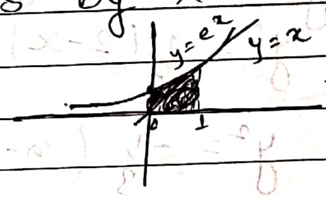
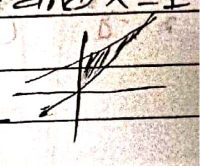
CHAPTER-6

Application Of Antiderivatives

Exercise :- 6.1

1) Find the area of the region bounded above by $y=e^x$, bounded below by $y=x$ and bounded on the sides by $x=0$ and $x=1$





Sol^n:- Here

Given

$y_1 = e^x$ —(i) &

$y_2 = x$ —(ii)

where the area bounded on the sides by $x=0$ & $x=1$ along x-axis, so that

Such that,

$$A = \int_0^1 (y_1 - y_2)\cdot dx$$

$$A = \int_0^1 (e^x - x)\cdot dx$$

$$A = \left[e^x - \frac{x^2}{2}\right]_0^1$$

$$A = \left[\left(e - \frac{1}{2}\right) - \left(e^0 - \frac{0}{2}\right)\right]$$

$$A = \left[e - \frac{1}{2} - 1\right]$$

$$A = [e - 0.5 - 1]$$

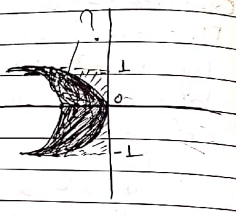
$$A = e - 1.5 \quad \text{Ans.}$$

2 Find the area of the region enclosed by $x+y^2=0$ & $x+3y^2=2$

Sol^n: Here

Given,

We know that



$x=-y^2$ $\quad y^2=-x$ — ① &

$y^2=\frac{1}{3}(2-x)$

$y^2=-\frac{1}{3}(x-2)$ — ②

Solving eq^n ① and ② we get,

$-y^2+3y^2=2$

$2y^2=2$

$y^2=1$

$y=\pm 1$

where

$x=-1$ If $y=1$

$x=-1$ If $y=-1$

So, that the curve enclosed the area of the region such that, along y-axis,

$$A=\int_{-1}^{0}(x_1-x_2)\,dy+\int_{0}^{1}(x_1-x_2)\cdot dy$$

$$A=\int_{-1}^{0}(-y^2-2+3y^2)\,dy+\int_{0}^{1}(-y^2-2+3y^2)\,dy$$

$$A=\int_{-1}^{0}2y^2-2\cdot dy+\int_{0}^{1}2y^2-2\,dy$$

$$A=2\left[\int_{-1}^{0}y^2-1\,dy+\int_{0}^{1}y^2-1\cdot dy\right]$$

$$A=2\left[\left[\frac{y^3}{3}-y\right]_{-1}^{0}+\left[\frac{y^3}{3}-y\right]_{0}^{1}\right]$$

$$A=2\left[0-\left(-\frac{1}{3}+1\right)+\left(\frac{1}{3}-1\right)\right]$$

$$A=2\left[\frac{1}{3}-1+\frac{1}{3}-1\right]$$

$$A=2\left[\frac{2}{3}-2\right]$$

$$A=2\left[\frac{2-6}{3}\right]$$

$$A=-\frac{8}{3}$$

$\therefore A=\frac{8}{3}$ sq. units Ans

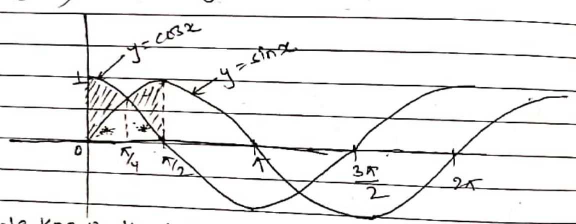
(3) Find the area of the region bounded by the curves $y = \sin x$, $y = \cos x$, $x = 0$ & $x = \frac{\pi}{2}$

Sol^n: here,

$y = \sin x$ — (i)

$y = \cos x$ — (ii)

This shows that the area of the region bounded by the curve $y = \sin x$, $y = \cos x$, along Y-axis such that



We know that

~~Solving~~ Dividing (i) and (ii) we get,

$\tan x = 1$

$x = \frac{\pi}{4}$ where the two curve intersect with each other,

Now,

Actual Area (A_T) bounded by the two curve is,

$$A_T = \int_0^{\pi/2} \cos x \cdot dx + \int_0^{\pi/2} \sin x \cdot dx - 4\int_0^{\pi/4} \sin x$$

$$A_T = [\sin x]_0^{\pi/2} + [-\cos x]_0^{\pi/2} - 4[-\cos x]_0^{\pi/4}$$

$$A_T = [\sin \tfrac{\pi}{2} - \sin 0] - [\cos \tfrac{\pi}{2} - \cos 0] + 4[\cos \tfrac{\pi}{4} - \cos 0]$$

$$A_T = (1-0) - (0-1) + 4\left(\tfrac{1}{\sqrt{2}} - 1\right)$$

30D extra.

$$A_T = 1 + 1 + \frac{4}{\sqrt{2}} - 4$$

$$A_T = \frac{4}{\sqrt{2}} - 4 + 2$$

$$A_T = \frac{4}{\sqrt{2}} - 2$$

$$A_T = \frac{4 \times \sqrt{2}}{2} - 2$$

$A_T = 2\sqrt{2} - 2$ is the required answer

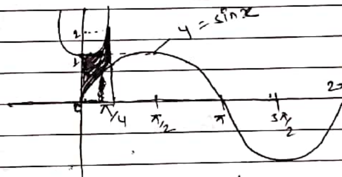
(4) Find the area between two curves $y = \sec^2 x$ & $y = \sin x$ from $x = 0$ to $x = \frac{\pi}{4}$

Sol^n: here,

Given,

$y = \sec^2 x$ — (i)

$y = \sin x$ — (ii)



Area bounded by the given curve along Y-axis such that $x = 0$ to $x = \frac{\pi}{4}$

$$A = \int_0^{\pi/4} \sec^2 x \, dx - \int_0^{\pi/4} \sin x \, dx$$

$$A = [\tan x]_0^{\pi/4} - [-\cos x]_0^{\pi/4}$$

$$A = (\tan \tfrac{\pi}{4} - \tan 0) + (\cos \tfrac{\pi}{4} - \cos 0)$$

$$A = (1 - 0) + \left(\tfrac{1}{\sqrt{2}} - 1\right)$$

$$A = 1 + \tfrac{1}{\sqrt{2}} - 1$$

$A = \frac{1}{\sqrt{2}}$ sq units Ans

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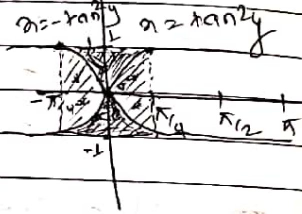
(5) Find the area between two curves $x = \tan^2 y$ and $x = -\tan^2 y$, $-\pi/4 \le y \le \pi/4$

Sol^n: Here,

$x = \tan^2 y$ — (i) &

$x = -\tan^2 y$



We know that

$A = \int_0^{1} \tan^2 y \, dy$

$A = \int_{-\pi/4}^{\pi/4} (\tan^2 y + \tan^2 y) \, dy = \int_{-\pi/4}^{\pi/4} (x_1 - x_2) \, dy$

$A = \int_{-\pi/4}^{\pi/4} 2\tan^2 y \, dy$

$A = 2\int_{-\pi/4}^{\pi/4} (\sec^2 y - 1) \, dy$

$A = 2\left[\tan y - y\right]_{-\pi/4}^{\pi/4}$

$A = 2\left[\tan \pi/4 - \pi/4 - \left(-\tan \pi/4 + \pi/4\right)\right]$

$A = 2\left[1 - \pi/4 + 1 - \pi/4\right]$

$A = 2\left[2 - \pi/2\right] = 4 - \pi$ sq unit

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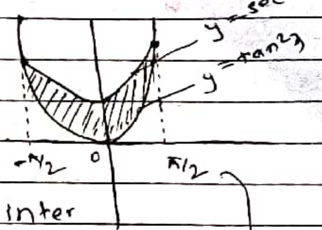
(6) Find the area between two curves $y = \sec^2 x$ & $y = \tan^2 x$, $-\pi/4 \le x \le \pi/4 = [-\pi/4, \pi/4]$.

Sol^n: Here.

$y = \sec^2 x$ — (i)

$y = \tan^2 x$ — (ii)



We have to findout the intersection point between two curves, i.e.

$\sec^2 x = \tan^2 x$

$1 + \tan^2 x = \tan^2 x$ $\frac{1}{\cos 2x} = \frac{\sin 2x}{\cos 2x}$

$\frac{1}{\tan^2 x} + 1 = 0$ $\sin 2x = 1$

$\frac{1}{\tan^2 x} = -1$ $\sin x = \pm 1$ $\therefore -\pi/2$ doesnot lie on the given interval so, that

$\tan^2 x = -1$ $x = \sin^{-1}(\pm 1)$

$x = \pi/2, -\pi/2$ taking $\boxed{x = \pi/4, -\pi/4}$

$A = \int_{-\pi/4}^{\pi/4} \sec^2 x \, dx - \int_{-\pi/4}^{\pi/4} \tan^2 x \, dx$

$A = \int_{-\pi/4}^{\pi/4} \sec^2 x \, dx - \int_{-\pi/4}^{\pi/4} (\sec^2 x - 1) \, dx$

$A = \left[\tan x\right]_{-\pi/4}^{\pi/4} - \left[\tan x - x\right]_{-\pi/4}^{\pi/4}$

$A = \left(\tan\frac{\pi}{4} + \tan\frac{\pi}{4}\right) - \left[\tan\frac{\pi}{4} - \frac{\pi}{4} - \left(\tan\left(-\frac{\pi}{4}\right) + \frac{\pi}{4}\right)\right]$

$A = \cancel{\tan\frac{\pi}{4}} + \cancel{\tan\frac{\pi}{4}} - \cancel{\tan\frac{\pi}{4}} + \frac{\pi}{4} - \cancel{\tan\frac{\pi}{4}} + \frac{\pi}{4}$

$A = \frac{\pi}{2} \cancel{+ \frac{\pi}{2}} = \frac{2\pi}{2} \cancel{\pi} \cdot \frac{\pi}{4} + \frac{\pi}{4} = \frac{2\pi}{4} = \frac{\pi}{2}$

Ans.

