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Exercise 7.3

(2) Half solution

We have

$A^4 = PD^4P^{-1}$

For P^{-1}

$$P^{-1} = \frac{Adj.P}{\det P} = \frac{\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}}{(1-2)} = -\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ -2 & -1 \end{bmatrix}$$

$\therefore A^4 = PD^4.P^{-1}$

$$= \begin{bmatrix} 1 & -1 \\ -2 & 1 \end{bmatrix} . \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix}^4 . \begin{bmatrix} -1 & -1 \\ -2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 81 & 0 \\ 0 & 625 \end{bmatrix} . \begin{bmatrix} -1 & -1 \\ -2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 81 & -625 \\ -162 & 625 \end{bmatrix} . \begin{bmatrix} -1 & -1 \\ -2 & -1 \end{bmatrix}$$

$$= \cancel{\begin{bmatrix} -81+1250 & 162-1250 \\ -81+625 & 162-625 \end{bmatrix}} \begin{bmatrix} -81+1250 & -81+625 \\ 162-1250 & 162-625 \end{bmatrix}$$

$$\cancel{\begin{bmatrix} 1169 & -1088 \\ 544 & -463 \end{bmatrix}} \begin{bmatrix} 1169 & 544 \\ -1088 & -463 \end{bmatrix}$$

Ans

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A. Diagonalize the following matrix if possible

(a) $\begin{bmatrix} 1 & 0 \\ 6 & -1 \end{bmatrix}$

Sol^n: Here,

let $A = \begin{bmatrix} 1 & 0 \\ 6 & -1 \end{bmatrix}$

$A - \lambda I = \begin{bmatrix} 1 & 0 \\ 6 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1-\lambda & 0 \\ 6 & -1-\lambda \end{bmatrix}$

The characteristic equation is

$|A - \lambda I| = 0$

$\begin{bmatrix} 1-\lambda & 0 \\ 6 & -1-\lambda \end{bmatrix} = 0$

$(1-\lambda)(-1-\lambda) = 0$

$\lambda = 1, -1$

For $\lambda = 1$

$A - \lambda I = \begin{bmatrix} 1-1 & 0 \\ 6 & -1-1 \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 \\ 6 & -2 \end{bmatrix}$

The augmented matrix is

$[A - \lambda I \; 0] = \begin{bmatrix} 0 & 0 & 0 \\ 6 & -2 & 0 \end{bmatrix}$

$R_1 \leftrightarrow R_2$

$= \begin{bmatrix} 6 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$R_1 \to R_1/2 \cdot \begin{bmatrix} 3 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

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Here x_2 is a free variable then corresponding equation is

$3x_1 - x_2 = 0$

$3x_1 = x_2$

$x_1 = \frac{1}{3} x_2$

$x_2 =$ free

$\therefore x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix} = 3x_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$\therefore$ The eigen space is $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$

For $\lambda = -1$

$A - \lambda I = \begin{bmatrix} 2 & 0 \\ 6 & 0 \end{bmatrix}$

The augmented matrix is

$[A - \lambda I \; 0] = \begin{bmatrix} 2 & 0 & 0 \\ 6 & 0 & 0 \end{bmatrix}$

$R_2 \to R_2 - 3R_1$

$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$R_1 \to \frac{1}{2} R_1$

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Here x_2 is a free variable then corresponding equation is

$x_1 = 0$

$x_2 =$ free

$\therefore x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \cdot x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

The eigen space is $\begin{bmatrix} 0 \\ \end{bmatrix}$

Thus, there are three basic vector in total which are linearly independent eigen vector and are,

$$P = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 \\ 6 & -1 \end{bmatrix}$$

$AP = DP$

$$AP = \begin{bmatrix} 1 & 0 \\ 6 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 6-3 & 0-1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 3 & -1 \end{bmatrix}$$

$$PDP = \cancel{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}} \cdot \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 \\ 0 & 0-1 \end{bmatrix}$$

$$= \cancel{\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}} \quad \begin{bmatrix} 1 & 0 \\ 3 & -1 \end{bmatrix}$$

$\therefore AP = DP$

proved

4(b) $\begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$

Soln: Here,

$$\text{let } A = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2-\lambda & 3 \\ 4 & 1-\lambda \end{bmatrix}$$

The characteristic equation is

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 2-\lambda & 3 \\ 4 & 1-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)(1-\lambda) - 12 = 0$$

$$(2-\lambda)(1-\lambda) = 12$$

$\cancel{2-\lambda = 12}$ $\qquad 2 - 2\lambda - \lambda + \lambda^2 = 12$

$\cancel{\lambda = 6 - 10}$ $\qquad 2 - 3\lambda + \lambda^2 = 12$

$$\lambda^2 - 3\lambda - 10 = 0$$

$$\lambda^2 - (5-2)\lambda - 10 = 0$$

$$\lambda^2 - 5\lambda + 2\lambda - 10 = 0$$

$$\lambda(\lambda - 5) + 2(\lambda - 5) = 0$$

$$(\lambda + 2)(\lambda - 5) = 0$$

$$\lambda = 5, -2$$

for $\lambda = 5$

$$A - \lambda I = \begin{bmatrix} -3 & 3 \\ 4 & -4 \end{bmatrix}$$

The augmented matrix is

$$[A - \lambda I \; 0] = \begin{bmatrix} -3 & 3 & 0 \\ 4 & -4 & 0 \end{bmatrix}$$

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$R_1 \to -\frac{1}{3}R_1$ and $R_2 \to \frac{1}{4}R_2$

$$\left[\begin{array}{cc|c} 1 & -1 & 0 \\ 1 & -1 & 0 \end{array}\right]$$

$R_2 \to R_2 - R_1$

$$\left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Here x_2 is $\to$ free variable then the corresponding eqn is

$x_1 - x_2 = 0$

$x_1 = x_2$

$x_2 = $ free

$$\therefore x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The eigen space is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

For $\lambda = -2$

$$A - \lambda I = \begin{bmatrix} 4 & 3 \\ 4 & 3 \end{bmatrix}$$

The augmented matrix is

$$[A - \lambda I \; 0] = \begin{bmatrix} 4 & 3 & 0 \\ 4 & 3 & 0 \end{bmatrix}$$

$R_2 \to R_2 - R_1$

$$\begin{bmatrix} 4 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Here x_2 is $\to$ free variable then the corresponding eqn is

$4x_1 + 3x_2 = 0$

$x_1 = -\frac{3x_2}{4}$

$x_2 = $ free

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$$\therefore x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -3/4\, x_2 \\ x_2 \end{bmatrix}$$

$$x = x_2 \begin{bmatrix} -3/4 \\ 1 \end{bmatrix}$$

$$x = -4x_2 \begin{bmatrix} 3 \\ -4 \end{bmatrix}$$ $\therefore$ The eigen space is $\begin{bmatrix} 3 \\ -4 \end{bmatrix}$

$\therefore$ There are three bases vector in total which are linearly independ then,

$AP = PD$

$$P = \begin{bmatrix} 1 & 3 \\ 1 & -4 \end{bmatrix}$$

$$D = \begin{bmatrix} 5 & 0 \\ 0 & -2 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$$

$$\therefore AP = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 1 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} 2+3 & 6-12 \\ 4+1 & 12-4 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & -6 \\ 5 & 8 \end{bmatrix}$$

$$PD = \begin{bmatrix} 1 & 3 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & -6 \\ 5 & 8 \end{bmatrix}$$

$\therefore AP = PD$ Ans

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(c) $\begin{bmatrix} 3 & -1 \\ 1 & 5 \end{bmatrix}$

Soln: Here,

let $A = \begin{bmatrix} 3 & -1 \\ 1 & 5 \end{bmatrix}$

$A - \lambda I = \begin{bmatrix} 3 & -1 \\ 1 & 5 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 3-\lambda & -1 \\ 1 & 5-\lambda \end{bmatrix}$

The characteristic equation is

$|A - \lambda I| = 0$

$\begin{vmatrix} 3-\lambda & -1 \\ 1 & 5-\lambda \end{vmatrix} = 0$

$(3-\lambda)(5-\lambda) + 1 = 0$

$15 - 3\lambda - 5\lambda + \lambda^2 + 1 = 0$

$\lambda^2 - 8\lambda + 16 = 0$

$\lambda^2 - 4\lambda - 4\lambda + 16 = 0$

$\lambda(\lambda-4) - 4(\lambda-4) = 0$

$\lambda = 4, 4$

Here the value of λ is 4 that means the value of λ is only one ~~the~~ existing in 2x2 order matrix so that the given matrix A cannot be diagonalize.

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(d) $\begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$

Soln: Here,

let $A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$

$A - \lambda I = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} -\lambda & 0 & -2 \\ 1 & 2-\lambda & 1 \\ 1 & 0 & 3-\lambda \end{bmatrix}$

The characteristic equation is

$(A - \lambda I) = 0$

$\begin{vmatrix} -\lambda & 0 & -2 \\ 1 & 2-\lambda & 1 \\ 1 & 0 & 3-\lambda \end{vmatrix} = 0$

$-\lambda \begin{bmatrix} 2-\lambda & 1 \\ 0 & 3-\lambda \end{bmatrix} + 0 - 2 \begin{bmatrix} 1 & 2-\lambda \\ 1 & 0 \end{bmatrix} = 0$

$-\lambda \left[(2-\lambda)(3-\lambda)\right] - 2\left(0 - (2-\lambda)\right) = 0$

$-\lambda \left[6 - 2\lambda - 3\lambda + \lambda^2\right] + 2(2-\lambda) = 0$

$-6\lambda + 5\lambda^2 - \lambda^3 + 4 - 2\lambda = 0$

$-\lambda^3 + 5\lambda^2 - 8\lambda + 4 = 0$

$\lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0$

$\lambda = 1, 2, 2,$

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For $\lambda = 2$

$$A - \lambda I = \begin{bmatrix} -2 & 0 & -2 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

The augmented matrix is

$$[A - \lambda I \; 0] = \begin{bmatrix} -2 & 0 & -2 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\begin{bmatrix} -2 & 0 & -2 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_1 \to -\frac{1}{2} R_1$

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_2 \to R_2 - R_1$

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Here x_2 and x_3 are free varfable then Corresponding equation is

$x_1 + x_3 = 0$

$x_1 = -x_3$

$x_2 = \text{free}$

$x_3 = \text{free}$

$$\therefore x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_3 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_3 \\ x_2 + 0 \cdot x_3 \\ 0 \cdot x_2 + x_3 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

The eigen space is $\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$

For $\lambda = 1$

$$A - \lambda I = \begin{bmatrix} -1 & 0 & -2 \\ 1 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix}$$

The augmented matrix is

$$[A - \lambda I \; 0] = \begin{bmatrix} -1 & 0 & -2 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 2 & 0 \end{bmatrix}$$

$R_3 \to R_3 + R_1$ and $R_2 \to R_2 + R_1$

$$\begin{bmatrix} -1 & 0 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Here x_3 is a free varsable then Corresponding equation is

$-x_1 - 2x_3 = 0$

$-x_1 = 2x_3$

$x_1 = -2x_3$

$x_2 - x_3 = 0$

$x_2 = x_3$

$x_3 = \text{free}$

$$\therefore x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2x_3 \\ x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

The eigen space is $\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$

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Thus there are three eigen bases in total which are linearly independent
then,

$$\cancel{A}P = \begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \Leftrightarrow \begin{bmatrix} -1 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$$

$$AP = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} -1 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 0 & -2 \\ -1+1 & 2 & -2+2+1 \\ -1+3 & 0 & -2+0+3 \end{bmatrix} = \begin{bmatrix} -2 & 0 & -2 \\ 0 & 2 & 1 \\ 2 & 0 & 1 \end{bmatrix}$$

$$PD = \begin{bmatrix} -1 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 0 & -2 \\ 0 & 2 & 1 \\ 2 & 0 & 1 \end{bmatrix}$$

$\therefore AP = PD$ Ans

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(e) $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ -3 & 5 & 2 \end{bmatrix}$

Soln: Here,
let $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ -3 & 5 & 2 \end{bmatrix}$

$$A - \lambda I = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ -3 & 5 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1-\lambda & 0 & 0 \\ 1 & 2-\lambda & 0 \\ -3 & 5 & 2-\lambda \end{bmatrix}$$

The characteristic equation is

$|A - \lambda I| = 0$

$$1-\lambda \begin{vmatrix} 2-\lambda & 0 \\ 5 & 2-\lambda \end{vmatrix} = 0$$

$(1-\lambda)(2-\lambda)^2 = 0$

$(1-\lambda)(4-4\lambda+\lambda^2) = 0$

$4-4\lambda+\lambda^2-4\lambda+4\lambda^2-\lambda^3 = 0$

$4-8\lambda+5\lambda^2-\lambda^3 = 0$

$\lambda^3-5\lambda^2+8\lambda-4 = 0$

$\lambda^3-5\lambda^2+8\lambda-4 = 0$

$\lambda = 1, 2, 2$

For $\lambda = 2$

$$A - \lambda I = \begin{bmatrix} -1 & 0 & 0 \\ 1 & 0 & 0 \\ -3 & 5 & 0 \end{bmatrix}$$

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The augmented matrix is

$$[A-\lambda I \; 0] = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ -3 & 5 & 0 & 0 \end{bmatrix}$$

$R_2 \to R_2 + R_1$ and $R_1 \to -R_1$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -3 & 5 & 0 & 0 \end{bmatrix}$$

Here x_2 and x_3 are free variable then the corresponding equation is

$x_1 = 0$

$x_2 = free$

$-3x_1 + 5x_2 = 0$

$x_3 = free$

$$\therefore x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0\cdot x_2 + 0\cdot x_3 \\ x_2 + 0\cdot x_3 \\ 0\cdot x_2 + x_3 \end{bmatrix} = x_2\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_3\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

The eigen space is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

For $\lambda = 1$

$$B - \lambda I = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ -3 & 5 & 1 \end{bmatrix}$$

$R_2 \to 5R_2$ The augmented matrix is

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -3 & 5 & 1 & 0 \end{bmatrix}$$

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$R_2 \to 5R_2$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 5 & 5 & 0 & 0 \\ -3 & 5 & 0 & 0 \end{bmatrix}$$

$R_2 \to R_2 - R_3$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 8 & 0 & 0 & 0 \\ -3 & 5 & 0 & 0 \end{bmatrix}$$

Here x_1 and x_2 are free variable and has eigen vector v_1 and v_2 so there are four bases vector for 3×3 order matrix, at this condition the matrix A is linear dependent So the given matrix is non-diagonal

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(1) $\begin{bmatrix} 2 & 4 & 3 \\ -4 & -6 & -3 \\ 3 & 3 & 1 \end{bmatrix}$

Sol: Here,

let $A = \begin{bmatrix} 2 & 4 & 3 \\ -4 & -6 & -3 \\ 3 & 3 & 1 \end{bmatrix}$

$A - \lambda I = \begin{bmatrix} 2-\lambda & 4 & 3 \\ -4 & -6-\lambda & -3 \\ 3 & 3 & 1-\lambda \end{bmatrix}$

The characteristic equation is,

$|A - \lambda I| = 0$

$2-\lambda \begin{vmatrix} -6-\lambda & -3 \\ 3 & 1-\lambda \end{vmatrix} - 4 \begin{vmatrix} -4 & -3 \\ 3 & 1-\lambda \end{vmatrix} + 3 \begin{vmatrix} -4 & -6-\lambda \\ 3 & 3 \end{vmatrix} = 0$

$(2-\lambda)\left[(1-\lambda)(-6-\lambda)+9\right] - 4\left[-4(1-\lambda)+9\right] + 3\left[-12-3(-6-\lambda)\right] = 0$

$(2-\lambda)\left[-6-\lambda+6\lambda+\lambda^2+9\right] - 4\left[-4+4\lambda+9\right] + 3\left[-12+18+3\lambda\right] = 0$

$(2-\lambda)\left[\lambda^2+5\lambda+3\right] - 4\left[4\lambda+5\right] + 3\left[6+3\lambda\right] = 0$

$2\lambda^2 - 10\lambda + 6 - \lambda^3 - 5\lambda^2 - 3\lambda - 16\lambda - 20 + 18 + 9\lambda = 0$

$-\lambda^3 - 3\lambda^2 + 4 = 0$

$\lambda^3 + 3\lambda^2 - 4 = 0$

$\lambda^3 + 3\lambda^2 = 4$

$\lambda^2(\lambda + 3) = 4$

$\lambda^2 = 4$

$\lambda = \pm 2, \lambda = 1$

$\therefore \lambda = 1, -2, -2$

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For $\lambda = 1$,

$A - \lambda I = \begin{bmatrix} 1 & 4 & 3 \\ -4 & -7 & -3 \\ 3 & 3 & 0 \end{bmatrix}$

The augmented matrix is

$[A - \lambda I \;\; 0] = \begin{bmatrix} 1 & 4 & 3 & 0 \\ -4 & -7 & -3 & 0 \\ 3 & 3 & 0 & 0 \end{bmatrix}$

$R_2 \to R_2 + 4R_1$ and $R_3 \to R_3 - 3R_1$

$\begin{bmatrix} 1 & 4 & 3 & 0 \\ 0 & 9 & 9 & 0 \\ 0 & -9 & -9 & 0 \end{bmatrix}$

$R_3 \to R_3 + R_2$

$\begin{bmatrix} 1 & 4 & 3 & 0 \\ 0 & 9 & 9 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$R_2 \to \frac{1}{9} R_2$

$\begin{bmatrix} 1 & 4 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Here x_3 is a free variable. then the corresponding eqn is.

$x_1 + 4x_2 + 3x_3 = 0$

$x_1 + 4x_2 + 3x_3 = 0$

$x_2 + x_3 = 0$

$x_2 = -x_3$

$x_3 = \text{free}$

$x_1 - 4x_3 + 3x_3 = 0$

$x_1 = x_3$

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$\therefore x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ -x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

The eigen space is $\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$

For $\lambda = -2$

$A+2I = \begin{bmatrix} 4 & 4 & 3 \\ -4 & -4 & -3 \\ 3 & 3 & 3 \end{bmatrix}$

The augmented matrix is

$[A+2I \;\; 0] = \begin{bmatrix} 4 & 4 & 3 & 0 \\ -4 & -4 & -3 & 0 \\ 3 & 3 & 3 & 0 \end{bmatrix}$

$R_2 \to R_2 + R_1$ and $R_3 \to R_3 - R_1$

$\begin{bmatrix} 4 & 4 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 \end{bmatrix}$

Here x_2 and x_3 are free variable then the corresponding eqn is

$4x_1 + 4x_2 + 3x_3 = 0$

$4x_1 = -4x_2 - 3x_3$

$x_1 = -4/4\, x_2 - 3/4\, x_3$

$= -x_2 - 3/4\, x_3$

$x_2 = $ free

$x_3 = $ free

$\therefore x = \begin{bmatrix} -x_2 - 3/4 x_3 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_2 - 3/4 x_3 \\ x_2 + 0.x_3 \\ 0x_2 + x_3 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3/4 \\ 0 \\ 1 \end{bmatrix}$

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$\therefore = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + 4x_3 \begin{bmatrix} -3 \\ 0 \\ 4 \end{bmatrix}$

The eigen space are $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 4 \end{bmatrix} \right\}$

There are three bases vector altogether which are linearly independent so,

$AP = PD$

$AP = \begin{bmatrix} 2 & 4 & 3 \\ -4 & -6 & -3 \\ 3 & 3 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 & -3 \\ -1 & 1 & 0 \\ 1 & 0 & 4 \end{bmatrix}$

$AP = \begin{bmatrix} 2-4+3 & -2+4+0 & -6+0+12 \\ -4+6-3 & 4-6+0 & 12+0-12 \\ 3-3+1 & -3+3+0 & -9+0+4 \end{bmatrix}$

$= \begin{bmatrix} 1 & 2 & 6 \\ -1 & -2 & 0 \\ 1 & 0 & -5 \end{bmatrix}$

$PD = \begin{bmatrix} 1 & -1 & -3 \\ -1 & 1 & 0 \\ 1 & 0 & 4 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$

$= \begin{bmatrix} 1 & 2 & 6 \\ -1 & -2 & 0 \\ 1 & 0 & -8 \end{bmatrix}$

$\therefore AP \neq PD$ so that A matrix cannot be diagonalize

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(g) $\begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$

Sol^n: Here

let $A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$

$A - \lambda I = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1-\lambda & 3 & 3 \\ -3 & -5-\lambda & -3 \\ 3 & 3 & 1-\lambda \end{bmatrix}$

The characteristic equation is

$|A - \lambda I| = 0$

$\begin{bmatrix} 1-\lambda & 3 & 3 \\ -3 & -5-\lambda & -3 \\ 3 & 3 & 1-\lambda \end{bmatrix} = 0$

$R_1 \to R_1 - R_3$

$\begin{bmatrix} -\lambda-2 & 0 & 2+\lambda \\ -3 & -5-\lambda & -3 \\ 3 & 3 & 1-\lambda \end{bmatrix} = 0$

$\lambda+2 \begin{bmatrix} -1 & 0 & 1 \\ -3 & -5-\lambda & -3 \\ 3 & 3 & 1-\lambda \end{bmatrix} = 0$

where $\lambda = -2$, for $(\lambda+2)$

$-1 \begin{vmatrix} -5-\lambda & -3 \\ 3 & 1-\lambda \end{vmatrix} + 0 + 1 \begin{vmatrix} -3 & -5-\lambda \\ 3 & 3 \end{vmatrix}$

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$-1 [(-5-\lambda)(1-\lambda)+9] + 1 [-9-3(-5-\lambda)] = 0$

$-1 [-5+5\lambda-\lambda+\lambda^2+9] + 1 [-9+15+3\lambda] = 0$

$-1 [4+4\lambda+\lambda^2] + 1 [6+3\lambda] = 0$

$-4-4\lambda-\lambda^2+6+3\lambda = 0$

$-\lambda^2-\lambda+2 = 0$

$-(\lambda^2+\lambda-2) = 0$

$\lambda^2+\lambda-2 = 0$

$\lambda^2+2\lambda-\lambda-2 = 0$

$\lambda(\lambda+2)-1(\lambda+2) = 0$

$\lambda = 1, -2$

$\therefore \lambda = 1, -2, -2$

For $\lambda = 1$

$A - I = \begin{bmatrix} 0 & 3 & 3 \\ -3 & -6 & -3 \\ 3 & 3 & 0 \end{bmatrix}$

The augmented matrix is

$[A - I \ \ 0] = \begin{bmatrix} 0 & 3 & 3 & 0 \\ -3 & -6 & -3 & 0 \\ 3 & 3 & 0 & 0 \end{bmatrix}$

$= R_3 \to R_3 + R_2$

$\begin{bmatrix} 0 & 3 & 3 & 0 \\ -3 & -6 & -3 & 0 \\ 0 & -3 & -3 & 0 \end{bmatrix}$

$R_3 \to R_3 + R_1$

$\begin{bmatrix} 0 & 3 & 3 & 0 \\ -3 & -6 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$R_1 \to \frac{1}{3} R_1$ and $R_2 \to -\frac{1}{3} R_2$

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$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xleftrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Here ~~x_2 and~~ x_3 are free variable then the corresponding equation is

