MATRIX ALGEBRA

Chapter – 3

\# Transpose of a matrix:- If A is $m \times n$ matrix then the transpose of A is denoted by ~~AT~~ A^T is the matrix $n \times m$ i.e. interchanging row and column.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 6 \end{bmatrix}_{2\times 3}, \quad A^T = \begin{bmatrix} 1 & 0 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}_{3 \times 2}$$

Properties.

1. $(A^T)^T = A$
2. $(A+B)^T = A^T + B^T$
3. $(AB)^T = A^T B^T$
4. $(\gamma A)^T = \gamma A^T$, where γ is any Scalar.

\# Singular and Non-singular matrix :-

A matrix A is said to be singular if $|A| = 0$ and non-singular if $|A| \neq 0$

* Inverse of a matrix

\- An $n \times n$ matrix A is ~~said~~ called invertible if there is an $n \times n$ matrix C such that ~~and~~ $AC = CA = I$ where I is a unit matrix and C is denoted & called inverse of A and denoted by $C = A^{-1}$

$$A \cdot A^{-1} = I = A^{-1} A$$

Inverse formula :-

let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

and $|A| \neq 0$

i.e. $\begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0$

$adj.(A) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$$A^{-1} = \frac{Adj(A)}{|A|}$$

Example.

Find the inverse of matrix.

$A = \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}$

Sol^n: Here

$|A| = 18 - 20$

$= -2 \neq 0$

$\therefore A^{-1}$ is exist

$adj.(A) = \begin{bmatrix} 6 & -4 \\ -5 & 3 \end{bmatrix}$

$$A^{-1} = \frac{Adj.(A)}{|A|}$$

$A^{-1} = -\frac{1}{2} \begin{bmatrix} 6 & -4 \\ -5 & 3 \end{bmatrix}$

$A^{-1} = \begin{bmatrix} -3 & 2 \\ 5/2 & -3/2 \end{bmatrix}$ Ans

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2075

let $A = \begin{bmatrix} -1 & -2 \\ 5 & 9 \end{bmatrix}$, $B = \begin{bmatrix} 9 & 2 \\ K & -1 \end{bmatrix}$ what value of K if any will have $AB = BA$?

Sol^n! Here,

$$A = \begin{bmatrix} -1 & -2 \\ 5 & 9 \end{bmatrix} \;\&\; B = \begin{bmatrix} 9 & 2 \\ K & -1 \end{bmatrix}$$

According to condition,

$$AB = BA$$

$$\begin{bmatrix} -1 & -2 \\ 5 & 9 \end{bmatrix} \cdot \begin{bmatrix} 9 & 2 \\ K & -1 \end{bmatrix} = \begin{bmatrix} 9 & 2 \\ K & -1 \end{bmatrix} \cdot \begin{bmatrix} -1 & -2 \\ 5 & 9 \end{bmatrix}$$

$$\begin{bmatrix} -9-2K & -2+2 \\ 45+9K & 10-9 \end{bmatrix} = \begin{bmatrix} -9+10 & -18+18 \\ -K-5 & -2K-9 \end{bmatrix}$$

$$\begin{bmatrix} -9-2K & 0 \\ 45+9K & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -K-5 & -2K-9 \end{bmatrix}$$

Now,

$$-9-2K = 1$$
$$-2K = 9+1$$
$$K = -5$$

$$\begin{bmatrix} -9-2.(-5) & 0 \\ 45+9(-5) & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -(-5)-5 & -2(-5)-9 \end{bmatrix}$$

$$\begin{bmatrix} -9+10 & 0 \\ 45-45 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 5-5 & 10-9 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$A = \begin{bmatrix} -1 & -2 \\ 5 & 9 \end{bmatrix}$ & $B = \begin{bmatrix} 9 & 2 \\ -5 & -1 \end{bmatrix}$ Ans

2075

If $A = \begin{bmatrix} 6 & 1 \\ 3 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$ then prov

that $\det(AB) = \det(A) + \det(B)$

Solⁿ: Here,

$$A = \begin{bmatrix} 6 & 1 \\ 3 & 2 \end{bmatrix} \ \& \ B = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

$$A \cdot B = \begin{bmatrix} 6 & 1 \\ 3 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 6+2 & 18+4 \\ 3+4 & 9+8 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 22 \\ 7 & 17 \end{bmatrix}$$

Now,

$$|AB| = 8 \times 17 - 22 \times 7$$

$$= 136 - 154$$

$$= -18 \neq 0$$

Again,

$$|A| = 12 - 3 = 9$$

$$|B| = 4 - 6 = -2$$

According to condition

$$\det(AB) = \det(A) \cdot \det(B)$$

$$\cancel{-118} \quad -18 = 9 \cdot (-2)$$

$$-18 = -18$$

Ans

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2075.

Find the Value of K, that the matrix $\begin{bmatrix} 3 & -9 \\ 2 & K \end{bmatrix}$ is not invertible.

Soln: Here,

let $A = \begin{bmatrix} 3 & -9 \\ 2 & K \end{bmatrix}$

For non-invertible matrix the determinant of A must be equal to zero such that,

$|A| = 0$

$A = \begin{bmatrix} 3 & -9 \\ 2 & K \end{bmatrix}$

$3K + 18 = 0$

$3K = -18$

$\boxed{K = -6}$ Ans

2076 Example,

let $A = \begin{bmatrix} 8 & 6 \\ 5 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 2 \\ 7 & 4 \end{bmatrix}$

Verify $(AB)^{-1} = B^{-1} \cdot A^{-1}$

Soln: Here,

Here

$A = \begin{bmatrix} 8 & 6 \\ 5 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 2 \\ 7 & 4 \end{bmatrix}$

$AB = \begin{bmatrix} 8 & 6 \\ 5 & 4 \end{bmatrix} \cdot \begin{bmatrix} 3 & 2 \\ 7 & 4 \end{bmatrix} = \begin{bmatrix} 24+42 & 16+24 \\ 15+28 & 10+16 \end{bmatrix}$

$AB = \begin{bmatrix} 66 & 40 \\ 43 & 26 \end{bmatrix}$

$|AB| = 66 \times 26 - 40 \times 43 = 1716 - 1720 = -4 \neq 0$

$\therefore (AB)^{-1}$ is exist

$Adj(AB) = \begin{bmatrix} 26 & -40 \\ -43 & 66 \end{bmatrix}$

$$(AB)^{-1} = \frac{Adj.(AB)}{|AB|}$$

$$= \frac{-1}{4}\begin{bmatrix} 26 & -40 \\ -43 & 66 \end{bmatrix}$$

$$= \begin{bmatrix} -13/2 & 10 \\ 43/4 & -33/2 \end{bmatrix}$$

Now,

For A^{-1}

$|A| = (32-30) = 2 \neq 0$

$\therefore A^{-1}$ is exist

$$Adj(A) = \begin{bmatrix} 4 & -6 \\ -5 & 8 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{Adj(A)}{|A|} = \frac{1}{2}\begin{bmatrix} 4 & -6 \\ -5 & 8 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ -5/2 & 4 \end{bmatrix}$$

For B^{-1}

$|B| = 12-14 = -2 \neq 0$

$\therefore B^{-1}$ is exist

$$Adj(B) = \begin{bmatrix} 4 & -2 \\ -7 & 3 \end{bmatrix}$$

$$\therefore B^{-1} = \frac{Adj(B)}{|B|}$$

$$B^{-1} = \frac{1}{-2}\begin{bmatrix} 4 & -2 \\ -7 & 3 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 7/2 & -3/2 \end{bmatrix}$$

$$B^{-1}.A^{-1} = \begin{bmatrix} -2 & 1 \\ 7/2 & -3/2 \end{bmatrix} \cdot \begin{bmatrix} 2 & -3 \\ -5/2 & 4 \end{bmatrix}$$

$$B^{-1}.A^{-1} = \begin{bmatrix} -4-5/2 & 6+4 \\ 7/2 . 2 + 3/2 . 5/2 & \frac{-21}{2} - 6 \end{bmatrix}$$

$$B^{-1}.A^{-1} = \begin{bmatrix} (-8-5)/2 & 10 \\ 7 + 15/4 & (-21-12)/2 \end{bmatrix}$$

$$B^{-1}.A^{-1} = \begin{bmatrix} -13/2 & 10 \\ (28+15)/4 & -33/2 \end{bmatrix}$$

$$B^{-1}.A^{-1} = \begin{bmatrix} -13/2 & 10 \\ 43/4 & -33/2 \end{bmatrix}$$

$$(AB)^{-1} = B^{-1}.A^{-1}$$

* Exercise:- 3.1

1. Let $A = \begin{bmatrix} 2 & 0 & -1 \\ 4 & -5 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 5 \\ -1 & 4 \end{bmatrix}$. then compute

i) BA (ii) AB.

