Function

Defination :- Let A and B be non empty sets. A function f from A to B is an assignment of exactly one element of B to each element of A. We write $f(a) = b$ if b is the unique element of B assigned by the function 'f' to the element a of A. If f is a function from A to B, we write $f: A \to B$.

Functions are sometimes also called mapping or transformations.

Eg:

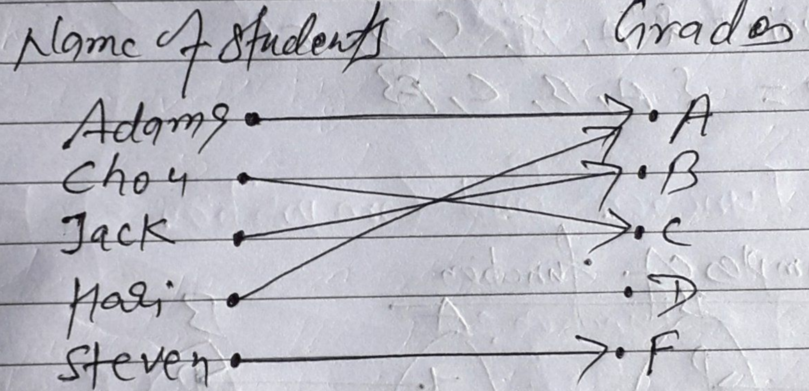


fig1 → Assignment of Grades in a Math

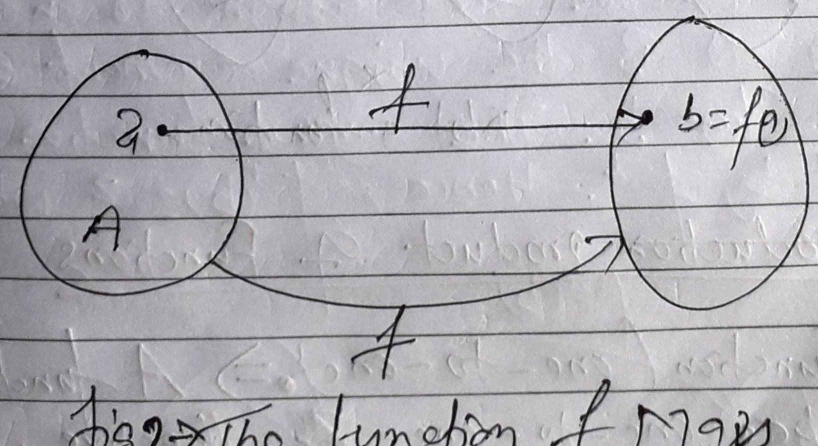


fig2 → The function f Maps A to B.

Note:- Functions can be specified by a formula, such as $f(x) = x+1$.

Domain, Co-domain, Image, preimage and Range.

⇒ If f is a function from A to B, we say that A is the domain of 'f' and B is the codomain of 'f'. If $f(a) = b$, we say that b is the image of a and a is a preimage of b. The range of f is the set of all images of elements of A. Also, if f is a function from A to B, we say that f maps A to B.

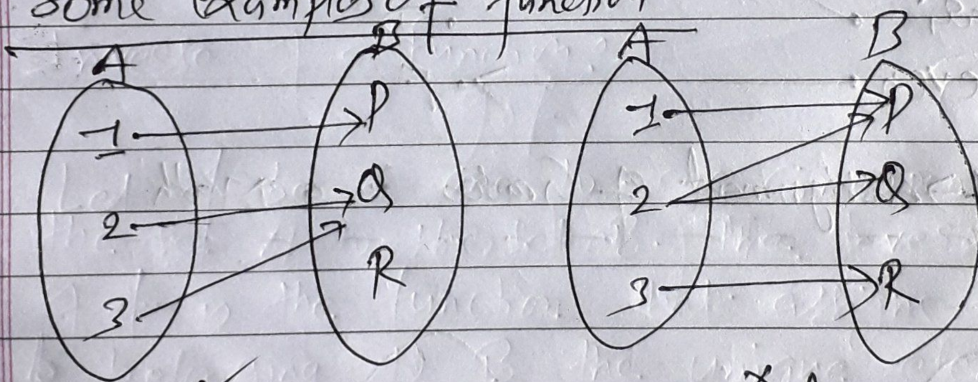
Eg: From fig1:

Domain = {Adams, Chou, Jack, Hari, Steven}

Codomain = {A, B, C, D, F}

Range = {A, B, C, F}

Some examples of function



Function   Not a function

Sum and ~~Production~~ product of Functions

① Injection Function (one-to-one) ⇒ A function 'f' is said to be one-to-one, or injective, if and only if $f(a) = f(b)$ implies that $a = b$ for all a and b in the domain of f. A function is said to be an injection if it is one-to-one.

