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CHAPTER: 7

Eigenvalues and Eigenvectors

Definition (Eigenvalue):

If A is $n \times n$ matrix, Then a scalar λ is called eigen value of matrix A. If The equation $Ax = \lambda x$ has a non-trivial solution. such x is called eigen vector corresponding to eigen vector λ.

If A is $n \times n$ matrix, Then a non-zero vector $x \in R^n$ is called an eigen value of matrix A if $Ax = \lambda x$, where λ is scalar.

Example I: Is $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ an eigen vector of $\begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$?

Solution:

$$A = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}, \quad x = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$Ax = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 + 0 \\ 8 \quad -2 \end{bmatrix}$$

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$$= \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

$$= 3 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\boxed{Ax = 3x}$$

Here, $x = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is eigen vector of $\begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$

Example 2: Is $x = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$ is eigen vector of $\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$?

<u>Solution:</u> Since, $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$,

$$x = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

Here, $Ax = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \end{bmatrix}$

$$= \begin{bmatrix} 3-12 \\ 15-4 \end{bmatrix} = \begin{bmatrix} -9 \\ 11 \end{bmatrix}$$

$$\boxed{Ax \neq \lambda x}$$

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Hence $x = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$ is not eigen vector of $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$

Example 3: Show that -2 is eigen value of $\begin{bmatrix} 7 & 3 \\ 3 & -1 \end{bmatrix}$

<u>Solution:</u> $\lambda = -2$, $A = \begin{bmatrix} 7 & 3 \\ 3 & -1 \end{bmatrix}$

If -2 is eigen vector, Then,

$$Ax = \lambda x$$

or $Ax - \lambda x = 0$

or, $(A - \lambda I)x = 0$

or, $(A + 2I)x = 0$ —————— ① has non trivial solution.

Here,

$$A + 2I = \begin{bmatrix} 7 & 3 \\ 3 & -1 \end{bmatrix} + 2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix}$$

<u>The augmented matrix is</u>

$$\begin{bmatrix} A+2I & 0 \end{bmatrix} = \begin{bmatrix} 9 & 3 & 0 \\ 3 & 1 & 0 \end{bmatrix}$$

$R_1 \longrightarrow R_1 \times \frac{1}{9}$

$$\begin{bmatrix} 1 & 1/3 & 0 \\ 3 & 1 & 0 \end{bmatrix}$$

$R_2 \longrightarrow R_2 - 3R_1$

$$\begin{bmatrix} 1 & 1/3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Here, x_2 is free variable so, The system has non-trival solution and hence $\lambda = -2$ is an eigen value

The corresponing equation is

$$x_1 + 1/3\, x_2 = 0$$

$$\Rightarrow x_1 = -1/3\, x_2$$

$$\Rightarrow x_2 = \text{free}$$

$$\therefore x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1/3\, x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -1/3 \\ 1 \end{bmatrix}$$

$\therefore$ eigen vector $x = \begin{bmatrix} -1/3 \\ 1 \end{bmatrix}$ Corresponding to $\lambda = -2$

Imp c.r.

Example 5: Find the basis for eigenspace corresponding to listed eigenvalue

Where

$$A = \begin{bmatrix} 4 & 2 & 3 \\ -1 & 1 & 3 \\ 2 & 4 & 9 \end{bmatrix} \text{ and } \lambda = 3$$

Solution:

If $\lambda = 3$ is eigen value, Then The equation.

$$Ax = \lambda x$$

or $Ax - \lambda x = 0$

or, $(A - \lambda I)x = 0$

or $(A - 3I)x = 0$ ——— ① has non-trival solution.

Here,

$$A - 3I = \begin{bmatrix} 4 & 2 & 3 \\ -1 & 1 & 3 \\ 2 & 4 & 9 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ -1 & -2 & 3 \\ 2 & 4 & 6 \end{bmatrix}$$

The augmented matrix is

$$[A - 3I \;\; 0] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & -2 & 3 & 0 \\ 2 & 4 & 6 & 0 \end{bmatrix}$$

$$[A - 3I \quad 0] = \begin{bmatrix} 1 & 2 & 3 & 0 \\ -1 & -2 & -3 & 0 \\ 2 & 4 & 6 & 0 \end{bmatrix}$$

$R_2 \to R_2 + R_1$, $R_3 \to R_3 - 2R_1$

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Here, x_2 and x_3 are free variable so
The system is non-trivial solution. so,
$\lambda = 3$ is eigen value

The corresponding equation is,

$$x_1 + 2x_2 + 3x_3 = 0$$

$\Rightarrow x_1 = -2x_2 - 3x_3$
$\Rightarrow x_2 = \text{free}$
$\Rightarrow x_3 = \text{free}$

$$\therefore x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2x_2 - 3x_3 \\ x_2 + 0.x_3 \\ 0x_2 + 0x_3 \end{bmatrix}$$

$$= x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

non-trival → no three value we

$\therefore$ The basis of eigen space are

$$\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Example 4: Is 5 an eigenvalue of A

$$A = \begin{bmatrix} 6 & -3 & 1 \\ 3 & 0 & 5 \\ 2 & 2 & 6 \end{bmatrix}$$

Solution: Here $\lambda = 5$

$$A = \begin{bmatrix} 6 & -3 & 1 \\ 3 & 0 & 5 \\ 2 & 2 & 6 \end{bmatrix}$$

If $Ax = \lambda x$

$Ax - \lambda x = 0$
$(A - \lambda I)x = 0$ ——— (1) has non-trival solution

Here,

$$A - 5I = \begin{bmatrix} 6 & -3 & 1 \\ 3 & 0 & 5 \\ 2 & 2 & 5 \end{bmatrix} - 5 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 61 & -3 & 1 \\ 3 & -5 & 5 \\ 2 & 2 & 0 \end{bmatrix}$$

