二阶线性非齐次微分方程

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$$a_0(x)y''(x) + a_1(x)y'(x) + a_2(x)y(x) = 0 \tag{1}$$

$$a_0(x)y''(x) + a_1(x)y'(x) + a_2(x)y(x) = f(x) \tag{2}$$

Theorem

If $y_1(x), y_2(x)$ 是 *(1)* 的两个线性无关解， $y^*(x)$ 是*(2)* 的一个特解， 则

(i) (1) 的通解为 $y = C_1y_1(x) + C_2y_2(x)$,

(ii) (2) 的通解为 $y = C_1y_1(x) + C_2y_2(x) + y^*(x)$。

先求 $ay'' + by' + cy = 0$ 的通解 $y_1(x), y_2(x)$,

"+" $ay'' + by' + cy = f(x)$ 的一个特解 $y^*(x)$。

$$y = C_1 y_1 + C_2 y_2 + y^*$$

求特解

1. $f(x) = P_m(x)e^{\alpha x}$

$$P_m(x) = a_0x^m + a_1x^{m-1} + \cdots + a_m$$

$$ay'' + by' + cy = (a_0x^m + a_1x^{m-1} + \cdots + a_m)e^{\alpha x},$$

猜测： $y^*(x) = Q(x)e^{\alpha x}$,

$$Q(x) = A_0x^k + A_1x^{k-1} + \cdots + A_k$$

$$
\begin{aligned}
(y^*)' &= Q'(x)e^{\alpha x} + \alpha Q(x)e^{\alpha x} \\
&= (\alpha A_0 x^k + (A_0 k + \alpha A_1)x^{k-1} + (A_1(k-1) + \alpha A_2)x^{k-2} \\
&\quad + \cdots + (2A_{k-2} + A_{k-1}\alpha)x + (A_{k-1} + \alpha A_k))e^{\alpha x}
\end{aligned}
$$

$$
\begin{aligned}
(y^*)'' &= Q''(x)e^{\alpha x} + 2\alpha Q'(x)e^{\alpha x} + \alpha^2 Q(x)e^{\alpha x} \\
&= (\alpha^2 A_0 x^k + (2A_0 k\alpha + \alpha^2 A_1)x^{k-1} \\
&\quad + (A_0 k(k-1) + 2\alpha A_1(k-1) + \alpha^2 A_2)x^{k-2} \\
&\quad + \cdots + (A_{k-2} + 2\alpha A_{k-1} + \alpha^2 A_k))e^{\alpha x}
\end{aligned}
$$

代入 $ay'' + by' + cy = P_m(x)e^{\alpha x}$

$$
\begin{aligned}
& (A_0(a\alpha^2 + b\alpha + c)x^k \\
& +(A_1(a\alpha^2 + b\alpha + c) + A_0 k(2a\alpha + b))x^{k-1} \\
& +(A_2(a\alpha^2 + b\alpha + c) + A_1(k-1)(2a\alpha + b) + A_0 ak(k-1))x^{k-2} \\
& + \cdots + (aA_{k-2} + (2a\alpha + b)A_{k-1} + (a^2 + b\alpha + c)A_k))e^{\alpha x} \\
= \; & (a_0 x^m + a_1 x^{m-1} + \cdots + a_m)e^{\alpha x}
\end{aligned}
$$

- If $a\alpha^2+b\alpha+c\neq 0$,

$$
\begin{aligned}
& (A_0(a\alpha^2+b\alpha+c)x^k \\
& +(A_1(a\alpha^2+b\alpha+c)+A_0k(2a\alpha+b))x^{k-1} \\
& +(A_2(a\alpha^2+b\alpha+c)+A_1(k-1)(2a\alpha+b) \\
& +A_0ak(k-1))x^{k-2} \\
& +\cdots+(aA_{k-2}+(2a\alpha+b)A_{k-1}+(a^2+b\alpha+c)A_k))e^{\alpha x} \\
= \; & (a_0x^m+a_1x^{m-1}+\cdots+a_m)e^{\alpha x}
\end{aligned}
$$

