

Group-A (10X2=20)

1. Verify Rolle's theorem for the function $f(x) = \frac{x^3}{3} - 3x$ on the interval $[-3,3]$

Here the given function is $f(x) = \frac{x^3}{3} - 3x$

The given interval is $[-3,3]$

According to Rolle's theorem,

$$f(-3) = \frac{(-3)^3}{3} - 3(-3) = 0$$

$$f(3) = \frac{(3)^3}{3} - 3(3) = 0$$

$$\text{Now, } f'(x) = \frac{3x^2}{3} - 3 = x^2 - 3$$

$$\text{We have, } f'(c) = 0$$

$$\text{or, } c^2 - 3 = 0$$

$$\text{or, } c = \sqrt{3}$$

Here, c falls between $[-3,3]$. Thus, Rolle's theorem is verified.

2. Obtain the area between two curves $y = \sec^2 x$ and $y = \sin x$ from $x = 0$ to $x = \frac{\pi}{4}$

$$\text{Let } f(x) = \sec^2 x \quad ; \quad 0 \leq x \leq \frac{\pi}{4}$$

$$g(x) = \sin x \quad ; \quad 0 \leq x \leq \frac{\pi}{4}$$

$$\text{we know that, } A = \int_a^b [f(x) - g(x)] dx$$

$$\text{or, } A = \int_0^{\frac{\pi}{4}} (\sec^2 x - \sin x) dx$$

$$\text{or, } A = [\tan x + \cos x]_0^{\frac{\pi}{4}}$$

$$\text{or, } A = [\tan \frac{\pi}{4} + \cos \frac{\pi}{4} - \tan 0 - \cos 0]$$

$$\text{or, } A = 1 + \frac{1}{\sqrt{2}} - 0 - 1$$

$$\text{or, } A = \frac{1}{\sqrt{2}}$$

3. Test the convergence of p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ for $p > 1$

Solution: If $p > 1$, then $f(x) = \frac{1}{x^p}$ is a positive decreasing function of x . Since

$$\begin{aligned} \int_1^{\infty} \frac{1}{x^p} dx &= \int_1^{\infty} x^{-p} dx = \lim_{b \rightarrow \infty} \left[\frac{x^{-p+1}}{-p+1} \right]_1^b \\ &= \frac{1}{1-p} \lim_{b \rightarrow \infty} \left(\frac{1}{b^{p-1}} - 1 \right) = \frac{1}{1-p} (0 - 1) = \frac{1}{p-1} \end{aligned}$$

Here, $b^{p-1} \rightarrow \infty$ as $b \rightarrow \infty$ because $p-1 > 0$. Thus, the series converges by the Integral test.

4. Find the eccentricity of the hyperbola $9x^2 - 16y^2 = 144$

Here the given hyperbola is

$$9x^2 - 16y^2 = 144$$

$$\text{or, } \frac{9x^2}{144} - \frac{16y^2}{144} = \frac{144}{144}$$

$$\text{or, } \frac{x^2}{16} - \frac{y^2}{9} = 1$$

$$\text{which is in the form of } \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$\text{where, } a^2 = 16 \rightarrow a = 4 \quad \text{and} \quad b^2 = 9 \rightarrow b = 3$$

$$\text{now, } c = \sqrt{a^2 + b^2} = \sqrt{16 + 9} = \sqrt{25} = 5$$

Eccentricity of the hyperbola is given as;

$$e = \frac{c}{a} = \frac{5}{4}$$

5. Find a vector perpendicular to the plane of $P(1, -1, 0), Q(2, 1, -1), R(-1, 1, 2)$

Here, $P(1, -1, 0), Q(2, 1, -1), R(-1, 1, 2)$

The vector $\overrightarrow{PQ} \times \overrightarrow{PR}$ is perpendicular to the plane because it is perpendicular to both the vector.