3, -2, 1

(7) Find the ~~area~~ area of region between the curve & x-axis.

(i) $f(x) = -x^2-2x$, $[-3,2]$,

Sol^n: Here,

$f(x) = y = -x^2 - 2x$

We have to findout the area of region between ~~to~~ the curve along x-axis

Such that,

$-x^2-2x=0$

$x^2+2x=0$

$x(x+2)=0$

$x=0,-2$

take modulus in each A_1, A_2, A_3 Area, i.e.

Now

$$A = \underbrace{\int_{-3}^{-2} (-x^2-2x)\,dx}_{A_1} + \underbrace{\int_{-2}^{0} (-x^2-2x)\,dx}_{A_2} + \underbrace{\int_{0}^{2} (-x^2-2x)}_{A_3}$$

$$A = -\left[\frac{x^3}{3}+\frac{2x^2}{2}\right]_{-3}^{-2} - \left[\frac{x^3}{3}+\frac{2x^2}{2}\right]_{-2}^{0} - \left[\frac{x^3}{3}+\frac{2x^2}{2}\right]_{0}^{2}$$

$$A = -\left[\frac{x^3}{3}+x^2\right]_{-3}^{-2} - \left[\frac{x^3}{3}+x^2\right]_{-2}^{0} - \left[\frac{x^3}{3}+x^2\right]_{0}^{2}$$

$$A = -\left\{-\frac{8}{3}+4-\left(-\frac{27}{3}+9\right)\right\} - \left\{-\left(-\frac{8}{3}+4\right)\right\} - \left\{\frac{8}{3}+4\right\}$$

$$A = -\left\{\left(-\frac{8}{3}+4+\frac{27}{3}-9\right)\right\} - \left\{\left(\frac{8}{3}-4\right)\right\} - \left\{\frac{8}{3}+4\right\}$$

$$A = \frac{8}{3} - \cancel{4} - \frac{27}{3} + 9 - \cancel{\frac{8}{3}} + \cancel{4} - \frac{8}{3} - 4$$

$$A = -\cancel{9} + \cancel{9} - \frac{8}{3} - 4$$

$$A = \frac{-8-12}{3}$$

$A = \frac{20}{3}$ sq units Ans//

(ii) $f(x) = x^2-6x+8$, $[0,3]$

Sol^n: Here

$y = x^2-6x+8$,

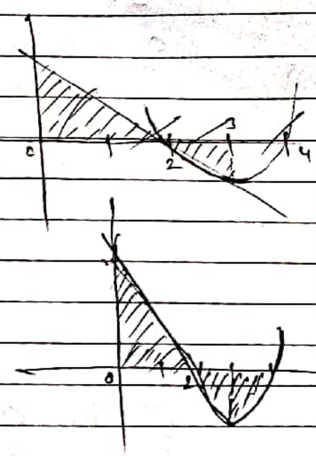
when $y=0$,

$x^2-6x+8=0$

$x^2-4x-2x+8=0$

$x(x-4)-2(x-4)=0$

$x=2,4$



Now,

$y = x^2 - 2\cdot x\cdot 3 + 3^2 - 9 + 8$

$y = (x-3)^2 - 1$

$(x-3)^2 = y+1$

$x=(3,-1)$

If $x=0$

$y=8$

the area bounded by the curve $y = x^2-6x+8$ along $x=0,2,3$ we get,

$$A = \int_0^2 (x^2-6x+8)\,dx + \int_2^3 (x^2-6x+8)\,dx$$

$$A = \left[\frac{x^3}{3} - \frac{6x^2}{2} + 8x\right]_0^2 + \left[\frac{x^3}{3} - \frac{6x^2}{2} + 8x\right]_2^3$$

$$A = \left(\frac{8}{3} - 12 + 16\right) + \left(\frac{27}{3} - 27 + 24 - \frac{8}{3} - 12 + 16\right)$$

$$A = \frac{8}{3} - \cancel{12} + \cancel{16} + 9 - 27 + 24 - \frac{8}{3} + \cancel{12} - \cancel{16}$$

$A = 33 - 27 = 6$ Ans

7 (iii) $y = x^3 - 4x$, $[-2, 2]$

Sol^n: Here,

$y = x^3 - 4x$

let $y = 0$,

$x^3 - 4x = 0$

$x(x^2 - 4) = 0$

$x = 0$

$x^2 = 4$

$x = \pm 2$

∴ The area bounded by the curve along x-axis of

$$A = \left| \int_{-2}^{0} (x^3 - 4x)\,dx \right| + \left| \int_{0}^{2} (x^3 - 4x)\,dx \right|$$

$$A = \left| \left[\frac{x^4}{4} - \frac{4x^2}{2} \right]_{-2}^{0} \right| + \left| \left[\frac{x^4}{4} - \frac{4x^2}{2} \right]_{0}^{2} \right|$$

$$A = \left| \left[\frac{x^4}{4} - 2x^2 \right]_{-2}^{0} \right| + \left| \left[\frac{x^4}{4} - 2x^2 \right]_{0}^{2} \right|$$

$$A = \left| (0 - 0) - \left\{ \frac{16}{4} - 8 \right\} \right| + \left| \left\{ \frac{16}{4} - 8 \right\} \right|$$

$$A = \left| \frac{-16}{4} + 8 \right| + \left| \frac{16}{4} - 8 \right|$$

$A = 4 + 4$

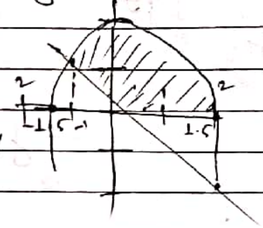
$A = 8$ Ans

(8) Q. Find the area of region enclosed by the parabola $y = 2 - x^2$ & line $y = -x$

Sol^n: Here,

$y = 2 - x^2$ —(i) &

$y = -x$ —(ii)



from eq^n (i) and (ii) we get,

$2 - x^2 = -x$

$x^2 - x - 2 = 0$

$x^2 - 2x + x - 2 = 0$

$x(x-2) + 1(x-2) = 0$

$x = -1, 2$

Where the region enclosed by $x = -1$ & 2 so, that the Area is,

$$A = \int_{-1}^{2} (2 - x^2 + x)\,dx$$

$$A = \left[2x - \frac{x^3}{3} + \frac{x^2}{2} \right]_{-1}^{2}$$

$$A = \left(4 - \frac{8}{3} + \frac{4}{2}\right) - \left(-2 + \frac{1}{3} + \frac{1}{2}\right)$$

$$A = 4 - \frac{8}{3} + 2 + 2 - \frac{1}{3} - \frac{1}{2}$$

$$A = 8 - \frac{8}{3} - \frac{1}{3} - \frac{1}{2}$$

$$A = \frac{48 - 16 - 2 - 3}{6}$$

$$A = \frac{48 - 21}{6} = \frac{27}{6}$$

$A = \frac{9}{2}$ square units Ans

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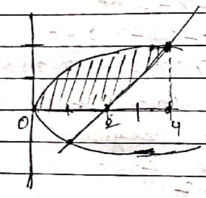
(9) Find the area of the region enclosed by parabola $x = y^2$ & line $x = y+2$ in first quadrant.

Sol^n: Here,

$x = y^2$ — (i) &

$x = y+2$ — (ii)



from eq^n (i) and (ii) we get,

$y+2 = y^2$

$y^2 - y - 2 = 0$

$y^2 - 2y + y - 2 = 0$

$y(y-2) + 1(y-2) = 0$

$y = -1, 2$

$y = -1, \ x = 1$

$y = 2 \quad x = 4$

Area of the region enclosed by parabola $x = y^2$ & line $x = y+2$ in first quadrant is

$A = \int_0^4 \sqrt{x}\, dx - \int_2^4 x - 2\, dx$

$A = \left[\frac{2}{3}(x)^{3/2}\right]_0^4 - \left[\frac{x^2}{2} - 2x\right]_2^4$

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$A = \left(\frac{2}{3} \times 8 - 0\right) - \left(8 - 8 - \left(\frac{4}{2} - 4\right)\right)$

$A = \frac{16}{3} - \left(0 - (2-4)\right)$

$A = \frac{16}{3} + (-2)$

$A = \frac{16 - 3\times 2}{3}$

$A = 10/3$ Ans

(12) Find the area bounded by x-axis & curve $y = 4 - x^2$

Sol^n: Here,

$x^2 - 4 = -y$

$x^2 = -y + 4$

$(x-0)^2 = -(y-4)$

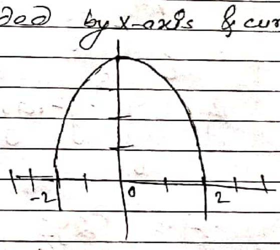
$h, k = (0, 4)$

put $y = 0$,

$4 - x^2 = 0$

$x^2 = 4$

$x = \pm 2$



Area bounded by the curve $y = 4 - x^2$ is

$A = \int_{-2}^{2} (4 - x^2)\, dx = \left[4x - \frac{x^3}{3}\right]_{-2}^{2}$

$A = 8 - 8/3 - \left(-8 + 8/3\right)$

$= 8 - 8/3 + 8 - 8/3$

$= 16 - 16/3 = \frac{48-16}{3} = 32/3$ Ans

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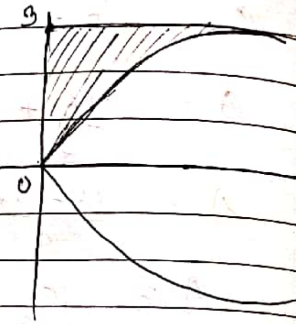
(11) Find the area of the region bounded by the curve $x = 2y^2$, $x = 0$ & $y = 3$



Sol^n: Here,

$y^2 = x/2$.

