Homework 9
APPM/MATH 4650 Fall '20 Numerical Analysis

Due date: Saturday, November 14, before midnight, via Gradescope. **Instructor**: Prof. Becker
Theme: ODEs and IVPs

Instructions Collaboration with your fellow students is OK and in fact recommended, although direct copying is not allowed. The internet is allowed for basic tasks (e.g., looking up definitions on wikipedia) but it is not permissible to search for proofs or to *post* requests for help on forums such as `http://math.stackexchange.com/` or to look at solution manuals. Please write down the names of the students that you worked with.

An arbitrary subset of these questions will be graded.

Turn in a PDF (either scanned handwritten work, or typed, or a combination of both) to **Gradescope**, using the link to Gradescope from our Canvas page. Gradescope recommends a few apps for scanning from your phone; see the Gradescope HW submission guide.

We will primarily grade your written work, and computer source code is *not* necessary except for when we *explicitly* ask for it (and you can use any language you want). If not specifically requested as part of a problem, you may include it at the end of your homework if you wish (sometimes the graders might look at it, but not always; it will be a bit easier to give partial credit if you include your code).

Problem 1: Consider the following initial value problem

$$y''(t) + 5y'(t) + 6y(t) = \cos(t), \quad y(0) = 1,\ y'(0) = 0,\ \text{for } t \in [0, 20] \tag{1}$$

Using the techniques you learned in your ODE class (APPM 2360 or MATH 3430), find an analytic solution. *Hint*: you are welcome to do the next problem and solve this numerically as a means to confirm that your derivation is correct.

Problem 2: Numerically solve the IVP in Eq. (1) by using the Runge-Kutta method given by the following Butcher array:

$$\begin{array}{c|cccc} 0 & 0 & 0 & 0 & 0 \\ 1/3 & 1/3 & 0 & 0 & 0 \\ 2/3 & -1/3 & 1 & 0 & 0 \\ 1 & 1 & -1 & 1 & 0 \\ \hline & 1/8 & 3/8 & 3/8 & 1/8 \end{array}$$

In particular:

a) Write out the IVP as a system of first-order ODEs.

b) Implement the given Runge-Kutta method (for a uniform stepsize h) and, choosing a reasonable value of h, plot y as a function of t for $t \in [0, 20]$. *Hint*: use `ode45` in Matlab or `scipy.integrate.solve_ivp` in Python to make sure that you've implemented the ODE correctly, then apply the given Runge-Kutta method.

c) Let $\hat{y}$ represent the numerical solution you found using the Runge-Kutta method, and let y be the true solution you found. Define the error ε to be the absolute value of the difference between your approximation at $t = 20$ and the true solution at $t = 20$. Plot ε as a function of the stepsize h, and use this to determine the order of the error (e.g., $O(h)$ or $O(h^3)$ etc.). Choose the scales (logarithmic or linear) of your plot wisely.

d) Include the code you used to solve the IVP (including your plotting code is optional)