$$\overrightarrow{PQ} = (2 - 1)\vec{i} + (1 + 1)\vec{j} + (-1 - 0)\vec{k} = \vec{i} + 2\vec{j} - \vec{k}$$

$$\overrightarrow{PR} = (-1 - 1)\vec{i} + (1 + 1)\vec{j} + (2 - 0)\vec{k} = -2\vec{i} + 2\vec{j} + 2\vec{k}$$

Now, vector perpendicular to the plane is given as;

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} i & j & k \\ 1 & 2 & -1 \\ -2 & 2 & 2 \end{vmatrix}$$

$$\text{or, } \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} 2 & -1 \\ 2 & 2 \end{vmatrix} \vec{i} - \begin{vmatrix} 1 & -1 \\ -2 & 2 \end{vmatrix} \vec{j} + \begin{vmatrix} 1 & 2 \\ -2 & 2 \end{vmatrix} \vec{k}$$

$$\text{or, } \overrightarrow{PQ} \times \overrightarrow{PR} = (4 + 2)\vec{i} - (2 - 2)\vec{j} + (2 + 4)\vec{k}$$

$$\text{or, } \overrightarrow{PQ} \times \overrightarrow{PR} = 6\vec{i} + 6\vec{k}$$

6. Find the area enclosed by the curve $r^2 = 4 \cos 2\theta$

Here, the equation of the curve is

$$r^2 = 4 \cos 2\theta \rightarrow r = 2\sqrt{\cos 2\theta}$$

If we draw typical ray from origin, we get

$$0 \leq r \leq 2\sqrt{\cos 2\theta} \quad \text{and} \quad 0 \leq \theta \leq \frac{\pi}{4}$$

Now,

$$\begin{aligned} \text{Area (A)} &= 4 \int_0^{\frac{\pi}{4}} \int_0^{2\sqrt{\cos 2\theta}} r \, dr \, d\theta \\ &= 4 \int_0^{\frac{\pi}{4}} \left[\frac{r^2}{2} \right]_0^{2\sqrt{\cos 2\theta}} d\theta \\ &= 4 \int_0^{\frac{\pi}{4}} \left[\frac{(2\sqrt{\cos 2\theta})^2}{2} \right] d\theta \\ &= 4 \int_0^{\frac{\pi}{4}} 2 \cos 2\theta \, d\theta \\ &= 8 \left[\frac{\sin 2\theta}{2} \right]_0^{\frac{\pi}{4}} \\ &= 8 \frac{\sin \frac{\pi}{2}}{2} \\ &= 4 \end{aligned}$$

7. Obtain the values of $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ at the point (4,-5) if $f(x, y) = x^2 + 3xy + y - 1$

Here, $f(x, y) = x^2 + 3xy + y - 1$

Now, $(\frac{\partial f}{\partial x})$ at point (4,-5) is given as

$$\frac{\partial f}{\partial x} = \frac{\partial (x^2 + 3xy + y - 1)}{\partial x} = 2x + 3y = 2 \times 4 + 3 \times (-5) = -7$$

Also, $(\frac{\partial f}{\partial y})$ at point (4,-5) is given as

$$\frac{\partial f}{\partial y} = \frac{\partial (x^2 + 3xy + y - 1)}{\partial y} = 3x + 1 = 3 \times 4 + 1 = 13$$

8. Using partial derivatives, find $\frac{dy}{dx}$ if $x^2 + \cos y - y^2 = 0$

Here, $f(x, y) = x^2 + \cos y - y^2$

Now, $f_x = 2x$ and $f_y = -\sin y - 2y$

we have, $\frac{dy}{dx} = \frac{f_x}{f_y} = \frac{-2x}{-\sin y - 2y} = \frac{2x}{\sin y + 2y}$

9. Find the partial differential equation of the function $(x - a)^2 + (y - b)^2 + z^2 = c^2$

Here, the given function is $(x - a)^2 + (y - b)^2 + z^2 = c^2$ ----- (i)