The augmented matrix is

$$[A - 5I \quad 0]$$

$$\begin{bmatrix} 1 & -3 & 1 & 0 \\ 3 & -5 & 5 & 0 \\ 2 & 2 & 1 & 0 \end{bmatrix}$$

$R_2 \longrightarrow R_2 - 3R_1$, $R_3 \longrightarrow R_3 - 2R_1$

$$\begin{bmatrix} 1 & -3 & 1 & 0 \\ 0 & 4 & 2 & 0 \\ 0 & 8 & -1 & 0 \end{bmatrix}$$

$R_3 \longrightarrow R_3 - 2R_2$

$$\begin{bmatrix} 1 & -3 & 1 & 0 \\ 0 & 4 & 2 & 0 \\ 0 & 0 & -5 & 0 \end{bmatrix}$$

Since, Thus the homogeneous system has no free variable. So, The equation ① has trival solution, which means $\lambda = 5$ is not eigenvalue of A.

Example 6: Find the eigenvalue of matrix

$$\begin{bmatrix} 4 & -2 & 1 \\ 0 & 1 & 5 \\ 0 & 0 & 6 \end{bmatrix}$$

Solution: Since, given matrix is upper triangular matrix, so eigenvalue of matrix are 4, 1, 6 ans

Exercise 7.1

1. Let $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$, $u = \begin{bmatrix} 6 \\ -5 \end{bmatrix}$. Is u is eigenvector of A? [2066 B.Sc.CSIT]

Solution: Here,

$$A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}, \quad u = \begin{bmatrix} 6 \\ -5 \end{bmatrix}$$

Then $Au = \underset{2\times 2}{\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}} \underset{2\times 1}{\begin{bmatrix} 6 \\ -5 \end{bmatrix}}$

$$= \begin{bmatrix} 6 + (-30) \\ 30 - 10 \end{bmatrix}$$

$$= \begin{bmatrix} -24 \\ 20 \end{bmatrix} = -4 \begin{bmatrix} 6 \\ -5 \end{bmatrix}$$

$$\boxed{\therefore Au = -4u}$$

Hence, $u = \begin{bmatrix} 6 \\ -5 \end{bmatrix}$ is eigenvector of $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$ Ans.

2. Is $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$ an eigenvector of $\begin{bmatrix} -3 & 1 \\ -3 & 8 \end{bmatrix}$?

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Solution: Here,

$$x = \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \quad A = \begin{bmatrix} -3 & 1 \\ -3 & 8 \end{bmatrix}$$

Then,

$$Ax = \begin{bmatrix} -3 & 1 \\ -3 & 8 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} -3 + 4 \\ -3 + 24 \end{bmatrix} = \begin{bmatrix} 1 \\ 21 \end{bmatrix}$$

$\boxed{\because Ax \neq \lambda x}$

Since, $x = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$ is eigenvector of $A = \begin{bmatrix} -3 & 1 \\ -3 & 8 \end{bmatrix}$

3. Is $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$ an eigenvector of $\begin{bmatrix} 3 & 1 \\ 2 & -1 \end{bmatrix}$?

Solution: Here,

$$A = \begin{bmatrix} 3 & 1 \\ 2 & -1 \end{bmatrix}, \quad x = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

Then,

$$Ax = \begin{bmatrix} 3 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 6 + 4 \\ \cancel{4} - 4 \end{bmatrix}$$

$$= \begin{bmatrix} 10 \\ 0 \end{bmatrix}$$

$\boxed{\text{Hence, } Ax \neq \lambda x.}$

Hence, $x = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$ is eigenvector of $A = \begin{bmatrix} 3 & 1 \\ 2 & -1 \end{bmatrix}$

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4. Is $\begin{bmatrix} 4 \\ -3 \\ 1 \end{bmatrix}$ an eigenvector of $\begin{bmatrix} 3 & 7 & 9 \\ -4 & -5 & 1 \\ 2 & 4 & 4 \end{bmatrix}$

Solution: Here,

$$A = \begin{bmatrix} 3 & 7 & 9 \\ -4 & -5 & 1 \\ 2 & 4 & 4 \end{bmatrix}, \quad x = \begin{bmatrix} 4 \\ -3 \\ 1 \end{bmatrix}$$

Then,

$$Ax = \begin{bmatrix} 3 & 7 & 9 \\ -4 & -5 & 1 \\ 2 & 4 & 4 \end{bmatrix} \begin{bmatrix} 4 \\ -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 12 - 21 + 9 \\ -16 + 15 + 1 \\ 8 - 12 + 4 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$Ax = 0, \quad Au = 0$

Since,

5. If $\lambda = 1$ an eigenvector of matrix $\begin{bmatrix} 2 & 2 \\ 3 & 5 \end{bmatrix}$?

Solution:

$$\lambda = 1, \quad A = \begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix}$$

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If $\lambda = 1$ an eigenvector, Then

$Ax = \lambda x$

or $Ax - \lambda x = 0$

or, $(A - I)x = 0$ —— ① has non-trivial solution.

Here,

$$A - I = \begin{bmatrix} 2 & 2 \\ 3 & 5 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

The augmented matrix:

$$[A - I \quad 0] = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 4 & 0 \end{bmatrix}$$

$R_2 \to R_2 - 3R_1$

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & -2 & 0 \end{bmatrix}$$

Since, The System has trivial solution, it's means it has not free variable. So, $(\lambda = 1)$ is not eigenvector of $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ Ans

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⑥ If $\lambda = 5$ an eigenvalue of matrix $\begin{bmatrix} 5 & 0 \\ 2 & 1 \end{bmatrix}$? if so find one corresponding eigenvector.

