CHAPTER - 9
PLANE AND SPACE VECTOR.

Find a unit vector that has the same direction as the given vector.

(a) $-3\hat{i}+7\hat{j}$

Soln: Here,

let $\vec{v} = -3\hat{i}+7\hat{j}$

$|\vec{v}| = \sqrt{(-3)^2+7^2} = \sqrt{9+49} = \sqrt{58}$

Thus,

$$\hat{v} = \frac{\vec{v}}{|\vec{v}|}$$

$$\hat{v} = \frac{-3\hat{i}+7\vec{j}}{\sqrt{58}}$$

$$\hat{v} = \frac{-3}{\sqrt{58}}\hat{i} + \frac{7}{\sqrt{58}}\vec{j}.$$ Ans.

(b) $(-4,2,4)$

Soln: Here,

$\vec{v} = -4\vec{i}+2\vec{j}+4\vec{k}$

Magnitude of vector $\vec{v}$ is

$|\vec{v}| = \sqrt{-4^2+2^2+4^2}$

$|\vec{v}| = \sqrt{16+4+16} = \sqrt{36} = 6$

then the unit ~~value~~ vector of $\vec{v}$ is,

$$\hat{v} = \frac{\vec{v}}{|\vec{v}|} = \frac{-4\vec{i}+2\vec{j}+4\vec{k}}{6} = -\frac{2}{3}\vec{i}+\frac{1}{3}\vec{j}+\frac{2}{3}\vec{k}$$

Ans

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(c) $8\vec{i} - \vec{j} + 4\vec{k}$

Sol^n: Here,

let $\vec{AB} = \vec{a} = 8\vec{i} - \vec{j} + 4\vec{k}$

Magnitude of vector $\vec{a}$ is,

$|\vec{a}| = \sqrt{8^2+1^2+4^2} = \sqrt{64+1+16} = \sqrt{81} = 9$

Unit vector along $\vec{a}$ is,

$$\hat{a} = \frac{\vec{a}}{|\vec{a}|}$$

$$\hat{a} = \frac{8\vec{i} - \vec{j} + 4\vec{k}}{9}$$

$$\hat{a} = \frac{8}{9}\vec{i} - \frac{1}{9}\vec{j} + \frac{4}{9}\vec{k}$$ Ans

(2) Find the angle between the vectors.

Sol^n: Here,

(a) $\vec{a} = (3,-1,5)$ and $\vec{b} = (-2,4,3)$

Sol^n: Here

~~$\vec{ab} = \vec{ob} - \vec{oa}$~~

~~$\vec{ab} = (-2-3, 4+1, 3-5)$~~

~~$\vec{ab} = (-5, 5, -2)$~~

2×2^2 $3\times3\times3$

~~The magnitude of vector $\vec{ab}$ is,~~

~~$|\vec{ab}| = \sqrt{5^2+5^2+2^2} = \sqrt{54} = 3\sqrt{6}$~~

~~Unit vector along $\vec{ab}$ is~~

~~$\hat{ab} = \frac{\vec{ab}}{|\vec{ab}|} = \frac{-5\vec{i}+5\vec{j}-2\vec{k}}{3\sqrt{6}}$~~

~~then,~~

~~$\hat{ab} = \frac{-5}{3\sqrt{6}}\vec{i} + \frac{5}{3\sqrt{6}}\vec{j} - \frac{2\vec{k}}{3\sqrt{6}}$~~

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~~where~~

~~$\cos\alpha = \frac{-5}{3\sqrt{6}}$~~

~~$\alpha = \cos^{-1}\left(\frac{-5}{3\sqrt{6}}\right)$~~

(a) $\vec{a} = (3,-1,5)$ and $\vec{b} = (-2,4,3)$

Sol^n: Here,

we know that $\cos\theta = \left(\frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}\right)$

so,

$\vec{a}\cdot\vec{b} = (3,-1,5)\cdot(-2,4,3)$

$= -6-4+15$

$= 5$

$|\vec{a}| = \sqrt{3^2+1^2+5^2} = \sqrt{9+25+1} = \sqrt{35}$

$|\vec{b}| = \sqrt{2^2+4^2+3^2} = \sqrt{4+16+9} = \sqrt{29}$

Now

$\cos\theta = \left(\frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}\right)$

$\cos\theta = \frac{5}{\sqrt{35}\times\sqrt{29}}$

$\theta = \cos^{-1}\left(\frac{5}{\sqrt{35\times29}}\right)$

$\theta = ~~80.98~~ \; 80.98°$

$\theta = 81°$

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2 (b) $a = (4,0,2)$, $b = (2,-1,0)$

Sol^n: Here,

$a = (4,0,2)$

$b = (2,-1,0)$

we know that,

$$\cos\theta = \left(\frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}\right)$$

then,

$\vec{a}\cdot\vec{b} = (8-0+0) = 8$

$|\vec{a}| = \sqrt{4^2+0^2+2^2} = \sqrt{20}$

$|\vec{b}| = \sqrt{2^2+1^2+0^2} = \sqrt{5}$

$$\cos\theta = \left(\frac{8}{\sqrt{20}\times\sqrt{5}}\right)$$

$$\cos\theta = \left(\frac{8}{10}\right)$$

$\theta = \cos^{-1}(8/10)$

$\theta = 37^\circ$ Ans

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(c) ~~$a = 4i - 3j, b =$~~ $a = 4\vec{i} - 3\vec{j} + \vec{k}$, $b = 2i - k$

Sol^n: Here,

$\vec{a} = 4\vec{i} - 3\vec{j} + \vec{k} = (4,-3,1)$

$\vec{b} = 2\hat{i} + 0\vec{j} - \vec{k} = (2,0,-1)$

$\vec{a}\cdot\vec{b} = 8 - 1 = 7$

$|\vec{a}| = \sqrt{4^2+3^2+1} = \sqrt{16+9+1} = \sqrt{26}$

$|\vec{b}| = \sqrt{0^2+2^2+1^2} = \sqrt{5}$

we know that,

$$\cos\theta = \left(\frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}\right)$$

$$\cos\theta = \left(\frac{7}{\sqrt{26}\times\sqrt{5}}\right)$$

$$\theta = \cos^{-1}\left(\frac{7}{\sqrt{26\times5}}\right)$$

$\theta = 52^\circ$ Ans

(d) $a = \vec{i} + 2\vec{j} - 2\vec{k}$, $b = 4\hat{i} - 0\vec{j} - 3\vec{k}$

Sol^n: Here

$\vec{a} = \vec{i} + 2\vec{j} - 2\vec{k} = (1,2,-2)$

$\vec{b} = 4\hat{i} - 0\vec{j} - 3\vec{k} = (4,0,-3)$

$\vec{a}\cdot\vec{b} = (4+0+6) = 10$

$|\vec{a}| = \sqrt{1+4+4} = 3$

$|\vec{b}| = \sqrt{16+9} = 5$

we know that

$$\theta = \cos^{-1}\left(\frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}\right) = \cos^{-1}\left(\frac{10}{15}\right)$$

$\theta = 48^\circ$ Ans

③ Determine whether the given vectors are orthogonal parallel or neither.

ⓐ $a = (-5, 3, 7)$, $b = (6, -8, 2)$

Sol^n: Here,

$\vec{a} \cdot \vec{b} = -30 - 24 + 14$

$\vec{a} \cdot \vec{b} = -40$

$|\vec{a}| = \sqrt{25+9+49} = \sqrt{83}$

$|\vec{b}| = \sqrt{36+64+4} = \sqrt{104}$

$\cos\theta = \left(\dfrac{\vec{a} \cdot \vec{b}}{|\vec{a}| \cdot |\vec{b}|}\right)$

$\cos\theta = \left(\dfrac{-40}{\sqrt{83} \cdot \sqrt{104}}\right)$

The given vector are neither orthogonal nor parallel.

ⓑ $a = (4, 6)$ $b = (-3, 2)$

Sol^n: Here,

$\vec{a} = (4, 6)$

$\vec{b} = (-3, 2)$

$\vec{a} \cdot \vec{b} = -12 + 12 = 0$, which is orthogonal

ⓒ $a = (-1, 2, 5)$ and $b = (3, 4, -1)$

Sol^n: Here

$\vec{a} = (-1, 2, 5)$ & $\vec{b} = (3, 4, -1)$

$\vec{a} \cdot \vec{b} = -3 + 8 - 5 = 0$ which is orthogonal

ⓓ $a = 2\vec{i} + 6\vec{j} - 4\hat{k}$, $b = -3\vec{i} - 9\vec{j} + 6k$

Sol^n: Here,

$\vec{a} = (2, 6, -4)$

$\vec{b} = (-3, -9, 6)$

$\vec{a} \cdot \vec{b} = (-6 - 54 - 24)$

$= (-104)$

$|\vec{a}| = \sqrt{4+36+16} = \sqrt{56}$

$|\vec{b}| = \sqrt{9+81+36} = \sqrt{126}$

$\theta = \cos^{-1}\left(\dfrac{\vec{a} \cdot \vec{b}}{|\vec{a}| \cdot |\vec{b}|}\right)$

$\theta = \cos^{-1}\left(\dfrac{-104}{\sqrt{56} \times \sqrt{126}}\right)$

④ Find the direction Cosines and direction angles of the vector.

ⓐ $(2, 1, 2)$

Sol^n: Here,

let $\vec{a} = (2, 1, 2) = 2\vec{i} + \vec{j} + 2\vec{k}$

$|\vec{a}| = \sqrt{4+1+4} = 3$,

we know that ~~the~~

$\hat{a} = \dfrac{2}{3}\vec{i} + \dfrac{1}{3}\vec{j} + \dfrac{2}{3}\vec{k}$

and the direction cosine of the given vector

$\cos\alpha = 2/3$ and $\cos\beta = 1/3$ and $\cos\gamma = 2$

The direction angles of the given vector is

$\alpha = 48$, $\beta = 79°$ and $\gamma = 48°$

(b) $(6,3,-2)$

Sol^n: Here

let $\vec{a} = (6,3,-2)$

the magnitude of $|\vec{a}|$ is

$|\vec{a}| = \sqrt{36+9+4} = \sqrt{49} = 7$

Now

The direction cosine of the given vector is

$\cos\alpha = \frac{6}{7}$, $\cos\beta = \frac{3}{7}$ and $\cos\gamma = -\frac{2}{7}$

The direction angles of the given vector is,

$\alpha = 31^\circ$, $\beta = 65^\circ$ and $\gamma = 107^\circ$ Ans

(c) $\vec{i} - 2\vec{j} - 3\vec{k}$

Sol^n: Here

let $\vec{a} = (1,-2,-3)$

Magnitude of $\vec{a}$ is

$|\vec{a}| = \sqrt{1+4+9} = \sqrt{14}$

The direction cosines of given vector is

$\cos\alpha = \frac{1}{\sqrt{14}}$ and $\cos\beta = \frac{-2}{\sqrt{14}}$ and $\cos\gamma = \frac{-3}{\sqrt{14}}$

The direction angle of given vectors

$\alpha = 75^\circ$, $\beta = 122^\circ$ $\gamma = 143^\circ$

(d) $\frac{1}{2}\vec{i} + \vec{j} + \vec{k}$

Sol^n: Here,

$\vec{a} = \frac{1}{2}\vec{i} + \vec{j} + \vec{k} = (\frac{1}{2}, 1, 1)$

The magnitude of vector $\vec{a}$ is

$|\vec{a}| = \sqrt{\frac{1}{4}+1+1} = \sqrt{\frac{1}{4}+2} = \sqrt{\frac{9}{4}} = \frac{3}{2}$

The direction cosines of ~~ang~~ of the given vector is.

$\cos\alpha = \frac{1}{2} \times \frac{2}{3} = \frac{1}{3}$

$\cos\beta = \frac{1}{\frac{3}{2}} = \frac{2}{3}$

$\cos\gamma = \frac{2}{3}$

Therefore the direction angle of given vector is

$\alpha = 70.5$ and $\beta = 48.2^\circ$ and $\gamma = 48.$

(e) (c,c,c), where $c>0$

Sol^n: Here

The magnitude of given vector is,

$|\vec{a}| = \sqrt{c^2+c^2+c^2} = \sqrt{3c^2} = c\sqrt{3}$

The direction cosines of given vector is,

$\cos\alpha = \frac{c}{c\sqrt{3}} = \frac{1}{\sqrt{3}}$

$\cos\beta = \frac{1}{\sqrt{3}}$

and $\cos\gamma = \frac{1}{\sqrt{3}}$

therefore the direction angle of given v is

$\alpha = 55^\circ$, $\beta = 55^\circ$ & $\gamma = 55^\circ$ Ans

3 + 12 + 6

(5) Find the scalar and vector projection of b onto a.

Sol: Here,

Dot product or Scalar product:-

let $\vec{a} = (a_1, a_2)$ and $\vec{b} = (b_1, b_2)$ be any two vectors, then the scalar product of two vector $\vec{a}$ and $\vec{b}$ is denoted by $\vec{a}\cdot\vec{b}$ and defined by:

$$\vec{a}\cdot\vec{b} = (a_1, a_2)\cdot(b_1, b_2)$$
$$= a_1b_1 + a_2b_2$$

Alternative Definition:-

The dot product of $\vec{a}$ and $\vec{b}$ is defined by $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$ where θ is the angle between two vector $\vec{a}$ and $\vec{b}$.

