复变函数

钟思佳

东南大学数学系

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复变函数

$$\begin{aligned} f:\mathbb{C} &\longrightarrow \mathbb{C} \\ z=x+yi &\longmapsto w=u+vi \end{aligned}$$

二元二维函数

$$w=f(z)=u(x,y)+iv(x,y)$$

例： $w=|z|$, u? v?

复变函数的极限: $\lim\limits_{z\to z_0} f(z)=w_0=u_0+iv_0$

$$\Longleftrightarrow \lim_{(x,y)\to(x_0,y_0)} u(x,y)=u_0,\ \lim_{(x,y)\to(x_0,y_0)} v(x,y)=v_0$$

复变函数f在z_0连续: $\lim\limits_{z\to z_0} f(z)=f(z_0)$

$$\Longleftrightarrow \lim_{(x,y)\to(x_0,y_0)} u(x,y)=u(x_0,y_0),\ \lim_{(x,y)\to(x_0,y_0)} v(x,y)=v(x_0,y_0),$$

$\Longleftrightarrow u(x,y),\ v(x,y)$都在$(x_0,y_0)$连续

极限的各个性质，依然成立，四则运算法则、无穷小量……

复变函数的导数

定义:

$$f'(z_0) = \lim_{\triangle z \to 0} \frac{f(z_0 + \triangle z) - f(z_0)}{\triangle z}$$

$$\frac{dw}{dz}|_{z=z_0}, \quad w'|_{z=z_0}$$

If f 在区域D内每一点可导，则称f在D内可导

例1. 求$f(z) = z^n \ (n \in \mathbb{Z}^+)$的导数

$$(C)' = 0$$

$$(f(z) \pm g(z))' = f'(z) \pm g'(z), \quad [f(z)g(z)]' = f'(z)g(z) + f(z)g'(z)$$

$$(\frac{f(z)}{g(z)})' = \frac{f'(z)g(z) - f(z)g'(z)}{g(z)^2}$$

$$(f(g(z)))' = f'(g(z))g'(z)$$

If $z = \varphi(w), \ w = f(z)$ 互为反函数，且 $\varphi'(w) \neq 0, \ f'(z) = \dfrac{1}{\varphi'(w)}$

例2. 讨论$f(z) = \bar{z}$的连续性与可导性

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$ —— C-R方程（条件）

Cauchy-Riemann

Theorem (1 可导的必要条件)

$f(z)=u(x,y)+iv(x,y)$在区域D上有定义，在$z_0=x_0+iy_0\in D$可导，$\Rightarrow u(x,y), v(x,y)$在$(x_0,y_0)$处存在偏导数，$\dfrac{\partial u}{\partial x},\dfrac{\partial u}{\partial y},\dfrac{\partial v}{\partial x},\dfrac{\partial v}{\partial y}$且满足$C$-$R$条件。

例3. 证明：$f(z)=\sqrt{|Rez \cdot Imz|}$ 在$z_0=0$ 满足C-R 条件，但不可导。

Theorem (2 可导的充要条件)

$f(z) = u(x, y) + iv(x, y)$ 在区域D 上有定义。$f(z)$在$z_0 = x_0 + iy_0 \in D$ 可导 $\Leftrightarrow$

1. $u(x, y)$, $v(x, y)$ 在(x_0, y_0) 处可微
2. u, v 在(x_0, y_0) 处满足C-R 条件。