Area bounded by the curve along y-axis such that $y = 0$ & $y = 3$

$$A = \int_0^3 x \cdot dy$$

$$A = \int_0^3 2y^2 \cdot dy$$

$$A = \left[\frac{2y^3}{3}\right]_0^3$$

$$A = \left(\frac{2 \times 27}{3} - 0\right)$$

$A = 18$ sq. units Ans

(10) Find the area of the region enclosed by parabola $y^2 - 4x = 4$ & line $4x - y = 16$

Sol^n: Here,

$y^2 - 4x = 4$ – eq^n of parabola

$y^2 = 4x + 4$

$(y-0)^2 = 4(x+1)$

$(h, k) = (-1, 0)$

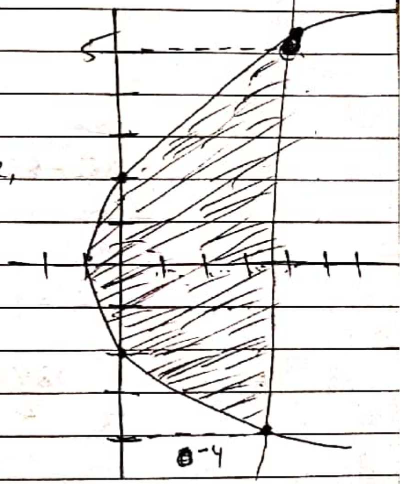
Intersection of the curve along y-axis,

$x = 0$

$y^2 = 4 \times 0 + 4$

$y^2 = 4$

$y = \pm 2$



Intersection pt betw^n curve & line,

$y^2 - 4x = 4$ —(i)

$4x = y + 16$ —(ii)

solving (i) and (ii) we get,

$y^2 - \cancel{y + 16}$

$y^2 - (y + 16) = 4$

$y^2 - y - 20 = 0$

$y^2 - 5y + 4y - 20 = 0$

$y(y-5) + 4(y-5) = 0$

$(y+4)(y-5) = 0$

$y = -4$, $y = 5$

If $y = -4$, $x = 3 \Rightarrow (3, -4)$

$y = 5$, $x = 5.25 \Rightarrow (5.25, 5)$

Area bounded by the curve & line is

$$A = \int_{-4}^{5} \frac{y + 16}{4} \cdot dy - \int_{-4}^{5} \frac{y^2 - 4}{4} \cdot dy$$

$$A = \left| \frac{1}{4}\left[\frac{y^2}{2} + 16y\right]_{-4}^{5} \right| - \left| \frac{1}{4}\left[\frac{y^3}{3} - 4y\right]_{-4}^{5} \right|$$

Vol^m of the sphere $= \int_{-r}^{r} A(x) \cdot dx$

$$V = \int_{-r}^{r} \pi (r^2 - x^2) \cdot dx$$

$$V = \pi \int_{-r}^{r} (r^2 - x^2) \cdot dx$$

$$V = \pi \left[r^2 x - \frac{x^3}{3} \right]_{-r}^{r}$$

$$V = \pi \left[\left(r^3 - \frac{r^3}{3} \right) - \left(-r^3 + \frac{r^3}{3} \right) \right]$$

$$V = \pi \left[\frac{2r^3}{3} - \left(\frac{-3r^3 + r^3}{3} \right) \right]$$

$$V = \pi \left[\frac{2r^3}{3} - \left(\frac{-2r^3}{3} \right) \right]$$

$$V = \pi \left[\frac{2r^3}{3} + \frac{2r^3}{3} \right]$$

$$V = \pi \frac{4r^3}{3}$$

$$V = \frac{4\pi r^3}{3} \quad \text{Ans}$$

Example:

Find the volume of the solid obtained rotating about x-axis the region under the curve $y = \sqrt{x}$ from 0 to 1

Sol^n: Here,

If we rotate the solid about x-axis then the cross-sectional radius $r = y = \sqrt{x}$

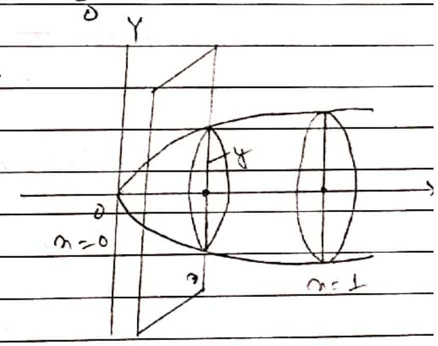
So, the cross-sectional Area $A(x) = \pi y^2$

$$= \pi (\sqrt{x})^2$$

$$= \pi x$$

Volume (V) $= \int_0^1 A(x) \cdot dx = \int_0^1 \pi x \cdot dx$

$$= \pi \int_0^1 x \, dx$$

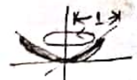


$$V = \pi \left[\frac{x^2}{2} \right]_0^1$$

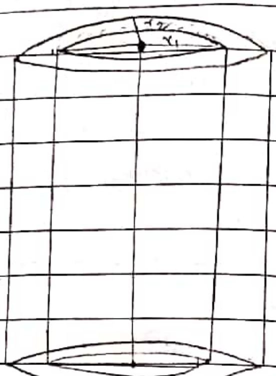
$$V = \pi \times \frac{1}{2}$$

$$\boxed{\therefore V = \frac{\pi}{2}}$$

Ans



6.2 Volume of Cylindrical Shells.



let r_1 & r_2 be the inner and outer radius of the cylindrical shell and h be the height of the cylindrical. If vol^m is calculated by subtracting the volume V_1 of inner cylinder from the volume V_2 of outer cylinder

i.e. $V = V_2 - V_1$

$V = \pi r_2^2 h - \pi r_1^2 h$

$V = \pi (r_2^2 - r_1^2) h$

$V = \pi (r_2 + r_1)(r_2 - r_1) h$

$V = \pi \left(\frac{r_2+r_1}{2}\right) \times 2 \,(r_2 - r_1) \cdot h$

$V = \pi r \times 2 (r_2 - r_1) \times h \quad \left(\frac{r_2+r_1}{2} = r\right)$ average radius

$V = 2\pi r \cdot h \cdot \Delta r$

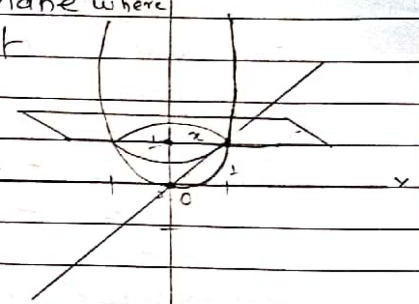
$\therefore$ vol^m = (circumference) (height). (thickness)

The vol^m of the solid obtained by rotating about y-axis the region under the curve $y = f(x)$ from $x = a$ to $x = b$ is

$$\boxed{V = \int_a^b 2\pi x \cdot y \cdot dx.}$$

(2) Find the vol^m of the solid obtained by rotating about the y-axis the region between $y = x$ & $y = x^2$

Sol^n: If we put the curve on the plane where ~~let~~ x be the radius of the curve/parabola $y = x^2$, & and x is the radius of sectional part



Now,

Area of the circle is

$A = \pi x^2$

$\therefore$ Volume of the whole curve is

$$V = \int_0^1 \pi x^2 \cdot dy$$

$$V = \int_0^1 \pi (x-1)^2 \cdot dy$$

$$V = \pi \int_0^1 (x^2 - 2x + 1)\, dy$$

$$V = \pi \int_0^1 (y - 2\sqrt{y} + 1)\, dy$$

$$V = \pi \left[\frac{y^2}{2} - \frac{2}{3} \times 2 \,(y)^{3/2} + y \right]_0^1$$

$$V = \pi \left[\frac{1}{2} - \frac{4}{3} + 1 \right]$$

$$V = \pi \left[\frac{3 - 8 + 6}{6} \right] = \pi \left[\frac{1}{6}\right] = \pi/6 \text{ Ans}$$

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Exercise:- 6.2

1. The region enclosed by the x-axis and the parabola $y = f(x) = 3x - x^2$ is revolved about the vertical line $x = -1$ to generate a solid. Find the volume of solid.

Sol^n: Here,

Given,

$y = 3x - x^2$ — (i)

$y = 0$ — (ii) because the region enclosed by the x-axis.