Differentiating equation (i) w.r.t. x keeping y constant.

$$\frac{\delta f}{\delta x} = 2(x - a) + 2z \frac{\partial z}{\partial x} = 0$$

$$\text{or, } 2(x - a) = -2z \frac{\partial z}{\partial x}$$

$$\text{or, } x - a = -zp ; [p = \frac{\partial z}{\partial x}]$$

$$\text{or, } a = x + zp$$

Differentiating equation (i) w.r.t. y keeping x constant.

$$\frac{\delta f}{\delta y} = 2(y - b) + 2z \frac{\partial z}{\partial y} = 0$$

$$\text{or, } 2(y - b) = -2z \frac{\partial z}{\partial y}$$

$$\text{or, } y - b = -zq ; [q = \frac{\partial z}{\partial y}]$$

$$\text{or, } b = y + zq$$

Putting the values of a and b in equation (i), we get

$$(x - x - zp)^2 + (y - y - zq)^2 + z^2 = c^2$$

$$\text{or, } z^2 p^2 + z^2 q^2 + z^2 = c^2$$

$$\text{or, } z^2(p^2 + q^2 + 1) = c^2$$

10. Solve the partial differential equation $x^2 p + q = z^2$

Here, the given partial differential equation is

$$x^2 p + q = z^2$$

Comparing it with: $Pp + Qq = R$

we get; $P = x^2 ; Q = 1 ; R = z^2$

$$\text{Now, } \frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \rightarrow \frac{dx}{x^2} = \frac{dy}{1} = \frac{dz}{z^2}$$

Taking first two equations, we have

$$\frac{dx}{x^2} = \frac{dy}{1}$$

Integrating both sides

$$\int \frac{1}{x^2} dx = \int 1 dy$$

$$\text{or, } -\frac{1}{x} = y + c_1$$

$$\text{or, } c_1 = y + \frac{1}{x}$$

Taking last two equations, we have

$$\frac{dy}{1} = \frac{dz}{z^2}$$

Integrating both sides

$$\int 1 \, dy = \int \frac{dz}{z^2}$$

$$\text{or, } y = -\frac{1}{z} + c_2$$

$$\text{or, } c_2 = y + \frac{1}{z}$$

Thus, $(c_1, c_2) = (y + \frac{1}{x}, y + \frac{1}{z})$

Group-B (5X4=20)**11. State and prove the Mean Value Theorem for a differentiable function****Statement:**

If $f(x)$ is continuous on $[a, b]$ and differentiable on (a, b) , then there exists a point c in (a, b) such that

$$f(b) - f(a)$$

$$f(b) - f(a) = f'(c)(b - a) \rightarrow f'(c) = \frac{b - a}{f(b) - f(a)}$$

Proof:

Consider the function $h(x) = f(x)(b - a) - x(f(b) - f(a))$, noting that $h(x)$ is continuous on $[a, b]$ because $f(x)$ is. We compute

$$h'(x) = f'(x)(b - a) - (f(b) - f(a))$$

which is differentiable on (a, b) since $f(x)$ is. Note that

$$\begin{aligned} h(a) &= b \cdot f(a) - a \cdot f(a) - a \cdot f(b) + a \cdot f(a) \\ &= b \cdot f(a) - a \cdot f(b) \\ &= b \cdot f(b) - a \cdot f(b) - b \cdot f(b) + b \cdot f(a) \\ &= f(b)(b - a) - b(f(b) - f(a)) \\ &= h(b) \end{aligned}$$

