CHAPTER-8.

INFINITE SEQUENCE AND SERIES.

Sequence:-

The arrangement of number with same fixed order is called sequence.

$1, 2, 3, 4, \ldots\ldots n$

$2, 4, 6, 8, \ldots\ldots 2n$

$1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \ldots \frac{1}{3^n}$

Convergent and Divergent of a Sequence.

An infinite sequence an is convergent if

$\lim_{n \to \infty} a_n = l$

= some fixed no.

and divergent if $\lim_{n \to \infty} a_n = \infty$ or,

no. fixed no.

Exercise base example.

Exercise → 8.1 Not important as exam point of view.

Exercise :- 8.2

Geometric series :-

$$a + ar + ar^2 + ar^3 + \dots + ar^{n-1} + \dots \sum_{n=1}^{\infty} ar^{n-1}$$

general term

The geometric series
$a + ar + ar^2 + ar^3 + \dots$ are convergent

if $|r| < 1$ and divergent if $|r| \geq 1$
and sum $= S_\infty = \frac{a}{1-r}$

Infinite series :-

The expression :-

$$a_1 + a_2 + a_3 + a_4 + \dots a_n + \dots = \sum_{n=1}^{\infty} a_n$$

is called infinite series

partial sum :-

$$S_k = \sum_{n=1}^{K} a_K$$

Convergence & divergence of infinite series.

If $\lim_{n \to \infty} a_n = L$ (some fixed number)

Then the series is convergent otherwise divergent

Telescopping series :-

A series in which ~~one~~ on the expansion if nth partial sum every term except first and last term is cancelled ~~and~~ is called telescoping series.

Exercise :- 8.2 :

Test the Convergency of the following infinite Series. If Convergent find the Sum.

① Applying geometric Series test.

(a) $\sum_{n=0}^{\infty}\left(\frac{(-1)^n \cdot 2}{3^n \cdot 5}\right)$

Sol^n: Here

$\sum_{n=0}^{\infty}\left(\frac{(-1)^n \cdot 2}{3^n \cdot 5}\right) = \frac{(-1)^0}{3^0 \cdot 5} \cdot 2 + \frac{(-1)^1}{3^1} \cdot \frac{2}{5} + \frac{(-1)^2}{3^2} \cdot \frac{2}{5} + \dots \infty$

$\sum_{n=0}^{\infty}\left(\frac{-1^n \cdot 2}{3^n \cdot 5}\right) = \frac{2}{5} - \frac{2}{15} + \frac{2}{45} - \frac{2}{135} \dots \infty$

Here $a = \frac{2}{5}$ and $r = -\frac{1}{3}$ which less than 1

$\therefore -\frac{1}{3} < 1$

So that series are convergent,

$S_\infty = \frac{a}{1-r} = \frac{2/5}{1+\frac{1}{3}} = \frac{2/5}{4/3} = \frac{2}{5} \times \frac{3}{4}$

$= \frac{3}{10}$ Ans

(b) $\sum_{n=0}^{\infty}\left(\frac{1}{\sqrt{2}}\right)^n$

Sol^n: Here,

$\sum_{n=0}^{\infty} = \left(\frac{1}{\sqrt{2}}\right)^0 + \left(\frac{1}{\sqrt{2}}\right)^1 + \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^3 + \dots \infty$

$\sum_{n=0}^{\infty} = 1 + \frac{1}{\sqrt{2}} + \frac{1}{2} + \frac{1}{2\sqrt{2}} + \dots \infty$

Here $a = 1$ and $r = \frac{1}{\sqrt{2}} < 1$

So that the series are convergent,

$S_\infty = \frac{a}{1-r} = \frac{1}{1-\frac{1}{\sqrt{2}}} = \frac{1}{\sqrt{2}-1} \times \sqrt{2} = \frac{\sqrt{2}}{\sqrt{2}-1}$

(c) $\sum_{n=2}^{\infty}(\sqrt{3})^n$

Sol^n: here,

$\sum_{n=2}^{\infty} = (\sqrt{3})^2 + (\sqrt{3})^3 + (\sqrt{3})^4 + (\sqrt{3})^5 + \dots \infty$

$\sum_{n=2}^{\infty} = 3 + 3\sqrt{3} + 9 + 9\sqrt{3}$

Here, $a = 3$ and $r = \sqrt{3} > 1$

the series are divergent, by the ~~test~~ geometric test ratio theorem

$S_\infty = \frac{a}{1-r} = \frac{3}{1-\sqrt{3}}$

(d) $\sum_{n=0}^{\infty} \frac{2^n+5}{3^n}$

Sol: Here,

$\sum_{n=0}^{\infty} = \frac{2^n}{3^n} + \frac{5}{3^n} = \left(\frac{2}{3}\right)^n + \frac{5}{3^n}$

$\sum_{n=0}^{\infty} = \left(\left(\frac{2}{3}\right)^0 + \frac{5}{3^0}\right) + \left(\left(\frac{2}{3}\right)^1 + \frac{5}{3}\right) + \left(\left(\frac{2}{3}\right)^2 + \frac{5}{9}\right) + \left(\left(\frac{2}{3}\right)^3 + \frac{5}{27}\right) + \cdots \infty$

$\sum_{n=0}^{\infty} = \cancel{1 + \frac{2}{3}} +$

$\sum_{n=0}^{\infty} = \left(\frac{2}{3}\right)^n + \frac{5}{3^n}$

~~So S = 2/3~~

$\sum_{n=0}^{\infty} = S_\infty = \left(\frac{2}{3}\right)^n$ & $\to S_1\infty$

$\sum_{n=0}^{\infty} = S_\infty = \frac{5}{3^n}$ and $\to S_2\infty$

$\sum_{n=0}^{\infty} = 1 + \frac{2}{3} + \frac{4}{9} + \frac{8}{27}$

Here $a=1$, and $r = \frac{2}{3} < 1$,

So that this series is convergent like that,

$\frac{5}{3} \cdot \frac{5}{5} = \frac{5}{3} \times \frac{1}{5}$

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$\sum_{n=0}^{\infty} = \frac{5}{3^0} + \frac{5}{3^1} + \frac{5}{3^2} + \frac{5}{3^3} + \cdots \infty$

$\sum_{n=0}^{\infty} = 5 + \frac{5}{3} + \frac{5}{9} + \frac{5}{27}$

Here $a=5$ and $r = \frac{1}{3} < 1$,

the series is convergent as

$S_\infty = S_{1\infty} + S_{2\infty}$

$S_\infty = \frac{a}{1-r} + \frac{a}{1-r}$

$= \frac{1}{1-\frac{2}{3}} + \frac{5}{1-\frac{1}{3}}$

$= \frac{3}{1} + \frac{5 \cdot 3}{2}$

$= 3 + \frac{15}{2}$

$= \frac{21}{2}$ Ans

(e) $\sum_{n=0}^{\infty}\left(\frac{1}{2^n}+\left(\frac{-1}{5}\right)^n\right)$

Sol^n Here

$\sum_{n=0}^{\infty} = \left(\frac{1}{2^n}+\left(\frac{-1}{5}\right)^n\right)$

$S_1 + S_2$

For S_1

$S_1 = \sum_{n=0}^{\infty} = \frac{1}{2^0}+\frac{1}{2^1}+\frac{1}{2^2}+\frac{1}{2^3}+\dots\infty$

$= 1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\dots\infty$

Here $a=1$, $r=\frac{1}{2}<1$,

Here the S_1 series is convergent,

$S_1 = \frac{a}{1-r} = \frac{1}{1-\frac{1}{2}} = \frac{2}{1} = 2$

Similarly for S_2

$S_2 = \sum_{n=0}^{\infty} = \left(\frac{-1}{5}\right)^0+\left(\frac{-1}{5}\right)^1+\left(\frac{-1}{5}\right)^2+\left(\frac{-1}{5}\right)^3+\dots\infty$

$S_2 = \sum_{n=0}^{\infty} = 1-\frac{1}{5}+\frac{1}{25}-\frac{1}{125}+\dots\infty$

Here $a=1$ and $r=-\frac{1}{5}<1$,

the Series S_2 is convergent,

$S_2 = \frac{1}{1+\frac{1}{5}} = \frac{5}{6}$

Then,

$S_\infty = S_1+S_2$

$S_\infty = 2+\frac{5}{6}$

$= \frac{17}{6}$ Ans

(f) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{3}{2^n}$

Sol^n: Here,

$\sum_{n=1}^{\infty} = (-1)^2\frac{3}{2^1}+(-1)^3\frac{3}{2^2}+(-1)^4\frac{3}{2^3}+(-1)^5\frac{3}{2^4}+\dots\infty$

$\sum_{n=1}^{\infty} = \frac{3}{2}-\frac{3}{4}+\frac{3}{8}-\frac{3}{16}+\dots+\infty$

Here $a=\frac{3}{2}$ and $r=-\frac{1}{2}<1$,

Here the Series are Convergent,

$S_\infty = \frac{a}{1-r} = \frac{\frac{3}{2}}{1+\frac{1}{2}} = \frac{3}{2}\times\frac{2}{3} = 1$

(g) $\sum_{n=0}^{\infty} \cos(n\pi)$

Sol: here.

$\sum_{n=0}^{\infty} = \cos(0\pi) + \cos\pi + \cos2\pi + \cos3\pi + \cos4\pi + \dots \infty$

$\sum_{n=0}^{\infty} = 1+1+1+1+1 \dots + \infty$

Here $a=1$, and $r = |1| \geq 1$,

Here the series are divergent.

(h) $\sum_{n=0}^{\infty} e^{-2n}$

Sol: here,

$\sum_{n=0}^{\infty} e^{-2n} = 1 + e^{-2} + e^{-4} + e^{-6} + e^{-8} + \dots$ to ∞

Here $a=1$ and $r = \frac{e^{-4}}{e^{-2}} = e^{-2} = \frac{1}{e^2} = \frac{1}{7.4} < 1$

$\therefore \frac{1}{7.4} < 1$,

Series are convergent

$S_\infty = \frac{a}{1-r} = \frac{1}{1-\frac{1}{e^2}} = \frac{e^2}{e^2-1}$

(i) $\sum_{n=1}^{\infty} \frac{2}{10^n}$

Sol: Here,

$\sum_{n=1}^{\infty} = \frac{2}{10} + \frac{2}{100} + \frac{2}{1000} + \frac{2}{10000} + \dots + \infty$

$= \frac{1}{5} + \frac{1}{50} + \frac{1}{5000} + \frac{1}{50000} + \dots + \infty$

Here $a = \frac{1}{5}$ and $r = \frac{1}{10} < 1$

the series is convergent

$S_\infty = \frac{a}{1-r} = \frac{\frac{1}{5}}{1-\frac{1}{10}} = \frac{\frac{1}{5}}{\frac{9}{10}} = \frac{1}{5} \times \frac{10}{9} = \frac{2}{9}$

(j) $\sum_{n=0}^{\infty} \frac{2^n - 1}{3^n}$

Sol: here,

$\sum_{n=0}^{\infty} = \left(\frac{2}{3}\right)^n - \frac{1}{3^n}$

$\sum_{n=0}^{\infty} = S_1 - S_2$

Here,

$S_1 = \sum_{n=0}^{\infty} = 1 + \frac{2}{3} + \frac{4}{9} + \frac{8}{27} + \dots \infty$

Here $a = 1$, $r = \frac{2}{3} < 1$,

the series S_1 is convergent

$S_{1\infty} = \frac{a}{1-r} = \frac{1}{1-\frac{2}{3}} = 3$

For S_2 ∞

$S_2 = \sum_{n=0} = \frac{1}{3^n}$

$S_2 = \sum_{n=0}^{\infty} = \frac{1}{1} + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \dots \infty$

Here $a = 1$ and $r = \frac{1}{3} < 1$.

the series is Convergent

$S_2 = \frac{a}{1-r} = \frac{1}{1-\frac{1}{3}} = \frac{3}{2}$

then

$S_\infty = S_1 + S_2 = 3 - \frac{3}{2} = \frac{6-3}{2} = \frac{6-3}{2} = \frac{3}{2}$ Ans

(3) Show that the following series are divergent.

If

$\lim_{n\to\infty} a_n = 0 \rightarrow$ that is Convergent,

but

$\lim_{n\to\infty} a_n \neq 0 \rightarrow$ that is divergent.

According to the Divergent test theorem. Contra positive of n^{th} term test

(a) $\sum_{n=1}^{\infty} n^2$

$\lim_{n\to\infty} \cdot n^2$

$\lim_{n\to\infty} = n^2$

∞^2

∞ divergent.

(b) $\sum_{K=1}^{\infty} \frac{K+1}{K}$

sol^n: here

$\sum_{K=1}^{\infty} \frac{K+1}{K}$

$\lim_{K\to\infty} \frac{K+1}{K}$

$\lim_{K\to\infty} 1 + \frac{1}{K}$

$1 + \frac{1}{\infty}$

$1 \neq 0$, so that the series is divergent.

(c) $\sum_{n=1}^{\infty} \frac{-n}{2n+5}$

Sol^n: Here,

$\sum_{n=1}^{\infty} \frac{-n}{2n+5}$

$\lim_{n\to\infty} \frac{-n}{2n+5}$

$\lim_{n\to\infty} \frac{-1}{\frac{2n}{n}+5/n}$

$\lim_{n\to\infty} \frac{-1}{2+5/n}$

$\frac{-1}{2+\frac{5}{\infty}}$

$-1/2 \neq 0$ So that this is divergent.

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(3) Show that following series are divergent.

(a) $\sum_{n=1}^{\infty} n^2$

Sol^n: here,

$\sum_{n=1}^{\infty} n^2 =$ ~~1+4+9+16~~

$=$ ~~$1^2+2^2+3^3+4^4 \ldots n$~~

$1+4+9+16+25+\ldots$

$1^2+2^2+3^2+4^2+5^2$

Here $a=1$, and $r = 2/1 = 2 > 1$, then it is divergent, according to GST

(b) $\sum_{K=1}^{\infty} \frac{K+1}{K}$

Sol^n: here,

$\sum_{K=1}^{\infty} = \frac{1+1}{1} + \frac{2+1}{2} + \frac{3+1}{3} + \frac{4+1}{4} + \ldots$

$= \frac{2}{1} + 3/2 + 4/3 + 5/4$

Exercise:- 8.3 :

The n^{th} test for convergent and divergent.
If $\lim_{n \to \infty} a_n = 0$, then the series is converge
& if $\lim_{n \to \infty} a_n \neq 0$, then the series is divergent.