Sol^n: Here,

$$A = \begin{bmatrix} 2 & 0 & -1 \\ 4 & -5 & 2 \end{bmatrix} \quad \& \quad B = \begin{bmatrix} 3 & 5 \\ -1 & 4 \end{bmatrix}$$

$$BA = \underset{2\times 2}{\begin{bmatrix} 3 & 5 \\ -1 & 4 \end{bmatrix}} . \underset{2\times 3}{\begin{bmatrix} 2 & 0 & -1 \\ 4 & -5 & 2 \end{bmatrix}}$$

$$BA = \begin{bmatrix} 6+20 & 0-25 & -3+10 \\ -2+16 & 0-20 & 1+8 \end{bmatrix}$$

$$BA = \begin{bmatrix} 26 & -25 & 7 \\ 14 & -20 & 9 \end{bmatrix}$$

Now, $AB = A_{2\times 3} \; B_{2\times 2}$

Not possible

AB is not possible.

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2. Compute $A-5I$ when $A = \begin{bmatrix} -9 & -1 & 3 \\ -8 & 7 & -6 \\ -4 & 1 & 8 \end{bmatrix}$

Sol^n: Here,

$$A = \begin{bmatrix} -9 & -1 & 3 \\ -8 & 7 & -6 \\ -4 & 1 & 8 \end{bmatrix}$$

$$A-5I = \begin{bmatrix} -9 & -1 & 3 \\ -8 & 7 & -6 \\ -4 & 1 & 8 \end{bmatrix} - 5\begin{bmatrix} -9 & -1 & 3 \\ -8 & 7 & -6 \\ -4 & 1 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} -9 & -1 & 3 \\ -8 & 7 & -6 \\ -4 & 1 & 8 \end{bmatrix} - \begin{bmatrix} -45 & -5 & 15 \\ -40 & 35 & -30 \\ -20 & 5 & 40 \end{bmatrix}$$

$$= \begin{bmatrix} -9+45 & -1+5 & 3-15 \\ -8+40 & 7-35 & -6+30 \\ -4+20 & 1-5 & 8-40 \end{bmatrix}$$

$$= \begin{bmatrix} 36 & 4 & -12 \\ 32 & -28 & 24 \\ 16 & -4 & -32 \end{bmatrix}$$

$$A-5I = \begin{bmatrix} -9 & -1 & 3 \\ -8 & 7 & -6 \\ -4 & 1 & 8 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -9 & -1 & 3 \\ -8 & 7 & -6 \\ -4 & 1 & 8 \end{bmatrix} - \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} -9-5 & -1-0 & 3-0 \\ -8-0 & 7-5 & -6-0 \\ -4-0 & 1-0 & 8-5 \end{bmatrix} = \begin{bmatrix} -14 & -1 & 3 \\ -8 & 2 & -6 \\ -4 & 1 & 3 \end{bmatrix}$$

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PAGE 1

3. let $A = \begin{bmatrix} 2 & -3 \\ -4 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 8 & 4 \\ 5 & 5 \end{bmatrix}$ and $C = \begin{bmatrix} 5 & -2 \\ 3 & 1 \end{bmatrix}$. verify that $AB = AC$.

Sol^n: Here,

$$A = \begin{bmatrix} 2 & -3 \\ -4 & 6 \end{bmatrix}, B = \begin{bmatrix} 8 & 4 \\ 5 & 5 \end{bmatrix}, \& C = \begin{bmatrix} 5 & -2 \\ 3 & 1 \end{bmatrix}$$

$$AB = \begin{bmatrix} 2 & -3 \\ -4 & 6 \end{bmatrix} \cdot \begin{bmatrix} 8 & 4 \\ 5 & 5 \end{bmatrix}$$

$$AB = \begin{bmatrix} 16-15 & 8-15 \\ -32+30 & -16+30 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 & -7 \\ -2 & 14 \end{bmatrix}$$

$$AC = \begin{bmatrix} 2 & -3 \\ -4 & 6 \end{bmatrix} \cdot \begin{bmatrix} 5 & -2 \\ 3 & 1 \end{bmatrix}$$

Sol^n.

$$AC = \begin{bmatrix} 2 & -3 \\ -4 & 6 \end{bmatrix} \cdot \begin{bmatrix} 5 & -2 \\ 3 & 1 \end{bmatrix}$$

$$AC = \begin{bmatrix} 10-9 & -4-3 \\ -20+18 & 8+6 \end{bmatrix}$$

$$AC = \begin{bmatrix} 1 & -7 \\ -2 & 14 \end{bmatrix}$$

$\therefore AB = AC$

4. Examine matrices are singular or non-Singular.

i) $\begin{bmatrix} 3 & 2 \\ 7 & 4 \end{bmatrix}$

Sol^n: Here

let $A = \begin{bmatrix} 3 & 2 \\ 7 & 4 \end{bmatrix} = 12 - 14 = -2 \neq 0$

Hence matrix A is a non-singular matrix

ii) $\begin{bmatrix} 3 & -4 \\ 7 & -8 \end{bmatrix}$

Sol^n: Here

let $A = \begin{bmatrix} 3 & -4 \\ 7 & -8 \end{bmatrix}$

$|A| = -24 + 28 = 4 \neq 0$

Hence, matrix A is a non-singular matrix

iii) $\begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$

Sol^n: Here.

$2\begin{bmatrix} 1 & 0 \\ 1 & 3 \end{bmatrix} - 0\begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix} - 1\begin{bmatrix} 5 & 1 \\ 0 & 1 \end{bmatrix}$

$2[3-0] - 0 - 1(5-0)$

$2.3 - 0 - 1.5$

$6 - 5$

$1 \neq 0$

iv) $\begin{bmatrix} 1 & 8 & -7 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$

Solⁿ: Here,

let $A = \begin{bmatrix} 1 & 8 & -7 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$

$|A| = 1 \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} - 8 \begin{bmatrix} 0 & 3 \\ 0 & 1 \end{bmatrix} - 7 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

$|A| = 1(1-0) - 8(0-0) - 7(0-0)$

$|A| = 1 - 0 - 0$

$|A| = 1 \neq 0$

v) $\begin{bmatrix} 1 & -2 & -1 \\ -1 & 5 & 6 \\ 5 & -4 & 5 \end{bmatrix}$

Solⁿ: Here,

let $A = \begin{bmatrix} 1 & -2 & -1 \\ -1 & 5 & 6 \\ 5 & -4 & 5 \end{bmatrix}$

$1 \begin{bmatrix} 5 & 6 \\ -4 & 5 \end{bmatrix} + 2 \begin{bmatrix} -1 & 6 \\ 5 & 5 \end{bmatrix} - 1 \begin{bmatrix} -1 & 5 \\ 5 & -4 \end{bmatrix}$

$(24+25) + 2(-5-30) - 1(4-25)$

$49 - 70 + 21$

$70 - 70$

$0 = 0$

Hence, the matrix A is a singular matrix.

(vi) $\begin{bmatrix} 1 & 0 & -2 \\ -3 & 1 & 4 \\ 2 & -3 & 4 \end{bmatrix}$

Sol^n: Here

$$\text{let } A = \begin{bmatrix} 1 & 0 & -2 \\ -3 & 1 & 4 \\ 2 & -3 & 4 \end{bmatrix}$$

$$|A| = 1 \begin{bmatrix} 1 & 4 \\ -3 & 4 \end{bmatrix} + 0 \begin{bmatrix} -3 & 4 \\ 2 & 4 \end{bmatrix} - 2 \begin{bmatrix} -3 & 1 \\ 2 & -3 \end{bmatrix}$$

$$|A| = (4+12) + 0 - 2(9-2)$$
$$|A| = 16 + 0 - 14$$
$$= 2 \neq 0$$

$\therefore$ Hence the matrix A is non-singular.

