Chapter 1

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Linear Equations in Linear Algebra

Linear equation

A linear equation in the variables $x_1, x_2 -- x_n$ is an equation in the form

$$a_1 x_1 + a_2 x_2 + --- a_n x_n = b$$

Where the coefficient $a_1, a_2 --- a_n$ and the value b are real or complex numbers.

The equation

i) $3x_1 - 5x_2 = -2$ (ii) $2x_1 + x_2 - x_3 = \sqrt{6}$

System of linear equation

A collection of one or more linear equations is called a system of linear equation or linear system.

i) $x_1 + 5x_1 = 7$ (ii) $2x_1 - x_2 + 1.5x_3 = 8$

$2x_1 + 7x_2 = 5$ $x_1 - 4x_3 = -7$.

are the system of linear equation.

Solution of the system of linear Equation.

A solution of the system is a list of values $(x_1, x_2, --- x_n)$ of numbers that ~~stat~~ satisfies the given system.

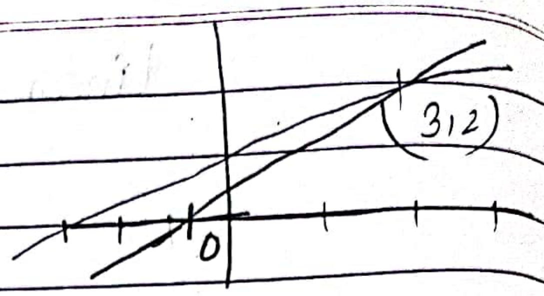
For example, the linear system $x_1 + 5x_2 = 7$
$2x_1 + 7x_2 = 5$

is satisfied by a list $(x_1, x_2) = (-8, 3)$, so $(-8, 3)$ is the solution of the system.

Ex: Consider a system

$x_1 - 2x_1 = -1$ — ①

$-x_1 + 3x_2 = 3$ — ②



① + ②

$x_2 = 2$ put in ① $x_1 - 4 = -1$ $x_1 = 3$

$\therefore (x_1, x_2)$ (3,2)

In the system, the solution is the ~~angle~~ single point (3,2) i.e the system has intersect exactly one point.

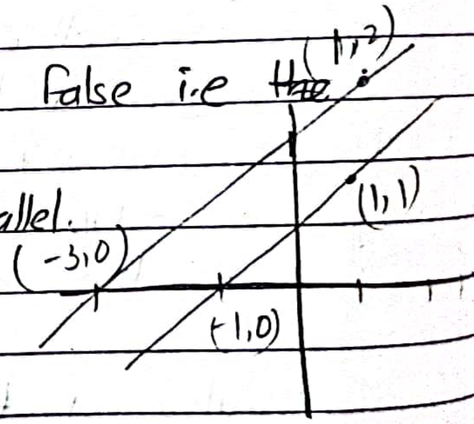
Ex: 2 Consider the system $x_1 - 2x_2 = -1$ — ①

$-x_1 + 2x_2 = 3$ — ②

① + ② $0+0=0$ which is false i.e ~~the~~

So the system has no soln.

The system of equation is parallel.



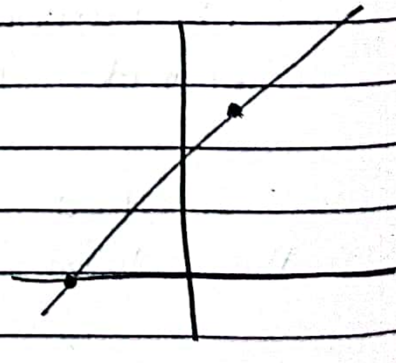
Ex: 3. Consider the system:

$x_1 - 2x_2 = -1$ — ①

$-x_1 + 2x_2 = 1$ — ②

① + ② $= 0 = 0$ which is true

So the system has infinity many soln i.e the system of equation has overlapping.



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Consistent and Inconsistent

A system of linear equation has either

i) exactly one solution or

(ii) no solution

iii) infinitely many sol^n

A system of linear equation is said to be consistent if it has either one solution or infinitely many solution and a system is inconsistent if it has no solution.

Ex: Why the system $x_1 - 3x_2 = 4$, $-3x_1 + 9x_2 = 8$ is consistent ? Give graphical representation.

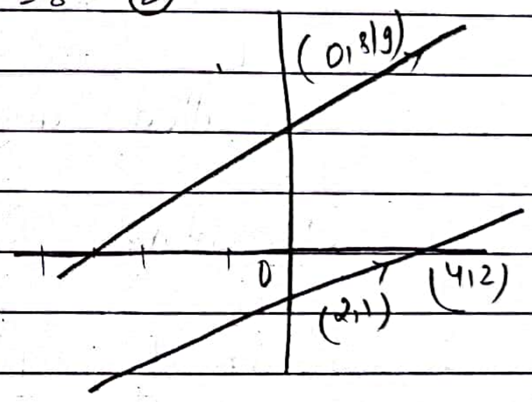
Given system is $x_1 - 3x_2 = 4$ — ①

$-3x_1 + 9x_2 = 8$ — ②

① × 3 + ②

$$3x_1 - 9x_2 = 12$$
$$-3x_1 + 9x_2 = 8$$
$$0 = 20 \quad \text{false}$$

∴ the system is inconsistent.



Matrix Notation of the system

A matrix term of coefficient and the consistant values of a linear system is known as matrix notation of the system. If the matrix involves only the coefficient then the matrix is called coefficient matrix and if the matrix notation involves the coefficient of linear system as well as constant value then the matrix is called augmented matrix.

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Ex: Consider a linear system

$$x_1 - 2x_2 + x_3 = 0$$
$$2x_2 - 8x_3 = 8$$
$$-4x_1 + 5x_2 + 9x_3 = -9$$

These the matrix notation with coefficient each variables in the form $\begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ -4 & 5 & 9 \end{bmatrix}$

is called coefficient matrix and the matrix notation of the system.

$$\left[\begin{array}{ccc:c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{array}\right]$$

is called augmented matrix.

Elementary Row Operation

1. Replacement: Replace one row by the sum of itself and a multiple of another row.

2. Interchange: Interchange two rows.

3. Scaling: Multiply all entries in a row by a non-constant.

Ex: Solve the system
$$x_1 - 2x_2 + x_3 = 0$$
$$2x_2 - 8x_3 = 8$$
$$-4x_1 + 5x_2 + 9x_3 = -9$$

Soln: The augmented matrix is

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$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{array}\right]$$

Apply $R_3 \rightarrow R_3 + 4R_1$

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{array}\right]$$

Apply $R_2 \rightarrow \frac{R_2}{2}$

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{array}\right]$$

Apply $R_3 \rightarrow R_3 + 3R_2$

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Apply $R_1 \rightarrow R_1 - R_3$ & $R_2 \rightarrow R_2 + 4R_3$

$$\left[\begin{array}{ccc|c} 1 & -2 & 0 & 0 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Apply $R_1 \rightarrow R_1 + 2R_2$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

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The system of eqn is

$x_1 - 2x_2$.

Ex: Determine if the following system is consistent.

$$x_2 - 4x_3 = 8$$
$$2x_1 - 3x_2 + 2x_3 = 1$$
$$5x_1 - 8x_2 + 7x_3 = 1$$

Soln:

The augmented matrix is.

$$\left[\begin{array}{ccc:c} 0 & 1 & -4 & 8 \\ 2 & -3 & 2 & 1 \\ 5 & -8 & 7 & 1 \end{array}\right]$$

Apply $R_1 \leftrightarrow R_2$

$$\left[\begin{array}{ccc:c} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 5 & -8 & 7 & 1 \end{array}\right]$$

Apply $R_3 \to R_3 - \frac{5}{2} R_1$

$$\left[\begin{array}{ccc:c} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & -\frac{1}{2} & 2 & -\frac{3}{2} \end{array}\right]$$

Apply $R_3 \to R_3 + \frac{1}{2} R_2$

$$\left[\begin{array}{ccc:c} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & \frac{5}{2} \end{array}\right]$$

The last row given

$0 = \frac{5}{2}$, false.

So the system have no solution i.e inconsistent.

Row Reduction and Echelon forms

A rectangular matrix is in echelon form (or row echelon form) if it has the following three properties

1) All nonzero rows are above any rows of all zero.
2) Each leading entry of a row is in a column to the right of the leading entry of the row above it.
3) All entries in a column below a leading entry are zero.

If a matrix in echelon form satisfies the following additional conditions, then it is in reduced echelon form (or reduced row echelon form)

4. The leading entry in each nonzero row is 1.
5. Each leading 1 is the only nonzero entry in its column.

The following matrices are in echelon form.

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & \frac{5}{2} \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Echelon matrix and Row Reduced Matrix.

Any matrix in echelon form is called echelon matrix. The following matrix are in echelon form. The leading entries [•] may have any non-zero value, the starred entries (*) may have any value (including zero).

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$$\begin{bmatrix} \bullet & * & * & * \\ 0 & \bullet & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & \bullet & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \bullet & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \bullet & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \bullet & * \end{bmatrix}$$

The following matrices are in reduced echelon form because the leading entries are 1's and there are 0's below and above each leading 1.

$$\begin{bmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & * & 0 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 1 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 1 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \end{bmatrix}$$

Pivot Position :-

A Pivot position in a matrix A is a location in A that corresponds to a leading 1 in the reduced echelon form of A. A pivot column is a column of A that contains a pivot position.

