V_2 = \begin{bmatrix} 1/3 \\ 2/3 \\ 0 \end{bmatrix}$

$V_1 \cdot V_2 = (-2/3, 1/3, 2/3) \cdot (1/3, 2/3, 0)$
$= (-2/9 + 2/9 + 0)$
$= 0$

$\therefore$ the set $\{V_1, V_2\}$ are orthogonal set,

$||V_1|| = \sqrt{4/9 + 1/9 + 4/9} = \sqrt{9/9} = 1$

$||V_2|| = \sqrt{1/9 + 4/9} = \sqrt{5/9} = \sqrt{5}/3 \neq 1$

Since, Here V_2 is not a unit vector ~~it shows to~~ that means, the set $\{V_1, V_2\}$ is not an orthonormal set.

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(iv) $\begin{bmatrix} 1/\sqrt{10} \\ 3/\sqrt{20} \\ 3/\sqrt{20} \end{bmatrix}, \begin{bmatrix} 3/\sqrt{10} \\ -1/\sqrt{20} \\ -1/\sqrt{20} \end{bmatrix}, \begin{bmatrix} 0 \\ -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

Sol^n: Here,

let $v_1 = \begin{bmatrix} 1/\sqrt{10} \\ 3/\sqrt{20} \\ 3/\sqrt{20} \end{bmatrix}, v_2 = \begin{bmatrix} 3/\sqrt{10} \\ -1/\sqrt{20} \\ -1/\sqrt{20} \end{bmatrix}, v_3 = \begin{bmatrix} 0 \\ -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

$v_1 \cdot v_2 = (1/\sqrt{10}, 3/\sqrt{20}, 3/\sqrt{20}) \; (3/\sqrt{10}, -1/\sqrt{20}, -1/\sqrt{20})$

$= \frac{3}{10} - \frac{3}{20} - \frac{3}{20}$

$= \frac{3}{10} - \frac{6}{20}$

$= \frac{6-6}{20} = \frac{0}{20} = 0$

$v_2 \cdot v_3 = (3/\sqrt{10}, -1/\sqrt{20}, -1/\sqrt{20}) \cdot (0, -1/\sqrt{2}, 1/\sqrt{2})$

$= 0 + \frac{1}{\sqrt{40}} - \frac{1}{\sqrt{40}} = \frac{1}{\sqrt{40}} - \frac{1}{\sqrt{40}} = 0$

$v_3 \cdot v_1 = (0, -1/\sqrt{2}, 1/\sqrt{2}) \cdot (1/\sqrt{10}, 3/\sqrt{20}, 3/\sqrt{20})$

$= (0 - 3/\sqrt{40} + 3/\sqrt{40})$

$= \frac{-3}{\sqrt{40}} + \frac{3}{\sqrt{40}} = 0$

The set $\{v_1, v_2, v_3\}$ are orthogonal set.

$\|v_1\| = \sqrt{\frac{1}{10} + \frac{9}{20} + \frac{9}{20}} = \sqrt{\frac{2+9+9}{20}} = \sqrt{20/20} = 1$

$\|v_2\| = \sqrt{\frac{9}{10} + \frac{1}{20} + \frac{1}{20}} = \sqrt{\frac{18+1+1}{20}} = \sqrt{20/20} = 1$

$\|v_3\| = \sqrt{0 + \frac{1}{2} + \frac{1}{2}} = \sqrt{2/2} = 1$

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which shows that $\{v_1, v_2, v_3\}$ are unit vector. Thus, the set $\{v_1, v_2, v_3\}$ is an orthonormal set since, every orthogonal of non-zero vector form a basis. So $\{v_1, v_2, v_3\}$ are an orthonormal basis,

(v) $\begin{bmatrix} 1/\sqrt{18} \\ 4/\sqrt{18} \\ 1/\sqrt{18} \end{bmatrix}, \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}, \begin{bmatrix} -2/3 \\ 1/3 \\ -2/3 \end{bmatrix}$

Sol^n: Here,

let $v_1 = \begin{bmatrix} 1/\sqrt{18} \\ 4/\sqrt{18} \\ 1/\sqrt{18} \end{bmatrix}, v_2 = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}, v_3 = \begin{bmatrix} -2/3 \\ 1/3 \\ -2/3 \end{bmatrix}$

$v_1 \cdot v_2 = \begin{bmatrix} 1/\sqrt{18} \\ 4/\sqrt{18} \\ 1/\sqrt{18} \end{bmatrix} \cdot \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix} = 1/\sqrt{36} + 0 - 1/\sqrt{36}$

$= 1/\sqrt{36} - 1/\sqrt{36} = 0$

$v_2 \cdot v_3 = (1/\sqrt{2}, 0, -1/\sqrt{2}) \cdot (-2/3, 1/3, -2/3)$

$= \frac{-2}{3\sqrt{2}} + 0 + \frac{2}{3\sqrt{2}} = \frac{-2}{3\sqrt{2}} + \frac{2}{3\sqrt{2}} = 0$

$v_3 \cdot v_1 = (-2/3, 1/3, -2/3) \cdot (1/\sqrt{18}, 4/\sqrt{18}, 1/\sqrt{18})$

$= \frac{-2}{3\sqrt{18}} + \frac{4}{3\sqrt{18}} - \frac{2}{3\sqrt{18}}$

$= \frac{-4}{3\sqrt{18}} + \frac{4}{3\sqrt{18}} = 0$

$\|v_1\| = \sqrt{\frac{1}{18} + \frac{16}{18} + \frac{1}{18}} = \sqrt{18/18} = 1$

$\|v_2\| = \sqrt{\frac{1}{2} + 0 + \frac{1}{2}} = \sqrt{2/2} = 1$

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$$\|V_3\| = \sqrt{\tfrac{4}{9}+\tfrac{1}{9}+\tfrac{4}{9}} = \sqrt{\tfrac{9}{9}} = 1$$

which shows that $\{v_1, v_2, v_3\}$ are unit vector thus, the set $\{v_1, v_2, v_3\}$ is an ~~orthogonal~~ orthonormal set. Since, every orthogonal of non-zero vector form a basis so $\{v_1, v_2, v_3\}$ are an orthonormal basis.

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Exercise:- 8.3

1. Assume that $\{u_1, \dots, u_4\}$ is an orthogonal basis for R^4. Let $u_1 = \begin{bmatrix}1\\2\\1\\1\end{bmatrix}$, $u_2 = \begin{bmatrix}-2\\1\\-1\\1\end{bmatrix}$, $u_3 = \begin{bmatrix}1\\1\\-2\\-1\end{bmatrix}$ and $u_4 = \begin{bmatrix}-1\\1\\1\\-2\end{bmatrix}$, $V = \begin{bmatrix}4\\5\\-3\\3\end{bmatrix}$. Write V as the sum of two vectors, one is span$\{u_1\}$ and the other in span $\{u_2, u_3, u_4\}$.

Soln: Here,

$V \cdot u_1 = (4,5,-3,3) \cdot (1,2,1,1)$
$= 4+10-3+3$
$= 14$

$u_1 \cdot u_1 = (1,2,1,1)(1,2,1,1)$
$= (1+4+1+1) = 7$

$V \cdot u_2 = (4,5,-3,3) \cdot (-2,1,-1,1)$
$= -8+5+3+3$
$= 3$

$u_2 \cdot u_2 = (-2,1,-1,1) \cdot (-2,1,-1,1)$
$= 4+1+1+1 = 7$

$V \cdot u_3 = (4,5,-3,3) \cdot (1,1,-2,-1)$
$= 4+5+6-3$
$= 12$

$u_3 \cdot u_3 = (1,1,-2,-1) \cdot (1,1,-2,-1)$
$= 1+1+4+1 = 7$

$V \cdot u_4 = (4,5,-3,3) \cdot (-1,1,1,-2)$
$= -4+5-3-6$
$= -8$

$u_4 \cdot u_4 = (-1,1,1,-2) \cdot (-1,1,1,-2) = 1+1+1+4 = 7$

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let x_1 be is in span $\{u_1\}$ and x_2 is in span $\{u_2, u_3, u_4\}$

$$x_1 + x_2 = \left\{\left(\frac{v.u_1}{u_1.u_1}\right).u_1\right\} + \left\{\left(\frac{v.u_2}{u_2.u_2}\right).u_2 + \left(\frac{v.u_3}{u_3.u_3}\right).u_3 + \left(\frac{v.u_4}{u_4.u_4}\right).u_4\right\}$$

$$= \left\{\frac{14}{7}.(1,2,1,1)\right\} + \left\{\frac{3}{7}(-2,1,-1,1) + \frac{12}{7}(1,1,-2,-1) + \frac{-8}{7}(-1,1,1,-2)\right\}$$

$$= \{(2,4,2,2)\} + \left\{\left(\frac{-6}{7}, \frac{3}{7}, \frac{-3}{7}, \frac{3}{7}\right) + \left(\frac{12}{7}, \frac{12}{7}, \frac{-24}{7}, \frac{-12}{7}\right) + \left(\frac{8}{7}, \frac{-8}{7}, \frac{-8}{7}, \frac{16}{7}\right)\right\}$$

$$= \{(2,4,2,2)\} + \left(\frac{-6}{7} + \frac{12}{7} + \frac{8}{7}, \frac{3}{7} + \frac{12}{7} - \frac{8}{7}, \frac{-3}{7} - \frac{24}{7} - \frac{8}{7}, \frac{3}{7} - \frac{12}{7} + \frac{16}{7}\right)\}$$

$$= \{(2,4,2,2)\} + \left\{\frac{14}{7}, \frac{7}{7}, \frac{-35}{7}, \frac{7}{7}\right\}$$

$$= \{(2,4,2,2)\} + \{2,1,-5,1\}$$

$$= \{(4, 5, -3, 3)\}$$

$$x_1 + x_2 = \begin{bmatrix} 4 \\ 5 \\ -3 \\ 3 \end{bmatrix}$$

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2. Verify that $\{u_1, u_2\}$ is an orthogonal set, and then find the orthogonal projection of y on to span $\{u_1, u_2\}$.

(1) $y = \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix}$, $u_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $u_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

Sol: Here,

$y = \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix}$, $u_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $u_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

$y.u_1 = (-1,4,3).(1,1,0)$
$= -1+4+0 = 3$

$u_1.u_1 = (1,1,0).(1,1,0) = 1+1+0 = 2$

$y.u_2 = (-1,4,3).(-1,1,0)$
$= 1+4+0$
$= 5$

$u_2.u_2 = (-1,1,0).(-1,1,0)$
$= 1+1+0$
$= 2$

$\therefore$ let the orthogonal projection of y onto span $\{u_1, u_2\}$ is,

$$\hat{y} = \left\{\left(\frac{y.u_1}{u_1.u_1}\right).u_1 + \left(\frac{y.u_2}{u_2.u_2}\right).u_2\right\}$$

$$= \left\{\frac{3}{2}.(1,1,0)\right\} + \left\{\frac{5}{2}(-1,1,0)\right\}$$

$$= \left(\frac{3}{2}, \frac{3}{2}, 0\right) + \left(\frac{-5}{2}, \frac{5}{2}, 0\right)$$

$$= \left(\frac{3}{2} - \frac{5}{2}, \frac{3}{2} + \frac{5}{2}, 0\right)$$

$$= (-1, 4, 0)$$ Ans

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$y = \begin{bmatrix} -1 \\ 2 \\ 6 \end{bmatrix}, u_1 = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}, u_2 = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$

Sol^n: Here,

$y = \begin{bmatrix} -1 \\ 2 \\ 6 \end{bmatrix}, u_1 = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}, u_2 = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$

$y.u_1 = (-1,2,6).(3,-1,2)$

$= -3-2+12$

$= 7$

$u_1.u_1 = (3,-1,2)(3,-1,2)$

$= 9+1+4$

$= 14$

$y.u_2 = (-1,2,6)(1,-1,-2)$

$= -1-2-12$

$= -15$

$u_2.u_2 = (1,-1,-2).(1,-1,-2)$

$= 1+1+4$

$= 6$

$\therefore$ the ~~proj~~ orthogonal projection of y on ~~to~~ span$\{u_1,u_2\}$ is

$$\hat{y} = \left\{\left(\frac{y.u_1}{u_1.u_1}\right).u_1\right\} + \left\{\left(\frac{y.u_2}{u_2.u_2}\right).u_2\right\}$$

$$= \left\{\frac{7}{14}(3,-1,2)\right\} + \left\{\left(\frac{-15}{6}(1,-1,-2)\right)\right\}$$

$$= \left(3/2, \frac{-1}{2}, 1\right) + \left(-5/2, 5/2, 5\right)$$

$$= (\cancel{-1,2,6})\ (-1,2,6)\ \text{Ans}$$

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(iii) $y = \begin{bmatrix} 6 \\ 4 \\ 1 \end{bmatrix}, u_1 = \begin{bmatrix} -4 \\ -1 \\ 1 \end{bmatrix}, u_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