$\cancel{x_2 + x_3 = 0}$

$\cancel{x_2 = -x_3}$

$\cancel{x_1 + 2x_2 + x_3 = 0}$

$\cancel{x_1 = \text{free}}$

$\cancel{x_2 = \text{free.}}$

$x_1 + 2x_2 + x_3 = 0$

$x_2 + x_3 = 0$

$x_2 = -x_3$

$x_1 - 2x_3 + x_3 = 0$

$x_1 = x_3$

$x_3 = \text{free}$

$$\begin{bmatrix} x_3 \\ -x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$\therefore x = \begin{bmatrix} x_1 \\ -x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 + 0 \cdot x_3 \\ \cancel{0 \cdot x_1 + (-1) x_3} \\ 0 \cdot x_1 + x_3 \end{bmatrix} = \cancel{x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}}$$

The eigen space are $\left\{ \cancel{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}} \right\}$ $\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$

Now

for $\lambda = -2$

$$A + 2I = \begin{bmatrix} 3 & 3 & 3 \\ -3 & -3 & -3 \\ 3 & 3 & 3 \end{bmatrix}$$

The augmented matrix is

$$[A + 2I \; \vdots \; 0] = \begin{bmatrix} 3 & 3 & 3 & 0 \\ -3 & -3 & -3 & 0 \\ 3 & 3 & 3 & 0 \end{bmatrix}.$$

$R_1 \to \frac{1}{3} R_1$, $R_2 \to -\frac{1}{3} R_2$ and $R_3 \to \frac{1}{3} R_3$

$$= \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

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$R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Here x_2 and x_3 are free variable then the corresponding equation is

$x_1 + x_2 + x_3 = 0$

$x_1 = -x_2 - x_3$

$x_2 = \text{free}$

$x_3 = \text{free}$

$$\therefore x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_2 - x_3 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_2 - x_3 \\ x_2 + 0 \cdot x_3 \\ 0 \cdot x_2 + x_3 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

The eigen vector are $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

$\therefore$ there are three basis vector in total which are linear independent then

$$P = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

then

$$AP = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 - 3 + 3 & -1 + 3 + 0 & -1 + 0 + 3 \\ -3 + 5 - 3 & 3 - 5 + 0 & 3 + 0 - 3 \\ 3 - 3 + 1 & -3 + 3 + 0 & -3 + 0 + 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 2 \\ -1 & -2 & 0 \\ 1 & 0 & -2 \end{bmatrix}$$

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$$PD = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 2 \\ -1 & -2 & 0 \\ 1 & 0 & -2 \end{bmatrix}$$

$\therefore AP = DP$. Ans

(4) $\begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 1 & 4 & -3 & 0 \\ -1 & -2 & 0 & -3 \end{bmatrix}$

Sol^n: Here,

let $A = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 1 & 4 & -3 & 0 \\ -1 & -2 & 0 & -3 \end{bmatrix}$

$A - \lambda I = \begin{bmatrix} 5-\lambda & 0 & 0 & 0 \\ 0 & 5-\lambda & 0 & 0 \\ 1 & 4 & -3-\lambda & 0 \\ -1 & -2 & 0 & -3-\lambda \end{bmatrix}$

where the given matrix is ~~upper~~ triangular matrix so it's diagonal element are the value of λ so,

$\lambda = 5, 5, -3, -3$

For $\lambda = 5$

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$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 4 & -8 & 0 \\ -1 & -2 & 0 & -8 \end{bmatrix}$$

The augmented matrix is

$[A - 5I \; 0] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 4 & -8 & 0 & 0 \\ -1 & -2 & 0 & -8 & 0 \end{bmatrix}$

$R_1 \leftrightarrow R_3$ and $R_2 \leftrightarrow R_4$

$= \begin{bmatrix} 1 & 4 & -8 & 0 & 0 \\ -1 & -2 & 0 & -8 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Here x_3 and x_4 ~~take~~ free Variable then the Corresponding equations are

$x_1 + 4x_2 - 8x_3 = 0$, $\quad -x_1 - 2x_2 - 8x_4 = 0$

$-2x_2 - 8x_4 + 4x_2 - 8x_3 = 0$, $\quad x_1 = -2x_2 - 8x_4$

$2x_2 = 8x_3 + 8x_4$ $\quad x_1 = -2(4x_3 + 4x_4) - 8x_4$

$x_2 = 4x_3 + 4x_4$ $\quad x_1 = -8x_3 - 8x_4 - 8x_4$

$x_3 = \text{free}$ $\quad x_1 = -8x_3 - 16x_4$

$x_4 = \text{free}$

$$\therefore x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -8x_3 - 16x_4 \\ 4x_3 + 4x_4 \\ x_3 + 0 \cdot x_4 \\ 0 \cdot x_3 + x_4 \end{bmatrix} = x_3 \begin{bmatrix} -8 \\ 4 \\ 1 \\ 0 \end{bmatrix} + x_4 \cdot \begin{bmatrix} -16 \\ 4 \\ 0 \\ 1 \end{bmatrix}$$

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For $\lambda = -3$

$$A+3I = \begin{bmatrix} 8 & 0 & 0 & 0 \\ 0 & 8 & 0 & 0 \\ 1 & 4 & 0 & 0 \\ -1 & -2 & 0 & 0 \end{bmatrix}$$

The augmented matrix is

$$= \begin{bmatrix} 8 & 0 & 0 & 0 & 0 \\ 0 & 8 & 0 & 0 & 0 \\ 1 & 4 & 0 & 0 & 0 \\ -1 & -2 & 0 & 0 & 0 \end{bmatrix}$$

$R_1 \to \frac{1}{8} R_1$ and $R_2 \to \frac{1}{8} R_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 4 & 0 & 0 & 0 & 0 \\ -1 & -2 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Here x_3 and x_4 are free variable then the corresponding equation is.

$x_1 = 0$

$x_2 = 0$

$x_3 =$ free

$x_4 =$ free.

$$\therefore x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$\therefore$ The Eigen vectors are $\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

$\therefore$ There are four bases vector in total which are linearly independent so,

$$P = \begin{bmatrix} -8 & -16 & 0 & 0 \\ 4 & 4 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix}$$

$$\therefore AP = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 1 & 4 & -3 & 0 \\ -1 & -2 & 0 & -3 \end{bmatrix} \cdot \begin{bmatrix} -8 & -16 & 0 & 0 \\ 4 & 4 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -40+0+0+0 & -80+0+0+0 & 0+0+0+0 & 0+0+0+0 \\ 0+20+0+0 & 0+20+0+0 & 0+0+0+0 & 0+0+0+0 \\ -8+16-3 & -16+16+0+0 & 0+0-3+0 & 0+0+0+0 \\ 8-8+0+0 & 16-8+0-3 & 0+0+0+0 & 0+0+0-3 \end{bmatrix}$$

$$\begin{bmatrix} -40 & -80 & 0 & 0 \\ 20 & 20 & 0 & 0 \\ 5 & 0 & -3 & 0 \\ 0 & 5 & 0 & -3 \end{bmatrix}$$

$$\therefore PD = \begin{bmatrix} -8 & -16 & 0 & 0 \\ 4 & 4 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix}$$

$$PD = \begin{bmatrix} -40 & -80 & 0 & 0 \\ 20 & 20 & 0 & 0 \\ 5 & 0 & -3 & 0 \\ 0 & 5 & 0 & -3 \end{bmatrix}$$

$\therefore AP = PD$ which diagonalized.

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1) $\begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 5 & -5 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$

Sol^n: Here

let $A = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 5 & -5 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$

$A - \lambda I = \begin{bmatrix} -2-\lambda & 0 & 0 & 0 \\ 0 & -2-\lambda & 5 & -5 \\ 0 & 0 & 3-\lambda & 0 \\ 0 & 0 & 0 & 3-\lambda \end{bmatrix}$

Here the given matrix is a Upper triangular matrix so its diagonal element gives the value of λ so,

$\lambda = -2, -2, 3, 3 \Rightarrow 3, 3, -2, -2$

For $\lambda = 3$

$A - 3I = \begin{bmatrix} -5 & 0 & 0 & 0 \\ 0 & -5 & 5 & -5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

The augmented matrix is

$[A - 3I \ 0] = \begin{bmatrix} -5 & 0 & 0 & 0 & 0 \\ 0 & -5 & 5 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

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$R_1 \to -\frac{1}{5}R_1, \ R_2 \to -\frac{1}{5}R_2$

$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Here x_2 and x_3 are free Variable then the corresponding equations are,

$x_1 = 0$

$x_2 - x_3 + x_4 = 0$

$x_2 = x_3 + x_4$

$x_3 =$ free

$x_4 =$ free

$\therefore x \begin{bmatrix} 0 \\ -x_3 + x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$

The eigen spaces are $\left\{ \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$

For $\lambda = -2$

$A + 2I = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & -5 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}$ The augmented matrix is

$R_2 \to \frac{1}{5}R_2, \ R_3 \to \frac{1}{5}R_3$ and $R_4 \to \frac{1}{5}R_4$

$= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

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Here x_1 and x_2 are free variable then the corresponding eqn zero

x_1 = free

x_2 = free

~~x_3 = 0~~

$x_3 = 0$

$x_4 = 0$

$$\therefore x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ 0 \\ 0 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

The eigen vectors are $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

Here, there are four basis vector in total which are linearly independent then,

$$P = \begin{bmatrix} 0 & 0 & 1 & 0 \\ -1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

$$AP = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 5 & -5 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 0 \\ -1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & -2 & 0 \\ 7 & -7 & 0 & -2 \\ 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \end{bmatrix}$$

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$$PD = \begin{bmatrix} 0 & 0 & 1 & 0 \\ -1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & -2 & 0 \\ -3 & 3 & 0 & -2 \\ 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \end{bmatrix}$$

$AP \neq DP$ so matrix A cannot be diagon

(J) $\begin{bmatrix} 5 & -3 & 0 & 9 \\ 0 & 3 & 1 & -2 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

Sol: Here

$$\text{let } A = \begin{bmatrix} 5 & -3 & 0 & 9 \\ 0 & 3 & 1 & -2 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 5-\lambda & -3 & 0 & 9 \\ 0 & 3-\lambda & 1 & -2 \\ 0 & 0 & 2-\lambda & 0 \\ 0 & 0 & 0 & 2-\lambda \end{bmatrix}$$

Here the matrix A is a Upper triangular matrix and Its diagonal element gives the eigen value i.e. λ.

so $\lambda = 5, 3, 2, 2$

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For $\lambda = 5$

$$A - 5I = \begin{bmatrix} 0 & -3 & 0 & 9 \\ 0 & 3-5 & 1 & -2 \\ 0 & 0 & 2-5 & 0 \\ 0 & 0 & 0 & 2-5 \end{bmatrix} = \begin{bmatrix} 0 & -3 & 0 & 9 \\ 0 & -2 & 1 & -2 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix}$$

The augmented matrix is

$$= \begin{bmatrix} 0 & -3 & 0 & 9 & 0 \\ 0 & -2 & 1 & -2 & 0 \\ 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 \end{bmatrix}$$

$R_1 \to -\frac{1}{3}R_1$, $R_3 \to -\frac{1}{3}R_3$ and $R_4 \to -\frac{1}{3}R_4$

$$= \begin{bmatrix} 0 & 1 & 0 & -3 & 0 \\ 0 & -2 & 1 & -2 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$R_2 \to R_2 - R_3$

$$\begin{bmatrix} 0 & 1 & 0 & -3 & 0 \\ 0 & -2 & 0 & -2 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$R_2 \to -\frac{1}{2}R_2$

$$\begin{bmatrix} 0 & 1 & 0 & -3 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$R_2 \to R_2 - R_4$ ~~and~~ $\begin{bmatrix} 0 & 1 & 0 & -3 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$

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$R_1 \to R_1 - R_2$ $\begin{bmatrix} 0 & 0 & 0 & -3 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$

Here x_1 is a free variable then the corresponding equations are

$x_1 =$ free

$x_2 = 0$

$x_3 = 0$

$x_4 = 0$

$$\therefore x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

For $\lambda = 3$

$$A - 3I = \begin{bmatrix} 2 & -3 & 0 & 9 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

The augmented matrix is

$$\begin{bmatrix} 2 & -3 & 0 & 9 & 0 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \end{bmatrix}$$

$R_2 \to R_2 + R_3$

$$\begin{bmatrix} 2 & -3 & 0 & 9 & 0 \\ 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \end{bmatrix}$$

Here x_2 is a free variable then the corresponding eqn are.

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$2x_1 - 3x_2 + 9x_4 = 0$

$-2x_4 = 0$

$x_4 = 0$

$x_3 = 0$

$x_2 = \text{free}$

$2x_1 - 3x_2 + 0 = 0$

$2x_1 - 3x_2 + 0 = 0$

$x_1 = \frac{3}{2} x_2$

$$\therefore x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \frac{3}{2}x_2 \\ x_2 \\ 0 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} 3/2 \\ 1 \\ 0 \\ 0 \end{bmatrix} = 2x_2 \begin{bmatrix} 3 \\ 2 \\ 0 \\ 0 \end{bmatrix}$$

For $\lambda = 2$

$$A - 2I = \begin{bmatrix} 3 & -3 & 0 & 9 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The A.M is

$R_1 \to \frac{1}{3} R_1$

$$\begin{bmatrix} 1 & -1 & 0 & 3 & 0 \\ 0 & 1 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Here x_3 and x_4 are free variable then the corresponding equations are:

$x_1 - x_2 + 3x_4 = 0$, $x_2 + x_3 - 2x_4 = 0$

$x_1 - x_2 = -3x_4$ $\quad$ $x_2 = -x_3 + 2x_4$

$x_1 - (-x_3 + 2x_4) = -3x_4$

$x_1 + x_3 - 2x_4 = -3x_4$

$x_1 = -x_3 - x_4$

$x_3 = \text{free}$ and $x_4 = \text{free}$

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Now:

$$\therefore x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -x_3 - x_4 \\ -x_3 + 2x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

The eigen vector are $\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 0 \\ 1 \end{bmatrix}$

Here, there are three basis vector in total which are linearly independent so

$$P = \begin{bmatrix} 1 & 3 & -1 & -1 \\ 0 & 2 & -1 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \ \& \ D = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

then

$$AP = \begin{bmatrix} 5 & -3 & 0 & 9 \\ 0 & 3 & 1 & -2 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 3 & -1 & -1 \\ 0 & 2 & -1 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 9 & -2 & -2 \\ 0 & 6 & -2 & 4 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

$$PD = \begin{bmatrix} 1 & 3 & -1 & -1 \\ 0 & 2 & -1 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 9 & -2 & -2 \\ 0 & 6 & -2 & 4 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \quad \therefore AP = DP \quad \text{Ans}$$

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(5) let $A = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$, $V_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $V_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Suppose V_1 and V_2 are eigen value of matrix A. Use this information to diagonalize A.

Sol: Here,

$A = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$

$V_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $V_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

we know that

$AV_1 = \lambda V_1$

$\lambda V_1 = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\lambda V_1 = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$

$\lambda V_1 = 3 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\lambda V_1 = 3 V_1$

$\lambda = 3$

Similarly

$AV_2 = \lambda V_2$

$\lambda V_2 = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$= \begin{bmatrix} 0 \\ 0-1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} = -1 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\lambda V_2 = -1 . V_2$

$\lambda = -1$

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$\lambda = 3, -1$

$D = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}$

$P = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$

$AP = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$

$= \begin{bmatrix} 3 & 0 \\ 6 & -1 \end{bmatrix}$

$PD = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}$

$= \begin{bmatrix} 3 & 0 \\ 6 & -1 \end{bmatrix}$

$\therefore$ $AP = PD$. which is diagonalized.

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UNIT-08.

Inner product :

If u and V are $n\times 1$ matrix, then u^TV be 1×1 matrix, when u^T be transpose of u, which is called inner product of u and V.

If $u = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}$ and $V = \begin{pmatrix} V_1 \\ V_2 \\ \vdots \\ V_n \end{pmatrix}$, then the inner product of u and V is

$$u^TV = \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

$$= u_1v_1 + u_2v_2 + \cdots + u_nv_n$$

$$= u \cdot v.$$

Scalar or dot product :

let $u = (u_1, u_2, \ldots, u_n)$ and $v = (v_1, v_2, \ldots, v_n)$, then the Scalar or dot product is denoted by $u \cdot v$ and defined as

$$u \cdot v = u_1v_1 + u_2v_2 + \cdots + u_nv_n.$$

Ex. compute $u \cdot v$, $v \cdot u$ where $u = (2,4,5)$, $v = (-1,3,-1)$

Sol^n: Here

$$u \cdot v = (2,4,5)\,(-1,3,-1) = -2+12,-5$$
$$= 12-7$$
$$= 5$$

$$v \cdot u = (-1,3,-1) \cdot (2,4,5) = -2+12-5$$
$$= 12-7$$
$$= 5$$

Ans

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Length of vector :-

The length of norm of a vector V is a non-negative scalar denoted by $\|V\|$ and defined as

$$\|V\| = \sqrt{V \cdot V} = \sqrt{v_1^2 + v_2^2 + \dots + v_n^2}$$

where $V = (v_1, v_2, \dots v_n)$

$\therefore \|V\|^2 = V \cdot V$

Ex. Find the length of vector $(4, 5, 6)$

Sol: Here,

let $V = (4, 5, 6)$

length of $V = \|V\| = \sqrt{16+25+36} = \sqrt{4^2+5^2+6^2}$

$= \sqrt{77}$

Unit vector :-

A vector having length 1, is called unit vector. If V is a non-zero vector, then the unit vector along the direction of V is $\frac{V}{\|V\|}$ in the direction of V

Ex. Find the unit vector along the vector $V = (-2, 1, 0)$ and verify it.

Sol:

$V = (-2, 1, 0)$

$\|V\| = \sqrt{(-2)^2 + (1)^2 + 0^2} = \sqrt{4+1} = \sqrt{5}$

Unit vector in the direction of V

$U = \frac{V}{\|V\|} = \frac{(-2, 1, 0)}{\sqrt{5}} = \left(\frac{-2}{\sqrt{5}}, \frac{1}{\sqrt{5}}, 0\right)$

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Verification.

$\|V\| = \sqrt{\left(\frac{-2}{\sqrt{5}}\right)^2 + \left(\frac{1}{\sqrt{5}}\right)^2 + 0^2} = \sqrt{\frac{4}{5} + \frac{1}{5}} = \sqrt{\frac{5}{5}} = 1$

verify.

Normalization of a vector :-

let V be a vector in R^n. Set $U = \frac{V}{\|V\|}$ then the process creating U is called normalization V.

* **Distances between two vectors :-**

let U and V are in R^n, then the distances between U and V is the length between them and denoted by $dis(U, V)$ and defined as

$dis(U, V) = \|U - V\|$

Ex. If $U = (2, 3)$, $V = (3, -1)$, then find the distance between U and V.

Sol: Here

$U - V = (2, 3) - (3, -1)$

$= (-1, 4)$

$\therefore dis(U, V) = \|U - V\| = \sqrt{(-1)^2 + (4)^2} = \sqrt{1+16} = \sqrt{17}$

* **Orthogonal vector**

Two vectors U and V in R^n are orthogonal to each other of $U \cdot V = 0$

Ex.

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Ex. Show that $u=(2,-3,3)$, $v=(12,3,-5)$ are Orthogonal.

Sol.

$u\cdot v=(2,-3,3),(12,3,-5)$

$=24-9-15$

$=24-24$

$=0$

$\therefore$ u and v are orthogonal.

Thm The pythagorean Theorem:
Two vectors u and v are orthogonal if and only if $\|u+v\|^2=\|u\|^2+\|v\|^2$.

proof: First Suppose that u and v are orthogonal i.e. $u\cdot v=0$ — ①

and $\|u\|^2=u\cdot u=u\cdot u$, $\|v\|^2=v\cdot v$

Now

$\|u+v\|^2=(u+v)\,(u+v)$

$=u\cdot u+u\cdot v+v\cdot u+v\cdot v$

$=\|u\|^2+2u\cdot v+\|v^2\|$

$=\|u\|^2+0+\|v\|^2$

$\|u+v\|^2=\|u\|^2+\|v\|^2$

~~Convergency~~ conversely

$\|u+v\|^2=\|u\|^2+\|v\|^2$

$\Rightarrow (u+v)\cdot(u+v)=\|u\|^2+\|v\|^2$

$u\cdot u+u\cdot v+v\cdot u+v^2=\|u\|^2+\|v\|^2$

$\|u\|^2+2u\cdot v+\|v\|^2=\|u\|^2+\|v\|^2$

$2u\cdot v=0$

$u\cdot v=0$

$\therefore$ u and v are Orthogonal Hence proved

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Exercise:- 8.1

1. Using these vector, compute the quantities where,

$u=\begin{bmatrix}-1\\2\end{bmatrix}$, $v=\begin{bmatrix}4\\6\end{bmatrix}$, $w=\begin{bmatrix}3\\-1\\-5\end{bmatrix}$, $x=\begin{bmatrix}6\\-2\\3\end{bmatrix}$

(i) $u\cdot u$

Here

$u=\begin{bmatrix}-1\\2\end{bmatrix}$

$u\cdot u=\|u\|^2$

$\|u\|=\sqrt{1^2+2^2}=\sqrt{1+4}=\sqrt{5}$

$u\cdot u=\|u\|^2=\begin{bmatrix}-1\\2\end{bmatrix}\cdot\begin{bmatrix}-1\\2\end{bmatrix}$

$=1+4$

$=5$

(ii) $v\cdot u=[4,6]\cdot[-1,2]$

$=-4+12$

$=8$

$\frac{v\cdot u}{u\cdot u}=\frac{8}{5}$ Ans

$w\cdot w=\begin{bmatrix}3\\-1\\5\end{bmatrix}\cdot\begin{bmatrix}3\\-1\\5\end{bmatrix}=9+1+25$

$=35$ Ans

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$X \cdot W = \begin{bmatrix} 6 \\ -2 \\ 3 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \\ -5 \end{bmatrix}$

$= 18+2-15$

$= 5$

then,

$\frac{X \cdot W}{W \cdot W} = \frac{5}{35} = \frac{1}{7}$ Ans

(iii) $W \cdot W = \begin{bmatrix} 3 \\ -1 \\ -5 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -1 \\ -5 \end{bmatrix} = \quad 9+1+25 = 35$

$\frac{1}{W \cdot W} \cdot W = -\frac{1}{35} \begin{bmatrix} 3 \\ -1 \\ -5 \end{bmatrix} = \begin{bmatrix} 3/35 \\ -1/35 \\ -5/35 \end{bmatrix} = \begin{bmatrix} 3/35 \\ -1/35 \\ -1/7 \end{bmatrix}$

Ans

(iv) $\frac{1}{u \cdot u} \quad u$

$u \cdot u = 5$

$\frac{1}{u \cdot u} \cdot u = \frac{1}{5} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1/5 \\ 2/5 \end{bmatrix}$

(v) $\left(\frac{u \cdot v}{v \cdot v}\right) \cdot v =$

$u \cdot v = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 6 \end{bmatrix} = -4+12 = 8$

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$v \cdot v = \begin{bmatrix} 4 \\ 6 \end{bmatrix} \begin{bmatrix} 4 \\ 6 \end{bmatrix} = 16+36 = 52$

$\therefore \left(\frac{u \cdot v}{v \cdot v}\right) \cdot v = \frac{8}{52} \begin{bmatrix} 4 \\ 6 \end{bmatrix} = \frac{2}{13} \begin{bmatrix} 4 \\ 6 \end{bmatrix} = \begin{bmatrix} 8/13 \\ 12/13 \end{bmatrix}$

(vi) $\left(\frac{X \cdot W}{X \cdot X}\right) X$

$X \cdot W = 5$

$X \cdot X = \begin{bmatrix} 6 \\ -2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 6 \\ -2 \\ 3 \end{bmatrix} = 36+4+9 = 49$

$\left(\frac{X \cdot W}{X \cdot X}\right) \cdot X = \frac{5}{49} \begin{bmatrix} 6 \\ -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 30/49 \\ -10/49 \\ 15/49 \end{bmatrix} = \begin{bmatrix} 30/49 \\ -10/49 \\ 15/49 \end{bmatrix}$

Ans

(vii) ~~$\|W\|^2$~~

~~$W \cdot W = \begin{bmatrix} 3 \\ -1 \\ -5 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -1 \\ -5 \end{bmatrix} = 9+1+25 = 35$ Ans~~

(vii) $\|W\|$

$\|W\| = \sqrt{9+1+25} = \sqrt{35}$

(viii) $\|X\| = \sqrt{6^2+(-2)^2+3^2} = \sqrt{36+4+9} = \sqrt{49}$

$\|X\| = 7$ Ans

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8 Find a unit vector in the direction of the given vector.