Geometric Interpretation:-

let $\vec{OA} = \vec{a}$ and $\vec{OB} = \vec{b}$ and $\angle AOB = \theta$,

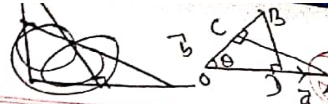
then $|\vec{OA}| = |\vec{a}| = a$

$|\vec{OB}| = |\vec{b}| = b$.

Draw AC $\perp^r$ OB and BD $\perp^r$ OA.

Now:

$$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$$
$$= ab\cos\theta$$
$$= (OA)(OB)\cos\theta$$
$$= (OA)\cdot(OD)$$
$$= (\text{Magnitude of } \vec{a})(\text{projection of } \vec{b} \text{ on } \vec{a})$$



Similarly,

$\vec{a}\cdot\vec{b} = (\text{magnitude } \vec{b})(\text{projection of } \vec{a} \text{ on } \vec{b})$

Orthogonal.

Two vector $\vec{a}$ and $\vec{b}$ are orthogonal if $\vec{a}\cdot\vec{b} = 0$

Direction angle and direction cosine:-

The α, β, γ made by non-zero vector with positive coordinate axes x-, y-, and z-axis are called direction angle, the value $\cos\alpha$, $\cos\beta$, and $\cos\gamma$ are called direction cosine of the vector.

let $\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$

$$|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

unit vector of $\hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{a_1\vec{i} + a_2\vec{j} + a_3\vec{k}}{\sqrt{a_1^2 + a_2^2 + a_3^2}}$

$$= \left(\frac{a_1}{\sqrt{a_1^2 + a_2^2 + a_3^2}}\right)\vec{i} + \left(\frac{a_2}{\sqrt{a_1^2 + a_2^2 + a_3^2}}\right)\vec{j} + \frac{a_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}}\vec{k}.$$

and $\vec{a}\cdot\vec{i} = |\vec{a}||\vec{i}|\cos\alpha$

$$\cos\alpha = \frac{\vec{a}\cdot\vec{i}}{|\vec{a}||\vec{i}|} = \frac{a_1}{|\vec{a}|}$$

$$\cos\beta = \frac{\vec{a}\cdot\vec{j}}{|\vec{a}||\vec{j}|} = \frac{a_2}{|\vec{a}|}$$

$$\cos\gamma = \frac{a_3}{|\vec{a}|}$$

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projection:

① Scalar ~~product~~ projection of $\vec{b}$ on $\vec{a}$: $\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}$

② vector projection of $\vec{b}$ on $\vec{a}$: $\left(\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}\right)\frac{\vec{a}}{|\vec{a}|}$

$$= \left(\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\right)\vec{a}.$$

Example.

prove that
$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1.$

proof

let $\vec{a} = a_1\vec{i} + a_2\vec{j}, a_3\vec{k}$.

then

$\cos\alpha = \frac{a_1}{|\vec{a}|} = \frac{a_1}{\sqrt{a_1^2+a_2^2+a_3^2}}$

$\cos\beta = \frac{a_2}{\sqrt{a_1^2+a_2^2+a_3^2}}$

$\cos\gamma = \frac{a_3}{\sqrt{a_1^2+a_2^2+a_3^2}}$

we have,

$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \frac{a_1^2}{a_1^2+a_2^2+a_3^2} + \frac{a_2^2}{a_1^2+a_2^2+a_3^2} + \frac{a_3^2}{a_1^2+a_2^2+a_3^2}$

$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \frac{\cancel{a_1^2+a_2^2+a_3^2}}{\cancel{a_1^2+a_2^2+a_3^2}} = 1.$

proved

$\begin{vmatrix} 9 & 10 & 9 \\ 5 & 1 & 4 \\ 2 & 4 & 1 \end{vmatrix}$

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cross product:-

let $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$ be any two vector, then the cross product of $\vec{a}$ and $\vec{b}$ is denoted by $\vec{a}\times\vec{b}$ and defined by

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$\vec{a}\times\vec{b} = (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1)$

In determinant form.

$$\vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

① Show that $\vec{a}\times\vec{a} = 0$.

Sol: $\vec{a}\times\vec{a} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 \end{vmatrix}$

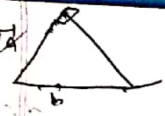
$R_2 = R_3$

② show that = 0

$\vec{a}\times\vec{b} = -\vec{b}\times\vec{a}$ (not Commutative)

$\vec{b}\times\vec{a} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \end{vmatrix}$ $R_2 \leftrightarrow R_3$

$$= -\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = -\vec{b}\times\vec{a}.$$

Area $= \frac{1}{2} b d$ 

$(\vec{a}\times\vec{b})\cdot\vec{a}$

$\hat{i}\times\hat{j} = \hat{k}$
$\hat{j}\times\hat{k} = \hat{i}$
$\hat{k}\times\hat{i} = \hat{j}$

③ The vector $\vec{a}\times\vec{b}$ is orthogonal to both vector $\vec{a}$ and $\vec{b}$:

Solⁿ: Here

$(\vec{a}\times\vec{b})\cdot\vec{a} = (a_2b_3 - a_3b_2,\ a_3b_1 - a_1b_3,\ a_1b_2 - a_2b_1)\cdot(a_1, a_2, a_3)$

$a_1a_2b_3 - a_1a_3b_2 + a_2a_3b_1 - a_1a_2b_3 + a_1a_3b_2 - a_2a_3b_1$

0

$\therefore \vec{a}\times\vec{b}$ is ~~oth~~ orthogonal to $\vec{a}$.

Unit vector $\hat{i}, \hat{j}, \hat{k}$

$\hat{i} = (1,0,0)$, $\hat{j} = (0,1,0)$, $\hat{k} = (0,0,1)$.

$\hat{i}\times\hat{j} = (1,0,0)\times(0,1,0)$
$= (0-0, 0-0, 1-0)$
$= (0,0,1)$.
$\hat{k}$.

$\hat{j}\times\hat{k} = (0,1,0)\times(0,0,1)$
$= (1-0, 0-0, 0-0)$
$= (1,0,0)$
$\hat{j}\times\hat{k} = \hat{i}$.

$\hat{k}\times\hat{i} = (0,0,1)\times(1,0,0)$
$= (0-0, 1-0, 0-0)$
$= (0,1,0)$
$\hat{k}\times\hat{i} = \hat{j}$

$\hat{j}\times\hat{i} = -\hat{k}$
$\hat{k}\times\hat{j} = -\hat{i}$
$\hat{i}\times\hat{k} = -\hat{j}$

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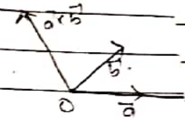
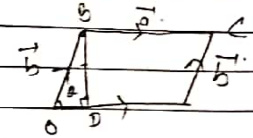
Alternative definition of Cross product:-
The cross product or vector product of two non-zero vector $\vec{a}$ and $\vec{b}$ is defined by
$\vec{a}\times\vec{b} = |\vec{a}||\vec{b}|\sin\theta\,\hat{n}$.
where $\hat{n}$ is the unit vector in the direction of $\vec{a}\times\vec{b}$ and θ be the angle between $\vec{a}$ and $\vec{b}$



Geometrical Interpretation:-
let $|\vec{OA}| = |\vec{a}| = a$
$|\vec{OB}| = |\vec{b}| = b$



$\angle AOB = \theta$.

Draw a parallelogram OACB, and draw a parallelogram OACB and draw BD ⊥r to OA.

Area of parallelogram = base × height.
$= OA \times BD$.
$= (OA)(OB\sin\theta)$
$= ab\sin\theta$

Also $\vec{a}\times\vec{b} = |\vec{a}||\vec{b}|\sin\theta\,\hat{n}$.
$|\vec{a}\times\vec{b}| = ab\sin\theta$.
Area of parallelogram $= |\vec{a}\times\vec{b}|$

Area of triangle $= \frac{1}{2}|\vec{a}\times\vec{b}|$

Scalar Triple product.

The Scalar triple product of three vectors $\vec{a}, \vec{b}$ & $\vec{c}$ is denoted by $[\vec{a}, \vec{b}, \vec{c}]$ or $\vec{a}.(\vec{b}+\vec{c})$ and defined by.

$$[\vec{a}, \vec{b}, \vec{c}] = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$$

The magnitude of $[\vec{a}\,\vec{b}\,\vec{c}]$ is the vol^m of parallelopiped determined by the vectors $\vec{a}, \vec{b}, \vec{c}$.

Vector triple product:

The vector triple product of three vector $\vec{a}, \vec{b}, \vec{c}$ is denoted by $\vec{a}\times(\vec{b}\times\vec{c})$. And defined by $\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}.\vec{c}).\vec{b} - (\vec{a}.\vec{b})\vec{c}$.

(5) Find the Scalar and vector projection of b onto a

Sol^n: Here,

$a = (3,6,-2)$, $b = (1,2,3)$

Sol^n: Here

$\vec{a}.\vec{b} =$ ~~pro~~ magnitude of $|a|$. (projection of ($\vec{b}$ on $\vec{a}$))

projection of ($\vec{b}$ on $\vec{a}$) $= \frac{\vec{a}.\vec{b}}{|\vec{a}|}$

$= \frac{(3,6,-2).(1,2,3)}{\sqrt{3^2+6^2+2^2}}$

$= \frac{3+12-6}{\sqrt{9+36+4}} = \frac{9}{7}$ Ans

(ii) vector projection of $\vec{b}$ onto a $- \frac{(\vec{a}.\vec{b})}{|\vec{a}|}.\frac{\vec{a}}{|\vec{a}|}$

$= \frac{(\vec{a}.\vec{b}).\vec{a}}{(|\vec{a}|)^2}$

$= \left(\frac{(3,6,-2).(1,2,3)}{(\sqrt{3^2+6^2+2^2})^2}\right).(\vec{a})$

$= \left(\frac{3+12-6}{(\sqrt{49})^2}\right).(3,6,-2)$

$= \frac{9(3\vec{i}+6\vec{j}-2\vec{k})}{49}$

$= \frac{27\vec{i}}{49} + \frac{54\vec{j}}{49} - \frac{18\vec{k}}{49}$

(b) $a = (-2,3,-6)$, $b = (5,-1,4)$

Sol^n: Here,

scalar projection of b on $\vec{a}$ $= \frac{(\vec{a}.\vec{b})}{|\vec{a}|} = \frac{(-2,3,-6).(5,-1,4)}{\sqrt{4+9+36}}$

$= \frac{-10-3-24}{\sqrt{49}}$

$= \frac{-37}{7}$ Ans

Similarly

vector projection of b on $\vec{a}$ $= \frac{(\vec{a}.\vec{b})}{|\vec{a}|^2}\times$

$= \frac{-37}{49}(-2,3,-6) = \frac{-37}{49}(-2\vec{i}+3\vec{j}-6\vec{k})$

$= \frac{74\vec{i}}{49} - \frac{111\vec{j}}{49} + \frac{222\vec{k}}{49}$ Ans

(c) $a = 2\vec{i} - \vec{j} + 4\vec{k}$, $b = 0\vec{i} + \vec{j} + \frac{1}{2}\vec{k}$

Solⁿ: Here,

$a = (2,-1,4)$ and $b = (0,1,\frac{1}{2})$

scalar projection of $(\vec{b}$ on $\vec{a})$ is $= \dfrac{(\vec{a}\cdot\vec{b})}{|\vec{a}|}$

$$= \frac{(2,-1,4)\cdot(0,1,\frac{1}{2})}{\sqrt{2^2+1^2+4^2}}$$

$$= \frac{(0-1+2)}{\sqrt{4+1+16}}$$

$$= \frac{1}{\sqrt{21}}$$

$$= \frac{\sqrt{21}}{21}$$

Now

vector projection of $(\vec{b}$ on $\vec{a}) = \dfrac{(\vec{a}\cdot\vec{b})}{|\vec{a}|^2}\cdot\vec{a}$

$$= \frac{1}{(\sqrt{21})^2}\cdot(2,-1,4)$$

$$= \frac{1}{21}(2,-1,4)$$

$$= \left(\frac{2}{21}, \frac{-1}{21}, \frac{4}{21}\right)$$

Ans.