Some function never assign the same value to two different domain elements. These functions are said to be one-to-one.

eg:

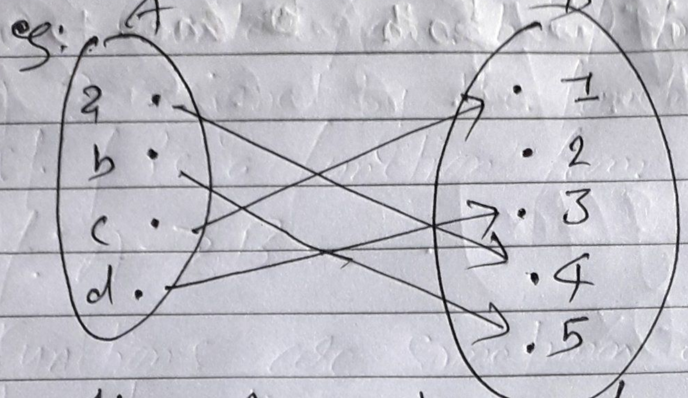


fig:- A one-to-one function

(2) **Surjective function (onto)** $\Rightarrow$ A function f from A to B is called onto, or surjective, if and only if for every element $b \in B$ there is an element $a \in A$ with $f(a) = b$. A function f is called a surjection if it is onto.

OR, For some functions the range and the codomain are equal. That is, every member of the codomain is the image of some element of the domain. Functions with this property are called onto function.

eg:

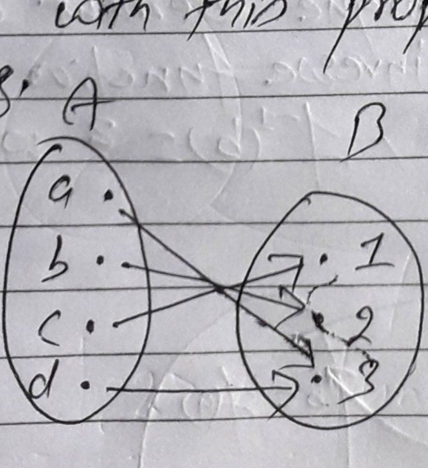
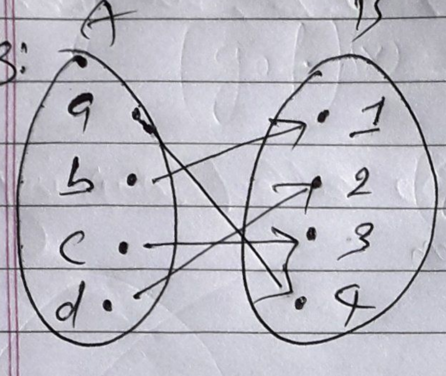


fig:- An onto function

(3) Bijection (one-to-one correspondence):-

The function f is a one-to-one correspondence or bijection, if it is both one-to-one and onto.

Eg:



f is:- one-to-one and onto.

Inverse Function and Composition of Function

(1) Inverse of a Function:-

(Bijection)

Let f be a one-to-one correspondence from the set A to the set B. The inverse function of f is the function that assigns to an element b belonging to B the unique element a in A such that $f(a) = b$. The inverse function of f is denoted by f^{-1}. Hence, $f^{-1}(b) = a$ when $f(a) = b$.

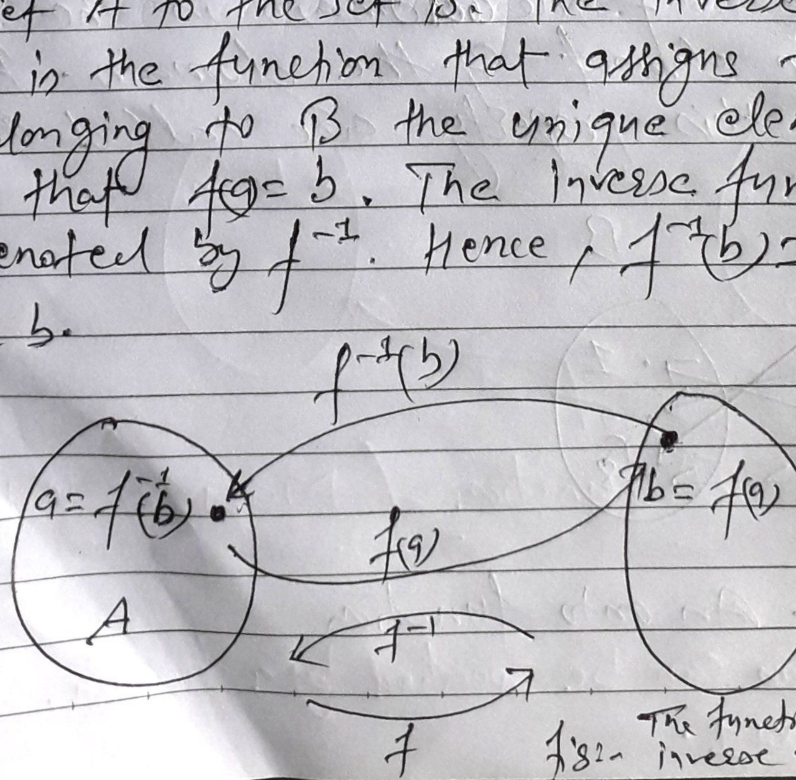


fig:- The function f^{-1} is the inverse of function f.