(i) $\begin{bmatrix} -30 \\ 40 \end{bmatrix}$

Sol^n: Here

let $V = \begin{bmatrix} -30 \\ 40 \end{bmatrix}$

$\|V\| = \sqrt{(-30)^2 + (40)^2} = \sqrt{900+1600} = \sqrt{2500} = 50$

We know that,
Unit vector in the direction of V is u

$u = \dfrac{V}{\|V\|} = \dfrac{(-30, 40)}{50} = (-3/5, 4/5)$

$u = \begin{bmatrix} -3/5 \\ 4/5 \end{bmatrix}$ Ans

(ii) $\begin{bmatrix} -6 \\ 4 \\ -3 \end{bmatrix}$

Sol^n: Here,

let $v = \begin{bmatrix} -6 \\ 4 \\ -3 \end{bmatrix}$

The unit vector in the direction of v is u
then,

$\|V\| = \sqrt{(-6)^2 + (4)^2 + (-3)^2} = \sqrt{36+16+9} = \sqrt{61}$

$u = \dfrac{v}{\|v\|} = \dfrac{(-6, 4, -3)}{\sqrt{61}} = \left(\dfrac{-6}{\sqrt{61}}, \dfrac{4}{\sqrt{61}}, \dfrac{-3}{\sqrt{61}}\right)$

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$\therefore u = \begin{bmatrix} -6/\sqrt{61} \\ 4/\sqrt{61} \\ -3/\sqrt{61} \end{bmatrix}$ Ans

(iii) $\begin{bmatrix} 7/4 \\ 1/2 \\ 1 \end{bmatrix}$

Sol^n: Here,

let $V = \begin{bmatrix} 7/4 \\ 1/2 \\ 1 \end{bmatrix}$

We know that
the unit vector along the direction of V is u.

$u = \dfrac{V}{\|V\|}$

$\|V\| = \sqrt{\dfrac{49}{16} + \dfrac{1}{4} + 1} = \sqrt{\dfrac{69}{16}} = \dfrac{\sqrt{69}}{4}$

$u = \dfrac{(7/4, 1/2, 1)}{\sqrt{\frac{69}{4}}} = \left(\dfrac{7}{\sqrt{69}}, \dfrac{2}{\sqrt{69}}, \dfrac{4}{\sqrt{69}}\right)$

$\therefore u = \begin{bmatrix} 7/\sqrt{69} \\ 2/\sqrt{69} \\ 4/\sqrt{69} \end{bmatrix}$ Ans

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(iv) $\begin{bmatrix} 8/3 \\ 2 \end{bmatrix}$

Sol^n: Here

let $V = \begin{bmatrix} 8/3 \\ 2 \end{bmatrix}$

we know that, the unit vector along the direction of V is u

$u = \frac{V}{\|V\|}$

$\|V\| = \sqrt{64/9 + 4} = \sqrt{64+36/9} = \sqrt{100/9} = \frac{10}{3}$

$u = \frac{V}{\|V\|} = \frac{\begin{bmatrix} 8/3 & 2 \end{bmatrix}}{10/3} = \begin{bmatrix} 8/10 & , & 6/10 \end{bmatrix}$

$\therefore u = \begin{bmatrix} 4/5 & 3/5 \end{bmatrix}$

(5) Find the distance between $x = \begin{bmatrix} 10 \\ -3 \end{bmatrix}$ and $y = \begin{bmatrix} -1 \\ -5 \end{bmatrix}$

Sol^n: Here,

$x - y = \begin{bmatrix} 10 & -3 \end{bmatrix} - \begin{bmatrix} -1 & -5 \end{bmatrix}$

$= \begin{bmatrix} 11 & 2 \end{bmatrix}$

dis $\|x-y\| = \sqrt{11^2 + 2^2} = \sqrt{121 + 4} = \sqrt{125}$

dis $\|x-y\| = 5\sqrt{5}$ Ans

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(4) Determine which pairs of vectors are orthogonal

(i) $a = \begin{bmatrix} 8 \\ -5 \end{bmatrix}, b = \begin{bmatrix} -2 \\ -3 \end{bmatrix}$

Sol^n: Here

$a = \begin{bmatrix} 8 \\ -5 \end{bmatrix}, b = \begin{bmatrix} -2 \\ -3 \end{bmatrix}$

For orthogonal, $a \cdot b = 0$

$a \cdot b = \begin{bmatrix} 8 \\ -5 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ -3 \end{bmatrix}$

$= -16 + 15$

$= -1 \neq 0$

$\therefore$ the given pair of vector ~~are~~ not orthogonal

(ii) $u = \begin{bmatrix} 12 \\ 3 \\ -5 \end{bmatrix}, v = \begin{bmatrix} 2 \\ -3 \\ 3 \end{bmatrix}$

Sol^n: Here

$u = \begin{bmatrix} 12 \\ 3 \\ -5 \end{bmatrix}, v = \begin{bmatrix} 2 \\ -3 \\ 3 \end{bmatrix}$

For Orthogonal, $u \cdot v = 0$

$u \cdot v = \begin{bmatrix} 12 \\ 3 \\ -5 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -3 \\ 3 \end{bmatrix} = 24 - 9 - 15$

$= 24 - 24$

$= 0$

$\therefore$ the given pair of vector ~~are~~ orthogonal

Ans

(iii) $u = \begin{bmatrix} 3 \\ 2 \\ -5 \\ 0 \end{bmatrix}, \quad v = \begin{bmatrix} -4 \\ 1 \\ -2 \\ 6 \end{bmatrix}$

Sol^n: Here,

$u = \begin{bmatrix} 3 \\ 2 \\ -5 \\ 0 \end{bmatrix}, \quad v = \begin{bmatrix} -4 \\ 1 \\ -2 \\ 6 \end{bmatrix}$

For Orthogonal, $u.v = 0$

$$u.v = \begin{bmatrix} 3 \\ 2 \\ -5 \\ 0 \end{bmatrix} . \begin{bmatrix} -4 \\ 1 \\ -2 \\ 6 \end{bmatrix}$$

$$= -12+2+10+0$$
$$= -10+10$$
$$= 0$$

$\therefore$ the given pair of vectors are orthogonal

(iv) $y = \begin{bmatrix} -3 \\ 7 \\ 4 \\ 0 \end{bmatrix}, \quad z = \begin{bmatrix} 1 \\ -8 \\ 15 \\ -7 \end{bmatrix}$

Sol^n: Here,

$y = \begin{bmatrix} -3 \\ 7 \\ 4 \\ 0 \end{bmatrix}, \quad z = \begin{bmatrix} 1 \\ -8 \\ 15 \\ -7 \end{bmatrix}$

For orthogonal, $y.z = 0$.

$$y.z = \begin{bmatrix} -3 \\ 7 \\ 4 \\ 0 \end{bmatrix} . \begin{bmatrix} 1 \\ -8 \\ 15 \\ -7 \end{bmatrix}$$

$$= -3-56+60+0$$
$$= -59+60$$
$$= 1 \neq 0$$

$\therefore$ the given pair of vectors are not orthogonal.

(5) Find the angle between given vectors.

(i) $u = (1,-3)$ and $v = (2,4)$

Sol^n: Here

$u = (1,-3)$

$v = (2,4)$

we know that

$u.v = \|u\|.\|v\| \cos\theta$

$\cos\theta = \dfrac{u.v}{\|u\| \, \|v\|}$

$u.v = (1,-3).(2,4) = 2-12 = -10$

$\|u\| = \sqrt{1+9} = \sqrt{10}$

$\|v\| = \sqrt{4+16} = \sqrt{20}$

$\therefore \cos\theta = \dfrac{u.v}{\|u\|.\|v\|} = \dfrac{-10}{\sqrt{10}.\sqrt{20}} = \dfrac{-10}{\sqrt{200}} = \dfrac{-\sqrt{2}}{2}$

$\theta = 135^\circ$ or $\dfrac{3\pi}{4}$ Ans.

(11) $u = \begin{bmatrix} 4 \\ 1 \\ 8 \end{bmatrix}$ and $v = \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}$

Sol: Here,

$u = \begin{bmatrix} 4 \\ 1 \\ 8 \end{bmatrix}$, $v = \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}$

We know that

$\cos\theta = \dfrac{u \cdot v}{\|u\|\|v\|}$

$u \cdot v = \begin{bmatrix} 4 \\ 1 \\ 8 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} = 4+0-24 = -20$

$\|u\| = \sqrt{16+1+64} = \sqrt{81} = 9$

$\|v\| = \sqrt{1+0+9} = \sqrt{10}$

$\therefore \cos\theta = \dfrac{-20}{9\sqrt{10}}$

$\theta = \cos^{-1}\left(\dfrac{-20}{9\sqrt{10}}\right)$ Ans.

(11) $x = (1,0,1,0)$ and $y = (-3,-3,-3,-3)$

Sol: Here

$x = (1,0,1,0)$

$y = (-3,-3,-3,-3)$

We know that

$\cos\theta = \dfrac{x \cdot y}{\|x\| \cdot \|y\|}$

$x \cdot y = (1,0,1,0) \cdot (-3,-3,-3,-3)$

$= -3-3 = -6$

$\|x\| = \sqrt{1+1} = \sqrt{2}$

$\|y\| = \sqrt{9+9+9+9} = \sqrt{36}$

$\therefore \cos\theta = \dfrac{x \cdot y}{\|x\| \cdot \|y\|} = \dfrac{-6}{\sqrt{2} \cdot \sqrt{36}} = \dfrac{-6}{\sqrt{2} \cdot 6} = -1/\sqrt{2}$

$\theta = \cos^{-1}\left(-1/\sqrt{2}\right)$

$\theta = 135^\circ$ or $\dfrac{3\pi}{4}$ Ans.

(6) If $u, v \in R^n$, prove that $[dis(u,-v)]^2 = [dis(u,v)]^2$ iff $u.v = 0$

Sol^n: Here,

$[dis(u,-v)]^2 = [dis(u,v)]^2$

where $u.v = 0$

~~We have,~~

$\cancel{[dis(u,-v)]^2 = (u+v)}$

taking L.H.S,

$dis(u,-v) = \|u+v\|$

$[dis(u,-v)]^2 = \|(u+v)\|^2$

$= u^2 + 2.u.v + v^2$

$= \|u\|^2 + 2.(u.v) + \|v\|^2$

$= \|u\|^2 + 2.0 + \|v\|^2$

$= \|u\|^2 + \|v\|^2$

taking R.H.S.

$dis(u,v) = \|u-v\|$

$[dis(u,v)]^2 = (u-v)^2$

$= u^2 - 2.u.v + v^2$

$= \|u\|^2 - 2.(u.v) + \|v\|^2$

$= \|u\|^2 - 2.0 + \|v\|^2$

$= \|u\|^2 + \|v\|^2$

$\therefore$ L.H.S = R.H.S

proved

Exercise:- 8.2

Orthogonal set

- A set of vectors $\{v_1, v_2, \ldots v_n\}$ in R^n is said to be an orthogonal if $u_i . u_j = 0$, for $i \neq j$, for $i \neq j$, for $i, j = 1, 2, \ldots n$

Ex. Find a set of vectors $\{u_1, u_2, u_3\}$ is an orthogonal set where $u_1 = (2,-7,-1)$, $u_2 = (-6,-3,9)$, $u_3 = (3,1,-1)$

Sol^n: Here

$u_1.u_2 = (2,-7,-1).(-6,-3,9)$

$= -12 + 21 - 9$

$= 0$

$u_2.u_3 = (-6,-3,9).(3,1,-1)$

$= -18 - 3 - 9$

$= -30 \neq 0$

$u_3.u_1 = (3,1,-1).(2,-7,-1)$

$= 6 - 7 + 1$

$= 0$

So, the set of vectors (u_1, u_2, u_3) are not orthogonal.

but $\{u_1, u_2\}$ and $\{u_3.u_1\}$ are orthogonal

Theorem:- If $S = \{u_1, u_2, \ldots u_n\}$ is an orthogonal set of non-zero vectors in R^n, then S is linearly independent and hence is a basis for the subspace spanned by S.

Theorem 1:

let $\{u_1, u_2, \cdots u_p\}$ be an Orthogonal basis for subspace W of R^n. For any y in W the weight in the linear combination $y = c_1u_1 + c_2u_2 + \cdots c_pu_p$ are given by

$$C_j = \frac{y.u_j}{u_j.u_j}, \text{ for } j = 1, 2, \cdots p.$$

Ex. The set $\{u_1, u_2, u_3\}$ where

$$u_1 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}, \quad u_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \quad u_3 = \begin{bmatrix} -1/2 \\ -2 \\ 7/2 \end{bmatrix}$$

is an Orthogonal basis in R^3. Express the vector $y = \begin{bmatrix} 6 \\ 1 \\ -8 \end{bmatrix}$ as a linear combination of vector.

Sol^n: Here,

$y.u_1 = (6, 1, -8), (3, 1, 1)$

$= 18 + 1 - 8 = 11$

$u_1.u_1 = (3, 1, 1)\ (3, 1, 1) = 9 + 1 + 1 = 11$

$C_1 = \frac{y.u_1}{u_1.u_1} = \frac{11}{11} = 1.$

$y.u_2 = (6, 1, -8).(-1, 2, 1)$

$= -6 + 2 - 8$

$= -12$

$u_2.u_2 = (-1, 2, 1)\ (-1, 2, 1)$

$C_2 = \frac{y.u_2}{u_2.u_2} = \frac{-12}{6} = -2$

For,

C_3

$y.u_3 = (6, 1, -8).(-1/2, -2, 7/2)$

$= -3 - 2 - 28$

$= -33$

$u_3.u_3 = (-1/2, -2, 7/2).(-1/2, -2, 7/2)$

$= 1/4 + 4 + 49/4$

$= \frac{1 + 16 + 49}{4} = \frac{66}{4} = \frac{33}{2}$

$C_3 = \frac{y.u_3}{u_3.u_3} = \frac{-33}{33} \times 2 = -2$

Now,

$c_1u_1 + c_2u_2 + c_3u_3 = 1(3, 1, 1) + -2(-1, 2, 1) + -2(-1/2, -2, 7/2)$

$= (3, 1, 1) + (2, -4, -2) + (1, 4, -7)$

$= (3+2+1, 1-4+4, 1-2-7)$

$= (6, 1, -8)$

$y = (6, 1, -8)$

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1. Determine which sets of vectors ~~are~~ are orthogonal

(i) $\begin{bmatrix}-1\\4\\-3\end{bmatrix}, \begin{bmatrix}5\\2\\1\end{bmatrix}, \begin{bmatrix}3\\-4\\-7\end{bmatrix}$

Sol^n: Here,

$u_1 = \begin{bmatrix}-1\\4\\-3\end{bmatrix}, u_2 = \begin{bmatrix}5\\2\\1\end{bmatrix}, u_3 = \begin{bmatrix}3\\-4\\-7\end{bmatrix}$

$u_1 \cdot u_2 = \begin{bmatrix}-1\\4\\-3\end{bmatrix} \cdot \begin{bmatrix}5\\2\\1\end{bmatrix} = -5+8-3 = 0$

$u_2 \cdot u_3 = (5,2,1)(3,-4,-7) = 15-8-7 = 0$

$u_3 \cdot u_1 = (3,-4,-7)(\cancel{+}-1,4,-3)$

$= -3-16+21$

$= -19+21 = 2 \neq 0$

Hence, the given sets of vectors are not orthogonal.

$\therefore$ The set $\{u_1 \cdot u_2\}$, and $\{u_2, u_3\}$ are orthogonal set.

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(ii) $\begin{bmatrix}1\\-2\\1\end{bmatrix} \begin{bmatrix}0\\1\\2\end{bmatrix} \begin{bmatrix}-5\\-2\\1\end{bmatrix}$

Sol^n: Here,

$u_1 = \begin{bmatrix}1\\-2\\1\end{bmatrix}, u_2 = \begin{bmatrix}0\\1\\2\end{bmatrix}, u_3 = \begin{bmatrix}-5\\-2\\1\end{bmatrix}$

$u_1 \cdot u_2 = (1,-2,1) \cdot (0,1,2)$

$= (0-2+2) = 0$

$u_2 \cdot u_3 = (0,1,2) \cdot (-5,-2,1)$

$= (0-2+2) = 0$

$u_3 \cdot u_1 = (-5,-2,1) \cdot (1,-2,1)$

$= -5+4+1$

$= -5+5 = 0$

$\therefore$ Hence, the given set of vectors are orthogonal.

(iii) $\begin{bmatrix}2\\-7\\-1\end{bmatrix}, \begin{bmatrix}-6\\-3\\9\end{bmatrix}, \begin{bmatrix}3\\1\\-1\end{bmatrix}$

Sol^n: Here,

$u_1 = \begin{bmatrix}2\\-7\\-1\end{bmatrix}, u_2 = \begin{bmatrix}-6\\-3\\9\end{bmatrix}, u_3 = \begin{bmatrix}3\\1\\-1\end{bmatrix}$

$u_1 \cdot u_2 = (2,-7,-1) \cdot (-6,-3,9)$

$= -12+21-9$

$21+21 = 0$

$u_2 \cdot u_3 = (-6,-3,9) \cdot (3,1,-1)$

$= -18-3-9$

$= -18-12$

$-30 \neq 0$

$u_3 \cdot u_1 = (3,1,-1) \cdot (2,-7,-1)$

$= 6-7+1$

$= 7-7 = 0$

$\therefore$ The given set of vectors $\{u_1, u_2, u_3\}$ are not orthogonal.

(iv) $\begin{bmatrix} 2 \\ -5 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ -2 \\ 6 \end{bmatrix}$

Sol^n: Here,

$u_1 = \begin{bmatrix} 2 \\ -5 \\ -3 \end{bmatrix}, u_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, u_3 = \begin{bmatrix} 4 \\ -2 \\ 6 \end{bmatrix}$

$u_1 \cdot u_2 = (2,-5,-3) \cdot (0,0,0)$

$= (0+0+0) = 0$

$u_2 \cdot u_3 = (0,0,0)(4,-2,6) = 0+0+0 = 0$

$u_3 \cdot u_1 = (4,-2,6) \cdot (2,-5,-3)$

$= 8+10-18$

$= 18-18$

0

$\therefore$ Hence the given set of vectors are orthogonal

(v) $\begin{bmatrix} 3 \\ -2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ -3 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 8 \\ 7 \\ 0 \end{bmatrix}$

Sol^n: Here,

$u_1 = \begin{bmatrix} 3 \\ -2 \\ 1 \\ 3 \end{bmatrix}, u_2 = \begin{bmatrix} -1 \\ 3 \\ -3 \\ 4 \end{bmatrix}, u_3 = \begin{bmatrix} 3 \\ 8 \\ 7 \\ 0 \end{bmatrix}$

$u_1 \cdot u_2 = (3,-2,1,3) \cdot (-1,3,-3,4)$

$= -3-6-3+12$

$= -12+12 = 0$

$u_2 \cdot u_3 = (3,8,7,0)(-1,3,-3,4)$

$= -3+24-21+0$

$= -24+24 = 0$

$u_3 \cdot u_1 = (3,8,7,0) \cdot (3,-2,1,3)$

$= 9-16+7$

$= 16-16 = 0$

$\therefore$ Hence, the given set of vectors $\{u_1, u_2, u_3\}$ are orthogonal.

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(vi) $\begin{bmatrix}5\\-4\\0\\3\end{bmatrix}, \begin{bmatrix}-4\\1\\-3\\8\end{bmatrix}, \begin{bmatrix}3\\3\\5\\-1\end{bmatrix}$

Sol^n Here.

$u_1 = \begin{bmatrix}5\\-4\\0\\3\end{bmatrix}, u_2 = \begin{bmatrix}-4\\1\\-3\\8\end{bmatrix}, u_3 = \begin{bmatrix}3\\3\\5\\-1\end{bmatrix}$

$u_1 \cdot u_2 = (5,-4,0,3)\cdot(-4,1,-3,8)$
$= -20-4+0+24$
$= -24+24 = 0$

$u_2 \cdot u_3 = (-4,1,-3,8)\cdot(3,3,5,-1)$
$= -12+3-15-8$
$= 3-35$
$= -32 \neq 0$

$u_3 \cdot u_1 = (3,3,5,-1)\cdot(5,-4,0,3)$
$= 15-12+0-3$
$= 15-15$
$= 0$

$\therefore$ Hence the given set of vectors $\{u_1, u_2, u_3\}$ ~~are~~ not Orthogonal.

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2. Show that $\{u_1, u_2\}$ or $\{u_1, u_2, u_3\}$ is an Orthogonal basis for R^2 or R^3, respectively. Then express x as a linear combination of the u's.

(i) $u_1 = \begin{bmatrix}2\\-3\end{bmatrix}, u_2 = \begin{bmatrix}6\\4\end{bmatrix}$, and $x = \begin{bmatrix}9\\-7\end{bmatrix}$

Sol^n Here,

$c_1 = \frac{y u_1}{u_1 \cdot u_1} = \frac{x \cdot u_1}{u_1 \cdot u_1}$

$y \cdot u_1 = x \cdot u_1 = (9,-7)\cdot(2,-3) = 18+21 = 39$

$u_1 \cdot u_1 = (2,-3)\cdot(2,-3) = 4+9 = 13$

$c_1 = \frac{x u_1}{u_1 \cdot u_1} = \frac{39}{13} = 3$

$c_2 = \frac{x u_2}{u_2 \cdot u_2}$

$x \cdot u_2 = (9,-7)\cdot(6,4) = 54-28 = 26$

$u_2 \cdot u_2 = (6,4)\cdot(6,4) = 36+16 = 52$ $\therefore c_2 = \frac{26}{52} = \frac{1}{2}$

$\therefore x = c_1 u_1 + c_2 u_2$
$= 3(2,-3) + 26(6,4)$
$= (6,-9) + (156,104)$ $\quad x = 3u_1 + \frac{1}{2} u_2$
$= (162,95)$

Ans

$\therefore x = \begin{bmatrix}167\\95\end{bmatrix}$ Ans

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(i) $u_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, $u_2 = \begin{bmatrix} -2 \\ 6 \end{bmatrix}$, and $x = \begin{bmatrix} -6 \\ 3 \end{bmatrix}$

Solⁿ: Here,

$u_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, $u_2 = \begin{bmatrix} -2 \\ 6 \end{bmatrix}$ and $x = \begin{bmatrix} -6 \\ 3 \end{bmatrix}$.

