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Binary Operation :-

let S be a non-empty Set. A binary operation $*$ on S is a function $f: S \times S \to S$. A binary operation $*$ On S is equivalent to the statement If $a, b \in S$, then $a * b \in S$ is known as closure axiom. In this case S is said to be closed under $*$.

commutative Law :-

A binary operation $*$ on S is commutative if and only if $a*b = b*a$, for all, $a, b \in S$

Associative Law.

A binary operation on a set S is associative if $(a*b)*c = a*(b*c)$, for all, $a, b, c \in S$

Identity element

An element e of S is an identity element for $*$ if $e*a = a*e = a$, for all $a \in S$

e.g $a + 0 = 0 + a = a$, so 0 is the identity element for addition.

$a \times 1 = 1 \times a = a$, so 1 is identity element for multiplication.

Inverse elements

An element a' of S is an inverse element for $*$ if $a*a' = a'*a = e$, where e is the identity element for all $a \in S$.

e.g $2 + (-2) = (-2) + 2 = 0$

So, -2 is the additive inverse of 2

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Group

let ~~total~~ $(G, *)$ be a binary structure then G is said to be a group with binary operation $*$ if the following condition ~~are~~ satisfied :-

(i) closure property :- $\forall a, b \in G \Rightarrow a*b \in G$

(ii) Associative property :- $\forall a, b, c \in G$
$\Rightarrow (a*b)*c = a*(b*c)$

(iii) Existence of identity element :-
$\forall a \in G$, there exist an element $e \in G$ Such that $a*e = e*a = a$

(iv) existence of inverse element :- $\forall a \in G$, there exist an element $a' \in G$ such that
$a*a' = a'*a = e$.

Abelian group

- A group G is said to be abelian if its binary operation is commutative.
i.e. if $a, b \in G \Rightarrow a*b = b*a$

Ex. Show that the set all integers Z form a group under addition.

Sol^n: let $Z = \{ \ldots -3, -2, -1, 0, 1, 2, 3 \ldots \}$

(i) closure property : $\forall a, b \in Z$
$a+b \in Z$
$\therefore Z$ is closed under addition

(ii) Associative property

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α, $-3, 1, 4 \in Z$

or, $(-3+1)+4 = -2+4 = 2$
$-3+(1+4) = -3+5 = 2$
$\therefore (-3+1)+4 = -3+(1+4)$
$\therefore Z$ is associative under addition

(iii) Existence of identity element
There exist an element $0 \in Z$
$0+2, = 2+0 = 2$

(iv) Existence of inverse element.
There exist an element $a \in Z$ such that
$a + (-a) = 0$
$-a$ is the inverse of a.

Here Z is a group under addition.

example 11.
Let $*$ be defined or Q^+ by
$a*b = \frac{ab}{2}$, then show that Q^+ form a group under the binary operation $*$.

Sol^n: Here,

(i) closure property:
$\forall a, b \in Q^+$ and $a*b = \frac{ab}{2} \in Q^+$
Hence Q^+ is closed under the operation $*$

(ii) Associative property
$\forall a, b, c \in Q^+$, then,
$(a*b)*c = a*(b*c)$
$(a*b)*c = \left(\frac{ab}{2}\right)*c = \frac{abc}{4}$
$a*(b*c) = a*\left(\frac{bc}{2}\right) = \frac{abc}{4}$

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Hence Q^+ is associative under the operation $*$

(iii) Existence of identity element

There is an element $e \in Q^+$ such that

$\forall a \in Q^+$

$a * e = a$

$\frac{ae}{2} = a$

$e = 2$

$\therefore a * e = \frac{ae}{2} = \frac{a \cdot 2}{2} = a$

2 is an identity element.

(iv) Existence of inverse element,

$\forall a \in Q^+$

There is an element $a' \in Q^+$ such that

$a * a' = e$

$\frac{aa'}{2} = e$

$aa' = 2 \times 2 \quad (\because e = 2)$

$\therefore a' = 4/a$

where $4/a$ is an inverse element of a

Hence Q^+ is a group under $*$.

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Show that set of all integers Z is not a group under multiplication

Sol^n: Here, let $Z = \{\dots -3, -2, -1, 0, 1, 2, \dots\}$

(i) closure property:-

$\forall a, b \in Z$

$a \times b \in Z$

Hence Z is closed under multiplication.