A one-to-one correspondence is called invertible because we can define an inverse of this function. A function is not invertible if it is not a one-to-one correspondence, because the inverse of such function does not exist. (11)

Example 1 ⇒

let f be a function from $\{a, b, c\}$ to $\{1, 2, 3\}$ such that $f(a) = 2$, $f(b) = 3$, and $f(c) = 1$. Is f invertible, and if it is, what is its inverse?

Sol^n The function f is invertible because it is one-to-one correspondence. The inverse function f^{-1} reverses the correspondence given by f, so $f^{-1}(1) = c$, $f^{-1}(2) = a$ and $f^{-1}(3) = b$.

Example 2 ⇒ let f be the function from R to R with $f(x) = x^2$. Is f invertible?

Sol^n Because $f(-2) = f(2) = 4$, f is not one-to-one. If an inverse function were defined, it would have to assign two elements to 4. Hence, f is not invertible.

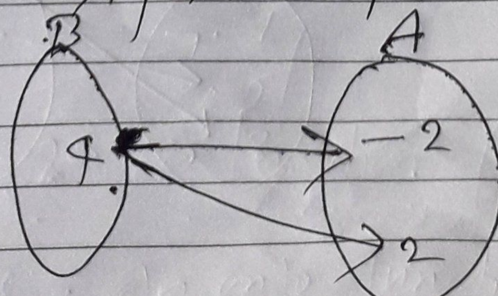


fig: Diagram of f^{-1} (Not a function)

Example

Let A = {1, 2, 3, 4}, B = {a, b, c, d} and let f = {(1, a), (2, a), (3, d), (4, c)}. Show that f is a function and state with a reason, whether f is invertible.

Solution

Here we have f(1) = a, f(2) = a, f(3) = d and f(4) = c. Clearly f is a function. Now f^{-1} = {(a, 1), (a, 2), (d, 3), (c, 4)}, which is not a function so f is not invertible since $f^{-1}(a)$ = {1, 2}.

Diagram of f:

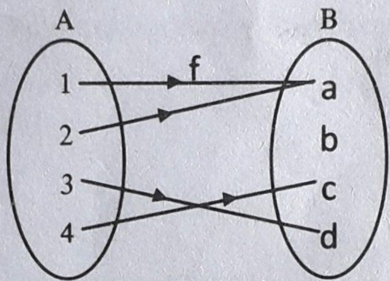


Fig. Function.

Diagram of f^{-1}:

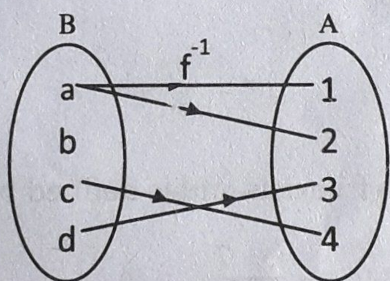


Fig.: Not a function

Example

Show that the mapping f: R → R is defined by f(x) = ax + b, where a, b, x ∈ R, a ≠ 0, is invertible. Define its inverse.

Solution

Inverse of a function exists if that function is both one-to-one and onto. Therefore, we first show f is one-to-one and then we show it is onto.

For, if $x_1, x_2 \in R$ then $f(x_1) = f(x_2)$

$\Rightarrow \quad ax_1 + b = ax_2 + b$

$\Rightarrow \quad x_1 = x_2$

This proves f is one-to-one.

Again, if y ∈ R, y = f(x) = ax + b

$\Rightarrow \quad x = \frac{(y - b)}{a} \in R$

and

$$f\left(\frac{(y-b)}{a}\right) = a\frac{(y-b)}{a} + b = y$$

$\therefore$ y is the image of $\frac{(y-b)}{a}$.

Thus, f is onto.

Hence f is one-to-one and onto therefore f^{-1} exists and is defined by $f^{-1}(y) = \frac{(y-b)}{a}$, is a formula defining the inverse function. Here, y is just a dummy variable and can be replaced by x, then $f^{-1}(x) = \frac{(x-b)}{a}$.