$c_1 = \dfrac{u_1 \cdot x}{u_1 \cdot u_1}$

$u_1 \cdot x = (3,1) \cdot (-6,3)$

$= -18 + 3 = -15$

$u_1 \cdot u_1 = (3,1) \cdot (3,1) = 9 + 1 = 10$

$c_1 = \dfrac{-15}{10} = \dfrac{-3}{2}$

$c_2 = \dfrac{u_2 \cdot x}{u_2 \cdot u_2}$

$u_2 \cdot x = (-2,6) \cdot (-6,3)$

$= 12 + 18 = 30$

$u_2 \cdot u_2 = (-2,6)(-2,6)$

$= 4 + 36 = 40$

$c_2 = \dfrac{u_2 \cdot x}{u_2 \cdot u_2} = \dfrac{30}{40} = 3/4$

$\therefore x = c_1 u_1 + c_2 u_2$

$= -3/2\, u_1 + 3/4\, u_2$ Ans

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(iii) $u_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $u_2 = \begin{bmatrix} \cancel{0}-1 \\ 4 \\ \cancel{0}1 \end{bmatrix}$, $u_3 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$, and $x = \begin{bmatrix} 8 \\ -4 \\ -3 \end{bmatrix}$

Solⁿ: Here,

$c_1 = \dfrac{u_1 \cdot x}{u_1 \cdot u_1}$

$u_1 \cdot x = (1,0,1) \cdot (8,-4,-3) = 8 + 0 - 3 = 5$

$u_1 \cdot u_1 = (1,0,1) \cdot (1,0,1) = 1 + 0 + 1 = 2$

$c_1 = 5/2$

$c_2 = \dfrac{u_2 \cdot x}{u_2 \cdot u_2}$

$u_2 \cdot x = (2,1,-2) \cdot (8,-4,-3) = 16 - 4 + 6 = 18$

$u_2 \cdot u_2 = (2,1,-2) \cdot (2,1,-2) = 4 + 1 + 4 = 9$

$c_2 = 18/9 = 2$

$c_3 = \dfrac{u_3 \cdot x}{u_3 \cdot u_3}$

$u_3 \cdot x = (2,1,-2) \cdot (8,-4,-3) = 16 - 4 + 6 = 18$

(iii) $u_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, u_2 = \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix}, u_3 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}, x = \begin{bmatrix} 8 \\ -4 \\ -3 \end{bmatrix}$

Soln: Here,

$u_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, u_2 = \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix}, u_3 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}, x = \begin{bmatrix} 8 \\ -4 \\ -3 \end{bmatrix}$

$c_1 = \frac{u_1 \cdot x}{u_1 \cdot u_1}$

$u_1 \cdot x = (1,0,1) \cdot (8,-4,-3) = 8+0-3 = 5$

$u_1 \cdot u_1 = (1,0,1) \cdot (1,0,1) = 1+0+1 = 2$

$\therefore c_1 = 5/2$

$c_2 = \frac{u_2 \cdot x}{u_2 \cdot u_2}$

$u_2 \cdot x = (-1,4,1) \cdot (8,-4,-3)$

$= -8-16-3$

$= -27$

$u_2 \cdot u_2 = (-1,4,1)(-1,4,1)$

$= (1+16+1) = 18$

$\therefore c_2 = \frac{-27}{18} = \frac{-3}{2}$

$c_3 = \frac{u_3 \cdot x}{u_3 \cdot u_3}$

$u_3 \cdot x = (2,1,-2) \cdot (8,-4,-3) = 16-4+6 = 18$

$u_3 \cdot u_3 = (2,1,-2)(2,1,-2) = 4+1+4 = 9$

$\therefore c_3 = 18/9 = 2$

$x = u_1 c_1 + u_2 c_2 + u_3 \cdot c_3 = 5/2\, u_1 - 3/2\, u_2 + 2 u_3$

(iv) $u_1 = \begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix}, u_2 = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}, u_3 = \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}$ and $x = \begin{bmatrix} 5 \\ -3 \\ 1 \end{bmatrix}$

Soln: Here,

$u_1 = \begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix}, u_2 = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}, u_3 = \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}$ and $x = \begin{bmatrix} 5 \\ -3 \\ 1 \end{bmatrix}$

$c_1 = \frac{u_1 \cdot x}{u_1 \cdot u_1}$

$u_1 \cdot x = (3,-3,0) \cdot (5,-3,1) = 15+9 = 24$

$u_1 \cdot u_1 = (3,-3,0)(3,-3,0) = 9+9 = 18$

$\therefore c_1 = \frac{24}{18} = \frac{4}{3}$

$c_2 = \frac{u_2 \cdot x}{u_2 \cdot u_2}$

$u_2 \cdot x = (2,2,-1) \cdot (5,-3,1) = 10-6-1 = 3.$

$u_2 \cdot u_2 = (2,2,-1) \cdot (2,2,-1) = 4+4+1 = 9$

$\therefore c_2 = 3/9 = 1/3$

$c_3 = \frac{u_3 \cdot x}{u_3 \cdot u_3}$

$u_3 \cdot x = (1,1,4) \cdot (5,-3,1) = 5-3+4 = 6$

$u_3 \cdot u_3 = (1,1,4) \cdot (1,1,4) = 1+1+16 = 18$

$c_3 = 6/18 = 1/3$

$\therefore x = c_1 u_1 + c_2 u_2 + c_3 u_3$

$= \frac{4}{3} u_1 + \frac{1}{3} u_2 + \frac{1}{3} u_3$

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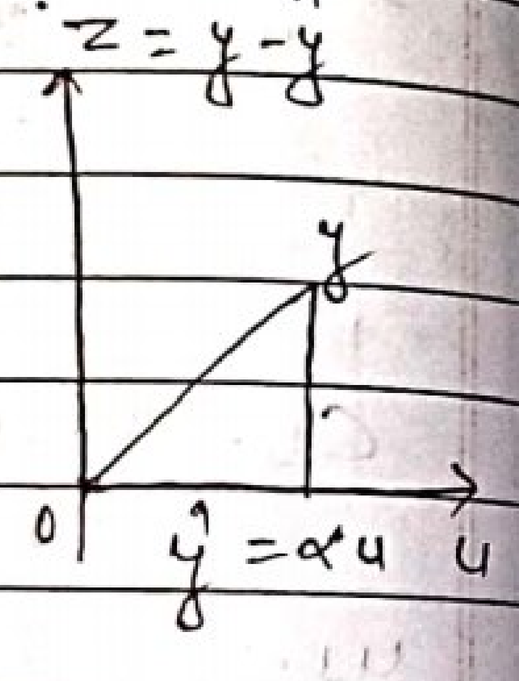
Orthogonal projection.

let u be a non-zero vector in R^n and y be the decomposition vector in R^n such that

$y = \hat{y} + z$



where $\hat{y}$ is the multiple of u and $z = y - \hat{y}$ is the Orthogonal vector of u then

$y - \hat{y}$ is the orthogonal to u if and only if

$(y-\hat{y}).u = 0$

$(y - \alpha u).u = 0$

$y.u - \alpha.u.u = 0$

$\alpha u.u = y.u$

$\Rightarrow \alpha = \dfrac{y.u}{u.u}$

Hence, the vector $\hat{y}$ is the Orthogonal projection of y on to u.

$\hat{y} = \left(\dfrac{y.u}{u.u}\right).u$

3. compute the Orthogonal projection of $\begin{bmatrix}1\\7\end{bmatrix}$ on to the line through $\begin{bmatrix}-4\\2\end{bmatrix}$ and the origin

Sol^n: Here,

let $y = \begin{bmatrix}1\\7\end{bmatrix}$ and $u = \begin{bmatrix}-4\\2\end{bmatrix}$

we Know that,

The orthogonal projection of y on to u is,

$\hat{y} = \left(\dfrac{y.u}{u.u}\right).u$

then,

$\therefore\ y.u = (1,7).(-4,2)$

$= -4+14 = 10$

$u.u = (-4,2).(-4,2) = 16+4 = 20$

$\hat{y} = \left(\dfrac{y.u}{u.u}\right).u = \left(\dfrac{10}{20}\right).u$

$= \frac{1}{2}.\begin{bmatrix}-4\\2\end{bmatrix}$

$= \begin{bmatrix}-2\\1\end{bmatrix}$ Ans

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4(i) Let $y = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $u = \begin{bmatrix} 4 \\ -7 \end{bmatrix}$. Write y as the Sum of two Orthogonal Vectors, One is span{u} and one orthogonal to u.

Sol^n: Here,

$y = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $u = \begin{bmatrix} 4 \\ -7 \end{bmatrix}$

$y \cdot u = (2,3)\cdot(4,-7) = 8-21 = -13$

$u \cdot u = (4,-7)(4,-7) = 16+49 = 65$

Now,

the orthogonal projection $\hat{y}$ of y on to u is

$\hat{y} = \left(\frac{u \cdot y}{u \cdot u}\right) \cdot u = \left(\frac{-13}{65}\right) u = -\frac{1}{5}\begin{pmatrix} 4 \\ -7 \end{pmatrix}$

Here

$y = \hat{y} + (y-\hat{y}) \Rightarrow y = \begin{bmatrix} -4/5 \\ 7/5 \end{bmatrix} + \left[(2,3) - (-4/5, 7/5) \right]$

$y = (-4/5, 7/5) + (2+4/5, 3-7/5)$

$= (-4/5, 7/5) + (14/5, 8/5)$

$y = \begin{bmatrix} -4/5 \\ 7/5 \end{bmatrix} + \begin{bmatrix} 14/5 \\ 8/5 \end{bmatrix}$

This shows that y can be written as Sum $\hat{y}$ and $(y-\hat{y})$, since $\alpha . u = \hat{y}$ for $\alpha = \left(\frac{y \cdot u}{u \cdot u}\right)$

So span $\{u\} = \hat{y}$ Here,

$\hat{y} \cdot (y-\hat{y}) = \begin{pmatrix} -4/5 \\ 7/5 \end{pmatrix} \cdot \begin{pmatrix} 14/5 \\ 8/5 \end{pmatrix} = -56/5 + 56/5 = 0$

This implies $\hat{y}$ is orthogonal to $(y-\hat{y})$, therefore $(y-\hat{y})$ is orthogonal to u,

4(ii) let $y = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$ and $u = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$. write y as the sum of a vector in span{u} and a vector Orthogonal to u

Sol^n: Here,

$y = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$ and $u = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$

$y \cdot u = (2,6)\cdot(7,1) = 14+6 = 20$

$u \cdot u = (7,1)(7,1) = 49+1 = 50$

The orthogonal projection ~~of~~ $\hat{y}$ of y on to u is

$\hat{y} = \left(\frac{y \cdot u}{u \cdot u}\right) \cdot u = \left(\frac{20}{50}\right) \cdot (7,1) = \begin{bmatrix} 14/5 \\ 2/5 \end{bmatrix}$

Here $y = \hat{y} + (y-\hat{y}) \Rightarrow y = \begin{bmatrix} 14/5 \\ 2/5 \end{bmatrix} + \left\{ \begin{bmatrix} 2 \\ 6 \end{bmatrix} - \begin{bmatrix} 14/5 \\ 2/5 \end{bmatrix} \right\}$

$y = \begin{bmatrix} 14/5 \\ 2/5 \end{bmatrix} + \begin{bmatrix} -4/5 \\ 28/5 \end{bmatrix}$

This show that y can be written as sum of $\hat{y}$ and $(y-\hat{y})$. Since $\alpha u = \hat{y} \Rightarrow \alpha = \left(\frac{y \cdot u}{u \cdot u}\right)$ so

Span $\{u\} = \hat{y}$ Here,

$\hat{y} \cdot (y-\hat{y}) = \begin{pmatrix} 14/5 \\ 2/5 \end{pmatrix} \cdot \begin{pmatrix} -4/5 \\ 28/5 \end{pmatrix} = -56/5 + 56/5 = 0$

This implies $\hat{y}$ is orthogonal to $(y-\hat{y})$ therefore $(y-\hat{y})$ is orthogonal to $\hat{y}$. and y as the sum of a vector in span{u} and a vector orthogonal to u.

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5(i) Let $y = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $u = \begin{bmatrix} 8 \\ 6 \end{bmatrix}$. compute the distance from y to the line through u and the origin.

Sol^n: Here

$y = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $u = \begin{bmatrix} 8 \\ 6 \end{bmatrix}$

$y \cdot u = (3,1) \cdot (8,6) = 24+6 = 30$

$u \cdot u = (8,6) \cdot (8,6) = 64+36 = 100$

$\hat{y} = \left(\frac{y \cdot u}{u \cdot u}\right) u = \frac{30}{100} \cdot (8,6) = \frac{3}{10}(8,6)$

then

let $z = y - \hat{y} = (3,1) - (24/10, 18/10)$

$= (3 - 24/10, 1 - 18/10)$

$= (6/10, -8/10)$

$\|z\| = \sqrt{\frac{36}{100} + \frac{64}{100}} = \sqrt{\frac{100}{100}} = \sqrt{1} = 1.$

∴ the distance from y to the line through the u and the origin is 1.

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(ii) Let $y = \begin{bmatrix} -3 \\ 9 \end{bmatrix}$ and $u = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ compute the distance from y to the line through u and the Origin.

Sol^n: Here,

$y = \begin{bmatrix} -3 \\ 9 \end{bmatrix}$ and $u = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$y \cdot u = (-3,9) \cdot (1,2) = -3+18 = 15$

$u \cdot u = (1,2) \cdot (1,2) = 1+4 = 5.$

$\hat{y} = \left(\frac{y \cdot u}{u \cdot u}\right) \cdot u = \frac{15}{5}(1,2)$

$= (3,6)$

Here

$z = y - \hat{y} = (-3,9) - (3,6)$

$z = (-6, 3)$

$\|z\| = \sqrt{36+9} = \sqrt{45} = 3\sqrt{5}$

∴ The distance from y to the line through u and the origin is $3\sqrt{5}$ Ans

Orthonormal set :- An orthogonal set of unit [vectors] is called orthonormal set.
orthonormal basis :- If every vector of an orthog[onal] basis of unit vector then the basis is orthonormal b[asis].

(6) Determine which sets of vectors are orth[onormal]. If a set is only orthogonal, normalize th[e] vector to produce an orthonormal set.

(i) $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$

Sol^n: Here,

let $V_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, V_2 = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$

$V_1 \cdot V_2 = (0,1,0) \cdot (0,-1,0) = 0-1+0 = -1 \neq 0$

since the given set of vector $\{V_1, V_2\}$ is not an orthogonal vector and hence it doesnot hold the orthonormal set

(ii) $\begin{bmatrix} -0.6 \\ 0.8 \end{bmatrix}, \begin{bmatrix} 0.8 \\ 0.6 \end{bmatrix}$

Sol^n: Here,

$V_1 = \begin{bmatrix} -0.6 \\ 0.8 \end{bmatrix}$ and $V_2 = \begin{bmatrix} 0.8 \\ 0.6 \end{bmatrix}$

$V_1 \cdot V_2 = \begin{bmatrix} -0.6 \\ 0.8 \end{bmatrix} \cdot \begin{bmatrix} 0.8 \\ 0.6 \end{bmatrix} = \overset{-0.48+0.48}{\cancel{-4.8+4.8}} = 0$

$\|V_1\| = \sqrt{(-0.6)^2+(0.8)^2} = \sqrt{\cancel{.36+.64}}\ \sqrt{0.36+0.64} = \sqrt{1} = 1$

$\|V_2\| = \sqrt{(0.8)^2+(0.6)^2} = \sqrt{0.36+0.64} = \sqrt{1} = 1$

which show that $\{V_1, V_2\}$ are unit vector

Thus the set $\{V_1, V_2\}$ is an orthonormal set. since every orthogonal of non-zero vector form a basis so $\{V_1, V_2\}$ are an orthonormal basis

(iii) $\begin{bmatrix} -2/3 \\ 1/3 \\ 2/3 \end{bmatrix}, \begin{bmatrix} 1/3 \\ 2/3 \\ 0 \end{bmatrix}$

Sol^n: Here,

let $V_1 = \begin{bmatrix} -2/3 \\ 1/3 \\ 2/3 \end{bmatrix}, V_2 = \begin{bmatrix} 1/3 \\ 2/3 \\ 0 \end{bmatrix}$

$V_1 \cdot V_2 = (-2/3, 1/3, 2/3) \cdot (1/3, 2/3, 0)$

$= (-2/9 + 2/9 + 0)$

$= 0$

$\therefore$ the set $\{V_1, V_2\}$ are orthogonal set,

$\|V_1\| = \sqrt{4/9 + 1/9 + 4/9} = \sqrt{9/9} = 1$

$\|V_2\| = \sqrt{1/9 + 4/9} = \sqrt{5/9} = \sqrt{5}/3 \neq 1$

Since, Here V_2 is not a unit vector ~~as~~ ~~show to~~ that means the set $\{V_1, V_2\}$ is not an orthonormal set.

(iv) $\begin{bmatrix} 1/\sqrt{10} \\ 3/\sqrt{20} \\ 3/\sqrt{20} \end{bmatrix}, \begin{bmatrix} 3/\sqrt{10} \\ -1/\sqrt{20} \\ -1/\sqrt{20} \end{bmatrix}, \begin{bmatrix} 0 \\ -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

Sol: Here,

let $v_1 = \begin{bmatrix} 1/\sqrt{10} \\ 3/\sqrt{20} \\ 3/\sqrt{20} \end{bmatrix}, v_2 = \begin{bmatrix} 3/\sqrt{10} \\ -1/\sqrt{20} \\ -1/\sqrt{20} \end{bmatrix}, v_3 = \begin{bmatrix} 0 \\ -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

$v_1 \cdot v_2 = (1/\sqrt{10}, 3/\sqrt{20}, 3/\sqrt{20})(3/\sqrt{10}, -1/\sqrt{20}, -1/\sqrt{20})$

$= \frac{3}{10} - \frac{3}{20} - \frac{3}{20}$

$= \frac{3}{10} - \frac{6}{20}$

$= \frac{6-6}{20} = \frac{0}{20} = 0$

$v_2 \cdot v_3 = (3/\sqrt{10}, -1/\sqrt{20}, -1/\sqrt{20}) \cdot (0, -1/\sqrt{2}, 1/\sqrt{2})$

$= 0 + 1/\sqrt{40} - 1/\sqrt{40} = 1/\sqrt{40} - 1/\sqrt{40} = 0$

$v_3 \cdot v_1 = (0, -1/\sqrt{2}, 1/\sqrt{2}) \cdot (1/\sqrt{10}, 3/\sqrt{20}, 3/\sqrt{20})$

$= (0 - 3/\sqrt{40} + 3/\sqrt{40})$

$= -\frac{3}{\sqrt{40}} + \frac{3}{\sqrt{40}} = 0$

The set $\{v_1, v_2, v_3\}$ are orthogonal set.

$\|v_1\| = \sqrt{\frac{1}{10} + \frac{9}{20} + \frac{9}{20}} = \sqrt{\frac{2+9+9}{20}} = \sqrt{20/20} = 1$

$\|v_2\| = \sqrt{\frac{9}{10} + \frac{1}{20} + \frac{1}{20}} = \sqrt{\frac{18+1+1}{20}} = \sqrt{20/20} = 1$

$\|v_3\| = \sqrt{0 + \frac{1}{2} + \frac{1}{2}} = \sqrt{2/2} = 1$

which shows that $\{v_1, v_2, v_3\}$ are unit vector

Thus, the set $\{v_1, v_2, v_3\}$ is an orthonormal set since, every orthogonal of non-zero vector form a basis so $\{v_1, v_2, v_3\}$ are an orthonormal basis,

(v) $\begin{bmatrix} 1/\sqrt{18} \\ 4/\sqrt{18} \\ 1/\sqrt{18} \end{bmatrix}, \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}, \begin{bmatrix} -2/3 \\ 1/3 \\ -2/3 \end{bmatrix}$

Sol: Here,

let $v_1 = \begin{bmatrix} 1/\sqrt{18} \\ 4/\sqrt{18} \\ 1/\sqrt{18} \end{bmatrix}, v_2 = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}, v_3 = \begin{bmatrix} -2/3 \\ 1/3 \\ -2/3 \end{bmatrix}$

$v_1 \cdot v_2 = \begin{bmatrix} 1/\sqrt{18} \\ 4/\sqrt{18} \\ 1/\sqrt{18} \end{bmatrix} \cdot \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix} = 1/\sqrt{36} + 0 - 1/\sqrt{36}$

$= 1/\sqrt{36} - 1/\sqrt{36} = 0$

$v_2 \cdot v_3 = (1/\sqrt{2}, 0, -1/\sqrt{2}) \cdot (-2/3, 1/3, -2/3)$

$= \frac{-2}{3\sqrt{2}} + 0 + \frac{2}{3\sqrt{2}} = -2/3\sqrt{2} + 2/3\sqrt{2} = 0$

$v_3 \cdot v_1 = (-2/3, 1/3, -2/3) \cdot (1/\sqrt{18}, 4/\sqrt{18}, 1/\sqrt{18})$

$= \frac{-2}{3\sqrt{18}} + \frac{4}{3\sqrt{18}} - \frac{2}{3\sqrt{18}}$

$= \frac{-4}{3\sqrt{18}} + \frac{4}{3\sqrt{18}} = 0$

$\|v_1\| = \sqrt{\frac{1}{18} + \frac{16}{18} + \frac{1}{18}} = \sqrt{18/18} = 1$

$\|v_2\| = \sqrt{\frac{1}{2} + 0 + \frac{1}{2}} = \sqrt{2/2} = 1$

$\|v_3\| = \sqrt{\frac{4}{9}+\frac{1}{9}+\frac{4}{9}} = \sqrt{\frac{9}{9}} = 1$

which shows that $\{v_1, v_2, v_3\}$ are unit vector. Thus, the set $\{v_1, v_2, v_3\}$ is an ~~orthogonal~~ orthonormal set. Since, every orthogonal of non-zero vector form a basis. So $\{v_1, v_2, v_3\}$ are an orthonormal basis.

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Exercise :- 8.3

1. Assume that $\{u_1, \dots u_4\}$ is an orthogonal basis for R^4. Let $u_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}$, $u_2 = \begin{bmatrix} -2 \\ 1 \\ -1 \\ 1 \end{bmatrix}$, $u_3 = \begin{bmatrix} 1 \\ 1 \\ -2 \\ -1 \end{bmatrix}$

and $u_4 = \begin{bmatrix} -1 \\ 1 \\ 1 \\ -2 \end{bmatrix}$, $v = \begin{bmatrix} 4 \\ 5 \\ -3 \\ 3 \end{bmatrix}$. write v as the sum of two vectors, one is span$\{u_1\}$ and the other in span $\{u_2, u_3, u_4\}$.