Solution: Here. $\lambda = 5$

$$A = \begin{bmatrix} 5 & 0 \\ 2 & 1 \end{bmatrix}$$

If 5 an eigenvalue of matrix

$Ax = \lambda x$

$Ax - \lambda x = 0$

$(A - 5I)x = 0$ —— ① has non trivial solution

Here,

$$A - 5I = \begin{bmatrix} 5 & 0 \\ 2 & 1 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ -2 & -4 \end{bmatrix}$$

The augmented matrix

$$[A - 5I \quad 0] = \begin{bmatrix} 0 & 0 & 0 \\ -2 & -4 & 0 \end{bmatrix}$$

$R_1 \longleftrightarrow R_2$

$$[A-5I \quad 0] = \begin{bmatrix} 2 & -4 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$R_1 \to \frac{1}{2}R_1$$

$$\begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Since, $\lambda = 5$ an eigenvalue of matrix A. since, The argumented matrix has non-trivial solution and x_2 is free variable.

So that,

The corresponding equation is,

$x_1 - 2x_2 = 0$

$\Rightarrow x_1 = +2x_2$

$x_2 = \text{free}$

$$\therefore x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Since, $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is ~~eig~~ basis of eigen vector of $A = \begin{bmatrix} 5 & 0 \\ 2 & 1 \end{bmatrix}$

ans

Q. If $\lambda = 3$ eigen value of $\begin{bmatrix} 1 & 2 & 2 \\ 3 & -2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

Solution: Here $\lambda = 3$,

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 3 & -2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

If $\lambda = 3$, is eigenvalue of The equation

$Ax = \lambda x$

$Ax - \lambda x = 0$

$(A - 3I)x = 0$ ——— (1) has non-trivial solution

Here,

$$A - 3I = \begin{bmatrix} 1 & 2 & 2 \\ 3 & -2 & 1 \\ 0 & 1 & 1 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 2 & 2 \\ 3 & -5 & 1 \\ 0 & 1 & -2 \end{bmatrix}$$

The argumented matrix is

$$[A - 3I \quad 0] = \begin{bmatrix} -2 & 2 & 2 & 0 \\ 3 & -5 & 1 & 0 \\ 0 & 1 & -2 & 0 \end{bmatrix}$$

$$R_1 \to R_1 \times -\frac{1}{2}$$

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$$[A - 3I \mid 0] = \left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 3 & -5 & 1 & 0 \\ 0 & 1 & -2 & 0 \end{array}\right]$$

$R_2 \longrightarrow R_2 - 3R_1$

$$\left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & -2 & 4 & 0 \\ 0 & 1 & -2 & 0 \end{array}\right]$$

$R_2 \longrightarrow -\frac{1}{2} R_2$

$$\left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 1 & -2 & 0 \end{array}\right]$$

$R_3 \longrightarrow R_3 - R_2$

$$\left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Since, $R_1 \longrightarrow R_1 + R_2$

$$\left[\begin{array}{ccc|c} 1 & 0 & -3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Here, x_3 is free variable so, The system has trival solution than $\lambda = 3$. Then.

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The corresponding equation is.

$x_1 - 3x_3 = 0$

$x_1 = 3x_3$

$x_2 - 2x_3 = 0$

$\Rightarrow x_2 = 2x_3$

$x_3 =$ free

$$\therefore x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3x_3 \\ 2x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

(8) Show that 2 is eigen value of matrix

$$\begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$$

Solution: Here, $\lambda = 2$ and $A = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$

If $Ax = \lambda x$

$Ax - \lambda x = 0$

$(A - 2I)x = 0$ ——— (1) has non-trival solution

Hence

$$A - 2I = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix} - 2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$(A - 2I) = \begin{bmatrix} 2 & -1 & 6 \\ 2 & -1 & 6 \\ 2 & -1 & 6 \end{bmatrix}$$

The Augmented matrix is

$$[A - 2I \quad 0] = \begin{bmatrix} 2 & -1 & 6 & 0 \\ 2 & -1 & 6 & 0 \\ 2 & -1 & 6 & 0 \end{bmatrix}$$

$$R_1 \longrightarrow \frac{R_1}{2}$$

$$\begin{bmatrix} 1 & -1/2 & 3 & 0 \\ 2 & -1 & 6 & 0 \\ 2 & -1 & 6 & 0 \end{bmatrix}$$

$$R_2 \longrightarrow R_2 - 2R_1, \quad R_3 \rightarrow R_3 - 2R_1$$

$-1+1$

$-1-2(-1/2)$

$-1+1/2$

$$\begin{bmatrix} 1 & -1/2 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since, x_2 and x_3 are free variable so the system has non trivial solution. Then the corresponding equation is

$$x_1 - \tfrac{1}{2}x_2 + 3x_3 = 0$$

$$\Rightarrow x_1 = \tfrac{1}{2}x_2 - 3x_3$$

x_2 = free

x_3 = free.

$$x_p = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \tfrac{1}{2}x_2 - 3x_3 \\ x_2 + 0.x_3 \\ 0.x_2 + x_3 \end{bmatrix} = x_2 \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

is eigen space and basis for eigen-space is $\begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$ ma

9. If λ is eigenvalue of invertible matrix A, corresponding eigenvector x Then $1/\lambda$ is an eigenvalue of A^{-1} and x is corresponding eigenvector.

Solution: $Ax = \lambda x$

$$\Rightarrow A^{-1}(Ax) = A^{-1}(\lambda x)$$
$$\Rightarrow (A^{-1}A)x = A^{-1}(\lambda x)$$
$$\Rightarrow Ix = A^{-1}(\lambda x)$$
$$\Rightarrow x = A^{-1}(\lambda x)$$
$$\Rightarrow \frac{x}{\lambda} = A^{-1}x$$
$$\Rightarrow A^{-1}x = \frac{1}{\lambda}x$$

$\therefore \frac{1}{\lambda}$ is the eigen value of A^{-1} ma

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10. Find a basis for the eigenspace corresponding to listed eigenvalue.