Ex:- Row reduce the matrix A below to echelon form and locate the pivot column of A.

$$A = \begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}$$

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Interchange R_1 and R_4 ie. $R_1 \leftrightarrow R_4$

pivot

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

pivot column

$R_2 \to R_2 + R_1$, $R_3 \to R_3 + 2R_1$

pivot

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

pivot column

$R_3 \to R_3 - \frac{5}{2} R_2$, $R_4 \to R_4 + \frac{3}{2} R_2$

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{bmatrix}$$

pivot column

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Ex: Reduce the augmented matrix of the system is reduced echelon form.

$$\begin{bmatrix} 0 & 1 & 0 & -3 & 3 \\ 1 & 0 & 3 & 0 & 2 \\ 0 & -2 & 3 & 2 & 1 \\ 3 & 0 & 0 & 7 & -5 \end{bmatrix}$$

Soln:

Apply $R_1 \to R_2$

$$\begin{bmatrix} 1 & 0 & 3 & 0 & 2 \\ 0 & 1 & 0 & -3 & 3 \\ 0 & -2 & 3 & 2 & 1 \\ 3 & 0 & 0 & 7 & -5 \end{bmatrix}$$

Apply $R_4 \to R.$

$$\begin{bmatrix} 1 & 0 & 3 & 0 & 2 \\ 0 & 1 & 0 & -3 & 3 \\ 0 & -2 & 3 & 2 & 1 \\ 0 & 0 & -9 & 7 & -11 \end{bmatrix}$$

Apply $R_3 \to R_3 + 2R_2$

$$\begin{bmatrix} 1 & 0 & 3 & 0 & 2 \\ 0 & 1 & 0 & -3 & 3 \\ 0 & 0 & 3 & -4 & 7 \\ 0 & 0 & -9 & 7 & -11 \end{bmatrix}$$

$R_4 \to R_4 + 3R_3$

$$\begin{bmatrix} 1 & 0 & 3 & 0 & 2 \\ 0 & 1 & 0 & -3 & 3 \\ 0 & 0 & 3 & -4 & 7 \\ 0 & 0 & 0 & -5 & 10 \end{bmatrix}$$

This is echelon form

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Now,

for reduced echelon form

$R_3 \to \frac{1}{3}R_3 \qquad R_4 \to -\frac{1}{3}R_4$

$$\begin{bmatrix} 1 & 0 & 3 & 0 & 2 \\ 0 & 1 & 0 & -3 & 3 \\ 0 & 0 & 1 & -\frac{4}{3} & \frac{7}{3} \\ 0 & 0 & 0 & 1 & -2 \end{bmatrix}$$

$R_2 \to R_2 + 3R_4 \qquad R_3 \to R_3 + \frac{4}{3}R_4$

$$\begin{bmatrix} 1 & 0 & 3 & 0 & 2 \\ 0 & 0 & 0 & 0 & -3 \\ 0 & 0 & 1 & 0 & -\frac{1}{3} \\ 0 & 0 & 0 & 1 & -2 \end{bmatrix}$$

$R_1 \to R_1 - 3R_3$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & -3 \\ 0 & 0 & 1 & 0 & -\frac{1}{3} \\ 0 & 0 & 0 & 1 & -2 \end{bmatrix}$$

This is reduced echelon form

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Solution of linear system.

The row reduction algorithm leads to the solution set of a linear system when the algorithm is applied to the augmented matrix of the system. Suppose for example that the augmented matrix of a linear system has been changed into the equivalent reduced echelon form.

$$\begin{bmatrix} 1 & 0 & -5 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

There are three variables because the augmented matrix has four columns. Here 1^{st} and 2^{nd} columns are pivot column. So the variables x_1 and x_2 in the matrix are basic variables and the 3^{rd} column is not a pivot column so x_3 is the free variables.

The associated system of equations is

$$x_1 - 5x_3 = 1$$
$$x_2 + x_3 = 4$$
$$0 = 0$$

Hence general soln is

$$\begin{cases} x_1 = 1 + 5x_3 \\ x_3 = 4 - x_3 \\ x_3 \text{ is free} \end{cases}$$

Ex: Given the matrix $\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix}$

discuss the forward phase and backward phase of the row reduction algorithm [2066].

Soln:-

$R_1 \leftrightarrow R_3$

pivot
$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$
pivot column

$R_2 \to R_2 - R_1$

pivot
$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$
pivot column

$R_3 \to R_3 - \frac{3}{2} R_2$

pivot
$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & -4 \end{bmatrix}$$
pivot column

This is echelon matrix

$R_1 \to \frac{1}{3} R_1$ and $R_2 \to \frac{1}{2} R_2$

$$\begin{bmatrix} 1 & -3 & 4 & -3 & 2 & 5 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

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$R_1 \rightarrow R_1 + 3R_2$

$$\begin{bmatrix} 1 & 0 & -2 & 3 & 5 & -4 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

$R_1 \rightarrow R_1 - 5R_3$ & $R_2 \rightarrow R_2 - R_3$

$$\begin{bmatrix} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

This is reduced echelon form
Step 1-3 is forward phase and step 4-6 is called backward phase.

Theorem 2 (Existence and Uniqueness Theorem)

A linear system is consistent if and only if the rightmost column of the augmented matrix is not a pivot column. (That is, a linear system has solution if and only if an echelon form the augmented matrix has no row of the form $[0 \cdot 0 \ldots 0 \; b]$ with $b \neq 0$). If the system consistent then the solution set contains either unique solution when there is no free variable or infinitely many solutions when there is at least one free variable.

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Ex: Find the general soln of the linear system whose augmented matrix has been reduced to

$$\begin{bmatrix} 1 & 6 & 2 & -5 & -2 & -4 \\ 0 & 0 & 2 & -8 & -1 & 3 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix}$$

Soln:-

The matrix is in echelon form but we want the reduced echelon form before solving for basic variables

Apply $R_1 \to R_1 + 2R_3$ & $R_2 \to R_2 + R_3$

$$\begin{bmatrix} 1 & 6 & 2 & -5 & 0 & 10 \\ 0 & 0 & 2 & -8 & 0 & 10 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix}$$

Apply $R_2 \to \frac{1}{2}R_2$

$$\begin{bmatrix} 1 & 6 & 2 & -5 & 0 & 10 \\ 0 & 0 & 1 & -4 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix}$$

Apply $R_1 \to R_1 - 2R_2$

$$\begin{bmatrix} 1 & 6 & 0 & 3 & 0 & 10 \\ 0 & 0 & 1 & -4 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix}$$

The associated system of equation is

$$x_1 + 6x_2 + 3x_4 = 0$$
$$x_3 - 4x_4 = 5$$
$$x_5 = 7$$

The pivot column of the matrix and I III and V are basic variable or x_2, x_4 are free variables.

Exercise 1.1

1. Determine the values of h such that the matrix is the augmented matrix of a consistent linear system.

i) $\begin{bmatrix} 2 & 3 & h \\ 4 & 6 & 7 \end{bmatrix}$

Soln:-

Apply : $R_2 \to R_2 - 2R_1$

$$\begin{bmatrix} 2 & 3 & h \\ 0 & 0 & 7-2h \end{bmatrix}$$

Since the system is consistence so the last row must be of the form $[0 \ 0 \ b]$

$$7 - 2h = 0$$
$$7 = 2h$$
$$\boxed{h = \frac{7}{2}}$$

(ii) $\begin{bmatrix} 1 & -2 & 3 \\ 3 & h & -2 \end{bmatrix}$

Soln:

Apply

$R_2 \to R_2 - 3R_1$

$\begin{bmatrix} 1 & -2 & 3 \\ 0 & h+6 & -11 \end{bmatrix}$

For consistency $h+6=0$

$\boxed{h \neq -6}$

2. Find the value of h and k so that the system has (a) no solution (b) a unique solution and (c) many solutions for the system of equation.

i) $x_1 + hx_2 = 2$

$4x_1 + 8x_2 = k$

Soln:

For augmented matrix

$\begin{bmatrix} 1 & h & 2 \\ 4 & 8 & k \end{bmatrix}$

Apply $R_2 \to R_2 - 4R_1$

$\begin{bmatrix} 1 & h & 2 \\ 0 & 8-4h & k-8 \end{bmatrix}$

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a) For no solution, the last row of the $[0 \quad 0 \quad b]$ with $b \neq 0$

$\therefore \; 8-4h=0$ & $k-8 \neq 0$

$h=2$ $\quad$ $k \neq 8$

b. For unique soln:

$8-4h \neq 0$

$\cancel{h=2}$ $h \neq 0$

c) For many soln

$8-4h=0$ and $k-8=0$

$h=2$ $\quad$ $k=8$

(ii) $x_1 + 3x_2 = 2$

$3x_1 + hx_2 = k$

soln:-

For augmented matrix

$$\begin{bmatrix} 1 & 3 & 2 \\ 3 & h & k \end{bmatrix}$$

Apply $R_2 \to R_2 - 3R_1$

$$\begin{bmatrix} 1 & 3 & 2 \\ 0 & h-9 & k-6 \end{bmatrix}$$

a) For no solution, the last row of the $[0 \quad 0 \quad b]$ with $b \neq 0$

$\therefore \; h-9=0$ & $k-6 \neq 0$

$h=9$ $\quad$ $k \neq 6$

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b. For unique solⁿ

$h - 9 \neq 0$

$h = 9$

c) For many solⁿ

$h - 9 = 0 \qquad k - 6 = 0$

$h = 9 \qquad k = 6$

3. Find the general solutions of the systems whose augmented matrices are:

i) $\begin{bmatrix} 1 & 3 & 4 & 7 \\ 3 & 9 & 7 & 6 \end{bmatrix}$

Solⁿ:

Apply: $R_2 \to R_2 - 3R_1$

$\begin{bmatrix} 1 & 3 & 4 & 7 \\ 0 & 0 & -5 & -15 \end{bmatrix}$

Here 1st column is pivot column so x_1 is basic corresponding variable and x_2 and x_3 are free.

Corresponding matrix is.

$x_1 + 3x_2 + 4x_3 = 7$

$-5x_3 = -15$

$x_3 = 3$

$x_1 + 3x_2 + 4 \times 3 = 7$

$x_1 + 3x_2 + 12 = 7$

$x_1 + 3x_2 = -5$

$x_1 = -5 - 3x_2$

$\boxed{\therefore\ x_1 = -5 - 3x_2 \ \& \ x_3 = 3}$

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(ii) $\begin{bmatrix} 0 & 1 & -6 & 5 \\ 1 & -2 & 7 & -6 \end{bmatrix}$

Soln:-

Apply $R_1 \leftrightarrow R_2$

$$\begin{bmatrix} 1 & -2 & 7 & -6 \\ 0 & 1 & -6 & 5 \end{bmatrix}$$

Here 1st and 2nd column are pivot column so x_1 and x_1 and x_2 are basic variable and x_3 is free.

The corresponding matrix is

$$x_1 - 2x_2 + 7x_3 \cancel{-6x_4} = -6 \qquad x_2 - 6x_3 + 0 = 5$$
$$x_1 = -6 + 2x_2 - 7x_3 \qquad x_2 = 5 + 6x_3$$
$$x_1 = -6 + 2(5 + 6x_3) - 7x_3$$
$$x_i = -6 + 10 + 12x_3 - 7x_3$$
$$x_1 = 4 + 5x_3$$

$\therefore$ $x_1 = 4 + 5x_3$ & $x_2 = 5 + 6x_3$

(iii) $\begin{bmatrix} 3 & -4 & 2 & 0 \\ -9 & 12 & -6 & 0 \\ -6 & 8 & -4 & 0 \end{bmatrix}$

Soln:-

Apply $R_2 \to R_2 + 3R_1$

$$\begin{bmatrix} 3 & -4 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ -6 & 8 & -4 & 0 \end{bmatrix}$$

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Apply $R_3 \to R_3 + 2R_1$

$$\begin{bmatrix} 3 & -4 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Apply $R_1 \to \frac{R_1}{3}$

$$\begin{bmatrix} 1 & -\frac{4}{3} & \frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Here 1st column is pivot column so x_1 is ~~pivot~~ basic variable and x_2 and x_3 is free.

