Ch 1: big-O notation

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Definition Case 1 $f(x) = O(g(x))$ as $x \to \infty$ if

usually $g(x) \to \infty$

$\exists\, x_0$ and $\exists\, M < \infty$ s.t. if $x > x_0$ then $|f(x)| \le M \cdot g(x)$

"exists"

Case 2 $f(x) = O(g(x))$ as $x \to a$ (usually $a = 0$) if

$\exists\, \delta > 0$ and $\exists\, M < \infty$ s.t. if $|x - a| < \delta$ then $|f(x)| \le M \cdot g(x)$

usually $g(x) \to 0$

(Both cases, equivalent def'n)

$$f = O(g) \text{ if } \limsup_{x \to \infty \text{ or } a} \left| \frac{f(x)}{g(x)} \right| < \infty$$

valid

or, if $\lim_{x \to \infty \text{ or } a} \left| \frac{f(x)}{g(x)} \right|$ exists and is $< \infty$

sufficient (since limit need not exist)

use L'Hôpital's rule if needed

if $\lim_{x \to \dots} \left| \frac{f(x)}{g(x)} \right| = \infty$ then f is not $O(g)$

Interpretations and Examples

∞ "infinite case"

$f = O(g)$ means, eventually, f grows no faster than g (up to constants)

(typically, $g(x) \to \infty$ as $x \to \infty$)

Use case: n = size of input, $f(n)$ = how long it takes to run

$f(n) = n^3 + 3n^2 - 4n + 7$, $\quad f(n) = O(n^3)$

$f(n) = 10n^3 + 3n^2 \dots$ $\quad f(n) = O(n^3)$, $f(n) \ne O(n^2)$

$f(n) = n^3$, $g(n) = n^2$, is $f = O(g)$? No

$$\lim_{n \to \infty} \frac{f(n)}{g(n)} = \lim_{n \to \infty} \frac{n^3}{n^2} = \lim n = \infty. \text{ No}$$

$n^2 = O(n^3)$ ✓

$n^2 = O(n^2)$

$x^2 = O(e^x)$, $e^x \ne O(x^2)$

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$a=0$ "infinitesimal case"

typically, $a=0$, write "h" instead of "x", $g(h) \to 0$ as $h \to 0$

discretization

$f=O(g)$ means, eventually, f decays to 0 at least as fast as g (up to a constant)

Ex: $3h^2 = O(h^2)$, $h^2 = O(h)$, $h^2 \neq O(h^3)$ } check: $\lim_{h\to 0} \frac{h^2}{h} = \lim_{h \to 0} h = 0 < \infty$

WARNING $x = O(x^2)$ as $x \to \infty$, $x^2 \neq O(x)$ as $x \to \infty$ — ∞ case

$h \neq O(h^2)$ as $h \to 0$, $h^2 = O(h)$ as $h \to 0$ — $a=0$ case

Smaller exponent is "better"

larger exponent is "better" (usually $f(h)$ is error term in a Taylor Series)

Ex: $f(x_0+h) = \underbrace{f(x_0) + f'(x_0)h + \frac{f''(x_0)h^2}{2!}}_{P} + \frac{f'''(\xi)h^3}{3!}$

if $|f'''(\xi)|$ is bounded for all $\xi \in [x_0, x_0+h]$

then we can say

$f(x_0+h) = P(h) + O(h^3)$

Variants

little-o notation:

$f=o(g)$ as $x \to \infty$ means (asymptotically) f grows slower than $c \cdot g(x)$ for all constants c

likewise, $f=o(g)$ as $x \to a$ means $\forall c>0, \exists \delta_0$ s.t. $|x-a|<\delta_0 \Rightarrow |f(x)| \le c \cdot g(x)$

i.e., $\forall c>0, \exists x_0$ s.t. $x > x_0$, $|f(x)| \le c \cdot g(x)$

If $g(x) \neq 0$, then you can show $f=o(g)$ by showing $\lim_{x \to \dots} \frac{f(x)}{g(x)} = 0$ (either a or ∞)

more 'precise' than big-O notation

(as $x \to \infty$) $x^2 = O(x^2)$, $x^2 \neq o(x^2)$, $x^2 = o(x^3)$

big-theta Θ

$f = \Theta(g)$ means $f = O(g)$ and $g = O(f)$

ex: $5x^3 = \Theta(10x^3)$ $x^3 \neq \Theta(x^2)$, $x^2 \neq \Theta(x^3)$

$f \sim g$ even stronger: means $\lim_{x \to a \text{ or } \infty} \frac{f(x)}{g(x)} = 1$

$5x^3 \not\sim 10x^3$, $5x^3 + 3x^2 \sim 5x^3 - x$

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$\tilde{O}$ ← ignore log factors

Ex: is $e^x = O(x!)$ or is $x! = O(e^x)$? $(x \to \infty)$

Use Stirling's formula: $x! \sim \sqrt{2\pi x}\,(x/e)^x$,

ie, $x! = \Theta(\sqrt{x}\,(\frac{x}{e})^x)$

x^x grows faster than e^x

So $x!$ grows faster than e^x, $e^x = O(x!)$, $x! \neq O(e^x)$

$x! \neq \Theta(e^x)$

⚠

$O(h^2) + O(h^2) = O(h^2)$

$O(h^2) + O(h^3) = O(h^2)$

$O(h^2) - O(h^2) = O(h^2)$ NOT zero!

Symmetry, Transitivity

Let p, q, s be functions

$p = O(q),\ q = O(s) \Rightarrow p = O(s)$

$p = o(q),\ q = o(s) \Rightarrow p = o(s)$] Transitive

$p = O(q)$ then it's possible (but not always) that $q = O(p)$

$p = o(q)$ then impossible for $q = o(p)$] never symmetric

$p = \Theta(q) \Leftrightarrow q = \Theta(p)$

$p \sim q \Leftrightarrow q \sim p$] Symmetric

$p = \Theta(q),\ q = \Theta(s) \Rightarrow p = \Theta(s)$

$p \sim q,\ q \sim s \Rightarrow p \sim s$] Transitive

in math, "$\Leftrightarrow$" means "iff" ie. "if and only if"