微分学基本定理及其应用

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Taylor 公式

$$\lim_{x\to 0}\frac{e^x-1}{x}=1 \;\Rightarrow e^x-1=x+o(x)$$

$$o(x)?$$

$$\lim_{x\to 0}\frac{e^x-1-x}{x^2}=\lim_{x\to 0}\frac{e^x-1}{2x}=\frac{1}{2}$$

$$e^x-1-x=\frac{1}{2}x^2+o(x^2)$$

$$\cdots$$

$$e^x=1+x+\frac{1}{2}x^2+\cdots+\frac{1}{n!}x^n+o(x^n)$$

$$e^x = 1 + x + \frac{1}{2}x^2 + \cdots + \frac{1}{n!}x^n + o(x^n)$$

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \cdots$$

Theorem (Taylor公式)

设$f(x)$在x_0处有n阶导数，则有

$$\begin{aligned} f(x) &= f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 \\ &\quad + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + o((x - x_0)^n). \end{aligned}$$

其中

$$P_n(x) = f(x_0) + f'(x_0)(x - x_0) + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

称为f在x_0处的n阶*Taylor*多项式，$r_n(x) = o((x - x_0)^n)$ 称为*Peano*余项。

特别地，称$x_0 = 0$的Taylor公式为Maclaurin公式，即

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + o(x^n)$$

$$e^x = 1 + x + \frac{1}{2}x^2 + \cdots + \frac{1}{n!}x^n + o(x^n)$$

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots + \frac{(-1)^n}{(2n+1)!}x^{2n+1} + o(x^{2n+1})$$

$$\cos x = 1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4 - + \frac{(-1)^n}{(2n)!}x^{2n} + o(x^{2n})$$

$$\frac{1}{1+x} = 1 - x + x^2 - \cdots + (-1)^n x^n + o(x^n)$$

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \cdots + \frac{(-1)^{n-1}}{n}x^n + o(x^n)$$

$$\begin{aligned}(1+x)^\alpha &= 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 \\ &\quad + \cdots + \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}x^n + o(x^n), \ \alpha \notin \mathbb{Z}\end{aligned}$$

例3.9 求$f(x) = \cos^2 x$的$2n$阶Maclaurin公式

例3.10 求$f(x) = \dfrac{1}{3-x}$在$x = 1$处的n阶Taylor公式

$$\lim_{x\to 0}\frac{\sin x-\tan x}{x^3}$$

例3.11 (1) 求 $\lim\limits_{x\to 0}\dfrac{e^{-\frac{1}{2}x^2}-\cos x}{x^4}$

(2) 假设$\alpha=e^{-\frac{x^2}{2}}-\cos x,\beta=x^k$，求$k$ s.t. $x\to 0$时，α,β为同阶无穷小

Theorem (Taylor公式II)

设$f(x)$在x_0的某邻域$N(x_0)$内具有$n+1$阶导数，则对$\forall x\in N(x_0)$，有

$$\begin{aligned} f(x) &= f(x_0)+f'(x_0)(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2 \\ &\quad +\cdots+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n+\frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}. \end{aligned}$$

其中ξ介于x_0与x之间，

$$\frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}=\frac{f^{(n+1)}(x_0+\theta(x-x_0))}{(n+1)!}(x-x_0)^{n+1},\ \theta\in(0,1)$$

称为*Lagrange*余项。

Remark:

- $n = 0$, $f(x) = f(x_0) + f'(\xi)(x - x_0)$, Lagrange 中值定理
- $n = 1$, $f(x) = f(x_0) + f'(x_0)(x - x_0) + o(x - x_0)$, 微分的定义

$$e^x = 1 + x + \frac{1}{2}x^2 + \cdots + \frac{1}{n!}x^n + \frac{e^\xi}{(n+1)!}x^{n+1}, \ x \in \mathbb{R}$$

$$\begin{aligned}\sin x &= x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots + \frac{(-1)^{(n-1)}}{(2n-1)!}x^{2n-1} \\ &\quad + \frac{\sin(\xi + \frac{(2n+1)\pi}{2})}{(2n+1)!}x^{2n+1}, \ x \in \mathbb{R}\end{aligned}$$

$$\cos x = 1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4 - + \frac{(-1)^n}{(2n)!}x^{2n} + \frac{(-1)^{n+1}\cos(\xi)}{(2n+2)!}x^{2n+2}, \ x \in \mathbb{R}$$

$$\frac{1}{1+x} = 1-x+x^2-\cdots+(-1)^n x^n+\frac{(-1)^{n+1}}{(1+\xi)^{n+2}}x^{n+1},\ -1<x<+\infty$$

$$\begin{aligned}\ln(1+x) &= x-\frac{1}{2}x^2+\frac{1}{3}x^3-\cdots+\frac{(-1)^{n-1}}{n}x^n \\ &\quad +\frac{(-1)^n}{(n+1)(1+\xi)^{n+1}}x^{n+1},\ -1<x<+\infty\end{aligned}$$

$$\begin{aligned}(1+x)^\alpha &= 1+\alpha x+\frac{\alpha(\alpha-1)}{2!}x^2+\cdots+\frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}x^n \\ &\quad +\frac{\alpha(\alpha-1)\cdots(\alpha-n)}{(n+1)!}(1+\xi)^{\alpha-n-1}x^{n+1}, \\ &\quad \alpha\notin\mathbb{Z},\ -1<x<+\infty\end{aligned}$$

例3.12 设f在$\mathbb{R}$上两阶可导，$f''(x) > 0$ (or < 0)，当$x \to 0$时，$f(x) \sim x$，证明：当$x \neq 0$时，$f(x) > x$ (or $< x$)

例3.13 设函数$f \in C^2_{[0,1]}$, 且$f(0) = f(1)$, $|f''(x)| \leq A$, 试证:

$$|f'(x)| \leq \frac{A}{2}, \ x \in [0, 1].$$