CHAPTER-7.

ORDINARY DIFFERENTIAL EQUATION.

Ordinary Differential Equation:-

A differential Equation is an equation involving one dependent variable and its derivative with respect to one or more independent variables.

Order of differential equation:-

The highest Order derivative involve in the equation is called order of differential Equation.

Degree of Differential equation:-

The power at the highest order derivative. after it has been from fractional power (or radical sign) is called degree of differential equation.

e.g. (i) $\left(\frac{dy}{dx}\right)^4 = \frac{d^2y}{dx^2} - 3y = 0$

degree.
order = 2, power = 1

(ii) $\left(\frac{d^3y}{dx^3}\right)^2 + 2x\left(\frac{d^2y}{dx^2}\right)^4 + 6x\left(\frac{dy}{dx}\right)^2 + 3xy = 12.$

degree.
order = 3, power = 2

(iii) $\left(\frac{d^2y}{dx^2}\right)^{3/2} + 3\left(\frac{dy}{dx}\right) = 0$

$$\left(\frac{d^2y}{dx^2}\right)^{3/2} = -3\left(\frac{dy}{dx}\right)$$

$$\left(\frac{d^2y}{dx^2}\right)^3 = 9\left(\frac{dy}{dx}\right)^2$$

order = 2
~~pow~~ degree = 3

Solution of differential equation:

The soln of a differential equation is the value of dependent variable interms of independent variable which is free from derivative and that satisfy the differential equations

example :-

Show that $y = \frac{x^4}{3} + 3x^2 + 1$ is a

Soln of differential equation.

$$\frac{dy}{dx} = \frac{4x^3 + 6x}{3}$$

Sol: We know,

$$y = \frac{x^4}{3} + 3x^2 + 2$$

$$\frac{dy}{dx} = \frac{4x^3 + 6x}{3}$$

$y = x^4/3 + 3x^2 + 2$ is the soln.

Equation of the first Order and first degree

A first Order and first degree Ordinary differential equation (ODE) is in the form $M\,dx + N\,dy = 0$

where M and N are constant or function x & y.

Initial value problem.

A differential equation with the given initial condition is called initial value problem

e.g. $dy/dx + xy = 0$ given $y(0) = 1$

e.g. $y' + 2y = 2$, $y(2) = 0$

Variable separation Equation.

Ex. solve the differential equation.

$$x\sqrt{1+y^2}\,.dx + y\,.\sqrt{1+x^2}\,.dy = 0$$

Soln: here,

$$\frac{x}{\sqrt{1+x^2}}\,dx + \frac{y}{\sqrt{1+y^2}}\,.dy = 0$$

Exercise 7.1.

Solve the following differential equations (direct Separation of variables).

① $(1+x)(1+y^2).dx+(1+y)(1+x^2)\,dy=0$

Sol^n: Here,

Given,

$(1+x)(1+y^2).dx+(1+y).(1+x^2).dy=0$

dividing by $(1+x^2)$ & $(1+y^2)$ on bothside we get,

$$\frac{1+x}{(1+x^2)}.dx+\frac{(1+y)}{(1+y^2)}.dy=0.$$

$$\frac{1}{1+x^2}dx+\frac{x}{1+x^2}.dx+\frac{1}{(1+y^2)}.dx+\frac{y}{1+y^2}.dx$$

~~$\tan^{-1}(x)$~~.

$$\frac{1}{(1+x^2)}.dx+\frac{1}{(1+y^2)}.dy+\left(\frac{x.dx}{1+x^2}\right)+\left(\frac{y}{1+y^2}\right)dy$$

$$\left(\frac{1}{1+x^2}\right).dx+\left(\frac{1}{1+y^2}\right).dy+\left(\frac{\tan x}{1+\tan^2 x}\right).dx+\left(\frac{\tan y}{1+\tan^2 y}\right).dy=0$$

$$\left(\frac{1}{1+x^2}\right)dx+\left(\frac{1}{1+y^2}\right).dy+\left(\frac{\tan x}{\sec^2 x}\right).dx+\frac{\tan y}{\sec^2 y}=0$$

$$\tan^{-1}x+\tan^{-1}y+\log(\sec x)+\log(\sec y)=c$$

$$\tan^{-1}x+\tan^{-1}y+\log\sqrt{1+\tan^2 x}+\log(\sqrt{1+\tan^2 y})=c$$

$$\tan^{-1}x+\tan^{-1}y+\log(\sqrt{1+x^2}\sqrt{1+y^2})=c$$

Ans//

② $e^{x-y}.dx + e^{y-x}.dy = 0$

Sol^n: Here

$e^x.e^{-y}.dx + e^y.e^{-x}.dy = 0$

$\frac{e^x}{e^y}.dx + \frac{e^y}{e^x}dy = 0$

$e^x.e^x.dx + e^y.e^y.dy = 0$

$e^{2x}.dx + e^{2y}.dy = 0$

By integration, we get

$\int e^{2x}.dx + \int e^{2y}.dy = 0$

$\frac{e^{2x}}{2} + \frac{e^{2y}}{2} = C$

$e^{2x} + e^{2y} = C$ Ans//

③ $(e^y+1)\cos x\,dx + e^y \sin x.dy = 0$

Sol^n: Here

$e^y.\cos x.dx + 1\cos x.dx + e^y \sin x.dy = 0$

$e^y.\cos x.dx + \cos x.dx + e^y \sin x\,dy = 0$

$e^y\cot x.dx + \cot x.dx + e^y.dy = 0$

$(e^y+1)\cot x.dx + e^y.dy = 0$

$\cot x.dx + \frac{e^y}{(e^y+1)}.dy = 0$

By integration we get

$\int \frac{\cos x}{\sin x}.dx + \int \frac{e^y}{(e^y+1)}.dy = 0$

$\log(\sin x) + \log(e^y+1) = C$

$\sin x(e^y+1) = C$

Ans

Date
Page 33

④ Find the particular solution of $y' = \sec y$, given that $y=0$ when $x=0$.

Sol^n: Here,

$y' = \sec y$. where $y=0, x=0$,

$\frac{dy}{dx} = \sec y$

$\frac{dy}{dx} = \frac{1}{\cos y}$

$dy.\cos y = dx$

$dx = \cos y.dy$

$dx - \cos y.dy = 0$

By integration, we get,

$\int dx - \int \cos y.dy = 0$

$x - \sin y = c$

$\sin y = x$ Ans//

⑤ $(xy^2+x)dx + (x^2y+y).dy = 0$

$x(y^2+1).dx + y(x^2+1).dy = 0$

$\frac{x}{(x^2+1)}.dx + \frac{y}{(y^2+1)}.dy = 0$

By integration we get,

$\int \frac{x}{x^2+1}.dx + \int \frac{y}{y^2+1}.dy = 0$

$\frac{1}{2}\int \frac{2x}{(x^2+1)}.dx + \frac{1}{2}\int \frac{2y}{y^2+1}.dy = 0$

$\frac{1}{2}\log(x^2+1) + \frac{1}{2}\log(y^2+1) = C$

$\frac{1}{2}\log\{(x^2+1)(y^2+1)\} = C$

$(x^2+1).(y^2+1) = C$ Ans

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(6) Solve the differential equation $\frac{dy}{dx} = \frac{x^2}{y^2}$