Sol^n: Here,

$y = \begin{bmatrix} 6 \\ 4 \\ 1 \end{bmatrix}, u_1 = \begin{bmatrix} -4 \\ -1 \\ 1 \end{bmatrix}, u_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

$y.u_1 = (6,4,1).(-4,-1,1)$

$= -24-4+1$

$= -27$

$u_1.u_1 = (-4,-1,1).(-4,-1,1)$

$= 16+1+1 = 18$

$y.u_2 = (6,4,1).(0,1,1)$

$= 0+4+1 = 5$

$u_2.u_2 = (0,1,1).(0,1,1)$

$= 0+1+1 = 2$

$\therefore$ The orthogonal projection of y on span$\{u_1,u_2\}$ is

$$\hat{y} = \left\{\left(\frac{y.u_1}{u_1.u_1}\right).u_1\right\} + \left\{\left(\frac{y.u_2}{u_2.u_2}\right).u_2\right\}$$

$$= \left(\frac{-27}{18}(-4,-1,1)\right) + \left(5/2\,(0,1,1)\right)$$

$$= \frac{-3}{2}(-4,-1,1) + 5/2\,(0,1,1)$$

$$= \left(\frac{12}{2}, 3/2, \frac{-3}{2}\right) + \left(0, 5/2, 5/2\right)$$

$$= \left(6+0,\ 3/2+5/2,\ -3/2+5/2\right)$$

$$= (6,4,1)\ \text{Ans}$$

3. Let W be the Subspace spanned by the u's and write y as the sum of a vector in W and a Vector Orthogonal to W.

① $y = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}, u_1 = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}, u_2 = \begin{bmatrix} 5 \\ 1 \\ 4 \end{bmatrix}$

Sol^n: Here,

$y \cdot u_1 = (1,3,5)(1,3,-2)$ and $y \cdot u_2 = (1,3,5)\cdot(5,1,4)$

$= 1+9-10$ $\qquad = 5+3+20 = 28$

$= 0$

$u_1 \cdot u_1 = (1,3,-2)(1,3,-2)$ and $u_2 \cdot u_2 = (5,1,4)^2$

$= 1+9-4$ $\qquad = 25+1+16$

$= 6$ $\qquad = 42$

The Orthogonal projection of $\hat{y}$ of y on to span$\{u_1, u_2\}$

$$\hat{y} = \left\{\left(\frac{y \cdot u_1}{u_1 \cdot u_1}\right) \cdot u_1\right\} + \left\{\left(\frac{y \cdot u_2}{u_2 \cdot u_2}\right) u_2\right\}$$

$$= \left\{\frac{0}{6}(1,3,-2) + \frac{28}{42}(5,1,4)\right\}$$

$$= 0 + \frac{2}{3}(5,1,4) = \left(\frac{10}{3}, \frac{2}{3}, \frac{8}{3}\right)$$

∴ Here $y = \hat{y} + (y - \hat{y})$

$$= \left(\frac{10}{3}, \frac{2}{3}, \frac{8}{3}\right) + \left\{(1,3,5) - \left(\frac{10}{3}, \frac{2}{3}, \frac{8}{3}\right)\right\}$$

$$= \left(\frac{10}{3}, \frac{2}{3}, \frac{8}{3}\right) + \left\{\left(-\frac{7}{3}, \frac{7}{3}, \frac{7}{3}\right)\right\}$$

$$\therefore y = \begin{bmatrix} 10/3 \\ 2/3 \\ 8/3 \end{bmatrix} + \begin{bmatrix} -7/3 \\ 7/3 \\ 7/3 \end{bmatrix}$$

This show that y can be written as the sum of $\hat{y}$ and $(y-\hat{y})$. where y as the sum of a vector in W and a vector orthogonal to W.

(ii) $y = \begin{bmatrix} 4 \\ 3 \\ 3 \\ -1 \end{bmatrix}, u_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, u_2 = \begin{bmatrix} -1 \\ 3 \\ 1 \\ -2 \end{bmatrix}, u_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$

Sol^n: Here,

$y \cdot u_1 = (4,3,3,-1) \cdot (1,1,0,1)$

$= 4+3+0-1$

$= 6$

$u_1 \cdot u_1 = (1,1,0,1) \cdot (1,1,0,1)$

$= 1+1+0+1$

$= 3$

$y \cdot u_2 = (4,3,3,-1) \cdot (-1,3,1,-2)$

$= -4+9+3+2$

$= 10$

$u_2 \cdot u_2 = (-1,3,1,-2) \cdot (-1,3,1,-2)$

$= 1+9+1+4$

$= 15$

$y \cdot u_3 = (4,3,3,-1) \cdot (-1,0,1,1)$

$= -4+0+3-1$

$= -2$

$u_3 \cdot u_3 = (-1,0,1,1) \cdot (-1,0,1,1)$

$= 1+0+1+1$

$= 3$

~~The~~ Here the Orthogonal projection of $\hat{y}$ of y on to span$\{u_1, u_2, u_3\}$ is

$y = \hat{y} + (y - \hat{y})$

For $\hat{y} = \left\{\left(\frac{y \cdot u_1}{u_1 \cdot u_1}\right) u_1 + \left(\frac{y \cdot u_2}{u_2 \cdot u_2}\right) \cdot u_2 + \left(\frac{y \cdot u_3}{u_3 \cdot u_3}\right) u_3\right\}$

$$\hat{y} = \left(\frac{6}{3}(1,1,0,1) + \frac{10}{15}(-1,3,1,-2) + \frac{-2}{3}(-1,0,1,1)\right\}$$

$$\hat{y} = (2,2,0,2) + \left(-\frac{2}{3}, 2, \frac{2}{3}, -\frac{4}{3}\right) + \left(\frac{2}{3}, 0, -\frac{2}{3}, -\frac{2}{3}\right)$$

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$y = (2,2,0,2) + (0,2,0,-2)$

$\hat{y} = (2,4,0,0)$

$\therefore\ y = \hat{y} + (y - \hat{y})$

$\therefore\ y = (2,4,0,0) + \{\cancel{(2,4,0,0)}\ (4,3,3,-1) - (2,4,0,0)\}$

$\therefore\ = (2,4,0,0) + (2,-1,3,-1)$

$$y = \begin{bmatrix} 2 \\ 4 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \\ 3 \\ -1 \end{bmatrix}$$

$\therefore$ Here this can be written as sum of $\hat{y}$ and $y-\hat{y}$. since $\hat{y}$ where $\hat{y}$ is in W when W be the subspace spanned by u_1 and u_2.

(iii) $$y = \begin{bmatrix} 3 \\ 4 \\ 5 \\ 6 \end{bmatrix},\ u_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix},\ u_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix},\ u_3 = \begin{bmatrix} 0 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$

Sol^n: Here,

$y \cdot u_1 = (3,4,5,6) \cdot (1,1,0,-1)$

$= 3+4+0-6 = 1$

$u_1 \cdot u_1 = (1,1,0,-1) \cdot (1,1,0,-1)$

$= 1+1+0+1 = 3$

$y \cdot u_2 = (3,4,5,6) \cdot (1,1,0,1) = 3+4+0+6 = 13$

$u_2 \cdot u_2 = (1,1,0,1) \cdot (1,1,0,1)$

$= 1+1+1+0 = 3$

$y \cdot u_3 = (3,4,5,6) \cdot (0,-1,1,-1)$

$= 0-4+5-6 = -5$

$u_3 \cdot u_3 = (0,-1,1,-1) \cdot (0,-1,1,-1)$

$= (0+1+1+1) = 3$

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The orthogonal projection $\hat{y}$ of y on to span $\{u_1, u_2, u_3\}$ is

$$\hat{y} = \left\{ \left(\frac{y \cdot u_1}{u_1 \cdot u_1}\right) u_1 + \left(\frac{y \cdot u_2}{u_2 \cdot u_2}\right) u_2 + \left(\frac{y \cdot u_3}{u_3 \cdot u_3}\right) u_3 \right\}$$

$= \frac{1}{3} (1,1,0,-1) + \frac{13}{3} (1,1,0,1) + -\frac{5}{3} (0,-1,1,-1)$

$= \left(\frac{1}{3}, \frac{1}{3}, 0, -\frac{1}{3}\right) + \left(\frac{13}{3}, \frac{13}{3}, 0, \frac{13}{3}\right) + \left(0, \frac{5}{3}, -\frac{5}{3}, \frac{5}{3}\right)$

$= \left(\frac{1}{3} + \frac{13}{3} + 0, \frac{1}{3} + \frac{13}{3} + \frac{5}{3}, 0+0-\frac{5}{3}, -\frac{1}{3} + \frac{13}{3} + \frac{5}{3}\right)$

$= \left(\frac{14}{3}, \frac{19}{3}, -\frac{5}{3}, \frac{17}{3}\right)$

Here,

$y = \hat{y} + (y - \hat{y})$

$= \left(\frac{14}{3}, \frac{19}{3}, -\frac{5}{3}, \frac{17}{3}\right) + \left\{ (3,4,5,6) - \left(\frac{14}{3}, \frac{19}{3}, -\frac{5}{3}, \frac{17}{3}\right) \right\}$

$= \left(\frac{14}{3}, \frac{19}{3}, -\frac{5}{3}, \frac{17}{3}\right) + \{ -\frac{6}{3},$

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(4) Find the closet point to y in the subspace W Spanned by v_1 and v_2.

Sol^n:

(i) $y = \begin{bmatrix} 3 \\ 1 \\ 5 \\ 1 \end{bmatrix}$, $v_1 = \begin{bmatrix} 3 \\ 1 \\ -1 \\ 1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$

Sol^n: Here,

let $\hat{y}$ be the closet point in $W = \text{span}\{u_1, u_2\}$ to y then

$y \cdot v_1 = (3,1,5,1) \cdot (3,1,-1,1)$
$= 9+1-5+1$
$= 6$

$v_1 \cdot v_1 = (3,1,-1,1) \cdot (3,1,-1,1)$
$= (9+1+1+1)$
$= 12$

$y \cdot v_2 = (3,1,5,1) \cdot (1,-1,1,-1)$
$= (3-1+5-1)$
$= 6$

$v_2 \cdot v_2 = (1,-1,1,-1) \cdot (1,-1,1,-1)$
$= 1+1+1+1$
$= 4$

$\therefore \hat{y} = \left(\frac{y \cdot v_1}{v_1 \cdot v_1}\right) \cdot v_1 + \left(\frac{y \cdot v_2}{v_2 \cdot v_2}\right) \cdot v_2$

$= \left(\frac{6}{12}\right) \cdot (3,1,-1,1) + \left(\frac{6}{4}\right) \cdot (1,-1,1,-1)$

$= \left(\frac{3}{2}, \frac{1}{2}, \frac{-1}{2}, \frac{1}{2}\right) + \left(\frac{3}{2}, -\frac{3}{2}, \frac{3}{2}, -\frac{3}{2}\right)$

$= (3,-1,1,-1) = \begin{bmatrix} 3 \\ -1 \\ 1 \\ -1 \end{bmatrix}$

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(ii) $y = \begin{bmatrix} 3 \\ -1 \\ 1 \\ 13 \end{bmatrix}$, $v_1 = \begin{bmatrix} 1 \\ -2 \\ -1 \\ 2 \end{bmatrix}$, $v_2 = \begin{bmatrix} -4 \\ 1 \\ 0 \\ 3 \end{bmatrix}$.

