

(12)

UNIT-3
MATRIX ALGEBRA

Matrix → It is a rectangular array of numbers enclosed between round or square brackets.

Matrix multiplication → Let, $A_{m \times n}$ and $B_{n \times p}$ be two matrices such that the number of columns in the first matrix equals to the number of rows in the second matrix. Then the product of A and B , AB is defined as:

The order of AB is $m \times p$; the number of rows in first matrix followed by number of columns in the second matrix.

i.e., $A_{m \times n} \cdot B_{n \times p}$

Example: Let $A = AB_{m \times p}$

$$A = \begin{bmatrix} 2 & -1 \\ 4 & 0 \\ 1 & 2 \end{bmatrix}_{3 \times 2} \quad \text{and } B = \begin{bmatrix} 1 & 0 & 2 \\ -1 & 4 & -1 \end{bmatrix}_{2 \times 3}$$

$$\text{where, } b_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, b_2 = \begin{bmatrix} 0 \\ 4 \end{bmatrix}, b_3 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$\text{Now, } AB = A[b_1 \ b_2 \ b_3]$$

$$AB = [Ab_1 \ Ab_2 \ Ab_3] \quad \text{--- (P)}$$

Here,

$$Ab_1 = 1 \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix} + (-1) \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ -1 \end{bmatrix}$$

$$Ab_2 = 0 \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix} + 4 \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} -4 \\ 0 \\ 8 \end{bmatrix}$$

$$Ab_3 = 2 \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix} + (-1) \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \\ 0 \end{bmatrix}$$

Now substituting these values in (P).

$$AB = \begin{bmatrix} 3 & -4 & 5 \\ 4 & 0 & 8 \\ -1 & 8 & 0 \end{bmatrix}_{3 \times 3}$$



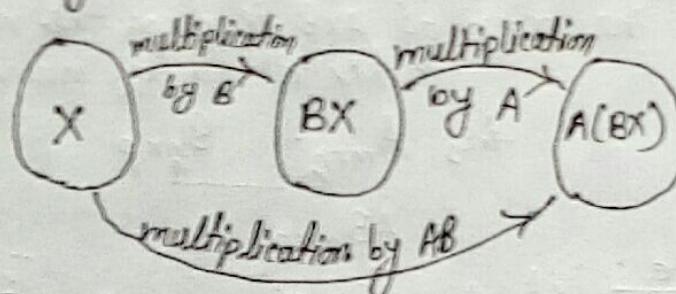
④ Matrix product as composition mapping:

Let $A_{m \times n}$, $B_{n \times p}$ and $X_{p \times 1}$ be the matrices of the order $m \times n$, $n \times p$ and $p \times 1$ respectively. Then the product of B and X i.e., image of X under the transformation multiplication by B is a matrix, BX of order $m \times 1$.

When the matrix BX is multiplied by A , $A(BX)$ is an image of BX under the mapping: multiplication by A .

Symbolically: $X \xrightarrow[\text{by } B]{\text{multiplication}} BX \xrightarrow[\text{by } A]{\text{multiplication}} A(BX)$.

Understanding diagram



$$\text{i.e., } A(BX) = (AB)X.$$

So, the matrix multiplication to a column matrix (vector) is composition mapping.

⑤ Invertible matrix:

Definition \rightarrow A $n \times n$ square matrix is said to be an invertible if there exists another matrix $B_{n \times n}$ (say), such that $AB=BA=I_n$.

The matrix B is called an inverse of A , denoted by A^{-1} and written as, $A^{-1}=B$.

Determination of an inverse of a matrix by row reduction algorithm
Let $A_{n \times n}$ be an $n \times n$ square matrix. Consider the augmented matrix $[A \ I]$.

i.e.,
$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} & 1 & \dots & 0 \\ \vdots & & & & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} & 0 & \dots & 1 \end{bmatrix}$$

Now, reduce the coefficient matrix of A into the identity matrix by the row operations. Let the augmented matrix assumes (takes) the form. $[I \ B]$ then, $B=A^{-1}$.

④ Uniqueness of inverse:

Let $A_{n \times n}$ be an invertible matrix with an inverse $B_{n \times n}$ (say). Then, from the definition, $AB = BA = I \dots (1)$.