Sol^n: here,

$$\frac{dy}{dx} = \frac{x^2}{y^2}$$

$$y^2 dy - x^2 dx = 0$$

By Integration we get

$$\int y^2 dy - \int x^2 . dx = 0$$

$$y^3/3 - x^3/3 = c$$

$$y^3 - x^3 = 3c$$

$$y^3 = 3c + x^3$$

$$y^3 = x^3 + 3c$$

$$y = \sqrt[3]{x^3+3c}$$

where c is an arbitary constant, so is 3k

(7) Solve the differential equation $\frac{dy}{dx} = \frac{6x^2}{2y+\cos y}$

Sol^n: here

$$\frac{dy}{dx} = \frac{6x^2}{2y+\cos y}$$

$$2y\,dy + \cos y\,dy - 6x^2 dx = 0$$

By the Integration we get,

$$2y^2/2 + \sin y - 6x^3/3 = c$$

$$y^2 + \sin y - 2x^3 = c$$

$$y^2 + \sin y = 2x^3 + c$$

Ans//

(8) Solve the equation $y' = x^2 y$

Sol^n: here

$$y' = x^2 . y$$

$$\frac{dy}{dx} = x^2 . y$$

$$\frac{1}{y} dy = x^2 . dx$$

$$\frac{1}{y} dy - x^2 . dx = 0$$

By Integration we get

$$\int \frac{1}{y} dy - \int x^2 . dx = 0$$

$$\log(y) - x^3/3 = c$$

$$\log y = c + x^3/3$$

$$y = e^{c + x^3/3} \qquad \log y = \frac{x^3}{3} + c$$

$$y = e^c . e^{x} \qquad y = e^c . e^{x^3/3}$$

where $e^c = A$ so,

$$y = A . e^{x^3/3}$$

Ans

(9) Solve $\frac{dy}{dx} = \frac{x}{y}$, $y(10) = -3$

Soln: here

$\frac{dy}{dx} = \frac{x}{y}$

$y \cdot dy = x \cdot dx$

$y \cdot dy - x \cdot dx = 0$

By integration we get

$\int y \cdot dy - \int x \cdot dx = 0$

$\frac{y^2}{2} - \frac{x^2}{2} = \frac{c^2}{2}$

$\frac{1}{2}(y^2 - x^2) = \frac{c^2}{2}$ where $x = 10$ & $y = -3$

$\frac{1}{2}(9 - 100) = c$

$c =$

$\frac{1}{2}(y^2 - x^2) = \frac{c^2}{2}$

$y^2 - x^2 = c^2$

$x^2 - y^2 = c^2$

$x^2 - c^2 = y^2$

$y = \sqrt{x^2 - c^2}$

$y(10) = \sqrt{100 - c^2}$

$(-3)^2 = 100 - c^2$

$c^2 = 100 - 9$

$c^2 = 91$

(9) Solve $\frac{dy}{dx} = \frac{x}{y}$, $y(10) = -3$

Soln: Here,

$\frac{dy}{dx} = \frac{x}{y}$

$y \cdot dy = x \cdot dx$

By Integration we get

$\frac{y^2}{2} = \frac{x^2}{2} + c$

$\frac{y^2}{2} - \frac{x^2}{2} = c$

$y^2 - x^2 = 2c$

$y^2 = x^2 + 2c$

$(-3)^2 = 10^2 + 2c$

$9 = 100 + 2c$

$91 = 2c$

$c = \frac{91}{2}$

$y^2 - x^2 = 91$

$y^2 = x^2 + 91$

$y = \sqrt{x^2 + 91}$ Ans

classmate Date Page

(10) $\frac{du}{dt} = \frac{2t+\sec^2 t}{2u}$, $u(0) = -5$

Sol^n: Here

$$\frac{du}{dt} = \frac{2t+\sec^2 t}{2u}, \quad u(0) = -5$$

$$2u\,du = (2t+\sec^2 t).dt$$

$$2u\,du - (2t+\sec^2 t).dt = 0$$

By integration we get,

$$\int 2u\,du - \int 2t\,dt - \int \sec^2 t.dt = 0$$

$$\frac{2u^2}{2} - \frac{2t^2}{2} - \tan t = C$$

$$u^2 - t^2 - \tan t = C$$

$$u^2 = t^2 + \tan t + C$$

$$C = u^2 - t^2 - \tan t$$

$$(-5)^2 = 0 + \tan 0 + C$$

$$25 = C$$

$$u^2 = t^2 + \tan t + 25$$

$$u = \sqrt{t^2+\tan t+25}$$

classmate Date Page 33

(11) $x \ln x = y(1+\sqrt{3+y^2})y'$, $y(1) = 1$

let $t = 3+y^2$

$\frac{dt}{dy} = 2y$

$dt = 2y.dy$

$\frac{1}{2}dt = y.dy$

Sol^n: Here

$$x.\ln x = y(1+\sqrt{3+y^2}).y'$$

$$x \ln x = y(1+\sqrt{3+y^2}).\frac{dy}{dx}$$

$$x.\ln x\,dx = y(1+\sqrt{3+y^2}).dy$$

$$x \ln x.dx = y\,dy + y\sqrt{3+y^2}.dy$$

By integration we get,

$$\int x \ln x.dx = \int y.dy + \int y\sqrt{3+y^2}.dy$$

$$\int x \ln x.dx = \int y.dy + \int (t)^{1/2}.\frac{1}{2}dt$$

$$\int x \ln x.dx = \int y.dy + \frac{1}{2}\int t^{1/2}.dt$$

$$-\frac{x^2}{4} + \frac{x^2}{2}\ln x = \frac{y^2}{2} + \frac{1}{2}\cdot\frac{2}{3}(t)^{3/2}$$

$$-\frac{x^2}{4} + \frac{x^2}{2}\ln x = \frac{y^2}{2} + \frac{1}{3}(t)^{3/2}$$

$$-\frac{x^2}{4} + \frac{x^2}{2}\ln x + C = \frac{y^2}{2} + \frac{1}{3}\sqrt{(3+y^2)^3}$$

$$\frac{y^2}{2} + \frac{1}{3}\sqrt{(3+y^2)^3} + \frac{x^2}{4} - \frac{x^2}{2}\ln x - C = 0$$

or, $\frac{y^2}{2} + \frac{1}{3}(3+y^2)^{3/2} + \frac{x^2}{4} - \frac{x^2}{2}\ln x - C = 0$

$$\frac{y^2}{2} + \frac{1}{3}(3+y^2)^{3/2} = \frac{x^2}{2}\ln x - \frac{x^2}{4} + C$$

$$\frac{1}{2}(y^2) + \frac{1}{3}(3+y^2)^{3/2} = \frac{x^2}{2}\ln x - \frac{x^2}{4} + C$$

$$\frac{1}{2}(1)^2 + \frac{1}{3}(3+1)^{3/2} = \frac{1}{2}\times 0 - \frac{1}{4} + C$$

$$\frac{1}{2} + \frac{8}{3} + \frac{1}{4} = C, \quad C = \frac{41}{12}$$

then, $\frac{y^2}{2} + \frac{1}{3}(3+y^2)^{3/2} = \frac{x^2}{2}\ln x - \frac{x^2}{4} + \frac{41}{12}$

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(12) $y' \tan x = a+y$, $y(\pi/2) = a$, $0 < x < \pi/2$

Sol^n: Here

$y' \tan x = a+y$.

$\frac{dy}{dx} \tan x = a+y$

$\cancel{(a+y)\,dy}$ $\frac{1}{(a+y)} \cdot dy = \frac{dx}{\tan x}$

$\frac{1}{a+y} \cdot dy = \frac{\cos x}{\sin x} \cdot dx$

$\frac{1}{a+y} \cdot dy - \frac{\cos x}{\sin x} \cdot dx = 0$

By integration we get,

$\int \frac{1}{a+y} \cdot dy - \int \frac{\cos x}{\sin x} \cdot dx = 0$

$t = a+y$

$dt = 0+1$

$dt = dy$

$\int \frac{1}{t} \cdot dt - \int \frac{\cos x}{\sin x} \cdot dx$

$\log(t) - \log(\sin x) = \log c$

$\log\left(\frac{t}{\sin x}\right) = \log(c)$

$\frac{t}{\sin x} = c$

$a+y = c \sin x$

$a+a = c \sin 60$

$2a = c \times \sqrt{3}/2$

$\frac{4a}{\sqrt{3}} = c$

then

$a+y = \frac{4a}{\sqrt{3}} \sin x$

$y = \frac{4a}{\sqrt{3}} \sin x - a$ Ans

33

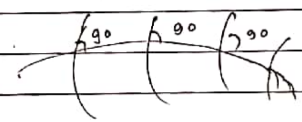
(13) Find the Orthogonal trajectories of the family of curves.

(i) $x^2+2y^2 = K^2$ (ii) $y = K/x$

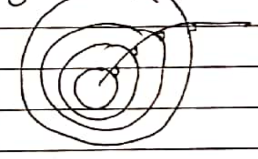
(iii) $y^2 = Kx^3$ (iv) $y = \frac{x}{1+Kx}$

* Orthogonal Trajectory

An orthogonal trajectory of a family of curve is a curve that intersect each curve of the family orthogonally



$x^2+y^2 = r^2$



(ii) $y = K/x$

Sol^n: Here,

$y = K/x$

diff. w. r. to. x on both side we get

$\frac{dy}{dx} = \frac{-K}{x^2} = \frac{-K}{x} \cdot \frac{1}{x} = -y/x$

where $\left(\frac{dy}{dx}\right)$ is the slope of the tangent at the general point of curve, which is

$\left(\frac{dy}{dx}\right) = -y/x$

On an orthogonal trajectory the slope of the tangent line must be the negative reciprocal of this slope. Therefore

The orthogonal trajectories must satisfy the differential eqn.

$$\frac{dy}{dx} = \frac{x}{y}$$

The variable separable form is, $y.dy = x.dx$

taking integration on both side we get

$$\int y.dy = \int x.dx$$

$$\frac{y^2}{2} - \frac{x^2}{2} = \frac{c}{2}$$

$$\frac{x^2}{2} - \frac{y^2}{2} = \frac{c}{2}$$

$$\boxed{x^2 - y^2 = c}$$

Thus the orthogonal trajectories are the family of the ellipse given by the eqn $x^2-y^2=c$ Ans

(iii) $y^2 = kx^3$

Soln: Here, $k = \frac{y^2}{x^3}$

$y^2 = kx^3$

diff. w.r.to. x on both side we get,

$$2y\frac{dy}{dx} = 3x^2.k$$

$$2y\frac{dy}{dx} = 3x^2.\frac{y^2}{x^3}$$

$$2\frac{dy}{dx} = \frac{3y}{x}$$

Now, the slope eqn of the tangent to the curve, is $\frac{dy}{dx} = \frac{3}{2}\frac{y}{x}$

On the orthogonal trajectory the slope of the tangent line must be the negative and reciprocal of this slope, then,

$$\frac{dy}{dx} = -\frac{2x}{3y}$$

then which is in variable separable form,

$$3y.dy = -2x.dx$$

$$2x.dx + 3y\,dy = 0$$

taking integration on both side weget

$$\int 2x.dx + \int 3y.dy = 0$$

$$\frac{2x^2}{2} + \frac{3y^2}{2} = c$$

$$x^2 + \frac{3y^2}{2} = c$$

$\boxed{2x^2+3y^2 = c}$ which is the required eqn of the ellipse to that trajectory.

(iv) $y = \frac{x}{1+Kx}$

Sol^n:- here

$y(1+Kx) = x$

$y + Kyx = x$

$y + Kyx = x$

$1 + Kx = x/y$

$Kx = \frac{x}{y} - 1$

$K = \frac{1}{y} - \frac{1}{x}$

diff. w.r.t x on both side we get,

$0 = -1y^{-2} + 1x^{-2}$

$0 = -\frac{1}{y^2}\frac{dy}{dx} + \frac{1}{x^2}$

$\frac{1}{x^2} - \frac{1}{y^2}\frac{dy}{dx} = 0$

~~Now the st~~ $\frac{1}{x^2} = \frac{1}{y^2}\frac{dy}{dx}$

$\frac{dy}{dx} = \frac{y^2}{x^2}$

Now the slope of the tangent to the curve is $\frac{dy}{dx} = \frac{y^2}{x^2}$

On the orthogonal trajectory the slope of the tangent line must be the negative and reciprocal of this slope then,

$\frac{dy}{dx} = \frac{-x^2}{y^2}$

Now, this can be clear that, the variable is separated form

$y^2 \cdot dy = -x^2 \cdot dx,$

taking integration on both side we get

$\int y^2 \cdot dy = -\int x^2 \cdot dx$

$\int x^2 dx + \int y^2 \cdot dy = 0$

$\frac{x^3}{3} + \frac{y^3}{3} = \frac{c}{3}$

$\boxed{x^3 + y^3 = c}$ which is the required eqn of the ellipse to that trajectory.

(1) $x^2 + 2y^2 = K^2$

Sol^n: Here,

$x^2 + 2y^2 = K^2$

diff. w.r.t. x on both side we get,

$\frac{d(x^2+2y^2)}{dx} = \frac{d(K^2)}{dx}$

$2x + 4y\frac{dy}{dx} = 0$

$4y\,dy/dx = -2x$

$\frac{dy}{dx} = \frac{-x}{2y}$

$\frac{dy}{dx}$ is the slope of the tangent curve, i.e.

$\frac{dy}{dx} = \frac{-x}{2y}$

On the orthogonal trajectory the slope of the tangent line must be the negate and reciprocal of this slope then,

$$\frac{dy}{dx} = \frac{2y}{x}$$

taking integration on both side we get,

$$\int \frac{1}{2y}\, dy = \int \frac{1}{x}\, dx$$

$$\int \frac{1}{x}.dx - \int \frac{1}{2y}.dy = 0$$

$$\log(x) - \frac{1}{2}\log(y) = \log c$$

$$2\log(x) - \log(y) = 2\log(c)$$

$$\log(x)^2 - \log(y) = \log c^2$$

$$\frac{x^2}{y} = c^2$$

$$\int \frac{1}{2y}.dy = \int \frac{1}{x}.dx$$

$$\frac{1}{2}\int \frac{1}{y}.dy = \int \frac{1}{x}.dx$$

$$2\int \frac{1}{x}.dx - \int \frac{1}{y}.dy = 0$$

$$2\log(x) - \log(y) = \log c$$

$$\log x^2 - \log y = \log c$$

$$\log(x^2/y) = \log c$$

$$x^2 = cy$$

Exercise 9.2

Linear Differential Equation.

