三重积分

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Outline

直角坐标系

三重积分换元法

一. 直角坐标系

$$\iiint_{\Omega} f(x,y,z)dxdydz = \iint_{D} dxdy \int_{z_1(x,y)}^{z_2(x,y)} f(x,y,z)dz \ \ --\text{先一后二}$$

$$\iint_{\Omega} f(x,y,z)dxdydz = \int_{c}^{d} dz \iint_{D_z} f(x,y,z)dxdy \ \ --\text{先二后一}$$

二. 三重积分换元法

Theorem (1)

设

1. $f(x,y,z) \in C_D$
2. 变换T : $\begin{cases} x = x(u,v,w) \\ y = y(u,v,w) \\ z = z(u,v,w) \end{cases}$ 把uvw平面上的区域$\Omega' \to \Omega$
3. T在Ω'上具有一阶连续偏导数,

   且$J(u,v,w) = \dfrac{\partial(x,y,z)}{\partial(u,v,w)} \neq 0, (u,v,w) \in D'$

则

$$\begin{aligned} & \iiint_{\Omega} f(x,y,z)dxdydz \\ = & \iiint_{\Omega'} f(x(u,v,w), y(u,v,w), z(u,v,w))|J|dudvdw \end{aligned}$$

1. 柱面坐标系

$$\begin{cases} x = r\cos\theta & 0 \le r < +\infty \\ y = r\sin\theta & 0 \le \theta \le 2\pi \\ z = z & -\infty < z < +\infty \end{cases}$$

$$J = \frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r$$

$$\iiint_{\Omega} f(x,y,z)dxdydz = \iiint_{\Omega'} f(r\cos\theta, r\sin\theta, z)\,{\color{red}r}\,drd\theta dz$$

例5. $\iiint_\Omega z\sqrt{x^2+y^2}dxdydz, \Omega: x^2+y^2=2x$, 及$z=0$, $z=a, y\geq 0$所围成。

例6. $\iiint_\Omega ze^{x^2+y^2}dxdydz, \Omega: x^2+y^2+z^2\leq 4, z\geq \sqrt{x^2+y^2}$

例7. $\iiint_\Omega (x^2+y^2)dxdydz, \Omega: 2z=x^2+y^2, z=2$围成

2. 球面坐标系

$$\begin{cases} x = r\sin\theta\cos\varphi & 0 \le r < +\infty \\ y = r\sin\theta\sin\varphi & 0 \le \theta \le \pi \\ z = r\cos\theta & 0 \le \varphi \le 2\pi \end{cases}$$

- $r = cost.$ 以原点为球心的球面
- $\theta = cost.$ 以原点为顶点，z轴为轴的圆锥面
- $\varphi = cost.$ 过z轴的半平面

$$
\begin{aligned}
J &= \frac{\partial(x,y,z)}{\partial(r,\theta,\varphi)} \\
&= \begin{vmatrix} \sin\theta\cos\varphi & r\cos\theta\cos\varphi & -r\sin\theta\sin\varphi \\ \sin\theta\sin\theta & r\cos\theta\sin\varphi & r\cos\theta\cos\varphi \\ \cos\theta & r\sin\theta & 0 \end{vmatrix} \\
&= r^2\sin\theta
\end{aligned}
$$

$$
\begin{aligned}
& \iiint_{\Omega} f(x,y,z)dxdydz \\
= & \iiint_{\Omega'} f(r\sin\theta\cos\varphi, r\sin\theta\sin\varphi, r\cos\theta)\textcolor{red}{r^2\sin\theta} drd\theta d\varphi
\end{aligned}
$$

例8. $I=\iiint_\Omega f(x,y,z)dxdydz, \Omega:$ 由$z=1$, $z=\sqrt{x^2+y^2}$围成

例9. 求由$2z=\sqrt{x^2+y^2}$与$x^2+y^2+(z-1)^2=1$围成球心所在部分的体积

广义柱面坐标变换

$$\begin{cases} x = ar\cos\theta & 0 \le r < +\infty \\ y = br\sin\theta & 0 \le \theta \le 2\pi \\ z = z & -\infty < z < +\infty \end{cases}$$

广义球面坐标变换

$$\begin{cases} x = ar\sin\theta\cos\varphi & 0 \le r < +\infty \\ y = br\sin\theta\sin\varphi & 0 \le \theta \le \pi \\ z = cr\cos\theta & 0 \le \varphi \le 2\pi \end{cases}$$

例10. $\iiint_\Omega x^2 dxdydz, z = ay^2, z = by^2, (y > 0, b > a > 0),$
$z = \alpha x, z = \beta x, \beta > \alpha > 0, z = h, h > 0$ 围成。

解：令$u = \dfrac{z}{y^2}, v = \dfrac{z}{x}, w = z,$

$$\Omega' : \ a \le u \le b,\ \alpha \le v \le \beta,\ 0 \le w \le h,$$

$$\begin{aligned} J &= \frac{\partial(x,y,z)}{\partial(u,v,w)} = \frac{1}{\frac{\partial(u,v,w)}{\partial(x,y,z)}} = \frac{1}{\begin{vmatrix} 0 & -2\frac{z}{y^3} & \frac{1}{y^2} \\ -\frac{z}{x^2} & 0 & \frac{1}{x} \\ 0 & 0 & 1 \end{vmatrix}} \\ &= -\frac{x^2y^3}{2z^2} = -\frac{w^{\frac{3}{2}}}{2v^2u^{\frac{3}{2}}} \end{aligned}$$

$$
\begin{aligned}
& \iiint_{\Omega} x^2 dxdydz \\
= \quad & \iiint_{\Omega'} \frac{w^2}{v^2} \frac{w^{\frac{3}{2}}}{2v^2 u^{\frac{3}{2}}} dudvdw = \frac{1}{2} \int_0^h dw \int_\alpha^\beta dv \int_a^b \frac{w^{\frac{7}{2}}}{v^4 u^{\frac{3}{2}}} du \\
= \quad & \frac{2}{27} h^{\frac{9}{2}} \left(\frac{1}{\beta^3} - \frac{1}{\alpha^3}\right)\left(\frac{1}{\sqrt{b}} - \frac{1}{\sqrt{a}}\right)
\end{aligned}
$$