Integral test.
The series $\sum_{n=1}^{\infty} a_n$ is convergent if and only
If the improper integral $\int_1^{\infty} f(x)\, dx$ is convergent.

The p- series.
If $f(x) = \frac{1}{x^p}$, then the series $\sum f(x)$ is at the
form $\frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \dots + \frac{1}{n^p}$ is

called p-series.

e.g.
$1 + 1 + 1 + \dots = \sum \frac{1}{n^0}$ is p-series with $p=0$

$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots = \sum \frac{1}{n^1}$ is with $p=1$.

$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \dots = \sum \frac{1}{\sqrt{n}}$ with $p = \frac{1}{2}$

$1+2+3+4+\dots = \sum \frac{1}{n^{-1}}$ with $p = -1$.

convergent of p- series,
The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent when $p > 1$

and divergent if $p \leq 1$

Exercise:- 8.3

1. Test the convergence of series by integral part. test

(a) $\sum_{n=1}^{\infty} 1/n^3$

Sol^n: here,

$a_n = \frac{1}{n^3}$, we take $f(x) = 1/x^3$

$$\sum_{n=1}^{\infty} \frac{1}{n^3} = \int_1^{\infty} 1/x^3 \, dx$$

$$\int_1^{\infty} x^{-3} \, d(x)$$

$$\left[\frac{x^{-3+1}}{-3+1}\right]_1^{\infty}$$

$$-\frac{1}{2}\left[\frac{1}{\infty} - \frac{1}{1}\right]$$

$$-\frac{1}{2}[0-1]$$

$$\frac{1}{2} \text{ Ans}$$

Therefore, ~~The series converge at~~ $\frac{1}{2}$

Hence by Integral test the given series is convergent

$(\ln x)_2^{\infty}$

(b) $\sum_{n=2}^{\infty} \frac{\ln n}{n}$

Sol^n: here,

$a_n = \frac{\ln n}{n}$, we take $f(x) = \frac{\ln x}{x}$

$$\sum_{n=2}^{\infty} \frac{\ln n}{n} = \int_2^{\infty} \frac{\ln x}{x} \, dx$$

$t = \log x$

$\frac{dt}{dx} = \frac{1}{x}$

$dt = \frac{1}{x} dx$

$n \to \infty$,

$t = \log n$

$$\int_{\log 2}^{\log n} t \, dt$$

$$\int_{\log 2}^{\log n} t \cdot dt$$

$$\left[\frac{t^2}{2}\right]_{\log 2}^{\log n}$$

$$\frac{1}{2}\left[(\log n)^2 - \log 2^2\right]$$

$$\frac{1}{2}\left[\infty - 0.0906\right]$$

∞ diverge

Hence ~~the~~ by the Integral test the given series is divergent

(c) $\sum_{n=1}^{\infty} \frac{e^n}{1+e^{2n}}$

Sol^n: here,

$a_n = \frac{e^n}{1+e^{2n}}$ we take $f(x) = \frac{e^x}{e^{2x}+1}$

$\sum_{n=1}^{\infty} \frac{e^n}{1+e^{2n}} = \int_1^{\infty} \frac{e^x}{e^{2x}+1} dx$

$= \int_1^{\infty} \frac{e^x}{(e^x)^2+1} dx \qquad \lim_{h \to \infty}$

let $t = e^x$ | when $x = 1$, $t = e^1$

$\frac{dt}{dx} = e^x$ | when $x = h$, $t = e^h$

$dt = e^x \cdot dx$

$\int_e^{e^h} \frac{dt}{t^2+1}$

$\left[\tan^{-1}(t)\right]_e^{e^h}$

$\tan^{-1}(e^h) - \tan^{-1}(e)$

$\tan^{-1}(\infty) - \tan^{-1}(e)$

$\frac{\pi}{2}$ - something,

converge.

Hence, by the Integral test the series is converged.

(d) $\sum_{n=1}^{\infty} \frac{8\tan^{-1}n}{1+n^2}$

Sol^n: here,

$a_n = \frac{8\tan^{-1}n}{1+n^2}$ we take $f(x) = \frac{8\tan^{-1}x}{1+x^2}$

$\sum_{n=1}^{\infty} \frac{8\tan^{-1}x}{1+n^2} = \int_1^{\infty} \frac{8\tan^{-1}x}{1+x^2} . dx$

$8\int_1^{\infty} \frac{\tan^{-1}x}{1+x^2} dx$ let $t = \tan^{-1}x$

$\frac{dt}{dx} = \frac{1}{1+x^2}$

$\int_{\pi/4}^{\pi/2} t \cdot dt$ $dt = \frac{dx}{1+x^2}$

when,

$x = 1$ $t = \pi/4$

$x = \infty$ $t = \pi/2$

$\left[\frac{t^2}{2}\right]_{\pi/4}^{\pi/2}$

$\frac{1}{2}\left[\left(\frac{\pi}{2}\right)^2 - \left(\frac{\pi}{4}\right)^2\right]$

$\frac{1}{2}\left[\frac{\pi^2}{4} - \frac{\pi^2}{16}\right]$

$\frac{4\pi^2 - \pi^2}{16 \times 2} = \frac{3\pi^2}{32}$ converge,

Here by the Integral test the series is converged.

(e) $\sum_{n=1}^{\infty} \frac{n^2}{n^3+1}$

Sol^n: Here,

$a_n = \frac{n^2}{n^3+1}$ We take $f(x) = \frac{x^2}{x^3+1}$

$\sum_{n=1}^{\infty} \frac{n^2}{n^3+1} = \int_1^{\infty} \frac{x^2}{x^3+1}\, dx$

$\int_1^{\infty} \frac{x^2}{x^3+1}\, dx$

let $t = x^3+1$

$\frac{dt}{dx} = 3x^2$

$\frac{1}{2} dt = dx\, x^2$

when $x = 1$
$t = 2$
when $x = n$
$t = n^3+1$

$\frac{1}{2} \int_2^{n^3+1} \frac{dt}{t}$

$\frac{1}{2} [\log t]_2^{n^3+1}$

$\frac{1}{2} [\log(n^3+1) - \log 2]$

$\frac{1}{2} [\log(\infty) - \log 2]$

∞ diverge

Here by the Integral test the series is divergent.

(f) $\sum_{n=1}^{\infty} \frac{1}{n^2+4}$

Sol^n: here,

$a_n = \frac{1}{n^2+4}$ we take $f(x) = \frac{1}{x^2+4}$

$\sum_{n=1}^{\infty} \frac{1}{n^2+4} = \int_1^{\infty} f(x)\, dx$

$= \int_1^{\infty} \frac{1}{x^2+4}\, dx$

let $t = x/2$
$\frac{dt}{dx} = \frac{1}{2}$
$\frac{1}{2} dt = dx$
when $n = 1, t = 1/2$
when $x = h$, $t = h/2$

$= \frac{1}{4} \int_1^{\infty} \frac{1}{(\frac{x^2}{4}+1)}\, dx$

$= \frac{1}{4} \int_1^{\infty} \frac{1}{[(x/2)^2+1]}\, dx$

$= \frac{1}{4} \int_{1/2}^{h/2} \frac{2\, dt}{[t^2+1]}$

$\frac{1}{4} \times 2 \int_{1/2}^{h/2} \frac{dt}{\tan^2 t + 1}$

$\frac{1}{2} [\tan^{-1}(\tan t)]_{1/2}^{h/2}$

$\frac{1}{8} [\tan^{-1}(\tan\infty) - \tan^{-1}(\tan 1/2)]$

$\frac{1}{8} [\tan^{-1}(\infty) - \tan^{-1}(0.00785)]$

$\frac{1}{8} [\pi/2 - 0.5]$

$\frac{1}{8} [\frac{\pi - 1}{2}] = \frac{\pi - 1}{16}$ which is converge.

$\frac{1}{2} [(\tan^{-1} t)]_{1/2}^{h/2}$

$\frac{1}{2} [\tan^{-1}(h/2) - \tan^{-1}(1/2)]$

$\frac{1}{2} [\pi/2 - \pi/6]$

$\frac{5\pi}{12}$ Ans

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(g) $\sum_{n=1}^{\infty} \frac{1}{n(\ln n)^2}$

sol^n: here,

$a_n = \frac{1}{n(\ln n)^2}$ we take $f(x) = \frac{1}{x(\log x)^2}$

$\sum_{n=1}^{\infty} \frac{1}{n(\ln n)^2} = \int_1^{\infty} \frac{1}{x} \frac{1}{(\log x)^2} dx$

$h \lim \infty$,

$\sum_{n=1}^{\infty} = \int_1^{h} \frac{1}{x} \cdot \frac{1}{(\log x)^2} dx$

let $t = \log x$

$\frac{dt}{dx} = \frac{1}{x}$

$dt = \frac{1}{x} dx$

when $x=1$, $t=0$

when $x=h$, $t=\log h$

$\int_0^{\log h} dt \frac{1}{t^2}$

$\left[\frac{t^{-2+1}}{-2+1}\right]_0^{\log h}$

$-\left[\frac{1}{t}\right]_0^{\log h}$

$-\left[\frac{1}{\log h} - \frac{1}{0}\right]$

$-\left[\frac{1}{\infty} - \frac{1}{0}\right]$ $[0-\infty] = \infty$

∞ diverge Ans

(h) $\sum_{n=1}^{\infty} n^2 e^{-n^3}$

sol^n: here,

$\sum_{n=1}^{\infty} n^2 e^{-n^3}$

$a_n = n^2 . e^{-n^3}$ we take $f(x) = x^2 . e^{-x^3}$

$\sum_{n=1}^{\infty} n^2 e^{-n^3} = \int_1^{\infty} x^2 . e^{-x^3} . dx$

let $t = e^{-x^3}$

$\frac{dt}{dx} = e^{-x^3} \times -3x^2$

$dt \times \frac{1}{e^{-x^3}} = dx(-3x^2)$

$= \frac{-1}{3} \int_1^{\infty} -3x^2 . dx . t$

~~$\frac{-1}{3} \int_1^{\infty} \frac{dt}{t} \times t$~~

when $x=1$, $t=e^{-1}$

$t = 1/e$

$x=h$, $t=e^{-h^3}$

$= \frac{-1}{3} \int_{1/e}^{e^{-h^3}} t . \frac{dt}{t}$

$\int_{1/e}^{e^{-h^3}} dt$.

$\left[\frac{t^2}{2}\right]_{e^{-1}}^{e^{-h^3}}$

$\frac{1}{2}\left[e^{-2h^3} - e^{-2}\right]$

$\frac{1}{2}\left[\frac{1}{e^{2h^3}} - \frac{1}{e^2}\right]$

$\frac{1}{2}\left[\frac{1}{\infty} - \frac{1}{e^2}\right]$

$\frac{-1}{2e^2}$ converge.

Direct comparision test:

let $\sum a_n$ be a series of positive test term

(a) If there is a convergent series $\sum b_n$ with $a_n \le b_n$, then $\sum a_n$ is convergent.

(b) If there is a divergent series $\sum b_n$ with $a_n \ge b_n$ then $\sum a_n$ is divergent

The limit comparision test.

Suppose $\sum a_n$ and $\sum b_n$ are a series of positive term and

$$\lim_{n \to \infty} \left(\frac{a_n}{b_n}\right) = l$$

(a) If $l > 0$ then both the series are convergent or divergent

(b) If $l = 0$ or ∞, then the series are divergent

3(a) Test the Convergence of series by comparision test.

(a) $\sum_{n=1}^{\infty} \frac{3}{3\sqrt{n}-2}$

Soln:-

$$a_n = \frac{3}{3\sqrt{n}-2} = \frac{1}{\sqrt{n}-2/3}$$

we take $b_n = \frac{1}{\sqrt{n}}$

$$(\sqrt{n}-2/3) < \sqrt{n}$$

$$\therefore \left(\frac{1}{\sqrt{n}-2/3}\right) > \frac{1}{\sqrt{n}}$$

then $a_n \ge b_n$

and $\sum b_n = \sum \frac{1}{\sqrt{n}}$ which is p-series with $p = 1/2 < 1$

$\therefore \sum b_n$ is divergent and hence the series $\sum a_n$ is divergent.

(c) $\sum_{n=1}^{\infty} \frac{1}{2^n+\sqrt{n}}$

soln: here,

$$a_n = \frac{1}{2^n+\sqrt{n}}$$

We take, $b_n = \frac{1}{2^n}$

$$(2^n+\sqrt{n}) > 2^n$$

$$\frac{1}{2^n+\sqrt{n}} < \frac{1}{2^n}$$

Then,

$$a_n < b_n$$

and $\sum b_n = \sum \frac{1}{2^n} = (1/2)^n$ which is geometrical g-series with ~~$p = 1 \le 1$~~, $r = 1/2 < 1$

$\therefore \sum b_n$ is ~~divergent~~ convergent and Hence the series $\sum a_n$ is convergent as well.

9 (a) $\sum_{n=1}^{\infty} \frac{4}{4n^{1/3}-5}$

Sol^n: Here,

$\sum_{n=1}^{\infty} \frac{4}{4n^{1/3}-5}$

Here

$a_n = \frac{4}{4(n^{1/3}-5/4)} = \frac{1}{n^{1/3}-5/4}$

we take $b_n = \frac{1}{n^{1/3}}$

~~then~~, $n^{1/3}-5/4 < n^{1/3}$

$\frac{1}{n^{1/3}-5/4} > \frac{1}{n^{1/3}}$

then

$a_n > b_n$

and $\sum b_n = \sum \frac{1}{n^{1/3}}$, which is p series with $p = \frac{1}{3}$ ~~that is~~ ~~less~~ $p \leq 1$ so, that

$\therefore \sum b_n$ is divergent and Hence $\sum a_n$ is also divergent.

(d) $\sum_{n=1}^{\infty} \frac{1}{n^2+5}$

Sol^n: Here,

$a_n = \frac{1}{n^2+5}$,

we take $b_n = 1/n^2$

$\therefore$ $n^2+5 > n^2$

$\frac{1}{n^2+5} < \frac{1}{n^2}$

then,

$a_n < b_n$

and $\sum b_n = \sum \frac{1}{n^2}$ which is p-series with $p = 2$ i.e. ≥ 1 so, that $\sum b_n$ is convergent Hence $\sum a_n$ is also convergent.

