Condition Number of a Problem

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Evaluate $f(x)$. Ex: $f(x) = x+1$

Conditioning = sensitivity to perturbations in the input

Evaluate f at $\tilde{x} = fl(x)$, $\tilde{x} = x\cdot(1+\varepsilon)$, $|\varepsilon| < \varepsilon_{mach}/2$

look at the rel. error $\dfrac{|f(\tilde{x}) - f(x)|}{|f(x)|}$

Ex: $f(x) = x+1$, $\dfrac{|\tilde{x}+1 - (x+1)|}{|x+1|} = \dfrac{|x(1+\varepsilon)+1 - (x+1)|}{|x+1|}$

$= \left|\dfrac{\varepsilon\cdot x}{x+1}\right|$, if $|x+1|$ is small, lot of rel. error

$x = 1 - \varepsilon_{m/2}$

Sensitivity: do small changes in input lead to small in output? well-conditioned

large in output? ill-conditioned

Conditioning is a property of f

NOT the implementation

best we could do

$\tilde{x} = x(1+\varepsilon)$

$\dfrac{|f(x) - f(\tilde{x})|}{|f(x)|} \Bigg/ \left(\dfrac{|x-\tilde{x}|}{|x|} = |\varepsilon|\right)$

Def Condition number of f at x is

$$K_f(x) = \lim_{\varepsilon\to 0} \frac{|f(x) - f(x+\varepsilon x)|}{|\varepsilon\cdot f(x)|} = \lim_{\varepsilon\to 0}\left|\frac{f(x+\varepsilon x) - f(x)}{\varepsilon x}\cdot\frac{x}{f(x)}\right|$$

$$= \left|\frac{x}{f(x)}\cdot f'(x)\right| \quad *$$

Rules: $h(x) = f(g(x))$

then $K_h(x) = K_f(g(x))\cdot K_g(x)$

If ε is small, $\left| \frac{f(\tilde{x})-f(x)}{f(x)} \right| \approx K_f(x) \cdot \varepsilon$

10^4

10^{-16} --- 10^{-12}

12 digits

$\log_{10}(K_f(x))$ is # of digits we'll likely lose