(d) $a = \vec{i} + \vec{j} + \vec{k}$ & $b = \vec{i} - \vec{j} + \vec{k}$

Solⁿ: Here,

$\vec{a} = (1,1,1)$ & $\vec{b} = (1,-1,1)$

scalar projection of $\vec{b}$ on $\vec{a} = \dfrac{(\vec{a}\cdot\vec{b})}{|\vec{a}|}$

$$= \frac{(1,1,1)\cdot(1,-1,1)}{\sqrt{1^2+1^2+1^2}}$$

$$= \frac{1-1+1}{\sqrt{3}}$$

$$= \frac{1}{\sqrt{3}}$$

Now

vector projection of $\vec{b}$ on $\vec{a} = \dfrac{(\vec{a}\cdot\vec{b})}{(|\vec{a}|)^2}\cdot\vec{a}$

$$= \frac{1}{3}(1,1,1)$$

$$= \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

⑥ Find two unit vectors Orthogonal to both $(3,2,1)$ and $(-1,1,0)$

Solⁿ: Here,
unit vector orthogonal to both given Vector $(3,2,1)$ and $(-1,1,0)$ is,
let (i,j,k) be any ~~give~~ vector,
then,

$$\vec{a}\times\vec{b} = \begin{vmatrix} i & j & k \\ 3 & 2 & 1 \\ -1 & 1 & 0 \end{vmatrix}$$

$$\vec{a}\times\vec{b} = i\left(2\times0-1\times1\right) - j\left(3\times0+1\times1\right) + k\left(3+2\right)$$

$$\vec{a}\times\vec{b} = i(-1) - j + 5k$$

$$\vec{a}\times\vec{b} = -\vec{i}-\vec{j}+5\vec{k}$$

$$|\vec{a}\times\vec{b}| = \sqrt{1^2+1^2+5^2} = \sqrt{27} = 3\sqrt{3}$$

unit vector orthogonal to both given vector is, $\left(\frac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}\right)$ i.e

$$\left(\frac{-1}{3\sqrt{3}}, \frac{-1}{3\sqrt{3}}, \frac{5}{3\sqrt{3}}\right)$$

Again

$$\vec{a}\times\vec{b} = \begin{vmatrix} i & j & k \\ -1 & 1 & 0 \\ 3 & 2 & 1 \end{vmatrix}$$

$$\vec{a}\times\vec{b} = i[1-0] + j(-1-0) + k(-2-3)$$

$$= \vec{i}+\vec{j}-5\vec{k}$$

$$|\vec{a}\times\vec{b}| = \sqrt{1^2+1^2+5^2} = \sqrt{27} = 3\sqrt{3}$$

unit vector along $\vec{a}\times\vec{b}$ is,

$$\left(\frac{1}{3\sqrt{3}}, \frac{1}{3\sqrt{3}}, \frac{-5}{3\sqrt{3}}\right)$$

Ans

⑦ Find two unit vectors Orthogonal to both $j-k$ and $i+j$.

Solⁿ: Here,
the given two vectors are
$\vec{a} = (0,1,-1)$ & $\vec{b} = (1,1,0)$

$$\vec{a}\times\vec{b} = \begin{vmatrix} i & j & k \\ 0 & 1 & -1 \\ 1 & 1 & 0 \end{vmatrix}$$

$$= i(0+1) - j(0+1) + k(0-1)$$

$$= \vec{i}-\vec{j}-\vec{k}$$

$$|\vec{a}\times\vec{b}| = \sqrt{1^2+1^2+1^2} = \sqrt{3}$$

unit vector along orthogonal to $\vec{a}\times\vec{b}$ is,

$$\left(\frac{1}{\sqrt{3}}\vec{i}, -\frac{1}{\sqrt{3}}\vec{j}, -\frac{1}{\sqrt{3}}\vec{k}\right) \text{ Ans}$$

$$\left(\frac{\sqrt{3}}{3}\vec{i} - \frac{\sqrt{3}}{3}\vec{j} - \frac{\sqrt{3}}{3}\vec{k}\right)$$

Again
$\vec{a}\times\vec{b} = \begin{vmatrix} i & j & k \\ 1 & 1 & 0 \\ 0 & 1 & -1 \end{vmatrix}$

$\vec{a}\times\vec{b} = \hat{i}(-1-0) - \vec{j}(-1-0) + \vec{k}(1-0)$
$= -1\vec{i} + \vec{j} + \vec{k}$

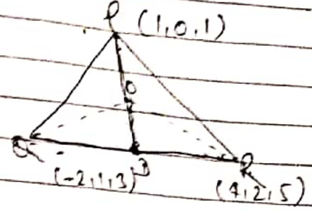
$|\vec{a}\times\vec{b}| = \sqrt{1^2+1^2+1^2} = \sqrt{3}$

Unit vector orthogonal to $(\vec{a}\times\vec{b})$ is,
$\frac{-1}{\sqrt{3}}\hat{i} + \frac{1}{\sqrt{3}}\vec{j} + \frac{1}{\sqrt{3}}\vec{k}$

$\therefore \left(-\frac{\sqrt{3}}{\sqrt{3}}\vec{i} + \frac{\sqrt{3}}{3}\vec{j} + \frac{\sqrt{3}}{3}\vec{k}\right)$ Ans

(3) Find a nonzero vector orthogonal to the plane through the points P, Q, and R and also find the area of the triangle PQR.

(a) $P(1,0,1)$, $Q(-2,1,3)$, $R(4,2,5)$



Soln: Here,
$\vec{PQ} = \vec{OQ} - \vec{OP}$

$\vec{PQ} = (-2,1,3) - (1,0,1)$
$= (-3,1,2)$

$\vec{PR} = (\vec{OR}) - (\vec{OP})$
$= (4,2,5) - (1,0,1)$
$= (3,2,4)$

The non-zero vector orthogonal to the plane through the points P, Q and R is,

$\vec{PQ}\times\vec{PR} = \begin{vmatrix} i & j & k \\ -3 & 1 & 2 \\ 3 & 2 & 4 \end{vmatrix}$

$\vec{PQ}\times\vec{PR} = \hat{i}(4-4) - \vec{j}(-12-6) + k(-6-3$

$= (0\vec{i} + 18\vec{j} - 9\vec{k})$
$= (0,18,-9)$

Now Again
$\vec{PR}\times\vec{PQ} = -(0,18,-9)$
$= (0,-18,9)$ Ans

Area of the triangle ~~QD~~ PQR is,

$\frac{1}{2}$ (base × height)

$\frac{1}{2} \left| \left(\vec{QR} \times \vec{PD} \right) \right|$

We know that

$\vec{QR} = \vec{OR} - \vec{OQ} = (4,2,5) - (-2,1,3)$
$= (6,1,2)$

and

$\vec{OD} = \left(\frac{-2+4}{2}, \frac{1+2}{2}, \frac{5+3}{2} \right)$

$\vec{OD} = \left(\frac{2}{2}, \frac{3}{2}, \frac{8}{2} \right)$

$= \left(1, \frac{3}{2}, 4\right)$

$\vec{PD} = \vec{OD} - \vec{OP} = \left(1, \frac{3}{2}, 4\right) - (1,0,1)$

$= \left(0, \frac{3}{2}, 3\right)$

Again

$\frac{1}{2} \left| \left((6,1,2) \times \left(0, \frac{3}{2}, 3\right) \right) \right|$

$\left(\begin{vmatrix} 6 & 1 & 2 \\ 0 & \frac{3}{2} & 3 \end{vmatrix} \right)$

$\frac{1}{2} \left[\begin{matrix} 1 \\ \frac{3}{2} \end{matrix} \times \begin{matrix} 2 \\ 3 \end{matrix} \times \begin{matrix} 6 \\ 0 \end{matrix} \times \begin{matrix} 1 \\ \frac{3}{2} \end{matrix} \right]$

$[3-3, 0-18, 9-0]$

$\frac{1}{2}[-18+9] \therefore \frac{9}{2}$ Ans

$(0, -18, 9)$

$|\vec{QR} \times \vec{PD}| = \sqrt{81+18^2} = 9\sqrt{5}$

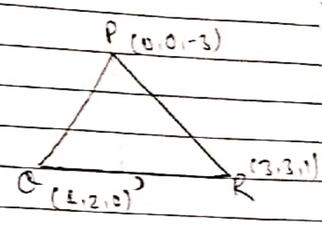
$\frac{1}{2} |(\vec{QR} \times \vec{PD})|$

$\frac{9\sqrt{5}}{2}$ sq units Ans //

5(b) $P(0,0,-3)$, $Q(4,2,0)$, $R(3,3,1)$

Sol^n: Here,

$\vec{PQ} = \vec{OQ} - \vec{OP} = (4,2,0) - (0,0,-3)$
$= (4,2,3)$



$\vec{PR} = \vec{OR} - \vec{OP} = (3,3,1)(0,0,-3)$
$= (3,3,4)$

$D(x,y,z) = \left(\frac{4+3}{2}, \frac{2+3}{2}, \frac{0+1}{2} \right)$

$\vec{OD} = \left(\frac{7}{2}, \frac{5}{2}, \frac{1}{2} \right)$

$\vec{PD} = \vec{OD} - \vec{OP} = \left(\frac{7}{2}, \frac{5}{2}, \frac{1}{2} \right) - (0,0,-3)$

$= \left(\frac{7}{2}, \frac{5}{2}, \frac{1}{2}+3 \right)$

$= \left(\frac{7}{2}, \frac{5}{2}, \frac{7}{2} \right)$

$\vec{QR} = \vec{OR} - \vec{OQ}$
$= (3,3,1) - (4,2,0)$
$= (-1,1,1)$

Area of triangle $= \frac{1}{2} |(\vec{PD} \times \vec{QR})|$

$(\vec{PD} \times \vec{QR}) = \left(\frac{7}{2}, \frac{5}{2}, \frac{7}{2} \right) \times (-1,1,1)$

$$\cancel{(\vec{PD}\times\vec{OR}) = \begin{pmatrix} 5/2 & 7/2 & 5/2 \\ 1 & 1 & 1 \end{pmatrix}}$$

$$(\vec{PD}\times\vec{QR}) = \begin{matrix} 5/2 & 7/2 & 7/2 & 5/2 \\ 1 & 1 & -1 & 1 \end{matrix}$$

$$(\vec{PD}\times\vec{QR}) = \left(5/2 - 7/2,\ -7/2 - 7/2,\ 7/2 + 5/2\right)$$

$$= \left(\frac{-2}{2}, \frac{-14}{2}, \frac{12}{2}\right)$$

$$= (-1, -7, 6)$$

$$|\vec{PD}\times\vec{QR}| = \sqrt{1^2+49+36} = \sqrt{86}$$

Area of $\Delta PQR = \frac{1}{2}\sqrt{86}$ Ans,

Again

Orthogonal to the plane through PQR is.

$$(\vec{PQ}\times\vec{PR}) = \begin{vmatrix} i & j & k \\ 4 & 2 & 3 \\ 3 & 3 & 4 \end{vmatrix}$$

$$(\vec{PQ}\times\vec{PR}) = \vec{i}(8-9) - \vec{j}(16-9) + k(12-6)$$

$$= -\vec{i} - 7\vec{j} + 6\vec{k}$$

$$= (-1, -7, 6)$$

$$\vec{PR}\times\vec{PQ} = -(-1, -7, 6)$$

$$= (1, 7, -6) \text{ Ans}$$

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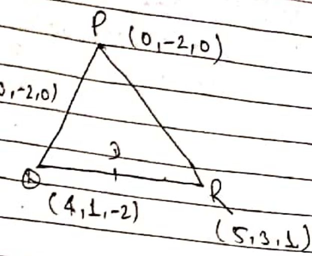
(c) $P(0,-2,0)$, $Q(4,1,-2)$, $R(5,3,1)$

Soln: Here,



$$\vec{PQ} = \vec{OQ} - \vec{OP} = (4,1,-2) - (0,-2,0)$$

$$\vec{PQ} = (4, 3, -2)$$

$$\vec{QR} = \vec{OR} - \vec{OQ} = (5,3,1) - (4,1,-2)$$

$$= (1, 2, 3)$$

$$\vec{PR} = \vec{OR} - \vec{OP} = (5,3,1) - (0,-2,0)$$

$$= (5, 5, 1)$$

$$\vec{OD} = \left(\frac{4+5}{2}, \frac{3+1}{2}, \frac{-2+1}{2}\right)$$

$$\vec{OD} = \left(9/2, 2, -1/2\right)$$

$$\vec{PD} = \left(9/2, 2, -1/2\right) - (0, -2, 0)$$

$$\vec{PD} = \left(9/2, 4, -1/2\right)$$

Area of ΔPQR is $= \frac{1}{2}|(\vec{PD}\times\vec{QR})|$

$$\vec{PD}\times\vec{QR} = \begin{pmatrix} 9/2 & 4 & -1/2 \\ 1 & 2 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & -1/2 & 9/2 & 4 \\ 2 & 3 & 1 & 2 \end{pmatrix}$$

$$= \left(12 + 1, -1/2 - 27/2, 9 - 4\right)$$

$$= (13, -28/2, 5)$$

$$= (13, -14, 5)$$

$$|\vec{PD}\times\vec{QR}| = \sqrt{169+196+25} = \sqrt{390}$$

Area of $\Delta PQR = \frac{1}{2}|\vec{PD}\times\vec{QR}| = \frac{1}{2}\sqrt{390}$ sq. units.