(e) $\sum_{n=1}^{\infty} \frac{5}{2n^2+4n+3}$

Sol^n: Here,

$\sum_{n=1}^{\infty} \frac{5}{2n^2+4n+3}$

$a_n = \frac{5}{2n^2+4n+3}$

we take, $b_n = \frac{1}{n^2}$

$\therefore$ $2n^2+4n+3 > n^2$

$\frac{5}{2n^2+4n+3} < \frac{1}{n^2}$

then, $a_n < b_n$ ~~with w~~

And $\sum b_n = \frac{1}{n^2}$ which is p series

with $p = 2 \geq 1$ so that,
$\sum b_n$ is convergence Hence,
$\sum a_n$ is also converges

(f) $\sum_{k=1}^{\infty} \frac{\ln k}{k}$

Sol^n: Here,
$a_k = \frac{\ln k}{k}$

we take $b_k = 1/k$

We know,
$\frac{\ln k}{k} < \frac{1}{k}$

then,
$a_k < b_k$,
$\sum b_k = 1/k$ which is the series of $p = 1 \leq 1$
So that,
$\sum b_k$ is divergent hence
$\sum a_k$ is also divergent.

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(g) $\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$

Sol^n: Here,
$\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$,

$a_n = \frac{1}{2^n - 1}$

and we take
$b_n = \frac{1}{2^n}$

$2^n - 1 < 2^n$

$\frac{1}{2^n - 1} > \frac{1}{2^n}$

then, $a_n > b_n$
And $\sum b_n = \frac{1}{2^n}$ which is geometric series of p

so here $p = 1/2$ i.e. < 1 so that,
$\sum b_n$ is convergent Hence,
$\sum a_n$ is also convergent.

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① $\sum_{n=1}^{\infty} \left(\frac{1}{n^3+1}\right)$

Sol^n: here,

$a_n = \frac{1}{n^3+1}$

We take, $b_n = \frac{1}{n^3}$

Now:-

$n^3+1 > n^3$

$\frac{1}{n^3+1} < \frac{1}{n^3}$

then

$a_n < b_n$

and $\sum b_n = \sum \frac{1}{n^3}$ which is the series of

$p = 3$ where $3 < 1$, so that,

$\sum b_n$ is convergent hence

$\sum a_n$ is also convergent

(A) Test the convergence of series.

(a) $\sum_{n=1}^{\infty} \frac{1}{n^3}$

Sol^n: here,

$a_n = \frac{1}{n^3}$ and we take $b_n = \frac{1}{n^3}$

$\lim_{n \to \infty} \left(\frac{a_n}{b_n}\right) = \lim_{n \to \infty} \left(\frac{1/n^3}{1/n^3}\right) = 1 > 0$

So both the series are convergent or divergent,

But $\sum b_n = \sum \frac{1}{n^3}$ which is p-series with $p = 3 > 1$

So $\sum b_n$ is convergent hence

$\sum a_n$ is also convergent.

(b) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

Sol^n: here

$a_n = \frac{1}{\sqrt{n}}$ & we take $b_n = \frac{1}{\sqrt{n}}$

By limit comparision test,

$\lim_{n \to \infty} \left(\frac{a_n}{b_n}\right) = \lim_{n \to \infty} \left(\frac{\frac{1}{\sqrt{n}}}{1/\sqrt{n}}\right) = 1 > 0.$

So both the series are convergent or divergent

But, $\sum b_n = \frac{1}{n^{1/2}}$ which is p-series with $p = \frac{1}{2} < 1$

So $\sum b_n$ is divergent hence

$\sum a_n$ is also divergent.

(c) $\sum_{n=1}^{\infty}\left(\frac{1}{n^{3/2}}\right)$

Sol^n: here

$a_n = \frac{1}{n^{3/2}}$ we take $b_n = \frac{1}{n^{3/2}}$

By limit comparision test,

$\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \lim_{n\to\infty}\left(\frac{1/n^{3/2}}{1/n^{3/2}}\right) = 1 > 0$

So that the both series are convergent or divergent But,

$\sum b_n = \sum \frac{1}{n^{3/2}}$ which is p series with $p = \frac{3}{2} > 1$

So that the $\sum b_n$ is convergent and $\sum a_n$ is also convergent

(d) $\sum_{n=1}^{\infty}\left(\frac{2n+1}{n^2+2n+1}\right)$

Sol^n: here,

$a_n = \left(\frac{2n+1}{n^2+2n+1}\right)$ we take $b_n = \frac{1}{n}$

By limit comparision test

$\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \lim_{n\to\infty}\left(\frac{\frac{2n+1}{n^2+2n+1}}{1/n}\right)$

$\lim_{n\to\infty} = \left(\frac{\cancel{n}(2+1/n)}{\cancel{n^2}(1+2/n+1/n^2)} \times \cancel{n}\right)$

$= \left(\frac{2+0}{1+0+0}\right) = 2 > 0$

So that Both the Series are convergent or divergent but

$\sum b_n = \sum \frac{1}{n}$ which is p series with $p = 1 \leq 1$ So that the $\sum b_n$ is divergent hence $\sum a_n$ is also divergent.

(e) $\sum_{n=1}^{\infty}\left(\frac{10n+1}{n(n+1)(n+2)}\right)$

Sol^n: here,

$a_n = \frac{10n+1}{n(n+1)(n+2)}$ we take $b_n = \frac{n}{n^3} = \frac{1}{n^2}$

By limit test comparision test

$\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \lim_{n\to\infty}\left(\frac{\frac{10n+1}{n(n+1)(n+2)}}{1/n^2}\right)$

$= \lim_{n\to\infty}\left(\frac{n(10+1/n)\times n^2}{n\, n(1+1/n)\, n(1+2/n)}\right)$

$\lim_{n\to\infty}\left(\frac{\cancel{n^3}}{\cancel{n^3}}\ \frac{(10+1/n)}{(1+1/n)(1+2/n)}\right)$

$= \frac{10}{1\times 1} = 10 > 0$

So that Both the series are Convergent or divergent but,

$\sum b_n = \sum 1/n^2$ which is the series of p with $p = 2 \geq 1$ so that, the $\sum b_n$ is convergent hence $\sum a_n$ is also convergent series

(i) $\sum_{n=1}^{\infty} \frac{n^2}{n^3-1}$

Sol^n: here

$a_n = \frac{n^2}{n^3-1}$ & we take $b_n = \frac{n^2}{n^3} = \frac{1}{n}$

By limit comparision test,

$\lim_{n\to\infty} \left(\frac{a_n}{b_n}\right) = \left(\frac{\frac{n^2}{n^3-1}}{1/n}\right)$

$\lim_{n\to\infty} \left(\frac{\frac{n^2}{n^3-1}}{\frac{1}{n}}\right) = \lim_{n\to\infty} \left(\frac{n^3}{n^3(1-1/n^3)}\right)$

$= \lim_{n\to\infty} \left(\frac{1}{1-1/n^3}\right)$

$\left(\frac{1}{1-0}\right) = 1 > 0$

Here Both the series are convergent or divergent but,

$\sum b_n = \sum 1/n$ which is the series of p with $p = 1 \leq 1$ so that

$\sum b_n$ is divergent series hence

$\sum a_n$ is also divergent series.

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(j) $\sum_{n=1}^{\infty} \frac{n^2+1}{n^4+4}$

Sol^n: here,

$a_n = \frac{n^2+1}{n^4+4}$ & we take $b_n = \frac{n^2}{n^4} = \frac{1}{n^2}$

By limit comparision test

$\lim_{n\to\infty} \left(\frac{a_n}{b_n}\right) = \lim_{n\to\infty} \left(\frac{\frac{n^2+1}{n^4+4}}{1/n^2}\right)$

$\lim_{n\to\infty} \left(\frac{n^2(1+1/n^2) \times n^2}{n^4(1+4/n^4)}\right)$

$\lim_{n\to\infty} \left(\frac{n^4}{n^4} \frac{(1+1/n^2)}{(1+4/n^4)}\right)$

$\frac{1+0}{1+0} = 1 > 0$

Here Both the series are convergent or divergent but

$\sum b_n = \sum \frac{1}{n^2}$ which is the series of p with $p = 2 \geq 1$ So that,

$\sum b_n$ is convergent series Hence

$\sum a_n$ is also convergent series.

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(h) $\sum_{n=1}^{\infty} \frac{3}{n+\sqrt{n}}$

Solⁿ: Here,

$a_n = \frac{3}{n+\sqrt{n}}$ and we take $b_n = \frac{1}{n}$

By limit comparision test,

$\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \lim_{n\to\infty}\left(\frac{\frac{3}{n+\sqrt{n}}}{\frac{1}{n}}\right)$

$\lim_{n\to\infty}\left(\frac{3n}{n+\sqrt{n}}\right)$

$\lim_{n\to\infty}\left(\frac{3n}{n(1+n^{1/2-1})}\right)$

$\lim_{n\to\infty}\left(\frac{3}{1+\frac{1}{\sqrt{n}}}\right)$

$\left(\frac{3}{1+0}\right) = 3 > 1$

Here Both the series are Convergent or divergent But,
$\sum b_n = \sum \frac{1}{n}$ which is the series of p with $p = 1$ i.e ≤ 1 so that,
$\sum b_n$ is ~~deo~~ divergent series Hence
$\sum a_n$ is also divergent series.

(i) $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+1}$

Solⁿ: here,

$a_n = \frac{\sqrt{n}}{n^2+1}$ and we take $b_n = \frac{n^{1/2}}{n^2} = n^{1/2-2}$

$b_n = n^{(1-4)/2} = n^{-3/2}$

$= \frac{1}{n^{3/2}}$

By limit Comparision test,

$\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \lim_{n\to\infty}\left(\frac{\frac{\sqrt{n}}{n^2+1}}{\frac{1}{n^{3/2}}}\right)$

$\lim_{n\to\infty}\left(\frac{n^{1/2+3/2}}{n^2+1}\right)$

$\lim_{n\to\infty}\left(\frac{n^2}{n^2(1+\frac{1}{n^2})}\right)$

$\lim_{n\to\infty}\left(\frac{1}{1+\frac{1}{n^2}}\right)$

$\frac{1}{1+0} = 1 > 0$

Here all the series are Convergent or divergent but
$\sum b_n = \sum \frac{1}{n^{3/2}}$ which is the series of p so that $p = \frac{3}{2} = 1.5 > 1$ therefore
$\sum b_n$ is a convergent series ~~and~~ hence
$\sum a_n$ is also convergent series.

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(1) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+\sqrt{n+1}}$

Sol^n : here,

$a_n = \frac{1}{\sqrt{n}+\sqrt{n+1}}$ & we take $b_n = \frac{1}{\sqrt{n}}$

By limit comparision test,

$\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \lim_{n\to\infty}\left(\frac{\frac{1}{\sqrt{n}+\sqrt{n+1}}}{1/\sqrt{n}}\right)$

$\lim_{n\to\infty}\left(\frac{1}{\sqrt{n}+\sqrt{n+1}} \times \sqrt{n}\right)$

$\lim_{n\to\infty}\left(\frac{1}{1+\sqrt{1+1/n}}\right)$

$\lim_{n\to\infty}\left(\frac{1}{1+\sqrt{1+1/n}}\right)$

$\frac{1}{1+\sqrt{1+0}} = \frac{1}{1+1} = \frac{1}{2} > 0$

Here the series are both convergent and divergent but,

$\sum b_n = \sum \frac{1}{\sqrt{n}}$ which is the series of p with $p = \frac{1}{2} \leq 1$ so that,

$\sum b_n$ is divergent series Hence $\sum a_n$ is also divergent series

(2) $\frac{1}{1.3} + \frac{2}{3.5} + \frac{3}{5.7} + \cdots \Rightarrow \frac{n}{(a+(n-1)d)\times(a+(n-1)d)}$

$\frac{n}{(1+(n-1)\times 2)\times(3+(n-1)\times 2)}$

$\frac{n}{(2n-1)\times(2n+1)}$

sol^n : Here,

$a_n = \frac{n}{(2n-1)(2n+1)}$ &

we take,

$b_n = \frac{n}{n^2} = \frac{1}{n}$

By limit comparision test

$\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \lim_{n\to\infty}\left(\frac{\frac{n}{(2n-1)(2n+1)}}{1/n}\right)$

$\lim_{n\to\infty}\left(\frac{n^2}{4n^2-1}\right) = \lim_{n\to\infty}\left(\frac{1}{4-1/n^2}\right)$

$\lim_{n\to\infty}\left(\frac{1}{4}\right) > 0$

Here both the series are convergent or divergent but,

$\sum b_n = \sum \frac{1}{n}$ which is the series of p with $p = 1 \leq 1$ so that

$\sum b_n$ is divergent series Hence $\sum a_n$ is also divergent series.

$$\text{(l)}\quad \frac{1}{1\cdot 4} + \frac{1}{2\cdot 5} + \frac{1}{3\cdot 6} + \ldots$$

Sol^n: Here

$$\sum_{n=1}^{\infty} \frac{1}{\{a+(n-1)d\}\times\{a+(n-1)d\}} = \frac{1}{(1+(n-1)\times1)\times(4+(n-1)1)}$$

$$\therefore \sum_{n=1}^{\infty} = \sum \frac{1}{n(n+3)}$$

$\therefore$

$$a_n = \frac{1}{n(n+3)} \quad \text{and} \quad b_n = \frac{1}{n^2}$$

By limit Comparison test,

$$\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \lim_{n\to\infty}\left(\frac{\frac{1}{n(n+3)}}{\frac{1}{n^2}}\right)$$

$$\lim_{n\to\infty}\left(\frac{n^2}{n^2+3n}\right) = \lim_{n\to\infty}\left(\frac{n^2}{n^2(1+3/n)}\right)$$

$$\lim_{n\to\infty}\left(\frac{1}{1+\frac{3}{\infty}}\right) = \frac{1}{1} = 1 > 0$$

Here both the series are convergent or divergent but,
$\sum b_n = \sum \frac{1}{n^2}$ which is the series of p with $p=2$ i.e >1 so that $\sum b_n$ is convergent series Hence $\sum a_n$ is also Convergent series.

$$\text{(m)}\quad \frac{1}{1\cdot2\cdot3} + \frac{3}{2\cdot3\cdot4} + \frac{5}{3\cdot4\cdot5} + \ldots$$

Sol^n: here,

$1,3,5 = a+(n-1)d = 1+(n-1)\times 2 = (2n-1)$

$1,2,3 = n$

$2,3,4 = 2+(n-1)\times 1 = n+1$

$3,4,5 = 3+(n-1) = n+2$

$$\therefore \sum_{n=1}^{\infty} = \sum \frac{(2n-1)}{n(n+1)(n+2)}$$

$$a_n = \frac{2n-1}{n(n+1)(n+2)} \quad \& \quad \text{we take } b_n = \frac{n}{n^3} = \frac{1}{n^2}$$

By limit comparison test

$$\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \lim_{n\to\infty}\left(\frac{\frac{2n-1}{n(n+1)(n+2)}}{\frac{1}{n^2}}\right)$$

$$\lim_{n\to\infty}\left(\frac{n^2(2n-1)}{n(n+1)(n+2)}\right) = \lim_{n\to\infty}\left(\frac{n^3(2-\frac{1}{n})}{n^3(1+\frac{1}{n})(1+\frac{2}{n})}\right)$$

$$\lim_{n\to\infty}\left(\frac{(2-\frac{1}{n})}{(1+\frac{1}{n})\cdot(1+\frac{2}{n})}\right)$$

$$\frac{2}{1\cdot1} = 2 > 0$$

Here both the series are Convergent or divergent But,
$\sum b_n = \sum \frac{1}{n^2}$ which is the series of p with $p=2$ i.e. >1 So that, $\sum b_n$ is convergent Series hence $\sum a_n$ is also convergent series

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(n) $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \cdots$

Sol^n: here,

$\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \cdots$

$\frac{n}{(n+1)}$

$a_n = \frac{n}{n+1}$ and we take $b_n = \frac{n}{n} = 1$.