The corresponding matrix is

$$x_1 - \frac{4}{3}x_2 + \frac{2}{3}x_3 = 0$$

$$x_1 = \frac{4}{3}x_2 - \frac{2}{3}x_3$$

$$\boxed{\therefore \quad x_1 = \frac{4}{3}x_2 - \frac{2}{3}x_3}$$

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A. Solve the following systems:

i) $2x + 3y + 4z = 20$

$3x + 4y + 5z = 26$

$3x + 5y + 6z = 31$

Soln:

Augmented matrix is

$$\left[\begin{array}{cccc} 2 & 3 & 4 & 20 \\ 3 & 4 & 5 & 26 \\ 3 & 5 & 6 & 31 \end{array}\right]$$

Apply $R_2 \to R_2 - \frac{3}{2}R_1 \quad R_3 \to R_3 - \frac{3}{2}R_1$

$$\left[\begin{array}{cccc} 2 & 3 & 4 & 20 \\ 0 & -\frac{1}{2} & -1 & -4 \\ 0 & \frac{1}{2} & 0 & 1 \end{array}\right]$$

Apply $R_3 \to R_3 + R_2$

$$\left[\begin{array}{cccc} 2 & 3 & 4 & 20 \\ 0 & -\frac{1}{2} & -1 & -4 \\ 0 & 0 & -1 & -3 \end{array}\right]$$

Apply $R_2 \to -2R_2 \quad R_3 \to -R_3$

$$\left[\begin{array}{cccc} 2 & 3 & 4 & 20 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

The corresponding system is

$2x_1 + 3x_2 + 4x_3 = 20$

$x_2 + 2x_3 = 8$

$x_3 = 3$

put $x_3 = 3$ in eqn (ii)

$x_2 + 2\times 3 = 8$

$x_2 + 6 = 8$

$x_2 = 2$

put x_2 and x_3 in

$2x_1 + 3\times 2 + 4\times 3 = 20$

$2x_1 + 6 + 12 = 20$

$2x_1 = 4$

$x_1 = 2$

$\therefore$ $\boxed{x_1, x_2, x_3 = 2, 2, 3}$

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(ii) $x_1 - 3x_3 = 8$

$2x_1 + 2x_2 + 9x_3 = 7$

$x_2 + 5x_3 = -2$

Soln:

The augmented matrix is

$$\begin{bmatrix} 1 & 0 & -3 & 8 \\ 2 & 2 & 9 & 7 \\ 0 & 1 & 5 & -2 \end{bmatrix}$$

Apply $R_2 \to R_2 - 2R_1$

$$\begin{bmatrix} 1 & 0 & -3 & 8 \\ 0 & 2 & 15 & -9 \\ 0 & 1 & 5 & -2 \end{bmatrix}$$

Apply $R_3 \to 2R_3 - R_2$

$$\begin{bmatrix} 1 & 0 & -3 & 8 \\ 0 & 2 & 15 & -9 \\ 0 & 0 & -5 & 5 \end{bmatrix}$$

Apply $R_2 \to \frac{1}{2}R_2$ $R_3 \to -\frac{1}{5}R_3$

$$\begin{bmatrix} 1 & 0 & -3 & 8 \\ 0 & 1 & \frac{15}{2} & -\frac{9}{2} \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

The corresponding system of eqn

$x_1 - 3x_3 = 8$

$x_2 + \frac{15}{2}x_3 = -\frac{9}{2}$

$x_3 = -1$

Put x_3 in eqn ①

$x_1 + 3 \times 1 = 8$

$x_1 = 5$

$x_2 + \frac{15}{2}x_3 = -\frac{9}{2}$

$x_2 + \frac{15}{2} \times -1 = -\frac{9}{2}$

$x_2 - \frac{15}{2} = -\frac{9}{2}$

$x_2 = -\frac{9}{2} + \frac{15}{2}$

$x_2 = \frac{6}{2} = 3$

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5. Determine if the system is consistent.

i) $x_1 + 3x_3 = 2$

$x_2 - 3x_4 = 3$

$-2x_2 + 3x_3 + 2x_4 = 1$

$3x_1 + 7x_4 = -5$

Soln:

→ The augmented matrix is

$$\begin{bmatrix} 1 & 0 & 3 & 0 & 2 \\ 0 & 1 & 0 & 3 & 3 \\ 0 & -2 & 3 & 2 & 1 \\ 3 & 0 & 0 & 7 & -5 \end{bmatrix}$$

Apply $R_4 \to R_4 - 3R_1$

$$\begin{bmatrix} 1 & 0 & 3 & 0 & 2 \\ 0 & 1 & 0 & 3 & 3 \\ 0 & -2 & 3 & 2 & 1 \\ 0 & 0 & 0 & 7 & -11 \end{bmatrix}$$

Apply $R_3 \to R_3 + 2R_2$

$$\begin{bmatrix} 1 & 0 & 3 & 0 & 2 \\ 0 & 1 & 0 & -3 & 3 \\ 0 & 0 & 3 & -4 & 7 \\ 0 & 0 & -9 & 7 & -11 \end{bmatrix} \xrightarrow{R_3 \to \frac{1}{3}R_3} \begin{bmatrix} 1 & 0 & 3 & 0 & 2 \\ 0 & 1 & 0 & -3 & 3 \\ 0 & 0 & 1 & -\frac{4}{3} & \frac{7}{3} \\ 0 & 0 & -9 & 7 & -11 \end{bmatrix}$$

$R_4 \to R_4 + 9R_3$

$$\begin{bmatrix} 1 & 0 & 3 & 0 & 2 \\ 0 & 1 & 0 & -3 & 3 \\ 0 & 0 & 1 & -\frac{4}{3} & \frac{7}{3} \\ 0 & 0 & 0 & -5 & 10 \end{bmatrix} \xrightarrow{R_4 \to \frac{1}{5}R_4} \begin{bmatrix} 1 & 0 & 3 & 0 & 2 \\ 0 & 1 & 0 & -3 & 3 \\ 0 & 0 & 1 & -\frac{4}{3} & \frac{7}{3} \\ 0 & 0 & 0 & 1 & -2 \end{bmatrix}$$

The corresponding system is

$$x_1 + 3x_3 = ?$$
$$x_2 - 3x_4 = 3$$
$$x_3 - \frac{4}{3}x_4 = \frac{7}{3}$$

Since the last row is not of the form $[0 \quad 0 \quad 0 \quad 0 \quad b]$

So the system is consistent.

(ii) ~~$x_2 - 8x_3 = 8$~~

(ii) $x_2 - 8x_3 = 8$

$2x_1 - 3x_2 + 2x_3 = 1$

$5x_1 - 8x_2 + 7x_3 = 1$

Soln:

The augmented matrix is,

$$\begin{bmatrix} 0 & 1 & -8 & 8 \\ 2 & -3 & 2 & 1 \\ 5 & -8 & 7 & 1 \end{bmatrix}$$

Apply $R_1 \leftrightarrow R_2$

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -8 & 8 \\ 5 & -8 & 7 & 1 \end{bmatrix}$$

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$R_1 \longrightarrow \frac{1}{2} R_1$

$$\begin{bmatrix} 1 & -\frac{3}{2} & -1 & \frac{1}{2} \\ 0 & 1 & -8 & 8 \\ 5 & -8 & 7 & 1 \end{bmatrix} \xrightarrow{R_3 \to R_3 - 5R_1} \begin{bmatrix} 1 & -\frac{3}{2} & -1 & \frac{1}{2} \\ 0 & 1 & -8 & 8 \\ 0 & -\frac{1}{2} & 12 & -\frac{3}{2} \end{bmatrix}$$

$R_3 \longrightarrow R_3 + \frac{1}{2} R_2 \quad , \quad R_1 \longrightarrow R_1 + \frac{3}{2} R_2$

$$\begin{bmatrix} 1 & 0 & -13 & \frac{25}{2} \\ 0 & 1 & -8 & 8 \\ 0 & 0 & 16 & \frac{5}{2} \end{bmatrix} \xrightarrow{R_3 \to \frac{1}{16} R_3} \begin{bmatrix} 1 & 0 & -13 & \frac{25}{2} \\ 0 & 1 & -8 & 8 \\ 0 & 0 & 1 & \frac{5}{32} \end{bmatrix}$$

The correspond system is

$$x_1 - 13x_3 = \frac{25}{2}$$

$$x_2 - 8x_3 = 8$$

$$x_3 = \frac{5}{32}$$

Since the last row is not of the form $\begin{bmatrix} 0 & 0 & 0 & b \end{bmatrix}$

So the system is consistent.

1.3 Vector Equations

Definition (column and Row vectors)
A matrix with only one column is called a column vector and a matrix with only one row is called a row vector.

Definition (Equal vectors)
Two vectors in R^2 are equal if and only if their corresponding entries are equal. Otherwise, the vectors are not equal.

Ex:- i) $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$, (ii) $u = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and $v = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$

Here in (i) $u = v$ if $u_1 = v_1$ and $u_2 = v_2$.
But in (ii) u and v are not equal because $3 \neq 2$ however $2 = 2$.

Sum of two vectors
The addition of two vectors in R^2 means the sum of corresponding entries of the vectors.

Ex:- If $u = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and $v = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$. Then

$$u + v = \begin{bmatrix} 3 \\ 2 \end{bmatrix} + \begin{bmatrix} -2 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 3-2 \\ 2+2 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}.$$

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Definition (linear combinations)

Given vectors $u_1, u_2 \ldots u_n$ in R^n and given scalars $c_1, c_2, \ldots c_n$ and if the vector y in R^n is defined by,

$$y = c_1 u_1 + c_2 u_2 - - - + c_n u_n$$

then y is called a linear combination of $u_1, u_2 \ldots u_n$ with $c_1, c_2 \ldots c_n$. In such case, the scalars $c_1, c_2 \ldots c_n$ are called weights for the combination.

Note:- A vector $y \in R^n$ is linear combination of vectors $u_1, u_2, u_3 \ldots u_n$ in R^n, if linear system with augumented matrix $[u_1 \ldots u_2 \ldots u_n \; y]$ represents consistence system i.e equation $x_1 u_1 + \ldots + x_n u_n = y$ is consistance.