(a) $A = \begin{bmatrix} 5 & 0 \\ 2 & 1 \end{bmatrix}$ and $\lambda = 1$.

Solution: Here, $A = \begin{bmatrix} 5 & 0 \\ 2 & 1 \end{bmatrix}$

$\lambda = 1$

If $\lambda = 1$ is eigenvalue of A. so,

$Ax = \lambda x$

$Ax - \lambda x = 0$

i.e. $(A - I)x = 0$ ——— (1)

has non-trivial solution.

Here,

$A - I = \begin{bmatrix} 5 & 0 \\ 2 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 4 & 0 \\ 2 & 0 \end{bmatrix}$

The corresponding equation is,

$[A - I \quad 0] = \begin{bmatrix} 4 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix}$

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$R_1 \rightarrow R_1/4$ // $R_2 \rightarrow R_2 - 2R_1$

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Since, x_2 is free variable so,
The system is non-trivial solution. Then
The corresponding equation is

$\Rightarrow x_1 = 0$

$x_2 = \text{free}$

$\therefore x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\therefore$ eigenspace of A is $= \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ ans.

(b) $A = \begin{bmatrix} 10 & -9 \\ 4 & -2 \end{bmatrix}$ and $\lambda = 4$.

Solution: $\lambda = 4$, $A = \begin{bmatrix} 10 & -9 \\ 4 & -2 \end{bmatrix}$

If $\lambda = 4$ is an eigenvector of A Then,

$Ax = \lambda x$.

$Ax - \lambda x = 0$

i.e $(A - 4I)x = 0$ ——— (1)

has non-trivial solution.

Here,

$$A - 4I = \begin{bmatrix} 10 & -9 \\ 4 & -2 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & -9 \\ 4 & -6 \end{bmatrix}$$

The augmented matrix is

$$[A - 4I \quad 0] = \begin{bmatrix} 6 & -9 & 0 \\ 4 & -6 & 0 \end{bmatrix}$$

$$R_1 \rightarrow R_1/6$$

$$\begin{bmatrix} 1 & -3/2 & 0 \\ 4 & -6 & 0 \end{bmatrix}$$

$$R_2 \rightarrow R_2 - 4R_1$$

$$\begin{bmatrix} 1 & -3/2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Since, x_2 is free variable so, the system is non trivial solution.
The corresponding equation is

$$x_1 - 3/2\, x_2 = 0$$

$$\Rightarrow x_1 = 3/2\, x_2$$

$$x_2 = \text{free}$$

$$\therefore x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3/2\, x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 3/2 \\ 1 \end{bmatrix}$$

Hence, eigenspace of matrix $A = \left\{ \begin{bmatrix} 3/2 \\ 1 \end{bmatrix} \right\}$

(d) $A = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$ and $\lambda = 2$.

Solution: Here, $\lambda = 2$

$$A = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$$

If λ is eigenvalue of A then,

$$Ax = \lambda x.$$

$$Ax - \lambda x = 0$$

i.e $(A - 2I)\, x = 0$ ——— (1) has non-trivial solution

Here, $(A - 2I) = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 2 & -1 & 6 \\ 2 & -1 & 6 \\ 2 & -1 & 6 \end{bmatrix}$$

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The augmented matrix

$$[A - 2I \quad 0] = \begin{bmatrix} 2 & -1 & 6 & 0 \\ 2 & -1 & 6 & 0 \\ 2 & -1 & 6 & 0 \end{bmatrix}$$

$$R_1 \rightarrow \frac{R_1}{2}$$

$$\begin{bmatrix} 1 & -1/2 & 3 & 0 \\ 2 & -1 & 6 & 0 \\ 2 & -1 & 6 & 0 \end{bmatrix}$$

$$R_2 \rightarrow R_2 - 2R_1, \; R_3 \rightarrow R_3 - 2R_1$$

$$\begin{bmatrix} 1 & -1/2 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Hence, x_2 and x_3 are free variable
Then the system has non-trivial solution.
Since $\lambda = 2$ is eigen value of A.

The corresponding equation is

$$x_1 - \tfrac{1}{2}x_2 + 3x_3 = 0$$

$$\Rightarrow x_1 = \tfrac{1}{2}x_2 - 3x_3$$
$$\Rightarrow x_2 = \text{free}.$$
$$\Rightarrow x_3 = \text{free}.$$

$$\therefore x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \tfrac{1}{2}x_2 - 3x_3 \\ x_2 + 0\cdot x_3 \\ 0\cdot x_2 + 1\cdot x_3 \end{bmatrix} = x_2 \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix} - x_3 \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$$

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Hence, eigenspace are $\left\{ \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$ ans.

Q. $A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$, and $\lambda = 1$ and $\lambda = 2$.

Solution: Here, $\lambda = 1$ and $\lambda = 2$.

For $\lambda = 1$

If $\lambda = 1$ is eigenvalue of A Then,

$$Ax = \lambda x.$$

$$Ax - \lambda x = 0$$

i.e. $(A - I)x = 0$ —— (1) has non trivial solution.

Here,

$$A - I = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 & -2 \\ 1 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix}$$

The augmented matrix is

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$$[A - I \quad 0] = \begin{bmatrix} -1 & 0 & -2 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 2 & 0 \end{bmatrix}$$

$R_1 \longleftrightarrow R_2, \; R_2 \to R_2 - R_1$
$R_3 \to R_3 + R_1$

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ -1 & 0 & -2 & 0 \\ 1 & 0 & 2 & 0 \end{bmatrix}$$

$R_1 \to R_1 - R_2$

$$\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Here, x_3 is a free variable then the system has non-trivial soln. Then $\lambda = 1$ is an eigen value of A.