Example

Let $f: R \to R$ be defined by $f(x) = 3x - 7$. Find a formula for the inverse function of f and also sketch, in a single diagram, the graphs of $y = f(x)$ and $y = f^{-1}(x)$, making clear relationships between the two graphs.

Solution

Suppose $x_1, x_2 \in R$ and $f(x_1) = f(x_2)$. Then

$f(x_1) = f(x_2) \Rightarrow 3x_1 - 7 = 3x_2 - 7$

$\Rightarrow x_1 = x_2$

Hence f is one-to-one

Also, let $y \in R$ then $y = f(x) = 3x - 7$

$\Rightarrow \quad x = \frac{y+7}{3}$

$\therefore \quad f\left(\frac{y+7}{3}\right) = 3\frac{(y+7)}{3} - 7 = y$

Thus f is onto.

Since f is both one-to-one and onto. So f^{-1} exists and is defined by $f^{-1}(y) = \frac{y+7}{3}$

In terms of x, the inverse function is, $f^{-1}(x) = \frac{x+7}{3}$

Example

Given function $g(x) = x^2$. Find the inverse of this function and plot both functions in XY–plane.

Solution

Suppose $x_1, x_2 \in R$ such that $g(x_1) = g(x_2)$

$\Rightarrow \quad x_1^2 = x_2^2$

$\Rightarrow \quad x_1 = x_2$

So g is one-to-one.

Also. Let $y \in R$. Then $y = g(x) = x^2$

$x = \sqrt{y}$

and $g(\sqrt{y}) = (\sqrt{y})^2 = y$

$\therefore \quad$ g is onto.

Hence, g is both one-to-one and onto, g^{-1} exists and is

$g^{-1}(y) = \sqrt{y}$.

In terms of x, the inverse is $g^{-1}(x) = \sqrt{x}$

(11) Composition of function

Function Composition is an operation that takes two function f and g and produces a function h such that: ~~h(x) = g(f(x))~~

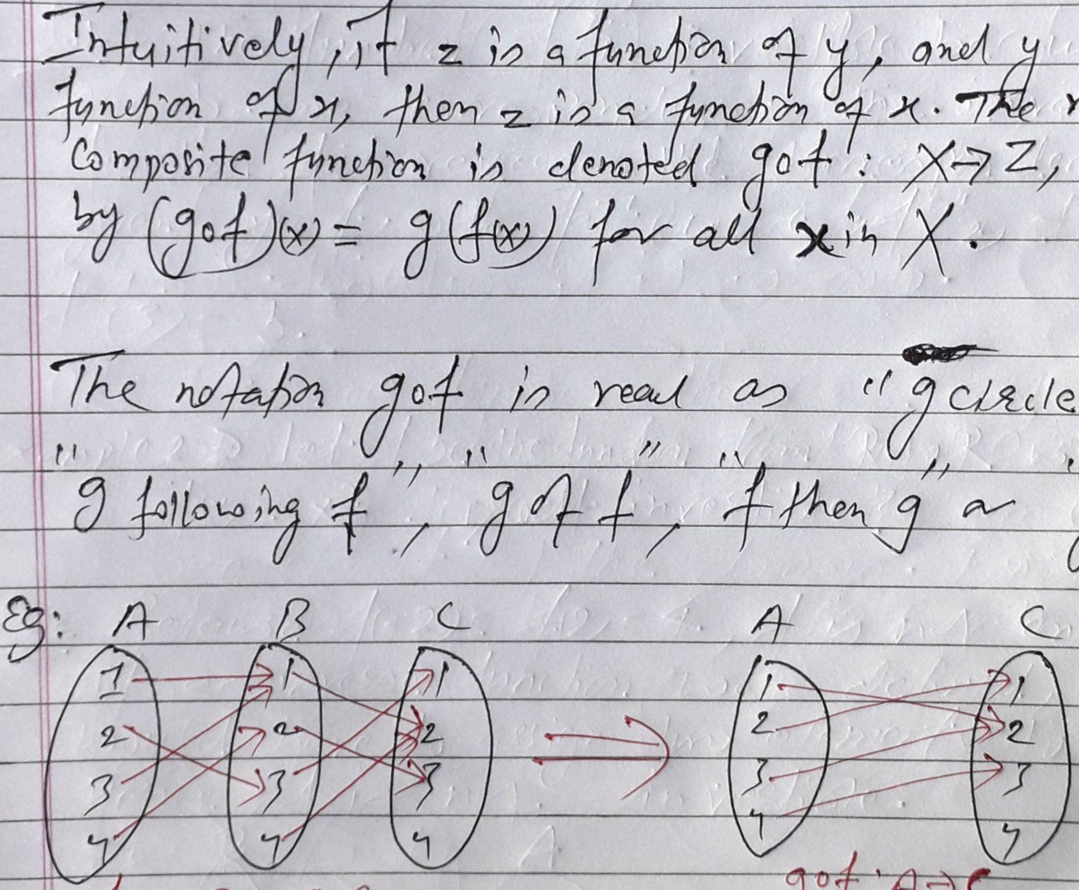
$$h(x) = g(f(x)).$$

In this operation, the function g is applied to the result of applying the function f to x. That is, the functions $f: X \to Y$ and $g: Y \to Z$ are Composed to yield a function that maps x in X to $g(f(x))$ in Z.

