Ch 1: Convergence Rates

Friday, August 29, 2025 9:28 AM

Book's definition isn't great. Here's a better one:

For a sequence $x_n \to x$, define $e_n = |x_n - x|$

(so we $\underline{\text{require}}$ $e_n \to 0$)

See Ch1_RatesOfConvergence.ipynb

① If $\exists\ C > 0$ and $\boxed{\alpha > 1}$ such that $\lim\limits_{n\to\infty} \dfrac{e_{n+1}}{e_n^{\alpha}} = C$

(or, if $\exists N$ s.t. $(\forall n \geq N)$ $e_{n+1}/e_n^{\alpha} \leq C$)

then we say x_n converges to x at order α

$\underline{\text{Ex}}$: $e_{n+1}/e_n^2 \leq C$ is quadratic convergence

We do not require $C < 1$

If $C > 1$, eg. $e_{n+1} = e_n^2$, then (e_n) could diverge if e_1 is large

if $e_1 > 1$, $e_n \to \infty$

$e_1 < 1$, $e_n \to 0$

... but recall, we $\underline{\text{assumed}}$ $e_n \to 0$ ← recall $p > 1$ for now

When e_n is sufficiently small (in particular, $e_n^{p-1} \cdot C < 1$)

then (e_n) becomes a strictly monotonic decreasing sequence

② $\boxed{\alpha = 1}$ (linear) case is a bit different

We need $\lim\limits_{n\to\infty} e_{n+1}/e_n = C$ (or $\exists N$ s.t. $(\forall n \geq N)$, $e_{n+1}/e_n \leq C$)

and need $\boxed{C < 1}$ (and $C > 0$) (this is unlike $\alpha > 1$ cases)] Book doesn't make this clear

* unlike the other cases, for linear convergence, the value of C matters a lot. Smaller is better

By requiring $0 < C < 1$, we no longer need to assume $e_n \to 0$ ... this follows automatically.

③a Superlinear if $\lim\limits_{n\to\infty} \dfrac{e_{n+1}}{e_n} = 0$ ex. $e_n = 1/n^n$, or any $\underline{\alpha > 1}$ convergence

③b Sublinear if $\lim\limits_{n\to\infty} \dfrac{e_{n+1}}{e_n} \geq 1$ ex. $e_n = 1/n^2$

* The book allows for $\alpha < 1$ but I've $\underline{\text{never}}$ seen this, so don't worry about this case

$\underline{\text{Ex}}$: prove $e_n = \dfrac{1}{n^\beta}$ is sublinear (for any fixed $\beta > 0$)

$\underline{\text{proof}}$: $\lim \left(\dfrac{e_{n+1}}{e_n} = \dfrac{n^\beta}{(n+1)^\beta} \right) = 1$ (you can show via L'Hôpital's rule ...)

or quick/slick proof: $f(g) = g^\beta$ is a continuous function for $g \geq 0$

So $\lim\limits_{n\to\infty} f\left(\frac{n}{n+1}\right) = f\left(\lim\limits_{n\to\infty} \frac{n}{n+1}\right)$ by continuity of f

$= f(1)$

$= 1^\beta = 1$

(Sometimes this exact property is called "sequential continuity")

Exercises 6, 7 in Section 2.4 of Burden and Faires get this wrong.

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Supplemental

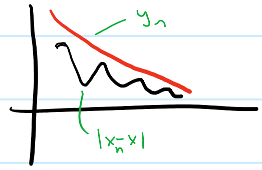
skip in lecture

Note: this is sometimes called Q-convergence (ex: $\alpha=1, c<1$ is Q-linear convergence), as it involves a Quotient

Sometimes we use a weaker notion, R-convergence (R for root), meaning

$x_n \to x$ R-linearly if $\exists (y_n)$ with $|x_n - x| \le y_n$ and $y_n \to 0$ Q-linearly

i.e., for R-convergence, error might actually go up! but trend is still correct.



Examples

* Exercises 6 and 7 in Section 2.4 Burden + Faires are 100% WRONG (they claim $1/n$ is linear convergence)

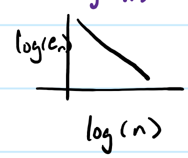
* $e_n = 1/\sqrt{n}$ so $e_n \to 0$ slowly. This is Q-sublinear

  to reach $e_n < \varepsilon$ takes $O(1/\varepsilon^2)$ iterations.

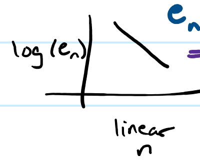
* $e_n = 1/n$ also Q-sublinear $O(1/\varepsilon)$
* $e_n = 1/n^2$ $O(1/\sqrt{\varepsilon})$

$e_n = n^{\alpha}$

$\Rightarrow \log(e_n) = \alpha \cdot \log(n)$



* $e_n = .9^n$ is Q-linear $O(-\log(\varepsilon))$



$e_n = c^n$

$\Rightarrow \log(e_n) = n \cdot c$

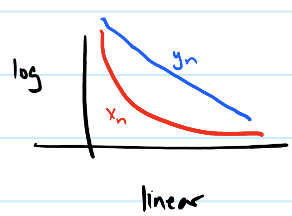
⚠ x_n sublinear, y_n linear, then asymptotically, $y_n \to 0$ faster than x_n ($\lim_{n\to\infty} y_n/x_n = 0$)

but for small n, maybe $x_n < y_n$

⚠ In matlab, or python, don't do the change of variables and then plot, rather tell the software to make axes logarithmic

ex: $x_n = 10^{-5} \cdot 1/n^2$

$y_n = (1 - 10^{-10})^n$



constants matter! linear convergence with $c = 1 - 10^{-10}$ is terrible but with $c = 1/2$ it's very good.

* $e_n = (.9)^{2^n}$ is Q-quadratic

  $O(\log(-\log(\varepsilon)))$

  Final accuracy doesn't reall matter it's so fast

Misc.

e_0 we often say "e - naught" instead of "e - zero"

"naught" (or "nought" in British English) is simply a synonym for "zero"

It's pronounced just like "knot" or "not"