Then, the corresponding equation is

$x_1 + 2x_3 = 0$
$\Rightarrow x_1 = -2x_3$
$\Rightarrow x_2 = x_3$
$\Rightarrow x_3 = \text{free}$

$$\therefore x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2x_3 \\ x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

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$\therefore$ The eigen vector of A is $x = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$

If $\lambda = 2$ then the eigen value of matrix A is

$Ax = \lambda x$
$Ax - \lambda x = 0$
for $(A - 2I)x = 0$ —— ① has non-trivial solution

$$\text{Here, } A - 2I = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix} - 2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 0 & -2 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

The augmented matrix of $(A - \lambda I)x = 0$ is,

$$[A - \lambda I \quad 0] = \begin{bmatrix} -2 & 0 & -2 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

$R_1 \to -\frac{R_1}{2}$

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

$R_2 \rightarrow R_2 - R_1$ and $R_3 \rightarrow R_3 - R_1$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Here, x_2 and x_3 are free variable then The system has non-trivial sol^n. Then The corresponding equations

$$x_1 + x_3 = 0$$
$$\Rightarrow x_1 = -x_3$$
$$x_2 = \text{free}$$
$$x_3 = \text{free}$$

$$\therefore x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0.x_2 - x_3 \\ x_2 + 0.x_3 \\ 0x_2 + x_3 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$\therefore$ The eigen vector of A is $x =$

$$x = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$\therefore$ The eigen space of A is $\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$

(11) Find the eigenvalue of following matrices.

Q. $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 5 \\ 0 & 0 & -1 \end{bmatrix}$

Solution: Since, Given matrix is upper triangular matrix, So eigenvalue of given matrix are $\lambda = 0, 2, -1$. ans

(12) If x is an eigen vector for matrix A correspond to eigenvalue λ, what is A^3x?

Solution: $Ax = \lambda x$

$$\Rightarrow A(Ax) = A(\lambda x)$$
$$\Rightarrow A^2x = \lambda(Ax)$$
$$\Rightarrow A^2x = \lambda(\lambda x)$$
$$\Rightarrow A^2x = \lambda^2 x$$
$$\Rightarrow A(A^2x) = A(\lambda^2 x)$$
$$\Rightarrow A^3x = \lambda^2(Ax)$$
$$\Rightarrow A^3x = \lambda^2 \lambda x = \lambda^3 x.$$
$$\Rightarrow A^3x = \lambda^3 x. \text{ Ans.}$$

(13). If $x = \begin{pmatrix} 6 \\ -5 \end{pmatrix}$ is an eigenvector for matrix $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$ correspond to $\lambda = -4$. Find A^3x?

Solution:

$$x = \begin{pmatrix} 6 \\ -5 \end{pmatrix}$$

$$A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} \text{ and } \lambda = -4$$

To find A^3x

$$\because A^3x = \lambda^3 x \quad \text{——— (1)}$$

$$= (-4)^3 \begin{pmatrix} 6 \\ -5 \end{pmatrix}$$

$$= -64 \begin{pmatrix} 6 \\ -5 \end{pmatrix}$$

$$= \begin{bmatrix} -384 \\ 320 \end{bmatrix} \text{ Ans.}$$

Imp.

7.2 The Characteristic Equation.

If λ be an eigenvalue of a square matrix A, then $\det(A - \lambda I)$ is called characteristic polynomial and $\det(A - \lambda I) = 0$ is called characteristic equation of matrix A.

det ⇒ $|A - \lambda I|$ characteristic polynomial.

det ⇒ $|A - \lambda I| = 0$ is the characteristic equation (polynomial).

Example 7: Find the characteristic polynomial of matrix $\begin{bmatrix} 2 & -1 \\ 1 & 4 \end{bmatrix}$. Also, find the eigenvalue.

Solution: Here,

$$A - \lambda I$$

$$\begin{bmatrix} 2 & -1 \\ 1 & 4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2-\lambda & -1 \\ 1 & 4-\lambda \end{bmatrix}$$

$\therefore$ The characteristic polynomial is

$$|A - \lambda I|$$

$$= \begin{vmatrix} 2-\lambda & -1 \\ 1 & 4-\lambda \end{vmatrix}$$

$$= (2-\lambda)(4-\lambda) + 1$$

$$= 8 - 2\lambda - 4\lambda + \lambda^2 + 1$$

$$= 9 - 6\lambda + \lambda^2.$$

The characteristic equation is

$$|A - \lambda I| = 0$$

or $9 - 6\lambda + \lambda^2 = 0$

or $(3-\lambda)^2 = 0$

$$\boxed{\lambda = 3.}$$

Therefore $\lambda = 3$ is eigenvalue of matrix

$$\begin{bmatrix} 1 & -1 \\ 1 & 4 \end{bmatrix}$$

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Example 8: Find characteristics equation and eigen value of A where $A = \begin{bmatrix} 1 & -4 \\ 4 & 2 \end{bmatrix}$

Solution:

$$A = \begin{bmatrix} 1 & -4 \\ 4 & 2 \end{bmatrix}$$

Then, $A - \lambda I = \begin{bmatrix} 1 & -4 \\ 4 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 1 & -4 \\ 4 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$$= \begin{bmatrix} 1-\lambda & -4 \\ 4 & 2-\lambda \end{bmatrix}$$

∴ The characteristic polynomial is

$$|A - \lambda I|$$

$$\begin{vmatrix} 1-\lambda & -4 \\ 4 & 2-\lambda \end{vmatrix}$$

or, $(1-\lambda)(2-\lambda) + 16$

or, $2 - \lambda - 2\lambda + \lambda^2 + 16$

or, $\lambda^2 - 3\lambda + 18$

or, $(\lambda - 3)$

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The characteristic equation is.

$$\lambda^2 - 3\lambda + 18 = 0$$

This give,

$$\lambda = \frac{3 \pm \sqrt{(-3)^2 - 4 \times 1 \times 18}}{2.1}$$

$$= \frac{3 \pm \sqrt{-63}}{2}$$

This gives the imaginary value of λ. Therefore, the matrix A has no real eigen-value.