Using limit comparision test,

$\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \lim_{n}\left(\frac{\frac{n}{n+1}}{1}\right)$

$\lim_{n\to\infty}\left(\frac{n}{n+1}\right) = \lim_{n\to\infty}\left(\frac{\cancel{n}}{\cancel{n}(1+1/n)}\right)$

$\lim_{n\to\infty}\left(\frac{1}{1+1/n}\right) = \frac{1}{1+0} = 1 > 0$

Here all the series are convergent or divergent

but,

$\Sigma b_n = \Sigma \frac{1}{n^0}$. which is series of p with

$p = 0$ i.e ≤ 1 so that,

Σb_n is ~~a~~ ~~Con~~ divergent series and

Σa_n is also a divergent series.

(o) $\sum_{n=1}^{\infty}\left(\frac{\sqrt{n+2}}{2n^2+n+1}\right)$

Sol^n: here,

$a_n = \frac{\sqrt{n+2}}{2n^2+n+1}$ and we take $b_n = \frac{n^{1/2}}{n^2} = n^{\frac{1}{2}-2} = n^{-3/2}$

$= \frac{1}{n^{3/2}}$

$b_n = \frac{1}{n^{3/2}}$

By limit comparision test,

$\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \lim_{n\to\infty}\left(\frac{\frac{\sqrt{n+2}}{2n^2+n+1}}{1/n^{3/2}}\right)$

$\lim_{n\to\infty}\left(\frac{\sqrt{n+2}}{2n^2+n+1} \times \sqrt{n^3}\right)$

$\lim_{n\to\infty}\left(\frac{n\sqrt{n}\,(\sqrt{n+2})}{n^2\left(2+\frac{1}{n}+\frac{1}{n^2}\right)}\right)$

$\lim_{n\to\infty}\left(\frac{\cancel{n^2}\left(\sqrt{1+2/n}\right)}{\cancel{n^2}\left(2+\frac{1}{n}+\frac{1}{n^2}\right)}\right)$

$\frac{\sqrt{1+0}}{2+0+0} = \frac{1}{2} > 0.$

Here all the series are convergent / divergent

but

$\Sigma b_n = \Sigma \frac{1}{n^{3/2}}$ which is the series of p with

$p = \frac{3}{2} > 1$ so that

Σb_n is convergent series hence

Σa_n is Convergent series.

(P) $\sum_{n=1}^{\infty} \sqrt{\frac{n^4+1}{n^3+n^2}}$ $\quad \frac{\sqrt{n^4+1}}{n^3+n^2}$

Soln: here,

$a_n = \sqrt{\frac{n^4+1}{n^3+n^2}}$ and $b_n = \frac{\sqrt{n^4}}{n^3} = \frac{n^2}{n^3} = \frac{1}{n}$

Now,
By limit comparision test,

$\lim_{n\to\infty} \left(\frac{a_n}{b_n}\right) = \lim_{n\to\infty} \left(\frac{\sqrt{n^4+1}}{n^3+n^2} \times n\right)$

$\lim_{n\to\infty} \left(\frac{n^2 \times n \left(\sqrt{1+1/n^4}\right)}{n^3\left(1+1/n\right)}\right)$

$\lim_{n\to\infty} \left(\frac{\sqrt{1+1/n^4}}{1+1/n}\right)$

$1 > 0,$

Here all the series are convergent or divergent but
$\sum b_n = \sum \frac{1}{n}$ which is the series of p
with $p=1$ i.e. ≤ 1 so that
$\sum b_n$ is ~~convergent~~ divergent series and hence
$\sum a_n$ is also divergent series.

$b_n = \frac{}{}$ $\quad b^n = \frac{1}{n^4}$

q. $\sum_{n=1}^{\infty} \frac{5+3^n}{(1+n^2)^2}$

Soln: here
$\sum_{n=1}^{\infty} \frac{5+3^n}{1+2n^2+n^4}$

Here
$a_n = \frac{5+3^n}{1+2n^2+n^4}$ and $b_n = \frac{1}{n^4}$

By limit comparision test,

$\lim_{n\to\infty} \left(\frac{a_n}{b_n}\right) = \lim_{n\to\infty} \left(\frac{5+3^n}{n^4+2n^2+1} \times \frac{n^4}{1}\right)$

$\lim_{n\to\infty} \left(\frac{(5+3^n)}{1} \times \frac{n^4}{n^4+2n^2+1}\right) = \frac{3^n\left(\frac{5}{3^n}+1\right) \times \cancel{n^4}}{\cancel{n^4}\left(1+\frac{2}{n^2}+\frac{1}{n^4}\right)}$

$\lim_{n\to\infty} \left(\frac{5}{3^n}+1\right) \lim_{n\to\infty} \left(\frac{1}{1+\frac{2}{n^2}+\frac{1}{n^4}}\right)$ $\quad \frac{3^\infty\left(\frac{5}{\infty}+1\right)}{1+\frac{2}{\infty}+\frac{1}{\infty}}$

$\lim_{n\to\infty} \left(\frac{5}{3^n}+1\right)$ & $\lim_{n\to\infty} \left(\frac{1}{1+\frac{2}{n^2}+\frac{1}{n^4}}\right)$ $\quad \frac{\infty \times 1}{1} = \infty$

$\left(\frac{5}{\infty}+1\right) \times \left(\frac{1}{1}\right)$

$1 \times 1 > 0$

$1 > 0.$

So all the series are ~~convergent or~~ divergent series, but
~~$\sum b_n = \frac{1}{n^4}$ which is geometric series who ratio r is 1 so that the series in divergent~~

$\sum a_n$ is convergent series.

D' Alembert Ratio test :-

let Σa_n be the series of positive term and
Suppose that,

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = l$$

(a). then, (a) If $l<1$, then the series is convergent.

(b) If $l>1$ or ∞ then the series is divergent.

(c) If $l=1$, the test is fail.

5 (a) $\sum_{n=0}^{\infty}$

Soln: Here

$$a_n = \frac{2^n+5}{3^n}$$

and

$$a_{n+1} = \frac{2^{n+1}+5}{3^{n+1}}$$

Now

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \left[\frac{\frac{2^{n+1}+5}{3^{n+1}}}{\frac{2^n+5}{3^n}}\right]$$

$$\lim_{n\to\infty}\left[\frac{2^n.2+5}{3^n.3} \times \frac{3^n}{2^n+5}\right]$$

$$= \lim_{n\to\infty} \frac{2^n(2+5/2^n)}{3^n.3} \times \frac{3^n}{2^n(1+5/2^n)}$$

$$= \frac{(2+0)}{3(1+0)}$$

$$= 2/3 < 1$$

So the given series is Convergent by ratio test

(b) $\sum_{n=1}^{\infty} \frac{(2n)!}{n!\,n!}$

Soln: here,

$$a_n = \frac{(2n)!}{n!\,n!}$$

and $a_{n+1} = \frac{(2(n+1))!}{(n+1)!\,(n+1)!}$

Now

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\left[\frac{\frac{(2n+2)!}{(n+1)!\,(n+1)!}}{\frac{(2n)!}{n!\,n!}}\right]$$

$$\lim_{n\to\infty}\left[\frac{(2n+2)(2n+1)(2n)!}{(n+1)\,n!\,(n+1)\,n!} \times \frac{n!\,n!}{(2n)!}\right]$$

$$\lim_{n\to\infty} \frac{n(2+2/n).n(2+1/n)}{n(1+1/n).n(1+1/n)}$$

$$= \frac{(2+0)(2+0)}{(1+0)(1+0)}$$

$$= 4 > 1$$

The given series is divergent by ratio test.

(h) $\sum_{n=1}^{\infty} \sqrt{\frac{2^n-1}{3^n-1}}$

Sol^n: Here:

$$a_n = \sqrt{\frac{2^n-1}{3^n-1}}$$

& $$a_{n+1} = \sqrt{\frac{2^{n+1}-1}{3^{n+1}-1}}$$

Now,

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \left[\frac{\sqrt{\frac{2^{n+1}-1}{3^{n+1}-1}}}{\sqrt{\frac{2^n-1}{3^n-1}}}\right]$$

$$\lim_{n\to\infty} \left[\sqrt{\frac{2^n 2-1}{3^n 3-1}} \times \sqrt{\frac{3^n-1}{2^n-1}}\right]$$

$$\lim_{n\to\infty} \sqrt{\frac{\cancel{2^n}(2-\frac{1}{2^n})}{\cancel{3^n}(3-\frac{1}{3^n})}} \times \sqrt{\frac{\cancel{3^n}(1-\frac{1}{3^n})}{\cancel{2^n}(1-\frac{1}{2^n})}}$$

$$\sqrt{2-0/3-0} \times \sqrt{1-0/1-0}$$

$$\sqrt{2/3} \times \sqrt{1}$$

$$\sqrt{2/3} < 1$$

the given Series is Convergent by Ratio test.

The n^{th} ~~test~~ root test.

let $\sum a_n$ be ~~a~~ series of positive term ~~and~~ ~~Opposite~~ suppose that

$$\lim_{n\to\infty} \sqrt[n]{a_n} = l.$$

then (a) If $l<1$, then the Series is converges
(b) If $l>1$ ~~then~~ or ∞, then the series is divergent
(c) If $l=1$, the test is fail.

6(a) $\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^{-n^2}$

Sol^n: here, $a_n = \left(1+\frac{1}{n}\right)^{-n^2}$

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \left[\left(1+\frac{1}{n}\right)^{-n^2}\right]^{1/n}.$$

$$\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^{-n}$$

$$\lim_{n\to\infty} \left(\frac{1}{(1+\frac{1}{n})^n}\right)$$

$$\frac{1}{e} < 1$$

the given series is ~~d~~ convergent by root test.

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(5) Investigate the Convergence of the following Series.

(a) $\sum_{n=0}^{\infty} \frac{2^n+5}{3^n}$

Sol^n: Here,

$a_n = \frac{2^n+5}{3^n}$ and we take $a_{n+1} = \frac{2^{n+1}+5}{3^{n+1}}$

Using ratio test,

$$\lim_{n\to\infty} \left(\frac{a_{n+1}}{a_n}\right) = \lim_{n\to\infty} \left(\frac{\frac{2^{n+1}+5}{3^{n+1}}}{\frac{2^n+5}{3^n}}\right)$$

$$\lim_{n\to\infty} \left(\frac{2^{n+1}+5}{3^{n+1}} \times \frac{3^n}{2^n+5}\right)$$

$$\lim_{n\to\infty} \left(\frac{\cancel{2^n}(2+5/2^n)}{\cancel{3^n}(3)}\right) \times \frac{\cancel{3^n}}{\cancel{2^n}(1+5/2^n)}$$

$$\lim_{n\to\infty} \left(\frac{(2+5/2^n)}{3(1+5/2^n)}\right)$$

$$\frac{2+0}{3(1+0)} = 2/3 < 1$$

So the given series is Convergent by ratio test.

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(b) $\sum_{n=1}^{\infty} \frac{(2n)!}{n!\cdot n!}$

Sol^n: here,

$a_n = \frac{(2n)!}{n!\cdot n!}$

$a_{n+1} = \frac{[2(n+1)]!}{(n+1)!\,(n+1)!}$

$a_{n+1} = \cancel{\frac{2!\,(n+1)!}{(n+1)!\,(n+1)!}}\ \frac{(2n+2)!}{(n+1)!\,(n+1)!}$

By ratio test,

$$\lim_{n\to\infty} \left(\frac{a_{n+1}}{a_n}\right) = \lim_{n\to\infty} \left(\frac{(2n+2)!}{(n+1)!\,(n+1)!} \times \frac{n! \times n!}{(2n)!}\right)$$

$$\lim_{n\to\infty} \left(\frac{(2n+2)(2n+1)\cancel{(2n)!} \times \cancel{n!} \times \cancel{n!}}{(n+1) \times \cancel{n!}\,(n+1) \times \cancel{n!}\,\cancel{(2n)!}}\right)$$

$$\lim_{n\to\infty} \left(\frac{2\cancel{(n+1)}(2n+1)}{(n+1)\cancel{(n+1)}}\right)$$

$$\lim_{n\to\infty} \left(\frac{4n+2}{n+1}\right)$$

$$\lim_{n\to\infty} \boxed{\frac{\cancel{n}(4+2/n)}{\cancel{n}(1+1/n)}}$$

$$\frac{4+0}{1+0} = 4 > 1,$$

So that the given series is divergent by ratio test

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(c) $\sum_{n=1}^{\infty} \frac{n!}{10^n}$

Sol^n: Here $\cancel{\lim_{n\to\infty}} \sum_{n=1}^{\infty} \frac{n!}{10^n}$

$a_n = \frac{n!}{10^n}$ and we take $a_{n+1} = \frac{(n+1)!}{10^{n+1}}$

By ratio test method,

$$\lim_{n\to\infty} \left(\frac{a_{n+1}}{a_n}\right) = \lim_{n\to\infty} \left(\frac{(n+1)!}{10^{n+1}} \times \frac{10^n}{n!}\right)$$

$$\lim_{n\to\infty} \left(\frac{(n+1) \times \cancel{n!} \times \cancel{10^n}}{\cancel{10^n} \cdot 10 \cdot \cancel{n!}}\right)$$

$$\lim_{n\to\infty} \left(\frac{n+1}{10}\right)$$

$$\lim_{n\to\infty} \left(\frac{n\left(1+\frac{1}{n}\right)}{10}\right)$$

$$\infty$$

So, that the given series is divergent by ratio test.