Ans: The non-zero vector Orthogonal to the plane through P.Q.R is

$\vec{PQ} \times \vec{PR} = \begin{vmatrix} i & j & k \\ 4 & 3 & -2 \\ 5 & 5 & 1 \end{vmatrix}$

$= i(3+10) - j(4+10) + k(20-15)$

$= 13\vec{i} - 14\vec{j} + 5\vec{k}$

$(13, -14, 5)$

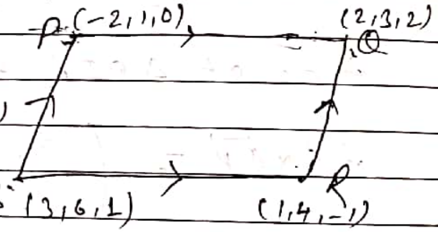
$\vec{PR} \times \vec{PQ} = -(13, -14, 5)$

$= (-13, 14, -5)$ Ans

(9) Find the vol^m of the parallelpiped with adjascent edges PQ, PR and PS.

(a) $P(-2,1,0), Q(2,3,2), R(1,4,-1), S(3,6,1)$

sol^n: Here,



$\vec{PQ} = \vec{OQ} - \vec{OP} = (2,3,2) - (-2,1,0)$

$= (4,2,2)$

$PS = \vec{OS} - \vec{OP} = (3,6,1) - (-2,1,0)$

$= (5,5,1)$

$\vec{QR} = \vec{OR} - \vec{OQ} = (1-2, 4-3, -1-2)$

$= (-1,1,-3)$

vol^m of the parallelpiped is,

$\vec{PQ} \times \vec{PS} \times \vec{QR} = \begin{vmatrix} 4 & 2 & 2 \\ 5 & 5 & 1 \\ -1 & 1 & -3 \end{vmatrix}$

$= 4(-15-1) - 2(-15+1) + 2(5+5)$

$= 4(-16) - 2(-14) + 2(10)$

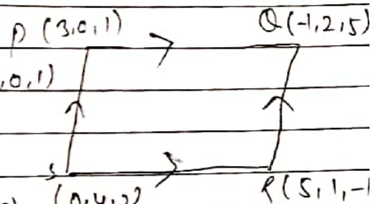
$= -64 + 28 + 20$

$= -16$

$\therefore$ 16 ~~cube~~ unit cubic

9(b) $P(3,0,1), Q(-1,2,5), R(5,1,-1), S(0,4,2)$

sol^n: Here,



$\vec{PQ} = \vec{OQ} - \vec{OP} = (-1,2,5) - (3,0,1)$

$= (-4,2,4)$

$\vec{QR} = \vec{OR} - \vec{OQ} = (5,1,-1)(-1,2,5)$

$= (6,-1,-6)$

$\vec{PS} = \vec{OS} - \vec{OP} = (0,4,2) - (3,0,1)$

$= (-3,4,1)$

Area of parallelogram is

$\vec{PQ} \times \vec{QR} \times \vec{PS} = \begin{vmatrix} -4 & 2 & 4 \\ 6 & -1 & -6 \\ -3 & 4 & 1 \end{vmatrix}$

$= -4(-1+24) - 2(6-18) + 4($

$= -4(23) - 2(-12) + 4(21)$

$= 16$ sq unit Ans

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(10) Use the scalar triple product to verify that the vector $u = \vec{i}+5\vec{j}-2\vec{k}$, $v = 3\vec{i}-\vec{j}+0\vec{k}$ and $w = 5\vec{i}+9\vec{j}-4\vec{k}$ are co-planar.

Sol^n: Here, To proof

Since, $a\cdot(b\times c) = 0$

$u\cdot(\vec{v}\times\vec{w}) = 0$

$$u\cdot(\vec{v}\times\vec{w}) = \begin{vmatrix} 1 & 5 & -2 \\ 3 & -1 & 0 \\ 5 & 9 & -4 \end{vmatrix}$$

$$= 1(4-0) - 5(-12-0) + -2(27+5)$$

$$4+60-64$$

$$64-64$$

$$0 \text{ proved}$$

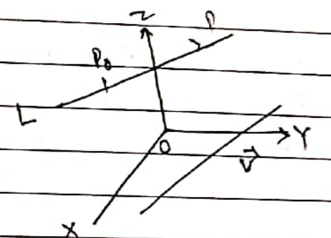
Hence they are co-planar.

Line & plane in space:

The vector equation of a line passing through a given point $P_0(x_0,y_0,z_0)$ and parallel to the given vector $\vec{v} = v_1\vec{i}+v_2\vec{j}+v_3\vec{k}$

let a line L in space passing through $P_0(x_0,y_0,z_0)$ and parallel to $\vec{v} = v_1\vec{i}+v_2\vec{j}+v_3\vec{k}$ and $p(x,y,z)$ be any point on the line, then $\vec{P_0P}$ is parallel to $\vec{v}$, so,

$$\vec{P_0P} = t\vec{v}$$



$$\Rightarrow (x-x_0)\vec{i} + (y-y_0)\vec{j} + (z-z_0)\vec{k} = t(v_1\vec{i}+v_2\vec{j}+v_3\vec{k})$$

$$x-x_0 = v_1t, \quad y-y_0 = v_2t, \quad z-z_0 = v_3t.$$

$$x = x_0+v_1t, \quad y = y_0+v_2t, \quad z = z_0+v_3t.$$

The equation of line in symmetric form is

$$\frac{x-x_0}{v_1} = \frac{y-y_0}{v_2} = \frac{z-z_0}{v_3}$$

Example: Find the eq^n of the line passing through the point $(-1,2,3)$ & parallel to the vector $\vec{v} = 2\vec{i}-\vec{j}+3\vec{k}$. Also find the other two points on the line.

Sol^n, give point $P_0(x_0,y_0,z_0) = (-1,2,3)$

given vector $\vec{v} = (2\vec{i}-\vec{j}+3\vec{k})$

The eq^n of the line v

$$x = x_0+v_1t, \quad y = y_0+v_2t, \quad z = z_0+v_3t.$$

$$x = -1+2t, \quad y = 2-t, \quad z = 3+3t$$

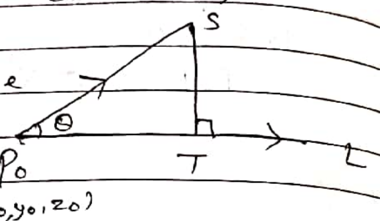
The eq^n of line in symmetric form is

$$\frac{x+1}{2} = \frac{y-2}{-1} = \frac{z-3}{3}$$

Again If $t = 1$,
$(x, y, z) = (1, 1, 6)$
if $t = -1$,
$(x, y, z) = (-3, 3, 0)$

The perpendicular distance from a point to a line in plane (shortest distance)

let s be a given point & L be a given line which is parallel to $\vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$



let the line passing through $P_0(x_0, y_0, z_0)$, Draw $ST \perp r$ P_0T and lies $\angle SP_0T = \theta$

$\vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$

than

$$\frac{ST}{P_0S} = \sin\theta$$

$$ST = P_0S \sin\theta$$

$$ST = |\vec{P_0S}| \frac{|\vec{P_0S} \times \vec{v}|}{|\vec{P_0S}||\vec{v}|}$$

$$ST = \left| \frac{\vec{P_0S} \times \vec{v}}{\vec{v}} \right|$$

The perpendicular distance

Exercise: 9.2

1 Find vector eqⁿ & parametric eqⁿ for the line.

(a) The line through the point $(6,-5,2)$ & parallel to the vector $(1,3,-2/3)$

Solⁿ: Here,

The line passes through the point is $P_0(6,-5,2)$

& the line become parallel to the vector at point $v(1,3,-2/3)$. then,

the ~~eqⁿ of~~ vector eqⁿ for the line is

$P_0(x_0,y_0,z_0) = P_0(6,-5,2)$

$P(x,y,z) = P_0(x,y,z)$ & $v = V(1,3,-2/3)$

then,

$\overrightarrow{P_0P} = v(t)$

$(x-x_0)+(y-y_0)+(z-z_0) = 1t + 3t - \frac{2}{3}t$

$x-x_0 = 1t$, $y-y_0 = 3t$ and $z-z_0 = -\frac{2}{3}t$

$x = x_0+1t$ $y = y_0+3t$ and $z = z_0 - \frac{2}{3}t$

$x = 6+t$, $y = -5+3t$ & $z = 2-\frac{2}{3}t$

& Symmetric ~~parametric~~ eqⁿ for the line is

$\frac{x-x_0}{v_1} = \frac{y-y_0}{v_2} = \frac{z-z_0}{v_3}$

$\frac{x-6}{1} = \frac{y+5}{3} = \frac{z-2}{-2/3} = \frac{3z-6}{-2}$ Ans

(b) The line through the point $(2,2.4,3.5)$ and parallel to the vector $3i+2j-k$

Solⁿ: Here,

The line passes through the point P_0 is $(2,2.4,3.5)$ and the line parallel to the vector through $(3,2,-1)$

then,

the eqⁿ of the line in parametric form is

$x = x_0+v_1t$, $y = y_0+v_2t$ & $z = z_0+v_3t$

$x = 2+3t$, $y = 2.4+2t$ & $z = 3.5-t$

Ans

(c) The line through the point $(0,14,-10)$ & parallel to the line $x=-1+2t$, $y=6-3t$ & $z=3+9t$

Solⁿ: Here,

The line passes through the point P_0 is $(0,14,-10)$ i.e.

$P_0(x_0,y_0,z_0) = P_0(0,14,-10)$

Here,

the given line is parallel to the line $x=-1+2t$,

$y = 6-3t$ & $z = 3+9t$ —(1)

Comparing eqⁿ (1) with, $x_1 = x+v_1t$, $y_1 = y+v_2t$ & $z_1 = z+v_3t$

$v_1 = 2$, $v_2 = -3$ and $v_3 = 9$

Now

the eqⁿ of the line in parametric form is

$x = x_0+v_1t$, $y = y_0+v_2t$ & $z = z_0+v_3t$

$x = 0+2t$, $y = 14-3t$ & $z = -10+9t$

$x = 2t$, $y = 14-3t$ & $z = -10+9t$.

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(d) The line through the point $(1,0,6)$ & $\perp$ to the plane $x+3y+z=5$

Sol^n: Here

Given $(x_0, y_0, z_0) = (1,0,6)$

Here $\vec{v}$: $x+3y+z=5$

(2) Find parametric equation and symmetric eq^n for the line:

(a) The line through the origin and the point $(4,3,-1)$

Sol^n: Here

Given points are $P(0,0,0)$ and $Q(4,3,-1)$

So the vector be

$\vec{PQ} = \vec{OQ} - \vec{OP} = (4,3,-1) - (0,0,0)$

$\vec{PQ} = (4,3,-1)$

which is parallel to the line with point P ~~(4,3,-1)~~ $(0,0,0)$

$x = x_0 + v_1 t$ $\quad y = y_0 + v_2 t$ $\quad , z = z_0 + v_3 t$

$x = 0 + 4t$ $\quad y = 0 + 3t$ $\quad z = 0 - t$

$x = 4t$ $\quad y = 3t$ $\quad z = -t$

The symmetric equation for the line is,

$$\frac{x - x_0}{v_1} = \frac{y - y_0}{v_2} = \frac{z - z_0}{v_3}$$

$$\frac{x-0}{4} = \frac{y-0}{4} = \frac{z-0}{-1}$$

$$x/4 = y/4 = -z \quad \text{Ans}$$

(b) The line through the point ~~(2,1,0) & perpendicular~~ $(-8,1,4)$ and $(3,-2,4)$

Sol^n: here,

Given points are $P(-8,1,4)$ & $Q(3,-2,4)$

~~Since~~ the line $\vec{PQ}$ is

$$\vec{PQ} = \vec{OQ} - \vec{OP} = (3,2,4)-(-8,1,4)$$
$$= (11,-3,0)$$

Since the line $\vec{PQ}$ is parallel to the point $P(-8,1,4)$ then the eq^n of line in parametric form is

$$x = x_0 + v_1 t, \quad y = y_0 + v_2 t, \quad z = z_0 + v_3 t$$
$$x = -8 + 11t, \quad y = 1 - 3t, \quad z = 4 \text{ Ans}$$

The line in symmetric form is,

$$\frac{x-x_0}{v_1} = \frac{y-y_0}{v_2} = \frac{z-z_0}{v_3}$$

$$\frac{x+8}{11} = \frac{y-1}{-3} = \frac{z-z_0}{v_3}$$

(c) The line through the point $(2,1,0)$ & perp^n to both $i+j$ and $j+k$.