Solⁿ: Here,

$v \cdot u_1 = (4, 5, -3, 3) \cdot (1, 2, 1, 1)$
$= 4 + 10 - 3 + 3$
$= 14$

$u_1 \cdot u_1 = (1, 2, 1, 1)(1, 2, 1, 1)$
$= (1 + 4 + 1 + 1) = 7$

$v \cdot u_2 = (4, 5, -3, 3) \cdot (-2, 1, -1, 1)$
$= -8 + 5 + 3 + 3$
$= 3$

$u_2 \cdot u_2 = (-2, 1, -1, 1) \cdot (-2, 1, -1, 1)$
$= 4 + 1 + 1 + 1 = 7$

$v \cdot u_3 = (4, 5, -3, 3) \cdot (1, 1, -2, -1)$
$= 4 + 5 + 6 - 3$
$= 12$

$u_3 \cdot u_3 = (1, 1, -2, -1) \cdot (1, 1, -2, -1)$
$= 1 + 1 + 4 + 1 = 7$

$v \cdot u_4 = (4, 5, -3, 3) \cdot (-1, 1, 1, -2)$
$= -4 + 5 - 3 - 6$
$= -8$

$u_4 \cdot u_4 = (-1, 1, 1, -2) \cdot (-1, 1, 1, -2) = 1 + 1 + 1 + 4 = 7$

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let x_1 be is in span $\{u_1\}$ and x_2 is in span $\{u_2, u_3, u_4\}$

$$x_1 + x_2 = \left\{\left(\frac{v.u_1}{u_1.u_1}\right).u_1\right\} + \left\{\left(\frac{v.u_2}{u_2.u_2}\right).u_2 + \left(\frac{v.u_3}{u_3.u_3}\right).u_3 + \left(\frac{v.u_4}{u_4.u_4}\right).u_4\right\}$$

$$= \left\{\frac{14}{7}.(1,2,1,1)\right\} + \left\{\frac{3}{7}(-2,1,-1,1) + \frac{12}{7}(1,1,-2,-1) + \frac{-8}{7}(-1,1,1,-2)\right\}$$

$$= \{(2,4,2,2)\} + \left\{\left(\frac{-6}{7}, \frac{3}{7}, \frac{-3}{7}, \frac{3}{7}\right) + \left(\frac{12}{7}, \frac{12}{7}, \frac{-24}{7}, \frac{-12}{7}\right) + \left(\frac{8}{7}, \frac{-8}{7}, \frac{-8}{7}, \frac{16}{7}\right)\right\}$$

$$= \{(2,4,2,2)\} + \left(\frac{-6}{7} + \frac{12}{7} + \frac{8}{7}, \frac{3}{7} + \frac{12}{7} - \frac{8}{7}, \frac{-3}{7} - \frac{24}{7} - \frac{8}{7}, \frac{3}{7} - \frac{12}{7} + \frac{16}{7}\right)$$

$$= \{(2,4,2,2)\} + \left\{\frac{14}{7}, \frac{7}{7}, \frac{-35}{7}, \frac{7}{7}\right\}$$

$$= \{(2,4,2,2)\} + \{2,1,-5,1\}$$

$$= \{(4,5,-3,3)\}$$

$$x_1 + x_2 = \begin{bmatrix} 4 \\ 5 \\ -3 \\ 3 \end{bmatrix}$$

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2. Verify that $\{u_1, u_2\}$ is an orthogonal set, and then find the orthogonal projection of y onto span $\{u_1, u_2\}$.

(i) $y = \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix}$, $u_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $u_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

Sol: Here,

$y = \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix}$, $u_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $u_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

$$y.u_1 = (-1,4,3).(1,1,0)$$
$$= -1+4+0 = 3$$

$$u_1.u_1 = (1,1,0).(1,1,0) = 1+1+0 = 2$$

$$y.u_2 = (-1,4,3).(-1,1,0)$$
$$= 1+4+0$$
$$= 5$$

$$u_2.u_2 = (-1,1,0).(-1,1,0)$$
$$= 1+1+0$$
$$= 2$$

$\therefore$ let ~~x_1~~ the orthogonal projection of y onto span $\{u_1, u_2\}$ is,

$$\hat{y} = \left\{\left(\frac{y.u_1}{u_1.u_1}\right).u_1 + \left(\frac{y.u_2}{u_2.u_2}\right).u_2\right\}$$

$$= \left\{\frac{3}{2}.(1,1,0)\right\} + \left\{\frac{5}{2}(-1,1,0)\right\}$$

$$= \left(\frac{3}{2}, \frac{3}{2}, 0\right) + \left(\frac{-5}{2}, \frac{5}{2}, 0\right)$$

$$= \left(\frac{3}{2} - \frac{5}{2}, \frac{3}{2} + \frac{5}{2}, 0\right)$$

$$= (-1, 4, 0) \text{ Ans}$$

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(11) $y = \begin{bmatrix} -1 \\ 2 \\ 6 \end{bmatrix}, u_1 = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}, u_2 = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$

Sol^n: Here.

$y = \begin{bmatrix} -1 \\ 2 \\ 6 \end{bmatrix}, u_1 = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}, u_2 = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$

$y \cdot u_1 = (-1,2,6) \cdot (3,-1,2)$
$= -3-2+12$
$= 7$

$u_1 \cdot u_1 = (3,-1,2)(3,-1,2)$
$= 9+1+4$
$= 14$

$y \cdot u_2 = (-1,2,6)(1,-1,-2)$.
$= -1-2-12$
$= -15$

$u_2 \cdot u_2 = (1,-1,-2) \cdot (1,-1,-2)$
$= 1+1+4$
$= 6$

$\therefore$ the ~~proj~~ orthogonal projection of y on to span$\{u_1, u_2\}$ is

$\hat{y} = \left\{ \left(\frac{y \cdot u_1}{u_1 \cdot u_1} \right) \cdot u_1 \right\} + \left\{ \left(\frac{y \cdot u_2}{u_2 \cdot u_2} \right) \cdot u_2 \right\}$

$= \left\{ \frac{7}{14} (3,-1,2) \right\} + \left\{ \frac{-15}{6} (1,-1,-2) \right\}$

$= \left(3/2, \frac{-1}{2}, 1 \right) + \left(-5/2, 5/2, 5 \right)$

$= (\cancel{-1,2,6})\ (-1,2,6)$ Ans

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(11) $y = \begin{bmatrix} 6 \\ 4 \\ 1 \end{bmatrix}, u_1 = \begin{bmatrix} -4 \\ -1 \\ 1 \end{bmatrix}, u_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

Sol^n: Here,

$y = \begin{bmatrix} 6 \\ 4 \\ 1 \end{bmatrix}, u_1 = \begin{bmatrix} -4 \\ -1 \\ 1 \end{bmatrix}, u_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

$y \cdot u_1 = (6,4,1) \cdot (-4,-1,1)$
$= -24-4+1$
$= -27$

$u_1 \cdot u_1 = (-4,-1,1) \cdot (-4,-1,1)$
$= 16+1+1 = 18$

$y \cdot u_2 = (6,4,1) \cdot (0,1,1)$
$= 0+4+1 = 5$

$u_2 \cdot u_2 = (0,1,1) \cdot (0,1,1)$
$= 0+1+1 = 2$

$\therefore$ The orthogonal projection of y in span$\{u_1, u_2\}$ is

$\hat{y} = \left\{ \left(\frac{y \cdot u_1}{u_1 \cdot u_1} \right) \cdot u_1 \right\} + \left\{ \left(\frac{y \cdot u_2}{u_2 \cdot u_2} \right) \cdot u_2 \right\}$

$= \left(\frac{-27}{18} (-4,-1,1) \right) + \left(\frac{5}{2} (0,1,1) \right)$

$= \frac{-3}{2} (-4,-1,1) + 5/2 (0,1,1)$

$= \left(\frac{12}{2}, 3/2, \frac{-3}{2} \right) + \left(0+5/2, 5/2 \right)$

$= \left(6+0, 3/2+5/2, -3/2+5/2 \right)$

$= (6, 4, 1)$ Ans

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3. let W be the subspace spanned by the u's and write y as the sum of a vector in W and a vector orthogonal to W.

① $y = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}, u_1 = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}, u_2 = \begin{bmatrix} 5 \\ 1 \\ 4 \end{bmatrix}$

Solⁿ: Here,

$y \cdot u_1 = (1,3,5)(1,3,-2)$ and $y \cdot u_2 = (1,3,5) \cdot (5,1,4)$

$= 1+9-10$ $\qquad = 5+3+20 = 28$

$= 0$

$u_1 \cdot u_1 = (1,3,-2)(1,3,-2)$ and $u_2 \cdot u_2 = (5,1,4)^2$

$= 1+9-4$ $\qquad = 25+1+16$

$= 6$ $\qquad = 42$

The Orthogonal projection of $\hat{y}$ of y on to span $\{u_1, u_2\}$

$$\hat{y} = \left\{\left(\frac{y \cdot u_1}{u_1 \cdot u_1}\right) \cdot u_1\right\} + \left\{\left(\frac{y \cdot u_2}{u_2 \cdot u_2}\right) u_2\right\}$$

$$= \left\{\frac{0}{6}(1,3,-2) + \frac{28}{42}(5,1,4)\right\}$$

$$= 0 + \frac{2}{3}(5,1,4) = \left(\frac{10}{3}, \frac{2}{3}, \frac{8}{3}\right)$$

$\therefore$ Here $y = \hat{y} + (y - \hat{y})$

$$= \left(\frac{10}{3}, \frac{2}{3}, \frac{8}{3}\right) + \left\{(1,3,5) - \left(\frac{10}{3}, \frac{2}{3}, \frac{8}{3}\right)\right\}$$

$$= \left(\frac{10}{3}, \frac{2}{3}, \frac{8}{3}\right) + \left\{\left(-\frac{7}{3}, \frac{7}{3}, \frac{7}{3}\right)\right\}$$

$$\therefore \quad y = \begin{bmatrix} 10/3 \\ 2/3 \\ 8/3 \end{bmatrix} + \begin{bmatrix} -7/3 \\ 7/3 \\ 7/3 \end{bmatrix}$$

This show that y can be written as the sum of $\hat{y}$ and $(y - \hat{y})$. where y as the sum of a vector in W and a vector orthogonal to W.

(11) $y = \begin{bmatrix} 4 \\ 3 \\ 3 \\ -1 \end{bmatrix}$, $u_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$, $u_2 = \begin{bmatrix} -1 \\ 3 \\ 1 \\ -2 \end{bmatrix}$, $u_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$

Sol^n: Here,

$y \cdot u_1 = (4,3,3,-1) \cdot (1,1,0,1)$
$= 4+3+0-1$
$= 6$

$u_1 \cdot u_1 = (1,1,0,1) \cdot (1,1,0,1)$
$= 1+1+0+1$
$= 3$

$y \cdot u_2 = (4,3,3,-1) \cdot (-1,3,1,-2)$
$= -4+9+3+2$
$= 10$

$u_2 \cdot u_2 = (-1,3,1,-2) \cdot (-1,3,1,-2)$
$= 1+9+1+4$
$= 15$

$y \cdot u_3 = (4,3,3,-1) \cdot (-1,0,1,1)$
$= -4+0+3-1$
$= -2$

$u_3 \cdot u_3 = (-1,0,1,1) \cdot (-1,0,1,1)$
$= 1+0+1+1$
$= 3$

~~The~~ Here the orthogonal projection of $\hat{y}$ of y on to span$\{u_1, u_2, u_3\}$ is

$y = \hat{y} + (y - \hat{y})$

For $\hat{y} = \left\{ \left(\frac{y \cdot u_1}{u_1 \cdot u_1}\right) u_1 + \left(\frac{y \cdot u_2}{u_2 \cdot u_2}\right) \cdot u_2 + \left(\frac{y \cdot u_3}{u_3 \cdot u_3}\right) u_3 \right\}$

$\hat{y} = \left(\frac{6}{3}(1,1,0,1) + \frac{10}{15}(-1,3,1,-2) + \frac{-2}{3}(-1,0,1,1) \right\}$

$\hat{y} = (2,2,0,2) + \left(\frac{-2}{3}, 2, \frac{2}{3}, \frac{-4}{3}\right) + \left(\frac{2}{3}, 0, \frac{-2}{3}, \frac{-2}{3}\right)$

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$\hat{y} = (2,2,0,2) + (0,2,0,-2)$

$\hat{y} = (2,4,0,0)$

$\therefore y = \hat{y} + (y-\hat{y})$

$\therefore y = (2,4,0,0) + \{\cancel{(2,4,0,0)}\ (4,3,3,-1) - (2,4,0$

$\therefore\ = (2,4,0,0) + (2,-1,3,-1)$

$$y = \begin{bmatrix} 2 \\ 4 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \\ 3 \\ -1 \end{bmatrix}$$

$\therefore$ Here this can be written as sum of $\hat{y}$ and $y-\hat{y}$. since $\hat{y}$ where $\hat{y}$ is in W when W be the subspace spanned by u_1 and u_2.

(iii) $y = \begin{bmatrix} 3 \\ 4 \\ 5 \\ 6 \end{bmatrix}$, $u_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}$, $u_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$, $u_3 = \begin{bmatrix} 0 \\ -1 \\ 1 \\ -1 \end{bmatrix}$

Sol^n: Here,

$y \cdot u_1 = (3,4,5,6) \cdot (1,1,0,-1)$

$= 3+4+0-6 = 1$

$u_1 \cdot u_1 = (1,1,0,-1) \cdot (1,1,0,-1)$

$= 1+1+0+1 = 3$

$y \cdot u_2 = (3,4,5,6) \cdot (1,1,0,1) = 3+4+0+6 = 13$

$u_2 \cdot u_2 = (1,1,0,1) \cdot (1,1,0,1)$

$= 1+1+1+0 = 3$

$y \cdot u_3 = (3,4,5,6) \cdot (0,-1,1,-1)$

$= 0-4+5-6 = -5$

$u_3 \cdot u_3 = (0,-1,1,-1) \cdot (0,-1,1,-1)$

$= (0+1+1+1) = 3$

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The orthogonal projection of $\hat{y}$ of y on to span $\{u_1, u_2, u_3\}$ is

$$\hat{y} = \left\{ \left(\frac{y \cdot u_1}{u_1 \cdot u_1}\right) \cdot u_1 + \left(\frac{y \cdot u_2}{u_2 \cdot u_2}\right) \cdot u_2 + \left(\frac{y \cdot u_3}{u_3 \cdot u_3}\right) \cdot u_3 \right\}$$

$= \frac{1}{3} \cdot (1,1,0,-1) + \frac{13}{3}(1,1,0,1) + -\frac{5}{3}(0,-1,1,-1)$

$= \left(\frac{1}{3}, \frac{1}{3}, 0, -\frac{1}{3}\right) + \left(\frac{13}{3}, \frac{13}{3}, 0, \frac{13}{3}\right) + \left(0, \frac{5}{3}, \frac{-5}{3}, \frac{5}{3}\right)$

$= \left(\frac{1}{3} + \frac{13}{3} + 0, \frac{1}{3} + \frac{13}{3} + \frac{5}{3}, 0 + 0 - \frac{5}{3}, -\frac{1}{3} + \frac{13}{3} + \frac{5}{3}\right)$

$= \left(\frac{14}{3}, \frac{19}{3}, -\frac{5}{3}, \frac{17}{3}\right)$

Here,

$y = \hat{y} + (y - \hat{y})$

$= \left(\frac{14}{3}, \frac{19}{3}, -\frac{5}{3}, \frac{17}{3}\right) + \left\{(3,4,5,6) - \left(\frac{14}{3}, \frac{19}{3}, -\frac{5}{3}, \frac{17}{3}\right)\right\}$

$= \left(\frac{14}{3}, \frac{19}{3}, -\frac{5}{3}, \frac{17}{3}\right) + \left\{-\frac{6}{3},\right.$

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(A) Find the closet point to y in the subspace W Spanned by V_1 and V_2.

Sol:

(i) $y = \begin{bmatrix} 3 \\ 1 \\ 5 \\ 1 \end{bmatrix}$, $V_1 = \begin{bmatrix} 3 \\ 1 \\ -1 \\ 1 \end{bmatrix}$, $V_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$.

Sol: Here,

let $\hat{y}$ be the closet point in $W = \text{span}\{u_1, u_2\}$ to y then

$y \cdot V_1 = (3,1,5,1) \cdot (3,1,-1,1)$

$= 9+1-5+1$

$= 6$

$V_1 \cdot V_1 = (3,1,-1,1) \cdot (3,1,-1,1)$

$= (9+1+1+1)$

$= 12$

$y \cdot V_2 = (3,1,5,1) \cdot (1,-1,1,-1)$

$= (3-1+5-1)$

$= 6$

$V_2 \cdot V_2 = (1,-1,1,-1) \cdot (1,-1,1,-1)$

$= 1+1+1+1$

$= 4$

$\hat{y} = \left(\frac{y \cdot V_1}{V_1 \cdot V_1}\right) \cdot V_1 + \left(\frac{y \cdot V_2}{V_2 \cdot V_2}\right) \cdot V_2$

$= \left(\frac{6}{12}\right) \cdot (3,1,-1,1) + \left(\frac{6}{4}\right) \cdot (1,-1,1,-1)$

$= \left(\frac{3}{2}, \frac{1}{2}, \frac{-1}{2}, \frac{1}{2}\right) + \left(\frac{3}{2}, -\frac{3}{2}, \frac{3}{2}, -\frac{3}{2}\right)$

$= (3,-1,1,-1) = \begin{bmatrix} 3 \\ -1 \\ 1 \\ -1 \end{bmatrix}$

Ans

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(ii) $y = \begin{bmatrix} 3 \\ -1 \\ 1 \\ 13 \end{bmatrix}$, $V_1 = \begin{bmatrix} 1 \\ -2 \\ -1 \\ 2 \end{bmatrix}$, $V_2 = \begin{bmatrix} -4 \\ 1 \\ 0 \\ 3 \end{bmatrix}$.

Sol: Here,

let $\hat{y}$ be the closet point in $W = \text{span}\{V_1, V_2\}$ to y then,

$y \cdot V_1 = (3,-1,1,13) \cdot (1,-2,-1,2)$

$= (3+2-1+26)$

$= 30$

$V_1 \cdot V_1 = (1,-2,-1,2) \cdot (1,-2,-1,2)$

$= 1+4+1+4$

$= 10$

$y \cdot V_2 = (3,-1,1,13) \cdot (-4,1,0,3)$

$= -12-1+0+39$

$= -13+39$

$= 26$

$V_2 \cdot V_2 = (-4,1,0,3) \cdot (-4,1,0,3)$

$= 16+1+0+9$

$= 26$

$\hat{y} = \left(\frac{y \cdot V_1}{V_1 \cdot V_1}\right) V_1 + \left(\frac{y \cdot V_2}{V_2 \cdot V_2}\right) \cdot V_2$

$= \frac{30}{10}(1,-2,-1,2) + \left(\frac{26}{26}\right) \cdot (-4,1,0,3)$

$= \cancel{(3,-6,-3,6) + \left(\frac{-2}{13}\right)(-4,1,0,3)}$

$= \cancel{\left(3+\frac{8}{13}, -6-\frac{2}{13}, -3-0, 6-\frac{6}{13}\right)}$

$= (3,-6,-3,6) + (-4,1,0,3)$

$= (-1,-5,-3,9) \Rightarrow \begin{bmatrix} -1 \\ -5 \\ -3 \\ 9 \end{bmatrix}$ Ans

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(5) Find the best approximation to z by vectors of the form $c_1v_1 + c_2v_2$ when

$$z = \begin{bmatrix} 3 \\ -7 \\ 2 \\ 3 \end{bmatrix}, \quad v_1 = \begin{bmatrix} 2 \\ -1 \\ -3 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}$$

Sol^n: Here

$z \cdot v_1 = (3,-7,2,3) \cdot (2,-1,-3,1)$
$= (6+7-6+3)$
$= 10$

$v_1 \cdot v_1 = (2,-1,-3,1) \cdot (2,-1,-3,1)$
$= 4+1+9+1$
$= 15$

$z \cdot v_2 = (3,-7,2,3) \cdot (1,1,0,-1)$
$= 3-7+0-3$
$= -7$

$v_2 \cdot v_2 = (1,1,0,-1) \cdot (1,1,0,-1)$
$= 1+1+0+1$
$= 3$

$\therefore \hat{z} = c_1v_1 + c_2v_2$

$= \left(\frac{z \cdot v_1}{v_1 \cdot v_1}\right) \cdot v_1 + \left(\frac{z \cdot v_2}{v_2 \cdot v_2}\right) \cdot v_2$

$= \left(\frac{10}{15}\right)(2,-1,-3,1) + \left(\frac{-7}{3}\right)(1,1,0,-1)$

$= \left(\frac{2}{3}\right)(2,-1,-3,1) + \left(\frac{-7}{3}\right)(1,1,0,-1)$

$= \left(\frac{4}{3}, -\frac{2}{3}, -2, \frac{2}{3}\right) + \left(-\frac{7}{3}, -\frac{7}{3}, 0, \frac{7}{3}\right)$

$= (-1, -3, -2, 3)$ Ans

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(6) Let $y = \begin{bmatrix} 5 \\ -9 \\ 5 \end{bmatrix}$, $u_1 = \begin{bmatrix} -3 \\ -5 \\ 1 \end{bmatrix}$, $u_2 = \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}$, find the distance from y to the plan in R^3 spanned by u_1 and u_2.

Sol^n: Here,

$y \cdot u_1 = (5,-9,5) \cdot (-3,-5,1)$
$= (-15+45+5)$
$= 35$

$u_1 \cdot u_1 = (-3,-5,1) \cdot (-3,-5,1)$
$= 9+25+1$
$= 35$

$y \cdot u_2 = (5,-9,5) \cdot (-3,2,1)$
$= -15-18+5$
$= -28$
$= -28$

$u_2 \cdot u_2 = (-3,2,1)(-3,2,1)$
$= 9+4+1$
$= 14$

$\therefore \hat{y} = \left(\frac{y \cdot u_1}{u_1 \cdot u_1}\right) \cdot u_1 + \left(\frac{y \cdot u_2}{u_2 \cdot u_2}\right) \cdot u_2$

$= \left(\frac{35}{35}\right) \cdot (-3,-5,1) + \left(\frac{-28}{14}\right)(-3,2,1)$

$= (-3,-5,1) + (-2)(-3,2,1)$

$= (-3,-5,1) + (6,-4,-2)$

$= (3,-9,-1)$

$\therefore$ let $z = y - \hat{y} = (5,-9,5) - (3,-9,-1)$
$= (2,0,6)$

$\|z\| = \sqrt{4+0+36} = \sqrt{40}$ Ans

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(7) Let $y = \begin{bmatrix} 4 \\ 8 \\ 1 \end{bmatrix}$, $u_1 = \begin{bmatrix} 2/3 \\ 1/3 \\ 2/3 \end{bmatrix}$, $u_2 = \begin{bmatrix} -2/3 \\ 2/3 \\ 1/3 \end{bmatrix}$ and

$W = \text{span}\{u_1, u_2\}$

(i) let $U = \{u_1, u_2\}$. compute U^TU and UU^T

(ii) compute $\text{proj}_W(y)$. and $(UU^T)y$

Sol: Here.