Claim: B is unique (i.e., inverse of A is unique).

If possible suppose that $C_{n \times n}$ is also an inverse of A .

$$C = IC$$

$$\text{or, } C = (BA) \cdot C$$

$$\text{or, } C = B(AC), \text{ by associativity}$$

$$\text{or, } C = BI \quad (\because C \text{ is an inverse of } A) \quad CA = AC = I$$

$$\text{or, } C = B.$$

⑤ Matrix product (Column row expansion):

$$\text{Let } A = \begin{bmatrix} 2 & -1 & 1 \\ 3 & 1 & 0 \end{bmatrix}_{2 \times 3} \text{ and } B = \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 2 & 1 \end{bmatrix}_{3 \times 2} \text{ be two matrices}$$

in which the product AB can be defined.

$$\text{Now, consider } \text{Col}_1 A \cdot \text{row}_1 B = \begin{bmatrix} 2 \\ 3 \end{bmatrix}_{2 \times 1} \cdot \begin{bmatrix} 1 & -1 \end{bmatrix}_{1 \times 2}$$

$$= \begin{bmatrix} 2 & -2 \\ 3 & -3 \end{bmatrix}_{2 \times 2}$$

$$\text{Col}_1 A \cdot \text{row}_1 B = \begin{bmatrix} 2 & -2 \\ 3 & -3 \end{bmatrix}_{2 \times 2}$$

$$\text{Also, } \text{Col}_2 A \cdot \text{row}_2 B = \begin{bmatrix} -1 \\ 1 \end{bmatrix}_{2 \times 1} \begin{bmatrix} 0 & 1 \end{bmatrix}_{1 \times 2} = \begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix}_{2 \times 2}$$

$$\text{& } \text{Col}_3 A \cdot \text{row}_3 B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}_{2 \times 1} \begin{bmatrix} 2 & 1 \end{bmatrix}_{1 \times 2} = \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}_{2 \times 2}$$

$$\therefore \text{Col}_1 A \cdot \text{row}_1 B + \text{Col}_2 A \cdot \text{row}_2 B + \text{Col}_3 A \cdot \text{row}_3 B = \begin{bmatrix} 2+0+2 & -2-1+1 \\ 3+0+0 & -3+1+0 \end{bmatrix}$$

$$\therefore \sum_{k=1}^3 \text{Col}_k A \cdot \text{row}_k B = \begin{bmatrix} 4 & -2 \\ 3 & -2 \end{bmatrix}$$

⊕ Partitioned matrix:

Let $A = \begin{bmatrix} 2 & 1 & -1 & 4 & 0 & 5 \\ -1 & 4 & 6 & 3 & 1 & 7 \\ 0 & 4 & 2 & 0 & 1 & 3 \end{bmatrix}_{3 \times 6}$ be a 3×6 matrix.

Let it be partitioned into sub matrices as follows:-

$$A = \left[\begin{array}{cc|cc|cc} 2 & 1 & -1 & 4 & 0 & 5 \\ -1 & 4 & 6 & 3 & 1 & 7 \\ \hline 0 & 4 & 2 & 0 & 1 & 3 \end{array} \right] \quad \begin{array}{l} \text{हमिलाई गाहिए} \\ \text{अनुसार जैसों तो} \\ \text{partition गर्ने सिलद} \end{array}$$

Now,

$$A_{11} = \begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} -1 & 4 \\ 6 & 3 \end{bmatrix}, \quad A_{13} = \begin{bmatrix} 0 & 5 \\ 1 & 7 \end{bmatrix}$$

$$A_{21} = \begin{bmatrix} 0 & 4 \end{bmatrix}, \quad A_{22} = \begin{bmatrix} 2 & 0 \end{bmatrix}, \quad A_{23} = \begin{bmatrix} 1 & 3 \end{bmatrix}$$

Here,

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \end{bmatrix} \text{ is the partitioned matrix.}$$

Partitioned matrix is the matrix whose elements are considered to be its sub-matrices.

Scalar product → The scalar product of a partition matrix is obtained by multiplying each block of the matrix with the scalar.