(d) $\sum_{n=1}^{\infty} \frac{(n+3)!}{3!\, n!\, 3^n}$

Sol^n: Here,

$a_n = \frac{(n+3)!}{3! \cdot n!\, 3^n}$ and we take $a_{n+1} = \frac{(n+4)!}{3!\,(n+1)!\,3^{n+1}}$

By ratio test method,

$$\lim_{n\to\infty} \left(\frac{a_{n+1}}{a_n}\right) = \lim_{n\to\infty} \left(\frac{(n+4)!}{3!\,(n+1)!\,3^{n+1}} \times \frac{3! \times n! \times 3^n}{(n+3)!}\right)$$

$$\lim_{n\to\infty} \left(\frac{(n+4)\,\cancel{(n+3)!} \times \cancel{n!} \times \cancel{3^n}}{(n+1) \times \cancel{n!}\, \cancel{3^n} \cdot 3\, \cancel{(n+3)!}}\right)$$

$$\lim_{n\to\infty} \left(\frac{(n+4)}{(n+1) \times 3}\right)$$

$$\lim_{n\to\infty} \left(\frac{\cancel{n}\left(1+\frac{4}{n}\right)}{3\cancel{n}\left(1+\frac{1}{n}\right)}\right)$$

$$\lim_{n\to\infty} \left(\frac{1}{3}\right) < 1$$

So that the given series is convergent by ratio test.

(e) $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$

Soln: Here,

$\sum_{n=1}^{\infty} \frac{n^2}{2^n}$

$a_n = \frac{n^2}{2^n}$ and we take $a_{n+1} = \frac{(n+1)^2}{2^{n+1}}$

By Ratio test

$\lim_{n \to \infty} \left(\frac{a_{n+1}}{a_n} \right) = \left(\frac{(n+1)^2}{2^{n+1}} \times \frac{2^n}{n^2} \right)$

$\lim_{n \to \infty} \left(\frac{(n^2+2n+1) \times \cancel{2^n}}{\cancel{2^n} \cdot 2 \times n^2} \right)$

$\lim_{n \to \infty} \left(\frac{\cancel{n^2}\left(1+\frac{2}{n}+\frac{1}{n^2}\right)}{2\cancel{n^2}} \right)$

$\left(\frac{1+0+0}{2} \right)$

$\left(\frac{1}{2}\right) < 1$

So that the given series is convergent by ratio test.

(f) $\sum_{n=1}^{\infty} \frac{2^n}{n^2}$

Soln: Here,

$a_n = \frac{2^n}{n^2}$ and $a_{n+1} = \frac{2^{n+1}}{(n+1)^2}$

By ratio test rule,

$\lim_{n \to \infty} \left(\frac{a_{n+1}}{a_n} \right) = \lim_{n \to \infty} \left(\frac{2^{n+1}}{(n+1)^2} \times \frac{n^2}{2^n} \right)$

$\lim_{n \to \infty} \left(\frac{\cancel{2^n} \cdot 2^1 \times n^2}{(n^2+2n+1) \times \cancel{2^n}} \right)$

$\lim_{n \to \infty} \left(\frac{2\cancel{n^2}}{\cancel{n^2}\left(1+\frac{2}{n}+\frac{1}{n^2}\right)} \right)$

$\frac{2}{1+0+0} = 2 > 1$

So that the given series is divergent by ratio test.

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(g) $\sum_{n=1}^{\infty} \frac{1}{2^n-1}$

Sol^n: Here,

$\sum_{n=1}^{\infty} \frac{1}{2^n-1}$

$a_n = \frac{1}{2^n-1}$ and we take $a_{n+1} = \frac{1}{2^{n+1}-1}$

By Ratio test,

$\lim_{n\to\infty} \left(\frac{a_{n+1}}{a_n}\right) = \lim_{n\to\infty} \left(\frac{1}{2^n.2-1} \times \frac{2^n-1}{1}\right)$

$\lim_{n\to\infty} \left(\frac{\cancel{2^n}\left(1-\frac{1}{2^n}\right)}{\cancel{2^n}\left(2-\frac{1}{2^n}\right)}\right)$

$\frac{1-0}{2-0}$

$\frac{1}{2} < 1$

$\therefore$ So that the given series is Convergent by Ratio test.

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(h) $\sum_{n=1}^{\infty} \sqrt{\frac{2^n-1}{3^n-1}}$

Sol^n: here,

$a_n = \sqrt{\frac{2^n-1}{3^n-1}}$ and we take $a_{n+1} = \sqrt{\frac{2^{n+1}-1}{3^{n+1}-1}}$

By Ratio test,

$\lim_{n\to\infty} \left(\frac{a_{n+1}}{a_n}\right) = \lim_{n\to\infty} \left(\sqrt{\frac{2^{n+1}-1}{3^{n+1}-1}} \times \sqrt{\frac{3^n-1}{2^n-1}}\right)$

$\lim_{n\to\infty} \left(\sqrt{\frac{2^n.2^1-1}{3^n.3^1-1}} \times \sqrt{\frac{3^n(1-1/3^n)}{2^n(1-1/2^n)}}\right)$

$\lim_{n\to\infty} \left(\sqrt{\frac{2^n(2-1/2^n)}{3^n(3-1/3^n)}} \times \sqrt{\frac{3^n}{2^n}\frac{(1-1/3^n)}{(1-1/2^n)}}\right)$

$\lim_{n\to\infty} \left(\sqrt{\frac{\cancel{2^n}}{3^n}} \times \sqrt{\frac{\cancel{3^n}}{2^n}} \times \sqrt{\frac{(2-1/2^n)}{(3-1/3^n)}} \times \sqrt{\frac{(1-1/3^n)}{(1-1/2^n)}}\right)$

$\left(\sqrt{\frac{2-0}{3-0}} \times \sqrt{\frac{1-0}{1-0}}\right)$

$\sqrt{2/3} < 1,$

So that given series are Convergent by Ratio test.

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(1) $\sum_{n=1}^{\infty} \left(\frac{4}{3^{2n}-1} \right)$

Solⁿ: Here,

$a_n = \left(\frac{4}{3^{2n}-1} \right)$ and we take $a_{n+1} = \left(\frac{4}{3^{2(n+1)}-1} \right)$

By ratio test method

$\lim_{n \to \infty} \left(\frac{a_{n+1}}{a_n} \right) = \lim_{n \to \infty} \left(\frac{\cancel{4}}{3^{2n+2}-1} \times \frac{3^{2n}-1}{\cancel{4}} \right)$

$\lim_{n \to \infty} \left(\frac{3^{2n}-1}{3^{2n}(3^2 - 1/3^{2n})} \right)$

$\lim_{n \to \infty} \left(\frac{\cancel{3^{2n}}(1 - 1/3^{2n})}{\cancel{3^{2n}}(9 - 1/3^{2n})} \right)$

$\frac{1-0}{9-0}$

$\frac{1}{9} < 1$

So that the given series is Convergent by ~~the~~ ratio test method.

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(2) $\sum_{n=1}^{\infty} \left(\frac{9^n}{3+10^n} \right)$

Solⁿ: Here,
$a_n = \left(\frac{9^n}{3+10^n} \right)$ and $a_{n+1} = \left(\frac{9^{n+1}}{3+10^{n+1}} \right)$

By ratio test,

$\lim_{n \to \infty} \left(\frac{a_{n+1}}{a_n} \right) = \lim_{n \to \infty} \left(\frac{9^{n+1}}{3+10^{n+1}} \times \frac{3+10^n}{9^n} \right)$

$\lim_{n \to \infty} \left(\frac{\cancel{9^n} \cdot 9 \; (3+10^n)}{(3+10^n \cdot 10) \times \cancel{9^n}} \right)$

$\lim_{n \to \infty} \left(\frac{9 \times \cancel{10^n}(1 + 3/10^n)}{\cancel{10^n}(3/10^n + 10)} \right)$

$\frac{9(1+0)}{(0+10)}$

$= \frac{9}{10} < 1$

So that the given series is converge by ratio test.

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(x) $\sum_{n=1}^{\infty} \frac{6^n}{5^n-1}$

Sol^n: Here,

$a_n = \frac{6^n}{5^n-1}$ and $a_{n+1} = \frac{6^{n+1}}{5^{n+1}-1}$

By ratio test,

$\lim_{n\to\infty}\left(\frac{a_{n+1}}{a_n}\right) = \lim_{n\to\infty}\left(\frac{6^{n+1}}{5^{n+1}-1} \times \frac{5^n-1}{6^n}\right)$

$\lim_{n\to\infty}\left(\frac{\cancel{6^n}.6 \cdot \cancel{5^n}(1-1/5^n)}{\cancel{5^n}(5-1/5^n)\cdot\cancel{6^n}}\right)$

$\lim_{n\to\infty}\left(\frac{6(1-1/5^n)}{(5-1/5^n)}\right)$

$\frac{6(1-0)}{(5-0)}$

$6/5 > 1,$

so that the Series is divergent by ratio test

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(l) $\sum_{n=1}^{\infty}\left(\frac{1+4^n}{1+3^n}\right)$

Sol^n: Here,

$a_n = \left(\frac{1+4^n}{1+3^n}\right)$ & we take $b_n = \left(\frac{1+4^{n+1}}{1+3^{n+1}}\right)$

By ratio test,

$\lim_{n\to\infty}\left(\frac{a_{n+1}}{a_n}\right) = \lim_{n\to\infty}\left(\frac{1+4^n.4}{1+3^n.3}\right) \times \left(\frac{1+3^n}{1+4^n.4}\right)$

$\lim_{n\to\infty}\left(\frac{\cancel{4^n}(1/4^n+4)}{\cancel{3^n}(1/3^n+3)} \times \frac{\cancel{3^n}(1/3^n+1)}{\cancel{4^n}(\frac{1}{4^n}+1)}\right)$

$\lim_{n\to\infty}\left(\frac{(1/4^n+4)(1/3^n+1)}{(1/3^n+3)(1/4^n+1)}\right)$

$\frac{(0+4)(0+1)}{(0+3)(0+1)}$

$4/3 > 1$

So that the series is divergent by ratio test,

(m) $\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2}$

Sol: Here,

$a_n = \frac{(2n)!}{(n!)^2}$ & we take $a_{n+1} = \frac{(2(n+1))!}{((n+1)!)^2}$

By ratio test,

$\lim_{n\to\infty} \left(\frac{a_{n+1}}{a_n}\right) = \lim_{n\to\infty} \left(\frac{(2n+2)!}{(n+1)! \times (n+1)!} \times \frac{n! \times n!}{(2n)!}\right)$

$\lim_{n\to\infty} = \left(\frac{(2n+2)(2n+1) \times \cancel{(2n)!} \times \cancel{n! \times n!}}{(n+1) \times \cancel{n!} \times (n+1) \times \cancel{n!} \times \cancel{(2n)!}}\right)$

$\lim_{n\to\infty} \left(\frac{2\cancel{(n+1)}(2n+1)}{(n+1)^{\cancel{2}}}\right)$

$\lim_{n\to\infty} \left(\frac{2\cancel{n}(2+1/n)}{\cancel{n}(1+1/n)}\right)$

$\frac{2 \cdot (2+0)}{(1+0)} = 2 \times 2 = 4 > 1$

So that the series is divergent by ratio test

(6) Investigate the convergence of the following series

The n^{th} root test.

⇒ let $\sum a_n$ be a series of positive term and suppose that

$\lim_{n\to\infty} \sqrt[n]{a_n} = l$

then (a) If $l < 1$ then the series is converge
(b) If $l > 1$ then the series is diverge
(c) If $l = 1$ the test fail.

(a) $\sum_{n=1}^{\infty} \left(1+1/n\right)^{-n^2}$

Sol: here

$\left(1+1/n\right)^{-n^2}$

By root test,

$\lim_{n\to\infty} \sqrt[n]{(1+1/n)^{-n^2}} = \lim_{n\to\infty} \left(1+1/n\right)^{\frac{-n^2}{n}}$

$\lim_{n\to\infty} \left(1+1/n\right)^{-n}$

$\lim_{n\to\infty} \left(\frac{1}{(1+1/n)^n}\right)$

$\frac{1}{e} < 1$

So that the given series is convergent by root test.

(b) $\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^{n^2}$

Sol^n: here,

$\left(1+\frac{1}{n}\right)^{n^2}$

By root test method

$\lim_{n\to\infty} \sqrt[n]{\left(1+\frac{1}{n}\right)^{n^2}}$

$\lim_{n\to\infty} \left(\left(1+\frac{1}{n}\right)\right)^{n^2/n}$

$\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^{n}$

$e > 1$

So that the given series is divergent by root test.

(c) $\sum \frac{(n!)^n}{(n^n)^2}$

Sol^n: here,

$\lim_{n\to\infty} \sqrt[n]{\frac{(n!)^n}{(n^n)^2}} = \left(\frac{(n!)^n}{(n)^{2n}}\right)^{1/n}$

$\lim_{n\to\infty} \frac{n!}{n^2} = \frac{1}{\left(\frac{n^2}{n!}\right)} = \frac{1}{0} = \infty$

∞

So that the given Series is divergent by root test method.