Definition (subset of R^n spanned by vectors)

If $u_1, u_2 \ldots u_n$ in R^n then the set of all linear combinations of the vectors is denoted by span $(u_1, u_2 \ldots u_n)$ and is called subset of R^n spanned (or generated by vectors $u_1, u_2 \ldots u_n$

Thus the span $(u_1, u_2 \ldots u_n)$ can be written with with weights $c_1, c_2 \ldots c_n$ as $c_1 u_1 + c_2 u_2 \ldots c_n u_n$.

Note 1: A vector b is in span $(u_1, u_2 \ldots u_n)$ if linear system with augmented matrix $(u_1, u_2 \ldots u_n \; b]$ represents the consistence system.

2) i) Span $(v_1, v_2 \ldots v_n)$ contain each v_i $i = 1, 2, \ldots n$ because

$$v_i = 0v_1 + 0v_2 + 1v_i - - - 0v_n$$

(ii) Span $(v_1, v_2 \ldots v_n)$ contain zero vector also, because

$$0 = 0v_1 + 0v_2 + \ldots + 0v_n$$

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Ex: let $u = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ and $v = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ show that $\begin{bmatrix} h \\ k \end{bmatrix}$ in the span $[u, v]$ for all h and k.

Soln, Given:

20. let $a_1 = \begin{bmatrix} 1 \\ 4 \\ -2 \end{bmatrix}$ $a_2 = \begin{bmatrix} -2 \\ -3 \\ 7 \end{bmatrix}$ and $b = \begin{bmatrix} 4 \\ 1 \\ h \end{bmatrix}$. For what value of h is b in the plane spanned by a_1 and a_2?

Soln:-

We have if $x_1 a_1 + x_2 a_2 = b$ has soln, then b is in the plan spanned by a_1 and a_2.

let b is in the plane spanned by a_1 and a_2. So, the eqn

$Ax = b$ has soln for $A = [a_1 \; a_2]$

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Here the augmented matrix of $Ax = b$ is,

$$\begin{bmatrix} 1 & -2 & 4 \\ 4 & -3 & 1 \\ -2 & 7 & h \end{bmatrix}$$

Apply $R_2 \to R_2 - 4R_1$ and $R_3 \to R_3 + 2R_1$ Then

$$\begin{bmatrix} 1 & -2 & 4 \\ 0 & 5 & -15 \\ 0 & 3 & h+8 \end{bmatrix}$$

Again apply $R_2 \to \frac{R_2}{5}$ and $R_3 \to \frac{R_3}{3}$. Then,

$$\begin{bmatrix} 1 & -2 & 4 \\ 0 & 1 & -3 \\ 0 & 1 & \frac{h+8}{3} \end{bmatrix}$$

Again Apply $R_3 \to R_3 - R_2$ then

$$\begin{bmatrix} 1 & -2 & 4 \\ 0 & 1 & -3 \\ 0 & 1 & \frac{h+17}{3} \end{bmatrix}$$

Clearly the matrix gives soln only if

$$\frac{h+17}{3} = 0$$

$$h + 17 = 0$$
$$h = -17$$

Thus for $h = -17$, the vector b is in the plane spanned by a_1 and a_2.

1.4

The Matrix equation $Ax = b$

Definition (Definition of Ax)

If A is an $m \times n$ matrix with columns $a_1, a_2 \dots a_n$ and if x is in R^n then Ax is defined as.

$$Ax = [a_1, a_2 \dots a_n] \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix} = x_1 a_1 + x_2 a_2 + \dots x_n a_n.$$

Note: The product Ax is possible only if the total number entries of A and x are same.

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Exercise 1.2

1. Compute $(u+v)$ and $(3u-2v)$ when

i) $u = \begin{bmatrix} 6 \\ -1 \\ 5 \end{bmatrix}$ and $v = \begin{bmatrix} -3 \\ 4 \\ 0 \end{bmatrix}$

Soln:

$$u+v = \begin{bmatrix} 6 \\ -1 \\ 5 \end{bmatrix} + \begin{bmatrix} -3 \\ 4 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \\ 3 \\ 5 \end{bmatrix}$$

$$3u-2v = 3\begin{bmatrix} 6 \\ -1 \\ 5 \end{bmatrix} - 2\begin{bmatrix} -3 \\ 4 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 18 \\ -3 \\ 15 \end{bmatrix} - \begin{bmatrix} -6 \\ 8 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 18+6 \\ -3-8 \\ 15-0 \end{bmatrix} = \begin{bmatrix} 24 \\ -11 \\ 15 \end{bmatrix}$$

(ii) $u = \begin{bmatrix} 1 \\ -7 \\ 5 \end{bmatrix}$ and $v = \begin{bmatrix} -3 \\ 4 \\ 0 \end{bmatrix}$

Soln:-

$$u+v = \begin{bmatrix} 1 \\ -7 \\ 5 \end{bmatrix} + \begin{bmatrix} -3 \\ 4 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$$

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$3u - 2v$

$$3\begin{bmatrix} 1 \\ -7 \\ 5 \end{bmatrix} - 2\begin{bmatrix} -3 \\ 4 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \\ -21 \\ 15 \end{bmatrix} - \begin{bmatrix} -6 \\ 8 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 3+6 \\ -21-8 \\ 15-0 \end{bmatrix} = \begin{bmatrix} 9 \\ -29 \\ 15 \end{bmatrix}$$

2. Determine if b is a linear combination of a_1, a_2 and a_3.

i) $a_1 = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$, $a_2 \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$, $a_3 = \begin{bmatrix} 5 \\ -6 \\ 8 \end{bmatrix}$, $b = \begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix}$

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Definition (Homogeneous linear systems)

A linear systems is called homogeneous if it can be written in the form $Ax=0$ where A is $m \times n$ matrix and 0 be a null matrix of order $m \times 1$.

Definition (Trivial and non-trivial solution of homogeneous linear systems)

let $Ax=0$ be a homogeneous linear systems. The equation $Ax=0$ always has one solution $x=0$ where 0 is zero vector (null vector), such solution is called trivial solution. And, the non-zero solution of the equation $Ax=0$ is called non-trivial solution.

Solutions of non-homogeneous systems

A linear systems of the form $Ax=b$ is a non-homogeneous linear systems. The solution of such non-homogeneous systems can be written in parametric vector form as a single vector plus a vector form that satisfy the corresponding homogeneous system.

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Exercise 1.3

A. Solve the following system of linear equations and write the solution in parametric form, possible.

1. $6x+4y=2$
$3x-5y=-34$

Soln:

The augmented matrix

$$\begin{bmatrix} 6 & 4 & 2 \\ 3 & -5 & -34 \end{bmatrix}$$

$R_1 \to \frac{1}{2}R_1$

$$\begin{bmatrix} 3 & 2 & 1 \\ 3 & -5 & -34 \end{bmatrix}$$

$R_2 \to R_2 - R_1$

$$\begin{bmatrix} 3 & 2 & 1 \\ 0 & -7 & -35 \end{bmatrix}$$

$R_2 \to -\frac{1}{7}R_2$

$$\begin{bmatrix} 3 & 2 & 1 \\ 0 & 1 & -5 \end{bmatrix}$$

$R_1 \to R_1 - 2R_2$

$$\begin{bmatrix} 3 & 0 & -9 \\ 0 & 1 & 5 \end{bmatrix}$$

The corresponding eqn

$3x=-9$ $\quad y=5$

$x=-3$

$\therefore x=-3$ & $y=5$

$y=5$

2. $x+y+z=6$
$x-y+z=5$
$3x+y+z=8$

soln:

The augmented matrix

$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & -1 & 1 & 5 \\ 3 & 1 & 1 & 8 \end{bmatrix}$$

$R_2 \to R_2 - R_1$ $\quad R_3 \to R_3 - 3R_1$

$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & -2 & 0 & 1 \\ 0 & -2 & 2 & 10 \end{bmatrix}$$

$R_2 \to -\frac{1}{2}R_2$ $\quad R_3 \to \frac{1}{2}R_3$

$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 0 & 2 \\ 0 & 1 & 1 & 5 \end{bmatrix}$$

$R_3 \to R_3 - R_2$ $\quad R_1 \to R_1 - R_2$

$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$R_1 \to R_1 - R_3$

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$x=1$ $\quad y=2$ & $z=3$

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3. $x + y - z = 9$
$8y + 6z = -6$
$-2x + 4y - 6z = 40$

Sol^n:

$$\begin{bmatrix} 1 & 1 & -1 & 9 \\ 0 & 8 & 6 & -6 \\ -2 & 4 & -6 & 40 \end{bmatrix}$$

$R_3 \rightarrow R_3 + 2R_1$

$$\begin{bmatrix} 1 & 1 & -1 & 9 \\ 0 & 8 & 6 & -6 \\ 0 & 6 & -8 & 58 \end{bmatrix}$$

$R_2 \rightarrow \frac{1}{8} R_2$ $\quad$ $R_3 \rightarrow \frac{1}{2} R_3$

$$\begin{bmatrix} 1 & 1 & -1 & 9 \\ 0 & 1 & \frac{3}{4} & -\frac{3}{4} \\ 0 & 3 & -4 & 29 \end{bmatrix}$$

$R_1 \rightarrow R_1 - R_2$ $\quad$ $R_3 \rightarrow R_3 - 3R_2$

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4. $4y + 3z = 8$

$2x - z = 2$

$3x + 2y = 5$

Soln:

The augmented matrix

~~0 4 3 8~~

$$\begin{bmatrix} 0 & 4 & 3 & 8 \\ 2 & 0 & -1 & 2 \\ 3 & 2 & 0 & 5 \end{bmatrix}$$

$R_1 \leftrightarrow R_3$

$$\begin{bmatrix} 3 & 2 & 0 & 5 \\ 2 & 0 & -1 & 2 \\ 0 & 4 & 3 & 8 \end{bmatrix}$$

$R_1 \to R_1 - R_2$

$$\begin{bmatrix} 1 & 2 & 1 & 3 \\ 2 & 0 & -1 & 2 \\ 0 & 4 & 3 & 8 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$

$$\begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & -4 & -3 & -4 \\ 0 & 4 & 3 & 8 \end{bmatrix}$$

$R_3 \to R_3 + R_2$

$$\begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & -4 & -3 & -4 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

Since last is of the $[0\ 0\ 0\ b]$ with $b \neq 0$ so the system is inconsistent.