(ii) Associative property:

$-2, 1, 3 \in Z$

$(-2 \times 1) \times 3 = -2 \times (1 \times 3)$

Hence Z is associative under multiplication

(iii) existence of identity element,

There is an element $1 \in Z$ such that

$2 \times 1 = 1 \times 2 = 2 \quad (\because e = 1)$

1 is the identity element

(iv) existence of inverse element,

There is an element $a' \in Z$ such that,

$a \times a' = e$

$a \times a' = 1 \quad (\because e = 1)$

$a' = 1/a \quad a' \notin Z$

let's we take $a = 2$ from the set of Z then inverse element of 2 according to above relation is that

$a' = 1/2$

where the fractional number $(\frac{1}{2} = a')$ doesnot exist in list of Z, so, the inverse element doesnot exist is set Z, Hence Z doesnot form a group under multiplication.

Ex.

Show that $G = \{1, -1, i, -i\}$ form a group under multiplication

Sol^n: Here,

$i^2 = -1$

×	1	-1	i	-i
1	1	-1	i	-i
-1	-1	1	-i	i
i	i	-i	-1	1
-i	-i	i	1	-1

(i) closure property:-

From the table

$\forall a, b \in G \Rightarrow ab \in G$

So G is closed under multiplication

(ii) Associative property

let $1, -1, i \in G$

$(1 \times -1) \times i = 1 \times (-1 \times i) = -i$

$\therefore (1 \times -1) \times i = -1 \times i = -i$

$\therefore 1 \times (-1 \times i) = 1 \times -i = -i$

$\therefore$ G is associative under multiplication

(iii) Existence of identity element.

1 is the identity element for multiplic^n and $1 \in G$

(iv) Existence of inverse element

inverse of $1 = 1$ $\quad\therefore$ So inverse is exist

" " $-1 = -1$ $\quad$ Hence G is a group

" " $i = -i$ $\quad$ Under multiplication

" " $-i = i$

$1 \times \omega^2$, 0

* Ex. show that $G = \{1, \omega, \omega^2\}$ form a group under multiplication, where ω is the cube root of unity.

Sol^n: Here,

$\omega^3 = 1$

(i) closure property:

$\forall a, b \in G \Rightarrow a \times b \in G$

Hence G is closed under multiplication

×	1	ω	ω²
1	1	ω	ω²
ω	ω	ω²	1
ω²	ω²	1	ω

(ii) Associative property:

$1, \omega, \omega^2 \in G$

$(1 \times \omega) \times \omega^2 = 1 \times (\omega \times \omega^2)$

$\therefore (1 \times \omega) \times \omega^2 = \omega \times \omega^2 = \omega^3 = 1$

$\therefore 1 \times (\omega \times \omega^2) = 1 \times \omega^3 = \omega^3 = 1$

Hence G is associative under multiplication.

(iii) existence of identity element

1 is the identity element for multiplication

i.e. $1 \in G$

(iv) existence of inverse element.

inverse of $1 = 1$

inverse of $\omega = \omega^2$

inverse of $\omega^2 = \omega$

$\therefore$ the inverse element is exist

Hence, the G form a group under multiplication

Ex.
Show that $Z_6 = \{0,1,2,3,4,5\}$ form a group under addition modulo 6.

Solⁿ: Here,
Addition modulo 6 table

$+_6$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

(i) closure property

$\forall a,b \in G \Rightarrow a +_6 b \in G$

$\therefore$ G is closed under $+_6$

(ii) Associative property.
$2,4,5 \in Z_6$, then
$(2 +_6 4) +_6 5 = 0 +_6 5 = 5$
& $2 +_6 (4 +_6 5) = 2 +_6 3 = 5$

$\therefore (2 +_6 4) +_6 5 = 2 +_6 (4 +_6 5)$

$\therefore$ G is associative under $+_6$

(iii) existence of identity element
0 is the identity element for addition and $0 \in Z_6$

(iv) existence of inverse element
inverse of $0 = 0$
" " $1 = 5$
" " $2 = 4$
" " $3 = 3$
" " $4 = 2$
" " $5 = 1$

$\therefore$ the inverse element is exist.
Here, the set Z_6 is a group under addition modulo 6

Ex. let $G = \left\{ \begin{bmatrix} a, b \\ c, d \end{bmatrix} ; a,b,c,d \in R \right\}$

prove that G is a group with respect to addition of matrices

Solⁿ: Here
(i) closure property :-

let $\forall A, B \in G$, s.t. where
$A = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$, $B = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}$

$\Rightarrow A+B = \begin{bmatrix} a_1+a_2 & b_1+b_2 \\ c_1+c_2 & d_1+d_2 \end{bmatrix} \in G$

G is closed under addition.