Note

(1) $\lim_{n\to\infty} \frac{\log n}{n} = 0$

(2) $\lim_{n\to\infty} \sqrt[n]{n} = \lim_{n\to\infty} n^{1/n} = 1$

(3) $\lim_{n\to\infty} x^{1/n} = 1 \ (x>0)$

(4) $\lim_{n\to\infty} x^n = 0 \ [|x|<1]$

(5) $\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e$

(6) $\lim_{n\to\infty} \left(1+\frac{x}{n}\right)^n = e^x$

(7) $\lim_{n\to\infty} \frac{x^n}{n!} = 0$

(d) $\sum_{n=1}^{\infty} \left(\frac{n^n}{2^{n^2}}\right)$

Sol^n: here,

$\sum_{n=1}^{\infty} \frac{n^n}{2^{n^2}}$

By using root test,

$\lim_{n\to\infty} \sqrt[n]{\frac{n^n}{2^{n\times n}}} = \lim_{n\to\infty} \left(\frac{n^n}{2^{n}}\right)^{1/n}$

$\lim_{n\to\infty} \frac{n}{2^n} \quad \frac{1}{2^n \log 2} = \frac{1}{2^\infty} = 0$

$P0 < 0,$

Q) $\sum_{n=1}^{\infty} \left(\frac{2n+3}{3n+2}\right)^n$

Sol^n: here

$\left(\frac{2n+3}{3n+2}\right)^n$

By Root test,

$\lim_{n \to \infty} \sqrt[n]{\left(\frac{2n+3}{3n+2}\right)^n}$

$\lim_{n \to \infty} \left(\left(\frac{2n+3}{3n+2}\right)\right)^{n/n}$

$\frac{2n+3}{3n+2}$

$\lim_{n \to \infty} \left(\frac{n\left(2+3/n\right)}{n\left(3+2/n\right)}\right)$

$2/3$ which is less than 1,

Hence the Given Series are convergent by Root test

Example

$\sum_{n=1}^{\infty} \left(\frac{n^n}{n!}\right)$

Sol^n: here,

$a_n = \frac{n^n}{n!}$ and we take $a_{n+1} = \frac{(n+1)^{n+1}}{(n+1)!}$

By ~~Root~~ ratio test,

$\lim_{n \to \infty} \left(\frac{a_{n+1}}{a_n}\right) = \lim_{n \to \infty} \left(\frac{(n+1)^{n+1}}{(n+1)!} \times \frac{n!}{n^n}\right)$

$\lim_{n \to \infty} \left(\frac{(n+1)^n \times \cancel{(n+1)} \times \cancel{n!}}{\cancel{(n+1)} \times \cancel{n!} \times n^n}\right)$

$= \lim_{n \to \infty} \left(\left(\frac{n+1}{n}\right)^n\right)$ and $\lim_{n \to \infty} (n+1)$

$\lim_{n \to \infty} \left(1+1/n\right)^n$, ~~$\lim_{n \to \infty} n\left(1+1/n\right)$~~

~~Req~~ $e > 1$

so that the given series is divergent by Ratio test.

Alternative Series

An infinite series $\sum_{n=0}^{\infty} (-1)^n a_n = a_0 - a_1 + a_2 - a_3 + \ldots$ is known as alternative series.

Leibritz's Theorem:-

The series $\sum_{n=0}^{\infty} (-1)^n a_n$

is converges if (a) $a_n > a_{n+1}$

(b) $\lim_{n\to\infty} a_n = 0$

Absolute Convergence:

A series $\sum (-1)^n a_n$ is absolutly Convergent if the Series $\sum |a_n|$ is convergent.

Conditionally Convergent:-

If a series $\sum (-1)^n a_n$ is convergent by $\sum (a_n)$ is not convergent, then the series is conditionally convergent.

Exercise:- 8.4

(a) $\sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{1}{n^2} = 1 - \frac{1}{4} + \frac{1}{9}$

Soln: Here,

$a_n = \frac{1}{n^2}$ and $a_{n+1} = \frac{1}{(n+1)^2}$

(a) $a_n > a_{n+1}$

(b) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n^2} = \frac{1}{\infty} = 0.$

So by leibritz's theorem, the given series is Convergent.

(b) $\sum_{n=1}^{\infty} (-1)^n \frac{\ln(n)}{n}$

Soln: Here,

$a_n = \frac{\ln(n)}{n}$ & $a_{n+1} = \frac{\ln(n+1)}{n+1}$

(a) $a_n > a_{n+1}$

(b) $\lim_{n\to\infty} \frac{\ln(n)}{n}$ $\left[\frac{\infty}{\infty} \text{ form}\right]$

$\lim_{n\to\infty} \frac{\ln(n)}{n} = \frac{\frac{1}{n}}{1} = \frac{1}{n}$

$\frac{1}{\infty} = 0$

So by leibritz's theorem the given series is convergent.

(c) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^{3/2}}$

Sol^n: Here,

$a_n = \frac{1}{n^{3/2}}$ & we take $a_{n+1} = \frac{1}{(n+1)^{3/2}}$

(a) $a_n > a_{n+1}$

(b) $\lim_{n\to\infty} a_n = \frac{1}{n^{3/2}}$

$\lim_{n\to\infty} \frac{1}{n^{3/2}}$

$\frac{1}{\infty} = 0,$

So by the Leibnitz's theorem the given series is convergent.

(d) $\sum_{n=2}^{\infty} (-1)^n \frac{\ln(n)}{\ln(n^2)}$

Sol^n: here,

$a_n = \frac{\ln(n)}{\ln n^2}$ & we take $a_{n+1} = \frac{\ln(n+1)}{\ln(n+1)^2}$

(a) $a_n > a_{n+1}$

(b) $\lim_{n\to\infty} a_n = \frac{\ln n}{\ln(n^2)}$ $\left[\frac{\infty}{\infty} \text{ form}\right]$

$= \frac{1/n}{1/n^2 \times 2n} = \frac{\frac{1}{n}}{\frac{1}{n}} \times \frac{1}{2}$

$= \frac{1}{2} \neq 0$

So that by the Leibnitz's theorem the given series is divergent.

(e) $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{\ln(n)}$

Sol^n: here,

$a_n = \frac{1}{\ln(n)}$ and we take $a_{n+1} = \frac{1}{\ln(n+1)}$

(a) $a_n > b\, a_{n+1}$

(b) $\lim_{n\to\infty} a_n = \frac{1}{\ln(n)} = \frac{1}{\infty} = 0$

So that by the Leibnitz's theorem the given series is convergent.

(f) $\sum_{n=1}^{\infty} (-1)^n \ln(1+1/n)$

Sol^n: Here,

(a) $a_n = \ln(1+1/n)$ and $a_{n+1} = \ln\left(1+\frac{1}{(n+1)}\right)$

(a) $a_n > a_{n+1}$

(b) $\lim_{n\to\infty} a_n = \ln(1+1/n)$

$= \ln(1+1/\infty)$

$= \ln(1)$

$= 0$

So that by the Leibnitz's theorem the given series is convergent.

(2) Test the absolute or conditional convergence of the following series

(a) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^2}$

Sol^n: Here

$|a_n| = \frac{1}{n^2}$

$\sum a_n = \sum \frac{1}{n^2}$ which is the p-series with $p=2$ i.e. >1 so that the given series is absolutely convergent

(b) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$

Sol^n: Here,

$|a_n| = \frac{1}{\sqrt{n}}$

$\sum a_n = \frac{1}{(n)^{1/2}}$ which is the p-series with $p=\frac{1}{2}$ i.e. <1 so that the given series is divergent ~~Red~~

Now.

$a_n = \frac{1}{\sqrt{n}}$ and we take $a_{n+1} = \frac{1}{\sqrt{n+1}}$

(a) $a_n > b_n$

(b) $\lim_{n\to\infty} a_n = \frac{1}{\sqrt{n}} = \frac{1}{\infty} = 0$

~~(ii)~~ so by Leibnitz's theorem the series is convergent so that the given series is conditionally convergent

(c) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n\sqrt{n}}$

Sol^n: Here

$a_n = \frac{1}{(n)^{3/2}}$ ~~(ii)~~

i.e. $\sum a_n = \sum \frac{1}{n^{3/2}}$ which is the p-series with $p = \frac{3}{2} > 1$ so that the given series absolutely convergent

(d) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{a+nb}$

Sol^n: Here,

$a_n = \frac{1}{a+nb}$ and we take $b_n = \frac{1}{n}$

and $\lim_{n\to\infty} \left|\frac{a_n}{b_n}\right| = \left[\frac{\frac{1}{a+nb}}{\frac{1}{n}}\right] = \frac{1}{a+nb} \times \frac{n}{1}$

$\lim_{n\to\infty} \frac{1}{\cancel{n}(b+\frac{a}{n})} \times \cancel{n}$

$\frac{1}{b}$ which is finite so that both the series are convergent/diver

$\sum b_n = \sum \frac{1}{n}$ which is the p series with $p=1 \leq 1$ so that the $\sum b_n$ is convergent series hence $\sum a_n$ is also convergent series,

now $a_n = \frac{1}{a+nb}$ and $a_{n+1} = \frac{1}{a+(n+1)b}$

(i) $a_n > a_{n+1}$

(b) $\lim_{n\to\infty} a_n = \frac{1}{a+nb}$

$a_n = \frac{1}{\infty} = 0$

So by the Leibnitz's theorem the given series is conditionally convergent

(e) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{\ln(n+1)}$

Sol^n: here,

$a_n = \frac{1}{\ln(n+1)}$ & we take $b_n = \frac{1}{\ln(n+2)}$

(a) $a_n > b_n$

(b) $\lim_{n\to\infty} a_n = \frac{1}{\ln(n+1)}$

$a_n = \frac{1}{\infty} = 0$

So by the Leibnitz theorem the given series is absolutely convergent.

(f) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n!}$

Sol^n: here

$a_n = \frac{1}{n!}$ and we take $a_{n+1} = \frac{1}{(n+1)!}$

By ratio test theorem,

$\lim_{n\to\infty} \left(\frac{a_{n+1}}{a_n}\right)$

$\lim_{n\to\infty} \frac{1}{(n+1)!} \times n!$

$\lim_{n\to\infty} \frac{1}{(n+1) \times \cancel{n!}} \times \cancel{n!}$

$\frac{1}{\infty} = 0 < 1$

So that the given series is absolutely convergent.

(g) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{3^{n-2}}$

Sol^n: here

$a_n = \frac{1}{3^{n-2}}$ and $a_{n+1} = \frac{1}{3^{n-1}}$

(a) $a_n > a_{n+1}$

(b) $\lim_{n\to\infty} a_n = \frac{1}{3^{n-2}}$

$\frac{1}{3^{\infty}} = \frac{1}{\infty} = 0$

So by the Leibnitz's theorem the given series is absolutely convergent.

By ratio test

$\lim_{n\to\infty} \left(\frac{a_{n+1}}{a_n}\right) = \lim_{n\to\infty} \left(\frac{1}{3^{n-1}} \times 3^{n-2}\right)$

$\lim_{n\to\infty} \left(\frac{1}{3^n \cdot 3^{-1}} \quad 3^n \cdot 3^{-2}\right)$

$\lim_{n\to\infty} \left(3^{1-2}\right) = 3^{-1}$

$\frac{1}{3} < 1$

So that the given series is absolute convergent.

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(h) $\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!}$

Sol: here

$a_n = \frac{1}{(2n)!}$ and $a_{n+1} = \frac{1}{[2(n+1)]!}$

By Ratio test,

$\lim_{n\to\infty} \left(\frac{a_{n+1}}{a_n}\right) = \lim_{n\to\infty} \frac{1}{(2n+2)!} \times (2n)!$

$\lim_{n\to\infty} \frac{1}{(2n+2)(2n+1)\times\cancel{(2n)!}} \cancel{(2n)!}$

$\lim_{n\to\infty} \frac{1}{\infty} = 0 < 1$ so that

the given series is absolutely convergent.

(i) $\sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{\sqrt{n}+1}{n+1}\right)$

Sol: here

$a_n = \frac{\sqrt{n}+1}{n+1}$ & $a_{n+1} = \frac{\sqrt{n+1}+1}{n+2}$

(a) $a_n > a_{n+1}$

(b) $\lim_{n\to\infty} a_n = \frac{\sqrt{n}+1}{n+1} \left[\frac{\infty}{\infty}\right]$

$\frac{\frac{1}{2\sqrt{n}}}{1} = \frac{1}{2\times\infty} = 0$

So by the Leibnitz theorem the given series is convergent.

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(j) $\sum_{n=1}^{\infty} (-1)^{n+1} \left(\sqrt{n+1} - \sqrt{n}\right)$

Sol: here,

$a_n = (\sqrt{n+1} - \sqrt{n})$ & we take $a_{n+1} = (\sqrt{n+2} - \sqrt{n+1})$

(a) $a_n > a_{n+1}$

(b) $\lim_{n\to\infty} a_n = \sqrt{n+1} - \sqrt{n}$

$= \infty$ which is divergent

~~Now~~

Now:

$a_n = \frac{\sqrt{n+1} - \sqrt{n}}{\sqrt{n+1} + \sqrt{n}} \times \sqrt{n+1} + \sqrt{n}$

$a_n = \frac{n+1-n}{\sqrt{n+1}+\sqrt{n}} = \frac{1}{\sqrt{n+1}+\sqrt{n}}$

$a_{n+1} = \frac{1}{\sqrt{n+2} + \sqrt{n+1}}$

By Ratio test

$\lim_{n\to\infty} \left(\frac{a_{n+1}}{a_n}\right) = \frac{1}{\sqrt{n+2}+\sqrt{n+}}$

Exercise :- 8.5

(1) ~~Find the interval, centre & radius of convergence of the following~~ s

(1) For what value of x, the following series convergence.

Sol^n: Here

(a) $\sum_{n=1}^{\infty} n^2 x^{n-1}$

Sol^n: here,

$$\sum_{n=1}^{\infty} n^2 x^{n-1}$$

$a_n = n^2 x^{n-1}$ & we take $a_{n+1} = (n+1)^2 x^n$

By D'Almbert Ratio test,

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^2 x^n}{n^2 x^{n-1}}\right|$$

$$\lim_{n\to\infty}\left|\frac{\cancel{n^2}(1+\frac{1}{n^2})\cancel{x^n}}{\cancel{n^2}\,\cancel{x^n}\cdot x^{-1}}\right|$$

$$\lim_{n\to\infty}\left|(1+\frac{1}{n^2})\cdot x\right|$$

$$|x|$$

Here the given series is convergent when $|x|<1$ and divergent when $|x|>1$ i.e. $-1<x<1$

~~For $x<-1$,~~ i.e. $-1<x<1$ the series is convergent.

For $x=-1$ the given series become

$$\sum_{n=1}^{\infty} n^2 \cdot (-1)^{n-1} = 1-2^2+3^2-4^2+\cdots$$

which is alternate series, and $a_n < a_{n+1}$

by leibnitz theorem, the series is divergent,

For $x=1$, the series become,

$\sum_{n=1}^{\infty} n^2$ and $\lim_{n\to\infty} a_n = \infty \neq 0$, divergent by the nth term test

Here the given series is convergent $-1<x<1$
i.e $(-1, 1)$ i.e. $|x|<1$.

