Homework 8
APPM/MATH 4650 Fall '20 Numerical Analysis

Due date: Saturday, November 7, before midnight, via Gradescope. **Instructor**: Prof. Becker
Theme: ODEs and IVPs, and Halloween

Instructions Collaboration with your fellow students is OK and in fact recommended, although direct copying is not allowed. The internet is allowed for basic tasks (e.g., looking up definitions on wikipedia) but it is not permissible to search for proofs or to *post* requests for help on forums such as `http://math.stackexchange.com/` or to look at solution manuals. Please write down the names of the students that you worked with.

An arbitrary subset of these questions will be graded.

Turn in a PDF (either scanned handwritten work, or typed, or a combination of both) to **Gradescope**, using the link to Gradescope from our Canvas page. Gradescope recommends a few apps for scanning from your phone; see the Gradescope HW submission guide.

We will primarily grade your written work, and computer source code is *not* necessary except for when we *explicitly* ask for it (and you can use any language you want). If not specifically requested as part of a problem, you may include it at the end of your homework if you wish (sometimes the graders might look at it, but not always; it will be a bit easier to give partial credit if you include your code).

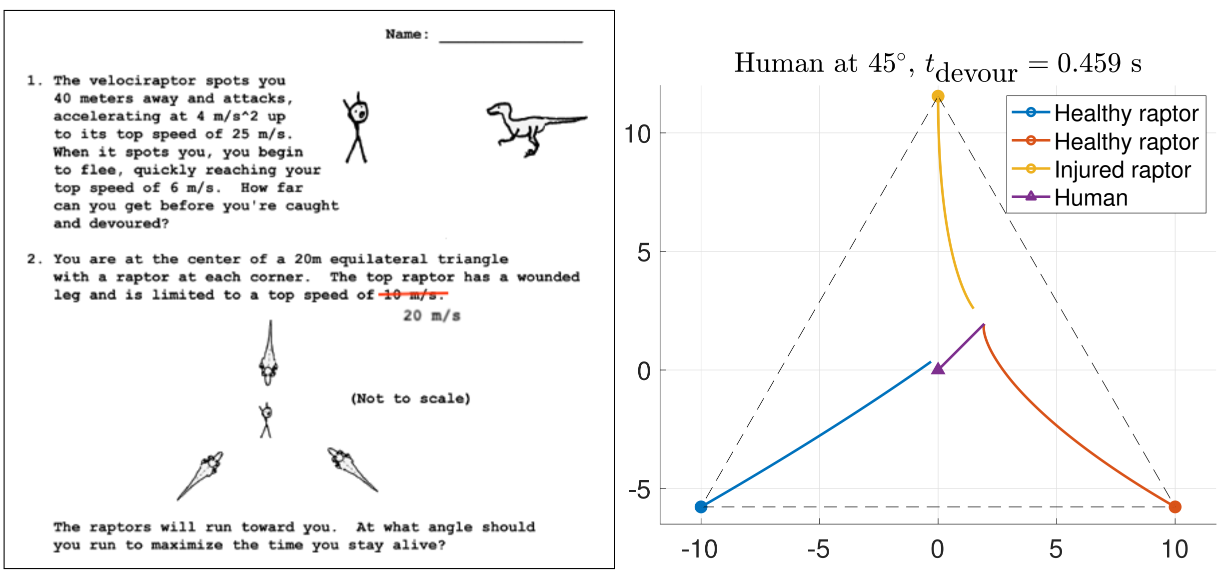

Figure 1: From the XKCD comic. **Left**: We'll solve problem 2. The full comic can be found at xkcd.com/135/. **Right**: An example of the output from solving the raptor ODE, when the human decides to run at a 45° angle above the horizontal. The healthy raptor in the bottom right corner reaches the human in 0.459 seconds. Your homework should look similar to this plot, but use an angle of 56°. All distances in meters.

Problem 1: **Raptors** Scientist/cartoonist Randall Munroe poses an interesting question about optimization in his XKCD comic. We can start to answer question 2 (see Figure 1) using Matlab or Python's builtin IVP solvers. First, formulate the problem as an ODE. Let the human position be $\mathbf{h}(t) \in \mathbb{R}^2$, and a raptor's position $\mathbf{r}(t) \in \mathbb{R}^2$. We assume a raptor is not good at predicting the human's future location, and at any time t, the raptor runs directly at the human's current position. If the raptor's speed is constant (for simplicity, we also assume instantaneous acceleration at the beginning) at, say, v_r, then we can model the raptor's motion as

$$\frac{d\mathbf{r}}{dt} = v_r \frac{\mathbf{h}(t) - \mathbf{r}(t)}{\|\mathbf{h}(t) - \mathbf{r}(t)\|_2} \tag{1}$$

where if $\mathbf{x} = (x, y)$ then we define $\|\mathbf{x}\|_2 = \sqrt{x^2 + y^2}$. This is an example of a pursuit curve, studied since at least Pierre Bouguer in 1732, although Bouguer did not focus much on velociraptors.

The three raptors each satisfy this ODE separately, i.e. their motions are not coupled. However, the x and y components of a raptor's motion are coupled, so we have 2-dimensional ODEs.