12. Find the length of the asteroid $x = \cos^3 t, y = \sin^3 t$ for $0 \leq t \leq 2\pi$

Here, $x = \cos^3 t, y = \sin^3 t$; $0 \leq t \leq 2\pi$

$$\left(\frac{dx}{dt}\right)^2 = [3 \cos^2 t (-\sin t)]^2 = 9 \cos^4 t \cdot \sin^2 t$$

$$\left(\frac{dy}{dt}\right)^2 = [3 \sin^2 t \cdot \cos t]^2 = 9 \sin^4 t \cdot \cos^2 t$$

$$\begin{aligned} \text{Length of astroid} &= \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \\ &= \sqrt{9 \cos^4 t \cdot \sin^2 t + 9 \sin^4 t \cdot \cos^2 t} \\ &= 3 \cos t \cdot \sin t \end{aligned}$$

13. Define a curvature of a curve. Prove that the curvature of a circle of radius a is $\frac{1}{a}$

Statement:

If T is the unit tangent vector of smooth curve, the curvature function of the curve is $k = \left| \frac{dT}{ds} \right|$

Proof:

$$\vec{r}(\theta) = (a \cos \theta) \vec{i} + (a \sin \theta) \vec{j} \text{----- (i)}$$

Substituting $\theta = \frac{s}{a}$ in equation (i) to parameterize in term of arc length s .

$$\vec{r} = \left(a \cos \frac{s}{a}\right) \vec{i} + \left(a \sin \frac{s}{a}\right) \vec{j}$$

Now,

$$T = \frac{d\vec{r}}{ds} = \left(-\sin \frac{s}{a}\right) \vec{i} + \left(\cos \frac{s}{a}\right) \vec{j}$$

$$\frac{dT}{ds} = \left(-\frac{1}{a} \cos \frac{s}{a}\right) \vec{i} - \left(\frac{1}{a} \sin \frac{s}{a}\right) \vec{j}$$

Now,

$$k = \left| \frac{dT}{ds} \right| = \sqrt{\left(-\frac{1}{a} \cos \frac{s}{a} \right)^2 + \left(\frac{1}{a} \sin \frac{s}{a} \right)^2} = \sqrt{\frac{1}{a^2}} = \frac{1}{|a|} = \frac{1}{a}$$

14. What is meant by direction derivative in the plane? Obtain the derivative of the function $f(x, y) = x^2 + xy$ at $P(1, 2)$ in the direction of the unit vector $v = \left(\frac{1}{\sqrt{2}}\right)\vec{i} + \left(\frac{1}{\sqrt{2}}\right)\vec{j}$

Statement:

The derivative of f at $P_0(x_0, y_0)$ in the direction of unit vector $u = u_1\vec{i} + u_2\vec{j}$ is

$$\left(\frac{df}{ds} \right)_{u, P_0} = \lim_{s \rightarrow 0} \frac{f(x_0 + su_1, y_0 + su_2) - f(x_0, y_0)}{s}$$

provided the limit exists. It is also denoted by $(D_u f)_{P_0}$.

Solution:

$$\begin{aligned} \left(\frac{df}{ds} \right)_{u, f} &= \lim_{s \rightarrow 0} \frac{f(x_0 + su_1, y_0 + su_2) - f(x_0, y_0)}{s} \\ &= \lim_{s \rightarrow 0} \frac{f\left(1 + \frac{s}{\sqrt{2}}, 2 + \frac{s}{\sqrt{2}}\right) - f(1, 2)}{s} \\ &= \lim_{s \rightarrow 0} \frac{\left(1 + \frac{s}{\sqrt{2}}\right)^2 + \left(1 + \frac{s}{\sqrt{2}}\right) \times \left(2 + \frac{s}{\sqrt{2}}\right) - (1^2 + 1 \times 2)}{s} \\ &= \lim_{s \rightarrow 0} \frac{5 \frac{s}{\sqrt{2}} + 2 \frac{s^2}{2}}{s} = \lim_{s \rightarrow 0} \frac{s \left(\frac{5}{\sqrt{2}} + s \right)}{s} = \lim_{s \rightarrow 0} \left(\frac{5}{\sqrt{2}} + s \right) = \frac{5}{\sqrt{2}} \end{aligned}$$