5. $7x - 4y - 2z = -6$
$16x + 2y + z = 3$

Soln:

The augmented matrix is

$$\begin{bmatrix} 7 & -4 & -2 & -6 \\ 16 & 2 & 1 & 3 \end{bmatrix}$$

$R_1 \to R_1 + 2R_2$

$$\begin{bmatrix} 39 & 0 & 0 & 0 \\ 16 & 2 & 1 & 3 \end{bmatrix}$$

$R_1 \to \frac{1}{39} R_1$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 16 & 2 & 1 & 3 \end{bmatrix}$$

$R_2 \to R_2 - 16R_1$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 3 \end{bmatrix}$$

1st and 2nd column are pivot column so x and y are basic but there are three variable so z is free variable.

The corresponding system,

$x = 0$ $\quad 2y + z = 3$

$2y = 3 - z$

$y = \frac{3}{2} - \frac{1}{2} z$

z = free.

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6. $2x_1 + 5x_2 + 6x_3 = 13$

$3x_1 + x_2 - 4x_3 = 0$

$x_1 - 3x_2 - 8x_3 = -10.$

Soln:

The augmented matrix is,

$$\begin{bmatrix} 2 & 5 & 6 & 13 \\ 3 & 1 & -4 & 0 \\ 1 & -3 & -8 & -10 \end{bmatrix}$$

$R_1 \leftrightarrow R_3$

$$\begin{bmatrix} 1 & -3 & -8 & -10 \\ 3 & 1 & -4 & 0 \\ 2 & 5 & 6 & 13 \end{bmatrix}$$

$R_2 \to R_2 - 3R_1$ & $R_3 \to R_3 - 2R_1$

$$\begin{bmatrix} 1 & -3 & -8 & -10 \\ 0 & 10 & 20 & 30 \\ 0 & 11 & 22 & 33 \end{bmatrix}$$

$R_2 \to \frac{1}{10} R_2$ & $R_3 \to \frac{1}{11} R_3$

$$\begin{bmatrix} 1 & -3 & -8 & -10 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\begin{bmatrix} 1 & -3 & -8 & -10 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_1 \to R_1 + 3R_2$

$$\begin{bmatrix} 1 & 0 & -2 & -1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

x_1 and x_2 are pivot and x_3 is variable.

Corresponding system is

$x - 2x_3 = -1$

$x_2 - 2x_3 = 3$

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B. Determine, if the following homogeneous system has a non-trival solution.

i) $3x_1 + 5x_2 - 4x_3 = 0$
$-3x_1 - 2x_2 + 4x_3 = 0$
$6x_1 + x_2 - 8x_3 = 0$

Soln:

$$\begin{bmatrix} 3 & 5 & -4 & 0 \\ -3 & -2 & 4 & 0 \\ 6 & 1 & -8 & 0 \end{bmatrix}$$

$R_2 \to R_2 + R_1$ & $R_3 \to R_3 - 2R_1$

$$\begin{bmatrix} 3 & 5 & -4 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & -9 & 0 & 0 \end{bmatrix}$$

$R_3 \to R_3 + 3R_2$

$$\begin{bmatrix} 3 & 5 & -4 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

1st and 2nd are pivot and x_1 and x_2 are basic x_3 is free variable.

The corresponding system is.

$3x_1 + 5x_2 - 4x_3 = 0$ $\quad$ $3x_2 = 0$ $\quad$ $x_3 =$ free

$3x_1 + 5\times0 - 4x_3 = 0$ $\quad$ $x_2 = 0$

$3x_1 - 4x_3 = 0$

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(ii) $x_1 + 3x_2 - 5x_3 = 0$

$x_1 + 4x_2 - 8x_3 = 0$

$-3x_1 - 7x_2 + 9x_3 = 0$

Soln:

The augmented matrix is,

$$\begin{bmatrix} 1 & 3 & -5 & 0 \\ 1 & 4 & -8 & 0 \\ -3 & -7 & 9 & 0 \end{bmatrix}$$

$R_2 \to R_2 - R_1$ & $R_3 \to R_3 + 3R_1$

$$\begin{bmatrix} 1 & 3 & -5 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 2 & -6 & 0 \end{bmatrix}$$

$R_1 \to R_1 - 3R_2$ & $R_3 \to 2R_2$

$$\begin{bmatrix} 1 & 0 & 4 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

1st and 2nd are pivot so x_1 and x_2 are basic and 3rd or x_3 is free variable.

$x_1 + 4x_3 = 0$ $\quad x_1 = -4x_3$

$x_1 - 3x_3 = 0$ $\quad x_2 = 3x_3$

$x_3 = 0$ $\quad x_3 = \text{free}$

$x_1 = -4x_3$

$x_2 = 3x_3$

$x_3 = \text{free}$

$$\begin{matrix} x_1 \\ x_2 \\ x_2 \end{matrix} = \begin{bmatrix} -4 \\ 3 \\ 1 \end{bmatrix} x_3$$

1.6 linear Independence

Definition (linearly independent and dependent)

The set of vectors $[v_1, \ldots v_n]$ in R^n is called linearly independent if the vector equation

$$x_1 v_1 + x_2 v_2 + \cdots + x_n v_n = 0$$

implies $x_1 = 0 = x_2 = \cdots = x_n$.

The set of vectors $[v_1 \ldots v_n]$ in R^n is called linearly dependent if the vector equation

$$x_1 v_1 + x_2 v_2 + \cdots + x_n v_n = 0$$

implies not all x_1 are zero for $j = 1, \ldots n$.

How to check the given set of vectors are linearly independent and linearly dependent?

- To check set of vectors $[v_1, v_2 \ldots v_k]$ linearly independent or linearly dependent.

i) Write the vector equation $x_1 v_1 + x_2 v_2 + \cdots + x_k v_k = 0$. where x_1 are scalar.

(ii) Write the augumented matrix $[v_1 \; v_2 \; v_3 \ldots v_k \; 0]$

(iii) Change the augumented matrix to echelon form

- If there has at least one free variable [vector equation having non-trival solution) then set of vector are linearly dependent.
- If there has no free variable (vector equation having trivial solution) then set of vectors are linearly independent.

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Theorem 7: If a set contains more vectors than there are entries in each vector, then the set is linearly dependent. That is, a set of vectors $\{v_1 \ldots v_n\}$ in R^m with $n>m$ then the set is linearly dependent.

Theorem 8: If a set $S=\{v_1 \ldots v_n\}$ in R^m contains the zero vector then the set is linearly dependent.

Ex: Determine of the given set is linearly dependent.

a) $\begin{bmatrix}1\\7\\6\end{bmatrix}, \begin{bmatrix}2\\0\\9\end{bmatrix}, \begin{bmatrix}3\\1\\5\end{bmatrix}, \begin{bmatrix}4\\1\\8\end{bmatrix}$; b) $\begin{bmatrix}-2\\4\\6\\10\end{bmatrix}, \begin{bmatrix}3\\-6\\-9\\15\end{bmatrix}$ c) $\begin{bmatrix}2\\3\\5\end{bmatrix} \begin{bmatrix}0\\0\\0\end{bmatrix} \begin{bmatrix}1\\1\\8\end{bmatrix}$

Soln:

a) Here, $V_1 V_2 V_3 V_4 \in R^3$ and $4>3$.
$\therefore$ Given set of vectors are linearly dependent.

b) let,

$$V_1=\begin{bmatrix}-2\\4\\6\\10\end{bmatrix} \text{ and } V_2=\begin{bmatrix}3\\-6\\-9\\15\end{bmatrix}$$

Here,

$$3V_1+2V_2 = 3\begin{bmatrix}-2\\4\\6\\10\end{bmatrix}+2\begin{bmatrix}3\\-6\\-9\\15\end{bmatrix}$$

$$=\begin{bmatrix}0\\0\\0\\60\end{bmatrix} \neq 0$$

$\therefore$ V_1 and V_2 are linearly independent.

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c) let,

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 3 & 0 & 1 \\ 5 & 0 & 8 \end{bmatrix}$$

Since the set of vectors A contains a zero vector (second column is zero vector). So A is linearly dependent.

Exercise 1.4.

1. Are the following set of vectors linearly dependent? If yes, find the relation between them.

i) $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

The augmented matrix of $Ax = 0$ is

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Here x_1, x_2 and x_3 are basic variable, so the homogeneous system has only trival sol^n $x_1 = 0$ $x_2 = 0$ $x_3 = 0$ Hence variable and linearly indipendent.

(ii) $\begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

Sol^n;

The augmented matrix of $Ax = 0$ is

$$\begin{bmatrix} 1 & 1 & 0 \\ 2 & 3 & 0 \end{bmatrix} \xrightarrow{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

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Here, x_1 and x_2 are both are basic variable so the homogeneous system has only trival soln $x_1=0$ $x_2=0$. Hence vector are linearly independent.

Alternatively: Vectors are not multiple of each other, so they are linearly independent.

(iii) $\begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 4 \\ 9 \\ 11 \end{bmatrix}$

Soln:

Vectors are not multiple of each other, so they are linearly independent.

iv) $\begin{bmatrix} 1 \\ 9 \\ 9 \\ 8 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 8 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 8 \end{bmatrix}$

Soln:

The augmented matrix $Ax=0$

$$\begin{bmatrix} 1 & 2 & 2 & 0 \\ 9 & 0 & 0 & 0 \\ 9 & 0 & 0 & 0 \\ 8 & 8 & 8 & 0 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - 9R_1 \\ R_3 \to R_3 - 9R_1 \\ R_4 \to R_4 - 8R_1 \end{matrix}$$

$$\begin{bmatrix} 1 & 2 & 2 & 0 \\ 0 & -2 & -2 & 0 \\ 0 & -2 & -2 & 0 \\ 0 & -8 & -8 & 0 \end{bmatrix}$$

$R_2 \to -\frac{1}{3} R_2$
$R_3 \to -\frac{1}{2} R_2$
$R_4 \to -\frac{1}{8} R_4$

$$\begin{bmatrix} 1 & 2 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

$R_3 \to R_3 - R_2$
$R_4 \to R_4 - R_3$

$$\begin{bmatrix} 1 & 2 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_1 \to R_1 - 2R_2 \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_1 = 0$
$x_2 + x_3 = 0$
$x_2 = -x_3$
if x_3 is
$x_1 = 0 \quad x_2 = -1 \quad x_3 = 1$
$x_1 v_1 + x_2 v_2 + x_3 v_3 = 0 v_1 - v_2 + v_3 = 0$

Here x_1 and x_2 are basic and x_3 are free variable, so the homogenous system has no trivial solution. Hence the vector are linear dependent.