5. Find the inverse of the matrices by elementary row reduce augmented matrix. If exist.

(i) $\begin{bmatrix} 3 & 2 \\ 7 & 4 \end{bmatrix}$

Sol^n: Here,

$$\text{let } A = \begin{bmatrix} 3 & 2 \\ 7 & 4 \end{bmatrix}$$

$$[A \; I] = \begin{bmatrix} 3 & 2 & 1 & 0 \\ 7 & 4 & 0 & 1 \end{bmatrix} \leftrightarrow \begin{bmatrix} 1 & 0 & \times & \times \\ 0 & 1 & \times & \times \end{bmatrix}$$

Invertible

$$= \begin{bmatrix} 7 & 4 & 0 & 1 \\ 3 & 2 & 1 & 0 \end{bmatrix} \quad R_2 \leftrightarrow R_1$$

$$\begin{bmatrix} 1 & 0 & -2 & 1 \\ 3 & 2 & 1 & 0 \end{bmatrix} \quad R_1 \to R_1 - 2R_2$$

$R_2 \to R_2 - 3R_1$

$$\begin{bmatrix} 1 & 0 & -2 & 1 \\ 0 & 2 & 7 & -3 \end{bmatrix}$$

$R_2 \to \frac{1}{2} R_2$

$$\begin{bmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & 7/2 & -3/2 \end{bmatrix}$$ Ans $= [I \cdot A^{-1}]$ where $A^{-1} = \begin{bmatrix} -2 & 1 \\ 7/2 & -3/2 \end{bmatrix}$

(11) $\begin{bmatrix} 3 & -4 \\ 7 & -8 \end{bmatrix}$

Soln: Here,

let $A = \begin{bmatrix} 3 & -4 \\ 7 & -8 \end{bmatrix}$

$$[A \; I] = \begin{bmatrix} 3 & -4 & 1 & 0 \\ 7 & -8 & 0 & 1 \end{bmatrix}$$

$= \quad R_1 \leftrightarrow R_2$

$$\begin{bmatrix} 7 & -8 & 0 & 1 \\ 3 & -4 & 1 & 0 \end{bmatrix}$$

$R_1 \to R_1 - 2R_2$

$$\begin{bmatrix} 1 & 0 & -2 & 1 \\ 3 & -4 & 1 & 0 \end{bmatrix}$$

$R_2 \to R_2 - 3R_1$

$$\begin{bmatrix} 1 & 0 & -2 & 1 \\ 0 & -4 & 7 & -3 \end{bmatrix}$$

$R_2 \xrightarrow{-1/4} \begin{bmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & -7/4 & 3/4 \end{bmatrix} = [I \; A^{-1}]$

where

$$A^{-1} = \begin{bmatrix} -2 & 1 \\ -7/4 & 3/4 \end{bmatrix}$$ Ans

(iii) $\begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$

Sol^n: Here,

$$\begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$$

$$[A\ I] = \begin{bmatrix} 2 & 0 & -1 & 1 & 0 & 0 \\ 5 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 3 & 0 & 0 & 1 \end{bmatrix}$$

$R_1 \to \frac{1}{2} R_1$

$$= \begin{bmatrix} 1 & 0 & -1/2 & 1/2 & 0 & 0 \\ 5 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 3 & 0 & 0 & 1 \end{bmatrix}$$

$= R_2 \to R_2 - 5R_1$

$$\begin{bmatrix} 1 & 0 & -1/2 & 1/2 & 0 & 0 \\ 0 & 1 & 5/2 & -5/2 & 1 & 0 \\ 0 & 1 & 3 & 0 & 0 & 1 \end{bmatrix}$$

$= R_3 \to R_3 - R_2$

$$= \begin{bmatrix} 1 & 0 & -1/2 & 1/2 & 0 & 0 \\ 0 & 1 & 5/2 & -5/2 & 1 & 0 \\ 0 & 0 & 1/2 & 5/2 & -1 & 1 \end{bmatrix}$$

$R_3 \to 2R_3$

$$= \begin{bmatrix} 1 & 0 & -1/2 & 1/2 & 0 & 0 \\ 0 & 1 & 5/2 & -5/2 & 1 & 0 \\ 0 & 0 & 1 & 5 & -2 & 2 \end{bmatrix}$$

$R_2 \to R_2 - \frac{5}{2} R_3$

$$= \begin{bmatrix} 1 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & -15 & 6 & -5 \\ 0 & 0 & 1 & 5 & -2 & 2 \end{bmatrix}$$

$R_1 \to R_1 + \frac{1}{2} R_3$

$$\begin{bmatrix} 1 & 0 & 0 & 3 & -1 & 1 \\ 0 & 1 & 0 & -15 & 6 & -5 \\ 0 & 0 & 1 & 5 & -2 & 2 \end{bmatrix}$$

$$= \cancel{A^{-1}} \, [I \quad A^{-1}]$$

where

$$A^{-1} = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$$ Ans

DATE

(iv)
$$\begin{bmatrix} 1 & 8 & -7 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

Sol^n: Here,

let $A = \begin{bmatrix} 1 & 8 & -7 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$

$$[A\ I] = \begin{bmatrix} 1 & 8 & -7 & 1 & 0 & 0 \\ 0 & 1 & 3 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

$R_2 \to \cancel{\frac{1}{3}R_2 - \frac{1}{7}R_1}$

$\cancel{R_3 \to \begin{bmatrix} 1 & 8 \\ 0 & 1 \\ 0 & \end{bmatrix}}$

$R_2 \to R_2 - 3R_3$

$$\begin{bmatrix} 1 & 8 & -7 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & -3 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

$R_1 \to R_1 + 7R_3$

$$\begin{bmatrix} 1 & 8 & 0 & 1 & 0 & 7 \\ 0 & 1 & 0 & 0 & 1 & -3 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

$R_1 \to R_1 - 8R_2$

$$\begin{bmatrix} 1 & 0 & 0 & 1 & -8 & 31 \\ 0 & 1 & 0 & 0 & 1 & -3 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} = [I \,.\, A^{-1}]$$

where $A^{-1} = \begin{bmatrix} 1 & 0 & 0 & 1 & -8 & 31 \\ 0 & 1 & 0 & 0 & 1 & -3 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$ Ans

(v) $\begin{bmatrix} 1 & -2 & -1 \\ -1 & 5 & 6 \\ 5 & -4 & 5 \end{bmatrix}$

soln: Here,

let $A = \begin{bmatrix} 1 & -2 & -1 \\ -1 & 5 & 6 \\ 5 & -4 & 5 \end{bmatrix}$

$$[A\ I] = \begin{bmatrix} 1 & -2 & -1 & 1 & 0 & 0 \\ -1 & 5 & 6 & 0 & 1 & 0 \\ 5 & -4 & 5 & 0 & 0 & 1 \end{bmatrix}$$

$R_2 \to R_2 + R_1$

$$= \begin{bmatrix} 1 & -2 & -1 & 1 & 0 & 0 \\ 0 & 3 & 5 & 1 & 1 & 0 \\ 5 & -4 & 5 & 0 & 0 & 1 \end{bmatrix}$$

$= R_3 \to R_3 - 5R_1$

$$\begin{bmatrix} 1 & -2 & -1 & 1 & 0 & 0 \\ 0 & 3 & 5 & 1 & 1 & 0 \\ 0 & 6 & 10 & -5 & 0 & 1 \end{bmatrix}$$

$R_2 \to \frac{1}{3} R_2$

$$\begin{bmatrix} 1 & -2 & -1 & 1 & 0 & 0 \\ 0 & 1 & 5/3 & 1/3 & 1/3 & 0 \\ 0 & 6 & 10 & -5 & 0 & 1 \end{bmatrix}$$

$R_3 \to R_3 - 6R_2 \Rightarrow R_3 - 6R_2$

$$\begin{bmatrix} 1 & -2 & -1 & 1 & 0 & 0 \\ 0 & 1 & 5/3 & 1/3 & 1/3 & 0 \\ 0 & 0 & 0 & -7 & -2 & 1 \end{bmatrix}$$

Here pivot column is first and second column but column 3 is not pivot column so that the inverse of the matrices doesnot exist.

(vi) $\begin{bmatrix} 1 & 0 & -2 \\ -3 & 1 & 4 \\ 2 & -3 & 4 \end{bmatrix}$

Soln: Here,

let $A = \begin{bmatrix} 1 & 0 & -2 \\ -3 & 1 & 4 \\ 2 & -3 & 4 \end{bmatrix}$

$$[A\ I] = \begin{bmatrix} 1 & 0 & -2 & 1 & 0 & 0 \\ -3 & 1 & 4 & 0 & 1 & 0 \\ 2 & -3 & 4 & 0 & 0 & 1 \end{bmatrix}$$

$R_2 \to R_2 + 3R_1$

$$[A\ I] = \begin{bmatrix} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 3 & 1 & 0 \\ 2 & -3 & 4 & 0 & 0 & 1 \end{bmatrix}$$

$R_3 \to R_3 - 2R_1$

$$= \begin{bmatrix} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 3 & 1 & 0 \\ 0 & -3 & 8 & -2 & 0 & 1 \end{bmatrix}$$

$R_3 \to R_3 + 3R_2$

$$= \begin{bmatrix} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 3 & 1 & 0 \\ 0 & 0 & 2 & 7 & 3 & 1 \end{bmatrix}$$

$R_3 \to \frac{1}{2} R_3$

$$= \begin{bmatrix} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 3 & 1 & 0 \\ 0 & 0 & 1 & 7/2 & 3/2 & 1/2 \end{bmatrix}$$