A differential eq^n of the form

$$\frac{dy}{dx} + P.y = Q \quad -(1)$$

where P & Q are constant or function of x is called linear different eq^n.

Multiply both side of (1) by $e^{\int p dx}$

$$e^{\int p dx}\frac{dy}{dx} + e^{\int p dx}.Py = Q.e^{\int p dx}$$

$$\frac{d}{dx}\left(y.e^{\int p.dx}\right) = Q.e^{\int p.dx}$$

Integrationg $y.e^{\int p dx} = \int Q.e^{\int P dx} + c$

where $e^{\int p dx}$ is called integrating factor or I.f.

$$\frac{d}{dx}\left(y.e^{\int p dx}\right)$$

$$y.e^{\int p dx}P + e^{\int p dx}.\frac{dy}{dx}$$

$\therefore$ Sol^n of (1) is

$$y \times I.f = \int (Q \times I.F)dx + c$$

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Exercise :- 7.2

(1) Determine whether the diff. eq^n is linear.

(a) $x - y' = xy$

Sol^n: Here

$$x - \frac{dy}{dx} = xy,$$

Setting the above eq^n in the form of,

$$\frac{dy}{dx} + P.y = Q$$

$$\frac{dy}{dx} = x - xy$$

$$\frac{dy}{dx} = x(1-y)$$

$$\frac{dy}{dx} - x + xy = 0$$

$$\frac{dy}{dx} + x(y-1) = 0$$

(2) Solve the following differential eq^n.

(i) $y' + y = 1$

Sol^n: Here,

$$\frac{dy}{dx} + y = 1$$

Comparing with $\frac{dy}{dx} + P.y = Q$ the above eq^n is satisfied so,

Here

$$P = 1 \ \& \ Q = 1,$$

$$I.f = e^{\int P dx} = e^{\int 1 dx} = e^x$$

sol^n of (a) is

$$y \times I.f = \int (Q \times I.f) dx + c$$

$$y \times e^x = \int 1.e^x.dx + c$$

$$y \times e^x = e^x + c$$

$$y = \frac{e^x + c}{e^x}$$

$$y = 1 + e^{-x}.c$$

$$y = 1 + e^{-x}.c$$

Ans

② (b) $y' = x - y$

Sol^n: Here,

$$\frac{dy}{dx} = x - y$$

$$\frac{dy}{dx} + y = x \quad -①$$

comparing eq - ① with $\frac{dy}{dx} + P.y = Q$ we get

$$P = 1 \ \& \ Q = x,$$

now

$$I.f = e^{\int p dx} = e^{\int 1.dx} = e^{\int dx} = e^{x}$$

Sol^n of eq^n ① is

$$I.f. y = \int If. Q \, dx + c$$

$$e^{x}.y = \int e^{x}. x . dx + c$$

$$e^{x}.y = x\int e^{x} .dx - \int \left[\frac{d(x)}{dx} \int e^{x} . dx\right] . dx$$

$$e^{x}.y = x.e^{x} - \int 1.e^{x} dx + c$$

$$e^{x}.y = x.e^{x} - e^{x} + c$$

$$y = x - 1 + e^{-x}c$$

$$y = x - 1 + ce^{-x}$$

Ans

ⓒ $xy' + y = \sqrt{x}$

Sol^n: Here,

$$xy' + y = \sqrt{x}$$

$$x.\frac{dy}{dx} + y = \sqrt{x}$$

$$\frac{dy}{dx} + \frac{1}{x} y = \frac{\sqrt{x}}{x}$$

$$\frac{dy}{dx} + \frac{1}{x}.y = (x)^{\frac{1}{2}-1}$$

$$\frac{dy}{dx} + \frac{1}{x}.y = (x)^{-\frac{1}{2}}$$

$$\frac{dy}{dx} + \frac{1}{x}.y = \frac{1}{\sqrt{x}}$$

comparing the eq^n with $\frac{dy}{dx} + P.y = Q$ we get

$$P = \frac{1}{x} \text{ and } Q = \frac{1}{\sqrt{x}}$$

$$I.f = e^{\int P.dx} = e^{\int \frac{1}{x}.dx} = e^{\log x} = x$$

Sol^n of eq^n ① is,

$$If. y = \int Q. If \, dx + c$$

$$x.y = \int \frac{1}{\sqrt{x}} . \cancel{x} \, dx + c$$

$$x.y = \int (x)^{-\frac{1}{2}+1} .dx + c$$

$$x.y = \int (\sqrt{x}) \, dx + c$$

$$x.y = \int (x)^{\frac{1}{2}}.dx + c$$

$$x.y = \frac{2}{3} (x)^{\frac{3}{2}} + c$$

$$y = \frac{1}{x} \frac{2}{3} (x)^{\frac{3}{2}} + x^{-1}c$$

$$y = \frac{2}{3} (x)^{\frac{1}{2}} + x^{-1}c$$

(d) $\sin x \frac{dy}{dx} + (\cos x).y = \sin(x^2)$

Sol^n Here,

$\sin x \frac{dy}{dx} + (\cos x).y = \sin(x^2)$

$\frac{dy}{dx} + \cot x.y = \frac{\sin(x^2)}{\sin x}$ — (1)

Comparing the eq^n (1) with $\frac{dy}{dx} + P.y = Q$ we get

$P = \cot x$, $Q = \frac{\sin(x^2)}{\sin x}$

$I.f = e^{\int P dx} = e^{\int \cot x . dx} = e^{\int \frac{\cos x}{\sin x} . dx}$

$= e^{\log(\sin x)}$

$= \sin x$

Sol^n of eq^n (1) is,

$I.f . y = \int Q . I.f \, dx + c$

$\sin x . y = \int \frac{\sin x^2}{\cancel{\sin x}} . \cancel{\sin x} \, dx + c$

$\sin x . y = \int (\sin x^2) \, dx + c$

$y = \frac{\int \sin(x^2) . dx + c}{\sin x}$ Ans

Page 342

(e) $(1+t) \frac{du}{dt} + u = 1+t$, $t > 0$

Sol^n: Here,

$(1+t) \frac{du}{dt} + u = 1+t$

$\frac{du}{dt} + \frac{1}{(1+t)} . u = \frac{1+t}{1+t}$

$\frac{du}{dt} + \frac{1}{(1+t)} . u = 1$ — (1)

Comparing eq^n (1) with $\frac{du}{dt} + P.u = Q$ we get

$P = \frac{1}{1+t}$, $Q = 1$,

$T.f = e^{\int P dt} = e^{\int \frac{1}{1+t} . dt} = e^{\log(1+t)}$

$= 1+t$

Sol^n of eq^n (1) is,

$y . If = \int Q . If \, dt + c$

$y . (1+t) = \int 1 . (1+t) . dt + c$

$y(1+t) = \int dt + \int t . dt + c$

$y(1+t) = t + \frac{t^2}{2} + c$

$y(1+t) = \frac{2t + t^2 + 2c}{2}$

$y = \frac{2t + t^2 + 2c}{2(1+t)}$ Ans

$y = \frac{t^2 + 2t + 2c}{2(t+1)}$

classmate
Date
Page

(f) $x^2y' + 2xy = \ln x$, $y(1) = 2$

Sol^n: Here,

$$x^2y' + 2xy = \ln x$$

$$y' + \frac{2xy}{x^2} = \frac{\ln x}{x^2}$$

$$y' + \frac{2}{x}\cdot y = \frac{\ln x}{x^2}$$

$$\frac{dy}{dx} + \frac{2}{x} y = \frac{\ln x}{x^2} \quad \text{--- (1)}$$

comparing eq^n(1) with $\frac{dy}{dx} + P.y = Q$ we get,

$P = \frac{2}{x}$ and $Q = \frac{\ln x}{x^2}$

Sol^n of eq^n (1) is -

$$If \times y = \int If\, Q\, dx + c$$

$$x^2 \times y = \int x^2 \times \frac{\ln x}{x^2}\, dx + c$$

$$x^2.y = \int \ln x . dx + c$$

LATE

$$x^2.y = \ln x \int dx - \int \left[\frac{d(\ln x)}{dx} \int dx \right] dx + c$$

$$x^2.y = \ln x . x - \int \left[\frac{1}{x} . x . dx + c \right.$$

$$x^2.y = x \ln x - \int dx + c$$

$$x^2 y = x \ln x - x + c$$

$$y = \frac{1}{x} \ln x - \frac{1}{x} + \frac{c}{x^2}$$

classmate
Date
Page 343

Now $y(1) = 2$

$x = 1$ and $y = 2$

$$2 = \frac{1}{1} \ln 1 - \frac{1}{1} + \frac{c}{1}$$

$$2 = 0 - 1 + c$$

$c = 3$ so, the ~~eq~~ sol^n become,

$$y = \frac{1}{x} \ln x - \frac{1}{x} + \frac{3}{x^2} \quad \text{Ans}$$

(g) $t.\frac{du}{dt} = t^2 + 3u$, $t > 0$, $u(2) = 4$,

Sol^n: Here,

$$\frac{du}{dt} . t = t^2 + 3u$$

$$\frac{du}{dt} . t - t^2 - 3u = 0$$

$$\frac{du}{dt} . t - 3u = t^2$$

$$\frac{du}{dt} - \frac{3}{t} . u = t$$

$$\frac{du}{dt} + (-3/t) . u = t \quad \text{--- (1)}$$

comparing eq^n (1) with $\frac{du}{dt} + P.u = Q$ we get

$P = \left(\frac{-3}{t}\right)$, $Q = t$

$$I.f = e^{\int P dt} = e^{\int -3/t . dt} = e^{-3\int \frac{1}{t} . dt} = e^{-3 \log t} = e^{\log t^{-3}} = t^{-3}$$

Sol^n of eq^n (1) is.

$$u \times If = \int Q . If . dt + c$$

$$u \times t^{-3} = \int t . t^{-3} . dt + c$$

$y(x) =$

$\therefore u(2) = 4$

$u = 4$

$t = 2$

$u t^{-3} = \int t^{-2} \cdot dt + c$

$u t^{-3} = \frac{t^{-2+1}}{-2+1} + c$

$u t^{-3} = \frac{t^{-1}}{-1} + c$

$u = \frac{-t^{-1} + c}{t^{-3}}$

$u = -t^{-1+3} + t^3 c$

$u = -t^2 + t^3 c$

$4 = -(2)^2 + 2^3 \cdot c$

$4 = -4 + 8 \cdot c$

$8 = 8c$

$c = 8/8 = 1$

$\therefore u = -t^2 + t^3$

$u = t^3 - t^2$ Ans

343 x_1

(2) $xy' = y + x^2 \sin x, \; y(\pi) = 0$

Sol^n: Here,

$xy' = y + x^2 \sin x, \; y(\pi) = 0,$

$\frac{dy}{dx} = \frac{y}{x} + x \sin x$

$\frac{dy}{dx} - \frac{y}{x} = x \sin x$

$\frac{dy}{dx} + \left(\frac{-1}{x}\right) \cdot y = x \sin x \quad -(1)$

comparing eq^n (1) ~~we g~~ with $\frac{dy}{dx} + P \cdot y = Q$ we get

$P = -\frac{1}{x}$ and $Q = x \sin x$

$I.f = e^{\int P dx} = e^{\int -\frac{1}{x} \cdot dx} = e^{-\int \frac{1}{x} \cdot dx} = e^{\log(x)^{-1}}$

$= x^{-1} = \frac{1}{x}$

$I.f = \frac{1}{x}$

Sol^n of eq^n (1) is

$y \cdot If = \int If \cdot Q \cdot dx + c$

$y \cdot \frac{1}{x} = \int \frac{1}{x} \cdot x \sin x \cdot dx + c$

$y \cdot \frac{1}{x} = \int \sin x \, dx + c$

$y \cdot \frac{1}{x} = -\cos x + c$

$y = -x \cos x + x c$

where

$y(\pi) = 0,$

$0 = -x \cos \pi + \pi c$

$x \cos \pi = \pi c$

$-x/\pi = c$

$c = -\pi/\pi = -1$

$y = -x \cos x + x \cdot -1$

$y = -x \cos x - x$

③ Solve the diff. eqⁿ.