Sum → The sum of two partition matrices is obtained by adding the corresponding blocks (sub-matrices) provided that the matrices are of the same order and way of partition is also same.

Multiplication → Let A and B be two partitioned matrices in which the number of columns in the first matrix equals to number of rows in the second matrix and partition of columns of A should be exactly the same to the row partition of B . Then the product is obtained by the sum of the products of block matrices as the element in ordinary way.

V. Imp
For example:-

(14)

$$A = \left[\begin{array}{ccc|cc} 2 & -3 & 1 & 0 & -4 \\ 1 & 5 & -2 & 3 & -1 \\ 0 & -4 & 2 & 7 & -1 \end{array} \right]_{3 \times 5}$$

$$\leftarrow B = \left[\begin{array}{cc} 6 & 4 \\ -2 & 1 \\ -3 & 7 \\ -1 & 3 \\ 5 & 2 \end{array} \right]_{5 \times 2}$$

$$\text{Now, } A = \left[\begin{array}{cc|c} A_{11} & A_{12} & \\ A_{21} & A_{22} & \end{array} \right]_{2 \times 2}, B = \left[\begin{array}{c} B_{11} \\ B_{21} \end{array} \right]_{2 \times 1}$$

$$\leftarrow AB_{2 \times 1} = \left[\begin{array}{c} A_{11}B_{11} + A_{12}B_{21} \\ A_{21}B_{11} + A_{22}B_{21} \end{array} \right]_{2 \times 2} \quad \text{--- (1)}$$

Here,

$$\begin{aligned} A_{11} \cdot B_{11} &= \left[\begin{array}{ccc} 2 & -3 & 1 \\ 1 & 5 & -2 \end{array} \right] \left[\begin{array}{c} 6 \\ -2 \\ -3 \end{array} \right] \\ &= \left[\begin{array}{cc} 12 + 6 - 3 & 8 - 3 + 7 \\ 6 - 15 + 6 & 4 + 5 - 14 \end{array} \right] \\ &= \left[\begin{array}{cc} 15 & 12 \\ -3 & -5 \end{array} \right] \end{aligned}$$

$$\begin{aligned} A_{12} \cdot B_{21} &= \left[\begin{array}{cc} 0 & -4 \\ -3 & -1 \end{array} \right] \left[\begin{array}{c} -1 \\ 5 \\ 2 \end{array} \right] \\ &= \left[\begin{array}{cc} 0 - 20 & 0 - 8 \\ -3 - 5 & -9 - 2 \end{array} \right] \\ &= \left[\begin{array}{cc} -20 & -8 \\ -8 & -7 \end{array} \right] \end{aligned}$$

$$\begin{aligned} A_{21} \cdot B_{11} &= \left[\begin{array}{ccc} 0 & -4 & 2 \end{array} \right] \left[\begin{array}{c} 6 \\ -2 \\ -3 \end{array} \right] \\ &= \left[\begin{array}{cc} 0 + 8 - 6 & 0 - 4 + 14 \end{array} \right] \\ &= [2 \ 10] \end{aligned}$$

$$A_{22} \cdot B_{21} = \begin{bmatrix} 7 & -1 \end{bmatrix} \begin{bmatrix} -1 & 3 \\ 5 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -7-5 & 21-2 \end{bmatrix}$$

$$= \begin{bmatrix} -12 & 19 \end{bmatrix}$$

Again,

$$A_{11} \cdot B_{11} + A_{12} \cdot B_{21} = \begin{bmatrix} 15 & 12 \\ -3 & -5 \end{bmatrix} + \begin{bmatrix} -20 & -8 \\ -8 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 15-20 & 12-8 \\ -3-8 & -5+7 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 4 \\ -11 & 2 \end{bmatrix}$$

4) $A_{21} \cdot B_{11} + A_{22} \cdot B_{21} = \begin{bmatrix} 2 & 10 \end{bmatrix} + \begin{bmatrix} -12 & 19 \end{bmatrix}$

$$= \begin{bmatrix} 2-12 & 10+19 \end{bmatrix}$$

$$= \begin{bmatrix} -10 & 29 \end{bmatrix}$$

Finally using these in ① we get multiplied partitioned matrix as follows:-

$$AB = \begin{bmatrix} 5 & 4 \\ -11 & 2 \\ \hline -10 & 29 \end{bmatrix}_{3 \times 2}$$