$U = \{u_1, u_2\} = \begin{bmatrix} 2/3 & -2/3 \\ 1/3 & 2/3 \\ 2/3 & 1/3 \end{bmatrix}$

$U^T = \begin{bmatrix} 2/3 & 1/3 & 2/3 \\ -2/3 & 2/3 & 1/3 \end{bmatrix}$

$\therefore U^T \cdot U = \begin{bmatrix} 2/3 & 1/3 & 2/3 \\ -2/3 & 2/3 & 1/3 \end{bmatrix} \cdot \begin{bmatrix} 2/3 & -2/3 \\ 1/3 & 2/3 \\ 2/3 & 1/3 \end{bmatrix}$

$= \begin{bmatrix} 4/9 + 1/9 + 4/9 & -4/9 + 2/9 + 2/9 \\ -4/9 + 2/9 + 2/9 & 4/9 + 4/9 + 1/9 \end{bmatrix}$

$= \begin{bmatrix} 9/9 & -4/9 + 4/9 \\ -4/9 + 4/9 & 9/9 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\therefore U^T \cdot U = I$ Ans

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$U \cdot U^T = \begin{bmatrix} 2/3 & -2/3 \\ 1/3 & 2/3 \\ 2/3 & 1/3 \end{bmatrix}_{3 \times 2} \cdot \begin{bmatrix} 2/3 & 1/3 & 2/3 \\ -2/3 & 2/3 & 1/3 \end{bmatrix}_{2 \times 3}$

$= \begin{bmatrix} 4/9 + 4/9 & 2/9 - 4/9 & 4/9 - 2/9 \\ 2/9 - 4/9 & 1/9 + 4/9 & 2/9 + 2/9 \\ 4/9 - 2/9 & 2/9 + 2/9 & 4/9 + 1/9 \end{bmatrix}$

$= \begin{bmatrix} 8/9 & -2/9 & 2/9 \\ -2/9 & 5/9 & 4/9 \\ 2/9 & 4/9 & 5/9 \end{bmatrix}$ Ans

(ii) $\text{proj}_W(y)$

$y \cdot u_1 = (4, 8, 1) \cdot (2/3, 1/3, 2/3)$

$= (8/3 + 8/3 + 2/3)$

$= (18/3)$

$= 6$

$u_1 \cdot u_1 = (2/3, 1/3, 2/3) \cdot (2/3, 1/3, 2/3)$

$= \left(\frac{4}{9} + \frac{1}{9} + \frac{4}{9}\right)$

$= 9/9 = 1$

$y \cdot u_2 = (4, 8, 1) \cdot (-2/3, 2/3, 1/3)$

$= -8/3 + 16/3 + 1/3$

$= 9/3$

$= 3$

$u_2 \cdot u_2 = (-2/3, 2/3, 1/3) \cdot (-2/3, 2/3, 1/3)$

$= 4/9 + 4/9 + 1/9$

$= 9/9 = 1$

$\hat{y} = \left(\frac{y \cdot u_1}{u_1 \cdot u_1}\right) \cdot u_1 + \left(\frac{y \cdot u_2}{u_2 \cdot u_2}\right) \cdot u_2 = \left(\frac{6}{1}\right) u_1 + \left(\frac{3}{1}\right) u_2$

$\hat{y} = 6u_1 + 3u_2$

$$\hat{y} = 6\left(\tfrac{2}{3}, \tfrac{1}{3}, \tfrac{2}{3}\right) + 3\left(-\tfrac{2}{3}, \tfrac{2}{3}, \tfrac{1}{3}\right)$$

$$= (4, 2, 4) + (-2, 2, 1)$$

$$= (4-2, 2+2, 4+1)$$

$$= (2, 4, 5)$$

$$\hat{y} = \begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix}$$

* $$(UU^T) \cdot y = \begin{bmatrix} 8/9 & -4/9 & 2/9 \\ -4/9 & 5/9 & 4/9 \\ 2/9 & 4/9 & 5/9 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 8 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{32}{9} - \frac{16}{9} + \frac{2}{9} \\ -8/9 + 40/9 + 4/9 \\ 8/9 + 32/9 + 5/9 \end{bmatrix}$$

$$= \begin{bmatrix} 18/9 \\ 36/9 \\ 45/9 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix}$$

$\therefore (UU^T) \cdot y = (2, 4, 5)$ Ans

Exercise: 8.4

The Gram Schmidt process:-

The Gram schmidt process is a simple process obtain an orthogonal or orthonormal basis for any non-zero subspace of R^n.

Theorem:-

Give a basis $\{x_1, x_2, \dots x_p\}$ for a subspace W of R^n, we define

$$V_1 = x_1$$

$$V_2 = x_2 - \left(\frac{x_2 \cdot V_1}{V_1 \cdot V_1}\right) \cdot V_1$$

$$V_3 = x_3 - \left(\frac{x_3 \cdot V_1}{V_1 \cdot V_1}\right) V_1 - \left(\frac{x_3 \cdot V_2}{V_2 \cdot V_2}\right) \cdot V_2$$

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$$V_p = x_p - \sum_{n=1}^{p-1} \left(\frac{x_p \cdot V_n}{V_n \cdot V_n}\right) \cdot V_n .$$

Then $V_p =$ ~~x_p~~ $\{V_1, V_2, \dots V_p\}$ is an orthogonal basis for W. In addition, Span $\{V_1, V_2, \dots V_p\}$ = span $\{x_1, x_2, \dots x_p\}$

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The given set is a basis for a Subspace W. Use the Gram-Schmidt process to produce an orthogonal basis for W.

① $\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 8 \\ 5 \\ -6 \end{bmatrix}$

Sol^n: here,

let $V_1 = x_1 = (3,0,1)$

then

$$V_2 = x_2 - \left(\frac{x_2 \cdot V_1}{V_1 \cdot V_1}\right) \cdot V_1$$

$x_2 V_1 = \cancel{(3,5,6) \cdot (3,0,1)}$ $(8,5,-6) \cdot (3,0,1)$

$= \cancel{9+0+1}$

$= 24+0-6$

$= 18$

$V_1 \cdot V_1 = (3,0,1) \cdot (3,0,1)$

$= 9+0+1 = 10$

$V_2 = (8,5,-6) - \left(\frac{18}{10}\right) \cdot (3,0,1)$

$= (8,5,-6) - \left(\frac{54}{10}, 0, \frac{18}{10}\right)$

$= (8,5,-6) - \left(\frac{27}{5}, 0, \frac{9}{5}\right)$

$= \left(\frac{13}{5}, 5, \frac{-39}{5}\right)$

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① $\begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 8 \\ 5 \\ -6 \end{bmatrix}$

Sol^n: Here,

let $x_1 = V_1 = \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}$

$x_2 = \begin{bmatrix} 8 \\ 5 \\ -6 \end{bmatrix}$

then

$$V_2 = x_2 - \left(\frac{x_2 \cdot V_1}{V_1 \cdot V_1}\right) \cdot V_1$$

$x_2 \cdot V_1 = (8,5,-6) \cdot (3,0,-1)$

$= 24+6$

$= 30$

$V_1 \cdot V_1 = (3,0,-1) \cdot (3,0,-1)$

$= 9+0+1$

$= 10$

$V_2 = (8,5,-6) - \left(\frac{30}{10}\right) \cdot (3,0,-1)$

$= (8,5,-6) - 3(3,0,-1)$

$= (8,5,-6) - (9,0,-3)$

$= (-1,5,-3)$

$V_2 = \begin{bmatrix} -1 \\ 5 \\ -3 \end{bmatrix}$

Now, the set $\{v_1, v_2\}$ is an orthogonal set for W

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(8) $\begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} 5 \\ 6 \\ -7 \end{bmatrix}$

Sol^n: Here,

let $x_1 = \begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix} = v_1 = \begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix}$

$x_2 = \begin{bmatrix} 5 \\ 6 \\ -7 \end{bmatrix}$

$\therefore v_2 = x_2 - \left(\frac{x_2 . v_1}{v_1 . v_1}\right) . v_1$

$x_2 . v_1 = (5, 6, -7)(0, 4, 2)$

$= (0 + 24 - 14)$

$= 10$

$v_1 . v_1 = \begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix} . \begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix}$

$= 0 + 16 + 4 = 20$

$\therefore v_2 = (5, 6, -7) - \left(\frac{10}{20}\right) . (0, 4, 2)$

$= (5, 6, -7) - \frac{1}{2}(0, 4, 2)$

$= (5, 6, -7) - (0, 2, 1)$

$= (5, 4, -8)$

$\therefore v_2 = \begin{bmatrix} 5 \\ 4 \\ -8 \end{bmatrix}$ Ans

Now, the set $\{v_1, v_2\}$ is an Orthogonal set for w

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(9) $\begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix}$

Sol^n: Here

let $x_1 = v_1 = \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix}$

$x_2 = \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix}$

$\therefore v_2 = x_2 - \left(\frac{x_2 . v_1}{v_1 . v_1}\right) . v_1$

$x_2 . v_1 = (4, -1, 2) . (2, -5, 1)$

$= 8 + 5 + 2$

$= 15$

$v_1 . v_1 = (2, -5, 1)(2, -5, 1)$

$= 4 + 25 + 1$

$= 30$

$\therefore v_2 = (4, -1, 2) - \left(\frac{15}{30}\right) . \cancel{(4, -1, 2)} (2, -5, 1)$

$= (4, -1, 2) - \frac{1}{2} \cancel{(4, -1, 2)} (2, -5, 1)$

$= (4, -1, 2) - \left(1, -\frac{5}{2}, \frac{1}{2}\right)$

$= \left(3, \frac{3}{2}, \frac{3}{2}\right)$

$v_2 = \begin{bmatrix} 3 \\ 3/2 \\ 3/2 \end{bmatrix}$

Now the set $\{v_1, v_2\}$ is an Orthogonal set for w

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(4) $\begin{bmatrix} 3 \\ -4 \\ 5 \end{bmatrix}$ $\begin{bmatrix} -3 \\ 14 \\ -7 \end{bmatrix}$

Sol: Here,

let $x_1 = v_1 = \begin{bmatrix} 3 \\ -4 \\ 5 \end{bmatrix}$

$x_2 = \begin{bmatrix} -3 \\ 14 \\ -7 \end{bmatrix}$

Now

$v_2 = x_2 - \left(\frac{x_2 \cdot v_1}{v_1 \cdot v_1}\right) \cdot v_1$

$x_2 \cdot v_1 = (-3, 14, -7) \cdot (3, -4, 5)$

$= (-9 - 56 - 35)$

$= -100$

$v_1 \cdot v_1 = (3, -4, 5) \cdot (3, -4, 5)$

$= 9 + 16 + 25$

$= 50$

$\therefore v_2 = x_2 - \left(\frac{x_2 \cdot v_1}{v_1 \cdot v_1}\right) \cdot v_1$

$= (-3, 14, -7) - \left(\frac{-100}{50}\right) \cdot (3, -4, 5)$

$= (-3, 14, -7) + 2(3, -4, 5)$

$= (-3, 14, -7) + (6, -8, 10)$

$= (3, 6, 3)$

$v_2 = \begin{bmatrix} 3 \\ 6 \\ 3 \end{bmatrix}$

$\therefore$ The set $\{v_1, v_2\}$ is a basis for a subspace W

(5) $\begin{bmatrix} 1 \\ -4 \\ 0 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 7 \\ -7 \\ -4 \\ 1 \end{bmatrix}$

let $x_1 = v_1 = \begin{bmatrix} 1 \\ -4 \\ 0 \\ 1 \end{bmatrix}$

$x_2 = \begin{bmatrix} 7 \\ -7 \\ -4 \\ 1 \end{bmatrix}$

Now

$v_2 = x_2 - \left(\frac{x_2 \cdot v_1}{v_1 \cdot v_1}\right) \cdot v_1$

$x_2 \cdot v_1 = (7, -7, -4, 1) \cdot (1, -4, 0, 1)$

$= 7 + 28 + 0 + 1$

$= 36$

$v_1 \cdot v_1 = (1, -4, 0, 1) \cdot (1, -4, 0, 1)$

$= (1 + 16 + 0 + 1)$

$= 18$

$\therefore v_2 = x_2 - \left(\frac{x_2 \cdot v_1}{v_1 \cdot v_1}\right) \cdot v_1$

$= (7, -7, -4, 1) - \left(\frac{36}{18}\right) \cdot (1, -4, 0, 1)$

$= (7, -7, -4, 1) - 2(1, -4, 0, 1)$

$= (7, -7, -4, 1) - (2, -8, 0, 2)$

$= (5, 1, -4, -1)$

$\therefore v_2 = \begin{bmatrix} 5 \\ 1 \\ -4 \\ -1 \end{bmatrix}$

The set $\{v_1, v_2\}$ is a basis for a subspace W.

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6. $\begin{bmatrix} 3 \\ -1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -5 \\ 9 \\ -9 \\ 3 \end{bmatrix}$

Sol^n: Here,

let $x_1 = v_1 = \begin{bmatrix} 3 \\ -1 \\ 2 \\ -1 \end{bmatrix}$

$x_2 = \begin{bmatrix} -5 \\ 9 \\ -9 \\ 3 \end{bmatrix}$

$\therefore v_2 = x_2 - \left(\frac{x_2 \cdot v_1}{v_1 \cdot v_1}\right) \cdot v_1$

$x_2 \cdot v_1 = (-5, 9, -9, 3) \cdot (3, -1, 2, -1)$

$= -15 - 9 - 18 - 3$

$= -45$

$v_1 \cdot v_1 = (3, -1, 2, -1) \cdot (3, -1, 2, -1)$

$= 9 + 1 + 4 + 1$

$= 15$

$\therefore v_2 = x_2 - \left(\frac{x_2 \cdot v_1}{v_1 \cdot v_1}\right) \cdot v_1$

$v_2 = (-5, 9, -9, 3) - \left(\frac{-45}{15}\right)(3, -1, 2, -1)$

$= (-5, 9, -9, 3) + 3(3, -1, 2, -1)$

$= (-5, 9, -9, 3) + (9, -3, 6, -3)$

$= (4, 6, -3, 0)$

$\therefore v_2 = \begin{bmatrix} 4 \\ 6 \\ -3 \\ 0 \end{bmatrix}$

$\therefore$ The set $\{v_1, v_2\}$ is a basis for subspace W.

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7. Find an Orthonormal basis of the Subspace spanned by the vectors $(2, -5, 1)$ and $(4, -1, 2)$

Sol^n: Here,

let $x_1 = v_1 = (2, -5, 1)$

$x_2 = (4, -1, 2)$

then

$v_2 = x_2 - \left(\frac{x_2 \cdot v_1}{v_1 \cdot v_1}\right) \cdot v_1$

$x_2 \cdot v_1 = (4, -1, 2) \cdot (2, -5, 1)$

$= (8 + 5 + 2)$

$= 15$

$v_1 \cdot v_1 = (2, -5, 1) \cdot (2, -5, 1)$

$= 4 + 25 + 1$

$= 30$

$\therefore v_2 = x_2 - \left(\frac{x_2 \cdot v_1}{v_1 \cdot v_1}\right) \cdot v_1$

$= (4, -1, 2) - \left(\frac{15}{30}\right)(2, -5, 1)$

$= (4, -1, 2) - \left(\frac{1}{2}\right)(2, -5, 1)$

$= (4, -1, 2) - \left(1, -5/2, 1/2\right)$

$= \left(3, 3/2, 3/2\right)$

$v_2' = \frac{3}{2}(2, 1, 1)$

$v_2' = (2, 1, 1)$

$\cancel{\|v_1\| = v_1}$ $u_1 = \frac{v_1}{\|v_1\|} = \frac{(2, -5, 1)}{\sqrt{4 + 25 + 1}} = \frac{(2, -5, 1)}{\sqrt{30}}$

$u_1 = \left(\frac{2}{\sqrt{30}}, \frac{-5}{\sqrt{30}}, \frac{1}{\sqrt{30}}\right)$

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$u_2 = \dfrac{v_2'}{\|v_2'\|} = \dfrac{(2,1,1)}{\sqrt{4+1+1}} = \dfrac{(2,1,1)}{\sqrt{6}}$

$= \left(\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$

Hence. $\{u_1, u_2\}$ is an orthogonal basis form

8. Find an Orthonormal basis of the Subspace spanned by the Vectors $(3,-4,5)$ and $(-3,14,-7)$

Sol: Here,

let $x_1 = v_1 = (3,-4,5)$

$x_2 = (-3,14,-7)$

$v_2 = x_2 - \left(\dfrac{x_2 \cdot v_1}{v_1 \cdot v_1}\right) \cdot v_1$

$x_2 \cdot v_1 = (-3,14,-7) \cdot (3,-4,5)$

$= -9-56-35$

$= -100$

$v_1 \cdot v_1 = (3,-4,5) \cdot (3,-4,5)$

$= 9+16+25$

$= 50$

$\therefore v_2 = x_2 - \left(\dfrac{x_2 \cdot v_1}{v_1 \cdot v_1}\right) \cdot v_1$

$v_2 = (-3,14,-7) - \left(\dfrac{-100}{50}\right)(3,-4,5)$

$= (-3,14,-7) + 2(3,-4,5)$

$= (-3,14,-7) + (6,-8,10)$

$= (3,6,3) = 3\,(1,2,1)$

Now $\therefore v_2' = (1,2,1)$

$u_1 = \dfrac{v_1}{\|v_1\|}$

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$u_1 = \dfrac{(3,-4,5)}{\sqrt{9+16+25}}$

$u_1 = \dfrac{(3,-4,5)}{\sqrt{50}}$

$u_1 = \left(\frac{3}{5\sqrt{2}}, \frac{-4}{5\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

$\therefore u_2 = \dfrac{v_2'}{\|v_2'\|} = \dfrac{(1,2,1)}{\sqrt{1+4+1}} = \dfrac{(1,2,1)}{\sqrt{6}}$

$u_2 = \left(\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$

$\therefore$ Hence $\{u_1, u_2\}$ is an orthogonal basis form.

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The Q.R factorization :

Theorem:

If A is an $m*n$ matrix with linearly independent column, then A can be factorized as $A = QR$, where Q is an $m*n$ matrix whose column form an orthonormal basis for Col$A$ and R is an $n*n$ upper triangular invertible matrix with positive entries on its diagonal.

Ex. Find Q.R factorization of a matrix A where

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Sol^n: Here,

let the column of A are x_1, x_2, x_3

so,

$x_1 = (1,1,1,1)$, $x_2 = (0,1,1,1)$ and $x_3 = (0,0,1,1)$

let $v_1 = x_1 = (1,1,1,1)$

$$v_2 = x_2 - \left(\frac{x_2 \cdot v_1}{v_1 \cdot v_1}\right) v_1 = (0,1,1,1) - \left(\frac{(0,1,1,1)\cdot(1,1,1,1)}{(1,1,1,1)\cdot(1,1,1,1)}\right)$$

$$v_2 = (0,1,1,1) - \left(\frac{3}{4}\right)(1,1,1,1)$$

$$v_2 = \left(-\tfrac{3}{4}, \tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4}\right)$$

$$v_2 = \tfrac{1}{4}(-3,1,1,1)$$

Set $v_2' = (-3,1,1,1)$

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$$v_3 = x_3 - \left(\frac{x_3 \cdot v_1}{v_1 \cdot v_1}\right) \cdot v_1 - \left(\frac{x_3 \cdot v_2'}{v_2' \cdot v_2'}\right) \cdot v_2'$$

$$= (0,0,1,1) - \left(\frac{(0,0,1,1)\cdot(1,1,1,1)}{(1,1,1,1)\cdot(1,1,1,1)}\right) \cdot v_1 - \left(\frac{(0,0,1,1)\cdot(-3,1,1,1)}{(-3,1,1,1)\cdot(-3,1,1,1)}\right) \cdot (-3,1,1,1)$$

$$= (0,0,1,1) - \left(\frac{2}{4}(1,1,1,1)\right) - \left(\frac{2}{12}(-3,1,1,1)\right)$$

$$= (0,0,1,1) - \left(\left(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}\right) - \left(\tfrac{1}{6}(-3,1,1,1)\right)\right)$$

$$= (0,0,1,1) - \left(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}\right) - \left(-\tfrac{1}{2}, \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}\right)$$

$$= \left(-\tfrac{1}{2}, -\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}\right) - \left(-\tfrac{1}{2}, \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}\right)$$

$$= \left(0, -\tfrac{4}{6}, \tfrac{2}{6}, \tfrac{2}{6}\right)$$

$$= \left(0, -\tfrac{2}{3}, \tfrac{1}{3}, \tfrac{1}{3}\right)$$

$$= \tfrac{1}{3}(0,-2,1,1)$$

Set $v_3' = (0,-2,1,1)$

let (u_1, u_2, u_3) is normalize of the orthogonal basis.

$$u_1 = \frac{v_1}{\|v_1\|} = \frac{(1,1,1,1)}{\sqrt{1+1+1+1}} = \frac{(1,1,1,1)}{\sqrt{4}} = \left(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}\right)$$

$$u_2 = \frac{v_2'}{\|v_2'\|} = \frac{(-3,1,1,1)}{\sqrt{9+1+1+1}} = \frac{(-3,1,1,1)}{\sqrt{12}} = \left(\frac{-3}{\sqrt{12}}, \frac{1}{\sqrt{12}}, \frac{1}{\sqrt{12}}, \frac{1}{\sqrt{12}}\right)$$

$$u_3 = \frac{v_3'}{\|v_3'\|} = \frac{(0,-2,1,1)}{\sqrt{4+1+1}} = \frac{(0,-2,1,1)}{\sqrt{6}} = \left(0, \frac{-2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$$

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Let Q be the matrix whose column are $u_1, u_2,$ u_3

$$Q = \begin{bmatrix} 1/2 & -3/\sqrt{12} & 0 \\ 1/2 & 1/\sqrt{12} & -2/\sqrt{6} \\ 1/2 & 1/\sqrt{12} & 1/\sqrt{6} \\ 1/2 & 1/\sqrt{12} & 1/\sqrt{6} \end{bmatrix}$$

Since $A = QR \Rightarrow Q^T.A = (Q^T.Q)R$

$Q^T.A = R$

then

$R = Q^T.A$

$$Q^T = \begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ -3/\sqrt{12} & 1/\sqrt{12} & 1/\sqrt{12} & 1/\sqrt{12} \\ 0 & -2/\sqrt{6} & 1/\sqrt{6} & 1/\sqrt{6} \end{bmatrix}$$

$$R = Q^T.A = \begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ -3/\sqrt{12} & 1/\sqrt{12} & 1/\sqrt{12} & 1/\sqrt{12} \\ 0 & -2/\sqrt{6} & 1/\sqrt{6} & 1/\sqrt{6} \end{bmatrix} . \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1/2 & 1/2 \\ -3/\sqrt{12} & \\ 0 & \end{bmatrix} \begin{bmatrix} 2 & 3/2 & 1 \\ 0 & 3/\sqrt{12} & 2/\sqrt{12} \\ 0 & 0 & 2/\sqrt{6} \end{bmatrix}$$

$\boxed{\therefore A = QR}$ Ans

9. Find the QR factorization.

(i) $A = \begin{bmatrix} 5 & 9 \\ 1 & 7 \\ -3 & -5 \\ 1 & 5 \end{bmatrix}$

Sol: Here

let $V_1 = x_1 = (5, 1, -3, 1)$

$x_2 = (9, 7, -5, 5)$

$$V_2 = x_2 - \left(\frac{x_2 . V_1}{V_1 . V_1}\right) . V_1$$

$$= (9,7,-5,5) - \left(\frac{(9,7,-5,5).(5,1,-3,1)}{(5,1,-3,1)\ (5,1,-3,1)}\right).(5,1,-3,1)$$

$$= (9,7,-5,5) - \left[\frac{(45+7+15+5)}{(25+1+9+1)}\right].(5,1,-3,1)$$

$$= (9,7,-5,5) - \left[\left(\frac{72}{36}\right).(5,1,-3,1)\right].$$

$$= (9,7,-5,5) - (2(5,1,-3,1))$$

$$= (9,7,-5,5) - (10,2,-6,2)$$

$V_2 = (-1, 5, 1, 3)$

let (u_1, u_2) is the normalize of the orthogonal basis

$$u_1 = \frac{V_1}{||V_1||} = \frac{(5,1,-3,1)}{\sqrt{25+1+9+1}} = \frac{(5,1,-3,1)}{\sqrt{36}} = \frac{1}{6}(5,1,-3,1)$$