(b) $\sum_{n=1}^{\infty}\left(\frac{2^n-2}{2^n+1}\right)x^{n-1}$

Soln: Here,

$a_n=\left(\frac{2^n-2}{2^n+1}\right)x^{n-1}$ & $a_{n+1}=\left(\frac{2^{n+1}-2}{2^{n+1}+1}\right)x^n$

By Almbert test

$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\left(\frac{2^{n+1}-2}{2^{n+1}+1}\right)x^n \Big/ \left(\frac{2^n-2}{2^n+1}\right)\right|$

$\lim_{n\to\infty}\left|\left(\frac{2^{n+1}-2}{2^{n+1}+1}\right)x^n \times \left(\frac{2^n+1}{2^n-2}\right)\frac{1}{x^{n-1}}\right|$

$\lim_{n\to\infty}\left|\frac{2^n(2-2/2^n).x^n}{2^n(2+1/2^n)} \times \frac{2^n(1+\frac{1}{2^n})}{2^n(1-2/2^n)x^n.x^{-1}}\right|$

$\lim_{n\to\infty}\left|\frac{(2-2/2^n)}{(2+1/2^n)} \times \frac{(1+1/2^n).x}{(1-2/2^n)}\right|$

$\lim \left|\left(\frac{2-0}{2+0}\right).\left(\frac{1+0}{1-0}\right)x\right|$

$|x|$

Here the given series is convergent when $|x|<1$ and divergent when $|x|>1$

Such that $-1<x<1$,

Now

for $x=-1$ the given series become,

$\sum_{n=1}^{\infty}\left(\frac{2^n-2}{2^n+1}\right)(-1)^{n-1}=0-\frac{2}{5}+\frac{2}{3}-\dots\infty$

which is alternative series.

in this series $a_n<a_{n+1}$ so by leibritz theorem test, the given series is divergent, at $x=-1$

Again

when $|x|=1$ the given series become,

$\sum_{n=1}^{\infty}\frac{(2^n-2)}{2^n+1}(1)^{n-1}$

By limit compa test,

$\lim_{n\to\infty}\frac{2^n(1-2/2^n)}{2^n(1+1/2^n)}$

$\frac{1}{1}$

$1\neq0$ so that the series is divergent

Hence the required interval in which the given series is convergent i.e. $(-1, 1)$ i.e. $|x|<1$

(c) $\sum_{n=1}^{\infty} \left(\frac{x^n}{3^n . n^2} \right)$ for $x > 0$

Solⁿ: here,

$a_n = \left(\frac{x^n}{3^n . n^2} \right)$ & we take $a_{n+1} = \left(\frac{x^{n+1}}{3^{n+1} . (n+1)^2} \right)$

By D-Alembert Ratio test

$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{x^{n+1}}{3^{n+1} . (n+1)^2} \cdot \frac{3^n . n^2}{x^n} \right|$

$\lim_{n \to \infty} \left| \frac{\cancel{x^n} . x}{\cancel{3^n} . 3 (n+1)^2} \cdot \frac{\cancel{3^n} . n^2}{\cancel{x^n}} \right|$

$\lim_{n \to \infty} \left| \frac{x}{3} \frac{1}{\cancel{n^2} (1 + 1/n)^2} . \cancel{n^2} \right|$

$\left| x/3 \right|$

the series is convergent when $|x/3| < 1$
and divergent when $|x/3| > 1$

Now $|x/3| < 1 \Rightarrow |x| < 3 \Rightarrow -3 < x < 3$ convergent

at $x = -3$, the ~~seri~~ given ~~g~~ series become,

$\sum_{n=1}^{\infty} \frac{(-3)^n}{3^n . n^2} \Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = -1 + 1/4 - 1/9 + \frac{1}{16}$
which is alternative series.

~~By leibnitz theorem,~~
(a) In this series, $a_n > a_{n+1}$
(b) $\lim_{n \to \infty} a_n = \frac{1}{n^2}$

$= 1/\infty = 0$ which is convergent,

By the Leibnitz theorem the given series is convergent.

at $x = -3$,
the given series become:-

$$\sum_{n=1}^{\infty} \frac{3^n}{3^n \cdot n^2} \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2}$$ which is p-series

in which $p = 2 > 1$ so that the given series is convergent,

∴ the value of x in which the series is convergence is $[-3, 3]$ i.e. $|x| < 3$

(d) $\sum_{n=1}^{\infty} \left(\frac{x^n}{n}\right)$ for $x > 0$

Sol^n: here,

$$\sum_{n=1}^{\infty} \left(\frac{x^n}{n}\right) \text{ for } x > 0$$

$$\sum_{n=1}^{\infty} \left(\frac{x^n}{n}\right)$$

$a_n = \frac{x^n}{n}$ and we take $a_{n+1} = \frac{x^{n+1}}{n+1}$

By D-Alembert Ratio test,

$$\lim_{n\to\infty} \left(\frac{a_{n+1}}{a_n}\right) = \lim_{n\to\infty} \left|\left(\frac{x^{n+1}}{n+1} \times \frac{n}{x^n}\right)\right|$$

$$\lim_{n\to\infty} \left|\left(\frac{\cancel{x^n} \cdot x^1 \cdot n}{n+1 \cdot \cancel{x^n}}\right)\right|$$

$$\lim_{n\to\infty} \left|\left(\frac{\cancel{n} \cdot x}{\cancel{n}(1 + 1/n)}\right)\right|$$

$|x|$

Here the given series become convergent when $|x| < 1$ and divergent when $|x| > 1$,

Now $|x| < 1 \Rightarrow -1 < x < 1$

at $x = -1$ the given series become,

$$\sum_{n=1}^{\infty} \left(\frac{(-1)^n}{n}\right) = -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \dots$$

which is alternative series,

~~In th~~ By leibnitz theorem test:

(a) $a > a_{n+1}$

(b) $\lim_{n\to\infty} a_n = \frac{1}{n} = \frac{1}{\infty} = 0$

~~Here~~ ~~By~~ the leibnitz theorem the given series is convergent at $x = -1$ and,

at $x = 1$ the series become,

$$\sum_{n=1}^{\infty} \frac{1}{n}$$ which is p-series in which $p \leq 1$ so that the given series divergent such that,

The value of x in which the series is convergent is $[-1, 1) \Rightarrow -1 \leq x < 1$

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(2) Find the interval, centre, & radius of convergence of the following series.

(a) $\sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!}$

Sol^n: Here

$a_n = \frac{x^{n-1}}{(n-1)!}$ & we take $a_{n+1} = \frac{x^n}{(n)!}$

By D-Alembert test

$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{x^n}{n!} \times \frac{(n-1)!}{x^n \cdot x^{-1}}\right|$

$\lim_{n\to\infty} \left|\frac{\cancel{x^n}}{n\cancel{(n-1)!}} \times \frac{\cancel{(n-1)!}}{\cancel{x^n} \cdot x^{-1}}\right|$

$\lim_{n\to\infty} \left|\frac{x}{n}\right|$

$\left|\frac{x}{\infty}\right| = 0$

Here the series is convergent to all value of x and interval is $(-\infty, \infty)$

centre $= 0$

radius $= \infty$

1, 3, 5, 7
$a+(n-1)\times d$
$1+(n-1)\times 2$
$2n-1$

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(b) $x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots$

Sol^n: here,

$x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots$

$\frac{(x)^{2n-1}}{2n-1}$

Now

$\sum_{n=1}^{\infty} \frac{(x)^{2n-1}}{2n-1} \; (-1)^{n-1}$.

$a_n = \frac{(x)^{2n-1}}{2n-1}$ & $a_{n+1} = \frac{(x)^{2(n+1)-1}}{2(n+1)-1} = \frac{x^{2n+1}}{2n+1}$

By D-Alembert Ratio test,

$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\left(\frac{x^{2n+1}}{2n+1} \times \frac{2n-1}{(x)^{2n-1}}\right)\right|$

$\lim_{n\to\infty} \left|\left(\frac{x^{2n+1-2n+1}}{2n+1} (2n-1)\right)\right|$

$\lim_{n\to\infty} \left|\frac{x^2}{\cancel{n}(2+1/n)} \times \cancel{n}(2-1/n)\right|$

$\left|\frac{x^2}{\cancel{2}} \times \cancel{2}\right|$

$|x^2|$

Here the given series is convergent when $|x^2| < 1$ and divergent when $|x^2| > 1$,

Now

$|x^2| < 1 \Rightarrow -1 < x < 1$

at $x=-1$, the series become

$a_n = \frac{1}{2n-1}$ and $a_{n+1} = \frac{1}{2n+1}$

$$\sum_{n=1}^{\infty} \frac{(-1)^{2n-1}}{2n-1} \Rightarrow -$$

By liberty theorem, (which alternative series)

(a) $a_n > a_{n+1}$

(b) $\lim_{n\to\infty} a_n = \frac{1}{2n-1}$

$$\frac{1}{\infty} = 0$$

By liberty theorem the given series is convergent at $x=-1$

at $x=1$, the series become,

$$\sum_{n=1}^{\infty} \frac{(1)^{2n-1}}{2n-1} \Rightarrow \frac{1}{2n-1}$$

$$\lim_{n\to\infty} \frac{1}{2n-1}$$

$$\frac{1}{\infty} = 0 \quad \text{which is covergent}$$

The series is convergent for the particular value of x so that the given interval is $[-1,1]$

centre $= \frac{-1+1}{2} = 0$

radius $= \frac{1+1}{2} = 1$

(c) $$\sum_{n=1}^{\infty} \left(\frac{n+1}{10^n}\right)(x-4)^n$$

sol^n: here,

$$\sum_{n=1}^{\infty} \left(\frac{n+1}{10^n}\right)(x-4)^n$$

$a_n = \left(\frac{n+1}{10^n}\right)(x-4)^n$ & $a_{n+1} = \left(\frac{n+2}{10^{n+1}}\right)(x-4)^{n+1}$

By D-Alembert ratio test,

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+2)}{10^{n+1}}(x-4)^{n+1} \cdot \frac{10^n}{(n+1)(x-4)^n}\right|$$

$$\lim_{n\to\infty} \left|\frac{\cancel{n}(1+2/n)\cancel{(x-4)^n}(x-4)^1}{\cancel{10^n}\cdot 10} \times \frac{\cancel{10^n}}{\cancel{n}(1+1/n)\cancel{(x-4)^n}}\right|$$

$$\left|\frac{(1+0)(x-4)}{10(1+0)}\right|$$

$$\left|\frac{x-4}{10}\right|$$

Here the given series is convergent when $\left|\frac{x-4}{10}\right| < 1$ and divergent as $\left|\frac{x-4}{10}\right| > 1$

Now

$\left|\frac{x-4}{10}\right| < 1 \Rightarrow |x-4| < 10 \Rightarrow -10 < x-4 < 10$

$-10+4 < x-4+4 < 10+4$

$-6 < x < 14$

at $x=-6$, the series become

$$\sum_{n=1}^{\infty} \left(\frac{n+1}{10^n}\right)(-6-4)^n = \left(\frac{n+1}{10^n}\right)(-10)^n$$

$$= \frac{(n+1)}{\cancel{10^n}}(-1)^n\cancel{(10)^n}$$

$= (n+1)(-1)^n$ which is alternative series,

$a_n = n+1$ and $a_{n+1} = n+2$

(a) $a_n < a_{n+1}$

By leibnitz theorem the given series is divergent

at $x = 14$

$$\sum_{n=1}^{\infty} \left(\frac{n+1}{10^n}\right)(10)^n$$

$$\sum_{n=1}^{\infty} (n+1)$$

Here $a_n = (n+1)$

$\lim_{n\to\infty} (a_n) = n+1$

∞ which is divergent,

the given series is convergent in between the interval $(-6, 14)$

centre $= \frac{-6+14}{2} = 8/2 = 4$

radius $= \frac{-6-14}{2} = \frac{-20}{2} = 10$

(d) $$\sum_{n=0}^{\infty} \frac{3^{2n}}{(n+1)} (x-2)^n$$

sol^n: here,

$$\sum_{n=0}^{\infty} \frac{3^{2n}}{(n+1)} (x-2)^n$$

$$a_n = \sum_{n=0}^{\infty} \left(\frac{3^{2n}}{n+1}\right)(x-2)^n$$

$$a_{n+1} = \frac{3^{2(n+1)}}{n+2} (x-2)^{n+1}$$

By D-Alembert ratio test

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{3^{2n+2}}{n+2}(x-2)^{n+1} \times \frac{(n+1)}{3^{2n}(x-2)^n}\right|$$

$$\lim_{n\to\infty} \left|\frac{\cancel{3^{2n}} \cdot 3^2}{n+2} \cancel{(x-2)^n} \cdot (x-2) \times \frac{(n+1)}{\cancel{3^{2n}} \cancel{(x-2)^n}}\right|$$

$$\lim_{n\to\infty} \left|\frac{9(x-2) \cancel{n}(1+1/n)}{\cancel{n}(1+2/n)}\right|$$

$|9(x-2)|$

Here the given series is convergent when $|9(x-2)| < 1$ and divergent when $|9x-18| > 1$

Now $|9x-18| < 1 \Rightarrow -1 < 9x-18 < 1$

$-1+18 < 9x-18+18 < 1+18$

$17 < 9x < 19$

$17/9 < x < 19/9$

at $x = 17/9$ the series become.

$$\sum_{n=0}^{\infty} \frac{3^{2n}}{(n+1)} (17/9 - 2)^n = \frac{3^{2n}}{(n+1)} \left(\frac{-1}{9}\right)^n$$

$a_n = \frac{3^{2n}}{n+1}\left(\frac{1}{9}\right)$ and $a_{n+1} = \frac{3^{2(n+1)}}{n+2} \times \frac{1}{9}$

$a_n = \frac{3^{2n}}{(n+1)} \frac{1}{9}$ and $\frac{3^{2n+2}}{n+2} \left(\frac{1}{9}\right)$

Here (i) $a_n < a_{n+1}$
by the leibnitz theorem the ~~series~~ series is divergent at $x = 17/9$ &

at $x = \frac{19}{9}$

$a_n = \frac{3^{2n}}{(n+1)} \left(\frac{19-2}{9}\right)^n$

$a_n = \frac{3^{2n}}{(n+1)} \left(\frac{19-18}{9}\right)^n$

$a_n = \frac{3^{2n}}{n+1} \left(\frac{1}{9}\right)^n$

$a_n = \frac{3^{2n}}{n+1} \frac{1}{3^{2n}}$

$a_n = \frac{1}{n+1}$

Now $\lim_{n\to\infty} a_n = \frac{1}{n+1}$ which is p-series with $p = 1 < 1$ so that
$\frac{1}{\infty} = 0$, the given series is divergent.