* Inverse of partitioned matrix:

Let, $A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$ be a partitioned matrix with the block matrices

$A_{11}, A_{12}, 0, A_{22}$ where A_{11} is a square matrix of order $p \times p$, A_{22} is of order $q \times q$ (say). Then its inverse A^{-1} is given by

$$A^{-1} = \begin{bmatrix} A_{11}^{-1} & -A_{11}^{-1} A_{12} A_{22}^{-1} \\ 0 & A_{22}^{-1} \end{bmatrix} \text{ such that } A \cdot A^{-1} = \begin{bmatrix} I_p & 0 \\ 0 & I_q \end{bmatrix}.$$

Example: Find the inverse of $A = \begin{bmatrix} 1 & 3 & 9 & 0 \\ 2 & 4 & 0 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 2 & -2 \end{bmatrix}$. (15)

Soln

Given,

$$A = \begin{bmatrix} 1 & 3 & 9 & 0 \\ 2 & 4 & 0 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 2 & -2 \end{bmatrix}$$

Here, $A_{11} = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$, $A_{12} = \begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix}$, $A_{21} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ & $A_{22} = \begin{bmatrix} 0 & 2 \\ 2 & -2 \end{bmatrix}$

Now,

$$|A_{11}| = 4 - 6 = -2 \neq 0$$

$$\text{adj. } A_{11} = \begin{bmatrix} 4 & -3 \\ -2 & 1 \end{bmatrix}$$

$$A_{11}^{-1} = \frac{1}{-2} \begin{bmatrix} 4 & -3 \\ -2 & 1 \end{bmatrix} \quad \rightarrow$$

$$= \begin{bmatrix} -2 & \frac{3}{2} \\ 1 & -\frac{1}{2} \end{bmatrix}$$

Again, $|A_{22}| = 0 - 4 = -4 \neq 0$

$$\text{adj. } A_{22} = \begin{bmatrix} -2 & -2 \\ -2 & 0 \end{bmatrix}$$

$$A_{22}^{-1} = \frac{1}{-4} \begin{bmatrix} -2 & -2 \\ -2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}$$

Here, $A_{12} \cdot A_{22}^{-1} = \begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}$

$$= \begin{bmatrix} \frac{9}{2} & \frac{9}{2} \\ \frac{1}{2} & 0 \end{bmatrix}$$

$$A_1 - (A_{11}^{-1} \cdot A_{12} \cdot A_{22}^{-1}) = - \left\{ \begin{bmatrix} -2 & \frac{3}{2} \\ 1 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{9}{2} & \frac{9}{2} \\ \frac{1}{2} & 0 \end{bmatrix} \right\}$$

$$= \begin{bmatrix} -9 + 3 & -9 \\ \frac{9}{2} - \frac{1}{4} & \frac{9}{2} \end{bmatrix}$$

$$= \begin{bmatrix} 6 & -9 \\ -\frac{17}{4} & \frac{9}{2} \end{bmatrix}$$

$$\therefore A^{-1} = \begin{bmatrix} A_{11}^{-1} & - (A_{11}^{-1} \cdot A_{12} \cdot A_{22}^{-1}) \\ 0 & A_{22}^{-1} \end{bmatrix} = \begin{bmatrix} -2 & \frac{3}{2} & 6 & 9 \\ 1 & -\frac{1}{2} & -\frac{17}{4} & -\frac{9}{2} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & 0 \end{bmatrix}$$

* LU factorization (V.Imp)

$$A = L U$$

→ Lower triangular matrix of echelon form
 → Upper triangular matrix = echelon form

While solving first we obtain echelon form of given matrix and as soon as we get pivot element we divide pivot column below pivot element (including pivot element) by pivot element, while for clear understanding below are the same examples.