The rotation is made about the vertical line $x = -1$ such that,

$y = 3x - x^2$

$x^2 - 3x = -y$

$x^2 - 2 \cdot x \cdot \frac{3}{2} + \left(\frac{3}{2}\right)^2 = -y + \left(\frac{3}{2}\right)^2$

$\left(x - \frac{3}{2}\right)^2 = -y + \frac{9}{4}$

$h, k = (1.5, -2.25)$

Intersection of the curve along x-axis can be obtained if $y = 0$,

$3x - x^2 = 0$

$x^2 - 3x = 0$

$x(x-3) = 0$

$x = 0, x = 3$

When $x = 0$, then $y = 0$ ∴ $(0, 0)$

When $x = 3$ then $y = 3$ ∴ $(0, 3)$

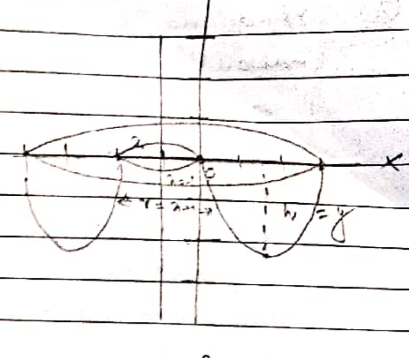
where

height(h) = y   because other y = 0,

radius (r) = x + 1

Now

the volume of the solid is



$V_1 = \int_0^{x=3} 2\pi \cdot h \cdot x \, dx$

$V_1 = \int_0^{(x+1)=3} 2\pi \cdot y \cdot x \, dx$

$V_1 = \int_0^{x+1} 2\pi \cdot (3x - x^2) \cdot x \cdot dx$

$V_1 = 2\pi \int_0^{x+1=3} (3x^2 - x^3) \cdot dx$

$V_1 = 2\pi \left[\frac{3x^3}{3} - \frac{x^4}{4}\right]_0^3$

$V_1 = 2\pi \left[3x^3 - \frac{x^4}{4}\right]_0$

$V_1 = 2\pi \left[27 - \frac{81}{4}\right]$

$V_1 = 2\pi \left[\frac{27 \times 4 - 81}{4}\right]$

$V_1 = 2\pi \left[\frac{27}{4}\right]$

$V_1 = \int_0^3 2\pi \cdot \text{height} \cdot r \cdot dx$

$V_1 = 2\pi \int_0^3 y \cdot (x+1) \cdot dx$

$V_1 = 2\pi \int_0^3 (3x - x^2)(x+1) \cdot dx$

$V_1 = 2\pi \int_0^3 (3x^2 - x^3 + 3x - x^2) \cdot dx$

$V_1 = 2\pi \int_0^3 (2x^2 - x^3 + 3x) \cdot dx$

$V_1 = 2\pi \left[\frac{2x^3}{3} - \frac{x^4}{4} + \frac{3x^2}{2}\right]_0^3$

$V_1 = 2\pi \left[\frac{2 \times 27}{3} - \frac{3^4}{4} + \frac{27}{2}\right] = 2\pi \left[18 - \frac{81}{4} + \frac{27}{2}\right]$

$= 2\pi \left[\frac{18 \times 4 - 81 + 54}{4 \, 2}\right]$

$= 45\pi/2$ Ans.

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(2) Find the volume of the solid obtained by rotating about the y-axis the region betw $y = x$ and $y = x^2$.

Sol^n: Here,

→ Rotation about y-axis –

$y = x$ — (i)

$y = x^2$ — (ii)

the ~~curve~~ parabola ~~pass~~ have centre at the origion (0,0)

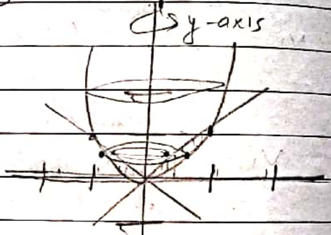
~~the~~ and, From eq^n (i) & (ii)

$x^2 = x$

$x^2 - x = 0$

$x(x-1) = 0$

$x = 0, 1$



when $x = 0$, $y = 0 \Rightarrow (0,0)$

when $x = 1$ $y = 1 \Rightarrow (1,1)$

radius for inner side $(r_1) = x_1 = y$ (For st. line)

radius for outer side $(r_2) = x_2 = \sqrt{y}$

Area of outer side $= \pi r_2^2 = \pi(\sqrt{y})^2 = \pi y$ — (A)

Area of inner side $= \pi r_1^2 = \pi(y)^2 = \pi y^2$ — (B)

Volume of the solid is obtained by.

$V = V_{outer} - V_{inner}$

$$V = \int_0^1 \pi y \, dy - \int_0^1 \pi y^2 . dy$$

$$V = \pi \int_0^1 y . dy - \pi \int_0^1 y^2 . dy$$

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$$V = \pi \left[\frac{y^2}{2} \right]_0^1 - \pi \left[\frac{y^3}{3} \right]_0^1$$

$$V = \pi \times \frac{1}{2} - \pi \times \frac{1}{3}$$

$$V = \frac{\pi}{2} - \frac{\pi}{3}$$

$$V = \frac{3\pi - 2\pi}{6}$$

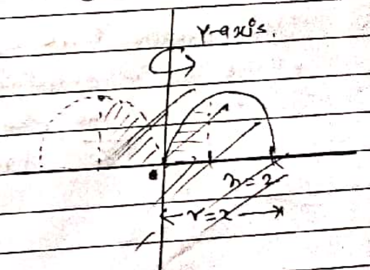
$\boxed{V = \frac{\pi}{6}}$ Ans//

(3) Find the vol^m of the solid obtained by rotating about the y-axis the region bounded by $y = 2x^2 - x^3$ & $y = 0$.

Sol^n: Here,

$y = 2x^2 - x^3$ — (i)

$y = 0$ — (ii)



From eq^n (i) and (ii) we get,

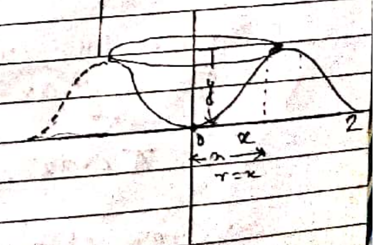
$2x^2 - x^3 = 0$

$x^2(2-x) = 0$

$x = 0, 2,$

radius $(r) = x$

height $(h) = y$.



Volume of the solid.

$$V = \int_0^2 2\pi . y . x . dx$$

$$V = 2\pi \int_0^2 x(2x^2 - x^3) . dx$$

$$V = 2\pi \int_0^2 2x^3 - x^4 . dx$$

$$V = 2\pi \left[\frac{2x^4}{4} - \frac{x^5}{5} \right]_0^2$$

$$V = 2\pi\left[\frac{x^4}{2} - \frac{x^5}{5}\right]_0^2$$

$$V = 2\pi\left[\frac{16}{2} - \frac{32}{5}\right]$$

$$V = 2\pi\left[\frac{80-64}{10}\right]$$

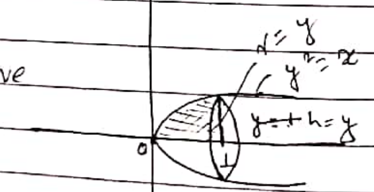
$$V = \pi\left[\frac{16}{5}\right]$$

$$\boxed{V = \frac{16\pi}{5}}$$ Ans

⑤ Use cylindrical shells to find the volume of the solid obtained by rotating about the x-axis the region under the curve $y = \sqrt{x}$ from 0 to 1.

Soln. Here,

$$y^2 = x$$



Area of the curve

$$A = \pi y^2 . dx$$

$$A = \pi x . dx,$$

volume of the solid,

$$V = \int_0^1 \pi x . dx$$

$$V = \pi\left[\frac{x^2}{2}\right]_0^1$$

$$V = \frac{\pi}{2} = \pi/2 \text{ Ans}$$

⑥ Find the volume of the solid obtained by rotating the region bounded by $y = x - x^2$ & $y = 0$ about the line $x = 2$

Soln: Here,

$y = x - x^2$ — (i)

$y = 0$ — (ii) x-axis

$x = 2$

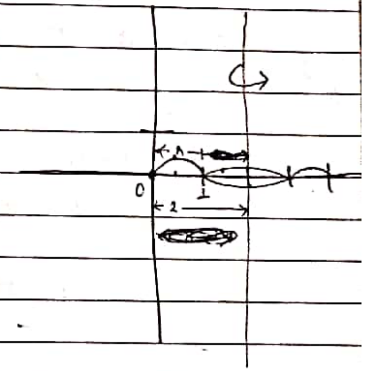
From eqn (i) and (ii) we get,

$$x - x^2 = 0$$

$$x^2 - x = 0$$

$$x(x-1) = 0$$

$$x = 0, 1$$



$$y = x - x^2$$

$$x^2 - x = -y$$

$$x^2 - 2 . x . \frac{1}{2} + (1/2)^2 = -y + (1/2)^2$$

$$(x - 1/2)^2 = -(y - 1/4)$$

$$(h, k) = (0.5, 0.25)$$

volm of the solid is

$$V = \int_0^1 2\pi x . y . \, dx$$

$$V = 2\pi\int_0^1 (2-x) . (x - x^2) . dx$$

$$V = 2\pi\int_0^1 (2x - 2x^2 - x^2 + x^3)\, dx$$

$$V = 2\pi\left[x^2 - \frac{2}{3}x^3 - \frac{x^3}{3} + \frac{x^4}{4}\right]_0^1$$

$$V = 2\pi\left[x^2 - x^3 + \frac{x^4}{4}\right]_0^1$$

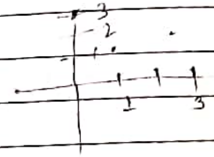
$$V = 2\pi\left[1 - 1 + \tfrac{1}{4}\right]$$

$$V = 2\pi\left[\tfrac{1}{4}\right]$$

$$\boxed{V = \pi/2} \text{ Ans}$$

(7) Use the method of cylindrical shell to find the vol^m of the solid obtained by rotating the region bounded by the given curved about the x-axis

(a) $xy = 1 \quad x = 0,\ y = 1,\ y = 3$



Sol^n: Here.