$\frac{dy}{dx} + 3x^2y = 6x^2$

Solⁿ: Here,

$\frac{dy}{dx} + 3x^2y = 6x^2$ — ①

Comparing eqⁿ ① with $\frac{dy}{dx} + Py = Q$ we get

$P = 3x^2$ & $Q = 6x^2$

$I.f = e^{\int P dx} = e^{\int 3x^2 . dx} = e^{\frac{3x^3}{3}} = e^{x^3}$

Solⁿ of eqⁿ ① is,

$y . I.f = \int Q . If\, dx + c$

$y . e^{x^3} = \int 6x^2 . e^{x^3} . dx + c$

$y . e^{x^3} = 2\int 3x^2 . e^{x^3} . dx + c$

$y . e^{x^3} = 2\int 3x^2 . e^{x^3} . dx + c$

$y . e^{x^3} = 2 . e^{x^3} + c$

$y = 2 + c\, e^{-x^3}$

$y = 2 + e^{-x^3} . c$ Ans

④ Solve $y' + 2xy = 1$

Solⁿ: Here,

$y' + 2xy = 1$

$\frac{dy}{dx} + 2xy = 1$ — ①

Comparing eqⁿ ① with $\frac{dy}{dx} + Py = Q$ we get

$P = 2x$ and $Q = 1$

$I.f = e^{\int P dx} = e^{\int 2x . dx} = e^{2\int x . dx} = e^{2 \times \frac{x^2}{2}} = e^{x^2}$

Now,

Solⁿ of eqⁿ ① is

$y . If = \int Q . If\, dx + c$

$y . e^{x^2} = \int 1 . e^{x^2} . dx + c$

$y = e^{-x^2} \int e^{x^2} . dx + e^{-x^2} . c$

Classmate Date Page

(5) Solve the following linear differential eqn.

(a) $\frac{dy}{dx} + \frac{y}{x^2} = \frac{1}{x^2}$

Sol: Here,

$\frac{dy}{dx} + \frac{1}{x^2} \cdot y = \frac{1}{x^2}$ — (1)

comparing eqn (1) with $\frac{dy}{dx} + P \cdot y = Q$ we get

$P = \frac{1}{x^2}$ and $Q = \frac{1}{x^2}$

$I.f = e^{\int P dx} = e^{\int x^{-2} \cdot dx} = e^{\frac{x^{-2+1}}{-2+1}} = e^{-x^{-1}}$

Sol of eqn (1) is

$y \cdot I.f = \int Q \cdot I.f \, dx + c$

$y \cdot e^{-x^{-1}} = \int \frac{1}{x^2} \cdot e^{-x^{-1}} \cdot dx + c$

$y e^{-x^{-1}} = \int x^{-2} \cdot e^{-x^{-1}} \cdot dx + c$

$y \cdot e^{-x^{-1}} = \int x^{-2} \cdot e^{-x^{-1}} \cdot dx + c$

$y \cdot e^{-x^{-1}} = e^{-x^{-1}} + c$

$y = \frac{e^{-x^{-1}} + c}{e^{-x^{-1}}}$

$y = 1 + e^{x^{-1}} \cdot c$

$y = 1 + e^{1/x} c$

Classmate Date Page 45

(b) $(1+x^2) \frac{dy}{dx} + y = e^{\tan^{-1}x}$

Sol: Here

$\frac{dy}{dx} + \left(\frac{1}{1+x^2}\right) \cdot y = \frac{e^{\tan^{-1}x}}{(1+x^2)}$ — (1)

comparing eqn (1) with $\frac{dy}{dx} + P \cdot y = Q$ we get

$P = \left(\frac{1}{1+x^2}\right)$ and $Q = e^{\tan^{-1}x} \times \frac{1}{1+x^2}$

$I.f = e^{\int P dx} = e^{\int \frac{1}{1+x^2} \cdot dx} = e^{\tan^{-1}x}$

Soln of eqn (1) is

$y \cdot If = \int Q \cdot If \cdot dx + c$

$y \cdot If = \int e^{\tan^{-1}x} \cdot e^{\tan^{-1}x} \frac{1}{1+x^2} dx + c$

$y \cdot e^{\tan^{-1}x} = \int e^{2\tan^{-1}x} \cdot dx \frac{1}{1+x^2} + c$

$y \cdot e^{\tan^{-1}x} = \frac{1}{2} \int 2 \cdot e^{2\tan^{-1}x} \times \frac{1}{1+x^2} \cdot dx + c$

$y \cdot e^{\tan^{-1}x} = \frac{1}{2} e^{2\tan^{-1}x} + c$

$y = \frac{\frac{1}{2} e^{2\tan^{-1}x} + c}{e^{\tan^{-1}x}}$

$y = \frac{1}{2} e^{2\tan^{-1}x} \cdot e^{-\tan^{-1}x} + e^{-\tan^{-1}x} c$

$y = \frac{1}{2} e^{\tan^{-1}x} + e^{-\tan^{-1}x} c$

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(c) $\sin x \frac{dy}{dx} + y\cos x = x\sin x.$

Sol^n: here,

$$\frac{dy}{dx} + y\cdot\cot x = \frac{x\sin x}{\sin x}$$

$$\frac{dy}{dx} + \cot x\cdot y = x \quad -①$$

Comparing eq^n ① with $\frac{dy}{dx} + P\cdot y = Q$

$P = \cot x$ & $Q = x$ we get

$$I.f = e^{\int P dx} = e^{\int \cot x . dx} = e^{\log \sin x} = \sin x$$

Sol^n of eq^n ① is

$$y\cdot If = \int Q\cdot If\, dx + c$$

$$y\cdot \sin x = \int x\cdot \sin x\cdot dx + c$$

$$y\sin x = x\int \sin x - \int\left[\frac{dx}{dx}\int \sin x . dx\right]dx + c$$

$$y\sin x = x\cdot -\cos x - \int 1\cdot -\cos x . dx + c$$

$$y\sin x = -x\cos x + \int \cos x\, dx + c$$

$$y\sin x = -x\cos x + \sin x + c$$

~~$y = -x\cot x + 1 + \sin x\, c$~~

$$y\sin x + x\cos x - \sin x = c$$

(d) $\frac{dy}{dx} + y\tan x = \sec x$

Sol^n: here,

$$\frac{dy}{dx} + y\tan x = \sec x \quad -①$$

Comparing eq^n ① with $\frac{dy}{dx} + P\cdot y = Q$ we get

$P = \tan x$ and $Q = \sec x$

Now

$$I.f = e^{\int P dx} = e^{\int \tan x . dx} = e^{-\int \frac{-\sin x}{\cos x}.dx} = e^{-\log(\cos x)}$$

$$I.f = e^{-\log(\cos x)} = e^{\log(\cos x)^{-1}} = (\cos x)^{-1}$$

Sol^n of eq^n ① is

$$y\cdot If = \int Q\cdot If\, dx + c$$

$$y\cdot\frac{1}{\cos x} = \int \frac{1}{\cos x}\cdot \sec x . dx + c$$

$$y\cdot \sec x = \int \sec^2 x\, dx + c$$

$$y\cdot \sec x = \tan x + c$$

$$y \times \frac{1}{\cos x} = \frac{\sin x}{\cos x} + c$$

$$y = \sin x + \cos x\, c$$

(e) $\frac{dy}{dx} + y = \cos x$

Sol^n: here,

$$\frac{dy}{dx} + y = \cos x \quad \text{--- (1)}$$

comparing eq^n (1) with $\frac{dy}{dx} + P.y = Q$ we get

$$P = 1 \;\&\; Q = \cos x$$

$$I.f = e^{\int P dx} = e^{\int 1.dx} = e^{\int dx} = e^x$$

Sol^n of eq^n (1) is,

$$y.If = \int Q.If\, dx + c$$

$$y.e^x = \int \cos x.e^x.dx + c$$

ILATE

$$\cancel{y.e^x = \cos x \int e^x - \int\left[\frac{d\cos x}{dx}\int e^x.dx\right].dx}$$

$$\cancel{y.e^x = \cos x.e^x - \int -\sin x\, e^x.dx}$$

$$= \cos x.e^x + \int \sin x.\, e^x.dx$$

$$\int \cos x.e^x.dx = \cos x \int e^x - \int\left[\frac{d\cos x}{dx}\int e^x dx\right].dx$$

$$= \cancel{\cos x \int e^x}$$

$$= \cos x\, e^x + \int \sin x.e^x.dx$$

$$\int \cos x.e^x.dx - \cos x.e^x + \sin x\int e^x.dx - \int\left[\frac{d(\sin x)}{dx}\int e^x dx\right]$$

$$\int \cos x.e^x dx = \cos x.e^x + \sin x.e^x - \int \cos x.e^x.dx$$

$$2\int \cos x.e^x.dx = \cos x.e^x + \sin x.e^x$$

$$\int \cos x.e^x.dx = \frac{\cos x.e^x + \sin x.e^x}{2}$$

we get

$$y.e^x = \frac{\cos x.e^x + \sin x.e^x}{2} + c$$

$$2y = \cos x + \sin x + 2c\,e^{-x}$$

Bernoulli's equation:

The differential of the form

$$\frac{dy}{dx} + P.y = Q.y^n$$

is called Bernoulli's equation.

Dividing by y^n

$$\frac{1}{y^n}.\frac{dy}{dx} + P.\frac{1}{y^{n-1}} = Q \quad \text{--- (2)}$$

(7) Solve the following Bernoulli equations.