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(ii) Associative Property.

$\forall A, B, C \in G$, then

$(A+B)+C = A+(B+C)$

$\therefore$ G is associative under addition

(iii) Existence of identity element

There exist an element $O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Such that ~~AI = IA = A~~, $A+O = O+A$

So identity element is exist.

(iv) Existence of inverse element

There exist an element $A = \begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix}$

Such that

$$A+(-A) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O$$

So, $-A$ is the inverse of A, so inverse is exist. Hence G is a group under matrix addition.

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Elementary properties of Group

(i) If G is a group with binary operation $*$ then left and right cancellation laws ~~help~~ hold in G that is

$$a*b = a*c \Rightarrow b=c$$

$$\& \quad b*a = c*a \Rightarrow b=c$$

(i) proof: let $a*b = a*c$

$$\Rightarrow a^{-1}*(a*b) = a^{-1}*(a*c)$$

$$(a^{-1}*a)*b = (a^{-1}*a)*c$$

$$e*b = e*c$$

$$\therefore b=c$$

(ii) proof: let $b*a = c*a \Rightarrow b=c$

$$\Rightarrow (b*a)*a^{-1} = (c*a)*a^{-1}$$

$$b*(a*a^{-1}) = c*(a*a^{-1})$$

$$b*e = c*e$$

$$\therefore b=c \quad \text{proved}$$

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Theorem: 2

If G is a group with binary operation $*$ and if a and b are any element of G, then the linear equation $a*x=b$ and $y*a=b$ have a unique solution x and y.

Sol^n Since,

$$a*x=b$$
$$a^{-1}*(a*x)=a^{-1}*b$$
$$(a^{-1}*a)*x=a^{-1}*b$$
$$e*x=a^{-1}*b$$
$$x=a^{-1}*b$$
$$x=a^{-1}*b$$

& $y*a=b$

$$(y*a)*a^{-1}=b*a^{-1}$$
$$y*(a*a^{-1})=b*a^{-1}$$
$$y*e=b*a^{-1}$$
$$y=b*a^{-1}$$

To show the Uniqueness

let y_1 and y_2 be two solution of the equation $y*a=b$, then

$$y_1*a=b$$
$$y_1*a=y_2*a$$
$$y_1=y_2$$ [Right cancellation law

Hence the solution is unique.

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Theorem: 3

The identity element and the inverse element are unique in a group.

proof:- let, if possible e_1 and e_2 be two identity element of the Group G.

$e_1*e_2=e_1$ [e_2 is identity element]

$e_1*e_2=e_2$ [e_1 is identity element].

Hence, the identity element is unique

Again if ~~inverse~~ possible let b and c are two inverse element of $a\in G$.

Sol^n: Here

$b*a=a*b=e$ [b is the inverse element]

and $c*a=a*c=e$ [c is the inverse ele.]

Now:

$\cancel{b=b*e}$

$\cancel{=b*[a*b]}$ $\cancel{b*(a*b)}$

$$b=b*e$$
$$=b*(a*c)$$
$$=(b*a)*c$$
$$=e*c$$
$$b=c$$

Hence, the inverse element is unique.

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Order of a group :-
The number of element in a finite group is called order of a group.
It is denoted by $O(G)$.

Subgroup :- A non-empty Subset H of a group G is called a subgroup of G if H is itself a group with respect to operation on G.

Theorem :- A Subset H of a group G is a subgroup of G if and only if
(i) H is closed under the binary operation on G.
(ii) The identity element e of G is in H
(iii) For all $a \in H \Rightarrow a^{-1} \in H$

Example
Show that $H = \{1, -1\}$ is a subgroup of a group $G = \{1, -1, i, -i\}$ under the multiplica

sol^n: Here,
(i) From the table
$\forall a, b \in H \Rightarrow a \cdot b \in H$
$\therefore$ H is closed under multiplication

×	1	-1
1	1	-1
-1	-1	1

(ii) 1 is the identity element and $1 \in H$
(iii) the inverse of 1 is 1
" " " -1 is -1
Hence $H = \{1, -1\}$ is a group
So H is a subgroup of the group G

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Cyclic group :-
A group G is said to be cyclic group if there exist an element $a \in G$ such that every element of G is a power of a, then a is called the generator of G and we write if $G = \langle a \rangle$.

e.g. (i) $G = \{1, \omega, \omega^2\}$ is a cyclic group.