$xy = 1,\ x = 0$

$$\int_1^3 2\pi \cdot x \cdot y \, dy$$

$$2\pi \int_1^3 \left(\tfrac{1}{y} - 0\right) \cdot y \cdot dy$$

$$2\pi \int_1^3 dy$$

$$2\pi \left[y \right]_1^3$$

$$2\pi [3-1]$$

$$2\pi \times (2) = 4\pi \text{ Ans}$$

(b) $y = x^3,\ y = 8,\ x = 0$

x-axis

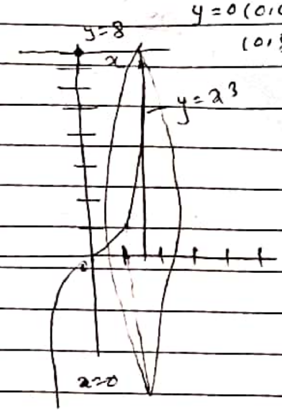
$x = 0,\ 8 = x^3,\ x = 2$

$(0,2)$

$x = 0,\ x = 2$

$y = 0\ (0,0)$

$(0,8)$



Sol^n: Here:

$y = x^3$ — (i)

$y = 8$ — (ii)

Now:

$$V = \int_0^8 2\pi\, x \cdot y\, dy.$$

$$V = 2\pi \int_0^8 2\pi\, x\, y \cdot dy$$

$$V = 2\pi \int_0^8 (y)^{1/3} \cdot y \cdot dy$$

$$V = 2\pi \int_0^8 (y)^{\frac{1}{3}+1} \cdot dy$$

$$V = 2\pi \int_0^8 (y)^{4/3} \cdot dy$$

$$V = 2\pi \left[\frac{y^{4/3+1}}{4/3+1} \right]_0^8$$

$$V = 2\pi \left[\tfrac{3}{7} \cdot y^{7/3} \right]_0^8$$

$$V = \frac{6\pi}{7} (8)^{7/3}$$

$$V = \frac{6\pi \times 128}{7}$$

$$V = \frac{768\pi}{7} \text{ Ans}$$

(c) $x = 1 + (y-2)^2,\ x = 2$

Sol^n: here

$x = 1 + (y-2)^2$

$(x-1) = (y-2)^2$

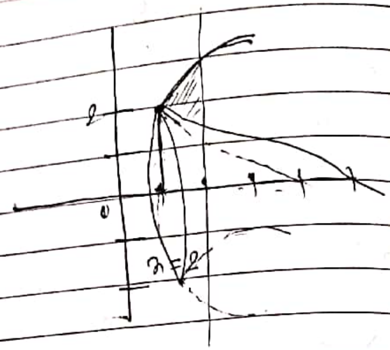
$(y-2)^2 = (x-1)$

$h, k = (1, 2)$

$x = 1 + (0-2)^2$

$x = 1 + 4$

$x = 5$



$L = \int_a^b \sqrt{1 + dy/dx}$

Arc length :-

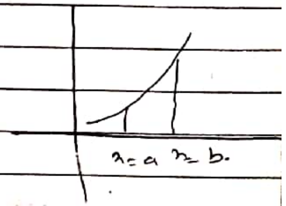
(1) The length of the curve $y = f(x)$ in $[a, b]$ is

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, .dx$$

if $\frac{dy}{dx}$ is not exist in $[a, b]$

then we use

$$L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)}\, .dy$$



Ex. Find the length of curve.

$y = \frac{4\sqrt{2}}{3} x^{3/2} - 1$ for $0 \leq x < 1$

Sol^n: here

$$\frac{dy}{dx} = \frac{4\sqrt{2}}{3} \cdot \frac{3}{2}\, x^{1/2}$$

$$= 2\sqrt{2}\sqrt{x}$$

& $1 + \left(\frac{dy}{dx}\right)^2 = 1 + (2\sqrt{2}\sqrt{x})^2 = 1 + 8x$

$\therefore$ The length of curve $= \int_0^1 \left(\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, .dx\right.$

$$= \int_0^1 \sqrt{1 + 8x}$$

$$L = \left[\frac{(1+8x)^{\frac{1}{2}+1}}{\left(\frac{1}{2}+1\right) \cdot 8}\right]_0^1$$

$$L = \left[(1+8x)^{3/2} \times \frac{2}{3\times 8} \right]_0^1$$

$$L = \left[(1+8)^{3/2} \times \frac{2}{24} \quad (-1)^{3/2} \right]$$

$$L = \frac{\cancel{27\times 2}}{\cancel{24}} \; \frac{26}{12}$$

$$L = \frac{13}{6} \quad \text{Ans}$$

Length in polar form

If $r = f(\theta)$, then

$$L = \int_{\theta=\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \, d\theta$$

6.4

(i) $r = 1 + \cos\theta$ [cardiod]

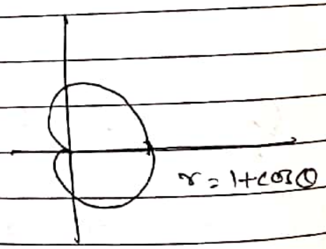
put $r = 0$

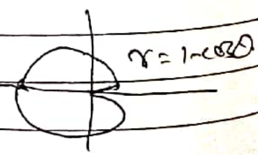
$1 + \cos\theta = 0$

$\cos\theta = -1$

$\theta = \pi$

Limit $\theta = 0, \theta = \pi$.





$r = 1 + \cos\theta$

$\frac{dr}{d\theta} = -\sin\theta$.

and $r^2 + \left(\frac{dr}{d\theta}\right)^2 = (1+\cos\theta)^2 + (-\sin\theta)^2$

$$x^2/_4 - 1/_{x^2} =$$

$1 + 2\cos\theta + \cos^2\theta + \sin^2\theta$

$1 + 2\cos\theta + 1$

$2 + 2\cos\theta$

$2(1+\cos\theta)$

Arc length

$$L = \int_0^{\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \, d\theta$$

$$L = \int_0^{\pi} \sqrt{2(1+\cos\theta)} \cdot d\theta$$

$$L = \int_0^{\pi} \sqrt{2 \cdot 2\cos^2\theta/_2} \cdot d\theta$$

$$L = 2\int_0^{\pi} \cos\theta/_2 \cdot d\theta$$

$$L = 2\left[\frac{\sin\theta/_2}{1/_2}\right]_0^{\pi}$$

$$L = 4\left[\sin 90 - \sin 0\right]$$

$$L = 4[1 - 0]$$

$$L = 4.$$

Total arc length $= 2 \times L = 2 \times 4 = 8$

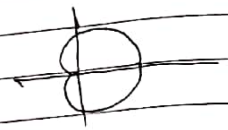
Exercise 6.4

1. Find the length of cardioids $r = 1+\cos\theta$.

Soln: Here

$r = 1+\cos\theta$

put $r = 0$ to determine the interval.



$1+\cos\theta = 0$

$\cos\theta = -1$

$\cos\theta = \cos\pi$

Limit $\theta = 0, \theta = \pi$.

then,

$$L = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2} = \int_0^\pi \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}$$

$$\frac{dr}{d\theta} = \frac{d(1+\cos\theta)}{d\theta} = 0+(-\sin\theta) = -\sin\theta$$

$$\left(\frac{dr}{d\theta}\right)^2 = (-\sin\theta)^2 = \sin^2\theta.$$

$$r^2+\left(\frac{dr}{d\theta}\right)^2 = (1+\cos\theta)^2+\sin^2\theta$$
$$= 1+2\cos\theta+\cos^2\theta+\sin^2\theta$$
$$= 1+2\cos\theta+1$$
$$= 2+2\cos\theta$$
$$= 2(1+\cos\theta).$$

$$L = \int_0^\pi \sqrt{(1+\cos\theta).2}\ .d\theta$$

$$L = \int_0^\pi \sqrt{2(1+\cos\theta)}\ .d\theta$$

$$L = \int_0^\pi \sqrt{2(1+\cos 2\theta/_2)}\ .d\theta$$

$$L = \int_0^\pi \sqrt{2(1+\cos^2\theta/_2-\sin^2\theta/_2)}.d\theta$$

$$L = \int_0^\pi \sqrt{2(1+\cos^2\theta/_2-(1-\cos^2\theta/_2)}\ .d\theta$$

$$L = \int_0^\pi \sqrt{2(\cancel{1}+\cos^2\theta/_2-\cancel{1}+\cos^2\theta/_2)}.d\theta$$

$$L = \int_0^\pi \sqrt{2.2\cos^2\theta/_2}.d\theta$$

$$L = \int_0^\pi 2\cos\theta/_2.d\theta.$$

$$L = \left[\frac{2\sin\theta/_2}{1/_2}\right]_0^\pi$$

$$L = 4\left[\sin\theta/_2\right]_0^\pi$$

$$L = 4[\sin 90-\sin 0]$$

$$L = 4.$$

Now!

The arc length $= 2\times 4$
$= 8$

(2) Find the length of the graph of
$f(x) = \frac{x^3}{12} + \frac{1}{x}$, $1 \le x \le 4$

Sol^n :- Here,
$f(x) = y = \frac{x^3}{12} + \frac{1}{x}$

We know that
Arc length $= \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}$ if $\frac{dy}{dx} \neq \infty$

Then,
$y = \frac{x^3}{12} + \frac{1}{x}$

$\frac{dy}{dx} = \frac{3x^2}{12} - \frac{1}{x^2}$

$\frac{dy}{dx} = \frac{x^2}{4} - \frac{1}{x^2}$

$\frac{dy}{dx} = \frac{x^4 - 4}{4x^2}$

$\left(\frac{dy}{dx}\right)^2 = \left(\frac{x^4-4}{4x^2}\right)^2 = \frac{x^8 - 2.x^4.4 + 16}{\cancel{4}\,16x^4}$

$\left(\frac{dy}{dx}\right)^2 = \frac{x^8 - 8x^4 + 16}{16x^4}$

$1+\left(\frac{dy}{dx}\right)^2 = \frac{1 + x^8 - 8x^4 + 16}{16x^4}$

$1+\left(\frac{dy}{dx}\right)^2 = \frac{16x^4 + x^8 - 8x^4 + 16}{16x^4}$

$1+\left(\frac{dy}{dx}\right)^2 = \frac{x^8 + 8x^4 + 16}{16x^4}$

$1+\left(\frac{dy}{dx}\right)^2 = \frac{(x^4)^2 + 2.x^4.4 + 4^2}{16x^4}$

$1+\left(\frac{dy}{dx}\right)^2 = \frac{(x^4+4)^2}{16x^4}$

$\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{\left(\frac{x^4+4}{16x^4}\right)^2}$

$\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \frac{(x^4+4)}{4x^2}$

Now :-

$\int_1^4 \sqrt{1+\left(\frac{dy}{dx}\right)^2}.dx = \int_1^4 \frac{(x^4+4)}{4x^2}.dx$