Sol^n: Here,

let $\hat{y}$ be the closet point in $W = \text{span}\{v_1, v_2\}$ to y then,

$y \cdot v_1 = (3,-1,1,13) \cdot (1,-2,-1,2)$
$= (3+2-1+26)$
$= 30$

$v_1 \cdot v_1 = (1,-2,-1,2) \cdot (1,-2,-1,2)$
$= 1+4+1+4$
$= 10$

$y \cdot v_2 = (3,-1,1,13) \cdot (-4,1,0,3)$
$= -12-1+0+39$
$= -13+39$
$= 26$

$v_2 \cdot v_2 = (-4,1,0,3) \cdot (-4,1,0,3)$
$= 16+1+0+9$
$= 26$

$\hat{y} = \left(\frac{y \cdot v_1}{v_1 \cdot v_1}\right) v_1 + \left(\frac{y \cdot v_2}{v_2 \cdot v_2}\right) \cdot v_2$

$= \frac{30}{10}(1,-2,-1,2) + \left(\frac{26}{26}\right) \cdot (-4,1,0,3)$

$= \cancel{(3,-6,-3,6) + \left(\frac{-2}{13}\right)(-4,1,0,3)}$

$= \cancel{\left(3+\frac{8}{13}, -6-\frac{2}{13}, -3-0, 6-\frac{6}{13}\right)}$

$= (3,-6,-3,6) + (-4,1,0,3)$

$= (-1,-5,-3,9) \Rightarrow \begin{bmatrix} -1 \\ -5 \\ -3 \\ 9 \end{bmatrix}$

Ans

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(7) Let $y = \begin{bmatrix} 4 \\ 8 \\ 1 \end{bmatrix}$, $u_1 = \begin{bmatrix} 2/3 \\ 1/3 \\ 2/3 \end{bmatrix}$, $u_2 = \begin{bmatrix} -2/3 \\ 2/3 \\ 1/3 \end{bmatrix}$ and

$W = \text{span}\{u_1, u_2\}$

(i) let $U = \{u_1, u_2\}$. compute U^TU and UU^T

(ii) compute $\text{proj}_W(y)$. and $(UU^T)y$

Sol^n: Here.

$$U = \{u_1, u_2\} = \begin{bmatrix} 2/3 & -2/3 \\ 1/3 & 2/3 \\ 2/3 & 1/3 \end{bmatrix}$$

$$U^T = \begin{bmatrix} 2/3 & 1/3 & 2/3 \\ -2/3 & 2/3 & 1/3 \end{bmatrix}$$

$$\therefore U^T.U = \begin{bmatrix} 2/3 & 1/3 & 2/3 \\ -2/3 & 2/3 & 1/3 \end{bmatrix} \cdot \begin{bmatrix} 2/3 & -2/3 \\ 1/3 & 2/3 \\ 2/3 & 1/3 \end{bmatrix}$$

$$= \begin{bmatrix} 4/9 + 1/9 + 4/9 & -4/9 + 2/9 + 2/9 \\ -4/9 + 2/9 + 2/9 & 4/9 + 4/9 + 1/9 \end{bmatrix}$$

$$= \begin{bmatrix} 9/9 & -4/9 + 4/9 \\ -4/9 + 4/9 & 9/9 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\therefore U^T.U = I$ Ans

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$$U.U^T = \begin{bmatrix} 2/3 & -2/3 \\ 1/3 & 2/3 \\ 2/3 & 1/3 \end{bmatrix}_{3\times2} \cdot \begin{bmatrix} 2/3 & 1/3 & 2/3 \\ -2/3 & 2/3 & 1/3 \end{bmatrix}_{2\times3}$$

$$= \begin{bmatrix} 4/9 + 4/9 & 2/9 - 4/9 & 4/9 - 2/9 \\ 2/9 - 4/9 & 1/9 + 4/9 & 2/9 + 2/9 \\ 4/9 - 2/9 & 2/9 + 2/9 & 4/9 + 1/9 \end{bmatrix}$$

$$= \begin{bmatrix} 8/9 & -2/9 & 2/9 \\ -2/9 & 5/9 & 4/9 \\ 2/9 & 4/9 & 5/9 \end{bmatrix}$$ Ans

(ii) $\text{proj}_W(y)$

$y \cdot u_1 = (4, 8, 1) \cdot (2/3, 1/3, 2/3)$

$= (8/3 + 8/3 + 2/3)$

$= (18/3)$

$= 6$

$u_1 \cdot u_1 = (2/3, 1/3, 2/3) \cdot (2/3, 1/3, 2/3)$

$= \left(\frac{4}{9} + \frac{1}{9} + \frac{4}{9}\right)$

$= 9/9 = 1$

$y \cdot u_2 = (4, 8, 1) \cdot (-2/3, 2/3, 1/3)$

$= -8/3 + 16/3 + 1/3$

$= \frac{9}{3}$

$= 3$

$u_2 \cdot u_2 = (-2/3, 2/3, 1/3) \cdot (-2/3, 2/3, 1/3)$

$= 4/9 + 4/9 + 1/9$

$= 9/9 = 1$

$\hat{y} = \left(\frac{y \cdot u_1}{u_1 \cdot u_1}\right) \cdot u_1 + \left(\frac{y \cdot u_2}{u_2 \cdot u_2}\right) \cdot u_2 = \left(\frac{6}{1}\right) u_1 + \left(\frac{3}{1}\right) u_2$

$\hat{y} = 6u_1 + 3u_2$

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$\hat{y} = 6\left(2/3, 1/3, 2/3\right) + 3\left(-2/3, 2/3, 1/3\right)$

$= (4, 2, 4) + (-2, 2, 1)$

$= (4-2, 2+2, 4+1)$

$= (2, 4, 5)$

$$\hat{y} = \begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix}$$

*. $(UU^T) \cdot y = \begin{bmatrix} 8/9 & -2/9 & 2/9 \\ -2/9 & 5/9 & 4/9 \\ 2/9 & 4/9 & 5/9 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 8 \\ 1 \end{bmatrix}$

$$= \begin{bmatrix} \frac{32}{9} - \frac{16}{9} + \frac{2}{9} \\ -8/9 + 40/9 + 4/9 \\ 8/9 + 32/9 + 5/9 \end{bmatrix}$$

$$= \begin{bmatrix} 18/9 \\ 36/9 \\ 45/9 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix}$$

$\therefore (UU^T) \cdot y = (2, 4, 5)$ Ans

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Exercise: 8.4

The Gram Schmidt Process:-

The Gram Schmidt process is a simple process obtain an orthogonal or orthonormal basis for any non-zero subspace of R^n.

Theorem:-

Give a basis $\{x_1, x_2, \ldots x_p\}$ for a subspace W of R^n, we define

$V_1 = x_1$

$V_2 = x_2 - \left(\frac{x_2 \cdot V_1}{V_1 \cdot V_1}\right) \cdot V_1$

$V_3 = x_3 - \left(\frac{x_3 \cdot V_1}{V_1 \cdot V_1}\right) V_1 - \left(\frac{x_3 \cdot V_2}{V_2 \cdot V_2}\right) \cdot V_2$

- - - - - - - - - - - - - - -

$$V_p = x_p - \sum_{n=1}^{p-1} \left(\frac{x_p \cdot V_n}{V_n \cdot V_n}\right) \cdot V_n$$

Then $\{V_1, V_2, \ldots V_p\}$ is an orthogonal basis for W. In addition, $\text{Span}\{V_1, V_2, \ldots V_p\} = \text{span}\{x_1, x_2, \ldots x_p\}$

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The given set is a basis for a Subspace W. Use the Gram-Schmidt process to produce an orthogonal basis for W.

(1) $\begin{bmatrix}3\\0\\1\end{bmatrix}, \begin{bmatrix}8\\5\\-6\end{bmatrix}$

Sol: Here,

let $v_1 = x_1 = (3,0,1)$

then

$v_2 = x_2 - \left(\frac{x_2 \cdot v_1}{v_1 \cdot v_1}\right) \cdot v_1$

$x_2 \cdot v_1 = \cancel{(3,5,6) \cdot (3,0,1)}\ (8,5,-6) \cdot (3,0,1)$

$= \cancel{9+0+1}$

$= 24+0-6$

$= 18$

$v_1 \cdot v_1 = (3,0,1) \cdot (3,0,1)$

$= 9+0+1 = 10$

$v_2 = (8,5,-6) - \left(\frac{18}{10}\right) \cdot (3,0,1)$

$= (8,5,-6) - \left(\frac{54}{10}, 0, \frac{18}{10}\right)$

$= (8,5,-6) - \left(\frac{27}{5}, 0, \frac{9}{5}\right)$

$= \left(\frac{13}{5}, 5, \frac{-39}{5}\right)$

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(1) $\begin{bmatrix}3\\0\\-1\end{bmatrix}, \begin{bmatrix}8\\5\\-6\end{bmatrix}$

Sol: Here,

let $x_1 = v_1 = \begin{bmatrix}3\\0\\-1\end{bmatrix}$

$x_2 = \begin{bmatrix}8\\5\\-6\end{bmatrix}$

then

$v_2 = x_2 - \left(\frac{x_2 \cdot v_1}{v_1 \cdot v_1}\right) \cdot v_1$

$x_2 \cdot v_1 = (8,5,-6) \cdot (3,0,-1)$

$= 24+6$

$= 30$

$v_1 \cdot v_1 = (3,0,-1) \cdot (3,0,-1)$

$= 9+0+1$

$= 10$

$v_2 = (8,5,-6) - \left(\frac{30}{10}\right) \cdot (3,0,-1)$

$= (8,5,-6) - 3(3,0,-1)$

$= (8,5,-6) - (9,0,-3)$

$= (-1,5,-3)$

$v_2 = \begin{bmatrix}-1\\5\\-3\end{bmatrix}$

Now, the set $\{v_1, v_2\}$ is an orthogonal set for W

(2) $\begin{bmatrix}0\\4\\2\end{bmatrix}, \begin{bmatrix}5\\6\\-7\end{bmatrix}$

Sol^n: Here,

let $x_1 = \begin{bmatrix}0\\4\\2\end{bmatrix} = v_1 = \begin{bmatrix}0\\4\\2\end{bmatrix}$

$x_2 = \begin{bmatrix}5\\6\\-7\end{bmatrix}$

$\therefore v_2 = x_2 - \left(\frac{x_2 \cdot v_1}{v_1 \cdot v_1}\right) \cdot v_1$

$x_2 \cdot v_1 = (5,6,-7)\,(0,4,2)$
$= (0+24-14)$
$= 10$

$v_1 \cdot v_1 = \begin{bmatrix}0\\4\\2\end{bmatrix} \cdot \begin{bmatrix}0\\4\\2\end{bmatrix}$

$= 0+16+4 = 20$

$\therefore v_2 = (5,6,-7) - \left(\frac{10}{20}\right) \cdot (0,4,2)$

$= (5,6,-7) - \frac{1}{2}(0,4,2)$

$= (5,6,-7) - (0,2,1)$
$= (5,4,-8)$

$\therefore v_2 = \begin{bmatrix}5\\4\\-8\end{bmatrix}$ Ans

Now, the set $\{v_1, v_2\}$ is an Orthogonal set form

(3) $\begin{bmatrix}2\\-5\\1\end{bmatrix}, \begin{bmatrix}4\\-1\\2\end{bmatrix}$

Sol^n: Here
let $x_1 = v_1 = \begin{bmatrix}2\\-5\\1\end{bmatrix}$

$x_2 = \begin{bmatrix}4\\-1\\2\end{bmatrix}$

$\therefore v_2 = x_2 - \left(\frac{x_2 \cdot v_1}{v_1 \cdot v_1}\right) \cdot v_1$

$x_2 \cdot v_1 = (4,-1,2) \cdot (2,-5,1)$
$= 8+5+2$
$= 15$

$v_1 \cdot v_1 = (2,-5,1)\,(2,-5,1)$
$= 4+25+1$
$= 30$

$\therefore v_2 = (4,-1,2) - \left(\frac{15}{30}\right) \cdot \cancel{(4,-1,2)}\,(2,-5,1)$

$= (4,-1,2) - \frac{1}{2}\cancel{(4,-1,2)}\,(2,-5,1)$

$= (4,-1,2) - \left(1, \frac{-5}{2}, \frac{1}{2}\right)$

$= \left(3, \frac{3}{2}, \frac{3}{2}\right)$

$v_2 = \begin{bmatrix}3\\3/2\\3/2\end{bmatrix}$

Now the set $\{v_1, v_2\}$ is an Orthogonal set form

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(4) $\begin{bmatrix} 3 \\ -4 \\ 5 \end{bmatrix} \begin{bmatrix} -3 \\ 14 \\ -7 \end{bmatrix}$

Sol: Here,

let $x_1 = v_1 = \begin{bmatrix} 3 \\ -4 \\ 5 \end{bmatrix}$

$x_2 = \begin{bmatrix} -3 \\ 14 \\ -7 \end{bmatrix}$

Now

$v_2 = x_2 - \left(\frac{x_2 \cdot v_1}{v_1 \cdot v_1}\right) \cdot v_1$

$x_2 \cdot v_1 = (-3, 14, -7) \cdot (3, -4, 5)$

$= (-9 - 56 - 35)$

$= -100$

$v_1 \cdot v_1 = (3, -4, 5) \cdot (3, -4, 5)$

$= 9 + 16 + 25$

$= 50$

$\therefore v_2 = x_2 - \left(\frac{x_2 \cdot v_1}{v_1 \cdot v_1}\right) \cdot v_1$

$= (-3, 14, -7) - \left(\frac{-100}{50}\right) \cdot (3, -4, 5)$

$= (-3, 14, -7) + 2(3, -4, 5)$

$= (-3, 14, -7) + (6, -8, 10)$

$= (3, 6, 3)$

$v_2 = \begin{bmatrix} 3 \\ 6 \\ 3 \end{bmatrix}$

$\therefore$ The set $\{v_1, v_2\}$ is a basis for a subspace W

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(5) $\begin{bmatrix} 1 \\ -4 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 7 \\ -7 \\ -4 \\ 1 \end{bmatrix}$

let $x_1 = v_1 = \begin{bmatrix} 1 \\ -4 \\ 0 \\ 1 \end{bmatrix}$

$x_2 = \begin{bmatrix} 7 \\ -7 \\ -4 \\ 1 \end{bmatrix}$

Now

$v_2 = x_2 - \left(\frac{x_2 \cdot v_1}{v_1 \cdot v_1}\right) \cdot v_1$

$x_2 \cdot v_1 = (7, -7, -4, 1) \cdot (1, -4, 0, 1)$

$= 7 + 28 + 0 + 1$

$= 36$

$v_1 \cdot v_1 = (1, -4, 0, 1) \cdot (1, -4, 0, 1)$

$= (1 + 16 + 0 + 1)$

$= 18$

$\therefore v_2 = x_2 - \left(\frac{x_2 v_1}{v_1 \cdot v_1}\right) \cdot v_1$

$= (7, -7, -4, 1) - \left(\frac{36}{18}\right) \cdot (1, -4, 0, 1)$

$= (7, -7, -4, 1) - 2(1, -4, 0, 1)$

$= (7, -7, -4, 1) - (2, -8, 0, 2)$

$= (5, 1, -4, -1)$

$\therefore v_2 = \begin{bmatrix} 5 \\ 1 \\ -4 \\ -1 \end{bmatrix}$

The set $\{v_1, v_2\}$ is a basis for a subspace W.