15. Find the centre of mass of a solid of constant density δ , bounded below by the disk: $x^2 + y^2 = 4$ in the plane $z = 0$ and above by the paraboloid $z = 4 - x^2 - y^2$

By symmetry, $\bar{x} = \bar{y} = 0$. To find \bar{z} , we first calculate

$$\begin{aligned}
 M_{xy} &= \int \int_R \int_{z=0}^{z=4-x^2-y^2} z \delta \, dz \, dy \, dx \\
 &= \int \int_R \left[\frac{z^2}{2} \right]_{z=0}^{z=4-x^2-y^2} \delta \, dy \, dx \\
 &= \frac{\delta}{2} \int \int_R (4 - x^2 - y^2)^2 \, dy \, dx \\
 &= \frac{\delta}{2} \int_0^{2\pi} \int_0^2 (4 - r^2)^2 r \, dr \, d\theta \\
 &= \frac{\delta}{2} \int_0^{2\pi} \left[-\frac{1}{6} (4 - r^2)^3 \right]_{r=0}^{r=2} d\theta \\
 &= \frac{16\delta}{3} \int_0^{2\pi} d\theta = \frac{32\pi\delta}{3}
 \end{aligned}$$

A similar calculation gives

$$M = \int \int_R \int_0^{4-x^2-y^2} \delta \, dz \, dy \, dx = 8\pi\delta$$

Therefore, $\bar{z} = \left(\frac{M_{xy}}{M} \right) = \frac{4}{3}$, and the center of mass is $(\bar{x}, \bar{y}, \bar{z}) = \left(0, 0, \frac{4}{3} \right)$

Group C (8X5=40)

16. Graph the function $f(x) = -x^3 + 12x + 5$ for $-3 \leq x \leq 3$

17. Define Taylor's Polynomial of order n. Obtain Taylor's Polynomial and Taylor's Series generated by the function $f(x) = e^x$ at $x = 0$

Statement:

Solution:

$$f(x) = e^x ; f(0) = 1$$

$$f'(x) = e^x ; f'(0) = 1$$

$$f''(x) = e^x ; f''(0) = 1$$

$$f^n(x) = e^x ; f^n(0) = 1$$

Therefore, Taylor's series is given as;

$$\begin{aligned} f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots \\ = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \end{aligned}$$

Therefore, Taylor's polynomial is given as;

$$\sum_{k=1}^{\infty} \frac{x^k}{k!}$$

18. Obtain the centroid of the region in the first quadrant that is bounded above by the line $y = x$ and below by the parabola $y = x^2$

Here,

$$M = \int_0^1 \int_{x^2}^x 1 \, dy \cdot dx = \int_0^1 [y]_{y=x^2}^{y=x} dx = \int_0^1 (x - x^2) dx = \left[\frac{x^2}{2} - \frac{x^3}{3} \right]_0^1 = \frac{1}{6}$$

$$M_x = \int_0^1 \int_{x^2}^x y \, dy \cdot dx = \int_0^1 \left[\frac{y^2}{2} \right]_{y=x^2}^{y=x} dx = \int_0^1 \left(\frac{x^2}{2} - \frac{x^4}{2} \right) dx = \left[\frac{x^3}{6} - \frac{x^5}{10} \right]_0^1 = \frac{1}{15}$$

$$M_y = \int_0^1 \int_{x^2}^x x \, dy \cdot dx = \int_0^1 [xy]_{y=x^2}^{y=x} dx = \int_0^1 (x^2 - x^3) dx = \left[\frac{x^3}{3} - \frac{x^4}{4} \right]_0^1 = \frac{1}{12}$$

From these values of M, M_x & M_y , we find

$$\bar{x} = \frac{M_y}{M} = \frac{\left(\frac{1}{12}\right)}{\left(\frac{1}{6}\right)} = \frac{1}{2} \quad \& \quad \bar{y} = \frac{M_x}{M} = \frac{\left(\frac{1}{15}\right)}{\left(\frac{1}{6}\right)} = \frac{2}{5}$$

The centroid is the point $\left(\frac{1}{2}, \frac{2}{5}\right)$

19. Find the maximum and minimum values of $f(x, y) = 2xy - 2y^2 - 5x^2 + 4x + 4y - 4$. Also find the saddle point if it exist.