Solve $xy' + y = -xy^2$

Sol^n: here,

$$xy' + y = -xy^2$$

Dividing y^2 on both side we get

$$\frac{x}{y^2}\frac{dy}{dx} + \frac{y}{y^2} = -x$$

$$\frac{x}{y^2}\frac{dy}{dx} + \frac{1}{y} = -x$$

Dividing by x on both side

$$\frac{1}{y^2}\frac{dy}{dx} + \frac{1}{xy} = -1$$

$$\frac{1}{y^2}\frac{dy}{dx} + \frac{1}{x}\cdot\frac{1}{y} = -1 \quad \text{—(i)}$$

let $\frac{1}{y} = v$

d.w.r.to x we get

$$\frac{d(y^{-1})}{dx} = \frac{dv}{dx}$$

$$-1\, y^{-2}\frac{dy}{dx} = \frac{dv}{dx}$$

$$\frac{-1}{y^2}\frac{dy}{dx} = \frac{dv}{dx}$$

$$\frac{1}{y^2}\frac{dy}{dx} = -\frac{dv}{dx} \quad \text{—(ii)}$$

now eq^n (i) becomes,

$$-\frac{dv}{dx} + \frac{1}{x}\cdot v = -1$$

$$\frac{dv}{dx} - \frac{1}{x}\cdot v = 1$$

$$\frac{dv}{dx} + \left(-\frac{1}{x}\right)\cdot v = 1$$

The eq^n is in the linear form, so that comparing the above eq^n with $\frac{dv}{dx} + P\cdot v = Q$ we get,

$P = -1/x$ and $Q = 1$

$$I.F = e^{\int P dx} = e^{-\int 1/x \cdot dx} = e^{\log x^{-1}} = x^{-1} = 1/x$$

Sol^n of eq^n (i) is,

$v \cdot I.F = \int Q \cdot I.F\, dx + c$

$v \cdot x^{-1} = \int 1 \cdot x^{-1} dx + c$

$v \cdot x^{-1} = \log(x) + c$

~~$v = \frac{x^2}{2x} + \frac{c}{x}$~~

~~$v = \frac{x}{2} + c x^{-1}$~~

$$v = x\log x + xc$$

$$\frac{1}{y} = x\log x + xc$$

Date
Page 349

8) $y' + \frac{2}{x}y = y^3/x^2$

Soln: Here

$\frac{dy}{dx} + \frac{2}{x}y = y^3/x^2$ — (i)

Dividing eqn (i) by y^3 on bothside we get

$\frac{1}{y^3}\frac{dy}{dx} + \frac{2}{x} \times \frac{y}{y^3} = \frac{1}{x^2}$

$\frac{1}{y^3}\frac{dy}{dx} + \frac{2}{x} \times \frac{1}{y^2} = \frac{1}{x^2}$ .... (ii)

let, $\frac{1}{y^2} = v$

$\frac{d(y^{-2})}{dx} = \frac{dv}{dx}$

$-2y^{-3}\frac{dy}{dx} = \frac{dv}{dx}$

$\frac{1}{y^3}\frac{dy}{dx} = \frac{-1}{2}\frac{dv}{dx}$ — (iii)

from eqn (ii) and (iii) we get,

$\frac{-1}{2}\frac{dv}{dx} + \frac{2}{x} \times v = \frac{1}{x^2}$

Multiplying by -2 on bothside we get

$\frac{dv}{dx} - \frac{4}{x}v = \frac{-2}{x^2}$

$\frac{dv}{dx} + (-4/x)v = -2/x^2$

comparing eqn with $\frac{dv}{dx} + p.v = Q$ we get

$p = -4/x$ and $Q = \frac{-2}{x^2}$,

$I.f. = e^{\int p dx} = e^{\int -4/x . dx} = e^{-4\int 1/x . dx} = e^{-4\log x}$

$I.f. = e^{\log(x)^{-4}} = x^{-4} = \frac{1}{x^4}$

soln of eqn (i) is,

$v. I.f. = \int (I.f \times Q) dx + c$

$v. \frac{1}{x^4} = \int x^{-4} . \frac{-2}{x^2} . dx + c$

$v. \frac{1}{x^4} = -2\int x^{-6} . dx + c$

$vx^{-4} = -2\left(\frac{x^{-5}}{-5}\right) + c$

$vx^{-4} = 2/5 (x^{-5}) + c$

$v = 2/5 \frac{x^{-5}}{x^{-4}} + \frac{c}{x^{-4}}$

$v = 2/5 x^{-1} + x^4 . c$

$\frac{1}{y^2} = 2/5 x^{-1} + x^4 . c$

$(y)^{-2} = \frac{2}{5x} + x^4 . c$

$y = \pm\left(\frac{2}{5x} + x^4 c\right)^{-1/2}$ Ans

Date
Page 50

Second Order Linear Differential Equation

A differential equation at the form.

$$P\frac{d^2y}{dx^2} + Q\frac{dy}{dx} + R.y = G(x).$$

where P, Q, R are function of x are constant is called second order linear equation.

If $G(x) = 0$, then the equation,

$$P\frac{d^2y}{dx^2} + Q.\frac{dy}{dx} + Ry = 0$$

is called homogeneous linear equation of second order.

Theorem:-

If $y(x)$ and $y_2(x)$ be the two solⁿ of the equation.

$$P\frac{d^2y}{dx^2} + Q\frac{dy}{dx} + Ry = 0 \quad -(1)$$

C_1 and C_2 are any constant, then $y(x) = C_1y_1(x) + \ldots(x_2)$ is also a solution of the equation (1)

Auxiliary eqⁿ.

$$P\frac{d^2y}{dx^2} + Q\frac{dy}{dx} + Ry = 0 \quad -(2)$$

If $y = e^{mx}$ be a solution, then

$$\frac{dy}{dx} = me^{mx}, \quad \frac{d^2y}{dx^2} = m^2e^{mx}$$

(1) given,

$$P.m^2e^{mx} + Q.m.e^{mx} + Re^{mx} = 0$$

$$e^{mx}(Pm^2 + Qm + R) = 0$$

$$\therefore Pm^2 + Qm + R = 0$$

This equation is called auxiliary equation.

Case-I : When two root are different i.e $m_1 \neq m_2$

Solⁿ of (1) is

$$y = C_1.e^{m_1x} + C_2e^{m_2x}$$

Case-II, When $m_1 = m_2 = m$

solⁿ (1) is $y = (C_1 + C_2x)e^{mx}$

Case-III

when $m = \alpha + i\beta$ (complex root).

$$y = e^{\alpha x}(A\cos\beta x + B\sin\beta x).$$

Exercise:- 7.3. $y'' - y = 0$

The auxiliary equation is

$m^2 - 1 = 0$

$m^2 = 1$

$m = \pm 1$

$\therefore$ Solⁿ is

$$y = C_1e^{x} + C_2e^{-x}$$

(ii) $\frac{d^2y}{dx^2} + 4y = 0 \quad -(1)$

The auxiliary equation is,

$m^2 + 4 = 0$

$m^2 = -4$

$m = \pm\sqrt{-4}$

$= 0 \pm 2i$.

$\therefore$ Solⁿ is

$$y = e^{0x}(A\cos 2x + B\sin 2x)$$

$y = A\cos 2x + B\sin 2x$ Ans.

Second Order Linear Differential Equation

A differential equation of the form

$$P\frac{d^2y}{dx^2} + Q\frac{dy}{dx} + Ry = G(x).$$

where P, Q, R are function of x are constant is called second order linear equation.

If $G(x)=0$ then the equation,

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is called homogeneous linear equation of second order.

Theorem:-

If $y_1(x)$ and $y_2(x)$ be the two solⁿ of the equation.

$$P\frac{d^2y}{dx^2} + Q\frac{dy}{dx} + Ry = 0 \quad -(1)$$

C_1 and C_2 are any constant, then $y(x) = C_1y_1(x) + C_2y_2(x)$ is also a solution of the equation (1)

Auxiliary eqⁿ:

$$P\frac{d^2y}{dx^2} + Q\frac{dy}{dx} + Ry = 0 \quad -(2)$$

If $y = e^{mx}$ be a solution, then

$$\frac{dy}{dx} = me^{mx}, \quad \frac{d^2y}{dx^2} = m^2e^{mx}$$

(1) given,

$$P.m^2e^{mx} + Q.m.e^{mx} + Re^{mx} = 0$$

$$e^{mx}(Pm^2 + Qm + R) = 0$$

$$\therefore Pm^2 + Qm + R = 0$$

This equation is called auxiliary equation.

Case-I : When two root are different i.e $m_1 \neq m_2$

Solⁿ of (1) is

$$y = C_1e^{m_1x} + C_2e^{m_2x}$$

Case-II, when $m_1 = m_2 = m$

solⁿ (1) is $y = (C_1 + C_2x)e^{mx}$

Case-III

when $m = \alpha + i\beta$ (complex root).

$$y = e^{\alpha x}(A\cos\beta x + B\sin\beta x).$$

Exercise:- 7.3. $y'' - y = 0$

The auxiliary equation is

$m^2 - 1 = 0$

$m^2 = 1$

$m = \pm 1$

$\therefore$ Solⁿ is

$$y = C_1e^{x} + C_2e^{-x}$$

(ii) $\frac{d^2y}{dx^2} + 4y = 0 \quad -(1)$

The auxiliary equation is,

$m^2 + 4 = 0$

$m^2 = -4$

$m = \pm\sqrt{-4}$

$= 0 \pm 2i$

$\therefore$ Solⁿ is

$$y = e^{0x}(A\cos 2x + B\sin 2x)$$

$$y = A\cos 2x + B\sin 2x$$

④ $3y'' + y' - y = 0$

The auxiliary equation is

$3m^2 + m - 1 = 0$

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$m = \frac{-1 \pm \sqrt{1 - 4.3(-1)}}{6}$$

$$m = \frac{-1 \pm \sqrt{13}}{6}$$

sol^n of ① is

$$y = c_1 e^{\left(\frac{-1+\sqrt{13}}{6}\right)x} + c_2 e^{\left(\frac{-1-\sqrt{13}}{6}\right)x}$$

Exercise:- 7.3.

① Solve $y'' - y = 0$.

Sol^n: Here

The auxiliary eq^n of the above eq^n is,

$$\frac{d^2y}{dx^2} - y = 0 \quad \text{—①}$$

$m^2 - 1 = 0$

$m^2 = 1$

$m_1 = 1$

$m_2 = -1$

Here the root are different so, applying 1st case to find the sol^n of the given equation such that.

$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$

$y = c_1 e^x + c_2 e^{-x}$ is the sol^n of the given equation.

② Solve $\frac{d^2y}{dx^2} + 4y = 0$

Sol^n:

$$\frac{d^2y}{dx^2} + 4y = 0$$

The auxiliary eq^n of the given eq^n is,

$m^2 + 4 = 0$

$m^2 = -4 = \sqrt{-4} = 0 \pm 2i$

$\cancel{m_1 = +2 \text{ \& } m_2 = -2}$

Here root are ~~different~~ imaginary So applying IInd case to find sol^n of the given eq^n such that

$\cancel{y = c_1 e^{m_1 x} + c_2 e^{m_2 x}}$

$\cancel{y = c_1 e^{2x} + c_2 e^{-2x}}$ ~~is the sol^n of the given eq^n~~

$y = e^{\alpha x}(A\cos\beta x + B\sin\beta x)$

$y = e^{0}(A\cos 2x + B\sin 2x)$

$y = 1(A\cos 2x + B\sin 2x)$

$\therefore y = A\cos 2x + B\sin 2x$ is the required soln of the above eqn.