(ii) $G = \{1, -1, i, -i\}$ is cyclic group.

Since $i = 1$, $i^1 = i$, $i^2 = -i$, $i^1 = i^2 \cdot i = -i$
with generator i.

UNIT-10

RING AND FIELD

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compute the product of the following ring

(i) $(12)(16)$ in Z_{24}

(ii) $(11)\ (-4)$ in Z_{15}

(iii) $(2,3)\ (3,5)$ in $Z_5 \times Z_9$

iv) $(16)(3)$ in Z_{32}

(v) $(-3,5)\ (2,-4)$ in $Z_4 \times Z_{11}$

vi) $(28)(19)$ in Z_4.

Solⁿ:

(i) $(12)\ (16) = 192$

If 192 is divided by 24 the remainder is zero

$\therefore$ required product is 0.

(ii) $(11).(-4) = -44$

If -44 is divided by 15, the remainder is -14

$\therefore$ The required product is $-14+15 = 1$. Ans

(iii) $(2,3).(3,5)$ in $Z_5 \times Z_9$.

Solⁿ: $(2).(3) = 6$

If 6 is divided by 5, the remained is 1.

$\therefore$ the required product is 1.

~~If~~ $(3).(5) = 15$

If 15 is divided by 9, the remained is 6

$\therefore$ The required product is 6

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(iv) $(16),(3)$ in Z_{32}

Solⁿ: Here,

$(16).(3) = 48$

If 48 is divided by 32, the remainder is 16.

$\therefore$ The required product is 16.

(v) $(-3,5),(2,-4)$ in $Z_4 \times Z_{11}$

Solⁿ: Here,

$(-3).(2) = -6$

If -6 is divided by 4, the remained is -2.

$\therefore$ The required product is $= -2+4 = 2$

$(5).(-4) = -20$

If -20 is divided by 11, the remainder is -9.

$\therefore$ the required product is $= -9+11 = 2 \Rightarrow (2,2)$

(vi) $(28)\ (19)$ in Z_4

Solⁿ: Here

$(28).(19) = 532$

If 532 is divided by 4, the remainder is 0.

$\therefore$ The required product is 0

Zero divisor

If a and b be two non-zero element of a ring R such that $ab = 0$, then a and b are called zero divisor or divisor of zero

Ex. If $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

and $A \cdot B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

A and B are zero divisor

ex. Find the zero divisor of a ring Z_{12}

Sol: Here,

$Z_{12} = \{0,1,2,3,4,5,6,7,8,9,10,11\}$.

Here

$2 \cdot 6 = 0$, $3 \cdot 4 = 0$, $4 \cdot 6 = 0$

$3 \cdot 8 = 0$, $8 \cdot 9 = 0$, $6 \cdot 8 = 0$

$4 \cdot 9 = 0$, $6 \cdot 6 = 0$, $6 \cdot 10 = 0$

Thus $2,3,4,6,8,9,10$ are zero divisor.

Ex show that matrix $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ is zero divisor of ring $M_2(Z)$

Sol: Here,

let $\begin{bmatrix} x & y \\ z & w \end{bmatrix} \in M_2(Z)$

so that

$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \cdot \begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\begin{bmatrix} 1x+2z & y+2w \\ 2x+4z & 2y+4w \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$x+2z = 0$ — (i)

~~y+2w~~ $2x+4z = 0$ (ii)

$y+2w = 0$ — (iii)

$2y+4w = 0$ — (iv)

From (1) and (2)

where $z = 1$, $x = -2$

From (3) and (4)

where $w = 1$, $y = -2$

$\therefore \begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} -2 & -2 \\ 1 & 1 \end{bmatrix}$

Here $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ is zero divisor of $M_2(Z)$

Ex. Solve the equation

$x^2+2x+4=0$ in Z_6

Sol^n: Here,

$Z_6 = \{0,1,2,3,4,5\}$

where

$x=0$,	$4 \neq 0$
$x=1$	$7 \neq 0$
$x=2$	$12=0$
$x=3$	$19 \neq 0$
$x=4$	$28 \neq 0$
$x=5$	$39 \neq 0$

$\therefore x=2$ is the Required Solution

Ex Solve

(1) $x^2-3x+6=0$ in Z_{12}

(2) $x^2+2x+2=0$ in Z_6

(3) $x^3-6x^2+11x-6=0$ in Z_8.