$= \int_1^4 \left(\frac{x^2}{4} + \frac{1}{x^2}\right).dx$

$\left[\frac{x^3}{12} + \frac{x^{-2+1}}{-2+1}\right]_1^4$

$\left[\frac{x^3}{12} - \frac{1}{x}\right]_1^4$

$\left(\frac{64}{12} - \frac{1}{4}\right) - \left[\frac{1}{12} - \frac{1}{1}\right]$

$\frac{64}{12} - \frac{1}{4} - \frac{1}{12} + 1 = \frac{64-3-1+12}{12} = \frac{72}{12} = 6$ Ans

(3) Find the length of the curve $y = (x/2)^{2/3}$ from $x=0$ to $x=2$

Sol^n: Here

$$y = \left(\frac{x}{2}\right)^{2/3}$$

Arc length $= \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2} \cdot dx$

then,

We know that,

$$y = \left(\frac{x}{2}\right)^{2/3} = \left(\frac{x}{2}\right)^{2/3}$$

$$\frac{dy}{dx} = \frac{2}{3}\left(\frac{x}{2}\right)^{2/3 - 1}$$

$$\frac{dy}{dx} = \frac{2}{3}\left(\frac{x}{2}\right)^{\frac{2-3}{3}}$$

$$\frac{dy}{dx} = \frac{2}{3}\left(\frac{x}{2}\right)^{-1/3}$$

$$\frac{dy}{dx} = \frac{2}{3}\left(\frac{2}{x}\right)^{1/3}$$

where if putting $x=0$, $\frac{dy}{dx} = \infty$,

So, we have to find the $\frac{dx}{dy}$ in the given function such that

$$L = \int_a^b \sqrt{1+\left(\frac{dx}{dy}\right)^2} \cdot dy$$

$$y = \left(\frac{x}{2}\right)^{2/3}$$

$$y^{3/2} = \left(\frac{x}{2}\right)$$

$$2(y)^{3/2} = x$$

$$\frac{dx}{dy} = 2 \times \frac{3}{2}(y)^{3/2 - 1}$$

$$\frac{dx}{dy} = 3(y)^{1/2}$$

$$\left(\frac{dx}{dy}\right)^2 = \left(3y^{1/2}\right)^2 = 9y$$

$$1+\left(\frac{dx}{dy}\right)^2 = 1+9y.$$

$$\sqrt{1+\left(\frac{dx}{dy}\right)^2} = \sqrt{1+9y}.$$

where when $x=0$

$y = (0/2)^{2/3} = 0$.

when $x=2$

$y = (2/2)^{2/3} = 1$

So limit is 0 to 1.

Now

Arc length $(L) = \int_0^1 \sqrt{1+9y} \cdot dy$

$$\left[\frac{(1+9y)^{1/2+1}}{(1/2+1) \cdot 9}\right]_0^1 = \left[\frac{1}{9} \cdot \frac{2}{3}(1+9y)^{3/2}\right]_0^1$$

$$= \left[\frac{2}{27}(1+9y)^{3/2}\right]_0^1$$

$$\frac{2}{27}\left\{(1+9)^{3/2} - (1)^{1/2}\right\}$$

$$\frac{2}{27}\left\{(10)^{3/2} - 1\right\} = \frac{2}{27}\left\{10\sqrt{10} - 1\right\}$$

$$= 2.27 \text{ Ans}$$

⑤ Find the length of the arc of the semicubical parabola $y^2 = x^3$ between the points $(1,1)$ and $(4,8)$

Sol^n: Here

$y^2 = x^3$

$y = (x)^{3/2}$

We know that

Arc length $= \int_a^b \sqrt{\left(1 + \left(\frac{dy}{dx}\right)^2\right)} \cdot dx$

then,

$\frac{dy}{dx} = \frac{3}{2}(x)^{3/2 - 1} = \frac{3}{2}(x)^{1/2}$

$\left(\frac{dy}{dx}\right)^2 = \left\{\frac{3}{2}(x)^{1/2}\right\}^2 = \frac{9}{4}.(x). = \frac{9x}{4}$

$1 + \left(\frac{dy}{dx}\right)^2 = \frac{1+9x}{4}$

$\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{\frac{1+9x}{4}}$

Now,

limit $x_1 = 1$ & $x_2 = 4$,

~~So~~. Again

$L = \int_1^4 \sqrt{\frac{1+9x}{4}} \cdot dx$

$L = \left[\frac{\left(1+\frac{9x}{4}\right)^{\frac{1}{2}+1}}{\left(\frac{1}{2}+1\right)\cdot\frac{9}{4}}\right]_1^4$

$L = \left[\frac{2}{3} \times \frac{4}{9}\left(1+\frac{9x}{4}\right)^{3/2}\right]_1^4$

$L = \left[\frac{8}{27}\left(1+\frac{9x}{4}\right)^{3/2}\right]_1^4$

$L = \frac{8}{27}\left[\left(1+\frac{36}{4}\right)^{3/2} - \left(1+\frac{9}{4}\right)^{3/2}\right]$

$L = \frac{8}{27}\left[(1+9)^{3/2} - \left(\frac{13}{4}\right)^{3/2}\right]$

$L = \frac{8}{27}\left[(10)^{3/2} - \left(\frac{13}{4}\right)^{3/2}\right]$

~~$L = \frac{8}{27}\left[\left(10 - \frac{13}{4}\right)^{3/2}\right]$~~

~~$L = \frac{8}{27}\left[\left(\frac{40-13}{4}\right)^{3/2}\right]$~~

~~$L = \frac{8}{27}\left(\frac{27}{4}\right)^{3/2}$~~

~~$L =$~~

$L = \frac{8}{27}\left[10\sqrt{10} - \frac{13}{8}\sqrt{13}\right]$

$L = \frac{1}{27}\left[80\sqrt{10} - 13\sqrt{13}\right)$

Ans

(A) Find the arc length function for the curve in Example 2 taking $A = (1, 13/12)$ as the starting point

Q. Mistake

Area Of Surface of Revolution

If the function f has continuous and differentiable then we define the surface area obtained by rotating the curve $y = f(x)$ $a < x < b$

about X-axis is

$$S = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \cdot dx$$

and about y-axis is

$$S = \int_c^d 2\pi x \left(\sqrt{1 + \left(\frac{dx}{dy}\right)^2}\right) \cdot dy.$$

Exercise 6.5

(0) $y = x^3, \; 0 < x \leq 2$

Exercise 6.5

1. Find the exact area of the surface obtained by rotating the curve about the x-axis.

(a) $y = x^3$, $0 \le x \le 2$

Sol^n: Here

We know that,

The surface area about x-axis is

$$S = \int_0^b \cancel{2\pi y} \, \cancel{d}$$

$$S = \int_0^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \cdot dx$$

$$S = \int_0^b 2\pi x^3 \left(\sqrt{1 + \left(\frac{dy}{dx}\right)^2} \cdot dx\right.$$

$$\frac{dy}{dx} = 3x^2$$

$$\left(\frac{dy}{dx}\right)^2 = 9x^4$$

$$\left[1 + \left(\frac{dy}{dx}\right)^2\right] = (1 + 9x^4)$$

Now

$$S = \int_0^2 2\pi x^3 \sqrt{(1 + 9x^4)} \cdot dx$$

put

$t = (1 + 9x^4)$

$$\frac{dt}{dx} = 0 + 36x^3 = 36x^3$$

$$dt = 36x^3 . dx$$

$$dx = \frac{1}{36x^3} . dt \qquad \frac{1}{36} dt = x^3 . dx$$

$$S = \int_0^2 2\pi (x^3 . dx) \sqrt{(1 + 9x^4)}$$

when $x = 0$ then $t = 1$

when $x = 2$ then $t = 145$

$$S = \int_1^{145} 2\pi \times \frac{1}{36} . dt \sqrt{t}$$

$$S = \frac{1}{18}\pi \int_1^{145} (t)^{1/2} . dt$$

$$S = \frac{\pi}{18} \left[\frac{2}{3} t^{3/2}\right]_1^{145}$$

$$S = \frac{\pi}{27} \left[(145)^{3/2} - (1)^{3/2}\right]$$

$$S = \frac{\pi}{27} \left[(145^3)^{1/2} - 1\right]$$

$$S = \frac{\pi}{27} \left[145\sqrt{145} - 1\right]$$ Ans

(b) $y = \sqrt{1+4x}, \quad 1 < x < 5$

Sol^n: Here

$y = \sqrt{1+4x} \quad 1 < x < 5$

We know that

The surface ~~area~~ about x-axis is

$$S = \int_a^b 2\pi y \sqrt{1+\left(\frac{dy}{dx}\right)^2} \cdot dx$$

$$S = \int_1^5 2\pi (1+4x)^{1/2} \sqrt{1+\left(\frac{dy}{dx}\right)^2} \cdot dx$$