$R_2 \to R_2 + 2R_3$

$$\begin{bmatrix} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 10 & 4 & 1 \\ 0 & 0 & 1 & 7/2 & 3/2 & 1/2 \end{bmatrix}$$

DATE

$R_1 \rightarrow R_1 + 2R_3$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 8 & 3 & 1 \\ 0 & 1 & 0 & 10 & 4 & 1 \\ 0 & 0 & 1 & 7/2 & 3/2 & 1/2 \end{array}\right]$$

$= [I . A^{-1}]$

where $A^{-1} = \begin{bmatrix} 8 & 3 & 1 \\ 10 & 4 & 1 \\ 7/2 & 3/2 & 1/2 \end{bmatrix}$ Ans

6. Solve the following system by using inverse matrix,

~~Soln: Here,~~

1) $8x_1 + 6x_2 = 2$

$5x_1 + 4x_2 = -1$

Soln: Here,

$8x_1 + 6x_2 = 2$

$5x_1 + 4x_2 = -1$

$$\begin{bmatrix} 8 & 6 \\ 5 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$\therefore \quad AX = B$

$X = A^{-1} B$

For A^{-1}

$|A| = 32 - 30 = 2 \neq 0$

$\therefore A^{-1}$ is exist

$$\text{adj}(A) = \begin{bmatrix} 4 & -6 \\ -5 & 8 \end{bmatrix}$$

$$A^{-1} = \frac{\text{adj}(A)}{|A|} = \frac{1}{2} \begin{bmatrix} 4 & -6 \\ -5 & 8 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ -5/2 & 4 \end{bmatrix}$$

Now $X = A^{-1} \cdot B$

$$X = \begin{bmatrix} 2 & -3 \\ -5/2 & 4 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4+3 \\ -5-4 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 7 \\ -9 \end{bmatrix}$$

$x_1 = 7$ and $x_2 = -9$ Ans.

(11) $3x_1 + 4x_2 = 3$
$5x_1 + 6x_2 = 7$

Sol: Here,

Given,

$3x_1 + 4x_2 = 3$
$5x_1 + 6x_2 = 7$

Sol: Here,

$$\begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$$

$$AX = B$$

$$X = A^{-1} \cdot B$$

for A^{-1}

$|A| = 18 - 20 = -2 \neq 0$

$$\text{Adj}(A) = \begin{bmatrix} 6 & -4 \\ -5 & 3 \end{bmatrix}$$

$$A^{-1} = \frac{\text{Adj}(A)}{|A|} = -\frac{1}{2} \begin{bmatrix} 6 & -4 \\ -5 & 3 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} -3 & 2 \\ 5/2 & -3/2 \end{bmatrix}$$

DATE

Now

$X = A^{-1} B$

$$X = \begin{bmatrix} -3 & 2 \\ 5/2 & -3/2 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 7 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -9+14 \\ 5/2 \cdot 3 - 3/2 \cdot 7 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \end{bmatrix}$$

$x_1 = 5$ and $x_2 = -3$ Ans

7. Determine which of the matrices are invertible by using Invertible Matrix theorem.

i) $\begin{bmatrix} 5 & 7 \\ -3 & -6 \end{bmatrix}$

Soln: Here

let $A = \begin{bmatrix} 5 & 7 \\ -3 & -6 \end{bmatrix}$

$= R_1 \to -\frac{1}{3} R_1$

$$= \begin{bmatrix} 5 & 7 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 7 \\ 1 & 2 \end{bmatrix}$$

$= R_1 \to R_1 - 4R_2$

$$\begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}$$

$R_2 \to R_2 - R_1$

$$\begin{bmatrix} 1 & -1 \\ 0 & 3 \end{bmatrix}$$

$R_2 \to \frac{1}{3} R_2$

$$\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

PAGE

$R_1 \to R_1 + R_2$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

This shows that A has 2-pivot positio[n]
Therefore, by invertible matrix theorem,
A is an invertible matrix.

(11)
$$\begin{bmatrix} 5 & 0 & 0 \\ -3 & -7 & 0 \\ 8 & 5 & -1 \end{bmatrix}$$

Soln: Here
let $A = \begin{bmatrix} 5 & 0 & 0 \\ -3 & -7 & 0 \\ 8 & 5 & -1 \end{bmatrix}$

$= R_1 \to \frac{1}{5} R_1$

$$\begin{bmatrix} 1 & 0 & 0 \\ -3 & -7 & 0 \\ 8 & 5 & -1 \end{bmatrix}$$

$R_2 \to R_2 + 3R_1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -7 & 0 \\ 8 & 5 & -1 \end{bmatrix}$$

$R_3 \to R_3 - 8R_1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -7 & 0 \\ 0 & 5 & -1 \end{bmatrix}$$

$R_2 \to -\frac{1}{7} R_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 5 & -1 \end{bmatrix}$$

$R_3 \to R_3 - 5R_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

This show that A has three- pivot position therefore by invertible matrix theorem, A is an invertible matrix.

(iii) $\begin{bmatrix} 0 & 3 & -5 \\ 1 & 0 & 2 \\ -4 & -9 & 7 \end{bmatrix}$

Soln: Here

$R_1 \leftrightarrow R_2$

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & -5 \\ -4 & -9 & 7 \end{bmatrix}$$

$R_3 \to R_3 + 4R_1$

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & -5 \\ 0 & -9 & 15 \end{bmatrix}$$

$R_3 \Rightarrow \frac{1}{3} R_3$

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & -5 \\ 0 & -3 & 5 \end{bmatrix}$$

$R_2 \to R_2 + R_3$

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & -3 & 5 \end{bmatrix}$$

This shows that A has only two- pivot position in 3x3 order matrix so that the A is an invertible matrix by using IMT.

8 Find a matrix A whose inverse is $\begin{bmatrix} 1 & -3 & 2 \\ -3 & 3 & -1 \\ 2 & -1 & 0 \end{bmatrix}$

Sol^n: Here,

$$A^{-1} = \begin{bmatrix} 1 & -3 & 2 \\ -3 & 3 & -1 \\ 2 & -1 & 0 \end{bmatrix}$$

$$[I . A^{-1}] = \begin{bmatrix} 1 & 0 & 0 & 1 & -3 & 2 \\ 0 & 1 & 0 & -3 & 3 & -1 \\ 0 & 0 & 1 & 2 & -1 & 0 \end{bmatrix}$$

$R_2 \to R_2 + 3R_1$

$$\begin{bmatrix} 1 & 0 & 0 & 1 & -3 & 2 \\ 3 & 1 & 0 & 0 & -6 & 5 \\ 0 & 0 & 1 & 2 & -1 & 0 \end{bmatrix}$$

$R_3 \to R_3 - 2R_1$

$$\begin{bmatrix} 1 & 0 & 0 & 1 & -3 & 2 \\ 3 & 1 & 0 & 0 & -6 & 5 \\ -2 & 0 & 1 & 0 & 5 & -4 \end{bmatrix}$$

$R_2 \Rightarrow R_2 + R_3$

$$\begin{bmatrix} 1 & 0 & 0 & 1 & -3 & 2 \\ 1 & 1 & 1 & 0 & -1 & 1 \\ -2 & 0 & 1 & 0 & 5 & -4 \end{bmatrix}$$

$R_2 \to -R_2$

$1+5$

$$\begin{bmatrix} 1 & 0 & 0 & 1 & -3 & 2 \\ -1 & -1 & -1 & 0 & 1 & -1 \\ -2 & 0 & 1 & 0 & 5 & -4 \end{bmatrix}$$

$R_3 \to R_3 - 5R_2$

$-2+5$

$-4+5$

$$\begin{bmatrix} 1 & 0 & 0 & 1 & -3 & 2 \\ -1 & -1 & -1 & 0 & 1 & -1 \\ 3 & 5 & 6 & 0 & 0 & 1 \end{bmatrix}$$

$1+5$
6

DATE

$R_2 \rightarrow R_2 + R_3$

$$\begin{bmatrix} 1 & 0 & 0 & 1 & -3 & 2 \\ 2 & 4 & 5 & 0 & 1 & 0 \\ 3 & 5 & 6 & 0 & 0 & 1 \end{bmatrix}$$

$R_1 \rightarrow R_1 - 2R_3$

$$\begin{bmatrix} -5 & -10 & -12 & 1 & -3 & 0 \\ 2 & 4 & 5 & 0 & 1 & 0 \\ 3 & 5 & 6 & 0 & 0 & 1 \end{bmatrix}$$

$R_1 \rightarrow R_1 + 3R_2$

$$\begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 4 & 5 & 0 & 1 & 0 \\ 3 & 5 & 6 & 0 & 0 & 1 \end{bmatrix}$$

$$= [A \quad I]$$

Where $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$ Ans

DATE

(9) Using Invertible Matrix theorem, show that A^T is invertible if $A = \begin{bmatrix} 1 & 0 & -2 \\ 3 & 1 & -2 \\ -5 & -1 & 9 \end{bmatrix}$

Sol^n: Here

$$A = \begin{bmatrix} 1 & 0 & -2 \\ 3 & 1 & -2 \\ -5 & -1 & 9 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & 3 & -5 \\ 0 & 1 & -1 \\ -2 & -2 & 9 \end{bmatrix}$$

$R_3 \to R_3 + 2R_1$

$$\begin{bmatrix} 1 & 3 & -5 \\ 0 & 1 & -1 \\ 0 & 4 & -1 \end{bmatrix}$$

$R_3 \to R_3 - 4R_2$

$$\begin{bmatrix} 1 & 3 & -5 \\ 0 & 1 & -1 \\ 0 & 0 & 3 \end{bmatrix}$$

$R_1 \to R_1 - 3R_2$

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 3 \end{bmatrix}$$

$R_3 \to \frac{1}{3} R_3$

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

This show that A^T has 3-pivot column positions. therefore, by invertible matrix theorem A is invertible matrix.