$$k=m$$

$$\text{设} \quad Q(x)=A_0x^m+A_1x^{m-1}+\cdots+A_{m-1}x+A_m$$

- If $a\alpha^2 + b\alpha + c = 0$, 但 α 是单根,

$$a\alpha^2 + b\alpha + c = 0, \quad 2a\alpha + b \neq 0,$$

$$
\begin{aligned}
& (A_0(a\alpha^2+b\alpha+c)x^k \\
& +(A_1(a\alpha^2+b\alpha+c)+A_0k(2a\alpha+b))x^{k-1} \\
& +(A_2(a\alpha^2+b\alpha+c)+A_1(k-1)(2a\alpha+b)+A_0ak(k-1))x^{k-2} \\
& +\cdots+(aA_{k-2}+(2a\alpha+b)A_{k-1}+(a^2+b\alpha+c)A_k))e^{\alpha x} \\
= \;& (a_0x^m+a_1x^{m-1}+\cdots+a_m)e^{\alpha x}
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow \qquad & (A_0k(2a\alpha+b)x^{k-1} \\
& +(A_1(k-1)(2a\alpha+b)+A_0ak(k-1))x^{k-2} \\
& +\cdots+(aA_{k-2}+(2a\alpha+b)A_{k-1}))e^{\alpha x} \\
= \;& (a_0x^m+a_1x^{m-1}+\cdots+a_m)e^{\alpha x}
\end{aligned}
$$

- If $a\alpha^2 + b\alpha + c = 0$, 但 α 是单根,

$$k - 1 = m$$

$$
\begin{aligned}
\text{设} \quad Q(x) &= A_0 x^{m+1} + A_1 x^m + \cdots + A_m x \\
&= x(A_0 x^m + A_1 x^{m-1} + \cdots + A_{m-1} x + A_m)
\end{aligned}
$$

- If $a\alpha^2 + b\alpha + c = 0$, 且 α 是二重根,

$$a\alpha^2 + b\alpha + c = 0, \quad 2a\alpha + b = 0,$$

$$
\begin{aligned}
& (A_0(a\alpha^2+b\alpha+c)x^k \\
& +(A_1(a\alpha^2+b\alpha+c)+A_0k(2a\alpha+b))x^{k-1} \\
& +(A_2(a\alpha^2+b\alpha+c)+A_1(k-1)(2a\alpha+b)+A_0ak(k-1))x^{k-2} \\
& +\cdots+(aA_{k-2}+(2a\alpha+b)A_{k-1}+(a^2+b\alpha+c)A_k))e^{\alpha x} \\
= \; & (a_0x^m+a_1x^{m-1}+\cdots+a_m)e^{\alpha x}
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow \qquad & (A_0ak(k-1)x^{k-2}+\cdots+aA_{k-2})e^{\alpha x} \\
= \; & (a_0x^m+a_1x^{m-1}+\cdots+a_m)e^{\alpha x}
\end{aligned}
$$

- If $a\alpha^2 + b\alpha + c = 0$, 且 α 是二重根,

$$k - 2 = m$$

$$\begin{aligned} \text{设} \quad Q(x) &= A_0x^{m+2} + A_1x^{m+1} + \cdots + A_mx^2 \\ &= x^2(A_0x^m + A_1x^{m-1} + \cdots + A_{m-1}x + A_m) \end{aligned}$$

f(x)=$P_m(x)e^{\alpha x}$总结:

- If $a\alpha^2 + b\alpha + c \neq 0$,

$$设\ \ Q(x) = A_0x^m + A_1x^{m-1} + \cdots + A_{m-1}x + A_m$$

- If $a\alpha^2 + b\alpha + c = 0$, 但 α 是单根,

$$设\ \ Q(x) = x(A_0x^m + A_1x^{m-1} + \cdots + A_{m-1}x + A_m)$$

- If $a\alpha^2 + b\alpha + c = 0$, 且 α 是二重根,

$$设\ \ Q(x) = x^2(A_0x^m + A_1x^{m-1} + \cdots + A_{m-1}x + A_m)$$

例 4.1. 求 $y'' + y' - 2y = x^2 + 3$ 的通解。

$$y = C_1 e^x + C_2 e^{-2x} - \frac{1}{2}x^2 - \frac{1}{2}x - \frac{9}{4}$$

例 4.2. 求 $y'' - 4y = e^{-2x}$ 的特解。

$$y^* = -\frac{1}{4}xe^{-2x}$$

例 4.3. 求 $y'' - 2y' + y = xe^x$ 的特解。

$$y^* = \frac{1}{6}x^3 e^x$$

例 4.4. 设 $f \in C^2$, 且 $f(x) = e^{2x} - \int_0^x (x-t)f(t)dt$, 求 $f(x)$。

$$f(x) = \frac{1}{5}\cos x + \frac{2}{5}\sin x + \frac{4}{5}e^{2x}$$

2. $f(x) = e^{\alpha x}(P_m(x)\cos(\beta x) + Q_n(x)\sin(\beta x))$

$$y^* = x^k e^{\alpha x}(R_{L1}\cos(\beta x) + R_{L2}\sin(\beta x)),$$

- k 取决于 $\alpha + i\beta$ 是否是 $ar^2 + br + c = 0$ 的根来确定 0 or 1,
- $R_{L_1}(x)$, $R_{L_2}(x)$ 是 $L = \max\{m,\ n\}$ 次多项式。