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Exercise 7.2.

1. Find the characteristics polynomial and eigenvalue of

i. $\begin{bmatrix} 2 & 7 \\ 7 & 2 \end{bmatrix}$

Soln: Here, $A = \begin{bmatrix} 2 & 7 \\ 7 & 2 \end{bmatrix}$

Then $A - \lambda I = \begin{bmatrix} 2 & 7 \\ 7 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 2 & 7 \\ 7 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$

$= \begin{bmatrix} 2-\lambda & 7 \\ 7 & 2-\lambda \end{bmatrix}$

$\therefore$ The characteristics polynomial is

$|A - \lambda I|$

$\begin{vmatrix} 2-\lambda & 7 \\ 7 & 2-\lambda \end{vmatrix}$

$= (2-\lambda)(2-\lambda) - 49$

$= 4 - 2\lambda - 2\lambda + \lambda^2 - 49$

$= \lambda^2 - 4\lambda - 45$

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The characteristic equation is

$|A - \lambda I| = 0$

or $\lambda^2 - 4\lambda - 45 = 0$

or, $\lambda^2 - (9-5)\lambda - 45 = 0$

or $\lambda^2 - 9\lambda + 5\lambda - 45 = 0$

or $\lambda(\lambda - 9) + 5(\lambda - 9) = 0$

or $(\lambda + 5)(\lambda - 9) = 0$

$\therefore \lambda = 9, -5$ Ans

b. $\begin{bmatrix} 5 & -3 \\ -4 & 3 \end{bmatrix}$

Solution: Here, $A = \begin{bmatrix} 5 & -3 \\ -4 & 3 \end{bmatrix}$

Then

$A - \lambda I = \begin{bmatrix} 5 & -3 \\ -4 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 5 & -3 \\ -4 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$

$= \begin{bmatrix} 5-\lambda & -3 \\ -4 & 3-\lambda \end{bmatrix}$

The characteristic polynomial is

$|A - \lambda I|$

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$$\begin{vmatrix} 5-\lambda & -3 \\ -4 & 3-\lambda \end{vmatrix}$$

$$= (5-\lambda)(3-\lambda) + 12$$

$$= 15 - 5\lambda - 3\lambda + \lambda^2 - 12$$

$$= \lambda^2 - 8\lambda + 3$$

The characteristic equation is,

$$|A - \lambda I| = 0$$

or, $\lambda^2 - 8\lambda + 3 = 0$

or, $\lambda^2 - 8\lambda + 3 = 0$

This gives,

$$\lambda = \frac{+8 \pm \sqrt{(-8)^2 - 4.1.3}}{2 \times 1}$$

$$= \frac{8 \pm \sqrt{64 - 12}}{2}$$

$$= \frac{8 \pm \sqrt{52}}{2}$$

$$= 4 \pm 52$$

Hence, $\lambda = 4 \pm 52$ Ans

(c) $\begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}$

Solution: $A = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}$

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Then, $A - \lambda I = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$

$$= \begin{bmatrix} 2-\lambda & 3 \\ 3 & -6-\lambda \end{bmatrix}$$

The characteristic polynomial equation is

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 2-\lambda & 3 \\ 3 & -6-\lambda \end{vmatrix} = 0$$

or, $(2-\lambda)(-6-\lambda) - 9 = 0$

or, $-12 - 2\lambda + 6\lambda + \lambda^2 - 9 = 0$

or, $\lambda^2 + 4\lambda - 21 = 0$

This give,

$$\lambda = \frac{-4 \pm \sqrt{(4)^2 + 4.1.21}}{2 \times 1}$$

$$= \frac{-4 \pm \sqrt{16 + 84}}{2}$$

$$= -2 \pm 10 = 8, -12$$

$$\lambda^2 + (7-3)\lambda - 21 = 0$$

or, $\lambda^2 + 7\lambda - 3\lambda - 21 = 0$

or, $\lambda(\lambda+7) - 3(\lambda+7) = 0$

or, $(\lambda-3)(\lambda+7) = 0$

$\boxed{\therefore \lambda = 3, -7}$

d. $\begin{bmatrix} 5 & 3 \\ -4 & 4 \end{bmatrix}$

Soln: Here, $A = \begin{bmatrix} 5 & 3 \\ -4 & 4 \end{bmatrix}$

Then $A - \lambda I = \begin{bmatrix} 5 & 3 \\ -4 & 4 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$

$= \begin{bmatrix} 5-\lambda & 3 \\ -4 & 4-\lambda \end{bmatrix}$

The characteristic equation is

$|A - \lambda I| = 0$

or $\begin{vmatrix} 5-\lambda & 3 \\ -4 & 4-\lambda \end{vmatrix} = 0$

or $(5-\lambda)(4-\lambda) + 12 = 0$

or $20 - 5\lambda - 4\lambda + \lambda^2 + 12 = 0$

or $20 - 9\lambda + \lambda^2 + 12 = 0$

or $\lambda^2 - 9\lambda + 32 = 0$

This gives,

$$\lambda = \frac{+(9) \pm \sqrt{(-9)^2 - 4 \cdot 1 \cdot 32}}{2 \cdot 1}$$

Hence, This gives imaginary value of λ.
Therefore, The matrix A has no real eigen value.

e. $\begin{bmatrix} 2 & 2 & -1 \\ 1 & 3 & -1 \\ -1 & -2 & 2 \end{bmatrix}$

Solution: Here, $A = \begin{bmatrix} 2 & 2 & -1 \\ 1 & 3 & -1 \\ -1 & -2 & 2 \end{bmatrix}$