Exercise :- 3.2

1. Let

$$A = \left[\begin{array}{ccc|cc|c} 3 & 0 & -1 & 5 & 9 & -2 \\ -5 & 2 & 4 & 0 & -3 & 1 \\ \hline -8 & -6 & 3 & 1 & 7 & -4 \end{array}\right] \text{ and } B = \left[\begin{array}{cc} 3 & 2 \\ 2 & 3 \\ 1 & 5 \\ \hline 4 & 1 \\ -1 & 2 \\ \hline 2 & 3 \end{array}\right]$$

Soln: Here

Given

$$A = \begin{bmatrix} 3 & 0 & -1 & 5 & 9 & -2 \\ -5 & 2 & 4 & 0 & -3 & 1 \\ -8 & -6 & 3 & 1 & 7 & -4 \end{bmatrix}, B = \left[\begin{array}{cc} 3 & 2 \\ 2 & 3 \\ 1 & 5 \\ \hline 4 & 1 \\ -1 & 2 \\ \hline 2 & 3 \end{array}\right]$$

Here

$$A_{11} = \begin{bmatrix} 3 & 0 & -1 \\ -5 & 2 & 4 \end{bmatrix}, A_{12} = \begin{bmatrix} 5 & 9 \\ 0 & -3 \end{bmatrix}$$

$$A_{13} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}, A_{21} = [-8, -6, 3], A_{22} = [1 \ 7] \ A_{23} = [-4]$$

$$B_1 = \begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 1 & 5 \end{bmatrix}, B_2 = \begin{bmatrix} 4 & 1 \\ -1 & 2 \end{bmatrix}, B_3 = [2 \ 3]$$

then,

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \end{bmatrix}, B = \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix}$$

So

$$AB = \begin{bmatrix} A_{11} B_1 + A_{12}.B_2 + A_{13} B_3 \\ A_{21}.B_1 + A_{22}.B_2 + A_{23}.B_3 \end{bmatrix}$$

Here

$$A_{11}.B_1 = \begin{bmatrix} 3 & 0 & -1 \\ -5 & 2 & 4 \end{bmatrix} . \begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 1 & 5 \end{bmatrix}$$

$$A_{11}.B_1 = \begin{bmatrix} 9+0-1 & 6+0-5 \\ -15+4+4 & -10+6+20 \end{bmatrix} = \begin{bmatrix} 8 & 1 \\ -7 & 16 \end{bmatrix}$$

DATE

$$A_{12} \cdot B_2 = \begin{bmatrix} 5 & 9 \\ 0 & -3 \end{bmatrix} \cdot \begin{bmatrix} 4 & 1 \\ -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 20-9 & 5+18 \\ 0+3 & 0-6 \end{bmatrix}$$

$$= \begin{bmatrix} 11 & 23 \\ 3 & -6 \end{bmatrix}$$

$$A_{13} \cdot B_3 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}_{2\times 1} \cdot \begin{bmatrix} 2 & 3 \end{bmatrix}_{1\times 2}$$

$$= \begin{bmatrix} -4 & -6 \\ 2 & 3 \end{bmatrix}$$

$$A_{21} \cdot B_1 = \begin{bmatrix} -8 & -6 & 3 \end{bmatrix} \cdot \begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 1 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} -24-12+3 & -16-18+15 \end{bmatrix}$$

$$= \begin{bmatrix} -33 & -19 \end{bmatrix}$$

$$A_{22} \cdot B_2 = \begin{bmatrix} 1 & 7 \end{bmatrix} \begin{bmatrix} 4 & 1 \\ -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 4-7 & 1+14 \end{bmatrix}$$

$$= \begin{bmatrix} -3 & 15 \end{bmatrix}$$

$$A_{23} \cdot B_3 = \begin{bmatrix} -4 \end{bmatrix}_{1\times 1} \begin{bmatrix} 2 & 3 \end{bmatrix}_{1\times 2}$$

$$= \begin{bmatrix} -8 & -12 \end{bmatrix}$$

Now

$$A \cdot B = \begin{bmatrix} \begin{bmatrix} 8 & 1 \\ -7 & 16 \end{bmatrix} & \begin{bmatrix} 11 & 23 \\ 3 & -6 \end{bmatrix} & \begin{bmatrix} -4 & -6 \\ 2 & 3 \end{bmatrix} \\ \begin{bmatrix} -33 & -19 \end{bmatrix} & \begin{bmatrix} -3 & 15 \end{bmatrix} & \begin{bmatrix} -8 & -12 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 8+11-4 & 1+23-6 \\ -7+3+2 & 16-6+3 \\ -33-3-8 & -19+15-12 \end{bmatrix}$$

$$= \begin{bmatrix} 15 & 18 \\ -2 & 13 \\ -44 & -16 \end{bmatrix}$$

$$AB = \begin{bmatrix} 15 & 18 \\ -2 & 13 \\ -44 & -16 \end{bmatrix}$$

2. let $A = \begin{bmatrix} -3 & 1 & 2 \\ 1 & -4 & 5 \end{bmatrix}$ and $C = \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}$. Find AB by column row expansion.

Sol^n: Here,

Column expansion for the matrix A is and B

$\cancel{A_{11} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}}$, B.

$col_1(A) \cdot row_1(B) = \begin{bmatrix} -3 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} a & b \end{bmatrix} = \begin{bmatrix} -3a & -3b \\ a & b \end{bmatrix}$

$col_2(A) \cdot row_2(B) = \begin{bmatrix} 1 \\ -4 \end{bmatrix} \begin{bmatrix} c & d \end{bmatrix} = \begin{bmatrix} c & d \\ -4c & -4d \end{bmatrix}$

$col_3(A) \cdot row_3(B) = \begin{bmatrix} 2 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} e & f \end{bmatrix} = \begin{bmatrix} 2e & 2f \\ 5e & 5f \end{bmatrix}$

Hence,

$$AB = \sum_{K=1}^{3} col_K(A)\, row_K(B) = \begin{bmatrix} -3a & -3b \\ a & b \end{bmatrix} + \begin{bmatrix} c & d \\ -4c & -4d \end{bmatrix} + \begin{bmatrix} 2e & 2f \\ 5e & 5f \end{bmatrix}$$

$$= \begin{bmatrix} -3a+c+2e & -3b+d+2f \\ a-4c+5e & b-4d+5f \end{bmatrix}$$

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3\. let $A = \begin{bmatrix} 3 & 0 & -1 \\ -5 & 2 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 1 & 5 \end{bmatrix}$. Obtain AB by applying column-row expansion method

Soln: Here,

$$A = \begin{bmatrix} 3 & 0 & -1 \\ -5 & 2 & 4 \end{bmatrix}_{2\times 3}, \quad B = \begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 1 & 5 \end{bmatrix}_{3\times 2}$$

By column-row expansion method

$$AB = \sum_{K=1}^{3} \text{column}_K(A) \cdot \text{row}_K(B), \text{ so,}$$

$$\text{col}_1(A) \cdot \text{row}_1(B) = \begin{bmatrix} 3 \\ -5 \end{bmatrix} \begin{bmatrix} 3 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 6 \\ -15 & -10 \end{bmatrix}$$

$$\text{col}_2(A) \cdot \text{row}_2(B) = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 4 & 6 \end{bmatrix}$$

$$\text{col}_3(A) \cdot \text{row}_3(B) = \begin{bmatrix} -1 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 1 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -5 \\ 4 & 20 \end{bmatrix}$$

$$\sum_{K=1}^{3} \text{col}_K(A) \cdot \text{row}_K(B) = \begin{bmatrix} 9 & 6 \\ -15 & -10 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 4 & 6 \end{bmatrix} + \begin{bmatrix} -1 & -5 \\ 4 & 20 \end{bmatrix}$$

$$= \begin{bmatrix} 9+0-1 & 6+0-5 \\ -15+4+4 & -10+6+20 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 1 \\ -7 & 16 \end{bmatrix}$$