(1) $x^2-3x+6=0$ in Z_{12}

let $Z_{12} = \{0,1,2,3,4,5,6,7,8,9,10,11\}$

where

$x=0$,	$6 \neq 0$
$x=1$,	$4 \neq 0$
$x=2$	$4 \neq 0$
$x=3$	$6 \neq 0$
$x=4$	$10 \neq 0$
$x=5$	$16 \neq 0$
$x=6$	$24=0$
$x=7$	$34 \neq 0$
$x=8$	$46 \neq 0$
$x=9$	$60=0$
$x=10$	$76 \neq 0$
$x=11$	$94 \neq 0$

the $x=6,9$ is the Required sol^n.

(2) $x^2+2x+2=0$ in Z_6

Sol^n: Here,

$x^2+2x+2=0$

let $Z = \{0,1,2,3,4,5\}$

where

$x=0$,	$2 \neq 0$
$x=1$	$5 \neq 0$
$x=2$	$10 \neq 0$
$x=3$	$17 \neq 0$
$x=4$	$26 \neq 0$
$x=5$	$37 \neq 0$.

(3) $x^3-6x^2+11x-6=0$ in Z_8

Sol^n: Here

$x^3-6x^2+11x-6=0$.

let $Z = \{0,1,2,3,4,5,6,7\}$

where

$x=0$,	$-6 \neq 0$
$x=1$.	$0=0$
$x=2$.	$0=0$
$x=3$.	$0=0$
$x=4$	$6 \neq 0$
$x=5$.	$24=0$
$x=6$	$60 \neq 0$
$x=7$	$120=0$

$x=1,2,3,5,7$ is the Required sol^n.

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Ring :- let R be a non-empty set. An abelian algebraic structure $(R, +, \cdot)$ together with binary operation addition and multiplication defined on R is called a ring if the following axioms are satisfies.

(i) closed under addition:
$a+b \in R, \forall a, b \in R$

(ii) Associative under addition
$a+(b+c) = (a+b)+c, \forall a, b, c \in R$

(iii) Existence of additive identity
$0+a = a+0 = a, \ 0 \in R$

(iv) Existence of additive inverse:
$-a+a = a+(-a) = 0 \ \forall a \in R$

(v) commutative under addition
$a+b = b+a \ \forall a, b \in R$

For multiplication,

(1) Closed under multiplication
$a \cdot b \in R, \forall a, b \in R$

(2) Associative under multiplication
$(a \cdot b) \cdot c = a(b \cdot c), \forall a, b, c \in R$

(3) Existence of multiplicative identity
$a \cdot 1 = 1 \cdot a = a, \ 1 \in R$

(4) Existence of multiplicative inverse
$a \cdot a' = a' \cdot a = 1 \ \forall a \in R$

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Distributive law
$a \cdot (b+c) = a \cdot b + a \cdot c$
and $(b+c) \cdot a = b \cdot a + c \cdot a, \ \forall a, b, c \in R$

Example:
The set Z of all integers with respect to addition and multiplication forms of ring
Sol^n: Here,
let $Z = \{ \ldots, -3, -2, -1, 0, 1, 2, 3, \ldots \}$

(i) closure property: $\forall a, b \in Z \Rightarrow a+b \in Z$
$\therefore Z$ is closed under addition

(ii) Associative law under addition
$\forall a, b, c \in Z$. then $a+(b+c) = (a+b)+c$
$\therefore Z$ is associative under addition

(iii) existence of additive identity
$\forall a \in Z$ there exist an element i.e $0 \in Z$
such that $a+0 = 0+a = a$

(iv) Existence of additive inverse
$\forall a \in Z$, then there exist an element $-a \in Z$
Such that $-a+a = a+(-a) = 0$

(v) commutative under addition
$\forall a, b \in Z \Rightarrow a+b = b+a$
$\therefore Z$ is commutative under addition

(vi) closed under multiplication
$\forall a, b \in Z \Rightarrow a \cdot b \in Z$
$\therefore Z$ is closed under multiplication.

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(vii) Associative under multiplication.

$\forall a,b,c \in R$ then $a.(b.c) = (a.b).c$

$5,2,-3 \in R$ $\quad 5.(2.-3) = 5.(-6) = -30$

$= (5.2).(-3) = 10.(-3) = -30$

$\therefore$ Z is associative under multiplication.