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v) $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Soln:

One matrix x_3 are zero value, so the vectors are linearly dependent.

vi) $\begin{bmatrix} -1 \\ 4 \end{bmatrix}, \begin{bmatrix} -2 \\ -8 \end{bmatrix}$

Soln:

The vectors are not multiple of each other, so they are linearly independent

2. Determine if the columns of the matrix form a linearly independent set. Justify each answer.

i) $\begin{bmatrix} 0 & -8 & 5 \\ 3 & -7 & 4 \\ -1 & 5 & -4 \\ 1 & 3 & 2 \end{bmatrix}$

Soln:

$$\begin{bmatrix} 0 & -8 & 5 & 0 \\ 3 & -7 & 4 & 0 \\ -1 & 5 & -4 & 0 \\ 1 & 3 & 2 & 0 \end{bmatrix}$$

$R_1 \leftrightarrow R_4$

$$\begin{bmatrix} 1 & 3 & 2 & 0 \\ 3 & -7 & 4 & 0 \\ -1 & 5 & -4 & 0 \\ 0 & -8 & 5 & 0 \end{bmatrix}$$

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$R_2 \to R_2 - 2R_1$ & $R_3 \to R_3 + R_1$

$$\begin{bmatrix} 1 & -3 & 2 & 0 \\ 0 & -7 & -2 & 0 \\ 0 & 2 & -2 & 0 \\ 0 & -8 & 5 & 0 \end{bmatrix}$$

$R_2 \to R_3 - \frac{2}{7} R_2$ & $R_4 \to R_4 +$

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$$\begin{bmatrix} 1 & 4 & -3 & 0 \\ -2 & -7 & 5 & 1 \\ -4 & -5 & 7 & 5 \end{bmatrix}$$

Soln:

The augmented matrix $Ax = 0$

$$\begin{bmatrix} 1 & 4 & -3 & 0 & 0 \\ -2 & -7 & 5 & 1 & 0 \\ -4 & -5 & 7 & 5 & 0 \end{bmatrix}$$

$R_2 \to R_2 + 2R_1$ & $R_3 \to R_3 + 4R_1$

$$\begin{bmatrix} 1 & 4 & -3 & 0 & 0 \\ \cancel{-2} & \cancel{-7} & \cancel{5} & \cancel{1} & \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 11 & -5 & 5 & 0 \end{bmatrix}$$

$R_3 \to R_3 - 11R_2$

$$\begin{bmatrix} 1 & 4 & -3 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 6 & -6 & 0 \end{bmatrix}$$

$R_3 \to \frac{1}{6} R_3$

$$\begin{bmatrix} 1 & 4 & -3 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{bmatrix}$$

Here x_1 ~~and~~ x_2 and x_3 are basic variable and x_2 is free variable, so the column of given matrix and linearly independent.

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3. Determine for what value of h, is $\{v_1, v_2, v_3\}$ linearly dependent?

i) $v_1 = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}$, $v_2 = \begin{bmatrix} -3 \\ 9 \\ -6 \end{bmatrix}$, $v_3 = \begin{bmatrix} 5 \\ -7 \\ h \end{bmatrix}$

Solⁿ:

Consider the homogenous system.

$x_1 v_1 + x_2 v_2 + x_3 v_3 = 0$

The augmented matrix is.

$$\begin{bmatrix} 1 & -3 & 5 & 0 \\ -3 & 9 & -7 & 0 \\ 2 & -6 & h & 0 \end{bmatrix}$$

$R_2 \to R_2 - 3R_1$ & $R_3 \to R_3 - 2R_1$

$$\begin{bmatrix} 1 & -3 & 5 & 0 \\ 0 & 0 & 8 & 0 \\ 0 & 0 & h-10 & 0 \end{bmatrix}$$

Here x_2 is free variable, so the vector are linearly dependent for all value of h.

(ii) $V_1 = \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$, $V_2 = \begin{bmatrix} 3 \\ -5 \\ 7 \end{bmatrix}$, $V_3 = \begin{bmatrix} -1 \\ 5 \\ h \end{bmatrix}$

Soln:

Consider the homogeneous system

$x_1 x_2 + x_2 V_2 + x_3 V_3 = 0$

The augmented matrix is

$$\begin{bmatrix} 1 & 3 & -1 & 0 \\ -1 & -5 & 5 & 0 \\ 4 & 7 & h & 0 \end{bmatrix}$$

$R_2 \to R_2 + R_1$ & $R_3 \to R_3 - 4R_1$

$$\begin{bmatrix} 1 & 3 & -1 & 0 \\ 0 & -2 & 4 & 0 \\ 0 & -5 & h+4 & 0 \end{bmatrix}$$

$R_2 \to -\frac{1}{2} R_2$

$$\begin{bmatrix} 1 & 3 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & -5 & h+4 & 0 \end{bmatrix}$$

$R_3 \to R_3 + 5R_2$

$$\begin{bmatrix} 1 & 3 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & h-6 & 0 \end{bmatrix}$$

Since the vector are linearly dependent, so x_3 must be free, so, i.e if $h-6=0$

$h=6$.

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(iii) $v_1 = \begin{bmatrix} 1 \\ 5 \\ -3 \end{bmatrix}, v_2 = \begin{bmatrix} -2 \\ -9 \\ 6 \end{bmatrix}, v_3 = \begin{bmatrix} 3 \\ h \\ 9 \end{bmatrix}$

Soln:

Consider the homogeneous system.

$x_1 v_1 + x_2 v_2 + x_3 v_3 = 0$

The augmented matrix:

$$\begin{bmatrix} 1 & -2 & 3 & 0 \\ 5 & -9 & h & 0 \\ -3 & 6 & 9 & 0 \end{bmatrix} \xrightarrow[R_3 \to R_3 + 3R_1]{R_2 \to R_2 - 5R_1} \begin{bmatrix} 1 & -2 & 3 & 0 \\ 0 & 1 & h-15 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since x_3 is free variable, so variable are linearly dependent, for any values of h.

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Chapter 2
Transformations

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Definition (Transformation or Function or Mapping)

A transformation T from R^n to R^m (noted as $T: R^n \to R^m$) is a rule that assigns each vector x in R^n to a vector $T(x)$ in R^m. In such condition, R^n is domain of T and R^m is co-domain of T.

Definition (Domain, Co-domain, Image and Range of a Transformation)

Let $T: R^n \to R^m$ is a transformation. Then the set R^n is domain and R^m is co-domain of T. And, for x is R^n, the value $T(x)$ in R^m is called image of x under the transformation T. The set of all such image of x under T (i.e $T(x)$) is called the range of T.

1. Let $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ be the given matrix and define $T: R^2 \to R^2$ by $T(x) = Ax$. Find the images under T of $u = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$ and $v = \begin{bmatrix} a \\ b \end{bmatrix}$

Soln:

Let $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ and $T(x) = Ax$

Then

$$T(u) = Au = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 2+0 \\ 0-6 \end{bmatrix} = \begin{bmatrix} 2 \\ -6 \end{bmatrix}$$

$$T(v) = Av = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 2a+0 \\ 0+2b \end{bmatrix} = \begin{bmatrix} 2a \\ 2b \end{bmatrix}$$

Then the image of u and v under T are

$\begin{bmatrix} 2 \\ -6 \end{bmatrix}$ and $\begin{bmatrix} 2a \\ 2b \end{bmatrix}$

Matrix transformation:-
In above example T is a transformation that transforms a matrix ~~u(v)~~ u (or v) to T(u) (or (T(v)) which is again a matrix.
The concept develops the idea of matrix transforma

Definition (Matrix Transformation): For each $x \in R^n$, T(x) is computed as $Ax \in R^m$, where A is $m \times n$ matrix behave as transformation operator.

Ex: let $A = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix}$, $u = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $b = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}$, $C = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$ and

define a transformation $T: R^2 \to R^3$ by $T(x) = Ax$ So th

a) Find T(u).
b) Find x in R^2 whose image under T is b.
c) Is there more than one x whose image under T is b
d) Determine if C is in the range of T.

Soln:-
let,
$$T(x) = Ax$$
$$T(u) = Au = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 2+3 \\ 6-5 \\ -2-7 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ -9 \end{bmatrix}$$

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b) let $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $T(x) = b$

$Ax = b$

The Augmented matrix of $Ax=b$ is

$$\begin{bmatrix} 1 & -3 & 3 \\ 3 & 5 & 2 \\ -1 & 7 & -5 \end{bmatrix} \xrightarrow[R_3 \to R_3 + R_1]{R_2 \to R_2 - 3R_1} \begin{bmatrix} 1 & -3 & 3 \\ 0 & 14 & -7 \\ 0 & 4 & -2 \end{bmatrix}$$

$R_2 \to \frac{1}{14} R_2$

$$\begin{bmatrix} 1 & -3 & 3 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 4 & -2 \end{bmatrix} \xrightarrow[R_3 \to R_3 - 4R_2]{R_1 \to R_1 + 3R_1} \begin{bmatrix} 1 & 0 & \frac{3}{2} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}$$

Here, x_1 and x_2 are basic variable and no free variable so, the solution is unique and

$x_1 = 3/2$

$x_2 = -\frac{1}{2}$

$\therefore x = \begin{bmatrix} \frac{3}{2} \\ -\frac{1}{2} \end{bmatrix}$

c) As above, the solution is unique i.e. There is exactly one x is R^2 whose image under T is b.

d) The augmented matrix of $Ax = c$ is

$$\begin{bmatrix} 1 & -3 & 3 \\ 3 & 5 & 2 \\ -1 & 7 & 5 \end{bmatrix} \xrightarrow[R_3 \to R_3 + R_1]{R_2 \to R_2 - 3R_1} \begin{bmatrix} 1 & -3 & 3 \\ 0 & 14 & -7 \\ 0 & 4 & 8 \end{bmatrix}$$

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$$\xrightarrow[R_3 \to \frac{1}{4}R_3]{R_2 \to \frac{1}{14}R_2} \begin{bmatrix} 1 & -3 & 3 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow[R_2 \to R_3 - R_2]{R_1 \to R_1 + 3R_2} \begin{bmatrix} 1 & 0 & \frac{3}{2} \\ 0 & 0 & -\frac{1}{2} \\ 0 & 0 & \frac{5}{2} \end{bmatrix}$$

Since the last row is of the form $[0, 0\ b]$ $b \neq$
So the system is inconsistent. Hence c is not the range of T.

The matrix of a linear Transformation.

A transformation (or mapping) T is linear if.
i) $A(u+v) = Au + Av$ and
(ii) $A(cu) = cA(u)$ for all $u\,v$ in R^n and for any scalar c.