Intuitively, if z is a function of y, and y is a function of x, then z is a function of x. The resulting composite function is denoted $g \circ f : X \to Z$, defined by $(g \circ f)(x) = g(f(x))$ for all x in X.

The notation $g \circ f$ is read as "g circle f", "g following f", "g of f", "f then g" or "g on f".

Eg:



$f: A \to B$ $g: B \to C$ $g \circ f: A \to C$

- Composition of functions on a finite set: if $f = \{(1,1), (2,3), (3,1), (4,2)\}$, and $g = \{(1,2), (2,3), (3,1), (4,2)\}$, then $g \circ f = \{(1,2), (2,1), (3,2), (4,3)\}$, as shown in the figure above.

OR, let, $f: A \to B$ and $g: B \to C$ be two functions. The composition of 'g' and 'f' denoted by $g \circ f$ is a new function from A to C, is defined by

$$(g \circ f)(x) = g(f(x))$$

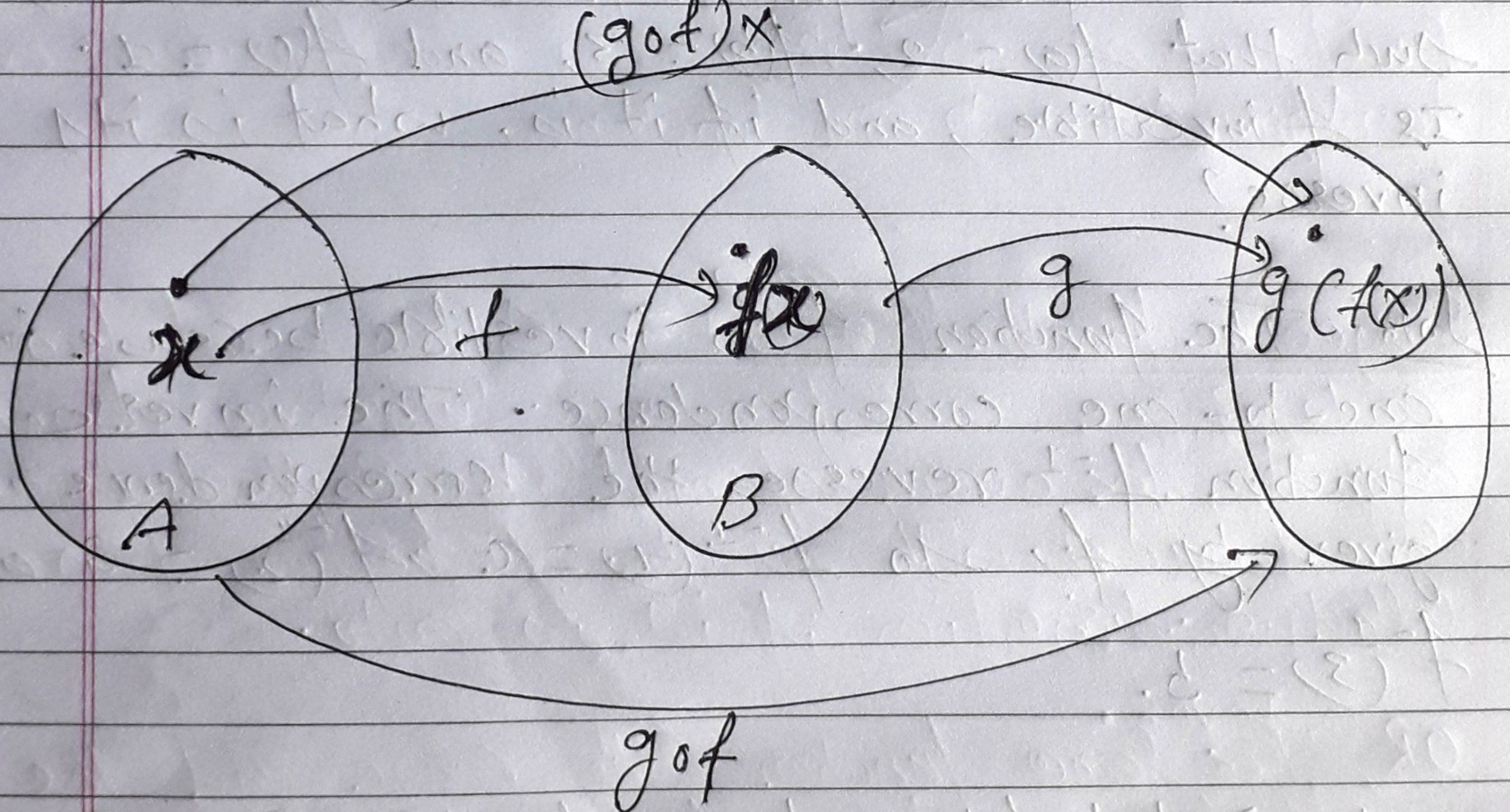


fig:- The composition of the function g and f

Example

Let A = {1, 2, 3}, B = {a, b} and C = {r, s) and $f : A \to B$ defined by f(1) = a, f(2) = a, f(3) = b and $g : B \to C$ defined by g(a) = s, g(b) = r. Find the composition function $gof : A \to C$.

Solution

Given that $f : A \to B$ and $g : B \to C$, then,

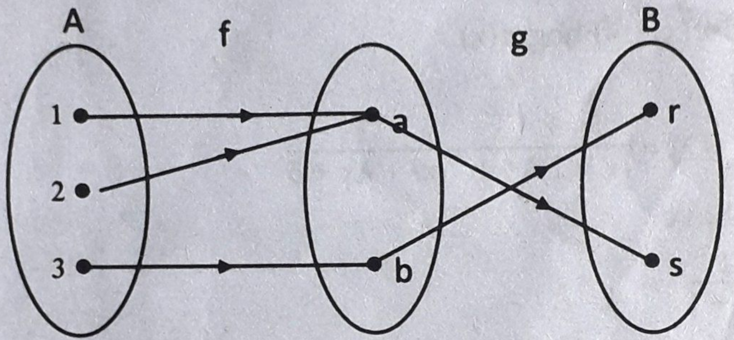


Figure:

Using the definition of composition of function,

$$gof(1) = g(f(1)) = g(a) = s$$
$$gof(2) = g(f(2)) = g(a) = s$$
and $$gof(3) = g(f(3)) = g(b) = r$$

Example

Show that the functions $f(x) = x^3$ and $g(x) = x^{1/3}$, for all $x \in R$ are inverse of one another.

Solution

Since $(fog)(x) = f(g(x)) = f(x^{1/3}) = x = I_x$

$(gof)(x) = g(f(x)) = g(x^3) = x = I_x$

