Convergence Rates

Thursday, August 27, 2020 10:57 PM

(see section 2.4 Burden and Faires) See Ch1_RatesOfConvergence.ipynb

Def let (x_n) be a sequence converging to x, $\lim_{n\to\infty} x_n = x$

If $\exists\ C>0$ and $\alpha>0$ s.t. $\lim_{n\to\infty} \frac{|x_{n+1}-x|}{|x_n-x|^{\alpha}} = C$

then we say (x_n) converges to x of order α

Book's definition. See below (end of notes) for a better definition

and in particular,

~~$\alpha<1$ $(c<1)$ "sublinear convergence"~~ See below for details

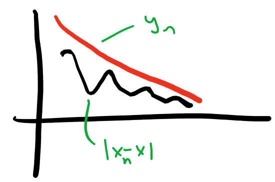
$\alpha=1$ $(c<1)$ "linear convergence"

$\alpha=2$ "quadratic convergence"

Note: this is sometimes called Q-convergence (ex: $\alpha=1, c<1$ is Q-linear convergence), as it involves a Quotient

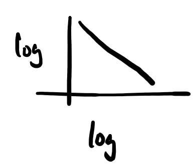
Sometimes we use a weaker notion, R-convergence (R for root), meaning $x_n \to x$ R-linearly if $\exists(y_n)$ with $|x_n-x|\le y_n$ and $y_n\to 0$ Q-linearly

i.e., for R-convergence, error might actually go up! but trend is still correct.

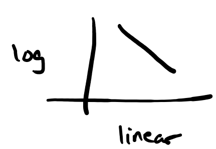


Examples

- $x_n = 1/\sqrt{n}$ so $x_n \to 0$ slowly. This is Q-sublinear
  to reach $x_n<\varepsilon$ takes $O(1/\varepsilon^2)$ iterations.
- $x_n = 1/n$ also Q-sublinear $O(1/\varepsilon)$
- $x_n = 1/n^2$ $O(1/\sqrt{\varepsilon})$



- $x_n = .9^n$ is Q-linear $O(-\log(\varepsilon))$



⚠ x_n sublinear, y_n linear then asymptotically $y_n\to 0$ faster than x_n $\left(\lim_{n\to\infty} y_n/x_n = 0\right)$

but for small n, maybe $x_n < y_n$

ex: $x_n = 10^{-5} \cdot 1/n^2$

$y_n = (1-10^{-10})^n$

constants matter!

linear convergence with $c = 1-10^{-10}$ is terrible

but with $c = 1/2$ it's very good.

$x_n = (.9)^{2^n}$ is Q-quadratic

$O(\log(-\log(\varepsilon)))$

Final accuracy doesn't really matter it's so fast

Update (after video)

Book's definition isn't great. Here's a better one:

For a sequence $x_n \to x$, define $e_n = |x_n - x|$

(so we require $e_n \to 0$)

① If $\exists\, C > 0$ and $\boxed{\alpha > 1}$ such that $\lim_{n\to\infty} \frac{e_{n+1}}{e_n^{\alpha}} = C$

(or, if $\exists N$ s.t. $(\forall n \geq N)$ $e_{n+1}/e_n^{\alpha} \leq C$)

then we say x_n converges to x at order α

Ex: $e_{n+1}/e_n^2 \leq C$ is quadratic convergence

We do not require $C < 1$

If $C > 1$, eg. $e_{n+1} = e_n^2$, then (e_n) could diverge if e_1 is large

if $e_1 > 1$, $e_n \to \infty$
$e_1 < 1$, $e_n \to 0$

... but recall, we assumed $e_n \to 0$ ← recall $p > 1$ for now

When e_n is sufficiently small (in particular, $e_n^{p-1} \cdot C < 1$)

then (e_n) becomes a strictly monotonic decreasing sequence

② $\boxed{\alpha = 1}$ (linear) case is a bit different

We need $\lim_{n\to\infty} e_{n+1}/e_n = C$ (or $\exists N$ s.t. $(\forall n \geq N)$, $e_{n+1}/e_n \leq C$)

and need $\boxed{C < 1}$ (and $C > 0$)
(this is unlike $\alpha > 1$ cases)] Book doesn't make this clear

By requiring $0 < C < 1$, we no longer need to assume $e_n \to 0$... this follows automatically.

* unlike the other cases, for linear convergence, the value of C matters a lot. Smaller is better

3a) Superlinear if $\lim_{n\to\infty} \frac{e_{n+1}}{e_n} = 0$ ex. $e_n = 1/n^n$, or any $\alpha > 1$ convergence

3b) Sublinear if $\lim_{n\to\infty} \frac{e_{n+1}}{e_n} \geq 1$ ex. $e_n = 1/n^2$

* The book allows for $\alpha < 1$ but I've <u>never</u> seen this, so don't worry about this case

Ex: prove $e_n = \frac{1}{n^\beta}$ is sublinear (for any fixed $\beta > 0$)

proof: $\lim \left(\frac{e_{n+1}}{e_n} = \frac{n^\beta}{(n+1)^\beta} \right) = 1$ (you can show via L'Hôpital's rule ...)

or quick/slick proof: $f(q) = q^\beta$ is a continuous function for $q \geq 0$

So $\lim_{n\to\infty} f\left(\frac{n}{n+1}\right) = f\left(\lim_{n\to\infty} \frac{n}{n+1}\right)$ by continuity of f

$= f(1)$

$= 1^\beta = 1$

(sometimes this exact property is called "sequential continuity")