(viii) Distributive law.

$\forall a,b,c \in Z$ the $a.(b+c) = a.b + ac$

$(b+c).a = b.a + c.a$

eg. $6,2,-4 \in Z$ then,

$6.(2-4) = 6.(-2) = -12$

~~$(2-4).6 = (-2).6 =$~~

$6.2 + 6.(-4) = 12-24 = -12$

also, $(2-4).6 = -2.6 = -12$

$2.6 + (-4).6 = 12-24 = -12$

So the distributive Law hold

Hence $(Z, +, \cdot)$ form a ring.

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TRANSFORMATIONS

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CHAPTER-02

1. let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and define $T: R^2$ by $T(x) = Ax$.

Find the image under T of $u = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$ and $v = \begin{bmatrix} a \\ b \end{bmatrix}$

Sol: Here,

let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $u = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$ & $v = \begin{bmatrix} a \\ b \end{bmatrix}$

the given function is

$$T(x) = Ax,$$

The image of u under T is,

$$T(x) = Ax$$

$$T(u) = Au$$

$$\therefore Au = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ -2 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ -2 \end{bmatrix}$$

Similarly,

The image of V under T is

$$T(x) = Ax$$

$$T(v) = Av$$

$$\therefore Av = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix}$$

$$= \begin{bmatrix} a \\ b \end{bmatrix}.$$

Ans

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(2) let $A = \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 0.5 \end{bmatrix}$, $u = \begin{bmatrix} 1 \\ 0 \\ -4 \end{bmatrix}$, and $v = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

Define $T: R^3 \to R^3$ by $T(x) = Ax$. find $T(u)$ and $T(v)$.

Sol^n: Here

$A = \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 0.5 \end{bmatrix}$, $u = \begin{bmatrix} 1 \\ 0 \\ -4 \end{bmatrix}$, $v = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

We know that,

The image of u under T is represented by

$T(u) = Au$

$Au = \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 0.5 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ -4 \end{bmatrix}$

$= \begin{bmatrix} 0.5 \\ 0 \\ -2.0 \end{bmatrix}$

Similarly,

$T(v) = Av$

$Av = \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 0.5 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

$= \begin{bmatrix} 0.5a \\ 0.5b \\ 0.5c \end{bmatrix}$

Ans

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(1) let $A = \begin{bmatrix} 1 & 0 & -2 \\ -2 & 1 & 6 \\ 3 & -2 & -5 \end{bmatrix}$, $b = \begin{bmatrix} -1 \\ 7 \\ -3 \end{bmatrix}$. Define T by $T(x) = Ax$. Find a vector x whose image under T is b.

Sol^n: Here,

$A = \begin{bmatrix} 1 & 0 & -2 \\ -2 & 1 & 6 \\ 3 & -2 & -5 \end{bmatrix}$, $b = \begin{bmatrix} -1 \\ 7 \\ -3 \end{bmatrix}$

$T(x) = Ax$,

Here, the image of b under T is,

$T(b) = Ab_x$

$Ab_x = T(b)$

$\begin{bmatrix} 1 & 0 & -2 \\ -2 & 1 & 6 \\ 3 & -2 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 7 \\ -3 \end{bmatrix}$

The augmented matrix is,

$\begin{bmatrix} 1 & 0 & -2 & : & -1 \\ -2 & 1 & 6 & : & 7 \\ 3 & -2 & -5 & : & -3 \end{bmatrix}$

$R_2 \to R_2 + 2R_1$ and $R_3 \to R_3 - 3R_1$

$\begin{bmatrix} 1 & 0 & -2 & : & -1 \\ 0 & 1 & 2 & : & 5 \\ 0 & -2 & 1 & : & 0 \end{bmatrix}$

$R_3 \to R_3 + 2R_2$

$\begin{bmatrix} 1 & 0 & -2 & : & -1 \\ 0 & 1 & 2 & : & 5 \\ 0 & 0 & 5 & : & 10 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 & -2 & : & -1 \\ 0 & 1 & 2 & : & 5 \\ 0 & 0 & 1 & : & 2 \end{bmatrix}$

$R_2 \to R_2 - 2R_3$

$$\left[\begin{array}{ccc:c} 1 & 0 & -2 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$R_1 \to R_1 + 2R_3$

$$\left[\begin{array}{ccc:c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$\therefore$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$$

4. let $A = \begin{bmatrix} 1 & -5 & -7 \\ -3 & 7 & 5 \end{bmatrix}$, $b = \begin{bmatrix} -2 \\ -2 \end{bmatrix}$. Define T by $T(x) = Ax$. find a vector x whose image under T is b.