For simplicity, we assume the human runs in a constant direction and at a constant speed; thus $\mathbf{h}(t) = v_h t \frac{\mathbf{c}}{\|\mathbf{c}\|_2} + \mathbf{h}(0)$, where $\mathbf{c} \in \mathbb{R}^2$ is an initial direction and $\mathbf{h}(0)$ is the human's initial position. Then, substitute $\mathbf{h}(t)$ into equation (1) to obtain $\frac{d\mathbf{r}}{dt} = F(t, \mathbf{r})$ for some function F that you must define.

We will use Munroe's setup, with a small modification (and assuming instantaneous acceleration). As in the comic, the human is at the center of a 20m equilateral triangle, and the human has a top speed of $v_h = 6$ m/s and the healthy raptors have a top speed of $v_r = 25$ m/s. However, if the injured raptor can only run at 10 m/s, then the best strategy for the human is to run directly at the injured raptor. To make the problem more interesting, assume the injured raptor can run at 20 m/s.

Your job is to setup the geometry and IVP, then using an existing IVP solver (such as `ode45` in Matlab or `scipy.integrate.solve_ivp` in Python), solve the IVP and determine how long the human will survive. We will define the human to be "caught" by the raptor when the raptor is within 0.01 m of the human. Specifically, assume the human starts running at a 56° angle above the horizontal (to the right, in the picture), and find which raptor reaches the human first, and how long it takes that raptor to reach the human (report your answer in seconds, with 3 decimal places). You should also include a plot, similar to the one in Figure 1 (right), and use interpolation to plot the lines (e.g., `deval` in Matlab, and set the `dense_output=True` flag for `solve_ivp` in Python). Please include your code.

Hints: it may be helpful to tighten the tolerance for the IVP solvers. Also, you do not know *a priori* the ending time of the IVP. One nice way to deal with this is to pick a conservative time (say, $t = 5$), and then define an "event", such that the IVP solver will stop when the event occurs. Both `ode45` and `scipy.integrate.solve_ivp` support "events"; see their documentation page, or ask in office hours. Note that you don't have to optimize the angle for this problem, though that is an interesting question and not much more difficult for us to answer numerically. Another interesting question is for which top speed of the wounded raptor does your optimal strategy become to run straight at the wounded raptor?

Problem 2: Consider the IVP

$$\begin{aligned} y' &= \frac{1+t}{1+y}, \quad t \in [1, 2] \\ y(1) &= 2 \end{aligned} \tag{2}$$

a) Prove there exists a unique solution y to this IVP for the time interval $[1, 1\frac{2}{3}]$. If you prove there exists a unique solution for the time interval $[1, 2]$, you get a small amount of extra credit. *Hint*: you may want to use the theorem provided at the end of this homework.

b) Find a formula for the true solution

c) Apply Euler's method with $h = 0.5$ to estimate $y(2)$. What is the estimate that Euler's method gives? You can include your code if you want (it might help with partial credit) but it's not necessary.

d) Compute the error bound from Theorem 5.9 in Burden and Faires, and compare the bound at $t = 2$ with the actual error at $t = 2$. If some of the constants in Theorem 5.9 are difficult to calculate, you are welcome to use a reasonable bound of them.

e) Now consider $t \in [1, 1 + 10^{-5}]$. Use Euler's method to estimate $y(1 + 10^{-5})$ using a variety of stepsizes h (between 10^{-4} and 10^{-14}). Plot the error as a function of stepsize h; choose the parameters of the plot to make it informative. Which choice of h minimizes the error, and is this expected?

f) Repeat the same exercise as above, but for $t \in [1, 2]$. Again, choose a variety of stepsizes, but don't make h so small that it takes more than a few seconds for your code to run. Plot the error. Which choice of h minimizes the error? Are there any differences with your result compared to the previous question?

Extra ODE existence theorem

There are many quite similar ODE existence theorems, such as Picard–Lindelöf theorem aka Cauchy–Lipschitz theorem. The naming conventions are not quite universal, and the slight differences in assumptions are important. The following theorem (adapted from Thm. 3.10 in Hunter and Nachtergaele's applied analysis) may be useful for us:

Theorem 1 (Local existence for ODEs) *Consider the IVP*

$$y' = f(t, y) \quad \text{and} \quad y(a) = y_0,$$

and define

$$\mathcal{D} = \{(t, y) \in \mathbb{R}^2 \mid a - T \le t \le a + T,\ y_0 - R \le y \le y_0 + R\}$$

for some $T > 0$. Let $f : \mathcal{D} \to \mathbb{R}$ be continuous; then since D is a closed set, $|f|$ is bounded over $\mathcal{D}$ (this follows from the Bolzano-Weierstrass theorem, an extension of the Extreme Value Theorem to higher dimension), and call this bound M, i.e., $\forall (t, y) \in \mathcal{D}$, $|f(t, y)| \le M$. Further suppose that on $\mathcal{D}$, f is Lipschitz continuous with respect to y uniformly in t. Then there is a unique solution y to the ODE within the time interval $[a, a + \delta]$ where $\delta = \min(T, R/M)$, and furthermore $|y(t) - y_0| \le R$ for all $t \in [a, a + \delta]$.