4. If $A = \left[\begin{array}{cc|cc} 2 & -3 & 1 & 3 \\ 1 & -2 & 5 & 2 \\ \hline 0 & 0 & 4 & 3 \\ 0 & 0 & -1 & 1 \end{array}\right]$ Find A^{-1}

Sol^n: Here

$$A = \left[\begin{array}{cc|cc} 2 & -3 & 1 & 3 \\ 1 & -2 & 5 & 2 \\ \hline 0 & 0 & 4 & 3 \\ 0 & 0 & -1 & 1 \end{array}\right] \Leftrightarrow \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

$$\therefore \left[\begin{array}{cc|cc} 2 & -3 & 1 & 3 \\ 1 & -2 & 5 & 2 \\ \hline 0 & 0 & 4 & 3 \\ 0 & 0 & -1 & 1 \end{array}\right]$$

we use the formula for inverse of partioned matrix, we obtain.

$$A^{-1} = \begin{bmatrix} A_{11}^{-1} & -A_{11}^{-1} \cdot A_{12} A_{22}^{-1} \\ 0 & A_{22}^{-1} \end{bmatrix}$$

Now

$$A_{11} = \begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix}, \quad A_{22} = \begin{bmatrix} 4 & 3 \\ -1 & 1 \end{bmatrix}$$

For A_{11}^{-1}

$\det(A_{11}) = (-4+3) = -1 \neq 0$

since, A_{11}^{-1} is exist,

$$\text{adj.}(A) = \begin{bmatrix} -2 & 3 \\ -1 & 2 \end{bmatrix}$$

then

$$A_{11}^{-1} = \frac{Adj(A)}{|A|} = -1 \begin{bmatrix} -2 & 3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix}$$

For $A_{22} = \begin{bmatrix} 4 & 3 \\ -1 & 1 \end{bmatrix}$

$\det(A_{22}) = (4+3) = 7 \neq 0$

So A_{22}^{-1} is exist,

$Adj(A_{22}) = \begin{bmatrix} 1 & -3 \\ 1 & 4 \end{bmatrix}$

then,

$$A^{-1}_{22} = \frac{Adj(A_{22})}{|A_{22}|} = \frac{\begin{bmatrix} 1 & -3 \\ 1 & 4 \end{bmatrix}}{7} = \begin{bmatrix} 1/7 & -3/7 \\ 1/7 & 4/7 \end{bmatrix}$$

then

$$-A_{11}^{-1} \cdot A_{12} \cdot A_{22}^{-1} = -\begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 3 \\ 5 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1/7 & -3/7 \\ 1/7 & 4/7 \end{bmatrix}$$

$$= -\begin{bmatrix} 2-15 & 6-6 \\ 1-10 & 3-4 \end{bmatrix} \cdot \begin{bmatrix} 1/7 & -3/7 \\ 1/7 & 4/7 \end{bmatrix}$$

$$= -\begin{bmatrix} -13 & 0 \\ -9 & -1 \end{bmatrix} \begin{bmatrix} 1/7 & -3/7 \\ 1/7 & 4/7 \end{bmatrix}$$

$$= -\begin{bmatrix} -13/7 & 39/7 \\ -9/7 - 1/7 & 27/7 - 4/7 \end{bmatrix}$$

$$= -\begin{bmatrix} -13/7 & 39/7 \\ -10/7 & 23/7 \end{bmatrix}$$

$$= \begin{bmatrix} 13/7 & -39/7 \\ 10/7 & -23/7 \end{bmatrix}$$

then $A^{-1} = \begin{bmatrix} 2 & -3 & 13/7 & -39/7 \\ 1 & -2 & 10/7 & -23/7 \\ 0 & 0 & 1/7 & -3/7 \\ 0 & 0 & 1/7 & 4/7 \end{bmatrix}$

Exercise:- 3.3.

Matrix factorization :-

The LU factorization

consider a matrix A of size $m*n$ that can be row reduced to echelon form without row interchanging. Then A can be written in the form $A=LU$, where L is $m*m$ Lower triangular matrix with 1's on the diagonal and U is $m \times n$ echelon matrix form of A such a factorization is called LU factorization of A.

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ * & 1 & 0 & 0 \\ * & * & 1 & 0 \\ * & * & * & 1 \end{bmatrix} \begin{bmatrix} \bullet & * & * & * & * \\ 0 & \bullet & * & * & * \\ 0 & 0 & 0 & \bullet & * \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

where $*$ = carry value including zero

$\bullet$ = any non-zero value

where $A=LU$ then the equation $Ax=b$

$$L(Ux)=b$$

Setting $Ux=y$, we can found x by solving the pair of equation

$$Ly=b$$

$$\& \; Ux=y$$

First solve $Ly=b$ for y and then solve $Ux=y$ for x.

DATE

Example :-

Find the LU factorization of

$$A = \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ -4 & -5 & 3 & -8 & 1 \\ 2 & -5 & -4 & 1 & 8 \\ -6 & 0 & 7 & -3 & 1 \end{bmatrix}$$

Sol: Here

$$A = \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ -4 & -5 & 3 & -8 & 1 \\ 2 & -5 & -4 & 1 & 8 \\ -6 & 0 & 7 & -3 & 1 \end{bmatrix}$$

Since A has 4×5 size so L should has 4×4 size.

$$A = \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ -4 & -5 & 3 & -8 & 1 \\ 2 & -5 & -4 & 1 & 8 \\ -6 & 0 & 7 & -3 & 1 \end{bmatrix}$$

$R_2 \to R_2 + 2R_1$, $R_3 \to R_3 - R_1$, $R_4 \to R_4 + 3R_1$

$$A = \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & -9 & -3 & -4 & 10 \\ 0 & 12 & 4 & 12 & -5 \end{bmatrix}$$

$R_3 \to R_3 + 3R_2$, $R_4 \to R_4 - 4R_2$

$$A = \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 4 & 7 \end{bmatrix}$$

$R_4 \to R_4 - 2R_3$

$$\begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix} = U$$

Now at each highligted column is divided by the corresponding pivot & get L.

$$\begin{bmatrix} 2 \\ -4 \\ 2 \\ -6 \end{bmatrix} \quad \begin{bmatrix} 3 \\ -9 \\ 12 \end{bmatrix} \quad \begin{bmatrix} 2 \\ 4 \end{bmatrix} \quad \begin{bmatrix} 5 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ -2 \\ 1 \\ -3 \end{bmatrix} \quad \begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \begin{bmatrix} 1 \end{bmatrix}$$

$$\therefore L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & -3 & 1 & 0 \\ -3 & 4 & 2 & 1 \end{bmatrix}$$

$\therefore A = LU$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & -3 & 1 & 0 \\ -3 & 4 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix}$$

Exercise :- 3.3

10. Find the LU factorization of the following matrices.

(a) $\begin{bmatrix} 3 & -1 & 2 \\ -3 & -2 & 10 \\ 9 & -5 & 6 \end{bmatrix}$

Sol^n: Here,

let, $A = \begin{bmatrix} 3 & -1 & 2 \\ -3 & -2 & 10 \\ 9 & -5 & 6 \end{bmatrix}$

Since A has 3x3 size, So that L has 3x3 size

$A = \begin{bmatrix} 3 & -1 & 2 \\ -3 & -2 & 10 \\ 9 & -5 & 6 \end{bmatrix}$

$R_2 \to R_2 + R_1, \quad R_3 \to R_3 - 3R_1$

$A = \begin{bmatrix} 3 & -1 & 2 \\ 0 & -3 & 12 \\ 0 & -2 & 0 \end{bmatrix}$

$R_3 \to \frac{R_3}{2} - \frac{2R_2}{3}$

$A = \begin{bmatrix} 3 & -1 & 2 \\ 0 & -3 & 12 \\ 0 & 0 & -4 \end{bmatrix} = U$

Now at each highlighted column is divided by the corresponding pivot and get L.