Ans

(3) Solve $y'' + y' - 6y = 0$

Soln: Here,

$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 0$

The auxiliary eqn of the above eqn is

$m^2 + m - 6 = 0$

$m^2 + (3-2)m - 6 = 0$

$m^2 + 3m - 2m - 6 = 0$

$m(m+3) - 2(m+3) = 0$

$m - 2 = 0$

$m + 3 = 0$

$\therefore m_1 = 2$ and $\therefore m_2 = -3$

Here roots are different so, applying 1st case to solve the given equation such that,

$y = C_1 e^{m_1 x} + C_2 e^{m_2 x}$

$y = C_1 e^{2x} + C_2 e^{-3x}$ is the required soln.

Ans

(4) Solve $3y'' + y' - y = 0$

Soln: Here,

$3y'' + y' - y = 0$

The auxiliary equation of the given eqn is,

$3\frac{d^2y}{dx^2} + \frac{dy}{dx} - y = 0$

$3m^2 + m - 1 = 0$

$3m^2 + (\cancel{3-2})m - 1 = 0$

$3m^2 + \cancel{2}m - 1 = 0$

$\cancel{3m(m+1)}$ $3m^2 + m - 1 = 0$

Comparing with $am^2 + bm + c = 0$ we get,

$(m_1, m_2) = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$(m_1, m_2) = \frac{-1 \pm \sqrt{1 - 4\times 3\times -1}}{2\times 3}$

$(m_1, m_2) = \frac{-1 + \sqrt{13}}{6}$

where,

$m_1 = \frac{-1+\sqrt{13}}{6}$ and $m_2 = \frac{-1-\sqrt{13}}{6}$

Here root are different so, applying case-1st to solve the given equation such that,

$y = C_1 e^{m_1 x} + C_2 e^{m_2 x}$

$y = C_1 e^{\frac{-1+\sqrt{13}}{6}x} + C_2 e^{\frac{(-1-\sqrt{13})x}{6}}$

Ans

(5) Solve $4y'' + 12y' + 9y = 0$

Sol^n: Here,

$4y'' + 12y' + 9y = 0,$

$$4\frac{d^2y}{dx^2} + 12\frac{dy}{dx} + 9y = 0$$

The auxiliary equation is

$4m^2 + 12m + 9 = 0$

$4m^2 + (6+6)m + 9 = 0$

$4m^2 + 6m + 6m + 9 = 0$

$2m(2m+3) + 3(2m+3) = 0$

$(2m+3)$ and $(2m+3) = 0$

$2m + 3 = 0$

$m = -3/2$ and $m = -3/2$

Here roots are same so, applying I_1^{nd} case to solve the above/given eq^n such that

$y = C_1 e^{mx} + xC_2 e^{mx}$

$y = (C_1 + C_2 x) e^{mx}$

$y = (C_1 + C_2 x) e^{-3/2 x}$ Ans

or,

$y = c_1 e^{-3x/2} + xc_2 e^{-3x/2}$ is the required sol^n.

(6) Solve $y'' - 6y' + 13y = 0$

Sol^n: Here,

$y'' - 6y' + 13y = 0$

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 13y = 0$$

The auxiliary eq^n of the given equation is,

$m^2 - 6m + 13 = 0$

comparing with $am^2 + bm + c = 0$ we get,

$$(m_1, m_2) = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\text{"} = \frac{-(-6) \pm \sqrt{36 - 4 \times 1 \times 13}}{2 \times 1}$$

$$\text{"} = \frac{6 \pm \sqrt{36 - 52}}{2}$$

$$\text{"} = \frac{6 \pm \sqrt{-16}}{2}$$

$$(m_1, m_2) = \frac{6 \pm 4i}{2}$$

$\alpha + \beta i$

$m_1 = \frac{6 + 4i}{2} \Rightarrow (3 + 2i),$

Now,

Here the root are ~~different~~ imaginary, so, applying IIIrd case to find the sol^n of the given eq^n such that,

$y = e^{\alpha x}(A\cos\beta x + B\sin\beta x)$

$y = e^{3x}(A\cos 2x + B\sin 2x)$ Ans

~~$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$~~

~~$y = c_1 e^{5x} + c_2 e^{\alpha}$~~ ~~is the required sol^n~~ ~~Ans~~

Date
Page 54

(7) Solve the initial value problem.

$y''+y'-6y=0,\ y(0)=1,\ y'(0)=0$

Sol^n: here.

$y''+y'-6y=0, \quad y(0)=1, y'(0)=0,$

$\frac{d^2y}{dx^2}+\frac{dy}{dx}-6y=0, \quad y(0)=1, y'(0)=0,$

The auxiliary eq^n of the given eq^n is,

$m^2+m-6=0$

$m^2+(3-2)m-6=0$

$m^2+3m-2m-6=0$

$m(m+3)-2(m+3)=0$

$(m-2)$ and $(m+3)=0$

where,

$m_1=2$ and $m_2=-3$,

Here roots are equal so, using 1st case to find the sol^n of the given equation such that,

$y=c_1e^{m_1x}+c_2e^{m_2x}$

$y=c_1e^{2x}+c_2e^{-3x}$ — (I)

Now Applying condition $y(0)=1$ where, $x=0$ and $y=1$, we get,

$1=c_1e^0+c_2e^0$.

$1=c_1+c_2$

$c_1=1-c_2$ — (II)

Again Applying IInd condition $y'(0)=0$

$\frac{dy}{dx}=y'=2c_1e^{2x}-3c_2e^{-3x}$

$0=2c_1e^0-3c_2e^0$

$0=2c_1-3c_2$

$2c_1=3c_2$

$2(1-c_2)=3c_2$

$2-2c_2=3c_2$

$2=5c_2$

$c_2=2/5$,

putting value of $c_2=2/5$ in eq^n (II) we get,

$c_1=1-2/5=3/5$

$\therefore$ The eq^n (I) become,

$y=\frac{3}{5}e^{2x}+\frac{2}{5}e^{-3x}$ Ans

(8) Solve the initial value problem.

$y''+y=0,\ y(0)=2,\ y'(0)=3,$

Sol^n: Here,

$y''+y=0, \quad y(0)=2, \quad y'(0)=3$

The auxiliary eq^n of the given equation is,

$\frac{d^2y}{dx^2}+y=0,$

$m^2+1=0$

$m^2=-1$ $\qquad \alpha+\beta i$

$m=\sqrt{-1}=(0\pm i)$,

Here roots are imaginary so applying IIIrd case to find the sol^n such that,

$y=e^{\alpha x}(A\cos\beta x+B\sin\beta x)$

$y=e^0(A\cos x+B\sin x)$

$y=A\cos x+B\sin x$, — (A)

Now, Applying the condition, (I) $y(0)=2$, we get,

$2=A\cos 0+B\sin 0$

$2=A$

Again Applying IInd condition (II) $y'(0)=3$

$y'=-A\sin x+B\cos x$

$3=-A\sin 0+B\cos 0 \Rightarrow 3=B$

Page 55

Now the eqⁿ (A) becomes,

$y = 2\cos x + 3\sin x$. is the required soln.

(9) Solve the boundary value problem $y'' + 2y' + y = 0$, $y(0) = 1$, $y(1) = 3$

Soln: Here,

$y'' + 2y' + y = 0$,

$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0$,

The auxiliary eqⁿ of the given equation is

$m^2 + 2m + 1 = 0$

$m^2 + 1m + 1m + 1 = 0$

$m(m+1) + (m+1)1 = 0$

$(m+1)(m+1) = 0$,

$m_1 = -1$ and $m_2 = -1$

Here roots are equal so, applying the ~~conditi~~ case-IInd to solve the given equation such that:

$y = C_1 e^{mx} + C_2 x e^{mx}$

$y = C_1 e^{-x} + x C_2 e^{-x}$ —(1)

Applying the condition (i) $y(0) = 1$ in eqⁿ (1) we get,

$1 = C_1 + C_2 \cdot 0$

$C_1 = 1$

And, appling IInd condition $y(1) = 3$ we get

$y = C_1 e^{-x} + C_2 e^{-x}$

$3 = C_1 e^{-1} + C_2 e^{-1}$

$3e = C_1 + C_2$

$3e = C_1 + 1 - C_1$

$3e = 1$

$e = \frac{1}{3}$

$3 = C_1 e^{-1} + C_2 e^{-1}$

$3 = e^{-1} + C_2 e^{-1}$

$3 = e^{-1}(0\,1 + C_2)$

$3e - 1 = C_2$

The eqⁿ becomes,

$y = e^{-x} + (3e-1)x e^{-x}$ Ans

(10) Solve the ~~boundary~~ value

$$100\frac{d^2p}{dt^2} + 200\frac{dp}{dt} + 101p = 0$$

Sol^n. Here,

The auxiliary eq^n of the given eq^n is

$$100m^2 + 200m + 101 = 0$$

By using formula of quadratic eq^n, we get,

$$(m_1, m_2) = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$(m_1, m_2) = \frac{-200 \pm \sqrt{(200)^2 - 4 \times 100 \times 101}}{2 \times 100}$$

$$(m_1, m_2) = \frac{-200 \pm \sqrt{40000 - 40400}}{200}$$

$$= \frac{-200 \pm \sqrt{-400}}{200}$$

$$= \frac{-200 \pm 20i}{200}$$

$$= \frac{20(-10 \pm i)}{200} \qquad \alpha + \beta$$

$$= \frac{-10 \pm i}{10} = -1 + \frac{i}{100}$$

Here the roots are in complex/imaginary form. So, applying the case - III to find the sol^n of the given eq^n such that

$$y = e^{\alpha t}(A\cos\beta t + B\sin\beta t)$$

$$y = e^{-t}\left(A\cos\frac{1}{10}t + B\sin\frac{t}{10}\right)$$

$$\therefore y = e^{-t}(A\cos t/10 + B\sin t/10) \text{ Ans}$$

Page 56

(11) Solve $y'' + 4y = 0$, $y(0) = 5$, $y(\pi/4) = 3$

Sol^n: Here,

$$y'' + 4y = 0$$

$$\frac{d^2y}{dx^2} + 4y = 0$$

The auxiliary eq^n of the given eq^n is

$$m^2 + 4 = 0$$

$$m^2 = -4$$

$$m = \sqrt{-2 \times 2} = \sqrt{-2^2 i^2} = 0 - 2i$$

Here the root are imaginary, so, applying the case - III to find the sol^n. Such that

$$y = e^{\alpha x}(A\cos\beta x + B\sin\beta x)$$

$$y = e^0(A\cos 2x + B\sin 2x)$$

$$y = A\cos 2x + B\sin 2x$$

Applying the given condition $y(0) = 5$

$$5 = A\cos 0 + B\sin 0$$

$$5 = A$$

Again $y(\pi/4) = 3$

$$3 = A\cos 2 \times \pi/4 + B\sin 2 \times \pi/4$$

$$3 = A\cos \pi/2 + B\sin \pi/2$$

$$3 = 0 + B$$

$$B = 3,$$

the required sol^n is

$$y = (5\cos 2x + 3\sin 2x)$$

(12) Solve $y''+4y'+4y=0$, $y(0)=2$, $y(1)=0$

Sol^n: Here,

$y''+4y'+4y=0$,

$\frac{d^2y}{dx^2}+4\frac{dy}{dx}+4y=0$

The auxiliary eq^n of the given eq^n is

$m^2+4m+4=0$

$m^2+2.2.m+2^2=0$

$(m+2)^2=0$

$(m+2)(m+2)=0$

$m_1=-2$ & $m_2=-2$

Here the roots are equal, so, applying II^n case to find out the sol^n of the given eq^n such that

$y=c_1e^{mx}+xc_2e^{mx}$

$y=c_1e^{-2x}+xc_2e^{-2x}$

Now, Applying the given condition i.e $y(0)=2$, we get

$2=c_1e^0+0c_2e^0$

~~$2=c_1+c_2$~~ – (i) $c_1=2$.