Thus, $g = f^{-1}$ or $f = g^{-1}$.

Example

Let the functions f and g be defined by $f(x) = 2x + 1$ and $g(x) = x^2 - 2$. Find the formula defining the composition function gof.

Solution

We have,

$$f(x) = 2x + 1 \text{ and } g(x) = x^2 - 2$$

Therefore,

$$\begin{aligned} gof(x) &= g(f(x)) = g(2x+1) \\ &= (2x+1)^2 - 2 \\ &= 4x^2 + 4x + 1 - 2 \\ &= 4x^2 + 4x - 1 \end{aligned}$$

Example

Let V = {1, 2, 3, 4} and let f = {(1, 3), (2, 1), (3, 4), (4, 3)} and g = {(1, 2), (2, 3), (3, 1), (4, 1)}. Find: (a) fog (b) gof (c) fof.

Solution

The composite function fog starts from the function g, so we have

fog = {(1, 1), (2, 4), (3, 3), (4, 3)}.

Similarly, gof starts from the function f, so we have,

gof = {(1, 1), (2, 2), (3, 1), (4, 1)}.

and

fof = {(1, 4), (2, 3), (3, 3), (4, 4)}.

Example

Let f, g and h : R → R be defined by

$$f(x) = x + 2,\ g(x) = \frac{1}{x^2+1},\ h(x) = 3.$$

Compute (i) gof(x) (ii) $gof^{-1}of(x)$ (iii) hogof(x)

Solution

(i) $gof(x) = g(f(x)) = g(x+2) = \frac{1}{(x+2)^2+1} = \frac{1}{x^2+4x+5}$

(ii) Since $f^{-1}of(x) = x$ we have

$$gof^{-1}of(x) = g(x) = \frac{1}{x^2+1}$$

(iii) Since $gof(x) = \frac{1}{x^2+4x+5}$, then

$$hogof(x) = h\left(\frac{1}{x^2+4x+5}\right) = 3. \text{ (since } h(x) = 3, \forall x)$$

Graph a Functions

We can associate a set of pairs in A×B to each of functions from A to B. This set of pairs is called graph of functions.

Let f be a function from set A to set B then graph of function f is set of ordered pairs {(a, b): a ∈ A and f(a) = b}.