TU.2075

Example:-1: Find the LU factorization of the matrix

Solution

$$\begin{bmatrix} 2 & 5 \\ 6 & -7 \end{bmatrix}$$

$$\text{Let } A = \begin{bmatrix} 2 & 5 \\ 6 & -7 \end{bmatrix}$$

Now, we reduce A into echelon form

$$R_2 \rightarrow R_2 + (-3)R_1$$

$$A = \begin{bmatrix} 2 & 5 \\ 0 & -22 \end{bmatrix} \quad \underline{\text{Echelon form.}}$$

$$\begin{aligned} \text{First pivot column is } & \begin{bmatrix} 2 \\ 6 \end{bmatrix} \div 2 \\ &= \begin{bmatrix} 1 \\ 3 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \text{Second pivot column is } & \begin{bmatrix} -22 \end{bmatrix} \div -22 \\ &= 1 \end{aligned}$$

Now,

$$L = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$$

Upper triangular so
in place of 5

$$U = \begin{bmatrix} 2 & 5 \\ 0 & -22 \end{bmatrix} \quad \leftarrow \text{Echelon form}$$

$$\begin{aligned} \therefore LU &= \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 5 \\ 0 & -22 \end{bmatrix} \\ &= \begin{bmatrix} 2+0 & 5-0 \\ 6+0 & 15-22 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 5 \\ 6 & -7 \end{bmatrix} = A // \end{aligned}$$

Example 2: Find the LU factorization of $A = \begin{bmatrix} 2 & -4 & 4 & -2 \\ 6 & -9 & 7 & -3 \\ -1 & -4 & 8 & 0 \end{bmatrix}$ (16)

Solⁿ

$$A = \begin{bmatrix} 2 & -4 & 4 & -2 \\ 6 & -9 & 7 & -3 \\ -1 & -4 & 8 & 0 \end{bmatrix}$$

$$R_2 \rightarrow R_2 + (-3)R_1, R_3 \rightarrow R_3 + R_1/2$$

$$A = \begin{bmatrix} 2 & -4 & 4 & -2 \\ 0 & 3 & -5 & 3 \\ 0 & -6 & 10 & -1 \end{bmatrix}, \text{ First pivot column is } \begin{bmatrix} 2 \\ 6 \\ -1 \end{bmatrix} \div 2 = \begin{bmatrix} 1 \\ 3 \\ -\frac{1}{2} \end{bmatrix}$$

$$R_3 \rightarrow R_3 + 2R_2$$

$$A = \begin{bmatrix} 2 & -4 & 4 & -2 \\ 0 & 3 & -5 & 3 \\ 0 & 0 & 0 & 5 \end{bmatrix}, \text{ Second pivot column is } \begin{bmatrix} 3 \\ -6 \end{bmatrix} \div 3 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

Now,

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -\frac{1}{2} & -2 & 1 \end{bmatrix}$$

Third pivot column is $\begin{bmatrix} 5 \end{bmatrix} \div 5 = 1$

$$U = \begin{bmatrix} 2 & -4 & 4 & -2 \\ 0 & 3 & -5 & 3 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

$$\therefore LU = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -\frac{1}{2} & -2 & 1 \end{bmatrix} \begin{bmatrix} 2 & -4 & 4 & -2 \\ 0 & 3 & -5 & 3 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 2+0+0 & -4+0+0 & 4+0+0 & -2+0+0 \\ 6+0+0 & -12+3+0 & 12-5+0 & -6+3+0 \\ -1+0+0 & -2-6+0 & -2+10+0 & 1-6+5 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -4 & 4 & -2 \\ 6 & -9 & 7 & -3 \\ -1 & -4 & 8 & 0 \end{bmatrix} = A //$$

Pivot element $\frac{3}{13}$ is taken
for factorization

-> तीसरे बारे में
किन्तु उपरी त्रिकोणल
भास्करों के लिए जैविक

Example 3: Find the LU factorization of

Soln

$$\begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ -4 & -5 & 3 & -8 & 1 \\ 2 & -5 & -4 & 1 & 8 \\ -6 & 0 & 7 & -3 & 1 \end{bmatrix} \quad 4 \times 5$$