Sol^n: here,

Given points is $P_0(2,1,0) = P_0(x_0,y_0,z_0)$

here,

vector $\vec{u} = i + j + 0k$

vector $\vec{v} = 0\vec{i} + \vec{j} + \vec{k}$

Here the vector $(\vec{u} \times \vec{v})$ is perpendicular to each of the vector $\vec{u}$ and $\vec{v}$ so that $\vec{u} \times \vec{v}$ is parallel to the given line such that,

$$\vec{u} \times \vec{v} = (1,1,0) \times (0,1,1)$$

$$\vec{u} \times \vec{v} = \begin{vmatrix} i & j & k \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix}$$

$$\vec{u} \times \vec{v} = i(1-0) - j(1-0) + k(1)$$

$$\vec{u} \times \vec{v} = \vec{i} - \vec{j} + \vec{k}$$

~~then the line~~

$v_1 = 1$, $v_2 = -1$ & $v_3 = 1$

Now the eq^n of the line in parametric form is,

$x = x_0 + v_1 t$ $y = y_0 + v_2 t$ and $z = z_0 + v_3 t$

$x = 2 + t$ $y = 1 - t$ & $z = 0 + t = t$

Now:

the line in symmetric form is

$$\frac{x-x_0}{v_1} = \frac{y-y_0}{v_2} = \frac{z-z_0}{v_3}$$

$$\frac{x-2}{1} = \frac{y-1}{-1} = \frac{z-0}{1}$$

$$x-2 = \frac{y-1}{-1} = z \text{ Ans}$$

(2d) The line through $(1,-1,1)$ & parallel to the line $x+2 = \frac{1}{2}y = z-3$

Sol^n: Here

the given point P_0 is $(x_0,y_0,z_0) = (1,-1,1)$

from above given symmetric form equation we get,

$v_1 = 1$, $v_2 = 2$ and $v_3 = 1$,

~~we get,~~ Now the equation of line through $(1,-1,1)$ in parametric form is

$x = x_0 + v_1 t$, $y = y_0 + v_2 t$, $z = z_0 + v_3 t$

$x = 1 + t$ $y = -1 + 2t$ $z = 1 + t$

the eq^n of line in symmetric form is

$$\frac{x-x_0}{v_1} = \frac{y-y_0}{v_2} = \frac{z-z_0}{v_3}$$

$$\frac{x-1}{1} = \frac{y+1}{2} = \frac{z-1}{1}$$

Exercise :- 9.4

Unit Tangent Vector, Curvature and TNB System

Derivative of vector function :-

A vector valued function

$\vec{r}(t) = f(t)\hat{i} + g(t)\vec{j} + h(t)\vec{k}$

is said to be differentiable at t if $f(t)$, $g(t)$ and $h(t)$ are differentiable at t.

$$\vec{r}'(t) = \frac{df(t)}{dt}\hat{i} + \frac{dg(t)}{dt}\vec{j} + \frac{dh(t)}{dt}\vec{k}$$

\# Show that $\vec{r}(t) = \sin t\,\vec{i} + \cos t\,\vec{j} + \sqrt{3}k$ has content length and is orthogonal to its derivation.

Sol :- Here,

$\vec{r}(t) = \sin t\,\vec{i} + \cos t\,\vec{j} + \sqrt{3}k$

$\frac{d\vec{r}(t)}{dt} = \cos t\,\vec{i} - \sin t\,\vec{j} + 0$

$\vec{r}'(t) = \cos t\,\vec{i} - \sin t\,\vec{j}$

$|\vec{r}(t)| = \sqrt{\cos^2 t + \sin^2 t + 3}$

$|\vec{r}(t)| = 2$

Now

$\vec{r}(t) \cdot \vec{r}'(t) = (\sin t\,\hat{i} + \cos\vec{j} + \sqrt{3}k) \cdot (\cos\hat{i} - \sin\vec{j} + 0\vec{k})$

$= \sin t . \cos t + (\cos\vec{j} . - \sin\vec{j})$

$= \sin t\cos t - \sin t\cos t.$

$= 0$ Ans//

$\vec{r}(t)$ and $\vec{r}'(t)$ are orthogonal proved

Arc length (Length of curve).

A vector valued function of a smooth curve

$\vec{r}(t) = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$

where $a \le t \le b$

then the length of curve

$$L = \int_a^b \sqrt{\left(\frac{df(t)}{dt}\right)^2 + \left(\frac{dg(t)}{dt}\right)^2 + \left(\frac{dh(t)}{dt}\right)^2} \, dt$$

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} \, dt$$

$$L = \int_a^b |\vec{r}'(t)| \, dt$$

$$\int_a^b |\vec{v}(t)| \, dt$$

Ex. Find the length of one turn of the circular helix $\vec{r}(t) = \cos t\,\vec{i} + \sin t\,\vec{j} + t\vec{k}$

Sol^n: here

$\vec{r}'(t) = -\sin t\,\vec{i} + \cos t\,\vec{j} + \vec{k}$

$|\vec{r}'(t)| = \sqrt{\sin^2 t + \cos^2 t + 1}$

$|\vec{r}'(t)| = \sqrt{2}$

Since the helix curve make full turn as t run from 0 to 2π

length of curve $= \int_0^{2\pi} |\vec{r}'(t)| \, dt$

$$\int_0^{2\pi} \sqrt{2} \, dt$$

$\left[\sqrt{2}t\right]_0^{2\pi}$

$\sqrt{2}(2\pi)$

$2\sqrt{2}\pi$

Unit tangent vector:-

The unit tangent vector of a smooth curve $\vec{r}(t)$ is denoted by $\vec{T}$ and defined by

$$\vec{T} = \frac{d\vec{r}}{ds} = \frac{\frac{d\vec{r}}{dt}}{\frac{ds}{dt}} = \frac{\vec{v}}{|\vec{v}|}$$

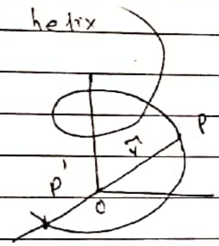
Ex. Find the unit tangent vector of the helix

$\vec{r}(t) = \cos t\,\vec{i} + \sin t\,\vec{j} + t\vec{k}$

Sol^n: here $\vec{v}(t) = \vec{r}'(t) = -\sin t + \cos t\,\vec{j} + \vec{k}$

$|\vec{v}(t)| = \sqrt{\sin^2 t + \cos^2 t + 1}$

$|\vec{v}(t)| = \sqrt{2}$



$\therefore$ Unit tangent vector $(\vec{T}) = \frac{\vec{v}}{|\vec{v}|}$

$$= \frac{-\sin t\,\vec{i} + \cos t\,\vec{j} + \vec{k}}{\sqrt{2}}$$

Ans

Curvature:

The ~~arc~~ rate of rotation of $\vec{T}$ is called the Curvature. Its magnitude is denoted by K (Kappa)

$$K = \left|\frac{d\vec{T}}{ds}\right|$$

$$K = \left|\frac{d\vec{T}/dt}{ds/dt}\right|$$

$$K = \left|\frac{d\vec{T}/dt}{|\vec{v}|}\right|$$

Note: The curvature of a straight line is zero

vector formula for curvature is,

$$K = \frac{|\vec{v} \times \vec{a}|}{|\vec{v}|} \quad \text{where } \vec{v} = \frac{d\vec{r}}{dt}$$

$$\vec{a} = d^2\vec{r}/dt^2$$

ex. Show that the curvature of a circle of radius a is $1/a$

proof. let us consider a circle with radius a then the parametric equation of the circle is

$\vec{r} = a\cos t\vec{i} + a\sin t\vec{j}$

$$\vec{v} = \frac{d\vec{r}}{dt} = -a\sin t\,\vec{i} + a\cos t\,\vec{j}$$

$$|\vec{v}| = \sqrt{a^2\sin^2 t + a^2\cos^2 t} = \sqrt{a^2} = a$$

$$\vec{T} = \frac{\vec{v}}{|\vec{v}|} = \frac{-a\sin t\,\vec{i} + a\cos t\,\vec{j}}{a} = -\sin t\,\vec{i} + \cos t\,\vec{j}$$

$$\frac{d\vec{T}}{dt} = -\cos t\,\vec{i} - \sin t\,\vec{j}$$

$$\left|\frac{d\vec{T}}{dt}\right| = \sqrt{\cos^2 t + \sin^2 t} = \sqrt{1} = 1$$

The curvature is

$$K = \frac{\left|\frac{d\vec{T}}{dt}\right|}{|\vec{v}|}$$

$$= 1/a. \quad \text{proved}$$

Radius of curvature

The radius of curvature of a smooth curve $\vec{r}(t)$ is denoted by ρ (radii) and given by $\rho = 1/K$

i.e. K

principle Unit Normal:-

The principle unit normal vector for a sm curve at point where $K \neq 0$ is denoted by $\vec{N}$ & given by

$$\vec{N} = \frac{1}{K}\frac{d\vec{T}}{dt} = \frac{1}{\left|\frac{d\vec{T}}{dt}\right|} \cdot \frac{d\vec{T}}{ds}$$

$$\vec{N} = \frac{\frac{d\vec{T}}{dt} \cdot \frac{dt}{dt}}{\left|\frac{d\vec{T}}{dt} \cdot \frac{dt}{dt}\right|}$$

$$\vec{N} = \frac{d\vec{T}/dt}{|d\vec{T}/dt|}$$

Binormal :-

let $\vec{T}$ & $\vec{N}$ be the unit tangent vector and principle unit normal vector at any point p of a space curve. then the binormal vector of a curve then the binormal vector of a curve in space at p is denoted by $\vec{B}$ and defined by

$\vec{B} = \vec{T} \times \vec{N}$

Note

① The binomial vector $\vec{B}$ is orthogonal to both $\vec{T}$ & $\vec{N}$.

② The three Orthogonal vector $\vec{T}, \vec{N}, \vec{B}$ defined coordinate system known as TNB system or frenet frame.

Exercise :- 9.4.

1 Find the unit tangent vector $\vec{T}(t)$ and unit normal vector $\vec{N}(t)$. Also find the curvature.

(a) $r(t) = (t, 3\cos t, 3\sin t) \Rightarrow t\vec{i} + 3\cos t\vec{j} + 3\sin t\vec{k}$

Soln:- Here,

Unit tangent vector $\vec{T}(t) = \dfrac{\frac{dr}{dt}}{\left|\frac{dr}{dt}\right|} = \dfrac{\vec{v}}{|\vec{v}|}$

$r'(t) \neq \vec{V}(t) = 1\vec{i} - 3\sin t\vec{j} + 3\cos t\vec{k}$

$|\vec{V}(t)| = \sqrt{1 + 9\sin^2 t + 9\cos^2 t}$

$|\vec{V}(t)| = \sqrt{1 + 9(\sin^2 t + \cos^2 t)}$

$|\vec{V}(t)| = \sqrt{10}$

$\therefore \vec{T}(t) = \dfrac{1\vec{i} - 3\sin t\vec{j} + 3\cos t\vec{k}}{\sqrt{10}}$

$\vec{N}(t) = \dfrac{\frac{d\vec{T}}{dt}}{\left|\frac{d\vec{T}}{dt}\right|}$

$\dfrac{d\vec{T}}{dt} = \dfrac{d(\vec{i} - 3\sin t\vec{j}}{dt} \quad \dfrac{1}{\sqrt{10}} \dfrac{d(\vec{i} - 3\sin t\vec{j} + 3\cos t\vec{k})}{dt}$

$\dfrac{d\vec{T}}{dt} = \dfrac{1}{\sqrt{10}}(0 - 3\cos t\vec{j} - 3\sin t\vec{k})$

$\dfrac{d\vec{T}}{dt} = \dfrac{-3}{\sqrt{10}}\cos t\vec{j} - \dfrac{3}{\sqrt{10}}\sin t\vec{k}$

$\left|\dfrac{dT}{dt}\right| = \sqrt{\dfrac{9}{10}\cos^2 t + \dfrac{9}{10}\sin^2 t}$

$\left|\dfrac{dT}{dt}\right| = \dfrac{3}{\sqrt{10}}$

$\vec{N}(t) = \dfrac{\frac{d\vec{T}}{dt}}{\left|\frac{d\vec{T}}{dt}\right|} = \dfrac{\frac{-3}{\sqrt{10}}\cos\vec{j} - \frac{3}{\sqrt{10}}\sin t\vec{k}}{\frac{3}{\sqrt{10}}}$

$\vec{N}(t) = 0\vec{i} - \cos\vec{j} - \sin t\vec{k}$

$\vec{N}(t) = (0, -\cos t, -\sin t)$ Ans

$K = \dfrac{\frac{d\vec{T}}{dt}}{|\vec{v}|} = \dfrac{\frac{1}{\sqrt{10}}(0 - 3\cos t\vec{j} - 3\sin t\vec{k})}{\sqrt{10}} \quad \dfrac{\frac{3}{\sqrt{10}}}{\sqrt{10}}$

$= \dfrac{1}{10}(0 - 3\cos t\vec{j} - 3\sin k)$

$= \dfrac{3}{10}$ Ans

(b) $r(t) = (t^2, \sin t - t\cos t, \cos t + t\sin t), \; t > 0$

Sol: here,

$r(t) = (t^2, \sin t - t\cos t, \cos t + t\sin t)$

$r(t) = t^2\vec{i} + (\sin t - t\cos t)\vec{j} + (\cos t + t\sin t)\vec{k}$

$\frac{dr(t)}{dt} = \vec{v}(t) = 2t\vec{i} + (\cos t - (\cos t \cdot 1 + t\sin t))\vec{j} + (-\sin t + (t\cos t + \sin t \cdot 1))\vec{k}$

$\vec{v}(t) = 2t\vec{i} + (\cos t - \cos t + t\sin t)\vec{j} + (-\sin t + t\cos t + \sin t)$

$\vec{v}(t) = 2t\vec{i} + t\sin t\vec{j} + (\cancel{\cdots} + t\cos t)\vec{k}$

$|\vec{v}(t)| = \sqrt{4t^2 + t^2\sin^2 t + t^2\cos^2 t}$

$|\vec{v}(t)| = \sqrt{4t^2 + t^2}$

$|\vec{v}(t)| = t\sqrt{5}$

$\vec{T}(t) = \frac{\vec{v}(t)}{|\vec{v}(t)|} = \frac{2t\vec{i} + t\sin t\vec{j} + t\cos t\vec{k}}{t\sqrt{5}}$

