Conditioning of rootfinding 2: multiple roots

Sunday, August 30, 2020 3:37 PM

Multiple roots

$f(x) = (x-3)^2$ has a double root at $x=3$

$\tilde{f}(x) = (x-3)^2 + 10^{-100}$ has no roots. Clearly this is bad

What does it mean to have a multiple root? (we want a definition to cover non-polynomials too)

let r be a root.

Write $f(x) = (x-r)\underbrace{\boxed{\frac{f(x)}{(x-r)}}}_{=g(x)}$

$= (x-r)\cdot g(x)$

(at $x=r$, $g(r) = \frac{f(r)}{x-r} = \frac{0}{0}$, but we can define $g(r) = \lim_{x\to r} \frac{f(r)}{x-r}$ which exists (L'Hôpital's rule) as long as $f'(r)$ exists.)

to be a simple root means $g(r) \neq 0$

... multiple root means $g(r) = 0$

... but there's a simpler way, no need for g

Observe $f'(x) = (x-r)g'(x) + g(x)$ (product rule)

so $f'(r) = 0\cdot g'(r) + g(r)$.

$= g(r)$. So

$$f'(r) = \begin{cases} \text{not zero} & \text{simple root} \\ 0 & \text{multiple root} \end{cases}$$

(we can also keep repeating for $f''(r)$ too ...

so r is a root of f with multiplicity m if

$f(r) = f'(r) = \ldots = f^{(m-1)} = 0$ and $f^{(m)} \neq 0$)

Conclusion: the absolute condition number for finding a multiple root $(m>1)$ is ∞

$\left| 1/f'(r) \right|$

$f(x) = (x-3)^2$ so $r=3$

$f' = 2(x-3)$ so $f'(3) = 0$, so $r=3$ is a double root

So condition # is ∞

$f(x) = (x-3)^2 - 10^{-6}$ has two single roots (both near $r=3$)

$f'(x) = 2\cdot(x-3)$ is 0 at $r=3$ but not 0 at the true roots

however, $f'(\text{roots})$ is small, so κ is very large.

Final recap: . .

if $|f'(r)|$ is small, it will be hard for <u>all</u> algorithms to accurately find the root

and if $f'(r)=0$, given floating point error, it may be completely impossible (ie., there may not exist any root!)