$A - \lambda I = \begin{bmatrix} 2 & 2 & -1 \\ 1 & 3 & -1 \\ -1 & -2 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 2 & 2 & -1 \\ 1 & 3 & -1 \\ -1 & -2 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$

$= \begin{bmatrix} 2-\lambda & 2 & -1 \\ 1 & 3-\lambda & -1 \\ -1 & -2 & 2-\lambda \end{bmatrix}$

The characteristic polynomial

$|A - \lambda I| = \begin{vmatrix} 2-\lambda & 2 & -1 \\ 1 & 3-\lambda & -1 \\ -1 & -2 & 2-\lambda \end{vmatrix}$

$= (2-\lambda)\begin{vmatrix} 3-\lambda & -1 \\ -2 & 2-\lambda \end{vmatrix} - 2\begin{vmatrix} 1 & -1 \\ -1 & 2-\lambda \end{vmatrix} +$

$(-1)\begin{vmatrix} 1 & 3-\lambda \\ -1 & -2 \end{vmatrix}$

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$= (2-\lambda)\{(3-\lambda)(1-\lambda)-2\} - 2\{(2-\lambda)-1\} +$
$(-1)\{-2 + (3-\lambda)\}$

$= (2-\lambda)\{(6-3\lambda-2\lambda+\lambda^2)\} - 4 + 2\lambda + 2 + 8 - 3 + \lambda$

$= 6 - 6\lambda - 4\lambda + 2\lambda^2 - 6\lambda + 3\lambda^2 + 4\lambda^2 - \lambda^3 - 4 - 2\lambda + 2 + 2 - 3 + \lambda$

$= -\lambda^3 + 2\lambda^2 + 3\lambda^2 + 4\lambda^2 - 6\lambda - 4\lambda - 6\lambda - 2\lambda + \lambda + 6 - 4 + 4 - 3$

$= -\lambda^3 + 7\lambda^2 - 11\lambda + 5$

<u>The characteristic equation is</u>

$-\lambda^3 + 7\lambda^2 - 11\lambda + 5 = 0$

$\therefore (\lambda-1)^2 + (\lambda+5) = 0$

$\boxed{\therefore \lambda = 1, 5 \text{ Ans.}}$

(6) $\begin{bmatrix} -1 & 4 & -2 \\ -3 & 4 & 0 \\ -3 & 1 & 3 \end{bmatrix}$

<u>Solution:</u> Here, $A = \begin{bmatrix} -1 & 4 & -2 \\ -3 & 4 & 0 \\ -3 & 1 & 3 \end{bmatrix}$

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$A - \lambda I = \begin{bmatrix} -1 & 4 & -2 \\ -3 & 4 & 0 \\ -3 & 1 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} -1-\lambda & 4 & -2 \\ -3 & 4-\lambda & 0 \\ -3 & 1 & 3-\lambda \end{bmatrix}$

<u>The characteristic polynomial is</u>

$|A - \lambda I| = \begin{vmatrix} -1-\lambda & 4 & -2 \\ -3 & 4-\lambda & 0 \\ -3 & 1 & 3-\lambda \end{vmatrix}$

<u>The characteristic equation is</u>

$|A - \lambda I| = 0$

$= (-1-\lambda)\begin{vmatrix} 4-\lambda & 0 \\ 1 & 3-\lambda \end{vmatrix} - 4 \begin{vmatrix} -3 & 0 \\ -3 & 3-\lambda \end{vmatrix} + 2 \begin{vmatrix} 3 & 4-\lambda \\ -3 & 1 \end{vmatrix}$

$= (-1-\lambda)(4-\lambda)(3-\lambda) - 4(-3(3-\lambda)) - 2\{3 + 3(4-\lambda)\}$

$= (-1-\lambda)(12-4\lambda-3\lambda+\lambda^2) - 4(-9+3\lambda) - 2(3+12-3\lambda)$

$= -\lambda^2 - 11\lambda + 6 + 6\lambda^2$

The characteristic equation is

$$|A - \lambda I| = 0$$

or, $-\lambda^3 + 6\lambda^2 - 11\lambda + 6 = 0$

or $\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$

or $(3-\lambda)(\lambda-1)(\lambda-2) = 0$

or $\lambda = 1, 2, 3$ ans

Q. $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ -3 & 5 & 2 \end{bmatrix}$

Solution: let $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ -3 & 5 & 2 \end{bmatrix}$

$$A - \lambda I = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ -3 & 5 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1-\lambda & 0 & 0 \\ 1 & 2-\lambda & 0 \\ -3 & 5 & 2-\lambda \end{bmatrix}$$

The characteristic polynomial is

$$|A - \lambda I| = (1-\lambda) \begin{vmatrix} 2-\lambda & 0 \\ 5 & 2-\lambda \end{vmatrix} - 0 + 0$$

$$= \boxed{-\lambda^3 + 5\lambda^2 - 8\lambda + 4}$$

The characteristic equation is

$$|A - \lambda I| = 0$$

$\cancel{-\lambda^3 + 5\lambda^2 - 8\lambda + 4 = 0}$

$\cancel{\text{or } \lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0}$

$\cancel{\text{or } \lambda^2(\lambda - 5) + 4(2\lambda - 1) = 0}$

$\cancel{\text{or } (\lambda^2 + 4)}$

$$(1-\lambda)(2-\lambda)^2 = 0$$

$\boxed{\therefore \lambda = 1, 2}$

Q. $\begin{bmatrix} -1 & 0 & 1 \\ -3 & 4 & 1 \\ 0 & 0 & 2 \end{bmatrix}$

Solution: Here, $A = \begin{bmatrix} -1 & 0 & 1 \\ -3 & 4 & 1 \\ 0 & 0 & 2 \end{bmatrix}$