Again Applying the given condition i.e. $y(1)=0$

$0=c_1e^{-2}+xc_2e^{-2}=e^{-2}(c_1+c_2)$

~~$0=(c_1+c_2)e^{-2}$~~

~~$c_1+c_2=0$~~

$0=e^{-2}(c_1+c_2)$

$c_1+c_2=0$

$2+c_2=0$

$c_2=-2$

The equation become,

$y=2e^{-2x}+x.-2e^{-2x}$

$y=2e^{-2x}-2xe^{-2x}$ Ans

(13) Solve $y''=y'$, $y(0)=1$, $y(1)=2$

Sol^n: Here,

$y''=y'$

$y''-y'=0$

or $\frac{d^2y}{dx^2}-\frac{dy}{dx}=0$,

The auxiliary eq^n of the given eq^n is

$m^2-m=0$

$m(m-1)=0$

$m_1=0$ and $m_2=1$,

Here the roots are different so applying Ist case to find the sol^n of the given equation such that

$y=c_1e^{m_1x}+c_2e^{m_2x}$

$y=c_1e^0+c_2e^x$

$y=c_1+c_2e^x$

Applying the condition to solve eq^n i.e. $y(0)=1$, we get

$1=c_1+c_2e^0$

$c_1+c_2=1$ $c_1=1-c_2$ – (i)

Again Applying IInd condition i.e. $y(1)=2$

$2=c_1+c_2e$ – (ii)

from (i) and (ii) we get,

$2 = 1 - c_2 + ec_2$

$1 = c_2(e-1)$

$c_2 = \frac{1}{e-1}$

Again,

$c_1 = 1 - c_2$

$c_1 = 1 - \frac{1}{e-1}$

$c_1 = \frac{e-1-1}{e-1}$

$c_1 = \frac{e-2}{e-1}$

Now the eqn becomes,

$y = \frac{e-2}{e-1} + \frac{1}{e-1} \times e^x$

$\therefore y = \frac{e-2}{e-1} + \frac{e^x}{e-1}$ is the required solⁿ.

(14) Solve $y'' + 4y' + 20y = 0$, $y(0) = 1$ and $y(\pi) = 2$

Solⁿ: Here,

$y'' + 4y' + 20y = 0$

$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 20y = 0$

The auxiliary eqⁿ is,

$m^2 + 4m + 20 = 0$

comparing the eqⁿ with quadratic form and using formula such that,

$(m_1, m_2) = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$= \frac{-4 \pm \sqrt{16 - 4 \times 1 \times 20}}{2 \times 1}$

$= \frac{-4 \pm \sqrt{16-80}}{2}$

$= \frac{-4 \pm \sqrt{-64}}{2}$

$= \frac{-4 + 8i}{2}$

$= -2 \pm 4i$,

where $\alpha = -2$ and $\beta = 4$

Here the roots are imaginary, so applying case IIIrd such that,

$y = e^{\alpha x}(A\cos\beta x + B\sin\beta x)$

$y = e^{-2x}(A\cos 4x + B\sin 4x)$

Now,

Applying the given condition, i.e. $y(0) = 1$

$1 = e^0(A\cos 0 + B\sin 0)$

$1 = A$

Applying the IInd condition i.e. $y(\pi) = 2$

$2 = e^{-2\pi}(A\cos 4\pi + B\sin 4\pi)$

$2 = e^{-2x} A$ i.e. No solⁿ because B has no value.

Non-Homogeneous Linear Equation.

The equation of the form:

$$y'' + Qy' + Ry = G(x)$$

Where $G(x) \neq 0$, P, Q, R are constant and $G(x)$ is function of x is called second order non-homogeneous linear equation with constant coefficient.

Theorem: The general solution of non-homogeneous equation (1) can be written as

$$y = y_c + y_p$$

where the first part y_c is the soln of homogeneous equation and is called complementary function (C.F.) and the second part y_p is called the particular solution of (1)

Method of Undetermined coefficient.

The differential equation

$$Py'' + Qy' + Ry = G(x)$$

(i) when $G(x)$ is polynomial,

Ex. solve: $y'' + y' - 2y = x^2$ —(1)

Soln:

The auxiliary equation is

$$m^2 + m - 2 = 0$$

$$m^2 + 2m - m - 2 = 0$$

$$m(m+2) - 1(m+2) = 0$$

$$m = 1, -2$$

$$C.F = y_c = C_1 e^x + C_2 e^{-2x}$$

for P.I.

we take $y_p = Ax^2 + Bx + C$

$$y'_p = 2Ax + B$$

$$y''_p = 2A$$

(1) become

$$2A + 2Ax + B - 2(Ax^2 + Bx + C) = x^2$$

$$2A + 2Ax + B - 2Ax^2 - 2Bx - 2C = x^2$$

$$-2Ax^2 + (2A - 2B)x + 2A + B - 2C = x^2$$

Equationg. the coefficient of the power of x,

$$-2A = 1 \Rightarrow A = -\tfrac{1}{2}$$

$$2A - 2B = 0$$

$$2 \times -\tfrac{1}{2} = 2B$$

$$B = -\tfrac{1}{2}.$$

or, $2A + B - 2C = 0$

$$2 \times -\tfrac{1}{2} - \tfrac{1}{2} = 2C$$

$$-1 - \tfrac{1}{2} = 2C$$

$$C = -\tfrac{3}{4}$$

$$P.I = y_c = -\tfrac{1}{2}x^2 - \tfrac{1}{2}x - \tfrac{3}{4}$$

The complete soln of (1) is

$$y = y_c + y_p$$

$$= C_1 e^x + C_2 e^{-2x} - \tfrac{1}{2}x^2 - \tfrac{1}{2}x - \tfrac{3}{4}.$$ Ans

Exercise : 7.4.

1. Solve the differential equation or initial value problems. Using the method of undetermined coefficients

(a) $y'' - y = x^3 - x$

Soln: Here,

The auxiliary equation of the given equation is,

$y'' - y = x^3 - x$

$\therefore m^2 - 1 = 0$

$m^2 = 1$

$m = \pm 1$

i.e. $m_1 = 1$ and m_2

Here m_1 and m_2 ~~are~~ are different so applying case-1 to determine the soln complementary equation i.e.

$y_c = c_1 e^{m_1 x} + c_2 e^{m_2 x}$

$y_c = c_1 e^{x} + c_2 e^{-x}$

Now:

For particular equation,

Since $G(x) = x^3$ is a polynomial of degree 2

~~So~~ the ~~par~~ soln of the particular equation is,

$y_p(x) = Ax^3 + Bx^2 + Cx + D$

$y'_p = 3Ax^2 + 2Bx + C$

$y''_p = 6Ax + 2B$

Substituting the value of y'_p and y''_p in eqn (1) we get,

$6Ax + 2B - Ax^3 - Bx^2 - Cx - D = x^3 - x$

$x(6A - C) - Ax^3 - Bx^2 + 2B - D = x^3 - x$

Equating the coefficient of power of x we get,

$6A - C = -1$	$-A = 1$	$-B = 0$	$2B - D = 0$
$6A - C = -1$	$A = -1$	$B = 0$	$2B = D$
$-6 = C - 1$			$D = 0$
$-5 = C$			

The particular soln is $y_p = -x^3 + 0 + -5x + 0$

$= -x^3 - 5x$

The general soln of the given differential equation is $= y_c + y_p$

$= c_1 e^{x} + c_2 e^{-x} - x^3 - 5x$ Ans.

(b) $y'' + 2y' + 5y = 1 + e^x$

Soln: here,

The auxiliary equation of the given eqn is

$m^2 + 2m + 5 = 0$

$m^2 + 2m + 5 = 0$

the roots ~~are~~ are,

$(m_1, m_2) = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$(m_1, m_2) = \frac{-2 \pm \sqrt{4 - 4 \times 1 \times 5}}{2 \times 1}$

$= \frac{-2 \pm \sqrt{4 - 20}}{2}$

$= \frac{-2 \pm 4i}{2}$

$= -1 \pm 2i$

$= \alpha \pm \beta = \alpha = -1$ & $\beta = 2$

Now:

The ~~comp~~ soln of the complementary eqn is

$y_c = e^{-x}(A\cos 2x + B\sin 2x)$

classmate

Date

Page

sol^n For particular equation, is,

Since $G(x) = 1+e^x$ where x is a polynomial of degree 1, so,

$y_c = Ae^x + B$

$y'(c) = Ae^x + C = Ae^x$

$y''(c) = Ae^x$

putting value of $y'(c)$ and $y''(c)$ in eq^n ①

we get

$Ae^x + Ae^x + Ae^x B - Ae^x + 1$

$B + 3Ae^x = Ae^x + 1$

$2Ae^x = 1$

$A = \frac{1}{2e^x}$

$y_c = \frac{1}{2e^x} \times e^x$

$y_c = \frac{1}{2}$

the general sol^n of the given eq^n is,

$y = y_p + y_c$

$y = \frac{1}{2} + e^x(A\cos 2x + B\sin 2x)$

classmate

Date

Page 36

For particular sol^n

Since $G(x) = 1+e^x$

$y_c = A + Be^x$

$y'c = \cancel{A} + Be^x = Be^x$

$y''c = Be^x$

eq^n ① become,

$Be^x + 2Be^x + 5(A+Be^x) = 1+e^x$

$Be^x + 2Be^x + 5A + 5Be^x = 1+e^x$

$e^xB + 2Be^x + 5Be^x + 5A = 1+e^x$

$e^x(B + 2B + 5B) = e^x$ $\qquad 5A = 1$

$B + 2B + 5B = 1$ $\qquad A = \frac{1}{5}$

$8B = 1$

$B = \frac{1}{8}$

$y_c = \frac{1}{5} + \frac{1}{8}e^x,$

the complete sol^n of the eq^n ① is,

$y = e^x(A\cos 2x + B\sin 2x) + \frac{1}{5} + \frac{1}{8}e^x$

(c) $y'' - 4y' + 5y = e^{-x}$

Sol^n: Here,

$y'' - 4y' + 5y = e^{-x}$ — (1)