$u_1 = (5/6, 1/6, -3/6, 1/6)$

$$u_2 = \frac{V_2}{||V_2||} = \frac{(-1,5,1,3)}{\sqrt{1+25+1+9}} = \frac{(-1,5,1,3)}{\sqrt{36}} = \frac{1}{6}(-1,5,1,3)$$

$$u_2 = \frac{1}{6}(-1,5,1,3) = \left(\frac{-1}{6}, \frac{5}{6}, \frac{1}{6}, \frac{3}{6}\right)$$

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let Q be the matrix whose column are u_1 and u_2

$$Q = \begin{bmatrix} 5/6 & -1/6 \\ 1/6 & 5/6 \\ -3/6 & 1/6 \\ 1/6 & 3/6 \end{bmatrix}$$

since $A = QR \Rightarrow Q^T A = (Q^T Q) \cdot R$

$Q^T A = IR$

$\therefore R = Q^T . A$

then

$$Q^T = \begin{bmatrix} 5/6 & 1/6 & -3/6 & 1/6 \\ -1/6 & 5/6 & 1/6 & 3/6 \end{bmatrix}.$$

$$R = Q^T . A = \begin{bmatrix} 5/6 & 1/6 & -3/6 & 1/6 \\ -1/6 & 5/6 & 1/6 & 3/6 \end{bmatrix} \cdot \begin{bmatrix} 5 & 9 \\ 1 & 7 \\ -3 & -5 \\ 1 & 5 \end{bmatrix}$$

$$R = \begin{bmatrix} 6 & 12 \\ 0 & 6 \end{bmatrix}$$

$\boxed{\therefore A = QR}$ Ans

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(ii) $A = \begin{bmatrix} -2 & 3 \\ 5 & 7 \\ 2 & -2 \\ 4 & 6 \end{bmatrix}$

Sol: Here,

let $V_1 = x_1 = (-2, 5, 2, 4)$

$x_2 = (3, 7, -2, 6)$

$$V_2 = x_2 - \left(\frac{x_2 . V_1}{V_1 . V_1} \right) \cdot V_1$$

$$= (3,7,-2,6) - \left[\frac{(3,7,-2,6) . (-2,5,2,4)}{(-2,5,2,4) . (-2,5,2,4)} \right] . (-2,5,2,4)$$

$$= (3,7,-2,6) - \left[\frac{-6+35-4+24}{(4+25+4+16)} ; (-2,5,2,4) \right]$$

$$= (3,7,-2,6) - \left[\left(\frac{49}{49} \right) \cdot (-2,5,2,4) \right]$$

$$= (3,7,-2,6) - (-2,5,2,4)$$

$V_2 = (5, 2, -4, 2)$

let (u_1, u_2) is the normalize of the orthogonal basis.

$$u_1 = \frac{V_1}{\|V_1\|} = \frac{(-2,5,2,4)}{\sqrt{4+25+4+16}} = \frac{(-2,5,2,4)}{\sqrt{49}} = \frac{1}{7}(-2,5,2,4)$$

$$u_1 = \left(\frac{-2}{7}, \frac{5}{7}, \frac{2}{7}, \frac{4}{7} \right)$$

$$u_2 = \frac{V_2}{\|V_2\|} = \frac{(5,2,-4,2)}{\sqrt{25+4+16+4}} = \frac{1}{\sqrt{49}}(5,2,-4,2) = \frac{1}{7}(5,2,-4,2)$$

$$u_2 = \left(\frac{5}{7}, \frac{2}{7}, \frac{-4}{7}, \frac{2}{7} \right)$$

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let Q be the matrix of whose column are u_1 and u_2

then

$$Q = \begin{bmatrix} -2/7 & 5/7 \\ 5/7 & 2/7 \\ 2/7 & -4/7 \\ 4/7 & 2/7 \end{bmatrix}$$

Since $A = QR \Rightarrow Q^T.A = (Q^T.Q).R$

$Q^T.A = IR$

$R = Q^T.A$

Now

$$Q^T = \begin{bmatrix} -2/7 & 5/7 & 2/7 & 4/7 \\ 5/7 & 2/7 & -4/7 & 2/7 \end{bmatrix}$$

$$R = \begin{bmatrix} -2/7 & 5/7 & 2/7 & 4/7 \\ 5/7 & 2/7 & -4/7 & 2/7 \end{bmatrix} . \begin{bmatrix} -2 & 3 \\ 5 & 7 \\ 2 & -2 \\ 4 & 6 \end{bmatrix}$$

$$R = \begin{bmatrix} 7 & 7 \\ 0 & 7 \end{bmatrix}$$

$\therefore$ $\boxed{A = QR}$ Ans

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Least Square problem :-

If A is $m \times n$ matrix and b is in R^n then a least square solution of $Ax = b$ is $\hat{x}$ in R^n such that $\|b - A\hat{x}\| < \|b - Ax\|$. for all x in R^n.

Theorem :-

The set of least square solution of $Ax = b$, coincides with the non-empty set of solution of the normal equation $A^TA\hat{x} = A^Tb$

Theorem :- The matrix A^TA is invertible if and only if the column of A are linearly independent. In this case, the equation $Ax = b$ has only one least square solution $\hat{x}$ and it is given by $\hat{x} = (A^TA)^{-1}. A^T.b$

NOTE :- The distance from b to $A\hat{x}$ is called least square error.

Ex:- Find a least square solution of the inconsistent system $Ax = b$ for.

$$A = \begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 2 \\ 0 \\ 11 \end{bmatrix}$$

Soln :- Here,

$$A^T = \begin{bmatrix} 4 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix}$$

$$A^T.A = \begin{bmatrix} 4 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 16+0+1 & 0+0+1 \\ 0+0+1 & 0+4+1 \end{bmatrix} = \begin{bmatrix} 17 & 1 \\ 1 & 5 \end{bmatrix}$$

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$$A^T b = \begin{bmatrix} 4 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 11 \end{bmatrix}$$

$$= \begin{bmatrix} 19 \\ 11 \end{bmatrix}$$

The normal equation is,

$$A^T A \hat{x} = A^T b$$

$$\therefore \hat{x} = (A^T A)^{-1} . (A^T b)$$

For $(A^T A^{-1})$

$$A^T A = \begin{bmatrix} 17 & 1 \\ 1 & 5 \end{bmatrix}$$

$\det.(A^T A) = 85 - 1 = 84$

$$\text{Adj.}(A^T A) = \begin{bmatrix} 5 & -1 \\ -1 & 17 \end{bmatrix}$$

$$(A^T A)^{-1} = \frac{1}{\det(A^T A)} \text{Adj}(A^T . A)$$

$$= \frac{1}{84} . \begin{bmatrix} 5 & -1 \\ -1 & 17 \end{bmatrix}$$

$$\therefore \hat{x} = (A^T . A)^{-1} . (A^T b)$$

$$= \frac{1}{84} \begin{bmatrix} 5 & -1 \\ -1 & 17 \end{bmatrix} . \begin{bmatrix} 19 \\ 11 \end{bmatrix}$$

$$= \frac{1}{84} \begin{bmatrix} 95 - 11 \\ -19 + 187 \end{bmatrix} = \frac{1}{84} \begin{bmatrix} 84 \\ 168 \end{bmatrix}$$

$$\hat{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Ans

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Also

$$A\hat{x} = \begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix} . \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 4+0 \\ 0+4 \\ 1+2 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 3 \end{bmatrix}$$

and $b - A\hat{x} = \begin{bmatrix} 2 \\ 0 \\ 11 \end{bmatrix} - \begin{bmatrix} 4 \\ 4 \\ 3 \end{bmatrix}$

$$= \begin{bmatrix} -2 \\ -4 \\ 8 \end{bmatrix}$$

$\therefore \|b - A\hat{x}\| = \sqrt{4+16+64} = \sqrt{84}$ is the least square error.

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Exercise :- 8.5

1. Find least squares solution $A\hat{x}=b$

ⓐ Constructing the normal equation for $\hat{x}$

and ⓑ solving for $\hat{x}$

So ① $A = \begin{bmatrix} -1 & 2 \\ 2 & -3 \\ -1 & 3 \end{bmatrix}$, $b = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$

Sol: Here,

$$A = \begin{bmatrix} -1 & 2 \\ 2 & -3 \\ -1 & 3 \end{bmatrix}.$$

$$A^T = \begin{bmatrix} -1 & 2 & -1 \\ 2 & -3 & 3 \end{bmatrix}$$

$$A^T.A = \begin{bmatrix} -1 & 2 & -1 \\ 2 & -3 & 3 \end{bmatrix} . \begin{bmatrix} -1 & 2 \\ 2 & -3 \\ -1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1+4+1 & -2-6-3 \\ -2-6-3 & 4+9+9 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & -11 \\ -11 & 22 \end{bmatrix}$$

$$A^Tb = \begin{bmatrix} -1 & 2 & -1 \\ 2 & -3 & 3 \end{bmatrix} . \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} -4+2-2 \\ 8-3+6 \end{bmatrix} = \begin{bmatrix} -4 \\ 11 \end{bmatrix}$$

The normal equation is

$$A^TA . \hat{x} = A^Tb$$

$$\therefore \hat{x} = (A^TA)^{-1}(A^Tb)$$

$$\therefore A^TA . \hat{x} = A^Tb$$

$$\begin{bmatrix} 6 & -11 \\ -11 & 22 \end{bmatrix} . \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -4 \\ 11 \end{bmatrix}$$

Solving for $\hat{x}$

$$\hat{x} = (A^TA)^{-1} . (A^Tb)$$

For $(A^TA)^{-1}$

$$\det(A^TA) = (132-121) = 11$$

$$Adj.(A^TA) = \begin{bmatrix} 22 & 11 \\ 11 & 6 \end{bmatrix}$$

$$(A^TA)^{-1} = \frac{Adj.(A^TA)}{\det(A^T.A)} = \frac{1}{11}\begin{bmatrix} 22 & 11 \\ 11 & 6 \end{bmatrix}$$

$$\therefore \hat{x} = \frac{1}{11}\begin{bmatrix} 22 & 11 \\ 11 & 6 \end{bmatrix} . \begin{bmatrix} -4 \\ 11 \end{bmatrix}$$

$$\hat{x} = \frac{1}{11}\begin{bmatrix} -88+121 \\ -44+66 \end{bmatrix}$$

$$\hat{x} = \frac{1}{11}\begin{bmatrix} 33 \\ 22 \end{bmatrix}$$

$$\hat{x} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$ Ans

(ii) $A = \begin{bmatrix} 2 & 1 \\ -2 & 0 \\ 2 & 3 \end{bmatrix}$, $b = \begin{bmatrix} -5 \\ 8 \\ 1 \end{bmatrix}$

Sol: Here,

$A = \begin{bmatrix} 2 & 1 \\ -2 & 0 \\ 2 & 3 \end{bmatrix}$,

$A^T = \begin{bmatrix} 2 & -2 & 2 \\ 1 & 0 & 3 \end{bmatrix}$

$A^T A = \begin{bmatrix} 2 & -2 & 2 \\ 1 & 0 & 3 \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 \\ -2 & 0 \\ 2 & 3 \end{bmatrix}$

$= \begin{bmatrix} 4+4+4 & 2+0+6 \\ 2+0+6 & 1+0+9 \end{bmatrix}$

$= \begin{bmatrix} 12 & 8 \\ 8 & 10 \end{bmatrix}$

$A^T b = \begin{bmatrix} 2 & -2 & 2 \\ 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} -5 \\ 8 \\ 1 \end{bmatrix}$

$= \begin{bmatrix} -10-16+2 \\ -5+0+3 \end{bmatrix}$

$\therefore \begin{bmatrix} -24 \\ -2 \end{bmatrix}$

The normal equation is

$A^T A \cdot \hat{x} = A^T b$.

$\begin{bmatrix} 12 & 8 \\ 8 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -24 \\ -2 \end{bmatrix}$

Solving for $\hat{x}$

$\hat{x} = (A^T A)^{-1} \cdot (A^T b)$

For $(A^T A)^{-1}$

$\det(A^T A) = (120-64) = 56$

$\text{Adj.}(A^T A) = \begin{bmatrix} 10 & -8 \\ -8 & 12 \end{bmatrix}$

$(A^T A)^{-1} = \frac{1}{\det(A^T A)} \cdot \text{Adj}(A^T A)$

$= \frac{1}{56} \begin{bmatrix} 10 & -8 \\ -8 & 12 \end{bmatrix}$

Then,

$\hat{x} = (A^T A)^{-1} \cdot (A^T b)$

$= \frac{1}{56} \begin{bmatrix} 10 & -8 \\ -8 & 12 \end{bmatrix} \cdot \begin{bmatrix} -24 \\ -2 \end{bmatrix}$

$= \frac{1}{56} \begin{bmatrix} -240+16 \\ 192-24 \end{bmatrix}$

$= \frac{1}{56} \begin{bmatrix} -224 \\ 168 \end{bmatrix}$

$\hat{x} = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$

Ans

(M1) $A = \begin{bmatrix} 1 & -2 \\ -1 & 2 \\ 0 & 3 \\ 2 & 5 \end{bmatrix}$, $b = \begin{bmatrix} 3 \\ 1 \\ -4 \\ 2 \end{bmatrix}$

Sol: Here,

$A = \begin{bmatrix} 1 & -2 \\ -1 & 2 \\ 0 & 3 \\ 2 & 5 \end{bmatrix}$

$A^T = \begin{bmatrix} 1 & -1 & 0 & 2 \\ -2 & 2 & 3 & 5 \end{bmatrix}$

$A^T.A = \begin{bmatrix} 1 & -1 & 0 & 2 \\ -2 & 2 & 3 & 5 \end{bmatrix} \cdot \begin{bmatrix} 1 & -2 \\ -1 & 2 \\ 0 & 3 \\ 2 & 5 \end{bmatrix}$

$= \begin{bmatrix} 1+1+0+4 & -2-2+0+10 \\ -2-2+0+10 & 4+4+9+25 \end{bmatrix}$

$= \begin{bmatrix} 6 & 6 \\ 6 & 42 \end{bmatrix}$

$A^Tb = \begin{bmatrix} 1 & -1 & 0 & 2 \\ -2 & 2 & 3 & 5 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 1 \\ -4 \\ 2 \end{bmatrix}$

$= \begin{bmatrix} 3-1+0+4 \\ -6+2-12+10 \end{bmatrix}$

$= \begin{bmatrix} 6 \\ -6 \end{bmatrix}$

The normal equation is

$A^TA.\hat{x} = A^Tb$

$\begin{bmatrix} 6 & 6 \\ 6 & 42 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6 \\ -6 \end{bmatrix}$

Solving for $\hat{x}$

$\hat{x} = (A^TA)^{-1}.(A^Tb)$

For $(A^TA)^{-1}$

det. $(A^TA) = \begin{bmatrix} 6 & 6 \\ 6 & 42 \end{bmatrix} = 252 - 36 = 216$

Adj. $(A^TA) = \begin{bmatrix} 42 & -6 \\ -6 & 6 \end{bmatrix}$

$(A^TA)^{-1} = \dfrac{1}{\det(A^TA)} \cdot \text{Adj.}(A^TA)$

$= \dfrac{1}{216} \cdot \begin{bmatrix} 42 & -6 \\ -6 & 6 \end{bmatrix}$

Now $\hat{x} = (A^TA)^{-1}.(A^Tb)$

$= \dfrac{1}{216} \begin{bmatrix} 42 & -6 \\ -6 & 6 \end{bmatrix} \cdot \begin{bmatrix} 6 \\ -6 \end{bmatrix}$

$= \dfrac{1}{216} \begin{bmatrix} 252+36 \\ -36-36 \end{bmatrix}$

$= \dfrac{1}{216} \begin{bmatrix} 288 \\ -72 \end{bmatrix}$

$\hat{x} = \begin{bmatrix} 4/3 \\ -1/3 \end{bmatrix}$ Ans

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(iv) $A = \begin{bmatrix} 1 & 3 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}$, $b = \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}$

Sol: Here,

$A = \begin{bmatrix} 1 & 3 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}$

$A^T = \begin{bmatrix} 1 & 1 & 1 \\ 3 & -1 & 1 \end{bmatrix}$

$A^TA = \begin{bmatrix} 1 & 1 & 1 \\ 3 & -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 3 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1+1+1 & 3-1+1 \\ 3-1+1 & 9+1+1 \end{bmatrix}$

$= \begin{bmatrix} 3 & 3 \\ 3 & 11 \end{bmatrix}$

$A^Tb = \begin{bmatrix} 1 & 1 & 1 \\ 3 & -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}$

$= \begin{bmatrix} 5+1+0 \\ 15-1+0 \end{bmatrix}$

$= \begin{bmatrix} 6 \\ 14 \end{bmatrix}$

The normal equation is

$(A^T.A)\hat{x} = A^Tb$

$\begin{bmatrix} 3 & 3 \\ 3 & 11 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6 \\ 14 \end{bmatrix}$

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Solving for $\hat{x}$

$\hat{x} = (A^TA)^{-1} . (A^Tb)$

For $(A^TA)^{-1}$

$\det(A^TA) = \begin{vmatrix} 3 & 3 \\ 3 & 11 \end{vmatrix} = 33 - 9 = 24$

$Adj. (A^TA) = \begin{bmatrix} 11 & -3 \\ -3 & 3 \end{bmatrix}$

$(A^TA)^{-1} = \dfrac{1}{\det(A^TA)} . Adj(A^TA)$

$= \dfrac{1}{24} \begin{bmatrix} 11 & -3 \\ -3 & 3 \end{bmatrix}$

Now

$(A^TA) . \hat{x} = (A^Tb)$

$\hat{x} = (A^TA)^{-1} . (A^Tb)$

$= \dfrac{1}{24} \begin{bmatrix} 11 & -3 \\ -3 & 3 \end{bmatrix} . \begin{bmatrix} 6 \\ 14 \end{bmatrix}$

$= \dfrac{1}{24} \begin{bmatrix} 66-42 \\ -18+42 \end{bmatrix}$

$= \dfrac{1}{24} \begin{bmatrix} 24 \\ 24 \end{bmatrix}$

$\hat{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Ans

2. Describe all least-squares solutions of the equation $Ax = b$.

S① $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$, $b = \begin{bmatrix} 1 \\ 3 \\ 8 \\ 2 \end{bmatrix}$

Sol^n: Here,

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+1+1+1 & 1+1+0+0 & 0+0+1+1 \\ 1+1+0+0 & 1+1+0+0 & 0+0+0+0 \\ 0+0+1+1 & 0+0+0+0 & 0+0+1+1 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 2 & 2 \\ 2 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix}$$

$$A^Tb = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 3 \\ 8 \\ 2 \end{bmatrix}$$

$$A^Tb = \begin{bmatrix} 1+3+8+2 \\ 1+3+0+0 \\ 0+0+8+2 \end{bmatrix} = \begin{bmatrix} 14 \\ 4 \\ 10 \end{bmatrix}$$

The normal equation is,

$$A^TA \cdot \hat{x} = A^Tb$$

$$\begin{bmatrix} 4 & 2 & 2 \\ 2 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 14 \\ 4 \\ 10 \end{bmatrix} \Rightarrow \text{For solving } \hat{x}$$

The augmented matrix is,

$$\begin{bmatrix} 4 & 2 & 2 & : & 14 \\ 2 & 2 & 0 & : & 4 \\ 2 & 0 & 2 & : & 10 \end{bmatrix}$$

$R_1 \to \frac{1}{2}R_1$ and $R_2 \to \frac{1}{2}R_2$ and $R_3 \to \frac{1}{2}R_3$

$$\begin{bmatrix} 2 & 1 & 1 & : & 7 \\ 1 & 1 & 0 & : & 2 \\ 1 & 0 & 1 & : & 5 \end{bmatrix}$$

$R_1 \leftrightarrow R_2$

$$\begin{bmatrix} 1 & 1 & 0 & : & 2 \\ 2 & 1 & 1 & : & 7 \\ 1 & 0 & 1 & : & 5 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 - R_1$

$$\begin{bmatrix} 1 & 1 & 0 & : & 2 \\ 0 & -1 & 1 & : & 3 \\ 0 & -1 & 1 & : & 3 \end{bmatrix}$$

$R_1 \to R_1 + R_2$ and $R_3 \to R_3 - R_2$

$$\begin{bmatrix} 1 & 0 & 1 & : & 5 \\ 0 & -1 & 1 & : & 3 \\ 0 & 0 & 0 & : & 0 \end{bmatrix}$$

$R_2 \to R_2 - R_1$

$$\begin{bmatrix} 1 & 0 & 1 & : & 5 \\ 0 & -1 & 0 & : & -2 \\ 0 & 0 & 0 & : & 0 \end{bmatrix} \quad R_2 \to (-1) \cdot R_2 \quad \begin{bmatrix} 1 & 0 & 1 & : & 5 \\ 0 & 1 & 0 & : & 2 \\ 0 & 0 & 0 & : & 0 \end{bmatrix}$$

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$$\left[\begin{array}{ccc:c} 1 & 0 & 1 & 5 \\ 0 & -1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Here x_1 and x_2 are basic and x_3 is free variable, then the associated system is,

$x_1 + x_3 = 5$

$-x_2 + x_3 = 3 \Rightarrow -3 + x_3 = x_2$

$x_3 =$ free and also $x_3 = 0$

then,

$x_1 + 0 = 5$ | $-x_2 + 0 = 3$ | $x_3 =$ free,

$x_1 = 5$ | $x_2 = -3$ |

For free variable (x_3)

$x_1 =$.

$$\therefore \hat{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 - x_3 \\ -3 + x_3 \\ 0 + x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

Ans

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 7 \\ 2 \\ 3 \\ 6 \\ 5 \\ 4 \end{bmatrix}$$

Sol: Here,

$$A^T = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

$$A^T \cdot A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

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$$A^T A = \begin{bmatrix} 1+1+1+1+1+1 & 1+1+1+0+0+0 & 0+0+0+1+1+1 \\ 1+1+1+0+0+0 & 1+1+1+0+0+0 & 0+0+0+0+0+0 \\ 0+0+0+1+1+1 & 0+0+0+0+0+0 & 0+0+0+1+1+1 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 6 & 3 & 3 \\ 3 & 3 & 0 \\ 3 & 0 & 3 \end{bmatrix}$$

$$A^T b = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 7 \\ 2 \\ 3 \\ 6 \\ 5 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} 7+2+3+6+5+4 \\ 7+2+3+0+0+0 \\ 0+0+0+6+5+4 \end{bmatrix}$$

$$= \begin{bmatrix} 27 \\ 12 \\ 15 \end{bmatrix}$$

Now,

The ~~other~~ normal equation is,

$$(A^T A) \cdot \hat{x} = (A^T b)$$

$$\begin{bmatrix} 6 & 3 & 3 \\ 3 & 3 & 0 \\ 3 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 27 \\ 12 \\ 15 \end{bmatrix}$$

Solving for $\hat{x}$

The augmented matrix is

$$\left[\begin{array}{ccc:c} 6 & 3 & 3 & 27 \\ 3 & 3 & 0 & 12 \\ 3 & 0 & 3 & 15 \end{array}\right]$$