$L = \begin{bmatrix} 3 \\ -3 \\ 9 \end{bmatrix} \begin{bmatrix} -3 \\ -2 \end{bmatrix} \begin{bmatrix} -4 \end{bmatrix}$

$$L = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2/3 \end{bmatrix} \begin{bmatrix} 1 \end{bmatrix}$$

Now

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 3 & 2/3 & 1 \end{bmatrix}$$

$\therefore A = LU$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 3 & 2/3 & 1 \end{bmatrix} \begin{bmatrix} 3 & -1 & 2 \\ 0 & -3 & 12 \\ 0 & 0 & -4 \end{bmatrix}$$

(b) $\begin{bmatrix} 1 & 3 & -5 & -3 \\ -1 & -5 & 8 & 4 \\ 4 & 2 & -5 & -7 \\ -2 & -4 & 7 & 5 \end{bmatrix}$

soln: Here,

let $A = \begin{bmatrix} 1 & 3 & -5 & -3 \\ -1 & -5 & 8 & 4 \\ 4 & 2 & -5 & -7 \\ -2 & -4 & 7 & 5 \end{bmatrix}$

Since A has 4x4 size so that L should has 4x4 size,

$$A = \begin{bmatrix} 1 & 3 & -5 & -3 \\ -1 & -5 & 8 & 4 \\ 4 & 2 & -5 & -7 \\ -2 & -4 & 7 & 5 \end{bmatrix}$$

$R_2 \to R_2 + R_1,\ R_3 \to R_3 - 4R_1,\ R_4 \to R_4 + 2R_1$

$$A = \begin{bmatrix} 1 & 3 & -5 & -3 \\ 0 & -2 & 3 & 1 \\ 0 & -10 & 15 & 5 \\ 0 & 2 & -3 & -1 \end{bmatrix}$$

$R_3 \to \frac{1}{5} R_3$

$$A = \begin{bmatrix} 1 & 3 & -5 & -3 \\ 0 & -2 & 3 & 1 \\ 0 & -2 & 3 & 1 \\ 0 & 2 & -3 & -1 \end{bmatrix}$$

$R_3 \to R_3 - R_2,\ R_4 \to R_4 + R_2$

$$A = \begin{bmatrix} 1 & 3 & -5 & -3 \\ 0 & -2 & 3 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Now, ~~at~~ each ~~column~~ highlighted column is divided by the corresponding pivot and get L

$$L = \begin{bmatrix} 1 \\ -1 \\ 4 \\ -2 \end{bmatrix} \quad \begin{bmatrix} -2 \\ -2 \\ 2 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 \\ -1 \\ 4 \\ -2 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

$$A = LU = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & \\ 4 & 1 & \\ -2 & -1 & \end{bmatrix}$$

(c) $\begin{bmatrix} 2 & -4 & -2 & 3 \\ 6 & -9 & -5 & 8 \\ 2 & -7 & -3 & 9 \\ 4 & -2 & -2 & -1 \\ -6 & 3 & 3 & 4 \end{bmatrix}$

Sol^n: Here,

$$A = \begin{bmatrix} 2 & -4 & -2 & 3 \\ 6 & -9 & -5 & 8 \\ 2 & -7 & -3 & 9 \\ 4 & -2 & -2 & -1 \\ -6 & 3 & 3 & 4 \end{bmatrix}$$

Since, A has 5×4 size so that the L should has 4×4 size

$$A = \begin{bmatrix} 2 & -4 & -2 & 3 \\ 6 & -9 & -5 & 8 \\ 2 & -7 & -3 & 9 \\ 4 & -2 & -2 & -1 \\ -6 & 3 & 3 & 4 \end{bmatrix}$$

$R_2 \to R_2 - 3R_1$, $R_3 \to R_3 - R_1$, $R_4 \to R_4 - 2R_1$, $R_5 \to R_5 + 3R_1$

$$A = \begin{bmatrix} 2 & -4 & -2 & 3 \\ 0 & 3 & 1 & -1 \\ 0 & -3 & -1 & 6 \\ 0 & 6 & 2 & -7 \\ 0 & -9 & -3 & 13 \end{bmatrix}$$

classmate

PAGE

$$L = \begin{bmatrix} 1 \\ 2 \\ -1/3 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 3 \end{bmatrix} \quad \begin{bmatrix} 1 \end{bmatrix}$$

$$\therefore A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1/3 & 3 & 1 \end{bmatrix} \cdot \begin{bmatrix} 3 & -6 & 3 \\ 0 & 5 & -4 \\ 0 & 0 & 15 \end{bmatrix} = LU$$

iv) $\begin{bmatrix} 2 & 5 \\ 6 & -7 \end{bmatrix}$

Sol^n: Here,

let $A = \begin{bmatrix} 2 & 5 \\ 6 & -7 \end{bmatrix}$

Since A has 2x2 size so that L should has 2x2 size

$$A = \begin{bmatrix} 2 & 5 \\ 6 & -7 \end{bmatrix}$$

$R_2 \to R_2 - 3R_1$

$$A = \begin{bmatrix} 2 & 5 \\ 0 & -22 \end{bmatrix} = U$$

Now at each highlighted column is divided by the corresponding pivot and Hence get L

$$L = \overset{\div 2}{\begin{bmatrix} 2 \\ 6 \end{bmatrix}} \quad \overset{\div -22}{\begin{bmatrix} -22 \end{bmatrix}}$$

$$L = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \quad \begin{bmatrix} 1 \end{bmatrix}$$

$$\therefore A = LU = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 5 \\ 0 & -22 \end{bmatrix}$$ Ans

DATE

(v) $\begin{bmatrix} 2 & -4 & 4 & -2 \\ 6 & -9 & 7 & -3 \\ -1 & -4 & 8 & 0 \end{bmatrix}$

Soln: Here,

let

$$A = \begin{bmatrix} 2 & -4 & 4 & -2 \\ 6 & -9 & 7 & -3 \\ -1 & -4 & 8 & 0 \end{bmatrix}$$

since A has 3x4 size ~~and~~ so that L should have 3x3 size

$$A = \begin{bmatrix} 2 & -4 & 4 & -2 \\ 6 & -9 & 7 & -3 \\ -1 & -4 & 8 & 0 \end{bmatrix}$$

$$R_2 \to R_2 - 3R_1, \quad R_3 \to 2R_3 + R_1$$

$$A = \begin{bmatrix} 2 & -4 & 4 & -2 \\ 0 & 3 & -5 & 3 \\ 0 & -12 & 20 & -2 \end{bmatrix}$$

$$R_3 \to R_3 + 4R_2$$

$$A = \begin{bmatrix} 2 & -4 & 4 & -2 \\ 0 & 3 & -5 & 3 \\ 0 & 0 & 0 & 10 \end{bmatrix}$$

At each highlighted column is divided by the corresponding pivot and here get L

$$L = \begin{bmatrix} 2 \\ 6 & \begin{bmatrix} 3 \\ -12 & \begin{bmatrix} 10 \end{bmatrix} \end{bmatrix} \\ -1 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 \\ 3 & \begin{bmatrix} 1 \\ -4 & \begin{bmatrix} 1 \end{bmatrix} \end{bmatrix} \\ -1/2 \end{bmatrix}$$

Now

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -1/2 & -4 & 1 \end{bmatrix}$$

then

$$A = LU = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -1/2 & -4 & 1 \end{bmatrix} \begin{bmatrix} 2 & -4 & 4 & -2 \\ 0 & 3 & -5 & 3 \\ 0 & 0 & 0 & 10 \end{bmatrix}$$

Example

Find the LU factorization of

$$A = \begin{bmatrix} 3 & -7 & -2 & 2 \\ -3 & 5 & 1 & 0 \\ 6 & -4 & 0 & -5 \\ -9 & 5 & -5 & 12 \end{bmatrix}$$

Use this LU factorization to solve $Ax = b$ where $b = \begin{bmatrix} -9 \\ 5 \\ 7 \\ 11 \end{bmatrix}$

Soln:-

Here

$$A = \begin{bmatrix} 3 & -7 & -2 & 2 \\ -3 & 5 & 1 & 0 \\ 6 & -4 & 0 & -5 \\ -9 & 5 & -5 & 12 \end{bmatrix}$$

$R_2 \to R_2 + R_1$, $R_3 \to R_3 - 2R_1$, $R_4 \to R_4 + 3R_1$

$$A = \begin{bmatrix} 3 & -7 & -2 & 2 \\ 0 & -2 & -1 & 2 \\ 0 & 10 & 4 & -9 \\ 0 & -16 & -11 & 18 \end{bmatrix}$$

$R_3 \to R_3 + 5R_2$, $R_4 \to R_4 - 8R_2$

$$A = \begin{bmatrix} 3 & -7 & -2 & 2 \\ 0 & -2 & -1 & 2 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & -3 & 2 \end{bmatrix}$$

DATE

$R_4 \to R_4 - 3R_3$

$$A = \begin{bmatrix} 3 & -7 & -2 & 2 \\ 0 & -2 & -1 & 2 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & \boxed{-1} \end{bmatrix} = U$$