Here,

$$f(x, y) = 2xy - 2y^2 - 5x^2 + 4x + 4y - 4$$

$$f_x = 2y - 10x + 4 ; f_{xx} = -10$$

$$f_y = 2x - 4y + 4 ; f_{yy} = -4$$

$$f_{xy} = 2 ; f_{yx} = 2$$

Now,

$$f_{xx} \cdot f_{yy} - f_{xy}^2 = (-10) \cdot (-4) - 4 = 36$$

we have, $f_{xx} < 0$; $f_{xx} \cdot f_{yy} - f_{xy}^2 > 0$. Thus there exists local maximum value.

$$2y - 10x + 4 = 0 \dots \dots (i)$$

$$2x - 4y + 4 = 0 \dots \dots (ii)$$

Solving equation (i) and (ii), we get;

$$x = \frac{2}{3} \text{ \& \; } y = \frac{4}{3}$$

Now,

$$f\left(\frac{2}{3}, \frac{4}{3}\right) = 2 \times \frac{2}{3} \times \frac{4}{3} - 2 \times \frac{16}{9} - 5 \times \frac{4}{9} + 4 \times \frac{2}{3} + 4 \times \frac{4}{3} - 4 = 0$$

OR 19. Evaluate the integral $\int_0^{\sqrt{2}} \int_0^{3y} \int_{x^2+3y^2}^{8-x^2-y^2} dz \cdot dx \cdot dy$

Here,

$$\int_0^{\sqrt{2}} \int_0^{3y} \int_{x^2+3y^2}^{8-x^2-y^2} dz \cdot dx \cdot dy$$

$$\begin{aligned}
&= \int_0^{\sqrt{2}} \int_0^{3y} 8 - x^2 - y^2 - x^2 - 3y^2 \, dx \, dy \\
&= \int_0^{\sqrt{2}} \int_0^{3y} 8 - 2x^2 - 4y^2 \, dx \, dy \\
&= \int_0^{\sqrt{2}} \left[8x - \frac{2x^3}{3} - 4xy^2 \right]_0^{3y} dy \\
&= \int_0^{\sqrt{2}} \left(8 \times 3y - \frac{2}{3} \times 27y^3 - 4y^2 \times 3y \right) dy \\
&= \int_0^{\sqrt{2}} 24y - 30y^3 \, dy \\
&= \left[\frac{24y^2}{2} - \frac{30y^4}{4} \right]_0^{\sqrt{2}} \\
&= 12 \times \sqrt{2}^2 - 30 \times \frac{\sqrt{2}^4}{4} \\
&= -6
\end{aligned}$$

20. What do you mean by D'Alembert's solution of the one-dimensional wave equation? Derive it.

Solution:

The one dimensional wave equation is

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 y}{\partial t^2} \dots \dots \dots (i)$$

whose solution is $y(x, t) = f(x + ct) + g(x - ct) \dots \dots \dots (ii)$

The general solution can be obtained by introducing new variables $\xi = x - ct$ and $\eta = x + ct$, and applying the chain rule to obtain

$$\begin{aligned}\frac{\partial}{\partial x} &= \frac{\partial \xi}{\partial x} \frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial x} \frac{\partial}{\partial \eta} \\ &= \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta} \\ \frac{\partial}{\partial t} &= \frac{\partial \xi}{\partial t} \frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial t} \frac{\partial}{\partial \eta} \\ &= -c \frac{\partial}{\partial \xi} + c \frac{\partial}{\partial \eta}.\end{aligned}$$