$\vec{T}(t) = \frac{1}{\sqrt{5}}(2, \sin t, \cos t)$ Ans

$\vec{N}(t) = \frac{\frac{d\vec{T}}{dt}}{\left|\frac{d\vec{T}}{dt}\right|}$

$\frac{d\vec{T}}{dt} = \frac{\frac{1}{\sqrt{5}}(2\vec{i} + \sin t\vec{j} + \cos t\vec{k})}{dt}$

$\frac{d\vec{T}}{dt} = \frac{1}{\sqrt{5}}(0 + \cos t\vec{j} - \sin t\vec{k})$

$\left|\frac{d\vec{T}}{dt}\right| = \sqrt{\left(\frac{1}{\sqrt{5}}\cos t\right)^2 + \left(\frac{1}{\sqrt{5}}\sin t\right)^2}$

$\left|\frac{d\vec{T}}{dt}\right| = \sqrt{\frac{1}{5}\cos^2 t + \frac{1}{5}\sin^2 t}$

$\left|\frac{dT}{dt}\right| = \frac{1}{\sqrt{5}}$

$\vec{N}(t) = \frac{\frac{1}{\sqrt{5}}(0, \cos t, -\sin t)}{1/\sqrt{5}}$

$\vec{N}(t) = (0, \cos t, -\sin t)$

curvature $(K) = \frac{\left|\frac{d\vec{T}}{dt}\right|}{|\vec{v}|} = \frac{1/\sqrt{5}}{t\sqrt{5}}$

$\frac{1}{5t}$ Ans

ⓒ $r(t) = (\sqrt{2}\,t, e^t, e^{-t})$

Sol^n: Here,

$r(t) = (\sqrt{2}\,t, e^t, e^{-t})$

$$\vec{T}(t) = \frac{\vec{v}(t)}{|\vec{v}(t)|}$$

$$\vec{T}(t) = \frac{\vec{v}(t)}{|\vec{v}(t)|}$$

$$\vec{v} = \frac{dr}{dt} = \frac{d(\sqrt{2}\,t\,\vec{i} + e^t\vec{j} + e^{-t}.\vec{k})}{dt}$$

$$\vec{v} = \sqrt{2}\,\vec{i} + e^t.\vec{j} - e^{-t}\vec{k}$$

$$|\vec{v}| = \sqrt{2 + e^{2t} + e^{-2t}}$$

$$|\vec{v}| = \sqrt{2.1 + (e^t)^2 + (e^{-t})^2}$$

$$|\vec{v}| = \sqrt{(e^t + e^{-t})^2}$$

$$|\vec{v}| = e^t + e^{-t} = \frac{e^{2t}+1}{e^t}$$

$$\vec{T}(t) = \frac{\vec{v}(t)}{|\vec{v}(t)|} = \frac{e^t(\sqrt{2}, e^t, -e^{-t})}{(e^{2t}+1)}$$

$$= \frac{(\sqrt{2}e^t, e^{2t}, -1)}{(e^{2t}+1)}$$

$$\vec{N}(t) = \frac{\frac{d\vec{T}}{dt}}{\left|\frac{d\vec{T}}{dt}\right|}$$

$$\frac{d\vec{T}(t)}{dt} = \sqrt{2}e^t\vec{i} + e^{2t}.2\vec{j} - 0$$

$$\frac{d\vec{T}(t)}{dt} = \sqrt{2}\,e^t\,\vec{i} + 2e^{2t}\vec{j} + 0\vec{k}.$$

$$\left|\frac{d\vec{T}(t)}{dt}\right| = \sqrt{2e^{2t} + 4e^{4t} + 0}$$

$$\left|\frac{d\vec{T}(t)}{dt}\right| = \sqrt{e^{2t}(2 + 4e^{2t})} = e^t(\sqrt{2+4e^{2t}})$$

$$\frac{d\vec{T}(t)}{dt} = \frac{d\left(\frac{(\sqrt{2}e^t, e^{2t}, -1)}{(e^{2t}+1)}\right)}{dt}$$

$$\frac{d\vec{T}}{dt} = \frac{(\sqrt{2}e^t, e^{2t}, -1)(2e^{2t}) + (e^{2t}+1).(\sqrt{2}e^t, e^{2t}, 0)}{(e^{2t}+1)^2}$$

$$\frac{d\vec{T}}{dt} = \frac{2\sqrt{2}e^{3t}, 2e^{4t}, -2e^{2t} + (\sqrt{2}e^{3t} + e^{4t} + 0 + \sqrt{2}e^t + e^{2t}}{(e^{2t}+1)^2}$$

$$\frac{d\vec{T}}{dt} = \frac{2\sqrt{2}e^{3t} + 2e^{4t} - 2e^{2t} + \sqrt{2}e^{3t} + e^{4t} + \sqrt{2}e^t + e^{2t}}{(e^{2t}+1)^2}$$

$$\frac{d\vec{T}}{dt} = 3e^{4t} - e^{2t} + 4\sqrt{2}e^{3t}$$

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(a) $r(t) = \left(t, \frac{1}{2}t^2, t^2\right)$

Sol^n: Here

$\vec{T}(t) = \dfrac{\vec{v}(t)}{|\vec{v}(t)|}$

$\vec{v}(t) = \left(1, \frac{1}{2}2t, 2t\right) = (1, t, 2t)$

$|\vec{v}(t)| = \sqrt{1+t^2+4t^2} = \sqrt{1+5t^2}$

$\vec{T}(t) = \dfrac{(1, t, 2t)}{\sqrt{1+5t^2}}$

$= \left((\sqrt{1+5t^2})^{-1}, (\sqrt{1+5t^2})^{-1} \cdot t, \sqrt{1+5t^2} \cdot 2t \right)$

$= \left[(1+5t^2)^{-1/2}, t(1+5t^2)^{-1/2}, 2t(1+5t^2)^{-1/2} \right]$

$\vec{N}(t) = \dfrac{\frac{d\vec{T}}{dt}}{\left|\frac{d\vec{T}}{dt}\right|}$

$\dfrac{d\vec{T}}{dt} = -\frac{1}{2} \cdot (1+5t^2)^{-3/2} \cdot (0+10t), -\frac{1}{2}(1+5t^2)^{-3/2}(10t) \cdot t + (1+5t^2)^{-1/2}, -1(1+5t^2)^{-3/2} \, 10t^2 + 2(1+5t^2)^{-1/2}$

$\dfrac{d\vec{T}}{dt} = -\frac{1}{2}(1+5t^2)^{-3/2} \cdot 10t, -\frac{1}{2}(1+5t^2)^{-3/2} \cdot 10t^2 + (1+5t^2)^{-1/2}, -1(1+5t^2)^{-3/2} \, 10t^2 + 2(1+5t^2)^{-1/2}$

$\dfrac{d\vec{T}}{dt} = (1+5t^2)^{-3/2} \left(-5t, -5t^2 + (1+5t^2)^{-1/2}, \right.$

$K = \dfrac{|d\vec{T}/dt|}{|\vec{v}|}$

(2) Find the curvature.

(a) $r(t) = t^3\vec{j} + t^2\vec{k}$

Sol^n: Here

$r(t) = t^3\vec{j} + t^2\vec{k}$

We know that

$K = \dfrac{\left|\frac{d\vec{T}}{dt}\right|}{|\vec{v}|}$

$\vec{T}(t) = \dfrac{\frac{d\vec{r}}{dt}}{\left|\frac{d\vec{r}}{dt}\right|} \Rightarrow \dfrac{\vec{v}(t)}{|\vec{v}(t)|}$

$\dfrac{d\vec{r}}{dt} = \vec{v} = (0, 3t^2\vec{j}, 2t\vec{k}) \Rightarrow |\vec{v}| = \sqrt{9t^4 + 4t^2}$

$\left|\dfrac{d\vec{r}}{dt}\right| = \sqrt{0+9t^4+4t^2} = \sqrt{9t^4+4t^2}$

$\left|\dfrac{d\vec{r}}{dt}\right| = \sqrt{t^2(9t^2+4)} = \sqrt{t^2(9t^2+4)}$

$= t\sqrt{9t^2+4}$

$\vec{T}(t) = \dfrac{1}{t\sqrt{9t^2+4}} (0, 3t^2\vec{j}, 2t\vec{k})$

$\vec{T}(t) = \dfrac{1}{\sqrt{9t^2+4}} (0, 3t\vec{j}, 2\vec{k})$

$\vec{T}(t) = \left(0, \dfrac{3t\,\vec{j}}{\sqrt{9t^2+4}}, \dfrac{2\,\vec{k}}{\sqrt{9t^2+4}}\right)$

$\dfrac{d\vec{T}(t)}{dt} = \left(0, 3\left(1 \cdot \sqrt{9t^2+4} + t \cdot \frac{1}{2}(9t^2+4)^{-1/2}\right), 2 \cdot -\frac{1}{2}(9t^2+4)\right.$

$\dfrac{d\vec{T}(t)}{dt} = \left(0, 3\sqrt{9t^2+4} + \dfrac{\frac{3}{2}t}{\sqrt{9t^2+4}}, \dfrac{-1}{(9t^2+4)^{3/2}}\right)$

$\dfrac{d\vec{T}(t)}{dt} = \left(0, 3\sqrt{9t^2+4} + \frac{3}{2}(9t^2+4)^{-1/2} \cdot t, -1(9t^2+4)^{-3/2}\right.$

$\left|\dfrac{d\vec{T}(t)}{dT}\right| = \sqrt{0 + 3(9t^2+4) + \frac{9}{4}(9t^2+4)^{-1} + (9t^2+4)^{-3}}$

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$$\frac{d\vec{T}}{dt} = \sqrt{3(9t^2+4) + \frac{9}{4}(9t^2+4)^{-1} + (9t^2+4)^{-3}}$$

$$\frac{dT^2}{dt} = \sqrt{3(9t^2+4) + \frac{9}{4}\frac{1}{(9t^2+4)} + \frac{1}{(9t^2+4)^3}}$$

$$\frac{d\vec{T}}{dt} = \frac{\sqrt{12(9t^2+4)^4 + 9(9t^2+4)^2 + 4}}{2(9t^2+4)\left(\sqrt{9t^2+4}\right)}$$

$$\frac{d\vec{t}}{dt} = \sqrt{\frac{12(9t^2+4)^4 + 9(9t^2+4)^2 + 4}{(9t^2+4)}} \times \frac{1}{2(9t^2+4)}$$

UNIT- 10.1.

Partial Derivative and multiple Integral.

limit :- let f be a function of two variables whose domain includes the point close to (a,b). Then we say that the limit of $f(x,y)$ as (x,y) approaches to (a,b) is L & we write.

$$\lim_{(x,y)\to(0,0)} f(x,y) = 1$$

Ex. evaluate.

(a) $\lim_{(x,y)\to(0,1)} \dfrac{x-xy+3}{x^2y+5xy-y^3}$.

$$= \frac{0-0+3}{0+0-1^3}$$

$$= \frac{3}{-1}$$

$$= -3 \text{ Ans}$$

(b) $\lim_{(x,y)\to(3,4)} \sqrt{x^2+y^2} =$

(c) $\lim_{(x,y)\to(0,0)} \left(\dfrac{x^2-xy}{\sqrt{x}-\sqrt{y}}\right) \times \dfrac{(\sqrt{x}+\sqrt{y})}{(\sqrt{x}+\sqrt{y})}$

$$\lim_{(x,y)\to(0,0)} \frac{(x^2-xy)(\sqrt{x}+\sqrt{y})}{(x-y)}$$

$$\lim_{(x,y)\to(0,0)} = \frac{x\cancel{(x-y)}(\sqrt{x}+\sqrt{y})}{\cancel{(x-y)}}$$

$$= 0 \text{ Ans}$$

Continuity of a function of two variable.

A function $f(x,y)$ is continuous at a point (x_0,y_0)

if (i) $f(x_0,y_0)$ i.e. exist [functional value]

(ii) $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ is exist [~~func~~ limiting value].

(iii) $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0)$ [equality].

The two-path test for non-existence of a limit.

If a function $f(x,y)$ has different value along two different path as (x,y) approaches to (x_0,y_0) then

$\lim_{(x,y)} f(x,y)$ doesnot exist.

Ex. show that $f(x,y) = \begin{cases} \dfrac{2xy}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$

is continuous at every point except the origin.

Sol^n:- here

Since the function is rational function It is continuous at every point except $(x,y) \neq (0,0)$

At the origin $(0,0)$

let $y = mx$, $x \neq 0$ & m is some finite value.

$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \left(\dfrac{2xy}{x^2+y^2}\right)$

$$\lim_{(x,y)\to(0,0)} \frac{2mx^2}{x^2(1+m^2)}$$

$$\lim_{(x,y)\to(0,0)} \frac{2m}{1+m^2}$$

Since the limiting value is not equal for different value of m so the function f(x,y) is discontinuous at origin.