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6. $\begin{bmatrix} 3 \\ -1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -5 \\ 9 \\ -9 \\ 3 \end{bmatrix}$

Sol^n: Here,

let $x_1 = v_1 = \begin{bmatrix} 3 \\ -1 \\ 2 \\ -1 \end{bmatrix}$

$x_2 = \begin{bmatrix} -5 \\ 9 \\ -9 \\ 3 \end{bmatrix}$

$\therefore v_2 = x_2 - \left(\frac{x_2 \cdot v_1}{v_1 \cdot v_1}\right) \cdot v_1$

$x_2 \cdot v_1 = (-5, 9, -9, 3) \cdot (3, -1, 2, -1)$

$= -15 - 9 - 18 - 3$

$= -45$

$v_1 \cdot v_1 = (3, -1, 2, -1) \cdot (3, -1, 2, -1)$

$= 9 + 1 + 4 + 1$

$= 15$

$\therefore v_2 = x_2 - \left(\frac{x_2 \cdot v_1}{v_1 \cdot v_1}\right) \cdot v_1$

$v_2 = (-5, 9, -9, 3) - \left(\frac{-45}{15}\right)(3, -1, 2, -1)$

$= (-5, 9, -9, 3) + 3(3, -1, 2, -1)$

$= (-5, 9, -9, 3) + (9, -3, 6, -3)$

$= (4, 6, -3, 0)$

$\therefore v_2 = \begin{bmatrix} 4 \\ 6 \\ -3 \\ 0 \end{bmatrix}$

$\therefore$ The set $\{v_1, v_2\}$ is a basis for subspace W.

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7. Find an Orthonormal basis of the subspace spanned by the vectors $(2, -5, 1)$ and $(4, -1, 2)$

Sol^n: Here,

let $x_1 = v_1 = (2, -5, 1)$

$x_2 = (4, -1, 2)$

then

$v_2 = x_2 - \left(\frac{x_2 \cdot v_1}{v_1 \cdot v_1}\right) \cdot v_1$

$x_2 \cdot v_1 = (4, -1, 2) \cdot (2, -5, 1)$

$= (8 + 5 + 2)$

$= 15$

$v_1 \cdot v_1 = (2, -5, 1) \cdot (2, -5, 1)$

$= 4 + 25 + 1$

$= 30$

$\therefore v_2 = x_2 - \left(\frac{x_2 \cdot v_1}{v_1 \cdot v_1}\right) \cdot v_1$

$= (4, -1, 2) - \left(\frac{15}{30}\right)(2, -5, 1)$

$= (4, -1, 2) - \left(\frac{1}{2}\right)(2, -5, 1)$

$= (4, -1, 2) - (1, -5/2, 1/2)$

$= (3, 3/2, 3/2)$

$v_2' = \frac{3}{2}(2, 1, 1)$

$v_2' = (2, 1, 1)$

$\|v_1\|$ $=$ ~~v_1~~ $u_1 = \frac{v_1}{\|v_1\|} = \frac{(2, -5, 1)}{\sqrt{4+25+1}} = \frac{(2, -5, 1)}{\sqrt{30}}$

$u_1 = \left(\frac{2}{\sqrt{30}}, \frac{-5}{\sqrt{30}}, \frac{1}{\sqrt{30}}\right)$

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$$u_2 = \frac{v_2'}{\|v_2'\|} = \frac{(2,1,1)}{\sqrt{4+1+1}} = \frac{(2,1,1)}{\sqrt{6}}$$

$$= \left(\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$$

Hence, $\{u_1, u_2\}$ is an orthogonal basis for W.

Find an orthonormal basis of the subspace spanned by the vectors $(3,-4,5)$ and $(-3,14,-7)$

Sol: Here,

let $x_1 = v_1 = (3,-4,5)$

$x_2 = (-3,14,-7)$

$$v_2 = x_2 - \left(\frac{x_2 \cdot v_1}{v_1 \cdot v_1}\right) \cdot v_1$$

$$x_2 \cdot v_1 = (-3,14,-7) \cdot (3,-4,5)$$
$$= -9-56-35$$
$$= -100$$

$$v_1 \cdot v_1 = (3,-4,5) \cdot (3,-4,5)$$
$$= 9+16+25$$
$$= 50$$

$$\therefore v_2 = x_2 - \left(\frac{x_2 \cdot v_1}{v_1 \cdot v_1}\right) \cdot v_1$$

$$v_2 = (-3,14,-7) - \left(\frac{-100}{50}\right)(3,-4,5)$$
$$= (-3,14,-7) + 2(3,-4,5)$$
$$= (-3,14,-7) + (6,-8,10)$$
$$= (3,6,3) = 3(1,2,1)$$

Now, $\therefore v_2' = (1,2,1)$

$$u_1 = \frac{v_1}{\|v_1\|}$$

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$$u_1 = \frac{(3,-4,5)}{\sqrt{9+16+25}}$$

$$u_1 = \frac{(3,-4,5)}{\sqrt{50}}$$

$$u_1 = \left(\frac{3}{5\sqrt{2}}, \frac{-4}{5\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

$$\therefore u_2 = \frac{v_2'}{\|v_2'\|} = \frac{(1,2,1)}{\sqrt{1+4+1}} = \frac{(1,2,1)}{\sqrt{6}}$$

$$u_2 = \left(\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$$

$\therefore$ Hence $\{u_1, u_2\}$ is an orthogonal basis for W.

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The Q-R factorization :-

Theorem:

If A is an $m \times n$ matrix with linearly independent column, then A can be factorized as $A = QR$, where Q is an $m \times n$ matrix whose column form an orthonormal basis for Col$A$ and R is an $n \times n$ upper triangular invertible matrix with positive entries on its diagonal.

Ex. Find Q.R factorization of a matrix A where

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Solⁿ: Here,

let the column of A are x_1, x_2, x_3

so,

$x_1 = (1,1,1,1)$, $x_2 = (0,1,1,1)$ and $x_3 = (0,0,1,1)$

let $v_1 = x_1 = (1,1,1,1)$

$$V_2 = x_2 - \left(\frac{x_2 \cdot v_1}{v_1 \cdot v_1}\right) v_1 = (0,1,1,1) - \left(\frac{(0,1,1,1).(1,1,1,1)}{(1,1,1,1)\ (1,1,1,1)}\right)(1,1,1,1)$$

$$V_2 = (0,1,1,1) - \left(\frac{3}{4}\right)(1,1,1,1)$$

$$V_2 = \left(\frac{-3}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$$

$$V_2 = \frac{1}{4}(-3,1,1,1)$$

set $V_2' = (-3,1,1,1)$

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$$V_3 = x_3 - \left(\frac{x_3 \cdot V_1}{V_1 \cdot V_1}\right) \cdot V_1 - \left(\frac{x_3 \cdot V_2'}{V_2' \cdot V_2'}\right) \cdot V_2'$$

$$= (0,0,1,1) - \left(\frac{(0,0,1,1)\cdot(1,1,1,1)}{(1,1,1,1)\cdot(1,1,1,1)}\right) \cdot V_1 - \left(\frac{(0,0,1,1)\cdot(-3,1,1,1)}{(-3,1,1,1)\cdot(-3,1,1,1)}\right) \cdot (-3,1,1,1)$$

$$= (0,0,1,1) - \left(\frac{2}{4}(1,1,1,1)\right) - \left(\frac{2}{12}(-3,1,1,1)\right)$$

$$= (0,0,1,1) - \left(\left(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}\right)\right) - \left(\frac{1}{6}(-3,1,1,1)\right)$$

$$= (0,0,1,1) - \left(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}\right) - \left(\tfrac{-1}{2}, \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}\right)$$

$$= \left(\tfrac{-1}{2}, \tfrac{-1}{2}, \tfrac{1}{2}, \tfrac{1}{2}\right) - \left(\tfrac{-1}{2}, \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}\right)$$

$$= \left(0, \tfrac{-4}{6}, \tfrac{2}{6}, \tfrac{2}{6}\right)$$

$$= \left(0, -\tfrac{2}{3}, \tfrac{1}{3}, \tfrac{1}{3}\right)$$

$$= \tfrac{1}{3}(0,-2,1,1)$$

set $V_3' = (0,-2,1,1)$

let (u_1, u_2, u_3) is normalize of the orthogonal basis.

$$u_1 = \frac{V_1}{\|V_1\|} = \frac{(1,1,1,1)}{\sqrt{1+1+1+1}} = \frac{(1,1,1,1)}{\sqrt{4}} = \left(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}\right)$$

$$u_2 = \frac{V_2'}{\|V_2'\|} = \frac{(-3,1,1,1)}{\sqrt{9+1+1+1}} = \frac{(-3,1,1,1)}{\sqrt{12}} = \left(\tfrac{-3}{\sqrt{12}}, \tfrac{1}{\sqrt{12}}, \tfrac{1}{\sqrt{12}}, \tfrac{1}{\sqrt{12}}\right)$$

$$u_3 = \frac{V_3'}{\|V_3'\|} = \frac{(0,-2,1,1)}{\sqrt{4+1+1}} = \frac{(0,-2,1,1)}{\sqrt{6}} = \left(0, \tfrac{-2}{\sqrt{6}}, \tfrac{1}{\sqrt{6}}, \tfrac{1}{\sqrt{6}}\right)$$

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Let Q be the matrix whose column are u_1, u_2, and u_3

$$Q = \begin{bmatrix} 1/2 & -3/\sqrt{12} & 0 \\ 1/2 & 1/\sqrt{12} & -2/\sqrt{6} \\ 1/2 & 1/\sqrt{12} & 1/\sqrt{6} \\ 1/2 & 1/\sqrt{12} & 1/\sqrt{6} \end{bmatrix}$$

Since $A = QR \Rightarrow Q^T.A = (Q^T.Q)R$

$Q^T.A = R$

then

$R = Q^T.A$

$$Q^T = \begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ -3/\sqrt{12} & 1/\sqrt{12} & 1/\sqrt{12} & 1/\sqrt{12} \\ 0 & -2/\sqrt{6} & 1/\sqrt{6} & 1/\sqrt{6} \end{bmatrix}$$

$$R = Q^T.A = \begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ -3/\sqrt{12} & 1/\sqrt{12} & 1/\sqrt{12} & 1/\sqrt{12} \\ 0 & -2/\sqrt{6} & 1/\sqrt{6} & 1/\sqrt{6} \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cancel{1/2} & \cancel{1/2} \\ \cancel{3/\sqrt{12}} & \\ 0 & \end{bmatrix} \begin{bmatrix} 2 & 3/2 & 1 \\ 0 & 3/\sqrt{12} & 2/\sqrt{12} \\ 0 & 0 & 2/\sqrt{6} \end{bmatrix}$$

$\boxed{\therefore A = QR}$ Ans

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9. Find the QR factorization.