Now,

$$\begin{aligned}\frac{\partial^2 y}{\partial x^2} &= \left(\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta} \right) \left(\frac{\partial y}{\partial \xi} + \frac{\partial y}{\partial \eta} \right) \\ &= \frac{\partial^2 y}{\partial \xi^2} + 2 \frac{\partial^2 y}{\partial \xi \partial \eta} + \frac{\partial^2 y}{\partial \eta^2} \\ \frac{\partial^2 y}{\partial t^2} &= \left(-c \frac{\partial}{\partial \xi} + c \frac{\partial}{\partial \eta} \right) \left(-c \frac{\partial y}{\partial \xi} + c \frac{\partial y}{\partial \eta} \right) \\ &= c^2 \frac{\partial^2 y}{\partial \xi^2} - 2c^2 \frac{\partial^2 y}{\partial \xi \partial \eta} + c^2 \frac{\partial^2 y}{\partial \eta^2}, \\ \frac{\partial^2 y}{\partial \xi \partial \eta} &= 0.\end{aligned}$$

This partial differential equation has general solution

$$\begin{aligned}y(x, t) &= f(\xi) + g(\eta) \\ &= f(x - ct) + g(x + ct),\end{aligned} \tag{i}$$

where f and g are arbitrary functions, with f representing a right-traveling wave and g a left-traveling wave.

The initial value problem for a string located at position $y(x, t=0) = y_0(x)$ as a function of distance along the string x and vertical speed $\left. \frac{\partial y}{\partial t} \right|_{t=0} = v_0(x)$ can be found as follows. From the initial condition and equation (i),

$$y_0(x) = f(x) + g(x). \tag{ii}$$

Taking the derivative with respect to t then gives

$$\begin{aligned} v_0(x) &= f'(x) \frac{\partial(x-ct)}{\partial t} + g'(x) \frac{\partial(x+ct)}{\partial t} \\ &= -c f'(x) + c g'(x), \end{aligned}$$

and integrating gives

$$\int_a^x v_0(s) ds = -c f(x) + c g(x). \quad (\text{iii})$$

Solving equation (ii) and (iii) simultaneously for f and g immediately gives

$$\begin{aligned} f(x) &= \frac{1}{2} y_0(x) - \frac{1}{2c} \int_a^x v_0(s) ds \\ g(x) &= \frac{1}{2} y_0(x) + \frac{1}{2c} \int_a^x v_0(s) ds, \end{aligned}$$

so plugging these into (ii) then gives the solution to the wave equation with specified initial conditions as

$$y(x, t) = \frac{1}{2} y_0(x-ct) + \frac{1}{2} y_0(x+ct) + \frac{1}{2c} \int_{x-ct}^{x+ct} v_0(s) ds.$$

OR 20. Find the particular integral of the equation $(D^2 - D^1)z = 2y - x^2$

where $D = \frac{\partial}{\partial x}$, $D' = \frac{\partial}{\partial y}$

Solution:

$$\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = 2y - x^2$$

ie; $(D^2 - D'^2)z = 2y - x^2$

$$PI = \frac{1}{D^2 - D'^2} (2y - x^2)$$

$$= \frac{\left[1 - \left(\frac{D'}{D}\right)^2\right]^{-1}}{D^2} (2y - x^2)$$

$$= \frac{1}{D^2} \left[2y - x^2 + \frac{D'^2}{D^2} (2y - x^2)\right]$$

$$= \frac{1}{D^2} \left[2y - x^2 + \frac{2yD'}{D^2} - \frac{D'}{D^2} x^2\right]$$

$$= \frac{2y}{D^2} - \frac{x^2}{D^2} + \frac{2yD'}{D^4} - \frac{D'}{D^4} x^2$$

$$= \frac{2x^2y}{2} - \frac{x^4}{12} + \frac{2x^4}{24} - 0$$

$$= x^2y - \frac{x^4}{12} + \frac{x^4}{12}$$

$$= x^2y$$