Sol: Here,

$A = \begin{bmatrix} 1 & -5 & -7 \\ -3 & 7 & 5 \end{bmatrix}$, $b = \begin{bmatrix} -2 \\ -2 \end{bmatrix}$

The image of $\cancel{x}$ under T is $b = \begin{bmatrix} -2 \\ -2 \end{bmatrix}$

We know that,

$T(x) = Ax$

$b = Ax$

$\therefore Ax = b$

$$\begin{bmatrix} 1 & -5 & -7 \\ -3 & 7 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \end{bmatrix}$$

The augmented matrix is,

$$\left[\begin{array}{ccc:c} 1 & -5 & -7 & -2 \\ -3 & 7 & 5 & -2 \end{array}\right]$$

$R_2 \to R_2 + 3R_1$

$$\left[\begin{array}{ccc:c} 1 & -5 & -7 & -2 \\ 0 & -8 & -16 & -8 \end{array}\right]$$

$$\left[\begin{array}{ccc:c} 1 & -5 & -7 & -2 \\ 0 & 1 & 2 & 1 \end{array}\right]$$

Here x_1 and x_2 are basic variable and x_3 is a free variable then the associated System is,

$x_1 - 5x_2 - 7x_3 = -2$, $\quad x_2 + 2x_3 = 1$

$x_2 = 1 - 2x_3$

$x_1 - 5(1 - 2x_3) - 7x_3 = -2$

$x_1 - 5 + 10x_3 - 7x_3 = -2$

$x_1 - 5 + 3x_3 = -2$

$x_1 = -3x_3 + 3$

$\therefore$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -3x_3 + 3 \\ -2x_3 + 1 \\ x_3 + 0 \end{bmatrix} = x_3 \begin{bmatrix} -3 \\ -2 \\ 1 \end{bmatrix} + \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$$

The required answer is $\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$ Ans

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5. Find all $x \in \mathbb{R}^4$ that are mapped in to zero vector by the transformation $x \to Ax$ for the given matrix.

Sol: Here,

$$A = \begin{bmatrix} 1 & -4 & 7 & -5 \\ 0 & 1 & -4 & 3 \\ 2 & -6 & 6 & -4 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & -4 & 7 & -5 \\ 0 & 1 & -4 & 3 \\ 2 & -6 & 6 & -4 \end{bmatrix}$$

The transformation of x under $T(x)$ is,

$x \to Ax$ where $x = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$Ax = x_T$

$$\begin{bmatrix} 1 & -4 & 7 & -5 \\ 0 & 1 & -4 & 3 \\ 2 & -6 & 6 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The augmented matrix is

$$\left[\begin{array}{cccc:c} 1 & -4 & 7 & -5 & 0 \\ 0 & 1 & -4 & 3 & 0 \\ 2 & -6 & 6 & -4 & 0 \end{array}\right]$$

$R_3 \to R_3 - 2R_1$

$$\left[\begin{array}{cccc:c} 1 & -4 & 7 & -5 & 0 \\ 0 & 1 & -4 & 3 & 0 \\ 0 & 2 & -8 & 6 & 0 \end{array}\right]$$

$R_3 \to R_3 - 2R_2$

$$\left[\begin{array}{cccc:c} 1 & -4 & 7 & -5 & 0 \\ 0 & 1 & -4 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Here x_3 and x_4 are free variable so the associated system is

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$x_1 - 4x_2 + 7x_3 - 5x_4 = 0$	$x_2 - 4x_3 + 3x_4 = 0$
$x_1 - 4(4x_3 - 3x_4) + 7x_3 - 5x_4 = 0$	$x_2 = 4x_3 - 3x_4$

$x_1 - 16x_3 + 12x_4 + 7x_3 - 5x_4 = 0$

$x_1 - 9x_3 + 7x_4 = 0$

$x_1 = 9x_3 - 7x_4$

$x_3 = \text{free}$

$x_4 = \text{free},$

Now

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 9x_3 - 7x_4 \\ 4x_3 - 3x_4 \\ x_3 + 0 \cdot x_4 \\ 0 \cdot x_3 + x_4 \end{bmatrix} = x_3 \begin{bmatrix} 9 \\ 4 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -7 \\ -3 \\ 0 \\ 1 \end{bmatrix}$$