(i) $A = \begin{bmatrix} 5 & 9 \\ 1 & 7 \\ -3 & -5 \\ 1 & 5 \end{bmatrix}$

Sol: Here

let $V_1 = x_1 = (5, 1, -3, 1)$

$x_2 = (9, 7, -5, 5)$

$$V_2 = x_2 - \left(\frac{x_2 \cdot V_1}{V_1 \cdot V_1}\right) \cdot V_1$$

$$= (9,7,-5,5) - \left(\frac{(9,7,-5,5)\cdot(5,1,-3,1)}{(5,1,-3,1)\,(5,1,-3,1)}\right)\cdot(5,1,-3,1)$$

$$= (9,7,-5,5) - \left[\frac{(45+7+15+5)}{(25+1+9+1)}\right]\cdot(5,1,-3,1)$$

$$= (9,7,-5,5) - \left[\left(\frac{72}{36}\right)\cdot(5,1,-3,1)\right]$$

$$= (9,7,-5,5) - (2(5,1,-3,1))$$

$$= (9,7,-5,5) - (10,2,-6,2)$$

$V_2 = (-1, 5, 1, 3)$

let (u_1, u_2) is the normalize of the orthogonal basis

$$u_1 = \frac{V_1}{\|V_1\|} = \frac{(5,1,-3,1)}{\sqrt{25+1+9+1}} = \frac{(5,1,-3,1)}{\sqrt{36}} = \frac{1}{6}(5,1,-3,1)$$

$u_1 = (5/6, 1/6, -3/6, 1/6)$

$$u_2 = \frac{V_2}{\|V_2\|} = \frac{(-1,5,1,3)}{\sqrt{1+25+1+9}} = \frac{(-1,5,1,3)}{\sqrt{36}} = \frac{1}{6}(-1,5,1,3)$$

$$u_2 = \frac{1}{6}(-1,5,1,3) : \left(\frac{-1}{6}, \frac{5}{6}, \frac{1}{6}, \frac{3}{6}\right)$$

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let Q be the matrix whose column are u_1 and u_2

$$Q = \begin{bmatrix} 5/6 & -1/6 \\ 1/6 & 5/6 \\ -3/6 & 1/6 \\ 1/6 & 3/6 \end{bmatrix}$$

Since $A = QR \Rightarrow Q^TA = (Q^T.Q).R$

$Q^TA = IR$

$\therefore R = Q^T.A$

then

$$Q^T = \begin{bmatrix} 5/6 & 1/6 & -3/6 & 1/6 \\ -1/6 & 5/6 & 1/6 & 3/6 \end{bmatrix}$$

$$R = Q^T.A = \begin{bmatrix} 5/6 & 1/6 & -3/6 & 1/6 \\ -1/6 & 5/6 & 1/6 & 3/6 \end{bmatrix} \begin{bmatrix} 5 & 9 \\ 1 & 7 \\ -3 & -5 \\ 1 & 5 \end{bmatrix}$$

$$R = \begin{bmatrix} 6 & 12 \\ 0 & 6 \end{bmatrix}$$

$\boxed{\therefore A = QR}$ Ans

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(11) $A = \begin{bmatrix} -2 & 3 \\ 5 & 7 \\ 2 & -2 \\ 4 & 6 \end{bmatrix}$

Sol: Here,

let $v_1 = x_1 = (-2,5,2,4)$

$x_2 = (3,7,-2,6)$

$$v_2 = x_2 - \left(\frac{x_2.v_1}{v_1.v_1}\right).v_1$$

$$= (3,7,-2,6) - \left[\frac{(3,7,-2,6).(-2,5,2,4)}{(-2,5,2,4).(-2,5,2,4)}.(-2,5,2,4)\right]$$

$$= (3,7,-2,6) - \left[\frac{-6+35-4+24}{(4+25+4+16)}.(-2,5,2,4)\right]$$

$$= (3,7,-2,6) - \left[\left(\frac{49}{49}\right).(-2,5,2,4)\right]$$

$$= (3,7,-2,6) - (-2,5,2,4)$$

$v_2 = (5,2,-4,2)$

let (u_1,u_2) is the normalize of the orthogonal basis.

$$u_1 = \frac{v_1}{\|v_1\|} = \frac{(-2,5,2,4)}{\sqrt{4+25+4+16}} = \frac{(-2,5,2,4)}{\sqrt{49}} = \frac{1}{7}(-2,5,2,4)$$

$$u_1 = \left(\frac{-2}{7}, \frac{5}{7}, \frac{2}{7}, \frac{4}{7}\right)$$

$$u_2 = \frac{v_2}{\|v_2\|} = \frac{(5,2,-4,2)}{\sqrt{25+4+16+4}} = \frac{1}{\sqrt{49}}(5,2,-4,2) = \frac{1}{7}(5,2,-4,2)$$

$$u_2 = \left(\frac{5}{7}, \frac{2}{7}, \frac{-4}{7}, \frac{2}{7}\right)$$

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let Q be the matrix of whose column are u_1 and u_2

then

$$Q = \begin{bmatrix} -2/7 & 5/7 \\ 5/7 & 2/7 \\ 2/7 & -4/7 \\ 4/7 & 2/7 \end{bmatrix}$$

Since $A = QR \Rightarrow Q^T.A = (Q^T.Q).R$

$$Q^T.A = IR$$

$R = Q^T.A$

Now

$$Q^T = \begin{bmatrix} -2/7 & 5/7 & 2/7 & 4/7 \\ 5/7 & 2/7 & -4/7 & 2/7 \end{bmatrix}$$

$$R = \begin{bmatrix} -2/7 & 5/7 & 2/7 & 4/7 \\ 5/7 & 2/7 & -4/7 & 2/7 \end{bmatrix} . \begin{bmatrix} -2 & 3 \\ 5 & 7 \\ 2 & -2 \\ 4 & 6 \end{bmatrix}$$

$$R = \begin{bmatrix} 7 & 7 \\ 0 & 7 \end{bmatrix}$$

$\boxed{\therefore A = QR}$ Ans

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Least Square Problem :-

If A is $m \times n$ matrix and b is in R^n then a least square solution of $Ax = b$ is $\hat{x}$ in R^n such that $\|b - A\hat{x}\| < \|b - Ax\|$ for all x in R^n.

Theorem :-

The set of least square solution of $Ax = b$, coincides with the non-empty set of solution of the normal equation $A^TA\hat{x} = A^Tb$

Theorem :- The matrix A^TA is invertible if and only if the column of A are linearly independent. In this case, the equation $Ax = b$ has only one least square solution $\hat{x}$ and it is given by $\hat{x} = (A^TA)^{-1}.A^T.b$

NOTE :- The distance from b to $A\hat{x}$ is called least square error.

Ex:- Find a least square solution of the inconsistent system $Ax = b$ for.

$$A = \begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 2 \\ 0 \\ 11 \end{bmatrix}$$

Sol^n: Here,

$$A^T = \begin{bmatrix} 4 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix}$$

$$A^T.A = \begin{bmatrix} 4 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 16+0+1 & 0+0+1 \\ 0+0+1 & 0+4+1 \end{bmatrix} = \begin{bmatrix} 17 & 1 \\ 1 & 5 \end{bmatrix}$$

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$$A^Tb = \begin{bmatrix} 4 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 11 \end{bmatrix}$$

$$= \begin{bmatrix} 19 \\ 11 \end{bmatrix}$$

The normal equation is,

$$A^TA\hat{x} = A^Tb$$

$$\therefore \hat{x} = (A^TA)^{-1}.(A^Tb)$$

For (A^TA^{-1})

$$A^TA = \begin{bmatrix} 17 & 1 \\ 1 & 5 \end{bmatrix}$$

$$det.(A^TA) = 85-1 = 84$$

$$Adj.(A^TA) = \begin{bmatrix} 5 & -1 \\ -1 & 17 \end{bmatrix}$$

$$(A^TA)^{-1} = \frac{1}{det(A^TA)} Adj(A^T.A)$$

$$= \frac{1}{84}.\begin{bmatrix} 5 & -1 \\ -1 & 17 \end{bmatrix}$$

$$\therefore \hat{x} = (A^T.A)^{-1}.(A^Tb)$$

$$= \frac{1}{84}\begin{bmatrix} 5 & -1 \\ -1 & 17 \end{bmatrix}.\begin{bmatrix} 19 \\ 11 \end{bmatrix}$$

$$= \frac{1}{84}\begin{bmatrix} 95-11 \\ -19+187 \end{bmatrix} = \frac{1}{84}\begin{bmatrix} 84 \\ 168 \end{bmatrix}$$

$$\hat{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$ Ans

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Also

$$A\hat{x} = \begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix}.\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 4+0 \\ 0+4 \\ 1+2 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 3 \end{bmatrix}$$

and

$$b - A\hat{x} = \begin{bmatrix} 2 \\ 0 \\ 11 \end{bmatrix} - \begin{bmatrix} 4 \\ 4 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} -2 \\ -4 \\ 8 \end{bmatrix}$$

$\therefore \|b - A\hat{x}\| = \sqrt{4+16+64} = \sqrt{84}$ is the least square error.

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Exercise:- 8.5

1. Find a least squares solution $Ax=b$ by (a) Constructing the normal equation for $\hat{x}$ and (b) solving for $\hat{x}$.

① $A = \begin{bmatrix} -1 & 2 \\ 2 & -3 \\ -1 & 3 \end{bmatrix}$, $b = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$

Sol^n: Here,

$$A = \begin{bmatrix} -1 & 2 \\ 2 & -3 \\ -1 & 3 \end{bmatrix}$$

$$A^T = \begin{bmatrix} -1 & 2 & -1 \\ 2 & -3 & 3 \end{bmatrix}$$

$$A^T.A = \begin{bmatrix} -1 & 2 & -1 \\ 2 & -3 & 3 \end{bmatrix} \cdot \begin{bmatrix} -1 & 2 \\ 2 & -3 \\ -1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1+4+1 & -2-6-3 \\ -2-6-3 & 4+9+9 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & -11 \\ -11 & 22 \end{bmatrix}$$

$$A^Tb = \begin{bmatrix} -1 & 2 & -1 \\ 2 & -3 & 3 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} -4+2-2 \\ 8-3+6 \end{bmatrix} = \begin{bmatrix} -4 \\ 11 \end{bmatrix}$$

The normal equation is

$$A^TA.\hat{x} = A^Tb$$

$$\therefore \hat{x} = (A^TA)^{-1}(A^Tb)$$

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$$\therefore A^TA.\hat{x} = A^Tb$$

$$\begin{bmatrix} 6 & -11 \\ -11 & 22 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -4 \\ 11 \end{bmatrix}$$

Solving for $\hat{x}$

$$\hat{x} = (A^TA)^{-1} \cdot (A^Tb)$$

For $(A^TA)^{-1}$

$$\det(A^TA) = (132-121) = 11$$

$$Adj.(A^TA) = \begin{bmatrix} 22 & 11 \\ 11 & 6 \end{bmatrix}$$

$$(A^TA)^{-1} = \frac{Adj.(A^TA)}{\det(A^T.A)} = \frac{1}{11} \begin{bmatrix} 22 & 11 \\ 11 & 6 \end{bmatrix}$$

$$\therefore \hat{x} = \frac{1}{11} \begin{bmatrix} 22 & 11 \\ 11 & 6 \end{bmatrix} \cdot \begin{bmatrix} -4 \\ 11 \end{bmatrix}$$

$$\hat{x} = \frac{1}{11} \begin{bmatrix} -88+121 \\ -44+66 \end{bmatrix}$$

$$\hat{x} = \frac{1}{11} \begin{bmatrix} 33 \\ 22 \end{bmatrix}$$

$$\hat{x} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$ Ans

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(11) $A = \begin{bmatrix} 2 & 1 \\ -2 & 0 \\ 2 & 3 \end{bmatrix}$, $b = \begin{bmatrix} -5 \\ 8 \\ 1 \end{bmatrix}$

Sol^n: Here,

$A = \begin{bmatrix} 2 & 1 \\ -2 & 0 \\ 2 & 3 \end{bmatrix}$,

$A^T = \begin{bmatrix} 2 & -2 & 2 \\ 1 & 0 & 3 \end{bmatrix}$

$A^T.A = \begin{bmatrix} 2 & -2 & 2 \\ 1 & 0 & 3 \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 \\ -2 & 0 \\ 2 & 3 \end{bmatrix}$

$= \begin{bmatrix} 4+4+4 & 2+0+6 \\ 2+0+6 & 1+0+9 \end{bmatrix}$

$= \begin{bmatrix} 12 & 8 \\ 8 & 10 \end{bmatrix}$

$A^T.b = \begin{bmatrix} 2 & -2 & 2 \\ 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} -5 \\ 8 \\ 1 \end{bmatrix}$

$= \begin{bmatrix} -10-16+2 \\ -5+0+3 \\ \cancel{0} \end{bmatrix}$

$= \begin{bmatrix} -24 \\ -2 \end{bmatrix}$

The normal equation is

$A^T A . \hat{x} = A^T b.$

$\begin{bmatrix} 12 & 8 \\ 8 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -24 \\ -2 \end{bmatrix}$