Ex. Examine whether f(x,y) is continuous at (0,0) for the function

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

Sol^n: let $y = mx$, $x \neq 0$ and m is finite value,

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to} \left(\frac{xy}{x^2+y^2}\right)$$

$$= \lim_{(x,y)\to(0,0)} \left(\frac{x \cdot mx}{x^2+m^2x^2}\right)$$

$$= \lim_{(x,y)\to(0,0)} \left(\frac{mx^2}{x(1+m^2x)}\right)$$

$$\lim_{(x,y)\to(0,0)} \left(\frac{mx}{(1+m^2x)}\right)$$

$= 0$, for any value of m.

Since the limiting value are zero for any value of m, i.e. the limit is exist.

and $f(0,0) = 0$.

$\therefore \lim_{(x,y)\to(0,0)} f(x,y) = f(0,0)$.

Hence the function f(x,y) is continuous at (0,0).

(Ex) Show that the function f(x,y) is discontinuous at (0,0)

$$\text{if } f(x,y) = \begin{cases} \frac{xy(x-2y)}{x^3+y^3}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0). \end{cases}$$

(2) Show that the function f(x,y) is continuous at (x,y) = (0,0) if

$$f(x,y) = \begin{cases} \frac{xy(x-2y)}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

Partial Derivative:

The partial derivative of $z = f(x,y)$ with respect to x, keeping y as constant at point (x_0, y_0) is

$$\frac{df}{dx} = \lim_{h\to 0} \frac{f(x_0+h, y_0) - f(x_0, y_0)}{h}$$

and with respect y, keeping x as constant.

$$\frac{df}{dy} = \lim_{h\to 0} \frac{f(x_0, y_0+h) - f(x_0, y_0)}{h}$$

$3x^2$
$3\frac{dz}{dx}$
$6x$

Notation $z = f(x, y)$

$\frac{\partial f}{\partial x} = \frac{\partial z}{\partial x}, f_x, z_x$

$\frac{\partial f}{\partial y}, \frac{\partial z}{\partial y}, f_y, z_y$.

Ex. If $z = x^3 + y^3 + 3x^2y$, find $\frac{dz}{dx}$ & $\frac{dz}{dy}$,

Solⁿ:
$z = x^3 + y^3 + 3x^2y$.

$\frac{dz}{dx} = 3x^2 + 6xy$

$\frac{dz}{dy} = 3y^2 + 3x^2$

Ex find $\frac{dz}{dx}$ & $\frac{dz}{dy}$ if z is defined as a function of x and y by the ~~funct~~ equation
$x^3 + y^3 + z^3 + 6xyz = 1$:-

Solⁿ: Diff ① partially w.r.to. x,
$3x^2 + 3z^2 \frac{dz}{dx} + 6xy\frac{dz}{dx} + 6yz.1 = 0$

$3\frac{dz}{dx}(z^2 + 2xy) = -3x^2 - 6yz$

$\frac{dz}{dx} = \frac{-3(x^2 + 2yz)}{3(z^2 + 2xy)}$

$\frac{1}{2}\frac{dz}{dy}$

Diff ① partially w.r.t. y.

$3y^2 + 3z^2\frac{dz}{dy} + 6xy\frac{dz}{dy} + 6xz \cdot 1 = 0$.

$3\frac{dz}{dy}(z^2 + 2xy) = -3y^2 - 6xz = -3(y^2 + 2xz)$

$\frac{dz}{dy} = \frac{-3(y^2 + 2xz)}{3(z^2 + 2xy)} = \frac{-(y^2 + 2xz)}{(z^2 + 2xy)}$ Ans.

Ex: If $yz - \log z = x + y$, $z = f(x, y)$ find $\frac{dz}{dx}$ & $\frac{dz}{dy}$.

Solⁿ: Diff ① partially w.r.to x.

$y.\frac{dz}{dx} - \frac{1}{z}\frac{dz}{dx} = 1$.

$\frac{dz}{dx}\left(y - \frac{1}{z}\right) = 1$

$\frac{dz}{dx} = \frac{z}{yz - 1}$ Ans.

Diff (1) partially w.r.to y.

$y.\frac{dz}{dy} + z.1 - \frac{1}{z}\frac{dz}{dy} = 1$.

$\frac{dz}{dy}\left(y - \frac{1}{z}\right) = 1 - z$

$\frac{dz}{dy} = \frac{(1 - z)z}{(yz - 1)}$

$\sqrt{e^{\log x}}$ $\frac{1}{2}$ $z = (y)x$ $m \log(x)$ $\boxed{\log_e e} = 1$ $e^{\log_e 2}$ $e^{-\log_e}$ $e^{\log_e (2^{-1})}$

For exercise 10.1

(2) Find the limits,

(i) $\lim_{(x,y) \to (0,0)} \left(\frac{3x^2 - y^2 + 5}{x^2 + y^2 + 2} \right)$

Sol^n: here,

$$\lim_{(x,y) \to (0,0)} \left(\frac{3x^2 - y^2 + 5}{x^2 + y^2 + 2} \right)$$

$$\frac{3 \times 0 - 0 + 5}{0 + 0 + 2}$$

$$\frac{5}{2} \text{ Ans}$$

(ii) $\lim_{(x,y) \to (0, \ln(2))} e^{x-y}$

Sol^n: here

$$\lim_{(x,y) \to (0, \ln(2))} . \quad e^{x-y} = e^x . e^{-y}$$

$$e^0 . e^{\ln(2)^{-1}}$$

$$1 . e^{\ln(\frac{1}{2})}$$

$$= \frac{1}{2} \text{ Ans}$$

(iii) $\lim_{(x,y) \to (3,4)} \left(\sqrt{x^2 + y^2 - 1} \right)$

Sol^n: Here,

$$\lim_{(x,y) \to (3,4)} \left(\sqrt{x^2 + y^2 - 1} \right)$$

$$\sqrt{9 + 16 - 1}$$

$$\sqrt{24}$$

$$2\sqrt{6} \text{ Ans}$$

(iv) $\lim_{(x,y) \to (1,0)} \left(\frac{e^y \sin x}{x} \right)$

Sol^n: here,

$$\lim_{(x,y) \to (1,0)} \left(\frac{e^y \sin x}{x} \right)$$

$$\frac{e^0 \sin 1}{1}$$

$$\sin 1 \text{ Ans}$$

(v) $\lim_{(x,y,z) \to (1,3,4)} \left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \right)$

Sol^n: here $\left(\frac{1}{1} + \frac{1}{3} + \frac{1}{4} \right)$

$$\frac{12 + 4 + 3}{12} = \frac{19}{12}$$

(vi) $\lim_{p \to (0,0,3)} \left(z e^{-2y} \cos 2x\right)$ where $p = x,y,z$

Solⁿ: Here,

$\lim_{(x,y,z) \to (0,0,3)} \left(z e^{-2y} \cdot \cos 2x\right)$

$3 e^{0} \cos 0$

3 Ans.

(3) Show that the limit of the following function doesnot exist

(i) $\lim_{(x,y) \to (0,0)} \left(\frac{x^2 + \sin^2 y}{2x^2 + y^2}\right)$ $\left[\frac{0}{0}\right]$

Solⁿ: Here,

while applying limit value i.e $(x,y) \to (0,0)$ the function is in $\left(\frac{0}{0}\right.$ form$)$.

so that,

let $y = mx$, $x \neq 0$ and m is finite

$\lim_{(x,y) \to (0,0)} \left(\frac{x^2 + \sin^2 mx}{2x^2 + m^2 x^2}\right)$

$\left(\frac{0 + \sin 2 0}{0 + 0}\right) = 0/0$

Since the limiting value are $0/0$ for many value of m i.e. the limit ~~does exist~~ doesnot exist.

(ii) $\lim_{(x,y) \to (0,0)} \left(\frac{x y \cos y}{3x^2 + y^2}\right)$

Solⁿ: Here,

$\lim_{(x,y) \to (0,0)} \left(\frac{x y \cos y}{3x^2 + y^2}\right)$

let $y = mx$, $x \neq 0$ & m is finite.

$\lim_{(x,y) \to (0,0)} \left(\frac{x \cdot mx \cos mx}{3x^2 + m^2 x^2}\right)$

$\lim_{(x,y) \to (0,0)} \left(\frac{x^2 m \cos x}{x^2(3 + m^2)}\right)$

$\lim_{(x,y) \to (0,0)} \left(\frac{m \cos x}{3 + m^2}\right)$

$\left(\frac{m}{3 + m^2}\right)$

Since the limiting value are different for ~~many~~ different value of m i.e. the limit doesnot exist

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(iii) $\lim_{(x,y)\to(0,0)} \left(\frac{6x^3y}{2x^4+y^4}\right)$

Sol^n: Here,

$\lim_{(x,y)\to(0,0)} \left(\frac{6x^3y}{2x^4+y^4}\right)$

let $y = mx$, $x \neq 0$, & m is finite

$\lim_{(x,y)\to(0,0)} \left(\frac{6x^3mx}{2x^4+m^4x^4}\right)$

$\lim_{(x,y)\to(0,0)} \left(\frac{6x^4m}{2x^4+x^4m^4}\right)$

$\lim_{(x,y)\to(0,0)} \left(\frac{6m}{2+m^4}\right)$

Since ~~for~~ any ~~of~~ limiting value ~~of~~ are different for different value of m. i.e. the limit doesnot exist.

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(4) Examine the continuity of the following functions:

(i) $f(x,y) \begin{cases} \frac{xy(x-2y)}{x^3+y^3} & \text{at } (x,y) \neq (0,0) \\ 0 & \text{at } (x,y) = (0,0) \end{cases}$

Sol^n: Here,

$\lim_{(x,y)\to(0,0)} \frac{xy(x-2y)}{x^3+y^3}$ i.e. $\left[\frac{0}{0} \text{ form}\right]$

let $y = mx$; $x \neq 0$, & m is finite.

$\lim_{(x,y)\to(0,0)} \frac{xmx(x-2mx)}{x^3+m^3x^3}$

$\lim_{(x,y)\to(0,0)} \frac{mx^2(x-2mx)}{x^3+m^3x^3}$

$\lim_{(x,y)\to(0,0)} \frac{\cancel{x^3}(m-2m^2)}{\cancel{x^3}(1+m^3)}$

$\frac{m-2m^2}{1+m^3}$

Since ~~for~~ limiting ~~of~~ value are different for different value of m i.e. the limit doesnot exist therefore the function is discontinuous.

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(ii) $f(x,y) = \begin{cases} \frac{xy(x-2y)}{x+y} & \text{at } (x,y) \neq (0,0) \\ 0 & \text{at } (x,y) = (0,0) \end{cases}$

Sol^n: Here,

$f(x,y) \lim_{(0,0)} \frac{xy(x-2y)}{x+y}$

let $y = mx$; $x \neq 0$ and m is finite.

$\lim_{(x,y)\to(0,0)} \frac{x\,mx\,(x-2mx)}{x+mx}$

$\quad " \quad \frac{x^2(mx-2m^2x)}{x(1+m)}$

$\quad " \quad \frac{x^{\cancel{3}}(m-2m^2)}{x(1+m)}$

$\lim_{(x,y)\to(0,0)} \frac{x^2(m-2m^2)}{1+m}$

$0.$

Since the limiting value are same for any value of m i.e. the limit is exist

& $f(0,0)=0$

$\lim_{(x,y)\to(0,0)} = f(x,y) = f(0,0)$

Hence the function is continuous

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(iii) $f(x,y) = \begin{cases} \frac{xy(x-2y)}{x^2-y^2} & \text{at } (x,y) \neq (0,0) \\ 0 & \text{at } (x,y) = (0,0) \end{cases}$

Sol^n: Here,

$\lim_{(x,y)\to(0,0)} \frac{xy(x-2y)}{x^2-y^2}$

let $y = mx$; $x \neq 0$, & m is the finite

$\lim_{(x,y)\to(0,0)} \frac{x\,mx\,(x-2mx)}{x^2-m^2x^2}$

$\lim_{(x,y)\to(0,0)} \frac{m\cancel{x^2}(x-2mx)}{\cancel{x^2}(1-m^2)}$

$\lim_{(x,y)\to(0,0)} \frac{x(1-2m)}{1-m^2}$

0

Since the limiting value are same or equal for any value of m i.e. the limit exist.

and $f(x,y) = f(0,0) = 0$

$\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0) = 0.$

Hence the function $f(x,y)$ is continuous.

Ans

(iv) $f(x,y) = \begin{cases} \dfrac{2xy^2}{x^2+3y^4} & \text{at } (x,y) \neq (0,0) \\ 0 & \text{at } (x,y) = (0,0) \end{cases}$

Sol^n: here,

$\lim_{(x,y)\to(0,0)} \dfrac{2xy^2}{x^2+3y^4}$

let $y = mx$; $x \neq 0$ & m is finite

$\lim_{(x,y)\to(0,0)} \dfrac{2x\,m^2x^2}{x^2+3m^4x^4}$

$\lim_{(x,y)\to(0,0)} \dfrac{\cancel{x^2}\, 2m^2}{\cancel{x^2}(1+3m^4x^2)}$

$\lim_{(x,y)\to(0,0)} \dfrac{2m^2}{1+3m^4x^2}$

Since ~~for all the~~ limiting value for all are different for different value of m so that the limit doesnot exist.

therefore the function $f(x,y)$ doesnot exist.

