Ch 1: Condition Number of a Problem

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cf. Driscoll + Braun

Evaluate $f(x)$. Ex: $f(x) = x+1$ — conceptually... could be a scientific problem, solving a PDE, predicting weather

Conditioning is about the sensitivity of the output to perturbations on the input (e.g. rounding error)

Suppose input x is perturbed to $\tilde{x} := (1+\varepsilon)x$ i.e. ε is relative error on input

look at the rel. error $\dfrac{|f(\tilde{x}) - f(x)|}{|f(x)|}$ } relative error of output

Ex: $f(x) = x+1$, $\dfrac{|\tilde{x}+1 - (x+1)|}{|x+1|} = \dfrac{|x(1+\varepsilon)+1 - (x+1)|}{|x+1|}$

$= \left|\dfrac{\varepsilon \cdot x}{x+1}\right|$, if $|x+1|$ is small, lot of rel. error

Sensitivity: do small relative changes in input lead to small relative changes in output? well-conditioned

large in output? ill-conditioned

Conditioning is a property of f (and x)

NOT the implementation

best we could do

$$\frac{|f(x) - f(\tilde{x})|}{|f(x)|} \Big/ \left(\frac{|x-\tilde{x}|}{|x|} = |\varepsilon|\right) \qquad \tilde{x} = x(1+\varepsilon)$$

Def. The relative condition number of f, at x, is

$$\kappa_f(x) := \left|\frac{x}{f(x)} \cdot f'(x)\right|$$

Why? $\kappa_f(x) = \lim\limits_{\varepsilon \to 0} \left(\dfrac{\text{rel. change in output}}{\text{rel. change in input}} = \left| \dfrac{\frac{f(\tilde{x})-f(x)}{f(x)}}{\underbrace{\frac{\tilde{x}-x}{x}}_{=\varepsilon}} \right| \right)$

$$= \lim_{\varepsilon \to 0} \left| \frac{f(x+\varepsilon x) - f(x)}{\varepsilon \cdot f(x)} \right|$$

$$= \lim_{\varepsilon \to 0} \left| \left(\frac{f(x+\varepsilon x) - f(x)}{\varepsilon x} \right) \cdot \frac{x}{f(x)} \right| = \left| f'(x) \cdot \frac{x}{f(x)} \right|.$$

Rules: $h(x) = f(g(x))$

then $\kappa_h(x) = \kappa_f(g(x)) \cdot \kappa_g(x)$

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(not on video)

⚠ The relative condition number is not the same as relative error (though they are related)

$$\kappa_f(x) = \lim_{\tilde{x}\to x} \left|\frac{f(x)-f(\tilde{x})}{f(x)}\right| \Big/ \left|\frac{x-\tilde{x}}{x}\right|$$

→ relative error (of output)

→ relative error (of input)

We can also define an absolute condition number based on absolute error

$$\kappa_f^{\text{absolute}}(x) = \lim_{\tilde{x}\to x} \frac{|f(x)-f(\tilde{x})|}{|x-\tilde{x}|}$$

→ absolute error (of output)

→ absolute error (of input)

(i.e., absolute condition number is just the slope!
$\kappa_f^{abs}(x) = |f'(x)|$)

Rule-of-thumb interpretation of relative condition number

If ε is small, $\left|\frac{f(\tilde{x})-f(x)}{f(x)}\right| \approx \kappa_f(x)\cdot\varepsilon$

$\kappa_f(x) \to 10^4$, $\varepsilon \to 10^{-16}$ --- 10^{-12}

12 digits

* $\boxed{\log_{10}(\kappa_f(x)) \text{ is \# of digits we'll likely lose}}$ (no matter how good the algorithm is)

Students ask... is there a precise definition of well-conditioned vs ill-conditioned?

Answer: No. $\kappa = 10$ is definitely "well-conditioned"

$\kappa = 10^{10}$ is definitely "ill-conditioned"

$\kappa = 10^5$ is less clear, depends on context, or say "Somewhat ill-conditioned"

Added after video for additional clarification

Aside: what if x is a vector, $\vec{x}$? $\vec{x}\in\mathbb{R}^d$

Less canonical, but one approach is to loop over all condition numbers and take max,

i.e. $$\kappa_f(\vec{x}) = \max_{i=1,2,\ldots,d} \left|\frac{x_i}{f(\vec{x})}\cdot\frac{\partial}{\partial x_i}f(\vec{x})\right|$$

SUPPLEMENT

(not in video / lecture)

If we ask about "digits of precision", this is a relative notion.

ex: to approximate π, 3.14 has 3 digits of precision

(whereas the absolute error is 10^{-2}, i.e., to the 2^{nd} decimal place)

to approximate $100 \cdot \pi$, 314 has 3 digits of precision

(but absolute error is 10^{-0}, i.e., no decimal places)

Usually we care about relative errors

A Cheetah can run 60 mph ± 10 mph

The speed of light is 670,616,629 ± 10 mph

it usually doesn't make sense to ask for a fixed absolute error for all problems

i.e. relative error adapts to the situation

$$K = \frac{\text{rel. error output}}{\text{rel. error input}}$$

So...

rel. error output $= K \cdot$ (rel. error input) — e.g. ε_m

Back to our ~~rule of thumb~~:

$$\underbrace{\frac{\overbrace{|f(x) - (\text{estimate})|}^{\text{absolute error}}}{|f(x)|}}_{\text{relative error}} \approx \varepsilon_{machine} \cdot \underbrace{K_f(x)}_{\text{relative condition \#}}$$

So $\varepsilon_{machine} = 10^{-16}$

$K_f = 10^{3}$

then relative error $\approx 10^{-16+3} = 10^{-13}$

(we "lost" 3 digits)

To empirically determine how many digits you lost,

think of ℓ = digits lost

so $K = 10^{\ell}$, and let $\varepsilon_{machine} = 10^{-16}$

then (relative error) $= 10^{16-\ell}$

or $\log_{10}(\text{rel. error}) = 16 - \ell$

so $\log_{10}(\text{rel. error})$ = correct digits and $16 - \log_{10}(\text{rel. error}) = \ell$ = # digits you "lost"