$$A - \lambda I = \begin{bmatrix} -1 & 0 & 1 \\ -3 & 4 & 1 \\ 0 & 0 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1-\lambda & 0 & 1 \\ -3 & 4-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{bmatrix}$$

$$|A - \lambda I| = (-1-\lambda) \begin{vmatrix} 4-\lambda & 1 \\ 0 & 2-\lambda \end{vmatrix} + 0 + 0$$

$$= (-1-\lambda)(4-\lambda)(2-\lambda)$$

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Hence, The characteristic equation

$$|A - \lambda I| = 0$$

$$(2-\lambda)\cdot(-1-\lambda)(4-\lambda) = 0$$

$$\lambda = -1, 2, 4 \text{ ans.}$$

① $\begin{bmatrix} 1 & 4 & 3 & 1 \\ 0 & 3 & 2 & 6 \\ 0 & 0 & -1 & 4 \\ 0 & 0 & 0 & 4 \end{bmatrix}$

Soln: Here, $A = \begin{bmatrix} 2 & 4 & 3 & 1 \\ 0 & 3 & 2 & 6 \\ 0 & 0 & -1 & 4 \\ 0 & 0 & 0 & 4 \end{bmatrix}$

Then, $|A - \lambda I| = \begin{bmatrix} 2 & 4 & 3 & 1 \\ 0 & 3 & 2 & 6 \\ 0 & 0 & -1 & 4 \\ 0 & 0 & 0 & 4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 2-\lambda & 4 & 3 & 1 \\ 0 & 3-\lambda & 2 & 6 \\ 0 & 0 & -1-\lambda & 4 \\ 0 & 0 & 0 & 4-\lambda \end{bmatrix}$$

The characteristic polynomial is

$$|A - \lambda I| = \begin{vmatrix} 2-\lambda & 4 & 3 & 1 \\ 0 & 3-\lambda & 2 & 6 \\ 0 & 0 & -1-\lambda & 4 \\ 0 & 0 & 0 & 4-\lambda \end{vmatrix}$$

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$$(4-\lambda)\begin{vmatrix} 2-\lambda & 4 & 3 \\ 0 & 3-\lambda & 2 \\ 0 & 0 & -1-\lambda \end{vmatrix}$$

$$= (4-\lambda)(-1-\lambda)\begin{vmatrix} 2-\lambda & 4 \\ 0 & 3-\lambda \end{vmatrix}$$

$$= (4-\lambda)(-1-\lambda)(2-\lambda)(3-\lambda)$$

The characteristics equation is

$$|A - \lambda I| = 0$$

$$(4-\lambda)(-1-\lambda)(2-\lambda)(3-\lambda) = 0$$

or $\lambda = -1, 2, 3, 4$

$$\boxed{\therefore \lambda = -1, 2, 3, 4}$$

① $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix}$

Solution: Here

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1-\lambda & 0 & 0 & 0 & 0 \\ 0 & 2-\lambda & 0 & 0 & 0 \\ 0 & 0 & -2-\lambda & 0 & 0 \\ 0 & 0 & 0 & 3-\lambda & 0 \\ 0 & 0 & 0 & 0 & 4-\lambda \end{bmatrix}$$

The characteristic polynomial is

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 0 & 0 & 0 & 0 \\ 0 & 2-\lambda & 0 & 0 & 0 \\ 0 & 0 & -2-\lambda & 0 & 0 \\ 0 & 0 & 0 & 3-\lambda & 0 \\ 0 & 0 & 0 & 0 & 4-\lambda \end{vmatrix}$$

$$= (1-\lambda) \begin{vmatrix} 2-\lambda & 0 & 0 & 0 \\ 0 & -2-\lambda & 0 & 0 \\ 0 & 0 & 3-\lambda & 0 \\ 0 & 0 & 0 & 4-\lambda \end{vmatrix}$$

$$= (1-\lambda)(2-\lambda)(-2-\lambda)(3-\lambda)(4-\lambda)$$

The characteristic equation is

$$|A - \lambda I| = 0$$

$$(1-\lambda)(2-\lambda)(-2-\lambda)(3-\lambda)(4-\lambda) = 0$$

$\therefore \lambda = 1, 2, -2, 3, 4$ ans.

7.3. Diagonalization:

A square matrix A is diagonalizable if there exist an invertible matrix P and diagonal matrix D such that $A = PDP^{-1}$

Theorem:

An $n \times n$ matrix A is diagonizable iff (if and only if) A has n linearly independent eigen vectors.

Procedure:

① Find n linearly independent eigen vectors say $v_1, v_2, \ldots, v_n$.

② Form matrix P having $v_1, v_2, \ldots, v_n$ as its column vectors.

③ The matrix D will be diagonal matrix with $\lambda_1, \lambda_2, \ldots, \lambda_n$ as its successive diagonal entries where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are eigen value. Here $A = PDP^{-1} \Rightarrow AP = PD$

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Example: Diagonalize, The matrix if possible.

$$A = \begin{bmatrix} 2 & 2 & -1 \\ 1 & 3 & -1 \\ -1 & -2 & 2 \end{bmatrix}$$

Here,

$$A - \lambda I = \begin{bmatrix} 2 & 2 & -1 \\ 1 & 3 & -1 \\ -1 & -2 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 2 & -1 \\ 1 & 3 & -1 \\ -1 & -2 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$= \begin{bmatrix} 2-\lambda & 2 & -1 \\ 1 & 3-\lambda & -1 \\ -1 & -2 & 2-\lambda \end{bmatrix}$$

The characteristics equation is,

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 2-\lambda & 2 & -1 \\ 1 & 3-\lambda & -1 \\ -1 & -2 & 2-\lambda \end{vmatrix} = 0$$