Now,

At each highlighted column is divided by the corresponding pivot and hence get L

$$L = \begin{bmatrix} 3 \\ -3 \\ 6 \\ -9 \end{bmatrix} \begin{bmatrix} -2 \\ 10 \\ -16 \end{bmatrix} \begin{bmatrix} -1 \\ -3 \end{bmatrix} \begin{bmatrix} -1 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 \\ -1 \\ 2 \\ -3 \end{bmatrix} \begin{bmatrix} 1 \\ -5 \\ 8 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} \begin{bmatrix} 1 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & -5 & 1 & 0 \\ -3 & 8 & 3 & 1 \end{bmatrix}$$

$$A = LU = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & -5 & 1 & 0 \\ -3 & 8 & 3 & 1 \end{bmatrix} \begin{bmatrix} 3 & -7 & -2 & 2 \\ 0 & -2 & -1 & 2 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

To solve $Ax = b$

$LUx = b$

where $y = Ux$ — (1)

then, $Ly = b$ — (1)

The augmented matrix is

$Ly = b$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -9 \\ -1 & 1 & 0 & 0 & 5 \\ 2 & -5 & 1 & 0 & 7 \\ -3 & 8 & 3 & 1 & 11 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -9 \\ -1 & 1 & 0 & 0 & 5 \\ 2 & -5 & 1 & 0 & 7 \\ -3 & 8 & 3 & 1 & 11 \end{array}\right]$$

$R_2 \to R_2 + R_1, \quad R_3 \to R_3 - 2R_1, \quad R_4 \to R_4 + 3R_1$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -9 \\ 0 & 1 & 0 & 0 & -4 \\ 0 & -5 & 1 & 0 & 25 \\ 0 & 8 & 3 & 1 & -16 \end{array}\right]$$

$R_3 \to R_3 + 5R_2, \quad R_4 \to R_4 - 8R_2$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -9 \\ 0 & 1 & 0 & 0 & -4 \\ 0 & 0 & 1 & 0 & \cancel{2}5 \\ 0 & 0 & 3 & 1 & 16 \end{array}\right]$$

$R_4 \to R_4 - 3R_3$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -9 \\ 0 & 1 & 0 & 0 & -4 \\ 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

From this we get

$$y = \begin{bmatrix} -9 \\ -4 \\ 5 \\ 1 \end{bmatrix}$$

Again the augmented matrix of $y = ux$ is

$$\left[\begin{array}{cccc|c} 3 & -7 & -2 & 2 & -9 \\ 0 & -2 & -1 & 2 & -4 \\ 0 & 0 & -1 & 1 & 5 \\ 0 & 0 & 0 & -1 & 1 \end{array}\right]$$

$R_3 \to R_3 + R_4$, $R_2 \to R_2 + 2R_4$, $R_1 \to R_1 + 2R_4$

$$\left[\begin{array}{cccc|c} 3 & -7 & -2 & 0 & -7 \\ 0 & -2 & -1 & 0 & -2 \\ 0 & 0 & -1 & 0 & 6 \\ 0 & 0 & 0 & -1 & 1 \end{array}\right]$$

$R_2 \to R_2 - R_3$, $R_1 \to R_1 - 2R_3$

$$\left[\begin{array}{cccc|c} 3 & -7 & 0 & 0 & -19 \\ 0 & -2 & 0 & 0 & -8 \\ 0 & 0 & -1 & 0 & 6 \\ 0 & 0 & 0 & -1 & 1 \end{array}\right]$$

$R_2 \to -\frac{1}{2} R_2$, $R_3 \to -R_3$, $R_4 \to -R_4$

$$\left[\begin{array}{cccc|c} 3 & -7 & 0 & 0 & -19 \\ 0 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & -6 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right] \xrightarrow{R_1 \to R_1 + 7R_2} \left[\begin{array}{cccc|c} 3 & 0 & 0 & 0 & 9 \\ 0 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & -6 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right]$$

$R_1 \to \frac{1}{3} R_1$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & -6 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right]$$

From this we get

$$X = \begin{bmatrix} 3 \\ 4 \\ -6 \\ -1 \end{bmatrix}$$

Leontief Input- Output Model

$x = cx + d$

$x - cx = d$

$x(I - c) = d$

where x is production vector

c is consumption

d is final demand.

① production model $x = cx + d$

$$C = \begin{bmatrix} 0.1 & 0.6 \\ 0.5 & 0.2 \end{bmatrix}, \quad d = \begin{bmatrix} 18 \\ 11 \end{bmatrix}$$

Use inverse matrix to find x

Sol^n:-

$x = cx + d$

$x - cx = d$

$(I - c)x = d$

Here $I - C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.1 & 0.6 \\ 0.5 & 0.2 \end{bmatrix}$

$$= \begin{bmatrix} 0.9 & -0.6 \\ -0.5 & 0.8 \end{bmatrix}$$

The augmented matrix of $(I - c)x = d$ is

$$\left[\begin{array}{cc|c} 0.9 & -0.6 & 18 \\ -0.5 & 0.8 & 11 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 9 & -6 & 180 \\ 5 & 8 & 110 \end{array}\right]$$

$R_1 = \frac{1}{9} R_1$

$$\left[\begin{array}{cc|c} 1 & -2/3 & 20 \\ -5 & 8 & 110 \end{array}\right]$$

$R_2 \to R_2 + 5R_1$

$$\left[\begin{array}{cc|c} 1 & -2/3 & 20 \\ 0 & 14/3 & 210 \end{array}\right]$$

$R_2 \to \frac{3}{14} R_2$

$$\left[\begin{array}{cc|c} 1 & -2/3 & 20 \\ 0 & 1 & 45 \end{array}\right]$$

$R_1 \to R_1 + \frac{2}{3} R_2$

$$\left[\begin{array}{cc|c} 1 & 0 & 50 \\ 0 & 1 & 45 \end{array}\right]$$

$$x = \begin{bmatrix} 50 \\ 45 \end{bmatrix}$$ Ans

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② $C = \begin{bmatrix} 0.2 & 0.2 & 0.0 \\ 0.3 & 0.1 & 0.3 \\ 0.1 & 0.0 & 0.2 \end{bmatrix}, d = \begin{bmatrix} 40 \\ 60 \\ 80 \end{bmatrix}$

$x = ?$

Solⁿ: Here,

The model is

$x = cx + d$

$x - cx = d$

$x(I - c) = d$

Here

$I - c = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.2 & 0.2 & 0.0 \\ 0.3 & 0.1 & 0.3 \\ 0.1 & 0.0 & 0.2 \end{bmatrix}$

$= \begin{bmatrix} 0.8 & -0.2 & 0.0 \\ -0.3 & 0.9 & -0.3 \\ -0.1 & 0.0 & 0.8 \end{bmatrix}$

The augmented matrix is

$\left[\begin{array}{ccc:c} 0.8 & -0.2 & 0.0 & 40 \\ -0.3 & 0.9 & -0.3 & 60 \\ -0.1 & 0.0 & 0.8 & 80 \end{array}\right]$

$\left[\begin{array}{ccc:c} 8 & -2 & 0 & 400 \\ -3 & 9 & -3 & 600 \\ 1 & 0 & -8 & -800 \end{array}\right]$

$R_1 \leftrightarrow R_3$

$\left[\begin{array}{ccc:c} 1 & 0 & -8 & -800 \\ -1 & 3 & -1 & 200 \\ 4 & -1 & 0 & 200 \end{array}\right]$

$R_2 \to R_2 + R_1$, & $R_3 \to R_3 - 4R_1$

$$\left[\begin{array}{ccc|c} 1 & 0 & -8 & -800 \\ 0 & 3 & -9 & -600 \\ 0 & -1 & 32 & 3400 \end{array}\right]$$

$R_2 \to \frac{1}{3} R_2$

$$\left[\begin{array}{ccc|c} 1 & 0 & -8 & -800 \\ 0 & 1 & -3 & -200 \\ 0 & -1 & 32 & 3400 \end{array}\right]$$

$R_3 \to R_3 + R_2$

$$\left[\begin{array}{ccc|c} 1 & 0 & -8 & -800 \\ 0 & 1 & -3 & -200 \\ 0 & 0 & 29 & 3200 \end{array}\right]$$

$R_3 \to \frac{1}{29} R_3$

$$\left[\begin{array}{ccc|c} 1 & 0 & -8 & -800 \\ 0 & 1 & -3 & -200 \\ 0 & 0 & 1 & \frac{3200}{29} \end{array}\right]$$

$R_1 \to R_1 + 8R_3$ & $R_2 \to R_2 + 3R_3$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2400/29 \\ 0 & 1 & 0 & 3800/29 \\ 0 & 0 & 1 & 3200/29 \end{array}\right]$$

The production is

$$x = \begin{bmatrix} 2400/29 \\ 3800/29 \\ 3200/29 \end{bmatrix} = \begin{bmatrix} 82.76 \\ 131.03 \\ 110.3 \end{bmatrix}$$