2. L'Hospital 法则.

Th3.5 ($\frac{0}{0}$型) 设 f 在 $(x_0, x_0+\delta)$, $(\delta>0)$ 内满足:

① $\lim\limits_{x\to x_0^+} f(x) = \lim\limits_{x\to x_0^+} g(x) = 0$

② f, g 在 $(x_0, x_0+\delta)$ 内可导, $g'(x)\neq 0$

③ $\lim\limits_{x\to x_0^+} \frac{f'(x)}{g'(x)} = A$. $(|A|\leq+\infty)$

$\Rightarrow \lim\limits_{x\to x_0^+} \frac{f(x)}{g(x)} = \lim\limits_{x\to x_0^+} \frac{f'(x)}{g'(x)} = A.$

Pf. ∵ 极限 $\lim\limits_{x\to x_0^+} \frac{f(x)}{g(x)}$ 的值与 $f(x), g(x)$ 在 x_0 点处的取值无关.

不妨设 $f(x_0)=g(x_0)=0$.

∴ 由 Cauchy 中值定理, $\forall x\in(x_0, x_0+\delta)$ $\exists \xi\in(x_0, x)$

s.t. $\frac{f(x)}{g(x)} = \frac{f(x)-f(x_0)}{g(x)-g(x_0)} = \frac{f'(\xi)}{g'(\xi)}$

∴ $\lim\limits_{x\to x_0^+} \frac{f(x)}{g(x)} = \lim\limits_{x\to x_0^+} \frac{f'(\xi)}{g'(\xi)} = \lim\limits_{\xi\to x_0^+} \frac{f'(\xi)}{g'(\xi)} = A.$ #

同样对 $x\to x_0$, $x\to x_0^-$, $x\to\infty$, $x\to\pm\infty$ 也成立.

Th3.6 ($\frac{\infty}{\infty}$型) 设 f 在 $(x_0, x_0+\delta)$, $(\delta>0)$ 内满足:

① $\lim\limits_{x\to x_0^+} f(x) = \lim\limits_{x\to x_0^+} g(x) = \infty$.

② f, g 在 $(x_0, x_0+\delta)$ 内可导, $g'(x)\neq 0$.

③ $\lim\limits_{x\to x_0^+}\dfrac{f'(x)}{g'(x)}=A.\quad (|A|\le+\infty)$

$\Rightarrow \lim\limits_{x\to x_0^+}\dfrac{f(x)}{g(x)}=\lim\limits_{x\to x_0^+}\dfrac{f'(x)}{g'(x)}=A.$

结论同样对 $x\to x_0^-$, $x\to x_0$, $x\to\infty$. $x\to\pm\infty$ 成立.

pf. 1. $|A|<+\infty$ 时.

$\because \lim\limits_{x\to x_0^+}\dfrac{f'(x)}{g'(x)}=A.$

$\therefore \forall\varepsilon>0$. $\exists\delta_1>0$, $\delta_1<\delta$ s.t. 当 $x_0<x<x_0+\delta_1$ 时,

$\left|\dfrac{f'(x)}{g'(x)}-A\right|<\varepsilon/4.$

取 $\eta=x_0+\frac{\delta_1}{2}$, 于是 $\forall x\in(x_0,\eta)$, $\exists\xi\in(x,\eta)$ s.t.

$$\frac{f(x)-f(\eta)}{g(x)-g(\eta)}=\frac{f'(\xi)}{g'(\xi)}$$

$$\therefore \left|\frac{f(x)}{g(x)}-A\right|=\left|\left(\frac{f(x)-f(\eta)}{g(x)-g(\eta)}-A\right)\frac{g(x)-g(\eta)}{g(x)}+\frac{f(\eta)}{g(x)}-\frac{f(\eta)}{g(x)}A\right|$$

$$\le\left|\frac{f(x)-f(\eta)}{g(x)-g(\eta)}-A\right|\left|\frac{g(x)-g(\eta)}{g(x)}\right|+\left|\frac{f(\eta)}{g(x)}\right||1-A|$$

$$\le\left|\frac{f'(\xi)}{g'(\xi)}-A\right|\left(1+\left|\frac{g(\eta)}{g(x)}\right|\right)+\left|\frac{f(\eta)}{g(x)}\right||1-A|$$

$$<\frac{\varepsilon}{4}\left(1+\left|\frac{g(\eta)}{g(x)}\right|\right)+\left|\frac{f(\eta)}{g(x)}\right||1-A|$$

$\because \lim\limits_{x\to x_0^+}g(x)=\infty.$

$\therefore$ 对 $G=\max\left\{|g(\eta)|+1,\ \dfrac{2|f(\eta)||1-A|}{\varepsilon}+1\right\}$,

$\exists\, 0<\delta_2<\delta/2$，当 $x\in(x_0, x_0+\delta_2)$ 时，$\subset(x_0, x_0+\eta)$. $|g(x)|>G$.

$$\therefore \left|\frac{f(x)}{g(x)}-A\right| < \frac{\varepsilon}{4}\left(1+\frac{|g(\eta)|}{G}\right)+\frac{|f(\eta)|}{G}|1-A|$$

$$\le \frac{\varepsilon}{4}\left(1+\frac{|g(\eta)|}{|g(\eta)|+1}\right)+\frac{|f(\eta)||1-A|}{\frac{2|f(\eta)||1-A|}{\varepsilon}+1}$$

$$< \frac{\varepsilon}{4}\cdot 2+\frac{\varepsilon}{2}=\varepsilon.$$

$$\therefore \lim_{x\to x_0^+}\frac{f(x)}{g(x)}=A.$$

2. $|A|=+\infty$ 时. i.e. $\lim\limits_{x\to x_0^+}\dfrac{f'(x)}{g'(x)}=\infty$.

则 $\lim\limits_{x\to x_0^+}\dfrac{g'(x)}{f'(x)}=0$

$\because g'(x)\neq 0$. $\therefore f'(x)\neq 0$.

则由1. $\lim\limits_{x\to x_0^+}\dfrac{g(x)}{f(x)}=\lim\limits_{x\to x_0^+}\dfrac{g'(x)}{f'(x)}=0$

$\therefore \lim\limits_{x\to x_0^+}\dfrac{f(x)}{g(x)}=\infty$. #

例3.7. (1) $\lim\limits_{x\to\infty}\dfrac{x+\sin x}{x}=\lim\limits_{x\to\infty}\dfrac{1+\cos x}{1}$ 极限不存在？

$=\lim\limits_{x\to\infty}1+\dfrac{\sin x}{x}=1.$

(2) $\lim\limits_{x\to+\infty}\dfrac{e^x-e^{-x}}{e^x+e^{-x}}=\lim\limits_{x\to+\infty}\dfrac{e^x+e^{-x}}{e^x-e^{-x}}=\lim\limits_{x\to+\infty}\dfrac{e^x-e^{-x}}{e^x+e^{-x}}$

L'Hospital 不是万能的！