Example:

Display the graph of the function $f(x) = x^2$ from the set of integers to set of integers.

Solution

The graph of f is the set of ordered pairs of the form $(x, f(x)) = (x.x^2)$ where x is an integer. This graph is displayed in figure.

EXAMPLE 21 Let f and g be the functions from the set of integers to the set of integers defined by $f(x) = 2x + 3$ and $g(x) = 3x + 2$. What is the composition of f and g? What is the composition of g and f?

Solution: Both the compositions $f \circ g$ and $g \circ f$ are defined. Moreover,

$$(f \circ g)(x) = f(g(x)) = f(3x + 2) = 2(3x + 2) + 3 = 6x + 7$$

and

$$(g \circ f)(x) = g(f(x)) = g(2x + 3) = 3(2x + 3) + 2 = 6x + 11.$$ ◀

Remark: Note that even though $f \circ g$ and $g \circ f$ are defined for the functions f and g in Example 21, $f \circ g$ and $g \circ f$ are not equal. In other words, the commutative law does not hold for the composition of functions.

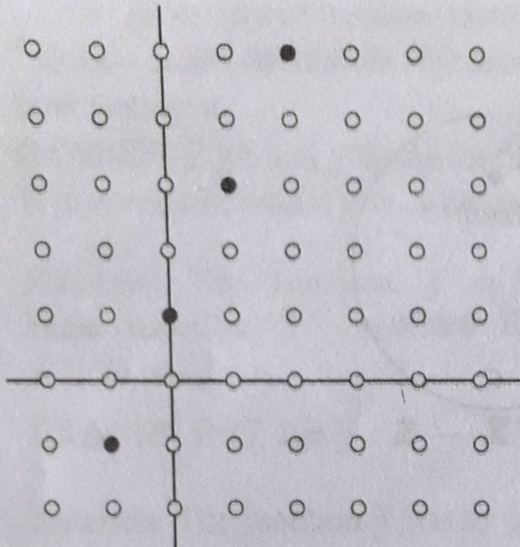


FIGURE 8 The Graph of $f(n) = 2n + 1$ from Z to Z.

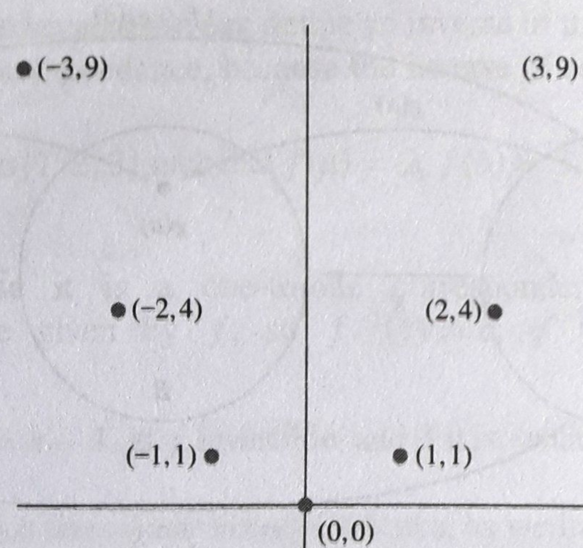


FIGURE 9 The Graph of $f(x) = x^2$ from Z to Z.

Consequently $f^{-1} \circ f = \iota_A$ and $f \circ f^{-1} = \iota_B$, where ι_A and ι_B are the identity functions on the sets A and B, respectively. That is, $(f^{-1})^{-1} = f$.

The Graphs of Functions

We can associate a set of pairs in $A \times B$ to each function from A to B. This set of pairs is called the **graph** of the function and is often displayed pictorially to aid in understanding the behavior of the function.

DEFINITION 11 Let f be a function from the set A to the set B. The *graph* of the function f is the set of ordered pairs $\{(a, b) \mid a \in A \text{ and } f(a) = b\}$.

From the definition, the graph of a function f from A to B is the subset of $A \times B$ containing the ordered pairs with the second entry equal to the element of B assigned by f to the first entry.

EXAMPLE 22 Display the graph of the function $f(n) = 2n + 1$ from the set of integers to the set of integers.

Solution: The graph of f is the set of ordered pairs of the form $(n, 2n + 1)$, where n is an integer. This graph is displayed in Figure 8. ◀

EXAMPLE 23 Display the graph of the function $f(x) = x^2$ from the set of integers to the set of integers.

Solution: The graph of f is the set of ordered pairs of the form $(x, f(x)) = (x, x^2)$, where x is an integer. This graph is displayed in Figure 9. ◀

Functions for Computer Science

(1) **Floor Function**:- Let x be a real number. The floor function rounds x down to the closest integer less than or equal to x. It is denoted by $\lfloor x \rfloor$.