(v) $f(x,y) = \begin{cases} \dfrac{x^2y}{x^4+y^2} & \text{at } (x,y) \neq (0,0) \\ 0 & \text{at } (x,y) = (0,0) \end{cases}$

Sol^n: here,

$\lim_{(x,y)\to(0,0)} \dfrac{x^2y}{x^4+y^2}$

let $y = mx^2$; $x \neq 0$ & m is finite

$\lim_{(x,y)\to(0,0)} \dfrac{x^2 \cdot mx^2}{x^4+m^2x^4}$

~~$\lim_{(x,y)\to(0,0)} \dfrac{x^4 m}{x^4(1+m^2)} = \dfrac{m}{1+m^2}$~~

~~$\lim_{(x,y)\to(0,0)} \dfrac{mx}{x^2+m^2} = 0$~~

since the limiting value for ~~all~~ ~~same~~ differ ~~or equal~~ for any value of m so that ~~the~~ the limit doesnot exist.

~~&, $f(x,y) = 0$,~~

~~$\lim_{(x,y)\to(0,0)} f(x,y)$~~

~~$f(0,0)$~~

~~0~~

~~Hence the function $f(x,y)$ is continuous at $f(0,0)$ at $(x,y)=(0,0)$~~

Exercise :- 10.2

1. Calculate the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$

(i) $f(x,y) = 2x^2 - 3y - 4$

Sol^n: Here,

(i) $f(x,y) = 2x^2 - 3y - 4$ — (i)

diff. w.r.t x of eq^n (i) we get,

$$\frac{\partial f}{\partial x} = 4x - 0 - 0$$

$$\frac{\partial f}{\partial x} = 4x$$

& again diff. w.r.t y of eq^n (i) we get,

$$\frac{\partial f}{\partial y} = \frac{\partial(2x^2-3y-4)}{\partial y} = 0 - 3 - 0$$

$$= -3 \text{ Ans}$$

(ii) $f(x,y) = (x^2-1)(y+1)$

Sol^n: Here,

$f(x,y) = x^2y + x^2 - 1 \cdot y$

$f(x,y) = x^2y + x^2 - y - 1$ — (i)

Diff w.r.to. x of eq^n (i) we get,

$$\frac{\partial f}{\partial x} = \frac{\partial(x^2y+x^2-y-1)}{\partial x}$$

$$\frac{\partial f}{\partial x} = y \cdot 2x + 2x$$

$$\frac{\partial f}{\partial x} = 2xy + 2x$$

$$2x(y+2)$$

diff w.r.to y of eq^n (i) we get,

$$\frac{\partial f}{\partial y} = \frac{\partial(x^2y+x^2-y-1)}{\partial y} = x^2 + 0 - 1 - 0$$

$$= x^2 - 1 \text{ Ans}$$

(iii) $f(x,y,z) = \frac{1}{\sqrt{x^2+y^2+z^2}}$

diff. w.r.to x of eq^n (i) we get

$$\frac{\partial f}{\partial x} = \frac{\partial\left((x^2+y^2+z^2)^{-1/2}\right)}{\partial x}$$

$$\frac{\partial f}{\partial x} = \cancel{2x+0+0} \; -\frac{1}{2}(x^2+y^2+z^2)^{-3/2} \cdot 2x$$

$$\frac{\partial f}{\partial x} = -x(x^2+y^2+z^2)^{-3/2} = \frac{-x}{(x^2+y^2+z^2)^{3/2}}$$

diff. w.r.to. y ① we get

$$\frac{\partial f}{\partial y} = \frac{\partial (x^2+y^2+z^2)^{-1/2}}{dy}$$

$$\frac{\partial f}{\partial y} = -\frac{1}{2}(x^2+y^2+z^2)^{-3/2}\cdot(0+2y+0)$$

$$= -\frac{1}{2}(x^2+y^2+z^2)^{-3/2}(2y)$$

$$= \frac{-y}{(x^2+y^2+z^2)^{3/2}} \text{ Ans}$$

(iv) $f(x,y) = e^{(x+y+1)}$ at $(3,4)$

Sol^n: Here,

$f(x,y) = e^{(x+y+1)}$. at $(3,4)$ — ①

diff. w.r.to. x eq ① we get,

$$\frac{\partial f}{\partial x} = \frac{\partial (e^x . e^y . e)}{\partial x}$$

$$\frac{\partial f}{\partial x} = e^y . e \; e^x.$$

$$\frac{df}{\partial x} = e^4 . e . e^3$$

$$= e^8 \text{ Ans}$$

diff. w.r.to. y on both side we get,

$$\frac{\partial f}{\partial y} = \frac{\partial (e^x . e^y . e)}{\partial y}$$

$$= e^x . e \frac{\partial (e^y)}{dy}$$

$$= e^x . e \, e^y$$

$$= e^3 . e . e^4$$

$$= e^8 \text{ Ans}$$

(v) $f(x,y) = \dfrac{x+y}{xy-1}$ — ①

Sol^n: Here,

diff. w.r.t. x of eq^n ① we get,

$$\frac{\partial (f)}{\partial x} = \frac{\partial (x+y)}{\frac{\partial (xy-1)}{\partial x}} = \frac{d(x+y).(xy-1) - (x+y)d(xy-1)}{(xy-1)}$$

$$\frac{\partial f}{\partial x} = \frac{(1+0)(xy-1) - (x+y).y}{(xy-1)^2}$$

$$\frac{xy-1-xy-y^2}{(xy-1)^2}$$

$$\frac{-1-y^2}{(xy-1)^2}$$

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diff. eqn ① w.r.to y we get

$$\frac{\partial f}{\partial y} = \partial\left(\frac{x+y}{xy-1}\right)$$

$$\frac{\partial f}{\partial x} = \frac{(xy-1)\frac{d(x+y)}{dy} - (x+y)\frac{d(xy-1)}{dy}}{(xy-1)^2}$$

$$\frac{\partial f}{\partial x} = \frac{(xy-1)(0+1) - (x+y)(x)}{(xy-1)^2}$$

$$\frac{\partial f}{\partial x} = \frac{xy-1-x^2-xy}{(xy-1)^2}$$

$$\frac{-1-x^2}{(xy-1)^2} \text{ Ans}$$

(vi) $f(x,y) = e^{xy} \ln y$ at $(2,1)$

Soln: Here

diff. w.r. to x we get,

$$\frac{\partial f}{\partial x} = \frac{\partial(e^{xy} \cdot \ln y)}{dx} = \ln y \frac{d(e^{xy})}{d(xy)} \times \frac{d(xy)}{dx}$$

$$= \ln y \, (e^{xy}) \cdot y$$

$$= \ln 1 \, (e^2) \cdot 1$$

$$= 0 \text{ Ans}$$

(vi) ~~f(x,y)~~ -

diff. w.r. to y we get,

$$\frac{\partial f}{\partial y} = \frac{\partial(e^{xy} \cdot \ln y)}{dy}$$

$$\frac{\partial f}{\partial y} = e^{xy} \cdot \frac{1}{y} + \ln y \frac{d(e^{xy})}{dy}$$

$$\frac{\partial f}{\partial y} = \frac{e^{xy}}{y} + \ln y \cdot e^{xy} \cdot x$$

$$\frac{\partial f}{\partial y} = e^2 + 0$$

$$e^2 \text{ Ans}$$

2. Calculate the partial derivatives f_x, f_y, f_z.

(i) $f(x,y,z) = 1+xy^2-2z^2$

Sol^n: Here,

(i) diff. w.r. to x.

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(1+xy^2-2z^2)$$

$$\frac{\partial f}{\partial x} = 0+y^2-0 = y^2$$

diff. w.r. to y

$$\frac{\partial f}{\partial y} = \frac{\partial(1+xy^2-2z^2)}{\partial y} = (0+2yx-0)$$

$$= 2xy$$

(iii) diff. w.r. to z

$$\frac{\partial f}{\partial z} = \frac{\partial(1+xy^2-2z^2)}{\partial z} = 0+0-4z$$

$$= -4z \text{ Ans}$$

(ii) $f(x,y,z) = x-\sqrt{y^2+z^2}$

Sol^n: here,

$$f(x,y,z) = x-\sqrt{y^2+z^2}$$

diff. w.r. to x we get,

$$\frac{\partial f}{\partial x} = 1-0 = 1$$

diff w.r. to y. we get

$$\frac{\partial f}{\partial y} = \frac{\partial(x-\sqrt{y^2+z^2})}{\partial y} = \left(0-\frac{1}{2}(y^2+z^2)^{-1/2}\cdot 2y\right.$$

$$= \frac{-y}{(y^2+z^2)^{1/2}}$$

diff. w.r. to z we get,

$$\frac{\partial f}{\partial z} = \frac{\partial(x-\sqrt{y^2+z^2})}{\partial z}$$

$$= 0-\frac{1}{2}(y^2+z^2)^{-1/2}\cdot(0+2z)$$

$$= \frac{-z}{(y^2+z^2)^{1/2}} \text{ Ans}$$

(iii) $f(x,y,z) = \frac{1}{\sqrt{x^2+y^2+z^2}}$

Sol^n: here,

diff. w.r. to x we get,

$$\frac{\partial f}{\partial x} = \frac{\partial(x^2+y^2+z^2)^{-1/2}}{\partial x} = -\frac{1}{2}(x^2+y^2+z^2)^{-3/2}\cdot 2x$$

$$= \frac{-x}{(x^2+y^2+z^2)^{3/2}}$$

Diff w.r. to y we get

$$\frac{\partial f}{\partial y} = \frac{\partial(x^2+y^2+z^2)^{-1/2}}{\partial y} = -\frac{1}{2}(x^2+y^2+z^2)^{-3/2}\cdot 2y$$

$$= \frac{-y}{(x^2+y^2+z^2)^{3/2}}$$

diff. w.r. to z we get,

$$\frac{\partial f}{\partial z} = \frac{-z}{(x^2+y^2+z^2)^{3/2}}$$ Ans

(iv) $f(x,y,z) = yz\ln(xy)$ at $(3,1,-1)$

Sol^n: Here,

$f(x,y,z) = yz\ln(xy)$ at $(3,1,-1)$ — (i)

diff. eq^n (i) w.r. to x we get

$$\frac{\partial f}{\partial x} = \frac{\partial(yz\ln(xy))}{\partial x} = yz\frac{\ln(xy)}{\partial(xy)} \times \frac{\partial(xy)}{dx}$$

$$yz \cdot \frac{1}{xy} \cdot y$$

$$yz/x$$

$$\frac{1 \cdot -1}{3}$$

$$-1/3$$

diff. w.r. to y eq^n (i) we get

$$\frac{\partial f}{\partial y} = \frac{\partial(yz\ln(xy))}{\partial y} = \ln(xy)\frac{d(yz)}{dy} + yz\frac{d(\ln xy)}{dy}$$

$$= \ln xy \cdot z + yz \cdot \frac{1}{xy} \times x$$

$$= \ln xy \cdot z + z$$

$$= -\ln 3 - 1 \text{ Ans}$$

diff. w.r. to z we get,

$$\frac{\partial f}{\partial z} = \frac{\partial}{\partial z}(yz\ln(xy))$$

$$= \ln(xy)\, y \cdot 1$$

$$= y\ln(xy)$$

$$= 1\ln(3)$$

$$\ln 3 \text{ Ans}$$

(v) $f(x,y,z) = e^{-(x^2+y^2+z^2)}$ at $(2,4,5)$

Sol^n: Here,

$$f(x,y,z) = e^{-(x^2+y^2+z^2)}$$

diff. w.r. to x we get

$$\frac{\partial f}{\partial x} = \frac{d(e^{-(x^2+y^2+z^2)})}{dx}$$

$$= \frac{d(e^{-(x^2+y^2+z^2)})}{d[-(x^2+y^2+z^2)]} \cdot \frac{d(-[x^2+y^2+z^2])}{dx}$$

$$= e^{-(x^2+y^2+z^2)} \cdot -2x$$

$$= -2x\, e^{-(x^2+y^2+z^2)}$$

$$= -2\times2\, e^{-(4+16+25)}$$

$$= -4e^{-45} \text{ Ans}$$

diff. w.r. to y we get,

$$\frac{\partial f}{\partial y} = \frac{\partial(e^{-(x^2+y^2+z^2)})}{dy} = e^{-(x^2+y^2+z^2)} \cdot -2y$$

$$= e^{-(4+16+25)} \cdot -2\times4$$

$$= -8e^{-45} \text{ Ans}$$

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diff. w. r. to z we get,

$$\frac{\partial f}{\partial z} = \frac{\partial}{dz}\left[e^{-(x^2+y^2+z^2)}\right]$$

$$= e^{-(x^2+y^2+z^2)} \cdot -2 \cdot z$$

$$= e^{-(4+16+25)} \quad -2 \times 5$$

$$= -10\left(e^{-45}\right)$$ Ans

(3) calculate the second order mixed derivatives and then verifies the Euler's mixed derivative theorem.