Let $A = \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ -4 & -5 & 3 & -8 & 1 \\ 2 & -5 & -4 & 1 & 8 \\ -6 & 0 & 7 & -3 & 1 \end{bmatrix}$

$$R_2 \rightarrow R_2 + 2R_1, R_3 \rightarrow R_3 + (-1)R_1, R_4 \rightarrow R_4 + 3R_1$$

$$A = \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & -9 & -3 & -4 & 10 \\ 0 & 12 & 4 & 12 & -5 \end{bmatrix}, \text{First pivot column is } \begin{bmatrix} 2 \\ 4 \\ 2 \\ -6 \end{bmatrix} \div 2$$

$$= \begin{bmatrix} 1 \\ -2 \\ 1 \\ -3 \end{bmatrix}$$

$$R_3 \rightarrow R_3 + 3R_2, R_4 \rightarrow R_4 + (-4)R_2$$

$$A = \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 4 & 7 \end{bmatrix}, \text{Second pivot column is } \begin{bmatrix} 3 \\ -9 \\ 12 \end{bmatrix} \div 3$$

$$= \begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix}$$

$$R_4 \rightarrow R_4 + (-2)R_3$$

$$A = \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix}, \text{Third pivot column is } \begin{bmatrix} 2 \\ 4 \end{bmatrix} \div 2$$

$$= \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Fourth pivot column is $\begin{bmatrix} 5 \end{bmatrix} \div 5 = 1$

$$\therefore L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & -3 & 1 & 0 \\ -3 & 4 & 2 & 1 \end{bmatrix} \quad \& U = \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix}$$

$$\therefore LU = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & -3 & 1 & 0 \\ -3 & 4 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ -4 & -5 & 3 & -8 & 1 \\ 2 & -5 & -4 & 1 & 8 \\ -6 & 0 & 7 & -3 & 1 \end{bmatrix}$$

$$= A$$

$\textcircled{*}$ Solve the system $2x_1 + x_2 + 5x_3 = 1$
 $-4x_1 + 4x_3 = 2$
 $6x_1 + 2x_2 + 3x_3 = 4$

Soln

The system is equivalent to the matrix equation, $AX=b$... \textcircled{P}
 where,

$$A = \begin{bmatrix} 2 & 1 & 5 \\ -4 & 0 & 4 \\ 6 & 2 & 3 \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ & } b = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$$

Let, $A = LU$

So, eqn \textcircled{P} becomes

$$(LU)X = b$$

$$L(UX) = b \quad \textcircled{PP}$$

again let

$$UX = Y \quad \textcircled{PPP}$$

Then eqn \textcircled{PPP} becomes

$$LY = b \quad \textcircled{P1}$$

Let $y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

So, from $\textcircled{P1}$

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & -\frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$$

$$\text{or, } \begin{bmatrix} y_1 + 0 + 0 \\ -2y_1 + y_2 + 0 \\ 3y_1 - \frac{1}{2}y_2 + y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$$

$$\Rightarrow y_1 = 1$$

$$y_2 = 2 + 2y_1$$

$$= 2 + 2 \times 1$$

$$= 4$$

$$y_3 = 4 - 3 \times y_1 + \frac{1}{2}y_2$$

$$= 4 - 3 \times 1 + \frac{1}{2} \times 4$$

$$= 4 - 3 + \frac{4}{2}$$

$$= 3$$

$$\therefore y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}$$

But we have to find x_1, x_2, x_3 so, from (PPT)

$$\begin{bmatrix} 2 & 1 & 5 \\ 0 & 2 & 14 \\ 0 & 0 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}$$

$$\Rightarrow 2x_1 + x_2 + 5x_3 = 1 \quad \textcircled{A}$$

~~$$\Rightarrow 2x_2 + 14x_3 = 4 \quad \textcircled{B}$$~~

$$\textcircled{A} - 5\textcircled{B} = 3$$

$$\text{or, } x_3 = -\frac{3}{5} \quad \textcircled{C}$$

Using value of x_3 in (B).

$$2x_2 + 14 \left(-\frac{3}{5} \right) = 4$$

$$\text{or, } x_2 = \frac{4 + \frac{42}{5}}{2}$$

$$\text{or, } x_2 = \frac{124}{5}$$

Again, using value of x_2 and x_3 in (A).

$$2x_1 + \frac{124}{5} + 5 \times \left(-\frac{3}{5} \right) = 1$$

$$\text{or, } x_1 = \frac{5}{218}$$

//