Now,

$$y = (1+4x)^{1/2}$$

$$\frac{dy}{dx} = 4 \cdot \frac{1}{2} (1+4x)^{1/2-1} = \frac{2}{\cancel{2}} \cdot (1+4x)^{-1/2}$$

$$\frac{dy}{dx} = \frac{2}{\sqrt{1+4x}}$$

$$\left(\frac{dy}{dx}\right)^2 = \frac{2}{(1+4x)}$$

$$1+\left(\frac{dy}{dx}\right)^2 = 1+\frac{2}{(1+4x)}$$

$$= \frac{(1+4x)+2}{4}$$

$$= \frac{5+16x}{4}$$

Now,

$$S = \int_1^5 2\pi (1+4x)^{1/2} \cdot \sqrt{\frac{5+16x}{4}} \cdot dx$$

$$S = 2\pi \int_1^5 \frac{1}{2} \sqrt{(1+4x) \times (5+16x)}$$

$$S = \pi \int_1^5 \sqrt{5+16x+20x+64x^2} \cdot dx$$

$$S = \pi \int_1^5 \sqrt{64x^2+36x+5} \cdot dx$$

$$S = \pi \int_1^5 \sqrt{(8x)^2 + 2 \cdot 8x \times \frac{9}{4} + (9/4)^2 - (9/4)^2 + 5}$$

$$S = \pi \int_1^5 \sqrt{(8x+9/4)^2 - \frac{81}{16} + 5}$$

$$S = \pi \int_1^5 \sqrt{(8x+9/4)^2 - 1^2}$$

$$1+\left(\frac{dy}{dx}\right)^2 = 1+\frac{2}{(1+4x)} = \frac{1+4x+2}{(1+4x)}$$

$$1+\left(\frac{dy}{dx}\right)^2 = \frac{3+4x}{1+4x}$$

$$S = \int_1^5 2\pi (1+4x)^{1/2} \sqrt{\frac{3+4x}{1+4x}}$$

$$S = 2\pi \int_1^5 \sqrt{\cancel{(1+4x)} \frac{(3+4x)}{\cancel{(1+4x)}}} \cdot dx$$

$$S = 2\pi \left[\frac{(3+4x)^2}{2\times 4}\right]_1^5$$

$$S = \pi/4 \left[(3+20)^2 - (3+4)^2\right]$$

$$S = \pi/4 \left[(23)^2 - 49\right]$$

$$S = \frac{\pi}{4} \times 480$$

(b) $y = \sqrt{1+4x}$, $1 \le x \le 5$

Soln. Here

$y = \sqrt{1+4x}$ $1 \le x \le 5$

We know that,

The surface area about x-axis is

$$S = \int_a^b 2\pi y \sqrt{1+\left(\frac{dy}{dx}\right)^2} \, dx$$

$$S = \int_1^5 2\pi \sqrt{1+4x} \sqrt{1+\left(\frac{dy}{dx}\right)^2} \, dx$$

$$y = (1+4x)^{1/2}$$

$$\frac{dy}{dx} = \frac{1}{2}(1+4x)^{-1/2} \cdot 4$$

$$= 2(1+4x)^{-1/2}$$

$$\frac{dy}{dx} = \frac{2}{\sqrt{1+4x}}$$

$$\left(\frac{dy}{dx}\right)^2 = \frac{4}{(1+4x)}$$

$$1+\left(\frac{dy}{dx}\right)^2 = 1 + \frac{4}{(1+4x)} = \frac{1+4x+4}{(1+4x)}$$

$$= \frac{5+4x}{1+4x}$$

then,

$$S = \int_1^5 2\pi \cdot \sqrt{1+4x} \cdot \sqrt{\frac{5+4x}{\cancel{1+4x}}} \, dx$$

$$S = 2\pi \int_1^5 \sqrt{\frac{(5+4x)\cancel{(1+4x)}}{\cancel{(1+4x)}}} \, dx$$

$$S = \cancel{2}\pi \left[\frac{\cancel{2}}{3} \frac{(5+4x)^{3/2}}{\times \cancel{4}} \right]_1^5$$

$$S = \pi/_3 \left[(5+20)^{3/2} - (5+4)^{3/2} \right]$$

$$S = \pi/_3 \left[(25)^{3/2} - (9)^{3/2} \right]$$

$$S = \pi/_3 \left[(5)^3 - (3)^3 \right]$$

$$S = \pi/_3 \left[125 - 27 \right]$$

$$S = \frac{98\pi}{3}$$ Ans

$y = \sin \pi x \quad 0 \le x \le 1$

Sol^n: Here

$y = \sin \pi x$

We know that the surface area about X-axis is

$$S = \int_0^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \cdot dx$$

$$S = \int_0^1 2\pi \sin \pi x \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \cdot dx$$

then,

$$\frac{dy}{dx} = \pi . \cos \pi x$$

$$\left(\frac{dy}{dx}\right)^2 = \pi^2 \cos^2 \pi x$$

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \pi^2 \cos^2 \pi x$$

Now

$$S = \int_0^1 2\pi \sin \pi x \sqrt{1 + \pi^2 \cos^2 \pi x} \cdot dx$$

Put $t = \pi \cos \pi x$	when $x = 0$
$\frac{dt}{dx} = -\pi^2 \sin \pi x$	$t = \pi \cos 0 = 1 \times \pi = \pi$
$dt = -\pi^2 \sin \pi x$	when $x = 1$
$-dt = \pi^2 \sin \pi x$	$t = \pi \cos \pi = -1 \times \pi = -\pi$

$\frac{-1}{\pi} dt \quad \pi \sin \pi x$

$$S = \int_{-\pi}^{\pi} 2(\pi \sin \pi x) . dx \sqrt{1 + \pi^2 \cos^2 \pi x}$$

$$S = \int_{-\pi}^{\pi} 2\sqrt{1 + t^2} . \frac{-1}{\pi} . dt$$

$$S = \frac{-2}{\pi} \int_{-\pi}^{\pi} \sqrt{1 + t^2} . dt$$

Using formula, of $\sqrt{a^2 + x^2} = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2} \log(x + \sqrt{x^2 + a^2})$

$$S = -\frac{2}{\pi}\left[\frac{t}{2}\sqrt{t^2+1} + \frac{1}{2}\log\left(t + \sqrt{t^2+1}\right)\right]_{-\pi}^{\pi}$$

$$S = -\frac{1}{\pi}\left[t\sqrt{t^2+1} + \log\left(t + \sqrt{t^2+1}\right)\right]_{-\pi}^{\pi}$$

$$S = -\frac{1}{\pi}\left[\pi\sqrt{\pi^2+1} + \log(\pi + \sqrt{\pi^2+1}) - (-\pi\sqrt{\pi^2+1} + \log(-\pi + \sqrt{\pi^2+1}\right]$$

$$S = -\frac{1}{\pi}\left[\pi\sqrt{\pi^2+1} + \log(\pi + \sqrt{\pi^2+1} + \pi\sqrt{\pi^2+1} - \log(-\pi + \sqrt{\pi^2+1})\right]$$

$$S = \left[-\sqrt{\pi^2+1} - \frac{1}{\pi}\log(\pi + \sqrt{\pi^2+1}) - \sqrt{\pi^2+1} + \frac{1}{\pi}\log(-\pi + \sqrt{\pi^2+1})\right]$$

$$S = -2\sqrt{\pi^2+1} - \frac{1}{\pi}\log(\pi + \sqrt{\pi^2+1}) + \frac{1}{\pi}\log(-\pi + \sqrt{\pi^2+1})$$

$$S = -2\sqrt{\pi^2+1} - \left(\frac{1}{\pi}\log(\pi + \sqrt{\pi^2+1} - \frac{1}{\pi}\log(-\pi + \sqrt{\pi^2+1})\right)$$

$$S = -2\sqrt{\pi^2+1} - \frac{1}{\pi}\left[\log\left(\frac{\pi + \sqrt{\pi^2+1}}{-\pi + \sqrt{\pi^2+1}}\right)\right]$$

$$S = -2\sqrt{\pi^2+1} - \frac{1}{\pi}\left[\log\left(\frac{\pi - \sqrt{\pi^2+1}}{\pi + \sqrt{\pi^2+1}}\right)\right]$$

$$\therefore S = -2\sqrt{\pi^2+1} - \frac{1}{\pi}\left[\log\left(\pi^2\right.\right.$$

② The following curve is rotated about the y-axis. Find the area of the resulting surface

(b) $x = \sqrt{a^2-y^2},\ 0 \le y \le \frac{a}{2}$

Soln: Here

$x = \sqrt{a^2-y^2},\ 0 \le y \le \frac{a}{2}$

Surface Area of the curve about y-axis is

$$S = \int_a^b 2\pi x \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy$$

Then,

We know that,

$$\frac{dx}{dy} = \frac{1}{2\sqrt{a^2-y^2}} \times -2y = \frac{-y}{\sqrt{a^2-y^2}}$$

$$\left(\frac{dx}{dy}\right)^2 = \frac{y^2}{a^2-y^2}$$

$$1+\left(\frac{dx}{dy}\right)^2 = 1+\frac{y^2}{a^2-y^2} = \frac{a^2-y^2+y^2}{a^2-y^2}$$

$$1+\left(\frac{dx}{dy}\right)^2 = \frac{a^2}{a^2-y^2}$$

$$S = \int_0^{a/2} 2\pi \cancel{\sqrt{a^2-y^2}} \times \sqrt{\frac{a^2}{\cancel{a^2-y^2}}}\, dy$$

$$S = 2\pi \int_0^{a/2} a \cdot dy = 2\pi \left[ay\right]_0^{a/2}$$

$$S = 2\pi \times \frac{a^2}{2} = \pi a^2 \text{ Ans}$$

(a) $y = \sqrt[3]{x},\ 1 \le y \le 2$

Soln: Here

$y^3 = x$

$x = y^3$

The surface area of the curve about y-axis is,

$$S = \int_a^b 2\pi x \cdot \sqrt{1+\left(\frac{dx}{dy}\right)^2}\cdot dy$$

We know that,

$$\frac{dx}{dy} = 3y^2$$

$$\left(\frac{dx}{dy}\right)^2 = (3y^2)^2 = 9y^4$$

$$1+\left(\frac{dx}{dy}\right)^2 = 1+9y^4$$

Now

$$S = \int_1^2 2\pi y^3 \sqrt{1+9y^4}\cdot dy$$

Put $t = 1+9y^4$ when $y=1,\ t=10$

$\frac{dt}{dy} = 0+36y^3$ when $y=2$ $t=145$

$\frac{dt}{dy} = 36y^3,\ \frac{1}{36}dt = y^3\, dy$

Again

$$S = \int_{10}^{145} 2\pi(y^3\cdot dy)\sqrt{1+9y^4} = 2\pi\int_{10}^{145} \frac{1}{36}\, dt\, \sqrt{t}$$

$$S = \frac{2\pi}{36\cdot 18}\left[\frac{2}{3} t^{3/2}\right]_{10}^{145}$$

$$= \frac{2\pi}{54}\left[t^{3/2}\right]_{10}^{145} = \frac{\pi}{}\left[145\sqrt{145} - 10\sqrt{\ }\right]$$

Exercise :- 6.3.