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solving for $\hat{x}$

$\hat{x} = (A^T A)^{-1} \cdot (A^T b)$

For $(A^T A)^{-1}$

$\det(A^T A) = (120-64) = 56$

$\text{Adj.}(A^T A) = \begin{bmatrix} 10 & -8 \\ -8 & 12 \end{bmatrix}$

$(A^T A)^{-1} = \dfrac{1}{\det(A^T A)} \cdot \text{Adj}(A^T A)$

$= \dfrac{1}{56} \begin{bmatrix} 10 & -8 \\ -8 & 12 \end{bmatrix}$

then,

$\hat{x} = (A^T A)^{-1} . (A^T b)$

$= \dfrac{1}{56} \begin{bmatrix} 10 & -8 \\ -8 & 12 \end{bmatrix} \cdot \begin{bmatrix} -24 \\ -2 \end{bmatrix}$

$= \dfrac{1}{56} \begin{bmatrix} -240+16 \\ 192-24 \end{bmatrix}$

$= \dfrac{1}{56} \begin{bmatrix} -224 \\ 168 \end{bmatrix}$

$\hat{x} = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$

Ans

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(iii) $A = \begin{bmatrix} 1 & -2 \\ -1 & 2 \\ 0 & 3 \\ 2 & 5 \end{bmatrix}$, $b = \begin{bmatrix} 3 \\ 1 \\ -4 \\ 2 \end{bmatrix}$

Sol: Here,

$A = \begin{bmatrix} 1 & -2 \\ -1 & 2 \\ 0 & 3 \\ 2 & 5 \end{bmatrix}$

$A^T = \begin{bmatrix} 1 & -1 & 0 & 2 \\ -2 & 2 & 3 & 5 \end{bmatrix}$

$A^T.A = \begin{bmatrix} 1 & -1 & 0 & 2 \\ -2 & 2 & 3 & 5 \end{bmatrix} \cdot \begin{bmatrix} 1 & -2 \\ -1 & 2 \\ 0 & 3 \\ 2 & 5 \end{bmatrix}$

$= \begin{bmatrix} 1+1+0+4 & -2-2+0+10 \\ -2-2+0+10 & 4+4+9+25 \end{bmatrix}$

$= \begin{bmatrix} 6 & 6 \\ 6 & 42 \end{bmatrix}$

$A^Tb = \begin{bmatrix} 1 & -1 & 0 & 2 \\ -2 & 2 & 3 & 5 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 1 \\ -4 \\ 2 \end{bmatrix}$

$= \begin{bmatrix} 3-1+0+4 \\ -6+2-12+10 \end{bmatrix}$

$= \begin{bmatrix} 6 \\ -6 \end{bmatrix}$

The normal equation is

$A^TA \cdot \hat{x} = A^Tb$

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$\begin{bmatrix} 6 & 6 \\ 6 & 42 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6 \\ -6 \end{bmatrix}$

Solving for $\hat{x}$

$\hat{x} = (A^TA)^{-1}.(A^Tb)$

For $(A^TA)^{-1}$

$\det.(A^TA) = \begin{vmatrix} 6 & 6 \\ 6 & 42 \end{vmatrix} = 252-36 = 216$

$Adj.(A^TA) = \begin{bmatrix} 42 & -6 \\ -6 & 6 \end{bmatrix}$

$(A^TA)^{-1} = \frac{1}{\det(A^TA)} \cdot Adj.(A^TA)$

$= \frac{1}{216} \begin{bmatrix} 42 & -6 \\ -6 & 6 \end{bmatrix}$

Now $\hat{x} = (A^TA)^{-1}.(A^Tb)$

$= \frac{1}{216} \begin{bmatrix} 42 & -6 \\ -6 & 6 \end{bmatrix} \cdot \begin{bmatrix} 6 \\ -6 \end{bmatrix}$

$= \frac{1}{216} \begin{bmatrix} 252+36 \\ -36-36 \end{bmatrix}$

$= \frac{1}{216} \begin{bmatrix} 288 \\ -72 \end{bmatrix}$

$\hat{x} = \begin{bmatrix} 4/3 \\ -1/3 \end{bmatrix}$ Ans

(iv) $A = \begin{bmatrix} 1 & 3 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}$, $b = \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}$

Soln: Here,

$$A = \begin{bmatrix} 1 & 3 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & 1 & 1 \\ 3 & -1 & 1 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 1 & 1 & 1 \\ 3 & -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 3 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+1+1 & 3-1+1 \\ 3-1+1 & 9+1+1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 3 \\ 3 & 11 \end{bmatrix}$$

$$A^Tb = \begin{bmatrix} 1 & 1 & 1 \\ 3 & -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 5+1+0 \\ 15-1+0 \end{bmatrix}$$

$$= \begin{bmatrix} 6 \\ 14 \end{bmatrix}$$

The normal equation is

$(A^T.A)\hat{x} = A^Tb$

$$\begin{bmatrix} 3 & 3 \\ 3 & 11 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6 \\ 14 \end{bmatrix}$$

Solving for $\hat{x}$

$$\hat{x} = (A^TA)^{-1} \cdot (A^Tb)$$

For $(A^TA)^{-1}$

$$\det(A^TA) = \begin{vmatrix} 3 & 3 \\ 3 & 11 \end{vmatrix} = 33-9 = 24$$

$$\text{Adj.}(A^TA) = \begin{bmatrix} 11 & -3 \\ -3 & 3 \end{bmatrix}$$

$$(A^TA)^{-1} = \frac{1}{\det(A^TA)} \cdot \text{Adj}(A^TA)$$

$$= \frac{1}{24} \begin{bmatrix} 11 & -3 \\ -3 & 3 \end{bmatrix}$$

Now

$$(A^TA) \cdot \hat{x} = (A^Tb)$$

$$\hat{x} = (A^TA)^{-1} \cdot (A^Tb)$$

$$= \frac{1}{24} \begin{bmatrix} 11 & -3 \\ -3 & 3 \end{bmatrix} \cdot \begin{bmatrix} 6 \\ 14 \end{bmatrix}$$

$$= \frac{1}{24} \begin{bmatrix} 66-42 \\ -18+42 \end{bmatrix}$$

$$= \frac{1}{24} \begin{bmatrix} 24 \\ 24 \end{bmatrix}$$

$$\hat{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Ans

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2. Describe all least-squares solutions of the equation $Ax=b$.

S. (1) $A=\begin{bmatrix}1&1&0\\1&1&0\\1&0&1\\1&0&1\end{bmatrix}$, $b=\begin{bmatrix}1\\3\\8\\2\end{bmatrix}$

Sol^n: Here,

$$A=\begin{bmatrix}1&1&0\\1&1&0\\1&0&1\\1&0&1\end{bmatrix}$$

$$A^T=\begin{bmatrix}1&1&1&1\\1&1&0&0\\0&0&1&1\end{bmatrix}$$

$$A^TA=\begin{bmatrix}1&1&1&1\\1&1&0&0\\0&0&1&1\end{bmatrix}\cdot\begin{bmatrix}1&1&0\\1&1&0\\1&0&1\\1&0&1\end{bmatrix}$$

$$=\begin{bmatrix}1+1+1+1&1+1+0+0&0+0+1+1\\1+1+0+0&1+1+0+0&0+0+0+0\\0+0+1+1&0+0+0+0&0+0+1+1\end{bmatrix}$$

$$\begin{bmatrix}4&2&2\\2&2&0\\2&0&2\end{bmatrix}$$

$$A^Tb=\begin{bmatrix}1&1&1&1\\1&1&0&0\\0&0&1&1\end{bmatrix}\cdot\begin{bmatrix}1\\3\\8\\2\end{bmatrix}$$

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$$A^Tb=\begin{bmatrix}1+3+8+2\\1+3+0+0\\0+0+8+2\end{bmatrix}=\begin{bmatrix}14\\4\\10\end{bmatrix}$$

The normal equation is,

$$A^TA\cdot\hat{x}=A^Tb$$

$$\begin{bmatrix}4&2&2\\2&2&0\\2&0&2\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}14\\4\\10\end{bmatrix}\Rightarrow \text{For solving } \hat{x}$$

The augmented matrix is,

$$\left[\begin{array}{ccc|c}4&2&2&14\\2&2&0&4\\2&0&2&10\end{array}\right]$$

$R_1\to\frac{1}{2}R_1$ and $R_2\to\frac{1}{2}R_2$ and $R_3\to\frac{1}{2}R_3$

$$\left[\begin{array}{ccc|c}2&1&1&7\\1&1&0&2\\1&0&1&5\end{array}\right]$$

$R_1\leftrightarrow R_2$

$$\left[\begin{array}{ccc|c}1&1&0&2\\2&1&1&7\\1&0&1&5\end{array}\right]$$

$R_2\to R_2-2R_1$ and $R_3\to R_3-R_1$

$$\left[\begin{array}{ccc|c}1&1&0&2\\0&-1&1&3\\0&-1&1&3\end{array}\right]$$

$R_1\to R_1+R_2$ and $R_3\to R_3-R_2$

$$\left[\begin{array}{ccc|c}1&0&1&5\\0&-1&1&3\\0&0&0&0\end{array}\right]$$

$R_2\to R_2-R_1$

$$\left[\begin{array}{ccc|c}1&0&1&5\\0&-1&0&-2\end{array}\right]$$

$R_2\to(-1)R_2$ $\left[\begin{array}{ccc|c}1&0&1&5\\0&1&0&2\\0&0&0&0\end{array}\right]$

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$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 5 \\ 0 & -1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Here x_1 and x_2 are basic and x_3 is free variable, then the associated system is,

$x_1 + x_3 = 5$

$-x_2 + x_3 = 3 \Rightarrow -3 + x_3 = x_2$

$x_3 = \text{free}$ and also $x_3 = 0$

then,

$x_1 + 0 = 5$, $x_1 = 5$ | $-x_2 + 0 = 3$, $x_2 = -3$ | $x_3 = \text{free}$,

For / free variable (x_3)

$x_1 =$

$$\therefore \hat{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 - x_3 \\ -3 + x_3 \\ 0 + x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

Ans

(ii) $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$, $b = \begin{bmatrix} 7 \\ 2 \\ 3 \\ 6 \\ 5 \\ 4 \end{bmatrix}$

Sol: Here,

$$A^T = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

$$A^T.A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

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$$A^TA = \begin{bmatrix} 1+1+1+1+1+1 & 1+1+1+0+0+0 & 0+0+0+1+1+1 \\ 1+1+1+0+0+0 & 1+1+1+0+0+0 & 0+0+0+0+0+0 \\ 0+0+0+1+1+1 & 0+0+0+0+0+0 & 0+0+0+1+1+1 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 6 & 3 & 3 \\ 3 & 3 & 0 \\ 3 & 0 & 3 \end{bmatrix}$$

$$A^Tb = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 7 \\ 2 \\ 3 \\ 6 \\ 5 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} 7+2+3+6+5+4 \\ 7+2+3+0+0+0 \\ 0+0+0+6+5+4 \end{bmatrix}$$

$$= \begin{bmatrix} 27 \\ 12 \\ 15 \end{bmatrix}$$

Now,

The ~~ahon~~ normal equation is,

$(A^TA) \cdot \hat{x} = (A^Tb)$

$$\begin{bmatrix} 6 & 3 & 3 \\ 3 & 3 & 0 \\ 3 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 27 \\ 12 \\ 15 \end{bmatrix}$$

Solving for $\hat{x}$

The augmented matrix is

$$\left[\begin{array}{ccc|c} 6 & 3 & 3 & 27 \\ 3 & 3 & 0 & 12 \\ 3 & 0 & 3 & 15 \end{array}\right]$$

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$R_1 = \frac{1}{3}R_1$, $R_2 \to \frac{1}{3}R_2$ and $R_3 \to \frac{1}{3}R_3$

$$\left[\begin{array}{ccc:c} 2 & 1 & 1 & 9 \\ 1 & 1 & 0 & 4 \\ 1 & 0 & 1 & 5 \end{array}\right]$$

$R_1 \leftrightarrow R_2$

$$\left[\begin{array}{ccc:c} 1 & 1 & 0 & 4 \\ 2 & 1 & 1 & 9 \\ 1 & 0 & 1 & 5 \end{array}\right]$$

$R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 - R_1$

$$\left[\begin{array}{ccc:c} 1 & 1 & 0 & 4 \\ 0 & -1 & 1 & 1 \\ 0 & -1 & 1 & 1 \end{array}\right]$$

$R_3 \to R_3 - R_2$ ~~and~~

$$\left[\begin{array}{ccc:c} 1 & 1 & 0 & 4 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$R_2 \to -1.R_2$

$$\left[\begin{array}{ccc:c} 1 & 1 & 0 & 4 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Here x_1 and x_2 are basic variable and x_3 is free variable then the associated system is,