The auxiliary eq^n of (1) is,

$m^2 - 4m + 5 = 0$

$m^2 - 5m + m + 5 = 0$

$m(m-5) + 1$

$m^2 - (5-1)m + 5 = 0$

$m^2 - 5m + m + 5$

The roots of the eq^n (1) is,

$(m_1, m_2) = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$= \frac{4 \pm \sqrt{16 - 4 \times 1 \times 5}}{2 \times 1}$

$= \frac{4 \pm \sqrt{16-20}}{2}$

$= \frac{4 \pm 2i}{2}$

$= 2 \pm i$

$\alpha = 2, \ \& \ \beta = 1$

$\therefore y_c = e^{2x}(A\cos x + B\sin x)$

The particular sol^n of eq^n (1) is,

$y'' - 4y' + 5y = e^{-x}$

$y_p = Ae^{-x}$

$y'_p = -Ae^{-x}$

$y''_p = Ae^{-x}$

eq^n (1) become,

$Ae^{-x} - 4(-Ae^{-x}) + 5e^{-x} = e^{-x}$

$Ae^{-x} + 4Ae^{-x} + 5e^{-x} = e^{-x}$

$A \cdot 10e^{-x} = e^{-x}$

$10A = 1$

$A = 1/10$

Now

$y_p = \frac{1}{10} e^{-x}$,

the complete solution of eq^n (1) is,

$y = y_c + y_p$

$y = e^{2x}(A\cos x + B\sin x) + \frac{1}{10} e^{-x}$

(d) $y'' - 4y' + 4y = x - \sin x$

Sol^n: here

$y'' - 4y' + 4y = x - \sin x$, — (1)

The auxiliary eq^n of (1) is,

$m^2 - 4m + 4 = 0$

$m^2 - 2 \cdot m \cdot 2 + 2^2 = 0$

$(m-2)^2 = 0$

$m = 2$,

$y_c = (c_1 + c_2 x) e^{2x}$

The particular solution of eq^n (1) is,

let $y_{p_1} = Ax + B$.

$y'_{p_1} = A$

$y''_{p_1} = 0$

then, $0 - 4A + 4(Ax+B) = x$

$-4A + 4Ax + 4B = x$

Now

$-4A + 4B = 0$ and $4A = 1$

$4A = 4B$ $\quad A = 1/4$

$A = B$ $\quad \therefore B = 1/4$

$y_{p_1} = \frac{1}{4}x + \frac{1}{4}$

Page

Again

$y_{P_2} = \cancel{A} \; \cancel{A - \sin x +}$

$y_{P_2} = A\cos x + B\sin x$

$y_{P_2}' = -A\sin x + B\cos x$

$y_{P_2}'' = -A\cos x - B\sin x$

Putting above value in eq^n (1)

$-A\cos x - B\sin x - 4(-A\sin x + B\cos x) + 4(A\cos x + B\sin x) = -\sin x$

$-A\cos x - B\sin x + 4A\sin x - 4B\cos x + 4A\cos x + 4B\sin x = -\sin x$

$3A\cos x + 4A\sin x - 4B\cos x + 3B\sin x = -\sin x$

$\cos x(3A - 4B) + \sin x(4A + 3B) = -\sin x$

$3A - 4B = 0$	$4A + 3B = -1$
$3A = 4B$	$4 \times \frac{4}{3}B + 3 \times B = -1$
$A = \frac{4}{3}B$	$\frac{16B}{3} + 3B = -1$
$A = \frac{4}{3} \times \frac{-3}{25}$	$16 + 9B = -3$
$A = -\frac{4}{25}$	$9B = -19$
	$B = -\frac{19}{9}$
	$25B = -3$
	$B = -\frac{3}{25}$

Now

$y_{P_2} = -\frac{4}{25}\cos x + \frac{-3}{25}\sin x$

$y_{P_2} = -\frac{4}{25}\cos x - \frac{3}{25}\sin x,$

The complete sol^n of eq^n (1) is,

$y = y_c + y_{P_1} + y_{P_2} = (C_1 + C_2 x)e^{2x} + \frac{x}{4} + \frac{1}{4} \cancel{+} \frac{4}{25}\cos x - \frac{3}{25}\sin x$

Date

Page 263

(2) $y'' + y = e^x + x^3$, $y(0) = 2$, $y'(0) = 0$

sol^n; Here,

$y'' + y = e^x + x^3$ —(1)

The auxiliary eq^n of (1) is,

$m^2 + 1 = 0$

$m^2 = -1$

$m = \sqrt{-1}$

$m = 0 \pm i$

$\alpha = 0$ and $\beta = 1$,

$y_c = e^0(A\cos x + B\sin x)$

when $y(0) = 2$

$2 = A\cos 0 + B\sin 0$

$A = 2$

$y'(0) = 0$

$y' = -A\sin x + B\cos x$

$0 = -A\sin 0 + B\cos 0$

$0 = B$

$B = 0$

The particular sol^n of eq^n (1) in term of x^3,

$y_{P_1} = Ax^3 + Bx^2 + Cx + D$

$y_{P_1}' = 3Ax^2 + 2Bx + C$

$y_{P_1}'' = 6Ax + 2B$

The eq^n (1) become,

$6Ax + 2B + Ax^3 + Bx^2 + Cx + D = x^3$

$6Ax + Cx + 2B + D + Ax^3 + Bx^2 = x^3$

$6A + C = 0$	$2B + D = 0$	$A = 1$	$B = 0$
$6A = -C$	$2B = -D$		
$A = -\frac{C}{6}$	$B = -\frac{D}{2}$		
$1 = -C$	$D = 0$	$A = 1$	$B = 0$
$C = -6$			

$y_{P_1} = x^3 + 0 + -6x + 0$

$y_{P_1} = x^3 - 6x,$

Again,

$y_{P_2} = Ae^x$

$y_{P_2}' = Ae^x$

$y_{P_2}'' = Ae^x$

Now eq^n (1) become,

$Ae^x + Ae^x = e^x$

$2Ae^x = e^x \Rightarrow A = \frac{1}{2}$

then,

$$y_{P_2} = \frac{e^x}{2}$$

The complete solⁿ of eqⁿ (1) is

$$y = y_c + y_{P_1} + y_{P_2}$$

$$y = (A\cos x + B\sin x) + x^3 - 6x + \frac{e^x}{2} \quad \underline{\underline{Ans}}$$

Now, According to the condition,

$y(0) = 2$

$$2 = A\cos 0 + B\sin 0 + 0 - 0 + \frac{e^0}{2}$$

$$2 - \frac{1}{2} = A$$

$$A = \frac{3}{2}$$

Again $y'(0) = 0$

$$y' = -A\sin x + B\cos x + 3x^2 - 6 + \frac{e^x}{2}$$

$$0 = -A\sin 0 + B\cos 0 + 0 - 6 + \frac{1}{2}$$

$$B = 6 - \frac{1}{2}$$

$$B = \frac{12-1}{2}$$

$$B = \frac{11}{2}$$

$$\therefore \; y = \frac{3}{2}\cos x + \frac{B}{2}\sin x + x^3 - 6x + \frac{e^x}{2} \quad Ans$$

(f) $y'' - 4y + e^x\cos x$, $y(0) = 1$, $y'(0) = 2$

Solⁿ: Here, $y'' - 4y = -e^x \cdot \cos x$

The auxiliary eqⁿ of (1) is,

$$m^2 - 4 = 0$$
$$m^2 = 4$$
$$m = \pm 2$$

The complement solⁿ of eqⁿ (1) is

$$y_c = c_1 e^{2x} + c_2 e^{-2x}$$

The particular solⁿ of eqⁿ (1) is,

$$y_{P_1} = -Ae^x$$
$$y'_{P_1} = -Ae^x$$
$$y''_{P_1} = -Ae^x$$

eqⁿ (1) become,

$$-Ae^x - 4(-Ae^x) = -e^x$$
$$-Ae^x + 4Ae^x = -e^x$$
$$3Ae^x = -e^x$$
$$3A = -1$$
$$A = -\frac{1}{3}$$

$$y_{P_1} = \frac{1}{3}e^x$$

Again,

$$y_{P_2} = A\cos x + B\sin x$$
$$y'_{P_2} = -A\sin x + B\cos x$$
$$y''_{P_2} = -A\cos x - B\sin x$$

eqⁿ (1) become,

$$-A\cos x - B\sin x - 4(A\cos x + B\sin x) = -\cos x$$
$$-A\cos x - B\sin x - 4A\cos x - 4B\sin x = -\cos x$$
$$\cos x(-A - 4A) - \sin x(B + 4B) = -\cos x$$

$-A - 4A = -1$	$B + 4B = 0$
$-5A = -1$	$5B = 0$
$A = \frac{1}{5}$	$B = 0$

then, $\frac{1}{5}\cos x$

the complement solⁿ of eqⁿ (1) is,

$y = y_c + y_{P_1} + y_{P_2}$

$y = C_1 e^{2x} + C_2 e^{-2x} + \frac{1}{3}e^{x} + \frac{1}{5}\cos x$

Jump 368

Page 365

(8) $y'' - y' = xe^{x}$, $y(0) = 2$, $y'(0) = 1$.

Solⁿ: Here

$y'' - y' = x.e^{x}$ —(1)

The auxiliary eqⁿ of (1) is,

$m^2 - m = 0$

$m(m-1) = 0$

$m = 0$ and $m_2 = 1$

$y_c = C_1 e^{0} + C_2 e^{x}$

$y_c = C_1 + C_2 e^{x}$

Now:

The ~~comp~~ particular solution of eqⁿ (1) is,

$y_p = (Ax+B)e^{x}$

$y_p = Axe^{x} + Be^{x}$

$y'_p = A(x.e^{x} + e^{x}) + B.e^{x}$

$y'_p = Axe^{x} + Ae^{x} + B.e^{x}$

$y''_p = A(x.e^{x} + e^{x}) + Ae^{x} + Be^{x}$

$y''_p = Axe^{x} + Ae^{x} + Ae^{x} + Be^{x}$

$y''_p = Axe^{x} + 2Ae^{x} + Be^{x}$

Now:- putting these value in eqⁿ (1) we get,

$Axe^{x} + 2Ae^{x} + Be^{x} - (Axe^{x} + Ae^{x} + B.e^{x}) = x.e^{x}$

$\cancel{Axe^{x}} + 2Ae^{x} + \cancel{Be^{x}} - \cancel{Axe^{x}} - Ae^{x} - \cancel{Be^{x}} = x.e^{x}$

$Ae^{x} = x.e^{x}$