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$R_1 \to \frac{1}{3}R_1,\ R_2 \to \frac{1}{3}R_2$ and $R_3 \to \frac{1}{3}R_3$

$$\left[\begin{array}{ccc:c} 2 & 1 & 1 & 9 \\ 1 & 1 & 0 & 4 \\ 1 & 0 & 1 & 5 \end{array}\right]$$

$R_1 \leftrightarrow R_2$

$$\left[\begin{array}{ccc:c} 1 & 1 & 0 & 4 \\ 2 & 1 & 1 & 9 \\ 1 & 0 & 1 & 5 \end{array}\right]$$

$R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 - R_1$

$$\left[\begin{array}{ccc:c} 1 & 1 & 0 & 4 \\ 0 & -1 & 1 & 1 \\ 0 & -1 & 1 & 1 \end{array}\right]$$

$R_3 \to R_3 - R_2$

$$\left[\begin{array}{ccc:c} 1 & 1 & 0 & 4 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$R_2 \to -R_2$

$$\left[\begin{array}{ccc:c} 1 & 1 & 0 & 4 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Here x_1 and x_2 are basic variable and x_3 is free variable for the associated system is,

$x_1 + x_2 = 4 \Rightarrow x_1 + x_3 - 1 = 4 \Rightarrow x_1 + x_3 = 5$

$x_2 - x_3 = -1 \Rightarrow x_2 = x_3 - 1$

$x_3 =$ free

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 - x_3 \\ -1 + x_3 \\ 0 + x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$ Ans

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(5) compute the least-squares error associated with the least square solution

$$A = \begin{bmatrix} 1 & -2 \\ -1 & 2 \\ 0 & 3 \\ 2 & 5 \end{bmatrix},\ b = \begin{bmatrix} 3 \\ 1 \\ -4 \\ 2 \end{bmatrix}$$

Sol^n: Here,

$$A = \begin{bmatrix} 1 & -2 \\ -1 & 2 \\ 0 & 3 \\ 2 & 5 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & -1 & 0 & 2 \\ -2 & 2 & 3 & 5 \end{bmatrix}$$

$$A^T \cdot A = \begin{bmatrix} 1 & -1 & 0 & 2 \\ -2 & 2 & 3 & 5 \end{bmatrix} \cdot \begin{bmatrix} 1 & -2 \\ -1 & 2 \\ 0 & 3 \\ 2 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 1+1+0+4 & -2-2+0+10 \\ -2-2+0+10 & 4+4+9+25 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 6 \\ 6 & 42 \end{bmatrix}$$

$$A^T b = \begin{bmatrix} 1 & -1 & 0 & 2 \\ -2 & 2 & 3 & 5 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 1 \\ -4 \\ 2 \end{bmatrix} = \begin{bmatrix} 3-1+0+4 \\ -6+2-12+10 \end{bmatrix}$$

$$\begin{bmatrix} 6 \\ -6 \end{bmatrix}$$

The normal equation is

$$A^TA\hat{x} = A^Tb$$

$$\hat{x} = (A^TA)^{-1}A^Tb$$

For $(A^TA)^{-1}$

$\det(A^TA) = 252-36 = 216$

$$\text{Adj}(A^TA) = \begin{bmatrix} 42 & -6 \\ -6 & 6 \end{bmatrix}$$

$$(A^TA)^{-1} = \frac{1}{\det(A^TA)} \cdot \text{Adj}(A^TA)$$

$$= \frac{1}{216}\begin{bmatrix} 42 & -6 \\ -6 & 6 \end{bmatrix}$$

Ier

$$\hat{x} = (A^TA)^{-1} \cdot A^Tb$$

$$= \frac{1}{216}\begin{bmatrix} 42 & -6 \\ -6 & 6 \end{bmatrix} \cdot \begin{bmatrix} 6 \\ -6 \end{bmatrix}$$

$$= \frac{1}{216}\begin{bmatrix} 252+36 \\ -36-36 \end{bmatrix}$$

$$= \frac{1}{216}\begin{bmatrix} 288 \\ -72 \end{bmatrix}$$

$$\hat{x} = \begin{bmatrix} 4/3 \\ -1/3 \end{bmatrix}$$

Also

$$A\hat{x} = \begin{bmatrix} 1 & -2 \\ -1 & 2 \\ 0 & 3 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} 4/3 \\ -1/3 \end{bmatrix}$$

$$= \begin{bmatrix} 4/3 + 2/3 \\ -4/3 - 2/3 \\ 0 - 3/3 \\ 8/3 - 5/3 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ -2 \\ -1 \\ 1 \end{bmatrix}$$

$$b - A\hat{x} = \begin{bmatrix} 3 \\ 1 \\ -4 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 \\ -2 \\ -1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 3 \\ -3 \\ 1 \end{bmatrix}$$

$\|b - A\hat{x}\| = \sqrt{1+9+9+1} = \sqrt{20} = 2\sqrt{5}$ is the least square error associated with the least square solution.

2. Compute the least-square error associated with the least-square solution.

$$A = \begin{bmatrix} 1 & 3 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}$$

Sol: Here,

$$A^T = \begin{bmatrix} 1 & 1 & 1 \\ 3 & -1 & 1 \end{bmatrix}$$

$$A^T \cdot A = \begin{bmatrix} 1 & 1 & 1 \\ 3 & -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 3 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 1+1+1 & 3-1+1 \\ 3-1+1 & 9+1+1 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 3 & 3 \\ 3 & 11 \end{bmatrix}$$

$$A^T \cdot b = \begin{bmatrix} 1 & 1 & 1 \\ 3 & -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 5+1+0 \\ 15-1+0 \end{bmatrix} = \begin{bmatrix} 6 \\ 14 \end{bmatrix}$$

The normal equation is

$$A^T A \cdot \hat{x} = A^T b$$

$$\hat{x} = (A^T A)^{-1} (A^T b)$$

For $(A^T A)^{-1}$

$$\det(A^T A) = (33-9) = 24$$

$$\operatorname{Adj}(A^T A) = \begin{bmatrix} 11 & -3 \\ -3 & 3 \end{bmatrix}$$

$$(A^T A)^{-1} = \frac{1}{\det(A^T A)} \cdot \operatorname{Adj}(A^T A)$$

$$(A^T A)^{-1} = \frac{1}{24} \begin{bmatrix} 11 & -3 \\ -3 & 3 \end{bmatrix}$$

then

$$\hat{x} = (A^T A)^{-1} \cdot (A^T b)$$

$$= \frac{1}{24} \begin{bmatrix} 11 & -3 \\ -3 & 3 \end{bmatrix} \cdot \begin{bmatrix} 6 \\ 14 \end{bmatrix}$$

$$= \frac{1}{24} \begin{bmatrix} 66-42 \\ -18+42 \end{bmatrix} = \frac{1}{24} \begin{bmatrix} 24 \\ 24 \end{bmatrix}$$

$$\therefore \hat{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Also,

$$A\hat{x} = \begin{bmatrix} 1 & 3 \\ 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1+3 \\ 1-1 \\ 1+1 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}$$

$$b - A\hat{x} = \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$$

$\|b - A\hat{x}\| = \sqrt{1+1+4} = \sqrt{6}$ is the ~~the~~ required least-square error associated with the least-squares solution

Ans

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5. Find (a) the Orthogonal projection of b onto col(A) and (b) a least-squares Solution of $Aa = b$.

(1) $A = \begin{bmatrix} 1 & 5 \\ 3 & 1 \\ -2 & 4 \end{bmatrix}$, $b = \begin{bmatrix} 4 \\ -2 \\ -3 \end{bmatrix}$

Sol: Here,

$A = \begin{bmatrix} 1 & 5 \\ 3 & 1 \\ -2 & 4 \end{bmatrix}$

let $u_1 = (1,3,-2)$ and $u_2 = (5,1,4)$

$y = (4,-2,3)$

then,

the orthogonal projection of b on to col(A) is,

$$\hat{y} = \left(\frac{y \cdot u_1}{u_1 \cdot u_1}\right) \cdot u_1 + \left(\frac{y \cdot u_2}{u_2 \cdot u_2}\right) \cdot u_2$$

$$= \left[\frac{(4,-2,3) \cdot (1,3,-2)}{(1,3,-2)(1,3,-2)}\right] \cdot (1,3,-2) + \left[\frac{(4,-2,-3)(5,1,4)}{(5,1,4) \cdot (5,1,4)}\right] \cdot u_2$$

$$= \left[\frac{(4-6+6)}{(1+9+4)}\right] \cdot u_1 + \left[\frac{(20-2-12)}{(25+1+16)}\right] u_2$$

$$= \left[\left(\frac{4}{14}\right) u_1 + \left(\frac{6}{42}\right) \cdot u_2\right]$$

Now, For L.S.S

$A^T = \begin{bmatrix} 1 & 3 & -2 \\ 5 & 1 & 4 \end{bmatrix}$

$A^T \cdot b = \begin{bmatrix} 1 & 3 & -2 \\ 5 & 1 & 4 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ -2 \\ -3 \end{bmatrix}$

$= \begin{bmatrix} 4-6+6 \\ 20-2-12 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$

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$A^T \cdot A = \begin{bmatrix} 1 & 3 & -2 \\ 5 & 1 & 4 \end{bmatrix} \cdot \begin{bmatrix} 1 & 5 \\ 3 & 1 \\ -2 & 4 \end{bmatrix}$

$= \begin{bmatrix} 1+9+4 & 5+3-8 \\ 5+3-8 & 25+1+16 \end{bmatrix}$

$= \begin{bmatrix} 14 & 0 \\ 0 & 42 \end{bmatrix}$

The normal equation is

$$\hat{x} = (A^T A)^{-1} \cdot A^T \cdot b$$

for $(A^T A)^{-1}$

$\det(A^T \cdot A) = 588$

$\mathrm{Adj} \cdot (A^T A) = \begin{bmatrix} 42 & 0 \\ 0 & 14 \end{bmatrix}$

$(A^T A)^{-1} = \frac{1}{\det(A^T A)} \mathrm{Adj}.(A^T A)$

$= \frac{1}{588} \begin{bmatrix} 42 & 0 \\ 0 & 14 \end{bmatrix}$

$\hat{x} = (A^T A)^{-1} \cdot (A^T b)$

$= \frac{1}{588} \begin{bmatrix} 42 & 0 \\ 0 & 14 \end{bmatrix} \begin{bmatrix} 4 \\ 6 \end{bmatrix}$

$= \frac{1}{588} \begin{bmatrix} 168+0 \\ 0+84 \end{bmatrix}$

$= \frac{1}{588} \begin{bmatrix} 168 \\ 84 \end{bmatrix}$

$\hat{x} = \begin{bmatrix} 2/7 \\ 1/7 \end{bmatrix}$ Ans

(11) $A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \\ 1 & 2 \end{bmatrix}$, $b = \begin{bmatrix} 3 \\ -1 \\ 5 \end{bmatrix}$

sol^n: Here, let $u_1 = (1,-1,1)$ and $u_2 = (2,4,2)$ and $y = (3,-1,5)$

The orthogonal projection of b onto col(A).

$$\hat{y} = \left[\left(\frac{y \cdot u_1}{u_1 \cdot u_1}\right) \cdot u_1\right] + \left[\left(\frac{y \cdot u_2}{u_2 \cdot u_2}\right) \cdot u_2\right]$$

$$\hat{y} = \left[\frac{(3,-1,5) \cdot (1,-1,1)}{(1,-1,1) \cdot (1,-1,1)}\right] \cdot u_1 + \left[\frac{(3,-1,5) \cdot (2,4,2)}{(2,4,2)(2,4,2)}\right] u_2$$

$$= \left[\left(\frac{3+1+5}{1+1+1}\right)\right] u_1 + \left[\left(\frac{6-4+10}{4+16+4}\right) \cdot u_2\right]$$

$$= \left[\left(\frac{9}{3}\right) u_1\right] + \left[\left(\frac{12}{24}\right) u_2\right]$$

$$= 3u_1 + \frac{1}{2} u_2$$

let $A^T = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 4 & 2 \end{bmatrix}$

$$A^T \cdot A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 4 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ -1 & 4 \\ 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1+1+1 & 2-4+2 \\ 2-4+2 & 4+16+4 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 0 \\ 0 & 24 \end{bmatrix}$$

$$A^T \cdot b = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 4 & 2 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -1 \\ 5 \end{bmatrix}$$

$$= \begin{bmatrix} 3+1+5 \\ 6-4+10 \end{bmatrix}$$

$$= \begin{bmatrix} 9 \\ 12 \end{bmatrix}$$

The normal equation is,

$$A^T \cdot A \cdot \hat{x} = A^T b$$

$$\hat{x} = (A^T A)^{-1} \cdot A^T b.$$

For $(A^T A)^{-1}$.

$\det(A^T A) = 72.$

$\operatorname{Adj}(A^T A) = \begin{bmatrix} 24 & 0 \\ 0 & 3 \end{bmatrix}$

$$(A^T A)^{-1} = \frac{1}{\det(A^T A)} \cdot \operatorname{Adj}(A^T A)$$

$$= \frac{1}{72} \begin{bmatrix} 24 & 0 \\ 0 & 3 \end{bmatrix}$$

then,

$$\hat{x} = (A^T A)^{-1} \cdot A^T b$$

$$= \frac{1}{72} \begin{bmatrix} 24 & 0 \\ 0 & 3 \end{bmatrix} \cdot \begin{bmatrix} 9 \\ 12 \end{bmatrix}$$

$$= \frac{1}{72} \begin{bmatrix} 216+0 \\ 0+36 \end{bmatrix} = \frac{1}{72} \begin{bmatrix} 216 \\ 36 \end{bmatrix}$$

$$\hat{x} = \begin{bmatrix} 3 \\ 1/2 \end{bmatrix}$$ Ans

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(iii) $A = \begin{bmatrix} 4 & 0 & 1 \\ 1 & -5 & 1 \\ 6 & 1 & 0 \\ 1 & -1 & -5 \end{bmatrix}, \quad b = \begin{bmatrix} 9 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

Sol: Here,

let $u_1 = (4,1,6,1)$, $u_2 = (0,-5,1,-1)$, $u_3 = (1,1,0,-5)$

and $y = (9,0,0,0)$

Now,

The orthogonal projection of b on to col A

$$\hat{y} = \left[\left(\frac{y\cdot u_1}{u_1\cdot u_1}\right)\cdot u_1\right] + \left[\left(\frac{y\cdot u_2}{u_2\cdot u_2}\right)\cdot u_2\right] + \left[\left(\frac{y\cdot u_3}{u_3\cdot u_3}\right)\cdot u_3\right]$$

$$\hat{y} = \left[\left(\frac{(9,0,0,0)\cdot(4,1,6,1)}{(4,1,6,1)\cdot(4,1,6,1)}\right)\cdot u_1\right] + \left[\left(\frac{(9,0,0,0)\cdot(0,-5,1,-1)}{(0,-5,1,-1)\cdot(0,-5,1,-1)}\right) u_2\right]$$

$$+ \left[\frac{(9,0,0,0)\cdot(1,1,0,-5)}{(1,1,0,-5)\cdot(1,1,0,-5)}\right]\cdot u_3$$

$$= \left[\left(\frac{36+0+0+0}{16+1+36+1}\right) u_1\right] + \left[\left(\frac{0}{27}\right) u_2\right] + \left[\left(\frac{9}{27}\right) u_3\right]$$

$$= \left(\frac{36}{54}\right)\cdot u_1 + 0\cdot u_2 + \frac{1}{3} u_3$$

$$= \frac{2}{3} u_1 + \frac{1}{3} u_3.$$

$A^T = \begin{bmatrix} 4 & 1 & 6 & 1 \\ 0 & -5 & 1 & -1 \\ 1 & 1 & 0 & -5 \end{bmatrix}$

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$R_2 \to R_2 + 2R_3$

$$\begin{bmatrix} 1 & 1 & 0 & -5 & : 9 \\ 0 & 1 & 0 & 1 & : 0 \\ 0 & 0 & 1 & 4 & : 0 \end{bmatrix} \xrightarrow{R_1 \to R_1 - R_2} \begin{bmatrix} 1 & 0 & 0 & -6 & : 9 \\ 0 & 1 & 0 & 1 & : 0 \\ 0 & 0 & 1 & 4 & : 0 \end{bmatrix}$$

then,

~~$x_1 + x_2 - 5x_4 = 9$~~, $x_2 + x_4 = 0$

~~$x_3 + 4x_4 = 0$~~

$$A^TA = \begin{bmatrix} 4 & 1 & 6 & 1 \\ 0 & -5 & 1 & -1 \\ 1 & 1 & 0 & -5 \end{bmatrix} \cdot \begin{bmatrix} 4 & 0 & 1 \\ 1 & -5 & 1 \\ 6 & 1 & 0 \\ 1 & -1 & -5 \end{bmatrix}$$

$$= \begin{bmatrix} 16+1+36+1 & 0-5+6-1 & 4+1+0-5 \\ 0-5+6-1 & 0+25+1+1 & 0-5+0+5 \\ 4+1+0-5 & 0-5+0+5 & 1+1+0+25 \end{bmatrix}$$

$$= \begin{bmatrix} 54 & 0 & 0 \\ 0 & 27 & 0 \\ 0 & 0 & 27 \end{bmatrix}$$

$$A^Tb = \begin{bmatrix} 4 & 1 & 1 \\ 0 & -5 & -1 \\ 1 & 1 & -5 \end{bmatrix} \cdot \begin{bmatrix} 9 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 36 \\ 0 \\ 9 \end{bmatrix}$$

The normal equation is

$A^TA \cdot \hat{x} = A^T b$

$$\begin{bmatrix} 54 & 0 & 0 \\ 0 & 27 & 0 \\ 0 & 0 & 27 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 36 \\ 0 \\ 9 \end{bmatrix}$$

For Solving $\hat{x}$

The augmented matrix is $\left[\begin{array}{ccc|c} 54 & 0 & 0 & 36 \\ 0 & 27 & 0 & 0 \\ 0 & 0 & 27 & 9 \end{array}\right]$

$R_1 \to \frac{1}{54} R_1$, $R_2 \to \frac{1}{27} R_2$ and $R_3 \to \frac{1}{27} R_3$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2/3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1/3 \end{array}\right]$$

$\therefore$

$x_1 = 2/3$

$x_2 = 0$

$x_3 = 1/3$

$\therefore \hat{x} = \begin{bmatrix} 2/3 \\ 0 \\ 1/3 \end{bmatrix}$ Ans

(iv) $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \\ -1 & 1 & -1 \end{bmatrix}$, $b = \begin{bmatrix} 2 \\ 5 \\ 6 \\ 6 \end{bmatrix}$

Sol: Here

let $u_1 = (1,1,0,-1)$, $u_2 = (1,0,1,1)$ and $u_3 = (0,-1,1,-1)$

and $y = (2,5,6,6)$

The orthogonal projection of b on col(A) is

$$\hat{y} = \left[\left(\frac{y \cdot u_1}{u_1 \cdot u_1}\right) u_1\right] + \left[\left(\frac{y \cdot u_2}{u_2 \cdot u_2}\right) \cdot u_2\right] + \left[\left(\frac{y \cdot u_3}{u_3 \cdot u_3}\right) \cdot u_3\right]$$

$$= \left[\frac{(2,5,6,6)\cdot(1,1,0,-1)}{(1,1,0,-1)(1,1,0,-1)} \cdot u_1\right] + \left[\frac{(2,5,6,6)(1,0,1,1)}{(1,0,1,1)\cdot(1,0,1,1)} \cdot u_2\right] + \left[\frac{(2,5,6,6)(0,-1,1,-1)}{(0,-1,1,-1)\cdot(0,-1,1,-1)} \cdot u_3\right]$$

$$= \left[\frac{(2+5+0-6)}{(1+1+0+1)} \cdot u_1\right] + \left[\left(\frac{2+0+6+6}{1+0+1+1}\right) \cdot u_2\right] + \left[\left(\frac{0-5+6-6}{0+1+1+1}\right) u_3\right]$$

$$= \frac{1}{3} u_1 + \frac{14}{3} u_2 + \frac{-5}{3} u_3$$

$$= \frac{1}{3} u_1 + \frac{14}{3} u_2 - \frac{5}{3} u_3 \text{ Ans}$$

$$A^T = \begin{bmatrix} 1 & 1 & 0 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & -1 & 1 & -1 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 1 & 1 & 0 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & -1 & 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \\ -1 & 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+1+0+1 & 1+0+0-1 & 0-1+0+1 \\ 1+0+0-1 & 1+0+1+1 & 0+0+1-1 \\ 0-1+0+1 & 0+0+1-1 & 0+1+1+1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$A^T.b = \begin{bmatrix} 1 & 1 & 0 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & -1 & 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 5 \\ 6 \\ 6 \end{bmatrix}$$

$$= \begin{bmatrix} 2+5+0-6 \\ 2+0+6+6 \\ 0-5+6-6 \end{bmatrix} = \begin{bmatrix} 1 \\ 14 \\ -5 \end{bmatrix}$$

The normal equation is,

$$A^TA \cdot \hat{x} = A^Tb$$

For solving $\hat{x}$

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 14 \\ -5 \end{bmatrix}$$

For solving $\hat{x}$, the augmented matrix is

$$\left[\begin{array}{ccc:c} 3 & 0 & 0 & 1 \\ 0 & 3 & 0 & 14 \\ 0 & 0 & 3 & -5 \end{array}\right]$$

$R_1 \to \frac{1}{3}R_1$, $\frac{1}{3}R_2$ and $R_3 \to \frac{1}{3}R_3$

$$\left[\begin{array}{ccc:c} 1 & 0 & 0 & 1/3 \\ 0 & 1 & 0 & 14/3 \\ 0 & 0 & 1 & -5/3 \end{array}\right]$$

$$\therefore x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 14/3 \\ -5/3 \end{bmatrix}$$ Ans

6. let $A = \begin{bmatrix} 3 & 4 \\ -2 & 1 \\ 3 & 4 \end{bmatrix}$, $b = \begin{bmatrix} 11 \\ -9 \\ 5 \end{bmatrix}$, $u = \begin{bmatrix} 5 \\ -1 \end{bmatrix}$ and $v = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$. compute Au and Av, and compare them with b. could u possibly be a least-squares solution of $A\hat{x} = b$?

Sol: Here,

$A = \begin{bmatrix} 3 & 4 \\ -2 & 1 \\ 3 & 4 \end{bmatrix}$, $b = \begin{bmatrix} 11 \\ -9 \\ 5 \end{bmatrix}$

$u = \begin{bmatrix} 5 \\ -1 \end{bmatrix}$, $v = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$

$$Au = \begin{bmatrix} 3 & 4 \\ -2 & 1 \\ 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 15-4 \\ -10-1 \\ 15-4 \end{bmatrix}$$

$$= \begin{bmatrix} 11 \\ -11 \\ 11 \end{bmatrix}$$

$$Av = \begin{bmatrix} 3 & 4 \\ -2 & 1 \\ 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ -2 \end{bmatrix}$$

$$= \begin{bmatrix} 15-8 \\ -10-2 \\ 15-8 \end{bmatrix}$$

$$= \begin{bmatrix} 7 \\ -12 \\ 7 \end{bmatrix}$$

Here $Au \neq Av$ so,

$$\|Au\| = \sqrt{11^2+11^2+11^2} = \sqrt{121+121+121} = \sqrt{363}$$

$$\|Av\| = \sqrt{7^2+12^2+7^2} = \sqrt{49+144+49} = \sqrt{242}$$

This shows $\|Au\| > \|Av\|$. Therefore v is the least square solution of $Ax = b$.