偏导数与微分

钟思佳

东南大学数学系

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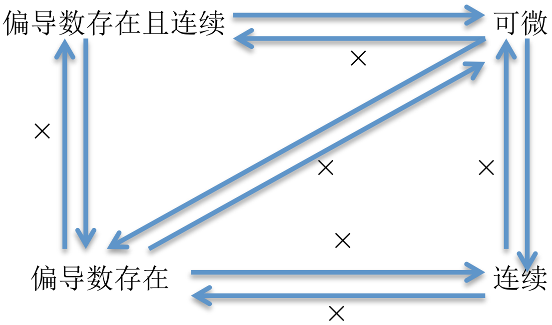
Outline

例8. $f(x,y)=\begin{cases}(x^2+y^2)\sin\dfrac{1}{x^2+y^2} & x^2+y^2\neq 0\\ 0 & x^2+y^2=0\end{cases}$ 问:
在$(0,0)$ 处，$f(x,y)$ 偏导数存在? 偏导数连续? 是否可微?


偏导数存在且连续
可微
偏导数存在
连续

方向导数

Definition

$z=f(x,y)$在 $M_0(x_0,y_0)$ 某邻域内有定义，向量$\vec{l}$ 的方向余弦为 $\{\cos\alpha,\cos\beta\}$, 若 $\lim\limits_{t\to 0}\dfrac{f(x_0+t\cos\alpha,y_0+t\cos\beta)-f(x_0,y_0)}{t}$ 存在，称此极限为$z=f(x,y)$ 在M_0 处沿$\vec{l}$ 的方向导数，记作$\dfrac{\partial z}{\partial l}|_{M_0}$

特别地，

if $\vec{l}=\{1,0\}$, 则 $\dfrac{\partial z}{\partial l}|_{M_0}=\dfrac{\partial z}{\partial x}|_{M_0}$

if $\vec{l}=\{0,1\}$, 则 $\dfrac{\partial z}{\partial l}|_{M_0}=\dfrac{\partial z}{\partial y}|_{M_0}$

Theorem (4)

If $z = f(x, y)$ 在$M_0(x_0, y_0)$ 可微，$\Rightarrow f(x, y)$ 在M_0 处沿任意方向$\vec{l}$ 的方向导数都存在，且

$$\frac{\partial z}{\partial l}\Big|_{M_0} = f_x(x_0, y_0)\cos\alpha + f_y(x_0, y_0)\cos\beta = gradf(M_0)\cdot\frac{\vec{l}}{|\vec{l}|},$$

其中$\{\cos\alpha, \cos\beta\} = \dfrac{\vec{l}}{|\vec{l}|}$.

$$gradf \cdot \frac{\vec{l}}{|\vec{l}|} = |\nabla f|\left|\frac{\vec{l}}{|\vec{l}|}\right| \cos\theta = |\nabla f| \cos\theta,\ \theta\text{为}gradf\text{与}\vec{l}\text{的夹角}$$

- $\theta = 0$ 时，即$gradf(M_0)$ 与$\vec{l}$ 同方向，因此z在M_0的方向导数取最大，且最大值为$|gradf(M_0)|$
- $\theta = \frac{\pi}{2}$, $gradf(M_0)$ 与$\vec{l}$ 垂直，方向导数=0
- $\theta = \pi$, $gradf(M_0)$ 与$\vec{l}$ 反方向，z在M_0的方向导数取最小，且最小值为$-|gradf(M_0)|$

例9. 求$z = x^2 - xy + y^2$ 在点$(-1, 1)$ 沿$\vec{l} = \{2, 1\}$ 的方向导数，并指出z 在该点沿那个方向的方向导数最大？最大导数是？z沿哪个方向减小最快？

多元函数的微分运算法则

Theorem (5)

设$u=\varphi(x)$，$v=\psi(x)$在x处可导。$z=f(u,v)$在(u,v)处可微。则复合函数$z=f(\varphi(x),\psi(x))$在x处可导，且

$$\frac{dz}{dx}=\frac{\partial z}{\partial u}\frac{du}{dx}+\frac{\partial z}{\partial v}\frac{dv}{dx}$$

可以推广到更多变量$z=f(u_1,u_2,\ldots,u_n)$, $\frac{dz}{dx}=\sum_{i=1}^{n}\frac{\partial z}{\partial u_i}\frac{du_i}{dx}$

例10. 设$z = e^u \sin(2v)$, $u = \sin t$, $v = \cos t$, 求$\dfrac{dz}{dt}$

例11. 设$z = f(u, v)$, $u = e^{-t}$, $v = \arctan t$, 求$\dfrac{dz}{dt}$

Theorem (6)

设$u=\varphi(x,y)$，$v=\psi(x,y)$ 在(x,y)处存在偏导数，$z=f(u,v)$在对应点(u,v)处存在偏导数，则复合函数$f(\varphi(x,y),\psi(x,y))$ 在(x,y) 处存在偏导数，且

$$\left\{\begin{array}{l}\dfrac{\partial z}{\partial x}=\dfrac{\partial z}{\partial u}\dfrac{\partial u}{\partial x}+\dfrac{\partial z}{\partial v}\dfrac{\partial v}{\partial x}\\ \dfrac{\partial z}{\partial y}=\dfrac{\partial z}{\partial u}\dfrac{\partial u}{\partial y}+\dfrac{\partial z}{\partial v}\dfrac{\partial v}{\partial y}\end{array}\right.$$

例12. 设$z=e^u\sin(2v)$, $u=xy$, $v=x+y$, 求$\dfrac{\partial z}{\partial x}$, $\dfrac{\partial z}{\partial y}$

例13. 设$z=f(x,y)$可微， $x=r\cos\theta$, $y=r\sin\theta$, 求$\dfrac{\partial z}{\partial r}$, $\dfrac{\partial z}{\partial \theta}$

例14. 设$u=f(x,y,z)$, $y=\varphi(x,t)$, $t=\psi(x,z)$均可微， 求$\dfrac{\partial u}{\partial x}$, $\dfrac{\partial u}{\partial z}$

例15. 设$z=f(x-y,xy^2)$, f有二阶连续偏导数，求$\dfrac{\partial z}{\partial x}$, $\dfrac{\partial^2 z}{\partial x\partial y}$, $\dfrac{\partial^2 z}{\partial x^2}$

例16. 设$z=f(e^x\sin y, x^2+y^2)$, f有二阶连续偏导数，求$\dfrac{\partial^2 z}{\partial x\partial y}$

例17. 设f 有二阶导数，g 有二阶连续偏导数，$z=f(2x-y)+g(x,xy)$, 求$\dfrac{\partial^2 z}{\partial x\partial y}$.

例18. f, g二阶连续可微，$u = yf(\dfrac{x}{y}) + xg(\dfrac{y}{x})$, 求$x\dfrac{\partial^2 u}{\partial x^2} + y\dfrac{\partial^2 u}{\partial x \partial y}$

例19. 设$u = u(\xi, \eta), \xi = x + ay, \eta = x + by, (a \neq b)$, 问$a, b$ 为何值时可使$u_{xx} + 4u_{xy} + 3u_{yy} = 0$ 变换为 $u_{\xi\eta} = 0$.