(11) **Ceiling Function**:- let x be a real number. The ceiling function rounds x up to the closest integer greater than or equal to x. It is denoted by $\lceil x \rceil$

Note: these functions are often used when objects are counted. They play an important role in the analysis of the number of steps used by procedures to solve problems of a particular size.

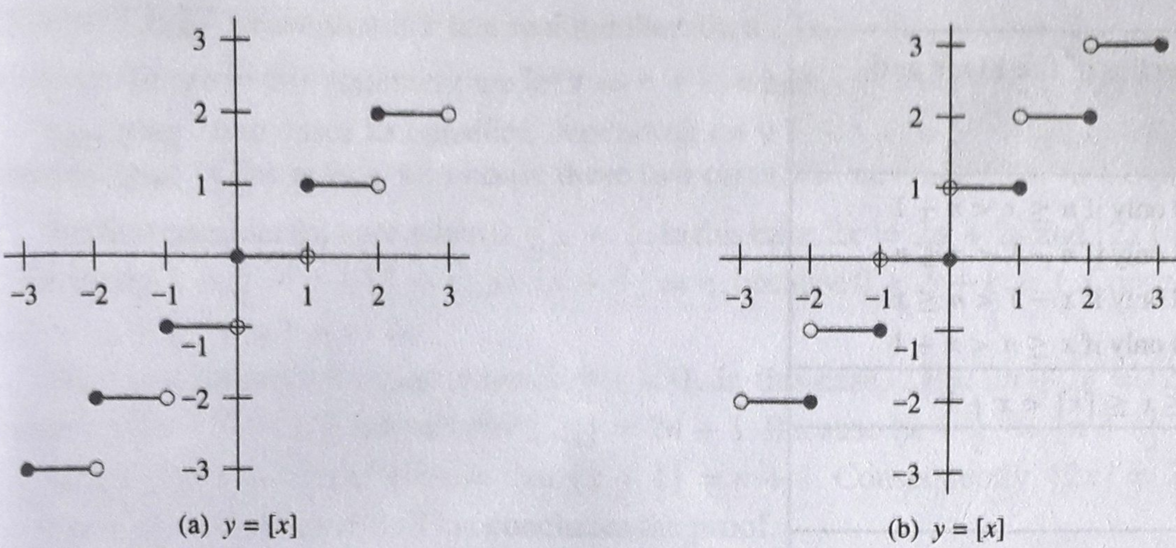


FIGURE 10 Graphs of the (a) Floor and (b) Ceiling Functions.

used when objects are counted. They play an important role in the analysis of the number of steps used by procedures to solve problems of a particular size.

DEFINITION 12 The *floor function* assigns to the real number x the largest integer that is less than or equal to x. The value of the floor function at x is denoted by $\lfloor x \rfloor$. The *ceiling function* assigns to the real number x the smallest integer that is greater than or equal to x. The value of the ceiling function at x is denoted by $\lceil x \rceil$.

Remark: The floor function is often also called the *greatest integer function*. It is often denoted by $[x]$.

EXAMPLE 21 These are some values of the floor and ceiling functions:

$$\lfloor \tfrac{1}{2} \rfloor = 0, \lceil \tfrac{1}{2} \rceil = 1, \lfloor -\tfrac{1}{2} \rfloor = -1, \lceil -\tfrac{1}{2} \rceil = 0, \lfloor 3.1 \rfloor = 3, \lceil 3.1 \rceil = 4, \lfloor 7 \rfloor = 7, \lceil 7 \rceil = 7.$$ ◀

Links

We display the graphs of the floor and ceiling functions in Figure 10. In Figure 10(a) we display the graph of the floor function $\lfloor x \rfloor$. Note that this function has the same value throughout the interval $[n, n + 1)$, namely n, and then it jumps up to $n + 1$ when $x = n + 1$. In Figure 10(b) we display the graph of the ceiling function $\lceil x \rceil$. Note that this function has the same value throughout the interval $(n, n + 1]$, namely $n + 1$, and then jumps to $n + 2$ when x is a little larger than $n + 1$.

The floor and ceiling functions are useful in a wide variety of applications, including those involving data storage and data transmission. Consider Examples 25 and 26, typical of basic calculations done when database and data communications problems are studied.

EXAMPLE 25 Data stored on a computer disk or transmitted over a data network are usually represented as a string of bytes. Each byte is made up of 8 bits. How many bytes are required to encode 100 bits of data?

Solution: To determine the number of bytes needed, we determine the smallest integer that is at least as large as the quotient when 100 is divided by 8, the number of bits in a byte. Consequently, $\lceil 100/8 \rceil = \lceil 12.5 \rceil = 13$ bytes are required. ◀