Convergence Rates

Thursday, August 27, 2020 10:57 PM

(see section 2.4 Burden and Faires)

See Ch1_RatesOfConvergence.ipynb

Def let (x_n) be a sequence converging to x, $\lim_{n\to\infty} x_n = x$

If $\exists\ C>0$ and $\alpha>0$ s.t. $\displaystyle\lim_{n\to\infty} \frac{|x_{n+1}-x|}{|x_n - x|^{\alpha}} = C$

then we say (x_n) converges to x of order α

and in particular,

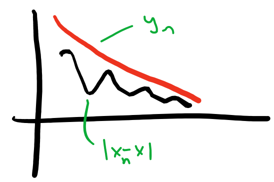
$\alpha < 1$ $(C<1)$ "sublinear convergence"
$\alpha = 1$ $(C<1)$ "linear convergence"
$\alpha = 2$ "quadratic convergence"

Note: this is sometimes called Q-convergence (ex: $\alpha=1, C<1$ is Q-linear convergence), as it involves a Quotient

Sometimes we use a weaker notion, R-convergence (R for root), meaning

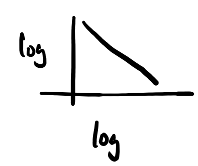
$x_n \to x$ R-linearly if $\exists (y_n)$ with $|x_n - x| \le y_n$ and $y_n \to 0$ Q-linearly

i.e., for R-convergence, error might actually go up! but trend is still correct.

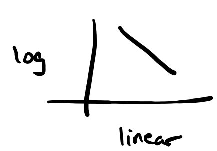


Examples

- $x_n = 1/\sqrt{n}$ so $x_n \to 0$ slowly. This is Q-sublinear
  to reach $x_n < \varepsilon$ takes $O(1/\varepsilon^2)$ iterations.
- $x_n = 1/n$ also Q-sublinear $O(1/\varepsilon)$
- $x_n = 1/n^2$ $O(1/\sqrt{\varepsilon})$



- $x_n = .9^n$ is Q-linear $O(-\log(\varepsilon))$

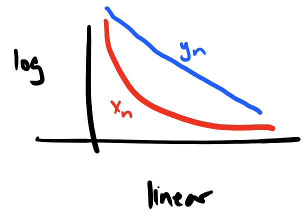


⚠ x_n sublinear then asymptotically
y_n linear $y_n \to 0$ faster than x_n
$(\lim_{n\to\infty} y_n / x_n = 0)$

but for small n, maybe $x_n < y_n$

ex: $x_n = 10^{-5} \cdot 1/n^2$

$y_n = (1 - 10^{-10})^n$



constants matter!

linear convergence with $c = 1 - 10^{-10}$ is terrible

but with $c = 1/2$ it's very good.

$x_n = (.9)^{2^n}$ is Q-quadratic

$O(\log(-\log(\varepsilon)))$

Final accuracy doesn't really matter it's so fast