NOTE: If T is linear transformation, Then.
$T(0) = 0$ and $T(cu + dv) = cT(u) + dT(v)$ for all vectors u, v in its domain of T and all scalars c and d.

Ex:- let $T: R^2 \to R^3$ be defined $T(x) = (x_1, x_2, x_3)$ show that T is linear.

Soln:

let $u_1(u_1, u_2, u_3)$ and $v\ (v_1, v_2, v_3)\ IR^3$

Thus, $T(u) = (-u_1\ u_2\ u_3)$, $T(v) = (-v_1, v_2, v_3)$

and $T(u+v) = (-u_1\ u_2\ u_3) + (-v_1\ v_2, v_3)$

$= -(u_1 + v_1)\ (u_2 + v_2)\ (u_3 + v_3)$

$= (-u_1\ u_2\ u_3) + (v_1\ v_2\ v_3)$

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$$T(u+v) = T(u) + T(v)$$

Again,

for any scalar c

$$T(cu) = (-cu_1, cu_2, cu_3)$$
$$= c(u_1, u_2, u_3)$$
$$= cT(u)$$

Hence T is linear.

Ex: Define a linear transformation $T: R^2 \to R^2$ by

$$T(x) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix}$$

Find the image under T of $u = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$, $v = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $u+v$.

Soln:

$$T(u) = Au \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \end{bmatrix} \begin{bmatrix} 0\times4 + (-1)\times4 \\ 1\times4 + 0\times1 \end{bmatrix} = \begin{bmatrix} -4 \\ 1 \end{bmatrix}$$

$$T(v) = Av \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0\times2 + (-1)\times2 \\ 1\times3 + 0\times3 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$$

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Ex:- Show that the transformation T defined by $T(x_1, x_2) = (2x_1 - 3x_2, x_1 + 4, 5x_2)$ is not linear.

Soln:-

let T be a transformation defined by.

$T(x_1, x_2) = (2x_1 - 3x_2, x_1 + 4, 5x_2)$

let $u = (u_1, u_2)$ $v = (v_1, v_2)$

and $u + v = (u_1 + v_1, u_2 + v_2)$.

Now,

$T(u+v) = T(u_1 + v_1, u_2 + v_2)$

$= 2(u_1 + v_1) - 3(u_2 + v_2)$, $u_1 + v_1 + 4$, $5(u_2 + v_2)$

$= (2u_1 + 2v_1 - 3u_2 - 3v_2, u_1 + v_1 + 4, 5u_2 + 5v_2)$

$T(u) + T(v) = T(u_1, u_2) + T(v_1, v_2)$

$= 2(u_1 + v_1) - 3(u_2 + v_2), u_1 + v_1 + 4, + 5(u_2 + v_2)$

$= (2u_1 + 2v_1 - 3u_2 - 3v_2, u_1 + v_1 + 4, 5u_2 + 5v_2)$

as,

$T(u) + T(v) = T(u_1, u_2) + T(v_1, v_2)$

$= (2u_1 - 3u_2, u_1 + 4, 5u_2) + (2v_1 - 3v_2, v_2 + 4 + 5v_2)$

$= (2u_1 + 2v_1 - 3u_2 - 3v_2, u_1 + v_1 + 8, + 5u_2 + 5v_2)$

$\therefore T(u+v) \neq T(u) + T(v)$

$\therefore T$ is not linear transformation.

Standard Matrix for the Linear Transformation

let $T: R^n \rightarrow R^m$ be a linear transformation denoted by $T(x) = Ax$ for all x is R^n, where A is $m \times n$ matrix Then $A = [T(e_1)\ T(e_2) \cdots T(e_n)]$ where e_j is its j^{th} column of the identity matrix is R^n Then the matrix A is called standard matrix for T.

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Ex. Find the standard matrix A for linear transformation. $T(x) = 2x$ for x is R^3

Soln:

Transformation $= 2x$ is R^3

$$T(e_1) = 2e_1 = 2\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}2\\0\\0\end{pmatrix}$$

$$T(e_2) = 2e_2 = 2\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}0\\2\\0\end{pmatrix}$$

$$T(e_3) = 2e_3 = 2\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{bmatrix}0\\0\\2\end{bmatrix}$$

Now,

The standard matrix is for $T(x) = 2x$ is

$$A = \cancel{T(e_1)}[T(e_1) \quad T(e_2) \quad T(e_3)]$$

$$A = \begin{bmatrix}2 & 0 & 0\\0 & 2 & 0\\0 & 0 & 2\end{bmatrix}$$

or,

Given, $T(x) = 2x$

$$T(x_1, x_2, x_3) = (2x_1, 2x_2, 2x_3)$$

$$T\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \begin{pmatrix}2x_1\\2x_2\\2x_3\end{pmatrix} = \begin{bmatrix}2x_1 + 0.x_2 & 0x_3\\0x_1 + 2x_2 & 0x_3\\0x_1 + 0x_2 & 2x_3\end{bmatrix}$$

$$\therefore \begin{bmatrix}2 & 0 & 0\\0 & 2 & 0\\0 & 0 & 2\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = Ax$$

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Where $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ is the standard matrix

The column of $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $e_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Suppose T is a linear transformation from R^2 into R^3 Such

$$Te_1 \begin{bmatrix} -5 \\ 7 \\ 2 \end{bmatrix} \qquad T(e_2) = \begin{bmatrix} -3 \\ 8 \\ 0 \end{bmatrix}$$

Find a formula for the image of an orbitory x is R^2

Soln:-

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
$$= x_1 e_1 + x_2 e_2$$

Since T is linear Transformation.

$$T(x) = x_1 T(e_1) + x_2 T(e_2)$$
$$= x_1 \begin{pmatrix} -5 \\ 7 \\ 2 \end{pmatrix} + x_2 \begin{pmatrix} -3 \\ 8 \\ 0 \end{pmatrix}$$

$$= \begin{bmatrix} 5x_1 - 3x_2 \\ -7x_1 + 8x_2 \\ 2x_1 + 0 \end{bmatrix}$$

$$T(x) = [T(e_1) + T(e_2)] \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = Ax.$$

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Theorem:

let $T: R^4 \to R^m$ be a linear transformation Then there exists a unique matrix A such that,

$T(x) = Ax$ for all x is R^n

In fact, A is $m \times n$ matrix, whose j^{th} column is the vector $T(e_j)$, where e_j is the j^{th} column of the identity matrix is R^n. Then,

$A = [T(e_1)\ T(e_2) \ldots T(e_n)]$ is called standard matrix for T.

Definition: Onto

A transformation $T: R^n \to R^m$ is said to be onto R^m if each b is R^m is image of at least one x is R^n.

One to one:

A transformation $T: R^n \to R^m$ is said to be one to one of each b is R^m is the image of at most one x is R^n.

Theorem:

let $T: R^n \to R^m$ be a linear transformation Then T is one to one and only if the equation $T(x) = 0$ has only the trival solution.

Proof:

let $T: R^n \to R^m$ is linear transformation.

Suppose that T is one to one - Then for any x is R^n. $T(x) = 0 = T(0)$

$x = 0$ [$\because$ T being one to one]

This means the equation $T(x) = 0$ has only the trival solution)

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Conversely: Suppose that the solution $T(x)=0$ has only the trivial solution and use with to show T is one to one.

Soln:-

$$T(u)=T(v) \quad \{\text{for, since } u,v \text{ is } R\}$$

$$T(u)-T(v)=0$$

$$T(u-v)=0 \quad [\text{being } T \text{ is linear}]$$

Since $T(x)=0$ has only trivial solution, so ~~we~~ we should have,

$$T(u-v)=0 \Rightarrow u-v=0$$

$$u=v$$

Thus $T(u)=T(u)=u=v$

This means T is one to one.

Theorem:

let $T: R^n \to R^m$ be a linear transformation and let A be the standard matrix for T; Then,

a) T matrix R^n onto R^m if and only if the column of A scalar R^m.

b) T is one to one if and only if the column of A linearly independence.

Exercise 2

1. Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and define $T: R^2 \to R^2$ by $T(x) = Ax$.
Find the image under T of $u = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$ and $v = \begin{bmatrix} a \\ b \end{bmatrix}$

Soln:

$$T(u) = Au = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ -2 \end{bmatrix}$$

$$= \begin{bmatrix} 1\times-1 + 0\times-1 \\ 0\times-2 + 1\times-2 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$$

$$T(v) = Av = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

$$= \begin{bmatrix} 1\times a + 0\times a \\ 0\times b + 1\times b \end{bmatrix} \quad \begin{bmatrix} a \\ b \end{bmatrix}$$ Ans.

2. let $A = \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 0.5 \end{bmatrix}$, $u = \begin{bmatrix} 1 \\ 0 \\ -4 \end{bmatrix}$ and $v = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ Define
$T: R^3 \to R^3$ by $T(x) = Ax$ find $T(u)$ and $T(v)$.

Soln:

$$T(u) = Au = \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 0.5 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -4 \end{bmatrix}$$

$$= \begin{bmatrix} 1\times0.5 + 1\times0 + 1\times0 \\ 0\times0 + 0\times0.5 + 0\times0 \\ -4\times0 + -4\times0 + -4\times0.5 \end{bmatrix}$$

$$= \begin{bmatrix} 0.5 \\ 0 \\ -2 \end{bmatrix}$$

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$$T(v) = AV = \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 0.5 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$= \begin{bmatrix} 0.5\times a + 0\times a + 0\times a \\ 0\times b + 0.5\times b + 0\times b \\ 0\times c + 0\times c + 0.5\times c \end{bmatrix} = \begin{bmatrix} 0.5a \\ 0.5b \\ 0.5c \end{bmatrix} = 0.5 \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

3. let $A = \begin{bmatrix} 1 & 0 & -2 \\ -2 & 1 & 6 \\ 3 & -2 & -5 \end{bmatrix}$, $b = \begin{bmatrix} -1 \\ 7 \\ -3 \end{bmatrix}$. Define T by $T(x) = Ax$. Find a vector x whose image under T is b.