例 4.5. 求 $y'' + 3y' - y = e^x \cos(2x)$ 的特解。

$$y = e^x\left(-\frac{1}{101}\cos(2x) + \frac{10}{101}\sin(2x)\right)$$

例 4.6. 求 $y'' + 4y = x + 1 + \sin x$ 的通解。

$$y = C_1 \cos(2x) + C_2 \sin(2x) + \frac{1}{3}\sin x + \frac{1}{4}x + \frac{1}{4}$$

其它 $ay'' + by' + cy = f(x)$

常数变异法: 齐次方程的基本解 $y_1(x), y_2(x)$,

$$y^*(x) = C_1(x)y_1(x) + C_2(x)y_2(x)$$

$$(y^*)'(x) = C_1'y_1 + C_1y_1' + C_2'y_2 + C_2y_2'$$

$$(y^*)''(x) = C_1''y_1 + 2C_1'y_1' + C_1y_1'' + C_2''y_2 + 2C_2'y_2' + C_2y_2''$$

$$\begin{aligned} f(x) &= ay^{*''} + by^{*'} + cy^* \\ &= C_1(ay_1'' + by_1' + cy_1) + C_2(ay_2'' + by_2' + cy_2) \\ &\quad + a(C_1''y_1 + 2C_1'y_1' + C_2''y_2 + 2C_2'y_2') + b(C_1'y_1 + C_2'y_2) \end{aligned}$$

$y_1(x), y_2(x)$分别是齐次方程的解， $\Rightarrow$

$$\begin{aligned} ay_1'' + by_1' + cy_1 &= 0 \\ ay_2'' + by_2' + cy_2 &= 0 \end{aligned}$$

$\Rightarrow$

$$\begin{aligned} f(x) &= C_1(ay_1'' + by_1' + cy_1) + C_2(ay_2'' + by_2' + cy_2) \\ &\quad + a(C_1''y_1 + 2C_1'y_1' + C_2''y_2 + 2C_2'y_2') + b(C_1'y_1 + C_2'y_2) \\ &= a(C_1''y_1 + 2C_1'y_1' + C_2''y_2 + 2C_2'y_2') + b(C_1'y_1 + C_2'y_2) \end{aligned}$$

令$C_1'y_1 + C_2'y_2 = 0$

$$C_1''y_1 + C_1'y_1' + C_2''y_2 + C_2'y_2' = 0 \Rightarrow C_1''y_1 + C_2''y_2 = -C_1'y_1' - C_2'y_2'$$

$\Rightarrow$

$$\begin{aligned} f(x) &= a(C_1''y_1 + 2C_1'y_1' + C_2''y_2 + 2C_2'y_2') + b(C_1'y_1 + C_2'y_2) \\ &= a(C_1'y_1' + C_2'y_2') \end{aligned}$$

$$\Rightarrow\ C_1'y_1' + C_2'y_2' = \frac{f(x)}{a}$$

In conclusion,

$$\begin{cases} C_1' y_1 + C_2' y_2 = 0 \\ C_1' y_1' + C_2' y_2' = \dfrac{f(x)}{a} \end{cases}$$

$y_1(x), y_2(x)$ 线性无关 $\Rightarrow \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \neq 0,$

$$C_1'(x) = \frac{-\dfrac{1}{a} f(x) y_2(x)}{y_1(x) y_2'(x) - y_1'(x) y_2(x)},$$

$$C_2'(x) = \frac{\dfrac{1}{a} f(x) y_1(x)}{y_1(x) y_2'(x) - y_1'(x) y_2(x)}$$

例 4.7. 求 $y'' - 2y' + y = \dfrac{e^x}{x}$ 的通解。

$$y = (C_1 + C_2 x)e^x + (\ln|x| - 1)xe^x$$