Approximate Integration.

1. Mid-point Rule:-

$$\int_a^b f(x)\,dx = \Delta x\left[f(\bar{x}_1) + f(\bar{x}_2) + \cdots + f(\bar{x}_n)\right]$$

where $\Delta x = \frac{b-a}{n}$ and $\bar{x}_i = \frac{x_{i-1}+x_i}{2}$ = Mid-pt.

2. ~~Trap~~ Trapezoidal Rule:

$$\int_a^b f(x).dx. = \frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots\right.$$

where $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\Delta x$

$x_n = b,\ x_0 = a$.

3. Simpson's Rule

$$\int_a^b f(x)\,dx = \frac{\Delta x}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\right].$$

where n is even & $\Delta x = \left(\frac{b-a}{n}\right)$

Exercise :- 6.3

Use (a) the Trapezoidal Rule (b) The mid-point Rule and (c) Simpson's Rule to approximate the given integral with the specified value of n (Round your answer to six decimal places

(a) $\int_1^2 \sqrt{(x^3-1)}\,dx,\ n=10$

Soln: here

Mid-point rule.

$\Delta x = \frac{b-a}{n} = \frac{2-1}{10} = 0.1$

$\int_1^2 \sqrt{(x^3-1)}.dx = \Delta x\left[f(x_1)+f(x_2)+f(x_3)+f(x_4) + f(x_5)+f(x_6)+f(x_7)+f(x_8)+f(x_9)+f(x_{10})\right.$

$= 0.1\left[f(1.1)+f(1.2)+f(1.3)+f(1.4)+f(1.5)+f(1.6) + f(1.7)+f(1.8)+f(1.9)+f(2.0)\right]$

$=(0.1)\left[0.1536897 + 0.314534 + 0.482228 + 0.656502 + 0.837117 + 1.023858 + 1.216529 + 1.414953 + 1.618969 + 1.828427\right]$

$= 0.1 \times 9.546807$

$0.1\,(0.575326 + 0.853229 + 1.0940 75 + 1.320606 + 1.541104 + 1.759545 + 1.978130 + 2.198181 + 2.420537 + 2.645751$

$(0.1) \times 16.386484$

1.6386484

(a) $\int_1^2 \sqrt{x^3-1}\,.dx,\quad n=10,$

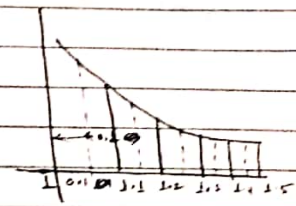
Soln: Here

Mid-point Rule,

$a=1$

$b=2$

$\Delta x = \frac{b-a}{n} = \frac{2-1}{10} = 0.1$



$x_1 = \frac{1+1.1}{2} = 1.05$

$x_2 = \frac{1.1+1.2}{2} = \frac{2.3}{2} = 1.15$

$x_3 = \frac{1.2+1.3}{2} = 1.25$

$x_4 = 1.35$

$x_5 = 1.45$

$x_6 = 1.55$

$x_7 = 1.65$

$x_8 = 1.75$

$x_9 = 1.85$

$x_{10} = 1.95$

$\int_1^2 \sqrt{x^3-1}\,.dx = \Delta x\left[f(1.05)+f(1.15)+f(1.25)+f(1.35) + f(1.45)+f(1.55)+f(1.65)+f(1.75) + f(1.85)+f(1.95)\right)$

$=\Delta x\left[0.397020 + 0.721717 + 0.976281 + 1.208459 + 1.431302 + 1.650417 + 1.868723 + 2.087912 + 2.309031 + 2.532760\right.$

0.1×15.1836228

1.51836228

By Trapezoidal Rule :-

$$\int_1^2 \sqrt{x^3-1}\,dx = \frac{\Delta x}{2}\left[f(x_0)+2f(x_1)+2f(x_2)\ldots+f(x_n)\right]$$

$$\Delta x = \frac{b-a}{2} = \frac{2-1}{2} = 0.1$$

$$\int_1^2 \sqrt{x^3-1}\,dx = \frac{\Delta x}{2}\left[f(1)+2f(1.1)+2f(1.2)+2f(1.3)+2f(1.4)+2\cdot f(1.5)+2f(1.6)+2f(1.7)+2f(1.8)+2f(1.9)+f(2.0)\right]$$

$$\frac{\Delta x}{2}\left[0+1.150052+1.706458+2.188149+2.641212+3.082207+3.519091+3.956209+4.396362+4.841074+2.6457\ldots\right]$$

$$\left(\frac{\Delta x}{2}\right)\times(30.1271659)$$

1.50636 Ans

By Simpson's Rule.

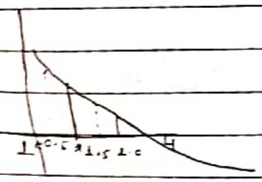
$$\int_1^2 \sqrt{x^3-1}\,dx = \frac{\Delta x}{3}\left[f(x_0)+4f(x_1)+2f(x_2)+\ldots f(x_n)\right]$$

$\Delta x = 0.1$

$$\frac{\Delta x}{3}\left[f(1)+4f(1.1)+2f(1.2)+4f(1.3)+2f(1.4)+4f(1.5)+2f(1.6)+4f(1.7)+2f(1.8)+4f(1.9)+f(2.0)\right]$$

$$\frac{0.1}{3}\left[0+2.301304+1.706458+4.376210+2.641212+6.164414+3.519091+7.912522+4.396362+9.682149+2.645751\right]$$

$$\frac{0.1}{3}\times 45.345473 = 1.511516 \text{ Ans}$$



(c) $\int_1^4 \sqrt{\ln x}\,dx,\ n=6$

Sol^n: Here,

from Mid-point Rule.

$$\Delta x = \frac{b-a}{n} = \frac{4-1}{6} = \frac{3}{6} = \frac{1}{2} = 0.5$$

$$x_1 = \frac{(1+1.5)}{2} = 1.25$$

$$x_2 = \frac{(1.5+2.0)}{2} = 1.75$$

$$x_3 = \cancel{2.0+1.75}\ \frac{2.0+2.5}{2} \quad 2.25$$

$x_4 = 2.75$

$x_5 = 3.25$

$x_6 = 3.75$

then,

$$\int_1^4 \sqrt{\ln x} = \Delta x\left[f(x_1)+f(x_2)+f(x_3)+f(x_4)+f(x_5)+f(x_6)\right]$$

$$= \Delta x\left[f(1.25)+f(1.75)+f(2.25)+f(2.75)+f(3.25)+f(3.75)\right]$$

$$= \Delta x\left[0.472381+0.748075+0.90052+1.005784+1.085659+1.149676\right)$$

$$= 0.5\times 5.362095$$

$$2.6810475$$

By

$$\frac{\Delta x}{2}\left[f(x_0)+2f(x_1)+2f(x_2)+2f(x_3)+2f(x_4)+2f(x_5)+\ldots f(x_n)\right]$$

$$\frac{\Delta x}{2}\left[f(1)+2f(1.5)+2f(2.0)+2f(2.5)+2f(3.0)+2f(3.5)+f(4.0)\right]$$

$$\frac{0.5}{2}\left[0+1.277440+1.2735228+1.665109+1.914462+2.096294+0.238538\right]$$

$$0.25\times 10.365338 = 2.6133345 \text{ Ans}$$

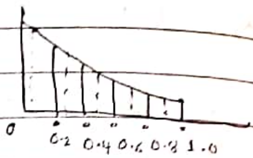
(b) $\int_0^2 \frac{e^x}{1+x^2}\,dx$, $n=10$,

Sol^n here,

By the formula of the Mid-pt Rule.

$$\int_0^2 \frac{e^x}{1+x^2}\,dx = \Delta x\left[f(x_1)+f(x_2)+f(x_3)+\dots f(x_n)\right]$$

$$\int_0^2 \frac{e^x}{1+x^2}\,dx =$$



where

$$\Delta x = \frac{b-a}{n} = \frac{2-0}{10} = \frac{2}{10} = 0.2$$

$$x_1 = \frac{0+0.2}{2} = \frac{0.2}{2} = 0.1$$

$$x_2 = \frac{0.2+0.4}{2} = \frac{0.6}{2} = 0.3$$

$$x_3 = \frac{0.4+0.6}{2} = \frac{1.0}{2} = 0.5$$

$x_4 = 0.7$

$x_5 = 0.9$

$x_6 = 1.1$

$x_7 = 1.3$

$x_8 = 1.5$

$x_9 = 1.7$

$x_{10} = 1.9$

$$\int_0^2 \frac{e^x}{1+x^2}\,dx = \Delta x\left[f(0.1)+f(0.3)+f(0.5)+f(0.7)+f(0.9)+f(1.1)+f(1.3)+f(1.5)+f(1.7)+f(1.9)\right]$$

$$= 0.2\left[1.094229+1.238403+1.318977+1.351512+1.358897+1.359351+1.3640508+1.378981+1.407184+1.450302\right]$$

$$= 0.2\times(13.3218868)$$

$$= 2.66437736$$

By Simpson's Rule,

$$\int_0^2 \frac{e^x}{1+x^2}\,dx = \frac{\Delta x}{2}\left[f(x_0)+2f(x_1)+2f(x_2)+2f(x_3)+2f(x_4)+2f(x_5)+2f(x_6)+2f(x_7)+2f(x_8)+f(x_9)\right]$$

$$= \frac{0.2}{2}\left[f(0)+2f(0.2)+2f(0.4)+2f(0.6)+2f(0.8)+2f(1.0)+2f(1.2)+2f(1.4)+2f(1.6)+2f(1.8)+f(2.0)\right].$$

$$= 0.1\left[1+\cancel{3.482806}+2.348851+2.572112+2.679586+2.714074+2.718282+2.721407+2.739999+2.782602+2.853607+2.955622\right]$$

$$= 0.1\times(28.0861429)$$

2.8086142 Ans