Ans

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7. Prove the following transformation T are linear also find the matrix that implements the mapping

(i) $T(x_1, x_2, x_3, x_4) = (0, x_1+x_2, x_2+x_3, x_3+x_4)$

(ii) $T(x_1, x_2, x_3) = (x_1 - 5x_2 + 4x_3, x_2 - 6x_3)$

Sol^n: Here,

$T(u+v+w+x) = T(u_1+v_1+w_1+x_1, u_2+v_2+w_2+x_2)$

$= (0,$

(ii) $T(x_1, x_2, x_3) = (x_1 - 5x_2 + 4x_3, x_2 - 6x_3)$

Since

$T(u+v+w) = T(u_1+v_1+w_1, u_2+v_2+w_2, u_3+v_3+w_3)$

$= T[u_1+v_1+w_1 - 5(u_2+v_2+w_2) + 4(u_3+v_3+w_3), u_2+v_2+w_2 - 6(u_3+v_3+w_3)]$

$= T[u_1+v_1+w_1 - 5u_2 - 5v_2 - 5w_2 + 4u_3 + 4v_3 + 4w_3, u_2+v_2+w_2 - 6u_3 - 6v_3 - 6w_3]$

Again

$T(u+v+w) = T(u_1, u_2, u_3) + T(v_1, v_2, v_3) + T(w_1, w_2, w_3)$

$= T(u_1 - 5u_2 + 4u_3, u_2 - 6u_3) + T(v_1 - 5v_2 + 4v_3, v_2 - 6v_3) + T(w_1 - 5w_2 + 4w_3, w_2 - 6w_3)$

$= T(u_1+v_1+w_1 - 5u_2 - 5v_2 - 5w_2 + 4u_3 + 4v_3 + 4w_3, u_2+v_2+w_2 - 6u_3 - 6v_3 - 6w_3)$

$\therefore T(u+v+w) = T(u) + T(v) + T(w)$

Hence the Transformation T is linear.

The matrix that implement the mapping is

$$\begin{bmatrix} 1 & -5 & 4 \\ 0 & 1 & -6 \end{bmatrix}$$ Ans

(i) $T(x_1, x_2, x_3, x_4) = (0, x_1+x_2, x_2+x_3, x_3+x_4)$

Sol^n: Here,

$T(u+v+w+x) = T[(u_1+v_1+w_1+x_1), (u_2+v_2+w_2+x_2), (u_3+v_3+w_3+x_3), (u_4+v_4+w_4+x_4)]$

$T(u+v+w+x) = T[u_1+v_1+w_1+x_1, u_2+v_2+w_2+x_2, u_3+v_3+w_3+x_3, u_4+v_4+w_4+x_4]$

$= T[0, u_1+v_1+w_1+x_1+u_2+v_2+w_2+x_2, u_2+v_2+w_2+x_2 + u_3+v_3+w_3+x_3, u_3+v_3+w_3+x_3+u_4+v_4+w_4+x_4]$

$= T[0, u_1+v_1+w_1+x_1+u_2+v_2+w_2+x_2, u_2+v_2+w_2+x_2+u_3+v_3+w_3+x_3, u_3+v_3+w_3+x_3+u_4+v_4+w_4+x_4]$

$T(u) + T(v) + T(w) + T(x) = T(0, u_1+u_2, u_2+u_3, u_3+u_4) + T(0, v_1+v_2, v_2+v_3, v_3+v_4) + T(0, w_1+w_2, w_2+w_3, w_3+w_4) + T(0, x_1+x_2, x_2+x_3, x_3+x_4)$

$= T(0, u_1+u_2+v_1+v_2+w_1+w_2+x_1+x_2, u_2+u_3+v_2+v_3+w_2+w_3+x_2+x_3, u_3+u_4+v_3+v_4+w_3+w_4+x_3+x_4)$

$= T(0, u_1+v_1+w_1+x_1+u_2+v_2+w_2+x_2, u_2+v_2+w_2+x_2 + u_3+v_3+w_3+x_3, u_3+v_3+w_3+x_3+u_4+v_4+w_4+x_4)$

$\therefore T(u+v+w+x) = T(u) + T(v) + T(w) + T(x)$.

Hence the Tranformation T is linear.

The matrix that implement the mapping is

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$ Ans