Soln:

$$T(x) = Ax = b$$

$$\begin{bmatrix} 1 & 0 & -2 \\ -2 & 1 & 6 \\ 3 & -2 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 7 \\ 3 \end{bmatrix}$$

The augmented matrix is, $Ax = b$ is,

$$\left[\begin{array}{ccc|c} 1 & 0 & -2 & -1 \\ -2 & 1 & 6 & 7 \\ 3 & -2 & -5 & 3 \end{array}\right]$$

$R_2 \to R_2 + 2R_1$, $R_3 \to R_3 - 3R_1$

$$\begin{bmatrix} 1 & 0 & -2 & -1 \\ 0 & 1 & 2 & 5 \\ 0 & -2 & 1 & 0 \end{bmatrix}$$

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$R_3 \rightarrow R_3 + 2R_2$

$$\begin{bmatrix} 1 & 0 & -2 & -1 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 5 & 10 \end{bmatrix}$$

$R_3 \rightarrow \frac{1}{5} R_3$

$$\begin{bmatrix} 1 & 0 & -2 & -1 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$R_1 \rightarrow R_1 + 2R_3$, $R_2 \rightarrow R_2 - 2R_3$

$$\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$$ Ans

4 let $A = \begin{bmatrix} 1 & -5 & -7 \\ -3 & 7 & 5 \end{bmatrix}$ $b = \begin{bmatrix} -2 \\ -2 \end{bmatrix}$. Define T by $T(x) = Ax$.

Find a vector x whose image under T is b.

soln:

$$T(x) = Ax = b$$

$$\begin{bmatrix} 1 & -5 & -7 \\ -3 & 7 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \end{bmatrix}$$

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The augmented matrix is,

$$\begin{bmatrix} 1 & -5 & -7 & -2 \\ -3 & 7 & 5 & -2 \end{bmatrix}$$

$R_2 \to R_2 + 3R_1$

$$\begin{bmatrix} 1 & -5 & -7 & -2 \\ 0 & -8 & -16 & -8 \end{bmatrix}$$

$R_2 \to -\frac{1}{8} R_2$

$$\begin{bmatrix} 1 & -5 & -7 & -2 \\ 0 & 1 & 2 & 1 \end{bmatrix}$$

$R_1 \to R_1 + 5R_2$

$$\begin{bmatrix} 1 & 0 & 3 & 3 \\ 0 & 1 & 2 & 1 \end{bmatrix}$$

Now,

x_3 is free variable. The corresponding system is

$x_1 + x_3 = 3$

$x_2 + 2x_3 = 1$

$x_3 =$ free

let $x_3 = 0$ then

$x_1 + 0 = 3$ $\qquad$ $x_2 + 2 \times 0 = 1$

$x_1 = 3$ $\qquad$ $x_2 = 1$

$$\therefore x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$$ Ans

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Find all $x \in R^4$ that are mapped in to zero vector by the transformation $x \rightarrow Ax$ for the given matrix

~~Soln:~~ $A = \begin{bmatrix} 1 & -4 & 7 & -5 \\ 0 & 1 & -4 & 3 \\ 2 & -6 & 6 & -4 \end{bmatrix}$ [Hint : find $x \in R^4$ for $Ax = 0$]

Soln:-

Given

$Ax = 0$

The augmented matrix of $Ax = 0$ is

$$\begin{bmatrix} 1 & -4 & 7 & -5 & 0 \\ 0 & 1 & -4 & 3 & 0 \\ 2 & -6 & 6 & -4 & 0 \end{bmatrix}$$

$R_3 \rightarrow R_3 - 2R_1$

$$\begin{bmatrix} 1 & -4 & 7 & -5 & 0 \\ 0 & 1 & -4 & 3 & 0 \\ 0 & 2 & -8 & 6 & 0 \end{bmatrix}$$

$R_1 \rightarrow R_1 + 4R_2$, $R_3 \rightarrow R_3 - 2R_2$

$$\begin{bmatrix} 1 & 0 & -9 & 7 & 0 \\ 0 & 1 & -4 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Here x_3 and x_4 are free variable, The corresponding system is,

$x_1 - 9x_3 + 7x_4 = 0$

$x_2 - 4x_3 + 3x_4 = 0$

$x_3 =$ free

$x_4 =$ free

$x_1 = 9x_3 - 7x_4$

$x_2 = 4x_3 - 3x_4$

$x_3 = 0$

$x_4 = 0$

$\therefore x = \begin{matrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{matrix} = \begin{bmatrix} 9x_3 - 7x_4 \\ 4x_3 - 3x_4 \\ 0 \\ 0 \end{bmatrix}$

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6. let $b = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ and $A = \begin{bmatrix} 1 & -4 & 7 & -5 \\ 0 & 1 & -4 & 3 \\ 2 & -6 & 6 & -4 \end{bmatrix}$. Is b in the range of linear transformation $x \rightarrow Ax$? Why or why not?

7. Prove the following transformation T are linear also find the matrix that implements the mapping.

i) $T(x_1, x_2, x_3, x_4) = (0, x_1 + x_2, x_2 + x_3, x_3 + x_4)$

(ii) $T(x_1, x_2, x_3) = (x_1 - 5x_2 + 4x_3, x_2 - 6x_3)$

Soln:-

$T(x_1, x_2, x_3, x_4) = (0, x_1 + x_2, x_2 + x_3, x_3 + x_4)$

$$T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ x_1 + x_2 \\ x_2 + x_3 \\ x_3 + x_4 \end{bmatrix}$$

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8. Assume that T is a linear transformation. Find the standard matrix of T.

i) $T: R^2 \to R^4$, $T(e_1) = (3,1,3,1)$ and $T(e_2) = (-5,2,0,0)$ where $e_1(1,0)$ and $e_2(0,1)$.

(ii) $T: R^3 \to R^2$, $T(e_1) = (1,3)$, $T(e_2) = (4,-7)$ and $T(e_3) = (-5,4)$ where e_1, e_2, e_3 are columns of 3x3 identity matrix.

Soln:

i) Standard matrix for T is

$$A = [T(e_1) \; T(e_2)]$$
$$= \begin{bmatrix} 3 & -5 \\ 1 & 2 \\ 3 & 0 \\ 1 & 0 \end{bmatrix}$$

(ii) The standard matrix for T is

$$A = [T(e_1), T(e_2) \; T(e_3)]$$
$$= \begin{pmatrix} 1 & 4 & -5 \\ 3 & -7 & 4 \end{pmatrix}$$

9. let $T: R^2 \to R^2$ be a linear transformation such that $T(x_1, x_2) = (x_1 + x_2, 4x_1 + 5x_2)$. Find x such that $T(x) = (3,8)$.

Soln:-

$$T(x_1, x_2) = (x_1 + x_2, 4x + 5x_2)$$

$$T\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ 4x_1 + 5x_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$Tx = Ax$$

$$Ax = \begin{bmatrix} 3 \\ 8 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} -3 \\ 8 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 8 \end{bmatrix}$$

$$R_2 \rightarrow R_2 - 4R_1$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

10. Let T be a linear transformation whose standard matrix is given. Describe if T is a one-to-one mapping. Justify your answer.

i) $\begin{bmatrix} -5 & 10 & -5 & 4 \\ 8 & 13 & -4 & 7 \\ 4 & -9 & 5 & -3 \\ -3 & -2 & 5 & 4 \end{bmatrix}$ (ii) $\begin{bmatrix} 7 & 5 & 4 & -9 \\ 10 & 6 & 16 & -4 \\ 12 & 8 & 12 & 7 \\ -8 & -6 & -2 & 5 \end{bmatrix}$

i) $\begin{bmatrix} -5 & 10 & -5 & 4 \\ 8 & 3 & -4 & 7 \\ 4 & -9 & 5 & -3 \\ -3 & -2 & 5 & 1 \end{bmatrix}$

Sol^n:-

$R_1 \to R_1 + R_3$

$$\begin{bmatrix} -1 & 1 & 0 & 1 \\ 8 & 3 & -4 & 7 \\ 4 & -9 & 5 & -3 \\ -3 & -2 & 5 & 4 \end{bmatrix} \xrightarrow{R_1 \to R_4} \begin{bmatrix} 1 & -1 & 0 & -1 \\ 8 & 3 & -4 & 7 \\ 4 & -9 & 5 & -3 \\ -3 & -2 & 5 & 4 \end{bmatrix}$$

$R_2 \to R_2 - 8R_1$, $R_3 \to R_3 - 4R_1$ $\quad R_3 \to R_3 + 3R_1$

$$\begin{bmatrix} 1 & -1 & 0 & -1 \\ 0 & 11 & -4 & 15 \\ 0 & -5 & 5 & 1 \\ 0 & -5 & 5 & 1 \end{bmatrix} \quad R_4 \to R_4 - R_3$$

$$\begin{bmatrix} 1 & -1 & 0 & -1 \\ 0 & 11 & -4 & 15 \\ 0 & -5 & 5 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_3 \to R_3 + \frac{5}{11}R_2} \begin{bmatrix} 1 & -1 & 0 & -1 \\ 0 & 11 & -4 & 15 \\ 0 & 0 & & \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since the system has one free variable then each b is the image of none

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(ii) $\begin{bmatrix} 7 & 5 & 4 & -9 \\ 10 & 6 & 16 & -4 \\ 12 & 8 & 12 & 7 \\ -8 & 6 & -2 & 5 \end{bmatrix}$

$R_1 \longrightarrow R_1 + R_4$ & $R_2 \longrightarrow$

$\begin{bmatrix} -1 & -1 & 2 & -4 \\ 5 & 3 & 8 & -2 \\ 12 & 8 & 12 & 7 \\ -8 & 6 & -2 & 5 \end{bmatrix}$

$R_1 \rightarrow -R_1$

$\begin{bmatrix} 1 & 1 & -2 & 4 \\ 5 & 3 & 8 & -2 \\ 12 & 8 & 12 & 7 \\ -8 & 6 & -2 & 5 \end{bmatrix}$

$R_2 \rightarrow R_2 - 5R_1$, $R_3 \rightarrow R_3 -$

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11. Let $T: R^5 \to R^5$ be a linear transformation whose standard matrix is given. Describe if T is a onto mapping. Justify your answer.

i)
$$\begin{bmatrix} 4 & -7 & 3 & 7 & 5 \\ 6 & -8 & 5 & 12 & -8 \\ -7 & 10 & -8 & -9 & 14 \\ 3 & -5 & 4 & 2 & -6 \\ -5 & 6 & -6 & -7 & 3 \end{bmatrix}$$

(ii)
$$\begin{bmatrix} 9 & 13 & 5 & 6 & -1 \\ 14 & 15 & -7 & -6 & 4 \\ -8 & -9 & 12 & -5 & -9 \\ -5 & -6 & -8 & 9 & 8 \\ 13 & 14 & 15 & 2 & 11 \end{bmatrix}$$

i)
$$\begin{bmatrix} 4 & -7 & 3 & 7 & 5 \\ 6 & -8 & 5 & 12 & -8 \\ -7 & 10 & -8 & -9 & 14 \\ 3 & -5 & 4 & 2 & -6 \\ -5 & 6 & -6 & -7 & 3 \end{bmatrix}$$

Sol: