偏导数与微分

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Outline

1. 偏导数
   - 高阶偏导数
2. 全微分

3.高阶偏导数

$$\frac{\partial^2 z}{\partial x^2} = f_{xx}(x, y) = \frac{\partial}{\partial x}(\frac{\partial z}{\partial x})$$

$$\frac{\partial^2 z}{\partial y^2} = f_{yy}(x, y) = \frac{\partial}{\partial y}(\frac{\partial z}{\partial y})$$

二阶混合偏导数:

$$\frac{\partial^2 z}{\partial x \partial y} = f_{xy}(x, y) = \frac{\partial}{\partial y}(\frac{\partial z}{\partial x})$$

$$\frac{\partial^2 z}{\partial y \partial x} = f_{yx}(x, y) = \frac{\partial}{\partial x}(\frac{\partial z}{\partial y})$$

3.高阶偏导数

$$\frac{\partial^2 z}{\partial x^2} = f_{xx}(x,y) = \frac{\partial}{\partial x}(\frac{\partial z}{\partial x})$$

$$\frac{\partial^2 z}{\partial y^2} = f_{yy}(x,y) = \frac{\partial}{\partial y}(\frac{\partial z}{\partial y})$$

二阶混合偏导数:

$$\frac{\partial^2 z}{\partial x \partial y} = f_{xy}(x,y) = \frac{\partial}{\partial y}(\frac{\partial z}{\partial x})$$

$$\frac{\partial^2 z}{\partial y \partial x} = f_{yx}(x,y) = \frac{\partial}{\partial x}(\frac{\partial z}{\partial y})$$

例3. 证明 $u = \dfrac{1}{\sqrt{x^2+y^2+z^2}}$ 满足拉普拉斯方程

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=0$$

例4. 设 $z = f(x, y) = x^y$ 求 $\dfrac{\partial^2 z}{\partial x \partial y}$ 与 $\dfrac{\partial^2 z}{\partial y \partial x}$

例5. 设$f(x, y) = \begin{cases} \dfrac{x^3 y}{x^2 + y^2} & x^2 + y^2 \neq 0 \\ 0 & x^2 + y^2 = 0 \end{cases}$ 求$f_{xy}(0, 0)$, $f_{yx}(0, 0)$

何时$\dfrac{\partial^2 z}{\partial x \partial y}$与$\dfrac{\partial^2 z}{\partial y \partial x}$相等?

Theorem (1)

若$f_{xy}(x, y)$ 与 $f_{yx}(x, y)$ 在点(x, y) 的某邻域内连续，则有$f_{xy}(x, y) = f_{yx}(x, y)$

全微分

If $\triangle z = f(x+\triangle x, y+\triangle y) - f(x,y) = A\triangle x + B\triangle y + o(\rho)$,

$\rho = \sqrt{(\triangle x)^2 + (\triangle y)^2}$, A, B 与 $\triangle x$, $\triangle y$ 无关，只与 x, y 有关，则称$z = f(x,y)$ 在(x,y) 处可微，$A\triangle x + B\triangle y$ 为$z = f(x,y)$ 在点(x,y) 的全微分，记为$dz = A\triangle x + B\triangle y = Adx + Bdy$

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = gradz \cdot \{dx, dy\}.$$

例6. 求全微分

(1) $z = e^{xy}$

(2) $z = x^4y^3 + 2x$ 在$(1, 2)$ 处

Theorem (2 必要条件)

If $z = f(x, y)$ 在点(x, y) 可微 $\Rightarrow$

1. $f(x, y)$ 在(x, y) 处连续
2. $f(x, y)$ 在 (x, y) 处存在偏导数$\dfrac{\partial z}{\partial x}, \dfrac{\partial z}{\partial y}$, 且

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy.$$

例7. 设$f(x,y)=\begin{cases}\dfrac{xy}{\sqrt{x^2+y^2}} & x^2+y^2\neq 0\\ 0 & x^2+y^2=0\end{cases}$ 求$f_x(0,0)$, $f_y(0,0)$, 并讨论$f(x,y)$ 在$(0,0)$ 处的可微性

Remark: 偏导数存在是可微的必要条件，而非充分，当$\dfrac{\partial z}{\partial x}, \dfrac{\partial z}{\partial y}$存在时，不一定可微。（$\dfrac{\partial z}{\partial x}dx+\dfrac{\partial z}{\partial y}dy$可以写出来，但不一定是全微分, dz只是形式上），必须验证"$\triangle z-(\dfrac{\partial z}{\partial x}\triangle x+\dfrac{\partial z}{\partial y}\triangle y)$ "是ρ的高阶无穷小。