the required intervals $\left(\frac{17}{9}, \frac{19}{9}\right)$

centre $= \left(\frac{17+19}{9 \times 2}\right) = \frac{36}{18} = 2$

radius $= \frac{17-19}{18} = \frac{2}{18} = \frac{1}{9}$

02

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(3) Using Maclaurin Series Expansion, show that;

(a) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots + (-1)^n \frac{x^{2n}}{(2n)!} + \dots$

Sol^n: Here,

$f(x) = \cos x$ $\quad$ $f'(x) = -\sin x$
$f''(x) = -\cos x$ $\quad$ $f'''(x) = \sin x$
$f^{iv}(x) = \cos x$ $\quad$ $f^{v}(x) = -\sin x$
$f^{vi}(x) = -\cos x$ $\quad$ $f^{vii}(x) = \sin x$
$f^{2n}(x) = (-1)^{n} \cos x$ $\quad$ $f^{2n+1}(x) = (-1)^{n} \sin x$

put $x = 0$,
$f(0) = 1$ $\quad$ $f'(0) = 0$
$f''(0) = -1$ $\quad$ $f'''(0) = 0$
$f^{iv}(0) = 1$ $\quad$ $f^{v}(0) = 0$
$f^{vi}(0) = -1$ $\quad$ $f^{vii}(0) = 0$

$f^{2n}(0) = (-1)^n$

By Maclaurin series,
$f(x) = f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \frac{f'''(0)}{3!} x^3 + \frac{f^{iv}(0)}{4!} x^4$

$f(x) = 1 + 0 - \frac{x^2}{2!} + 0 + \frac{1 x^4}{4!} + \dots + \frac{x^{2n}}{(2n)!} (-1)^n + \dots$

$f(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots + \frac{x^{2n}}{(2n)!} (-1)^n + \dots$

Ans

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(b) $\frac{1}{1-x} = 1+x+x^2+\ldots$ for $|x|<1$

Sol^n: here,

$f(x) = \frac{1}{1-x}$ $\quad$ $f'(x) = (1-x)^{-1} = -1(1-x)^{-2}\cdot -1$

$f'(x) = 1(1-x)^{-2}$

$f''(x) = -2(1-x)^{-3}\cdot -1$ $\quad$ $f'''(x) = -6(1-x)^{-4}\cdot -1$

$= 2(1-x)^{-3}$ $\quad$ $= 6(1-x)^{-4}$

$f^{iv}(x) = 24(1-x)^{-5}$ $\quad$ $f^{v}(x) = 120(1-x)^{-6}$

put $(x=0)$ we get,

$f(0) = 1$ $\quad$ $f'(0) = 1$

$f''(0) = 2$ $\quad$ $f'''(0) = 6$

$f^{iv}(x) = 24$ $\quad$ $f^{v}(0) = 120$

Now,

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \frac{f^{iv}(0)x^4}{4!} + \ldots$$

$$f(x) = 1 + 1\cdot x + \frac{2\cdot x^2}{2!} + \frac{6x^3}{3!} + \frac{24x^4}{4!} + \ldots$$

$$f(x) = 1+x+x^2+x^3+x^4+\ldots+x^n+\ldots$$ Ans.

(c) $e^{ax} = 1 + ax + \frac{a^2x^2}{2!} + \ldots \frac{a^nx^n}{n!} + \ldots$

Sol^n: here,

$f(x) = e^{ax}$ $\quad$ $f'(x) = ae^{ax}$

$f''(x) = a^2e^{ax}$ $\quad$ $f'''(x) = a^3e^{ax}$

$f^{iv}(x) = a^4e^{ax}$ $\quad$ $f^{v}(x) = a^5e^{ax}$

put $x=0$,

$f(0) = 1$ $\quad$ $f'(0) = a$

$f''(0) = a^2$ $\quad$ $f'''(0) = a^3$

$f^{iv}(0) = a^4$ $\quad$ $f^{v}(0) = a^5$,

We know that,

$$f(x) = f(0) + \frac{f'(0)\cdot x}{1!} + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \frac{f^{iv}(0)x^4}{4!} + \ldots$$

$$f(x) = 1 + ax + \frac{a^2x^2}{2!} + \frac{a^3x^3}{3!} + \frac{a^4x^4}{4!} + \ldots \frac{a^nx^n}{n!} + \ldots$$

Ans.

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(d) $\cos 2x = 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \dots (-1)^n \frac{(2x)^{2n}}{(2n)!}$

Sol^n: Here,

$f(x) = \cos 2x$ $\quad f'(x) = -2\sin 2x$

$f''(x) = -4\cos 2x$ $\quad f'''(x) = 8\sin 2x = 8\sin 2x$

$f^{iv}(x) = 16\cos 2x$, $\quad f^{v}(x) = -32\sin 2x$

$f^{vi}(x) = -64\cos 2x$ $\quad f^{vii}(x) = 128\sin 2x$

put $x=0$

$f(0) = 1$ $\quad f'(0) = 0$

$f''(0) = -4$ $\quad f'''(0) = 0$

$f^{iv}(0) = 16$ $\quad f^{v}(0) = 0$

$f^{vi}(0) = -64$ $\quad f^{vii}(0) = 0$

Now

we know that,

$$f(x) = f(0) + \frac{f'(0)x}{1!} + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \frac{f^{iv}(0)x^4}{4!} + \frac{f^{v}(0)x^5}{5!} + \frac{f^{vi}(0)x^6}{6!} + 0 \dots$$

$$f(x) = 1 + 0 + \frac{(-4)x^2}{2!} + 0 + \frac{16x^4}{4!} + 0 + \frac{(-64)x^6}{6!} + \dots$$

$$f(x) = 1 - \frac{4x^2}{2!} + \frac{16x^4}{4!} - \frac{64x^6}{6!} + \dots$$

$$f(x) = 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \dots$$

$$f(x) = 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \dots \frac{(2x)^{2n}}{(2n)!}(-1)^n$$

Ans

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(e) $\sin 2x = 2x - \frac{(2x)^3}{3!} + \dots + (-1)^n \frac{(2x)^{2n+1}}{(2n+1)!} + \dots$

Sol^n: Here,

$\sin 2x = 2x - \frac{(2x)^3}{3!} + \dots + (-1)^n \frac{(2x)^{2n+1}}{(2n+1)!} + \dots$

$f(x) = \sin 2x$ $\quad f'(x) = 2\cos 2x$

$f''(x) = -4\sin 2x$ $\quad f'''(x) = -8\cos 2x$

$f^{iv}(x) = 16\sin 2x$ $\quad f^{v}(x) = 32\cos 2x$

$f^{vi}(x) = -64\sin 2x$ $\quad f^{vii}(x) = -128\cos 2x$

put $x=0$, we get,

$f(0) = 0$, $\quad f'(0) = 2$

$f''(0) = 0$ $\quad f'''(0) = -8$

$f^{iv}(0) = 0$ $\quad f^{v}(0) = 32$

$f^{vi}(0) = 0$ $\quad f^{vii}(x) = -128$

Now

$$f(x) = f(0) + \frac{f'(0)x}{1!} + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \frac{f^{iv}(0)x^4}{4!} + \frac{f^{v}(0)x^5}{5!} + \frac{f^{vi}(0)x^6}{6!} + \dots$$

$$f(x) = 0 + \frac{2x}{1} + 0 - \frac{8x^3}{3!} + 0 + \frac{32x^5}{5!} + 0 - \frac{128x^7}{7!} + \dots$$

$$f(x) = \frac{2x}{1!} - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} + \dots (-1)^n \frac{(2x)^{2n+1}}{(2n+1)!} + \dots$$

Ans,

① $x\cos x = x - \frac{x^3}{2!} + \frac{x^5}{4!} + \dots (-1)^n \frac{x^{2n+1}}{(2n)!} + \dots$

Solⁿ: Here,

$f(x) = x\cos x$ $\qquad f'(x) = -x\sin x + \cos x$

$f''(x) = -x\cos x - 1\cdot\sin x$ $\qquad f'''(x) = x\sin x - \cos x - \cos x$

$= x\sin x - 2\cos x$

$f^{iv}(x) = x\cos x + \sin x + 2\sin x$ $\qquad f^{v}(x) = -x\cos x + \cos x + 3\cos x$

$= x\cos x + 3\sin x$ $\qquad = -x\sin x + 4\cos x$

put $x = 0$; we get,

$f(0) = 0$; $f'(0) = 1$

$f''(0) = 0$; $f'''(0) = -2$

$f^{iv}(0) = 0$; $f^{v}(0) = 4$

Now, we know that,

$$f(x) = f(0) + \frac{f'(0)\cdot x}{1!} + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \frac{f^{iv}(0)x^4}{4!} + \frac{f^{v}(0)x^5}{5!} + \frac{f^{vi}(0)\cdot x^6}{6!} + \dots$$

$$f(x) = 0 + \frac{x}{1!} + 0 + \frac{-2\cdot x^3}{3!} + 0 + \frac{4x^5}{5!} + \dots$$

$$f(x) = x - \frac{2x^3}{3!} + \frac{4x^5}{5!} + \dots$$

$$f(x) = x - \frac{2x^3}{3!} + \frac{x^5}{5!} + \dots$$

$$f(x) = x - \frac{2x^3}{3!} + \frac{4x^5}{5!} + \dots$$

$x\cos x = x - \frac{x^3}{2!} + \frac{x^5}{4!} + \dots + (-1)^n \frac{x^{2n+1}}{(2n)!}$

Sol: here

let $f(x) = \cos x$,

$f(x) = \cos x$ & $f'(x) = -\sin x$

$f''(x) = -\cos x$ & $f'''(x) = \sin x$

$f^{iv}(x) = \cos x$ & $f^{v}(x) = -\sin x$

$f^{2n}(x) = (-1)^n \cos x$ & $f^{2n+1}(x) = (-1)^n \sin x$

put $x = 0$,

$f(0) = 1$ & $f'(0) = 0$

$f''(0) = -1$ & $f'''(0) = 0$

$f^{iv}(0) = 1$ & $f^{v}(0) = 0$

$f^{2n}(0) = (-1)^n$ & $f^{2n+1}(0) = (-1)^n\cdot 0$

$= 0.$

We know that

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \frac{f^{iv}(0)x^4}{4!} + \dots$$

$$f(x) = 1 + 0 + \frac{-1x^2}{2!} + 0 + \frac{1x^4}{4!} + 0 + \dots$$

$$f(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + -\frac{x^6}{6!} + \dots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

$$x\cos x = x - \frac{x^3}{2!} + \frac{x^5}{4!} - \frac{x^7}{6!} + \dots$$ Ans.

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④ Find the Taylor Series expansion.

ⓞ $f(x) = \frac{1}{x}$ at $a=2$ and at $a=-1$.

Sol^n: Here

let $f(x) = \frac{1}{x} = x^{-1}$

$f(x) = x^{-1}$ & $f'(x) = -1x^{-2}$
$f''(x) = 2x^{-3}$ & $f'''(x) = -6x^{-4}$.
$f^{iv}(x) = 24x^{-5}$ & $f^{v}(x) = -120x^{-6}$.

Now:-

$f(2) = \frac{1}{2}$ & $f'(2) = \frac{-1}{4}$

$f''(2) = 2 \times \frac{1}{8} = \frac{1}{4}$ & $f'''(2) = \frac{-6}{16} = \frac{-3}{8}$

$f^{iv}(2) = 24 \times \frac{1}{32} = \frac{3}{4}$

Now,

$f(2) = f(2) + \frac{f'(2)(x-2)}{1!} + \frac{f''(2)(x-2)^2}{2!} + \frac{f'''(2)(x-2)^3}{3!} + \frac{f^{iv}(2)(x-2)^4}{4!}$

$f(2) = \frac{1}{2} - \frac{1}{4}(x-2) + \frac{1}{4}\frac{(x-2)^2}{2!} - \frac{3}{8}\times\frac{(x-2)^3}{3!} + \frac{3}{4}\cdot\frac{(x-2)^4}{4!} + \cdots$

$f(2) = \frac{1}{2} - \frac{x}{4} + \frac{x^2}{8} - \frac{1}{16}x^3 + \frac{3}{32}x^4$

$f(2) = \frac{1}{2} - \frac{1}{4}x\left(\frac{x}{2} - 1\right) + \frac{1}{8}x^3\left(\frac{3x}{4} - \frac{1}{2}\right) + \cdots$

Ans

$f(x) = \frac{1}{2} - \frac{(x-2)}{4} + \frac{1}{8}(x-2)^2 - \frac{3}{8}\frac{(x-2)^3}{3!} + \frac{3}{4}\frac{(x-2)^4}{4!} \cdots$

$\frac{1}{x} = \frac{1}{2} - \frac{(x-2)}{4} + \frac{(x-2)^2}{8} - \frac{3}{8}\frac{(x-2)^3}{3!} + \frac{3}{4}\frac{(x-2)^4}{4!} + \cdots$

$\frac{1}{x} = \frac{1}{2} - \frac{(x-2)}{4} + \frac{(x-2)^2}{8} - \frac{1}{16}(x-2)^3 + \frac{1}{32}(x-2)^4 + \cdots + \frac{(x-2)^{n-1}(-1)^{n+1}}{2^n}$

Similarly at $a=-1$,

$f(-1) = -1$ & $f'(-1) = -1$
$f''(-1) = -2$ & $f'''(-1) = -6$
$f^{iv}(-1) = -24$ & $f^{v}(x) = -120$

Now

$f(x) = f(-1) + \frac{f'(-1)(x-a)}{1!} + \frac{f''(-1)(x-a)^2}{2!} + \frac{f'''(-1)(x-a)^3}{3!}$
$+ \frac{f^{iv}(-1)(x-a)^4}{4!} + \frac{f^{v}(-1)(x-a)^5}{5!} + \cdots$

$f(x) = -1 - \frac{(x+1)}{1!} + \frac{-2(x+1)^2}{2!} - \frac{6(x+1)^3}{3!}$
$- \frac{24(x+1)}{4!} + \frac{-120(x+1)^5}{5!}$

$f(x) = -1 - \frac{(x+1)}{1!} - \frac{2(x+1)^2}{2!} - \frac{6(x+1)^3}{3!} - \frac{24(x+1)^4}{4!}$
$- \frac{120(x+1)^5}{5!}$

$f(x) = -1 - (x+1) - (x+1)^2 - (x+1)^3 - (x+1)^4 - (x+1)^5$
$+ \cdots (-1)^{n}(x+1)^{n-1}$.

$f(x)$