$x_1 + x_2 = 4 \Rightarrow x_1 + x_3 - 1 = 4 \Rightarrow x_1 + x_3 = 5$

$x_2 - x_3 = -1 \Rightarrow x_2 = x_3 - 1$

$x_3 =$ free

$$\therefore X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 - x_3 \\ -1 + x_3 \\ 0 + x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$ Ans

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(9) compute the least-squares error associated with the least square solution

$$A = \begin{bmatrix} 1 & -2 \\ -1 & 2 \\ 0 & 3 \\ 2 & 5 \end{bmatrix}, \quad b = \begin{bmatrix} 3 \\ 1 \\ -4 \\ 2 \end{bmatrix}$$

Sol^n: Here,

$$A = \begin{bmatrix} 1 & -2 \\ -1 & 2 \\ 0 & 3 \\ 2 & 5 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & -1 & 0 & 2 \\ -2 & 2 & 3 & 5 \end{bmatrix}$$

$$A^T.A = \begin{bmatrix} 1 & -1 & 0 & 2 \\ -2 & 2 & 3 & 5 \end{bmatrix} . \begin{bmatrix} 1 & -2 \\ -1 & 2 \\ 0 & 3 \\ 2 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 1+1+0+4 & -2-2+0+10 \\ -2-2+0+10 & 4+4+9+25 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 6 \\ 6 & 42 \end{bmatrix}$$

$$A^T.b = \begin{bmatrix} 1 & -1 & 0 & 2 \\ -2 & 2 & 3 & 5 \end{bmatrix} . \begin{bmatrix} 3 \\ 1 \\ -4 \\ 2 \end{bmatrix} = \begin{bmatrix} 3-1+0+4 \\ -6+2-12+10 \end{bmatrix}$$

$$\begin{bmatrix} 6 \\ -6 \end{bmatrix}$$

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The normal equation is

$$A^TA\hat{x} = A^Tb$$

$$\hat{x} = (A^TA)^{-1}A^Tb$$

For $(A^TA)^{-1}$

$\det(A^TA) = 252 - 36 = 216$

$$\text{Adj.}(A^TA) = \begin{bmatrix} 42 & -6 \\ -6 & 6 \end{bmatrix}$$

$$(A^TA)^{-1} = \frac{1}{\det(A^TA)} \cdot \text{Adj}(A^TA)$$

$$= \frac{1}{216}\begin{bmatrix} 42 & -6 \\ -6 & 6 \end{bmatrix}$$

then

$$\hat{x} = (A^TA)^{-1} \cdot A^Tb$$

$$= \frac{1}{216}\begin{bmatrix} 42 & -6 \\ -6 & 6 \end{bmatrix} \cdot \begin{bmatrix} 6 \\ -6 \end{bmatrix}$$

$$= \frac{1}{216}\begin{bmatrix} 252+36 \\ -36-36 \end{bmatrix}$$

$$= \frac{1}{216}\begin{bmatrix} 288 \\ -72 \end{bmatrix}$$

$$\hat{x} = \begin{bmatrix} 4/3 \\ -1/3 \end{bmatrix}$$

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Also

$$A\hat{x} = \begin{bmatrix} 1 & -2 \\ -1 & 2 \\ 0 & 3 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} 4/3 \\ -1/3 \end{bmatrix}$$

$$= \begin{bmatrix} 4/3 + 2/3 \\ -4/3 - 2/3 \\ 0 - 3/3 \\ 8/3 - 5/3 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ -2 \\ -1 \\ 1 \end{bmatrix}$$

$$b - A\hat{x} = \begin{bmatrix} 3 \\ 1 \\ -4 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 \\ -2 \\ -1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 3 \\ -3 \\ 1 \end{bmatrix}$$

$\|b - A\hat{x}\| = \sqrt{1+9+9+1} = \sqrt{20} = 2\sqrt{5}$ is the least square error associated with the least square solution.

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1. Compute the least-square error associated with the least-square solution.

$$A = \begin{bmatrix} 1 & 3 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}$$

Sol: Here,

$$A^T = \begin{bmatrix} 1 & 1 & 1 \\ 3 & -1 & 1 \end{bmatrix}$$

$$A^T.A = \begin{bmatrix} 1 & 1 & 1 \\ 3 & -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 3 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 1+1+1 & 3-1+1 \\ 3-1+1 & 9+1+1 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 3 & 3 \\ 3 & 11 \end{bmatrix}$$

$$A^T.b = \begin{bmatrix} 1 & 1 & 1 \\ 3 & -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 5+1+0 \\ 15-1+0 \end{bmatrix} = \begin{bmatrix} 6 \\ 14 \end{bmatrix}$$

The normal equation is

$$A^TA \cdot \hat{x} = A^Tb$$

$$\hat{x} = (A^TA)^{-1}(A^Tb)$$

For $(A^TA)^{-1}$

$$\det(A^TA) = (33-9) = 24$$

$$\text{Adj}(A^TA) = \begin{bmatrix} 11 & -3 \\ -3 & 3 \end{bmatrix}$$

$$(A^TA)^{-1} = \frac{1}{\det(A^TA)} \cdot \text{Adj}(A^TA)$$

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$$(A^TA)^{-1} = \frac{1}{24}\begin{bmatrix} 11 & -3 \\ -3 & 3 \end{bmatrix}$$

then

$$\hat{x} = (A^TA)^{-1} \cdot (A^Tb)$$

$$= \frac{1}{24}\begin{bmatrix} 11 & -3 \\ -3 & 3 \end{bmatrix} \cdot \begin{bmatrix} 6 \\ 14 \end{bmatrix}$$

$$= \frac{1}{24}\begin{bmatrix} 66-42 \\ -18+42 \end{bmatrix} = \frac{1}{24}\begin{bmatrix} 24 \\ 24 \end{bmatrix}$$

$$\therefore \hat{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Also,

$$A\hat{x} = \begin{bmatrix} 1 & 3 \\ 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1+3 \\ 1-1 \\ 1+1 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}$$

$$b - A\hat{x} = \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$$

$\|b - A\hat{x}\| = \sqrt{1+1+4} = \sqrt{6}$ is the the required least-square error associated with the least-squares solution

Ans

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5. Find ⓐ the Orthogonal projection of b onto col(A) and ⓑ a least-squares solution of $Aa = b$.

① $A = \begin{bmatrix} 1 & 5 \\ 3 & 1 \\ -2 & 4 \end{bmatrix}$, $b = \begin{bmatrix} 4 \\ -2 \\ -3 \end{bmatrix}$

Sol: Here,

$A = \begin{bmatrix} 1 & 5 \\ 3 & 1 \\ -2 & 4 \end{bmatrix}$

let $u_1 = (1,3,-2)$ and $u_2 = (5,1,4)$

$y = (4,-2,3)$

then,

the orthogonal projection of b on to col (A) is,

$$\hat{y} = \left(\frac{y \cdot u_1}{u_1 \cdot u_1}\right) \cdot u_1 + \left(\frac{y \cdot u_2}{u_2 \cdot u_2}\right) \cdot u_2$$

$$= \left[\frac{(4,-2,3)\cdot(1,3,-2)}{(1,3,-2)(1,3,-2)}\right] \cdot (1,3,-2) + \left[\frac{(4,-2,-3)(5,1,4)}{(5,1,4)\cdot(5,1,4)}\right] \cdot u_2$$

$$= \left[\frac{(4-6+6)}{(1+9+4)}\right] \cdot u_1 + \left[\frac{(20-2-12)}{(25+1+16)}\right] u_2$$

$$= \left[\left(\frac{4}{14}\right) u_1 + \left(\frac{6}{42}\right) \cdot u_2\right]$$

Now, For L.S.S

$A^T = \begin{bmatrix} 1 & 3 & -2 \\ 5 & 1 & 4 \end{bmatrix}$

$$A^T \cdot b = \begin{bmatrix} 1 & 3 & -2 \\ 5 & 1 & 4 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ -2 \\ -3 \end{bmatrix}$$

$$= \begin{bmatrix} 4-6+6 \\ 20-2-12 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$$

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$$A^T \cdot A = \begin{bmatrix} 1 & 3 & -2 \\ 5 & 1 & 4 \end{bmatrix} \cdot \begin{bmatrix} 1 & 5 \\ 3 & 1 \\ -2 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1+9+4 & 5+3-8 \\ 5+3-8 & 25+1+16 \end{bmatrix}$$

$$= \begin{bmatrix} 14 & 0 \\ 0 & 42 \end{bmatrix}$$

The normal equation is

$$\hat{x} = (A^T A)^{-1} \cdot A^T \cdot b$$

for $(A^T A)^{-1}$

$\det(A^T \cdot A) = 588$

$\mathrm{Adj} \cdot (A^T A) = \begin{bmatrix} 42 & 0 \\ 0 & 14 \end{bmatrix}$

$$(A^T A)^{-1} = \frac{1}{\det(A^T A)} \mathrm{Adj}.(A^T A)$$

$$= \frac{1}{588} \begin{bmatrix} 42 & 0 \\ 0 & 14 \end{bmatrix}$$

$$\hat{x} = (A^T A)^{-1} \cdot (A^T b)$$

$$= \frac{1}{588} \begin{bmatrix} 42 & 0 \\ 0 & 14 \end{bmatrix} \begin{bmatrix} 4 \\ 6 \end{bmatrix}$$

$$= \frac{1}{588} \begin{bmatrix} 168+0 \\ 0+84 \end{bmatrix}$$

$$= \frac{1}{588} \begin{bmatrix} 168 \\ 84 \end{bmatrix}$$

$$\hat{x} = \begin{bmatrix} 2/7 \\ 1/7 \end{bmatrix}$$ Ans

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(11) $A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \\ 1 & 2 \end{bmatrix}$, $b = \begin{bmatrix} 3 \\ -1 \\ 5 \end{bmatrix}$

Soln: Here,
let $u_1 = (1,-1,1)$ and $u_2 = (2,4,2)$ and
$y = (3,-1,5)$

The orthogonal projection of b on to col(A).

$$\hat{y} = \left[\left(\frac{y \cdot u_1}{u_1 \cdot u_1}\right) \cdot u_1\right] + \left[\left(\frac{y \cdot u_2}{u_2 \cdot u_2}\right) \cdot u_2\right]$$

$$\hat{y} = \left[\frac{(3,-1,5)\cdot(1,-1,1)}{(1,-1,1)\cdot(1,-1,1)}\right] \cdot u_1 + \left[\frac{(3,-1,5)\cdot(2,4,2)}{(2,4,2)(2,4,2)}\right] u_2$$

$$= \left[\left(\frac{3+1+5}{1+1+1}\right) \cdot\right] u_1 + \left[\left(\frac{6-4+10}{4+16+4}\right) \cdot u_2\right]$$

$$= \left[\left(\frac{9}{3}\right) u_1\right] + \left[\left(\frac{12}{24}\right) u_2\right]$$

$$= 3u_1 + \frac{1}{2} u_2$$

let $A^T = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 4 & 2 \end{bmatrix}$

$$A^T . A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 4 & 2 \end{bmatrix} . \begin{bmatrix} 1 & 2 \\ -1 & 4 \\ 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1+1+1 & 2-4+2 \\ 2-4+2 & 4+16+4 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 0 \\ 0 & 24 \end{bmatrix}$$

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$$A^T . b = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 4 & 2 \end{bmatrix} . \begin{bmatrix} 3 \\ -1 \\ 5 \end{bmatrix}$$

$$= \begin{bmatrix} 3+1+5 \\ 6-4+10 \end{bmatrix}$$

$$= \begin{bmatrix} 9 \\ 12 \end{bmatrix}$$

The normal equation is,

$$A^T . A . \hat{x} = A^T b$$

$$\hat{x} = (A^T A)^{-1} . A^T b .$$

For $(A^T A)^{-1}$.

$\det(A^T A) = 72.$

$$Adj(A^T A) = \begin{bmatrix} 24 & 0 \\ 0 & 3 \end{bmatrix}$$

$$(A^T A)^{-1} = \frac{1}{\det(A^T A)} . Adj(A^T A)$$

$$= \frac{1}{72} \begin{bmatrix} 24 & 0 \\ 0 & 3 \end{bmatrix}$$

then,

$$\hat{x} = (A^T A)^{-1} . A^T b$$

$$= \frac{1}{72} \begin{bmatrix} 24 & 0 \\ 0 & 3 \end{bmatrix} . \begin{bmatrix} 9 \\ 12 \end{bmatrix}$$

$$= \frac{1}{72} \begin{bmatrix} 216+0 \\ 0+36 \end{bmatrix} = \frac{1}{72} \begin{bmatrix} 216 \\ 36 \end{bmatrix}$$

$$\hat{x} = \begin{bmatrix} 3 \\ 1/2 \end{bmatrix}$$ Ans

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(iii) $A = \begin{bmatrix} 4 & 0 & 1 \\ 1 & -5 & 1 \\ 6 & 1 & 0 \\ 1 & -1 & -5 \end{bmatrix}, \quad b = \begin{bmatrix} 9 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

Sol: Here,

let $u_1 = (4,1,6,1)$, $u_2 = (0,-5,1,-1)$, $u_3 = (1,1,0,-5)$

and $y = (9,0,0,0)$

Now,

The orthogonal projection of b on to col A

$$\hat{y} = \left[\left(\frac{y \cdot u_1}{u_1 \cdot u_1}\right) \cdot u_1\right] + \left[\left(\frac{y \cdot u_2}{u_2 \cdot u_2}\right) \cdot u_2\right] + \left[\left(\frac{y \cdot u_3}{u_3 \cdot u_3}\right) \cdot u_3\right]$$

$$\hat{y} = \left[\left(\frac{(9,0,0,0) \cdot (4,1,6,1)}{(4,1,6,1) \cdot (4,1,6,1)}\right) \cdot u_1\right] + \left[\left(\frac{(9,0,0,0) \cdot (0,-5,1,-1)}{(0,-5,1,-1) \cdot (0,-5,1,-1)}\right) u_2\right]$$

$$+ \left[\frac{(9,0,0,0) \cdot (1,1,0,-5)}{(1,1,0,-5) \cdot (1,1,0,-5)}\right] \cdot u_3$$

$$= \left[\left(\frac{36+0+0+0}{16+1+36+1}\right) u_1\right] + \left[\left(\frac{0}{27}\right) u_2\right] + \left[\left(\frac{9}{27}\right) u_3\right]$$

$$= \left(\frac{36}{54}\right) \cdot u_1 + 0 \cdot u_2 + \frac{1}{3} u_3$$

$$\frac{2}{3} u_1 + \frac{1}{3} u_3.$$

$$A^T = \begin{bmatrix} 4 & 1 & 6 & 1 \\ 0 & -5 & 1 & -1 \\ 1 & 1 & 0 & -5 \end{bmatrix}$$

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$R_2 \to R_2 + 2R_3$

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & -5 & 9 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 4 & 0 \end{array}\right] \xrightarrow{R_1 \to R_1 - R_2} \begin{bmatrix} 1 & 0 & 0 & -6 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

then,

$x_1 + \cancel{x_2} - 5x_4 = 9$, $\quad x_2 + x_4 = 0$

$x_3 + 4x_4 = 0$

$$A^T A = \begin{bmatrix} 4 & 1 & 6 & 1 \\ 0 & -5 & 1 & -1 \\ 1 & 1 & 0 & -5 \end{bmatrix} \cdot \begin{bmatrix} 4 & 0 & 1 \\ 1 & -5 & 1 \\ 6 & 1 & 0 \\ 1 & -1 & -5 \end{bmatrix}$$

$$= \begin{bmatrix} 16+1+36+1 & 0-5+6-1 & 4+1+0-5 \\ 0-5+6-1 & 0+25+1+1 & 0-5+0+5 \\ 4+1+0-5 & 0-5+0+5 & 1+1+0+25 \end{bmatrix}$$

$$= \begin{bmatrix} 54 & 0 & 0 \\ 0 & 27 & 0 \\ 0 & 0 & 27 \end{bmatrix}$$

$$A^T b = \begin{bmatrix} 4 & 1 & 1 \\ 0 & -5 & -1 \\ 1 & 1 & -5 \end{bmatrix} \cdot \begin{bmatrix} 9 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 36 \\ 0 \\ 9 \end{bmatrix}$$

The normal equation is

$$A^T A \cdot \hat{x} = A^T b$$

$$\begin{bmatrix} 54 & 0 & 0 \\ 0 & 27 & 0 \\ 0 & 0 & 27 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 36 \\ 0 \\ 9 \end{bmatrix}$$

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For Solving $\hat{x}$

The augmented matrix is $\left[\begin{array}{ccc:c} 54 & 0 & 0 & 36 \\ 0 & 27 & 0 & 0 \\ 0 & 0 & 27 & 9 \end{array}\right]$

$R_1 \to \frac{1}{54} R_1$, $R_2 \to \frac{1}{27} R_2$ and $R_3 \to \frac{1}{27} R_3$

$$\left[\begin{array}{ccc:c} 1 & 0 & 0 & 2/3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1/3 \end{array}\right]$$

$\therefore$

$x_1 = 2/3$

$x_2 = 0$

$x_3 = 1/3$

$\therefore \hat{x} = \begin{bmatrix} 2/3 \\ 0 \\ 1/3 \end{bmatrix}$ Ans

(iv) $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \\ -1 & 1 & -1 \end{bmatrix}$, $b = \begin{bmatrix} 2 \\ 5 \\ 6 \\ 6 \end{bmatrix}$

Sol: Here

let $u_1 = (1,1,0,-1)$, $u_2 = (1,0,1,1)$ and $u_3 = (0,-1,1,-1)$

and $y = (2,5,6,6)$

The orthogonal projection of b on col(A) is

$$\hat{y} = \left[\left(\frac{y \cdot u_1}{u_1 \cdot u_1}\right) u_1\right] + \left[\left(\frac{y \cdot u_2}{u_2 \cdot u_2}\right) \cdot u_2\right] + \left[\left(\frac{y \cdot u_3}{u_3 \cdot u_3}\right) \cdot u_3\right]$$

$$= \left[\frac{(2,5,6,6)\cdot(1,1,0,-1)}{(1,1,0,-1)(1,1,0,-1)} \cdot u_1\right] + \left[\frac{(2,5,6,6)(1,0,1,1)}{(1,0,1,1)\cdot(1,0,1,1)} \cdot u_2\right] + \left[\frac{(2,5,6,6)(0,-1,1,-1)}{(0,-1,1,-1)\cdot(0,-1,1,-1)} \cdot u_3\right]$$

$$= \left[\left(\frac{2+5+0-6}{1+1+0+1}\right) \cdot u_1\right] + \left[\left(\frac{2+0+6+6}{1+0+1+1}\right) \cdot u_2\right] + \left[\left(\frac{0-5+6-6}{0+1+1+1}\right) u_3\right]$$

$$= \frac{1}{3} u_1 + \frac{14}{3} u_2 + \frac{-5}{3} u_3$$

$$= \frac{1}{3} u_1 + \frac{14}{3} u_2 - \frac{5}{3} u_3 \text{ Ans}$$

$A^T = \begin{bmatrix} 1 & 1 & 0 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & -1 & 1 & -1 \end{bmatrix}$

$A^T A = \begin{bmatrix} 1 & 1 & 0 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & -1 & 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \\ -1 & 1 & -1 \end{bmatrix}$

$$= \begin{bmatrix} 1+1+0+1 & 1+0+0-1 & 0-1+0+1 \\ 1+0+0-1 & 1+0+1+1 & 0+0+1-1 \\ 0-1+0+1 & 0+0+1-1 & 0+1+1+1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$A^T \cdot b = \begin{bmatrix} 1 & 1 & 0 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & -1 & 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 5 \\ 6 \\ 6 \end{bmatrix}$

$$= \begin{bmatrix} 2+5+0-6 \\ 2+0+6+6 \\ 0-5+6-6 \end{bmatrix} = \begin{bmatrix} 1 \\ 14 \\ -5 \end{bmatrix}$$

The normal equation is,

$$A^T A \cdot \hat{x} = A^T b$$

For solving $\hat{x}$

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$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 14 \\ -5 \end{bmatrix}$$

For solving $\hat{x}$, the augmented-matrix is

$$\left[\begin{array}{ccc|c} 3 & 0 & 0 & 1 \\ 0 & 3 & 0 & 14 \\ 0 & 0 & 3 & -5 \end{array}\right]$$

$R_1 \to \frac{1}{3}R_1$, $\frac{1}{3}R_2$, and $R_3 \to \frac{1}{3}R_3$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1/3 \\ 0 & 1 & 0 & 14/3 \\ 0 & 0 & 1 & -5/3 \end{array}\right]$$

$$\therefore x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 14/3 \\ -5/3 \end{bmatrix}$$ Ans

6. let $A = \begin{bmatrix} 3 & 4 \\ -2 & 1 \\ 3 & 4 \end{bmatrix}$, $b = \begin{bmatrix} 11 \\ -9 \\ 5 \end{bmatrix}$, $u = \begin{bmatrix} 5 \\ -1 \end{bmatrix}$

and $v = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$. compute Au and Av,

and compare them with b. could u possibly be a least-squares solution of $A\hat{x} = b$?

sol: Here,

$A = \begin{bmatrix} 3 & 4 \\ -2 & 1 \\ 3 & 4 \end{bmatrix}$, $b = \begin{bmatrix} 11 \\ -9 \\ 5 \end{bmatrix}$

$u = \begin{bmatrix} 5 \\ -1 \end{bmatrix}$, $v = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$

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$$Au = \begin{bmatrix} 3 & 4 \\ -2 & 1 \\ 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 15-4 \\ -10-1 \\ 15-4 \end{bmatrix}$$

$$= \begin{bmatrix} 11 \\ -11 \\ 11 \end{bmatrix}$$

$$Av = \begin{bmatrix} 3 & 4 \\ -2 & 1 \\ 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ -2 \end{bmatrix}$$

$$= \begin{bmatrix} 15-8 \\ -10-2 \\ 15-8 \end{bmatrix}$$

$$= \begin{bmatrix} 7 \\ -12 \\ 7 \end{bmatrix}$$

Here $Au \neq Av$ so,

$\|Au\| = \sqrt{11^2+11^2+11^2} = \sqrt{121+121+121} = \sqrt{363}$

$\|Av\| = \sqrt{7^2+12^2+7^2} = \sqrt{49+144+49} = \sqrt{242}$

This shows $\|Au\| > \|Av\|$. Therefore v is the least square solution of $Ax=b$.

Exercise :- 8.6

The least square lines :-

Suppose β_0 and β_1 are fixed and consider the equation

$y = \beta_0 + \beta_1 x$ is called least square line. It is also known as line of regression of y on x and β_0 and β_1 are the regression coefficient.

Example :-

Find the equation of least square line that best fit the data point $(2,1)$, $(5,2)$, $(7,3)$ and $(8,3)$.

Sol^n:

let the equation be

$$y = \beta_0 + \beta_1 x \quad -① $$

At point $(2,1)$, $(5,2)$, $(7,3)$ and $(8,3)$, ① gives.

$$\left.\begin{aligned} 1 &= \beta_0 + 2\beta_1 \\ 2 &= \beta_0 + 5\beta_1 \\ 3 &= \beta_0 + 7\beta_1 \\ 3 &= \beta_0 + 8\beta_1 \end{aligned}\right\} -②$$

The matrix form of ②

$$Y = X\beta \quad -③$$

where

$$y = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 3 \end{bmatrix}, \quad X = \begin{bmatrix} 1 & 2 \\ 1 & 5 \\ 1 & 7 \\ 1 & 8 \end{bmatrix}, \quad \beta = \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix}.$$

The normal equation of ③ is

$$X^T X \beta = X^T y$$

$$\beta = (X^T X)^{-1} . X^T y$$

Here,

$$(X^T X) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 5 & 7 & 8 \end{bmatrix} . \begin{bmatrix} 1 & 2 \\ 1 & 5 \\ 1 & 7 \\ 1 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} 1+1+1+1 & 2+5+7+8 \\ 2+5+7+8 & 4+25+49+64 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 22 \\ 22 & 142 \end{bmatrix}$$

$$|X^T X| = 4 \times 142 - 22 \times 22$$

$$= 84.$$

$$\text{adj}(X^T X) = \begin{bmatrix} 142 & -22 \\ -22 & 4 \end{bmatrix}$$

$$\therefore (X^T X)^{-1} = \frac{1}{|X^T X|} \text{adj}(X^T X)$$

$$= \frac{1}{84} \begin{bmatrix} 142 & -22 \\ -22 & 4 \end{bmatrix}$$

and

$$X^T . y = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 5 & 7 & 8 \end{bmatrix} . \begin{bmatrix} 1 \\ 2 \\ 3 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1+2+3+3 \\ 2+10+21+24 \end{bmatrix} = \begin{bmatrix} 9 \\ 57 \end{bmatrix}$$

$$\beta = (X^T X)^{-1} . X^T y$$

$$= \frac{1}{84} \begin{bmatrix} 142 & -22 \\ -22 & 4 \end{bmatrix} . \begin{bmatrix} 9 \\ 57 \end{bmatrix}$$

$$= \frac{1}{84} \begin{bmatrix} 1278 - 1254 \\ -198 + 288 \end{bmatrix} = \frac{1}{84} \begin{bmatrix} 24 \\ 30 \end{bmatrix} = \begin{bmatrix} 2/7 \\ 5/14 \end{bmatrix}$$

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$\beta_0 = 2/7 , \beta_1 = 5/14$

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